1. Let $ABCD$ be a quadrilateral. Let M and N be the midpoints of two opposite sides respectively. Let P be the midpoint of $[MN]$. Prove that $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 0$.

2. Let ABC and $A'B'C'$ be two triangles in $\mathbb{E}^3$ with centroids G and G' respectively. Show that

$$\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = 3\overrightarrow{GG'}.$$

3. Let ABC be a triangle. Consider the points $C' \in [AB]$ and $B' \in [AC]$ such that $\overrightarrow{AC'} = \lambda \overrightarrow{BC'}$ and $\overrightarrow{AB'} = \mu \overrightarrow{CB'}$. Let M be the intersection point of BB' and CC'. Show that

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} - \lambda\overrightarrow{OB} - \mu\overrightarrow{OC}}{1-\lambda-\mu}.$$

for any point O outside the plane of the triangle.

4. Let ABC be a triangle with centroid G, orthocenter H, incenter I and circumcenter Q. Show that

1. $\overrightarrow{OG} = \frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$
2. $\overrightarrow{OI} = \frac{a\overrightarrow{OA}+b\overrightarrow{OB}+c\overrightarrow{OC}}{a+b+c}$
3. $\overrightarrow{OH} = \frac{\tan(\hat{A})\overrightarrow{OA}+\tan(\hat{B})\overrightarrow{OB}+\tan(\hat{C})\overrightarrow{OC}}{\tan(\hat{A})+\tan(\hat{B})+\tan(\hat{C})}$
4. $\overrightarrow{OQ} = \frac{\sin(2\hat{A})\overrightarrow{OA}+\sin(2\hat{B})\overrightarrow{OB}+\sin(2\hat{C})\overrightarrow{OC}}{\sin(2\hat{A})+\sin(2\hat{B})+\sin(2\hat{C})}$

for any point O outside the plane of the triangle.

5. Let BOB' be an angle. Consider $A \in [O,B]$, $A' \in [O,B']$ and $m.n \in \mathbb{R}$ such that $\overrightarrow{OB} = m\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\overrightarrow{OA'}$. Let $M = AB' \cap A'B$ and $N = AA' \cap BB'$. Show that

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\overrightarrow{OA} + n\frac{1-m}{1-mn}\overrightarrow{OA'}$$

and

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\overrightarrow{OA} + n\frac{m-1}{m-n}\overrightarrow{OA'}.$$

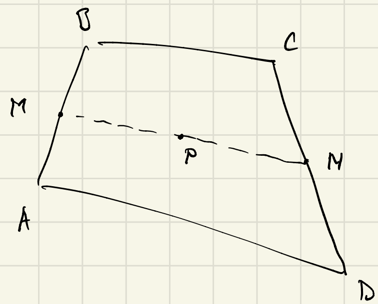
6. Let $OAEBDC$ be a complete quadrilateral. Let M, N, P be the midpoints of the diagonals $[OB]$, $[AC]$ and $[ED]$ respectively. Show that M, N, P are collinear.

7. Let ABC be a triangle with centroid G, orthocenter H and circumcenter Q. Let A' be such that $[AA']$ is a diameter of the circumcenter. Show that

1. $\overrightarrow{QA} + \overrightarrow{QB} + \overrightarrow{QC} = \overrightarrow{QH}$,
2. $\overrightarrow{HA} + \overrightarrow{HC} = \overrightarrow{HA'}$,
3. $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\overrightarrow{HQ}$,
4. $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 3\overrightarrow{HG}$,
5. the points H, G, Q are collinear and $2|GQ| = |HG|$.

1. Let $ABCD$ be a quadrilateral. Let M and N be the midpoints of two opposite sides respectively. Let P be the midpoint of $[MN]$. Prove that $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 0$.

