Unprovabiliy of Continuation-Passing Style Transformation in Lambek Calculus[1]

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Outline

1. Background: incremental parsing
2. Continuation in lambda calculus
3. Categorial grammar and lambek calculus
4. Barker's CPS transformation is provable.
5. Plotkin's CPS transformation is unprovable.
6. Type-restricted CPS transformation is provable.
7. Formal proof in Isabelle/HOL.

Introduction
Scrambling

- Categorial Grammar is Context-Free Grammar (CFG).
- However, not all of natural language is CFG.

[4] Continuation Passing Style

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- Some grammar rule was introduced to be richer than CFG.
- For example, the scrambling is not a CFG rule.

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- For example, the scrambling is not a CFG rule.
- The haphazard introduction potentially exceeds the original grammar class.
- Our target is the extension of type-raising; CPS[4] transformation.

$$
\dfrac{\textit{that} : \mathrm{SBAR}/\mathrm{S} \qquad \dfrac{\textit{cat} : \mathrm{NP} \qquad \textit{walks} : \mathrm{NP}\backslash\mathrm{S}}{\textit{cat walks} : \mathrm{S}} <}{\textit{that cat walks} : \mathrm{SBAR}} >
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[4]Continuation Passing Style

- In “Bridges and Gaps between Formal and Computational Linguistics”, we give an implementation of incremental parser of CCG.
- The parser relies on *Q-combinator*. $(X \to Y) \to ((Y \to Z) \to (X \to Z))$
- Is the combinator rule safe grammar rule?

Continuation and QCCG

Let f be a function $X \to Y$. After the execution of f, we expect the value of Y is consumed by another function $g : Y \to Z$.

$$g(f(x)) \qquad (f : X \to Y)$$

Instead of passing the value, we consider passing g called a *continuation* to f. This is called CPS-transformation.

$$f^*(g, x) \qquad (f^* : (Y \to Z) \to (X \to Z))$$

Example (Continuations in $0 + 2$)

- The call-by-value interpreter evaluates '$\mathrm{add}\ 1\ 2$' as follows.
- The continuation of Step 1 is Step 2–3.
- Generally, the continuation of Step i is Step $(i + 1)$–3.
- This form is called Direct Style (DS).

$$\underbrace{\mathrm{add}}_{3}\ \underbrace{1}_{2}\ \underbrace{2}_{1}$$

1. Evaluate the integer 2
2. Evaluate the integer 1.
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Example (Continuation Passing Style of $1+2$)

Let add' be transformed to CPS recursively,
then the following is the CPS of $1+2$

$$\lambda k.(\lambda l.(\lambda s.s\,\mathrm{add}')(\lambda h.(\lambda t.t1)(\lambda i.hil)))(\lambda m.(\lambda u.u2)(\lambda n.mnk))$$

1. $\lambda n.mnk$ is the continuation of 2.

Preliminaries

Conitnuations in Lambda Calculus

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2. $\lambda i.hil$ is the continuation of 1.
3. $\lambda h.(\lambda t.t1)(\lambda i.hil)$ is the continuation of add'.

Definition (Plotkin's CPS transformation, Plotkin (1975))

$$[\![x]\!] \equiv \lambda k.kx \qquad [\![\lambda x.M]\!] \equiv \lambda k.k(\lambda x.[\![M]\!]) \qquad [\![MN]\!] \equiv \lambda k.[\![M]\!](\lambda m.[\![N]\!](\lambda n.mnk))$$

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Definition (Barker's CPS transformation, Barker und Shan (2014))

$$\{\!\!\{a\}\!\!\} \equiv \lambda k.ka \qquad \{\!\!\{MN\}\!\!\} \equiv \lambda k.\{\!\!\{M\}\!\!\}(\lambda m.\{\!\!\{N\}\!\!\}(\lambda n.kmn))$$

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Plotkin's CPS transformation vs Barker's CPS transformation ?

We employ a category instead of the Part-of-Speech (POS) tags.

Definition (Category)

- The category consists of an atomic category and functional categories defined inductively as follows.
- The functional categories are the biting (left/ right) relations.
- The left functional is X/Y and the right functional is $X\backslash Y$.

Definition (Categorial Grammar (CG), Steedman (2001))

The lexicon is a map from a word to a category.
The parsing tree is built with the two rules.

$$X/Y \quad Y \Rightarrow X \qquad\qquad X \quad X\backslash Y \Rightarrow Y$$

Definition (Lambek Calculus LC, Lambek (1958); Kanazawa (1992))

$$\frac{}{\alpha \vdash \alpha}\ \mathsf{Id}$$

Preliminaries

Lambek Calculus

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$$\frac{}{\alpha \vdash \alpha}\ \mathsf{Id} \qquad \frac{\Sigma \vdash \alpha \qquad \Gamma, \alpha, \Delta \vdash \beta}{\Gamma, \Sigma, \Delta \vdash \beta}\ \mathsf{Cut}$$

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Example (Application of CG in LC)

LC is the mathematical formalization of categorial grammar (CG). Hence, some of CCG rules are provable in this system. For instance, the application is provable.

$$\frac{\alpha \vdash \alpha \qquad \beta \vdash \beta}{\alpha, \alpha\backslash\beta \vdash \beta} \backslash \boldsymbol{I}$$

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Equivalence of language classes

Acceptable language class generated with a grammar rule is the same as LC's if the grammar rule is probable in LC.

