Changing reference frames.

1. We consider two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}')$ (see Fig. 0.1) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.

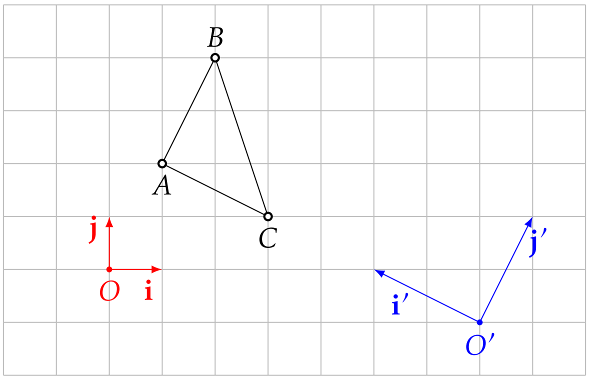


Figure 0.1: Coordinate systems 2D.

2. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the lines AB, AC, BC both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

3. Consider the tetrahedron $ABCD$ (see Fig. 0.2) and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

1. the coordinates of the vertices of the tetrahedron in the three coordinate systems,
2. the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,
3. the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.

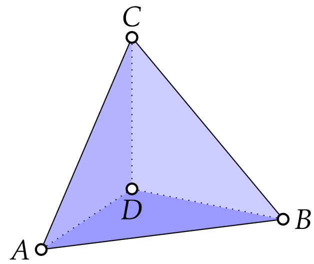


Figure 0.2: Tetrahedron

4. We consider the coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}', \mathbf{k}')$ (see Fig. 0.3) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{k}']_{\mathcal{K}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.

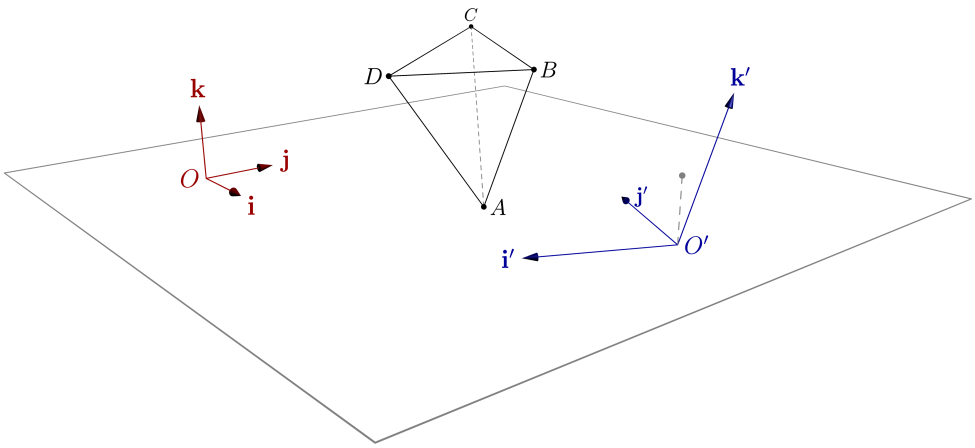


Figure 0.3: Coordinate systems 3D.

5. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the line AB and the plane ACD both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

Projections and reflections on/in hyperplanes.

6. Consider $\mathbf{v}(2,1,1) \in \mathbb{V}^3$ and $Q(2,2,2) \in \mathbb{E}^3$.

1. Give the matrix form for the parallel projection on the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.
2. Give the matrix form for the parallel reflection in the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.

7. Write down the vector forms and matrix forms for parallel projections and reflections in $\mathbb{E}^3$.

8. In $\mathbb{E}^2$, for the lines/hyperplanes

$$\pi : ax + by + c = 0, \quad \ell : \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2}$$

with $\pi \nparallel \ell$, deduce the matrix forms of $\mathrm{Pr}_{\pi,\ell}$ and $\mathrm{Ref}_{\pi,\ell}$.

9. Let H be a hyperplane and let $\mathbf{v}$ be a vector. Use the deduced compact matrix forms to show that

1. $\mathrm{Pr}_{H,\mathbf{v}} \circ \mathrm{Pr}_{H,\mathbf{v}} = \mathrm{Pr}_{H,\mathbf{v}}$ and
2. $\mathrm{Ref}_{H,\mathbf{v}} \circ \mathrm{Ref}_{H,\mathbf{v}} = \mathrm{Id}$.