Let M and N be as in the picture



$$\overrightarrow{PA} + \overrightarrow{PB} = 2\,\overrightarrow{PM}$$

$$\overrightarrow{PC} + \overrightarrow{PD} = 2\,\overrightarrow{PN}$$

$$\oplus \quad \overline{\hphantom{\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD}}}$$

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 0$$

2. Let ABC and $A'B'C'$ be two triangles in $\mathbb{E}^3$ with centroids G and G' respectively. Show that

$$\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = 3\,\overrightarrow{GG'}.$$

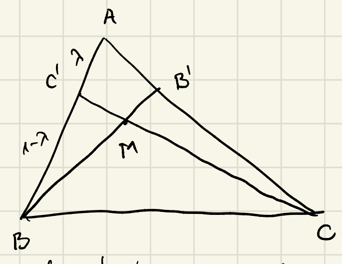
Fix $O \in \mathbb{E}^3$ then $3\overrightarrow{GG'} = 3\overrightarrow{OG'} - 3\underbrace{\overrightarrow{OG}}_{= \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}} = \overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} - \overrightarrow{OA} - \overrightarrow{OB} - \overrightarrow{OC}$

$$= \overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}$$

3. Let ABC be a triangle. Consider the points $C' \in [AB]$ and $B' \in [AC]$ such that $\overrightarrow{AC'} = \lambda\overrightarrow{BC'}$ and $\overrightarrow{AB'} = \mu\overrightarrow{CB'}$. Let M be the intersection point of BB' and CC'. Show that

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} - \lambda\overrightarrow{OB} - \mu\overrightarrow{OC}}{1 - \lambda - \mu}.$$

for any point O outside the plane of the triangle.



$\overrightarrow{OM} = \overrightarrow{OC} + t\,\overrightarrow{CC'} = (1-t)\,\overrightarrow{OC} + t\,\overrightarrow{OC'}$ for some $t \in \mathbb{R}$

$\overrightarrow{OM} = \overrightarrow{OB} + s\,\overrightarrow{BB'} = (1-s)\,\overrightarrow{OB} + s\,\overrightarrow{OB'}$ for some $s \in \mathbb{R}$

We express $\overrightarrow{OM}$ w.r.t the basis $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ in two ways and set the coefficients equal.

$\overrightarrow{OC'} = \overrightarrow{OB} + \overrightarrow{BC'}$ and $\lambda\,\overrightarrow{BC'} = \overrightarrow{AC'} \;\big|\; + \overrightarrow{C'B}$

$$(\lambda - 1)\,\overrightarrow{BC'} = \overrightarrow{AB} \Rightarrow \overrightarrow{BC'} = \frac{1}{\lambda - 1}\,\overrightarrow{OB} - \frac{1}{\lambda - 1}\,\overrightarrow{OA}$$

$$\Rightarrow \overrightarrow{OC'} = \overrightarrow{OB} + \frac{1}{\lambda - 1}\,\overrightarrow{OB} - \frac{1}{\lambda - 1}\,\overrightarrow{OA} = \frac{1}{1-\lambda}\,\overrightarrow{OA} - \frac{\lambda}{1-\lambda}\,\overrightarrow{OB}$$

$$\Rightarrow \vec{OM} = \frac{t}{1-\lambda}\vec{OA} - \frac{t\lambda}{1-\lambda}\vec{OB} + (1-t)\vec{OC}$$

similar $\vec{OM} = \frac{s}{1-\mu}\vec{OA} + (1-s)\vec{OC} - \frac{s\mu}{1-\mu}\vec{OC}$

$\vec{OA}, \vec{OB}, \vec{OC}$ is a basis

so, identifying coefficients we have

$$\begin{cases} \frac{t}{1-\lambda} = \frac{s}{1-\mu} \\ \frac{t\lambda}{1-\lambda} = s-1 \\ \frac{s\mu}{1-\mu} = t-1 \end{cases} \Rightarrow t = \frac{1-\lambda}{1-\mu}s \Bigg\} \Rightarrow \frac{\lambda}{1-\mu}s = s-1 \Rightarrow 1 = \frac{1-\mu-\lambda}{1-\mu}s$$

$$\Rightarrow s = \frac{1-\mu}{1-\mu-\lambda}$$

$$\left(\Rightarrow t = \frac{1-\lambda}{1-\mu-\lambda}\right)$$

$$\Rightarrow \vec{OM} = \frac{1}{1-\mu-\lambda}\vec{OA} + \left(1 - \frac{1-\mu}{1-\mu-\lambda}\right)\vec{OB} - \frac{\mu}{1-\mu-\lambda}\vec{OC} = \frac{1}{1-\mu-\lambda}\left(\vec{OA} - \lambda\vec{OB} - \mu\vec{OC}\right)$$

