Differential equations

In this section some of the common definitions and concepts in a differential equations course are introduced including order, linear vs. nonlinear, initial conditions, initial value problem and interval of validity.

A **differential equation** is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

Examples: Newton's second law of motion is written as a differential equation in terms of either the velocity, v, or the position, u, of the object as follows

$$m\frac{dv}{dt} = F(t, v), \tag{1}$$

$$m\frac{d^2u}{dt^2} = F\left(t, u, \frac{du}{dt}\right). \tag{2}$$

More examples of differential equations:

$$x''(t) + x(t) = \sin t, \tag{3}$$

$$x'(t) = tx^2(t) + 2tx(t), \tag{4}$$

$$x'''(t) + t^2x(t) - x'(t) = 2t^4, \tag{5}$$

$$x\,(t) - tx'\,(t) = \sqrt{1 + (x'\,(t))^2}, t \in \mathbb{R}, \tag{6}$$

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, u = u(t, x), \tag{7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, u = u(x, y). \tag{8}$$

The order of a differential equation is the largest derivative present in the differential equation.

So equations (1), (4), (6), (8) are of first order, equations (2), (3), (7) are of second order and equation (5) is of third order.

Ordinary and Partial Differential Equations

A differential equation is called an **ordinary differential equation**, abbreviated by ODE, if it has ordinary derivatives in it, equations (1),(2), (3), (4), (5) and (6). Likewise, a differential equation is called a **partial differential equation**, abbreviated by PDE, if it has partial derivatives in it, equations (7) and (8).

Linear Differential Equations

A **linear differential equation** is any differential equation that can be written in the following form

$$x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \cdots + a_{n-1}(t)x'(t) + a_n(t)x(t) = f(t) \tag{9}$$

The important thing to note about linear differential equations is that there are no products of the function, $x(t)$, and its derivatives and neither the function or its derivatives occur to any power other than the first power. Also note that neither the function or its derivatives are "inside" another function, for example, $\sqrt{x(t)}$ or $\ln x(t)$.

The coefficients $a_1(t)$ $a_2(t), \ldots,$ $a_n(t)$ and $f(t)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions.

Only the function, $x(t)$ and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the form (9) then it is called a non-linear differential equation. Equations (4) and (6) are nonlinear.

A solution of a differential equation on an interval $t \in (a, b)$ is any function $x(t)$ which satisfies the differential equation in question on the interval (a, b). It is important to note that solutions are often accompanied by intervals and these intervals can give some important information about the solution.

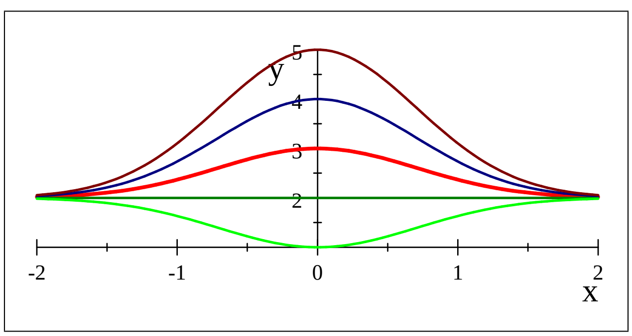
General Solution

The general solution to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

The solution $x(t) = 2 + Ce^{-t^2}$ is the general solution of equation

$$x(t) + 2tx(t) = 4t. \tag{10}$$

The solution to an ordinary differential equation is a family of parametric curve. So there are an infinite number of solutions to the differential equation. The set of solutions that we've graphed below are for some values of C, $C = -1, 0, 1, 2, 3,$



(11)

We can ask a natural question. Which is the solution that we want or does it matter which solution we use? This question leads us to the next ideas,

Initial Condition(s)

Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions are of the form,

$$x(t_0) = x_0 \text{ or } x^{(k)}(t_0) = x_{0,k}.$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points.

The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation.

If we consider the initial condition $x(0) = 3$ for equation (10) the solution will be $x(t) = 2+e^{-t^2}$.It is a particular solution. This solution is colored on the figure (11) red.

Initial Value Problem

An Initial Value Problem (or IVP) is a differential equation along with an appropriate number of initial conditions.

The following problems are IVP

$$\begin{cases} x(t) + 2tx(t) = 4t, \\ x(0) = 3, \end{cases}$$

and

$$\begin{cases} x''' + 3x'' - x' - 3x = 0, \\ x(0) = 0, x'(0) = 1, x''(0) = -1. \end{cases}$$

As I noted earlier the number of initial condition required will depend on the order of the differential equation.

Interval of Validity

The interval of validity for an IVP with initial condition(s) is the largest possible interval on which the solution is valid and contains t_0.

A **Cauchy problem** can be an Initial Value Problem.

The simplest Cauchy problem is to find a function $x(t)$ defined on the half-line $t \geq t_0$, satisfying a first-order ordinary differential equation

$$\frac{dx}{dt} = f(x,t) \tag{12}$$

(f is a given function) and taking a specified value x_0 at $t = t_0$:
$x(t_0) = x_0$.
We write this

$$\begin{cases} \dfrac{dx}{dt} = f(x,t), \\ x(t_0) = x_0. \end{cases}$$

In geometrical terms this means that, considering the family of integral curves of equation (12) in the (t, x)- plane, one wishes to find the curve passing through the point (t_0, x_0).

Author: Ariadna Lucia Pletea