The Complexity of Helly-B_1-EPG graph Recognition[*]

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Golumbic, Lipshteyn, and Stern defined in 2009 the class of EPG graphs, the intersection graph class of edge paths on a grid. An EPG graph G is a graph that admits a representation where its vertices correspond to paths in a grid Q, such that two vertices of G are adjacent if and only if their corresponding paths in Q have a common edge. If the paths in the representation have at most k bends, we say that it is a B_k-EPG representation. A collection C of sets satisfies the Helly property when every sub-collection of C that is pairwise intersecting has at least one common element. In this paper, we show that given a graph G and an integer k, the problem of determining whether G admits a B_k-EPG representation whose edge-intersections of paths satisfy the Helly property, so-called Helly-B_k-EPG representation, is in NP, for every k bounded by a polynomial function of $|V(G)|$. Moreover, we show that the problem of recognizing Helly-B_1-EPG graphs is NP-complete, and it remains NP-complete even when restricted to 2-apex and 3-degenerate graphs.

Keywords: paths, grid, EPG, Helly, intersection graphs, NP-completeness, single bend.

1   Introduction

An EPG graph G is a graph that admits a representation in which its vertices are represented by paths of a grid Q, such that two vertices of G are adjacent if and only if the corresponding paths have at least one common edge.

The study of EPG graphs has motivation related to the problem of VLSI design that combines the notion of edge intersection graphs of paths in a tree with a VLSI grid layout model, see Golumbic et al. (2009). The number of bends in an integrated circuit may increase the layout area, and consequently,

increase the cost of chip manufacturing. This is one of the main applications that instigate research on the EPG representations of some graph families when there are constraints on the number of bends in the paths used in the representation. Other applications and details on circuit layout problems can be found in Bandy and Sarrafzadeh (1990); Molitor (1991).

A graph is a B_k-EPG graph if it admits a representation in which each path has at most k bends. As an example, Figure 1(a) shows a C_3, Figure 1(b) shows an EPG representation where the paths have no bends and Figure 1(c) shows a representation with at most one bend per path. Consequently, C_3 is a B_0-EPG graph. More generally, B_0-EPG graphs coincide with interval graphs.
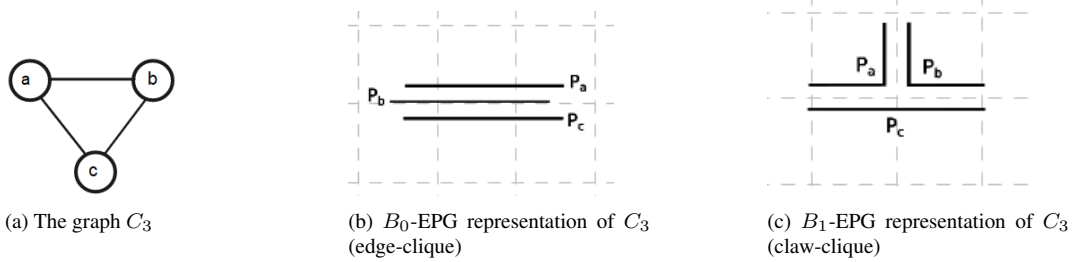

(a) The graph C_3 (b) B_0-EPG representation of C_3 (edge-clique)   (c) B_1-EPG representation of C_3 (claw-clique)

Fig. 1: The graph C_3 and representations without bends and with 1 bend

The *bend number* of a graph G is the smallest k for which G is a B_k-EPG graph. Analogously, the bend number of a class of graphs is the smallest k for which all graphs in the class have a B_k-EPG representation. Interval graphs have bend number 0, trees have bend number 1, see Golumbic et al. (2009), and outerplanar graphs have bend number 2, see Heldt et al. (2014a). The bend number for the class of planar graphs is still open, but according to Heldt et al. (2014a), it is either 3 or 4.

The class of EPG graphs has been studied in several papers, such as Alcón et al. (2016); Asinowski and Suk (2009); Cohen et al. (2014); Golumbic et al. (2009); Heldt et al. (2014b); Pergel and Rzążewski (2017); Golumbic and Morgenstern (2019), among others. The investigations regarding EPG graphs frequently approach characterizations concerning the number of bends of the graph representations. Regarding the complexity of recognizing B_k-EPG graphs, only the complexity of recognizing a few of these sub-classes of EPG graphs have been determined: B_0-EPG graphs can be recognized in polynomial time, since it corresponds to the class of interval graphs, see Booth and Lueker (1976); in contrast, recognizing B_1-EPG and B_2-EPG graphs are NP-complete problems, see Heldt et al. (2014b) and Pergel and Rzążewski (2017), respectively. Also, note that the paths in a B_1-EPG representation have one of the following shapes: ⌞, ⌟, ⌜ and ⌝. Cameron et al. (2016) showed that for each $S \subset \{⌞, ⌟, ⌜, ⌝\}$, it is NP-complete to determine if a given graph G has a B_1-EPG representation using only paths with shape in S.

A collection C of sets satisfies the Helly property when every sub-collection of C that is pairwise intersecting has at least one common element. The study of the Helly property is useful in diverse areas of science. We can enumerate applications in semantics, code theory, computational biology, database, image processing, graph theory, optimization, and linear programming, see Dourado et al. (2009).

The Helly property can also be applied to the B_k-EPG representation problem, where each path is considered a set of edges. A graph G has a Helly-B_k-EPG representation if there is a B_k-EPG representation of G where each path has at most k bends, and this representation satisfies the Helly property. Figure 2(a)

presents two B_1-EPG representations of a graph with five vertices. Figure 2(b) illustrates 3 pairwise intersecting paths ($P_{v_1}, P_{v_2}, P_{v_5}$), containing a common edge, so it is a Helly-B_1-EPG representation. In Figure 2(c), although the three paths are pairwise intersecting, there is no common edge in all three paths, and therefore they do not satisfy the Helly property.

The Helly property related to EPG representations of graphs has been studied in Golumbic et al. (2009) and Golumbic et al. (2013).

Let $\mathcal{F}$ be a family of subsets of some universal set U, and $h \geq 2$ be an integer. Say that $\mathcal{F}$ is *h-intersecting* when every group of h sets of $\mathcal{F}$ intersect. The *core* of $\mathcal{F}$, denoted by $core(\mathcal{F})$, is the intersection of all sets of $\mathcal{F}$. The family $\mathcal{F}$ is *h-Helly* when every h-intersecting subfamily $\mathcal{F}'$ of $\mathcal{F}$ satisfies $core(\mathcal{F}') \neq \emptyset$, see e.g. Duchet (1976). On the other hand, if for every subfamily $\mathcal{F}'$ of $\mathcal{F}$, there are h subsets whose core equals the core of $\mathcal{F}'$, then $\mathcal{F}$ is said to be *strong h-Helly*. Note that the Helly property that we will consider in this paper is precisely the property of being 2-Helly.

The *Helly number* of the family $\mathcal{F}$ is the least integer h, such that $\mathcal{F}$ is h-Helly. Similarly, the *strong Helly number* of $\mathcal{F}$ is the least h, for which $\mathcal{F}$ is strong h-Helly. It also follows that the strong Helly number of $\mathcal{F}$ is at least equal to its Helly number. In Golumbic et al. (2009) and Golumbic et al. (2013), they have determined the strong Helly number of B_1-EPG graphs.
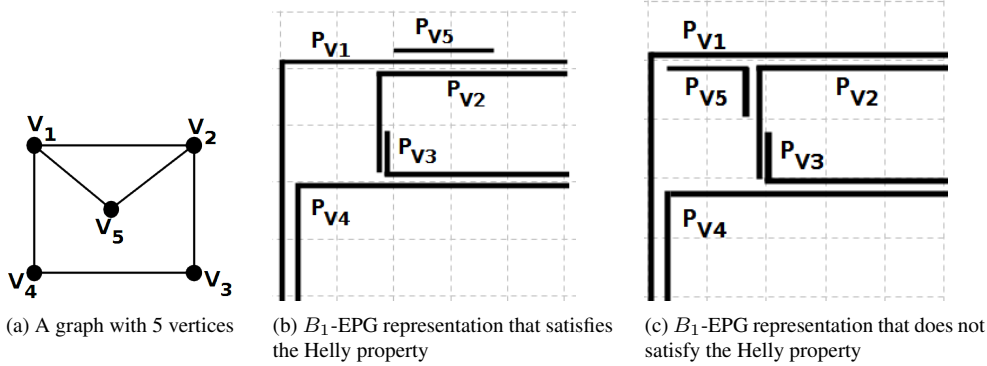

(a) A graph with 5 vertices   (b) B_1-EPG representation that satisfies the Helly property   (c) B_1-EPG representation that does not satisfy the Helly property

Fig. 2: A graph with 5 vertices in (a) and some single bend representations: Helly in (b) and not Helly in (c)

Next, we describe some terminology and notation.

