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Cross product and related properties

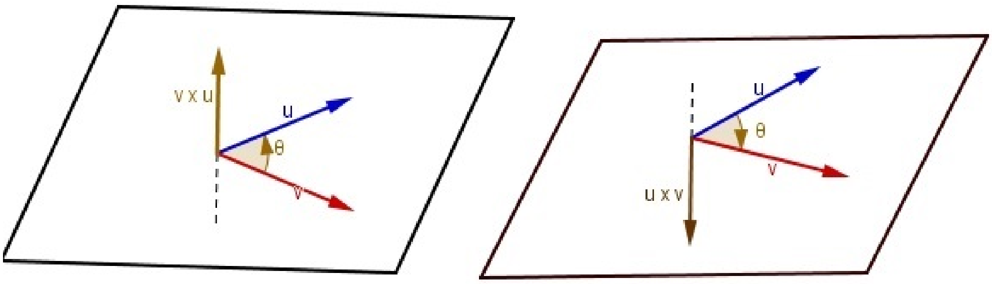
Vector Product (Cross product)

The vector product of two vectors u and v is a vector $u \times v$ that is at right angles to both and is defined by

$$u \times v = ||u||||v||sin(\widehat{uv})n, \quad \text{with} \quad ||n|| = 1 \quad \text{and} \quad u, v \perp n.$$

Specifically,

1. $u \times v$ is perpendicular to the vectors u and v;
2. $||u \times v|| = ||u|| \cdot ||v||| \sin(\widehat{uv})|$;
3. $u \times v$ has sense determined by the right hand (follow with the fingers of the right hand, the rotation movement of the vector u to approach v and consider the direction of the thumb).



Notice that:

- $u \times v$ is orthogonal to the plane containing the vectors;
- $u \times v = 0$ when vectors u and v point in the same, or opposite, direction.

In the 3-dimensional Cartesian system, the vector product of vectors $u = (u_1, u_2, u_3)$ e $v = (v_1, v_2, v_3)$ is defined as

$$u \times v = (u_2v_3 - v_2u_3, v_1u_3 - u_1v_3, u_1v_2 - v_1u_2).$$

It is a vector perpendicular to the vectors u and v and can more easily be represented matrix-wise as:

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - v_2u_3)i - (u_1v_3 - v_1u_3)j + (u_1v_2 - v_1u_2)k.$$

Example: $(1, 2, -1) \times (2, 0, 1) = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & 0 & 1 \end{vmatrix} = 2i - 3j - 4k = (2, -3, -4)$

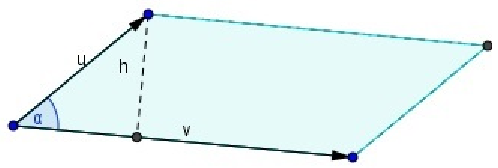
Regarding the previous example, note that $(2,-3,-4)\cdot(1,2,-1) = 2-6+4 = 0$ and $(2,-3,-4)\cdot(2,0,1) = 4-4 = 0$.

Properties: Be the vectors $u, v, w \in \mathbb{R}^3$. We have

1. $u \times v \times w = u \times (v \times w)$ (associative);
2. $u \times v = -v \times u$ (anti-commutative);
3. $u \times v = 0 \Leftrightarrow u = 0 \vee v = 0 \vee \widehat{uv} = 0^{\text{o}} \vee \widehat{uv} = 180^{\text{o}}$.

Example: $(1,-2,3) \times (-2,4,-6) = \begin{vmatrix} i & j & k \\ 1 & -2 & 3 \\ -2 & 4 & -6 \end{vmatrix} = (0,0,0)$,
because the vectors $(1,-2,3)$ e $(-2,4,-6)$ are collinear.

The norm of the vector product $||u \times v|| = ||u|| \cdot ||v|| |\sin(\widehat{uv})|$ the area of the parallelogram determined by u and v.



In effect, according to the figure above, the area of the parallelogram is given by $A = ||v|| \cdot h$. Besides that, $||u|| \sin(\widehat{uv}) = h$.