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## Constrained Extrema and Lagrange Multipliers

**Definition 1** *Let $D \subset \mathbb{R}^n$ and $f, g : D \subset \mathbb{R}$. An extreme value of $f$ subject to the condition $g(\mathbf{x}) = 0$, is called a* ***constrained extreme value*** *and $g(\mathbf{x}) = 0$ is called the constraint.*

**Definition 2** *Let $D \subset \mathbb{R}^n$ and $f, g : D \subset \mathbb{R}$.* ***The Lagrangian function*** *of $f$ subject to the constraint $g(\mathbf{x}) = 0$ is the function of $n + 1$ variables*

$$L(\mathbf{x}; \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x}),$$

*where $\lambda$ is known as* ***the Lagrange multiplier****.*

**Definition 3** ***The Lagrangian function*** *of $f$ subject to the $k$ constraints $g_i(\mathbf{x}) = 0$, $i = \overline{1,k}$ is the function with $k$ Lagrange multipliers, $\lambda_i$, $i = \overline{1,k}$,*

$$L(\mathbf{x}; \lambda) = f(\mathbf{x}) + \sum_{i=1}^{k} \lambda_i g_i(\mathbf{x}).$$

**Theorem 4** ***(The Extreme Value Theorem for Functions of $n$ Variables)*** *Let $D \subset \mathbb{R}^n$ and $f : D \subset \mathbb{R}$ be a continuous $n$ variable real-valued function. If $D(f)$ is a closed and bounded set in $\mathbb{R}^n$ then $R(f)$ is a closed and bounded set in $\mathbb{R}$ and there exists $\mathbf{x}$,$\mathbf{y} \in D(f)$ such that $f(\mathbf{x})$ is an absolute maximum value of $f$ and $f(\mathbf{y})$ is an absolute minimum value of $f$.*

To find the extreme values of $f$ subject to the constraint $g(\mathbf{x}) = 0$:

1. calculate $\dfrac{\partial L}{\partial x_1}, ..., \dfrac{\partial L}{\partial x_n}, \dfrac{\partial L}{\partial \lambda}$, remembering that $L$ it is a function of the $n + 1$ variables $\mathbf{x} = (x_1, ..., x_n)$ and $\lambda$,
2. solve the equations
$\dfrac{\partial L}{\partial x_1} = 0, ..., \dfrac{\partial L}{\partial x_n} = 0$ and $g(\mathbf{x}) = 0$,
3. evaluate $f$ at these points to find the required extrema.

**Example 5** *Find the extreme values of*

$$f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x^2 + y^2 - x - y + 1$$

*on the set $S = \left\{(x, y) : x^2 + y^2 = 1\right\}$.*

*Solution.*
Let $g(x,y) = x^2+y^2-1, L(x,y;\lambda) = x^2+y^2-x-y+1+\lambda\left(x^2+y^2-1\right)$.
Compute
$\dfrac{\partial L}{\partial x} = 2x-1+\lambda 2x, \dfrac{\partial L}{\partial y} = 2y-1+\lambda 2y, \dfrac{\partial L}{\partial \lambda} = x^2+y^2-1$
Solve the system
$\left\{\begin{array}{l} 2x-1+\lambda 2x = 0 \\ 2y-1+\lambda 2y = 0 \\ x^2+y^2-1=0 \end{array}\right., \left\{\begin{array}{l} x = \frac{1}{2(\lambda+1)} \\ y = \frac{1}{2(\lambda+1)} \\ x^2+y^2-1=0 \end{array}\right.$
Hence $x=y$.