The term *grid* is used to denote the Euclidean space of integer orthogonal coordinates. Each pair of integer coordinates corresponds to a *point* (or vertex) of the grid. The *size* of a grid is its number of points. The term *edge of the grid* will be used to denote a pair of vertices that are at a distance one in the grid. Two edges e_1 and e_2 are *consecutive edges* when they share exactly one point of the grid. A (simple) path in the grid is as a sequence of distinct edges $e_1, e_2, \leq, e_m$, where consecutive edges are adjacent, i.e., contain a common vertex, whereas non-consecutive edges are not adjacent. In this context, two paths only intersect if they have at least a common edge. The first and last edges of a path are called *extremity edges*.

The *direction of an edge* is vertical when the first coordinates of its vertices are equal, and is horizontal when the second coordinates are equal. A *bend* in a path is a pair of consecutive edges e_1, e_2 of that path, such that the directions of e_1 and e_2 are different. When two edges e_1 and e_2 form a bend, they are called *bend edges*. A *segment* is a set of consecutive edges with no bends. Two paths are said to be *edge-intersecting*, or simply *intersecting* if they share at least one edge. Throughout the paper, any time

we say that two paths intersect, we mean that they edge-intersect. If every path in a representation of a graph G has at most k bends, we say that this graph G has a B_k-*EPG* representation. When $k = 1$ we say that this is a *single bend* representation.

In this paper, we study the Helly-B_k-EPG graphs. First, we show that every graph admits an EPG representation that is Helly, and present a characterization of Helly-B_1-EPG representations. Besides, we relate Helly-B_1-EPG graphs with L-shaped graphs, a natural family of subclasses of B_1-EPG. Finally, we prove that recognizing Helly-B_k-EPG graphs is in NP, for every fixed k. Besides, we show that recognizing Helly-B_1-EPG graphs is NP-complete, and it remains NP-complete even when restricted to 2-apex and 3-degenerate graphs.

The rest of the paper is organized as follows. In Section 2, we present some preliminary results, we show that every graph is a Helly-EPG graph, present a characterization of Helly-B_1-EPG representations, and relate Helly-B_1 EPG with L-shaped graphs. In Section 3, we discuss the NP-membership of HELLY-B_k EPG RECOGNITION. In Section 4, we present the NP-completeness of recognizing Helly-B_1-EPG graphs.

2   Preliminaries

The study starts with the following lemma.

Lemma 1 (Golumbic et al. (2009)). *Every graph is an EPG graph.*

We show that this result extends to Helly-EPG graphs.

Lemma 2. *Every graph is a Helly-EPG graph.*

Proof: Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$ and μ maximal cliques $C_1, C_2, \ldots, C_\mu$. We construct a Helly-EPG representation of G using a $\mu + 1 \times \mu + 1$ grid Q. Each maximal clique C_i of G is mapped to an edge of Q as follow:

- if i is even then the maximal clique C_i is mapped to the edge in column i between rows i and $i + 1$;

- if i is odd then the maximal clique C_i is mapped to the edge in row i between columns i and $i + 1$.

The following describes a descendant-stair-shaped construction for the paths.

Let $v_l \in V(G)$ and C_i be the first maximal clique containing v_l according to the increasing order of their indices. If i is even (resp. odd) the path P_l starts in column i (resp. in row i), in the point (i, i). Then P_l extends to at least the point $(i + 1, i)$ (resp. $(i, i + 1)$) proceeding to the until the row (resp. column) corresponding to next maximal clique of the sequence containing v_l, we say C_j. At this point, we bend P_l, which goes to the point (j, j) and repeat the process previously described. Figure 3 shows the Helly-EPG representation of the octahedral graph O_3, according to the construction previously described.

By construction, each path travels only rows and columns corresponding with maximal cliques containing its respective vertex. And, every path crosses the edges of the grid to which your maximal cliques were mapped. Thus, the previously described construction results in an EPG representation of G, which is Helly since every set $\mathcal{P}$ of paths representing a maximal clique has at least one edge in its core. $\square$

Definition 3. *The* Helly-bend number *of a graph G, denoted by $b_H(G)$, is the smallest k for which G is a Helly-B_k-EPG graph. Also, the bend number of a graph class $\mathcal{C}$ is the smallest k for which all graphs in $\mathcal{C}$ have a B_k-EPG representation.*

(a) The octahedral graph O_3

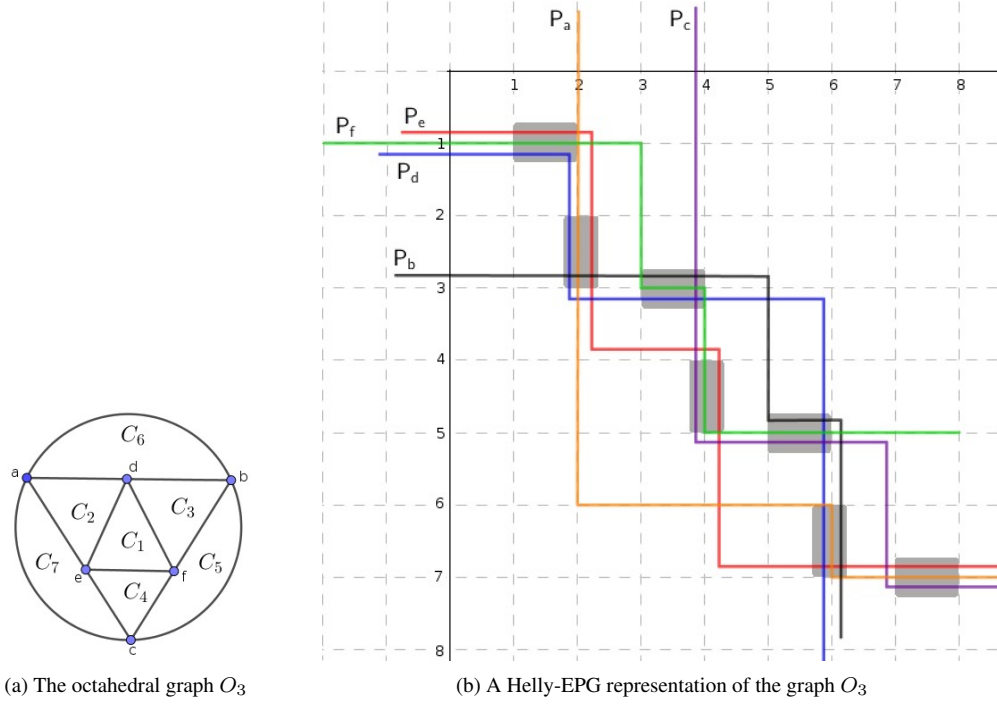(b) A Helly-EPG representation of the graph O_3

Fig. 3: Helly-EPG representation of the graph O_3 according to the construction of Lemma 2. The paths have been extended to the first/last row or column to improve the presentation.

Corollary 4. *For every graph G containing μ maximal cliques, it holds that $b_H(G) \leq \mu - 1$.*

Proof: From the construction presented in Lemma 2, it follows that any graph admits a Helly-EPG representation where its paths have a descendant-stair shape. Since the number of bends in such a stair-shaped path is the number of maximal cliques containing the represented vertex minus one, it holds that $b_H(G) \leq \mu - 1$ for any graph G. ☐

Next, we examine the B_1-EPG representations of a few graphs that we employ in our constructions.

Given an EPG representation of a graph G, for any grid edge e, the set of paths containing e is a clique in G; such a clique is called an edge-clique. A claw in a grid consists of three grid edges meeting at a grid point. The set of paths that contain two of the three edges of a claw is a clique; such a clique is called a claw-clique, see Golumbic et al. (2009). Fig. 1 illustrates an edge-clique and a claw-clique.

Lemma 5 (Golumbic et al. (2009))**.** *Consider a B_1-EPG representation of a graph G. Every clique in G corresponds to either an edge-clique or a claw-clique.*

Next, we present a characterization of Helly-B_1-EPG representations.

Lemma 6. *A B_1-EPG representation of a graph G is Helly if and only if each clique of G is represented by an edge-clique, i.e., it does not contain any claw-clique.*

Proof: Let R be a B_1-EPG representation of a graph G. It is easy to see that if R has a claw-clique, it does not satisfy the Helly property. Now, suppose that R does not satisfy the Helly property. Thus it has a set $\mathcal{P}$ of pairwise intersecting paths having no common edge. Note that the set $\mathcal{P}$ represents a clique of G, and by Lemma 5, every clique in G corresponds to either an edge-clique or a claw-clique. Since $\mathcal{P}$ represents a clique, but its paths have no common edge, then it has a claw-clique. $\qquad\square$

Now, we consider EPG representations of C_4.

Definition 7. *Let Q be a grid and let $(a_1, b), (a_2, b), (a_3, b), (a_4, b)$ be a 4-star as depicted in Figure 4(a). Let $\mathcal{P} = \{P_1, \ldots, P_4\}$ be a collection of distinct paths each containing exactly two edges of the 4-star.*

- *A* true pie *is a representation where each P_i of $\mathcal{P}$ forms a bend in b.*

- *A* false pie *is a representation where two of the paths P_i do not contain bends, while the remaining two do not share an edge.*

Fig. 4 illustrates true pie and false pie representations of a C_4.

