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HELLY AND STRONG HELLY NUMBERS OF B_K-EPG AND B_K-VPG GRAPHS

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Abstract

EPG graphs were introduced by Golumbic, Lypshteyn, and Stern (2009) and consist of the intersection graphs of sets of paths on the orthogonal grid, whose intersections are taken considering the edges of the paths. If the intersections of the paths consider the vertices and not the edges, the resulting graph class is called VPG graphs. A path P is a B_k-path if it contains at most k bends. B_k-EPG and B_k-VPG graphs are the intersection graphs of B_k-paths on the orthogonal grid, considering the intersection of edges and vertices, respectively. A family $\mathcal{F}$ is h-Helly when every h-intersecting subfamily $\mathcal{F}'$ of it satisfies $core(\mathcal{F}') \neq \emptyset$. If for every subfamily $\mathcal{F}'$ of $\mathcal{F}$, there are h subsets whose core equals the core of $\mathcal{F}'$, then $\mathcal{F}$ is said to be strong h-Helly. The Helly number of the family $\mathcal{F}$ is the least integer h, such that $\mathcal{F}$ is h-Helly. Similarly, the strong Helly number of $\mathcal{F}$ is the least h, for which $\mathcal{F}$ is strong h-Helly. In this paper, we solve the problem of determining both the Helly and strong Helly numbers, for B_k-EPG, and B_k-VPG graphs, for each value k.

Keywords: EPG, VPG, path, grid, bend, Helly number, strong Helly.

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1. Introduction

EPG graphs were introduced by Golumbic, Lypshteyn, and Stern (2009) and consist of the intersection graphs of sets of paths on the orthogonal grid, whose intersections are taken considering the edges of the paths. If the intersections of the paths consider the vertices and not the edges, the resulting graph class is called VPG graphs. Such a class was introduced in 2011 [1] and [2]. In the present paper, we study two graph parameters of both EPG and VPG graphs, namely the Helly number and the strong Helly number.

Let $\mathcal{F}$ be a family of subsets of some universal set U, and h an integer ≥ 1. Say that $\mathcal{F}$ is *h-intersecting* when every group of h sets of $\mathcal{F}$ intersect. The *core* of $\mathcal{F}$ is the intersection of all sets of $\mathcal{F}$, denoted $core(\mathcal{F})$.

The family $\mathcal{F}$ is *h-Helly* when every h-intersecting subfamily $\mathcal{F}'$ of it satisfies $core(\mathcal{F}') \neq \emptyset$, see e.g. [4]. On the other hand, if for every subfamily $\mathcal{F}'$ of $\mathcal{F}$, there are h subsets whose core equals the core of $\mathcal{F}'$, then $\mathcal{F}$ is said to be *strong h-Helly*. Clearly, if $\mathcal{F}$ is h-Helly then it is h'-Helly, for $h' \geq h$. Similarly, if $\mathcal{F}$ is strong h-Helly then it is strong h'-Helly, for $h' \geq h$.

Finally, the *Helly number* of the family $\mathcal{F}$ is the least integer h, such that $\mathcal{F}$ is h-Helly. Similarly, the *strong Helly number* of $\mathcal{F}$ is the least h, for which $\mathcal{F}$ is strong h-Helly. It also follows that the strong Helly number of $\mathcal{F}$ is at least equal to its Helly number.

A *class* $\mathcal{C}$ of families $\mathcal{F}$ of subsets of some universal set U is a subcollection of the families $\mathcal{F}$ of U. Say that $\mathcal{C}$ is a *hereditary* class when it closed under inclusion. The *Helly number* of a class $\mathcal{C}$ of families $\mathcal{F}$ of subsets is the largest Helly number among the families $\mathcal{F}$. Similarly, the *strong Helly number* of a class $\mathcal{C}$ is the largest strong Helly number of the families of $\mathcal{C}$.

If $\mathcal{F}$ is a family of subsets and $\mathcal{C}$ a class of families, denote by $H(\mathcal{F})$ and $H(\mathcal{C})$, the Helly numbers of $\mathcal{F}$ and $\mathcal{C}$, respectively, while $sH(\mathcal{F})$ and $sH(\mathcal{C})$ represent the strong Helly numbers of $\mathcal{F}$ and $\mathcal{C}$.

