Math-E ~ Tópico Integrais Triplas.

1) $\int_0^1\int_2^3\int_5^7 xyz\, dx\, dy\, dz = \int_0^1\int_2^3 \frac{x^2}{2}\Big|_5^7 yz\, dy\, dz = \int_0^1\int_2^3 \left(\frac{49}{2} - \frac{25}{2}\right) yz\, dy\, dz$

$\int_0^1 12\frac{y^2}{2}\Big|_2^3 z\, dz = \int_0^1 6z(9-4)\, dz = \int_0^1 30z\, dz = 30\frac{z^2}{2}\Big|_0^1 = 15$

2) $\int_2^3\int_1^4\int_0^2 2x^2 y \frac{z^2}{3}\, dx\, dz\, dy = \int_2^3\int_1^4 2\frac{x^3}{3}\Big|_0^2 y\frac{z^2}{3}\, dz\, dy = \int_2^3\int_1^4 \frac{2}{9} y z^2 (8-0)$

$\int_2^3\int_1^4 \frac{16}{9} y z^2\, dz\, dy = \int_2^3 \frac{16}{9} y \frac{z^3}{3}\Big|_1^4 dy = \int_2^3 \frac{16}{27} y (64-1)\, dy = \int_2^3 \frac{16}{27}\cdot 63 y\, dy$

$\frac{16\cdot 63}{27}\frac{y^2}{2}\Big|_2^3 = \frac{16\cdot 63}{27}\left(\frac{9}{2} - \frac{4}{2}\right) = \frac{16\cdot 63}{27}\cdot\frac{5}{2} = \frac{8\cdot 21\cdot 5}{9} = \frac{280}{3}$

3) $\int_0^1\int_2^4\int_{-2}^3 \frac{x^2}{2} y^2 z^3\, dz\, dy\, dx = \int_0^1\int_2^4 \frac{x^2}{2} y^2 \frac{z^4}{4}\Big|_{-2}^3 dy\, dx = \int_0^1\int_2^4 \frac{x^2y^2}{8}(81-16)\, dy\, dx$

$\int_0^1\int_2^4 \frac{x^2y^2\cdot 65}{8}\, dy\, dx = \int_0^1 \frac{x^2\cdot 65}{8}\cdot\frac{y^3}{3}\Big|_2^4 dx = \int_0^1 \frac{x^2\cdot 65}{24}(64-8)\, dx$

$\int_0^1 \frac{x^2\, 65}{24}(56)\, dx = \int_0^1 \frac{3640}{24} x^2\, dx = \int_0^1 \frac{455}{3} x^2\, dx = \frac{455}{3}\cdot\frac{x^3}{3}\Big|_0^1 = \frac{455}{9}$

4) $\int_0^2\int_{-1}^2\int_{-\pi}^0 \sin(x)\, y z^2\, dx\, dy\, dz = \int_0^2\int_{-1}^2 -\cos(x)\Big|_{-\pi}^0 y z^2\, dy\, dz = \int_0^2\int_{-1}^2 -2yz^2\, dy\, dz$

$\int_0^2 -2\frac{y^2}{2}\Big|_{-1}^2 z^2\, dz = \int_0^2 -z^2(4-1)\, dz = \int_0^2 -3z^2\, dz = -3\frac{z^3}{3}\Big|_0^2 = -(8-0) = -8$

5) $\int_2^5\int_1^2\int_3^4 yz\sqrt{x}\, dx\, dy\, dz = \int_2^5\int_1^2 yz\cdot\frac{2}{3}\cdot x^{3/2}\Big|_3^4 dy\, dz = \int_2^5\int_1^2 \frac{2}{3} yz\left(4^{3/2} - 3^{3/2}\right) dy\, dz$

$\int_2^5\int_1^2 \left(\frac{2\sqrt{64}}{3} - \frac{2\sqrt{27}}{3}\right) yz\, dy\, dz = \int_2^5\int_1^2 yz\cdot\frac{16}{3} - \frac{2\cdot 3\sqrt{3}}{3}\, dy\, dz = \int_2^5\int_1^2 yz\left(\frac{16}{3} - 2\sqrt{3}\right) dy\, dz = \int_2^5 \left(\frac{16}{3} - 2\sqrt{3}\right)\cdot\frac{y^2}{2}\Big|_1^2 z\, dz$

