Teaching material. Systems of linear equations

Exercise. Perform Gauss-Jordan elimination on the following matrices:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 2 & -1 \end{bmatrix}.$$

Then, for each matrix, solve the system of linear equation whose augmented matrix is the given matrix.

Solution.

(1) By row operations (*i.e.* left multiplication by elementary matrices) we reduce the matrix in row echelon form:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{E_{31}(-1)} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & -1 \end{bmatrix} \xrightarrow{E_{32}(-2)} \begin{bmatrix} 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & \boxed{-5} \end{bmatrix}$$

To achieve its reduced row echelon form, we divide each non zero row by its pivot and we perform row operations to set to zero the entries above each pivot:

$$\dots \xrightarrow{E_3(-\frac{1}{5})} \begin{bmatrix} 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow{E_{13}(-1)} \begin{bmatrix} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow{E_{23}(-2)} \begin{bmatrix} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

The corresponding system is therefore equivalent to $\begin{cases} y = 0 \\ z = 0 \\ \boxed{0=1} \end{cases}$, inconsistent: solution set $\emptyset$.

(2) As before:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{E_{13}} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{E_{21}(-1)} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{E_{31}(-1)} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & -2 \end{bmatrix} \xrightarrow{E_{23}} \begin{bmatrix} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & -1 \end{bmatrix}$$

Which is already in reduced row echelon form. The corresponding system is $\begin{cases} x = 3 \\ y = -1 \\ z = -1 \end{cases}$, solution set: $\{(3; -1; -1)\}$.

(3) Row echelon form:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 2 & -1 \end{bmatrix} \xrightarrow{E_{21}(-2)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 3 & -2 \\ 1 & 1 & 2 & -1 \end{bmatrix} \xrightarrow{E_{31}(-1)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 3 & -2 \end{bmatrix} \xrightarrow{E_{32}(-1)} \begin{bmatrix} \boxed{1} & 1 & -1 & 1 \\ 0 & 0 & \boxed{3} & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon form:

$$\dots \xrightarrow{E_2(\frac{1}{3})} \begin{bmatrix} \boxed{1} & 1 & -1 & 1 \\ 0 & 0 & \boxed{1} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{E_{12}(1)} \begin{bmatrix} \boxed{1} & 1 & 0 & \frac{1}{3} \\ 0 & 0 & \boxed{1} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system is $\begin{cases} x + y = \frac{1}{3} \\ z = -\frac{2}{3} \\ 0 = 0 \end{cases} \iff$ (choosing y as free variable) $\begin{cases} x = \frac{1}{3} - y \\ y = y \\ z = -\frac{2}{3} \end{cases}$. Thus, its solution set is $\left\{\left(\frac{1}{3} - y; y; -\frac{2}{3}\right) \mid y \in \mathbb{R}\right\}$.