Exponentials

Given a real number $a > 0$

$$a^n = \underbrace{a \cdot a \cdot \quad \cdots a}_{n}$$

- $$a^{n+m} = a^n \cdot a^m$$
- $$(a^n)^m = \underbrace{a^n \cdot a^n \cdots \quad \cdot a^n}_{m} = \underbrace{a \cdot \cdots \quad \cdot a}_{m \cdot n} = a^{m \cdot n}$$

True for every $n, m \in \mathbb{N}$

$a^{\frac{p}{q}}$ is the unique positive real number b such that $b^q = a^p$.

$$= (a^p)^{\frac{1}{q}} = \sqrt[q]{a^p}$$

How to extend the exponential to exponents which are real numbers?

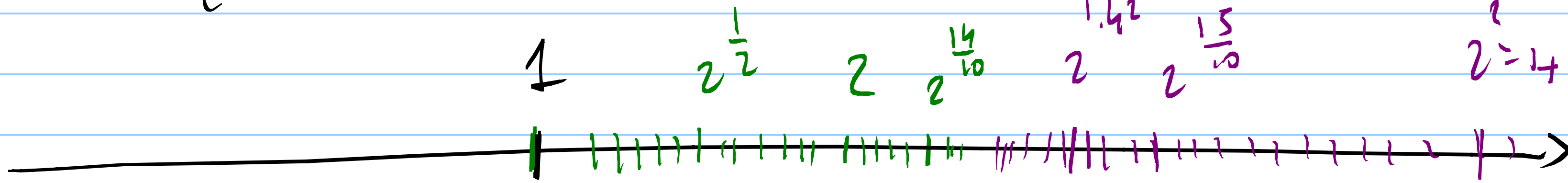
(Ex How can I define $2^{\sqrt{2}}$?)

Let us consider $A, B \subseteq \mathbb{R}$

$$A = \{ 2^r : r \in \mathbb{Q} \ \sqrt{2} > r > 0 \}$$

$$B = \{ 2^r : r \in \mathbb{Q} \quad r > \sqrt{2} \}$$

$A = \{ 2^1, 2^{\frac{14}{10}}, 2^{\frac{1}{2}}, \dots \}$



$B = \{ 2^2, 2^{\frac{15}{10}}, 2^{1,42}, 2^{1,415}, \dots, 2^{100} \}$

A is on the left of B

$\exists\ c$ s.t. $a \le c \le b \quad \forall\ a \in A,\ b \in B$

and also c is unique $c = 2^{\sqrt{2}}$

This defines a^b for every $a>0$, $b>0$, moreover $a^{-b} = \frac{1}{a^b}$. This will define the exp. for all real numbe b.

It can be seen that actually the algebraic properties of the exponential on $\mathbb{N}$ carries on on $\mathbb{R}$.

(1) $a^{b+c} = a^b \cdot a^c \quad \forall\, a>0,\ \forall\, b,c \in \mathbb{R}$

(2) $(a^b)^c = a^{b\cdot c} \quad \forall\, a>0 \quad \forall\, b,c \in \mathbb{R}$

In particular from (1), using $c = -b$

$$a^{b+(-b)} = a^b \cdot a^{(-b)}$$

$$\shortparallel \qquad\qquad \shortparallel$$

$$a^0 = 1 = a^b \cdot a^{-b} \iff a^{-b} = \frac{1}{a^b}$$

(3) $(a \cdot b)^c = a^c \cdot b^c$

(3') $\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$

Logarithms.

Q. what is the number x that I have to put at the exponent of $a > 0$ in order to get y.

$$a^x = y$$

Answ. This is possible only if $y > 0$

And there is a unique solution, that we call $x = \log_a(y)$

(1) $\log_a(b \cdot c) = \log_a(b) + \log_a(c) \quad \forall a,b,c>0$

(1') $\log_a(b/c) = \log_a(b) - \log_a(c) \quad \forall a,b,c>0$

(2) $\log_a(b^c) = c \cdot \log_a(b) \quad \begin{cases} \forall a,b>0 \\ \forall c \in \mathbb{R} \end{cases}$

(3) $\log_a(c) = \dfrac{\log_b(c)}{\log_b(a)}$

<u>WARNING</u> IT COULD BE that $\log_a(y) < 0$,

but always we have to have $y>0$

<u>example.</u> $\log_2(2) = 1$ (since $2^1 = 2$)

$\log_2\left(\frac{1}{2}\right) = -1$ (since $2^{-1} = \frac{1}{2}$)

(TAUTOLOGY) $\quad a^{\log_a(y)} = y \quad \forall\ a, y > 0$

(TAUTOLOGY II) $\quad \log_a\left(a^y\right) = y \quad \forall\ a, y > 0$