10. Give Cartesian equations for the line passing through the point $M(1,0,7)$, parallel to the plane $\pi : 3x - y + 2z - 15 = 0$ and intersecting the line

$$\ell : \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

11. In $\mathbb{E}^3$, show that the orthogonal reflection $\mathrm{Ref}_{\pi}^{\perp}(x)$ in the plane $\pi : \langle n, x \rangle = p$ is given by

$$\mathrm{Ref}_{\pi}(x) = Ax + b$$

where $A = \left(I - 2\frac{nn^t}{\|n\|^2}\right)$ and $b = \frac{2p}{\|n\|^2} n$.

12. Give the matrix form for the orthogonal reflections in the planes

$$\pi_1 : 3x - 4z = -1 \quad \text{and} \quad \pi_2 : 10x - 2y + 3z = 4 \quad \text{respectively.}$$

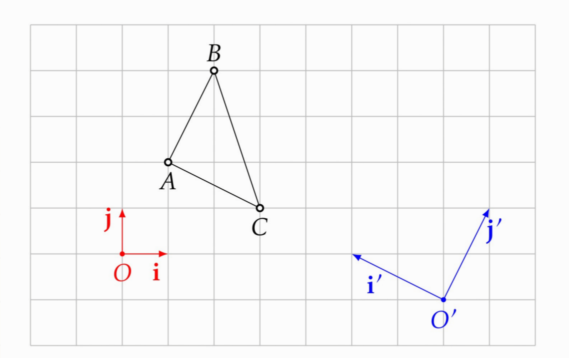
1. We consider two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}')$ (see Fig. 1) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.



• Let $v = (i, j)$ and $w = (i', j')$

• The base change matrix from K' to K is the matrix $M_{v,w}(Id)$

$$M_{K,K'} = M_{v,w} = \begin{array}{c} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \\ \begin{matrix} [i']_K & [j']_K \end{matrix} \end{array}$$

are given

• In order to change coordinates, we have $[A]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}([A]_{\mathcal{K}} - [O']_{\mathcal{K}}) = \mathrm{M}_{\mathcal{K}',\mathcal{K}}[A]_{\mathcal{K}} + [O]_{\mathcal{K}'}$

• $M_{K',K} = M_{K,K'}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \cdot \frac{1}{-5}$

So $[A]_{K'} = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \frac{1}{-5} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \frac{1}{-5} \begin{bmatrix} -6 \\ 3 \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} -15 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$[B]_{K'} = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$[C]_{K'} = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

• In order to change coordinates backwards we use $[A]_K = M_{K,K'}([A]_{K'} - [O]_{K'}) =$

$= M_{K,K'}[A]_{K'} + [O']_K$

Notice that $[A]_K = M_{K,K'}(M_{K',K}[A]_K + [O]_{K'}) + [O']_K$

$= [A]_K + M_{K,K'}[O]_{K'} + [O']_K = [A]_K$

2. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the lines AB, AC, BC both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

- In K we have $\overrightarrow{AB} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow AB_K : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{= [v]_K} \quad t \in \mathbb{R} \qquad (*)$

  In K': $AB_{K'}: \begin{bmatrix} x' \\ y' \end{bmatrix} = M_{K',K}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right)$

  $$= \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \left(\begin{bmatrix} t \\ 2t \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \end{bmatrix} \right)$$

  $$= \frac{1}{5} \left(\begin{bmatrix} 0 \\ 5t \end{bmatrix} + \begin{bmatrix} 15 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

  $$AB_{K'}: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + t \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{= [v]_{K'}}$$

  Notice that if $\ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ in K then

  in K' we have $\ell : \begin{bmatrix} x' \\ y' \end{bmatrix} = M_{K',K} \left(\begin{bmatrix} x_A \\ y_A \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \end{bmatrix} - [O']_K \right)$

  $$= \underbrace{M_{K',K}\left(\begin{bmatrix} x_A \\ y_A \end{bmatrix} - [O']_K \right)}_{= [A]_{K'}} + t \underbrace{M_{K',K} \begin{bmatrix} v_x \\ v_y \end{bmatrix}}_{= [v]_{K'}}$$

- $(*) \Rightarrow \begin{cases} x = 1 + t \\ y = 2 + 2t \end{cases} \Rightarrow t = x - 1 = \frac{y-2}{2} \Rightarrow \ell : 2x - \cancel{2} - y + \cancel{2} = 0$

