Author: Edite Martins Cordeiro
September 2020

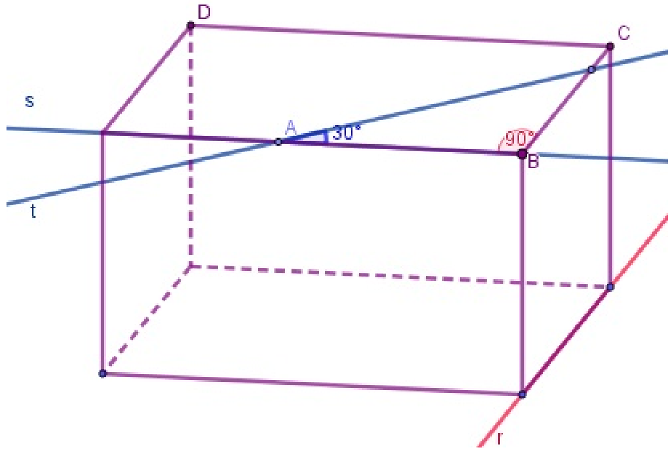
Angle between planes or lines

Angle between two lines

The angle between two lines is defined as as smallest angle between their directions.

In the figure to the side we can see that:

- The angle of the straight lines s and t belonging to the ABC plane measures 30º.
- The angle of the reverse lines r and s is of 90º (equal to the angle between lines BC and s in the same plane).



So, the angle between two reverse lines (which do not intersect and are not parallel to each other) is the acute angle that one forms with a line parallel to the other.

Example: Let us consider the lines

$$r : (x, y, z) = (1, 2, 0) + k(2, 1, 3), k \in \mathbb{R} \quad \text{and} \quad s : (x, y, z) = (0, -1, -1) + t(3, 2, 1), t \in \mathbb{R}$$

of $\mathbb{R}^3$, whose directions are those of the non-collinear vectors $u = (2, 1, 3)$ and $v = (3, 2, 1)$, respectively. We can see that r and s do not intersect. In fact,

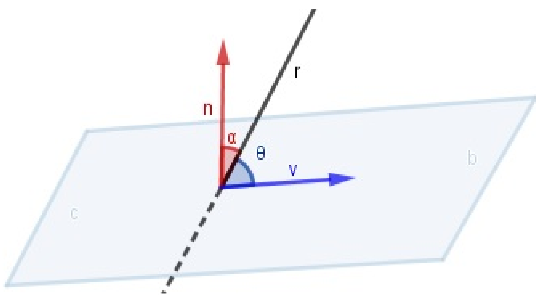
$$(1, 2, 0) + k(2, 1, 3) = (0, -1, -1) + t(3, 2, 1) \Leftrightarrow \begin{cases} 2k - 3t = -1 \\ k - 2t = -3 \\ 3k - t = -1 \end{cases} \Leftrightarrow \begin{cases} k = -\dfrac{2}{7} \\ k = \dfrac{1}{5} \\ t = 3k + 1 \end{cases}$$

So r and s are reverse lines.

Besides that, $cos(\hat{rs}) = |cos(\hat{uv}| = \dfrac{|u \cdot v|}{|u||v|} = \dfrac{6 + 2 + 3}{\sqrt{14}\sqrt{14}} = \dfrac{11}{14}$, that is, $\hat{rs} = 23, 6^{\text{o}}$.

Angle between a line and a plane

The angle θ between a line r and a plane p is defined as the angle complementary to the acute angle between the direction vector on this line and the vector normal to the plane, according to the image on the side.



If u has the direction line r and n is a vector normal to the plane p, then

$$sin(\theta) = sin(\frac{\pi}{2} - \alpha) = cos(\alpha) = \frac{|u \cdot n|}{|u||n|}.$$

Example: The line $r : (x, y, z) = (1, -2, 0) + k(2, 2, 0), k \in \mathbb{R}$ has a direction given by $u = (2, 2, 0)$ and the plane $p : 2x + y + z - 1 = 0$ is orthogonal to $n = (2, 1, 1)$.

Then the angle formed by r and p is such that

$$sin(\hat{rp}) = cos(\hat{nu}) = \frac{(4, 2, 2) \cdot (2, 2, 0)}{\sqrt{16 + 4 + 4}\sqrt{4 + 4}} = \frac{\sqrt{3}}{2}.$$

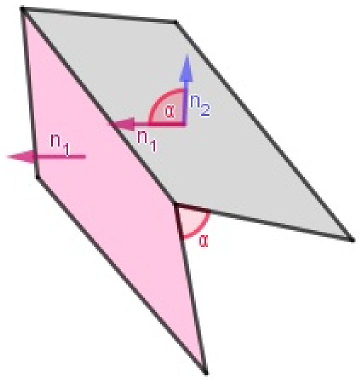
That is, $\hat{rp} = \frac{\pi}{3}$.

Angle between two planes

Definition: The angle between planes is equal to a angle between their normal vectors.

Consider the equation plans $\pi_1 : a_1x + b_1y + c_1z + d_1 = 0$ and $\pi_2 : a_2x + b_2y + c_2z + d_2 = 0$.
In this case, $n_1 = (a_1, b_1, c_1)$ and $n_2 = (a_2, b_2, c_2)$. Then, by the scalar product, we have:

$$cos(\alpha) = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}.$$



Example: The angle α formed by the $p_1 : 2x - y + z = 0$ and $p_2 = x + 2y - z + 1 = 0$ planes is such that

$$cos(\alpha) = \frac{(2, -1, 1) \cdot (1, 2, -1)}{\sqrt{6}\sqrt{6}} = \frac{-1}{6}.$$

Then $\alpha = 163^o$.