Dept of Electrical and Electronic Eng.
University of Cagliari

5th Workshop
on
Structural Dynamical Systems: Computational Aspects

Capitolo (BA) – Italy

Sliding mode:
Basic theory and new perspectives

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Summary

- Switching systems
- Introductive examples
- *Zeno* behaviour
- Sliding modes in switching systems
- Regularization of sliding a mode trajectory
- Sliding modes in control systems
- Stability of a sliding mode control system
- Invariance of a sliding mode control system
- Recovering the uncertain dynamics
- Approximability
- Discrete time implementation
- Higher order sliding modes

Switching systems

Switching systems are dynamical systems such that their behaviour is characterised by different dynamics in different domains

$$\dot{\mathbf{x}}(t) = f_i(\mathbf{x}(t), t, \mathbf{u}(t)), \quad \mathbf{x} \in X_i \subseteq \mathsf{R}^n, \quad \mathbf{u} \in U_i \subseteq \mathsf{R}^m, \quad t \in \mathsf{R}^+$$

$$i \in Q \subseteq \mathsf{N}$$

f_i is a smooth vector field $\quad f_i$: $\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^+ \to \mathbf{R}^n$

The state dynamics is invariant until a switch occurs

The system dynamics is represented by a differential equation with discontinuous right-hand side

Switching systems

Switching between different dynamics

$$g_i^{sw}\left(\mathbf{x}(\tau_k),\tau_k,\mathbf{u}(\tau_k),\mathbf{v}(\tau_k)\right)=0$$

$$\Downarrow$$

$$\dot{\mathbf{x}}\left(\tau_k^-\right)=f_i\left(\mathbf{x}\left(\tau_k^-\right),\tau_k^-,\mathbf{u}\left(\tau_k^-\right)\right),\qquad \dot{\mathbf{x}}\left(\tau_k^+\right)=f_j\left(\mathbf{x}\left(\tau_k^+\right),\tau_k^+,\mathbf{u}\left(\tau_k^+\right)\right)$$

$$\mathbf{x}\in X_i\subseteq \mathbb{R}^n,\quad \mathbf{u}\in U_i\subseteq \mathbb{R}^m,\quad \mathbf{v}\in\{0,1\}^l\quad t\in\mathbb{R}^+$$

$$i,j\in Q\subseteq\mathbb{N}$$

The reaching of the guard g_i^{sw} cause the switching from the dynamics f_i to the dynamics f_j, according to proper rules

Switching systems

What does it happen on the guard? $g_i^{sw}\left(\mathbf{x}(\tau_k), \tau_k, \mathbf{u}(\tau_k), \mathbf{v}(\tau_k)\right) = 0$

$\mathbf{v}(\tau_k) \in \phi$ Autonomous switching

$\mathbf{v}(\tau_k) \in \{0,1\}$ Forced switching

$\mathbf{x}(\tau_k^-) = \mathbf{x}(\tau_k^+)$ Continuous state variables

$\mathbf{x}(\tau_k^-) \neq \mathbf{x}(\tau_k^+)$ Jumps in state variables

$\mathbf{u}(\tau_k^-) = \mathbf{u}(\tau_k^+)$ Continuous control variables

$\mathbf{u}(\tau_k^-) \neq \mathbf{u}(\tau_k^+)$ Discontinuous control variables

Most interesting……….. the time evolution of the guard function!

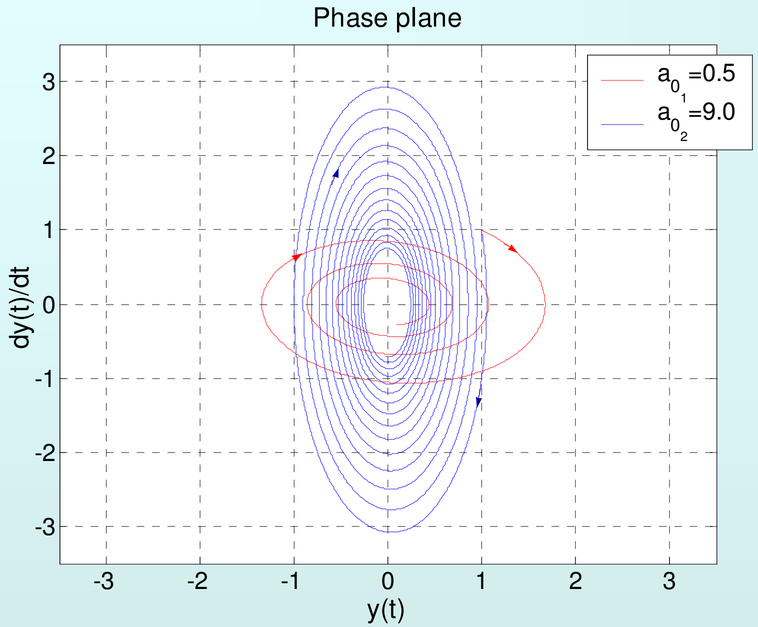
Introductive examples

Switching systems may behave very differently from each of the constituting ones.

$$\ddot{y}(t)+a_1\dot{y}(t)+a_{01}y(t)=0 \quad system\,1$$
$$\ddot{y}(t)+a_1\dot{y}(t)+a_{02}y(t)=0 \quad system\,2$$
$$0<a_{01}<a_{02}$$

- $a_1>0$ the systems are both asymptotically stable
- $a_1=0$ the systems are both marginally stable
- $a_1<0$ the systems are both unstable

Introductive examples



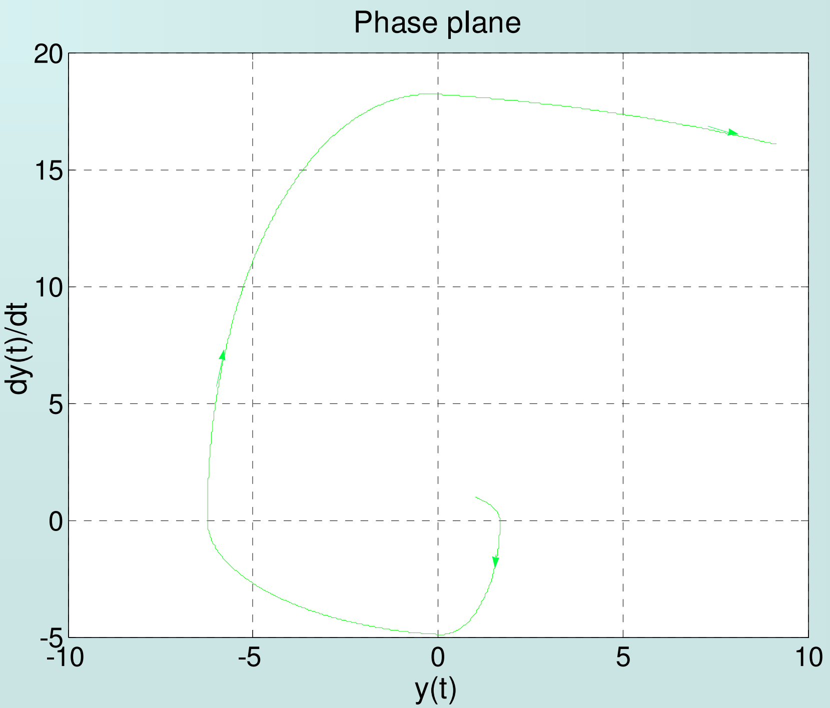
$$a_1 = 0.1$$

Both dynamics are asymptotically stable

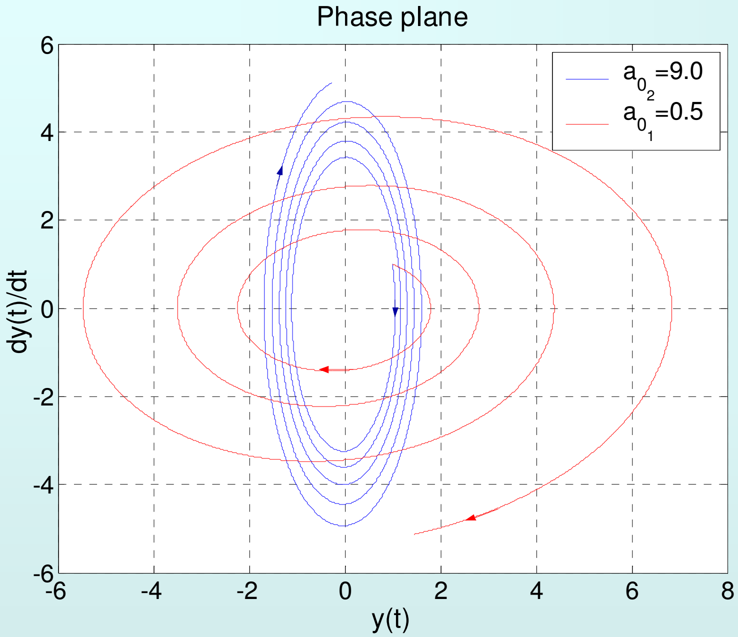
$$\ddot{y} + a_1 \dot{y} + a_0 y - \Delta a_0 |y| \operatorname{sgn}(\dot{y}) = 0$$

$$a_1 = 0.1 \quad a_0 = 0.7 \quad \Delta a_0 = 0.2$$

Switched unstable dynamics



Introductive examples



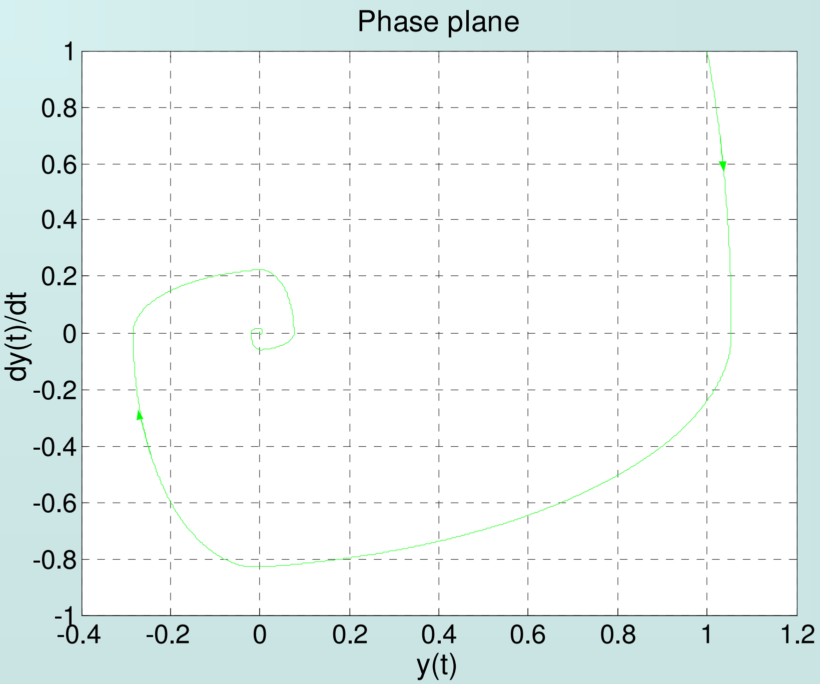
$$a_1 = -0.1$$

Both dynamics are unstable

$$\ddot{y} + a_1 \dot{y} + a_0 y + \Delta a_0 |y| \operatorname{sgn}(\dot{y}) = 0$$