4. Let ABC be a triangle with centroid G, orthocenter H, incenter I and circumcenter Q. Show that

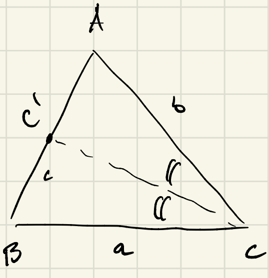
1. $\overrightarrow{OG} = \frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$
2. $\overrightarrow{OI} = \frac{a\overrightarrow{OA}+b\overrightarrow{OB}+c\overrightarrow{OC}}{a+b+c}$
3. $\overrightarrow{OH} = \frac{\tan(\hat{A})\overrightarrow{OA}+\tan(\hat{B})\overrightarrow{OB}+\tan(\hat{C})\overrightarrow{OC}}{\tan(\hat{A})+\tan(\hat{B})+\tan(\hat{C})}$
4. $\overrightarrow{OQ} = \frac{\sin(2\hat{A})\overrightarrow{OA}+\sin(2\hat{B})\overrightarrow{OB}+\sin(2\hat{C})\overrightarrow{OC}}{\sin(2\hat{A})+\sin(2\hat{B})+\sin(2\hat{C})}$

We use the previous exercise and identify λ an μ

1. G = intersection of medians $\Rightarrow \lambda = -1, \mu = -1$

$$\overset{ex3}{\Rightarrow} OG = \frac{\vec{OA} + \vec{OB} + \vec{OC}}{3}$$

2. I = incenter = intersection of angle bisectors



$$\frac{|AC'|}{|BC'|} = \frac{b}{a} \Rightarrow \|\vec{AC'}\| = \frac{b}{a}\|\vec{BC'}\| \rightarrow \vec{AC'} = -\frac{b}{a}\vec{BC'}$$

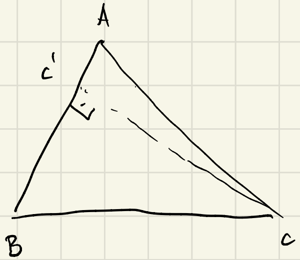
parallel and opposite

$$\Rightarrow \lambda = -\frac{b}{a}$$

$$\Rightarrow \mu = -\frac{c}{a}$$

$$\overset{ex3}{\Rightarrow} \vec{OI} = \frac{\vec{OA} - (-\frac{b}{a})\vec{OB} - (-\frac{c}{a})\vec{OC}}{1 - (-\frac{b}{a}) - (-\frac{c}{a})} = \frac{a\vec{OA} + b\vec{OB} + c\vec{OC}}{a+b+c}$$

3. H = orthocenter = intersection of heights / altitudes



$$\left.\begin{aligned}\tan\hat{A} &= \frac{|CC'|}{|AC'|}\\ \tan\hat{B} &= \frac{|CC'|}{|BC'|}\end{aligned}\right\} \Rightarrow \frac{\tan\hat{B}}{\tan\hat{A}} = \frac{|AC'|}{|BC'|} = \frac{\|\overrightarrow{AC'}\|}{\|\overrightarrow{BC'}\|}$$

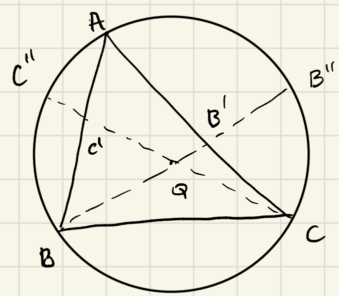
$$\Rightarrow \overrightarrow{AC'} = -\underbrace{\frac{\tan\hat{B}}{\tan\hat{A}}}_{=\lambda}\overrightarrow{BC'}$$

