MathE project

Limit for real functions of several variables

1 The characterization with sequences

Let $E \subseteq \mathbb{R}^k$ be a nonemty set, let a be a cluster point for E and let us consider a real function $f : E \to \mathbb{R}$.

Definition 1.1 *(with neighbourhoods) One says that $\ell \in \overline{\mathbb{R}}$ is the limit of f at the point a if for any $U \in \mathcal{V}(\ell)$ from $\mathbb{R}$, there exists $V \in \mathcal{V}(a)$ from $\mathbb{R}^k$, such that for any $x \in V \cap E$ with $x \neq a$, we have $f(x) \in U$. We denote this by*

$$\ell = \lim_{x \to a} f(x).$$

Proposition 1.1 *(with sequences)* Let $\ell \in \overline{\mathbb{R}}$. The limit of f at the point a is ℓ if and only if for any sequence $(x_n)_n \subset E$ with $x_n \neq a$ and $\lim\limits_{n \to +\infty} x_n = a$, we have that

$$\lim_{n \to +\infty} f(x_n) = \ell.$$

For a two-variables function $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}$, $f = f(x, y)$ the above proposition is:

Proposition 1.2 *(with sequences)* Let $\ell \in \overline{\mathbb{R}}$. The limit of f at the point (a, b) is ℓ if and only if for any sequence $(x_n, y_n)_n \subset E$ with $(x_n, y_n) \neq (a, b)$ $\lim\limits_{n \to +\infty} x_n = a$ and $\lim\limits_{n \to +\infty} y_n = b$, we have that

$$\lim_{n \to +\infty} f(x_n, y_n) = \ell.$$

Remark 1.1 If there are two sequences $(x_n, y_n)_n, (x'_n, y'_n)_n \subset E$ with $(x_n, y_n) \neq (a, b)$, $(x'_n, y'_n) \neq (a, b)$ and $\lim\limits_{n \to +\infty} (x_n, y_n) = \lim\limits_{n \to +\infty} (x'_n, y'_n) = (a, b)$ such that

$$\lim_{n \to +\infty} f(x_n, y_n) = \ell \text{ and } \lim_{n \to +\infty} f(x'_n, y'_n) = \ell, \quad \ell \neq \ell',$$

then the limit of the function f does not exist.

Example 1.1 *Find $\ell_1 = \lim\limits_{x \to 0} (\lim\limits_{y \to 0} f(x, y))$, $\ell_2 = \lim\limits_{y \to 0} (\lim\limits_{x \to 0} f(x, y)$ and $\ell = \lim\limits_{(x,y) \to (0,0)} f(x, y)$, where*

$$f(x, y) = \frac{x + y}{x - y}, \quad x \neq y.$$

Solution.

$$\ell_1 = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{x+y}{x-y}\right) = \lim_{x\to 0}\frac{x}{x} = 1$$

and

$$\ell_2 = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x+y}{x-y}\right) = \lim_{y\to 0}\frac{y}{-y} = -1.$$

Since the both limits ℓ_1 and ℓ_2 exist and $\ell_1 \neq \ell_2$ it follows that the global limit ℓ of the function doesn't exist. Indeed, we may choose the next two sequences

$$(x_n, y_n) = \left(\frac{1}{n}, 0\right) \to (0,0) \quad \text{and} \quad (x_n', y_n') = \left(0, \frac{1}{n}\right) \to (0,0),$$

for which we have

$$\lim_{n\to+\infty} f(x_n, y_n) = 1 \quad \text{and} \quad \lim_{n\to+\infty} f(x_n', y_n') = -1.$$

Example 1.2 *Find* $\ell_1 = \lim\limits_{x\to 0}(\lim\limits_{y\to 0} f(x,y))$, $\ell_2 = \lim\limits_{y\to 0}(\lim\limits_{x\to 0} f(x,y))$ *and* $\ell = \lim\limits_{(x,y)\to(0,0)} f(x,y)$, *where*

$$f(x,y) = \frac{2xy}{x^2+y^2}, \quad (x,y) \neq (0,0).$$

Solution.

$$\ell_1 = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{2xy}{x^2+y^2}\right) = \lim_{x\to 0} 0 = 0$$

and

$$\ell_2 = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{2xy}{x^2+y^2}\right) = \lim_{y\to 0} 0 = 0.$$

