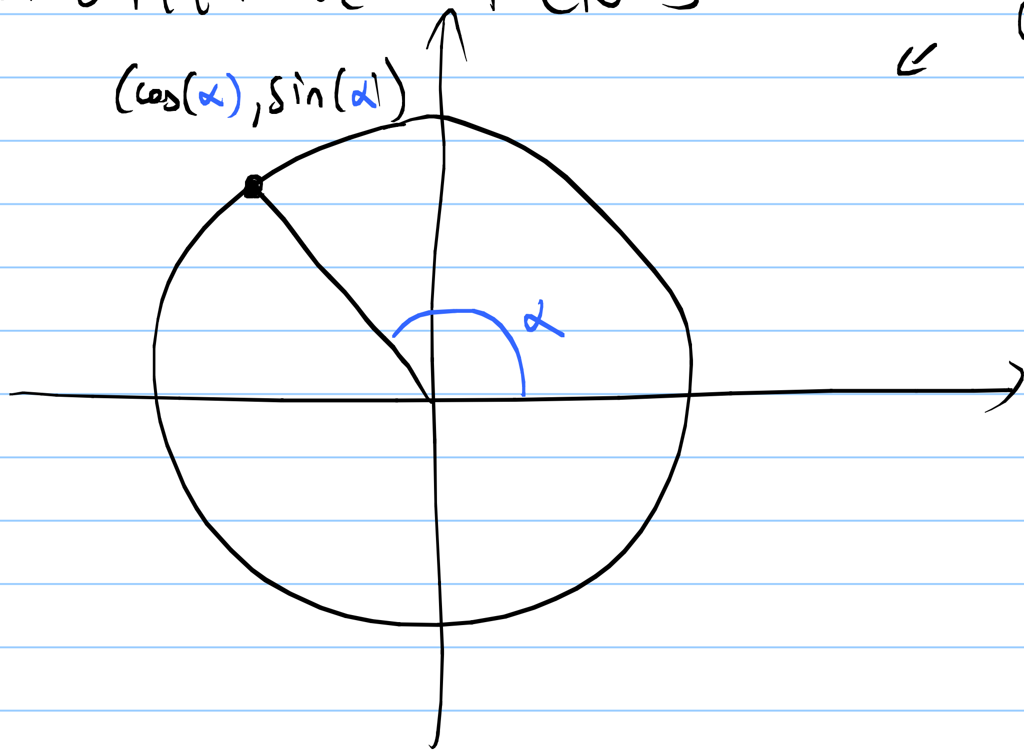
TRIGONOMETRIC FCN'S



GONIOMETRIC CIRCLE (center in (0,0) and radius = 1)

Def. $tg(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)}$ $\quad$ $cotg = \dfrac{\cos(\alpha)}{\sin(\alpha)}$

Basic properties

1) $\cos(\alpha)^2 + \sin(\alpha)^2 = 1$ (Pythagora's thm)

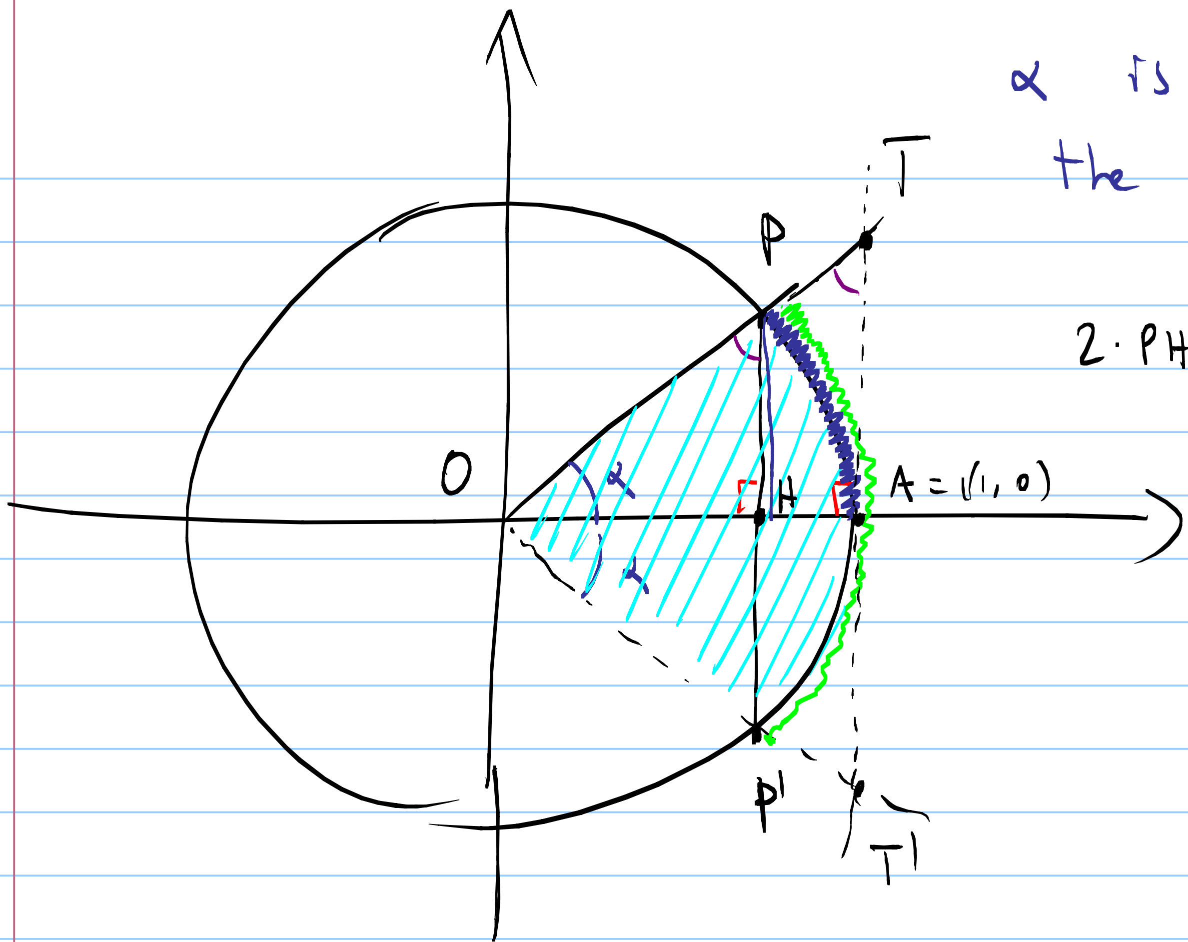
2) $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

