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Cycles in dynamic economic modelling

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Abstract

This paper offers a unified perspective of the analytical detection of Hopf bifurcation, which is a crucial tool in dynamic economic modelling. We clarify the relations between stability theorems and the notions of simple and general Hopf bifurcations. A Liénard–Chipart-type theorem for detecting bifurcations, which appears of considerable usefulness in applications, is proved. Subsequently we show how to use the notions of 'stability boundary' and 'bifurcation boundary', providing a new, surprisingly straightforward, tool for detecting bifurcations in economics. An economic illustration is given by two models with time-delay: a Solow-type demo-economic model and a Kaleckian extension of the Lotka–Volterra–Goodwin model.

JEL classifications: E0

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1. Introduction

Persistent oscillations are one of the most ubiquitous forms by which economic phenomena may be observed. Thus, it does not come as a surprise that a principal aim of the scholars in the fields of economic growth and business cycle, ranging from the 'endogenous growth theory' to the 'real business cycle theory', and to Goodwinian and Keynesian macrodynamics, is the search for mechanisms leading

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to persistent oscillations of the economy. From the standpoint of applied modellers there is one mathematical notion of cycle which is of major relevance (independent of the physical, biological or economic actual context): that of (asymptotically) stable limit cycles. This is one of the reasons why the well known conservative Lotka–Volterra cycle, despite its indisputable role as paradigm of non-linear oscillations in applied sciences, is not a 'good' fluctuation model.

Limit cycles are the simplest non-linear phenomena, e.g. they are the simplest example of how the interaction between economic forces may compel a system to abandon its steady state and start to steadily oscillate. Moreover, in many cases limit cycles are the door through which more complex oscillations patterns arise (Guckenheimer and Holmes, 1983).

The detection of stable oscillations, e.g. stable limit cycles, in continuous-time systems, is intimately related with the notion of Hopf (or Andronov-Hopf) bifurcation. The Hopf bifurcation is a 'fundamental' model for such oscillations: it is '…the generic mathematical model of the phenomenon how a real world system depending on a parameter is losing the stability of an equilibrium state as the parameter is varied, giving rise to small, stable or unstable, oscillations' (Farkas, 1995, pp. 399). In other words the Hopf bifurcation is thus the typical way in which instability, mainly observable as oscillations, arises in physical, economical and social processes. Why is the notion of Hopf bifurcation so important in economics? There are at least three reasons. First, it is always the outcome of a fully endogenous interaction between (non-linear) economic forces. Second, it is a 'local' bifurcation, thus much in spirit with the common belief of our science by which economic systems are generally close to their equilibrium state. Third, because it implies 'local' oscillations, which are the normal route through which disequilibrium manifests itself when the equilibrating forces operating in the economy are relaxed (e.g. the adjustment process of a Walrasian market). For instance, when oscillations persist in a market normally in equilibrium (in the absence of stochastic and seasonal perturbations), it is very likely these oscillations are the outcome of a Hopf bifurcation.

Contrary to the other main tool for the detection of oscillations, the Poincaré–Bendixson theorem, the applicability of which is confined to planar systems, the Hopf theorem can, in principle, establish the existence of (local) periodic behaviours in any dimension. Bifurcation theory, especially the Hopf theorem, is considered (Semmler, 1994) a major factor behind the recent advances in the area of both optimised and non-optimised business cycle models.

Given an n-dimensional system tuned by a parameter μ, and having an isolated equilibrium E_1, a Hopf bifurcation occurs at E_1 when a simple pair of complex eigenvalues of the linearised system crosses the imaginary axis. To analytically detect a Hopf bifurcation one therefore has to investigate the behaviour of eigenvalues as functions of μ. As n increases, 'direct' detection of the bifurcation, via explicit computation of the bifurcating eigenvalues, is already awkward for $n=3$ and impossible for $n>4$. Therefore, though some important problems have been solved by explicitly finding eigenvalues,[1] resorting to 'indirect' methods to detect the bifurcation is often unavoidable. Indirect methods often exploit the Routh–Hurwitz (RH) stability theorem (instances with $n=3$ or $n=4$ are Farkas and Kotsis, 1992; Asada and Semmler, 1995; Fanti and Manfredi, 1998). Important questions are then as follows: what is the relation between the Hopf bifurcation and the RH criterion? Can stability criteria be used to detect Hopf bifurcations in systems of any dimension? In effect also indirect methods lead to computational difficulties, as proved by the fact that most macro-economic applications of the Hopf theorem have considered oversimplified models of very low dimension.[2]

These questions have been clarified by Liu (1994), who distinguishes between simple Hopf bifurcations (SHBs), occurring when a pair of complex eigenvalues crosses the imaginary axis while all other eigenvalues have negative real parts, and general Hopf bifurcations (GHBs), in which some other eigenvalues are on the right half plane. SHBs deal with traditional modelling approaches, in which persistent oscillations are investigated as the outcome of the destabilisation of a previously stable equilibrium. SHBs pervade dynamic economic theory: also the limit cycles recently discovered in optimal intertemporal neo-classical models (for instance Boldrin and Rustichini, 1994; Benhabib and Perli, 1994; Greiner and Semmler, 1996) appear through SHBs. We note, however, that such optimal cycles imply the occurrence of 'indeterminacy', which is a well-debated issue in the context of the recent neo-classical literature. Obviously, such a loss of 'determinacy' is a serious difficulty for neo-classical theory. When 'saddle-path stability' holds (the number of positive eigenvalues[3] is identical to the number of 'jump' variables), and therefore the 'indeterminacy' problem is avoided, cycles may only emerge through GHBs. In Fanti and Manfredi (2001) we demonstrate, however, that the GHBs arising in neo-classical models with optimising agents (for instance Wirl, 1997) are, in the final analysis, 'reducible' to SHBs. Thus in some cases such bifurcations can be detected via the apparatus developed in this paper.

It is the aim of the present paper to provide a unifying perspective of the analytical detection of SHBs, in order to make the Hopf theorem an operative tool at any dimensions.

First, by exploiting the Liénard–Chipart (LC) stability condition we extend Liu's theorem[4] by proving a LC-type result for the detection of SHBs which appears quite

[1] This was allowed by special forms of the Jacobian matrix. Instances are: (i) the optimal economic models with some rate of future discount, thanks to the zero trace of the Hamiltonian matrix (Dockner, 1985; Wirl, 1991; Dockner and Feichtinger, 1991), and (ii) the multisector neoclassical model (Gandolfo, 1996, ch. 25), where the Jacobian matrix is triangular.

