CHAPTER 10

Quaternions and rotations

Contents

10.1 Another description of rotations in dimension 3

Remark 10.1. Rotations around the coordinate axes are standard examples

$$[\mathrm{Rot}_x(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$[\mathrm{Rot}_x(\theta)] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$[\mathrm{Rot}_x(\theta)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The composition of these maps is $\mathrm{Rot}_x(\psi) \circ \mathrm{Rot}_y(\theta) \circ \mathrm{Rot}_z(\phi) =$

$$\begin{bmatrix} \cos\psi\cos\theta\cos\psi - \sin\psi\sin\phi & -\cos\psi\cos\theta\sin\phi - \sin\psi\cos\phi & \cos\psi\sin\theta \\ \sin\psi\cos\theta\cos\phi + \cos\psi\sin\phi & -\sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta \\ -\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}.$$

All orientation preserving rotations are of this form, i.e. any matrix in SO(3) can be written in this form for some $\psi \in [0, 2\pi[$, $\theta \in [0, \pi[$ and $\phi \in [0, 2\pi[$. The angles ψ, θ and ϕ are called *Euler's angles*. We will see later, in the next section, that there is a better way of dealing with rotations in $\mathbb{E}^3$.

Proposition 10.2 (Euler-Rodrigues)**.** *Let* $\mathbf{v}$ *be a unit vector and* $\theta \in \mathbb{R}$. *The rotation of angle* θ *and axis* $\mathbb{R}\mathbf{v}$ *(here* $\|\mathbf{v}\| = 1$*) is given by*

$$\mathrm{Rot}_{\mathbf{v},\theta}(x) = \cos(\theta)x + \sin(\theta)(\mathbf{v} \times x) + (1 - \cos(\theta))\langle \mathbf{v}, x\rangle \mathbf{v}. \tag{10.1}$$

Proof. We assume that $\mathbf{v}$ is a unit vector. Let $x = x' + x''$ where x' is parallel to $\mathbf{v}$ and x'' is orthogonal to $\mathbf{v}$. We know that

$$x' = \langle v, x\rangle v \quad \text{and} \quad x'' = x - x'.$$

Let $x''' = \mathbf{v} \times x$. A direct calculation shows that

$$\mathrm{Rot}_{\mathbf{v},\theta}(x) = x' + \cos(\theta)x'' + \sin(\theta)x'''. \tag{10.2}$$

The vector x''' is orthogonal to x' and x'' and has the same norm as x'' since

$$\begin{aligned} \|x'''\|^2 &= \langle v \times x, v \times x\rangle \\ &= \langle v, v\rangle\langle x, x\rangle - \langle v, x\rangle\langle v, x\rangle \\ &= \|x\|^2 - \|x'\|^2 \\ &= \|x''\|. \end{aligned}$$

For the second equality we used the fact that

$$\langle v \times x, v \times x\rangle = \|v \times x\|^2 = \|v\|^2\|x\|^2(\sin\alpha)^2 = \|v\|^2\|x\|^2 - \|v\|^2\|x\|^2(\cos\alpha)^2 = \langle v, v\rangle\langle x, x\rangle - \langle v, x\rangle^2.$$

It follows that the vector $\cos(\theta)x'' + \sin(\theta)x'''$ has the same lenght as x'', it is orthogonal to v and forms an angle θ with x''. Therefore, formula (10.2) shows that $\mathrm{Rot}_{\mathbf{v},\theta}$ rotates x'' by θ around the axis $\mathbb{R}v$. □

10.2 Quaternions

10.2.1 Algebraic considerations

Definition. Denote the standard basis of $\mathbb{R}^4$ by 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and consider the bilinear form

$$\_ \cdot \_ : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$$

given on the basis vectors by

$$\begin{array}{c|cccc} & 1 & \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \hline 1 & 1 & \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{i} & \mathbf{i} & -1 & \mathbf{k} & -\mathbf{j} \\ \mathbf{j} & \mathbf{j} & -\mathbf{k} & -1 & \mathbf{i} \\ \mathbf{k} & \mathbf{k} & \mathbf{j} & -\mathbf{i} & -1 \end{array}.$$

We denote $\mathbb{R}^4$ with the above multiplication by $\mathbb{H}$. The elements of $\mathbb{H}$ are called *quaternions*. The product is the *Hamilton product*.

