Intersection and sum of two vector spaces and the relationship between their dimensions

Intersection and sum of two vector subspaces

Definition: Let V be a vector space, and let U and W be subspaces of V . Then:

1. $U + W = \{u + w : u \in U \wedge w \in W\}$ and is called the sum of U and W.
2. $U \cap W = \{v : v \in U \wedge v \in W\}$ and is called the intersection of U and W.

Example: Consider the plans $P_1 = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and $P_2 = \{(x, y, z) \in \mathbb{R}^3 : x-y+z = 0\}$. These are both subspaces of $\mathbb{R}^3$, that we can define by its generic vectors as:

- $P_1 = \{(x_1, y_1, 0) : x_1, y_1 \in \mathbb{R}\}$;
- $P_2 = \{(x_2, y_2, y_2 - x_2) : x_2, y_2 \in \mathbb{R}\}$.

Their intersection is the subspace

$$P_1 \cap P_2 = \{(x, y, z) \in \mathbb{R}_3 : z = 0 \wedge x - y = 0\} = \{(x, x, 0) : x \in \mathbb{R}\}.$$

The sum,

$$P_1 + P_2 = \{(x_1 + x_2, y_1 + y_2, y_2 - x_2) : x_1, x_2, y_1, y_2 \in \mathbb{R}\}$$

is the vector space $\mathbb{R}^3$.

Relationship between the dimensions of the vector spaces sum and intersection of two vector spaces

In relation to the previous example, we easily determine the size of each of the spaces P_1, P_2, $P_1 \cap P_2$ and $P_1 + P_2$.

Just consider the number of free variables of the generic vector and we conclude that both subspaces $P_1 = \{(x, y, 0) : x, y \in \mathbb{R}\}$ and $P_2 = (\{x, y, y - x) : x, y \in \mathbb{R}\}$ of $\mathbb{R}^3$ have dimension 2.

Their intersection is the subspace $P_1 \cap P_2 = \{(x, x, 0) : x \in \mathbb{R}\}$ whith dimension 1 and the subspace $P_1 + P_2$ has dimension 3, because $P_1 + P_2$ spans $\mathbb{R}^3$. We have

$$dim(P_1 + P_2) = dim(P_1) + dim(P_2) - dim(P_1 \cap P_2).$$

It will be always like this? The answer is affirmative:

Theorem Dimension of sum: Let V be a vector space with subspaces U and W, each one of them have finite dimension. Then $U + W$ also has finite dimension which is given by

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W).$$

Example: Let the vector subspaces $S_1 = \left\{ \begin{bmatrix} a & 0 \\ 3a & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ and $S_2 = \left\{ \begin{bmatrix} c & d \\ -d & e \end{bmatrix} : c, d, e \in \mathbb{R} \right\}$ of the space of the square matrices of order 2.

S_1 has dimension 2, S_2 has dimension 3 and their intersection, $S_1 \cap S_2 = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{R} \right\}$, has dimension 2.

Then the sum, $S_1 + S_2$ is a subspace with dimension $2 + 3 - 2 = 3$.

Notice that

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis of S_1 and

$$C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis of S_2.

Then

$$D = \left\{ \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

spans $S_1 + S_2$.

As the equality

$$k_1 \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + k_3 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + k_5 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

represents a system, whose expanded matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

We can conclude that the system is doubly indeterminate, so D is linearly dependent. Thus, the minimum set that generates $S_1 + S_2$ has cardinality 3.