LIMITS OF FUNCTIONS

$f: \mathbb{R} \to \mathbb{R}$

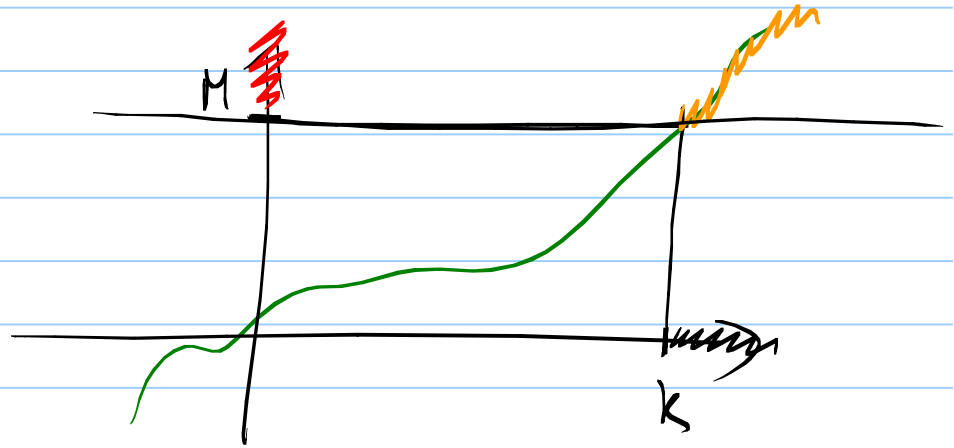
from a subset

$\lim_{x \to x_0} f(x)$ $\lim_{x \to +\infty} f(x)$ $\lim_{x \to -\infty} f(x)$

$\lim_{x \to x_0^+} f(x)$ $\lim_{x \to x_0^-} f(x)$

For each of them there are 4 possibilities

$$\lim_{x \to +\infty} f(x) = \begin{cases} \ell & \text{1} \quad \ell \in \mathbb{R} \text{ finite real number} \\ +\infty & \text{2} \\ -\infty & \text{3} \\ \text{IT DOESN'T EXIST} & \text{4} \end{cases}$$



