Determine a basis for a subspace

- **Find one basis for the subspace $F = \{(x, y, x) \in \mathbb{R}^3 : y - 3x + 5z = 0\}$.**

<u>Remember</u>**:** A subset A of a vector space V is a basis of V if A is a linearly independent set and A spans V.

We have $y - 3x + 5z = 0 \Leftrightarrow y = 3x - 5z$. So, $(x, 3x - 5z, z)$ represents any vector of F.

Like $(x, 3x - 5z, z) = x(1,3,0) + z(0, -5,1)$, we conclude the vectors $(1,3,0)$ and $(0, -5,1)$ spans F.

Now, we must verify if $(1,3,0)$ and $(0, -5,1)$ are linearly independents.

$$c_1(1,3,0) + c_2(0, -5,1) = (0,0,0)$$

$$\begin{cases} c_1 & = & 0 \\ 3c_1 - 5c_2 & = & 0 \\ c_2 & = & 0 \end{cases} \Leftrightarrow \begin{cases} c_1 & = & 0 \\ 0 & = & 0 \\ c_2 & = & 0 \end{cases}$$

We conclude the vectors $(1,3,0)$ and $(0, -5,1)$ are linearly independents.

Thus, $\{(1,3,0), (0, -5,1)\}$ spans F and is a linearly independent set.

<u>Conclusion</u>: $\{(\mathbf{1}, \mathbf{3}, \mathbf{0}), (\mathbf{0}, -\mathbf{5}, \mathbf{1})\}$ **is a basis of F.**

We can say, $\boldsymbol{dim}(\boldsymbol{F}) = \mathbf{2}$.

- **Find one basis for the subspace $H = \left\{\begin{bmatrix} a + b & -3b \\ 2c - 4a & c \end{bmatrix} \in M(\mathbb{R})_{2\times 2}\right\}$.**

Notice that

$$\begin{bmatrix} a + b & -3b \\ 2c - 4a & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ -4a & 0 \end{bmatrix} + \begin{bmatrix} b & -3b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2c & c \end{bmatrix}$$
$$= a\begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$$

We conclude $\begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$ spans H.

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We must verify if $\begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$ are linearly independents:

$$c_1 \begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Multiplying by the scalar and adding the matrices we have,

$$\begin{bmatrix} c_1 + c_2 & -3c_2 \\ -4c_1 + 2c_3 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since the matrices are equal if their corresponding entries are equal, we have

$$\begin{cases} c_1 + c_2 & = & 0 \\ -3c_2 & = & 0 \\ -4c_1 + 2c_3 & = & 0 \\ c_3 & = & 0 \end{cases}$$

Solving the system, we obtain $c_1 = c_2 = c_3 = 0$. Thus, $\begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$ are linearly independents.

Thus, $\left\{\begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}\right\}$ spans H and is a linearly independent set.

Conclusion: $\left\{\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{-4} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \mathbf{-3} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \textbf{ and } \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}\right\}$ **is a basis of $\boldsymbol{H}$.**

We can say, $\boldsymbol{dim(H) = 3}$.

To think: **Find another bases for F and H subspaces!**

Remember that any basis of a subspace always has the same number of vectors.