  $\ell : 2x - y = 0$

  So, in K we have $\ell : \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

  $\Rightarrow$ in K' we have $\ell : \begin{bmatrix} 2 & -1 \end{bmatrix} \left(M_{K,K'} \begin{bmatrix} x' \\ y' \end{bmatrix} + [O']_K \right)$

  $$\Leftrightarrow \begin{bmatrix} 2 & -1 \end{bmatrix} \left(\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = 0$$

  $$\Leftrightarrow \begin{bmatrix} -5, & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + 15 = 0$$

  AC and BC are treated similarly

  $$\Leftrightarrow -5x' + 15 = 0$$

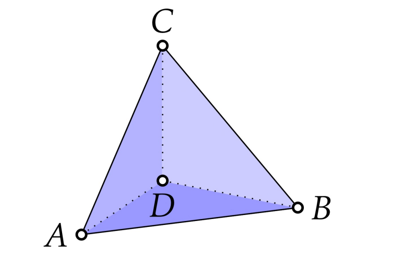
  $$\ell_{K'} : x' - 3 = 0$$

3. Consider the tetrahedron $ABCD$ (see Fig. 3) and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

1. the coordinates of the vertices of the tetrahedron in the three coordinate systems,
2. the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,
3. the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.



a.) $[A]_{K_A} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $[B]_{K_A} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $[C]_{K_A} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $D_{K_A} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$[A]_{K'_A} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $[B]_{K'_A} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $[C]_{K'_A} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $[D]_{K'_A} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$[A]_{K_B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $[B]_{K_B} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $[C]_{K_B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $[D]_{K_B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

b.) Let $M = M_{K'_A, K_A}$ $\quad M[A]_{K_A} = [A]_{K'_A}$

$M\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $M\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

basis vectors in K'_A relative to K_A

$$M\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Leftrightarrow M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

c.) $\overrightarrow{BA} = -\overrightarrow{AB}$ $\quad [\overrightarrow{BA}]_{K_A} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$ $\quad [\overrightarrow{BC}]_{K_A} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$ $\quad [\overrightarrow{BD}]_{K_A} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

So $M_{K_A, K_B} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the base change matrix from K_B to K_A

4. We consider the coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}', \mathbf{k}')$ (see Fig. 4) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{k}']_{\mathcal{K}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.

$$M_{KK'} = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow M_{K'K} = M_{KK'}^{-1} = \begin{bmatrix} 2 & 4 & 0 \\ 4 & -2 & 0 \\ -2 & 1 & -5 \end{bmatrix}^T \cdot \frac{1}{-10} = \frac{1}{10}\begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\det M_{KK'} = -2 - 8$$

$$\Rightarrow [A]_{K'} = \frac{1}{10}\begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$[B]_{K'} = \frac{1}{10}\begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

similarly for $[C]_{K'}$ and $[D]_{K'}$

Notice that $[A]_K = M_{K,K'}\left(M_{K',K}[A]_K + [O]_{K'}\right) + [O']_K = [A]_K + M_{K,K'}[O]_{K'} + [O']_K = [A]_K$

$$M_{K,K'}[O]_{K'} = [O']_K$$

5. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the line AB and the plane ACD both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

line AB in K

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underset{A}{\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}} + t \underset{\overrightarrow{AB}}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}$$

$\Rightarrow$ line AB in K'

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = M_{K'K}\left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \overset{[O']_K}{\begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}} \right)$$

$$= M_{K'K}\left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} \right) + t\, M_{K'K} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= \underset{[A]_{K'}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} + t\, M_{K'K} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \ldots\ldots$$

If the plane ACD in K is $ax+by+cz+d=0 \iff [a\ b\ c] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -d$

Then the plane ACD in K' is $[a\ b\ c]\left(M_{K,K'} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + \overset{[O']_K}{\begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}} \right) = -d$

6. Consider $\mathbf{v}(2,1,1) \in \mathbb{V}^3$ and $Q(2,2,2) \in \mathbb{E}^3$.

1. Give the matrix form for the parallel projection on the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.
2. Give the matrix form for the parallel reflection in the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.

