Orthogonal sets and basis

Recall that the dot product:

$$u \cdot v = ||u||||v||cos(\theta), \quad \theta = \hat{uv} \in [0, \pi].$$

If, for example, $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$, we also have

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

An inner product of 2 vectores is a generalization of the dot product, it is a way to multiply vectors, whose product being a scalar. More precisely, an inner product, $/$, in a real vector space V is any operator that satisfies the following properties:

1. $(u + v)/w = (u/w) + (v/w), \forall u, v, w \in V$.
2. $(ku)/v = k(u/v)u/w + v/w, \forall u, v \in V, \forall k \in \mathbb{R}$.
3. $u/v = v/u, \forall u, v \in V$.
4. $u/u \leq 0$ if and only if $u = 0$.

The vector space V together with $/$ is called an **inner product space**.

An inner product space V induces a norm, that is, a notion of length of a vector:

$$||v|| = \sqrt{v/v}.$$

In particular, the usual inner product (scalar product) in the vector space $\mathbb{R}^2$ induces a norm

$$||v|| = \sqrt{v \cdot v}.$$

That is, if $v = (v_1, v_2)$, then $v \cdot v = v_1^2 + v_2^2$ and $||v|| = \sqrt{v_1^2 + v_2^2}$

Definition: The set $A = \{v_1, v_2, \ldots, v_k\} \in V \backslash \{0\}$ is an orthogonal set if each vector of A is orthogonal to each of the other vectors in the set, that is,

$$v_i \cdot v_j = 0 \quad \text{for} \quad i \neq j.$$

If, in addition, all vectors are of unit norm,$||v_i|| = 1$, then $\{v_1, v_2, \ldots, v_k\}$ is called an **orthonormal set**.

Examples:

1. The set $A = \{(1, -2), (4, 2)\} \subset \mathbb{R}^2$ is orthogonal but is not orthonormal.
   In fact $(1, -2) \cdot (4, 2) = 1 \times 4 - 2 \times 2 = 0$, but $||(1, -2)|| = \sqrt{(1, -2) \cdot (1, -2)} = \sqrt{5} \neq 1$.
2. The set $B = \{(1, 0, 0), (0, 1, 0)\} \subset \mathbb{R}^3$ is orthonormal.

Theorem: Any orthogonal set is linearly independent.

Definition: An orthonormal basis for an inner product space V with finite dimension is a basis for V whose vectors are orthonormal to each other and are all unit vectors.

The following are examples of orthonormal bases:

- The standard basis of $\mathbb{R}^2$, $A = \{(1,0),(0,1)\}$;
- The standard basis of $\mathbb{R}^3$, $B = \{(1,0,0),(0,1,0),(0,0,1)\}$;
- The basis $C = \left\{\left(\frac{3}{5},\frac{4}{5}\right),\left(\frac{4}{5},-\frac{3}{5}\right)\right\}$ of $\mathbb{R}^2$;

Let V an inner product space whith an inner product, " $\cdot$ ". The following properties give us the coordinates of V vectors with respect to orthogonal bases.

Theorem: If $B = \{w_1, w_2, \ldots, w_n\}$ is a *orthonormal basis* of V, then for any $v \in V$ we have

$$v = (v \cdot w_1)\, w_1 + (v \cdot w_2)\, w_2 + \cdots + (v \cdot w_n)\, w_n.$$

Example: $B = \left\{\left(\frac{3}{5},\frac{4}{5}\right),\left(\frac{4}{5},-\frac{3}{5}\right)\right\}$ is an orthonormal basis of $\mathbb{R}^2$. We can write any vector $v = (x,y)$ as

$$\begin{aligned}(x,y) &= \left((x,y)\cdot\left(\frac{3}{5},\frac{4}{5}\right)\right)\left(\frac{3}{5},\frac{4}{5}\right) + \left((x,y)\cdot\left(\frac{4}{5},-\frac{3}{5}\right)\right)\left(\frac{4}{5},-\frac{3}{5}\right)\\ &= \frac{3x+4y}{5}\left(\frac{3}{5},\frac{4}{5}\right) + \frac{4x-3y}{5}\left(\frac{4}{5},-\frac{3}{5}\right)\end{aligned}$$

That is,

$$v_B = \left(\frac{3x+4y}{5}, \frac{4x-3y}{5}\right).$$

Theorem: Let $A = \{w_1, w_2, \ldots, w_r\} \subset \mathbb{R}^n \backslash \{0\}$ such that $w_i \cdot w_j = 0$, for $i \neq j$ and $i, j \in \{1, 2, \ldots, r\}$. Then:

1. A is a basis of $\langle A \rangle$;
2. For any $v \in \langle A \rangle$, we have $v = k_1 w_1 + k_2 w_2 + \cdots + k_r w_r$, with
$$k_i = \frac{v \cdot w_i}{||w_i||^2}.$$

Example: $A = \{(1,-1,0),(0,0,2)\} \subset \mathbb{R}^3$ is an orthogonal basis, because $(1,-1,0)\cdot(0,0,2) = 0$.
The norms of vectors are $||(1,-1,0)|| = \sqrt{2}$ and $||(0,0,2)|| = 2$.
Besides that, $\langle A \rangle = \{(x,-x,z) : x, z \in \mathbb{R}\}$ and any $v = (x,-x,z) \in \langle A \rangle$ is such that

$$(x,-x,z) = \frac{(x,-x,z)\cdot(1,-1,0)}{||(1,-1,0)||^2}(1,-1,0) + \frac{(x,-x,z)\cdot(0,0,2)}{||(0,0,2)||^2}(0,0,2) = \frac{2x}{2}(1,-1,0) + \frac{2z}{4}(0,0,2).$$