$\int_2^5 \left(\frac{16}{3} - 2\sqrt{3}\right)(2 - 0{,}5) z\, dz = \int_2^5 (8 - 3\sqrt{3}) z\, dz = (8-3\sqrt{3})\cdot\frac{z^2}{2}\Big|_2^5 = (8-3\sqrt{3})\cdot\frac{(25-4)}{2} = (8-3\sqrt{3})\cdot\frac{21}{2}$

6) $\int_5^7 \int_1^2 \int_{-2}^0 xy - z \, dz\, dy\, dx = \int_5^7 \int_1^2 xy - \frac{z^2}{2} \Big|_{-2}^0 dy\, dx = \int_5^7 \int_1^2 xy + 2$

$2 + \int_5^7 x \frac{y^2}{2} \Big|_1^2 dx = 2 + \int_5^7 x\,(2 - 0{,}5)\, x\, dx = 2 + 1{,}5 \cdot \frac{x^2}{2} \Big|_5^7 = 2 + \frac{3}{4}(49 - 25) =$

6) $\int_2^0 \int_{-1}^1 \int_0^1 (x \cdot y \cdot e^z)\, dx\, dy\, dz = \iint \frac{x^2}{2}\Big|_0^1 ye^z\, dy\, dz = \int_2^0 \frac{1}{2} \cdot \frac{y^2}{2}\Big|_{-1}^1 e^z\, dy\, dz$

$\int_0^2 \frac{1}{2} e^z \left(\frac{1}{2} - \frac{1}{2}\right) dz = 0$

Rectangular Coordinates

7) lower bound $z = 0$ $\qquad T = \{(x, y, z) \mid 0 \leq x \leq 1,$

upper bound $z = 1 - x - y$ $\qquad 0 \leq y \leq 1 - x,$

$z // z \qquad 1 - x - y = 0 \qquad 0 \leq z \leq 1 - x - y \}$

$y = 1 - x$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \frac{z^2}{2}\Big|_0^{1-x-y} dy\, dx = \frac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2\, dy\, dx$$

$$= \frac{1}{2} \int_0^1 \left[-\frac{(1 - x - y)^3}{3} \right]_0^{y = 1-x} dx = \frac{1}{6} \int_0^1 (1 - x^3)\, dx = \frac{1}{6} \left[\frac{-(1-x)^4}{4} \right]_0^1 = \frac{1}{24}$$

$= 0.0416$

8) $A = (1, 3, 0)$ $\qquad T = \{(x, y, z \mid 0 \leq x \leq 1,$

$B = (0, 3, 0)$ $\qquad 3x \leq y \leq 3,$ $\qquad y^2 + z^2 = 9$

$C = (0, 0, 3)$ $\qquad 0 \leq z \leq \sqrt{9 - y^2} \}$ $\qquad z^2 = 9 - y^2$

$D = (0, 0, 0)$

$$\int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z \, dz\, dy\, dx = \int_0^1 \int_{3x}^3 \frac{z^2}{2} \Big|_0^{\sqrt{9-y^2}} dy\, dx = \int_0^1 \int_{3x}^3 \frac{1}{2}(9 - y^2)\, dy\, dx$$

$$\int_0^1 \left[\frac{9}{2} y - \frac{1}{6} y^3 \right]_{y=3x}^{y=3} dx = \int_0^1 \left[9 - \frac{27}{2} x + \frac{9}{2} x^3 \right] dx = \left[9x - \frac{27}{4} x^2 + \frac{9}{8} x^4 \right]_0^1$$

$= 9 - 27/4 + 9/8 = 27/8$

$x + z = 1 \qquad x = y^2$

$z = 1 - x \qquad \pm\sqrt{x} = y$

$T = \{ (x,y,z) \mid 0 \le x \le 1, -\sqrt{x} \le y \le \sqrt{x}, 0 \le z \le 1-x \}$

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{1-x} z \, dz\, dy\, dx = \int_0^1 \int_{\sqrt{x}}^{\sqrt{x}} (1-x)\, dy\, dx = \int_0^1 2\sqrt{x}(1-x)\, dx$$

$$\int_0^1 2(\sqrt{x} - x^{3/2})\, dx = 2\left(\frac{2}{3}x^{3/2} - 2/5\, x^{5/2}\right)\Big|_0^1 = 2(2/3 - 2/5) = 8/15$$

Cylindrical Coordinates

10)