- Let $\ell = Q + \langle v \rangle$

Consider a point $P(x_0, y_0, z_0) \in \mathbb{A}^3(\mathbb{R})$

the line containing P and parallel to ℓ is $\ell_P : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_0 + 2t \\ y_0 + t \\ z_0 + t \end{bmatrix}$

$\ell_P \cap \pi : \quad z_0 + t = 0 \iff t = -z_0$ so $\ell \cap \pi = \begin{bmatrix} x_0 - 2z_0 \\ y_0 - z_0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

So, $\quad Pr_{\pi, \ell} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- For the reflection let $P'(x_1, y_1, z_1)$ be the reflection of P in π along ℓ

Then $\quad \frac{1}{2}\left(\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \right) = \begin{bmatrix} x_0 - 2z_0 \\ y_0 - z_0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_0 - 4z_0 \\ y_0 - 2z_0 \\ -z_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

<u>Rem</u> clearly, one can also apply the general formulas that we deduced

$\hookrightarrow$ do this and check that you get the same answer.

7. Write down the vector forms and matrix forms for parallel projections and reflections in $\mathbb{E}^3$.

- We show this for projections in hyperplanes (the other cases are similar)
- the vector forms don't change:

$$\mathrm{Pr}_{H,\mathbf{v}}(P) = P - \frac{\varphi(P)}{\operatorname{lin}\varphi(\mathbf{v})}\mathbf{v}.$$

but H is in this case a plane so it has an equation of the form

$H: ax+by+cz+d=0$

and $v = v(v_x, v_y, v_z)$

- For the matrix form we have

$$[\mathrm{Pr}_{H,\mathbf{v}}(P)]_{\mathcal{K}} = \left(\mathrm{Id}_n - \frac{\mathbf{v}\cdot\mathbf{a}^t}{\mathbf{v}^t\cdot\mathbf{a}}\right)\cdot[P]_{\mathcal{K}} - \frac{a_{n+1}}{\mathbf{v}^t\cdot\mathbf{a}}[\mathbf{v}]_{\mathcal{K}}$$

which in our case becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{av_x+bv_y+cv_z}\begin{pmatrix} v_x a & v_x b & v_x c \\ v_y a & v_y b & v_y c \\ v_z a & v_z b & v_z c \end{pmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{d}{av_x+bv_y+cv_z}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$$= \frac{1}{av_x+bv_y+cv_z}\begin{bmatrix} bv_y+cv_z & v_x b & v_x c \\ v_y a & av_x+cv_z & v_y c \\ v_z a & v_z b & av_x+bv_y \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{d}{av_x+bv_y+cv_z}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

8. In $\mathbb{E}^2$, for the lines/hyperplanes

$$\pi : ax + by + c = 0, \quad \ell : \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2}$$

with $\pi \nparallel \ell$, deduce the matrix forms of $\mathrm{Pr}_{\pi,\ell}$ and $\mathrm{Ref}_{\pi,\ell}$.

As in the previous exercise we consider $\mathrm{Pr}_{\pi,\ell}$ since $\mathrm{Ref}_{\pi,\ell}$ is similar

$$\mathrm{Pr}_{\pi,\ell}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{av_1 + bv_2}\begin{bmatrix} bv_2 & cv_2 \\ bv_1 & cv_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \frac{d}{av_1 + bv_1}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

9. Let H be a hyperplane and let $\mathbf{v}$ be a vector. Use the deduced compact matrix forms to show that

1. $\mathrm{Pr}_{H,\mathbf{v}} \circ \mathrm{Pr}_{H,\mathbf{v}} = \mathrm{Pr}_{H,\mathbf{v}}$ and
2. $\mathrm{Ref}_{H,\mathbf{v}} \circ \mathrm{Ref}_{H,\mathbf{v}} = \mathrm{Id}$.

Rem from the definition of these maps it should be clear that the indicated relations are true

1.
$$[\mathrm{Pr}_{H,\mathbf{v}}(P)]_{\mathcal{K}} = \left(\mathrm{Id}_n - \frac{\mathbf{v}\cdot\mathbf{a}^t}{\mathbf{v}^t\cdot\mathbf{a}}\right)\cdot[P]_{\mathcal{K}} - \frac{a_{n+1}}{\mathbf{v}^t\cdot\mathbf{a}}[\mathbf{v}]_{\mathcal{K}}$$

