Linear differential equation of first order

Let's see how to solve a linear first order differential equation. The goal is to obtain a solution in the form $x = x(t)$.

The **general form** of a linear first order differential equation is:

$$x'(t) + a(t)x(t) = b(t), t \in I, I \subset \mathbb{R}, \tag{1}$$

where both $a(t)$ and $b(t)$ are *continuous functions* on the interval I.

If the differential equation is not in this form then the process we are going to use will not work.

We assume that there is some function called an integrating factor, $\mu(t)$, to multiply equation (1) such that

$$\mu(t)x'(t) + \mu(t)a(t)x(t) = \mu(t)b(t), \tag{2}$$

If $\mu(t) \neq 0, t \in I$, exist, it will satisfy the following relation:

$$\mu(t)a(t) = \mu'(t) \tag{3}$$

So substituting (3) into (2) we now arrive at

$$\mu(t)x'(t) + \mu'(t)x(t) = \mu(t)b(t). \tag{4}$$

But $\mu(t)x'(t) + \mu'(t)x(t) = (\mu(t)x'(t))'$ and we replace at (4) so we obtain

$$(\mu(t)x(t))' = \mu(t)b(t) \tag{5}$$

We integrate both sides:

$$\mu(t)x(t) = \int \mu(t)b(t) + C$$

where C is a real constant of integration.

We obtain the general solution

$$x(t) = \mu^{-1}(t)\left(\int \mu(t)b(t)dt + C\right), t \in I. \tag{6}$$

We need to determine the function $\mu(t)$. We start relation (3). Divide both sides by $\mu(t)$ and integrate

$$\frac{\mu'(t)}{\mu(t)} = a(t) \Leftrightarrow (\ln \mu(t))' = a(t) \Leftrightarrow \ln \mu(t) + C_1 = \int a(t)dt$$

$$\ln \mu(t) = \int a(t)dt - C_1 \Leftrightarrow \mu(t) = e^{\int a(t)dt - C_1}$$

It is inconvenient to have the C_1 in the exponent so we're going to get it out of the exponent in the following way.

$$\mu(t) = e^{\int a(t)dt - C_1} = e^{-C_1}e^{\int a(t)dt},$$

where e^{-C_1} is a constant. Because we need a function, not all the function with this propriety, we can choose $C_1 = 0$, so

$$\mu(t) = e^{\int a(t)dt.} \tag{7}$$

So substituting (7) into (6) we now arrive at

$$x(t) = e^{-\int a(t)dt} \left(\int b(t) e^{\int a(t)dt} dt + C \right). \tag{8}$$

Solution Process

The solution process for linear differential equation of first order is as follows:

1. Put the differential equation in the correct initial form, (1).
2. Find the function $\mu(t)$, using (7).
3. Multiply the both sides of differential equation by $\mu(t)$ and verify that the left side becomes the product rule $(\mu(t)x(t))'$ and write it as such (5).
4. Integrate both sides and take care to the constant of integration.
5. Solve for the solution $x(t)$.

Example 1 *Find the solution to the following differential equation*

$$x(t) + 2tx(t) = 4t, t \in R.$$

Solution.

1. We observe that the differential equation is in the correct form.
2. Find the integrating factor, $\mu(t)$.
   $a(t) = 2t$, $\mu(t) = e^{\int 2tdt} = e^{t^2}$.
3. Multiply the both sides of the differential equation by $\mu(t) = e^{t^2}$,
   $x'(t)e^{t^2} + 2te^{t^2}x(t) = 4te^{t^2}$.
   Verify that the left side becomes the product rule $(\mu(t)x(t))'$,
   $\left(x(t)e^{t^2}\right)' = x'(t)e^{t^2} + 2te^{t^2}x(t)$,
   and write it as such
   $\left(x(t)e^{t^2}\right)' = 4te^{t^2}$.
4. Integrate both sides
   $x(t)e^{t^2} = 2e^{t^2} + C$.
5. Solve for the solution $x(t)$
   $x(t) = \left(2e^{t^2} + C\right) e^{-t^2}, x(t) = 2 + Ce^{-t^2}$.

Exercise 2 *Find the solution to the following differential equation*

$$tx'(t) + x(t) = 3t^2, t \neq 0.$$

Solution.

1. We convert this equation into the standard form. So we divide both part by t :
   $x'(t) + \frac{1}{t}x(t) = 3t, t \neq 0.$
2. Find the integrating factor, $\mu(t)$.
   $a(t) = \frac{1}{t}$, $\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t.$
3. Multiply both sides of the differential equation by $\mu(t) = t$,
   $x'(t)t + x(t) = 3t^2.$
   Verify that the left side becomes the product rule $(\mu(t)x(t))', (x(t)t)' = x'(t)t + x(t)$, and write it as such
   $(x(t)t)' = 3t^2.$
4. Integrate both sides
   $x(t)t = t^3 + C.$
5. Solve for the solution $x(t) = t^2 + \dfrac{C}{t}.$

Exercise 3 *Find the solution to the following differential equation*

$$\begin{cases} x'(t)\cos t + x(t)\sin t + 4\cos^3 t = 0, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \\ x(0) = 1 \end{cases}$$

Solution.

1. We convert this equation into the standard form. So we divide both part by $\cos t$:
   $x'(t) + \frac{\sin t}{\cos t}x(t) = -4\cos^2 t, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$
2. Find the integrating factor, $\mu(t)$.
   $a(t) = \frac{\sin t}{\cos t}$, $\mu(t) = e^{\int \frac{\sin t}{\cos t} dt} = e^{-\ln \cos t} = \frac{1}{\cos t}.$
3. Multiply both sides of the differential equation by $\mu(t) = \frac{1}{\cos t}$,
   $x'(t)\frac{1}{\cos t} + \frac{\sin t}{\cos^2 t}x(t) = -4\cos t, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$
   Verify that the left side becomes the product rule $(\mu(t)x(t))', \left(x(t)\frac{1}{\cos t}\right)' = x'(t)\frac{1}{\cos t} + \frac{\sin t}{\cos^2 t}x(t)$, and write it as such
   $\left(x(t)\frac{1}{\cos t}\right)' = -4\cos t.$
4. Integrate both sides
   $x(t)\frac{1}{\cos t} = -4\sin t + C.$
5. Solve for the solution $x(t) = (-4\sin t + C)\cos t.$

Use condition to find the constant C,
$x(0) = C = 1 \Rightarrow x(t) = (-4\sin t + 1)\cos t.$

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