INSTITUTO POLITÉCNICO DE BRAGANÇA Escola Superior de Tecnologia e Gestão

Author: Edite Martins Cordeiro

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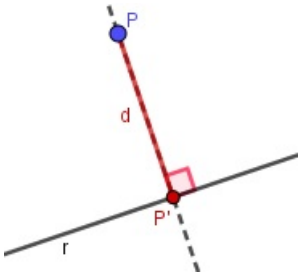
Distance from a point to a line and distance between two parallel lines

Distance from a point to a line

The distance from a point A to a line r is equal to the distance from A to its orthogonal projection A' on the line r, according to the figure beside.

We calculate the distance d by doing:



1. Determine the line PP' that is perpendicular to r containing P;
2. Determine $P' = PP' \cap r$;
3. Determine $d = \overline{PP'}$.

Example:

Consider in $\mathbb{R}^3$, $P = (2,1,1)$ and $r : (x,y,z) = (0,0,-1) + k(1,-1,1), \quad k \in \mathbb{R}$. Let us determine the distance from P to r.

For example, $u = (1,2,1)$ is orthogonal to $v = (1,-1,1)$ because $u \cdot v = 0$.

Then, $PP' : (x,y,z) = (2,1,1) + t(1,2,1), t \in \mathbb{R}$.

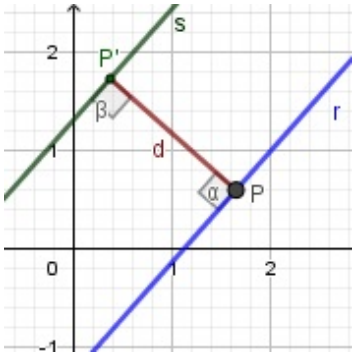
Besides that $P' = (x,y,z) = PP' \cap r$ is such that $\begin{cases} x = \dfrac{y}{-1} = z+1 \\ x-2 = \dfrac{y-1}{2} = z-1 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = -1 \\ z = 0 \end{cases}$.

That is $P' = (1,-1,0)$.

Finally $d = \overline{PP'} = \sqrt{(2-1)^2 + (1+1)^2 + (1-0)^2} = \sqrt{6}$.

Distance between two parallel lines

The distance between two parallel lines r and s is equal to the distance between a point P of the line r and its orthogonal projection on s.



- P' is the orthogonal projection of $P \in r$ on s;
- d is the distance of P to P'.

Example: The lines $r : 2x - y + 2 = 0$ and $s : -4x + 2y + 1 = 0$ are parallel, both have the direction of $v = (1,2)$. The distance from r to s is equal to the distance from a point $P \in r$ to s.

Consider $P = (-1,0) \in r$ and $u = (2,-1)$ orthogonal to $v = (1,2)$.

Then the line $t : y - 0 = -\frac{1}{2}(x+1)$ contains P and is perpendicular to the lines r and s. Thus

$$P'(x,y) = t \cap s : \left\{ \begin{array}{c} y = -\frac{1}{2}(x+1) \\ -4x + 2y + 1 = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} y = -\frac{1}{2}(x+1) \\ 4x - x - 1 + 1 = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} y = \frac{1}{2} \\ x = 0 \end{array} \right. .$$

Finally, the distance from r to s is equal to the distance from $P = (-1, 0)$ to $P'(0, \frac{1}{2})$,

$$d(r,s) = d(P,P') = \sqrt{(-1+0)^2 + (0 - \frac{1}{2})^2} = \frac{\sqrt{5}}{2}.$$