Linear Combination

- **The vector $(-10,13,-14)$ is a linear combination of the vectors $(1,5,-7)$ and $(4,-1,0)$?**

Can we write $(-10,13,-14)$ as the sum resulting from the product of scalars by the vectors $(1,5,-7)$ and $(4,-1,0)$? This is, there will be $c_1, c_2 \in \mathbb{R}$, such that

$$(-10,13,-14) = c_1(1,5,-7) + c_2(4,-1,0)?$$

We must write the system and solve it.

$$\begin{cases} c_1 + 4c_2 & = & -10 \\ 5c_1 - c_2 & = & 13 \\ -7c_1 & = & -14 \end{cases} \Leftrightarrow \begin{cases} c_1 + 4c_2 & = & -10 \\ 5c_1 - c_2 & = & 13 \\ c_1 & = & 2 \end{cases} \Leftrightarrow$$

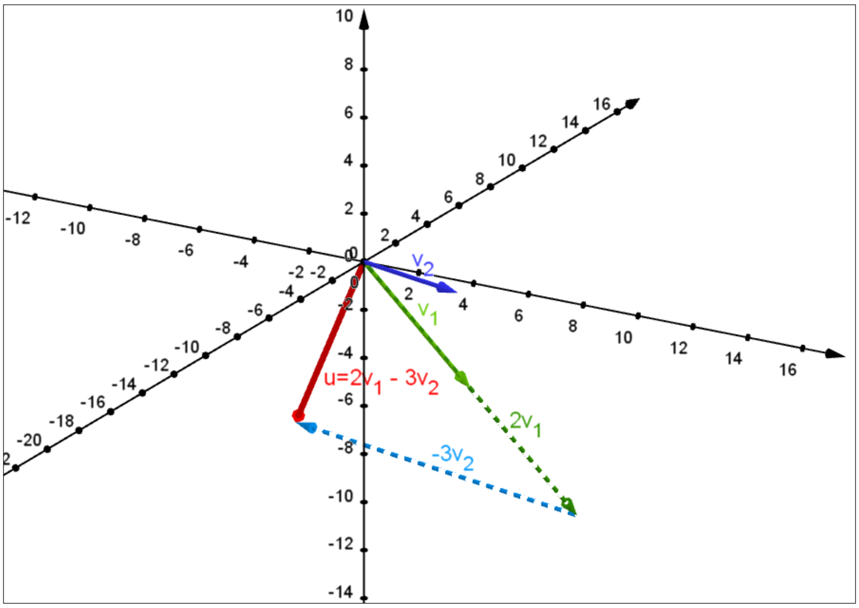
$$\Leftrightarrow \begin{cases} c_1 + 4c_2 & = & -10 \\ c_2 & = & -13 + 5\times 2 \\ c_1 & = & 2 \end{cases} \Leftrightarrow \begin{cases} c_1 + 4c_2 & = & -10 \\ c_2 & = & -3 \\ c_1 & = & 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2 + 4\times(-3) & = & -10 \\ c_2 & = & -3 \\ c_1 & = & 2 \end{cases}$$

<u>Conclusion</u>**:** $(\mathbf{-10},\mathbf{13},\mathbf{-14}) = \mathbf{2}(\mathbf{1},\mathbf{5},\mathbf{-7}) - \mathbf{3}(\mathbf{4},\mathbf{-1},\mathbf{0})$,

so $(\mathbf{-10},\mathbf{13},\mathbf{-14})$ is a linear combination of $(\mathbf{1},\mathbf{5},\mathbf{-7})$ and $(\mathbf{4},\mathbf{-1},\mathbf{0})$.

Geometrically, we can observe the way to obtain the vector $u = (-10,13,-14)$ from the vectors $v_1 = (1,5,-7)$ and $v_2 = (4,-1,0)$.



