Higher Order Linear Equations

Definition *An n-th linear differential equation is an equation of the form*

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t) \tag{3.1}$$

We assume that the functions $P_0, ..., P_n$ and G are continuous real-valued functions on some interval $I : \alpha < t, < \beta$, and that P_0 is nowhere zero in this interval. Then, dividing Eq. (3.1) by $P_0(t)$, we obtain

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t) \tag{3.2}$$

In developing the theory of linear differential equations, it is helpful to introduce a differential operator notation. Let $p_1, \cdots p_n$ be continuous functions on an open interval I , that is, for $\alpha < t < \beta$. The cases $\alpha = -\infty$, or $\beta = \infty$, or both, are included. Then, for any function f that is n-th differentiable on I , we define the differential operator L by the equation

$$L[y](t) = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y.$$

First by direct calculation it can prove the next assertion

Proposition 3.1 *The differential operator L is linear transform, that is*

$$L[ay_1 + by_2] = aL[y_1] + bL[y_2]$$

where $a, b \in R$ and y_1, y_2 are continuous functions.

Since Eq. (3.2) involves the n-th derivative of y with respect to t, it will, so to speak, require n integrations to solve Eq. (3.2). Each of these integrations introduces an arbitrary constant. Hence we can expect that, to obtain a unique solution, it is necessary to specify n initial conditions,

$$y(t_0) = y_0, \quad y'(t_0) = y_0', ..., y^{(n-1)}(t_0) = y_0^{(n-1)}, \tag{3.3}$$

where t_0 may be any point in the interval I and $y_0,\ y_0', ..., y_0^{(n-1)}$ is any set of prescribed real constants. That there does exist such a solution and that it is unique are assured by the following existence and uniqueness theorem.

Theorem 3.1 *If the functions $p_1, p_2, ..., p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = \psi(t)$ of the differential equation* (3.2) *that also satisfies the initial conditions* (3.3). *This solution exists throughout the interval* I .

We will not give a proof of this theorem here. However, if the coefficients $p_1, ..., p_n$ are constants, then we can construct the solution of the initial value problem (3.2), (3.3).

The Homogeneous Equation. We first discuss the homogeneous equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \ + p_{n-1}(t)y' + p_n(t)y = 0. \tag{3.4}$$

If we denote by V_0 the set of all solution of the eq.(3.4) then it is easy to see that V_0 is subset of the set V of n-th differentiable function which is a linear space. By direct calculation it can prove

Proposition 3.2 *V_0 is a real linear subspace of V of dimension n.*

If the functions $y_1, y_2, ..., y_n$ are solutions of Eq. (3.4), then it follows by direct computation that the linear combination

$$y = c_1y_1(t) + c_2y_2(t) + \ + c_ny_n(t), \tag{3.5}$$

where $c_1, ..., c_n$ are arbitrary constants, is also a solution of Eq. (3.4). It is then natural to ask whether every solution of Eq. (3.4) can be expressed as a linear combination of $y_1, ..., y_n$. This will be true if, regardless of the initial conditions (3.3) that are prescribed, it is possible to choose the constants $c_1, ..., c_n$ so that the linear combination (3.5) satisfies the initial conditions. Specifically, for any choice of the point t_0 in I, and for any choice of $y_0, y_0', ..., y_0^{(n-1)}$, we must be able to determine $c_1, ..., c_n$ so that the equations

$$
\begin{cases}
c_1 y_1(t_0) + \quad + c_n y_n(t_0) = y_0 \\
c_1 y_1'(t_0) + \quad + c_n y_n'(t_0) = y_0' \\
\vdots \quad \vdots \\
c_1 y_1^{(n-1)}(t_0) + \quad + c_n y_n^{(n-1)}(t_0) = y_0^{(n-1)}.
\end{cases}
\tag{3.6}
$$

