

4 **HELLY AND STRONG HELLY NUMBERS OF  $B_K$ -EPG AND**  
5  **$B_K$ -VPG GRAPHS**

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22 **Abstract**

23 EPG graphs were introduced by Golumbic, Lypshteyn, and Stern (2009)  
24 and consist of the intersection graphs of sets of paths on the orthogonal grid,  
25 whose intersections are taken considering the edges of the paths. If the in-  
26 tersections of the paths consider the vertices and not the edges, the resulting  
27 graph class is called VPG graphs. A path  $P$  is a  $B_k$ -path if it contains at  
28 most  $k$  bends.  $B_k$ -EPG and  $B_k$ -VPG graphs are the intersection graphs of  
29  $B_k$ -paths on the orthogonal grid, considering the intersection of edges and  
30 vertices, respectively. A family  $\mathcal{F}$  is  $h$ -Helly when every  $h$ -intersecting sub-  
31 family  $\mathcal{F}'$  of it satisfies  $\text{core}(\mathcal{F}') \neq \emptyset$ . If for every subfamily  $\mathcal{F}'$  of  $\mathcal{F}$ , there

are  $h$  subsets whose core equals the core of  $\mathcal{F}'$ , then  $\mathcal{F}$  is said to be strong  $h$ -Helly. The Helly number of the family  $\mathcal{F}$  is the least integer  $h$ , such that  $\mathcal{F}$  is  $h$ -Helly. Similarly, the strong Helly number of  $\mathcal{F}$  is the least  $h$ , for which  $\mathcal{F}$  is strong  $h$ -Helly. In this paper, we solve the problem of determining both the Helly and strong Helly numbers, for  $B_k$ -EPG, and  $B_k$ -VPG graphs, for each value  $k$ .

**Keywords:** EPG, VPG, path, grid, bend, Helly number, strong Helly.

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## 1. INTRODUCTION

EPG graphs were introduced by Golumbic, Lypshteyn, and Stern (2009) and consist of the intersection graphs of sets of paths on the orthogonal grid, whose intersections are taken considering the edges of the paths. If the intersections of the paths consider the vertices and not the edges, the resulting graph class is called VPG graphs. Such a class was introduced in 2011 [1] and [2]. In the present paper, we study two graph parameters of both EPG and VPG graphs, namely the Helly number and the strong Helly number.

Let  $\mathcal{F}$  be a family of subsets of some universal set  $U$ , and  $h$  an integer  $\geq 1$ . Say that  $\mathcal{F}$  is  $h$ -intersecting when every group of  $h$  sets of  $\mathcal{F}$  intersect. The *core* of  $\mathcal{F}$  is the intersection of all sets of  $\mathcal{F}$ , denoted  $\text{core}(\mathcal{F})$ .

The family  $\mathcal{F}$  is  $h$ -Helly when every  $h$ -intersecting subfamily  $\mathcal{F}'$  of it satisfies  $\text{core}(\mathcal{F}') \neq \emptyset$ , see e.g. [4]. On the other hand, if for every subfamily  $\mathcal{F}'$  of  $\mathcal{F}$ , there are  $h$  subsets whose core equals the core of  $\mathcal{F}'$ , then  $\mathcal{F}$  is said to be *strong  $h$ -Helly*. Clearly, if  $\mathcal{F}$  is  $h$ -Helly then it is  $h'$ -Helly, for  $h' \geq h$ . Similarly, if  $\mathcal{F}$  is strong  $h$ -Helly then it is strong  $h'$ -Helly, for  $h' \geq h$ .

Finally, the *Helly number* of the family  $\mathcal{F}$  is the least integer  $h$ , such that  $\mathcal{F}$  is  $h$ -Helly. Similarly, the *strong Helly number* of  $\mathcal{F}$  is the least  $h$ , for which  $\mathcal{F}$  is strong  $h$ -Helly. It also follows that the strong Helly number of  $\mathcal{F}$  is at least equal to its Helly number.

A *class*  $\mathcal{C}$  of families  $\mathcal{F}$  of subsets of some universal set  $U$  is a subcollection of the families  $\mathcal{F}$  of  $U$ . Say that  $\mathcal{C}$  is a *hereditary* class when it closed under inclusion. The *Helly number* of a class  $\mathcal{C}$  of families  $\mathcal{F}$  of subsets is the largest Helly number among the families  $\mathcal{F}$ . Similarly, the *strong Helly number* of a class  $\mathcal{C}$  is the largest strong Helly number of the families of  $\mathcal{C}$ .

If  $\mathcal{F}$  is a family of subsets and  $\mathcal{C}$  a class of families, denote by  $H(\mathcal{F})$  and  $H(\mathcal{C})$ , the Helly numbers of  $\mathcal{F}$  and  $\mathcal{C}$ , respectively, while  $sH(\mathcal{F})$  and  $sH(\mathcal{C})$  represent the strong Helly numbers of  $\mathcal{F}$  and  $\mathcal{C}$ .

In this work, we study families of subsets  $\mathcal{F}$  of edge and vertex paths in a grid. In the context of edge paths, a path consists of a sequence of consecutive edges in the orthogonal grid. We call a collection of such paths an *EPG representation*, i.e., a collection of paths that represent a graph via its intersection graph (considering edge intersections). *EPG graphs* are the class of graphs that admit an EPG representation. Similarly, for vertex paths, a path consists of a sequence of consecutive vertices of the orthogonal grid and a collection of these paths form a *VPG representation* and correspond to a *VPG graph*.

