Bases and dimension of a vector space

Bases of a vector space

Definition (basis): Let V be a vector space. Then the subset $A = \{v_1, v_2, \ldots, v_t\}$ of V is said to be a basis for V if:

1. A is a linearly independent set of vectors;
2. A spans V, that is, $V = \langle A \rangle$.

Example: The set $A = \{(1,1), (1,-2)\}$ is a basis of $\mathbb{R}^2$.
In fact:

1. $k_1(1,1) + k_2(1,-2) = (0,0) \Leftrightarrow \begin{cases} k_1 + k_2 = 0 \\ k_1 - 2k_2 = 0 \end{cases} \Leftrightarrow \begin{cases} k_2 = 0 \\ k_1 = 0 \end{cases}$, that is, A is linearly independent;
2. A spans $\mathbb{R}^2$, because for any $(x,y) \in \mathbb{R}^2$ there are $k_1, k_2 \in \mathbb{R}$ such that $k_1(1,1) + k_2(1,-2) = (x,y)$.
   Indeed $\begin{cases} k_1 + k_2 = x \\ k_1 - 2k_2 = y \end{cases} \Leftrightarrow \begin{cases} k_2 = (x+y)/3 \\ k_1 = (2x-y)/3 \end{cases}$.

Example: The set $A = \{(1,1,2), (1,-2,0), (2,-1,2)\}$ is not a basis of $\mathbb{R}^3$, because A is linearly dependent.
In fact, $(1,1,2) = -(1,-2,0) + (2,-1,2)$, that is, the first is a linear combination of the others.

Theorem: If V is a vector space, then a smallest spanning set is a basis of V.

Example: The set $B = \{(1,1,2), (1,-2,0), (1,-1,1)\}$ is a basis of $\mathbb{R}^3$.
In fact,
$k_1(1,1,2) + k_2(1,-2,0) + k_3(1,-1,1) = (0,0,0) \Leftrightarrow$

$$\begin{cases} k_1 + k_2 + k_3 = 0 \\ k_1 - 2k_2 - k_3 = 0 \\ 2k_1 + k_3 = 0 \end{cases} \Leftrightarrow \begin{cases} -k_1 + k_2 = 0 \\ 3k_1 - 2k_2 = 0 \\ k_3 = -2k_1 \end{cases} \Leftrightarrow \begin{cases} k_1 = k_2 \\ k_1 = 0 \\ k_3 = -2k_1 \end{cases}$$

That is, the vector equation has the unique solution $k_1 = k_2 = k_3 = 0$.

In addition, B spans $\mathbb{R}^3$. In fact, any $(x,y,z) \in \mathbb{R}^3$ is a linear combination of the vectors of B. That is, the linear system $k_1(1,1,2) + k_2(1,-2,0) + k_3(1,-1,1) = (x,y,z)$ in variables k_1, k_2, k_3 is possible for any values of $x, y, z \in \mathbb{R}$.

Note that $B \cup \{v\}$, for any $v \in \mathbb{R}^3$, is not a basis of $\mathbb{R}^3$. In fact, $B \cup \{v\}$ is linearly dependent.

Dimension of a vector space

Theorem: Consider V a vector space with a basis $B = \{v_1, v_2, \ldots, v_n\}$ of n vectors. Then any set of $n+1$ vectors is linearly dependent. Besides that, any basis of V has exactly n vectors.

Example: The set $C = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$, because C is linearly independent and any $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$.

The set $D = C \cup \{(-1,2,0)\}$ is not a basis of $\mathbb{R}^3$. In fact,

$$k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) + k_4(-1,2,0) = (0,0,0)$$

represents a system, whose expanded matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

which is linearly dependent, that is, there are other solutions besides the null solution.

Also the space generated for $S = \{(1,0,0),(0,1,0)\} \subset C$ is its own subspace that represents a plane, it is not $\mathbb{R}^3$. Indeed, the vectors (x,y,z) that $(x,y,z) = k_1(1,0,0) + k_2(0,1,0)$ are those for which the system whose expanded matrix

$$\left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{array}\right]$$

is possible. This happens if $z = 0$, that is, $\langle S \rangle = \{(x,y,0) : x, y \in \mathbb{R}\}$.

Definition (dimension): The number n of vectors of any basis of vector space V is called the dimension of V and is denoted by $dim(V)$.

Examples:

- We say that $\mathbb{R}^2$ is a space two-dimensional because any of its bases has 2 vectors of $\mathbb{R}^2$;
- We say that $\mathbb{R}^3$ is a space three-dimensional because any of its bases has 3 vectors of $\mathbb{R}^3$.

Example: The set $A = \{(1,1),(1,-2)\}$ is a basis of $\mathbb{R}^2$ and the set $C = \{(1,0),(0,1)\}$ is other basis of $\mathbb{R}^2$, called the canonical basis of $\mathbb{R}^2$.

Furthermore, all bases of $\mathbb{R}^2$ are sets of two linearly independent vectors of $\mathbb{R}^2$, so $\mathbb{R}^2$ has dimension 2.

Note that:

- $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$ has dimension 2 and the generic vector (x,y) has 2 free variables.
- $\mathbb{R}^3 = \{(x,y,z) : x, y, z \in \mathbb{R}\}$ has dimension 3 and the generic vector (x,y,z) has 3 free variables.

Generally, the dimension of a vector space is equal to the number of free variables in its generic vector.

Example: The set $A = \{(1,-1,0),(2,0,1)\}$ spans

$$\{(x,y,z) \in \mathbb{R}^3 : (x,y,z) = k_1(1,-1,0) + k_2(2,0,1)\}.$$

In fact, the condition that defines $\langle A \rangle$ represents a system whose expanded matrix is:

$$\left[\begin{array}{cc|c} 1 & 2 & x \\ -1 & 0 & y \\ 0 & 1 & z \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 2 & y+x \\ 0 & 1 & z \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 2 & y+x \\ 0 & 0 & 2z-x-y \end{array}\right]$$

This system is possible, if $x + y - 2z = 0$. This means that $\langle A \rangle = \{(2z-y, y, z) : y, z \in \mathbb{R}\}$. The generic vector has 2 free variables and therefore $\langle A \rangle$ has dimension 2.