Remark 10.3. From the definition we observe

1. The multiplication map on arbitrary quaternions $p = a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$ and $q = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$ is
$$\begin{aligned} pq &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + a_2b_1 + c_1d_2 - c_2d_1)\mathbf{i} \\ &+ (a_1c_2 + a_2c_1 - b_1d_2 + b_2d_1)\mathbf{j} + (a_1d_2 + a_2d_1 + b_1c_2 - b_2c_1)\mathbf{k} \end{aligned} \tag{10.3}$$

2. Direct calculations show that $\mathbb{H}$ is an algebra, usually called *quaternion algebra*.

3. $\mathbb{H}$ is not commutative, $\mathbf{i} \cdot \mathbf{j} = \mathbf{k} = -\mathbf{j} \cdot \mathbf{i}$.

4. $\mathbb{R} \cdot 1$ is a subfield of $\mathbb{H}$ so we just write $\mathbb{R}$ for it.

5. $\mathbb{C} = \mathbb{R} \cdot 1 + \mathbb{R} \cdot \mathbf{i}$ is a subfield of $\mathbb{H}$.

Definition. For a quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$, a is the *real part* $\mathfrak{Re}(q)$ *of* q and $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ the *imaginary part* $\mathfrak{Im}(q)$ *of* q. We say that q is *real* if it equals its real part. We say that q is *purely imaginary* if it equals its imaginary part.

Proposition 10.4. *A quaternion is real if and only if it commutes with all quaternions, i.e. the center of* $\mathbb{H}$ *is* $\mathbb{R}$.

Proof. That $\mathbb{R} \subseteq \mathbb{H}$ commutes with any other quaternion is clear from the definition. Consider $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ with $a, b, c, d \in \mathbb{R}$. Since q commutes with $\mathbf{i}$ we have

$$a\mathbf{i} - b + c\mathbf{k} - d\mathbf{j} = \mathbf{i}q = q\mathbf{i} = a\mathbf{i} - b + c\mathbf{k} + d\mathbf{j}$$

hence $2(c\mathbf{k} - d\mathbf{j}) = 0$, so $c = d = 0$. Since q commutes with $\mathbf{j}$ we have

$$a\mathbf{j} - b\mathbf{k} = \mathbf{j}q = q\mathbf{j} = a\mathbf{j} + b\mathbf{k}$$

hence $b = 0$ and we obtain $q = a \in \mathbb{R}$. $\square$

Proposition 10.5. *A quaternion is purely imaginary if and only if its square is real and non-pozitive.*

Proof. If $q = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ with $b, c, d \in \mathbb{R}$ then $q^2 = -(b^2 + c^2 + d^2)$.

In the other direction consier $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Then $q^2 = a^2 - b^2 - c^2 - d^2 + 2a(b\mathbf{i} + c\mathbf{j} + d\mathbf{k})$. If q^2 is real then either $a = 0$ in which case q is purely imaginary or $b = c = d = 0$ and $a \neq 0$ in which case q is real and $q^2 = a^2$ is positive. So, q is purely imaginary if and only if q^2 is real and non-positive. $\square$

Definition. For a quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$, the *conjugate of* q is

$$\overline{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k} = \mathfrak{Re}(q) - \mathfrak{Im}(q) \in \mathbb{H}.$$

Proposition 10.6. *For $p, q \in \mathbb{H}$ and $a \in \mathbb{R}$ we have*

1. $\overline{p+q} = \overline{p} + \overline{q}$
2. $\overline{ap} = a\overline{p}$
3. $\overline{\overline{p}} = p$
4. $\overline{p \cdot q} = \overline{q} \cdot \overline{p}$
5. $p \in \mathbb{R} \Leftrightarrow \overline{p} = p$
6. p *is purely imaginary* $\Leftrightarrow \overline{p} = -p$
7. $\mathfrak{Re}(p) = \frac{1}{2}(p + \overline{p})$
8. $\mathfrak{Im}(p) = \frac{1}{2}(p - \overline{p})$