Each edge has an associated direction in the grid, which can be either horizontal or vertical. A *bend* in a path is a pair of consecutive edges that have different directions. A *segment* of a path is a sequence of consecutive edges of the path, with no bends. Say that a path  $P_i$  is a  $B_k$ -path if it contains at most  $k$  bends. Say that  $\mathcal{F}$  is a  $B_k$ -paths family, or simply a  $B_k$ -family, if each path of  $\mathcal{F}$  is a  $B_k$ -path.

In this paper, we solve the problem for determining the Helly and strong Helly numbers, for both  $B_k$ -EPG and  $B_k$ -VPG graphs, for each value  $k$ .

For EPG graphs, the Helly number of  $B_0$ -families is well known and is equal to 2, since  $B_0$ -EPG graphs coincide with interval graphs. It is also simple to conclude that the strong Helly number of  $B_0$ -EPG graphs are also equal to 2. For  $k = 1$ , we prove that both the Helly number and the strong Helly number of the class of  $B_1$ -families are equal to 3. For the class of  $B_2$ -families, we prove that these two parameters are equal to 4. The Helly and strong Helly number for  $B_3$ -families equal 8, and finally, these parameters are unbounded for  $k \geq 4$ .

As for VPG graphs, it is simple to verify that the Helly number of  $B_0$ -VPG graphs equals 2, and we prove that  $B_1$ -VPG have Helly number 4,  $B_2$ -VPG graphs have Helly number 6,  $B_3$ -VPG has Helly number 12, while the Helly number for  $B_4$ -VPG graphs is again unbounded.

Finally, the strong Helly number equals the Helly number of  $B_k$ -EPG graphs, for each  $k$ . Similarly, for  $B_k$ -VPG graphs.

As for existing results, Golumbic, Lipshteyn, and Stern [9] have already shown that the strong Helly number for  $B_1$ -EPG graphs equal 3, and for  $B_1$ -VPG graphs is equal to 4. employing a different proof technique. See [11], Theorem 11.13, below:

**Theorem 1.** [11] *Let  $P$  be a collection of single bend paths on a grid. If every two paths in  $P$  share at least one grid-edge, then  $P$  has strong Helly number 3. Otherwise,  $P$  has strong Helly number 4.*

No other results concerning the strong Helly number, or no results for the Helly number of  $B_k$ -EPG graphs seem to have been reported in the literature. As for other classes, Golumbic and Jamison have determined the strong Helly number of the intersection of edge paths of a tree [8]. Finally, Asinowski, Cohen,

Golumbic, Limouzy, Lipshteyn, and Stern have reported that the strong Helly number of  $B_0$ -VPG graphs equals two [1].

Deciding whether a given hypergraph is  $k$ -Helly can be done in polynomial time for fixed  $k$ , employing the characterization by Berge and Duchet [3]. For arbitrary  $k$ , the problem is co-NP-complete [7]. For the corresponding problems for strong  $k$ -Helly see [6, 7].

The paper is organized as follows. Section 2 contains some preliminary propositions and further notation. Section 3 describes the results for the Helly number of  $B_k$ -EPG graphs, while Section 4 contains the results of this parameter for  $B_k$ -VPG graphs. The strong Helly number is considered in Section 5. Final remarks are presented in the last section.

## 2. PRELIMINARIES

The following theorem characterizes  $h$ -Helly families of subsets.

**Theorem 2.** ([3]): *A family  $\mathcal{F}$  of subsets of the universal set  $U$  is  $h$ -Helly if and only if for every subset  $U' \subseteq U$ ,  $|U'| = h + 1$ , the subfamily  $\mathcal{F}'$  of  $\mathcal{F}$ , formed by the subsets containing at least  $h$  of the  $h + 1$  elements of  $U'$ , has a non-empty core.*

The next theorem is central to our results.

**Theorem 3.** *Let  $\mathcal{C}$  be a hereditary class of families  $\mathcal{F}$  of subsets of the universal set  $U$ , whose Helly number  $H(\mathcal{C})$  equals  $h$ . Then there exists a family  $\mathcal{F}' \in \mathcal{C}$  with exactly  $h$  subsets, satisfying the following condition:*

*For each subset  $P_i \in \mathcal{F}'$ , there is exactly one distinct element  $u_i \in U$ , such that*

$$u_i \notin P_i,$$

*but  $u_i$  is contained in all subsets*

$$P_j \in \mathcal{F}' \setminus P_i.$$

**Proof:** Let  $\mathcal{C}$  be a class of families  $\mathcal{F}$  of subsets  $P$ , each subset formed by elements  $u \in U$ , such that the Helly number  $H(\mathcal{C})$  equals  $h$ . Then each family  $\mathcal{F} \in \mathcal{C}$  satisfies  $H(\mathcal{F}) \leq h$ . Consider a family  $\mathcal{F}' \in \mathcal{C}$  whose Helly number is exactly  $h$ , and containing exactly  $h$  subsets. Such a family must exist since  $\mathcal{C}$  is hereditary. Since  $H(\mathcal{F}') = h$ ,  $\mathcal{F}'$  is  $h$ -intersecting, and therefore  $(h - 1)$ -intersecting. Furthermore,  $\mathcal{F}'$  is not  $(h - 1)$ -Helly. Applying Theorem 2, we conclude that there are  $h$  elements  $U' = \{u_1, \dots, u_h\} \subset U$ , such that each set of  $\mathcal{F}'$  contains at least  $h - 1$  elements of  $U'$ . Since  $H(\mathcal{F}') > h - 1$ ,  $\text{core}(\mathcal{F}') = \emptyset$  and