are satisfied. Equations (3.6) can be solved uniquely for the constants $c_1, ..., c_n$, provided that the determinant of coefficients is not zero. On the other hand, if the determinant of coefficients is zero, then it is always possible to choose values of $y_0, y_0', ..., y_0^{(n-1)}$ such that Eqs. (3.6) do not have a solution. Hence a necessary and sufficient condition for the existence of a solution of Eqs. (3.6) for arbitrary values of $y_0, y_0', ..., y_0^{(n-1)}$ is that the Wronskian

$$
W(y_1, ..., y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}
\tag{3.7}
$$

is not zero at $t = t_0$. Since t_0 can be any point in the interval I , it is necessary and sufficient that $W(y_1, y_2, ..., y_n)$ be nonzero at every point in the interval. It can be shown that if $y_1, y_2, ..., y_n$ are solutions of Eq. (3.4), then $W(y_1, y_2, ..., y_n)$ is either zero for every t in the interval I or else is never zero there. Hence we have the following theorem.

Theorem 3.2 *If the functions $p_1, p_2, ..., p_n$ are continuous on the open interval I , if the functions $y_1, y_2, ..., y_n$ are solutions of Eq.* (3.4), *and if for at least $W(y_1, y_2, ..., y_n) \neq 0$ one point in I , then every solution of Eq.* (3.4) *can be expressed as a linear combination of the solutions $y_1, y_2, ..., y_n$.*)

A set of solutions $y_1, y_2, ..., y_n$ of Eq. (3.4) whose Wronskian is nonzero is referred to as a fundamental set of solutions. Since all solutions of Eq. (3.4) are of the form (3.5), we use the term general solution to refer to an arbitrary linear combination of any fundamental set of solutions of Eq. (3.4).

The discussion of linear dependence and independence can also be generalized. The functions $f_1, f_2, ..., f_n$ are said to be linearly dependent on I if there exists a set of constants $k_1, k_2, ..., k_n$, not all zero, such that

$$k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0 \tag{3.8}$$

for all t in I . The functions $f_1, f_2, ..., f_n$ are said to be linearly independent on I if they are not linearly dependent there. If $y_1, ..., y_n$ are solutions of Eq. (4), then it can be shown that a necessary and sufficient condition for them to be linearly independent is that $W(y_1, y_2, ..., y_n)(t_0) \neq 0$ for some t_0 in I. Hence a fundamental set of solutions of Eq. (3.4) is linearly independent, and a linearly independent set of n solutions of Eq. (3.4) forms a fundamental set of solutions.

The Non homogeneous Equation.

Now consider the non homogeneous equation (3.2),

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \ + p_{n-1}(t)y' + p_n(t)y = g(t).$$

If Y_1 and Y_2 are any two solutions of Eq. (3.2), then it follows immediately from the linearity of the operator L that

$$L[Y_1 - Y_2](t) = L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0.$$

Hence the difference of any two solutions of the non homogeneous equation (3.2) is a solution of the homogeneous equation (3.4). Since any solution of the homogeneous equation can be expressed as a linear combination of a fundamental set of solutions $y_1, y_2, ..., y_n$ it follows that any solution of Eq. (3.2) can be written as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \ + c_n y_n(t) + Y(t), \tag{3.9}$$

where Y is some particular solution of the non homogeneous equation (3.2). The linear combination (3.9) is called the general solution of the non homogeneous equation (3.2). Thus the primary problem is to determine a fundamental set of solutions $y_1, y_2, ..., y_n$, of the homogeneous equation (3.4). If the coefficients are constants, this is a fairly simple problem.

Homogeneous Equations with Constant Coefficients

Consider the nth order linear homogeneous differential equation

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \;\; + a_{n-1} y' + a_n y = 0, \tag{3.10}$$

where $a_0, a_1, ..., a_n$ are real constants. By using the Euler's method we find a solution of the equations with constant coefficients of the form $y = e^{\lambda t}$, for suitable values of λ. Indeed,

$$L[e^{\lambda t}] = e^{\lambda t}(a_0 \lambda^n + a_1 \lambda^{n-1} + \;\; + a_{n-1}\lambda + a_n) = e^{\lambda t} P(\lambda) \tag{3.11}$$

for all t, where

$$P(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \;\; + a_{n-1}\lambda + a_n. \tag{3.12}$$

