Finding the analytical expression of linear transformation from the image vectors of a basis of the starting space

- **Consider the basis $\boldsymbol{S=\{(1,4),(-2,1)\}}$ for $\mathbb{R}^2$.**
  Let $\boldsymbol{T}:\mathbb{R}^2\to\mathbb{R}^3$ be the linear transformation for which
  $$\boldsymbol{T(1,4)=(4,-1,1)\ \text{ and }\ T(-2,1)=(0,-2,3)}$$
  (a) Determine $\boldsymbol{T(-13,-7)}$.
  (b) Find a formula for $\boldsymbol{T(x,y)}$.

Notice that

Let $T: U\to V$ be a linear transformation, where U is finite-dimensional. If $S=\{u_1,u_2,\dots,u_n\}$ is a basis for U and $u=c_1u_1+c_2u_2+\cdots+c_nu_n$, for $u\in U$, $c_1,c_2,\dots,c_n\in\mathbb{R}$, then $$\boldsymbol{T(u)=c_1T(u_1)+c_2T(u_2)+\cdots+c_nT(u_n)}$$

(a) Attend to

$$(-13,-7)=-3(1,4)+5(-2,1)$$

we have

$$\begin{aligned}T(-13,-7)&=-3T(1,4)+5T(-2,1)\\&=-3(4,-1,1)+5(0,-2,3)\\&=(-12,3,-3)+(0,-10,15)\\&=(-12,-7,12)\end{aligned}$$

Thus, $T(-13,-7)=(-12,-7,12)$.

(b) We must begin to find the coordinates of (x,y) on S basis, this is α and β.

$$(x,y)=\alpha(1,4)+\beta(-2,1)$$

$$\begin{cases}\alpha-2\beta&=&x\\4\alpha+\beta&=&y\end{cases}\Leftrightarrow\begin{cases}\alpha&=&x+2\beta\\4\alpha+\beta&=&y\end{cases}\Leftrightarrow\begin{cases}\alpha&=&x+2\beta\\4(x+2\beta)+\beta&=&y\end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha & = & x+2\beta \\ 4x+9\beta & = & y \end{cases} \Leftrightarrow \begin{cases} \alpha & = & x+2\beta \\ \beta & = & \dfrac{y-4x}{9} \end{cases} \Leftrightarrow \begin{cases} \alpha & = & x+\dfrac{2y-8x}{9} \\ \beta & = & \dfrac{y-4x}{9} \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha & = & \dfrac{2y+9x-8x}{9} \\ \beta & = & \dfrac{y-4x}{9} \end{cases} \Leftrightarrow \begin{cases} \alpha & = & \dfrac{2y+x}{9} \\ \beta & = & \dfrac{y-4x}{9} \end{cases}$$

So

$$(x,y) = \frac{2y+x}{9}(1,4) + \frac{y-4x}{9}(-2,1)$$

And

$$T(x,y) = \frac{2y+x}{9}T(1,4) + \frac{y-4x}{9}T(-2,1)$$

This is,

$$T(x,y) = \frac{2y+x}{9}(4,-1,1) + \frac{y-4x}{9}(0,-2,3)$$

$$= \left(\frac{8y+4x}{9}, \frac{-2y-x-2y+8x}{9}, \frac{2y+x+3y-12x}{9}\right)$$

$$= \left(\frac{4x+8y}{9}, \frac{7x-4y}{9}, \frac{-11x+5y}{9}\right)$$

Thus,

$$\boldsymbol{T(x,y) = \left(\frac{4x+8y}{9}, \frac{7x-4y}{9}, \frac{-11x+5y}{9}\right)}$$

Suggestion: Determine $T(-13,-7)$ using that formula.