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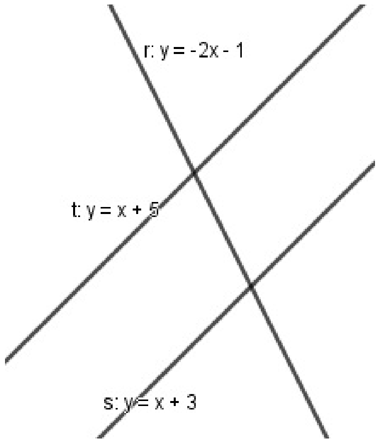
Relative position of straight lines and planes

Relative position of straight lines in the plane

Two straight lines on the plane can be secant, parallel or coincidental.

Let $r : y = ax + b$ and $s : y = cx + d$. Notice that:

- If $a = b$ and $b \neq d$ then r and s are parallel;
- If $a = b$ and $b = d$ then r and s coincidental
- If $a \neq b$ then r and s are secant;

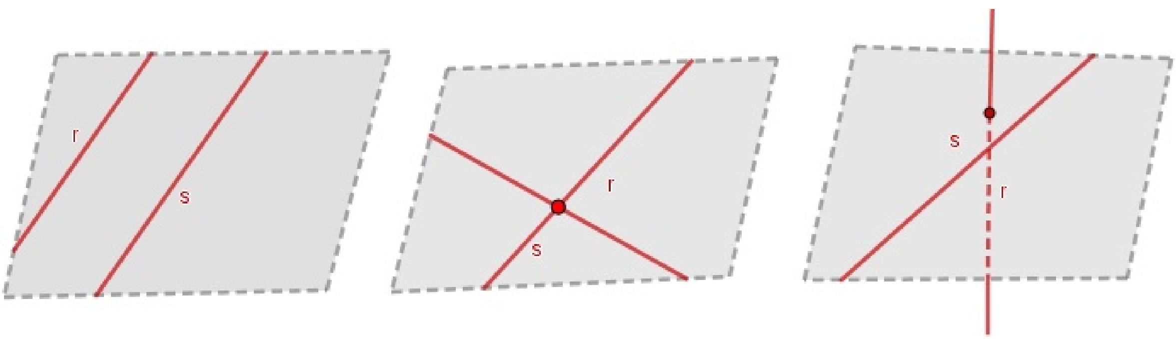


Relative position of straight lines in $\mathbb{R}^3$

In space one more case is added to the cases in the plane, if the lines AB and CD don't lie in the same plane.

So, for the relative position of two straight lines in space, the following cases are possible:

1. Lines lie in one plane and have no common points (parallel lines).
2. Lines lie in the same plane and have a common point (intersecting lines).
3. Lines do not lie in any plane (skew lines).



Two lines belong to the same plane if they are parallel lines or have a common point. Otherwise, they are skew lines.

Let be the vectorial equations of the lines r and s, respectively:

$r : (x, y, z) = (a_1, a_2, a_3) + k(v_1, v_2, v_3), \quad k \in \mathbb{R}$ and $s : (x, y, z) = (b_1, b_2, b_3) + t(u_1, u_2, u_3) \quad t \in \mathbb{R}$.

Note that:

1. If $v = (v_1, v_2, v_3)$ and $u = (u_1, u_2, u_3)$ are collinear vectors then r and s are parallel lines.
2. If $v = (v_1, v_2, v_3)$ and $u = (u_1, u_2, u_3)$ are not collinear vectors and $r \cap s \neq \emptyset$, then r and s are intersecting lines.
3. If $v = (v_1, v_2, v_3)$ and $u = (u_1, u_2, u_3)$ are not collinear vectors and $r \cap s \neq \emptyset$, then r and s are not not lines that lie in any plane.

In $\mathbb{R}^3$, we consider the line $r : (x, y, z) = (a_1, a_2, a_3) + k(v_1, v_2, v_3), k \in \mathbb{R}$ whose direction is that of the vector $v = (v_1, v_2, v_3)$ and the plane $\pi : Ax + By + C = 0$ orthogonal to the vector $n = (A, B, C)$. Notice that r is either parallel to a plane $\pi : Ax + By + C = 0$ or intersects it in a single point. Specifically:

1. If $n \cdot v = 0$, then r is parallel to π;
2. If $n \cdot v \neq 0$, then r intersects π in a single point

Even more, if r is parallel to π we have that $r \subset \pi$ (if all points on the r belong to π) or $r \cap \pi = \emptyset$

Relative position of planes

Let two planes, π_1 and π_2, be given by their general equations:

$$\pi_1 : A_1x + B_1y + C_1z + D_1 = 0 \quad \text{and} \quad \pi_2 : A_2x + B_2y + C_2z + D_2 = 0$$

Consider the system of two linear equations:

$$S : \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

- If the system S is inconsistent then the planes are parallel, and so the coordinates of the normal vectors $n_1 = (A_1, B_1, C_1)$ and $n_2 = (A_2, B_2, C_2)$ are proportional, that is, $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2} \neq \dfrac{D_1}{D_2}$.
- If system S is consistent and the equations are proportional to each other, then π_1 is just the same plane as π_2, that is, $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2} = \dfrac{D_1}{D_2}$.
- If system S is consistent, and the rank of the coefficient matrix equals 2, then π_1 and π_2 are intersecting planes.

  The locus of these distinct intersecting planes is exactly one line r whose direction is given by $v = n_1 \times n_2$ (cross product of vectors n_1 and n_2 that are orthogonal to the planes π_1 and π_2, respectively), according to the image beside.