[2] This difficulty is emphasised in the most influential textbooks in economic dynamics. Lorenz (1993, p. 101) state: '…the conditions for the existence of the bifurcation can be shown to be fulfilled without difficulty only in two and three dimensional cases. In higher dimensions the bifurcation values can often be calculated only by numerical algorithms' (see also Gabisch and Lorenz (1989, p. 166)). And Gandolfo (1996, pp. 478–479): '…also the existence part of the Hopf theorem often becomes analytically intractable for systems of dimension higher than the third… except in particular cases'.

[3] We recall that at least one real and positive eigenvalue always emerge in such systems, as inherited from the basic Hamiltonian problem.

[4] Liu (1994) has proved, in particular, a general 'RH-type' theorem for the detection of SHBs.

useful in applications, as it correspondingly reduces the number of conditions needed to detect an SHB. Its usefulness is evident at ‘intermediate’ dimensions (say $n=4$, 5, 6). For instance, we show that the detection of an SHB in a 4D system only needs the annihilation of a third-order (Routh–Hurwitz) determinant, which is surely a feasible task.

But in effect we can go further. We show that much simpler conditions are obtained if we consider systems obtained as parametric perturbations of a known stable system. In this case, which is quite frequent for modellers, who usually consider ‘hierarchical’ or ‘step by step’ complications of ‘basic’ stable models, all we need is to find the subset of the ‘stability boundary’ over which stability is lost due to the movements of a simple complex pair. We will argue that, whenever we start from an ‘initial’ parameter constellation μ_S in which the system is stable, then, in most cases (e.g. with the only requirement that we can rule out zero-eigenvalue bifurcations, which is a non-generic condition), the points belonging to the set $\Delta_{n-1}=0$ (Δ_{n-1} is the higher order RH determinant), are SHB points. The extent of this simplification for the detection of SHBs is amazing: we no longer need to evaluate a large number of RH determinants, but only one! The approach is especially fruitful for detecting SHBs in systems in which the ‘initial’ stable parameter constellation μ_S ‘naturally’ exists. Moreover, the analytical detection of the bifurcation may be simplified to a great extent when the aforementioned parametric perturbation does not affect the equilibria of the system but only their stability properties.

Fortunately, already in the very core of dynamic economic modelling we find noteworthy classes of problems that meet the previous requirements. In particular, three among these classes belong to the standard approach to ‘disequilibrium analysis’: (1) ‘traditional’ macro-economic models (e.g. open-economies in the IS-LM frame) in which the stability of equilibrium is investigated by assuming that quantities and/or prices adjust, according to Marshallian or Walrasian adjustment rules, in order to equilibrate demand and supply in various markets (e.g. goods, labour, money, financial); (2) optimal models in which, though the economic agents can be assumed to be in equilibrium at all times (e.g. they are on their optimal supply, demand or price schedules), the presence of risk and/or costs of adjustment, mathematically leads to the appearance of (approximately) optimal linear partial adjustment rule (typical of the behaviour of stock variables such as physical capital, financial wealth, durable goods, labour supply etc.), see Barnett et al. (1996); (3) delay models in which some economic variables enter the model with lagged, rather than current values (for instance in many famous macro-models current consumption also depends on past consumption or past income). In all these cases our results suggest that the existence of cycles also in high-dimensional systems can be investigated through the inspection of a unique function of parameters, in contrast with the common practice in literature. Since, in effect, also the first and second aforementioned classes of disequilibrium relationships imply, in some sense, a distributed lag function in the adjustment process, we focus here on the case of delay models.

An important class of delay models is that of distributed delay systems governed by Erlangian kernels. Such systems, which represent a broad and flexible class, are 'reducible' to (higher dimensional) ordinary differential equation (ODE) systems (MacDonald, 1989; Farkas and Kotsis, 1992). This new ODE system preserves the equilibria of the underlying unlagged system (actually this feature, e.g. preservation of equilibria, is a general feature of standard delay systems). If the unlagged system is stable, every point in the parameter space in which the delay parameter (say T) is set equal to zero, may be taken as the 'initial' stable parameter constellation μ_S, allowing the application of the aforementioned methodology based on the notion of 'stability boundary'.

These facts are illustrated by means of two noteworthy examples: (i) a delayed Solow (1956) model (ii) a 5D extension of Goodwin's (1967) model. These examples show how the notion of stability boundary may be applied to detect SHBs not only when the underlying unlagged model is stable (example (i)), but also in the case of neutral stability (example (ii)).

Areas of economic analysis which could substantially benefit from the results discussed here seem to be those of delayed systems and systems incorporating heterogeneity of agents. Erlangian lags allow a sound representation of two realistic elements, so far often neglected in economics mostly because of the involved analytical complexity: the heterogeneity of agents and their tendency to react to economic stimuli with different patterns of lag. The equivalence (Invernizzi and Medio, 1991) between a single 'representative' agent reacting along a continuous gamma-type lag and an indefinitely large number of agents reacting with different discrete lags whose lengths are randomly distributed among agents according to a gamma distribution, allow us to avoid the usual 'rough' dynamic aggregation (which is implicit in all models based on a unique fixed lag).

In many cases, starting from an existing 'roughly' aggregated model whose stability is known, we need to investigate whether stability is preserved under more general and realistic assumptions, such as heterogeneity and/or delayed responses of economic agents. The treatment discussed in this paper allow us to deal with this task, and to detect endogenous oscillations, with much less effort than usually believed, and often permit substantive economic interpretations.

The paper is organised as follows. Section 2 reviews the notions of SHBs and GHBs. Section 3 reports Liu's RH-type theorem and our LC extension for the detection of SHBs. Section 4 deals with stability boundaries. Economic illustrations are reported in Section 5. Conclusive remarks follow.

2. Simple vs. general Hopf bifurcations

A standard formulation of the Hopf theorem (for rigorous formulations see Guckenheimer and Holmes (1983) and Marsden and MacCracken (1976)) states that a dynamical system: $\dot{X}=f_\mu(X)$ (f of class C^∞), parametrised by the scalar parameter μ, and having an isolated equilibrium $E_0=(X_0(\mu))$, undergoes a Hopf bifurcation for $\mu=\mu_0$ (e.g. at: $X_0(\mu_0)$), if: (a1) a simple pair of purely imaginary eigenvalues $\lambda(\mu_0)$, $\bar{\lambda}(\mu_0)$ exist at (X_0, μ_0), and no other eigenvalues have zero real parts; (a2) the complex pair $\lambda(\mu)$, $\bar{\lambda}(\mu)$ which becomes purely imaginary at μ_0 satisfies the 'nonzero speed' condition:

$$\left[\frac{\mathrm{d}Re(\lambda(\mu))}{\mathrm{d}\mu}\right]_{\mu_0} \neq 0 \tag{1}$$

The detection of the bifurcation is, therefore, to be solved in two steps: first by checking for the existence of a simple pair of purely imaginary eigenvalues of the characteristic equation; second by applying the 'test of nonzero speed' (Eq. (1)). Notice that the nonzero speed condition is actually not necessary for having a Hopf bifurcation: it is purely a genericity requirement (Farkas, 1995).

