Eulerian graphs

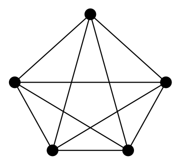
Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once. We can also say that G is an **Eulerian graph**.

Theorem 1. A graph (with at least two vertices) is Eulerian if and only if it is connected and every vertex is even.

Proof. $\Rightarrow$ In walking along an Eulerian circuit, every time we meet a vertex (other than the one where we started), either we leave on a loop and return immediately, never traversing that loop again, or we leave on an edge different from that by which we entered and traverse neither edge again. So the edges (other than loops) incident with any vertex in the middle of the circuit can be paired. So also can the edges incident with the first (and last) vertex since the edge by which we left it at the beginning can be paired with the edge by which we returned at the end. Thus, an Eulerian graph must not only be connected, but also have vertices of even degree. Conversely, a connected graph all of whose vertices are even must be Eulerian.
$\Leftarrow$ For the converse, suppose that G is a connected pseudograph with all vertices of even degree. We must prove that G has an Eulerian circuit. Let v be any vertex of G. If there are any loops incident with v, follow these first, one after the other without repetition. Then, since we are assuming that G has at least two vertices and since G is connected, there must be an edge vv_1 (with $v_1 \neq v$) incident with v. If there are loops incident with v_1, follow these one after the other without repetition. Then, since $deg(v_1)$ is even and bigger than 0, there must be an edge v_1v_2 different from vv_1 . Thus we have a trail from v to v_2 which we continue if possible. Each time we arrive at a vertex not encountered before, follow all the loops without repetition. Since the degree of each vertex is even, we can leave any vertex different from v on an edge not yet covered. Remembering that we are considering that graphs are always finite, we see that the process just described cannot continue indefinitely; eventually, we must return to v, having traced a circuit C_1. Notice that every vertex in C_1 is even since we entered and left on different edges each time it was encountered. At this point, it may happen that every edge has been covered; in other words, that C_1 is an Eulerian circuit, in which case we are done. If C_1 is not Eulerian, as in the preceding example, we delete from G all the edges of C_1 and all the vertices of G which are left isolated (that is, acquire degree 0) by this procedure. All vertices of the remaining graph GI are even (since both g and Cl have only even vertices) and of positive degree. Also, G_1 and C_1 have a vertex u in common, because G is connected. Starting at u, and proceeding in G_1 as we did in G, we construct a circuit C in G_1 which returns to u. Now combine C and C_1 by starting at v, moving along C_1 to u, then through C back to u, and then back to v on the remaining edges of C_1. We obtain a circuit C_2 in G which contains more edges than C_1. If it contains all the edges of G, it is Eulerian; otherwise, we repeat the process, obtaining a sequence of larger and larger circuits. Since our graph is finite, the process must eventually stop, and it stops only

with a circuit through all edges and vertices, that is, with an Eulerian circuit. □

Example 1. The complete graph K_5 is an Eulerian graph.

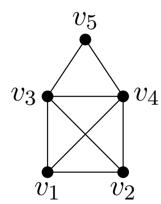


K_5

Proposition 1. A graph G possesses an Eulerian trail between two (different) vertices u and v if and only if G is connected and all vertices except u and v are even.

Proof. $\Rightarrow$ If G possesses an Eulerian trail that is not a circuit, then because the star vertex and the end vertex are different, only that two vertices have odd degree. The other vertices have even degree because if they belong to the trail we use one edge to get in the vertex and another (different) edge to get out. $\Leftarrow$ Suppose that all, but exactly two vertices in G have even degree, then from the previous theorem the is no Euler circuit. Consider that u and v are vertices from G with odd degree and consider the graph G', $E(G') = E(G) \cup \{uv\}$, thus all vertices in G' have even degree, then G' admits an Euler circuit. If we add the edge uv to the circuit we obtain a trail. □

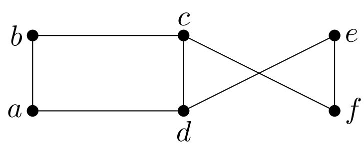
Example 2. The graph pictured is not an Eulerian graph, because there exists vertices with odd degree, but admits an eulerian trail: $v_1v_2v_3v_4v_5v_3v_1v_4v_2$.