For those values of λ for which $P(\lambda) = 0$, it follows that $L[e^{\lambda t}] = 0$ and $y = e^{\lambda t}$ is a solution of Eq. (3.10). The polynomial $P(\lambda)$ is called the characteristic polynomial, and the equation $P(\lambda) = 0$ is the characteristic equation of the differential equation (3.10). Next we have the next situation:

Real and Unequal Roots. If the roots of the characteristic equation are real and no two are equal, then we have n distinct solutions $e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_n t}$ of Eq. (3.10). If these functions are linearly independent, then the general solution of Eq. (3.10) is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t}. \tag{3.13}$$

One way to establish the linear independence of $e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_n t}$ is to evaluate their Wronskian determinant.

Example Find the general solution of

$$y^{(4)} + y^{(3)} - 7y'' - y' + 6y = 0. \tag{3.14}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 1,\ y'(0) = 0,\ y''(0) = -2, y'''(0) = -1 \tag{3.15}$$

The characteristic equation of the differential equation (3.16) is the polynomial equation

$$r^4 + r^3 - 7r^2 - r + 6 = 0. \tag{3.16}$$

The roots of this equation are $r_1 = 1, r_2 = -1, r_3 = 2, r_4 = -3$. Therefore the general solution of Eq. (3.15) is

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}. \tag{3.17}$$

The initial conditions (3.15) require that $c_1, ..., c_4$ satisfy the four equations

$$\begin{cases} c_1 + c_2 + c_3 + c_4 = 1, \\ c_1 - c_2 + 2c_3 - 3c_4 = 0, \\ c_1 + c_2 + 4c_3 + 9c_4 = -2, \\ c_1 - c_2 + 8c_3 - 27c_4 = -1. \end{cases} \tag{3.18}$$

By solving this system of four linear algebraic equations, we find that $c_1 = 11/8, c_2 = 5/12, c_3 = -2/3, c_4 = -1/8$. Therefore the solution of the initial value problem is

$$y = \frac{11}{8}e^t + \frac{5}{12}e^{-t} - 23e^{2t} - 18e^{-3t}. \tag{3.19}$$

Complex Roots. If the characteristic equation has complex roots, they must occur in conjugate pairs, $\lambda \pm j\mu,\ j^2 = -1$, since the coefficients $a_0, \cdots, a_n$ are real numbers. Provided that none of the roots is repeated, the general solution of Eq. (3.10) is still of the form (3.15). However, we can replace the complex-valued solutions $e^{(\lambda+j\mu)t}$ and $e^{(\lambda-j\mu)t}$ by the real-valued solutions

$$e^{\lambda t}\cos\mu t, \qquad e^{\lambda t}\sin\mu t \tag{3.20}$$

obtained as the real and imaginary parts of $e^{(\lambda+j\mu)t}$. Thus, even though some of the roots of the characteristic equation are complex, it is still possible to express the general solution of Eq. (3.10) as a linear combination of real-valued solutions.

Example. Find the general solution of

$$y^{(iv)} - y = 0. \tag{3.21}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 7/2, \quad y'(0) = -4, \quad y''(0) = 5/2, y'''(0) = -2 \tag{3.22}$$

We find that the characteristic equation of the differential equation (3.21)

$$r^4 - 1 = (r^2 - 1)(r^2 + 1) = 0.$$

Therefore the roots are $r = 1, -1, j, -j$, and the general solution of Eq. (3.22)

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

If we impose the initial conditions (3.22), we find that $c_1, = 0,\ \ c_2 = 3,\ \ c_3 = 1/2,\ \ c_4 = -1$, thus the solution of the given initial value problem is

$$y = 3e^{-t} + 12\cos t - \sin t. \tag{3.23}$$

Observe that the initial conditions (3.22) cause the coefficient c_1 of the exponentially growing term in the general solution to be zero. Therefore this term is absent in the solution (3.23), which describes an exponential decay to a steady oscillation. However, if the initial conditions are changed slightly, then c_1 is likely to be nonzero and the nature of the solution changes enormously. For example, if the first three initial conditions remain the same, but the value of $y'''(0)$ is changed from -2 to -15/8, then the solution of the initial value problem becomes

$$y = \frac{1}{32}e^{t} + \frac{95}{32}e^{-t} + \frac{1}{2}\cos t - \frac{17}{16}\sin t. \tag{3.24}$$

