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Lines and planes in the Cartesian coordinate system

Lines in a Cartesian plane can be described algebraically by linear equations.

Lines in $\mathbb{R}^2$

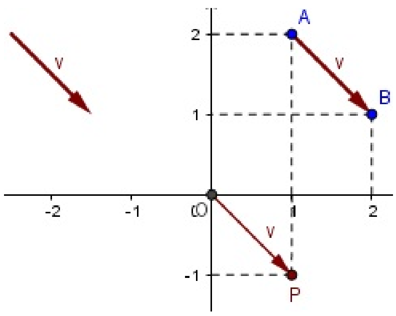
In the two-dimensional Cartesian referential, a point $P \in \mathbb{R}^2$ is determined by its distance each axis and by the quadrant in which it is located, that is, by its Cartesian coordinates (x, y).

Given a vector v (through its length and direction) and a point A, the sum $A + v$ corresponds to a point B, end of v when applied to A.

For example, let the vector v in the figure on the side and be $A = (1, 2) \in \mathbb{R}^2$. We observe that $A + v = B = (2, 1)$, or

$$\vec{v} = \vec{AB} = B - A = (1, -1).$$

Also note that $\vec{v} = \vec{OP}$ with $P = (1, -1)$ and $O = (0, 0)$.



In the Cartesian coordinate system, at each point P we associate the vector $\vec{OP} = P - O = P$ with the same coordinates of P.

A line in the Cartesian plane can be defined by:

- Two points on the line;
- A point on the line and its slope;
- A point and a straight line vector.

Given two fixed points on the plane, $A = (a_1, a_2)$ and $B = (b_1, b_2)$, the line AB has the direction of the vector

$$\vec{AB} = B - A = (b_1 - a_1, b_2 - a_2) = (v_1, v_2)$$

and has a slope

$$m = \frac{b_2 - a_2}{b_1 - a_1} = \frac{v_2}{v_1}.$$

The vector equation of the line AB is

$$(x, y) = (a_1, a_2) + k(v_1, v_2), \quad k \in \mathbb{R},$$

From the vector equation we deduce the Cartesian equation

$$\frac{x-a_1}{v_1}=\frac{y-a_2}{v_2}$$

or we can still transform into the reduced equation

$$y=\frac{v_2}{v_1}x+b.$$

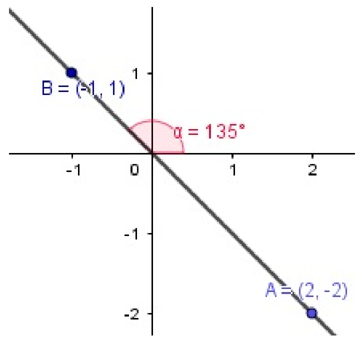
Example: The vector equation of the line AB, with $A=(2,-2)$ e $B=(-1,1)$, is given by

$$(x,y)=(-1,1)+k(3,-3),\quad k\in\mathbb{R}.$$

By eliminating the parameter k, we obtain the reduced equation

$$y=-x$$

This line intersects the Oy axis at the origin and has a slope of -1.



Lines and planes in $\mathbb{R}^3$

We represent elements in space using the three-dimensional Cartesian framework $Oxyz$

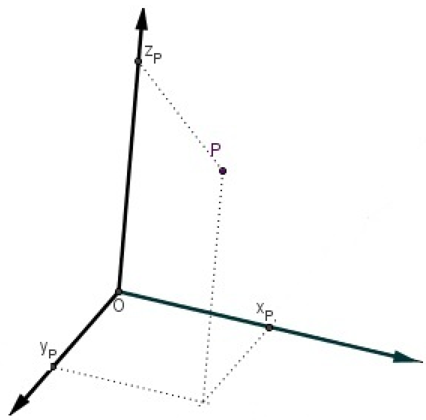
For example, to say that a point $P\in\mathbb{R}^3$ has Cartesian coordinates $(2,3,2)$, means to say that P is written as the linear combination

$$P=2i+3j+2k$$

of unit vectors

$$i=(1,0,0),j=(0,1,0),k=(0,0,1),$$

oriented according to the reference axes, as the image suggests.



Like the vector equation of the line in the plane, also if $A=(a_1,a_2,a_3),B=(b_1,b_2,b_3)\in\mathbb{R}^3$, the line AB is the locus of the points $P=(x,y,z)$ to the space, such that

$$P=A+k\vec{AB},\quad \text{to any}\quad k\in\mathbb{R}.$$

From the vector equation of the line AB,

$$AB:(x,y,z)=(a_1,a_2,a_3)+k(b_1-a_1,b_2-a_2,b_3-a_3),\quad k\in\mathbb{R}.$$

If $\vec{AB}(b_1-a_1,b_2-a_2,b_3-a_3)=(v_1,v_2,v_3)$ is such that $v_1,v_2,v_3\neq 0$, eliminating the parameter k, we obtain the Cartesian equation:

$$AB:\frac{x-a_1}{v_1}=\frac{y-a_2}{v_2}=\frac{z-a_3}{v_3}.$$

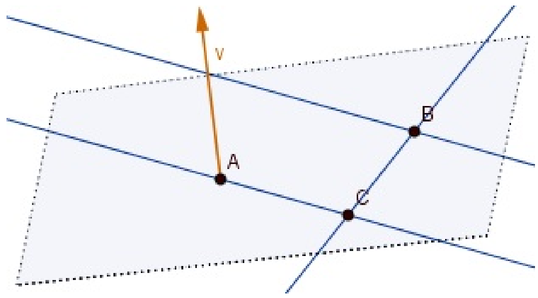
If, for example $v_1,v_3\neq 0$, but $v_2=0$ we have

$$AB:\frac{x-a_1}{v_1}=\frac{z-a_3}{v_3}\wedge y=a_2.$$

Example: The equation $\frac{x+1}{2} = \frac{y-1}{3} = z$ represents the line that contains $A = (-1, 1, 0)$ and has the direction of $v = (2, 3, 1)$.

A plane in space can be defined by:

- Three non-collinear points;
- Two cross lines;
- Two parallel lines;
- A point and a vector perpendicular to the plane.



Definition: The plan containing A and is perpendicular to v is the locus of the points $P = (x, y, z)$, such that the scalar product $\vec{AP} \cdot v$ is zero,

$$\vec{AP} \cdot v = 0.$$

Example: The plan containing $A = (1, 0, 2)$ and is perpendicular to $v = (-1, 3, 2)$ has the equation $(x-1, y, z-2) \cdot (-1, 3, 2) = 0 \Leftrightarrow -x + 1 + 3y + 2z - 4 = 0 \Leftrightarrow -x + 3y + 2z - 3 = 0$.

Thus, we obtained the general equation of the plane, that is, a linear equation in the variables x, y, z,

$$Ax + By + Cz + D = 0.$$

Example: Let's determine the plane containing the points $A = (1, 1, -1), B = (2, 1, 0)$ and $C = (3, 0, 1)$. We start by calculating a vector orthogonal to the plane ABC, considering

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 2 & -1 & 2 \end{vmatrix} = (1, 0, -1)$$

Then, the equation of ABC is $(x-1, y-1, z+1) \cdot (1, 0, -1) = 0 \Leftrightarrow x - z - 2 = 0$.