Method of Partial Fractions

It is used to

integrate rational functions $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions and the power of $P(x)$ is less than the power of $Q(x)$.

Examples

1. $\displaystyle\int \frac{x}{x^3 - x}\,\mathrm{d}x$
2. $\displaystyle\int \frac{x}{x^2 - x - 6}\,\mathrm{d}x$
3. $\displaystyle\int \frac{1}{x^2 - 1}\,\mathrm{d}x$

Method of Partial Fractions

Alert

If a rational function $\frac{R(x)}{Q(x)}$ is such that the power of $R(x)$ is greater than the power of $Q(x)$, then one must use long division and write the rational function in the form

$$\frac{R(x)}{Q(x)} = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n + \frac{P(x)}{Q(x)}$$

where now P(x) is a remainder term with the power of P(x) less than the power of Q(x) and our objective is to integrate each term of the above representation.

Case 1: power of $P(x) \geq Q(x)$

Consider $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$

1. If power of $P(x) \geq$ power of $Q(x)$

   Apply a long division.

$$\begin{array}{l|l} P(x) & Q(x) \\ \hline R(x) & q(x) \end{array}$$

And, then

$$\int \frac{P(x)}{Q(x)}\,dx = \underbrace{\int q(x)\,dx}_{\textit{formula table}} + \int \frac{R(x)}{Q(x)}\,dx$$

Case 2: power of $P(x) < Q(x)$

Consider $\displaystyle\int \frac{P(x)}{Q(x)}\,\mathrm{d}x$

1. The denominator $Q(x)$ has only first power factors, none of which are repeated. Then, $Q(x)$ has the form

$$Q(x) = (x - x_0)(x - x_1)(x - x_2)\cdots(x - x_n)$$

where $x_0 \neq x_1 \neq x_2 \neq \cdots \neq x_n$. One can then write

$$\frac{P(x)}{Q(x)} = \frac{A}{x - x_0} + \frac{B}{x - x_1} + \cdots + \frac{C}{x - x_n}$$

where $A, B, \cdots, C$ are constants to be determined.

Case 2: power of $P(x) < Q(x)$

Consider $\displaystyle\int \frac{P(x)}{Q(x)}\,\mathrm{d}x$

2. The denominator $Q(x)$ has only first power factors, but some of these factors may be repeated factors. For example, the denominator $Q(x)$ might have a form such as

$$Q(x) = (x - x_0)^k (x - x_1)^p \cdots (x - x_n)^m$$

where $k, p, \cdots, m$ are integers.
Here the denominator has repeated factors of orders $k, p, \cdots, m$.
In this case one can write the rational function in the form:

Case 2: power of $P(x) < Q(x)$

2. (continue)

$$\begin{aligned}\frac{P(x)}{Q(x)} = &\quad \frac{A_1}{x-x_0} + \frac{A_2}{(x-x_0)^2} + \cdots + \frac{A_k}{(x-x_0)^k} + \\ + &\quad \frac{B_1}{x-x_1} + \frac{B_2}{(x-x_1)^2} + \cdots + \frac{A_p}{(x-x_1)^p} + \\ + &\quad \cdots + \\ + &\quad \frac{C_1}{x-x_n} + \frac{C_2}{(x-x_n)^2} + \cdots + \frac{C_m}{(x-x_n)^m}\end{aligned}$$

where $A_1, \cdots, A_k, B_1, \cdots, B_p, \cdots, C_1, \cdots, C_m$ are constants to be determined.

Case 2: power of $P(x) < Q(x)$

3. The denominator $Q(x)$ has one or more quadratic factors of the form $ax^2 + bx + c$ none of which are repeated. In this case, for each quadratic factor there corresponds a partial fraction of the form

$$\frac{P(x)}{Q(x)} = \frac{A_0 x + B_0}{ax^2 + bx + c}$$

where A_0 and B_0 are constants to be determined.

Case 2: power of $P(x) < Q(x)$

4. The denominator $Q(x)$ has one or more quadratic factors, some of which are repeated quadratic factors. In this case, for each repeated quadratic factor $(ax^2 + bx + c)^k$ there corresponds a sum of partial fractions of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

where $A_1, B_1, \cdots, A_k, B_k$ are constants to be determined.