Repeated Roots. If the roots of the characteristic equation are not distinct, that is, if some of the roots are repeated, then the solution (3.13) is clearly not the general solution of Eq. (3.10). Recall that if r_1, has multiplicity s (where $s \leq n$), then $e^{tr_1}, te^{tr_1}, t^2e^{tr_1}, \cdots, t^{s-1}e^{tr_1}$(18) are corresponding solutions of Eq. (3.10). If a complex root $\lambda + j\mu$ is repeated s times, the complex conjugate $\lambda + j\mu$ is also repeated s times. Corresponding to these $2s$ complex-valued solutions, we can find $2s$ real-valued solutions by noting that the real and imaginary parts of $e^{(\lambda+j\mu)t}, te^{(\lambda+j\mu)t}, t^2e^{(\lambda+j\mu)t}, \cdots, t^{s-1}e^{(\lambda+j\mu)t}, e^{(\lambda-j\mu)t}$, $te^{(\lambda-j\mu)t}, t^2e^{(\lambda-j\mu)t}, \cdots, t^{s-1}e^{(\lambda+j\mu)t}, t^{s-1}e^{(\lambda-j\mu)t}$, are also linearly independent solutions: $e^{\lambda t}\cos\mu t,\ \ e^{\lambda}\sin\mu t,\ \ te^{\lambda t}\cos\mu t,\ \ te^{\lambda t}\sin\mu t,\ \ t^2e^{\lambda t}\cos\mu t,$ $t^2e^{\lambda t}\sin\mu t,\ \cdots,\ t^{s-1}e^{\lambda t}\cos\mu t,\ \ t^{s-1}e^{\lambda t}\sin\mu t$. Hence the general solution of Eq. (3.10) can always be expressed as a linear combination of n real-valued solutions. Consider the following example.

Example. Find the general solution of

$$y^{iv} + 2y'' + y = 0. \tag{3.25}$$

The characteristic equation is $r^4 + 2r^2 + 1 = (r^2 + 1)^2 = 0$. The roots are $r = j, j, -j, -j$, and the general solution of Eq. (19) is

$$y = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t.$$

Exercises. Find the general solution of the given differential equation.

1. $y''' - y'' - y' + y = 0$, 2. $y''' - 3y'' + 3y' + y = 0$,
3. $2y''' - 4y'' - 4y' + 4y = 0$, 4. $y''' - y'' - y' + y = 0$,
5. $y^{(iv)} - 4y''' + 4y'' = 0$ 6. $y^{(vi)} + y = 0$
7. $y^{(iv)} - 5y'' + 4y = 0$ 8. $y^{(vi)} - 3y^{(iv)} + 3y'' - y = 0$ 9. $y^(vi) - y'' = 0$
10. $y^{(v)} - 3y^{(iv)} + 3y''' - 3y'' + 2y' = 0$ 11. $y^{(iv)} - 8y' = 0$
12. $y^{(6)} + 8y^{(4)} + 16y = 0$. 13. $18y''' + 21y'' + 14y' + 4y = 0$,
14. $y^{(4)} - 7y''' + 6y'' + 30y' - 36y = 0$,
15. $12y^{(4)} + 31y''' + 75y'' + 37y' + 5y = 0$.

In each of the given initial value problem, find the particular solution.

