Systems of linear equations

Exercise. Let

$$A_\alpha = \begin{pmatrix} 1 & 1 & -1 & \alpha \\ 2 & 3 & -2 & \alpha \\ \alpha & \alpha+1 & 0 & 0 \end{pmatrix} \in M_{34}(\mathbb{R})$$

where $\alpha \in \mathbb{R}$.

1. Find the rank $\varrho(A_\alpha)$ for each $\alpha \in \mathbb{R}$.
2. Let $\alpha = 0$: decide whether the system of linear equations
$$A_0 \cdot {}^t\begin{pmatrix} x & y & z & t \end{pmatrix} = {}^t\begin{pmatrix} -1 & 1 & 2 \end{pmatrix}$$
is consistent or not and, in positive case, how many solutions it has.
3. Let $\alpha = 1$: decide whether the system of linear equations
$$A_1 \cdot {}^t\begin{pmatrix} x & y & z & t \end{pmatrix} = {}^t\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$
is consistent or not and, in positive case, how many solutions it has.

Solution.

1. We reduce A_α in row echelon form:
$$A_\alpha = \begin{pmatrix} 1 & 1 & -1 & \alpha \\ 2 & 3 & -2 & \alpha \\ \alpha & \alpha+1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & \alpha \\ 0 & 1 & 0 & -\alpha \\ 0 & 1 & \alpha & -\alpha^2 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 1 & -1 & \alpha \\ 0 & \boxed{1} & 0 & -\alpha \\ 0 & 0 & \alpha & \alpha(1-\alpha) \end{pmatrix}$$
If $\alpha = 0$, the matrix becomes
$$A_0 = \begin{pmatrix} \boxed{1} & 1 & -1 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
hence $\varrho(A_0) = 2$. If, otherwise, $\alpha \neq 0$, then the α in the last row is a pivot. Therefore:
$$\begin{pmatrix} \boxed{1} & 1 & -1 & \alpha \\ 0 & \boxed{1} & 0 & -\alpha \\ 0 & 0 & \boxed{\alpha} & \alpha(1-\alpha) \end{pmatrix},$$
hence $\varrho(A_\alpha) = 3$ for each $\alpha \neq 0$.
2. We found above $\varrho(A_0) = 2$. For the augmented matrix we get instead:
$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & -1 \\ 2 & 3 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & \boxed{-1} \end{array}\right),$$
hence the corresponding linear system is not consistent (as the last row translates into the impossible equation $0 = -1$).

3. By Gauss-Jordan elimination we find the reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & -1 & 1 & | & 1 \\ 2 & 3 & -2 & 1 & | & 1 \\ 1 & 2 & 0 & 0 & | & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 & | & 1 \\ 0 & 1 & 0 & -1 & | & -1 \\ 0 & 1 & 1 & -1 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 & | & 1 \\ 0 & 1 & 0 & -1 & | & -1 \\ 0 & 0 & 1 & 0 & | & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & 1 & | & 2 \\ 0 & 1 & 0 & -1 & | & -1 \\ 0 & 0 & 1 & 0 & | & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 & | & 3 \\ 0 & 1 & 0 & -1 & | & -1 \\ 0 & 0 & 1 & 0 & | & 1 \end{pmatrix}$$

whose associated linear system is

$$\begin{cases} x + 2t = 3 \\ y - t = -1 \\ z = 1 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x = 3 - 2t \\ y = -1 + t \\ z = 1 \end{cases} .$$

Hence, its solution set is

$$\{(3 - 2t;\ t - 1;\ 1;\ t) \mid t \in \mathbb{R}\}.$$