Evaluate $\int_1^2 \frac{x^2-1}{x(x^2+1)} dx$

* All the conditions for Fundamental theorem of calculus are met.

Since, $m<n$, the partial fractions should be obtained.

For, $I(x) = \int \frac{x^2-1}{x(x^2+1)} dx$, the partial fractions are,

A.c.i

$$\frac{x^2-1}{x(x^2+1)} = \frac{A}{x} + \frac{Cx+D}{x^2+1}$$

$$\Leftrightarrow \quad x^2-1 = A(x^2+1) + (Cx+D)x$$

$$\Leftrightarrow \quad x^2-1 = Ax^2 + A + Cx^2 + Dx$$

$$\Leftrightarrow \quad x^2-1 = (A+C)x^2 + Dx + A$$

$$\Leftrightarrow \quad x^2 + 0.x - 1 = (A+C)x^2 + Dx + A$$

comparing coefficients of left and right hand side.

$$\begin{cases} A = -1 \\ D = 0 \\ A+C = 1 \end{cases} \Leftrightarrow \begin{cases} A = -1 \\ D = 0 \\ -1+C = 1 \end{cases} \Leftrightarrow \begin{cases} A = -1 \\ D = 0 \\ C = 2 \end{cases}$$

So, $\dfrac{x^2-1}{x(x^2+1)} = -\dfrac{1}{x} + \dfrac{2x}{x^2+1}$

A.C.I

$$I(x) = \int -\frac{1}{x}\,dx + \int \frac{2x}{x^2+1}\,dx$$

$$= -\ln|x| + \ln|x^2+1| + C$$

Now,

$$\int_1^2 \frac{x^2-1}{x(x^2+1)}\,dx = \Big[I(x)\Big]_1^2$$

$$= \Big[-\ln|x| + \ln|x^2+1|\Big]_1^2$$

$$= \left(-\ln|2| + \ln|5|\right) - \left(\ln|1| + \ln|2|\right)$$

$$= -\ln(2) + \ln(5) - \ln(2)$$

$$= \ln(5) - 2\ln(2)$$

$$= \ln(5) - \ln(2^2)$$

$$= \ln(5) - \ln(4)$$

$$= \ln\left(\frac{5}{4}\right)$$