Systems of linear equations

Test. Consider the following two systems of linear equations:

$$(I)\begin{cases} x+y+z+t=1 \\ x+y+5z+7t=5 \end{cases} \qquad (II)\begin{cases} x+y+z+t=1 \\ 2z+3t=2 \\ x+y-z-2t=-1 \end{cases}$$

Decide whether the following sentences are true or false. Provide full explanation of your answers.

(i) The two systems can't be equivalent as they do not have the same number of equations.

(ii) Every solution of the first system is a solution of the second, but not the converse.

(iii) The two systems are equivalent.

Solution

(i) The sentence is **false**. In general, two equivalent systems can have, each one, any number of equations. What is true is that equivalent consistent linear systems must have the same number of *maximum (linearly) independent* equations, that is to say that their matrices must have the same rank. We will see that the two given systems are actually equivalent, albeit they do not have the same number of equations.

(ii) The sentence is **false**. We will show that the two given systems are equivalent, hence every solution of the first system is a solution of the second, and the converse is also true.

(iii) The sentence is **true**. We will show the two systems are equivalent in two ways: working with the corresponding (augmented) matrices and also by direct inspection of the equations.

Using the matrices, we get

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 7 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 & 3 \\ 1 & 1 & -1 & -2 & -1 \end{pmatrix}$$

by row elementary operations the two matrices reduce to

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 6 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 & 2 \\ 0 & 0 & -2 & -3 & -2 \end{pmatrix}$$

and eventually to

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The last two matrices represent the same system of linear equations, hence the given two systems are equivalent.

The same result can be found as follows. Let us call E_1, E_2 the two linear equations of the first system, and E_3, E_4, E_5 those of the last one. Then we can observe: $E_1 = E_3$, $E_2 = E_3 + 2E_4$, $E_5 = E_3 - E_4$ (hence the equation E_5 is actually redundant in the second system and therefore can be discarded), and also $E_3 = E_1$ and $E_4 = \frac{1}{2}(E_2 - E_1)$. Thus every equation of the first system is a linear combination of equations from the second and viceversa, hence the two systems are equivalent.