In effect the above-mentioned formulation is not unique in the literature. There are also other textbooks formulations (Farkas, 1995) which are based on the stronger requirement that the $(n-2)$ 'non-bifurcating' eigenvalues have negative real part. The latter formulation, though apparently special if compared with the previous one, is the more relevant from the modelling viewpoint. The following Definition 1 is useful in order to organise the present discussion:

Definition 1. (Simple Hopf bifurcation)
A dynamical system with an equilibrium point E_1 undergoes an SHB at E_1 when a simple pair of complex conjugate eigenvalues of the Jacobian $J(E_1)$ crosses the imaginary axis from left to right, while all other eigenvalues have negative real parts.

The Definition 1 distinguishes SHB from, say, GHB, in which some eigenvalues may have positive real part. The SHB is the most relevant type of Hopf bifurcation in traditional approaches, as it deals with the case in which, for those parameter constellations for which the bifurcation is supercritical, the emerging periodic orbit will be asymptotically stable, and hence 'observable', physically or numerically (Liu, 1994). Moreover, it largely corresponds to the typical approach of modellers, who usually have in mind a 'basically stable' world. Indeed, given a non-linear system tuned by a parameter μ and having at least an equilibrium point E_1, modellers usually discuss first the condition for its local stability in terms of μ. Subsequently, since they also worry about the possibility that instability (mainly observable as oscillations) occurs, they look at those parameters, which may be responsible for stability losses. From this perspective an SHB is one of the simplest routes to instability of an equilibrium.

Finally, the notion of SHB is operative from the 'detectability' point of view: to detect an SHB at E_1 one just needs to check for stability losses of E_1 governed by 'movements' of a simple complex pair. This fills the bridge between Hopf bifurcations and the theorems for local stability such as the RH criterion. On the contrary, the detection of GHB is not necessarily related to the problem of stability.

3. Stability vs. bifurcations: detection of SHBs via RH-type theorems

3.1. Necessary conditions for stability; the Liénard–Chipart conditions

Let $P_J(\lambda)$ be a characteristic polynomial (CP) ascertaining the local stability of an equilibrium point E_1 of an n-dimensional dynamical system:

$$P_J(\lambda)=\det(J(E_1)-\lambda I)=\lambda^n+a_1(\mu)\lambda^{n-1}+\cdots+a_{n-1}(\mu)\lambda+a_n(\mu) \quad (2)$$

where $J(E_1)$ is the underlying Jacobian matrix. We write $a_i=a_i(\mu)$ to denote that the coefficients are functions of some (scalar) parameter μ. The equilibrium E_1 is said to be locally asymptotically stable (LAS) (alternatively $P_J(\lambda)$ is strictly Hurwitz (MacDonald, 1989, pp. 60)) if all its eigenvalues have negative real parts. The RH theorem (Gantmacher, 1959) gives a necessary and sufficient condition for the local stability of the polynomial $P_J(\lambda)$. Given the Routh table (Gantmacher, 1959), the RH test says that $P_J(\lambda)$ is LAS if (and only if) the determinants Δ_i of the first n principal minors of the Routh table are strictly positive.

A necessary condition for stability is $a_n>0$. As: $\Delta_n=a_n\Delta_{n-1}$, we only need to consider $(n-1)$ RH determinants. A more powerful necessary condition comes from the fundamental theorem of algebra: E_1 is LAS if all the coefficients a_i are strictly positive (Gantmacher, 1959). This gives a simple test for stability: if only one coefficient a_i is negative, then the system is unstable. Hence, the set of strictly Hurwitz CPs is a subset of the set of CPs with positive coefficients. This often neglected fact is quite useful. It shows that in the set of the parameter space in which the CP is LAS, all the coefficients a_i are 'forced' to be strictly positive. Obviously the converse is not true: positive coefficients are not sufficient to imply stability.

When some of the a_is are positive, then the n conditions of the RH theorem are no longer independent and the RH test may be replaced by the more 'economical' LC test, expressed by any one of the following four alternative versions (Gantmacher, 1959).

a $a_n>0$; $a_{n-2}>0$;...; $\Delta_1>0$; $\Delta_3>0$;...
b $a_n>0$; $a_{n-2}>0$;...; $\Delta_2>0$; $\Delta_4>0$;...
c. $a_n>0$; $a_{n-1}>0$; $a_{n-3}>0$;...; $\Delta_1>0$; $\Delta_3>0$;...
d $a_n>0$; $a_{n-1}>0$; $a_{n-3}>0$;...; $\Delta_2>0$; $\Delta_4>0$;...

A point relevant to our subsequent discussion is that the necessary condition $a_i>0$ for all i becomes an IFF condition for stability when all the eigenvalues are real[5]. This implies that, if we 'start' from a parameter constellation allowing the local stability of the equilibrium, then parametric perturbations which do not violate the condition $a_i>0$, can only lead to instability through 'movements' of complex pairs.

[5] If $a_i>0$ $\forall i$, then Descartes' rule says that if there are real roots, they are always negative. This implies that if a given CP has only real roots, then the positivity of its coefficients becomes an IFF condition for stability (rather than simply a necessary one).

3.2. Routh–Hurwitz and Liénard–Chipart-type theorems for the detection of SHBs

The relation between the RH theorem and the SHB has long been used by modellers. The following result by Liu (1994) is an RH-type theorem for the detection of an SHB which states this relation in formal terms:

Theorem 1. The conditions (a1), (a2) (Section 2) for an SHB at the point μ_0 are equivalent to the following conditions

$$\text{(b1)} \quad \Delta_1(\mu_0)>0, \quad \Delta_2(\mu_0)>0,\ldots, \ \Delta_{n-2}(\mu_0)>0, \quad \Delta_{n-1}(\mu_0)=0 \tag{3a}$$

$$\text{(b2)} \quad \left(\mathrm{d}\Delta_{n-1}/\mathrm{d}\mu\right)_{\mu=\mu_0}\neq 0 \tag{3b}$$

Liu's result fills the bridge between the body of theorems for the local stability of equilibria and the notion of SHB. A useful consequence of Liu's criterion is that we do not need to check for the presence of complex eigenvalues as a necessary condition for the bifurcation. This fact was repeatedly stressed by Lorenz (1993, 1994) who recommends, as regards the third order case, to study the sign of the discriminant of the resolvent formula of the characteristic equation. Such a Sisyphus fatigue (impossible at dimensions higher than four) is bypassed by Liu's theorem.

