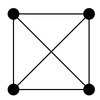
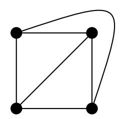
Planar graphs

A graph is called **plane** if it is drawn in the plane without any edges crossing, where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint. A graph is called **planar** if it can be drawn in the plane without any edges crossing. Such a drawing is called a **planar representation** of the graph.

Example 1. The complete graph K_4 is a planar graph because it can be drawn without crossings.



K_4

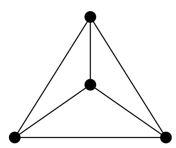


K_4

A planar graph divides the plane into various connected regions, one of which is called **the exterior region**. Every region, including the exterior, is bounded by edges.

Planar graphs were first studied by Euler because of their connections with polyhedra. A convex regular polyhedron is a geometric solid all of whose faces are congruent. There are in all just five of these-the cube, the tetrahedron, the octahedron, the icosahedron, and the dodecahedron-and they are popularly known as the Platonic solids because they were regarded by Plato as symbolizing earth, fire, air, water, and the universe, respectively.

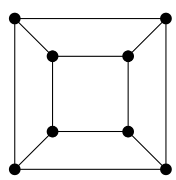
Example 2. Planar representation of tethaedro.



This representation splits the plane into four region, three in the interior and one exterior region.

Exercise 1. Drawn a planar representation of a cube.

Solution:

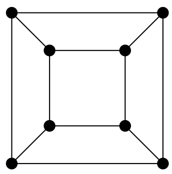


In 1752, Euler published the remarkable formula $V - E + F = 2$, which holds for any convex polyhedron with V vertices, E edges, and F faces. (A polygon is convex if the line joining any pair of nonadjacent vertices lies entirely within the polygon.) Euler showed that all planar representations of a graph split the plane into the same number of regions. He accomplished this by finding a relationship among the number of regions, the number of vertices, and the number of edges of a planar graph.

Theorem 1 (Euler's Formula)**.** Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $v - e + r = 2$.

Proof. We use induction on e, the number of edges.
If $e = 0$, then $v = r = 1$ (because G is connected) and the formula is true.
Now assume the formula holds for connected plane graphs G' with $e' = e - 1$ edges, where $e \geq 1$. We must show that G is a connected plane graph with e edges, v vertices, and r regions. We must show that $v - e + r = 2$. We have two cases:
1st case: the edge connect to existent vertices, then the number v remains the same but the number of the regions increase one unity; $v - e + r = v' - (e' + 1) + r' + 1 = v' - e' + r'$ and the formula is true because G' is a planar graph;
2nd case: the edge connect one existent vertex to another new vertex, then $e = e' + 1$ and $v = v' + 1$, but $r = r'$, $v - e + r = v' + 1 - (e' + 1) + r' = v' - e' + r' = 2$, again because G' is planar. □

Example 3. The following graph has 12 edges, 8 vertices and 6 regions, thus the formula is verified: 8-12+6=2.



Corollary 1. If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Proof. A connected planar simple graph drawn in the plane divides the plane into regions, say r of them. Each region has at least three edges on the boundary. (Because the graphs discussed here are simple graphs, no multiple edges that could produce regions of degree two, or loops that could produce regions of degree one, are permitted.) In particular, note that of the unbounded region is at least three because there are at least three vertices in the graph.
Note that the sum S of the number of boundary edges of the regions is exactly twice the number of edges in the graph, because each edge occurs on the boundary of a region exactly twice (either in two different regions, or twice in the same region). Because each region has at least three edges on the boundary, it follows that $2e = S \geq 3r$. Hence, $\frac{2}{3}e \geq r$. Using $r = e - v + 2$ (Eulers formula), we obtain $e - v + 2 \leq \frac{2}{3}e$. It follows that $\frac{e}{3} \leq v - 2$. This shows that $e \leq 3v - 6$. □

Exercise 2. Show that K_5 is nonplanar using Corollary 1.

Solution:

The graph K_5 has five vertices and 10 edges. However, the inequality $e \leq 3v - 6$ is not satisfied for this graph because $e = 10$ and, because $v = 5$, $3 \times 5 - 6 = 9$. Therefore, K_5 is not planar.

Corollary 2. If a connected planar, bipartite and simple graph has e edges and v vertices with $v \geq 3$, then $e \leq 2v - 4$.

