Examples of linear transformations

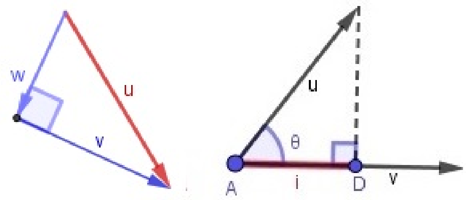
Computer graphics deals with the manipulation of images, through their positioning through linear transformations such as orthogonal projections and rotations, among others.

Orthogonal projections

Any vector v can be written as the sum of two orthogonal vectors, called components of v.

Besides that, if $u, v \in \mathbb{R}^2$ or $u, v \in \mathbb{R}^3$, then the orthogonal projection of a vector $v \in E$ in u is

$$proj_v u = ||u||cos(\theta)\frac{v}{||v||} = \frac{u \cdot v}{||v||^2}v,$$



according to the image beside.

We can say that if $W = \langle v \rangle$ is a subspace of a vector space V, then the orthogonal projection of a vector $u \in V$ in W is given by

$$proj_W(u) = \frac{u \cdot v}{||v||^2}v.$$

Similarly, if $B = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal base of $W \subseteq V$ and $u \in V$, then

$$proj_W(u) = \frac{u \cdot v_1}{||v_1||^2}v_1 + \frac{u \cdot v_2}{||v_2||^2}v_2 + \cdots + \frac{u \cdot v_n}{||v_n||^2}v_n.$$

In particular, $B = \{(1,0,0), (0,1,0)\}$ is an orthogonal base of the plan $\pi : z = 0$ which is a subspace of $\mathbb{R}^3$. To any $u = (u_1, u_2, u_3) \in V$, we have

$$\begin{aligned} proj_\pi(u) &= ((u_1, u_2, u_3) \cdot (1,0,0))(1,0,0) + ((u_1, u_2, u_3) \cdot (0,1,0))(0,1,0) \\ &= u_1(1,0,0) + u_2(0,1,0). \end{aligned}$$

That is, the orthogonal projection onto the xy-plane drops of the z coordinate. Formally, this can be written in matrix form as the following:

$$proj_{xy}(u) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}.$$

Notice that this transformation preserves u_1 and u_2 but drops the last coordinate.

Also the orthogonal projection onto the yz-plane drops of the x coordinate. Formally, that is:

$$proj_{yz}(u) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix}.$$

Example: $proj_{yz}(-1, 2, 3) = (0, 2, 3)$

Rotation

An operator that rotates a vector in $\mathbb{R}^2$ through a given angle θ is called a rotation operator in $\mathbb{R}^2$ and is defined by

$$\begin{array}{rccc} f_R\colon & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x, y) & \longrightarrow & (xcos(\theta) - ysin(\theta), xsen(\theta) + ycos(\theta)) \end{array}$$

or in the matrix form,

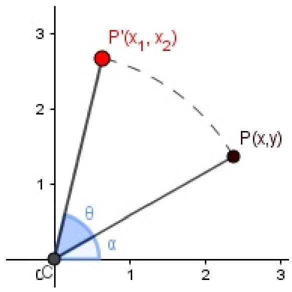
$$f_R(x, y) = \begin{bmatrix} cos(\theta) - sin(\theta) \\ sen(\theta) + cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

according to the following:

Let $(x_1, y_1) = f_R(x, y)$ and check the diagram. we can write $x_1 = rcos(\theta + \alpha)$, $y_1 = rsin(\theta + \alpha)$.

Also $x = rcos(\alpha)$, $y_1 = rsin(\alpha)$.
Using trigonometric identities we have

$$x_1 = xcos(\theta) - ysin(\theta) \quad \text{and} \quad y_1 = xsen(\theta) + ycos(\theta).$$



The operator that rotates a vector in $\mathbb{R}^3$ about the positive x-axis through a given angle θ is called a rotation operator in $\mathbb{R}^3$ and is defined by the matrix form,

$$f_R(x, y, z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sen(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

The operator that rotates a vector in $\mathbb{R}^3$ about the positive y-axis through a given angle θ is called a rotation operator in $\mathbb{R}^3$ and is defined by the matrix form,

$$f_R(x, y, z) = \begin{bmatrix} cos(\theta) & 0 & -sin(\theta) \\ 0 & 1 & 0 \\ sen(\theta) & 0 & cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The operator that rotates a vector in $\mathbb{R}^3$ about the positive z-axis through a given angle θ is called a rotation operator in $\mathbb{R}^3$ and is defined by the matrix form,

$$f_R(x, y, z) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sen(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$