therefore there is no common element among the sets of  $\mathcal{F}'$ . In particular, since each set  $P_i \in \mathcal{F}'$  contains at least  $h - 1$  elements of  $U'$ , and  $\text{core}(\mathcal{F}') = \emptyset$ , we can choose  $h$  subsets  $P_i$ , in which each of them misses a distinct element  $u_i \in U'$ . Then for each subset  $P_i \in \mathcal{F}'$ , there exists some element  $u_i \notin P_i$ , but  $u_i \in P_j$ , for all  $P_j \in \mathcal{F}'$ ,  $j \neq i$ .  $\square$

Let  $\mathcal{F}'$  be as in the previous theorem. It is simple to conclude that the removal of any subset from  $\mathcal{F}'$  makes it an  $(h - 1)$ -Helly family. Therefore we call  $\mathcal{F}'$  a *minimal non- $(h - 1)$ -Helly family*. Moreover, the element  $u_i \notin P_i$ , contained in all subsets  $P_j \in \mathcal{F}' \setminus P_i$ , except  $P_i$ , is the  *$h$ -non-representative* of  $P_i$ .

We can apply this notion of minimal families of subsets for the  $B_k$ -EPG and  $B_k$ -VPG representations. Recall that  $B_k$ -EPG and  $B_k$ -VPG graphs are hereditary classes.

### 3. THE HELLY NUMBER OF $B_k$ -EPG GRAPHS

In this section, we determine the Helly number of the classes of  $B_1$ -EPG,  $B_2$ -EPG and  $B_3$ -EPG graphs, and show that for  $B_k$ -EPG graphs,  $k \geq 4$ , the Helly number is unbounded. We prove the following result.

**Theorem 4.** *The Helly number of  $B_k$ -EPG graphs satisfy:*

- (i)  $H(B_1\text{-EPG}) = 3$
- (ii)  $H(B_2\text{-EPG}) = 4$
- (iii)  $H(B_3\text{-EPG}) = 8$
- (iv)  $H(B_k\text{-EPG})$  is unbounded, for  $k \geq 4$ .

The proof consists in determining tight lower and upper bounds, as shown in the next two subsections.

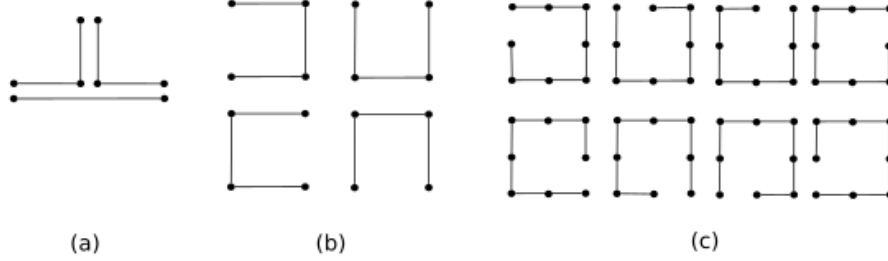
#### 3.1. Lower Bounds

We present lower bounds for the Helly number, as a function of the number  $k$  of bends.

**Claim 5.** *The following are lower bounds for  $B_k$ -EPG graphs.*

- (i)  $H(B_1\text{-EPG}) \geq 3$
- (ii)  $H(B_2\text{-EPG}) \geq 4$
- (iii)  $H(B_3\text{-EPG}) \geq 8$
- (iv)  $H(B_k\text{-EPG})$  is unbounded for  $k \geq 4$ .

**Proof.** For each value of  $k$ , we exhibit a  $B_k$ -family of edge paths whose Helly number is the corresponding stated value. We refer to the pair of coordinates of grid points, to describe the paths.

Figure 1. Minimal non-Helly sub-families for the  $B_1$ ,  $B_2$  and  $B_3$  -families.

For  $k = 1$ , let  $\mathcal{F}$  be a family of three 1-bend paths that pairwise intersect but which have no common edge, as depicted in Figure 1(a). Then  $\mathcal{F}$  is a 2-intersecting  $B_1$ -EPG family of three paths, having an empty core. Furthermore, removing any of the paths from  $\mathcal{F}$  makes its core become non-empty. Therefore  $\mathcal{F}$  is a minimal non-2-Helly family and  $H(B_1\text{-EPG}) \geq 3$ .

Let  $S$  be the 4-cycle formed by the four edge segments, with bends at the grid points  $(0, 0)$ ,  $(0, 2)$ ,  $(2, 2)$ ,  $(2, 0)$ , respectively. For  $k = 2$ , consider  $\mathcal{F}$  to be the family of four 2-bend paths formed when we remove exactly one of the two-edge segments that form the 4-cycle, as depicted in Figure 1(b). It follows that  $\mathcal{F}$  is 3-intersecting and there is no common edge to all four paths. Hence  $H(B_2\text{-EPG}) \geq 4$ .

