End digression.

Numerical sets

$\mathbb{N} = \{\boxed{0}, 1, 2, 3, \ldots\}$ natural numbers

$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ integer numbers

$\mathbb{Q} = \{\frac{m}{n} ; m \in \mathbb{Z}, n \in \mathbb{N}, n \neq 0\}$ rational numbers

$\mathbb{R} =$ "every number"

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

Alg. properties of $\mathbb{R}$

in $\mathbb{R}$, we have defined two operations:

$+$ and $\cdot$ (sum and product), with the following properties:

also in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ commutativity

(S1) $a + b = b + a \quad \forall a \in \mathbb{R}, \forall b \in \mathbb{R}$

(for each real numbers a, b)

also in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ associativity

(S2) $a + (b+c) = (a+b) + c \quad \forall a, b, c \in \mathbb{R}$

(S3) $\exists$ of a "special element" (identity element
neutral element)

$$0 + a = a + 0 = a$$ also in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

(S4) $\forall a \in \mathbb{R}$ there exists $(-a) \in \mathbb{R}$ ~~NOT IN~~ $\mathbb{Z}$ ok, $\mathbb{Q}$, $\mathbb{R}$

inverse with respect to $+$

$$a + (-a) = (-a) + a = 0$$

(M1) $a \cdot b = b \cdot a \quad \forall a, b \in \mathbb{R}$ (also in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$)

(M2) $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in \mathbb{R}$, also (also in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$)

(M3) $\exists$ "special elem" $1 \in \mathbb{R}$ (also in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$)

$$a \cdot 1 = 1 \cdot a = a$$

(M4) $\forall \; a \in \mathbb{R}$ $\boxed{a \neq 0}$ then exist $\frac{1}{a} \in \mathbb{R}$ ~~NOT IN~~ $\mathbb{Z}$ ~~NOT in~~ $\mathbb{N}$ ok in $\mathbb{Q}$

s.t. $a \cdot \left(\frac{1}{a}\right) = \left(\frac{1}{a}\right) \cdot a = 1$

(D) $a \cdot (b+c) = a \cdot b + a \cdot c \quad \forall\, a,b,c \in \mathbb{R}$

(distributivity)

(also in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$.)

ORDERING OF $\mathbb{R}$ $\quad \forall\, a,b \in \mathbb{R}$

EITHER $a \leq b$ OR $b \leq a$

(OA1) $a \leq b \quad c \in \mathbb{R}$

$$a + c \leq b + c$$

(I can add the same quantity to both sides and the order doesn't change)

(OA2) $a \leq b$

$a \cdot c \leq b \cdot c$ if $c > 0$

$a \cdot c \geq b \cdot c$ if $c < 0$

(O.0) $\forall a, b \in \mathbb{R}$ $a \leq b$ OR $b \leq a$

(O1) $a \leq b$ and $b \leq c$ $\Rightarrow$ $a \leq c$ (transitivity)

(O2) $a \leq b$ and $b \leq a$ $\Rightarrow$ $a = b$

So, up to now $\mathbb{R}$ and $\mathbb{Q}$ aren't different

$$\mathbb{Q} = \left\{ \frac{m}{n} : \quad m \in \mathbb{Z}, \quad n \in \mathbb{N}, \quad n \neq 0 \right\}$$

$$\frac{5}{3} \in \mathbb{Q} \qquad 5 \in \mathbb{Z} \quad 3 \in \mathbb{N} \quad 3 \neq 0$$

$$\mathbb{R} \setminus \mathbb{Q} = \text{"irrational numbers"} \ni \sqrt{2}$$

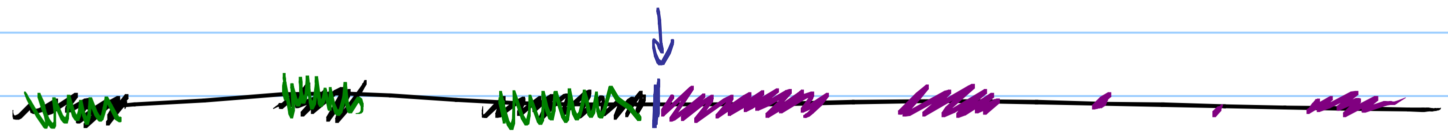
$$\exists\, q \in \mathbb{Q} \quad \text{s.t.} \quad q^2 = 2 \qquad \text{No!}$$

COMPLETENESS / (CONTINUITY AXIOM)

$A \subseteq \mathbb{R}$ $\quad B \subseteq \mathbb{R}$

Let us suppose that $\forall a \in A$, $\forall b \in B$ we have $a \leq b$: the set $\underline{A}$ is "on the left" of $\underline{B}$

$\exists\, c \in \mathbb{R}$ "in the middle"?



c is a separator for A and B if

- $\forall a \in A \quad a \leq c$
- $\forall b \in B \quad c \leq b$

Key property of $\mathbb{R}$: for every A, B such that A is "on the left" of B, there exists a separator for A and B. (Continuity axiom)

<u>Obs.</u> $\mathbb{Q}$ does NOT satisfy the cont. axiom.

$$A = \{ x \in \mathbb{Q} : x > 0 , \ x^2 < 2 \}$$

$$B = \{ x \in \mathbb{Q} : x > 0 , \ x^2 > 2 \}$$

A is "on the left" of B. But it doesn't exists a separator $c \in \mathbb{Q}$, because if it would exist, we should have $c^2 = 2$.