Vector spaces

Definition: A vector space is a set V on which two operations $+$ and $\cdot$ are defined, called vector addition and scalar multiplication, respectively.

The operation $+$ (vector addition) must satisfy the following conditions:

1. Closure: If u and v are any vectors in V, then the sum $u + v$ belongs to V;
2. Commutative law: For all vectors $u, v \in V$, $u + v = v + u$;
3. Associative law: For all vectors $u, v, w \in V$, $u + (v + w) = (u + v) + w$;
4. Additive identity: The set V contains an additive identity element, denoted by 0, such that for any vector $v \in V$, $0 + v = v$ and $v + 0 = v$.
5. Additive inverses: For each vector $v \in V$, the equations $v + x = 0$ and $x + v = 0$ have a solution $x \in V$, called an additive inverse of v, and denoted by $-v$.

The operation $\cdot$ (scalar multiplication) is defined between real numbers (or scalars) and vectors, and must satisfy the following conditions:

1. Closure: If $v \in V$, and $c \in \mathbb{R}$, then the product $c \cdot v \in V$.
2. Distributive law: For all $c \in \mathbb{R}$ and all vectors $u, v \in V$, $c \cdot (u + v) = c \cdot u + c \cdot v$;
3. Distributive law: For all $c, d \in \mathbb{R}$ and all vectors $v \in V$, $(c + d) \cdot v = c \cdot v + d \cdot v$;
4. Associative law: For all real numbers c,d and all vectors $v \in V$, $c \cdot (d \cdot v) = (c \cdot d) \cdot v$;
5. Unitary law: For all vectors $v \in V$, $1 \cdot v = v$.

Examples: Some sets that equipped with scalar addition and multiplication have a structure of vector spaces:

1. $A = \{(x, y, z) \in \mathbb{R}^3 : 2x - y + 3z = 0\}$;
2. $\mathbb{R}^n$, with $n \in \mathbb{N}$;
3. $P(n) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, with $a_i \in \mathbb{R}$;
4. $M = [a_{i,j}]_{m \times n}$, with $a_{i,j} \in \mathbb{R}, \quad i = 1, \ldots, m \quad j = 1, \ldots, n.$

Note that $B = \{(x, y, z) \in \mathbb{R}^3 : x + y - z + 1 = 0\}$ is not a vector space.

Indeed, $u = (u_1, u_2, u_1 + u_2 + 1), v = (v_1, v_2, v_1 + v_2 + 1) \in B$, but

$$u + v = (u_1 + v_1, u_2 + v_2, u_1 + u_2 + v_1 + v_2 + 2) \notin B.$$

Besides that, $(0, 0, 0) \notin B$.

Edite Martins Cordeiro Flora Silva Paula Maria Barros