Definition 8. *Consider a rectangle of any size with 4 corners at points (x_1, y_1); (x_2, y_1); (x_2, y_2); (x_1, y_2), positioned as in Fig. 5(a).*

- *A* frame *is a representation containing 4 paths $\mathcal{P} = \{P_1, \ldots, P_4\}$, each having a bend in a different corner of a rectangle, and such that the sub-paths $P_1 \cap P_2, P_1 \cap P_3, P_2 \cap P_4, P_3 \cap P_4$ share at least one edge. While $P_1 \cap P_4$ and $P_2 \cap P_3$ are empty sets.*

- *A* square-frame *is a frame where P_1, P_2, P_3 and P_4 have respectively point of bend (x_1, y_1), (x_2, y_1), (x_1, y_2) and (x_2, y_2), and are of the shape $\llcorner$, $\lrcorner$, $\ulcorner$ and $\urcorner$. (see Fig.5)*

Fig. 5 illustrates some frame representations of a C_4.

(a) 4-star in grid   (b) True pie   (c) False pie
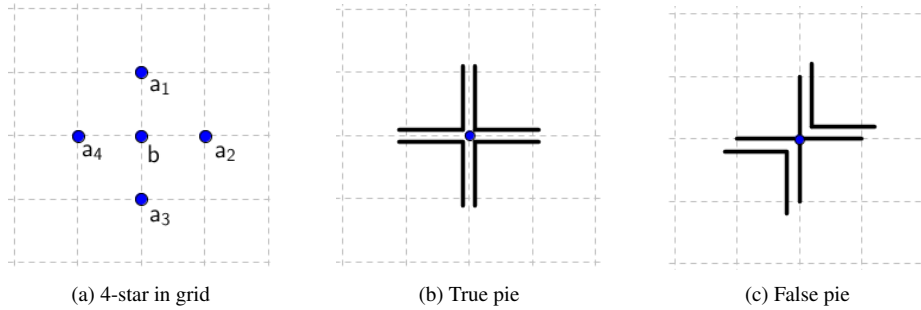
Fig. 4: B_1-EPG representation of the induced cycle of size 4 as pies with emphasis in center b

Lemma 9 (Golumbic et al. (2009))**.** *Every C_4 that is an induced subgraph of a graph G corresponds, in any representation, to a true pie, a false pie, or a frame.*

The following is a claim of Heldt et al. (2014b) which a reasoning can be found in Asinowski and Suk (2009).
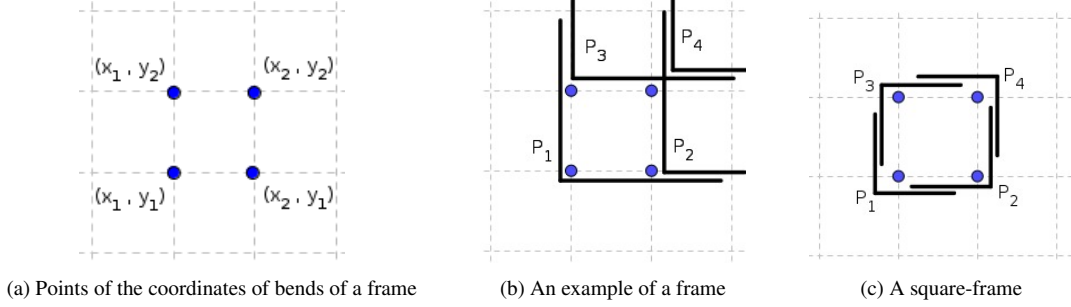
(a) Points of the coordinates of bends of a frame   (b) An example of a frame   (c) A square-frame

Fig. 5: B_1-EPG representation of the induced cycle of size 4 as frame

Lemma 10 (Heldt et al. (2014a) and Asinowski and Suk (2009)). *In every single bend representation of a $K_{2,4}$, the path representing each vertex of the largest part has its bend in a false pie.*

By creating four $K_{2,4}$ and identifying a vertex of the largest part of each one to a distinct vertex of a C_4, we construct the graph called bat graph (see Fig 6). Regarding to such a graph, the following holds.
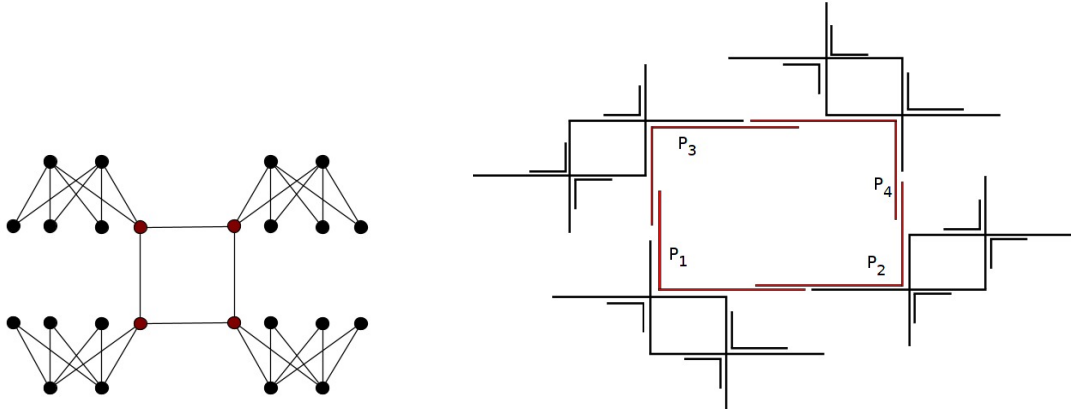

Fig. 6: A bat graph G and a Helly-B_1-EPG representation of G.

Corollary 11. *In every single bend representation of the bat graph, G presented in Fig. 6, the C_4 that is a transversal of all $K_{2,4}$ is represented by a square-frame.*

Proof: By Lemma 10, it follows that in every single bend representation of the bat graph, each path representing a vertex of the C_4 (transversal to all $K_{2,4}$) has its bend in a false pie in which paths represent vertices of a $K_{2,4}$ (Fig. 7 illustrates a B_1-EPG representation of a $K_{2,4}$). Thus, the intersection of two paths representing vertices of this C_4 does not contain any edge incident to a bend point of such paths, which implies that such a C_4 must be represented by a frame (see Lemma 9). Note that for each path of the frame, we have four possible shapes (∟, ⌟, ⌜, and ⌝). Let P_1 be the path having the bottom-left bend point, P_2 be the path having the bottom-right bend point, P_3 be the path having the top-left bend point

and P_4 be the path having the top-right bend point. Note that to prevent P_2 and P_3 from containing edges incident at the bend point of P_1, the only shape allowed for P_1 is ⌞. Similarly, the only shape allowed for P_2 is ⌟ as well as for P_3 is ⌜ and for P_4 is ⌝. Thus, the C_4 is represented by a square-frame.   □
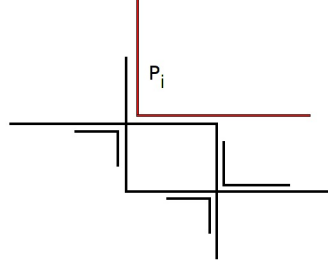

Fig. 7: Helly-B_1-EPG representation of a $K_{2,4}$.

Definition 12. *A B_k-EPG representation is* minimal *when its set of edges does not properly contain another B_k-EPG representation.*

The *octahedral* graph is the graph containing 6 vertices and 12 edges, depicted in Figure 8(a). Next, we consider representations of the octahedral graph.

The next lemma follows directly from the discussion presented in Heldt et al. (2014b).

Lemma 13. *Every minimal B_1-EPG representation of the octahedral graph O_3 has the same shape.*

Proof: Note that the octahedral graph O_3 has an induced C_4 such that the two vertices of the octahedral graph that are not in such a cycle are false twins whose neighborhood contains the vertices of the induced C_4.

If in an EPG representation of the graph O_3 such a C_4 is represented as a frame, then no single bend path can simultaneously intersect the four paths representing the vertices of the induced C_4. Therefore, we conclude that the frame structure cannot be used to represent such a C_4 in a B_1-EPG representation of the O_3. Now, take a B_1-EPG representation of such a C_4 shaped as a true pie or false pie. By adding the paths representing the false twin vertices, which are neighbors of all vertices of the C_4, in both cases (from a true or false pie), we obtain representations with the shape represented in Fig. 8(b).   □

2.1 Subclasses of B_1-EPG graphs

By Lemma 13, every minimal B_1-EPG representation of the octahedral graph O_3 has the same shape, as depicted in Fig. 8(b). Since in any representation of the graph O_3 there is always a triple of paths that do not satisfy the Helly property, paths P_a, P_b and P_c in the case of Fig. 8(b), it holds that $O_3 \notin$ Helly-B_1 EPG, which implies that the class of Helly-B_1-EPG graphs is a proper subclass of B_1-EPG.

Also, B_0-EPG and Helly-B_0-EPG graphs coincide. Hence, Helly-B_0 EPG can be recognized in polynomial time, see Booth and Lueker (1976).