1. $y''' + y' = 0;\quad y(0) = 0,\quad y'(0) = 1,\quad y''(0) = 2$
2. $y^{(4)} + y = 0;\quad y(0) = 0,\quad y'(0) = 0,\quad y''(0) = -1,\quad y'''(0) = 0$
3. $y^{(4)} - 4y''' + 4y'' = 0;\quad y(1) = -1,\quad y'(1) = 2,\quad y''(1) = 0,\quad y'''(1) = 0$,
4. $y''' - y'' + y' - y = 0;\quad y(0) = 2,\quad y'(0) = -1,\quad y''(0) = -2$
5. $2y^{(4)} - y''' - 9y'' + 4y' + 4y = 0;\quad y(0) = -2,\quad y'(0) = 0,\quad y''(0) = -2$,
$y'''(0) = 0$ 6. $4y''' + y' + 5y = 0;\quad y(0) = 2,\quad y'(0) = 1,\quad y''(0) = -1$
7. $6y''' + 5y'' + y' = 0;\quad y(0) = -2,\quad y'(0) = 2,\quad y''(0) = 0$
8. $y^{(4)} + 6y''' + 17y'' + 22y' + y = 0;\quad y(0) = 1,\quad y'(0) = -2, y''(0) = 0$,
$y^{(3)}(0) = 0$,

The Method of Undetermined Coefficients

A particular solution y of the non homogeneous nth order linear equation with constant coefficients

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \ + a_{n-1} y' + a_n y = g(t) \tag{4.1}$$

can be obtained by the method of undetermined coefficients, provided that g(t) is of an appropriate form. While the method of undetermined coefficients is not as general as the method of variation of parameters described in the next section, it is usually much easier to use when applicable.

When the constant coefficient linear differential operator L is applied to a polynomial $A_0 t^m + A_1 t^{m-1} + \ + A_m$, an exponential function $e^{\alpha t}$, a sine function $\sin \beta t$, or a cosine function $\cos \beta t$, the result is a polynomial, an exponential function, or a linear combination of sine and cosine functions, respectively. Hence, if g(t) is a sum of polynomials, exponentials, sines, and cosines, or products of such functions, we can expect that it is possible to find Y(t) by choosing a suitable combination of polynomials, exponentials, and so forth, multiplied by a number of undetermined constants. The constants are then determined so that Eq. (4.1) is satisfied. More exactly if $g(t) = e^{at}(P_n(t)\cos bt + Q_m(t)\sin bt)$ and if $r = a + jb$ is a root of the characteristic polinom of the eq. 4.1, of multiplicity s then it can be determine a particular solution of the form $y(t) = t^s e^{at}(P_u(t)\cos bt + Q_u(t)\sin bt)$, where $u = max\{m, n\}$ and P_u, Q_u are the undetermined polinoms which are determinate if y(t) is a solution of eq.1

Example Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^t. \tag{4.2}$$

The characteristic polynomial for the homogeneous equation corresponding to Eq. (4.2) is $P(r) = r^3 - 3r^2 + 3r - 1 = (r-1)^3$, so the general solution of the homogeneous equation is

$$y_0(t) = c_1e^t + c_2te^t + c_3t^2e^t. \tag{4.3}$$

To find a particular solution Y(t) of Eq. (4.2), we start by assuming that $Y(t) = Ae^t$. However, since e^t, te^t, t^2e^t are all solutions of the homogeneous equation, we must multiply this initial choice by t^3. Thus our final assumption is that $Y(t) = At^3e^t$, where A is an undetermined coefficient. To find the correct value for A, we differentiate Y(t) three times, substitute for y and its derivatives in Eq. (4.2), and collect terms in the resulting equation. In this way we obtain $6Ae^t = 4e^t$. Thus A = 2/3 and the particular solution is

$$Y_p(t) = \frac{2}{3}t^3e^t. \tag{4.4}$$

The general solution of Eq. (4.2) is the sum of $y_0(t)$ from Eq. (4.3) and Y(t) from Eq. (4.4), that is

$$y(t) = Y_0(t) + y_p(t) = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t.$$

