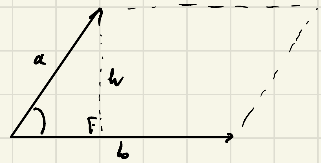
- The norm $\|\mathbf{a} \times \mathbf{b}\|$ equals the area of the parallelogram spanned by the two vectors.

$\shortparallel$

$\|a\| \cdot \|b\| \cdot \sin \measuredangle(a,b)$

$$\sin \measuredangle(a,b) = \frac{h}{\|a\|}$$



$\Rightarrow \|a\| \cdot \|b\| \cdot \sin \measuredangle(a,b) = \|a\| \cdot \|b\| \cdot \frac{h}{\|a\|} = \|b\| \cdot h =$ area of the corresponding parallelogram

Proposition 2.5. For any $\mathbf{a}, \mathbf{b} \in \mathbb{V}^3$ and any $\lambda \in \mathbb{R}$ we have

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
2. $(\lambda \mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$.
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ and $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$.

1. follows from the definition of the vector product:

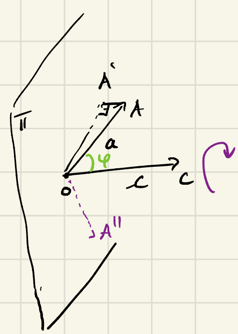
for $a \times b$, $(a, b, a \times b)$ has to be right oriented
for $b \times a$, $(b, a, b \times a)$ —— $\shortparallel$ ——
$\}$ $a \times b$ and $b \times a$ have opposite orientation

2. also follows from the definition

3. because of 1. it is enough to show $(a+b) \times c = a \times c + b \times c$

because of 2. it is enough to show this when $\|c\| = 1$

For this let us describe $a \times c$ in a different way



Fix O and let A, C be such that $a = \vec{OA}$, $c = \vec{OC}$

Consider the plane π passing through O and orthogonal to c

Let A' be the orthogonal projection of A on π

Rotate A' clockwise around OC by $90°$ to obtain A''

If $\varphi = \measuredangle(a,c)$ then

$$\|\overrightarrow{OA''}\| \underset{\substack{\uparrow \\ \text{since we} \\ \text{used a rotation}}}{=} \|\overrightarrow{OA'}\| = \|\overrightarrow{OA}\| \cdot \cos(90^\circ - \varphi) = \|a\| \cdot \sin\varphi = \|a\| \cdot \|c\| \sin\varphi = \|a \times c\|$$

By construction $\overrightarrow{OA''} = a \times c$

With the same construction we obtain B', B'' such that $\overrightarrow{OB''} = b \times c$

and X', X'' such that $\overrightarrow{OX''} = (a+b) \times c$

Then $$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{OB} \underset{\substack{\uparrow \\ \text{since we} \\ \text{project}}}{\Rightarrow} \overrightarrow{OX'} = \overrightarrow{OA'} + \overrightarrow{OB'} \underset{\substack{\uparrow \\ \text{since we} \\ \text{rotate}}}{\Rightarrow} \overrightarrow{OX''} = \overrightarrow{OA''} + \overrightarrow{OB''}$$

$$\Updownarrow$$

$$(a+b) \times c = a \times c + b \times c$$

Theorem 2.6 (Grassmann's vector product formula)**.** For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}^3$ we have

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}. \qquad (*)$$

- If $a \parallel b$ then $\begin{cases} b = \lambda a \text{ and } (a\cdot c)b - (b \cdot c)a = (a \cdot c)\cdot\lambda\cdot a - (\lambda a \cdot c)a = 0 \\ a \times b = 0 \end{cases}$ $(*)$ is true $0 = 0$

- if $a \nparallel b \Rightarrow a, b, a\times b$ is a basis of $\mathbb{V}^3$

$\Rightarrow \exists\, \alpha, \beta, \gamma \in \mathbb{R}$ such that $c = \alpha a + \beta b + \gamma (a \times b)$

$$\Rightarrow (a\times b) \times c = (a \times b) \times (\alpha a + \beta b + \gamma\, a \times b)$$

$$= \alpha (a \times b) \times a + \beta (a \times b) \times b + \gamma \underbrace{(a \times b) \times (a \times b)}_{=0}$$

$$= \alpha (a \times b) \times a + \beta (a \times b) \times b$$

<u>Suppose we</u> prove our claim for $c = a$ and $c = b$ then

$$(a \times b) \times c = \alpha[(a\cdot a)b - (a \cdot b)a] + \beta[(a\cdot b)b - (b \cdot b)a]$$

$$= [-\alpha(a\cdot b) - \beta(b\cdot b)]\,a + [\alpha(a\cdot a) + \beta(a\cdot b)]\,b$$

$$= [-\alpha(a\cdot b) - \beta(b\cdot b) - \gamma\overbrace{(a\times b, b)}^{0}]\,a + [\alpha(a\cdot a) + \beta(b\cdot a) + \gamma\overbrace{(a\times b, a)}^{0}]\,b$$

$$= (-\alpha a - \beta - \gamma\, a\times b, b)\cdot a + (\alpha a + \beta b + \gamma\, a\times b, a)\cdot b$$

$$= -(c, b)\,a + (c, a)\cdot b$$ which proves the claim in general.