In a B_1-EPG representation of a graph, the paths can be of the following four shapes: ⌞, ⌟, ⌜ and ⌝. Cameron et al. (2016) studied B_1-EPG graphs whose paths on the grid belong to a proper subset of the

(a) The octahedral graph O_3 (b) B_1-EPG representation of the graph O_3
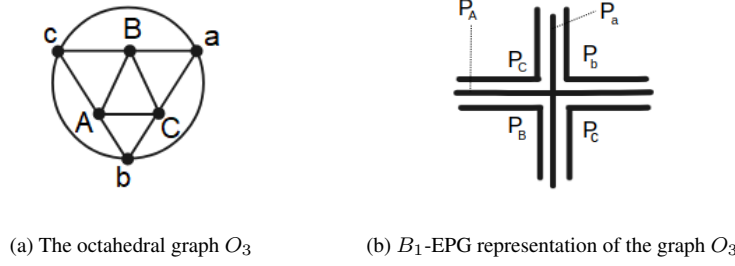
Fig. 8: The octahedral graph O_3 graph and its B_1-EPG representation

four shapes. If S is a subset of $\{\llcorner, \lrcorner, \ulcorner, \urcorner\}$, then $[S]$ denotes the class of graphs that can be represented by paths whose shapes belong to S, where zero-bend paths are considered to be degenerate $\llcorner$'s. They consider the natural subclasses of B_1-EPG: $[\llcorner], [\llcorner, \ulcorner], [\llcorner, \urcorner]$ and $[\llcorner, \ulcorner, \urcorner]$, all other subsets are isomorphic to these up to 90 degree rotation. Cameron et al. (2016) showed that recognizing each of these classes is NP-complete.

The following shows how these classes relate to the class of Helly-B_1-EPG graphs.
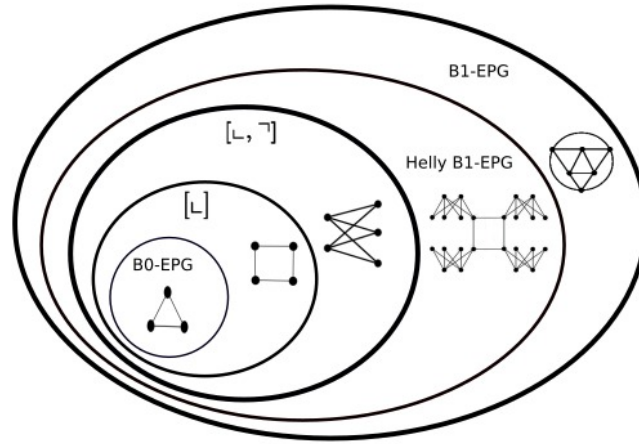

Fig. 9: Hierarchical diagram of some EPG classes

Theorem 14. $[\llcorner] \subsetneq [\llcorner, \urcorner] \subsetneq$ *Helly-B_1 EPG, and Helly-B_1 EPG is incomparable with* $[\llcorner, \ulcorner]$ *and* $[\llcorner, \ulcorner, \urcorner]$.

Proof: Cameron et al. (2016) showed that $[\llcorner] \subsetneq [\llcorner, \urcorner]$. Also, it is easy to see that $\llcorner$'s and $\urcorner$'s cannot form a claw-clique, thus, by Lemma 6, it follows that $[\llcorner, \urcorner] \subseteq$ Helly-B_1 EPG. In order to observe that $[\llcorner, \urcorner]$ is a proper subclass of Helly-B_1 EPG, it is enough to analyze the bat graph (see Fig. 6): by Corollary 11 follows that any B_1-EPG representation of a bat graph contains a square-frame, thus it is not in $[\llcorner, \urcorner]$. In addition, the bat graph is bipartite which implies that any B_1-EPG representation of that graph does not contain claw-cliques and therefore is Helly.

Now, it remains to show that Helly-B_1 EPG is incomparable with $[\llcorner, \ulcorner]$ and $[\llcorner, \ulcorner, \urcorner]$. Again, since any B_1-EPG representation of a bat graph contains a square-frame, bat graph is a Helly-B_1-EPG graph that is not in $[\llcorner, \ulcorner, \urcorner]$. On the other hand, the S_3 (3-sun) is a graph in $[\llcorner, \ulcorner]$ such that any of its B_1-EPG representations have a claw-clique, see Observation 7 in Cameron et al. (2016). Therefore, S_3 is a graph in $[\llcorner, \ulcorner]$ that is not Helly-B_1 EPG.   □

Figure 9 depicts example of graphs of the classes B_0-EPG, $[\llcorner]$, $[\llcorner, \urcorner]$, Helly-B_1 EPG, and B_1-EPG that distinguish these classes.

It is known that recognizing $[\llcorner]$, $[\llcorner, \urcorner]$, and B_1-EPG are NP-complete while recognizing B_0-EPG and EPG graphs can be done in polynomial time (c.f. Booth and Lueker (1976), Heldt et al. (2014b), and Cameron et al. (2016)).

In this paper, we show that it is NP-complete to recognize Helly-B_1-EPG graphs.

3   Membership in NP

The HELLY-B_k EPG RECOGNITION problem can be formally described as follows.

HELLY-B_k EPG RECOGNITION
Input: A graph G and an integer $k \leq |V(G)|^c$, for some fixed c.
Goal: Determine if there is a set of k-bend paths $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ in a grid Q such that: • $u, v \in V(G)$ are adjacent in G if only if P_u, P_v share an edge in Q; and • $\mathcal{P}$ satisfies the Helly property.

A (positive) certificate for the HELLY-B_k EPG RECOGNITION consists of a grid Q, a set $\mathcal{P}$ of k-bend paths of Q, which is in one-to-one correspondence with the vertex set $V(G)$ of G, such that, for each pair of distinct paths $P_i, P_j \in \mathcal{P}$, $P_i \cap P_j \neq \emptyset$ if and only if the corresponding vertices are adjacent in G. Furthermore, $\mathcal{P}$ satisfies the Helly property.

The following are key concepts that make it easier to control the size of an EPG representation. A *relevant edge* of a path in a B_k-EPG representation is either an extremity edge or a bend edge of the path. Note that each path with at most k bends can have up to $2(k+1)$ relevant edges, and any B_k-EPG representation contains at most $2|\mathcal{P}|(k+1)$ distinct relevant edges.

To show that there is a non-deterministic polynomial-time algorithm for HELLY-B_k EPG RECOGNITION, it is enough to consider as certificate a B_k-EPG representation R containing a collection $\mathcal{P}$ of paths, $|\mathcal{P}| = |V(G)|$, such that each path $P_i \in \mathcal{P}$ is given by its set of relevant edges along with the relevant edges, that intersects P_i, of each path P_j intersecting P_i, where $P_j \in \mathcal{P}$. The relevant edges for each path are given in the order that they appear in the path, to make straightforward checking that the edges correspond to a unique path with at most k bends. This representation is also handy for checking that the paths form an intersection model for G.

To verify in polynomial time that the input is a positive certificate for the problem, we must assert the following:

(i) The sequence of relevant edges of a path $P_i \in \mathcal{P}$ determines P_i in polynomial time;

(ii) Two paths $P_i, P_j \in \mathcal{P}$ intersect if and only if they intersect in some relevant edge;

(iii) The set $\mathcal{P}$ of relevant edges satisfies the Helly property.

The following lemma states that condition (i) holds.

Lemma 15. *Each path P_i can be uniquely determined in polynomial time by the sequence of its relevant edges.*

Proof: Consider the sequence of relevant edges of some path $P_i \in \mathcal{P}$. Start from an extremity edge of P_i. Let t be the row (column) containing the last considered relevant edge. The next relevant edge e' in the sequence, must be also contained in row (column) t. If e' is an extremity edge, the process is finished, and the path has been determined. It contains all edges between the considered relevant edges in the sequence. Otherwise, if e' is a bend edge, the next relevant edge is the second bend edge e'' of this same bend, which is contained in some column (row) t'. The process continues until the second extremity edge of P_i is located.

With the above procedure, we can determine in $\mathcal{O}(k \cdot |V(G)|)$ time, whether path P_i contains any given edge of the grid Q. Therefore, the sequence of relevant edges of P_i uniquely determines P_i. $\square$

Next, we assert property (ii).

Lemma 16. *Let $\mathcal{P}$ be the set of paths in a B_k-EPG representation of G, and let $P_1, P_2 \in \mathcal{P}$. Then P_1, P_2 are intersecting paths if and only if their intersection contains at least one relevant edge.*

Proof: Assume that P_1, P_2 are intersecting, and we show they contain a common relevant edge. Without loss of generality, suppose P_1, P_2 intersect at row i of the grid, in the B_k-EPG representation R. The following are the possible cases that may occur:

- **Case 1:** Neither P_1 nor P_2 contain bends in row i.

  Then P_1 and P_2 are entirely contained in row i. Since they intersect, either P_1, P_2 overlap, or one of the paths contains the other. In any of these situations, they intersect in a common extremity edge, which is a relevant edge.

- **Case 2:** P_1 does not contain bends in i, but P_2 does.

  If some bend edge of P_2 also belongs to P_1, then P_1, P_2 intersect in a relevant edge. Otherwise, since P_1, P_2 intersect, the only possibility is that the intersection contains an extremity edge of P_1 or P_2. Hence the paths intersect in a relevant edge.

