Equation Reducible to Exact Differential Equation. Integrating Factors.

Integrating Factors. It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor. To investigate the possibility of implementing this idea more generally, let us multiply the equation

$$P(x,y)dx + Q(x,y)dy = 0 \tag{1.20}$$

by a function $\mu(x,y)$ and then try to choose $\mu(x,y)$ so that the resulting equation

$$\mu(x,y)P(x,y)dx + \mu(x,y)Q(x,y)dy = 0 \tag{1.21}$$

is exact, that is taking into account (1.15),eq. (23) is exact if and only if

$$\frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x}. \tag{1.22}$$

Since P and Q are given functions, Eq. (1.22) states that the integrating factor μ must satisfy the first order partial differential equation

$$P\mu_y - Q\mu_x + (P_y - Q_x)\mu = 0. \tag{1.23}$$

If a function μ satisfying Eq. (1.25) can be found, then Eq. (1.21) will be exact.

A partial differential equation of the form (1.23) may have more than one solution; if this is the case, any such solution may be used as an integrating factor of Eq. (1.20). This possible non uniqueness of the integrating factor is illustrated in further example.

Unfortunately, Eq. (1.23), which determines the integrating factor μ, is ordinarily at least as difficult to solve as the original equation (1.20). Therefore, while in principle integrating factors are powerful tools for solving differential equations, in practice they can be found only in special cases. The most important situations in which simple integrating factors can be found occur when μ is a function of only one of the variables x or y, instead of both. Let us determine necessary conditions on P and Q so that Eq. (1.20) has an integrating factor μ that depends on x only. Assuming that μ is a function of x only, we have

$$(\mu P)_y = \mu P_y, \qquad (\mu Q)_x = \mu Q_x + Q\frac{d\mu}{dx}.$$

Thus, if $(\mu P)_y$ is to equal $(\mu Q)_x$, it is necessary that

$$\frac{d\mu}{dx} = \frac{P_y - Q_x}{Q}\mu. \tag{1.24}$$

If $\frac{P_y - Q_x}{Q}$ is a function of x only, then there is an integrating factor μ that also depends only on x; further, $\mu(x)$ can be found by solving Eq. (26), which is both linear and separable.

A similar procedure can be used to determine a condition under which Eq. (1.20) has an integrating factor depending only on y.

Example. Find an integrating factor for the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0 \tag{1.25}$$

and then solve the equation. We showed that this equation is not exact. Let us determine whether it has an integrating factor that depends on x only. On computing the quantity $\frac{P_y - Q_x}{Q}$, we find that

$$\frac{P_y(x,y) - Q_x(x,y)}{Q(x,y)} = \frac{(3x + 2y) - (2x + y)}{x^2 + xy} = \frac{1}{x}.$$

Thus there is an integrating factor μ that is a function of x only, and it satisfies the differential equation

$$\frac{d\mu}{dx} = \frac{\mu}{x}.$$

Hence $\mu(x) = x$. Multiplying Eq. (1.25) by this integrating factor, we obtain

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0.$$

The latter equation is exact and it is easy to show that its solutions are given implicitly by

$$x^3y + 12x^2y^2 = C.$$

You may also verify that a second integrating factor of Eq. (1.25) is $\mu(x,y) = \frac{1}{xy(2x+y)}$, and that the same solution is obtained, though with much greater difficulty, if this integrating factor is used.

Determine whether or not each of the next equations is exact. If it is exact, find the solution.

$$1.\ (2x+3)+(2y-2)y'=0, \quad 2.\ (2x+4y)+(2x-2y)y'=0,$$

$$3.\ (3x^2-2xy+2)dx+(6y^2-x^2+3)dy=0, \quad 4.\ (2xy^2+2y)+(2x^2y+2x)y'=0,$$

$$5.\ (e^x\sin y-2y\sin x)dx+(e^x\cos y+2\cos x)dy=0$$

$$6.\ \frac{dy}{dx}=\frac{ax+by}{bx+cy}, \quad 7.\ \frac{dy}{dx}=-\frac{ax-by}{bx-cy} \quad 8.\ (e^x\sin y+3y)dx-(3x-e^x\sin y)dy=0,$$

$$9.\ (ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x)dx+(xe^{xy}\cos 2x-3)dy=0,$$

$$10.\ (\frac{y}{x}+6x)dx+(\ln x-2)dy=0, x>0,$$

$$11.\ (x\ln y+xy)dx+(y\ln x+xy)dy=0; x>0, y>0.$$

In each of the next equation solve the given initial value problem and determine at least approximately where the solution is valid.

$$(2x-y)dx+(2y-x)dy=0,\ y(1)=3, \quad (9x^2+y-1)dx-(4y-x)dy=0,\ y(1)=0.$$

In each of the next problems find the value of b for which the given equation is exact and then solve it using that value of b.

$$(xy^2+bx^2y)dx+(x+y)x^2dy=0, \qquad (ye^{2xy}+x)dx+bxe^{2xy}dy=0.$$

Show that the next equations are not exact, but become exact when multiplied by the given integrating factor. Then solve the equations.

$$1.\ x^2y^3+x(1+y^2)y'=0, \quad \mu(x,y)=\frac{1}{xy^3},$$

$$2.\ (\frac{\sin y}{y}-2e^{-x}\sin x)dx+(\frac{\cos y+2e^{-x}\cos x}{y})dy=0,\ \mu(x,y)=ye^x,$$

$$3.\ ydx+(2x-ye^y)dy=0, \mu(x,y)=y,$$

$$4.\ (x+2)\sin ydx+x\cos ydy=0, \mu(x,y)=xe^x$$

Show that the equation

$$y' + f(x)y = 0$$

has an integrating factor of the form

$$\mu(x) = e^{-\int f(x)dx}$$

Show that if $(Q_x - P_y)/(xP - yQ) = R$, where R depends on the quantity xy only, then the differential equation

$$P(x, y) + Q(x, y)y' = 0$$

has an integrating factor of the form $\mu(xy)$. A general formula for this integrating factor is of the form

$$\mu(xy) = e^{\int R(xy)d(xy)}.$$

In each of the next problems find an integrating factor and solve the given equation.

$$1. \ (3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0, \qquad 2. \ y' = e^{2x} + y - 1,$$

$$3. dx + (x/y - \sin y)dy = 0,$$

$$4. ydx + (2xy - e^{-2y})dy = 0,$$

$$5. e^x dx + (e^x \cot y + 2ycosecy)dy = 0, \quad 6. \ (4(x^3/y^2) + 3/y)dx + 3x/y^2 + 4y)dy = 0,$$

$$7. \ (3x + 6/y) + (x^2/y + 3y/x)(dy/dx) = 0$$