Isomorphism

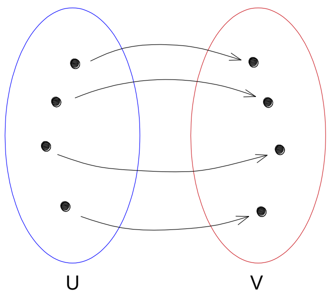
<u>Definition</u>: Let U and V be vector spaces and let the linear transformation $T: U \rightarrow V$. Then:

(i) T is an **injection** if and only if:

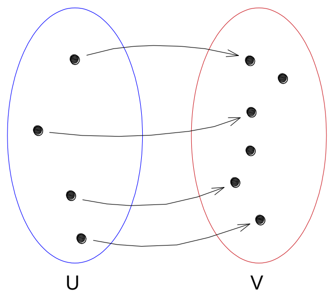
$$\forall\, x, y \in U, T(x) = T(y) \Rightarrow x = y \text{ or } \forall\, x, y \in U, x \neq y \Rightarrow T(x) \neq T(y)$$

(ii) T is a **sobrejection** if $Range\ (T) = V$

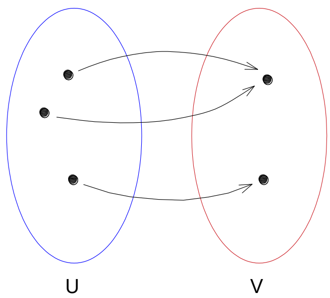
Examples:



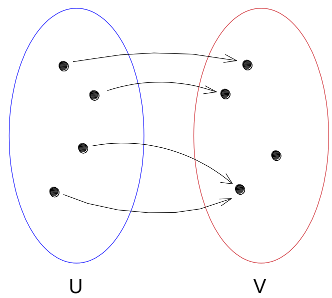
T is an injection and a sobrejection



T is an injection but not a sobrejection



T is a sobrejection but not an injection



T is neither an injection nor a sobrejection

- T is a **monomorphism** if it is an **injection**;
- T is an **epimorphism** if it is a **sobrejection**;
- T is an **isomorphism** if it is a **bijection** (an injection and a sobrejection);
- T is an **endomorphism** if $U = V$;
- T is an **automorphism** if it is also an isomorphism and an endomorphism.

The following statements are equivalent:

- T is an injection;
- $\ker(T) = \{0_U\}$.

1. The linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x+2y+3z, y+2z, -z)$ is an endomorphism. Verify it is an isomorphism.

We will determine the **kernel** to verify if the linear transformation is an **injection**:

$$\ker(T) = \{(x,y,z) \in \mathbb{R}^3 : T(x,y,z) = (0,0,0)\}$$

Then,

$$T(x,y,z) = (0,0,0) \Leftrightarrow (x+2y+3z, y+2z, -z) = (0,0,0)$$

$$\Leftrightarrow \begin{cases} x+2y+3z & = & 0 \\ \quad y+2z & = & 0 \\ \quad -z & = & 0 \end{cases} \Leftrightarrow \begin{cases} x & = & 0 \\ y & = & 0 \\ z & = & 0 \end{cases}$$

☺ Like $\ker(T) = \{(0,0,0)\}$, T is an **injection**. Consequently T is a **monomorphism**.

Let us now determine the **range** to verify if the linear transformation is a **sobrejection**:

$$range(T) = \{(a,b,c) \in \mathbb{R}^3 : T(x,y,z) = (a,b,c)\ with\ (x,y,z) \in \mathbb{R}^3\}$$

We have:

$$T(x,y,z) = (a,b,c) \Leftrightarrow (x+2y+3z, y+2z, -z) = (a,b,c)$$

$$\Leftrightarrow \begin{cases} x+2y+3z & = & a \\ \quad y+2z & = & b \\ \quad -z & = & c \end{cases}$$

The matrix of the system is: $\begin{bmatrix} 1 & 2 & 3 & | & a \\ 0 & 1 & 2 & | & b \\ 0 & 0 & -1 & | & c \end{bmatrix}$

Considering that A is the matrix of the coefficients, $A|B$ is the augmented matrix of the system and n is the number of unknowns, we observed that:

$$rank(A) = 3; rank(A|B) = 3; n = 3$$

As $rank(A) = rank(A|B) = n$, the system is possible (and determined).

Therefore, there are no restrictions to be imposed on variables a and b, so we conclude that the $range(T) = \mathbb{R}^3$.

☺ As $range\ (T) = \mathbb{R}^3 = V$, then T is a **sobrejection**, this is, T is an **epimorphism**.

Conclusion: As T is a **bijection** (an injection and a sobrejection), so it is an **isomorphism**.

2. Verify if the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, x + y)$ is an automorphism.

- We must verify if T is an **endomorphism** and an **isomorphism**.

☺ T is a **endomorphism** because the starting set and the finishing set are the same, this is, $\mathbb{R}^2$.

Remember T is an **isomorphism**, if T is a monomorphism and an epimorphism.

Let´s see if T is a monomorphism:

Considering the vectors $u = (2,3)$ and $v = (3,2)$:

$$T(2,3) = (5,5) \text{ and } T(3,2) = (5,5)$$

We have $(2,3) \neq (3,2)$, but $T(2,3) = T(3,2) = (5,5)$. Then, T is not a monomorphism.

☹ **Consequently, T is not an isomorphism.**

Note: To verify if T is a monomorphism, alternatively we can determine the kernel:

$$\ker(T) = \{(x, y) \in \mathbb{R}^2 : T(x, y) = (0,0)\}$$

Then,

$$T(x, y) = (0,0) \Leftrightarrow (x + y, x + y) = (0,0)$$

$$\Leftrightarrow \begin{cases} x + y & = & 0 \\ x + y & = & 0 \end{cases} \Leftrightarrow \begin{cases} x & = & -y \\ -y + y & = & 0 \end{cases} \Leftrightarrow \begin{cases} x & = & -y \\ 0 & = & 0 \end{cases}$$

Like $\ker(T) = \{(-y, y): y \in \mathbb{R}\}$, we concluded again that T is not a monomorphism.

Alternatively, we can begin to verify if T is an epimorphism. For this we determine the range of T:

$$range(T) = \{(a,b) \in \mathbb{R}^2: T(x,y) = (a,b) \; with \; (x,y) \in \mathbb{R}^2\}$$

We have:

$$T(x,y) = (a,b) \Leftrightarrow (x+y, x+y) = (a,b)$$

$$\Leftrightarrow \begin{cases} x+y & = & a \\ x+y & = & b \end{cases}$$

The matrix of the system is: $\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 1 & b \end{array}\right] \underset{L_2 \leftarrow -L_1 + L_2}{\longrightarrow} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 0 & -a+b \end{array}\right]$

Considering that A is the matrix of the coefficients, $A|B$ is the augmented matrix of the system and n is the number of unknowns, we observed that:

If, $-a+b=0$:

$$rank(A) = 1; rank(A|B) = 1; n = 2$$

As $rank(A) = rank(A|B) < n$, the system is possible (and indeterminate).

If , $-a+b \neq 0$, the system is impossible ($rank(A) \neq rank(A|B)$).

Therefore:

$$range(T) = \{(a,b) \in \mathbb{R}^2: a = b\}$$

Like $range(T) \neq \mathbb{R}^2$, T is not an epimorphism. And we can conclude that T is not an isomorphism.

Note: Is sufficient to fail one condition (to be a monomorphism or to be an epimorphism) for we conclude that T is not an isomorphism.

<u>Conclusion</u>: Despite *T* being an endomorphism is not an isomorphism, so *T* is not an automorphism.