- **The vector $(5,6)$ is a linear combination of the vectors $(1,2)$, $(2,4)$ and $(-1,-2)$?**

There will be $c_1, c_2, c_3 \in \mathbb{R}$, such that

$$(5,6) = c_1(1,2) + c_2(2,4) + c_3(-1,-2)?$$

$$\begin{cases} c_1 + 2c_2 - c_3 &= 5 \\ 2c_1 + 4c_2 - 2c_3 &= 6 \end{cases} \Leftrightarrow \begin{cases} c_1 &= -2c_2 + c_3 \\ 2c_1 + 4c_2 - 2c_3 &= 6 \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 &= -2c_2 + c_3 \\ 2(-2c_2 + c_3) + 4c_2 - 2c_3 &= 6 \end{cases}$$

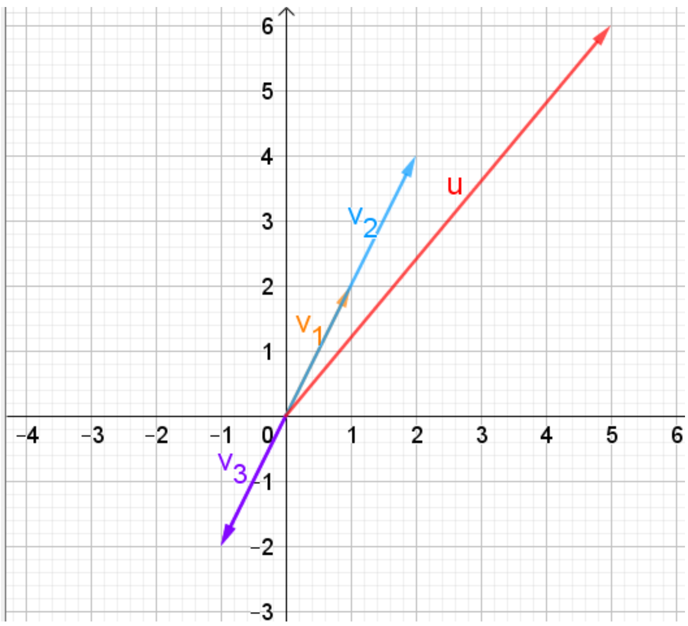
$$\Leftrightarrow \begin{cases} c_1 &= -2c_2 + 3c_3 \\ -4c_2 + 2c_3 + 4c_2 - 2c_3 &= 6 \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 &= -2c_2 + c_3 \\ 0 &= 6 \end{cases} \qquad \text{False proposition}$$

The system doesn't have any solution.

<u>Conclusion</u>: **$(5,6)$ is not a linear combination of the vectors $(1,2)$, $(2,4)$ and $(-1,-2)$.**

Geometrically, representing the vectors, we can see that it is not possible to obtain the vector $u = (5,6)$ from a linear combination of the vectors $v_1 = (1,2)$, $v_2 = (2,4)$ and $v_3 = (-1,-2)$. Note that the vectors v_1, v_2 and v_3 have the same direction, so any linear combination of one or more of these vectors is a vector in that direction.



Observation:

The vector $v_2 = (2,4)$ is a linear combination of $v_1 = (1,2)$ and vice versa, since

$$(2,4) = 2(1,2) \text{ and } (1,2) = \frac{1}{2}(2,4)$$

Similarly $v_3 = (-1,-2)$ is a linear combination of $v_1 = (1,2)$, of $v_2 = (2,4)$, or of v_1 and v_2. In this case, we can write this linear combination in several ways:

$$(-1,-2) = -1(1,2) + 0(2,4)$$
$$(-1,-2) = (1,2) - (2,4)$$
$$(-1,-2) = -4(1,2) + \frac{3}{2}(2,4)$$
$$(\dots)$$

Generalizing, $(-1,-2) = (-1-2b)(1,2) + b(2,4)$, for $b \in \mathbb{R}$.

In summary, to analyze whether a vector is a linear combination of other vectors, we can use the

Definition: The vector $u \in \mathbb{R}^n$ is a **linear combination** of the vectors $v_1, v_2, \dots, v_j \in \mathbb{R}^n$ if

$$\exists\, \boldsymbol{c_1, c_2, \dots, c_j} \in \mathbb{R}\boldsymbol{:}\ \boldsymbol{u = c_1 v_1 + c_2 v_2 + \cdots + c_j v_j.}$$