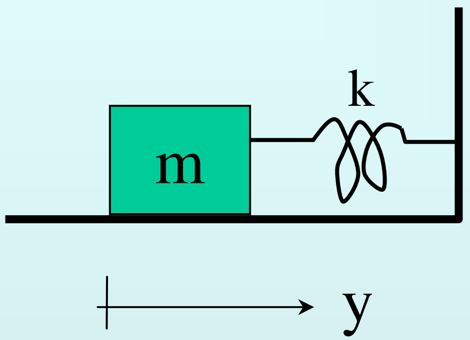
$$a_1 = -0.1 \quad a_0 = 0.7 \quad \Delta a_0 = 0.2$$

Switched asymptotically stable dynamics



Introductive examples

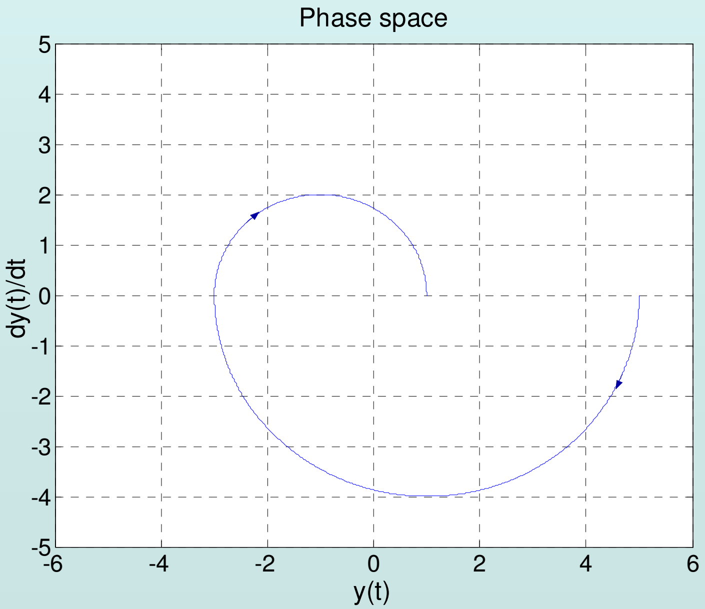
Mass-spring and Coulomb friction

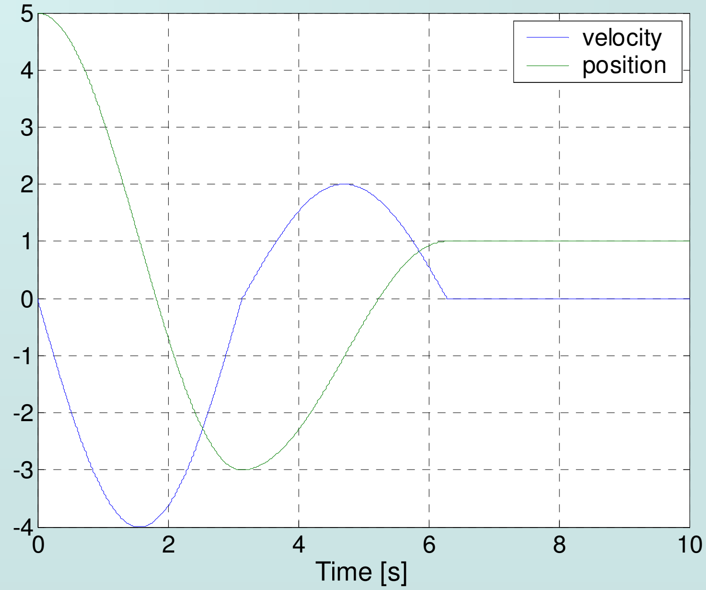


$$m\ddot{y}(t)+b\,\mathrm{sgn}(\dot{y}(t))+ky(t)=0$$

Switching time instants

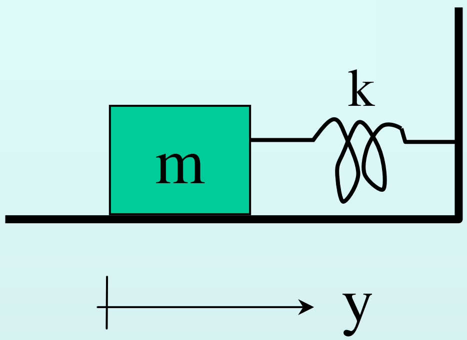
$$\dot{y}(\tau_k)=0 \rightarrow \quad \tau_k = k\pi\sqrt{\frac{m}{k}}$$





Introductive examples

Mass-spring and Coulomb friction



$$m\ddot{y}(t)+b\,\mathrm{sgn}(\dot{y}(t))+ky(t)=0$$

The guard is $\dot{y}(t)=0$

The state is continuous

The switching is autonomous

In a finite time τ_∞ the mass stops!

$$\left.\begin{array}{l}\dot{y}(t)=0\\ y(t)=y_\infty\in\left[-\frac{b}{k},+\frac{b}{k}\right]\end{array}\right. \quad t\geq\tau_\infty$$

$$\ddot{y}(t)=-\frac{b}{m}\mathrm{sgn}(\dot{y}(t))+\frac{k}{m}y(t)=0 \quad t\geq\tau_\infty$$

?

Introductive examples

Singular optimal control

$$\dot{\mathbf{x}} = f(\mathbf{x}) + b(\mathbf{x})u$$

$$J = \int_0^T \left[f_0(\mathbf{x}) + b_0(\mathbf{x})u \right] dt$$

$$|u| \le 1, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(T) = \mathbf{x}_T$$

$$H = f_0(\mathbf{x}) + b_0(\mathbf{x})u + \mathbf{p}^T \left[f(\mathbf{x}) + b(\mathbf{x})u \right]$$

$$\dot{\mathbf{p}} = -\left[\frac{\partial H}{\partial \mathbf{x}} \right]^T$$

$$u = -\operatorname{sgn}\left(b_0(\mathbf{x}) + \mathbf{p}^T b(\mathbf{x}) \right)$$

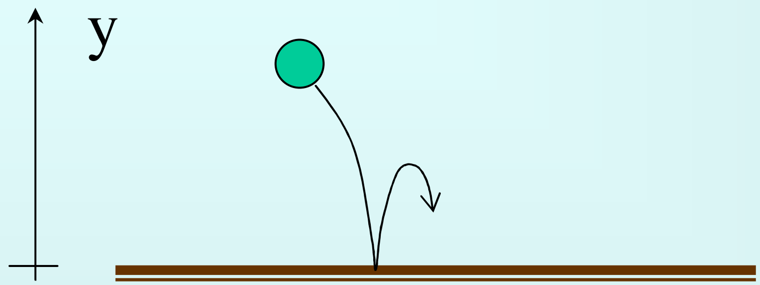
$b_0(\mathbf{x}) + \mathbf{p}^T b(\mathbf{x}) = 0 \quad t \in [t_1, t_2]$ $\Longrightarrow$

u switches at infinite frequency

u is defined looking at the higher derivatives of the guard

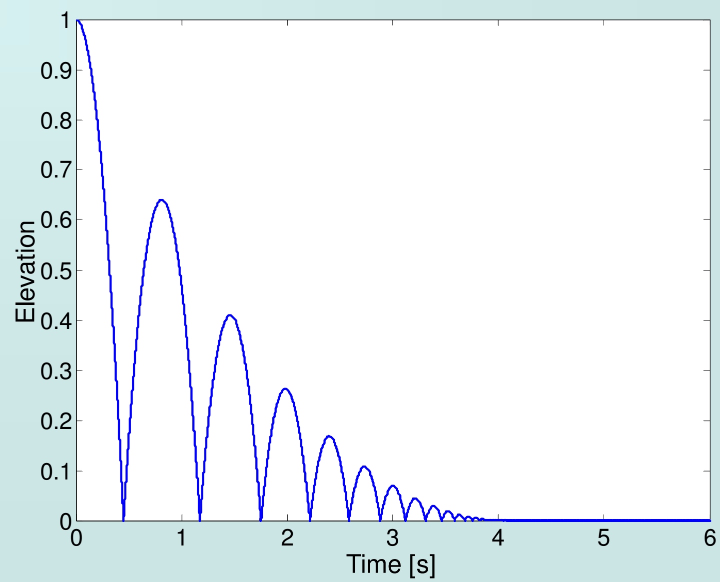
Introductive examples

The bouncing ball



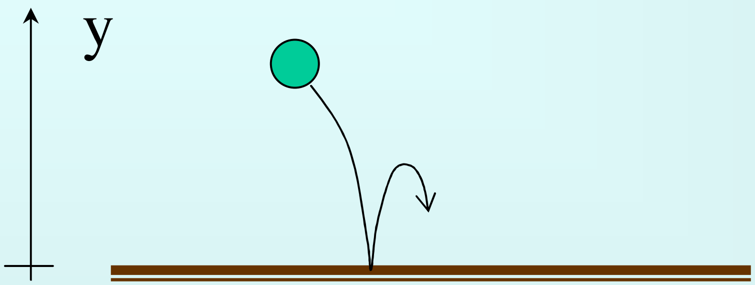
$$\ddot{y}(t) = -g \qquad y \geq 0$$
$$\tau_i : y(\tau_i) = 0 \quad (i = 1, 2, \ldots)$$
$$\dot{y}(\tau_i^+) = -\alpha \dot{y}(\tau_i^-) \quad \alpha \in [0,1]$$

$$\tau_{k+1}^- - \tau_k^+ = \tfrac{2}{g} \alpha^k \sqrt{\dot{y}_0^2 + 2 g y_0}$$
$$\dot{y}_k = \dot{y}(\tau_k^+) = \alpha^k \sqrt{\dot{y}_0^2 + 2 g y_0}$$
$$\tau_\infty = \tfrac{1}{g}\dot{y}_0 + \tfrac{1}{g}\sqrt{\dot{y}_0^2 + 2 g y_0}\,\frac{1+\alpha}{1-\alpha}$$



