Orthogonal linear operator

For any linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, for some n, you can find a matrix which implements the mapping.

For example, consider $T : \mathbb{R}^3 \to \mathbb{R}^3$, such that $T(x, y, z) = (3x - y, y + 2z, x + 3z)$.
The standard matrix $A = [a_{ij}]_{3\times 3}$ for the transformation T have 3 columns. The first is the vector $T(1, 0, 0) = (3, 0, 1)$, the second vector is $T(0, 1, 0) = (-1, 1, 0)$ and the third column is $T(0, 0, 1) = (0, 2, 3)$. Therefore,

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}.$$

Notice that $T(v) = Av$, that is, if $v = (v_1, v_2, v_3)$, then

$$T(v_1, v_2, v_3) = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix} . \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Also remember that a square matrix is orthogonal if its inverse is equal to its transposed matrix, that is $A^{-1} = A^T$. The matrix A of the liner operator T above is not orthogonal. In fact,

$$\begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix} . \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & -1 & 3 \\ -1 & 5 & 6 \\ 3 & 6 & 10 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

An orthogonal linear operator is one which preserves not only sums and scalar multiples, but dot products and other related metrical properties such as distances, lengths and angles.

Since metrical properties can all be described in terms of dot products, we use the following definition.

Definition: A linear operator T on 2-space or on 3-space is called orthogonal if it preserves inner products, that is, if $T(u) \cdot T(v) = u \cdot v$ for all vectors u and v. This means that T preserves the norm,

$$||T(v)|| = ||v||, \quad \forall v \in V.$$

So we say that T is an orthogonal operator if it is an isometry.

Example: The linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(x, y) = (y, -x)$ is orthogonal.
Indeed, $||T(x, y)|| = ||(y, -x)|| = \sqrt{y^2 + x^2} = ||(x, y)||$.

Notice that the standard matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

of the linear transformation T ($T(v) = Av$) is an ortogonal matrix. In fact $A^{-1} = A^T$.

Example: The linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(x, y) = (x - y, 3y)$ is not orthogonal. Indeed, $||T(1, -1)|| = ||(2, -3)|| = \sqrt{13} \neq ||(1, -1)|| = \sqrt{2}$.

Properties:

- The inverse of an orthogonal transformation is also orthogonal.
- The composition of orthogonal transformations is orthogonal.

Example: The linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$, such that

$$T(x, y, z) = \left(\frac{3}{5}x + \frac{4}{5}y, -\frac{4}{5}x + \frac{3}{5}y, z\right)$$

is orthogonal. Indeed,

$$||T(x, y, z)|| = \sqrt{\left(\frac{3}{5}x + \frac{4}{5}y\right)^2 + \left(-\frac{4}{5}x + \frac{3}{5}y\right)^2 + z^2} = \sqrt{\frac{9}{25}x^2 + \frac{16}{25}y^2 + \frac{16}{25}x^2 + \frac{9}{25}y^2 + z^2} = ||(x, y, z)||.$$

Consider the standard matrix $A = [a_{ij}]_{3\times 3}$ of this linear operator T. Is A^T orthogonal?

Yes, because if T is an orthogonal linear operator, then $A^{-1} = A^T$, that is, $AA^T = A^T A = I_3$. Just now note that $(A^T)^T = A$.