$z = x^2 + y^2$

$z = 18 - x^2 - y^2$

$T = \{ (x,y,z) \mid x^2 + y^2 \le 9, x^2 + y^2 \le z \le 18 - x^2 - y^2 \}$

intersecção $x^2 + y^2 = 18 - x^2 - y^2$

$0 = 18 - 2x^2 - 2y^2$

$9 = x^2 + y^2$

Área:

$$\int \int \int_{x^2+y^2}^{18-x^2-y^2} z\, dz = \iint 18 - 2x^2 - 2y^2 \, dA.$$

Volume

$$\int_0^{2\pi} \int_0^3 (18 - 2r^2) r\, dr\, d\theta = \int_0^{2\pi} \left[9r^2 - \frac{1}{2}r^4\right]_0^3 d\theta = \int_0^{2\pi} \frac{81}{2}\, d\theta = 81\pi$$

11) $T = \{ (r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4, r \le z \le 4 \}$

$z = r$ → cone bounded

$z = 4$ → horizontal bounded

$$\int_0^4 \int_0^{2\pi} \int_r^4 r\, dz\, d\theta\, dr = \int_0^4 \int_0^{2\pi} [rz]\Big|_r^4 d\theta\, dr = \int_0^4 \int_0^{2\pi} r(4-r)\, d\theta\, dr$$

$$\int_0^4 (4r - r^2)\, dr \int_0^{2\pi} d\theta = \left[2r^2 - \frac{1}{3}r^3\right]_0^4 [\theta]_0^{2\pi} = \left(32 - \frac{64}{9}\right)(2\pi)$$

$$= \frac{64\pi}{3}$$

12) $T = \{(r,\theta,z) \mid 0 \le r \le 2, \quad 0 \le \theta \le 2\pi \quad , -1 \le z \le 2\}$

$$\iiint_T (x^2+y^2)\,dV = \int_{-1}^{2}\int_0^{2\pi}\int_0^{2} \underbrace{x^2+y^2}_{r^2}\, r\,dr\,d\theta\,dz$$

$$\int_{-1}^{2}\int_0^{2\pi}\int_0^{2} r^3\,dr\,d\theta\,dz = \int_{-1}^{2}\int_0^{2\pi} \frac{r^4}{4}\Big|_0^{2}\,d\theta\,dz = \int_{-1}^{2}\int_0^{2\pi} 4\,d\theta\,dz$$

$$\int_{-1}^{2}\int_0^{2\pi} 4\,d\theta\,dz = \int_{-1}^{2} 4\theta\Big|_0^{2\pi}\,dz = \int_{-1}^{2} 4\cdot 2\pi\,dz = 8\pi z\Big|_{-1}^{2} = 8\pi(2-(-1))$$

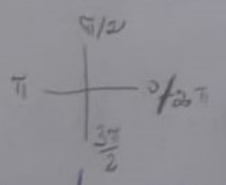
$$= 24\pi$$

13)
$x = r\cos\theta \qquad r^2 = x^2+y^2$
$y = r\,\text{sen}\,\theta \qquad \text{tg}\,\theta = y/x$
$z = z \qquad z = z$

$$\int_0^1\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy = \int_0^{\pi/2}\int_0^1\int_{r^2}^{r} \underbrace{r^2\sin\theta\cos\theta\cdot z\cdot r}_{r^3}\,dz\,dr\,d\theta$$

$\sqrt{x^2+y^2} = \sqrt{r^2} = r$
$x^2+y^2 = r^2$

14) $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\underbrace{x^2+y^2}_{r^2}}^{\overbrace{2-x^2-y^2}^{2-r^2}} (\underbrace{x^2+y^2}_{r^2})^{3/2}\,dz\,dy\,dx$$



$$\int_0^{2\pi}\int_0^{1}\int_{r^2}^{2-r^2} (r^2)^{3/2}\, r\,dz\,dr\,d\theta.$$

$r\sqrt{(r^2)^3} =$

(15) rect cylindrical
$(-3, 3\sqrt{3}, 5) \rightarrow (6, \frac{2}{3}\pi, 5)$

$r^2 = 9 + 27 = 36 \rightarrow r = \pm 6 \rightarrow r = 6$

$tg\theta = \frac{y}{x} = \frac{3\sqrt{3}}{-3} = -\sqrt{3} \rightarrow \theta = \frac{2}{3}\pi$

(16) $(\sqrt{3}, 1, 7) \rightarrow (2, \pi/6, 7)$

$r^2 = 3 + 1 = 4 \rightarrow r = \pm 2 \rightarrow r = 2$

$tg\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = 30° \rightarrow \pi/6$

