DUALITY THEORY

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Given the standard form for the primal problem at the left, its dual problem has the form shown at the right.

Primal Problem

$$\text{Max } Z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i,\ i = 1, \ldots, m$$

$$x_j \geq 0,\ j = 1, \ldots, n.$$

Dual Problem

$$\text{Min } W = \sum_{i=1}^{m} b_i y_i$$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j,\ j = 1, \ldots, n$$

$$y_i \geq 0,\ i = 1, \ldots, m.$$

The primal problem is in maximization form while the dual problem is in minimization form.

Moreover, the dual problem uses the same parameters as the primal problem, but in different positions, as described below:

- The coefficients in the objective function $c_j,\ j = 1, \ldots, n$ of the primal problem are the right-hand sides of the functional constraints in the dual problem.
- The right-hand sides of the functional constraints $b_i,\ i = 1, \ldots, m$ in the primal problem are the coefficients in the objective function of the dual problem.
- The coefficient matrix of the functional constraints of the dual problem is the transpose of the coefficient matrix of the functional constraints of the primal problem.

To highlight the comparison between the primal and the dual problems, the matrix notation can be used to define them. Set $c = [c_1 \ c_2 \ \ldots \ c_n]$ and $y = [y_1 \ y_2 \ \ldots \ y_m]$ the row vectors, considering that b, x and 0 are column vectors while A is a matrix, as described below:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

Matrix notation

Primal Problem	Dual Problem
Maximize $Z = cx$	Minimize $W = yb$
subject to	subject to
$Ax \leq b$	$A^T y \geq c$
$x \geq 0.$	$y \geq 0.$

Example

Primal Problem

$$\text{Max } Z = 8x_1 + 5x_2$$

subject to

$$\begin{aligned} 3x_1 + 6x_2 &\leq 20 \\ 5x_1 + 2x_2 &\leq 40 \\ x_2 &\leq 60 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Dual Problem

$$\text{Min } W = 20y_1 + 40y_2 + 60y_3$$

subject to

$$\begin{aligned} 3y_1 + 5y_2 &\geq 8 \\ 6y_1 + 2y_2 + y_3 &\geq 5 \\ y_1, y_2, y_3 &\geq 0. \end{aligned}$$

Reference

Hillier F. S., & Lieberman G.R. (2010). Introduction to operations research (9th ed.). New York: McGraw-Hill.