- **Case 3:** Both P_1, P_2 contain bends in i

  Again, if the intersection occurs in some bend edge of P_1 or P_2, the lemma follows. Otherwise, the same situation as above must occur: P_1, P_2 must intersect in an extremity edge.

In any of the cases, P_1 and P_2 intersect in some relevant edge. $\square$

The two previous lemmas let us check that a certificate is an actual B_k-EPG representation of a given graph G. The next lemma says we can also verify in polynomial time that the representation encoded in the certificate is a Helly representation. Fortunately, we do not need to check every subset of intersecting paths of the representation to make sure they have a common intersection.

Lemma 17. *Let $\mathcal{P}$ be a collection of paths encoded as a sequence of relevant edges that constitute a B_k-EPG representation of a graph G. We can verify in polynomial time if $\mathcal{P}$ has the Helly property.*

Proof: Let T be the set of relevant edges of $\mathcal{P}$. Consider each triple T_i of edges of T . Let P_i be the set of paths of $\mathcal{P}$ containing at least two of the edges in the triple T_i. By Gilmore's Theorem, see Berge and Duchet (1975), $\mathcal{P}$ has the Helly property if an only if the subset of paths P_i corresponding to each triple T_i has a non-empty intersection. By Lemma 16, it suffices to examine the intersections on relevant edges. Therefore a polynomial algorithm for checking if $\mathcal{P}$ has the Helly property could examine each of the subsets P_i, and for each relevant edge e of a path in P_i, to compute the number of paths in P_i that contain e. Then $\mathcal{P}$ has the Helly property if and only if for every P_i, there exists some relevant edge that is present in all paths in P_i, yielding a non-empty intersection. $\qquad\square$

Corollary 18. *Let $\mathcal{P}'$ be a set a pairwise intersecting paths in a Helly-B_k-EPG representation of a graph G. Then the intersection of all paths of $\mathcal{P}'$ contains at least one relevant edge.*

Note that the property described in Corollary 18 is a consequence of Gilmore's Theorem, see Berge and Duchet (1975), and it applies only to representations that satisfy Helly's property.

From Corollary 18, the following theorem concerning the Helly-bend number of a graph holds.

Theorem 19. *For every graph G containing n vertices and μ maximal cliques, it holds that*

$$\frac{\mu}{2n} - 1 \leq b_H(G) \leq \mu - 1.$$

Proof: The upper bound follows from Corollary 4. For the lower bound first notice that each path with at most k bends can have up to $2(k+1)$ relevant edges, and any B_k-EPG representation with a set of paths $\mathcal{P}$ contains at most $2|\mathcal{P}|(k+1)$ distinct relevant edges. Now, let G be a graph with n vertices, μ maximal cliques, and $b_H(G) = k$. From Corollary 18, it follows that in a Helly-B_k-EPG representation of G every maximal clique of G contains at least one relevant edge. By maximality, two distinct maximal cliques cannot share the same edge-clique. Thus, in a Helly-B_k-EPG representation of G every maximal clique of G contains at least one distinct relevant edge, which implies that $\mu \leq 2n(k+1)$, so $\frac{\mu}{2n} - 1 \leq b_H(G)$. $\quad\square$

Lemma 20. *Let G be a (Helly-)B_k-EPG graph. Then G admits a (Helly-)B_k-EPG representation on a grid of size at most $4n(k+1) \times 4n(k+1)$.*

Proof: Let R be a B_k-EPG representation of a graph G on a grid Q with the smallest possible size. Let $\mathcal{P}$ be the set of paths of R. Note that $|\mathcal{P}| = n$. A counting argument shows that there are at most $2|\mathcal{P}|(k+1)$ relevant edges in R. If Q has a pair of consecutive columns c_i, c_{i+1} neither of which contains relevant edges of R, and such that there is no relevant edge crossing from c_i to c_{i+1}, then we can contract each edge crossing from c_i to c_{i+1} into single vertices so as to obtain a new B_k-EPG representation of G on a smaller grid, which is a contradiction. An analogous argument can be applied to pairs of consecutive rows of the grid. Therefore the grid Q is such that each pair of consecutive columns and consecutive rows of Q has at least one relevant edge of R or contains a relevant edge crossing it. Since Q is the smallest possible grid for representing G, then the first row and the first column of Q must contain at least one point belonging to some relevant edge of R. Thus, if G is B_k-EPG then it admits a B_k-EPG representation on a grid of size at most $4|\mathcal{P}|(k+1) \times 4|\mathcal{P}|(k+1)$. Besides, by Corollary 18, it holds that the contraction operation previously described preserves the Helly property, if any. Hence, letting R be a Helly-B_k-EPG

representation of a graph G on a grid Q with the smallest possible size it holds that Q has size at most $4|\mathcal{P}|(k+1) \times 4|\mathcal{P}|(k+1)$. □

Given a graph G with n vertices and an EPG representation R, it is easy to check in polynomial time with respect to $n + |R|$ whether R is a B_k-EPG representation of G. By Lemma 20, if G is a B_k-EPG graph then there is a positive certificate (an EPG representation) R of polynomial size with respect to $k + n$ to the question "$G \in B_k$-EPG?". Therefore, Corollary 21 holds.

Corollary 21. *Given a graph G and an integer $k \geq 0$, the problem of determining whether G is a B_k-EPG graph is in NP, whenever k is bounded by a polynomial function of $|V(G)|$.*

At this point, we are ready to demonstrate the NP-membership of HELLY-B_k EPG RECOGNITION.

Theorem 22. HELLY-B_k EPG RECOGNITION *is in NP.*

Proof: By Lemma 20 and the fact that k is bounded by a polynomial function of $|V(G)|$, it follows that the collection $\mathcal{P}$ can be encoded through its relevant edges with $n^{\mathcal{O}(1)}$ bits.

Finally, by Lemmas 15, 16 and 17, it follows that one can verify in polynomial-time in the size of G whether $\mathcal{P}$ is a family of paths encoded as a sequence of relevant edges that constitute a Helly-B_k-EPG representation of a graph G. □

4   NP-hardness

Now we will prove that HELLY-B_1 EPG RECOGNITION is NP-complete. For this proof, we follow the basic strategy described in the prior hardness proof of Heldt et al. (2014b). We set up a reduction from POSITIVE (1 IN 3)-3SAT defined as follows:

POSITIVE (1 IN 3)-3SAT	
Input:	A set X of positive variables; a collection C of clauses on X such that for each $c \in C$, $|c| = 3$.
Goal:	Determine if there is an assignment of values to the variables in X so that every clause in C has exactly one true literal.

POSITIVE (1 IN 3)-3SAT is a well-known NP-complete problem (see Garey and Johnson (1979), problem [L04], page 259). Also, it remains NP-complete when the incidence graph of the input CNF (Conjunctive Normal Form) formula is planar, see Mulzer and Rote (2008).

Given a formula F that is an instance of POSITIVE (1 IN 3)-3SAT we will present a polynomial-time construction of a graph G_F such that $G_F \in$ Helly-B_1 EPG if and only if F is satisfiable. This graph will contain an induced subgraph G_{C_i} with 12 vertices (called *clause gadget*) for every clause $C_i \in \mathcal{C}$, and an induced subgraph (*variable gadget*) for each variable x_j, containing a special vertex v_j, plus a *base gadget* with 55 additional vertices.

We will use a graph H isomorphic to the graph presented in Figure 10, as a gadget to perform the proof. For each clause C_i of F of the target problem, we will have a *clause gadget* isomorphic to H, denoted by G_{c_i}.

The reduction of a formula F from POSITIVE (1 IN 3)-3SAT to a particular graph G_F (where G_F has a Helly-B_1-EPG representation if only if F is satisfiable) is given below.
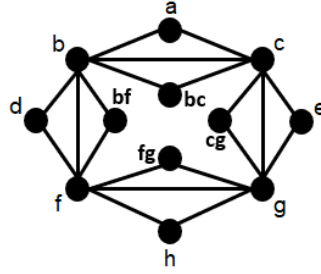
Fig. 10: The partial gadget graph H

Definition 23. *Let F be a CNF-formula with variable set $\mathcal{X}$ and clause set $\mathcal{C}$ with no negative literals, in which every clause has exactly three literals. The graph G_F is constructed as follows:*

1. *For each clause $C_i \in \mathcal{C}$ create a clause gadget G_{C_i}, isomorphic to graph H;*

2. *For each variable $x_j \in \mathcal{X}$ create a* variable vertex *v_j that is adjacent to the vertex a, e, or h of G_{C_i}, when x_j is the first, second or third variable in C_i, respectively;*

3. *For each variable vertex v_j, construct a* variable gadget *formed by adding two copies of H, H_1 and H_2, and making v_j adjacent to the vertices of the triangles (a, b, c) in H_1 and H_2.*

4. *Create a vertex V, that will be used as a vertical reference of the construction, and add an edge from V to each vertex d of a clause gadget;*

5. *Create a bipartite graph $K_{2,4}$ with a particular vertex T in the largest stable set. This vertex is* nominated *true vertex. Vertex T is adjacent to all v_j and also to V;*

6. *Create two graphs isomorphic to H, G_{B1} and G_{B2}. The vertex T is connected to each vertex of the triangle (a,b,c) in G_{B1} and G_{B2};*

7. *Create two graphs isomorphic to H, G_{B3} and G_{B4}. The vertex V is connected to each vertex of the triangle (a,b,c) in G_{B3} and G_{B4};*

8. *The subgraph induced by the set of vertices $\{V(K_{2,4}) \cup \{T, V\} \cup V(G_{B1}) \cup V(G_{B2}) \cup V(G_{B3}) \cup V(G_{B4})\}$ will be referred to as the* base gadget.

