Systems of linear equations

Test. Let $A\underline{x} = \underline{b}$ be a system of linear equations, where A is a square matrix of order n with coefficients in a field $\mathbb{K}$.

Decide whether the following sentences are true or false. Provide full explanation of your answers.

(i) If $\det A \neq 0$, then the given system is equivalent to one with same variables but with the identity matrix as coefficient matrix.

(ii) If $\det A = 0$, then the system is not consistent.

(iii) If $\det A \neq 0$, then the system has a unique solution.

(iv) If $\det A \neq 0$, then $\underline{x} = A^{-1}\underline{b}$ is a solution of the system.

Solution

(i) The sentence is **true**. As $\det A \neq 0$, then A has matrix inverse A^{-1} and by left multiplication by it we get the equivalent system

$$A^{-1}(A\underline{x}) = A^{-1}\underline{b} \iff (A^{-1}A)\underline{x} = A^{-1}\underline{b} \iff I\underline{x} = A^{-1}\underline{b}.$$

(ii) The sentence is **false in general**. If $\det A = 0$ we can only say that the rank of A is not full. In this case the system can either have infinitely many solutions, as for

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \iff \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1+h \\ h \end{pmatrix} \text{ for each } h \in \mathbb{K},$$

or no solution at all, as for

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ (no solution)} .$$

(iii) The sentence is **true**. If $\det A \neq 0$, then the rank of the square matrix A is equal to the number of its (rows and) columns, which in turns is equal to the number of variables; therefore the system has a unique solution by the Rouché-Capelli Theorem.

One can also argue, directly, that if $\underline{x}^*$ is any solution of the given system, that is $A\underline{x}^* = \underline{b}$, then by left multiplication by A^{-1} one finds

$$A^{-1}(A\underline{x}^*) = A^{-1}\underline{b} \iff (A^{-1}A)\underline{x}^* = A^{-1}\underline{b} \iff I\underline{x}^* = A^{-1}\underline{b} \iff \underline{x}^* = A^{-1}\underline{b},$$

hence the system, if it has solution, has a unique solution, on the other hand one can easily check that $A^{-1}\underline{b}$ is a solution.

(iv) The sentence is **true**. As $\det A \neq 0$, then A has matrix inverse A^{-1} and by substitution we find

$$A(A^{-1}\underline{b}) = (AA^{-1})\underline{b} = I\underline{b} = \underline{b},$$

hence $\underline{x} := A^{-1}\underline{b}$ is a solution of the system.