Example Find a particular solution of the equation

$$y^{(4)} + 2y'' + y = 3sint - 5cost. \tag{4.5}$$

The characteristic polynomial for the homogeneous equation corresponding to eq. 4.5 is $r^4 + 2r^2 + 1 = (r^2+1)^2$, and the general solution of the homogeneous equation is

$$y_0(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t, \tag{4.6}$$

corresponding to the roots $r = j, j, -j, -j$ of the characteristic equation. Our initial assumption for a particular solution is $y_p(t) = A \sin t + B \cos t$, but we must multiply this choice by t^2 to make it different from all solutions of the homogeneous equation. Thus our final assumption is

$$y_p(t) = At^2 \sin t + Bt^2 \cos t.$$

Next, we differentiate $y_p(t)$ four times, substitute into the differential equation (4.5), and collect terms, obtaining finally $-8A \sin t - 8B \cos t = 3 \sin t - 5 \cos t$. Thus $A = -38, B = 58$, and the particular solution of Eq. (4.5) is

$$y_p(t) = -38t^2 \sin t + 58t^2 \cos t. \tag{4.7}$$

The general solution of Eq. (4.5) is the sum of $y_0(t)$ from Eq. (4.6) and Y (t) from Eq. (4.6), that is

$$y(t) = Y_0(t) + y_p(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t - 38t^2 \sin t + 58t^2 \cos t.$$

If g(t) is a sum of several terms, it is often easier in practice to compute separately the particular solution corresponding to each term in g(t). As for the second order equation, the particular solution of the complete problem is the sum of the particular solutions of the individual component problems. This is illustrated in the following example.

Example Find a particular solution of

$$y''' - 4y' = t + 3 \cos t + e^{-2t}. \tag{4.8}$$

First we solve the homogeneous equation. The characteristic equation is $r^3 - 4r = 0$, and the roots are $0, \pm 2$, hence

$$y_0(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}.$$

We can write a particular solution of Eq. (4.8) as the sum of particular solutions of the differential equations

$$y''' - 4y' = t, \qquad y''' - 4y' = 3 \cos t, \qquad y''' - 4y' = e^{-2t}.$$

Our initial choice for a particular solution $y_1(t)$ of the first equation is $A_0 t + A_1$, but since a constant is a solution of the homogeneous equation, we multiply by t. Thus $y_1(t) = t(A_0t + A_1)$. For the second equation we choose $y_2(t) = B\cos t + C\sin t$, and there is no need to modify this initial choice since $\cos t$ and $\sin t$ are not solutions of the homogeneous equation. Finally, for the third equation, since e^{-2t} is a solution of the homogeneous equation, we assume that $y_3(t) = Ete^{-2t}$. The constants are determined by substituting into the individual differential equations; they are $A_0 = -\frac{1}{8}$, $A_1 = 0$, $B = 0$, $C = -\frac{3}{5}$, $E = \frac{1}{8}$. Hence a particular solution of Eq. (4.8) is

$$y_p(t) = -\frac{1}{8}t^2 - \frac{3}{5}\sin t + \frac{1}{8}te^{-2t}.$$

Exercises In each of the next problems determine the general solution of the given differential equation.

1. $y''' - y'' - y' + y = 2e^{-t} + 3$ 2. $y^{(4)} - y = 3t + \cos t$
3. $y''' + y'' + y' + y = e^{-t} + 4t$ 4. $y''' - y' = 2\sin t$ 5.$y^{(4)} - 4y'' = t^2 + e^t$
6. $y^{(4)} + 2y'' + y = 3 + cos2t$ 7. $y^{(4)} + y^{(3)} = t$ 8.$y^{(4)} + y^{(3)} = sin2t$

In each of Problems which follows find the solution of the given initial value problem.

9. $y^{(3)} + 4y' = t,\ y(0) = y'(0) = 0,\ y''(0) = 1$
10. $y^{(4)} + 2y'' + y = 3t + 4,\ y(0) = y'(0) = 0,\ \ y''(0) = y^{(3)}(0) = 1$
11. $y^{(3)} - 3y'' + 2y' = t + e^t,\ \ y(0) = 1,\ \ y'(0) = -1/4, y''(0) = -32$
12. $y^{(4)} + 2y''' + y'' + 8y' - 12y = 12\sin t - e^{-t}\ y(0) = 3,\ y'(0) = 0,$ $y''(0) = -1,\ y'''(0) = 2$

The Method of Undetermined Coefficients In each of Problems determine a suitable form for y(t) if the method of undetermined coefficients is to be used. Do not evaluate the constants.