Figure 11 illustrates how this construction works on a small formula.

Lemma 24. *Given a satisfiable instance F of* POSITIVE (1 IN 3)-3SAT*, the graph G_F constructed from F according to Definition 23 admits a Helly-B_1-EPG representation.*

Proof: We will use the true pie and false pie structures to represent the *clause gadgets* G_C (see Figure 12), but the construction could also be done with the frame structure without loss of generality.

The *variable gadgets* will be represented by structures as of Figure 13.

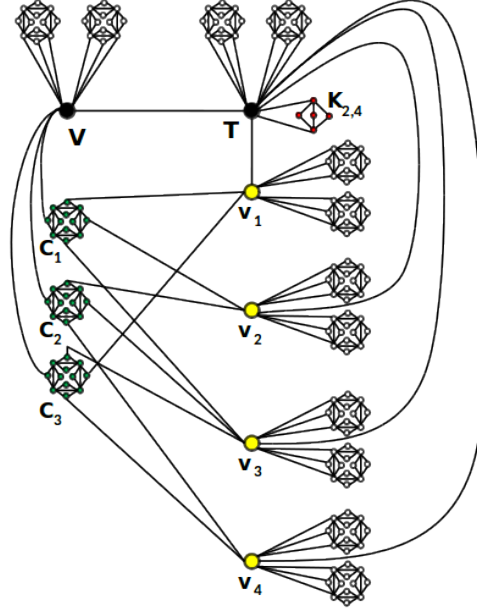The *base gadget* will be represented by the structure of Figure 14.

Fig. 11: The G_F graph corresponding to formula $F = (x_1 + x_2 + x_3) \cdot (x_2 + x_3 + x_4) \cdot (x_3 + x_1 + x_4)$

It is easy to see that the representations of the clause gadgets, variable gadgets, and base gadgets are all Helly-B_1 EPG. Now, we need to describe how these representations can be combined to construct a single bend representation R_{G_F}.

Given an assignment A that satisfies F, we can construct a Helly-B_1-EPG representation R_{G_F}. First we will fix the representation structure of the base gadget in the grid to guide the single bend representation, see Figure 14. Next we will insert the variable gadgets with the following rule: if the variable x_i related to the path P_{v_i} had assignment *True*, then the adjacency between the path P_{v_i} with P_T is horizontal, and vertical otherwise. For example, for an assignment $A = \{x_1 = False; x_2 = False; x_3 = True; x_4 = False\}$ to variables of the formula F that generated the gadget G_F of Figure 11, it will give us a single bend representation (base gadget + variables gadget) according to the Figure 15(a).

When a formula F of POSITIVE (1-IN-3)-3SAT has clauses whose format of the assignment is $(False, True, False)$ or $(False, False, True)$ then we will use false pie to represent these clauses. When the clause has format $(True, False, False)$, we will use true pie to represent this clause (the use of true pie in the last case is only to illustrate that the shape of the pie does not matter in the construction). To insert a *clause gadget* G_C, we introduce a horizontal line l_h in the grid between the horizontal rows used by the paths for the two false variables in C. Then we connect the path $P_{d_{c_i}}$ of G_{C_i} to P_V vertically using the bend of $P_{d_{c_i}}$. We introduce a vertical line l_v in the grid, between the vertical line of the grid used by P_V and the path to the true variable in C_i, *i.e.* between P_V and the path of the true variable $x_j \in C_i$. At the point where l_h and l_v cross, to insert the center of the *clause gadget* as can be seen in Figure 15(b). The complete construction of this single bend representation for the G_F can be seen in Figure 16.

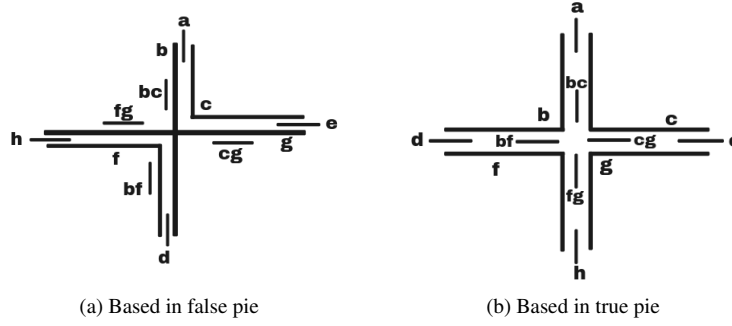Note that when we join all these representations of gadgets that form R_{G_F}, we do not increase the

(a) Based in false pie   (b) Based in true pie

Fig. 12: Single bend representations of a clause gadget isomorphic to graph H

Fig. 13: Single bend representation of a variable gadget

number of bends. Then the representation necessarily is B_1-EPG. Let us show that it satisfies the Helly property.

A simple way to check that R_{G_F} satisfies the Helly property is to note that the particular graph G_F never forms triangles between variable, clause, and base gadgets. Thus, any triangle of G_F is inside a variable, clause, or base gadget. As we only use Helly-B_1-EPG representations of such gadgets, R_{G_F} is a Helly-B_1-EPG representation of G_F. □

Now, we consider the converse. Let R be a Helly-B_1-EPG representation of G_F.

Definition 25. *Let H be the graph shown in Figure 10, such that the 4-cycle $H[\{b, c, f, g\}]$ corresponds*
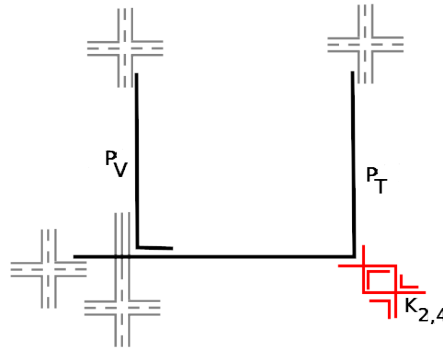

Fig. 14: Single bend representation of the base gadget

in R to a false pie or true pie, then:

- *the* center *is the unique grid-point of this representation which is contained in every path representing 4-cycle $\{b, c, f, g\}$;*

- *a* central ray *is an edge-intersection between two of the paths corresponding to vertices b, c, f, g, respectively.*

Note that every B_1-EPG representation of a C_4 satisfies the Helly property, see Lemma 9, and triangles have B_1-EPG representations that satisfy the Helly property, *e.g.* the one shown in Figure 1(b). The graph H is composed by a 4-cycle $C_4^H = H[b, c, f, g]$ and eight cycles of size 3.

As C_4^H has well known representations (see in Lemma 9), then we can start drawing the Helly-B_1-EPG representation of H from these structures. Figure 17 shows possible representations for H.

If C_4^H is represented by a pie, then the paths P_b, P_c, P_f, P_g share the center of the representation. On the other hand, if C_4^H is represented by a frame, then the bends of the four paths correspond to the four distinct corners of a rectangle, *i.e.* all paths representing the vertices of C_4^H have distinct bend points, see Golumbic et al. (2009).
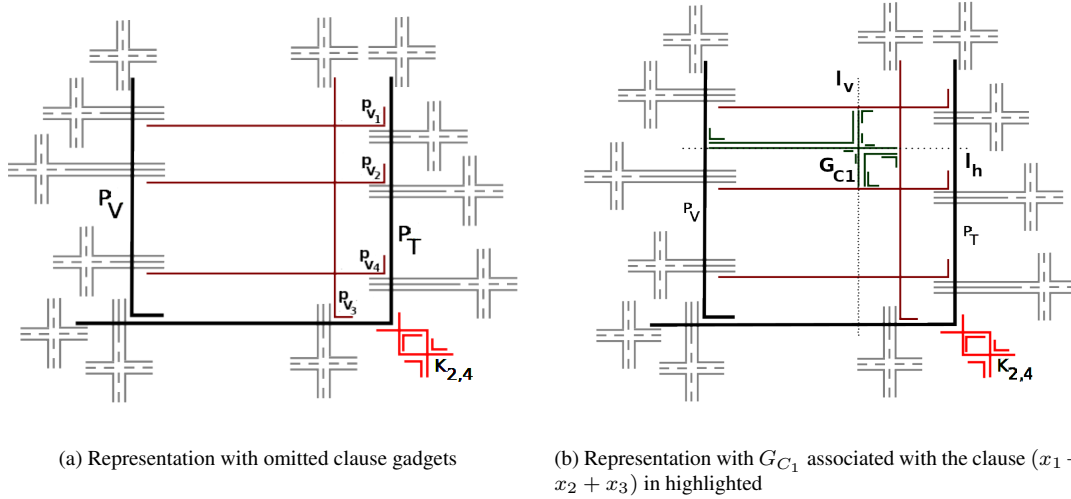

(a) Representation with omitted clause gadgets

(b) Representation with G_{C_1} associated with the clause $(x_1 + x_2 + x_3)$ in highlighted

Fig. 15: Single bend representation of the base and variables gadgets associated with the assignment $x_1 = False, x_2 = False, x_3 = True, x_4 = False$

Next, we examine the use of the frame structure.