As pointed out in Section 3.1, the RH theorem is not the most economic IFF condition for stability, which is actually given by the LC conditions. Liu's theorem can be reformulated by replacing the 'structure' of RH conditions with the corresponding LC conditions. A simplified version assuming the strict positivity of all the coefficients a_i is the following:

Theorem 2. Provided $a_i>0$, the requirements (a1), (a2) for an SHB are equivalent to one or the other of the following two sets of conditions
Set (i):

$$\text{(c1)} \quad \Delta_2(\mu_0)>0, \ \Delta_4(\mu_0)>0,\ldots, \ \Delta_{n-3}(\mu_0)>0, \ \Delta_{n-1}(\mu_0)=0 \tag{4a}$$

$$\text{(c2)} \quad (d\Delta_{n-1}/d\mu)_{\mu=\mu_0}\neq 0 \tag{4b}$$

Set (ii)

$$\text{(c1 bis)} \quad \Delta_3(\mu_0)>0, \ \Delta_5(\mu_0)>0,\ldots, \ \Delta_{n-3}(\mu_0)>0, \ \Delta_{n-1}(\mu_0)=0 \tag{5a}$$

$$\text{(c2 bis)} \quad \left(\mathrm{d}\Delta_{n-1}/\mathrm{d}\mu\right)_{\mu=\mu_0}\neq 0 \tag{5b}$$

We omit the proof of Theorem 2. Theorem 2 considerably reduces the computations involved in the detection of SHBs, as shown in the following low-dimensional examples (Example 1).

Example 1. Let the necessary condition $a_i > 0$ for all i be satisfied. At dimension four (Fanti and Manfredi, 1998) an SHB occurs when:

(i) $\Delta_{n-1}(\mu) = \Delta_3(\mu) = 0$ (ii) $\left(d\Delta_3/d\mu\right)_{\mu=\mu_0} \neq 0$

At dimension five the LC test would require: (a) $\Delta_2 > 0$, $\Delta_4 > 0$, or alternatively (b) $\Delta_3 > 0$, $\Delta_5 > 0$. Theorem 2 (we choose the simpler set of conditions) thus leads to the conditions

$$\Delta_2(\mu) > 0,\ \Delta_4(\mu) = 0; \quad \left(d\Delta_4(\mu)/d\mu\right) \neq 0$$

At dimension six an SHB occurs when $\Delta_3(\mu) > 0$, $\Delta_5(\mu) > 0$; $(d\Delta_5(\mu)/d\mu) \neq 0$. And so on.

4. Boundaries of stability and SHB: 'physiology' of the bifurcation process

Though Theorem 2 leads, compared to Theorem 1, to simpler conditions, its usefulness decays as the dimension of the system increases. Fortunately, as far as SHBs are concerned, a powerful tool comes from the notion of (local) 'stability boundary'. Here we show how to use this notion to detect Hopf bifurcations.[6]

The notion of stability boundary allows us to define a tool which, under conditions often met in macro-economic models, allows SHBs to be detected with surprisingly little effort (see next section), even compared to Theorem 1 and Theorem 2, and also clarifies the distinction between SHBs and GHBs. As pointed out by MacDonald (1989, ch. 4), what really matters when we study the local stability of an equilibrium: '…is to find, in the space of the involved parameters, the curves, or surfaces, that bound regions of stability. Typically, one can start from some point in the parameter space in which stability is known to prevail. Then, since the eigenvalues depend continuously on the parameters, a change of stability can only happen by way of the appearance of a zero (real) eigenvalue ($\lambda = 0$) or of a purely imaginary pair $\lambda = \pm i\omega$.'

The previous remark extends, mutatis mutandis, to SHBs. The key-tools are the 'stability switch' indicators, i.e. functions which change their sign in correspondence of the stability boundary. The simplest indicator of the appearance of a zero eigenvalue is

$$a_n = (-1)^n \det(J(E_1)) = (-1) \prod_{j=1}^{n} \lambda_j.$$

Since we are not concerned with stability losses caused by real eigenvalues,[7] we

[6] This notion has been extensively discussed by MacDonald (1989), but only limited to the context of stability rather than bifurcation analysis.

[7] This leads to the so-called fold bifurcations (see Guckenheimer and Holmes, 1983, and, for economic applications, Lorenz, 1993).

always assume $a_n > 0$. The simplest indicator of the appearance of a purely imaginary pair is Δ_{n-1}, as shown by Orlando's formula (Gantmacher, 1959)[8]

$$\Delta_{n-1} = (-1)^{n(n-1)/2} \prod_{i=1}^{n} \prod_{j=i+1}^{n} (\lambda_i + \lambda_j) \tag{6}$$

We now show how the locus $\Delta_{n-1}=0$ becomes, under suitable conditions, a true 'Hopf bifurcation boundary'. The previous formula Eq. (6) shows that $\Delta_{n-1}=0$ occurs in the following cases: (i) $P_J(\lambda)$ has a zero real root with algebraic multiplicity at least two; (ii) $P_J(\lambda)$ has (at least) two real roots of identical absolute value but opposite sign; (iii) $P_J(\lambda)$ has (at least) a purely imaginary pair; (iv) $P_J(\lambda)$ has (at least) two complex pairs with opposite real parts. If the necessary condition $a_i > 0$ for all i is fulfilled, then non-negative real roots are impossible and cases (i) and (ii) are ruled out. Thus, only possibilities (iii) and (iv) remain. Let us take stability as the starting point of the story, by considering an 'initial' parameter constellation μ_S in which the system is stable. Consider then parametric perturbations obtained through continuous displacements from μ_S, which do not violate the condition $a_i > 0$ for all i. In this case losses of stability can only occur through the crossing of the imaginary axis by one (or more) previously stable complex pair (case (iii)). But, as previously mentioned, this necessarily implies a crossing, in the parameter space, of the locus $\Delta_{n-1}=0$. Such considerations can be used as follows to detect SHBs. Let us consider Fig. 1a and b,[9] reporting two distinct forms of the locus $\Delta_{n-1}=0$ (we obviously assume that Fig. 1a and b represent the $\Delta_{n-1}=0$ locus of an equilibrium point of a non-linear system) in a 2D parameter space labeled as (p, q). Let the point P represent the 'initial' parameter constellation μ_S at which stability prevails and assume that $a_n > 0$ in the whole parameter space, so that $\Delta_{n-1}=0$ defines the SHB boundary. In Fig. 1a, the whole external region is a region of stability, while the whole inner region is an instability region. Clearly, in general, all the points of the curve $\Delta_{n-1}=0$ are[10] SHB points.[11]

[8] The search for 'alternative' tools for detecting stability boundaries and bifurcations is still an area of active research in mathematics, see Guckenheimer et al. (1997) or the use of compound matrices by Li and Wang (1999). Compound matrices, for instance, offer an 'alternative' to the basic RH which could be used for defining an alternative criterion for Hopf bifurcation, along the lines of this paper. We also notice that the notion of stability and bifurcation boundary is the starting point of the algorithms for the numerical detection of bifurcations (Kuznetsov, 1994). But numerical methods usually prevent sound economic interpretation of the bifurcation conditions.

