Systems of linear equations

Example. Consider the system

$$\begin{cases} x+y+z=a \\ ax+y+2z=2 \\ x+ay+z=4 \end{cases}$$

Decide whether the system is consistent and find the number of solutions in dependence on parameter $a \in \mathbb{R}$. Find its solution set for each $a \in \mathbb{R}$.

Solution

The system has augmented matrix $\overline{A}$, where A is the coefficient matrix, and $\underline{b}$ is the constant column term:

$$\overline{A} := (A|\underline{b}) = \begin{pmatrix} \boxed{1} & 1 & 1 & | & a \\ a & 1 & 2 & | & 2 \\ 1 & a & 1 & | & 4 \end{pmatrix}$$

performing row operations ($R_2 \to R_2 - aR_1$, $R_3 \to R_3 - R_1$, $R_3' \to R_3' + R_2'$) we get a row echelon form

$$\overline{A} \longrightarrow \begin{pmatrix} \boxed{1} & 1 & 1 & | & a \\ 0 & 1-a & 2-a & | & 2-a^2 \\ 0 & a-1 & 0 & | & 4-a \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 1 & 1 & | & a \\ 0 & 1-a & 2-a & | & 2-a^2 \\ 0 & 0 & 2-a & | & 6-a-a^2 \end{pmatrix}$$

that is (changing sign in the last two rows):

$$\overline{A} \longrightarrow \begin{pmatrix} \boxed{1} & 1 & 1 & | & a \\ 0 & a-1 & a-2 & | & a^2-2 \\ 0 & 0 & a-2 & | & (a+3)(a-2) \end{pmatrix}$$

Now, if $a \neq 1 \wedge a \neq 2$, then A and $\overline{A}$ have equal rank $\varrho(A) = \varrho(\overline{A}) = 3$, hence the system is consistent and it has exactly one solution, which can be found by Gauss-Jordan elimination:

$$\overline{A} \longrightarrow \begin{pmatrix} \boxed{1} & 1 & 1 & | & a \\ 0 & \boxed{a-1} & a-2 & | & a^2-2 \\ 0 & 0 & \boxed{a-2} & | & (a+3)(a-2) \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 1 & 1 & | & a \\ 0 & \boxed{1} & \frac{a-2}{a-1} & | & \frac{a^2-2}{a-1} \\ 0 & 0 & \boxed{1} & | & a+3 \end{pmatrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} \boxed{1} & 1 & 0 & | & -3 \\ 0 & \boxed{1} & 0 & | & \frac{a^2-2}{a-1} - \frac{(a-2)(a+3)}{a-1} \\ 0 & 0 & \boxed{1} & | & a+3 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & 0 & | & -3 - \frac{4-a}{a-1} \\ 0 & \boxed{1} & 0 & | & \frac{4-a}{a-1} \\ 0 & 0 & \boxed{1} & | & a+3 \end{pmatrix} = \begin{pmatrix} \boxed{1} & 0 & 0 & | & \frac{2a+1}{1-a} \\ 0 & \boxed{1} & 0 & | & \frac{a-4}{1-a} \\ 0 & 0 & \boxed{1} & | & a+3 \end{pmatrix}$$

Hence the solution set of the associated system is $\left\{ \left(\frac{2a+1}{1-a}; \frac{a-4}{1-a}; a+3 \right) \right\}$.

The exceptional cases $a = 1$ and $a = 2$ must be studied directly.

If $a = 1$, A becomes ($R_3 \to R_3 - R_2$):

$$\overline{A} \longrightarrow \begin{pmatrix} \boxed{1} & 1 & 1 & | & 1 \\ 0 & 0 & -1 & | & -1 \\ 0 & 0 & -1 & | & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 1 & 1 & | & 1 \\ 0 & 0 & \boxed{-1} & | & -1 \\ 0 & 0 & 0 & | & \boxed{5} \end{pmatrix},$$

hence the system is not consistent, as $\varrho(A) = 2$ while $\varrho(\overline{A}) = 3$ (the last equation, $0 = 5$, has no solution).

If $a = 2$, $\overline{A}$ becomes (in reduced row echelon form)

$$\overline{A} \longrightarrow \begin{pmatrix} \boxed{1} & 1 & 1 & | & 2 \\ 0 & \boxed{1} & 0 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & 1 & | & 0 \\ 0 & \boxed{1} & 0 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{cases} x + z = 0 \\ y = 2 \end{cases}$$

therefore A and $\overline{A}$ have equal rank $\varrho(A) = \varrho(\overline{A}) = 2$, hence the system is consistent and it has $\infty^{3-2} = \infty^1$ solutions:

$$\{(-z;\, 2;\, z) \mid z \in \mathbb{R}\}\,.$$

Observe that, for $a = 2$, we have $\left(\frac{2a+1}{1-a};\, \frac{a-4}{1-a};\, a+3\right) = (-5;\, 2;\, 5)$ and, on the other hand

$$\begin{cases} -\frac{2a+1}{1-a} = a + 3 \\ \frac{a-4}{1-a} = 2 \end{cases} \quad \Longleftrightarrow \quad a = 2.$$

Hence, the generic solution we found for $a \neq 1 \wedge a \neq 2$ is coherent with that found for $a = 2$.