Introductive examples

The bouncing ball



with abuse of notation

$$\ddot{y}(t) = -g - (1+\alpha)\dot{y}(t) \cdot \delta(y) \quad y \geq 0$$

$$\alpha \in [0,1]$$

$\delta(\bullet)$ is the Dirac's function

In a finite time τ_∞ the mass stops!

$$\left.\begin{array}{l} y(t) = 0 \\ \dot{y}(t) = 0 \end{array}\right. \quad t \geq \tau_\infty$$

$$\Downarrow$$

$$\ddot{y}(t) = -g - (1+\alpha)\dot{y}(t) \cdot \delta(y) = 0 \quad t \geq \tau_\infty$$

?

Zeno behaviour

The switching behaviour of a hybrid system can be described by its execution set

$$\chi^{\mathrm{H}} = \{T, In, Ed\}$$

$T = \{\tau_i\}_{i \in \mathrm{N}}$: set of switching/jump time instants
$In = \{\mathbf{x}_i\}_{i \in \mathrm{N}}$ $\mathbf{x}_i \subseteq D$: set of initial states sequence
$Ed = \{\eta_i\}_{i \in \mathrm{N}}$ $\eta_i = (i,j) \subseteq Q \times Q$: set of edge sequence

The execution χ^{H} of the hybrid system H can be constituted by finite or infinite elements.

Executions with infinite number of elements may be due to the **Zeno** phenomenon

Zeno behaviour

The Zeno phenomenon appears when the execution χ^H of the hybrid system is such that

$$\lim_{i\to\infty}\tau_i = \sum_{i=0}^{\infty}\left(\tau_{i+1}-\tau_i\right) = \tau_\infty < \infty$$

τ_∞ (**Zeno time**) is a right accumulation point for the time instants sequence

$$\left(\tau_{i+1}-\tau_i\right) \underset{i\to\infty}{\longrightarrow} 0$$

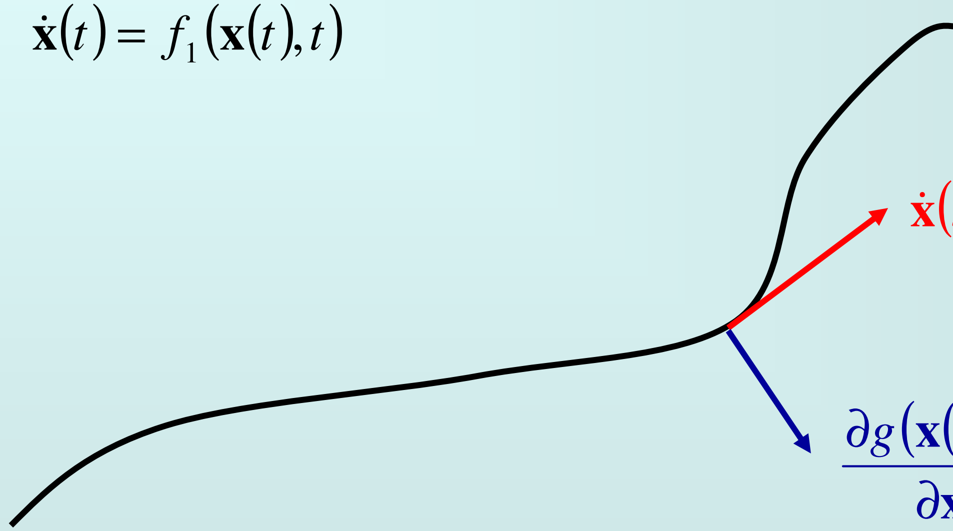
The switching frequency tends to be infinite

In a Zeno condition the system evolves along a guard

$$\frac{\partial g\left(\mathbf{x}(t),t\right)}{\partial \mathbf{x}}\cdot\dot{\mathbf{x}}(t) = 0 \quad \forall t \geq \tau_\infty$$

Sliding modes in switching systems

The Zeno phenomenon is mainly related to the switching frequency on a guard, but previous relationship shows the relation between the guard and the system dynamics



The motion of the system on a discontinuity surface is called sliding mode

Sliding modes in switching systems

- ✓ Sliding modes are Zeno behaviours in switching systems
- ✓ The system is constrained onto a surface in the state space, the *sliding surface*
- ✓ When the system is constrained on the sliding surface, the system modes differ from those of the original systems
- ✓ The system is invariant when constrained on the sliding surface
- ✓ Any system on the sliding surface behaves the same way

Sliding modes in switching systems

$$\ddot{y} - a_1 \dot{y} + a_0 y = 0$$

$$a_1 > 0 \quad a_0 = \pm\alpha$$

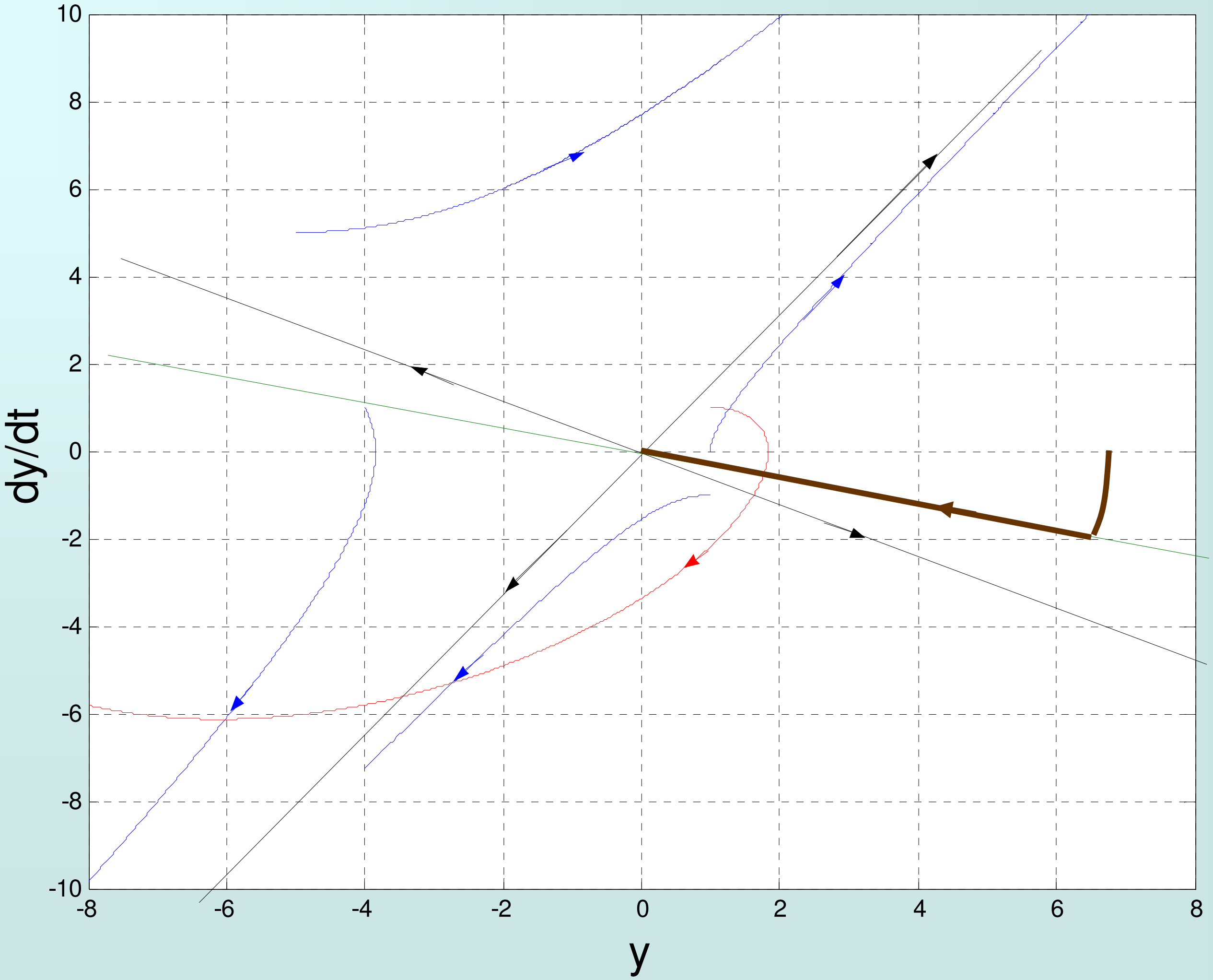
$a_0=\alpha$ $a_0=-\alpha$

$\alpha>0$

$$\ddot{y} - a_1 \dot{y} + a_0 \operatorname{sgn}(y\sigma) = 0$$

$$\sigma = \dot{y} + cy$$

$$a_1 = 1, \quad a_2 = 1, \quad c = 0.2,$$



Sliding modes in switching systems

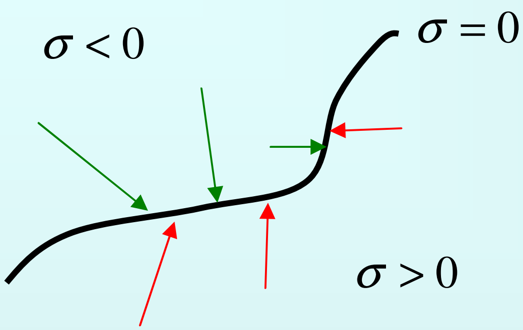
When considering switched systems, the system dynamics can be represented by a discontinuous right-hand side differential equation

$$\dot{\mathbf{x}}(t) = \begin{cases} f_1(\mathbf{x}(t), t) & \sigma(\mathbf{x}(t), t) < 0 \\ f_2(\mathbf{x}(t), t) & \sigma(\mathbf{x}(t), t) > 0 \end{cases}$$

