# LINEAR PROGRAMMING: Sensitivity analysis example


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## Example of LP model:

A company produces three products (1, 2 and 3), using 110, 150 and 200 hours (h) in machines M1, M2 and M3, respectively, according to the following linear programming (LP) model:

$$\text{Max } Z = 3x_1 + 3x_2 + 2x_3$$

subject to

$$2x_1 + 3x_2 + 4x_3 \leq 110$$
$$3x_1 + 2x_2 + 3x_3 \leq 150$$
$$4x_1 + 2x_2 + 3x_3 \leq 200$$
$$x_1, x_2, x_3 \geq 0.$$

The decision variables $x_1$, $x_2$ and $x_3$ represent the units produced of A, B and C products, respectively.

The *Answer Report* provided by the Excel Solver Add-in indicates that the optimum profit is 156€ which is achieved if the company produces 46 units of product A and 6 units of product B. There are 4 hours available in the machine M3.

**Answer Report**

Objective Cell (Max)

| Cell | Name | Original Value | Final Value |
|---|---|---|---|
| $E$8 | Z | 0 | 156 |

Variable Cells

| Cell | Name | Original Value | Final Value | Integer |
|---|---|---|---|---|
| $B$4 | x1 | 0 | 46 | Contin |
| $C$4 | x2 | 0 | 6 | Contin |
| $D$4 | x3 | 0 | 0 | Contin |

Constraints

| Cell | Name | Cell Value | Formula | Status | Slack |
|---|---|---|---|---|---|
| $E$12 | Hours used in M1 | 110 | $E$12<=$G$12 | Binding | 0 |
| $E$13 | Hours used in M2 | 150 | $E$13<=$G$13 | Binding | 0 |
| $E$14 | Hours used in M3 | 196 | $E$14<=$G$14 | Not Binding | 4 |

# Optimum solutions of the primal and dual problems from the Answer and Sensitivity Reports:

**Sensitivity Report**

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$4 | $x_1$ | 46 | 0 | 3 | 1,5 | 1 |
| $C$4 | $x_2$ | 6 | 0 | 3 | 1,5 | 1 |
| $D$4 | $x_3$ | 0 | -2,2 | 2 | 2,2 | 1E+30 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $E$12 | Hours used in M1 | 110 | 0,6 | 110 | 115 | 10 |
| $E$13 | Hours used in M2 | 150 | 0,6 | 150 | 2,5 | 76,66666667 |
| $E$14 | Hours used in M3 | 196 | 0 | 200 | 1E+30 | 4 |

| Primal problem | $x_1^*$ | $x_2^*$ | $x_3^*$ | $x_4^*$ | $x_5^*$ | $x_6^*$ |
|---|---|---|---|---|---|---|
| Primal Solution | 46 | 6 | 0 | 0 | 0 | 4 |
| Dual problem | $y_4^*$ | $y_5^*$ | $y_6^*$ | $y_1^*$ | $y_2^*$ | $y_3^*$ |
| Dual Solution | 0 | 0 | 2.2 | 0.6 | 0.6 | 0 |

## **Economic interpretation from the Sensitivity Report**:

**Reduced costs of the products produced** :

As the products A and B are produced (46 units and 6 units, respectively), their reduced costs (1st table) are equal to $0$.
As the product C is not produced, it has a reduced cost of $2.2$, indicating that for each unit produced of the product C there will be a reduction of $2.2$€ in the optimum profit obtained.

**Shadow prices of the available resources**:

The shadow price (2nd table) of the machine M3 is $0$ due to it available hours (only $196$ h were used from the $200$ h available).
The shadow prices of the machines M2 and M3 are both equal to $0.6$, indicating that for each extra hour available on that machines, the optimum profit has an increase of $0.6$ €.

**Sensitivity analysis of the unit profit of the products**: to determine the allowable range to keep the original optimal solution (see the last two columns in the 1st table - Sensitivity report)

- Profit of product A:
  - Current value of profit of product A: $c_1 = 3$
  - Allowable increase of profit of product A: $1.5$, so $c_1 \leq 3 + 1.5 = 4.5$
  - Allowable decrease of profit of product A: $1$, so $c_1 \geq 3 - 1 = 2$
  - Allowable range for the profit of product A: $2 \leq c_1 \leq 4.5$
- Allowable range for the profit of product B: $2 \leq c_2 \leq 4.5$
- Allowable range for the profit of product C: $-\infty \leq c_3 \leq 4.2$

Calculation of the allowable range in the total profit due to the allowable range on profit of each product. For the product A:

- The allowable range on the profit of product A implies an allowable range in total profit equal to $156 - 1 * 46 \leq z^* \leq 156 + 1.5 * 46$.

**Sensitivity analysis of the available capacity of resources**: to determine the allowable range to keep the original optimal solution (see the last two columns in the 2nd table - Sensitivity report)

- Available capacity of machine M3:
  - Current value of capacity of M3: $b_1 = 110$
  - Allowable increase of capacity of M3: $115$, so $b_1 \leq 110 + 115 = 225$
  - Allowable decrease of capacity of M3: $10$, so $b_1 \geq 110 - 10 = 100$
  - Allowable range for the capacity of M3: $100 \leq b_1 \leq 225$
- Allowable range for the profit of product B: $73.33 \leq b_2 \leq 152.5$
- Allowable range for the profit of product C: $196 \leq b_3 \leq +\infty$

## Reference

Hillier F. S., & Lieberman G.R. (2010). Introduction to operations research (9th ed.). New York: McGraw-Hill.