[9] We use, in order to better illustrate the problem, the same type of figures used by MacDonald (1989, p. 74).

[10] We say 'in general' because the equality $\Delta_{n-1}=0$ cannot discriminate whether just one previously stable pair crosses the imaginary axis, or more than one. In low-dimensional cases this can be checked directly: at dimension four, the condition $a_i > 0$ for all i, implies $\Delta_1 = a_1 > 0$, therefore preventing the simultaneous crossing by two pairs. In higher order cases to make sure that we are facing a SHB we should check that the remaining Liénard–Chipart determinants are strictly positive for those parameter values causing $\Delta_{n-1}=0$. But of course this is little worrying: the simultaneous crossing by several pairs is certainly a less likely event (less generic) because it would require (i) the existence of some complex pair with algebraic multiplicity greater than one, or (ii) complex pairs sharing dependencies of their real parts on the bifurcation parameters leading to exactly the same bifurcation value.

[11] This will not be true in a few special cases, for instance conservative systems, such as Lotka–Volterra predator-prey systems, where the bifurcation will be degenerate.

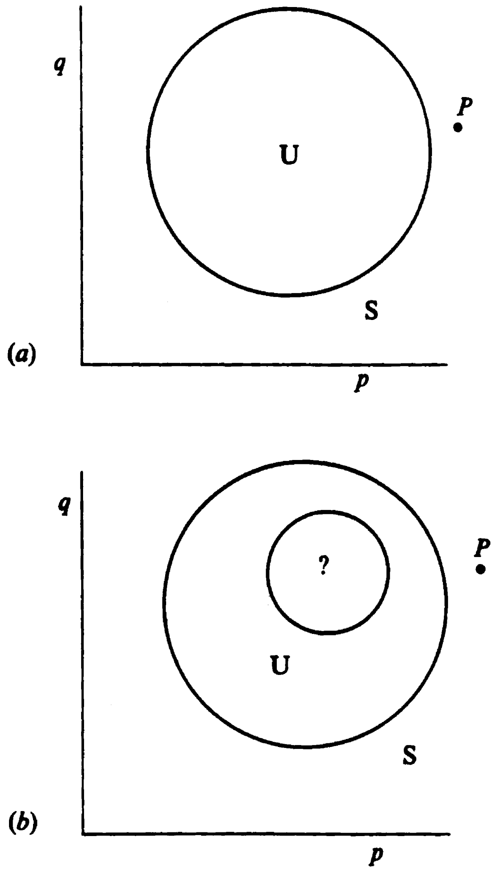

Fig. 1. (a) A SHB boundary in a 2D parameter space; (b) a SHB followed by a GHB in a 2D parameter space.

Let us now move to Fig. 1b. Consider a continuous movement of some parameter implying a movement from P in the parameter space leading to a crossing of the external curve: all the points of $\Delta_{n-1}=0$ host an SHB, as in Fig. 1a. Suppose now that a further movement leads to a crossing of the internal curve. We claim that a GHB has occurred. In fact two cases are possible: (i) the crossing is governed by the same complex pair responsible for the crossing of the external curve; (ii) the crossing is governed by a different complex pair. In the former case we face a re-switch of stability. In order to check, which of the two cases actually occurred we must check the stability 'status' of the points inside the inner curve.

The 'bifurcation boundary' approach is particularly useful for systems for which the 'initial' parameter constellation μ_S in which the system is stable, 'naturally' exists and is identifiable. This situation is not at all uncommon: modellers usually study the effects of parametric perturbations of known models. For a parametric perturbation we mean a 'complication' of a known model (which is assumed to be stable in some subset of the parameter space) which is obtained by adding to the original model new terms depending on some extra-parameters ϑ. These enlarged models usually reduce to the 'old' simpler model when $\vartheta=0$. Hence we often know a 'natural' initial parameter constellation. Moreover, the analytical detection of the bifurcation may be simplified to a great extent when the aforementioned parametric perturbation does not affect the equilibria of the system but only their stability properties.

Fortunately, as we pointed out in the introduction, there are in dynamic economic modelling major classes of models (classes (1), (2) and (3) in Section 1) to which the previous theory applies. We have chosen here to present our economic illustrations focusing on delay models. As is well known, standard formulations of delay models (i.e. delay models obtained by introducing a time-delay in an unlagged one) do not affect the equilibria but only the stability properties of the underlying unlagged model. The key question is: given a system having a stable equilibrium in the absence of the delay, how is stability affected by the introduction of delays? In this case, the 'natural' initial parameter constellation μ_S in which the system is stable corresponds to the case in which the delay is absent. These aspects are illustrated in the next section by means of some ODE models derived from an underlying distributed delay model.

5. Economic illustrations

This section aims to illustrate how the tool-box developed in the previous section can be used in actual modelling problems. We first consider a 3D Solow-type model with a time-delay mirroring the demographic process of recruitment into the labour force. Second we consider a 5D Goodwin–Kalecki delayed model. In both cases, we apply the 'stability boundary' tool-box which gives a simple geometric view of the bifurcation, thus making the detection of the bifurcation surprisingly simple. Essentially all that is necessary are the following steps: (a) to draw in the space of economic parameters which are of interest for the problem at hand, the locus identifying the stability boundary; (b) to find and locate in the parameter space the 'initial parameter constellation' in which the system is stable. This second step is simple for delay systems: we only need to draw in the parameter space the set of points with zero delay (identifying the underlying unlagged model). The reader will easily understand, by navigating in the parameter space starting from the stable initial constellation, that SHBs can only occur by a crossing of the stability boundary.

The second illustration is more complex because it is 5D, and moreover, there is the complication that the underlying unlagged system is not locally but neutrally stable. In this case, we will introduce a surprisingly simple strategy for solving the problem of finding the required stable initial parameter constellation. Moreover, we also apply, as a control step, the detection theorems of Section 3, in order to provide full algebraic confirmation of the results obtained via the stability boundary approach.

Before starting our illustrations we introduce a noteworthy concept, that of Erlangian probability density, which is the fundamental tool for the theory of reducible delay systems. A density function is Erlangian-type with parameters (r, a) when its density function $G_{r,a}(x)$ has the form

$$G_{r,a}=\frac{r^{a}}{(r-1)!}x^{r-1}e^{-ax} \quad x>0;\ a>0,\ r=1,\ 2,\ldots \tag{7}$$

By varying r the Erlangian family describes a flexible family of densities: for $r=1$ we have the classical exponentially fading memory, while for $r=2, 3,\ldots$ we have typical 'humped' memories with a Gamma shape.