and similar $\mu = -\dfrac{\tan\hat{C}}{\tan\hat{A}}$

ex3 $\Rightarrow$

$$\overrightarrow{OH} = \frac{\overrightarrow{OA} + \frac{\tan\hat{B}}{\tan\hat{A}}\overrightarrow{OB} + \frac{\tan\hat{C}}{\tan\hat{A}}\overrightarrow{OC}}{1 + \frac{\tan\hat{B}}{\tan A} + \frac{\tan C}{\tan A}} = \frac{\tan\hat{A}\cdot\overrightarrow{OA} + \tan\hat{B}\,\overrightarrow{OB} + \tan\hat{C}\cdot\overrightarrow{OC}}{\tan\hat{A} + \tan\hat{B} + \tan\hat{C}}$$

4. Q = circumcenter = intersection of perpendicular bisectors



$$\begin{aligned}\sphericalangle ACC' &= \sphericalangle ACC'' \\ &= \tfrac{1}{2}\sphericalangle AQC'' \\ &= \tfrac{1}{2}\left(180^\circ - \sphericalangle AQC\right) \\ &= 90^\circ - \sphericalangle\hat{B}\end{aligned}
\qquad
\begin{aligned}\sphericalangle BCC' &= \sphericalangle BCC'' \\ &= \tfrac{1}{2}\sphericalangle BQC'' \\ &= \tfrac{1}{2}\left(180^\circ - \sphericalangle BQC\right) \\ &= 90^\circ - \sphericalangle\hat{A}\end{aligned}$$

by the law of sines in $\Delta ACC'$ and in $\Delta BCC'$ we have

$$\frac{|CC'|}{\sin\hat{A}} = \frac{|AC'|}{\sin\sphericalangle ACC'} = \frac{|AC'|}{\sin(90-\hat{B})} = \frac{|AC'|}{\cos\hat{B}}$$

and similar

$$\frac{|CC'|}{\sin\hat{B}} = \frac{|BC'|}{\cos\hat{A}}$$

$$\Rightarrow \frac{|AC'|}{|BC'|} = \frac{\cos\hat{B}\cdot\sin\hat{B}}{\sin\hat{A}\quad\cos\hat{B}} = \frac{\sin 2\hat{B}}{\sin 2\hat{A}} \Rightarrow \lambda = -\frac{\sin 2B}{\sin 2A}$$

and similar by $\mu = -\dfrac{\sin 2C}{\sin 2A}$

so, the expression of $\overrightarrow{OQ}$ follows from ex. 3

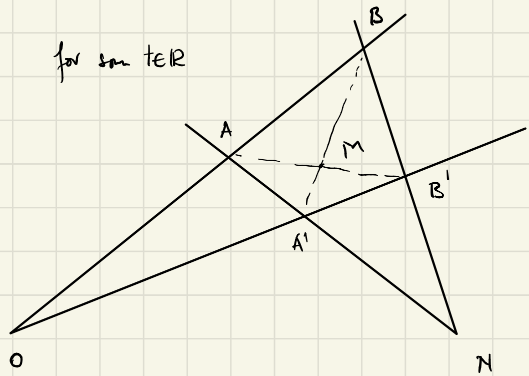
5. Let BOB' be an angle. Consider $A \in [O,B]$, $A' \in [O,B']$ and $m.n \in \mathbb{R}$ such that $\overrightarrow{OB} = m\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\overrightarrow{OA'}$. Let $M = AB' \cap A'B$ and $N = AA' \cap BB'$. Show that

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\overrightarrow{OA} + n\frac{1-m}{1-mn}\overrightarrow{OA'}$$

and

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\overrightarrow{OA} + n\frac{m-1}{m-n}\overrightarrow{OA'}.$$

$M \in AB' \Rightarrow \overrightarrow{OM} = \overrightarrow{OA} + t\overrightarrow{AB'} = (1-t)\overrightarrow{OA} + t\overrightarrow{OB'}$ for some $t \in \mathbb{R}$



$M \in A'B \Rightarrow \overrightarrow{OM} = \overrightarrow{OB} + s\overrightarrow{BA'} = (1-s)\overrightarrow{OB} + s\overrightarrow{OB'}$

$$\Rightarrow \begin{cases} \overrightarrow{OM} = (1-t)\overrightarrow{OA} + t \cdot n\overrightarrow{OA'} \\ \overrightarrow{OM} = (1-s)m\overrightarrow{OA} + s\overrightarrow{OA'} \end{cases}$$

