Dimension theorem

Theorem: Let $T: U \to V$ be a linear transformation between the vector spaces of finite dimension U and V, then:

$$dim(U) = dim(ker(T)) + dim(range(T))$$

1. Let $\boldsymbol{T}: \mathbb{R}^4 \to \mathbb{R}^6$ be a linear trasformation.

a) Knowing that $\boldsymbol{dim\big(ker(T)\big) = 2}$, determine the dimension of the $\boldsymbol{range(T)}$.

Applying the dimension theorem:

$$dim(\mathbb{R}^4) = dim(ker(T)) + dim(range(T))$$

Then,

$$4 = 2 + dim\big(range(T)\big) \Leftrightarrow \boldsymbol{dim\big(range(T)\big) = 2}$$

b) Knowing that $\boldsymbol{dim\big(range(T)\big) = 3}$, determine the dimension of the $\boldsymbol{ker(T)}$.

Applying again the dimension theorem:

$$dim(\mathbb{R}^4) = dim(ker(T)) + dim(range(T))$$

Then,

$$4 = dim(ker(T) + 3 \Leftrightarrow \boldsymbol{dim\big(ker(T)\big) = 1}$$

2. Verify the dimension theorem for the linear transformation $\boldsymbol{T}: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\boldsymbol{T(x, y) = (x, x + y, y)}$.

Let us first determine the **kernel** of the transformation T. By definition we have:

$$ker(T) = \{(x, y) \in \mathbb{R}^2 : T(x, y) = (0,0,0)\}$$

Then,

$$T(x, y) = (0,0,0) \Leftrightarrow (x, x + y, y) = (0,0,0)$$

$$\Leftrightarrow \begin{cases} x & = & 0 \\ x + y & = & 0 \\ y & = & 0 \end{cases} \Leftrightarrow \begin{cases} x & = & 0 \\ 0 & = & 0 \\ y & = & 0 \end{cases}$$

Therefore,

$$ker(T) = \{(0,0)\}$$

As the $ker(T) = \{(0,0)\}$, then the $dim(ker(T)) = 0$.

Let us now determine the **range** of the transformation T:

$$range(T) = \{(a,b,c) \in \mathbb{R}^3 : T(x,y) = (a,b,c) \; with \; (x,y) \in \mathbb{R}^2\}$$

We have:

$$T(x,y) = (a,b,c) \Leftrightarrow (x, x+y, y) = (a,b,c) \Leftrightarrow \begin{cases} x & = & a \\ x+y & = & b \\ y & = & c \end{cases}$$

The matrix of the system is: $\left[\begin{array}{cc|c} 1 & 0 & a \\ 1 & 1 & b \\ 0 & 1 & c \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & -a+b \\ 0 & 1 & c \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & -a+b \\ 0 & 0 & a-b+c \end{array}\right]$

$$L_2 \longleftarrow -L_1 + L_2 \qquad L_3 \longleftarrow -L_2 + L_3$$

For the system to be possible:

$$a - b + c = 0 \Leftrightarrow c = b - a$$

Then,

$$range(T) = \{(a,b,c) \in \mathbb{R}^3 : c = b - a\}$$

Let's determine a basis for the $range(T)$:

$$\begin{aligned} (a,b,b-a) &= (a,0,-a) + (0,b,b) \\ &= a(1,0,-1) + b(0,1,1) \end{aligned}$$

The vectors $(1,0,-1)$ and $(0,1,1)$ generate the $range(T)$. So, let's verify if they are linearly independent:

$$c_1(1,0,-1) + c_2(0,1,1) = (0,0,0)$$

$$\Leftrightarrow \begin{cases} c_1 & = & 0 \\ c_2 & = & 0 \\ -c_1 + c_2 & = & 0 \end{cases} \Leftrightarrow \begin{cases} c_1 & = & 0 \\ c_2 & = & 0 \\ 0 & = & 0 \end{cases}$$

Therefore, the vectors are linearly independent.

Thus, the set formed by the vectors $(1,0,-1)$ and $(0,1,1)$ is a basis for the $range(T)$ and the $dim\big(range(T)\big) = 2$.

Let's verify the dimension theorem for this linear transformation:

$$dim(\mathbb{R}^2) = dim(ker(T)) + dim(range(T))$$

Thus,

$$2 = 0 + 2$$