Proof. First notice that $\mathfrak{Re}$ and $\mathfrak{Im}$ are linear maps, being projections on vector subspaces.
1. We have $\overline{p+q} = \mathfrak{Re}(p+q) - \mathfrak{Im}(p+q) = (\mathfrak{Re}(p) - \mathfrak{Im}(p)) + (\mathfrak{Re}(q) - \mathfrak{Im}(q)) = \overline{p} + \overline{q}$.
2. $\overline{aq} = \mathfrak{Re}(aq) - \mathfrak{Im}(aq) = a(\mathfrak{Re}(q) - \mathfrak{Im}(q)) = a\overline{q}$.
3. $\overline{\overline{q}} = \mathfrak{Re}(\overline{q}) - \mathfrak{Im}(\overline{q}) = \mathfrak{Re}(q) - (-\mathfrak{Im}(q)) = q$
4. As in (10.3) we have

$$\begin{aligned} pq &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + a_2b_1 + c_1d_2 - c_2d_1)\mathbf{i} \\ &+ (a_1c_2 + a_2c_1 - b_1d_2 + b_2d_1)\mathbf{j} + (a_1d_2 + a_2d_1 + b_1c_2 - b_2c_1)\mathbf{k} \end{aligned}$$

So

$$\begin{aligned} \overline{pq} &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) - (a_1b_2 + a_2b_1 + c_1d_2 - c_2d_1)\mathbf{i} \\ &- (a_1c_2 + a_2c_1 - b_1d_2 + b_2d_1)\mathbf{j} - (a_1d_2 + a_2d_1 + b_1c_2 - b_2c_1)\mathbf{k} \end{aligned}$$

and

$$\begin{aligned} \overline{q}\,\overline{p} &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (-a_1b_2 - a_2b_1 + c_2d_1 - c_1d_2)\mathbf{i} \\ &+ (-a_1c_2 - a_2c_1 - b_2d_1 + b_1d_2)\mathbf{j} + (-a_1d_2 - a_2d_1 + b_2c_1 - b_1c_2)\mathbf{k} \end{aligned}$$

5., 6., 7. and 8. are clear from the definition of $\mathfrak{Im}$ and $\mathfrak{Re}$. ☐

10.2.2 $\mathbb{H}$ and $\mathbb{E}^4$

By construction $\mathbb{H}$ is $\mathbb{R}^4$ as real vector space, so we may view it as a 4-dimensional real affine space. If in addition we consier the 4-dimensional Euclidean structure we may identify $\mathbb{H}$ with $\mathbb{E}^4$. In particular, we may consider the standard scalar product $\langle \_, \_ \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \cong \mathbb{R}^4$.

Proposition 10.7 (Compare this with the similar statements for $\mathbb{C} \cong \mathbb{E}^2$)**.** *For $p, q \in \mathbb{H}$ we have*

1. $\langle p, q \rangle = \frac{1}{2}(\overline{p}q + \overline{q}p)$

2. $\langle p,p\rangle = \overline{p}p$
3. $||p|| = \sqrt{\overline{p}p}$

If in addition p and q are purely imaginary, we have

4. $\langle p,q\rangle = -\frac{1}{2}(pq+qp) = -\mathfrak{Re}(pq)$
5. $\langle p,p\rangle = -p^2$
6. $||p|| = \sqrt{-p^2}$
7. $\langle p,q\rangle = 0 \Leftrightarrow pq = -qp.$

Proof. 2. and 3. are particular cases of 1. which can be checked directly with (10.3) or from

$$\langle p,q\rangle = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2$$

$$\overline{p}q = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 + \mathfrak{Im}(\overline{p}q)$$
$$\overline{q}p = \overline{\overline{p}q} = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 - \mathfrak{Im}(\overline{p}q)$$

where we used 4. in the previous proposition.

The other statements follow since for p purely imaginary we have $\overline{p} = -p$. □

Definition. With our identification $||q|| = (\overline{q}\,q)^{\frac{1}{2}}$ is the *norm* of the quaternion q. If $||q|| = 1$ we say that q is a *unit quaternion*.

Proposition 10.8. *For any $p,q \in \mathbb{H}$ we have*

$$||pq|| = ||p|| \cdot ||q||.$$

In particular, left and right multiplication by unit quaternions are isometries.