2') $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$

3) $\sin(\alpha + 2\pi) = \sin(\alpha)$, $\cos(\alpha + 2\pi) = \cos(\alpha)$

4) (Analytic property) if $0 < \alpha < \frac{\pi}{2}$

$$\boxed{0 < \sin(\alpha) < \alpha < tg(\alpha)}$$



α is the length of the blue arc. (proportions)

$2 \cdot PH = \overline{PP'} <$ length of the arc PAP'

$\shortparallel$

2α

$$\sin(\alpha) = PH = \frac{PP'}{2} < \frac{\cancel{2}\alpha}{\cancel{2}} = \alpha$$

POH is similar to TOA. In particular

$$PH : OH = TA : OA$$

$$TA = \frac{PH \cdot OA}{OH} = \frac{\sin(\alpha)}{\cos(\alpha)} \cdot 1 = tg(\alpha)$$

Area of the whole circle $= \pi r^2 = \pi$

Area of $POP'A$ $=$ Area of the circle $\cdot \frac{2\alpha}{2\pi} = \cancel{\pi} \cdot \frac{\cancel{2}\alpha}{\cancel{2\pi}} = \alpha$

Area of $\hat{TOT'}$ $= \frac{1}{2} TT' \cdot AO = \frac{1}{2} \cdot 2\, tg(\alpha) \cdot 1 = tg(\alpha)$

But Area of $\hat{TOT'} >$

$$\tan(\alpha) > \alpha.$$

Exercises 1

Given

$$\cos\left(\frac{\pi}{2}\right) = 0 \qquad \sin\left(\frac{\pi}{2}\right) = 1$$

$$\cos(\pi) = -1 \qquad \sin(\pi) = 0$$

Prove.

- $\cos(\pi + x) = -\cos(x)$
- $\sin(\pi + x) = -\sin(x)$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$
- $\cos(2x) = \cos(x)^2 - \sin(x)^2$
- $\sin(2x) = 2\sin(x)\cos(x)$

- $\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
- $\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

(*) try $\alpha = \frac{x+y}{2}$, $\beta = \pm\left(\frac{x-y}{2}\right)$ in (2), (2)'.

- $\cos(3x) = 4\cos(x)^3 - 3\cos(x)$
- $tg(2\alpha) = \dfrac{2tg(\alpha)}{1 - tg(\alpha)^2}$

