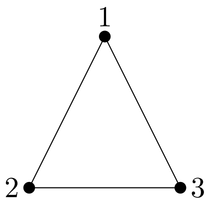
Matrix representation of graphs

A graph G (or multigraph, directed or not) with n vertices $V(G) = \{v_1, v_2, \cdots, v_n\}$ can be represented by a square matrix of order n, $A_G = [a_{ij}]$ called **adjacency matrix**, such us a_{ij} is equal to the number of edges between vertices v_i and v_j.

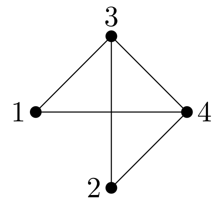
Observe that if the graph is simple then $a_{ij} = \begin{cases} 1 & \text{if } \{v_i,\ v_j\} \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$

Example 1. A adjacency matrix of the graph



is $\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$

Exercise 1. Find the adjacency matrix for the following graph.

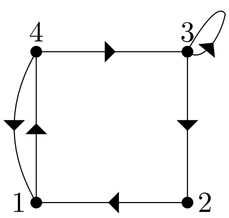


Solution:

Considering the same order of vertices considered in the lables $\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$.

Notice that the sum of entries in each line (or column, since the matrix is symmetric) is the degree of the vertex, that is, for example $deg(1) = 2$ *and* $deg(3) = 3$.

Example 2. A adjacency matrix of the direct graph



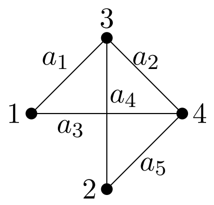
is $\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix} \end{array}$

The **incidence matrix** of a undirected graph G with n vertices $V(G) = \{v_1, v_2, \cdots, v_n\}$ and m edges, $A(G) = \{a_1, a_2, \ldots, a_m\}$ is a matrix of type $n \times m$, $M_G = [m_{ij}]$, $1 \leq i \leq n$, $1 \leq j \leq m$

such that

$$a_{ij} = \begin{cases} 0, & \text{se } a_j = v_p v_q, \text{ com } i \notin \{p,q\} \\ 1, & \text{se } a_j = v_i v_k, \text{ com } k \neq i \\ 2, & \text{se } a_j = v_i v_i \end{cases}$$

Example 3. The incidence marix of the following graph

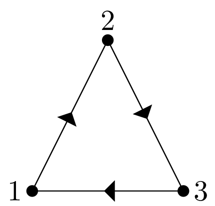


$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

The incidence matrix of a directed graph is similar to the undirected graphs but considering the direction of each edge, that is, the edgev_1v_2 is diferent from the edge v_2v_1. Thus, the incidence matrix of a directed graph without loops is a matrix $M_G = [m_{ij}]$ where

$$m_{ij} = \begin{cases} 0, & \text{if } a_j = v_p v_q, \text{ with } i \notin \{p,q\} \\ -1, & \text{if } a_j = v_k v_i, \text{ for some vertex } v_k \\ 1, & \text{if } a_j = v_j v_k, \text{ for some vertex } v_k \end{cases}$$

Example 4. A incidence matrix of the graph



is
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \end{array}$$

Exercise 2. Consider the adjacency matrix $A_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

a. Draw a simple graph G represented by A_G;

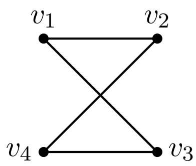
b. Draw the complementary graph $\bar{G}$;

c. Draw an directed graph D represented by A_G;

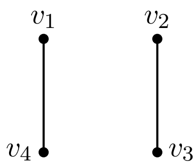
d. Determine the incidence matrix of the directed graph D.

Solution:

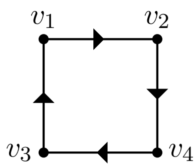
a. Nominating the vertices by v_1, v_2, v_3 e v_4 the graph G could be



b. Considering the graph G pictured in the previous question, the complementary graph $\bar{G}$ is



c. An directed graph D represented by A_G could be



d. Considering the directed graph D pictured in the previous question, and that the edges $a_1 = v_1v_2$, $a_2 = v_2v_4$, $a_3 = v_4v_3$, $a_4 = v_3v_1$, then the incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

References

[1] Domingos Cardoso, Jerzy Szymański, and Mohammad Rostami. *Matemática Discreta: Combinatória, Teoria dos Grafos, Algoritmos*. Escolar Editora, 2009.

[2] Edgar Goodair and Michael Parmenter. *Discrete Mathematics with Graph Theory.* (3rd Ed.) Pearson, 2006.

Exercises in MathE platform