* Every definite integration problem begins with checking if the conditions for Fundamental theorem of calculus are met or not.

Evaluate $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{4-x^2}}{x}\,dx$

converting to form * $R(x, \sqrt{1-f^2})$

Sol

* $$I(x) = \int \frac{\sqrt{4-x^2}}{x}\,dx = \int \frac{\sqrt{\left(\frac{4-x^2}{4}\right)\times 4}}{x}\,dx$$

$$= 2\int \frac{\sqrt{1-\left(\frac{x}{2}\right)^2}}{x}\,dx$$

A.C.I

Performing trigonometric substitution,

$$\frac{x}{2} = \sin(t)$$

$$\Leftrightarrow x = 2\sin(t)$$

$$\Leftrightarrow dx = 2\cos(t)\,dt$$

A.C.I

$$= 2\int \frac{\sqrt{1-\sin^2(t)}}{\cancel{2}\sin(t)}\,\cancel{2}\cos(t)\,dt$$

$$= 2\int \frac{\sqrt{\cos^2(t)}}{\sin(t)}\cos(t)\,dt$$

$$= 2\int \frac{\cos(t)}{\sin(t)}\cdot\cos(t)\,dt$$

$$= 2\int \frac{\cos^2(t)}{\sin(t)}\,dt$$

$$= 2\int \frac{1-\sin^2(t)}{\sin(t)}\,dt$$

$$= 2\int \operatorname{cosec}(t) - \sin(t)\, dt$$

$$= 2\int \operatorname{cosec}(t)\, dt - 2\int \sin(t)\, dt$$

$$= 2\ln\left|\operatorname{cosec}(t) - \operatorname{ctg}(t)\right| + 2\cancel{\sin}\cos(t) + C$$

converting to function dependent on 'x'

A.C.L,

$$\sin(t) = \frac{x}{2}$$

$$\operatorname{cosec}(t) = \frac{2}{x}$$

$$\cos(t) = \sqrt{1-\sin^2 t}$$

$$= \sqrt{1-\frac{x^2}{4}}$$

$$= \frac{\sqrt{4-x^2}}{2}$$

$$\cot(t) = \frac{\cos(t)}{\sin(t)}$$

$$= \frac{\sqrt{4-x^2}}{x}$$

A.C.L

$$= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \cancel{2}\cdot\frac{\sqrt{4-x^2}}{\cancel{2}} + C$$

$$= 2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C$$

Now, $$\int_{\sqrt{3}}^{2} \frac{\sqrt{4-x^2}}{x}\,dx = \Big[I(x)\Big]_{\sqrt{3}}^{2}$$

$$= \left[2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2}\right]_{\sqrt{3}}^{2}$$

$$= \left(\cancelto{0}{2\ln\left|\frac{2-\sqrt{4-4}}{2}\right|} + \cancelto{0}{\sqrt{4-4}}\right) - \left(2\ln\left|\frac{2-\sqrt{4-3}}{\sqrt{3}}\right| + \sqrt{4-3}\right)$$

$$= -\left(2\ln\frac{1}{\sqrt{3}} + 1\right)$$

$$= -2\ln\frac{1}{\sqrt{3}} - 1$$

$$= -2\ln(3)^{-1/2} - 1$$

$$= -2\times -\frac{1}{2}\ln(3) - 1$$

$$= \ln(3) - 1$$