MathE project

Continuity for real functions of several variables

Example 1.1. *Study the continuity of the function* $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \sqrt{1-x^2-y^2}, & x^2+y^2 \le 1 \\ \\ \lambda, & x^2+y^2 > 1, \quad \lambda \in \mathbb{R}. \end{cases}$$

Solution. On the set $\{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 < 1\}$ the function f is a composition of elementary continuous functions, so f is continuous. On the set $\{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 > 1\}$ the function f is continuous being a constant. We study the continuity at the points from the circle $x^2+y^2=1$. Let $(x_0,y_0) \in \mathbb{R}^2$ such that $x_0^2+y_0^2=1$. Then $f(x_0,y_0) = \sqrt{1-x_0^2-y_0^2} = 1$. Obviously we have

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ x^2+y^2<1}} f(x,y) = \lim_{\substack{(x,y)\to(x_0,y_0)\\ x^2+y^2<1}} \sqrt{1-x^2-y^2} = 0,$$

and

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ x^2+y^2>1}} f(x,y) = \lim_{\substack{(x,y)\to(x_0,y_0)\\ x^2+y^2>1}} \lambda = \lambda.$$

The function f is continuous at (x_0,y_0), so on $\mathbb{R}^2$, if and only if $\lambda = 0$. If $\lambda \neq 0$ the function f is continuous only on $\mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2=1\}$.

Example 1.2. *Study the continuity of the function* $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{(x^4-y^2)^2}{x^6}, & y^2 < x^4 \text{ and } x \neq 0 \\ 0, & y^2 \ge x^4 \text{ or } x = 0. \end{cases}$$

Solution. Let us denote the sets

$$D_1 = \{(x,y) \in \mathbb{R}^2 \,|\, y^2 < x^4 \text{ and } x \neq 0\} = \{(x,y) \in \mathbb{R}^2 \,|\, -x^2 < y < x^2 \text{ and } x \neq 0\}$$

and

$$D_2 = \{(x,y) \in \mathbb{R}^2 \,|\, y^2 \ge x^4 \text{ or } x \neq 0\} = \{(x,y) \in \mathbb{R}^2 \,|\, y \le -x^2 \text{ or } y \ge x^2 \text{ or } x = 0\}.$$

Obviously we have $\mathbb{R}^2 = D_1 \cup D_2$ and

$$f(x,y) = \begin{cases} \dfrac{(x^4-y^2)^2}{x^6}, & (x,y) \in D_1 \\ \\ 0, & (x,y) \in D_2. \end{cases}$$

For the points (x_0, y_0) with $x_0^4 = y_0^2$ and $x_0 \neq 0$ we have $f(x_0, y_0) = 0$ and

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in D_1}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\(x,y)\in D_1}} \frac{(x^4-y^2)^2}{x^6} = \lim_{\substack{(x,y)\to(x_0,y_0)\\(x,y)\in D_1}} f(x,y) = \frac{0}{x_0^6} = 0 \text{ and } \lim_{\substack{(x,y)\to(x_0,y_0)\\(x,y)\in D_2}} f(x,y) = 0.$$

For $(x_0, y_0) = (0,0)$ we have $f(0,0) = 0$ and

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in D_1}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\(x,y)\in D_1}} \frac{(x^4-y^2)^2}{x^6} = \lim_{\substack{(x,y)\to(0,0)\\(x,y)\in D_1}} \left(x - \frac{y^2}{x^3}\right)^2 = 0,$$

because for $(x,y) \in D_1$ we can write $0 \leq \left|\frac{y^2}{x^3}\right| \leq \frac{x^4}{|x|^3} = |x|$ so, $\lim\limits_{\substack{(x,y)\to(0,0)\\(x,y)\in D_1}} \frac{y^2}{x^3} = 0$. It results that f is continuous at $(0,0)$. At the other points f is an elementary continuous function. Finally f is continuous on $\mathbb{R}^2$.

Example 1.3. *Find the real constant λ such that the function $f : D \to \mathbb{R}$, $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < \pi/2\}$ given by*

$$f(x,y) = \begin{cases} \dfrac{1-\cos\sqrt{x^2+y^2}}{tg\,(x^2+y^2)}, & (x,y) \neq (0,0) \\ \lambda, & (x,y) = (0,0) \end{cases}$$

be continuous on D.

Solution. On the set $D \setminus \{(0,0)\}$ the function f is a composition of elementary continuous functions, so f is continuous. We calculate the limit at the point $(0,0)$. If we denote $\sqrt{x^2+y^2} = t$ and use $\lim\limits_{t\to 0} \frac{\sin t}{t} = \lim\limits_{t\to 0} \frac{\operatorname{tg} t}{t} = 1$, the limit can be calculate as

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{1-\cos\sqrt{x^2+y^2}}{\operatorname{tg}\,(x^2+y^2)} = \lim_{t\to 0} \frac{1-\cos t}{\operatorname{tg}\,(t^2)} = \lim_{t\to 0} \frac{2\sin^2\left(\frac{t}{2}\right)}{\left(\frac{t}{2}\right)^2} \cdot \frac{t^2}{\operatorname{tg}\,(t^2)} \cdot \frac{1}{4} = \frac{1}{2}.$$

It results that f is continuous at $(0,0)$, and so on D, if and only if $\lambda = \dfrac{1}{2}$.