# *Kernel and Range of a Linear Transformation*

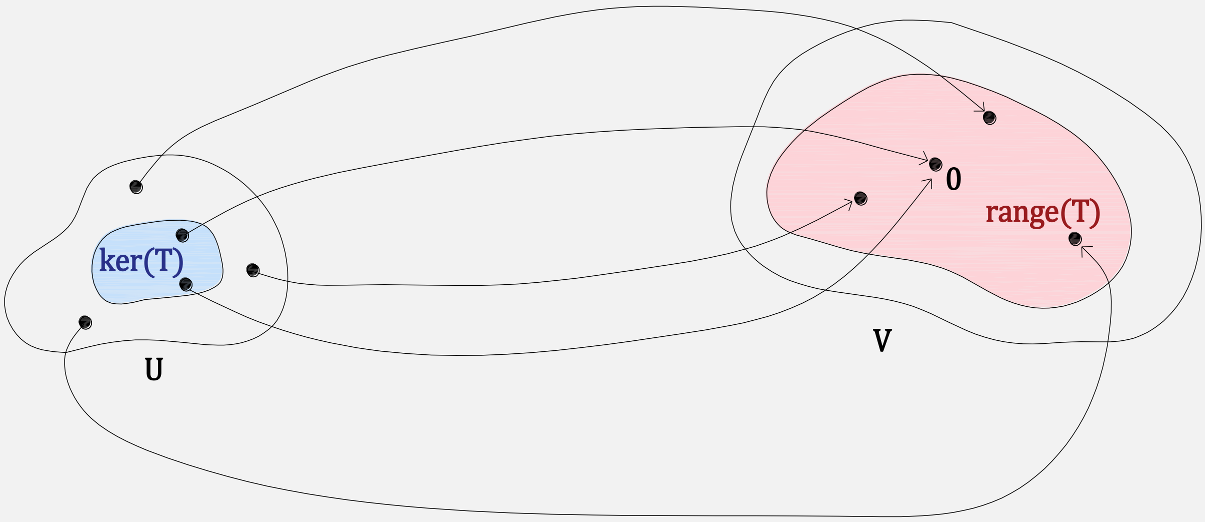
**Definition**: Let $T: U \to V$ be a linear transformation.

The kernel of $T$ ($ker(T)$) is the set of vectors of $U$ that $T$ transforms into the null element of V:

$$ker(T) = \{u \in U : T(u) = 0_V\}$$

The range of $T$ ($range(T)$) is the set of vectors of $V$ that are image by $T$ of at least one vector of $U$:

$$range(T) = \{v \in V : T(u) = v, u \in U\}$$




Kernel and range of a linear transformation $T$

## 1. Determine the kernel and the range of the linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^2$ defined by $T(x, y, z, w) = (x - z, y + 2w)$.

Let us first determine the **kernel** of the transformation $T$. By definition we have:

$$\ker(T) = \{(x, y, z, w) \in \mathbb{R}^4 : T(x, y, z, w) = (0,0)\}$$

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Then,

$$T(x,y,z,w) = (0,0) \Leftrightarrow (x-z, y+2w) = (0,0)$$

$$\Leftrightarrow \begin{cases} x-z & = & 0 \\ y+2w & = & 0 \end{cases} \Leftrightarrow \begin{cases} x & = & z \\ y & = & -2w \end{cases}$$

Therefore,

$$\ker(T) = \{(x,y,z,w) \in \mathbb{R}^4 : x = z \land y = -2w\}$$

$$= \{(z,-2w,z,w) : z,w \in \mathbb{R}\}$$

Let us now determine the **range** of the transformation $T$:

$$range(T) = \{(a,b) \in \mathbb{R}^2 : T(x,y,z,w) = (a,b) \; with \; (x,y,z,w) \in \mathbb{R}^4\}$$

We have:

$$T(x,y,z,w) = (a,b) \Leftrightarrow (x-z, y+2w) = (a,b) \Leftrightarrow \begin{cases} x\ -z & = & a \\ y+2w & = & b \end{cases}$$

The matrix of the system is: $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & a \\ 0 & 1 & 0 & 2 & b \end{array}\right]$

Considering that $A$ is the matrix of the coefficients, $A|B$ is the augmented matrix of the system and $n$ the number of unknowns, we observed that:

$$rank(A) = 2; rank(A|B) = 2; n = 4$$

As $rank(A) = rank(A|B) < n$, the system is possible (and indeterminate).

Therefore, there are no restrictions to be imposed on variables $a$ and $b$.

<u>**Conclusion**</u>: $\boldsymbol{range(T) = \mathbb{R}^2}$**.**

**Note**: $ker\ (T)$ is a vectorial subspace of $\mathbb{R}^4$ (starting set) and $range\ (T)$ is a vectorial subspace of $\mathbb{R}^2$ (finishing set).