$\sigma(\mathbf{x}(t), t) = 0$ Represents the boundary between two distinct regions of the state space, possibly time varying

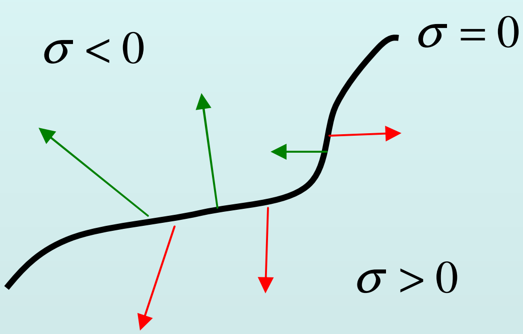
The behaviour of the system on/across the guard σ, defining the two distinct regions of the state space, depends on how the dynamics f_1 and f_2 are related to the switching surface

Sliding modes in switching systems



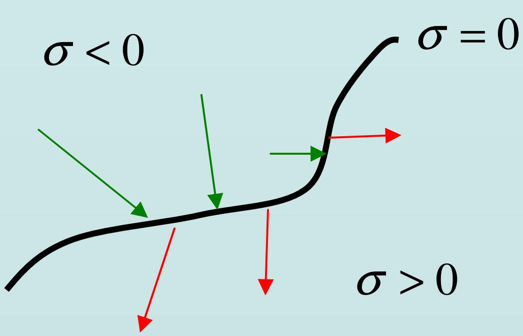
$$\begin{cases} \dfrac{\partial\sigma(\mathbf{x}(t),t)}{\partial\mathbf{x}} \cdot f_2(\mathbf{x}(t),t) < 0 \\ \dfrac{\partial\sigma(\mathbf{x}(t),t)}{\partial\mathbf{x}} \cdot f_1(\mathbf{x}(t),t) > 0 \end{cases}$$

attractive switching surface



$$\begin{cases} \dfrac{\partial\sigma(\mathbf{x}(t),t)}{\partial\mathbf{x}} \cdot f_2(\mathbf{x}(t),t) > 0 \\ \dfrac{\partial\sigma(\mathbf{x}(t),t)}{\partial\mathbf{x}} \cdot f_1(\mathbf{x}(t),t) < 0 \end{cases}$$

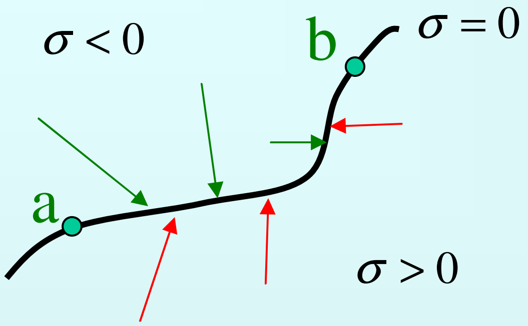
repulsive switching surface



$$\begin{cases} \dfrac{\partial\sigma(\mathbf{x}(t),t)}{\partial\mathbf{x}} \cdot f_2(\mathbf{x}(t),t) < 0 \\ \dfrac{\partial\sigma(\mathbf{x}(t),t)}{\partial\mathbf{x}} \cdot f_1(\mathbf{x}(t),t) > 0 \end{cases}$$

across switching surface

Sliding modes in switching systems



$$\begin{cases} \dfrac{\partial \sigma(\mathbf{x}(t),t)}{\partial \mathbf{x}} \cdot f_2(\mathbf{x}(t),t) < 0 \\ \dfrac{\partial \sigma(\mathbf{x}(t),t)}{\partial \mathbf{x}} \cdot f_1(\mathbf{x}(t),t) > 0 \end{cases}$$

attractive switching surface

A Sliding Mode appears in the segment [a, b] of the switching surface if

$$\begin{cases} \lim\limits_{\sigma \to 0^+} \dfrac{\partial \sigma(\mathbf{x}(t),t)}{\partial \mathbf{x}} \cdot f_2(\mathbf{x}(t),t) < 0 \\ \lim\limits_{\sigma \to 0^-} \dfrac{\partial \sigma(\mathbf{x}(t),t)}{\partial \mathbf{x}} \cdot f_1(\mathbf{x}(t),t) > 0 \end{cases} \qquad \forall \mathbf{x} \in [\mathrm{a}, \mathrm{b}] \subseteq \{\mathbf{x} : \sigma(\mathbf{x}(t),t) = 0\}$$

The Sliding Mode is stable in the segment [a, b] of the switching surface

Regularization of a sliding mode trajectory

Considering the switched dynamics

$$\dot{\mathbf{x}}(t) = \begin{cases} f_1(\mathbf{x}(t),t) & \sigma(\mathbf{x}(t),t) < 0 \\ f_s(\mathbf{x}(t),t) & \sigma(\mathbf{x}(t),t) = 0 \\ f_2(\mathbf{x}(t),t) & \sigma(\mathbf{x}(t),t) > 0 \end{cases}$$

Regularization of the sliding mode implies to find the continuous vector function f_s such that the state trajectory remains on the switching surface

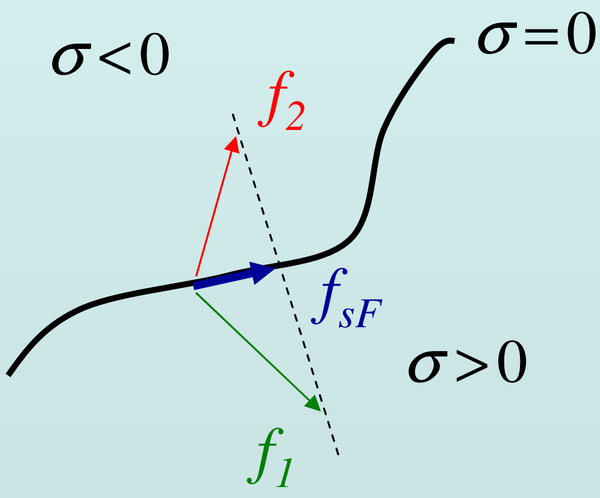
- ✓Filippov's continuation method
- ✓Equivalent dynamics method ("equivalent control")

Regularization of a sliding mode trajectory

Filippov's continuation method

$$f_{sF}(\mathbf{x}(t),t)=\alpha\, f_2(\mathbf{x}(t),t)+(1-\alpha)f_1(\mathbf{x}(t),t),$$
$$\alpha\in[0,1]:\ \frac{\partial\sigma(\mathbf{x}(t),t)}{\partial\mathbf{x}}f_{sF}(\mathbf{x}(t),t)=0\qquad\forall$$

The dynamics on the sliding surface is defined by a convex combination of the vector fields defined in the two separated regions of the state space



$$\alpha=\frac{grad(\sigma)\cdot f_1}{grad(\sigma)\cdot(f_1-f_2)},$$

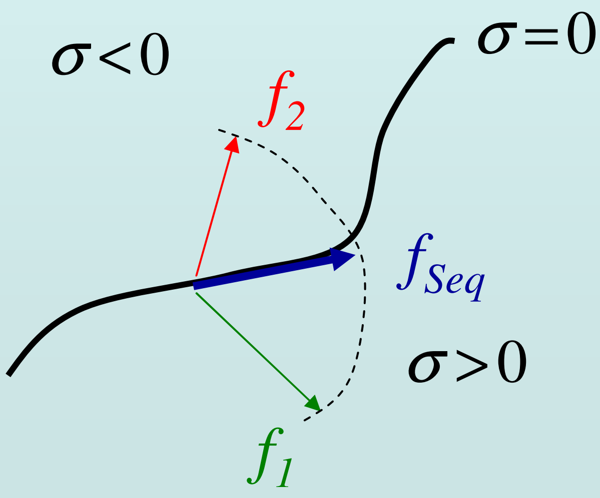
$$1-\alpha=\frac{grad(\sigma)\cdot f_2}{grad(\sigma)\cdot(f_1-f_2)},$$

Regularization of a sliding mode trajectory

Equivalent dynamics method

It is used when the discontinuity is due to switching of an independent variable, the “control”, $\boldsymbol{u}(t)$

$$\dot{\mathbf{x}}(t) = \begin{cases} f\left(\mathbf{x}(t), t, \mathbf{u}_1(t)\right) & \sigma(\mathbf{x}(t), t) < 0 \\ f\left(\mathbf{x}(t), t, \mathbf{u}_{eq}(t)\right) & \sigma(\mathbf{x}(t), t) = 0 \\ f\left(\mathbf{x}(t), t, \mathbf{u}_2(t)\right) & \sigma(\mathbf{x}(t), t) > 0 \end{cases}$$



$$f_{s\,eq}(\mathbf{x}(t), t, \mathbf{u}(t)) = f\left(\mathbf{x}(t), t, \mathbf{u}_{eq}(t)\right)$$

$$\mathbf{u}_{eq} : \frac{\partial \sigma(\mathbf{x}(t), t)}{\partial \mathbf{x}} f_{s\,eq}(\mathbf{x}(t), t, \mathbf{u}(t)) = 0$$

Regularization of a sliding mode trajectory

Filippov's continuation method and the equivalent control method can give different solutions in nonlinear systems

⇩

Not uniqueness problems in finding the solution of the sliding mode dynamics

⇧

Presence of more than a single switching surface

In real systems such motions appear in representing constrained systems, or in control systems in which a variable is forced to be zero

⇩

In most cases the uniqueness problem does not arise

Sliding modes in control systems

Usually sliding mode control systems consider dynamics with affine control

$$\dot{\mathbf{x}} = f(\mathbf{x},t) + B(\mathbf{x},t)\mathbf{u}$$

$$\boldsymbol{\sigma}(\mathbf{x},t) = 0$$

$$\mathbf{x} \in R^n \quad \mathbf{u} \in R^m \quad \boldsymbol{\sigma} \in R^m$$