Proof. We have $\overline{q}q = q\overline{q} \in \mathbb{R}$ so

$$||pq||^2 = \overline{pq}pq = \overline{q}(\overline{p}q)p = \overline{q}(q\overline{p})p = ||p||^2||q||^2.$$

Hence, since $||q|| > 0$, the statement follows. □

Proposition 10.9. *$\mathbb{H}$ is a skew field. The inverse of $q \in \mathbb{H} \setminus \{0\}$ is*

$$q^{-1} = \frac{\overline{q}}{||q||^2}.$$

Proof. Indeed

$$\frac{\overline{q}}{||q||^2}q = \frac{\overline{q}q}{||q||^2} = \frac{||q||^2}{||q||^2} = 1 = \frac{||q||^2}{||q||^2} = \frac{q\overline{q}}{||q||^2} = q\frac{\overline{q}}{||q||^2}.$$

□

10.2.3 $\mathbb{H}$ and rotations in $\mathbb{E}^3$

We identified $\mathbb{H}$ with $\mathbb{E}^4$. Next we view $\mathbb{E}^3$ as a subspace of $\mathbb{H}$ identifying it with purely imaginary quaterions $\mathfrak{Im}(\mathbb{H}) = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$.

Proposition 10.10. *Let q_1, q_2 be two quaternions with $a_i = \mathfrak{Re}\, q_i$, $v_i = \mathfrak{Im}\, q_i$. Making use of the scalar product and the vector product in $\mathbb{E}^3$ me have*

$$q_1 q_2 = (a_1 + v_1)(a_2 + v_2) = a_1 a_2 - \langle v_1, v_2\rangle + a_2 v_1 + a_1 v_2 + v_1 \times v_2. \tag{10.4}$$

Proof. This is a direct consequence of the explicit form of the Hamilton multiplication (10.3). □

Proposition 10.11. *Let $v = v_i\mathbf{i} + v_j\mathbf{j} + v_k\mathbf{k} \in D(\mathbb{E}^3) \cong \mathfrak{Im}(\mathbb{H})$ be a unit quaternion and $p \in \mathbb{E}^3 \cong \mathfrak{Im}(\mathbb{H})$ a point. The rotation of p around the axis $\mathbb{R}v$ by an angle θ is given by*

$$p' = qpq^{-1}$$

where

$$q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) v$$

Proof. By (10.1) we are done if we show that

$$p' = \cos(\theta)p + \sin(\theta)(v \times p) + (1 - \cos(\theta))\langle v, p\rangle v.$$

Notice that q is unitary so $q^{-1} = \overline{q}$. Hence, we need to show

$$p' = qp\overline{q}.$$

Now, by (10.4), we have

$$qp = -\sin\frac{\theta}{2}\langle v, p\rangle + \cos\frac{\theta}{2} p + \sin\frac{\theta}{2} v \times p.$$

Again, by (10.4), we have

$$qp\overline{q} = \left(-\sin\frac{\theta}{2}\langle v,p\rangle + \cos\frac{\theta}{2}p + \sin\frac{\theta}{2}v\times p\right)\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}v\right)$$

$$= -\cos\frac{\theta}{2}\sin\frac{\theta}{2}\langle v,p\rangle + \langle \cos\frac{\theta}{2}p + \sin\frac{\theta}{2}v\times p, \sin\frac{\theta}{2}v\rangle + \left(\sin\frac{\theta}{2}\right)^2\langle v,p\rangle v + \left(\cos\frac{\theta}{2}\right)^2 p + \cos\frac{\theta}{2}\sin\frac{\theta}{2}v\times p$$
$$-\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}p + \sin\frac{\theta}{2}v\times p\right)\times v$$

$$= \left(\sin\frac{\theta}{2}\right)^2\langle v,p\rangle v + \left(\cos\frac{\theta}{2}\right)^2 p + \cos\frac{\theta}{2}\sin\frac{\theta}{2}v\times p - \sin\frac{\theta}{2}\cos\frac{\theta}{2}p\times v - \left(\sin\frac{\theta}{2}\right)^2 \underbrace{(v\times p)\times v}_{\langle v,v\rangle p - \langle p,v\rangle v}$$

$$= \underbrace{\left[\left(\cos\frac{\theta}{2}\right)^2 - \left(\sin\frac{\theta}{2}\right)^2\right]}_{\cos(\theta)} p + \underbrace{2\left(\sin\frac{\theta}{2}\right)^2}_{1-\cos(\theta)}\langle v,p\rangle v + \underbrace{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}_{\sin(\theta)} v\times p$$

□