Integral of a Rational function — Summary

Let's consider the case 2 (power of denominator bigger than the power of numerator) and the following general form of partial fractions:

$$
\begin{aligned}
\frac{P(x)}{Q(x)} = & \frac{A_p}{(x-\alpha)^p} + \frac{A_{p-1}}{(x-\alpha)^{p-1}} + \cdots + \frac{A_2}{(x-\alpha)^2} + \frac{A_1}{x-\alpha} + \\
+ & \frac{B_q}{(x-\beta)^q} + \frac{B_{q-1}}{(x-\beta)^{q-1}} + \cdots + \frac{B_2}{(x-\beta)^2} + \frac{B_1}{x-\beta} + \cdots + \\
+ & \frac{C_r+D_r x}{((x-a)^2+b^2)^r} + \frac{C_{r-1}+D_{r-1} x}{((x-a)^2+b^2)^{r-1}} + \cdots + \frac{C_2+D_2 x}{((x-a)^2+b^2)^2} + \frac{C_1+D_1 x}{(x-a)^2+b^2}
\end{aligned}
\tag{1}
$$

Important:

All the constants must be determined before applying the integral calculus.

Techniques to find the constants

Consider Case 2.

Remark:

The 3 bellow sub-cases can appear in the integration. For each factor we should write the partial fractions as described above

Techniques to find the constants

- Undetermined coefficients method
- The cover-up method
- Differentiation Rule

Cover-up method

- If $Q(x) = (x-\alpha)^p Q_1(x)$ and $\alpha \in \mathbb{R}$

$$A_p = \left[\frac{P(x)}{Q(x)}\right]_{x=\alpha}$$

- If $Q(x) = ((x-a)^2 + b^2)^r Q_1(x)$ and $x = a + bi \in \mathbb{C}$

$$\left[C_r + D_r x = \frac{P(x)}{Q_1(x)}\right]_{x=a+bi}$$

Cover-up method

Example

$$\frac{6x-1}{x^2-4x+3}=\frac{6x-1}{(x-3)(x-1)}=\frac{A}{x-3}+\frac{B}{x-1}$$

The constants A and B can be determined by cover-up method:

$$A=\left[\frac{6x-1}{x-1}\right]_{x=3}=\frac{17}{2}$$

$$B=\left[\frac{6x-1}{x-3}\right]_{x=1}=\frac{5}{-2}=-\frac{5}{2}$$

Cover-up method

Example

$$\frac{6x-1}{(x-3)^2(x-1)} = \frac{A_1}{x-3} + \frac{A_2}{(x-3)^2} + \frac{B_1}{x-1}$$

The constants A_2 e B_1 can be determined by cover-up method:

$$A_2 = \left[\frac{6x-1}{x-1}\right]_{x=3} = \frac{26}{2} = 13$$

$$B_1 = \left[\frac{6x-1}{(x-3)^2}\right]_{x=1} = \frac{6}{4} = -\frac{3}{2}$$

Be careful:

The constant A_1 can not be determined using cover-up method

Differentiation method

Useful when the denominator has roots (real or complex) with multiplicity greater than one.

1. Put the same denominator in the equation 1 (page 46).
2. Assign to x the roots values and find some coefficients.
3. Differentiate both sides of the equation and repeat step 2.
4. Differentiate again both sides of the equation until all the coefficients are found.

Differentiation method

For real roots

If $Q(x) = (x-\alpha)^p Q_1(x)$ and $\alpha \in \mathbb{R}$

$$\left[\frac{1}{r!} \cdot \frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\frac{P(x)}{Q_1(x)}\right)\right]_{x=\alpha} = A_{p-r}, \quad 0 \leq r \leq p-1$$

Remark: The derivative of order 0 is $\frac{\mathrm{d}^0}{\mathrm{d}x^0}(f(x)) = f(x)$

Differentiation method

Examples

1. Consider $\dfrac{x^2+1}{(x-1)^3(x-2)} = \dfrac{A_3}{(x-1)^3} + \dfrac{A_2}{(x-1)^2} + \dfrac{A_1}{x-1} + \dfrac{B_1}{x-2}$. Determine all the constants using the differentiation rule.

