Concept of Linear Transformation

Definition: Let U and V be two real vector spaces. $T: U \to V$ is a linear transformation if:

(i) $\forall\, x, y \in U,\ T(x+y) = T(x) + T(y)$
(ii) $\forall\, x \in U,\ \forall\, \alpha \in \mathbb{R},\ T(\alpha x) = \alpha\, T(x)$

1. Prove that the transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x,y) = (2x, y, -y)$ is linear.

(i) Considering $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we have:

$$\begin{aligned} T\big((x_1,y_1)+(x_2,y_2)\big) &= T(x_1+x_2, y_1+y_2) \\ &= (2(x_1+x_2), y_1+y_2, -(y_1+y_2)) \\ &= (2x_1+2x_2, y_1+y_2, -y_1-y_2) \end{aligned}$$

On the other side,

$$\begin{aligned} T(x_1,y_1)+T(x_2,y_2) &= (2x_1, y_1, -y_1) + (2x_2, y_2, -y_2) \\ &= (2x_1+2x_2, y_1+y_2, -y_1-y_2) \end{aligned}$$

We concluded that,

$$T\big((x_1,y_1)+(x_2,y_2)\big) = T(x_1,y_1)+T(x_2,y_2), \forall\, (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$$

☺ **The first condition of linearity of a transformation is proved.**

(ii) Considering $(x_1, y_1) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, we have:

$$\begin{aligned} T\big(\alpha(x_1,y_1)\big) &= T(\alpha x_1, \alpha y_1) = (\alpha 2x_1, \alpha y_1, -\alpha y_1) \\ &= \alpha(2x_1, y_1, -y_1) = \alpha T(x_1,y_1) \end{aligned}$$

We concluded that,

$$T\big(\alpha(x_1,y_1)\big) = \alpha T(x_1,y_1), \forall\, (x_1,y_1) \in \mathbb{R}^2, \forall\, \alpha \in \mathbb{R}$$

☺ **The second condition of linearity is also verified.**

Conclusion: **Since both linearity conditions are verified, *T* is a linear transformation.**

2. The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2,\ T(x,y) = (x, 1+y)$ is linear?

(i) Considering $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we have:

$$\begin{aligned} T\big((x_1,y_1)+(x_2,y_2)\big) &= T(x_1+x_2, y_1+y_2) \\ &= (x_1+x_2, 1+y_1+y_2) \end{aligned}$$

On the other side,

$$\begin{aligned} T(x_1,y_1)+T(x_2,y_2) &= (x_1, 1+y_1)+(x_2, 1+y_2) \\ &= (x_1+x_2, 2+y_1+y_2) \end{aligned}$$

We concluded that,

$$\exists\ (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2 : T\big((x_1,y_1)+(x_2,y_2)\big) \neq T(x_1,y_1)+T(x_2,y_2)$$

☹ **The first condition of linearity of a transformation is not verified.**

Conclusion: **As the first linearity condition is not verified, we concluded that *T* is not a linear transformation.**