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Products of vectors and orthogonal projection of one vector over another

Let's talk about two types of products of two vectors, the scalar product and the vector product.

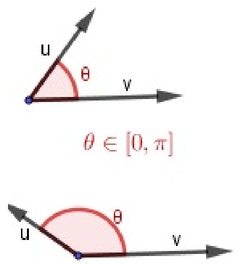
Scalar product of two vectors and orthogonal projection of one vector over another

Definition: The scalar product (or dot product) $u \cdot v$ of two vectors u and v is a number defined by

$$||u||||v||cos(\theta),$$

with $\theta = \hat{uv} \in [0, \pi]$.

- If u and v are parallel vectors, then $u \cdot v = ||u||||v||$;
- If u and v are antiparallel vectors, then $u \cdot v = -||u||||v||$;
- If u and v are two orthogonal vectors, then $u \cdot v = 0$.



Notice that $\hat{vv} = 0$ and this implies that the dot product of a vector a with itself is $v \cdot v = ||v||||v||$, which gives

$$||v|| = \sqrt{v \cdot v}.$$

In $\mathbb{R}^n$ we have the alternative definition of scalar product:

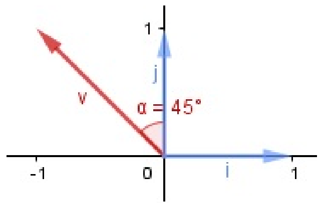
Definition: The dot product of two vectors $v = (v_1, v_2, \ldots, v_n), u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ is

$$v \cdot u = v_1u_1 + v_2u_2 + v_3u_3 + \cdots v_nu_n.$$

Example: On the Cartesian plane, consider the vectors $i = (1, 0), j = (0, 1)$ and $v(-1, 1)$.

On the one hand, we have $i \cdot j = (1, 0) \cdot (0, 1) = 0$ and $j \cdot v = (0, 1) \cdot (-1, 1) = 1 + 0 = 1$. On the other hand, we also have

$$i \cdot j = ||(1, 0)||||(0, 1)||cos(\frac{\pi}{2}) = 0$$



Also

$$j \cdot v = ||(0, 1)||||(-1, 1)||cos(\frac{\pi}{2}) = \sqrt{2} \times (\frac{\sqrt{2}}{2}) = 1.$$

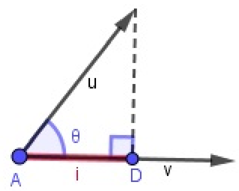
$$i \cdot v = ||(1, 0)||||(-1, 1)||cos(\tfrac{3\pi}{2}) = \sqrt{2} \times (-\tfrac{\sqrt{2}}{2}) = -1.$$

The dot product fulfills the following properties if u, v, and w are vectors and k is a real scalar:

1. $v \cdot u = u \cdot v$;
2. $v \cdot (u + w) = (v \cdot u) + (v \cdot w)$;
3. $v \cdot (ku + w) = k(v \cdot u) + (v \cdot w)$;
4. $k_1 v \cdot (k_2 u) = k_1 k_2 (v \cdot u)$.

An inner product is a generalization of the dot product, is any operator who checks the properties above.

One important use of dot products is in projections.



The orthogonal projection of u onto v is the length of the segment $[AD]$ shown in the figure beside, $||\vec{AD}||$.
The **vector projection** of u onto v is the vector $\vec{AD}$

Note that $|proj_v u| = ||v|||cos(\theta)|$ and therefore:

$$|proj_v u| = ||\overline{AD}|| = \frac{|u \cdot v|}{||v||} \quad \text{and} \quad proj_v u = \vec{AD} = \frac{|u \cdot v|}{||v||^2} v.$$

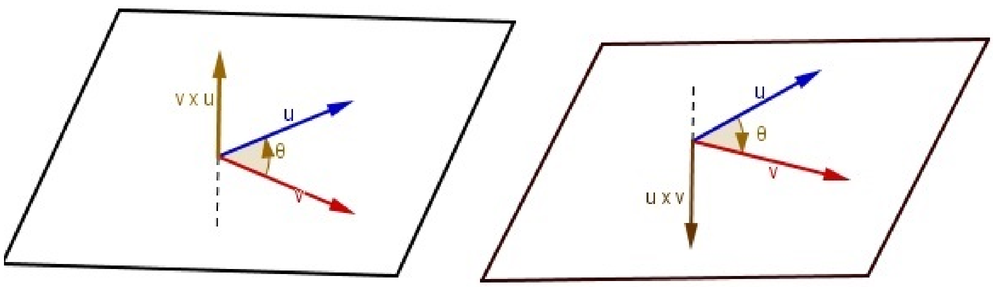
Vector Product (Cross Product)

The vector product of two vectors u and v is a vector $u \times v$ that is at right angles to both and is defined by

$$u \times v = ||u||||v||sin(\hat{uv})n, \quad \text{with} \quad ||n|| = 1 \quad \text{and} \quad u, v \perp n.$$

Specifically,

1. $u \times v$ is perpendicular to the vectors u and v;
2. $||u \times v|| = ||u|| \cdot ||v||| \sin((u, \hat{v}))|$;
3. $u \times v$ has sense determined by the right hand (follow with the fingers of the right hand, the rotation movement of the vector u to approach v and consider the direction of the thumb).



Notice that:

- $u \times v$ is orthogonal to the plane containing the vectors;
- $u \times v = 0$ when vectors u and v point in the same, or opposite, direction.

In the 3-dimensional Cartesian system, the vector product of vectors $u = (u_1, u_2, u_3)$ e $v = (v_1, v_2, v_3)$ is defined as

$$u \times v = (u_2v_3 - v_2u_3, v_1u_3 - u_1v_3, u_1v_2 - v_1u_2).$$

It is a vector perpendicular to the vectors u and v and can more easily be represented matrix-wise as:

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - v_2u_3)i - (u_1v_3 - v_1u_3)j + (u_1v_2 - v_1u_2)k.$$

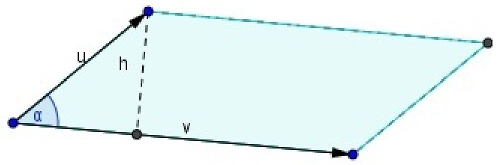
Example: $(1, 2, -1) \times (2, 0, 1) = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & 0 & 1 \end{vmatrix} = 2i - 3j - 4k = (2, -3, -4)$

Properties: Be the vectors $u, v, w \in \mathbb{R}^3$. We have

1. $u \times v \times w = u \times (v \times w)$ (associative);
2. $u \times v = -v \times u$ (anti-commutative);
3. $u \times v = 0 \Leftrightarrow u = 0 \vee v = 0 \vee (u \hat{,} v) = 0^{\text{o}} \vee (u \hat{,} v) = 180^{\text{o}}$.

Example: $(1, -2, 3) \times (-2, 4, -6) = \begin{vmatrix} i & j & k \\ 1 & -2 & 3 \\ -2 & 4 & -6 \end{vmatrix} = (0, 0, 0)$,
because the vectors $(1, -2, 3)$ e $(-2, 4, -6)$ are collinear.

The norm of the vector product $||u \times v|| = ||u|| \cdot ||v|| |\sin(\angle(u, v))|$ the area of the parallelogram determined by u and v.



In effect, according to the figure above, the area of the parallelogram is given by $A = ||v|| \cdot h$. Besides that, $||u|| \sin(\angle(u, v)) = h$.