# Bernoulli differential equation

In this section we'll see how to solve the Bernoulli differential equation.

**The general form** of Bernoulli differential equation is,

$$x'(t) + a(t)x(t) = b(t)x^n(t), t \in I, I \subset \mathbb{R}, \tag{1}$$

where both $a(t)$ and $b(t)$ are *continuous functions* on the interval $I$.

**Remark.** If $n = 0$ or $n = 1$ then the equation is linear and we already know how to solve it in these case. Therefore, in this section we're going to obtain solutions for values of $n$ other than these two.

In order to solve these we'll first divide the differential equation by $x^n(t)$ to get,

$$x'(t)x^{-n}(t) + a(t)x^{1-n}(t) = b(t). \tag{2}$$

We use the substitution $y = x^{1-n}$ to convert this into a differential equation in terms of $y$. As we will see this will lead to a differential equation that we can solve.

We must be careful when we determine $y'$. We'll need to use the chain rule for differentiation,

$y'(t) = \left(x^{1-n}(t)\right)' = (1-n)\, x^{-n}(t)x'(t).$

We obtain

$x^{-n}(t)x'(t) = \dfrac{1}{1-n}y'(t).$

Now, using substitution into the differential equation, we

$$\frac{1}{1-n}y'(t) + a(t)y(t) = b(t). \tag{3}$$

This is a linear differential equation that we can solve for $y$ and once we have this in hand we can also get the solution to the original differential equation by plugging $y$ back into our substitution and solving for $x$.

**Solution Process**

The solution process for Bernoulli differential equation is as follows.

1. We observe if the differential equation is in the correct form (1). If the differential equation is not in the correct form we do it.

2. Observe that $n$ must be different of 0 or 1.

3. We divide the differential equation by $x^n(t)$ to get (2).

4. We use the substitution $y = x^{1-n}$. We determine $y'$,
$$y'(t) = (1-n)\, x^{-n}(t)x'(t). \tag{4}$$
5. We make the substitution (4) on equation (2) and obtain (3).
6. This is a linear differential equation.

**Example 1** *Find the solution to the following differential equation:*

$$\begin{cases} x'(t) + \frac{4}{t}x(t) = t^3x^2(t), t > 0, \\ x(2) = -1. \end{cases}$$

*Solution.*

1. We observe if the differential equation is in the correct form (1).
2. Observe that $n = 2$.
3. We divide the differential equation by $x^2(t)$
$x'(t)x^{-2}(t) + \frac{4}{t}x^{-1}(t) = t^3$.
4. We determine $y'$.
$y'(t) = -x^{-2}(t)x'(t)$.
5. We use the substitution $y = x^{-1}$. We make the substitution on equation
$x'(t)x^{-2}(t) + \frac{4}{t}x^{-1}(t) = t^3$
and obtain
$-y'(t) + \frac{4}{t}y(t) = t^3$
6. This is a linear differential equation.
$y'(t) - \frac{4}{t}y(t) = -t^3$.
The solution is
$y(t) = Ct^4 - t^4 \ln t \Rightarrow x^{-1}(t) = Ct^4 - t^4 \ln t$.
Now we need to determine the constant of integration.
$x^{-1}(2) = C2^4 - 2^4 \ln 2 \Rightarrow -1 = C2^4 - 2^4 \ln 2 \Rightarrow C = \ln 2 - \frac{1}{16}$
$x^{-1}(t) = \left(\ln 2 - \frac{1}{16}\right) t^4 - t^4 \ln t \Rightarrow x(t) = \frac{1}{\left(\ln 2 - \frac{1}{16} - \ln t\right)t^4}$.

**Example 2** *Find the solution to the following differential equation:*

$$\begin{cases} tx'(t) + x(t) + 3t\,(\ln t)\, x^2(t), t > 0, \\ x(1) = 5. \end{cases}$$

*Solution.*

1. We observe if the differential equation is not in the correct form (1). So we divide both part by $t$ :

   $x'(t) + \frac{1}{t}x(t) = -3\left(\ln t\right)x^2(t)$

2. Observe that $n = 2$.

3. We divide the differential equation by $x^2(t)$

   $x'(t)x^{-2}(t) + \frac{1}{t}x^{-1}(t) = -3\left(\ln t\right).$

4. We determine $y'$.

   $y'(t) = -x^{-2}(t)x'(t).$

5. We use the substitution $y = x^{-1}$. We make the substitution on equation

   $x'(t)x^{-2}(t) + \frac{1}{t}x^{-1}(t) = -3\left(\ln t\right)$

   and obtain

   $-y'(t) + \frac{1}{t}y(t) = -3\left(\ln t\right).$

6. This is a linear differential equation.

   $y'(t) - \frac{1}{t}y(t) = 3\left(\ln t\right).$

   The solution is

   $y(t) = Ct + \frac{3}{2}t\ln t \Rightarrow x^{-1}(t) = Ct + \frac{3}{2}t\ln t.$

   Now we need to determine the constant of integration.

   $x^{-1}(1) = C \Rightarrow C = \frac{1}{5}$

   $x^{-1}(t) = \frac{1}{5}t + \frac{3}{2}t\ln t \Rightarrow x(t) = \frac{10}{(2+15\ln t)t}, t > 0.$

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