2. Consider $\dfrac{2}{(x^2+1)^2 x} = \dfrac{C_2+D_2x}{(x^2+1)^2} + \dfrac{C_1+D_1x}{x^2+1} + \dfrac{B_1}{x}$. Determine all the constants using the undetermined coefficients method.

Solution of First Example I

It is intended to find $\displaystyle\int \frac{x^2+1}{(x-1)^3(x-2)}\,\mathrm{d}x$

First, write the rational function as sums of partial fractions:

$$\frac{x^2+1}{(x-1)^3(x-2)} = \frac{A_3}{(x-1)^3} + \frac{A_2}{(x-1)^2} + \frac{A_1}{x-1} + \frac{B_1}{x-2}.$$

By comparison we obtain the following:

- $P(x) = x^2 + 1$.
- $(x-\alpha)^p = (x-1)^3$, then $\alpha = 1 \wedge p = 3$.
- $Q_1(x) = x - 2$.

Solution of First Example II

Since $p = 3 \wedge 0 \leq r \leq p-1 \Longrightarrow 0 \leq r \leq 2$.

$$A_{p-r} = \left[\frac{1}{r!} \cdot \frac{d^r}{dx^r}\left(\frac{P(x)}{Q_1(x)}\right)\right]_{x=\alpha}, \quad 0 \leq r \leq 2.$$

Then, for:

- $r = 0,\ A_3 = \left[\frac{1}{0!} \cdot \frac{d^0}{dx^0}\left(\frac{x^2+1}{x-2}\right)\right]_{x=1} = \left[\left(\frac{x^2+1}{x-2}\right)\right]_{x=1} = -2.$
- $r = 1,\ A_2 = \left[\frac{1}{1!} \cdot \left(\frac{x^2+1}{x-2}\right)'\right]_{x=1} = \left[\left(\frac{x^2-4x-1}{(x-2)^2}\right)\right]_{x=1} = -4$
- $r = 2,\ A_1 = \left[\frac{1}{2!} \cdot \left(\frac{x^2+1}{x-2}\right)''\right]_{x=1} = \left[\frac{1}{2!}\left(\frac{x^2-4x-1}{(x-2)^2}\right)'\right]_{x=1} = \ldots = -5$

Solution of First Example III

B_1 can be determined by hidden rule:

$$B_1 = \left[\frac{x^2+1}{(x-1)^3}\right]_{x=2} = \frac{5}{1} = 5$$

Second, replace the constants and find the related integrals:

$$\int \frac{x^2+1}{(x-1)^3(x-2)}\,\mathrm{d}x = \int \frac{A_3}{(x-1)^3}\,\mathrm{d}x + \int \frac{A_2}{(x-1)^2}\,\mathrm{d}x + \int \frac{A_1}{x-1}\,\mathrm{d}x + \int \frac{B_1}{x-2}\,\mathrm{d}x.$$

$$\begin{aligned}
\int \frac{x^2+1}{(x-1)^3(x-2)}\,\mathrm{d}x &= \int \frac{-2}{(x-1)^3}\,\mathrm{d}x + \int \frac{-4}{(x-1)^2}\,\mathrm{d}x + \int \frac{-5}{x-1}\,\mathrm{d}x + \int \frac{5}{x-2}\,\mathrm{d}x \\
&= \int -2(x-1)^{-3}\,\mathrm{d}x + \int -4(x-1)^{-2}\,\mathrm{d}x + \int \frac{-5}{x-1}\,\mathrm{d}x + \int \frac{5}{x-2}\,\mathrm{d}x \\
&= -2\int (x-1)^{-3}\,\mathrm{d}x - 4\int (x-1)^{-2}\,\mathrm{d}x - 5\int \frac{1}{x-1}\,\mathrm{d}x + 5\int \frac{1}{x-2}\,\mathrm{d}x \\
&= \cdots
\end{aligned}$$