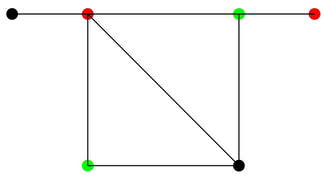
# Coloring (the vertices) of a graph

A **coloring of a graph** is an assignment of colors to the vertices so that adjacent vertices have different colors. An $n$-coloring is a coloring with $n$ colors. The **chromatic number** of a graph $G$, denoted $\chi(G)$, is the minimum value of $n$ for which an $n$-coloring of $G$ exists.

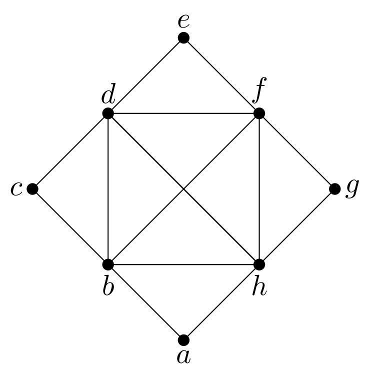
To realize a coloring of a graph, in order to determine the cromatic number, we most:

1. start for the vertex with maximum degree, $v_1$, color it with a color;
2. use the same color to coloring the vertices non adjacents to $v_1$;
3. choose the non colored vertex with maximum degree, $v_2$, and color with a color not already used;
4. use the same color to coloring all vertices non adjacents to $v_2$;
5. Repeat that procedure until all vertices are colored.

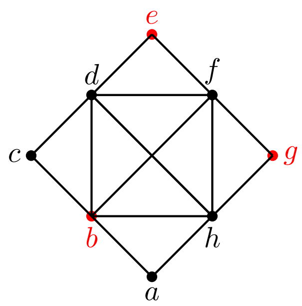
**Example 1.** The cromatic number of graph following graph $G$ is 3, that is, $\chi(G) = 3$.



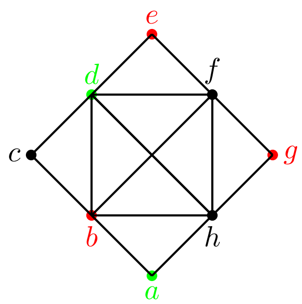
**Exercise 1.** Determine the cromatic number of the following graph.



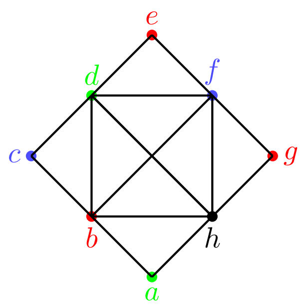

*Solution:*
Let's start coloring a vertex with maximum degree. We can choose vertex $b$, $d$, $f$ or $h$. Let's choose the vertex $b$ and color it and the non adjacent vertices with red.




Now, we can choose between the vertices with maximum degree, $d$, $f$ or $h$. Let's choose the vertex $d$ and color it and the non adjacent vertices with green.




Continuing with the procedure we obtain



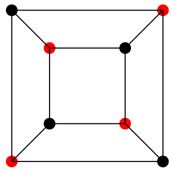

Then, the cromatic number is 4, that is, $\chi = 4$

**Theorem 1.** Let $\Delta(G)$ be the maximum of the degrees of the vertices of a graph $G$. Then $\chi(G) < 1 + \Delta(G)$.

*Proof.* The proof is by induction on $V$, the number of vertices of the graph. When $V = 1$, $\Delta(G) = 0$ and $\chi(G) = 1$, so the result clearly holds. Now let $k$ be an integer, $k > 1$, and assume that the result holds for all graphs with $|V| = k$ vertices. Suppose $G$ is a graph with $k + 1$ vertices. Let $v$ be any vertex of $G$ and let $G_0 = G \backslash \{v\}$ be the subgraph with $v$ (and all edges incident with it) deleted. Note that $\Delta(G_0) \leq \Delta(G)$. Now $G_0$ can be colored with $\chi(G_0)$ colors. Since $G_0$ has $k$ vertices, we can use the induction hypothesis to conclude that $\chi(G_0) \leq 1 + \Delta(G_0)$. Thus, $\chi(G_0) \leq 1 + \Delta(G)$, so go can be colored with at most $1 + \Delta(G)$ colors. Since there are at most $\Delta(G)$ vertices adjacent to $v$, one of the available $1 + \Delta(G)$ colors remains for $v$. Thus, $G$ can be colored with at most $1 + \Delta(G)$ colors. $\square$

**Theorem 2** (Four-Color Theorem)**.** For any planar graph $G$, $\chi(G) \leq 4$.

**Example 2.** The planar representation of a cube has cromatic number 2



## References

[1] Edgar Goodair and Michael Parmenter. *Discrete Mathematics with Graph Theory.* (3rd Ed.) Pearson, 2006.

[2] Susanna Epp. *Discrete Mathematics and Applications.* (4th Ed.) Brooks/Cole CENGAGE Learning, 2011.

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