13. $y^{(3)} - 2y'' + y' = t^3 + 2e^t$ 14. $y^{(3)} - y' = te^{-t} + 2\cos t$
15. $y^{(4)} - 2y'' + y' = e^t + \sin t$ 16. $y^{(4)} + 4y'' = \sin 2t + te^t + 4$
17. $y^{(4)} - y^{(3)} - y'' + y' = t^2 + 4 + t\sin t$
18. $y^{(4)} + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$

The Method of Variation of Parameters

The method of variation of parameters for determining a particular solution of the non homogeneous n-th order linear differential equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t) \qquad (4.9)$$

is known as the Lagrange method. As before, to use the method of variation of parameters, it is first necessary to solve the corresponding homogeneous differential equation. In general, this may be difficult unless the coefficients are constants. However, the method of variation of parameters is still more general than the method of undetermined coefficients in that it leads to an expression for the particular solution for any continuous function g, whereas the method of undetermined coefficients is restricted in practice to a limited class of functions g. Suppose then that we know a fundamental set of solutions $y_1, y_2, ..., y_n$ of the homogeneous equation. Then the general solution of the homogeneous equation is

$$y_0(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t). \tag{4.10}$$

The method of variation of parameters for determining a particular solution of Eq. (4.9) rests on the possibility of determining n functions $u_1, u_2, ..., u_n$ such that y(t) is of the form

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t). \tag{4.11}$$

Since we have n functions to determine, we will have to specify n conditions. One of these is clearly that y satisfy Eq. (4.9). The other n - 1 conditions are chosen so as to make the calculations as simple as possible. Since we can hardly expect a simplification in determining y if we must solve high order differential equations for $u_1, \cdots, u_n$, it is natural to impose conditions to suppress the terms that lead to higher derivatives of $u_1, \cdots, u_n$. From Eq. (4.11) we obtain

$$y' = (u_1 y_1' + u_2 y_2' + \cdots, +u_n y_n') + (u_1' y_1 + u_2' y_2 + \cdots, +u_n' y_n), \tag{4.12}$$

where we have omitted the independent variable t on which each function in Eq. (4.12) depends. Thus the first condition that we impose is that

$$u_1' y_1 + u_2' y_2 + \cdots, +u_n' y_n = 0. \tag{4.13}$$

Continuing this process in a similar manner through n - 1 derivatives of Y gives

$$y^{(m)} = u_1 y_1^{(m)} + u_2 y_2^{(m)} + \cdots, +u_n y_n^{(m)}, \quad m = 0, 1, 2, \cdots, n-1, \tag{4.14}$$

and the following n - 1 conditions on the functions $u_1, \cdots, u_n$,

$$u_1' y_1^{(m-1)} + u_2' y_2^{(m-1)} + \cdots, +u_n' y_n^{(m-1)} = 0, \quad m = 1, 2, \cdots, n-1. \tag{4.15}$$

The n-th derivative of y is

$$y^{(n)} = (u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)}) + (u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)}). \tag{4.16}$$

Finally, we impose the condition that y must be a solution of Eq. (4.9). On substituting for the derivatives of y from Eqs. (4.14) and (4.16), collecting terms, and making use of the fact that $L[y_i] = 0, i = 1, 2, \cdots, n,$ we obtain

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g. \tag{4.17}$$

Equation (4.17), coupled with the n - 1 equations (4.15), gives n simultaneous linear non homogeneous algebraic equations for $u_1', u_2', \cdots, u_n'$:

$$\begin{cases} u_1' y_1 + u_2' y_2 + \cdots, +u_n' y_n = 0 \\ u_1' y'' + u_2' y_2'' + \cdots, +u_n' y_n'' = 0 \\ u_1' y_1^{(3)} + u_2' y_2^{(3)} + \cdots, +u_n' y_n^{(3)} = 0 \\ \vdots \\ u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots, +u_n' y_n^{(n-1)} = g \end{cases} \tag{4.18}$$