$\overrightarrow{OA}, \overrightarrow{OA'}$ is a basis of $\mathbb{V}^2$

thus, identifying coefficients we have $\begin{cases} 1-t = m(1-s) \\ nt = s \end{cases}$ $\Rightarrow 1-t = m - mnt$

$\Rightarrow 1-m = (1-mn)t$

$\Rightarrow t = \dfrac{1-m}{1-mn}$

$\left(\Rightarrow s = n\dfrac{1-m}{1-mn}\right)$

$-mn+m$

$$\Rightarrow \overrightarrow{OM} = \left(1 - \frac{1-m}{1-mn}\right)\overrightarrow{OA} + n\frac{1-m}{1-mn}\overrightarrow{OA'}$$

$$= m\frac{1-n}{1-mn}\overrightarrow{OA} + n\frac{1-m}{1-mn}\overrightarrow{OA'}$$

$N \in AA' \Rightarrow \overrightarrow{ON} = (1-t)\overrightarrow{OA} + t\overrightarrow{OA'}$

$N \in BB' \Rightarrow \overrightarrow{ON} = (1-s)m\overrightarrow{OA} + sn\overrightarrow{OA'}$

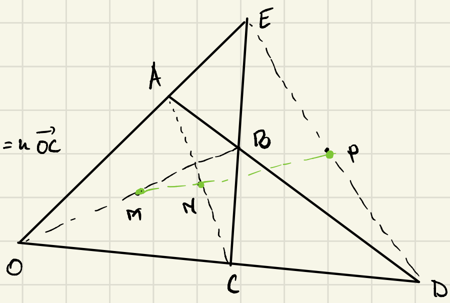
$\Rightarrow \begin{cases} 1-t = (1-s)m \\ t = sn \end{cases}$ $\Rightarrow s = \dfrac{1-m}{n-m}$

$\Rightarrow \overrightarrow{ON} = \ldots$

6. Let $OAEBDC$ be a complete quadrilateral. Let M, N, P be the midpoints of the diagonals $[OB]$, $[AC]$ and $[ED]$ respectively. Show that M, N, P are collinear.

We express everything in terms of $\vec{OA}$ and $\vec{OC}$

$$\vec{ON} = \frac{1}{2}(\vec{OA} + \vec{OC})$$



Let m, n be such that $\vec{OE} = m\,\vec{OA}$ and $\vec{OD} = n\,\vec{OC}$

$$\vec{OP} = \frac{1}{2}(m\vec{OA} + n\vec{OC})$$

$$\vec{OM} = \frac{1}{2}\vec{OB}$$

By exercise 5 we have

$$\vec{OB} = m\frac{1-n}{1-mn}\vec{OA} + n\frac{1-m}{1-mn}\vec{OC}$$

$$\Rightarrow \vec{MP} = \vec{OP} - \vec{OM} = \left(\frac{1}{2}m - \frac{1}{2}m\frac{1-n}{1-mn}\right)\vec{OA} + \left(\frac{1}{2}n - \frac{1}{2}n\frac{1-m}{1-mn}\right)\vec{OC}$$

$$= \frac{-mn}{2(1-mn)}\left[(1-m)\vec{OA} + (1-n)\vec{OC}\right]$$

$$\Rightarrow \vec{NP} = \vec{OP} - \vec{ON} = \frac{1}{2}\left[(m-1)\vec{OA} + (n-1)\vec{OC}\right]$$

$$\Rightarrow \vec{MP} = \frac{mn}{mn-1}\vec{NP} \Rightarrow M, N, P \text{ are collinear.}$$

