Change of Basis

In a vector space, the coordinates of a vector is always with respect to a basis and if we omit the basis, we naturally assume it to be the standard basis.

Without loss of generality, consider $A = \{v_1, v_2, v_3\}$ and $B = \{u_1, u_2, u_3\}$ two bases of three-dimensional space $\mathbb{R}^3$. For all $v \in V$, $v_A = (k_1, k_2, k_3)$ means that $v = k_1v_1 + k_2v_2 + k_3v_3$ and $v_B = (t_1, t_2, t_3)$ means that $v = t_1u_1 + t_2u_2 + t_3u_3$.

In particular, we can write the vectors u_1, u_2, u_3 of B in base A as follows:

$$\begin{cases} u_1 &= a_{11}v_1 + a_{21}v_2 + a_{31}v_3 \\ u_2 &= a_{12}v_1 + a_{22}v_2 + a_{32}v_3 \\ u_3 &= a_{13}v_1 + a_{23}v_2 + a_{33}v_3 \end{cases}$$

Then,

$$t_1u_1 + t_2u_2 + t_3u_3 = t_1(a_{11}v_1 + a_{21}v_2 + a_{31}v_3) + t_2(a_{12}v_1 + a_{22}v_2 + a_{32}v_3) + t_3(a_{13}v_1 + a_{23}v_2 + a_{33}v_3)$$

Associating the terms in v_i, we have:

$$t_1u_1 + t_2u_2 + t_3u_3 = (t_1a_{11} + t_2a_{12} + t_3a_{13})v_1 + (t_1a_{21} + t_2a_{22} + t_3a_{23})v_2 + (t_1a_{31} + t_2a_{32} + t_3a_{33})v_3,$$

As the coordinates in relation to a base are unique, we have

$$t_1a_{11} + t_2a_{12} + t_3a_{13} = k_1, t_1a_{21} + t_2a_{22} + t_3a_{23} = k_2 \quad \text{and} \quad t_1a_{31} + t_2a_{32} + t_3a_{33} = k_3.$$

That is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

In short, we can write

$$P_A^B \cdot v_A = v_B,$$

where P_A^B is called the **change matrix from B to base A**.

In particular, if $B = \{v_1, \ldots, v_n\}$ is a basis of a vector space V and the matrix whose columns are the vectors of B,

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix},$$

is a square matrix, then its determinant is nonzero.

Remember that:

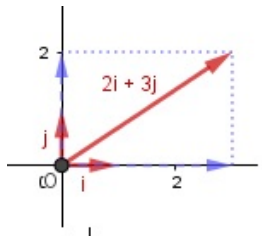
- The standard basis of $\mathbb{R}^2$ is $\{(1, 0), (0, 1)\}$;
- The standard basis of $\mathbb{R}^3$ is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

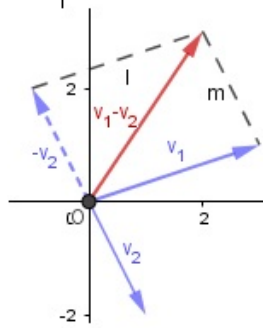
For example, in $\mathbb{R}^2$, $v = (2,3)$ means

$v = 2(1,0) + 3(0,1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ or

$v = 1(3,1) - 1(1,-2) = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or

$v = -\frac{4}{3}(1,-1) + \frac{5}{3}(2,1) = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{4}{3} \\ \frac{5}{3} \end{bmatrix}$.





So, the coordinates of v with respect to:

- the standard basis are $v = (2,3)$;
- the basis $A = \{(3,1),(1,-2)\}$ are $v_A = (1,-1)$;
- the basis $B = \{(1,-1),(2,1)\}$ are $v_B = \left(-\frac{4}{3}, \frac{5}{3}\right)$.

Notice that

$$v_A = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \cdot v \quad \text{and} \quad v_B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}^{-1} \cdot v.$$

Besides that,

$$\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{4}{3} \\ \frac{5}{3} \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{4}{3} \\ \frac{5}{3} \end{bmatrix}.$$

That is,

$$v_A = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \cdot v_B.$$

So, if A and B are two bases of a n dimentional vector space and the matrices $A = [a_{i,j}]_{n\times n}$ and $B = [b_{i,j}]_{n\times n}$, whose columns are the vectors of bases A and B (respectively) are square matrices, then the coordinates of any vector $v \in V$ in bases A and B are related as follows:

$$v_A = A^{-1} \cdot B \cdot v_B \quad \text{and} \quad v_B = B^{-1} \cdot A \cdot v_A$$

The product $A^{-1} \cdot B$ corresponds to the change matrix from B to base A, that is:

$$P_A^B = A^{-1} \cdot B.$$

We still have

Properties: If A and B are basis of a V vector space of n dimension, then:

1. $P_A^B = (P_B^A)^{-1}$.
2. Given $v \in V$, we have $[v]_A = P_A^B \cdot [v]_B$;
3. Given $v \in V$, we have $[v]_B = (P_A^B)^{-1} \cdot [v]_A$;
4. $P_C^B = P_A^B \cdot P_C^A$.