Proposition 26. *In a frame-shaped B_1-EPG representation of a C_4, every path P_i that represents a vertex of the C_4 intersects exactly two other paths P_{i-1} and P_{i+1} of the frame so that one of the intersections is horizontal and the other is vertical.*

Proposition 27. *Given a Helly-B_1-EPG representation of a graph G that has an induced C_4 whose representation is frame-shaped. If there is a vertex v of G, outside the C_4, that is adjacent to exactly two*
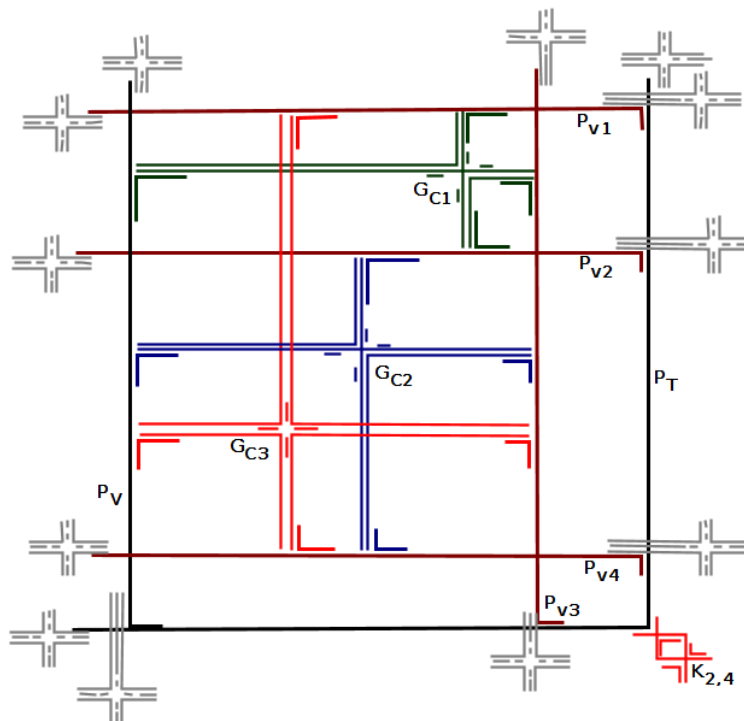
Fig. 16: Single bend representation of G_F

consecutive vertices of this C_4, then the path representing v shares at least one common edge-intersection with the paths representing both of these vertices.

Proof: By assumption, G has a triangle containing v and two vertices of a C_4. Therefore the path representing v shares at least one common edge intersecting with the paths representing these neighbors, otherwise the representation does not satisfy the Helly property. $\square$

By Proposition 26 and Proposition 27 we can conclude that for every vertex $v_i \in V(H)$ such that $v_i \neq V(C_4^H)$, when we use a frame to represent the C_4^H, P_{v_i} will have at least one common edge-intersection with the pair of paths representing its neighbors in H. Figure 17(c) presents a possible Helly-B_1-EPG representation of H. Note that we can apply rotations and mirroring operations while maintaining it as a Helly-B_1-EPG representation of H.

Definition 28. *In a frame-shaped single bend representation of a C_4 graph, the paths that represent consecutive vertices in the C_4 are called* consecutive paths *and the segment that corresponds to the intersection between two consecutive paths is called* side intersection.

Lemma 29. *In any minimal single bend representation of a graph isomorphic to H, there are two paths in $\{P_a, P_e, P_d, P_h\}$ that have horizontal directions and the other two paths have vertical directions.*

Proof: If the $C_4^H = [b, c, f, g]$ is represented by a true pie or false pie, then each path of C_4^H shares two central rays with two other paths of C_4^H, where each central ray corresponds to one pair of consecutive vertices in C_4^H.

As the vertices a, e, d and h are adjacent to pairs of consecutive vertices in C_4^H so the paths P_a, P_e, P_d and P_h have to be positioned in each one of the different central rays, 2 are horizontal and 2 are vertical.

If the C_4^H is represented by a frame, then each path of the C_4^H has a bend positioned in the corners of the frame. In the frame, the adjacency relationship of pairs of consecutive vertices in the C_4^H is represented by the edge-intersection of the paths that constitute the frame. Thus, since a frame has two parts in the vertical direction and two parts in the horizontal direction, then there are two paths in $\{P_a, P_e, P_d, P_h\}$ that have horizontal direction and two that have vertical direction.

Note that no additional edge is needed on the different paths by the minimality of the representation. □

Corollary 30. *In any minimal single bend representation of a graph isomorphic to H, the following paths are on the same central ray or side intersection: P_a and P_{bc}; P_e and P_{cg}; P_h and P_{fg}; P_d and P_{bf}.*

(a) Based in false pie   (b) Based in true pie   (c) Based in frame
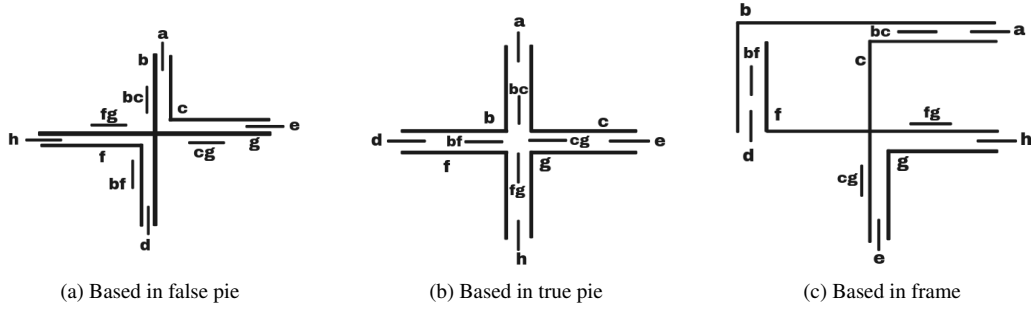
Fig. 17: Different single bend representations of the graph H using a false pie (a), a true pie (b) and a frame (c) for representing C_4^H
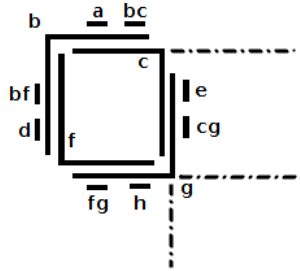

Fig. 18: A frame representation where the bend of dashed paths change directions

The following proposition helps us in the understanding of the NP-hardness proof.

Proposition 31. *In any Helly-B_1 representation of the graph G', presented in Figure 19(a), the path P_x has obstructed extremities and bends.*

Proof: Consider G' consisting of a vertex x together with two graphs, H_1 and H_2, isomorphic to H and a bipartite graph $K_{2,4}$, such that: x is a vertex of the largest stable set of the $K_{2,4}$; x is adjacent to an induced cycle of size 3 of H_1, $C_3^{H_1}$ and to an induced cycle of size 3 of H_2, $C_3^{H_2}$, see Figure 19(a).

   We know that the paths belonging to the largest stable set of a $K_{2,4}$ always will bend into a false pie, see Fact 10. Since P_x is part of the largest stable set of the $K_{2,4}$, then P_x has an *obstructed bend*, see Figure 19(b).

   The vertex x is adjacent to $C_3^{H_1}$ and $C_3^{H_2}$, so that its path P_x intersects the paths representing them. But in a single bend representations of a graph isomorphic to H there are pairs of paths that always are on some segment of a central ray or a side intersection, see Corollary 30, and the representation of $C_3^{H_1}$ (similarly $C_3^{H_2}$) has one these paths. Therefore, there is an edge in the set of paths that represent H_1 (similarly in H_2) that has a intersection of 3 paths representing $C_3^{H_1}$ (and $C_3^{H_2}$), otherwise the representation would not be Helly. There is another different edge in the same central ray or side intersection that contains three other paths and one of them is not in the set of paths $C_3^{H_1}$ (similarly $C_3^{H_2}$). Thus in a single bend representation of G', the paths that represent $C_3^{H_1}$ (similarly $C_3^{H_2}$) must intersect in a bend edge or an extremity edge of P_x, because P_x intersects only one of the paths that are on some central ray or side intersection where $C_3^{H_1}$ (similarly $C_3^{H_2}$) is. As the bend of G' is already obstructed by structure of $K_{2,4}$, then H_1 (similarly in H_2) must be positioned at an extremity edge of P_x. This implies that P_x has a condition of *obstructed extremities*, see Figure 19(b). □
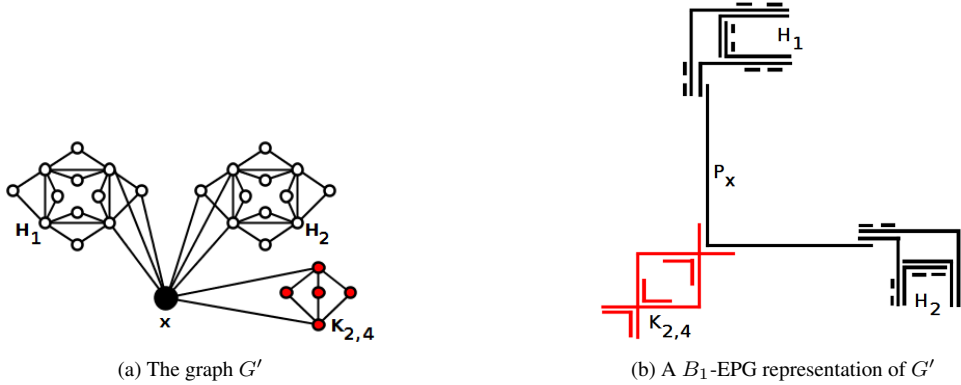

(a) The graph G' (b) A B_1-EPG representation of G'

Fig. 19: The sample of obstructed extremities and bend.