$\cos(\pi + x) = ?$

$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$ $\qquad \alpha = \pi$

$\beta = x$

$\cos(\pi + x) = \cos\pi \cdot \cos x - \sin\pi \cdot \sin x$

$= (-1) \cdot \cos x - \cancel{0 \cdot \sin x} = -\cos(x)$

$\cos\left(\frac{\pi}{2} - x\right)$

$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$\alpha = \frac{\pi}{2}$

$\beta = -x$

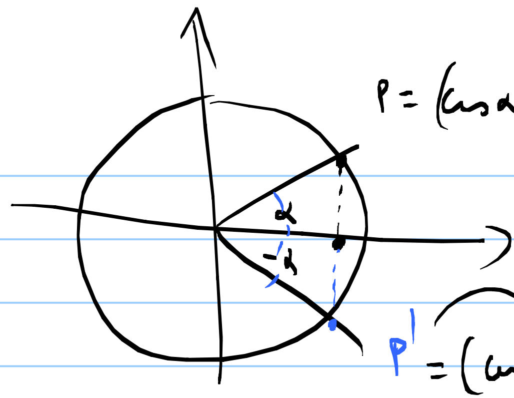
$\cos\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} + (-x)\right) =$

$= \cos\left(\frac{\pi}{2}\right) \cdot \cos(-x) - \sin\left(\frac{\pi}{2}\right) \sin(-x)$

$= 0 \cdot \cancel{\cos(-x)} - 1 \cdot \sin(-x)$

$= -\sin(-x) = -(-\sin(x)) = \sin(x)$

[Another route: use formula for difference of cos]



$P = (\cos\alpha, \sin\alpha)$

since P' is the symm. w.r.t. x axis

$$P' = (\cos\alpha, -\sin\alpha) = (\cos(-\alpha), \sin(-\alpha))$$

def. of cos, sin

$$\cos(-\alpha) = \cos(\alpha) \quad , \quad \sin(-\alpha) = -\sin(\alpha)$$

Remark.

- If a function f has the property that $f(x) = f(-x)$ $\forall x \in \mathbb{R}$ $\Leftrightarrow$ f is EVEN
- If $f(-x) = -f(x)$ $\forall x \in \mathbb{R}$ $\Leftrightarrow$ f is ODD

$$\cos(3x) = \cos(2x + x) =$$

$$= \underbrace{\cos(2x)}_{\cos(x+x)} \cos(x) - \underbrace{\sin(2x)}_{\sin(x+x)} \sin x$$

$$= [\cos x \cdot \cos x - \sin x \cdot \sin x] \cos x - [\sin x \cdot \cos x + \sin x \cdot \cos x] \cdot \sin x$$

$$= [(\cos(x))^2 - (\sin(x))^2] \cdot \cos(X) - [2 \cdot \cos(x) \cdot \sin(x)] \cdot \sin(x)$$

$$= \cos(x)^3 - \sin(x)^2 \cdot \cos(x) - 2 \cdot \sin(x)^2 \cdot \cos(x)$$

$$= \cos(x)^3 - 3 \sin(x)^2 \cdot \cos(x)$$

$$\sin(x)^2 + \cos(x)^2 = 1$$
$$\sin(x)^2 = 1 - \cos(x)^2$$

$$= \cos(x)^3 - 3 \cdot (1 - \cos(x)^2) \cdot \cos(x)$$

$$= 4 \cdot (\cos(x))^3 - 3 \cdot \cos(x)$$

$$tg(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin(\alpha)\cdot\cos(\beta) + \sin(\beta)\cdot\cos(\alpha)}{\cos(\alpha)\cdot\cos(\beta) - \sin(\alpha)\cdot\sin(\beta)}$$

$$= \frac{\cancel{\cos(\alpha)\cos(\beta)}\left(\frac{\sin(\alpha)}{\cos(\alpha)} + \frac{\sin(\beta)}{\cos(\beta)}\right)}{\cancel{\cos(\alpha)\cos(\beta)}\cdot\left(1 - \frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}\right)}$$

$$= \frac{tg(\alpha) + tg\beta}{1 - tg\alpha \cdot tg\beta}$$

$\alpha = \beta = x$

$$tg(2x) = \frac{2\,tg(x)}{1 - tgx \cdot tgx} = \frac{2tg(x)}{1 - (tg(x))^2}$$

SUM - TO - PRODUCT FORMULA

$$\cos(x) + \cos(y) = 2 \cdot \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$\alpha = \frac{x+y}{2}$ $\quad$ $\beta = \frac{x-y}{2}$

$$\alpha + \beta = \frac{x+\cancel{y}}{2} + \frac{x-\cancel{y}}{2} = x$$

$$\alpha - \beta = \frac{x+y}{2} - \frac{x-y}{2} = y$$

$$\cos(x) = \cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \cancel{\sin\alpha \sin\beta}$$

$$+ \qquad\qquad + \qquad\qquad +$$

$$\cos(y) = \cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \cancel{\sin\alpha \sin\beta}$$

$$\parallel$$

$$\cos(x) + \cos(y) = 2\cos(\alpha)\cos(\beta) = 2 \cdot \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

PRODUCT-TO-SUM FORMULA

- $\cos(\alpha)\cos(\beta) = \dfrac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2}$

- $\sin(\alpha)\cos(\beta) = \dfrac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$