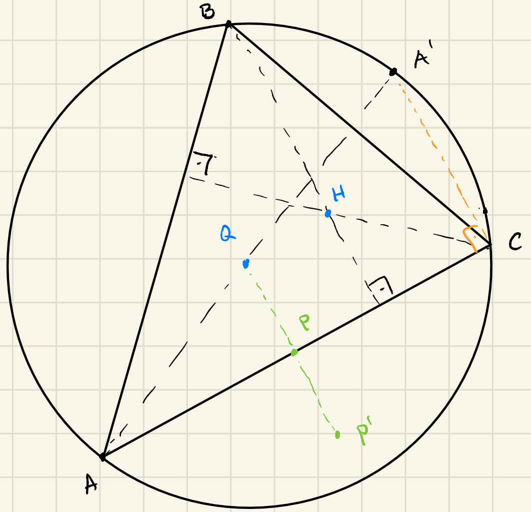
7. Let ABC be a triangle with centroid G, orthocenter H and circumcenter Q. Let A' be such that $[AA']$ is a diameter of the circumcenter. Show that

1. $\overrightarrow{QA} + \overrightarrow{QB} + \overrightarrow{QC} = \overrightarrow{QH}$,
2. $\overrightarrow{HA} + \overrightarrow{HC} = \overrightarrow{HA'}$,
3. $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\overrightarrow{HQ}$,
4. $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 3\overrightarrow{HG}$,

G = intersection of medians

H = intersection of heights / altitudes

Q = intersection of perpendicular bisectors



Let P be the midpoint of AC

Let P' be such that $\overrightarrow{QA} + \overrightarrow{QC} = \overrightarrow{QP'}$

then $1. \iff \overrightarrow{QP'} + \overrightarrow{QB} = \overrightarrow{QH}$

$\iff OP'HB$ is a parallelogram

$\iff 2\overrightarrow{QP} + \overrightarrow{QB} = \overrightarrow{QH}$

Notice that $QP \perp AC$ and $BH \perp AC$

$\Rightarrow QP \parallel BC$ So in order to show 1. is suffices to show that $2|QP| = |BH|$ $(*)$

AA' is a diameter $\Rightarrow$ $\measuredangle ACA'$ is $90°$

$\Rightarrow \Delta APQ \sim \Delta ACA'$ and $2|QP| = |CA'|$, i.e $2\overrightarrow{QP} = \overrightarrow{A'C}$ $(**)$

AA' is a diameter $\Rightarrow A'C \perp AC$ and $A'B \perp AB$

$\Rightarrow A'C \parallel BH$ and $A'B \parallel CH$

$\Rightarrow A'CHB$ is a parallelogram $\Rightarrow |A'C| = |BH|$ $(***)$

$(*)$ & $(**)$ & $(***)$ $\Rightarrow BOP'H$ is a parallelogram and 1. true.

2. $\vec{HB} + \vec{HC} = \vec{HA'}$ from the above argument, $A'CHB$ is a parallelogram

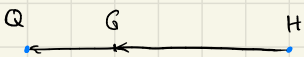
3. $$\underbrace{\vec{HA} + \vec{HB} + \vec{HC}}_{} = 2\vec{HQ}$$

$$\vec{QA} - \vec{QH} + \vec{QB} - \vec{QH} + \vec{QC} - \vec{QH} = \underbrace{\vec{QA} + \vec{QB} + \vec{QC}}_{\overset{\shortparallel}{\vec{QH}} \text{ by 1.}} - 3\vec{QH} = -2\vec{QH} = 2\vec{HQ}$$

4. $$\vec{HA} + \vec{HB} + \vec{HC} = 3\vec{HG}$$

$$\overset{\shortparallel}{\vec{QG} - \vec{QH}} \Rightarrow 3\vec{HG} = \underbrace{3\,\frac{\vec{QA} + \vec{QB} + \vec{QC}}{3}}_{\overset{\shortparallel}{\vec{QH}} \text{ by 1.}} + 3\vec{HQ} = 2\vec{HQ}$$

So $3\vec{HG} = 2\vec{HQ} = \vec{HA} + \vec{HB} + \vec{HC}$
by 1.



$$3\vec{HG} = 2\vec{HQ} \Leftrightarrow -3\vec{GH} = 2\vec{GQ} - 2\vec{GH}$$

$$\Leftrightarrow -2\vec{GQ} = \vec{GH}$$

$$\Rightarrow 2|QG| = |GH|$$