The control vector is such that

$$u_i(\mathbf{x},t) = \begin{cases} u_i^+(\mathbf{x},t) & if \quad \sigma_i(\mathbf{x},t) > 0 \\ u_i^-(\mathbf{x},t) & if \quad \sigma_i(\mathbf{x},t) < 0 \end{cases} \qquad i = 1,2,\ldots,m$$

the sliding manifold is constituted by the intersection of m switching surfaces, and uniqueness problems can arise

Stability of a sliding mode control system

Local stability of a sliding mode on the domain

$S(t)=\{\boldsymbol{\sigma}: \sigma_i=0; i=1,2,\ldots,m\}$

can be stated by the following Lyapunov-like theorem

The subspace $S(t)$ is a sliding domain if a function $V(\boldsymbol{\sigma},\mathbf{x}, t)$ exists in a domain Ω of the space $\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ containing the origin such that:

- $V(\boldsymbol{\sigma},\mathbf{x}, t)$ is positive definite with respect to $\boldsymbol{\sigma}$, for all $\mathbf{x}$ and any t
- On the sphere $\|\boldsymbol{\sigma}\|=R$, for all $\mathbf{x}$ and any t, the function $V(\boldsymbol{\sigma},\mathbf{x}, t)$ remains bounded
- The total time derivative of function $V(\boldsymbol{\sigma},\mathbf{x}, t)$

$$\dot{V}=\left(\frac{\partial V}{\partial \boldsymbol{\sigma}}\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}}+\frac{\partial V}{\partial \mathbf{x}}\right)(f+B\mathbf{u})+\frac{\partial V}{\partial t}$$

  is negative everywhere except on the discontinuity surfaces where it is not defined
- On the sphere $\|\boldsymbol{\sigma}\|=R$, for all $\mathbf{x}$ and any t, the total time derivative of $V(\boldsymbol{\sigma},\mathbf{x}, t)$ is upper bounded by a negative constant

Stability of a sliding mode control system

The sufficient conditions for stability of the sliding mode are not easy to verify in general but some specific results can be given

$$\dot{\mathbf{x}} = f(\mathbf{x},t) + B(\mathbf{x},t)\mathbf{u} \qquad \mathbf{x} \in R^n \quad \mathbf{u} \in R^m \quad \boldsymbol{\sigma} \in R^m$$

$$\boldsymbol{\sigma}(\mathbf{x},t) = 0$$

$$V(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma}^T \left(\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot B\right)^{-1} \boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\sigma}^T (C \cdot B)^{-1} \boldsymbol{\sigma} \quad \Rightarrow \quad \dot{V}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}^T (C \cdot B)^{-1} C(\mathbf{x},t) \cdot \left(f(\mathbf{x},t) + B(\mathbf{x},t)\mathbf{u}\right)$$

$$u_i = -\alpha \operatorname{sgn}(\sigma_i); \quad \alpha b_i > |f_i| + k \quad \forall i \quad \Rightarrow \quad \operatorname{sgn}(f + \alpha B\mathbf{u}) = \operatorname{sgn}(\mathbf{u}) = -\operatorname{sgn}(\boldsymbol{\sigma})$$

$$\Downarrow$$

$$\dot{V}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}^T (CB)^{-1} C \cdot \left(f - B \operatorname{sgn}(\boldsymbol{\sigma})\right) \le -k\boldsymbol{\sigma}^T \operatorname{sgn}(\boldsymbol{\sigma}) = -\beta\sqrt{V(\boldsymbol{\sigma})}$$

Convergence to the sliding mode in finite time

Stability of a sliding mode control system

SISO systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u(t), t)$$
$$\sigma = \mathbf{h}(\mathbf{x}, t)$$

$\mathbf{x} \in R^n \quad \sigma \in R \quad u \in R$

r is the relative-degree

$$y^{(r)} = \varphi(\mathbf{x}, u, t); \quad \frac{\partial y^{(i)}}{\partial u} = 0 \quad i = 1,2,\ldots,r-1 < n$$

The control can stabilise the output vector $\mathbf{y}$, but the internal dynamics $\mathbf{w}$ can be unstable for $\mathbf{y}$ =0 (zero-dynamics)

$$\sigma = \varphi'(\mathbf{y}, \mathbf{w}, u, t);$$
$$\dot{\mathbf{w}} = \psi(\mathbf{y}, \mathbf{w}, t)$$

$$\mathbf{y} = \left[\sigma, \dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r-1)}\right]^T,$$
$$\mathbf{w} = \left[w_1, w_2, \ldots, w_{n-r}\right]^T$$
$$\mathbf{x} = \Theta(\mathbf{y}, \mathbf{w}),$$
$$\varphi' = \varphi(\Theta(\mathbf{y}, \mathbf{w}), u, t)$$

Stability of a sliding mode control system

Example

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - x_2|x_2| + u$$

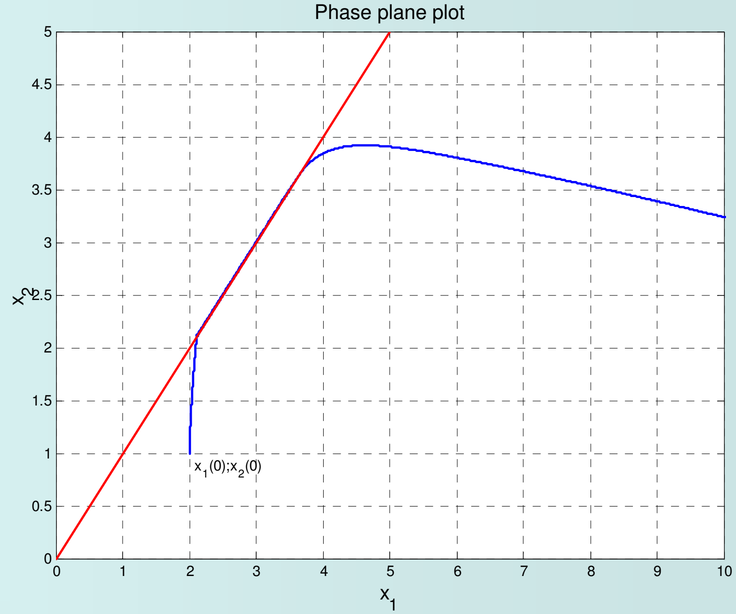
$$\sigma = x_2 - x_1$$

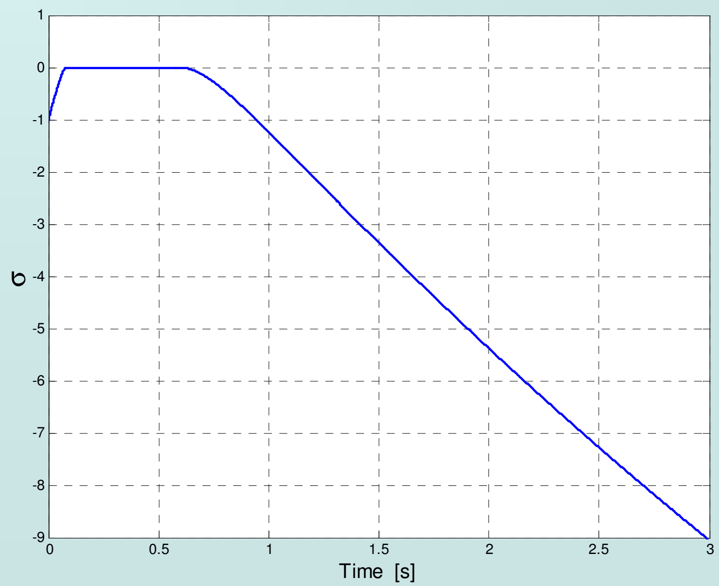
$$\dot{\sigma} = -x_2(1+|x_2|) - x_1 + u$$

$$x_1(0) = 2; x_2(0) = 1$$

$$u = -20\,\mathrm{sgn}\,\sigma$$

The unstable zero-dynamics causes the loss of the sliding mode behaviour





Invariance of a sliding mode control system

The system is invariant when constrained on the sliding manifold σ

$$\dot{x}_i = x_{i+1} \quad i = 1,2,\ldots n-1$$
$$\dot{x}_n = f(\mathbf{x},t) + b(\mathbf{x},t)u$$

The system is uncertain with known bounds

$$|f(\mathbf{x},t)| \le F(\mathbf{x}), \qquad 0 < b_m(\mathbf{x}) \le b(\mathbf{x},t)$$

c_i are chosen such that the corresponding polynomial is Hurwitz

$$\sigma = x_n + \sum_{i=1}^{n-1} c_i x_i$$
$$\dot{\sigma} = f(\mathbf{x},t) + b(\mathbf{x},t)u + \sum_{i=1}^{n-1} c_i x_{i+1}$$

Finite time convergence to the sliding manifold is assured

$$u = -\operatorname{sgn}\frac{F(\mathbf{x}) + \left|\sum_{i=1}^{n-1} c_i x_{i+1}\right| + k^2}{b_m(\mathbf{x})} \quad \Rightarrow \quad \sigma\dot{\sigma} \le -k^2\sigma$$

Invariance of a sliding mode control system

The system is invariant when constrained on the sliding manifold σ

$$\dot{x}_i = x_{i+1} \quad i = 1,2,\ldots n-2$$
$$\dot{x}_{n-1} = -\sum_{i=1}^{n-1} c_i x_i + \sigma$$
$$x_n = -\sum_{i=1}^{n-1} c_i x_i + \sigma$$

The system behaves as a reduced order system with prescribed eigenvalues

Matching uncertainties, included in the uncertain function f, are completely rejected

