Systems of linear equations

Exercise. Solve the following system of linear equations:

$$\begin{cases} 2x_1 - x_2 - x_3 - 4x_4 = 9 \\ 4x_1 - 3x_3 - x_4 = 0 \\ 8x_1 - 2x_2 - 5x_3 - 9x_4 = 18 \end{cases}$$

Solution The system has augmented matrix $\overline{A}$, where A is the coefficient matrix, and $\underline{b}$ is the constant column term:

$$\overline{A} := (A|\underline{b}) = \begin{pmatrix} \boxed{2} & -1 & -1 & -4 & 9 \\ 4 & 0 & -3 & -1 & 0 \\ 8 & -2 & -5 & -9 & 18 \end{pmatrix}$$

performing row operations ($R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$) we get

$$\overline{A} \longrightarrow \begin{pmatrix} \boxed{2} & -1 & -1 & -4 & 9 \\ 0 & \boxed{2} & -1 & 7 & -18 \\ 0 & 2 & -1 & 7 & -18 \end{pmatrix}.$$

Eventually, performing $R_3 \to R_3 - R_1$ we find the row echelon form

$$\overline{A} \longrightarrow \begin{pmatrix} \boxed{2} & -1 & -1 & -4 & 9 \\ 0 & \boxed{2} & -1 & 7 & -18 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

from which, by the Rouché-Capelli Theorem, we deduce that the system is consistent, as $\overline{A}$ and A both have rank 2 (number of non zero lines of the row echelon form), and that the number of its solutions is $\infty^{4-2} = \infty^2$: the solution set will depends on 2 reals parameters, which correspond to the free (i. e. non-pivotal) variables x_3, x_4.

To write down the solution set we can go ahead to a complete reduction performing the following row operations: $R_2 \to (1/2)R_2 =: R_2'$, $R_1 \to R - 1 + R_2' =: R_1'$:

$$\begin{pmatrix} \boxed{2} & -1 & -1 & -4 & 9 \\ 0 & \boxed{2} & -1 & 7 & -18 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{2} & -1 & -1 & -4 & 9 \\ 0 & \boxed{1} & -1/2 & 7/2 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{2} & 0 & -3/2 & -1/2 & 0 \\ 0 & \boxed{1} & -1/2 & 7/2 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and finally $R_1' \to (1/2)R_1'$:

$$\overline{A} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & -3/4 & -1/4 & 0 \\ 0 & \boxed{1} & -1/2 & 7/2 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence the starting system is equivalent to the reduced system

$$\begin{cases} x_1 - \frac{3}{4}x_3 - \frac{1}{4}x_4 = 0 \\ x_2 - \frac{1}{2}x_3 + \frac{7}{2}x_4 = -9 \end{cases} \iff \begin{cases} x_1 = \frac{3}{4}h + \frac{1}{4}k \\ x_2 = -9 + \frac{1}{2}h - \frac{7}{2}k \\ x_3 = h \\ x_4 = k \end{cases}.$$

Therefore, its solution set is

$$S = \left\{ \begin{pmatrix} 0 \\ -9 \\ 0 \\ 0 \end{pmatrix} + h \begin{pmatrix} \frac{3}{4} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + k \begin{pmatrix} \frac{1}{4} \\ -\frac{7}{2} \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^4 \mid h,\, k \in \mathbb{R} \right\}$$