Tree

A simple graph G is called a **forest** if it contains no circuits. A **tree** is a connected graph which contains no circuits. In other words a tree is a connected component of a forest. A **trivial tree** is a graph that consists of a single vertex. The vertices with degree one are called **leafs**.

Example 1. The graph picture is a forest with two trees.

Proposition 1. Let $G = (V,\ E)$ be a graph with n vertices, then the following statements are equivalent:

(a) G is a tree;

(b) G is an acyclic graph and has $n-1$ edges;

(c) G is connected and and has $n-1$ edges;

(d) G is connected and each edge is a bridge;

(e) $\forall v, w \in V$ there is precisely one path between v and w;

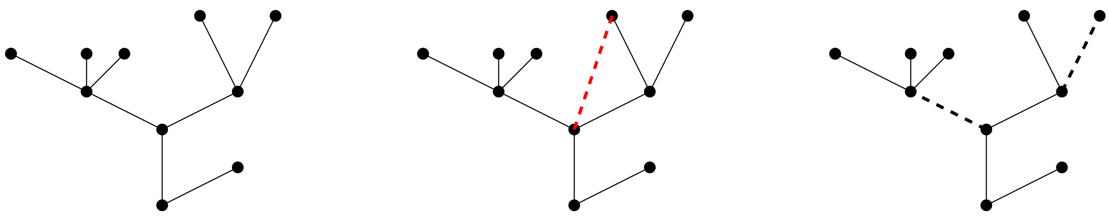
(f) G is an acyclic graph but adding an edge we obtain a cycle.

Proof. Let $G = (V,\ E)$ be a graph with n vertices.
(a) $\Rightarrow$ (b) We will prove by mathematical induction on the number of the vertices n. If $n = 1$, the only tree with one vertex is the trivial tree and $0 = n-1$ edges, then the implication is true. Suppose now, that the implication is true for all trees with less than $n \geq 2$ vertices. Since, by definition, G does not contain cycles, the removal of any edge subdivides the graph into two components G_1 and G_2, each of which is a tree. Considering that G_1 has n_1 vertices nad G_2 has n_2 vertices, in witch $n = n_1 + n_2$, by induction hypothesis, G_1 has $n_1 - 1$ edges, G_2 has $n_2 - 1$ edges, thus G has $n_1 - 1 + n_2 - 1 + 1 = n - 1$ edges.
(b) $\Rightarrow$ (c) Suppose that G is not connected. Then, each component of G is a connected graph without circuits, so, by hypothesis, the number of vertices of each component exceeds the number of edges by one unit. Hence, the total number of vertices of G, exceeds that in a total number of edges of G by at least two units, contradicting the hypothesis that G has $n-1$ edges.
(c) $\Rightarrow$ (d) As G is connected, with $n-1$ edges, the removal of any edge produces a graph with n vertices and $n-2$ edges and, consequently, this graph is not connected, since a connected graph of order n has at least $n-1$ edges.
(d) $\Rightarrow$ (e) Given two arbitrary vertices u and v, by definition of the connected graph, there is a

path between u and v. Since, by hypothesis, any edge of that path is a bridge, we can conclude that the path is unique.
(e) $\Rightarrow$ (f) Assuming that G contains a cycle, then any two vertices of that cycle are connected by at least two paths and, consequently, there are vertices of G that are connected by more than one path. Therefore, if there is a single path between any two vertices of G, then G does not contain cycles. However, adding an edge between two vertices u and v, as, for hypothesis, there is already a path between u and v, we create a cycle
(f) $\Rightarrow$ (a) Note that it is sufficient to prove that if G satisfies the hypothesis then it is connected. Suppose G satisfies the hypothesis, but it is not connected. If we add an edge to G, connecting two vertices belonging to different components, no cycle is created, which is a contradiction.

$\square$

Example 2. The following connected graph is a tree with 10 vertices and 9 edges. Adding an edge we obtain a cycle. Deleting an edge we obtain a disconnected graph



Proposition 2. Each non-trivial tree contains at least two vertices of degree one (which are called leaf).

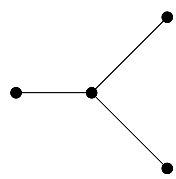
Proof. Let $G = (V, E)$ be a non-trivial tree with n vertices. As the tree is connected then for all vertex $v \in V$ has degree $deg(v) \geq 1$. Recall that $\sum_{v \in V} \deg(v) = 2|E|$ nad because G is a tree $|E| = n - 1$. Thus

$$\sum_{v \in V} \deg(v) = 2n - 2.$$

As a consequence, at least two vertices are grade one (otherwise, $\sum_{v \in V} \deg(v) \geq 2n - 2$.) $\square$

Example 3. A non trivial tree has, at least, 2 vertices connected by an edge.

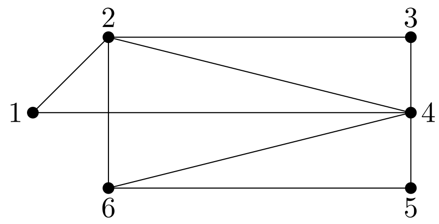
T_1



T_2

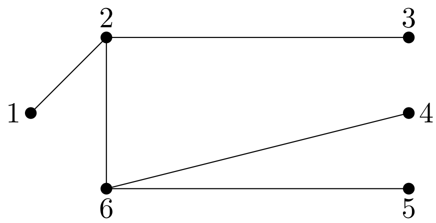
A **spanning tree** of a simple graph G is a subgraph of G that is a tree containing every vertex of G.

Example 4. Considering the graph G pictured

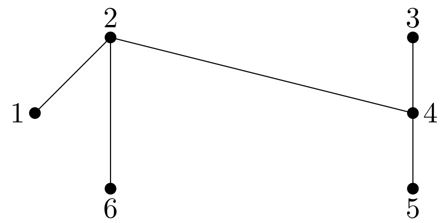


G

the following graphs are spanning trees of G



ST_1



ST_2

References

[1] Domingos Cardoso, Jerzy Szymanski, and Mohammad Rostami. *Matemática Discreta: Combinatória, Teoria dos Grafos, Algoritmos*. Escolar Editora, 2009.

Exercises in MathE platform