Points of extremum for functions of several variables

One of the main uses of ordinary derivatives is in finding maximum and minimum values (extreme values). We see how to use partial derivatives to locate maximum and minimum of functions of two variables.

Maximum, minimum and saddle points

Look at the hills and valleys in the graph of shown in Figure 1.

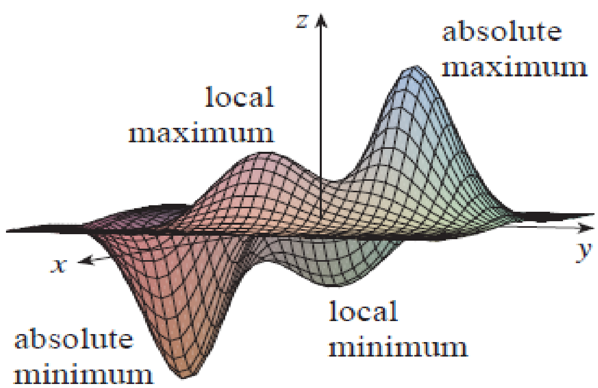

Figure 1

There are two points (a_1, a_2) where f has a local maximum, that is, where $f(a_1, a_2)$ is larger than nearby values of $f(x, y)$. The larger of these two values is the absolute maximum. Likewise, f has two local minimum, where $f(x, y)$ is smaller than nearby values. The smaller of these two values is the absolute minimum.

Definition 1 *Let $D \subset \mathbb{R}^n$ and $f : D \subset \mathbb{R}$. The point $a = (a_1, a_2, ..., a_n) \in D$ whose coordinates verify the equations*

$$\frac{\partial f}{\partial x_i}(a_1, a_2, ..., a_n) = 0, i = \overline{1, n} \tag{1}$$

is called a ***critical*** *or* ***stationary point*** *for f.*

Definition 2 *Let $D \subset \mathbb{R}^n$ and $f : D \subset \mathbb{R}$. The point $a \in D$ is said to be:*

(1) a ***local maximum*** *if $f(x) \leq f(a)$ for all points x sufficiently close to a;*

(2) a ***local minimum*** *if $f(x) \geq f(a)$ for all points x sufficiently close to a;*

(3) a ***global*** *(or absolute)* ***maximum*** *if $f(x) \leq f(a)$ for all points $x \in D$;*

*(4) a **global** (or absolute) **minimum** if $f(x) \geq f(a)$ for all points $x \in D$;*
(5) a local or global extremum if it is a local or global maximum or minimum.

Definition 3 *A critical point a which is neither a local maximum nor minimum is called a **saddle point.***

There are three types of stationary points possible, these being a maximum point, a minimum point, and a saddle point.

An analogous of Fermat Theorem is the following:

Theorem 4 *A point of local extremum for a function f, belonging to interior of the domain, is a stationary point for f.*

Fermat's theorem gives only a necessary condition for extreme function values.

Procedure to determine maxima, minima and saddle points for functions of several variables

We can located the point of extremum. Then we must classifying them in points of maximum, points of minimum or saddle point.

Theorem 5 *Let be a stationary point $a = (a_1, a_2, ..., a_n) \in D$ for $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and suppose f has continuous partial derivatives of second older in a neighborhood of a*
(1) if $d^2 f(a)$ is a positive quadric form, then a is a local minimum point,
(2) if $d^2 f(a)$ is a negative quadric form, then a is a local maximum point,
(3) if $d^2 f(a)$ is a undefined quadric form, then a is not a point of extremum.

In order to establish if a quadric form is positive, negative or undefined we turn it to canonical expression, by using an algebraic method. While using the method of Jacobi (when is possible) the above theorem can be rewritten as follows:

Definition 6 *If $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a function of two variables such that all second order partial derivatives exist at the point (a_1, a_2), then the Hessian matrix of f at (a_1, a_2) is the matrix*

$$H_2 = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} \tag{2}$$

where the derivatives are evaluated at (a_1, a_2).

Definition 7 *If $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is a function of three variables such that all second order partial derivatives exist at the point (a_1, a_2, a_3), then the Hessian* *of f at (a_1, a_2, a_3) is the matrix*

$$H_3 = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} \tag{3}$$

where the derivatives are evaluated at (a_1, a_2, a_3).