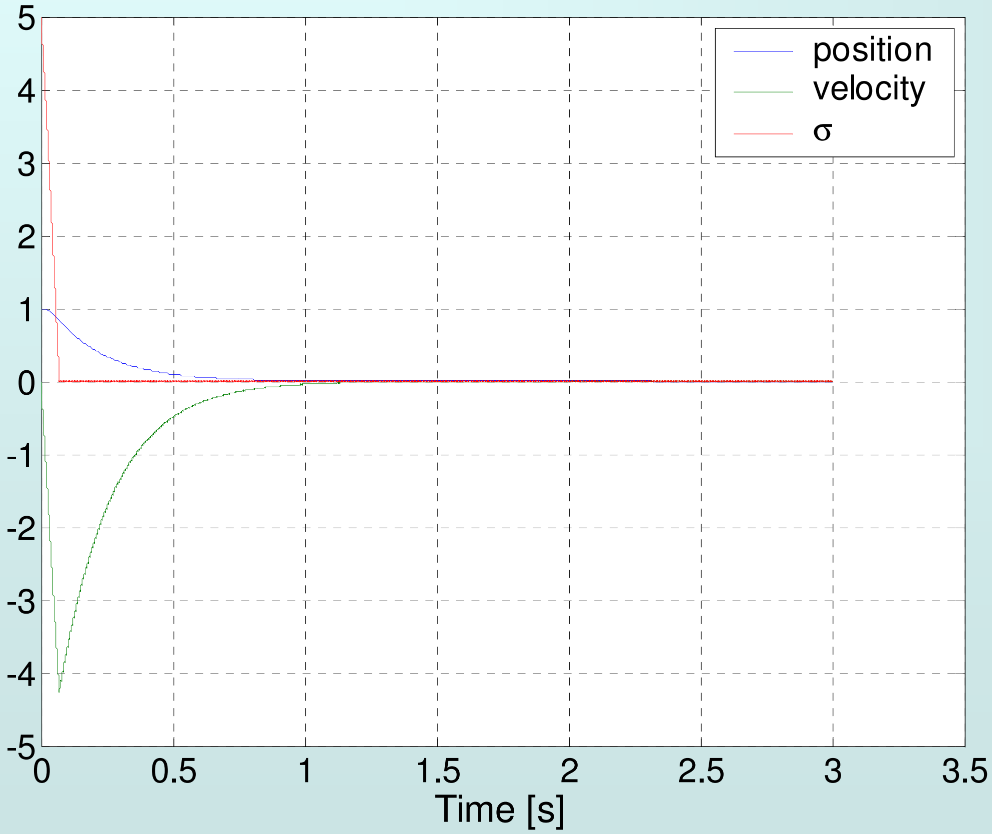
In the sliding mode it is not possible to “recover” the original system dynamics (*semi-group property*)

Take care of unstable zero-dynamics

Invariance of a sliding mode control system

$$m(t)\ddot{y}+\left(b_1+b_2\left|\dot{y}\right|\right)\dot{y}+b_3\,\mathrm{sgn}\left(\dot{y}\right)+\left(k_1+k_3y^2\right)y=u-\sin\left(\pi t\right)$$

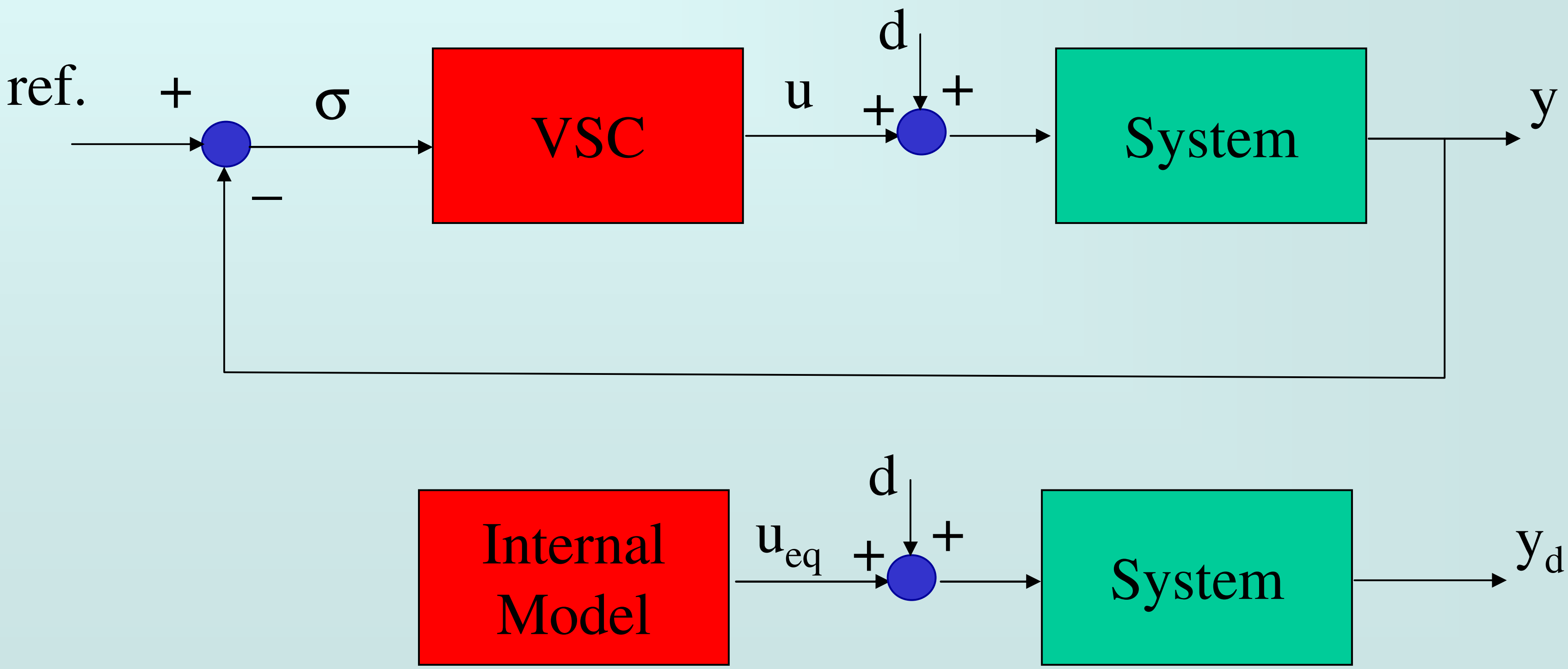
$$\sigma=\dot{y}+cy$$



$$u=-U\,\mathrm{sgn}\left(\sigma\right)$$

Invariance of a sliding mode control system

The invariance property during the sliding mode means that the "**Internal Model Principle**" is fulfilled



Invariance of a sliding mode control system

The controlled system dynamics belongs to a differential inclusion

$$\dot{\mathbf{x}} \in \mathsf{F} = f(\mathbf{x},t) + b(\mathbf{x},t)[-U, +U]$$

The sliding variable σ can be considered as a performance index to be nullified to find the “right” solution

$$\dot{\mathbf{x}}^{*} = f(\mathbf{x},t) + b(\mathbf{x},t)u_{eq} \in \mathsf{F}$$

Recovering the uncertain dynamics

$$\tau u_{av} + u_{av} = u$$

If u_{eq} is bounded with its time derivative then

$$\lim_{\substack{\tau \to 0 \\ \frac{\Delta}{\tau} \to 0}} u_{av} = u_{eq} \qquad |\sigma| \leq \Delta$$

The *cut-off frequency* of the low-pass filter must be

- Greater than the bandwidth of the equivalent control
- Lower than the real switching frequency

In practice only an estimate of u_{eq} can be evaluated

Recovering the uncertain dynamics

The *equivalent control* is the control signal that assures $\dot{\sigma} \equiv 0$

$$\sigma = x_n + \sum_{i=1}^{n-1} c_i x_i$$

$$\dot{\sigma} = f(\mathbf{x},t) + b(\mathbf{x},t)u + \sum_{i=1}^{n-1} c_i x_{i+1}$$

$$u_{eq} = -\frac{f(\mathbf{x},t) + \sum_{i=1}^{n-1} c_i x_{i+1}}{b(\mathbf{x},t)} \quad \Rightarrow \quad \dot{\sigma} \equiv 0$$

The *equivalent control* contains the information on the uncertain dynamics

$$U(j\omega) = U_{eq}(j\omega) + \overline{U}(j\omega)\Big|_{\omega=\infty}$$

The equivalent control can be estimated by a low-pass filter

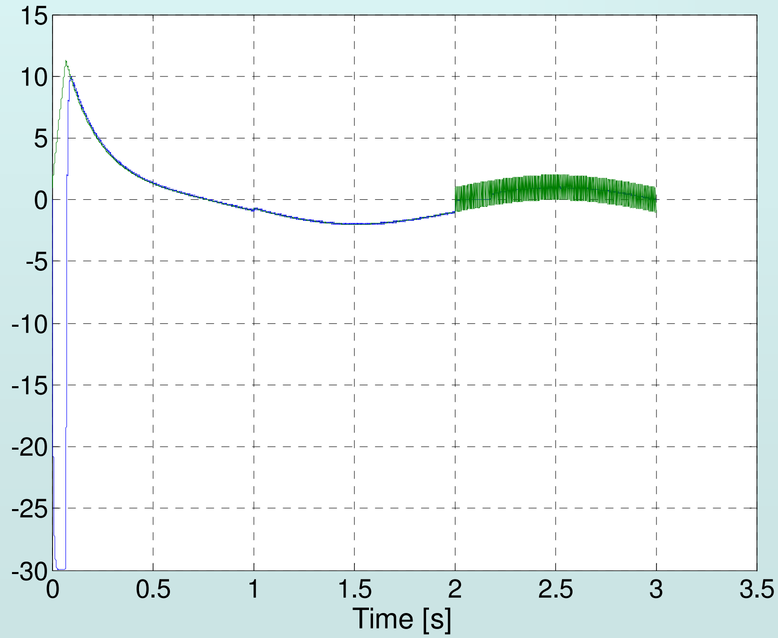
$$\tau u_{av} + u_{av} = u$$

Recovering the uncertain dynamics

$$m(t)\ddot{y}+\left(b_1+b_2\left|\dot{y}\right|\right)\dot{y}+b_3\,\mathrm{sgn}\left(\dot{y}\right)+\left(k_1+k_3y^2\right)y=u-\sin(\pi t)$$

$$\sigma=\dot{y}+cy$$

$$u_{eq}=+\left(b_1+b_2\left|\dot{y}\right|\right)\dot{y}+b_3\,\mathrm{sgn}\left(\dot{y}\right)+\left(k_1+k_3y^2\right)y+\sin(\pi t)+cm(t)\dot{y}$$