2 $\forall M \in \mathbb{R} \quad \exists K \in \mathbb{R}$

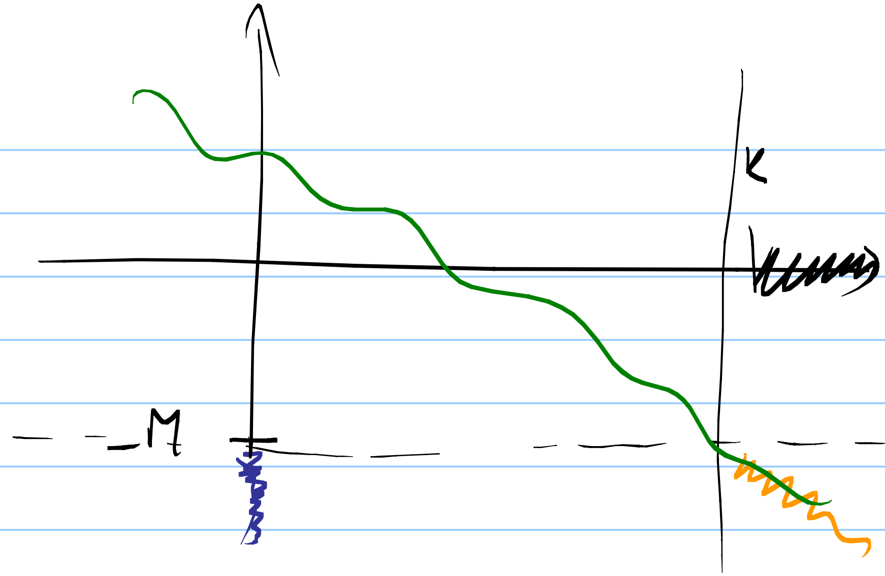
such that $\forall x \geq K$

we have $f(x) \geq M$

③ $\forall \; M \in \mathbb{R} \quad \exists \; K \in \mathbb{R}$

such that $\forall \; x \geq K$

$f(x) \leq -M$

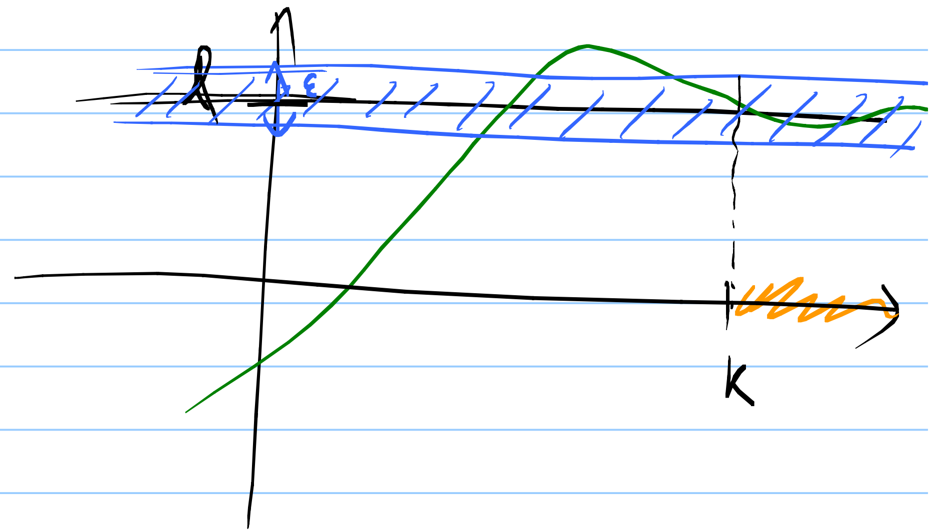


① $\forall \; \varepsilon > 0 \; \exists \; K \in \mathbb{R}$ s.t.

$\forall \; x \geq K \qquad |f(x) - \ell| < \varepsilon$

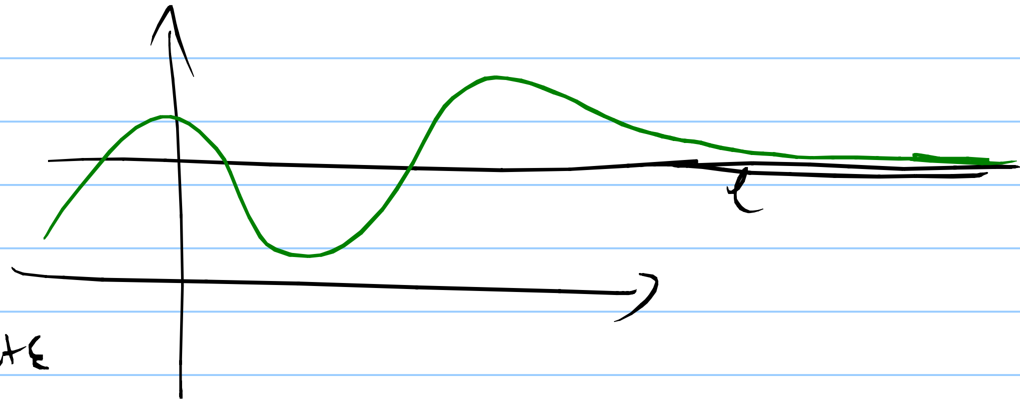
$\Updownarrow$

$\ell - \varepsilon \leq f(x) \leq \ell + \varepsilon \qquad (*)$



1bis $\lim_{x \to \infty} f(x) = \ell^+$ it means that we are converging "from above"

Def. is the same as before but the condition (*) becomes $\ell \le f(x) \le \ell + \varepsilon$



1ter Similar $\lim_{x \to \infty} f(x) = \ell^-$

((*) becomes $\ell - \varepsilon \le f(x) \le \ell$)

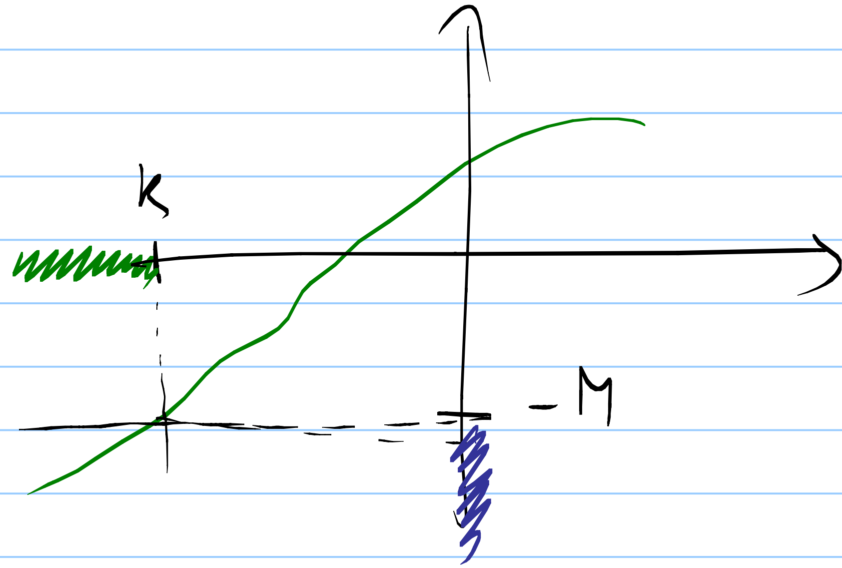
$$\lim_{x \to -\infty} f(x) = \begin{cases} l & ① \\ +\infty & ② \\ -\infty & ③ \\ \text{DOESN'T EXIST} & ④ \end{cases}$$

