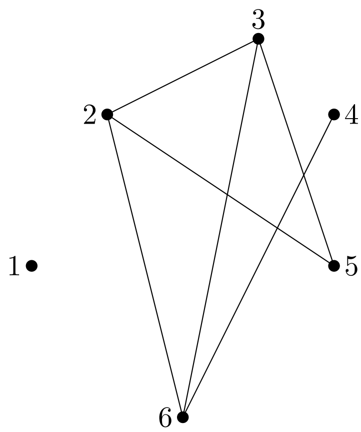
Definitions and basic properties

A **graph** G consists of two finite sets $V(G)$ and $E(G)$, where $V(G)$ is nonempty and each element of $E(G)$ is an unordered pair of distinct elements of $V(G)$. The elements of $V(G)$ are called **vertices** and the elements of E are called **edges**. Thus, if e is an edge, then e is a set of the form $\{v, w\}$, where v and w are different elements of V called the **end vertices or endpoints** of edge e. We often omit the braces and refer to the edge vw, and this is of course the same as teh edge wv.

Example 1. Consider the sets

$$V_1 = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\} \text{ and } E_1 = \{\{2,\ 3\},\ \{3,\ 5\},\ \{3\ 6\},\ \{2,\ 6\},\ \{4,\ 6\},\ \{2,\ 5\}\}.$$

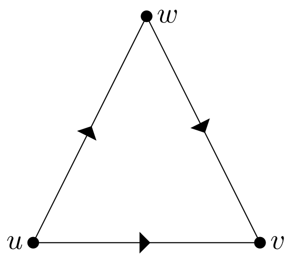
$G_1 = (V, E)$ is a (simple) graph with vertices V and edges E and could be pictured as showned bellow



We can also write the vertex set $E_1 = \{23,\ 35,\ 36,\ 26,\ 46,\ 25\}$.

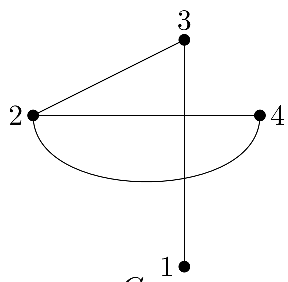
A **directed graph**, or **digraph**, consists of two finite sets: a nonempty set $V(G)$ of vertices and a set $D(G)$ of directed edges, where each is associated with an ordered pair of vertices, its endpoints. If edge e is associated with the pair (v, w) of vertices, then e is said to be the **(directed) edge or arc** from v to w.

Example 2. Consider the set of the vertices $V_2 = \{u, v, w\}$ and the set of the edges $D_2 = \{(u, v); (v, w); (u, w)\}$. $G_2 = (V_2, D_2)$ is a directed graph.

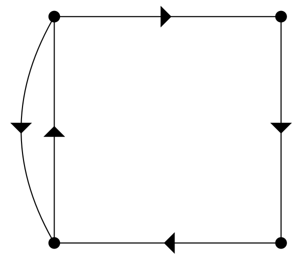


A graph that have two or more edges connecting the same pair of vertices, **multiple edges or parallel edges**, is called **multigraph**. An edge with just one endpoint is called a **loop**.

Example 3. The following graphs are not simple, because three is two different edges connecting the same pair of vertices. They are multigraph.



G_3



G_4

A **simple graph** is a graph that does not have any loops or multiple (parallel) edges.
The **order of a graph** is the number of vertices in the graph and the **size of a graph** is the number of edges in the graph.
An edge vw is said to be **incident** on each of its endpoints, v and w, and two edges incident on the same endpoint are called **adjacent**.
Two vertices are called **adjacent** if they are the endpoints of same edge. A vertex on which no edges are incident is called **isolated** vertex.

Example 4. In the graph G_1 in the Example **??** the edge 23 is adjacent to the edge 35 and the vertex 4 is adjacent to vertex 6.

The **degree** of v, denoted $deg(v)$, equals the number of edges that are incident on v, with an edge that is a loop counted twice. The total degree of G is the sum of the degrees of all the vertices of G. A vertex with degree zero is an **isolated** vertex.

Example 5. In the graph G_1 in the Example **??** $deg(6) = 3$ and $deg(1) = 0$. The vertex 1 is an isolated vertex.

In a digraph, each vertex has two kinds of degree, the **indegree** of vertex v is the number of edges which are coming into the vertex v and **outdegree** of vertex v is the number of edges which are going out from the vertex v, with notation $deg^-(v)$ and $deg^+(v)$, respectivaly.

Example 6. In the graph G_2 in the Example **??** $deg^-(u) = 0$ and $deg^+(u) = 2$ and the $deg^-(w) = deg^+(u) = 1$.

Proposition 1. Let $G = (V,\, E)$ be a undirected graph. Then the sum of the vertex degrees of the graph is equal to twice of the number of edges, i.e,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Proof. Adding the degrees of all the vertices involves counting one for each edge incident with each vertex. If it is not a loop, it is incident with two different vertices and so gets counted twice, once at each vertex. On the other hand, a loop at a vertex is also counted twice, by convention, in the degree of that vertex. $\square$

Corollary 1. The number of vertices with odd degree is always even.

Proof. The total degree of a graph $G = (V,\ E)$ is equal two times the number of edges, that is,

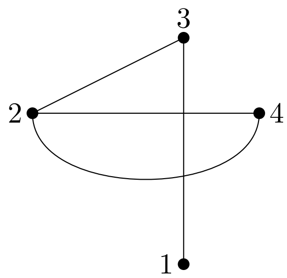
$$\begin{aligned} \sum_{v\in V} \deg(v) = 2|E| \quad &\Rightarrow \quad \sum_{v\in V} \deg(v) \text{ is even} \\ &\Rightarrow \quad \text{the number of vertices with odd degree is always even.} \end{aligned}$$

$\square$

Example 7. A graph has six vertices each of degree 3. Since $\sum_{v\in V} \deg(v) = 6 \times 3 = 18 = 2 \times 9$, the graph must have 9 edges.

A **degree sequence** of the vertex degrees of a graph G is a nonincreasing sequence of degrees of its graph vertices.

Example 8. (3, 2, 2, 1) is the degree sequence of the graph G_3 in the Example **??**



Example 9. A graph with degree sequence (3, 2, 2, 1) has 4 edges because $\sum_{v\in V} \deg(v) = 3 + 2 \times 2 + 1 = 8$. (You can check in the graph!)

References

[1] Edgar Goodair and Michael Parmenter. *Discrete Mathematics with Graph Theory.* (3rd Ed.) Pearson, 2006.

[2] Susanna Epp. *Discrete Mathematics and Applications.* (4th Ed.) Brooks/Cole CENGAGE Learning, 2011.

Exercises in MathE platform