Subsets that spans $\mathbb{R}^2$

- **The subset $\boldsymbol{A=\{(2,-8),(-1,4)\}}$ spans $\mathbb{R}^2$?**

Attend to the

> Definition: Let V a vector space. Consider $A=\{v_1,v_2,\dots,v_j\}$ a subset of V. ***A* spans *V*** if
> $$\forall u\in V\ \exists c_1,c_2,\dots,c_j\in\mathbb{R}: c_1v_1+c_2v_2+\cdots+c_jv_j=u$$

and applying it to our question, we have to check if

$$\forall(x,y)\in\mathbb{R}^2\ \exists c_1,c_2\in\mathbb{R}: c_1(2,-8)+c_2(-1,4)=(x,y)$$

Solving the system resulting from this expression:

$$\begin{cases}2c_1-c_2=x\\-8c_1+4c_2=y\end{cases}\Leftrightarrow\begin{cases}c_2=-x+2c_1\\-8c_1+4c_2=y\end{cases}$$
$$\Leftrightarrow\begin{cases}c_2=-x+2c_1\\-8c_1+4(-x+2c_1)=y\end{cases}$$
$$\Leftrightarrow\begin{cases}c_2=-x+2c_1\\-8c_1-4x+8c_1=y\end{cases}$$
$$\Leftrightarrow\begin{cases}c_2=-x+2c_1\\-4x=y\end{cases}$$

<u>Conclusion</u>: For $y\neq-4x$, the system doesn't have any solution, therefore ***A* does not spans $\mathbb{R}^2$.**

In this case, we can conclude that A spans the subset $\{(x,y)\in\mathbb{R}^2: y=-4x\}$.

- **The subset $\boldsymbol{B=\{(2,-10),(0,2),(4,-1)\}}$ spans $\mathbb{R}^2$?**

As in the previous case, we must check if

$$\forall(x,y)\in\mathbb{R}^2\ \exists c_1,c_2,c_3\in\mathbb{R}: c_1(2,10)+c_2(0,2)+c_3(4,-1)=(x,y)$$

Solving the system resulting from this expression:

$$\begin{cases}2c_1+4c_3=x\\-10c_1+2c_2-c_3=y\end{cases}\Leftrightarrow\begin{cases}c_1=\dfrac{x-4c_3}{2}\\-10c_1+2c_2-c_3=y\end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = \dfrac{x - 4c_3}{2} \\ -10\left(\dfrac{x - 4c_3}{2}\right) + 2c_2 - c_3 = y \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = \dfrac{x - 4c_3}{2} \\ -5x + 20c_3 + 2c_2 - c_3 = y \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = \dfrac{x - 4c_3}{2} \\ -5x + 19c_3 + 2c_2 = y \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = \dfrac{x - 4c_3}{2} \\ c_2 = \dfrac{5x - 19c_3 + y}{2} \end{cases}$$

Conclusion: For all $(x, y) \in \mathbb{R}^2$, the system has always a solution. Therefore $\boldsymbol{B}$ **spans** $\mathbb{R}^2$**.**

Alternatively, we can use the Gaussian elimination method to solve the system:

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & x \\ -10 & 2 & -1 & y \end{array}\right] \xrightarrow{L_2 \leftarrow 5L_1 + L_2} \left[\begin{array}{ccc|c} 2 & 0 & 4 & x \\ 0 & 2 & -19 & 5x + y \end{array}\right]$$

Observe that for all $x, y \in \mathbb{R}$ the system is always possible. Note that, considering that A is the matrix of the coefficients, $A|B$ the augmented matrix of the system and n the number of unknowns, we have

$rank(A) = 2;\ rank(A|B) = 2; n = 3$, this is, $rank(A) = rank(A|B) < n$.

As previously, we can conclude that $\boldsymbol{B}$ **spans** $\mathbb{R}^2$**.**

To think:

Note that the system has an infinite number of solutions.

Couldn't two vectors of *B* be enough to generate $\mathbb{R}^2$?