We note $H_1 = \left(\frac{\partial^2 f}{\partial x^2} \right)$.

Theorem 8 *Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a function of two variables having partial derivatives of order three on D and consider (a_1, a_2) a stationary point of f and H_2 the Hessian of f at (a_1, a_2). If $det(H_2) \neq 0$, then (a_1, a_2) is:*

(1) a local maximum if $\frac{\partial^2 f}{\partial x^2} < 0$ and $det(H_2) > 0$.

(2) a local minimum if $\frac{\partial^2 f}{\partial x^2} > 0$ and $det(H_2) > 0$.

(3) a saddle point if neither of the above hold,

where the partial derivatives are evaluated at (a_1, a_2).

Theorem 9 *If $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is a function of three variables having partial derivatives of order three on D and consider (a_1, a_2, a_3) a stationary point of f and H_3 the Hessian of f at (a_1, a_2, a_3). If $det(H_3) \neq 0$, then (a_1, a_2, a_3) is:*

(1) a local maximum if $det(H_1) < 0$, $det(H_2) > 0$ and $det(H_3) < 0$;

(2) a local minimum if $det(H_1) > 0$, $det(H_2) > 0$ and $det(H_3) > 0$;

(3) a saddle point if neither of the above hold,

where the partial derivatives are evaluated at (a_1, a_2, a_3).

In each case, if $det\,(H_i) = 0, i = 1, 2$, then a can be either a local extremum or a saddle point.

Example 10 *Find the points of extremum for the function*

$$f(x, y, z) = y + \frac{z^2}{4y} + \frac{x^2}{z} + \frac{2}{x}, x \neq 0, y \neq 0, z \neq 0.$$

Solution.

Find the stationary (critical) point.

$\frac{\partial f}{\partial x} = \frac{2x}{z} - \frac{2}{x^2}; \frac{\partial f}{\partial y} = 1 - \frac{z^2}{4y^2}; \frac{\partial f}{\partial z} = \frac{z}{2y} - \frac{x^2}{z^2}$

$$\begin{cases} \frac{2x}{z} - \frac{2}{x^2} = 0 \\ 1 - \frac{z^2}{4y^2} = 0 \\ \frac{z}{2y} - \frac{x^2}{z^2} = 0 \end{cases},$$

The solutions are: $\left[x=1, y=\frac{1}{2}, z=1\right], \left[x=-1, y=-\frac{1}{2}, z=-1\right]$.

Establish the sign of the quadric form $d^2f\left(1,\frac{1}{2},1\right)$ and $d^2f\left(-1,-\frac{1}{2},-1\right)$.

$\dfrac{\partial^2 f}{\partial x^2}=\dfrac{2}{z}+\dfrac{4}{x^2}; \dfrac{\partial^2 f}{\partial y^2}=\dfrac{z^2}{2y^3}; \dfrac{\partial^2 f}{\partial z^2}=\dfrac{1}{2y}+\dfrac{2x^2}{z^3};$

$\dfrac{\partial^2 f}{\partial x\partial y}=0; \dfrac{\partial^2 f}{\partial y\partial z}=-\dfrac{z}{2y^2}; \dfrac{\partial^2 f}{\partial x\partial z}=-\dfrac{2x}{z^2};$

$\dfrac{\partial^2 f}{\partial x^2}\left(1,\frac{1}{2},1\right)=6; \dfrac{\partial^2 f}{\partial y^2}\left(1,\frac{1}{2},1\right)=4; \dfrac{\partial^2 f}{\partial z^2}\left(1,\frac{1}{2},1\right)=3;$

$\dfrac{\partial^2 f}{\partial x\partial y}\left(1,\frac{1}{2},1\right)=0; \dfrac{\partial^2 f}{\partial y\partial z}\left(1,\frac{1}{2},1\right)=-2; \dfrac{\partial^2 f}{\partial x\partial z}\left(1,\frac{1}{2},1\right)=-2.$

$d^2f\left(1,\frac{1}{2},1\right)=6dx^2+4dy^2+3dz^2-4dydz-4dxdz,$

$H_3=\begin{pmatrix} 6 & 0 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 3 \end{pmatrix},$

$\det(H_3)=32\neq 0$, so we can apply Theorem 9.