Approximability

The dynamics of the original system can be reduced to

$$\dot{x}_i = x_{i+1} \quad i = 1,2,\ldots n-2$$
$$\dot{x}_{n-1} = -\sum_{i=1}^{n-1} c_i x_i + \sigma$$
$$x_n = -\sum_{i=1}^{n-1} c_i x_i + \sigma$$

In the case of switching errors – the switching frequency is $f_s \leq \infty$ – the real trajectory $\mathbf{x}(t)$ is near to the ideal one $\mathbf{x}^*(t)$ and

$$\left\| \mathbf{x}(t) - \mathbf{x}^*(t) \right\| \xrightarrow[f_{s\to\infty}]{} 0$$

Assuming that the dynamics of the original system is bounded

$$\dot{\mathbf{x}} = f(\mathbf{x},t) + b(\mathbf{x},t)[-U,+U]$$
$$\left\| f(\mathbf{x},t) + b(\mathbf{x},t)u(\mathbf{x},t) \right\| \leq M + N\left\| \mathbf{x} \right\|$$

Approximability

Starting from a vicinity of the sliding surface

$$\sigma = C(\mathbf{x}) \cdot \mathbf{x}$$

$$\|\mathbf{x}_0\| \leq \Delta$$

By the Bellman-Gronwall lemma

$$\|\mathbf{x}(t)\| \leq M\|\mathbf{x}_0\| + MT + \int_0^T N\|\mathbf{x}\|dt$$

$$\|\mathbf{x}(t) - \mathbf{x}^*(t)\| \leq S\Delta + L\int_0^T \|\mathbf{x}(t) - \mathbf{x}^*(t)\|dt \leq H\Delta$$

If τ is the equivalent delay of the switching

$$\|\dot{\sigma}\| \leq \Sigma$$

$$\|\sigma\| \leq \|\sigma_0\| + \Sigma\tau$$

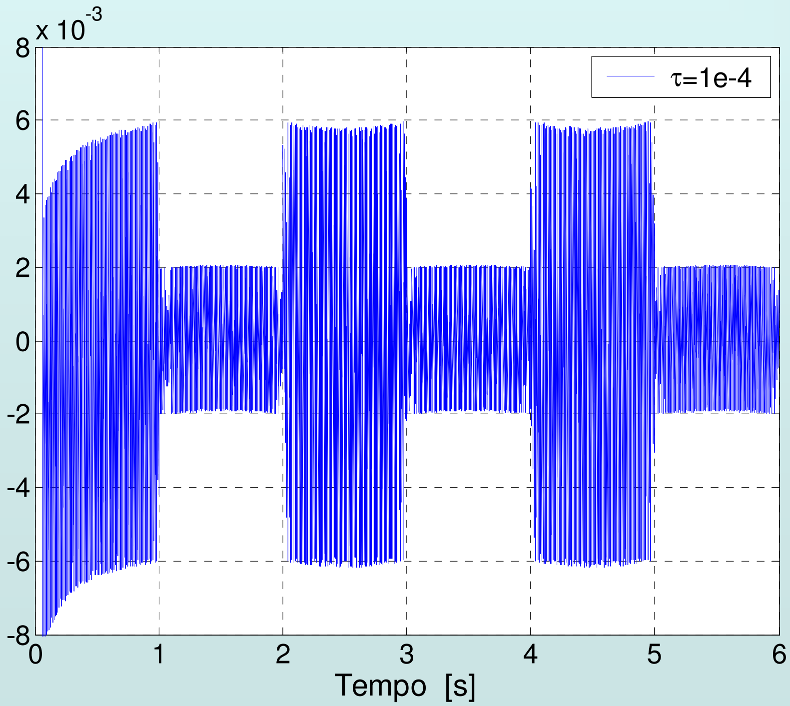
$$\Delta = D\tau$$

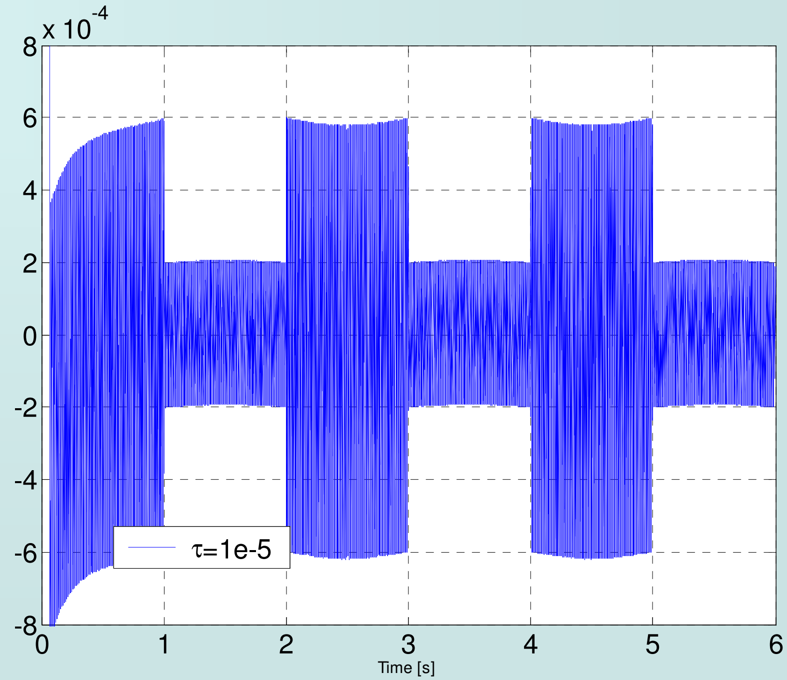
Approximability

The most common cause of delay switching is the digital implementation of the controller

$$m(t)\ddot{y} + (b_1 + b_2|\dot{y}|)y + (k_1 + k_3 y^2)y = u - \sin(\pi t)$$

$$\sigma = \dot{y} + cy$$





Discrete time implementation

Software implementation of the controller

Discrete-time sliding mode control

$$\bar{\mathbf{x}}[k+1] = \mathbf{A}_d \bar{\mathbf{x}}[k] + \mathbf{B}_d \boldsymbol{\sigma}[k]$$

$$\boldsymbol{\sigma}[k+1] = \boldsymbol{\Phi}_d(\bar{\mathbf{x}}[k], \boldsymbol{\sigma}[k], k) + \boldsymbol{\Gamma}_d \mathbf{u}[k]$$

$$\mathbf{u}[k] = -\alpha \,\mathrm{sgn}(\boldsymbol{\sigma}[k])$$

Discrete-time sliding mode control can appear also in systems with continuous right-hand-side (e.g., deadbeat control)

Discrete time sliding mode control is sensible only in the presence of uncertainties or disturbances

Discrete time implementation

What is a discrete time sliding mode?

$$\boldsymbol{\sigma}[k+1] = \mathbf{0} \qquad k = 0,1,2,\ldots$$

$$\boldsymbol{\sigma}[k+1] - \boldsymbol{\sigma}[k] = \mathbf{0} \qquad k = 0,1,2,\ldots$$

The second is not convincing and does not imply the first

The usual implementation of the control law has two parts

* the nominal part
* the discontinuous part to cope with uncertainties

Discrete time implementation

Usual accuracy of discrete time sliding modes is $O(T_c)$

Learning and adaptive methods $\longrightarrow$ $O(T_c^2)$

Multirate sampling allows for output-feedback implementation of the controller

Effective approach is constituted by continuous time design and subsequent discretization analysis

$\Downarrow$

Can chaotic behaviour appear ?!

Higher order sliding modes

The sliding mode concept can be extended to more complete integral manifolds

$$\sigma = \dot{\sigma} = \ddot{\sigma} = \cdots = \sigma^{(r-1)} = 0$$

- r constraints are introduced in the system dynamics
- The control acts on the r-th derivative of the sliding variable σ
- Accuracy is improved

$$\left|\sigma^{(i)}\right| = H_i \tau^{r-i}, \quad i = 0,1,2,\ldots,r-1$$

H_i does not depend on τ

Higher order sliding modes

- The internal dynamics has a reduced dimension n-r
- The robustness properties are preserved
- The approximability property is preserved
- Allows for chattering attenuation
- No general Lyapunov-like methods for the stability analysis are available (only very recent results for r=2 were presented)
- Phase plane of homogeneity approaches to state the closed-loop stability
- Derivatives of the sliding variable σ are needed

Higher order sliding modes

Higher Order Sliding Modes can be used to regularise nonlinear variable structure systems with sliding modes

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u(t), t) \qquad \mathbf{x} \in R^n \quad \sigma \in R \quad u \in R$$
$$\sigma = \mathbf{h}(\mathbf{x}, t)$$

$$\dot{\sigma} = \frac{\partial h(\mathbf{x}, t)}{\partial t} + \frac{\partial \mathbf{h}(\mathbf{x}, t)}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, u(t), t) = \phi(\mathbf{x}, u(t), t)$$
$$\ddot{\sigma} = \frac{\partial \phi(\mathbf{x}, u(t), t)}{\partial t} + \frac{\partial \phi(\mathbf{x}, u(t), t)}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, u(t), t) + \frac{\partial \phi(\mathbf{x}, u(t), t)}{\partial u} \dot{u} = \varphi(\mathbf{x}, u(t), t) + \gamma(\mathbf{x}, u(t), t) v$$

If a discontinuous v constrains the system onto the 2nd order manifold

$$\sigma = \dot{\sigma} = 0$$

Filippov's solution and the "Equivalent Control" coincide, and the control u is continuous (**Chattering attenuation**)

Higher order sliding modes

The control algorithms for Higher-Order Sliding Modes require derivatives of σ

Relative degree	Feedback signals	VSS controller
1	$sign(s)$	*Classical 1-SMC, Super-Twisting*
2	$s, \dot{s}$	*Dynamic / Terminal 2-SMC*
	$s, \{sign(\dot{s})\}$	*Sub-optimal 2-SMC*
	$sign(s), sign(\dot{s})$	*Twisting 2-SMC*
3	$sign(s), sign(\dot{s}), sign(\ddot{s})$	*Hybrid 3-VSC*
Any r	$s, \dot{s}, \ldots, s^{(r-2)}, sign(s^{(r-1)})$	*Universal HOSM*