5.1. First illustration: the unlagged system is locally stable; a delayed Solow-type model

Here, we consider the following delayed Solow-type model (Fanti and Manfredi, 2003) aiming to embody the age structure of the population in the Solow (1956) model

$$\dot{k}=sk^{\alpha}-\delta k-\left(\int_{-\infty}^{t} n(k^{\alpha}(\tau))G(t-\tau)\mathrm{d}\tau\right)k \tag{8}$$

where k is the capital-labour ratio, k^{α} denotes output per capita per unit time under a Cobb–Douglas production function $(0<\alpha<1)$, s is the saving rate $(0\leqslant s\leqslant 1)$, δ the rate of capital depreciation. Compared to Solow's original model, the constant exogenous rate of growth of the supply of labour (usually denoted by n) has been replaced by an integral term dependent on past per-capita income $k^{\alpha}(\tau)$, $\tau<t$, the purpose of which is to mimic the effects of past wage-related fertility on the current labour force. The function $G(\cdot)$ is the delaying kernel, usually chosen as a probability density function. The function $n(\cdot)$ is assumed positive and non-decreasing with per-capita income, according to a Malthusian mechanism. For simplicity we assume that $n(\cdot)$ is linear: $n(k^{\alpha}(\tau))=nk^{\alpha}(\tau)$, where the parameter n tunes the past fertility rate of individuals. Model (Eq. (8)) is a delayed variant of the following model with endogenous population, already considered by Solow (1956)

$$\dot{k}=sk^{\alpha}-\delta k-n(k^{\alpha})k \tag{9}$$

The intuitive idea underlying formulation (Eq. (8)) is that the current rate of change of the supply of labour is related to past fertility, and thus to past levels of the wage, following a prescribed pattern of delay. Indeed, as pointed out in the biological literature (MacDonald, 1978) time-delays allow simpler representations of age structures while preserving at the same time their main dynamical implications.

Thus, model (Eq. (8)) introduces a long lasting economic argument, e.g. the role of past population patterns (e.g. fertility plus age structure) in determining the current supply of labour. This argument was lucidly pointed out by McCulloch already in 1854: 'the supply of labourers in the market can neither be speedily increased when wage rise, nor speedily diminished when they fall. When wages rise a period of eighteen or twenty years must elapse before the stimulus, given the principle of population, can be felt in the market' (McCulloch 1854, p. 34). The previous McCulloch's sentence contains, according to the current literature in population economics, a terse definition of Malthusian cycles. Malthusian cycles are a firm theoretical point in population economics. They are the consequence: '…of the lags between the response of fertility to current labour market conditions and the time when the resulting births actually enter the labour force' (Lee, 1997, p. 1097).

It is easy to check (Fanti and Manfredi, 2003) that model (Eq. (9)) has a unique positive equilibrium E_1 which is globally asymptotically stable (GAS). The behaviour of the integro-differential (IDE) model (Eq. (8)) is expected to depend more crucially on the structure of the kernel G. When G belongs to the Erlangian family, (Eq. (8)) may be reduced to an ODE system via the linear trick (MacDonald, 1978). In particular, the mean delay implied by an Erlangian density of parameters (r, a) is given by: $T = r/a$, while its variance is $\mathrm{Var} = r/a^2$. When we let $T \to 0$ in an Erlangian kernel, the underlying unlagged system, e.g. Eq. (9), is recovered. The reduction of Eq. (8) to ODEs under Erlangian kernels (r, a) is carried out by introducing r auxiliary variables:

$$S_j(t) = \int_{-\infty}^{t} n(k^{\alpha}(\tau)) G_{j,a}(t-\tau) \mathrm{d}\tau \quad j = 1,2\ldots,r \tag{10}$$

A time differentiation of Eq. (10) transforms[12] the model (Eq. (8)) into its 'augmented' ODE form, in which the delay is replaced by a 'cascade' of r adaptive equations characterised by the same speed of adjustment a.

Thus, for $r = 1$, which is the case, well known to scholars in economic dynamics, of the exponentially fading memory, Eq. (8) reduces to a 2D ODE system having the same equilibria of Eqs. (8) and (9). It may be shown that the unique positive equilibrium E_1 of the model, remains LAS irrespective of the delay. More interesting things occur in the case $r = 2$. This case is much more interesting compared to the previous one, because for $r = 2$ the Erlangian Kernel is humped (it is the simplest example of humped Erlangian kernel), e.g. much more apt to represent a phenomenon as the contribution of past fertility to the current labour supply (e.g. a contribution due to individuals born in a rather narrow interval of time in the past). In this case, we obtain the 3D system

[12] For sake of precision, further formal requirement is needed, in order to make the 'distributed' initial condition of the original IDE system compatible with the 'concentrated' initial condition of the ODE system.

$$\dot{Z}=\alpha Z(sZ^{(\alpha-1)/\alpha}-nX)$$

$$\dot{X}=a(R-X) \tag{11}$$

$$\dot{R}=a(Z-R)$$

where the further change of variable $Z=k^{\alpha}$ has been used. The characteristic polynomial in correspondence of the unique positive equilibrium E_1: $P_J(E_1)(X)=X^3+a_1X^2+a_2X+a_3$, has the coefficients

$$a_1=2a+(1-\alpha)nZ_1;\quad a_2=2a(1-\alpha)nZ_1+a^2;\quad a_3=a^2nZ_1 \tag{12}$$

which are strictly positive (Z_1 is the equilibrium value of Z at E_1). The necessary condition for stability ($a_i>0$ for all i) is therefore always satisfied. The locus: $\Delta_{n-1}=0$, i.e. $\Delta_2=a_1a_2-a_3=0$ is given by

$$\Delta_2=2a^2+nZ_1(4-5\alpha)a+2((1-\alpha)nZ_1)^2=0 \tag{13}$$

Eq. (13) defines thus a Hopf bifurcation boundary. In order to view, the bifurcation process let us concentrate on a pair of economic parameters of Eq. (13). The parameters of major interest are the average demographic delay ($T=2/a$), and the fertility parameter n. In the (2D space) of these two parameters the stability boundary $\Delta_2=0$ is represented (Fig. 2) by the union of the lines L_1 and L_2. Now, in order to fruitfully apply the concept of stability boundary we need to find the 'natural' initial parameter constellation μ_S in which the system is stable. This is easy: all the points of the n-axis ($T=0$) can be chosen as our stable 'initial parameter constellation', since they correspond to the unique positive equilibrium E_1 of the unlagged model (Eq. (9)), which is globally asymptotically stable. For fixing the ideas consider, in Fig. 2, the initial parameter constellation represented by point P (at which stability prevails since P is located on the n-axis) and 'navigate' across the (n, T) space by a continuously increasing movement of the T parameter from P, while keeping n fixed. In all the points of the (n, T) space below line L_1 stability prevails (no stability boundary is crossed). Thus the line L_1 necessarily is a locus of SHB points. The region in between the lines L_1 and L_2 is an instability region. What happens at L_2? As the necessary condition for stability is satisfied in the whole parameter space, also the line L_2 is a Hopf bifurcation boundary. This second bifurcation is, according to the definition, a GHB, at which a stability reswitch occurs, due to a further crossing of the previously unstable pair. This is exactly case (ii) in Fig. 1b, which has been discussed in the previous section.