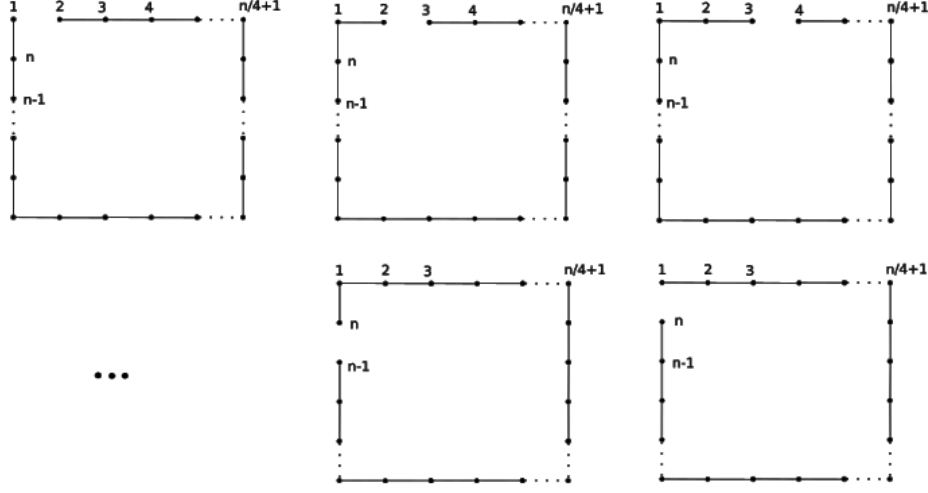
For  $k = 3$ , consider again the same cycle  $S$  as above. Note that  $S$  contains 8 grid edges. Let  $\mathcal{F}$  consist of the 8 paths  $P_i$ ,  $1 \leq i \leq 8$ , obtained by removing from  $S$ , exactly one of these distinct 8 edges, as depicted in Figure 1(c). Consequently,  $\mathcal{F}$  is 7-intersecting, but  $\text{core}(\mathcal{F}) = \emptyset$ . Therefore,  $H(B_3\text{-EPG}) \geq 8$ .

Finally, for  $k = 4$ , let  $\mathcal{F}$  be the family of  $n$  paths  $P_i$ , described as follows:

- $P_1$  is formed by the segments connecting:  
 $(0, 0), (0, 1), (1, 1), (1, 0), (n, 0)$ ;
- for  $2 \leq i \leq n - 1$ ,  $P_i$  contains the segments connecting:  
 $(0, 0), (0, i - 1), (i - 1, 1), (i, 1), (i, 0), (n, 0)$ ;
- $P_n$  is formed by the segments connecting:  
 $(0, 0), (n - 1, 0), (n - 1, 1), (n - 1, 0)$ .

Observe that  $\mathcal{F}$  is  $(n - 1)$ -intersecting, while  $\text{core}(\mathcal{F}) = \emptyset$  (see Figure 2). Therefore  $H(B_4\text{-EPG})$  is unbounded. Clearly the same holds for  $k > 4$ .  $\square$

Next, we consider upper bounds for the Helly number  $B_k$ -EPG graphs.


 Figure 2.  $B_4$  has an unlimited Helly number.

### 3.2. Upper Bounds

In order to obtain tight upper bounds for the Helly number, in terms of the number of bends, we introduce below more notation and lemmas.

Say that a set of edges of a grid is *co-linear* if all edges of the set belong to the same line of the grid, horizontal or vertical. The set of edges is called *parallel* if all its edges lie on parallel lines of the grid, but no two of them are co-linear.

**Lemma 6.** *Let  $\mathcal{F}$  be a minimal non- $(h - 1)$ -Helly family of paths on a grid containing three co-linear non-representative edges. Then  $\mathcal{F}$  must contain paths with at least four bends.*

**Proof.** Let  $u_i$  be the middle one of the three co-linear non-representative edges. It corresponds to the path  $P_i$  of  $\mathcal{F}$ , not containing  $u_i$ . Then  $P_i$  must go through the other two non-representative edges, but it cannot include the middle edge. Therefore path  $P_i$  must leave the common line of the grid, containing those three representative edges, and return to that same line, thus requiring at least four bends.  $\square$

**Lemma 7.** *Let  $\mathcal{F}$  be a minimal non- $(h - 1)$ -Helly family of paths on a grid, containing three parallel edges, and having Helly number  $H(\mathcal{F}) \geq 4$ . Then  $\mathcal{F}$  must contain paths with at least four bends.*

**Proof.** Since  $H(\mathcal{F}) \geq 4$  and  $\mathcal{F}$  is a minimal  $(h - 1)$ -family, it follows that  $\mathcal{F}$  must contain at least four paths,  $P_1, P_2, P_3, P_4$ . Without loss of generality, let

217  $u_1, u_2, u_3$  be non-representative edges which are parallel and correspond to the  
 218 paths  $P_1, P_2$  and  $P_3$  respectively. Then  $P_4$  must go through all the three parallel  
 219 non-representative edges  $u_1, u_2, u_3$ , thus requiring at least four bends.  $\square$

220 **Lemma 8.** *Let  $\mathcal{F}$  be a minimal non- $(h-1)$ -Helly family of paths on a grid with*  
 221 *Helly number  $H(\mathcal{F}) \geq 4$ . If  $\mathcal{F}$  contains three non-representative edges that lie on*  
 222 *a common  $B_1$ -subpath  $P_i$ , then  $\mathcal{F}$  must have some path with at least three bends.*

223 **Proof.** Since  $\mathcal{F}$  is a minimal  $(h-1)$ -family having Helly number  $\geq 4$ , it contains  
 224 at least four paths. Without loss of generality, let  $u_1, u_2, u_3$  be the three non-  
 225 representative edges contained in  $P_4$  and such that  $u_2$  lies between  $u_1$  and  $u_3$  in  
 226  $P_4$ . Then path  $P_2$  must contain  $u_1$  and  $u_3$ , but avoid  $u_2$ , thus requiring at least  
 227 three bends.  $\square$

228 The following are tight upper bounds for the Helly numbers of  $B_k$ -EPG paths,  
 229 for  $k = 1, 2, 3$ .

230 **Claim 9.**  $H(B_1\text{-EPG}) \leq 3$ .

