Subspace spanned by a subset of vectors

- **Which is the subspace spanned by the subset $A = \{(-6, 4),\ (9, -6)\}$ of $\mathbb{R}^2$?**

Multiply each vector by a scalar and sum the resultants vectors. What kind of vectors do we get?

For example:

$$2 \times (-6{,}4) + 1 \times (9, -6) = (-3, 2)$$
$$-3 \times (-6{,}4) + 0 \times (9, -6) = (18, -12)$$
$$\frac{1}{2} \times (-6{,}4) - \frac{5}{6} \times (9, -6) = \left(-\frac{21}{2}, 7\right)$$
$$\sqrt{3} \times (-6{,}4) + \frac{1}{3} \times (9, -6) = \left(-6\sqrt{3} + 3, 4\sqrt{3} - 2\right)$$

We say that the vectors $(-3, 2), (18, -12), \left(-\frac{21}{2}, 7\right)$ and $\left(-6\sqrt{3} + 3, 4\sqrt{3} - 2\right)$ belong to the subspace spanned by A [denoted by $< A >$].

How do you can meet all vectors of the $< A >$?

> Let V a vector space. Consider $A = \{v_1, v_2, \dots, v_j\}$ a subset of V and $c_1, c_2, \dots, c_j \in \mathbb{R}$. We can meet all vectors of the $< A >$ if we determine all vectors resulting from the linear combination of the elements of A, this is
>
> $$c_1 v_1 + c_2 v_2 + \cdots + c_j v_j, \forall\, c_1, c_2, \dots, c_j \in \mathbb{R}$$

Attend to this, and considering $\alpha, \beta \in \mathbb{R}$ we have:

$$\alpha(-6{,}4) + \beta(9, -6) = (-6\alpha + 9\beta, 4\alpha - 6\beta)$$

For each achievement of α and β, we have a vector belong to the $< A >$, so $(-6\alpha + 9\beta, 4\alpha - 6\beta)$ represents a general vector of the subspace spanned by A. Analysing the vector, we can see that their coordinates depend on each other. For determining the relationship between its coordinates we can consider

$(-6\alpha + 9\beta, 4\alpha - 6\beta) = (x, y)$ and solve the resultant system.

$$\begin{cases} -6\alpha + 9\beta & = & x \\ 4\alpha - 6\beta & = & y \end{cases} \Leftrightarrow \begin{cases} -6\alpha + 9\beta & = & x \\ \alpha & = & \dfrac{6\beta + y}{4} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} -6\dfrac{6\beta+y}{4}+9\beta = x \\ \alpha = \dfrac{6\beta+y}{4} \end{cases} \Leftrightarrow \begin{cases} -6\dfrac{6\beta+y}{4}+9\beta = x \\ \alpha = \dfrac{6\beta+y}{4} \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{-36\beta-6y}{4}+\dfrac{36\beta}{4} = x \\ \alpha = \dfrac{6\beta+y}{4} \end{cases} \Leftrightarrow \begin{cases} -\dfrac{3}{2}y = x \\ \alpha = \dfrac{6\beta+y}{4} \end{cases}$$

$$\Leftrightarrow \begin{cases} y = -\dfrac{2}{3}x \\ \alpha = \dfrac{6\beta+y}{4} \end{cases}$$

Conclusion:

We say $< \boldsymbol{A} >= \left\{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^2 : \boldsymbol{y} = -\frac{2}{3}\boldsymbol{x}\right\}$ or $\boldsymbol{A}$ **spans** $\left\{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^2 : \boldsymbol{y} = -\frac{2}{3}\boldsymbol{x}\right\}$.

Geometrically the subspace is a line that passes through the origin.
The vectors $v_1 = (-6,4)$ and $v_2 = (9,-6)$ are collinear, so all linear combination of this vectors give rise to a vector contained on the same line, this is $y = -\frac{2}{3}x$.