$\det(H_1)=6, \det(H_2)=\det\begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}=24,\ \det(H_3)=32.$

Since $\det(H_1)>0, \det(H_2)>0, \det(H_3)>0$ then $\left(1,\frac{1}{2},1\right)$ is a maximum point.

Similarly

$\dfrac{\partial^2 f}{\partial x^2}\left(-1,-\frac{1}{2},-1\right)=2; \dfrac{\partial^2 f}{\partial y^2}\left(-1,-\frac{1}{2},-1\right)=-4; \dfrac{\partial^2 f}{\partial z^2}\left(-1,-\frac{1}{2},-1\right)=-3;$

$\dfrac{\partial^2 f}{\partial x\partial y}\left(-1,-\frac{1}{2},-1\right)=0; \dfrac{\partial^2 f}{\partial y\partial z}\left(-1,-\frac{1}{2},-1\right)=2; \dfrac{\partial^2 f}{\partial x\partial z}\left(-1,-\frac{1}{2},-1\right)=2.$

$H_3=\begin{pmatrix} 2 & 0 & 2 \\ 0 & -4 & 2 \\ 2 & 2 & -3 \end{pmatrix}.$

$\det(H_3)=32\neq 0$, so we can apply Theorem 9.

$\det(H_1)=2, \det(H_2)=\det\begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}=-8, \det(H_3)=32.$

Since $\det(H_1)>0, \det(H_2)<0, \det(H_3)>0$ then $\left(-1,-\frac{1}{2},-1\right)$ is a saddle point.

Example 11 *Find the points of extremum for the function $f:\mathbb{R}^2\to\mathbb{R}$,*

$$f(x,y):=x^4+y^4-x^2-2xy-y^2.$$

Solution.

Find the stationary (critical) point.

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y)=0 \\ \dfrac{\partial f}{\partial y}(x,y)=0 \end{cases} \Leftrightarrow \begin{cases} 4x^3-2x-2y=0 \\ 4y^3-2x-2y=0. \end{cases}$$

The solutions are: $[x = 1, y = 1], [x = 0, y = 0], [x = -1, y = -1]$.

Next, we will establish the sign of the quadratic forms $d^2f(0,0), d^2f(1,1), d^2f(-1,-1)$,

$\dfrac{\partial^2 f}{\partial x^2}(x,y) = 12x^2 - 2; \dfrac{\partial^2 f}{\partial y^2}(x,y) = 12y^2 - 2; \dfrac{\partial^2 f}{\partial x \partial y}(x,y) = -2.$

$\dfrac{\partial^2 f}{\partial x^2}(0,0) = -2; \dfrac{\partial^2 f}{\partial y^2}(0,0) = -2; \dfrac{\partial^2 f}{\partial x \partial y}(x,y) = -2.$

$\dfrac{\partial^2 f}{\partial x^2}(1,1) = 10; \dfrac{\partial^2 f}{\partial y^2}(1,1) = 10; \dfrac{\partial^2 f}{\partial x \partial y}(x,y) = -2.$

$\dfrac{\partial^2 f}{\partial x^2}(-1,-1) = 10; \dfrac{\partial^2 f}{\partial y^2}(-1,-1) = 10; \dfrac{\partial^2 f}{\partial x \partial y}(-1,-1) = -2.$

For $(0,0)$ we have

$H_2 = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}, \det(H_2) = 0$

and we cannot use Theorem 8. We construct the quadratic form. Thus,

$d^2f(0,0) = -2dx^2 - 4dxdy - 2dy^2 = -2(dx + dy)^2$.

The quadratic form is negative semidefinite, so we cannot use Theorem 5. We observe that, for $\varepsilon > 0$ small, we have $f(\varepsilon, 0) = \varepsilon^4 - \varepsilon^2 < 0 = f(0,0)$, and $f(\varepsilon, -\varepsilon) = 2\varepsilon^4 > 0 = f(0,0)$, so the point $(0,0)$ is not extremum point.

For $(1,1)$ we have

$H_2 = \begin{pmatrix} 10 & -2 \\ -2 & 10 \end{pmatrix}, \det(H_2) \neq 0.$

We can apply Theorem 8. So $\dfrac{\partial^2 f}{\partial x^2}(1,1) = 10 > 0$ and $\det(H_2) = 36 > 0$. The point $(1,1)$ is a minimum point. Similarly for $(-1,-1)$,

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