Higher order sliding modes

Dynamical sliding modes

$$s = x_n + \sum_{1}^{n-1} c_i x_i$$

$$\sigma = s^{(r-1)} + \sum_{1}^{r-2} \gamma_i s^{(i-1)}$$

Starting from a classical sliding variable s, a new switching variable σ is defined

Sliding mode is achieved asymptotically

Terminal sliding modes

$$s = x_n + \sum_{1}^{n-1} c_i x_i$$

$$1 < \frac{p}{q} < 2, \;\; p, q \;\; \text{odd}$$

$$\sigma = s + \gamma \dot{s}^{\frac{p}{q}}$$

A nonlinear switching surface is defined

Sliding mode is achieved in finite time

Higher order sliding modes

Super-Twisting controller

$$\dot{\sigma} = \varphi(\bullet) + \gamma(\bullet)u$$

$$u = -\alpha\sqrt{|\sigma|} + z$$

$$\dot{z} = -\lambda \,\mathrm{sgn}(\sigma)$$

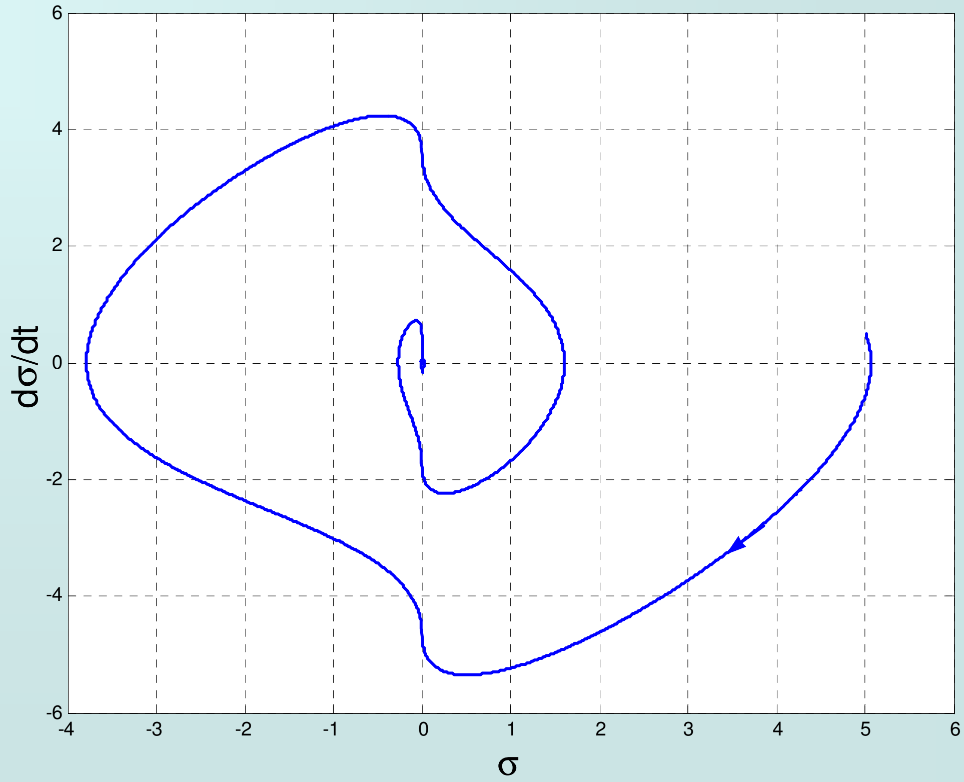
$$\lambda > \frac{\Phi}{\Gamma_m}$$

$$\alpha^2 \geq \frac{4\Phi}{\Gamma_m^2}\frac{\Gamma_M(\lambda+\Phi)}{\Gamma_m(\lambda-\Phi)}$$

Φ: upper bound of the drift term

Γ_m, Γ_M: positive lower and upper bounds of the gain term

The control is continuous



Higher order sliding modes

Suboptimal-controller

$$\ddot{\sigma} = \varphi(\bullet) + \gamma(\bullet)u$$

$$u = -\alpha(t)U\,\mathrm{sgn}(\sigma - \beta\sigma_M)$$

$$\alpha(t) = \begin{cases} 1 & (\sigma - \beta\sigma_M)\sigma_M \geq 0 \\ \overline{\alpha} & (\sigma - \beta\sigma_M)\sigma_M < 0 \end{cases}$$

$$\beta \in (0;1)$$

$$U > \frac{\Phi}{\Gamma_m}$$

$$\overline{\alpha} \in [1;+\infty) \cap \left(\frac{2\Phi + \Gamma_M U(1-\beta)}{\Gamma_m U(1+\beta)};+\infty \right)$$

Φ: upper bound of the drift term

Γ_m, Γ_M: positive lower and upper bounds of the gain term

The control is discontinuous

Higher order sliding modes

Suboptimal-controller

$$u = -\alpha(t) U \,\mathrm{sgn}(\sigma - \beta\sigma_M)$$

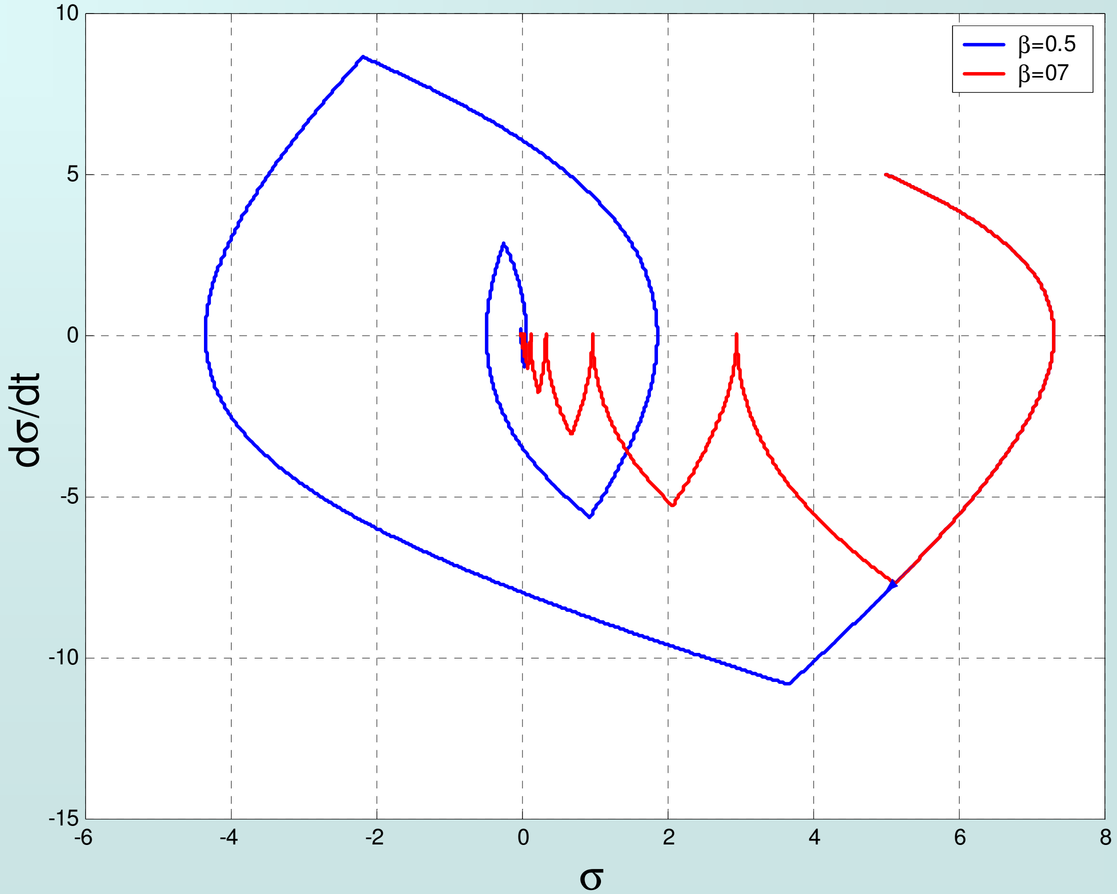
$$\alpha(t) = \begin{cases} 1 & (\sigma - \beta\sigma_M)\sigma_M \geq 0 \\ \overline{\alpha} & (\sigma - \beta\sigma_M)\sigma_M < 0 \end{cases}$$

$$\beta \in (0;1)$$

β: anticipation factor

α: modulation factor

U: control gain



Higher order sliding modes

Universal quasi-continuous HOSM-controller

$$u = -\alpha \Psi_{n-1,n}(e_y, \dot{e}_y, \ldots, e_y^{(n-1)}).$$

$e_y \equiv \sigma$

$$\begin{aligned}
\varphi_{0,n} &= e_y, \quad N_{0,n} = |e_y| \\
\Psi_{0,n} &= \varphi_{0,n}/N_{0,n} = \text{ sign } e_y, \\
\varphi_{i,n} &= e_y^{(i)} + \beta_i N_{i-1,n}^{(n-i)/(n-i+1)} \Psi_{i-1,n}, \\
N_{i,n} &= |e_y^{(i)}| + \beta_i N_{i-1,n}^{(n-i)(n-i+1)}, \\
\Psi_{i,n} &= \varphi_{i,n}/N_{i,n}
\end{aligned}$$

i=1, 2,.., n-1

Homogeneous nested implementations of discontinuous controllers

Final remarks

- ✓ **Sliding Modes** are a usual behaviour in switching systems
- ✓ Sliding Modes are a useful tool for controlling **uncertain dynamical systems**
- ✓ Switching control with Sliding Modes is a simple way of applying the **Internal Model Principle**
- ✓ By resorting to the **Equivalent Control** definition it is possible to retrieve some information about an uncertain system by low-pass filters
- ✓ Ideal Sliding Modes are not implemented in practice because they require infinite frequency switching, and only an **approximate sliding** can be achieved
- ✓ **Higher Order Sliding Modes** generalise the concept of sliding modes to integral manifolds
- ✓ Real Higher Order Sliding Modes improve the **accuracy** and attenuate the **chattering** effect