**Proposition 1.1.** For two ordered pairs of points $(A,B)$ and $(C,D)$ the following statements are equivalent:

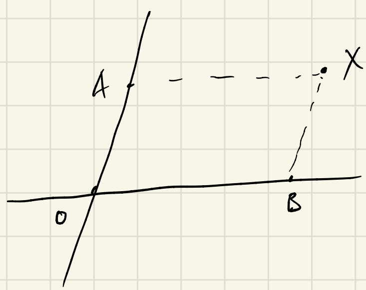
1. $(A,B) \sim (C,D)$.
2. $ABDC$ is a parallelogram.
3. $(A,B)$ and $(C,D)$ have the same distance and direction.

Idea: The three equivalences follow from the fact that

ABCD is a parallelogram if and only if the diagonals AC and BD intersect in their midpoints

**Proposition 1.2.** For any ordered pair of points $(A,B)$ and any point $O$, there is a unique point $X$ such that $(A,B) \sim (O,X)$.

Idea: There is a unique parallelogram with vertices AOBX




**Proposition 1.3.** The equipollence relation is an equivalence relation.

We need to show

Reflexivity $(A,B) \sim (A,B)$

Symmetry $(A,B) \sim (C,D) \Rightarrow (C,D) \sim (A,B)$

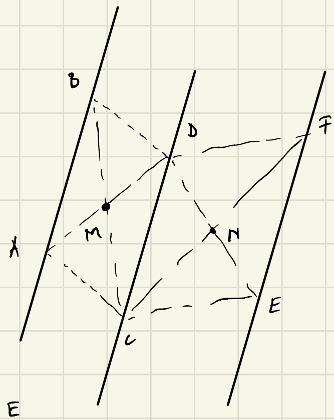
Transitivity $\left.\begin{array}{l}(A,B) \sim (C,D) \\ (C,D) \sim (E,F)\end{array}\right\} \Rightarrow (A,B) \sim (C,D)$

The first two are easy to check.

For the third one, we need to consider all possible cases (positions of points)

Let us look at one case, the others are similar:

Suppose the lines $AB$, $CD$, $EF$ are distinct (as in the picture)




$(A,B) \sim (C,D)$ so $ABDC$ is a parallelogram with center $M$

$(C,D) \sim (E,F)$ so $CDFE$ —— '' —— $N$

$\Rightarrow$ $M$ midpoint of $[AD]$, $N$ —''— $[ED]$ $\Rightarrow$ $[MN]$ is a midsegment in $\Delta ADE$ $\Rightarrow MN \parallel AE$

$M$ midpoint of $[BC]$, $N$ —''— $[CF]$ $\Rightarrow$ $[MN]$ is a midsegment in $\Delta BCF$ $\Rightarrow MN \parallel BF$

$\Rightarrow ABFE$ parallelogram $\Rightarrow (A,B) \sim (E,F)$

**Proposition 1.4.** For any point $O$, the map $\phi_O$ defined by $\phi_O(A) = \overrightarrow{OA}$ is a bijection between points and vectors.

<u>Idea</u> This is a restatement of Prop 1.2 which says that

Let $\overrightarrow{PQ}$ be a vector. By Prop 1.2 $\exists! A$ s.t $(P,Q) \sim (O,A)$

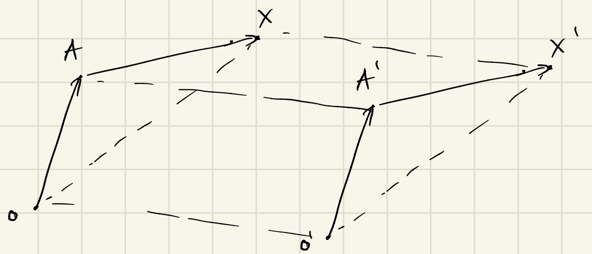
i.e $\exists! A$ such that $\overrightarrow{PQ} = \overrightarrow{OA}$

there exists $A \Rightarrow \phi_o$ is surjective

$A$ is unique $\Rightarrow \phi_o$ is injective

**Proposition 1.5.** The addition of vectors is well defined.

- Notice that we defined the sum of two vectors $a$ and $b$ by choosing representatives $(O,A) \in a$ and $(A,X) \in b$
- So, the question is, does this definition depend on the choices of these representatives? The answer is no but it requires proof
- Idea one shows that if we choose $\overrightarrow{O'A'} = a$ and $\overrightarrow{A'X'} = b$ then $\overrightarrow{O'X'} = \overrightarrow{OX}$ in other words the construction produces the same vector.



