Author: Edite Martins Cordeiro

September 2020

Scalar product of two vectors and orthogonal projection of one vector over another

Let's talk about scalar product of two vectors and and its application for the calculation of the orthogonal projection of one vector over another.

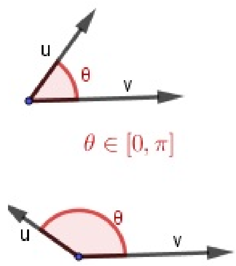
Scalar product of two vectors and related properties

Definition: The scalar product (or dot product) $u \cdot v$ of two vectors u and v is a number defined by

$$||u||||v||cos(\theta),$$

with $\theta = \widehat{uv} \in [0, \pi]$.

- If u and v are parallel vectors, then $u \cdot v = ||u||||v||$;
- If u and v are antiparallel vectors, then $u \cdot v = -||u||||v||$;
- If u and v are two orthogonal vectors, then $u \cdot v = 0$.



Notice that $\hat{vv} = 0$ and this implies that the dot product of a vector a with itself is $v \cdot v = ||v||||v||$, which gives

$$||v|| = \sqrt{v \cdot v}.$$

In $\mathbb{R}^n$ we have the alternative definition of scalar product:

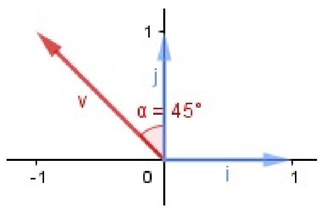
Definition: The dot product of two vectors $v = (v_1, v_2, \dots, v_n), u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ is

$$v \cdot u = v_1 u_1 + v_2 u_2 + v_3 u_3 + \cdots v_n u_n.$$

Example: On the Cartesian plane, consider the vectors $i = (1, 0), j = (0, 1)$ and $v(-1, 1)$.

On the one hand, we have $i \cdot j = (1, 0) \cdot (0, 1) = 0$ and $j \cdot v = (0, 1) \cdot (-1, 1) = 1 + 0 = 1$. On the other hand, we also have

$$i \cdot j = ||(1, 0)||||(0, 1)||cos(\frac{\pi}{2}) = 0$$



Also

$$j \cdot v = ||(0, 1)||||(-1, 1)||cos(\frac{\pi}{2}) = \sqrt{2} \times (\frac{\sqrt{2}}{2}) = 1.$$

$i \cdot v = ||(1,0)||||(-1,1)||cos(\frac{3\pi}{2}) = \sqrt{2} \times (-\frac{\sqrt{2}}{2}) = -1.$

The dot product fulfills the following properties if u, v, and w are vectors and k is a real scalar:

1. $v \cdot u = u \cdot v$;
2. $v \cdot (u + w) = (v \cdot u) + (v \cdot w)$;
3. $v \cdot (ku + w) = k(v \cdot u) + (v \cdot w)$;
4. $k_1 v \cdot (k_2 u) = k_1 k_2 (v \cdot u)$.

Examples: Consider $u = (-1, 2, 3)$ e $v = (2, 0, -1)$. We have:

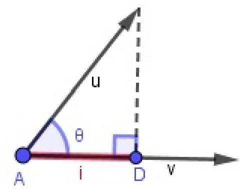
- $u \cdot 3v = 3(-1, 2, 3) \cdot (2, 0, -1) = 3(-2 - 3) = -10.$
- $(u + v) \cdot (u - v) = u \cdot u - u \cdot v + v \cdot u - v \cdot v = ||u||^2 + ||v||^2 = 14 + 5.$

An inner product is a generalization of the dot product. Is any operator who checks the properties above.

Orthogonal projection of one vector over another

One important use of dot products is in projections.

The orthogonal projection of u onto v is the length of the segment $[AD]$ shown in the figure beside, $\overline{AD}$.
The **vector projection** of u onto v is the vector $\overrightarrow{AD}$



Note that $|proj_v u| = ||v|||cos(\theta)|$ and therefore:

$$|proj_v u| = \overline{AD} = \frac{|u \cdot v|}{||v||} \quad \text{and} \quad proj_v u = \overrightarrow{AD} = \frac{|u \cdot v|}{||v||^2} v.$$

Exemplo: The orthogonal projection of $u = (-2, 1)$ onto $v = (-3, -1)$ is the vector:

$proj_v u = \dfrac{|(-2,1) \cdot (-3,-1)|}{10}(-3,-1) = \dfrac{5}{10}(-3,-1).$

We can also decompose u as the sum of two vectors w_1, w_2, such that $w_1 || u$ e $w_2 \perp u$. In fact,

$w_1 = proj_v u = (-\dfrac{3}{2}, -\dfrac{1}{2})$

and

$w_2 = u - w_1 = (-2, 1) - (-\dfrac{3}{2}, -\dfrac{1}{2}) = (-\dfrac{1}{2}, \dfrac{3}{2}).$