Hamiltonian graphs

An **Hamilton path** in a graph is a path which contains every vertex of the graph. G. If the path is close, taht is the star vertex and the final vertex are the same, is a **Hamiltonian cycle**. A **Hamiltonian graph** is one with a Hamiltonian cycle.

Example 3. The following graph is Hamiltonian, because it admits an Hamiltonian cycle: $abcfeda$



Proprieties of cycles

Suppose H is a cycle in a graph G.

- For each vertex v of H, precisely two edges incident with v are in H; hence, if H is a Hamiltonian cycle of G and a vertex v in G has degree 2, then both edges incident with v must be part of H.
- The only cycle contained in H is H itself. (We say that H contains no proper cycles.)

Theorem 2 (Ore)**.** If G is a simple graph with n vertices with $n \geq 3$ such that $deg(u)+deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G is Hamiltonian.

Proof. (for reduction to absurdity) Let G be a graph that satisfies the hypotheses of the theorem and suppose that G is not Hamiltonian. Suppose further that G is such that adding an edge gives a Hamilton cycle (which contains all the vertices). Since G it is not complete (otherwise it would admit a Hamilton cycle) there is a pair of non-adjacent vertices u and w such that adding the uw edge to G gives a cycle of Hamilton. Thus G will contain a path between u and w that traverses all other vertices of G, that is, $u = v_1 \quad v_2$ ... $v_{n-1}\ v_n = w$. If u is adjacent to v_i, w is not adjacent to v_{i-1}, otherwise there would be a cycle of Hamilton in the form $w \quad v_{i-1} \quad v_{i-2}$ ... $v_1\ (= u)v_i \quad v_{i+1}$ ... $v_{n-1} \quad v_n\ (= w)$. If $v_1 = u$ is adjacent to r vertices, $v_n = w$ can only be adjacent to $(n-1) - r$ vertices and therefore, as $deg(u) + deg(v) = r + (n-1) - r \leq n$, which contradicts the theorem. □

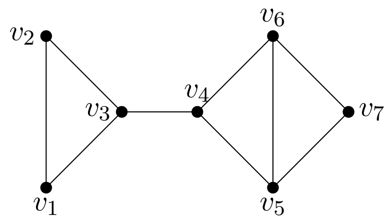
Theorem 3 (Dirac)**.** If a graph G has $n > 3$ vertices and every vertex has degree at least n then G is Hamiltonian.

Proof. Among the possible paths in G, consider $P = v_1 v_2 \cdots v_t$, the longest path, in the sense that it contains the greater number of vertices. So there will be no path in G that uses more than t vertices. If there was a vertex w adjacent to v_1 that was not in P then P would not be the biggest path in G. Then all vertices adjacent to v_1 are in P. If $deg(v_1) \geq \frac{n}{2}$ then $t \geq \frac{n}{2} + 1$, where " $+1$ " is relative to the vertex v_1.

Note that there is a pair of vertices v_k, v_{k+1} in P $(1 \leq k \leq t)$ such that v_1 is adjacent to v_{k+1} e v_t is adjacent to v_k. If not, then each P vertex adjacent to v_1 will determine a non-vertex adjacent to v_t. Like all vertices $v_2, \cdots, v_t$ are different exist in G at least $\ fracn2$ vertices that are not adjacent to v_t. These vertices together with the vertices adjacent to v_t will be at least n. With the apex v_t, G will then have more than n vertices, which is false. So the observation is true. It follows that G has a cycle $C = v_1 v_{k+1} v_{k+2} \cdots v_t \cdots v_k v_{k-1} \cdots v_1$.

It remains to be shown that C contains all vertices of G and is therefore a Hamiltonian cycle. We know that C contains at least $\frac{n}{2} + 1$ vertices, and therefore may exist $\frac{n}{2}$ vertices of G that are not in C. If so some vertex w that is not in C will be adjacent to a vertex v_s of C. So w, v_s, and the rest vertices of the cycle will define a path greater than P, which contradicts the choice of P. □

Example 4. The graph pictured is not Hamiltonian. $v_3v_1v_2v_3$ is a cycle and also $v_4v_6v_7v_5v_4$. The vertex v_3 and v_4 already have two edges incidente in each cycle, so is not possible to connect this two cycles in order to obtain an Hamiltonian cycle.



References

[1] Domingos Cardoso, Jerzy Szymański, and Mohammad Rostami. *Matemática Discreta: Combinatória, Teoria dos Grafos, Algoritmos*. Escolar Editora, 2009.

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