Hence, the acknowledgement of a realistic pattern of change of the labour supply leads to two distinct bifurcation values of the delay, a smaller one occurring on a

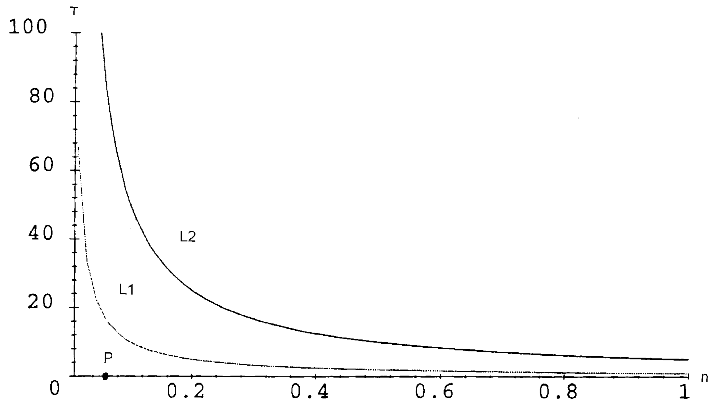

Fig. 2. The Hopf bifurcation boundary for the positive equilibrium of a 3D Solovian model.

typical demographic time scale, whereas the larger one occurs on a much longer time scale.

5.2. The unlagged system is neutrally stable: a Goodwin–Kalecki-type model

We consider now bifurcations arising in the following Goodwin-type model (Goodwin, 1967)

$$\dot{V} = (-(\alpha+\gamma)+\rho U)V$$
$$\dot{U} = \left[(c+k)m(1-V)-(\alpha+n)-km\left(1-\int_{-\infty}^{t} V(\tau)G(t-\tau)d\tau\right)\right]U \tag{14}$$

In Eq. (14) $U=U(t)$ is the employment rate at time t, defined as the ratio between the total labour force actually employed $L(t)$ and the supply of labour $N(t)$, $V=V(t)$ is the distributive share of labour, given by the ratio $w(t)L(t)/Q(t)$, where w is the real wage and Q the total product per unit time. Moreover, m is the (constant) output–capital ratio, α and n respectively denote exogenous growth rates of the average productivity of labour and the labour force, c is the saving rate of the capitalists; finally γ, ρ are characteristic parameters of the (linear) Phillips curve governing the labour market $(0<\gamma<\rho)$. The model (Eq. (14)) embeds Kaleckian effects via the lagged term. Indeed in (Eq. (14)) $m(1-V)$ is the current profit rate, whereas the term

$$m\left(1-\int_{-\infty}^{t} V(\tau)G(t-\tau)d\tau\right)$$

represents past profit. According to Kalecki, also future expected profit rates matter for investors, to the extent to which, according to Robinson's definition, the investors are 'rash' or 'cautious' (representing past profitability as a proxy for future profitability). The parameter $k>0$ and the delaying kernel G tune the 'rashness' of investors. When $k=1$ and the mean delay is set equal to zero, (Eq. (14)) collapses in the original Goodwin (1967) model, exhibiting the classical Lotka–Volterra conservative oscillation, e.g. the positive equilibrium of the model is neutrally stable. Thus, the model of the present section contrasts with the model of the preceding one, where the underlying unlagged system was 'locally' stable. Model (Eq. (14)) has the same equilibria as those of the original Goodwin model (notwithstanding the introduction of rash and cautious behaviours of the capitalists), namely the zero equilibrium $E_0=(0, 0)$, and the positive equilibrium $E_1=(U^*, V^*)=(\gamma/\rho, (m-\alpha-n)/m)$. E_1 is economically meaningful provided that $m-\alpha-n>0$.

The dynamical properties of Eq. (14) depend on the structure of the delaying kernel G. Under the action of an exponentially fading memory (i.e. G is assumed Erlangian $(1, a)$), system (Eq. (14)) expands as:

$$\dot{V}=(-(\alpha+\gamma)+\rho U)V$$

$$\dot{U}=[(c+k)m(1-V)-(\alpha+n)-km(1-S)]U \tag{15}$$

$$\dot{S}=a(V-S)$$

It is easy to show that the positive equilibrium E_1 of Eq. (15) is always LAS independent of the delay. In other terms a Kaleckian expected profitability with exponentially fading memory always stabilises the conservative center of Goodwin's model. This is a nice instance of the fact that delays can also be stabilising, and not only destabilising, as often claimed in the literature (Farkas and Kotsis, 1992). It is of interest to ascertain whether this stability is preserved under different forms of the delaying kernel. In many cases, systems which are stable under an exponentially fading memory are destabilised under the simplest type of 'hump' memory, i.e. a kernel Erlangian $(2, a)$ (Fanti and Manfredi, 1998). This effect is usually explained with the different qualitative action played by a humped memory as opposed to an exponentially fading one. Assuming a kernel Erlangian $(2, a)$, (Eq. (14)) transforms to the 4D ODE system

$$\dot{V}=(-(\alpha+\gamma)+\rho U)V \tag{16}$$

$$\dot{U}=[(c+k)m(1-V)-(\alpha+n)-km(1-S)]U$$

$$\dot{S}=a(Z-S)$$

$$\dot{Z}=a(V-Z)$$

with the same equilibria as in Eq. (14). The local stability analysis at E_1 leads to the characteristic polynomial

$$P_{J(E_1)}(X)=X^4+2aX^3+(a^2+Bk)X^2+2aB(1+k)X+Ba^2 \tag{17}$$

whose coefficients are always positive (we denoted $B=m\rho U^*V^*=(m-\alpha-n)(\alpha+\gamma)$). The LC test for stability requires that $\Delta_1>0$ (always satisfied as $\Delta_1=a_1$) and $\Delta_3>0$. But: $\Delta_3=4a^4Bk>0$, i.e. E_1 remains stable independent of the delay in the simplest humped case as well. It is therefore of interest to look for the possibility that destabilisation occurs by delays of higher order. Let us then consider the effects of the next element of the Erlangian family, e.g. the kernel Erlangian (3, a). The economic meaning of this fact is that entrepreneurs form their expectations by focusing on a more narrow time window in the past, compared to the previous examples.