(similar to before)

③ $\forall M \in \mathbb{R} \ \exists K \in \mathbb{R}$

s.t. $\forall x \in \mathbb{R} \ x \leq K$

we have $f(x) \leq -M$



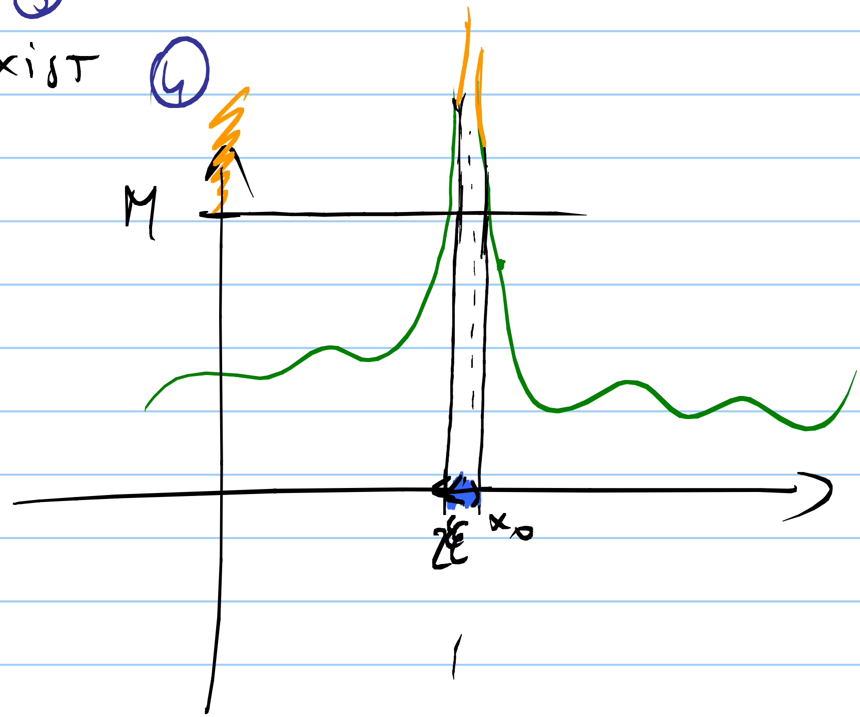
and so on...

$$\lim_{x \to x_0} f(x) = \begin{cases} \ell \in \mathbb{R} & (1) \\ +\infty & (2) \\ -\infty & (3) \\ \text{DOESN'T EXIST} & (4) \end{cases}$$

(2) $\forall M \quad \exists \, \varepsilon \quad$ s.t.

