MathE project

Limit for real functions of several variables

1 The $\varepsilon - \delta$ characterization

Let $E \subseteq \mathbb{R}^k$ be a nonemty set, let a be a cluster point for E and let us consider a real function $f : E \to \mathbb{R}$.

Definition 1.1 *(with neighbourhoods) One says that $\ell \in \overline{\mathbb{R}}$ is the limit of f at the point a if for any $U \in \mathcal{V}(\ell)$ from $\mathbb{R}$, there exists $V \in \mathcal{V}(a)$ from $\mathbb{R}^k$, such that for any $x \in V \cap E$ with $x \neq a$, we have $f(x) \in U$. We denote this by*

$$\ell = \lim_{x \to a} f(x).$$

Proposition 1.1 *(with $\varepsilon - \delta$)*

(i) Let $\ell \in \mathbb{R}$. The limit of f at the point a is ℓ if and only if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for $x \in E$ with $x \neq a$ and $\|x - a\| < \delta$ we have $|f(x) - \ell| < \varepsilon$.

(ii) The limit of f at the point a is $+\infty$ if and only if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for $x \in E$ with $x \neq a$ and $\|x - a\| < \delta$ we have $f(x) > \varepsilon$.

(iii) The limit of f at the point a is $-\infty$ if and only if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for $x \in E$ with $x \neq a$ and $\|x - a\| < \delta$ we have $f(x) < -\varepsilon$.

For a two-variables function $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}$, $f = f(x, y)$ we obtain:

Proposition 1.2 *(with $\varepsilon - \delta$)*

(i) Let $\ell \in \mathbb{R}$. The limit of f at the point (a, b) is ℓ if and only if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for $(x, y) \in E$, $(x, y) \neq (a, b)$ with $|x - a| < \delta$ and $|y - b| < \delta$ we have $|f(x) - \ell| < \varepsilon$.

(ii) The limit of f at the point (a, b) is $+\infty$ if and only if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for $(x, y) \in E$, $(x, y) \neq (a, b)$ with $|x - a| < \delta$ and $|y - b| < \delta$ we have $f(x) > \varepsilon$.

(iii) The limit of f at the point a is $-\infty$ if and only if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for $(x, y) \in E$, $(x, y) \neq (a, b)$ with $|x - a| < \delta$ and $|y - b| < \delta$ we have $f(x) < -\varepsilon$.

Example 1.1 *Using the $\varepsilon - \delta$ criterion of the limit, show that*

$$\lim_{(x,y) \to (3,1)} (2x - y) = 5.$$

Solution. Consider $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that for any $(x, y) \in \mathbb{R}^2$ with $(x, y) \neq (3, 1)$, $|x - 3| < \delta$ and $|y - 1| < \delta$ we have $|2x - y - 5| < \varepsilon$. Indeed, we can write

$$|2x - y - 5| = |2(x - 3) - (y - 1)| \leq 2|x - 3| + |y - 1| < 4\delta$$

and for $\delta \leq \dfrac{\varepsilon}{4}$ the inequality is fulfilled.

Example 1.2 *Using the $\varepsilon - \delta$ criterion of the limit, show that*

$$\lim_{(x,y)\to(4,+\infty)} \frac{xy-1}{y+2} = 4.$$

Solution. Consider $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that for any $(x,y) \in \mathbb{R}^2$ with $|x-4| < \delta$ and $y > \dfrac{1}{\delta}$ we have

$$\left| \frac{xy-1}{y+2} - 4 \right| < \varepsilon.$$

Indeed, we can write

$$\left| \frac{xy-1}{y+2} - 4 \right| = \left| \frac{(x-4)y-9}{y+2} \right| \leq |x-4| \cdot \frac{y}{y+2} + 9 \cdot \frac{1}{y} < \delta + 9\delta = 10\delta$$

and for $\delta \leq \dfrac{\varepsilon}{10}$ the inequality is fulfilled.

Example 1.3 *Find the limit of the following functions:*

a) $\lim\limits_{(x,y)\to(2,0)} \frac{\sin(xy)}{y}$ *b)* $\lim\limits_{(x,y)\to(+\infty,2)} \left(1+\frac{y^2}{x}\right)^{xy}$

c) $\lim\limits_{(x,y)\to(+\infty,+\infty)} \left(\frac{xy}{x^2+y^2}\right)^{y^2}$ *d)* $\lim\limits_{(x,y)\to(+\infty,+\infty)} \frac{x+y}{x^2-xy+y^2}$

Solution.

a) Since $\lim\limits_{t\to 0} \frac{\sin t}{t}$, we have $\displaystyle\lim_{(x,y)\to(2,0)} \frac{\sin(xy)}{y} = \lim_{(x,y)\to(2,0)} \frac{\sin(xy)}{xy} \cdot x = 2.$

b) Using the fundamental limit $\lim\limits_{t\to 0} (1+t)^{\frac{1}{t}} = e$, we deduce

$$\lim_{(x,y)\to(+\infty,2)} \left(1+\tfrac{y^2}{x}\right)^{xy} = \lim_{(x,y)\to(+\infty,2)} \left[\left(1+\tfrac{y^2}{x}\right)^{\frac{x}{y^2}}\right]^{y^3} = e^{\lim\limits_{y\to 2} y^3} = e^8.$$

c) We have $0 < \left(\dfrac{xy}{x^2+y^2}\right)^{y^2} \leq \left(\dfrac{1}{2}\right)^{y^2}$ for each $x, y > 0$, so we get

$$0 \leq \lim_{(x,y)\to(+\infty,+\infty)} \left(\frac{xy}{x^2+y^2}\right)^{y^2} \leq \lim_{(x,y)\to(+\infty,+\infty)} \left(\frac{1}{2}\right)^{y^2} = 0, \text{ thus}$$

$$\lim_{(x,y)\to(+\infty,+\infty)} \left(\frac{xy}{x^2+y^2}\right)^{y^2} = 0.$$

d) Since

$$\frac{x+y}{x^2-xy+y^2} \leq \frac{1}{x} + \frac{1}{y}$$

for each $x, y > 0$, and

$$\lim_{(x,y)\to(+\infty,+\infty)} \left(\frac{1}{x} + \frac{1}{y}\right) = 0$$

it follows

$$\lim_{(x,y)\to(+\infty,+\infty)} \frac{x+y}{x^2-xy+y^2} = 0.$$