(17) $(3, 3, 3) \rightarrow (3\sqrt{2}, \pi/4, 3)$

$r^2 = 9 + 9 = 18 \rightarrow r = \pm 3\sqrt{2} \rightarrow r = 3\sqrt{2}$

$tg\theta = \frac{3}{3} = 1 \rightarrow 45° = \pi/4$

$z = 3$

(18) $\int_{-1}^{1} \int_{0}^{\pi/2} \int_{0}^{1-r^2} r^2 \sin\theta \, dz \, d\theta \, dr.$

$\int_{-1}^{1} \int_{0}^{\pi/2} r^2 \sin\theta (1-r^2) \, d\theta \, dr = -\int_{-1}^{1} r^2(1-r^2) \cos\theta \Big|_0^{\pi/2} dr.$

$-\int_{-1}^{1} (r^2 - r^4)(0-1) dr = -\int_{-1}^{1} r^4 - r^2 \, dr = -\left[\frac{r^5}{5} - \frac{r^3}{3}\right]_{-1}^{1}$

$-\left[\left(\frac{1}{5} - \frac{1}{3}\right) - \left(-\frac{1}{5} + \frac{1}{3}\right)\right] = -\left[\left(\frac{3-5}{15}\right) - \left(\frac{(-3+5)}{15}\right)\right] = -\left[-\frac{2}{15} - \frac{2}{15}\right] = -\left[-4/15\right]$

$= + \frac{4}{15}$

(19) $\int_{-2}^{3}\int_{0}^{\pi}\int_{0}^{r-1} z \sin\theta \, dr \, d\theta \, dz = \int_{-2}^{3}\int_{0}^{\pi} z \sin\theta \,(r-1)\, d\theta \, dz$

$-\int_{-2}^{3} z \cos\theta \Big|_{0}^{\pi} (r-1)\, d\theta \, dz = \int_{-2}^{3} z(r-1)(-1-1)\, dz$

$-\int_{-2}^{3} z(r-1)(-2)\, dz = -\frac{z^2}{2}\Big|_{-2}^{3} (r-1)(-2) = -\left(\frac{9}{2}-\frac{4}{2}\right)(-2r+2)$

$-\frac{5}{2}(-2r+2) = -5(-r+1) = 5r-5 = 5(r-1)$

Spherical Coordinates

20) $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$

Relações

$x = \rho \sin\theta \cos\phi$ $\qquad \rho = \sqrt{x^2+y^2+z^2}$ $\qquad dV = \rho^2 \sin\phi$

$y = \rho \sin\theta \sin\phi$ $\qquad \phi = \text{arctang } y/x$ $\qquad 0 \le \rho \le \rho_0$

$z = \rho \cos\phi$ $\qquad \theta = \text{arc tan } \frac{\sqrt{x^2+y^2}}{z}$ $\qquad 0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

~ " ~

$dz \to 0 \sim 2$

$dy \to 0 \sim \pi/2$

$dx \to 0 \sim 2\pi$

$\left.\begin{array}{l} z^2 = \rho^2\cos^2\phi \\ \sqrt{x^2+y^2+z^2} = \rho \\ dV = \rho^2 \sin\phi \end{array}\right\} \rho^5 \cos^2\theta \sin\phi.$

$\int_{0}^{2\pi}\int_{0}^{\pi/2}\int_{0}^{2} \rho^5 \cos^2\phi \sin\phi \, d\rho\, d\phi\, d\theta = \int_{0}^{2\pi}\int_{0}^{\pi/2} \frac{\rho^6}{6}\Big|_{0}^{2} \cos^2\phi \sin\phi \, d\phi\, d\theta.$

$\int_{0}^{2\pi}\int_{0}^{\pi/2} \frac{64}{6} \cdot \cos^2\phi \sin\phi \, d\phi\, d\theta =$

$\frac{32}{3}\int_{0}^{2\pi} \left[-\frac{1}{3}\cos^3\phi\right]_{0}^{\pi/2} d\theta = \int_{0}^{2\pi} \frac{32}{3}\cdot\frac{1}{3}\, d\theta = \left[\theta\right]_{0}^{2\pi} \cdot \frac{32}{9} = \frac{64}{9}\pi$