From the last equation, it now follows that $2x^2=1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$. Thus we have two points to consider for extreme values: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Since $S$ is closed and bounded, we know from the Extreme Value Theorem that one of these values is an absolute maximum of $f$ on $S$ and the other an absolute minimum of $f$ on $S$. Now
$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2-\sqrt{2}$
and
$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2+\sqrt{2}$
so $f$ has an absolute maximum value of $2+\sqrt{2}$ at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and an absolute minimum value of $2-\sqrt{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

**Example 6** *Find the shortest distance from the origin to the curve* $x^6+3y^2=1$.

*Solution.*
We extremize $f(x,y) = x^2+y^2$ under the constraint $g(x,y) = x^6+3y^2-1$.
$L(x,y;\lambda) = x^2+y^2+\lambda\left(x^6+3y^2-1\right)$
Compute
$\dfrac{\partial L}{\partial x} = 2x+6\lambda x^5, \dfrac{\partial L}{\partial y} = 2y+\lambda 6y, \dfrac{\partial L}{\partial \lambda} = x^6+3y^2-1$
Solve the system
$\left\{\begin{array}{l} 2x+6\lambda x^5 = 0 \\ 2y+\lambda 6y = 0 \\ x^6+3y^2-1 \end{array}\right..$

Solutions are: $\left[x=0, y=\frac{1}{3}\sqrt{3}, \lambda=-\frac{1}{3}\right], \left[x=0, y=-\frac{1}{3}\sqrt{3}, \lambda=-\frac{1}{3}\right]$, $\left[x=-1, y=0, \lambda=-\frac{1}{3}\right], \left[x=1, y=0, \lambda=-\frac{1}{3}\right]$.

So, we have the solutions $(0, \pm\sqrt{1/3})$ and $(1,0), (-1,0)$. To see which is the minimum, just evaluate $f$ on each of the points. We see that $(0, \pm\sqrt{1/3})$ are the minima.

**Example 7** *Find the rectangular box with the largest volume that fits inside the ellipsoid* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, *given that it sides are parallel to the axes.*

*Solution.*

Clearly the box will have the greatest volume if each of its corners touch the ellipsoid. Let one corner of the box be point $(x, y, z)$ in the positive octant, then the box has corners $(\pm x; \pm y; \pm z)$ and its volume is $V = 8xyz$.

We want to maximize $V$ given that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$. (Note that since the constraint surface is bounded the max does exist). The Lagrangian is

$L(x, y, z; \lambda) = 8xyz + \lambda \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right)$.

The critical points are solutions of the system

$$\begin{cases} \frac{\partial L}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = 0 \\ \frac{\partial L}{\partial z} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \Leftrightarrow \begin{cases} 8yz + \lambda \frac{2x}{a^2} = 0 \\ 8xz + \lambda \frac{2y}{b^2} = 0 \\ 8xy + \lambda \frac{2z}{c^2} = 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \end{cases}$$

(Note that $\frac{\partial L}{\partial \lambda}$, will always be the constraint equation.) As we want to maximize $V$ we can assume that $xyz \neq 0$ so that $x, y, z \neq 0$.) Hence, eliminating $\lambda$, we get

$\lambda = -\frac{4yza^2}{x}, \lambda = -\frac{4xzb^2}{y}, \lambda = -\frac{4xyc^2}{z}$

so that $\frac{4yza^2}{x} = \frac{4xzb^2}{y} \Rightarrow y^2a^2 = x^2b^2, \frac{4xzb^{2^2}}{y} = \frac{4xyc^2}{z} \Rightarrow z^2b^2 = y^2c$

But then $\frac{x^2}{a^2} = \frac{y^2}{b^2}, \frac{y^2}{b^2} = \frac{z^2}{c^2}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \Rightarrow 3\frac{y^2}{b^2} = 1 \Rightarrow y = \frac{b}{\sqrt{3}}$ which implies that $x = \frac{a}{\sqrt{3}}$ and $z = \frac{c}{\sqrt{3}}$ (they are all positive by assumption). So there is only one stationary point $\left( \frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}} \right)$.

Being the only stationary point it is the required point of maximum and consequently, the largest volume of the rectangular box inscribed in the ellipsoid is

$V = \frac{8abc}{3\sqrt{3}}$.

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