In spite of the fact that the both limits ℓ_1 and ℓ_2 exist and $\ell_1 = \ell_2$, the global limit ℓ of the function doesn't exist. Indeed, we may choose the next two sequences

$$(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0) \quad \text{and} \quad (x_n', y_n') = \left(\frac{2}{n}, \frac{1}{n}\right) \to (0,0),$$

for which we have

$$\lim_{n\to+\infty} f(x_n, y_n) = \lim_{n\to+\infty}\frac{\frac{2}{n^2}}{\frac{2}{n^2}} = 1 \quad \text{and} \quad \lim_{n\to+\infty} f(x_n', y_n') = \lim_{n\to+\infty}\frac{\frac{4}{n^2}}{\frac{5}{n^2}} = \frac{4}{5}.$$

Example 1.3 *Find* $\ell_1 = \lim\limits_{x\to 0}(\lim\limits_{y\to 0} f(x,y))$, $\ell_2 = \lim\limits_{y\to 0}(\lim\limits_{x\to 0} f(x,y))$ *and* $\ell = \lim\limits_{(x,y)\to(0,0)} f(x,y)$, *where*

$$f(x,y) = \frac{xy^2}{2x^2+y^4}, \quad (x,y) \neq (0,0).$$

Solution.

$$\ell_1 = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy^2}{2x^2+y^4}\right) = \lim_{x\to 0} 0 = 0$$

and

$$\ell_2 = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy^2}{2x^2+y^4}\right) = \lim_{y\to 0} 0 = 0.$$

In spite of the fact that the both limits ℓ_1 and ℓ_2 exist and $\ell_1 = \ell_2$, the global limit ℓ of the function doesn't exist. Indeed, we may choose the next two sequences

$$(x_n, y_n) = \left(\frac{1}{n^2}, \frac{1}{n}\right) \to (0,0) \quad \text{and} \quad (x_n', y_n') = \left(\frac{2}{n^2}, \frac{1}{n}\right) \to (0,0),$$

for which we have

$$\lim_{n\to+\infty} f(x_n, y_n) = \lim_{n\to+\infty} \frac{\frac{1}{n^4}}{\frac{3}{n^4}} = \frac{1}{3} \quad \text{and} \quad \lim_{n\to+\infty} f(x_n', y_n') = \lim_{n\to+\infty} \frac{\frac{2}{n^4}}{\frac{9}{n^4}} = \frac{2}{9}.$$

Example 1.4 *Consider the function*

$$f(x,y) = x\cos\frac{1}{y}, \quad y \neq 0.$$

Study the existence of the limits $\ell_1 = \lim\limits_{x\to 0}(\lim\limits_{y\to 0} f(x,y))$, $\ell_2 = \lim\limits_{y\to 0}(\lim\limits_{x\to 0} f(x,y))$ *and* $\ell = \lim\limits_{(x,y)\to(0,0)} f(x,y)$.

Solution. As $\lim\limits_{y\to 0}\cos\frac{1}{y}$ does not exist, it follows that the limit $\ell_1 = \lim\limits_{x\to 0}(\lim\limits_{y\to 0} f(x,y))$ does not exist, while the limit $\ell_2 = 0$. Indeed

$$\ell_2 = \lim_{y\to 0}\left(\lim_{x\to 0} x\cos\frac{1}{y}\right) = \lim_{y\to 0}\cos\frac{1}{y}\cdot 0 = 0.$$

For the study of the global limit ℓ, let us observe that we have

$$\left|\cos\frac{1}{y}\right| \leq 1$$

for each $y \neq 0$, hence $|f(x,y)| \leq |x|$ and, because $\lim\limits_{(x,y)\to(0,0)} |x| = 0$, we deduce that the limit ℓ exists and $\ell = \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Example 1.5 *Consider the function*

$$f(x,y) = (x+y)\sin\frac{1}{x}\sin\frac{1}{y}, \quad x \neq 0, y \neq 0.$$

Show that the limits $\ell_1 = \lim\limits_{x\to 0}(\lim\limits_{y\to 0} f(x,y))$ *and* $\ell_2 = \lim\limits_{y\to 0}(\lim\limits_{x\to 0} f(x,y))$ *do not exist, but the global limit* $\ell = \lim\limits_{(x,y)\to(0,0)} f(x,y)$ *exists.*

Solution. As $\lim\limits_{y\to 0}\sin\frac{1}{y}$ and $\lim\limits_{x\to 0}\sin\frac{1}{x}$ do not exist, it follows that the both limits ℓ_1 and ℓ_2 do not exist. Regarding to the global limit ℓ, let us observe that we have

$$\left|\sin\frac{1}{x}\sin\frac{1}{y}\right| \leq 1$$

for each $x \neq 0$, $y \neq 0$, hence $|f(x,y)| \leq |x| + |y|$ and, because $\lim\limits_{(x,y)\to(0,0)}(|x| + |y|) = 0$ we deduce that the global limit ℓ exists and $\ell = \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.