Operations on Linear Transformations

Let the linear transformations:

$$f: \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \longrightarrow & (2x-y, x+3y) \end{array} \qquad g: \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \longrightarrow & (x+y, x-2y) \end{array}$$

we can always add vectors from the same vector space, so

$$(f+g)(x,y) = f(x,y) + g(x,y) = (2x-y, x+3y) + (x+y, x-2y) = (3x, 2x+y).$$

Also note that $f(x,y) = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ and $g(x,y) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Then:

$$(f+g)(x,y) = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Conclusion: we can only add linear transformations with the same starting space and the same finishing space, and note that the sum is still a linear application.

Theorem: Consider V and E vector spaces and $f : V \to E$ and $g : V \to E$ linear transformations. Then the following functions are also linear transformations:

- $f+g$ defined by $(f+g)(v) = f(v) + g(v)$;
- kf defined by $(kf)(v) = kf(v)$;
- $f \circ g$ defined by $(f \circ g)(v) = (f[g(v)])$, since $g(V) \subseteq D_f$.

Example: Let the linear transformations:

$$f: \begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{R}^2 \\ (x,y,z) & \longrightarrow & (y, x-z) \end{array} \qquad g: \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \longrightarrow & (x+y, x-2y) \end{array}$$

Note that the application sum $f+g$ does not exist, because the starting space of f is different from the starting space of g.

The aplication $-5f$ is defined by $-5f(x,y,z) = -5(y, x-z) = (-5y, -5x+5z)$.

The composite application $g \circ f : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$g(f(x,y,z)) = g(f(x,y,z)) = g(y, x-z) = (x+y-z, -2x+y+2z)$$

and is linear, such that are f and g.

Also given that $f(x,y,z)=\begin{bmatrix}0&1&0\\1&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $g(x,y)=\begin{bmatrix}1&1\\1&-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$, it is proved that

$$g\circ f=\begin{bmatrix}1&1\\1&-2\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1&1&-1\\-2&1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}.$$

Notice that $f\circ g$ does not exist.

Theorem: Consider V and E vector spaces and $f:V\to E$ and $g:E\to W$ linear transformations, such that

$$f(v)=[a_{ij}]_{n\times m}v,\quad v\in V\quad\text{and}\quad g(u)=[b_{ij}]_{m\times r}u,\quad u\in E.$$

Then the composite function $f\circ g:V\to W$ is defined by

$$(g\circ f)(v)=[a_{ij}]_{n\times m}[b_{ij}]_{m\times r}v.$$