LINEAR PROGRAMMING (LP): Formulation

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"The development of linear programming (LP) has been ranked among the most important scientific advances of the mid-20th century" (Hillier & Lieberman, 2010)

Today, the LP is a standard tool that enables to save many thousands or millions of euros for many companies or businesses in the various industrialized countries of the world.

Briefly, the most common type of application of LP involves selecting the optimum level of certain activities that compete for scarce resources that are necessary to perform those activities.

The large diversity of situations in which the LP applies:

- allocation of production facilities to products
- allocation of national resources to domestic needs
- portfolio selection
- selection of shipping patterns
- agricultural planning
- design of radiation therapy, ...

LP uses a mathematical model to describe the problem to solve:

- The adjective linear means that all the mathematical functions in this model are required to be linear functions.
- The word programming is essentially a synonym for planning.

Thus, the LP involves the planning of activities to obtain an optimal solution among all feasible alternatives.

So, a LP problem includes the following components:

- Decision variables which are the quantities to be determined.
- Constraints which define the admissible (and not admissible) values for the decision variables, being defined by the various resources available and the technical limitations of the problem; the optimal solution will have to respect them.
- Objective function defines the evaluation criteria for the various admissible solutions which should be minimized or maximized.

Consider the LP problem to allocate m resources to n activities:

- The amount available of each resource i is limited by b_i $(i = 1, \ldots, m)$, being necessary a careful allocation of resources to activities j. Set a_{ij} the amount of resource i consumed by each unit of activity j.
- To achieve the best possible value of the overall measure of performance Z, it is necessary to chose the levels of the activities x_j for $j = 1, \ldots, n$. Set c_j the increase in Z that would result from each unit increase in level of activity j.

Thus, the standard formulation of LP model is the following:

$$\text{Max } Z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \ i = 1, \ldots, m$$

$$x_j \geq 0, \ j = 1, \ldots, n.$$

Components of this standard formulation of LP model:

- **Objective function**: $Z = \sum_{j=1}^{n} c_j x_j$ which should be maximized.
- **Functional constraints** (or structural constraints): $\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \ i = 1, \ldots, m$. The restrictions normally are referred as constraints.
- **Nonnegativity constraints** (or nonnegativity conditions): $x_j \geq 0, \ j = 1, \ldots, n$.

Assumptions of LP:

- **Proportionality**: it is an assumption about both the objective function and the functional constraints
- **Additivity**: every function in a LP model (the objective function or the function on the left-hand side of a functional constraint) is the sum of the individual contributions of the respective activities.
- **Divisibility**: the decision variables in a LP model are allowed to have any values, including noninteger values, that satisfy the functional and nonnegativity constraints.
- **Certainty**: the value assigned to each parameter of a LP model is assumed to be a known constant, being important to conduct the sensitivity analysis after an optimum solution is found.

Reference

Hillier F. S., & Lieberman G.R. (2010). Introduction to operations research (9th ed.). New York: McGraw-Hill.