So
$$\left[\mathrm{Pr}_{H,v} \circ \mathrm{Pr}_{H,v}(P)\right]_K = \left[\mathrm{Pr}_{H,v}\left(\mathrm{Pr}_{H,v}([P]_K)\right)\right]$$

$$= \left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)\mathrm{Pr}_{H,v}([P]_K) - \frac{a_{n+1}}{v^t\cdot a}v$$

$$= \left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)\left[\left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)[P]_K - \frac{a_{n+1}}{v^t\cdot a}v\right] - \frac{a_{n+1}}{v^t\cdot a}v$$

$$(*) = \left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)^2[P]_K - \left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)\frac{a_{n+1}}{v^t\cdot a}v - \frac{a_{n+1}}{v^t\cdot a}v$$

$$- \frac{a_{n+1}}{(v^t\cdot a)^2}\left(2v^t\cdot a\cdot\mathrm{Id}_n - v\cdot a^t\right)\cdot v$$

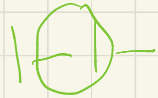
$$\underbrace{2v^t\cdot a\cdot v - \underbrace{v\cdot a^t\cdot v}_{\substack{\parallel \\ v^t\cdot a\cdot v}}}_{v^t\cdot a\cdot v}$$

$$= -\frac{a_{n+1}}{v^t\cdot a}v$$

so

$$(*) = \underbrace{\left(Id_n - \frac{v \cdot a^t}{v^t \cdot a} \right)^2}_{\substack{\shortparallel \\ \underbrace{Id_n - 2\frac{v \cdot a^t}{v^t \cdot a} + \underbrace{\frac{v \cdot a^t}{v^t \cdot a} \cdot \frac{v \cdot a^t}{v^t \cdot a}}_{\substack{\shortparallel \\ \frac{\cancel{(v^t a)}(v \cdot a^t)}{\cancel{(v^t a)} \cdot (v^t \cdot a)}}}}_{= Id_n - \frac{v \cdot a^t}{v^t \cdot a}}}} [p]_K - \frac{a_{n+1}}{v^t \cdot a} v$$

$v^t a \cdot v \cdot a^t$

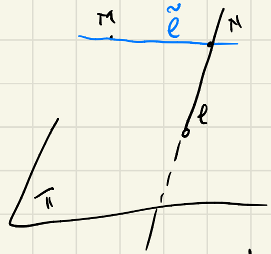


so $(*) = \left(Id - \frac{v \cdot a^t}{v^t \cdot a} \right) [p]_k - \frac{a_{n+1}}{v^t a} v = [Pr_{H,v}(p)]_k$ $\square$

for 2. the calculation is similar

10. Give Cartesian equations for the line $\tilde{\ell}$ passing through the point $M(1,0,7)$, parallel to the plane $\pi: 3x - y + 2z - 15 = 0$ and intersecting the line

$$\ell: \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$



$\boxed{\text{Method 1}}$ Determine the point $N = \ell \cap \tilde{\ell}$ using a projection

$N = Pr_{\ell,\pi}(M)$

$$Pr_{\ell,\pi}(p) = p' = q - \frac{\varphi(q)}{(\text{lin}\,\varphi)(v)}\, v \quad \text{where}$$

- q is a point on ℓ
- v is a direction vector of ℓ
- $\varphi^{-1}(0)$ is the plane parallel to π and containing p

in our case $\ell: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}}_{q} + t\underbrace{\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}}_{v}$ and, if $\tilde{\pi} \parallel \pi$ and $\tilde{\pi} \ni p$ then

$\text{lin}\,\varphi(x,y,z) = \text{lin}\,\psi(x,y,z)$

$$\tilde{\pi}: \underbrace{3x - y + 2z + D}_{\varphi(x,y,z)} = 0$$

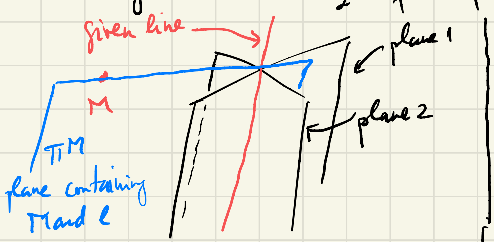
D is determined by the condition $\varphi(p) = 0 \Rightarrow \varphi(x,y,z) = (\text{lin}\,\varphi)(x,y,z) - (\text{lin}\,\varphi)(p)$