LC is two-directional. Thus, the CPS transformation is not unique in LC.

Definition (Type of Barker's CPS tr., Barker und Shan (2014))

$$\langle\!\langle X \rangle\!\rangle_A = (X \to A) \to A$$

Definition (Barker's CPS Transfromation in LC)

$$\alpha \vdash \beta/(\alpha\backslash\beta) \qquad \alpha \vdash (\beta/\alpha)\backslash\beta$$

- $NP \Rightarrow S/(NP\backslash S)$ is a type-raising rule.
- It switches the biting relation from 'NP $NP\backslash S$' to '$S/(NP\backslash S)$ $NP\backslash S$'.

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Provability of type-raising (Barker's CPS Transformation)

The sequent $\alpha \vdash \beta/(\alpha\backslash\beta)$ and $\alpha \vdash (\beta/\alpha)\backslash\beta$ are provable in LC.

Proof.

The following is a proof of the type-raising rules in LC.

$$\cfrac{\cfrac{\alpha \vdash \alpha \qquad \beta \vdash \beta}{\alpha, \alpha\backslash\beta \vdash \beta}\,\backslash I}{\alpha \vdash \beta/(\alpha\backslash\beta)}\,I/ \qquad\qquad \cfrac{\cfrac{\alpha \vdash \alpha \qquad \beta \vdash \beta}{\alpha/\beta, \beta \vdash \beta}\,/I}{\alpha \vdash (\beta/\alpha)\backslash\beta}\,I\backslash$$

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Definition (Plotkin's CPS transformation in LC)

$$\frac{\alpha \in A}{\alpha \xrightharpoondown{\gamma} \alpha}$$

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Definition (Plotkin's CPS transformation in LC)

$$\frac{\alpha \in A}{\alpha \xrightarrow{\gamma} \alpha} \qquad \frac{\alpha \xrightarrow{\gamma} \tau \qquad \beta \xrightarrow{\gamma} \upsilon}{\alpha \backslash \beta \xrightarrow{\gamma} \tau \backslash \upsilon}$$

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Theorem (Unprovability of Plotkin's CPS transformation in LC)

There exists an unprovable sequent $\varphi \vdash \psi$ *even if* $\varphi \xrightarrow{\delta} \psi$.

$$\exists \varphi \psi \delta. \quad \varphi \xrightarrow{\delta} \psi \land \varphi \nvdash \psi$$

Sketch of Proof.

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- We run the proof search exhaustively.
- All leaves are unprovable sequent, *e.g.*, $\quad \vdash \alpha$.

$\square$

Provability of CPS

Plotkin's CPS Transformation

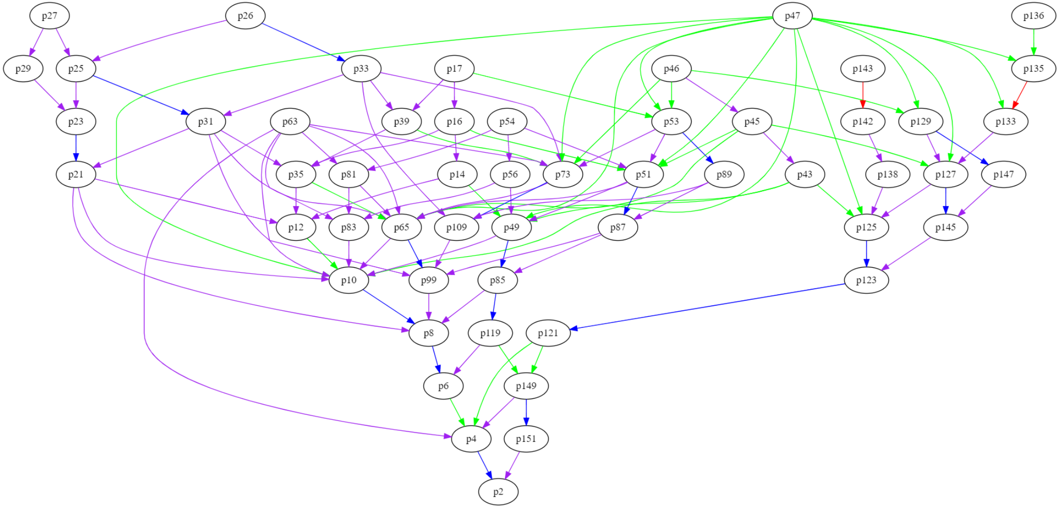

Figure: Proof of unprovability

Type-restricted CPS Transformation

Why is Plotkin's CPS-transformation unprovable?

- Lambek Calculus is intuitionistic.
- $(\alpha \to \beta) \to \gamma$ is $\neg(\neg\alpha \vee \beta) \vee \gamma$, that is, $(\neg\neg\alpha \wedge \neg\beta) \vee \gamma$.

