Can a binary relation be both symmetric and anti-symmetric?

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Binary Relations

Cartesian Product

For any two sets X and Y, the Cartesian product of X by Y is defined as :

$$X \times Y = \{(a, b) : a \in X \wedge b \in Y\}$$

Binary Relations

- If X and Y are two sets, then a binary relation from X to Y is a subset of $X \times Y$.
- A subset of $X \times X$ is called a binary relation in X.
- The empty set ($\emptyset$) and the cartesian product $X \times Y$ are binary relations from X to Y.
- If R is a binary relation from X to Y and $a \in X$, $b \in Y$, we write $(a, b) \in R$ or aRb.

Reflexive Relations

Definition

A binary relation R defined in set X is reflexive if it relates every element of X to itself.

$$R \text{ is reflexive iff } \forall a \in X \implies aRa$$

Example

For a set $A = \{1, 2, 3\}$, the relation R on A defined as $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\}$, is reflexive because $(1, 1), (2, 2), (3, 3)$ are in the relation.

Transitive Relations

Definition

A binary relarion R defined in a set X such that for all a, b and c in X, if aRb and bRc then aRc, is said to be transitive.

$$R \text{ is transitive iff } \forall a, b, c \in X, aRb \wedge bRc \implies aRc$$

Example

For a set $A = \{1, 2, 3\}$, $R = \{(1, 1), (1, 2), (2, 3), (1, 3), (3, 3)\}$ is transitive because:

- For every a, b, c, aRb and bRc implies aRc. Actually, $(1, 2)$ and $(2, 3)$ are in R and so is $(1, 3)$, $(1, 1)$ and $(1, 2)$ are in R and so is $(1, 2)$, $(1, 1)$ and $(1, 3)$ are in R and so is $(1, 3)$,$(2, 3)$ and $(3, 3)$ are in R and so is $(2, 3)$, $(1, 3)$ and $(3, 3)$ are in R and so does $(1, 3)$.

Note : If only aRb exists without bRc then it is not necessary

Symmetric Relations

Definition

A binary relation R defined in a set X is said to be symmetric in X if and only if for any a and b in X, aRb implies bRa.

$$R \text{ is Symmetric iff } \forall a, b \in X, aRb \implies bRa$$

Example

For a set $A = \{1, 2, 3\}$, relation $R = \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 3)\}$ is symmetric because:

- For every aRb there exists bRa. Actually, $(1, 2)$ and $(2, 1)$ both exist in R,$(2, 3)$ and $(3, 2)$ both exist in R.
- For $(3, 3)$ the symmetric is also $(3, 3) \in R$.

Anti-Symmetric Relations

Definition

A binary relation Rdefined in a set X is said to be anti-symmetric in X if and only if for any a and b in X, aRb, bRa implies $a = b$.

$$R \text{ is anti-symmetric iff } \forall a, b \in X, aRb \wedge bRa \implies a = b$$

If only aRb exist and bRa does not, then it is not necessary for $a = b$ for the relation R to be anti-symmetric.

Note: anti-symmetric doesn't mean not symmetric.

Anti-Symmetric Relations

Example

For a set $A = \{1, 2, 3\}$, relation $R = \{(1, 1), (2, 1), (1, 3)(3, 3)\}$ is anti-symmetric because:

- $(1, 1)$ and $(3, 3)$ both fit in the condition if aRb and bRa then $a = b$.
- Furthermore, $(2, 1)$ and $(1, 3)$, their symmetric ones doesn't exist in R so they do not need to be equal for R to be symmetric.

Equivalence Relations

Definition

A binary relation that is reflexive, symmetric and transitive is called an equivalence relation.
The equivalence class of an element a of X is the set of the elements of X that relate to a :

$$[a]_R = \{x \in A : xRa\}$$

Element a is is said to represent such class.

Equivalence Relations

Example

Let us consider a set $A = \{a, b, c\}$. Is $\{R = (a, a), (b.b), (c, c), (a, c), (c, a)\}$ an equivalence relation in A?

- Since $(a, a), (b, b)$ and (c, c) are all in R, R is reflexive.
- For all the pairs in R, the symmetric pair is also in R. For example (a, c) has (c, a), and the same happens for th other pairs of R. So, R is also symmetric.
- If aRb and bRc there is also aRc. For example, there is aRa and aRc and there is also aRc. This applies for all other possible combinations of pairs so, R is also transitive.

As R is reflexive, symmetric and transitive, then R is an equivalence Relation.

Equivalence Relations

Example - Equivalent Classes

In the relation R above, what are the equivalent classes of $[a], [b]$ and $[c]$?

1. In R, a is related with a and c, so, $[a] = \{a, c\}$
2. In R, b is related with b only, so $[b] = \{b\}$
3. In R, c is related with a and c, so, $[c] = \{a, c\}$.

Therefore, the set of all equivalence classes for the equivalence relation R is $\{\{a, c\}, \{b\}\}$.

Partial Order

Definition

A binary relation that is reflexive, anti-symmetric and transitive is called a partial order.

Example

Is a relation $R = \{(1,2),(1,1),(2,2),(2,3),(1,3),(3,3)\}$ in $X = \{1,2,3\}$ a partial order?

- R is reflexive as $(1,1),(2,2)$ and $(3,3)$ all belong in R.
- The only pairs whose symmetric also exists in R are $(1,1),(2,2),(3,3)$. so, here for all aRb and bRa then $a = b$. so, R is anti-symmetric
- If for all $a, b, c \in X$, aRb and bRc, there is also aRc. Like there is aRa and aRc then there is also aRc, so R is also transitive.

R is reflexive, anti-symmetric and transitive. so, R is a partial order.

Symmetric and Anti-Symmetric at same time?

Yes, A relation can be both symmetric and anti-symmetric at same time. Or it can can be neither as well.

Explanation

Let us consider a set $A = \{1, 2, 3\}$ and relation $R = \{(1, 1), (2, 2), (3, 3)\}$ in A. Let's see if R can be both symmetric and anti-symmetric:

- For $(1, 1)$, the symmetric pair is also $(1, 1)$. The same happens for all the pairs (x, x) in R, so the relation R is symmetric.
- Since the elements of R are pairs of the type (x, x), they satisfy the requirement 'if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$' which is the condition required for anti-symmetry, so R is also anti-symmetric.