$\forall \quad x \in (x_0 - \varepsilon, x_0 + \varepsilon)$

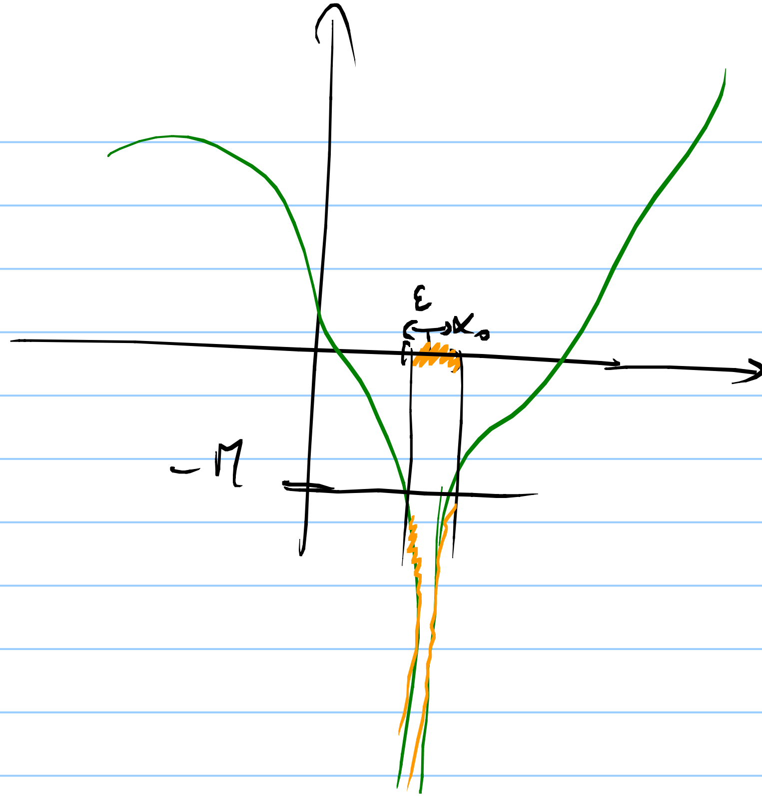
we have $\quad f(x) > M$



③ $\forall M \in \mathbb{R} \quad \exists \varepsilon > 0$ s.t.

$\forall \boxed{x \in (-\varepsilon + x_0, x_0 + \varepsilon)}$

we have $f(x) \leq -M$



③bis) $\lim_{x \to x_0^+} f(x) = +\infty$

Very similar to ③, the only thing that changes is $\boxed{x \in (x_0, x_0 + \varepsilon)}$

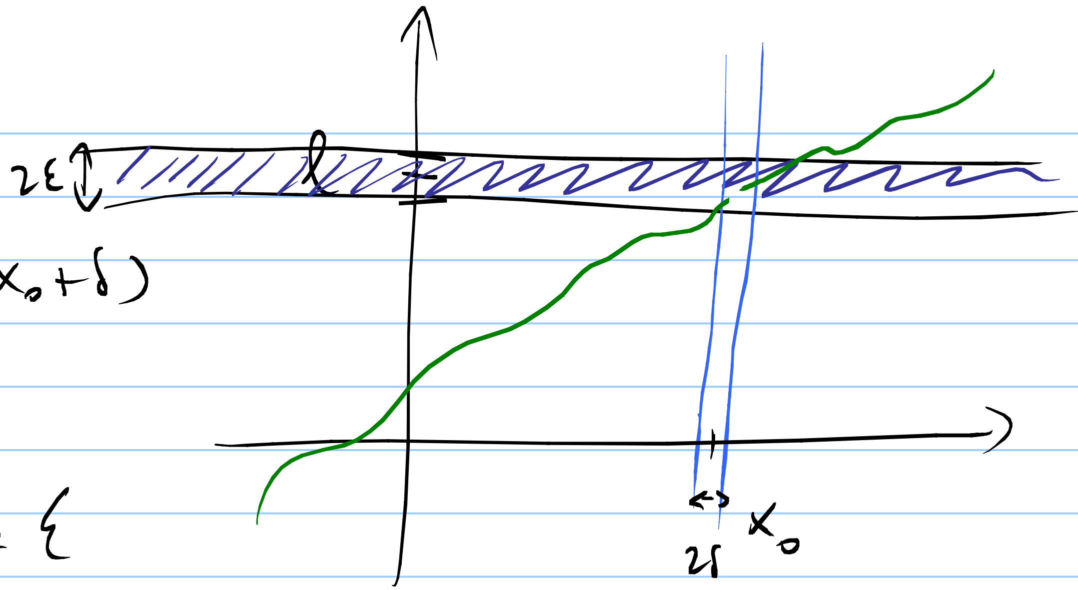
① $\forall \varepsilon > 0 \quad \exists \delta > 0$

s.t. $\forall x \in (x_0 - \delta, x_0 + \delta)$

we have

$$|f(x) - \ell| \leq \varepsilon$$

(that is, $\ell - \varepsilon \leq f(x) \leq \ell + \varepsilon$)



Why do we care about limits?

1) See next lecture ... in order to do derivatives

2) It can happen that we are faced with the problem of finding "the asymptotic behavior of a function".

Classical example $f(x) = \frac{\sin(x)}{x}$ | $f(x) = x^2$