Linear Transformation and matrices

In a linear application the coordinates of the image vector are a linear combination of the coordinates of the object vector.

For example, a linear application $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z)$$

can be represented in matrix form in the following way:

$$T(x, y, z) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Specifically, if $T(x, y, z) = (x - y, 2y + z)$, then:

$$T(x, y, z) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Any matrix $A = [a_{ij}]_{m \times n}$ represents an application of $\mathbb{R}^n$ in $\mathbb{R}^m$, which depends on the bases considered for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If those are the canonical bases, then T is defined by:

$$\begin{array}{cccc} T_A: & \mathbb{R}^n & \to & \mathbb{R}^m \\ & v & \to & A \cdot v \end{array}$$

Example:

The matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

induces a linear application $T_A: \mathbb{R}^4 \to \mathbb{R}^3$, defined by:

$$T_A\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x + 2z + 3w \\ -y + z \\ x - z + 2w \end{bmatrix}$$

This is, A defines the application $T(x, y, z, w) = (x + 2z + 3w, -y + z, x - z + 2w)$, when considering the canonical bases of $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. In fact, considering other bases of the vector spaces involved, matrix A would define another linear transformation.

If $T: U \rightarrow V$ be a linear transformation and $U = \{u_1, u_2, \dots, u_n\}$ be a base of U and $V = \{v_1, v_2, \dots, v_m\}$ be a base of V, the following procedure allows to determine the matrix of the T transformation from the base U to the base V, denoted by $M(T, U, V)$:

(i) Determine $T(u_1), T(u_2), \dots, T(u_n)$;

(ii) Determine the coordinates of $T(u_1), T(u_2), \dots, T(u_n)$ in the base V:

$$\begin{aligned} T(u_1) &= a_{11}v_1 + \cdots + a_{m1}v_m \\ T(u_2) &= a_{12}v_1 + \cdots + a_{m2}v_m \\ &\cdots \\ T(u_n) &= a_{1n}v_1 + \cdots + a_{mn}v_m \end{aligned}$$

(iii) Write these coordinates as columns of a matrix (which will be of the type $m \times n$):

$$M(T, U, V) = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

1. Consider the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(x, y, z) = (x - 3y, 2z)$. Determine the matrix of T from the base $A = \{(1,2,4), (0,3,0), (3,0,0)\}$ of $\mathbb{R}^3$ to the base $B = \{(1,2), (0,3)\}$ of $\mathbb{R}^2$.

(i) Calculate $T(1,2,4), T(0,3,0)$ and $T(3,0,0)$:

$$\begin{aligned} T(1,2,4) &= (1 - 3 \times 2, 2 \times 4) = (-5, 8) \\ T(0,3,0) &= (0 - 3 \times 3, 2 \times 0) = (-9, 0) \\ T(3,0,0) &= (3 - 3 \times 0, 2 \times 0) = (3, 0) \end{aligned}$$

(ii) Write $T(1,2,4)$, $T(0,3,0)$ and $T(3,0,0)$ as a linear combination of the B vectors:

$$(-5,8) = c_1(1,2) + c_2(0,3)$$

$$\begin{cases} c_1 + 0c_2 = -5 \\ 2c_1 + 3c_2 = 8 \end{cases} \Leftrightarrow \begin{cases} c_1 = -5 \\ c_2 = 6 \end{cases}$$

$$(-9,0) = c_1(1,2) + c_2(0,3)$$

$$\begin{cases} c_1 + 0c_2 = -9 \\ 2c_1 + 3c_2 = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = -9 \\ c_2 = 6 \end{cases}$$

$$(3,0) = c_1(1,2) + c_2(0,3)$$

$$\begin{cases} c_1 & +0c_2 & = & 3 \\ 2c_1 & +3c_2 & = & 0 \end{cases} \Leftrightarrow \begin{cases} c_1 & = & 3 \\ c_2 & = & -2 \end{cases}$$

(iii) Write the coefficients of each of the previous linear combinations as columns of a matrix:

$$M(T, A, B) = \begin{bmatrix} -5 & -9 & 3 \\ 6 & 6 & -2 \end{bmatrix}$$