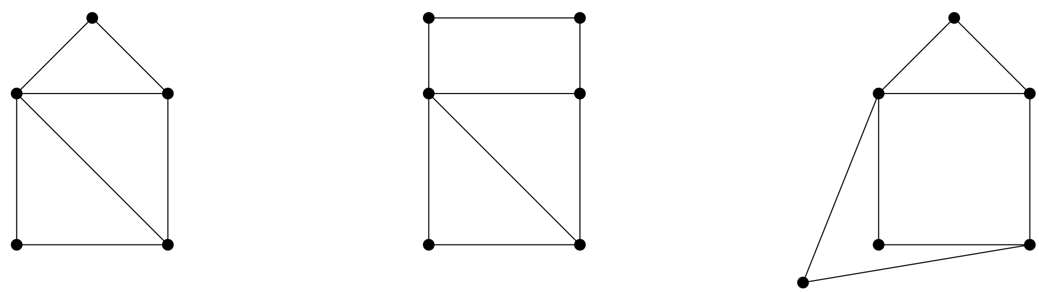
Proof. Notice that if the graph is bipartite the graph contains no cycles of odd lenght. Thus, each region is bounded at least for four edges. Analogous to the proof of Corollary 1 we obtain $2e \geq 4r$ and by the Euler's Formula $4r = 4(e - v + 2)$, then $e \leq 2v - 4$. $\square$

Corollary 3. If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

Proof. If G has one or two vertices, the result is true. If G has at least three vertices, by Corollary 1 we know that $e \leq 3v - 6$, so $2e \leq 6v - 12$. If the degree of every vertex were at least six, then because $2e = S$ (S is the sum of the number of boundary edges of all regions), we would have $2e \geq 6v$. But this contradicts the inequality $2e \leq 6v - 12$. It follows that there must be a vertex with degree no greater than five. $\square$

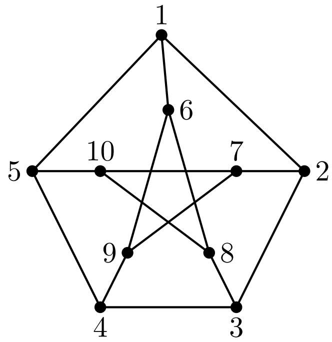
It was the Polish mathematician Kazimierz Kuratowski (1896-1980) who discovered the crucial role played by $K_{3,3}$ and K_5 in determining whether or not a graph is planar. He defined that two graphs are **homeomorphic** if and only if each can be obtained from the same graph by adding vertices (necessarily of degree 2) to edges.

Example 4. The following graphs are homeomorphic.



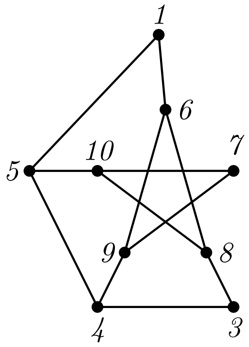
Theorem 2 (Kuratowski)**.** A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

Exercise 3. Is the Petersen graph planar?

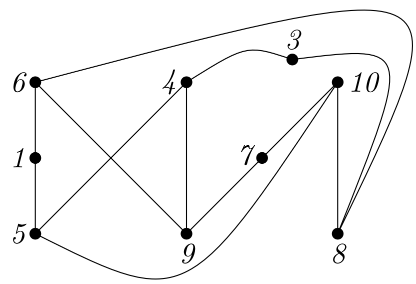


Solution:

The subgraph H of the Petersen graph is obtained deleting the vertex 2 and the three edges that have 2 as an endpoint.

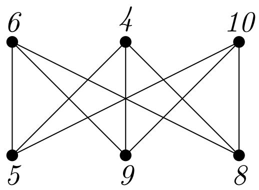


H



H

This subgraph is homeomorphic to $K_{3,3}$ considering the vertex sets $\{6,\ 4,\ 10\}$ and $\{5,\ 9,\ 8\}$ because it can be obtained by a sequence of elementary subdivisions, deleting $\{4,\ 8\}$ and adding $\{3,\ 8\}$ and $\{3,\ 4\}$, deleting $\{5,\ 6\}$ and adding $\{1,\ 5\}$ and $\{1,\ 6\}$, and deleting $\{9,\ 10\}$ and adding $\{7,\ 9\}$ and $\{7,\ 10\}$.



$K_{3,3}$

Thus, the Petersen graph is not a planar graph.

References

Exercises in MathE platform