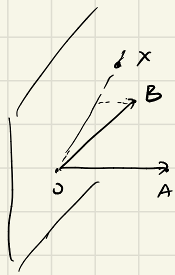
We show the case $c = a$ (the other case, $c = b$, is similar)

so, we show that $(a\times b)\times a = (a\cdot a)\,b - (a\cdot b)\cdot a$

Fix O and let A and B be such that $a = \vec{OA}$ and $b = \vec{OB}$

Let X be such that $\vec{OX} = (a\times b)\times a$

then X is determined by the following properties



1.) $OX \perp OA$ (since $\vec{OX} \perp a$)

and X lies in the plane OAB (since $\vec{OX} \perp a\times b$)

2.) X lies on the same side of OA as B

Indeed $(a\times b, a, \vec{OX})$ is right oriented

$(a, b, a\times b)$ is right oriented $\Rightarrow (a\times b, a, b)$ right oriented

$\Rightarrow$ $\vec{OX}$ and $\vec{OB}$ point in the same direction when observed from U with $\vec{OU} = a\times b$

3.) Let $\theta = \sphericalangle(AOB)$ then

$$\|\vec{OX}\| \overset{def}{=} \|a\times b\|\cdot\|a\|\cdot \sin \overset{\sphericalangle(a\times b, a)}{90^\circ} \underset{def}{=} \|a\|^2\cdot\|b\|\cdot\sin\theta$$

These properties determine X uniquely and we show that

Y with $\vec{OY} = (a\cdot a)\,b - (a\cdot b)\,a$ also satisfies these properties

$$\vec{OY} = (\vec{OA}\cdot\vec{OA})\,\vec{OB} - (\vec{OA}\cdot\vec{OB})\,\vec{OA}$$

then $X = Y$, so $\vec{OX} = \vec{OY}$ and the proof is finished.

1.) Y lies in the plane OAB

moreover $\vec{OY}\cdot\vec{OA} = \left(\vec{OA}^2\cdot\vec{OB} - (\vec{OA}\cdot\vec{OB})\,\vec{OA}\right)\cdot\vec{OA}$

$= \vec{OA}^2\cdot(\vec{OB}\cdot\vec{OA}) - (\vec{OA}\cdot\vec{OB})\cdot\vec{OA}^2 = 0$ so $OY \perp OA$

2.) Y lies on the same side as B with respect to OA

since the coefficient of $\vec{OB}$ in the expression of $\vec{OY}$ is positive, it is $\vec{OA}^2 = \|OA\|^2$

3.) The norm of $\vec{OY}$ is

$$\|\vec{OY}\| = \vec{OY}^2$$

$$= \left[\vec{OA}^2\cdot\vec{OB} - (\vec{OA}\cdot\vec{OB})\,\vec{OA}\right]^2$$

$$= \|\vec{OA}\|^4\cdot\|\vec{OB}\|^2 - 2\,\|\vec{OA}\|^2\,(\vec{OA}\cdot\vec{OB})^2 + (\vec{OA}\cdot\vec{OB})^2\,\|\vec{OA}\|^2$$

$$= \|\vec{OA}\|^4\,\|\vec{OB}\|^2\left(1 - 2\cos\theta^2 - \cos(\theta)^2\right)$$

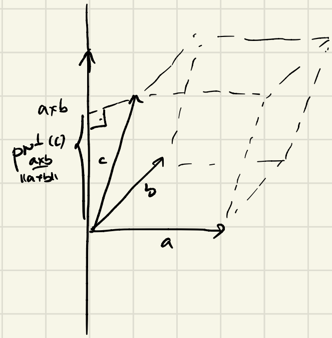
$$= \|\vec{OA}\|^4\cdot\|\vec{OB}\|^2\,\sin\theta$$

So $X = Y \iff \vec{OX} = \vec{OY} \iff (a\times b)\times a = (a\cdot a)b - (a\cdot b)a$

Theorem 2.8. Let $\mathbf{a} = \overrightarrow{OA}, \mathbf{b} = \overrightarrow{OB}, \mathbf{c} = \overrightarrow{OC} \in \mathbb{V}^3$ be non-collinear vectors and let $\mathcal{P}$ be the parallelepiped spanned by the three vectors (i.e. $[OA]$, $[OB]$ and $[OC]$ are sides of the parallelepiped $\mathcal{P}$). Then $[\mathbf{a},\mathbf{b},\mathbf{c}]$ is the volume of $\mathcal{P}$ if $(\mathbf{a},\mathbf{b},\mathbf{c})$ is right oriented; it is minus the volume of $\mathcal{P}$ if $(\mathbf{a},\mathbf{b},\mathbf{c})$ is left oriented.

$$[a,b,c] = (a\times b)\cdot c$$

$$= \underbrace{\|a\times b\|}_{\left(\text{area of parallelogram spanned by } a \text{ and } b\right)} \cdot \underbrace{pr^{\perp}_{\frac{a\times b}{\|a\times b\|}}(c)}_{\pm\left(\text{height of the parallelepiped}\right)}$$



$= \pm$ Volume of the parallelepiped spanned by a, b and c

↳ "+" if a, b, c are right oriented

↳ "−" if a, b, c are left oriented