Riccati differential equation

The Riccati equation is one of the most interesting nonlinear differential equations of first order. The general form is:

$$x'(t) = a(t)x(t) + b(t)x^2(t) + c(t), \tag{1}$$

where $a(t), b(t)$ and $c(t)$ are *continuous functions* of $t \in \mathbb{I} \subset \mathbb{R}$. If $b \equiv 0$ equation (1) is linear differential equation of first order and for $c \equiv 0$ the equation (1) is a Bernoulli equation with $n = 2$.

It can be solved if a particular solution $x_1(t)$ of a Riccati equation is known. Unfortunately, there is no strict algorithm to find the particular solution, which depends on the types of the functions $a(t)$, $b(t)$ and $c(t)$.

If a particular solution $x_1(t)$ of a Riccati equation is known, the general solution of the equation is given by

$$x(t) = x_1(t) + y(t). \tag{2}$$

Indeed, substituting the solution into Riccati equation, we have

$$(x_1(t) + y(t))' = a(t)\left(x_1(t) + y(t)\right) + b(t)\left(x_1(t) + y(t)\right)^2 + c(t)$$

$$\underline{x_1'(t)} + y'(t) = \underline{a(t)x_1(t)} + a(t)y(t) + \underline{b(t)x_1^2(t)} + 2b(t)x_1(t)y(t) + b(t)y(t)^2 + \underline{c(t)}$$

The underlined terms in the left and in the right side can be canceled because x_1 is a particular solution satisfying the equation. As a result we obtain the differential equation for the function

$$y'(t) = \left(a(t) + 2b(t)x_1(t)\right)y(t) + b(t)y^2(t) \tag{3}$$

which is a Bernoulli equation.

Substitution of $z(t) = y^{-1}(t)$ converts the given Bernoulli equation into a linear differential equation that allows integration.

Besides the general Riccati equation, there is an infinite number of particular cases of Riccati equation at certain coefficients of $a(t)$, $b(t)$ and $c(t)$. Many of these particular cases have integrable solutions.

Solution Process

The solution process for linear differential equation of first order is as follows:

1. Put the differential equation in the correct initial form, (1).
2. If a particular solution $x_1(t)$ of a Riccati equation is known, we use the substitution (2).
3. We obtain the Bernoulli equation (3) with $n = 2$.

4. Solve for the Bernoulli equation (3).

Example 1 *Find the solution to the following differential equation:*

$$tx'(t) - 3x(t) + 2x^2(t) = 2, t \neq 0.$$

A particular solution $x_1(t) = 2$ of a Riccati equation is known.

1. We convert this equation into the standard form:
   $x'(t) = \frac{3}{t}x(t) - \frac{2}{t}x^2(t) + \frac{2}{t}$.
   It is a Riccati equation.

2. We use the particular solution $x_1(t) = 2$ in the substitution (2),
   $x(t) = 2 + y(t)$.

3. We get the following differential equation for the new function

   $(2 + y(t))' = \frac{3}{t}(2 + y(t)) - \frac{2}{t}\left(t^2 + y(t)\right)^2 + \frac{2}{t}$,

   $y'(t) = \frac{6}{t} - \frac{1}{t}y(t) + t + \frac{2}{t}y(t) + \frac{1}{t^3}y^2(t) + 2t$,

   $y'(t) = \frac{1}{t}y(t) + \frac{1}{t^3}y^2(t)$.

   It is a Bernoulli equation with $n = 2$.

4. Solve for the Bernoulli equation. The substitution $z(t) = y^{-1}(t)$ converts it to a linear differential equation.
   We divide the differential equation by $y^2(t)$

   $$y^{-2}(t)y'(t) = \frac{1}{t}y^{-1}(t) + \frac{1}{t^3}. \tag{4}$$

   We make the substitution $z(t) = y^{-1}(t)$ and $z'(t) = -y^{-2}(t)y'(t)$ on equation (4) and obtain $-z'(t) = \frac{1}{t}z(t) + \frac{1}{t^3} \Rightarrow z'(t) + \frac{1}{t}z(t) = -\frac{1}{t^3}$ a linear differential equation.

5. We solve it and $z(t) = \dfrac{1 + ct}{t^2} \Rightarrow y(t) = \dfrac{t^2}{1 + ct} \Rightarrow x(t) = t^2 + \dfrac{t^2}{1 + ct}$.

Example 2 *Find the solution to the following differential equation:*

$$tx'(t) + x(t) - \frac{1}{t^2}x^2(t) = 2t^2, t \neq 0.$$

1. We convert this equation into the standard form:

   $x'(t) = -\frac{1}{t}x(t) + \frac{1}{t^3}x^2(t) + 2t$.

   It is a Riccati equation.

2. Try to find a particular solution in the form $x_1(t) = ct^2$. Substituting this into the Riccati equation, we can determine the coefficient c.

   (a) $2ct = -ct + c^2t + 2t \Rightarrow c \in \{1, 2\}$.

   Thus, there are even two particular solutions. However, we need only one of them. So we choose, for example, $c = 1$, $x_1(t) = t^2$. We use the substitution (2),

   $x(t) = t^2 + y(t)$.

   We get the following differential equation for the new function

   $\left(t^2 + y(t)\right)' = -\frac{1}{t}\left(t^2 + y(t)\right) + \frac{1}{t^3}\left(t^2 + y(t)\right)^2 + 2t,$

   $2t + y'(t) = -t - \frac{1}{t}y(t) + t + \frac{2}{t}y(t) + \frac{1}{t^3}y^2(t) + 2t,$

   $y'(t) = \frac{1}{t}y(t) + \frac{1}{t^3}y^2(t).$

3. It is a Bernoulli equation with Bernoulli equation $n = 2$. The substitution $z(t) = y^{-1}(t)$ converts it to a linear differential equation

   $z'(t) = -y^{-2}(t)y'(t).$

   We divide the differential equation by $y^2(t)$

   $$y^{-2}(t)y'(t) = \frac{1}{t}y^{-1}(t) + \frac{1}{t^3}. \qquad (5)$$

4. We make the substitution $z(t) = y^{-1}(t)$ and $z'(t) = -y^{-2}(t)y'(t)$ on equation (5) and obtain

   $-z'(t) = \frac{1}{t}z(t) + \frac{1}{t^3} \Rightarrow z'(t) + \frac{1}{t}z(t) = -\frac{1}{t^3}$ a linear differential equation.

   We solve it and $z(t) = \dfrac{1+ct}{t^2} \Rightarrow y(t) = \dfrac{t^2}{1+ct} \Rightarrow x(t) = t^2 + \dfrac{t^2}{1+ct}$.

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