Subspace spanned

Recall that:

Definition (linear combination): For vectors $v_1, v_2, \ldots, v_k$ in a vector space V, the vector

$$v = a_1v_1 + a_2v_2 + \cdots + a_kv_k$$

is called a linear combination of the vectors $v_1, v_2, \ldots, v_k$. The scalars a_i are called coefficients.

Example: Consider the vector space $\mathbb{R}^3$.

The vector $v = (1, 2, 0)$ is a linear combination of the vector set $A = \{(3, 1, 2), (2, -1, 2)\}$, because $(1, 2, 0) = (3, 1, 2) - (2, -1, 2)$.

Also $u = (0, -5, 2)$ is a linear combination of the vector set $A = \{(3, 1, 2), (2, -1, 2)\}$, because $(0, -5, 2) = -2(3, 1, 2) + 3(2, -1, 2)$.

The set S of all vectors that are a linear combination of $A = \{(3, 1, 2), (2, -1, 2)\}$ are all vectors $(x, y, z) \in \mathbb{R}^3$ such that

$$(x, y, z) = k_1(3, 1, 2) + k_2(2, -1, 2), \quad k_1, k_2 \in \mathbb{R}.$$

This equality represents the system

$$\begin{cases} 3k_1 + 2k_2 = x \\ k_1 - k_2 = y \\ 2k_1 + 2k_2 = z \end{cases} \Leftrightarrow \begin{cases} 3k_1 + 2k_2 = x \\ k_1 = k_2 + y \\ 2k_1 + 2k_2 = z \end{cases} \Leftrightarrow \begin{cases} 3(k_2 + y) + 2k_2 = x \\ k_1 = k_2 + y \\ 2(k_2 + y) + 2k_2 = z \end{cases} \Leftrightarrow \begin{cases} k_2 = \dfrac{x - 3y}{5} \\ k_1 = k_2 + y \\ k_2 = \dfrac{z - 2y}{4} \end{cases}$$

Note that this system is only possible if

$$\frac{x - 3y}{5} = \frac{z - 2y}{4}.$$

In conclusion, only the vectors that check the condition $4x - 2y - 5z = 0$ are a linear combination of A.

Definition (linear span): Let V be a vector space and $A = \{v_1, v_2, \ldots, v_k\} \subset V$. The linear span of A is the set of all linear combinations of the vectors $v_1, v_2, \ldots, v_k$, denoted by $\langle A \rangle$, that is:

$$\langle A \rangle = \{a_1v_1 + a_2v_2 + \cdots + a_kv_k : a_1, a_2, \ldots, a_k \in \mathbb{R}\}.$$

Theorem (subspace spanned): If $A = \{v_1, v_2, \ldots, v_k\}$ is a set of vectors of a vector space V, then $\langle A \rangle$ is a subspace of V and is also called the subspace spanned by A. It is the smallest subspace containing the vectors $v_1, v_2, \ldots, v_k$.

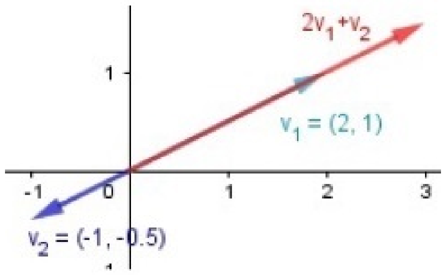
Note that any vector of $\mathbb{R}^2$ spans a line of the plane that contains $(0, 0)$.

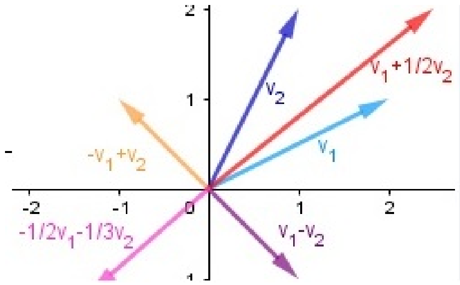
Example: If $A = \{(2,1)\}$, then $\langle A \rangle = \{(2k,k) : k \in \mathbb{R}\} = \{(x,y) \in \mathbb{R}^2 : x - 2y = 0\}$.

Two vectors of $\mathbb{R}^2$ can define a straight line of the plane or the entire plane $\mathbb{R}^2$.
Example:

- $$\begin{aligned} \langle \{(2,1),(-1,-1/2)\} \rangle &= \langle \{(2,1)\} \rangle \\ &= \{(2k,k) : k \in \mathbb{R}\} \end{aligned}$$

  because $\{(2,1),(-1,-1/2)\}$ is linearly dependent. According to the figure, the two vectors are collinear





- If $A = \{(2,1),(1,2)\}$, then $\langle A \rangle = \mathbb{R}^2$, because A is linearly independent and has cardinality 2. According to the figure beside, $v_1 = (2,1)$ and $v_2 = (1,2)$ have different directions and any vector of $\mathbb{R}^2$ can be written as the sum of a scalar multiple of v_1 with a scalar multiple of v_2.

Example: The linear space $\langle B \rangle$ such that $B = \{(1,0,1),(1,2,0),(0,1,1)\}$ is de set

$$S = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z) = k_1(1,0,1) + k_2(1,2,0) + k_3(0,2,-1), k_1,k_2,k_3 \in \mathbb{R}\}.$$

That is, the set of vectores $(x,y,z) \in \mathbb{R}^3$ such that the system

$$\begin{cases} k_1 + k_2 = x \\ 2k_2 + k_3 = y \\ k_1 + k_3 = z \end{cases}$$

is possible. Through its expanded matrix

$$\begin{bmatrix} 1 & 1 & 0 & | & x \\ 0 & 2 & 2 & | & y \\ 1 & 0 & -1 & | & z \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & x \\ 0 & 2 & 2 & | & y \\ 0 & -1 & -1 & | & z-x \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & x \\ 0 & 2 & 2 & | & y \\ 0 & 0 & 0 & | & y+2z-2x \end{bmatrix}$$

we can conclude that the system is possible if $-2x + y + 2z = 0$. That is,

$$S = \{(x,y,z) \in \mathbb{R}^3 : -2x + y + 2z = 0\},$$

which represents a plan of $\mathbb{R}^3$.