231 **Proof.** Assume by contradiction that the Helly number of  $B_1$ -EPG paths is  
 232  $h > 3$ . In this case, consider a minimal non- $(h-1)$ -Helly family of  $\mathcal{F}$  of  $B_1$ -EPG  
 233 paths. Then  $\mathcal{F}$  contains at least  $h$  paths. Any path  $P_1 \in \mathcal{F}$  must contain  $h-1$   
 234 non-representative edges corresponding to the  $h-1$  distinct paths of  $\mathcal{F}$  other  
 235 than  $P_1$ . Since  $h-1 \geq 3$ ,  $P_1$  contains at least three distinct non-representative  
 236 edges  $u_2, u_3, u_4 \in P_i$ , with  $u_3$  lying between  $u_2$  and  $u_4$  in the path.

237 If  $u_2, u_3$  and  $u_4$  are co-linear then by Lemma 6  $P_3 \in \mathcal{F}$  must contain at least  
 238 four bends. Otherwise, the edges must lie on  $P_1$  which has a single bend. Thus, it  
 239 follows from Lemma 8 that  $P_3$  has three bends. In any situation, a contradiction  
 240 arises, implying that  $H(\mathcal{F}) \leq 3$ .  $\square$

241 **Claim 10.**  $H(B_2\text{-EPG}) \leq 4$ .

242 **Proof.** Assume by contradiction that the Helly number of  $B_2$ -EPG families of  
 243 paths is  $h > 4$ . Consider a minimal non- $(h-1)$ -Helly family  $\mathcal{F}$  of  $B_2$ -EPG  
 244 paths. The family  $\mathcal{F}$  must contain at least  $h \geq 5$  distinct paths, each of them  
 245 corresponding to a distinct non-representative edge. Choose arbitrarily 5 of these  
 246 non-representative edges.

247 By Lemmas 6 and 7 any three of these chosen edges can neither be co-linear  
 248 nor parallel. Therefore, at least one of the five chosen non-representative edges  
 249 must be in a different direction from the majority of the chosen edges. Call the  
 250 direction of this edge vertical and the direction of the majority of the chosen  
 251 edges horizontal. Consider a path  $P_1$  from the family  $\mathcal{F}$  that goes through this  
 252 vertical edge. The path  $P_1$  contains at least four of the chosen non-representative  
 253 edges, at least one of which is vertical. Since  $P_1$  has at most two bends, then it



must have at most three segments. Since we have three segments and four non-representative edges which  $P_1$  must contain, by the pigeon hole principle, one of these segments must have two non-representative edges. If this pair of edges are in a horizontal segment of  $P_1$ , then such pair of edges, along with the vertical edge are in two consecutive path segments, forming a  $B_1$ -subpath in  $\mathcal{F}$ . Then Lemma 8 implies that some path of  $\mathcal{F}$  must have at least three bends. Otherwise, the two edges are vertical. But the others must be horizontal, and again we have at least three edges in a pair of consecutive segments forming a subpath in  $\mathcal{F}$  having one bend. Again, Lemma 8 implies that some path has at least three bends.  $\square$

**Claim 11.**  $H(B_3\text{-EPG}) \leq 8$ .

**Proof.** Assume by contradiction that the Helly number of  $B_3$ -EPG paths is  $h > 8$ . In this case, consider a minimal non- $(h - 1)$ -Helly family  $\mathcal{F}$  of  $B_3$ -EPG paths. Then  $\mathcal{F}$  contains at least  $h$  distinct non-representative edges, corresponding to  $h$  distinct paths. By Lemma 7, since we can have at most three bends in any path, then these  $h$  non-representative edges must lie in at most two vertical and two horizontal lines of the grid. Therefore one of these four possible lines must contain at least three distinct non-representative edges. By Lemma 6, that would imply the existence of a path with four bends.  $\square$

This completes the proof of Theorem 4.

#### 4. HELLY NUMBER OF $B_k$ -VPG GRAPHS

In this section, we determine the Helly number of  $B_k$ -VPG graphs. We prove the following results.

**Theorem 12.** *The Helly numbers for  $B_k$ -VPG graphs satisfy:*

1.  $H(B_1\text{-VPG}) = 4$
2.  $H(B_2\text{-VPG}) = 6$
3.  $H(B_3\text{-VPG}) = 12$
4.  $H(B_4\text{-VPG})$  is unbounded.

Again, we prove the theorem by showing tight lower and upper bounds.

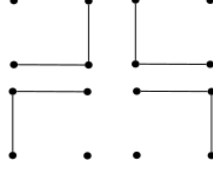


Figure 3. Lower bound for  $B_1$ -VPG graphs

#### 283 4.1. Lower Bounds

284 We start by describing some sets of paths that achieve our lower bounds.  
 285 Figure 3 shows a set of 4  $B_1$ -paths of a graph  $G$ , in a  $2 \times 2$  grid, such that each  
 286 path covers three vertices of the grid, and avoids exactly one of the vertices.

