Systems of linear equations

Exercise. Let

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 1 \\ -5 & 2 & 3 \end{bmatrix} \text{ and } b = {}^t\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$$

Solve the system of linear equations $AX = b$ in three ways: (1) by Cramer rule, (2) via Gauss-Jordan elimination and (3) using the inverse matrix A^{-1}.

Solution. We observe, to begin with, that $\det(A) = 14$, therefore A is invertible (hence $\varrho(A) = 3$ and the system has a unique solution).

Cramer: the solution $X = {}^t\begin{bmatrix} x & y & z \end{bmatrix}$ is given as

$$x = \frac{\det \begin{bmatrix} -1 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}}{14} = \frac{4}{7}.$$

$$y = \frac{\det \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ -5 & 0 & 3 \end{bmatrix}}{14} = \frac{11}{14}.$$

$$z = \frac{\det \begin{bmatrix} 1 & -2 & -1 \\ 1 & 0 & 1 \\ -5 & 2 & 0 \end{bmatrix}}{14} = \frac{3}{7}.$$

Gauss-Jordan elimination: by performing row operations we get

$$\begin{bmatrix} 1 & -2 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ -5 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{E_{21}(-1)} \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 2 & 1 & 2 \\ -5 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{E_{31}(5)} \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 2 & 1 & 2 \\ 0 & -8 & 3 & -5 \end{bmatrix}$$

$$\xrightarrow{E_{32}(4)} \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 7 & 3 \end{bmatrix} \xrightarrow{E_{2}(1/2)} \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 7 & 3 \end{bmatrix} \xrightarrow{E_{3}(1/7)} \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 1 & 3/7 \end{bmatrix}$$

$$\xrightarrow{E_{12}(2)} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 1 & 3/7 \end{bmatrix} \xrightarrow{E_{13}(-1)} \begin{bmatrix} 1 & 0 & 0 & 4/7 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 1 & 3/7 \end{bmatrix} \xrightarrow{E_{23}(-1/2)} \begin{bmatrix} 1 & 0 & 0 & 4/7 \\ 0 & 1 & 0 & 11/14 \\ 0 & 0 & 1 & 3/7 \end{bmatrix}$$

The linear system associated to the last matrix gives the same solution as before.

Using the inverse: the inverse matrix of A can be obtained via the (classical) adjoint matrix, as well as, again, via Gauss-Jordan reduction. Using the adjoint matrix:

$$A^* = \begin{bmatrix} -2 & -8 & 2 \\ 6 & 3 & 8 \\ -2 & -1 & 2 \end{bmatrix},$$

we get

$$A^{-1} = \frac{{}^t(A^*)}{\det(A)} = \begin{bmatrix} -1/7 & 3/7 & -1/7 \\ -4/7 & 3/14 & -1/14 \\ 1/7 & 4/7 & 1/7 \end{bmatrix}$$

Thus

$$X = A^{-1}b = \begin{bmatrix} -1/7 & 3/7 & -1/7 \\ -4/7 & 3/14 & -1/14 \\ 1/7 & 4/7 & 1/7 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 11/14 \\ 3/7 \end{bmatrix}$$