MathE project

Continuity for real functions of several variables

Let $D \subseteq \mathbb{R}^k$ be a nonemty set, $a \in D$ and let us consider a real function $f : D \to \mathbb{R}$.

Definition 1.1 *If $a \in D$ is a cluster point of D, we say that the function f is continuous at a if the limit of f at the point a exists and*

$$\lim_{x \to a} f(x) = f(a).$$

If $a \in D$ is a isolated point, f is continuous at a. We say that the function f is continuous on the set D if it is continuous at the each point of D.

Proposition 1.1 *(with sequences)* Let $D \subseteq \mathbb{R}^k$ be a nonemty set and let $a \in D$ a cluster point of D. The function $f : D \to \mathbb{R}$ is continuous at the point a if and only if for any sequence $(x_n)_n \subset D$ with $\lim\limits_{n \to +\infty} x_n = a$, we have that

$$\lim_{n \to +\infty} f(x_n) = f(a).$$

Remark 1.1 If there exists a sequence $(x_n)_n \subset D$ with $\lim\limits_{n \to +\infty} x_n = a$ such that

$$\lim_{n \to +\infty} f(x_n) \neq f(a),$$

then the function f is not continuous at the point a.

Definition 1.2 *Let $a = (a_1, a_2, \ldots, a_k) \in D$. Consider the function $f_i : D_i \to \mathbb{R}$ of variable x_i, $i = \overline{1,k}$, given by*

$$f_i(x_i) = f(a_1, a_2, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_k)$$

defined on the set $D_i = \{x_i \in \mathbb{R} \,|\, (a_1, a_2, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_k) \in D\}$. If the function f_i is continuous at $a_i \in D$, one says that the function f is partially continuous with respect to variable x_i at the point a.

Remark 1.2 If the function f is continuous at the point $a \in D$ (on D), then it is partially continuous with respect to each variable x_i, $i = \overline{1,k}$, at the point $a \in D$ (on D, respectively). The partially continuous of f at the point a does not involve the global continuity of f at a.

For a two-variables function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, $f = f(x, y)$ the above proposition is:

Proposition 1.2 *(with sequences)* Let $D \subseteq \mathbb{R}^2$ be a nonemty set and let $(a, b) \in D$ a cluster point of D. The function $f : D \to \mathbb{R}$ is continuous at (a, b) if and only if for any sequence $(x_n, y_n)_n \subset D$ with $\lim\limits_{n \to +\infty} (x_n, y_n) = (a, b)$, we have that

$$\lim_{n \to +\infty} f(x_n, y_n) = f(a, b).$$

Remark 1.3 If there exists a sequence $(x_n, y_n)_n \subset D$ with $\lim\limits_{n\to+\infty}(x_n, y_n) = (a, b)$ and $\lim\limits_{n\to+\infty} f(x_n, y_n) \neq f(a, b)$ then f is not continuous at (a, b).

Example 1.1 *Study the continuity of the function* $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^3)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

<u>Solution</u>. On the set $\mathbb{R}^2 \setminus \{(0,0)\}$ the function f is a composition of elementary continuous functions, so f is continuous. We study the continuity at $(0,0)$. We use the known limit $\lim\limits_{t\to 0}\dfrac{\sin t}{t} = 1$ and we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^3+y^3} \cdot \frac{x^3+y^3}{x^2+y^2} = 0 = f(0,0).$$

It follows that f is continuous at $(0,0)$ and so it is continuous on $\mathbb{R}^2$. To prove that

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$$

let us observe that we can write

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| \leq |x| \cdot \frac{x^2}{x^2+y^2} + |y| \cdot \frac{y^2}{x^2+y^2} \leq |x| + |y| \to 0.$$

Example 1.2 *Study the continuity of the function* $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{xy}{\ln(1+x^2+y^2)}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

<u>Solution</u>. On the set $\mathbb{R}^2 \setminus \{(0,0)\}$ the function f is a composition of elementary continuous functions, so f is continuous. We study the continuity at $(0,0)$. We use the known limit $\lim\limits_{t\to 0}\dfrac{\ln(1+t)}{t} = 1$ and we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\ln(1+x^2+y^2)} \cdot \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2},$$

and the last one is not equal with $f(0,0) = 0$ (it does not exist). Using the Remark 1.3, we can choose the sequence $(x_n, y_n) = \left(\dfrac{1}{n}, \dfrac{1}{n}\right) \to (0,0)$ and

$$\lim_{n\to+\infty} f(x_n, y_n) = \lim_{n\to+\infty} \frac{\frac{1}{n^2}}{\frac{2}{n^2}} = \frac{1}{2} \neq f(0,0).$$

In conclusion f is not continuous at $(0,0)$.

Example 1.3 *Prove that the function* $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{xy^2 + \sin(x^3 + y^5)}{x^2 + y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is partially continuous with respect to both variables at the point $(0,0)$*, but it doesn't continuous at this point.*

<u>Solution</u>. One of the partial function at the point $(0,0)$ is $f_1 : \mathbb{R} \to \mathbb{R}$,

$$f_1(x) = f(x,0) = \begin{cases} \dfrac{\sin x^3}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

It is known that $\lim\limits_{t \to 0} \dfrac{\sin t}{t} = 1$ and we have

$$\lim_{x \to 0} f_1(x) = \lim_{x \to 0} \frac{\sin x^3}{x^2} = \lim_{x \to 0} \frac{\sin x^3}{x^3} \cdot x = 0 = f_1(0).$$

So the function f_1 is continuous. The partial function $f_2 : \mathbb{R} \to \mathbb{R}$,

$$f_2(y) = f(0,y) = \begin{cases} \dfrac{\sin y^4}{y^3}, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

is continuous due to the relation

$$\lim_{y \to 0} f_2(y) = \lim_{y \to 0} \frac{\sin y^4}{y^3} = \lim_{y \to 0} \frac{\sin y^4}{y^4} \cdot y = 0 = f_2(0).$$

But the function f is not continuous at $(0,0)$. To prove this, let us observe that we can write

$$f(x,y) = \frac{\sin(x^3 + y^5)}{x^2 + y^4} + \frac{xy^2}{x^2 + y^4}$$

and, for the first term, we have

$$\lim_{(x,y) \to (0,0)} \frac{\sin(x^3 + y^5)}{x^2 + y^4} = \lim_{(x,y) \to (0,0)} \frac{\sin(x^3 + y^5)}{x^3 + y^5} \cdot \frac{x^3 + y^5}{x^2 + y^4} = 0.$$

Indeed, we have

$$\left| \frac{x^3 + y^5}{x^2 + y^4} \right| \leq |x| \frac{x^2}{x^2 + y^4} + |y| \frac{y^4}{x^2 + y^4} \leq |x| + |y| \to 0$$

when $(x,y) \to (0,0)$.

For the second term, let us observe that there exists a sequence $(x_n, y_n) = \left(\dfrac{1}{n^2}, \dfrac{1}{n} \right) \to (0,0)$ on which the limit of this function

$$(x,y) \mapsto \frac{xy^2}{x^2 + y^4}$$

is not equal with 0. Indeed we have

$$\lim_{n \to +\infty} \frac{\frac{1}{n^4}}{\frac{2}{n^4}} = \frac{1}{2}.$$

So $\lim\limits_{n \to +\infty} f\left(\dfrac{1}{n^2}, \dfrac{1}{n} \right) = \dfrac{1}{2} \neq f(0)$. In conclusion f is not globally continuous at $(0,0)$.