Exact Differential Equations

Definition *Let $D \subseteq R^2$ be a connex set and P, Q the differential equations so that*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{1.15}$$

then the equation

$$P(x,y)dx + Q(x,y)dy = 0 \tag{1.16}$$

is called the exact differential equation*, and its solution is implicitely defined by the next equation*

$$\int_{x_0}^{x} P(t, y_0)dt + \int_{y_0}^{y} P(x,t)dt = C, \quad \text{or equivalently}$$

$$\int_{y_0}^{y} Q(x_0, t)dt + \int_{x_0}^{x} Q(t,y)dt = C$$

Remark. We now show that if P and Q satisfy relation (1.15) then we can find the solution of differentiable (1.16). More exactly let $\psi(x,y) = C$ (this fact is ensured by condition (1.15)), so that

$$\frac{\partial \psi}{\partial x} = P \quad \text{and} \quad \frac{\partial \psi}{\partial y} = Q. \tag{1.17}$$

Next we have

$$\frac{\partial \psi}{\partial x} = P \quad \rightarrow \quad \psi(x,y) = \int P(x,y)dx + f(y) \tag{1.18}$$

Taking into account the second relation of (1.17), is obtain

$$\frac{\partial \psi}{\partial y} = Q \quad \rightarrow \quad Q(x,y) = \frac{\partial}{\partial y}(\int P(x,y)dx) + f'(y) = \int (\frac{\partial}{\partial y} P(x,y)dx) + f'(y).$$

This above relation determine the function ψ as follows

$$f'(y) = Q(x,y) - \int (\frac{\partial}{\partial y} P(x,y)dx.$$

To determine h(y), it is essential that, despite its appearance, the right side of above equation be a function of y only. This fact, is ensured by condition (1.15), and it can by proved by direct calculation. It should be noted that this proof contains a method for the computation of $\psi(x,y)$ and thus for solving the original differential equation (1.16). Note also that the solution is obtained in implicit form; it may or may not be feasible to find the solution explicitly.

Example Solve the differential equation

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0. \tag{1.19}$$

It is easy to see that

$$(y\cos x + 2xe^y)dx + (\sin x + x^2e^y - 1)dy = 0,$$

and thus

$$P(x,y) = \cos x + 2xe^y = y\cos x + 2xe^y, \quad Q(x,y) = \sin x + x^2e^y - 1 \quad \to$$

$$\frac{\partial P}{\partial y} = 2xye^y = \frac{\partial Q}{\partial x}$$

so the given equation is exact. Thus there is a $\psi(x,y)$ such that

$$\psi_x(x,y) = y\cos x + 2xe^y,$$

$$\psi_y(x,y) = \sin x + x^2e^y - 1$$

Integrating the first of these equations, we obtain

$$\psi(x,y) = ysinx + x^2e^y + h(y)$$

Setting $\psi_y = Q(x,y)$ gives

$$\psi_y(x,y) = sinx{+}x2ey{+}h'(y) = sinx{+}x^2e^y{-}1, \ \to \ h'(y) = -1 \ \to \ h(y) = -y.$$

The constant of integration can be omitted since any solution of the preceding differential equation is satisfactory; we do not require the most general one. Substituting for h(y) gives $\psi(x,y) = y\sin x + x^2e^y - y$. Hence solutions of Eq. (1.19) are given implicitly by

$$y\sin x + x^2e^y - y = c.$$