Subspace of a vector space

Definition: Let V be a vector space, and let W be a subset of V. If W is a vector space with respect to the operations in V, then W is called a subspace of V.

For example, the vector space $A = \{(x, y, z) \in \mathbb{R}^3 : 2x - y + 3z = 0\}$ is a subspace of $\mathbb{R}^3$.

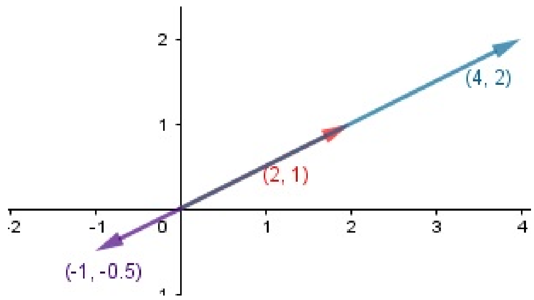
Theorem: Let V be a vector space, with operations $+$ and $\cdot$, and let W be a subset of V. Then W is a subspace of V if and only if the following conditions hold.

- W is nonempty;
- If u and v are any vectors in W, then $u + v \in W$ (closure under $+$);
- If $v \in W$, and $c \in \mathbb{R}$, then $c \cdot v \in W$ (closure under $\cdot$).

Note that if W is a vector subspace, then the null vector must belong to W.

Example:
$A = \{(x, y) \in \mathbb{R}^2 : x - 2y = 0\}$ is a subspace of $\mathbb{R}^2$;

Indeed, $A = \{(2y, y) : y \in \mathbb{R}\}$ and we have:



- $(0, 0) \in A$;
- If $u = (2u_2, u_2)$ and $v = (2v_2, v_2)$, then

  $u+v = (2u_2+2v_2, u_2+v_2) = (2(u_2+v_2), u_2+v_2) \in A$;
- If $u = (2u_2, u_2)$ and $k \in \mathbb{R}$, then

$$ku = (2(ku_2), ku_2) \in A.$$

Example: $S = \left\{ \begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $M_{2\times 2}$.

However, the set of polynomials $P_1 = \{a_0 + a_1 x + a_2 x^2 : a_0 + a_1 - a_2 = 3\}$ is not a subspace of de vector space $P = \{a_0 + a_1 x + a_2 x^2 : \quad a_0, a_1, a_2 \in \mathbb{R}\}$. In fact, the null polynomial does not belong to P_1.

Also $A = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$ is not a subspace of $\mathbb{R}^2$. In fact, $u = (u_1, u_1^2), v = (v_1, v_1^2) \in A$, but $u + v = (u_1 + v_1, u_1^2 + v_1^2)$ does not always belong to A. There are vectors $(u_1, u_2), (v_1, v_2) \in \mathbb{R}^2$ such that

$$u_1^2 + v_1^2 \neq (u_1 + v_1)^2.$$

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