In this case, the reduced ODE system has the form:

$$\begin{aligned}
\dot V&=(-(\alpha+\gamma)+\rho U)V\\
\dot U&=[(c+k)m(1-V)-(\alpha+n)-km(1-S)]U\\
\dot S&=a(Z-S)\\
\dot Z&=a(W-Z)\\
\dot W&=a(V-W)
\end{aligned} \tag{18}$$

The characteristic polynomial evaluated at E_1, $P_J(E_1)$, has the coefficients

$$a_1=3a;\quad a_2=B(1+k)+3a^2;\quad a_3=a(a^2+3B(1+k));$$

$$a_4=3Ba^2(c+k);\quad a_5=Ba^3$$

which are always positive. In particular,

$$\Delta_{n-1}=\Delta_4=Bka^6(24a^2-B(8+9k)) \tag{19}$$

The condition $\Delta_{n-1}=0$ gives as unique admissible bifurcating value of a

$$a_H=\sqrt{\frac{B(8+9k)}{24}}=\sqrt{\frac{(m-\alpha-n)(\alpha+\gamma)(8+9k)}{24}} \tag{20}$$

to which a mean delay $T=3/a_{\mathrm{H}}$ corresponds. Note that Eq. (20) exhibits a simple dependency of the bifurcation threshold from (some of) the economic parameters appearing in the model, thus allowing a meaningful economic interpretation. Can we now ensure that the locus (Eq. (20)) is a Hopf bifurcation locus? A possible difficulty to the use of the sole condition $\Delta_{n-1}=0$ lies in the fact that the 'natural initial parameter constellation' corresponding to the case of no-delay ($T=0$), corresponds to the original Goodwin model, for which the E_1 equilibrium is not (linearly) stable, but only neutrally stable. The difficulty is solved as follows. It can be immediately ascertained that the system (Eq. (18)) is the ODE system that would have been obtained by delaying (Eq. (14)) by an Erlangian kernel $G_{2,a}$. In general, we may say that the application of a delaying kernel $G_{r,\mathrm{a}}$ is equivalent to r sequential applications of a kernel $G_{1,\mathrm{a}}$: this is a standard result on exponential density functions.[13]

This suggests that we do not need to necessarily refer to the original unlagged system as the 'natural' initial parameter constellation in which the system is stable. Let us reconsider our problem. We have to perform the stability analysis of the fifth order system (Eq. (18)) obtained by delaying with a kernel $G_{3,a}$ the original conservative system. But the stability analysis of Eq. (18) is equivalent, for instance, to the stability analysis of the system (Eq. (15)) which—as shown above—is stable, when the S variable therein is further delayed by $G_{2,a}$. It is also equivalent to the stability analysis of Eq. (16), which is stable, when the S variable therein is further delayed by $G_{1,a}$. This implies that both systems Eq. (15) or Eq. (16) provide an initial parameter constellation in which the system is stable. Therefore, once the bifurcation boundary (Eq. (20)) is crossed, a unique switch between stability and instability occurs: this implies, without the need for any further inquiry, that all the points of the stability boundary are SHB points.[14]

The following substantive remarks follow:

(i) Our treatment allows a clear interpretation of the role played in the bifurcation process by the 'rashness' parameters k, as clear from Eq. (20). In other words: the bifurcation process in the 5D model (Eq. (18)) is perfectly interpretable in terms of the underlying economic theory.

(ii) We have shown that, although the simplest humped memory is not sufficient to raise instability, this can be caused by a sufficiently concentrated humped memory. A more concentrated humped memory is economically interpretable as a greater homogeneity in the rashness pattern of the investors.

[13] This result may be further extended: the application of a delaying kernel $G_{r,a}$ is equivalent to the sequential application of a kernel $G_{s,a}$, followed by that of a kernel $G_{r-s,a}$ (s in an integer $\leqslant r$).

[14] In order to check the latter statement let us use 'in toto' Theorem 2 (Section 3) and compute Δ_2. We have:

$$\Delta_2=a_1a_2-a_3=8a^3$$

which is always positive for $a>0$. We can thus confirm that all the points of the locus $a_{\mathrm{H}}=\sqrt{B(8+9k)/24}$ are SHB points for E_1.

(iii) Moreover, this result is of interest in itself: it proves that (distributed) delays may not only stabilise a conservative Lotka–Volterra–Goodwin (LVG) system for slightly concentrated delays, and destabilise it for highly concentrated memories, but they may also lead conservative LVG systems to persistently oscillate. This fact has not been, as far as we know, pointed out in the literature on prey-predator models.

(iv) The analysis of Eqs. (14)–(16) and Eq. (18) has shown that, in order to apply the bifurcation boundary approach, it is not necessary that the underlying unlagged system is stable. Also conservative, e.g. neutrally stable, systems, of which the LVG family is an important example, are tractable by the same tools.

6. Conclusions

The detection of endogenous fluctuations, which is a prime concern in macroeconomic dynamic modelling, is strictly related to the detection of Hopf bifurcations. Most of the papers concerned with the subject in the economic literature have been concerned with systems of very low dimension. This is usually due to the analytic complexity involved in detection. But realistic economic modelling increasingly requires the use of systems of larger dimension. Therefore, the availability of a usable 'detection tool-box' is a fundamental need for applied scientists. This paper aims to provide a unified treatment of SHBs, which is the only 'true' observable notion of cycle. We first discuss how stability theorems can be used to detect SHBs. Second we extend a recent mathematical result by Liu (1994), proving a LC-type result which appears quite useful in applications.

Moreover, via the notion of bifurcation boundary, we show that in some cases the conditions for the detection of an SHB can be stated in an amazingly parsimonious way, especially compared to the standard belief. This result appears to be useful for all the disequilibrium economic models in which the stability of equilibrium depends on vectors of parameters, which in turn do not alter the structure of equilibria itself. Among these there are for instance: (1) macro-economic models out of equilibrium, with simple Marshallian or Walrasian adjustments rules; (2) optimising models of agents behaviour when there are risks or adjustments costs; (3) heterogeneous agents with different 'reaction times', leading to distributed delay models.

Our economic illustrations, which are based on variants of the well-known Solow (1956) and Goodwin (1967) models, finally show how bifurcation boundaries can be used to detect cycles in dynamic economic modelling, in both the cases of locally and neutrally stable underlying unlagged systems.

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