287 Figure 4 shows a set of 6  $B_2$ -paths of a graph  $G$ , in a  $2 \times 3$  grid, such that  
 288 each path covers five vertices of the grid, and avoids exactly one.

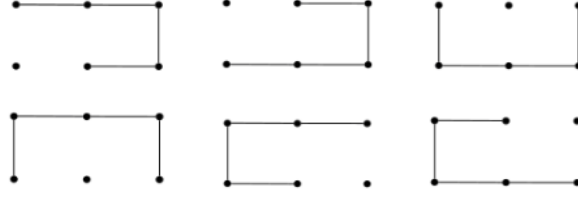


Figure 4. Lower bound for  $B_2$ -VPG graphs

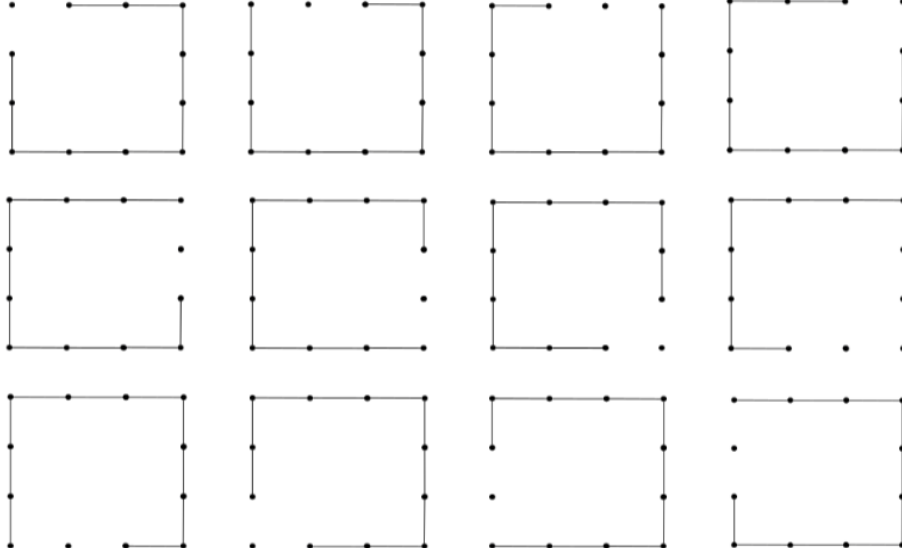
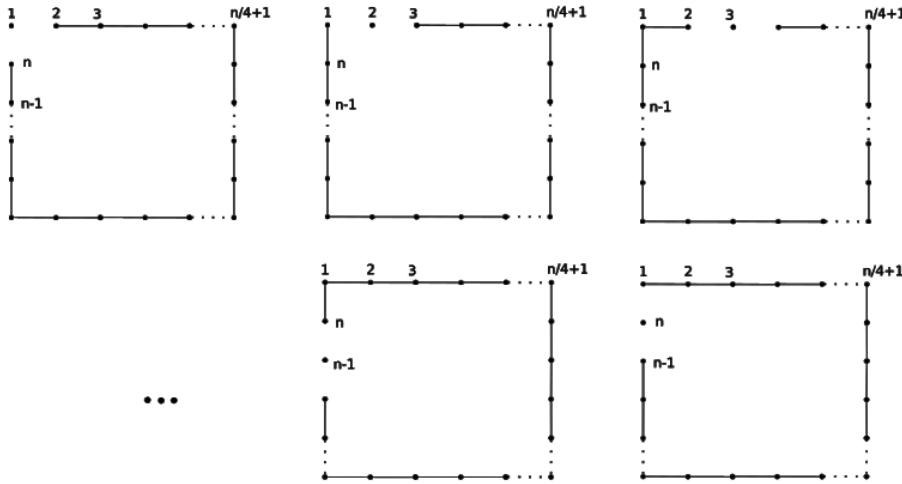
289 Figure 5 shows 12  $B_3$ -paths of a graph  $G$ , in a grid, of perimeter 12, such  
 290 that each path covers 11 vertices of the grid, avoiding one of them.

291 Figure 6 shows a set of  $n$   $B_4$ -paths of a  $n$ -vertex graph  $G$ , in a grid having  
 292 perimeter  $n$ , such that each path covers  $n - 1$  vertices of  $G$ , avoiding one of them.

293 Applying Theorem 3, we can then conclude that the number of vertices of  
 294 each of the above-described graphs is lower bound for the corresponding class.  
 295 Then, we can claim the following bounds.

296 **Claim 13.** *The following are lower bounds for the Helly numbers of  $B_k$ -VPG*  
 297 *graphs.*

- 298 1.  $H(B_1\text{-VPG}) \geq 4$
- 299 2.  $H(B_2\text{-VPG}) \geq 6$
- 300 3.  $H(B_3\text{-VPG}) \geq 12$
- 301 4.  $H(B_4\text{-VPG})$  is unbounded.


