Orthogonal projection of a vector v over a vector space S and distance from v to S

We calculate the distance from v to S, using the following theorem:

Theorem of the best aproximation: Consider the Euclidean space E and a subspace W of E. If $v \in E$ is such that $v \notin W$ then the vector $proj_W(v)$ is the best approach of v to W. That is,

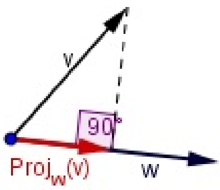
$$||v - proj_w(v)|| \leq ||v - w||, \quad \text{to any} \quad w \in W.$$

Thus, $proj_W(v)$ is the vector of W that best approximates v. Then, the distance between v and the vector space W is is given by $||v - proj_W(v)||$.

Orthogonal projection of one vector over another

Consider two vectors u and v. The orthogonal projection of v over $w \neq 0$ is the scalar multiple of w,

$$proj_w(v) = \frac{v \bullet w}{|w|^2} w$$



Example: Consider, in $\mathbb{R}^2$, $v = (-1, -1)$ and $u = (3, 4)$. The orthogonal projection of v over u is

$$proj_u(v) = \frac{(-1,1) \bullet (2,-1)}{||(2,-1)||^2}(2,-1) = \frac{-3}{5}(2,-1) = (-\frac{6}{5}, \frac{3}{5}).$$

So the distance from v to the subspace generated by u, $\langle u \rangle$, is

$$||v - proj_u(v)|| = ||(-1,1) - (-\frac{6}{5}, \frac{3}{5})|| = ||(\frac{1}{5}, \frac{2}{5})|| = \sqrt{\frac{5}{25}} = \frac{\sqrt{5}}{5}.$$

Orthogonal projection of one vector over a vector space

Let E be a Euclidean space, W a subspace of E and $B = \{w_1, w_2, \ldots, w_n\}$ an orthogonal basis of W. Then

$$proj_W(v) = \frac{v \bullet w_1}{|w_1|^2} w_1 + \frac{v \bullet w_2}{|w_2|^2} w_2 + \cdots + \frac{v \bullet w_n}{|w_n|^2} w_n.$$

Scalars

$$k_i = \frac{v \bullet w_i}{|w_i|^2}$$

are said color ipb Fourier coefficients v in relation to w_i.

Example: Consider the subspace S of $\mathbb{R}^3$ generated by $A = \{(1,-1,2),(1,0,1)\}$ and $v = (1,2,3) \notin S$. The orthogonal projection of v over S is

$$proj_S(v) = \frac{(1,2,3) \bullet (1,-1,2)}{||(1,-1,2)||^2}(1,-1,2) + \frac{(1,2,3) \bullet (1,0,1)}{||(1,0,1)||^2}(1,0,1) = \frac{5}{4}(1,-1,2) + \frac{2}{2}(1,0,1).$$

We have $proj_S(v) = (\frac{9}{4}, -\frac{5}{4}, \frac{7}{2})$. So the distance from v to the subspace S is

$$||v - proj_u(v)|| = ||((1,2,3) - (\frac{9}{4}, -\frac{5}{4}, \frac{7}{2})|| = ||(-\frac{5}{4}, \frac{3}{4}, -\frac{1}{2})|| = \sqrt{\frac{25}{16} + \frac{9}{16} + \frac{1}{4}} = \frac{19}{8}.$$