Systems of linear equations

Test. Consider a system of n linear equations.

Decide whether the following sentences are true or false. Provide full explanation of your answers.

(i) Multiplying by a non zero scalar its first equation, one gets an equivalent system.

(ii) If it is homogeneous, then it is consistent.

(iii) By substitution of one of its equation by a linear combination of the others, one gets an equivalent system.

(iv) If it is homogeneous, then it has a unique solution.

Solution

(i) The sentence is **true**, as the new equation one gets in this way has exactly the same solutions of the starting one.

(ii) The sentence is **true**. As the system is homogeneous (*i.e.* its constant term is zero), then the zero vector whose size equals to the number of the variables is certainly a solution. Hence the system is necessarily consistent.

From a different perspective: if the system is homogeneous, then its augmented matrix and coefficient matrix have the same rank, hence the system is necessarily consistent by the Rouché-Capelli Theorem.

(iii) The sentence is **false** in general. It would be true only if the discarded equation is itself a linear combination of the others and it is replaced with a *non-zero* linear combination of the others. Le us consider the following examples:

$$(a)\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \qquad\qquad (b)\begin{cases} x + y = 0 \\ y = 0 \\ z = 0 \end{cases}$$

In (a), one cannot replace $x = 0$ with $y + z = 0$: the new system is not equivalent to system (a):

$$\begin{cases} y + z = 0 \\ y = 0 \\ z = 0 \end{cases} \iff \begin{cases} y = 0 \\ z = 0 \end{cases}$$

has solutions $\{(x;\ 0;\ 0) \mid x \in \mathbb{R}\}$ while (a) has solution $\{(0;\ 0;\ 0)\}$, hence the two systems are not equivalent.

In (b), one cannot replace $x + y = 0$ with $0y + 0z = 0$ (trivial linear combination): the new system is not equivalent to system (b):

$$\begin{cases} 0 = 0 \\ y = 0 \\ z = 0 \end{cases} \iff \begin{cases} y = 0 \\ z = 0 \end{cases}$$

has solutions $\{(x;\ 0;\ 0) \mid x \in \mathbb{R}\}$ while (b) has solution $\{(0;\ 0;\ 0)\}$.

(iv) The sentence is **false** in general. It would be true only if its coefficient matrix is a nondegenerate square matrix (that is a square matrix with non zero determinant).

To show that the sentence is false in general consider the system given by a unique homogeneous equation $x + y = 0$, which has the infinitely many solutions $\{(t;\ -t) \mid t \in \mathbb{R}\}$.

Also the homogeneous linear system consisting of the single equation $x = 0$ in two variables x and y has the infinitely many solutions $(\{(0;\ t) \mid t \in \mathbb{R}\})$.