 Figure 5. Lower bound for  $B_3$ -VPG graphs

 Figure 6. Lower bound for  $B_4$ -VPG graphs

## 302 4.2. Upper Bounds

303 Next, we provide upper bounds for the Helly number of  $B_k$ -VPG graphs.  
 304 The following lemmas are employed.

305 **Lemma 14.** *Let  $\mathcal{F}$  be a minimal non- $(h-1)$ -Helly family of paths, for some  $h$ ,  
 306 containing  $k \in \{3, 4, 5\}$  distinct co-linear non-representative points of the grid.*

307 Then  $\mathcal{F}$  contains a path having at least  $k - 1$  bends.

308 **Proof.** For  $k \in \{3, 5\}$ , the path avoiding the middle point has at least  $k - 1$   
 309 bends, while for  $k = 4$ , the path avoiding one of the middle points also has this  
 310 same property.  $\square$

311 **Lemma 15.** *Let  $\mathcal{F}$  be a minimal non- $(h - 1)$ -Helly family of paths, on a grid  
 312 containing  $k < h$  distinct pairwise non-co-linear non-representative points. Then  
 313  $\mathcal{F}$  must contain a path with at least  $k - 1$  bends.*

314 **Proof.** Since  $k < h$ ,  $\mathcal{F}$  must contain a path that visits all such  $k$  pairwise non-  
 315 co-linear points. Such a path requires at least one bend, between two consecutive  
 316 non-co-linear points. Therefore  $\mathcal{F}$  contains a path with at least  $k - 1$  bends.  $\square$

317 We also employ some additional concepts and notation, below described.

318 Let  $\mathcal{F}$  be a minimal non- $(h - 1)$ -Helly family of  $B_{k-1}$ -paths on a grid  $Q$ . By  
 319 Theorem 3, we can choose  $h$  paths  $P_i \in \mathcal{F}$ , each of them associated to a distinct  
 320 non-representative grid point  $p_i$ , such that  $P_i$  avoids  $p_i$ , but contains all the other  
 321  $h - 1$  distinct non-representative points  $p_j \in P_j$ , for each  $j \neq i$ . Denote by  $P_N$ ,  
 322  $|P_N| = h$ , the subset of grid points of  $Q$ , restricted to the chosen set of distinct  
 323 non-representative points  $p_i$ . By Lemmas 14 and 15, the grid points of  $P_N$  are  
 324 contained in at most  $k$  columns (lines), and each column (line) contains at most  
 325  $k$  points of  $P_N$ . Consequently, the cardinalities of the points of  $P_N$ , contained in  
 326 the columns (lines) of  $Q$ , form a partition of the integer  $h$ , into at most  $k$  parts,  
 327 such that each part is at most  $k$ . Call such a partition as a *feasible partition of*  
 328  *$h$ , relative to  $P_N$* . Therefore, each non-representative point  $p_i \in P_N$  contributes  
 329 with one unit to some part of the partition, which is then referred to, as the part  
 330 of the partition *corresponding* to  $p_i$ .

332 The following lemma describes sufficient conditions for an integer  $h$  to be an  
 333 upper bound for the Helly number.

334 **Lemma 16.** *Let  $\mathcal{F}$  be a minimal non- $(h - 1)$ -Helly family of  $B_{k-1}$ -paths on a  
 335 grid  $Q$ , and  $P_N$  the set of non-representative points of  $Q$ . Let  $k, h$  be integers,  
 336  $1 \leq k \leq 3$  and  $k < h$ . The following conditions imply  $H(B_k\text{-VPG}) \leq h$*

- 337 (i) *there is no feasible partition of  $h + 1$ , relative to  $P_N$ , or*
- 338 (ii) *for any possible feasible partition, and for any arrangement of the grid points*  
 339 *of  $P_N$  in  $Q$ , there is some non-representative point  $p_i \in P_N$ , such that no*  
 340 *path exists in  $Q$ , having at most  $k$  bends, containing all points of  $P_N$ , except*  
 341  *$p_i$ .*

342 **Proof:** The proof of (i) follows from Lemmas 14 and 15, while the proof of  
 343 (ii) is a consequence of Theorem 3.  $\square$

344

345 The following are upper bounds for the Helly number of  $B_k$ -VPG graphs, for  
 346 each  $k$ ,  $1 \leq k \leq 3$ , obtained by applying Lemma 16.

347 **Claim 17.**  $H(B_1\text{-VPG}) \leq 4$ .

348 **Proof.** There is no partition of the integer 5, into two parts, in which each part  
 349 is at most 2. Consequently, the result follows from Lemma 16 (i).  $\square$

350 **Claim 18.**  $H(B_2\text{-VPG}) \leq 6$ .

351 **Proof.** Assume the contrary. Then  $H(B_2\text{-VPG}) \geq 7$ , let  $\mathcal{F}$  be a minimal non-  
 352 6-Helly family of  $B_2$ -paths, and  $P_N$  be the set of non-representative points of  $\mathcal{F}$   
 353 in  $Q$ . There are two possible partitions of the integer 7, in three parts, each  
 354 of them of size at most 3, namely  $(3, 3, 1)$  and  $(3, 2, 2)$ . In any of these cases,  
 355 it is always possible to choose some point  $p_i \in P_N$ , belonging to a part of the  
 356 partition of size 3, such that a path in  $\mathcal{F}$  which avoids  $p_i$  and covers the other six  
 357 non-representative points, must contain at least three bends. Then by Lemma  
 358 16, indeed  $H(B_2\text{-VPG}) \leq 6$ .  $\square$

359 **Claim 19.**  $H(B_3\text{-VPG}) \leq 12$ .