$\Rightarrow$ We obtain $N = Pr_{\ell,\pi}(M) = \begin{pmatrix} -\frac{14}{3} \\ \frac{1}{6} \\ -\frac{17}{2} \end{pmatrix}$ $\Rightarrow$ we have two points (M and N) on $\tilde{\ell}$, so we can write down eq. for $\tilde{\ell}$

$\boxed{\text{Method 2}}$ Determine $\tilde{\ell}$ using the pencil of planes passing through the line ℓ

$$\ell: \frac{x-1}{4} = \frac{y-3}{2} = z \Leftrightarrow \ell: \begin{cases} \frac{x-1}{4} = z \\ \frac{y-3}{2} = z \end{cases} \Leftrightarrow \begin{cases} x - 4z - 1 = 0 \\ y - 2z - 3 = 0 \end{cases}$$

and any other plane containing ℓ has an eq. of the form $\pi_{\alpha,\beta}: \alpha(x - 4z - 1) + \beta(y - 2z - 3) = 0$, $\alpha^2 + \beta^2 > 0$, $\alpha, \beta \in \mathbb{R}$



- if we determine π^M then the line $\tilde{\ell}$ is $\pi^M \cap \tilde{\pi}$ (where $\tilde{\pi}$ is as before the plane parallel to π and containing M)
- $\pi^M = \pi_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{R}$

$$M \in \pi_{\alpha,\beta} \Rightarrow \alpha(1 - 28 - 1) + \beta(-14 - 3) = 0 \Rightarrow \alpha = -\frac{17}{28}\beta \Rightarrow \pi^M = \pi_{-\frac{17}{28}\beta, \beta}: -17x + 28y + 12z - 67 = 0$$

$$M \in \tilde{\pi} \Rightarrow M = \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} \text{ satisfies } 3x - y + 2z + D = 0 \Rightarrow D = -17$$

$$\Rightarrow \tilde{\ell}: \begin{cases} -17x + 28y + 12z - 67 = 0 \\ 3x - y + 2z - 17 = 0 \end{cases}$$

11. Consider the Euclidean space $\mathbb{E}^3$. Show that the orthogonal reflection $\mathrm{Ref}_{\pi}^{\perp}(x)$ in the plane $\pi : \langle n, x \rangle = p$ is given by

$$\mathrm{Ref}_{\pi}(x) = Ax + b$$

where $A = \left(I - 2\frac{nn^t}{\|n\|^2}\right)$ and $b = \frac{2p}{\|n\|^2} n$.

if $n = n(n_1, n_2, n_3)$ and $x = (x_1, x_2, x_3)$

then $\langle n \cdot x \rangle = p \iff n_1 x_1 + n_2 x_2 + n_3 \cdot x_3 - p = 0$

Use the compact matrix form and notice that

$\langle n, n \rangle = n^t \cdot n = \|n\|^2$

12. Give the matrix form for the orthogonal reflections in the planes

$$\pi_1 : 3x - 4z = -1 \quad \text{and} \quad \pi_2 : 10x - 2y + 3z = 4 \quad \text{respectively.}$$

$\boxed{\text{for } \pi_1}$

- $\pi_1 = \varphi^{-1}(0)$ where $\varphi(x,y,z) = 3x - 4z + 1$
- the direction of the reflection is given by the normal vector to π_1 $\quad n = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$

$$\Rightarrow \mathrm{Ref}^{\perp}_{\pi_1}(p) = p + 2\mu n \quad \text{with } \mu = -\frac{\varphi(p)}{(\mathrm{lin}\,\varphi)(n)} = \frac{3x - 4z + 1}{9 + 16} \quad \text{if } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \mathrm{Ref}^{\perp}_{\pi_1}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{6x - 8z + 2}{25}\begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 25x - 18x + 24z - 6 \\ 25y \\ 25z + 24x - 32z + 8 \end{pmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 7x & & 24z \\ & 25y & \\ 24x & & -7z \end{bmatrix} + \frac{2}{25}\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = \frac{1}{25}\begin{bmatrix} 7 & 0 & 24 \\ 0 & 25 & 0 \\ 24 & 0 & -7 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{2}{25}\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$$

Check:
- this needs to be the same as replacing in the generic matrix form which we deduced for dimension 3
- the points in π_1 do not change if they are reflected with $\mathrm{Ref}^{\perp}_{\pi}$. Check this on an affine basis of π_1

$\boxed{\text{for } \pi_2}$ the calculations are similar