(21)
$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\;dz\,dy\,dx$$

$$\left.\begin{aligned}\sqrt{x^2+y^2+z^2} &= \rho\\ z &= \rho\cos\phi\\ dV &= \rho^2\sin\phi\end{aligned}\right\rangle\ \rho^4\cos\phi\sin\phi$$

$(\rho^2\cos\phi)(\rho^2\sin\phi)$

$0\le\rho\le 3$
$0\le\phi\le\pi/2$
$0\le\theta\le 2\pi$

$$\int_0^{\pi/6}\int_0^{\pi/3}\int_0^{2}\rho^5\cos\phi\sin\phi\;d\rho\,d\phi\,d\theta = \frac{2\pi}{3}$$

$$\int_0^{\pi/6}\int_0^{\pi/3}\frac{\rho^6}{6}\Big|_0^2\cos\phi\sin\phi\;d\phi\,d\theta = \int_0^{\pi/6}\int_0^{\pi/3}\frac{32}{3}\cos\phi\sin\phi\;d\phi\,d\theta$$

$$\frac{32}{3}\int_0^{\pi/6}\sin\phi\cos\phi\Big|_0^{\pi/3}d\theta = \frac{32}{3}\int_0^{\pi/6}\left[\frac{\sqrt3}{2}\cdot\frac12-0\right]d\theta = \int_0^{\pi/6}\frac{32}{3}\,\frac{\sqrt3}{4}\;d\theta$$

$$\frac{8\sqrt3}{3}\left[\frac{\pi}{6}-0\right] = \frac{4}{9}\sqrt3\,\pi$$

(22)
$$\int_0^{\pi/4}\int_0^{\pi}\int_1^{3}\rho\sin\phi\;d\rho\,d\phi\,d\theta = \int_0^{\pi/4}\int_0^{\pi}\frac{\rho^2}{2}\Big|_1^3\sin\phi\;d\phi\,d\theta$$

$$\int_0^{\pi/4}\int_0^{\pi}\left(\frac92-\frac12\right)\sin\phi\;d\phi\,d\theta = \int_0^{\pi/4}-4\cos\phi\Big|_0^{\pi}d\theta$$

$$\int_0^{\pi/4}-4(\underbrace{\cos\pi}_{-1}-\underbrace{\cos 0}_{1})\;d\theta = \frac{8\pi}{4} = 2\pi$$

(23) $(1,0,1) \longrightarrow \left(\sqrt2, \frac{\pi}{4}, 0\right)$

$\rho = \sqrt{1^2+1^2} = \sqrt2$

$\cos\phi = \frac{1}{\sqrt2} = \frac{\sqrt2}{2} \Rightarrow \frac{\pi}{4}$

$\cos\theta = \frac{1}{\sqrt2\sin\pi/4} = \frac{1}{\sqrt2\cdot\frac{\sqrt2}{2}} = \frac11 = 1 \Rightarrow 0$

(24) $(0,0,1) \rightarrow (1, \pi/2, \pi/2)$

$\rho = \sqrt{1^2} = 1$

$\cos\phi = 0 \rightarrow \pi/2$

$\cos\theta = 0 \rightarrow \pi/2.$

(25) $\int_0^{2\pi}\int_0^{\pi}\int_0^{3} \rho^3 \sin\phi \, d\rho \cdot d\phi \, d\theta$

$$\int_0^{2\pi}\int_0^{\pi} \frac{\rho^4}{4}\Big|_0^3 \sin\phi \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{81}{4} \sin\phi \, d\phi \, d\theta = \frac{81}{4}\int_0^{2\pi} \cos\phi \Big|_0^{\pi} d\theta$$

$$\frac{81}{4}\int_0^{2\pi} -(-1-1)\, d\theta = +\cancel{2}\cdot\frac{81}{\cancel{4}}(2\pi) = +81\pi$$

(26) $f(x, y, z) = x + y + z^2$

$r \sin\phi \cos\theta + r \sin\phi \sin\theta + r^2 \cos^2\phi$

$r(\sin\phi \cos\theta + \sin\phi \sin\theta + r\cos^2\phi)$

(27) $f(x, y, z) = (2x - y)z$

$= [2(r \sin\phi \cos\theta) - (r \sin\phi \sin\theta)] \cdot r\cos\phi$

$= 2r^2 \sin\phi \cos\theta \cos\phi - r^2 \sin\phi \sin\theta \cos\phi$

$= r^2(2\sin\phi \cos\theta \cos\phi - \sin\phi \sin\theta \cos\phi)$