The system (4.18) is a linear algebraic system for the unknown quantities $u_1', u_2', \cdots, u_n'$. By solving this system and then integrating the resulting expressions, you can obtain the coefficients $u_1, \cdots, u_n,$. A sufficient condition for the existence of a solution of the system of equations (4.18) is that the determinant of coefficients is nonzero for each value of t. However, the determinant of coefficients is precisely $W(y_1, y_2, ..., y_n)$, and it is nowhere zero since $y_1, ..., y_n$ are linearly independent solutions of the homogeneous equation. Hence it is possible to determine $u_1', u_2', \cdots, u_n'$. Using Cramers rule, we find that the solution of the system of equations (4.18).

Remark. If when we determine, by integrating, the functions u_i it is considered and the additional arbitrary constants, then we obtain the general solution of non homogeneous equation (4.9).

Example Find the general solution for the next equation

$$y^{(3)} - y'' - y' + y = 4e^t, \tag{4.19}$$

First we determine the general solution for the homogeneous equation corresponding to the eq.(4.19)

$$y^{(3)} - y'' - y' + y = 0 \tag{4.20}$$

The characteristic algebraic equation is $r^3 - r^2 - r + 1 = 0$, and its roots are $r_1 = -1,\ r_2 = r_3 = 1$. The fundamental system of solutions is $y_1(t) = e^t, y_2(t) = te^t, y_3(t) = e^{-t}$. The general solution of eq.(4.20) is

$$y_0(t) = c_1e^t + c_2te^t + c_3e^{-t}.$$

Next we find the particular (general) solution of eq (4.19) of the form

$$y_p(t) = u_1(t)e^t + u_2(t)te^t + u_3(t)e^{-t}, \tag{4.21}$$

where the unknown functions u_i are solutions of the algebraic system

$$u_1'(t)e^t + u_2'(t)te^t + u_3'(t)e^{-t} = 0$$
$$u_1'(t)e^t + u_2'(t)(t+1)e^t - u_3'(t)e^{-t} = 0$$
$$u_1'(t)e^t + u_2'(t)(t+2)e^t + u_3'(t)e^{-t} = 4e^t.$$

By solving this system we find

$$u_1'(t) = -2t - 1,\quad u_2'(t) = 2,\quad u_3'(t) = e^{2t}.$$

Now we integrate the resulting expressions and obtain the coefficients

$$u_1(t) = -\int(2t+1)dt = -t^2 - t + c_1,\quad u_2(t) = 2t + c_2,\quad u_3(t) = \frac{1}{2}e^{2t} + c_3.$$

Now from eq.(4.21) we obtain the general solution of non homogeneous equation (4.19)

$$y(t) = (-t^2-t+c_1)e^t+(2t+c_2)te^t+\frac{1}{2}e^{2t}+c_3)e^{-t} = c_1e^t+c_2te^t+c_3e^{-t}+(t^2-t+3/2)e^t.$$

Exercises In each of the next problems use the method of variation of parameters to determine the general solution of the given differential equation.

1. $y''' + y' = \tan t, 0 < t < \pi$, 2. $y''' - y' = t$, 3. $y''' - 2y'' + y' + 2y = e^{4t}$,

4. $y''' + y' = \sec t,\ -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$ 5. $y''' - y'' + y' - y = e^{-t}\sin t$,

6. $y^{(4)} + 2y'' + y = \sin t$

Find the solution of the given initial value problem.

7. $y''' + y' = \sec t,\ y(0) = 2,\ y'(0) = 1,\ y''(0) = .2$

8. $y^{(4)} + 2y'' + y = \sin t,\quad y(0) = 2,\quad y'(0) = 0,\quad y''(0) = -1,\quad y'''(0) = 1$

9. $y''' - y'' + y' - y = \sec t,\quad y(0) = 2, y'(0) = -1, y''(0) = 1$

10. $y''' - y' = \csc t,\ y(\pi/2) = 2,\quad y'(\pi/2) = 1,\quad y''(\pi/2) = -1$