Concept of Linear Transformation

Definition: Let U and V be two real vector spaces. $T: U \to V$ is a linear transformation if:

i. $\forall\, x, y \in U,\ T(x+y) = T(x) + T(y)$
ii. $\forall\, x \in U,\ \forall\, \alpha \in \mathbb{R},\ T(\alpha x) = \alpha\, T(x)$

1. Prove that the transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x, y) = (2x, y, -y)$ is linear.

i. Considering $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we have:

$$\begin{aligned} T\big((x_1, y_1) + (x_2, y_2)\big) &= T(x_1 + x_2, y_1 + y_2) \\ &= (2(x_1 + x_2), y_1 + y_2, -(y_1 + y_2)) \\ &= (2x_1 + 2x_2, y_1 + y_2, -y_1 - y_2) \end{aligned}$$

On the other side,

$$\begin{aligned} T(x_1, y_1) + T(x_2, y_2) &= (2x_1, y_1, -y_1) + (2x_2, y_2, -y_2) \\ &= (2x_1 + 2x_2, y_1 + y_2, -y_1 - y_2) \end{aligned}$$

We concluded that,

$$T\big((x_1, y_1) + (x_2, y_2)\big) = T(x_1, y_1) + T(x_2, y_2), \forall\, (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$$

☺ **The first condition of linearity of a transformation is proved.**

ii. Considering $(x_1, y_1) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, we have:

$$\begin{aligned} T\big(\alpha(x_1, y_1)\big) &= T(\alpha x_1, \alpha y_1) = (\alpha 2x_1, \alpha y_1, -\alpha y_1) \\ &= \alpha(2x_1, y_1, -y_1) = \alpha T(x_1, y_1) \end{aligned}$$

We concluded that,

$$T\big(\alpha(x_1, y_1)\big) = \alpha T(x_1, y_1), \forall\, (x_1, y_1) \in \mathbb{R}^2, \forall\, \alpha \in \mathbb{R}$$

☺ **The second condition of linearity is also verified.**

Conclusion: **Since both linearity conditions are verified, T is a linear transformation.**

2. The transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(x, y) = (x, 1 + y)$ is linear?

i. Considering $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we have:

$$\begin{aligned} T\big((x_1, y_1) + (x_2, y_2)\big) &= T(x_1 + x_2, y_1 + y_2) \\ &= (x_1 + x_2, 1 + y_1 + y_2) \end{aligned}$$

On the other side,

$$\begin{aligned} T(x_1, y_1) + T(x_2, y_2) &= (x_1, 1 + y_1) + (x_2, 1 + y_2) \\ &= (x_1 + x_2, 2 + y_1 + y_2) \end{aligned}$$

We concluded that,

$$\exists\ (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 : T\big((x_1, y_1) + (x_2, y_2)\big) \neq T(x_1, y_1) + T(x_2, y_2)$$

☹ **The first condition of linearity of a transformation is not verified.**

Conclusion: **As the first linearity condition is not verified, we concluded that *T* is not a linear transformation.**