Type-restricted CPS Transformation

Definition (Recap. Plotkin's CPS transformation in LC)

$$
\frac{\alpha \in A}{\alpha \xrightharpoondown{\gamma} \alpha}
\qquad
\frac{\alpha \xrightharpoondown{\gamma} \tau \qquad \beta \xrightarrow{\gamma} \upsilon}{\alpha\backslash\beta \xrightharpoondown{\gamma} \tau\backslash\upsilon}
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\frac{\alpha \xrightharpoondown{\gamma} \tau \qquad \beta \xrightarrow{\gamma} \upsilon}{\beta/\alpha \xrightharpoondown{\gamma} \upsilon/\tau}
$$

$$
\frac{\alpha \xrightharpoondown{\gamma} \tau}{\alpha \xrightarrow{\gamma} \gamma/(\tau\backslash\gamma)}
\qquad
\frac{\alpha \xrightharpoondown{\gamma} \tau}{\alpha \xrightarrow{\gamma} (\gamma/\tau)\backslash\gamma}
$$

Type-restricted CPS Transformation

We remove nested implications in presumptions.

Definition (Type-restricted CPS transformation in LC)

$$\frac{\alpha \in \mathsf{A}}{\alpha \stackrel{\gamma}{\not\rightharpoonup} \alpha} \qquad \frac{\alpha \in \mathsf{A} \qquad \beta \stackrel{\gamma}{\nrightarrow} \upsilon}{\alpha\backslash\beta \stackrel{\gamma}{\not\rightharpoonup} \alpha\backslash\upsilon} \qquad \frac{\alpha \in \mathsf{A} \qquad \beta \stackrel{\gamma}{\nrightarrow} \upsilon}{\beta/\alpha \stackrel{\gamma}{\not\rightharpoonup} \upsilon/\alpha}$$

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Type-restricted CPS Transformation is provable in LC.
The proof is available in proceedings.

Bonus: Formalization in Isabelle/HOL

Provability and Unprovability of CPS

- We need to search all the possible derivation-paths.
- The search space is exponentially grown.
- We translate the proof to the formal proof in Isabelle/ HOL.
- The fragment of formal proof is available in the proceedings.

```
datatype 'a category =
  Atomic 'a ("^")
  | RightFunctional "'a category" "'a category" (infix "→" 60)
  | LeftFunctional "'a category" "'a category" (infix "←" 60)

inductive LC::
  "'a category list ⇒ 'a category ⇒ bool" (infix "⊢" 55)
  where
    r1: "([x] ⊢ x)"
  | r2: "(X @ [x] ⊢ y) ⟹ (X ⊢ y←x)"
  | r3: "([x] @ X ⊢ y) ⟹ (X ⊢ x→y)"
  | r4: "⟦(Y ⊢ y); (X @ [x] @ Z ⊢ z)⟧ ⟹ (X @ Y @ [y→x] @ Z ⊢ z)"
  | r5: "⟦(X @ [x] @ Z ⊢z); (Y ⊢ y)⟧ ⟹ (X @ [x←y] @ Y @ Z ⊢ z)"
```

```
lemma
  assumes "distinct [a,b,c,d]"
  shows "¬(
    [^a←(^b←^c)] ⊢^d←(((^d←(^a→^d))←((^d←(^b→^d))←^c))→^d)
  )"
proof -
have p17: "¬([^a,^d]⊢^a)"
  apply auto apply(subst (asm) LC.simps) apply auto
  by (simp add: Cons_eq_append_conv)+

...

show p2: "¬([^a← ^b,^a→ ^c]⊢^c← ^b)"
  apply auto apply(subst(asm)LC.simps)
  apply auto
  apply (simp_all add: Cons_eq_append_conv)+
  using p10 p23 p4 p25 p9 by fastforce+
qed
```

```
theorem rCPS_transformation:
  fixes a x :: "'a category"
  shows "⋀y. rCPS' a x y ⟹ [x]⊢y"
    and "⋀y. rCPS a x y ⟹ [x]⊢y"
proof (induct x)
case (Atomic x)
{case 1 show ?case using "1.prems"  rCPS'.cases identity by blast}
{case 2 have "⋀y. rCPS' a (^x) y ⟹ [^x]⊢y"
        using "2.prems" rCPS'.cases identity by blast ...
next case (LeftFunctional x1 x2) ...
next case (RightFunctional x1 x2) ...
thus "⋀y. rCPS a (x1→x2) y ⟹ [x1→x2]⊢y" by (metis rCPS.cases)
qed
```

Conclusion

- We investigated Barker's and Plotkin's CPS transformations on LC.

Table: Comparison: CPS transformations

	Lambek calculus	Lambda calculus
Plotkin's CPS tr.	unprovable	any lambda terms
Barker's CPS tr.	provable	no lambda abstraction
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- Otherwise, Plotkin's CPS transformation was unprovable.
- We proposed a syntactic restriction on CPS transformation, which ensured provability in LC.

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