Teaching material. Systems of linear equations

Example. Consider the system

$$\begin{cases} abx + y + t = 1 \\ (b+1)y + t = a \\ (b+1)y + a(b-1)z + t = b \end{cases}$$

For each $(a; b) \in \mathbb{R}^2$, decide whether the system is consistent and find its "number" of solutions. Find, if possible, an $a \in \mathbb{R}$ such that the system is consistent for every $b \in \mathbb{R}$, and an $a \in \mathbb{R}$ such that the system is inconsistent for every $b \in \mathbb{R}$.

Solution The system has augmented matrix $\overline{A}$, where A is the coefficient matrix, and $\underline{v}$ is the constant column term:

$$\overline{A} := (A|\underline{v}) = \begin{pmatrix} ab & 1 & 0 & 1 & 1 \\ 0 & b+1 & 0 & 1 & a \\ 0 & b+1 & a(b-1) & 1 & b \end{pmatrix}$$

performing row operations ($R_3 \to R_3 - R_2$) we get a row echelon form

$$\overline{A} \longrightarrow \begin{pmatrix} ab & 1 & 0 & 1 & 1 \\ 0 & b+1 & 0 & 1 & a \\ 0 & 0 & a(b-1) & 0 & b-a \end{pmatrix}$$

Now, if $a \neq 0 \wedge b \neq -1 \wedge b \neq 0 \wedge b \neq 1$, A and $\overline{A}$ have equal rank $\varrho(A) = \varrho(\overline{A}) = 3$, hence the system is consistent and it has $\infty^{4-3} = \infty^1$ solutions, with t as free variable.

If $a = 0$ we get

$$\overline{A} \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & b+1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & b & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}$$

whose corresponding system is inconsistent for $b \neq 0$ (because of the last row) as well as for $b = 0$ (because of the second row). Thus for $a = 0$ the system is inconsistent for every $b \in \mathbb{R}$.

If $b = -1$ we get

$$\overline{A} \longrightarrow \begin{pmatrix} -a & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & a \\ 0 & 0 & -2a & 0 & -1-a \end{pmatrix} \longrightarrow \begin{pmatrix} -a & 1 & 0 & 1 & 1 \\ 0 & 0 & -2a & 0 & -1-a \\ 0 & 0 & 0 & 1 & a \end{pmatrix}.$$

If $b = -1 \wedge a \neq 0$ the matrix is in row echelon form with $\varrho(A) = \varrho(\overline{A}) = 3$, hence the corresponding system is consistent and it has $\infty^{4-3} = \infty^1$ solutions, with y as free variable.

If $b = -1 \wedge a = 0$ the matrix becomes

$$\overline{A} \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(no longer in row echelon form) which corresponds to an inconsistent system (because of the second row), coerently with what we found above.

If $b = 0$ we get

$$\overline{A} \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & a \\ 0 & 0 & -a & 0 & -a \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & a-1 \\ 0 & 0 & -a & 0 & -a \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -a & 0 & -a \\ 0 & 0 & 0 & 0 & a-1 \end{pmatrix}$$

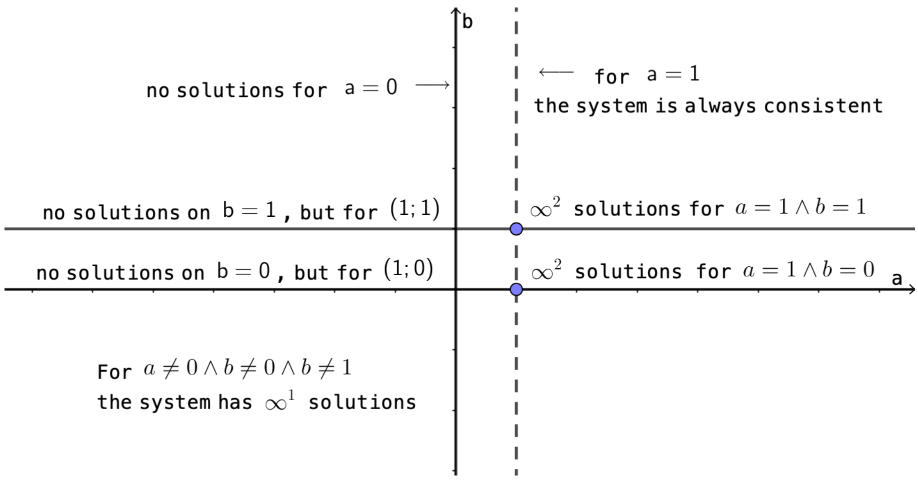
If $b = 0 \wedge a \neq 1$ the corresponding system is inconsistent (because of the last row).

If $b = 0 \wedge a = 1$ the matrix becomes
$$\overline{A} \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
which is in row echelon form with $\varrho(A) = \varrho(\overline{A}) = 2$, hence the corresponding system is consistent and it has $\infty^{4-2} = \infty^2$ solutions, with x and t as free variables.

If $b = 1$ we get
$$\overline{A} \longrightarrow \begin{pmatrix} a & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & a \\ 0 & 0 & 0 & 0 & 1-a \end{pmatrix}$$
If $b = 1 \wedge a \neq 1$ the corresponding system is inconsistent (because of the last raw).

If $b = 1 \wedge a = 1$ the matrix is in row echelon form with $\varrho(A) = \varrho(\overline{A}) = 2$, hence the corresponding system is consistent and it has $\infty^{4-2} = \infty^2$ solutions, with z and t as free variables.

We can graphically summarize the previous discussion as follows:



Thus for $a = 1$ the system is consistent for every $b \in \mathbb{R}$.