Definition 32. *We say that a segment s is* internally contained *in a path P_x if s is contained in P_x, and it does not intersect a relevant edge of P_x.*

   Some of the vertices of G_F have highly constrained B_1-EPG representations. Vertex T has its bend and both extremities obstructed by its neighbors in G_{B1}, G_{B2} and in the $K_{2,4}$ subgraphs. Vertex V and each variable vertex v_i must have one of its segments internally contained in T, and also have its extremities and bends obstructed. Therefore, vertex V and each variable vertex has only one segment each that can be used in an EPG representation to make them adjacent to the clause gadget. The direction of this segment, being either horizontal or vertical, can be used to represent the true or false value for the variable. The clause gadgets, on the other hand, are such that exactly two of its adjacencies to the variable vertices and

V can be realized with a horizontal intersection, whereas the other two must be realized with a vertical intersection. If we consider the direction used by V as a truth assignment, we get that exactly one of the variables in each clause will be true in any possible representation of G_F. Conversely, it is fairly straightforward to obtain a B_1-EPG representation for G_F when given a truth assignment for the formula F. Therefore, Lemma 33 holds.

Lemma 33. *If a graph G_F, constructed according to Definition 23, admits a Helly-B_1-EPG representation, then the associated CNF-formula F is a yes-instance of* POSITIVE (1 IN 3)-3SAT.

Proof: Suppose that G_F has a Helly-B_1-EPG representation, R_{G_F}. From R_{G_F} we will construct an assignment that satisfies F.

First, note that in every single bend representation of a $K_{2,4}$, the path of each vertex of the largest stable set, in particular, P_T (in R_{G_F}), has bends contained in a false pie (see Lemma 10).

The vertex T is adjacent to the vertices of a triangle of G_{B1} and G_{B2}. As the $K_{2,4}$ is positioned in the bend of P_T, then in R_{G_F} the representations of G_{B1} and G_{B2} are positioned at the extremities of P_T, see Proposition 4.3.

Without loss of generality assume that $P_V \cap P_T$ is a horizontal segment in R_{G_F}.

We can note in R_{G_F} that: the number of paths P_d with segment internally contained in P_V is the number of clauses in F; the intersection between each P_a, P_e, P_h in the gadget clause and each path P_{v_j} indicates the variables composing the clause. Thus, we can assign to each variable x_j the value *True* if the edge intersecting P_{v_j} and P_T is horizontal, and *False* otherwise.

In Lemma 29 it was shown that any minimal B_1-EPG representation of a clause gadget has two paths in $\{P_a, P_d, P_e, P_h\}$ with vertical direction and the other two paths have horizontal direction. Since P_d intersects P_V, it follows that in a single bend representation of G_F, we must connect two of these to represent a false assignment, and exactly one will represent a true assignment. Thus, from R_{G_F}, we construct an assignment to F such that every clause has exactly one variable with a true value. $\square$

Recall that a B_1-EPG representation is Helly if and only if each clique is represented by an edge-clique (and not by a claw-clique). Thus, an alternative way to check whether a representation is Helly is to note that all cliques are represented as edge-cliques.

Theorem 34. HELLY-B_1 EPG RECOGNITION *is NP-complete.*

Proof: By Theorem 22, Lemma 24, Lemma 33. $\square$

We say that a k-apex graph is a graph that can be made planar by the removal of k vertices. A d-degenerate graph is a graph in which every subgraph has a vertex of degree at most d. Recall that POSITIVE (1 IN 3)-3SAT remains NP-complete when the incidence graph of the input formula is planar, see Mulzer and Rote (2008). Thus, the following corollary holds.

Corollary 35. HELLY-B_1 EPG RECOGNITION *is NP-complete on 2-apex and 3-degenerate graphs.*

Proof: To prove that G_F is 3-degenerate, we apply the d-degenerate graphs recognition algorithm, consisting of repeatedly removing the vertices of a minimum degree from the graph. Note that each vertex to be removed at each iteration of the algorithm always has a degree at most three, and therefore the graphs G_F constructed according to Definition 23 is 3-degenerate.

Now, recall that POSITIVE (1 IN 3)-3SAT remains NP-complete when the incidence graph of F is planar, see Mulzer and Rote (2008). Let F be an instance of PLANAR POSITIVE (1 IN 3)-3SAT, we

know that the incidence graph of the formula F is planar. By using the planar embedding of the incidence graph, we can appropriately replace the vertices representing variables and clauses by variables gadgets and clauses gadgets. As each variable gadget, clause gadget, and base gadget are planar, then something not planar may have arisen only from the intersection that was made between them. As the incidence graph assures that there is a planar arrangement between the intersections of the variable gadgets and clause gadgets, then from that one can construct a graph G_F such that the removal of V and T results into a planar graph, see Figura 20. Thus G_F is 2-apex. □
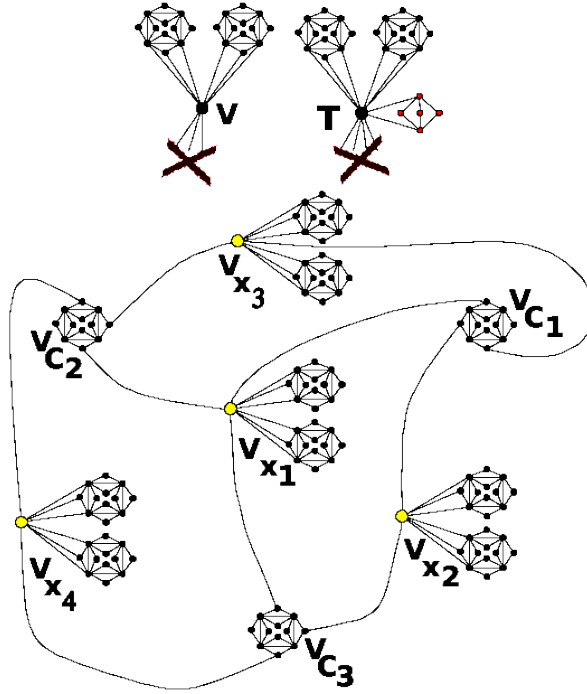

Fig. 20: Planar graph built from $F = (x_1 + x_2 + x_3) \cdot (x_1 + x_3 + x_4) \cdot (x_1 + x_2 + x_4)$, after removing V and T.

5 Concluding Remarks

In this paper, we show that every graph admits a Helly-EPG representation, and $\frac{\mu}{2n} - 1 \le b_H(G) \le \mu - 1$. Besides, we relate Helly-B_1-EPG graphs with L-shaped graphs, a natural family of subclasses of B_1-EPG. Also, we prove that recognizing (Helly-)B_k-EPG graphs is in NP, for every fixed k. Finally, we show that recognizing Helly-B_1-EPG graphs is NP-complete, and it remains NP-complete even when restricted to 2-apex and 3-degenerate graphs.

Now, let r be a positive integer and let K_{2r}^- be the cocktail-party graph, i.e., a complete graph on $2r$ vertices with a perfect matching removed. Since K_{2r}^- has 2^r maximal cliques, by Theorem 19 follows that $\frac{2^r}{4r} - 1 \le b_H(K_{2r}^-)$. This implies that, for each k, the graph $K_{2(k+5)}^-$ is not a Helly-B_k-EPG graph.

Therefore, as Pergel and Rzążewski (2017) showed that every cocktail-party graph is in B_2-EPG, we conclude the following.

Lemma 36. *Helly-B_k-EPG $\subsetneq B_k$-EPG for each $k > 0$.*

The previous lemma suggests asking about the complexity of recognizing Helly-B_k-EPG graphs for each $k > 1$. Also, it seems interesting to present characterizations for Helly-B_k-EPG representations similar to Lemma 6 (especially for $k = 2$) as well as considering the h-Helly-B_k EPG graphs. Regarding L-shaped graphs, it also seems interesting to analyse the classes Helly-$[\llcorner, \ulcorner]$ and Helly-$[\llcorner, \ulcorner, \urcorner]$ (recall Thereom 14).

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