360 **Proof.** Assume the contrary,  $H(B_3\text{-VPG}) \geq 13$ . Let  $\mathcal{F}$  be a minimal non-12-  
 361 Helly family of  $B_3$ -paths, and  $P_N$  be the set of non-representative points of  $\mathcal{F}$   
 362 in  $Q$ . There are three possible partitions of the integer 13, into four parts, each  
 363 of them of size at most 4, namely  $(4, 4, 4, 1)$ ,  $(4, 4, 3, 2)$  and  $(4, 3, 3, 3)$ . In this  
 364 case, choose  $p_i \in P_N$  to be a non-representative point, corresponding to a part  
 365 of size 4 of the partition. The path of  $\mathcal{F}$ , which avoids  $p_i$ , must cover the other  
 366 12 non-representative points. These points are located in 4 distinct columns, of  
 367 cardinalities 4,4,3,1, 4,3,3,2, or 3,3,3,3, considering the three possible partitions,  
 368 respectively. Such a path must contain at least four bends, a contradiction. Then  
 369 by Lemma 16,  $H(B_3\text{-VPG}) \leq 12$ .  $\square$

370 From the lower and upper bounds described in the previous subsections, we  
 371 obtain the results for the Helly numbers of  $B_k$ -VPG graphs, completing the proof  
 372 of Theorem 12.

## 373 5. STRONG HELLY NUMBER

374 In this section, we first consider determining the strong Helly number of  $B_k$ -EPG  
 375 graphs.

376 We start by describing a theorem similar to Theorem 3.

377 **Theorem 20.** *Let  $\mathcal{C}$  be a hereditary class of families  $\mathcal{F}$  of subsets of the universal  
 378 set  $U$ , whose strong Helly number  $sH(\mathcal{C})$  equals  $h$ . Then there exists a family  
 379  $\mathcal{F}' \in \mathcal{C}$  with exactly  $h$  subsets satisfying the following condition:*

For each subset  $P_i \in \mathcal{F}'$ , there is exactly one distinct element  $u_i \in U$ , such that

$$u_i \notin P_i,$$

but  $u_i$  is contained in all subsets

$$P_j \in \mathcal{F}' \setminus P_i.$$

Proof: The strong Helly number of  $\mathcal{C}$  is  $h$  and not  $h - 1$ , so that there must exist some family  $\mathcal{F} \in \mathcal{C}$  whose strong Helly number is exactly  $h$ , i.e.,  $\mathcal{F}$  contains  $h$  subsets  $P_i$  whose intersection equals  $\text{core}(\mathcal{F}')$  but is such that no  $h - 1$  of its subsets have the same intersection. In particular, let  $\mathcal{F}'$  be the family containing exactly the  $h$  subsets  $P_i$  described above. Such a family must exist, since  $\mathcal{C}$  is hereditary. Then each  $P_i$  does not contain at least one element  $u_i$  in the intersection of the remaining  $h - 1$  subsets  $P_j$ ,  $j \neq i$ , since the intersection of these  $h - 1$  subsets must not be equal to the  $\text{core}(\mathcal{F}')$ .  $\square$

Again, if we consider the family  $\mathcal{F}'$  described in the theorem above it is simple to conclude that the removal of any subset from  $\mathcal{F}'$  turns it  $(h - 1)$ -strong Helly. Then call  $\mathcal{F}'$  a *minimal non- $(h - 1)$ -strong Helly family*. Moreover, the element  $u_i \notin P_i$ , contained in all subsets  $P_j \in \mathcal{F}' \setminus P_i$ , except  $P_i$ , is the  $h$  *non-representative* of  $P_i$ .

As before, we employ the above minimal families of subsets, applied to paths in a grid.

We prove that the strong Helly number of  $B_k$ -EPG graphs coincide with the Helly number, for each corresponding value of  $k$ . Similarly, for  $B_k$ -VPG graphs. For  $k = 0$ , it is simple to show that if a set of intervals  $\mathcal{I}$  in a line pairwise intersect, then there exist two intervals of  $\mathcal{I}$ , whose intersection equals the intersection of all intervals of  $\mathcal{I}$ . Consequently, the  $k$ -strong Helly number of  $B_0$ -EPG graphs equals 2. Similarly, for  $B_0$ -VPG graphs. Recall that the strong Helly number is at least equal to the Helly number of a family so that the lower bounds presented in Claim 5 also hold for the strong Helly number. The proofs for the strong Helly numbers for  $k \geq 1$  are similar to those described in Section 3.

## 6. CONCLUDING REMARKS

We have determined the Helly number and strong Helly number of  $B_k$ -EPG graphs and  $B_k$ -VPG graphs, for  $k \geq 0$ .

Table 1 summarizes the results obtained.

We leave two questions to be investigated concerning the presented results.

$k$	$B_k$ -EPG	$B_k$ -VPG
0	2	2
1	3	4
2	4	6
3	8	12
$\geq 4$	unbounded	unbounded

Table 1. Helly and Strong Helly Numbers for  $B_k$ -EPG and  $B_k$ -VPG Graphs

1. Given a *specific* EPG or VPG graph, the question is to formulate an algorithm to determine its Helly and strong Helly numbers. See [5], for instance, for such algorithms, applied to general graphs.
2. The values of the Helly and strong Helly numbers, which were determined in this paper, coincided in all cases. Clearly, in general, this is not the case. We leave as an open question, to find the conditions for such equality to occur.

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