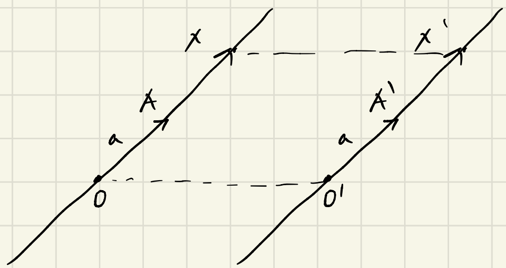

$$\left.\begin{array}{l} |OO'| = |AA'| = |XX'| \\ \text{and } OO' \parallel AA' \parallel XX' \end{array}\right\} \Rightarrow OXX'O' \text{ parallelogram} \Rightarrow \overrightarrow{O'X'} = \overrightarrow{OX}$$

**Proposition 1.6.** The set of vectors $\mathbb{V}^2$ with addition is a commutative group. Similarly, $(\mathbb{V}^3, +)$ is a commutative group.

- commutativity is best observed with the parallelogram rule
- the neutral element is $0 = \overrightarrow{AA}$ $\forall$ point $A$
- the inverse element of $\overrightarrow{AB}$ is $\overrightarrow{BA}$ since $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = 0$

**Proposition 1.7.** The multiplication of scalars with vectors is well defined.

- Again, the definition of multiplication with a scalar depends on the representative of the vector. So, one needs to show that



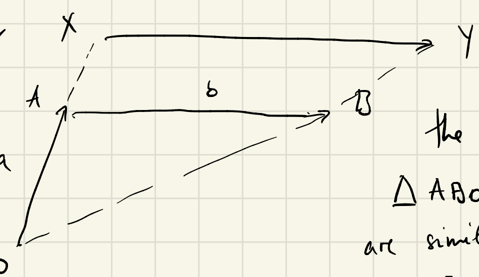

$\vec{OX} = \vec{O'X'}$

**Proposition 1.8.** For $\mathbf{a}, \mathbf{b} \in \mathbb{V}^3$ and $x, y \in \mathbb{R}$ we have

1. $(x+y)\cdot\mathbf{a} = x\cdot\mathbf{a} + y\cdot\mathbf{a}$
2. $x\cdot(\mathbf{a}+\mathbf{b}) = x\cdot\mathbf{a} + x\cdot\mathbf{b}$
3. $x\cdot(y\cdot\mathbf{a}) = (xy)\cdot\mathbf{a}$
4. $1\cdot\mathbf{a} = \mathbf{a}$.

1, 3, 4 follow directly from the definition

for 2. consider




the triangles $\Delta ABO$ and $\Delta OXY$ are similar with similarity factor $x$

so $\quad x a + x b = \vec{OX} + \vec{XY} = \vec{OY} = x \cdot \vec{OB} = x(a+b)$

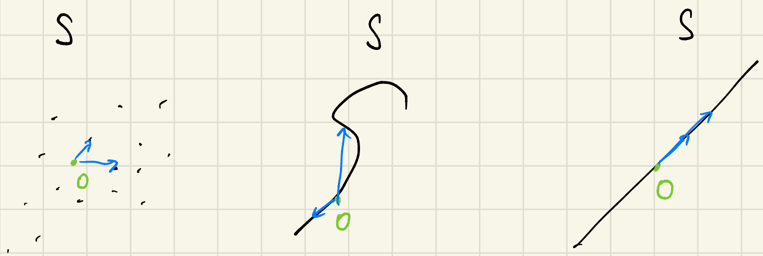
**Theorem 1.9.** The set of vectors $\mathbb{V}^2$ with vector addition and scalar multiplication is a vector space. Similarly, $(\mathbb{V}^3, +, \cdot)$ is a vector space.

Follows from Proposition 1.6 and Proposition 1.8

**Theorem 1.10.** Let $S$ be a subset of $\mathbb{E}^2$ and let $O$ be a point in $S$.

1. The set $S$ is a line if and only if $\phi_O(S)$ is a 1-dimensional vector subspace of $\mathbb{V}^2$.
2. If $S$ is a line then the vector subspace $\phi_O(S)$ is independent of the choice of $O$ in $S$.
3. Two vectors $\overrightarrow{OA}, \overrightarrow{OB}$ are linearly dependent $\Leftrightarrow O, A, B$ are collinear.
4. $\dim \mathbb{V}^2 = 2$, i.e. $\mathbb{V}^2 \cong \mathbb{R}^2$.

Idea: Fix a set of points in the plane $\mathbb{E}^2$




and $O \in S$. Next consider all vectors $V_S$ that you can represent as $\vec{OA}$ with $A \in S$

- is the sum of any two such vectors an element of $V_S$?
- is any scalar multiple of such a vector an element of $V_S$?

If Yes, then $V_S$ is a vector subspace of $\mathbb{V}^2$

Since $\dim \mathbb{V}^2 = 2$ $\quad \dim V_S = \begin{cases} 0 & \Rightarrow S \text{ is a point} \\ 1 & \Rightarrow \text{(straight) line} \\ 2 & \Rightarrow V_S = \mathbb{V}^2 \end{cases}$

The interpretation of the second part of the theorem is similar

- Think about the set of vectors that can be represented with points in $S$