Discussing a linear system depending on a parameter

Consider the following linear system in the variables x, y, z, t and depending on the parameter $k \in \mathbb{R}$:

$$\begin{cases} x & - & ky & + & z & + & (k-1)t & = & 1 \\ -x & + & ky & - & kz & & & = & -1 \\ (k+1)x & - & 2y & + & 2z & & & = & 2 \end{cases}$$

1. Discuss the solutions of the linear system with respect to k.
2. Find the solution of the linear system for $k = 1$.
3. Add an equation so that the linear system has no solutions for every k.

Solution.

1. By reducing the complete matrix of linear the system (using $R_2 \to R_2 + R_1; R_3 \to R_3 - (k+1)R_1; R_2 \leftrightarrow R_3$) one finds

$$\begin{pmatrix} 1 & -k & 1 & k-1 & | & 1 \\ -1 & k & -k & 0 & | & -1 \\ k+1 & -2 & 2 & 0 & | & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -k & 1 & k-1 & | & 1 \\ 0 & (k+2)(k-1) & 1-k & 1-k^2 & | & 1-k \\ 0 & 0 & 1-k & k-1 & | & 0 \end{pmatrix}$$

   from which one deduces that the ranks of the incomplete matrix and of the complete matrix are equal for every value of k, so the system has a solution for every $k \in \mathbb{R}$. More precisely: if $k \neq -2 \wedge k \neq 1$ the rank of both matrices is 3 and the system has $\infty^{4-3} = \infty^1$ solutions (depending on t); if $k = -2$, the rank is 3 and the system has $\infty^{4-3} = \infty^1$ solutions (depending on y); if $k = -1$, the rank of both matrices is 1 and the system has $\infty^{4-1} = \infty^3$ solutions (depending on y, z and t).

2. If $k = 1$ we have

$$\begin{pmatrix} 1 & -1 & 1 & 0 & | & 1 \\ -1 & 1 & -1 & 0 & | & -1 \\ 2 & -2 & 2 & 0 & | & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

   so $x = y - z + 1$; the solutions of the system are then

$$(x;\, y;\, z;\, t) = (y - z + 1;\, y;\, z;\, t) = (1;\, 0;\, 0;\, 0) + y(1;\, 1;\, 0;\, 0) + z(-1;\, 0;\, 1;\, 0) + t(0;\, 0;\, 0;\, 1),$$

   for every $y,\, z,\, t \in \mathbb{R}$.

3. It is enough to add to the linear system an impossible equation which does not depend on k (like $0 = 1$), or an equation which is incompatible with the previous ones. As instance, the sum of the left hand sides equals a value which is different from the some of the right hand sides: $(k+1)x - 2y + (3-k)z + (k-1)t = A$ for any $A \neq 2$, or $x - ky + z + (k-1)t = 0$.