In this work, we study families of subsets $\mathcal{F}$ of edge and vertex paths in a grid. In the context of edge paths, a path consists of a sequence of consecutive edges in the orthogonal grid. We call a collection of such paths an *EPG representation*, i.e., a collection of paths that represent a graph via its intersection graph (considering edge intersections). *EPG graphs* are the class of graphs that admit an EPG representation. Similarly, for vertex paths, a path consists of a sequence of consecutive vertices of the orthogonal grid and a collection of these paths form a *VPG representation* and correspond to a *VPG graph*.

Each edge has an associated direction in the grid, which can be either horizontal or vertical. A *bend* in a path is a pair of consecutive edges that have different directions. A *segment* of a path is a sequence of consecutive edges of the path, with no bends. Say that a path P_i is a B_k-*path* if it contains at most k bends. Say that $\mathcal{F}$ is a B_k-paths family, or simply a B_k-family, if each path of $\mathcal{F}$ is a B_k-path.

In this paper, we solve the problem for determining the Helly and strong Helly numbers, for both B_k-EPG and B_k-VPG graphs, for each value k.

For EPG graphs, the Helly number of B_0-families is well known and is equal to 2, since B_0-EPG graphs coincide with interval graphs. It is also simple to conclude that the strong Helly number of B_0-EPG graphs are also equal to 2. For $k = 1$, we prove that both the Helly number and the strong Helly number of the class of B_1-families are equal to 3. For the class of B_2-families, we prove that these two parameters are equal to 4. The Helly and strong Helly number for B_3-families equal 8, and finally, these parameters are unbounded for $k \geq 4$.

As for VPG graphs, it is simple to verify that the Helly number of B_0-VPG graphs equals 2, and we prove that B_1-VPG have Helly number 4, B_2-VPG graphs have Helly number 6, B_3-VPG has Helly number 12, while the Helly number for B_4-VPG graphs is again unbounded.

Finally, the strong Helly number equals the Helly number of B_k-EPG graphs, for each k. Similarly, for B_k-VPG graphs.

As for existing results, Golumbic, Lipshteyn, and Stern [9] have already shown that the strong Helly number for B_1-EPG graphs equal 3, and for B_1-VPG graphs is equal to 4. employing a different proof technique. See [11], Theorem 11.13, below:

Theorem 1. *[11] Let P be a collection of single bend paths on a grid. If every two paths in P share at least one grid-edge, then P has strong Helly number 3. Otherwise, P has strong Helly number 4.*

No other results concerning the strong Helly number, or no results for the Helly number of B_k-EPG graphs seem to have been reported in the literature. As for other classes, Golumbic and Jamison have determined the strong Helly number of the intersection of edge paths of a tree [8]. Finally, Asinowski, Cohen, Golumbic, Limouzy, Lipshteyn, and Stern have reported that the strong Helly number of B_0-VPG graphs equals two [1].

Deciding whether a given hypergraph is k-Helly can be done in polynomial time for fixed k, employing the characterization by Berge and Duchet [3]. For arbitrary k, the problem is co-NP-complete [7]. For the corresponding problems for strong k-Helly see [6, 7].

The paper is organized as follows. Section 2 contains some preliminary propositions and further notation. Section 3 describes the results for the Helly number of B_k-EPG graphs, while Section 4 contains the results of this parameter for B_k-VPG graphs. The strong Helly number is considered in Section 5. Final remarks are presented in the last section.

2. Preliminaries

The following theorem characterizes h-Helly families of subsets.

Theorem 2. *([3]): A family $\mathcal{F}$ of subsets of the universal set U is h-Helly if and only if for every subset $U' \subseteq U$, $|U'| = h + 1$, the subfamily $\mathcal{F}'$ of $\mathcal{F}$, formed by the subsets containing at least h of the $h + 1$ elements of U', has a non-empty core.*

The next theorem is central to our results.

Theorem 3. *Let $\mathcal{C}$ be a hereditary class of families $\mathcal{F}$ of subsets of the universal set U, whose Helly number $H(\mathcal{C})$ equals h. Then there exists a family $\mathcal{F}' \in \mathcal{C}$ with exactly h subsets, satisfying the following condition:*

For each subset $P_i \in \mathcal{F}'$, there is exactly one distinct element $u_i \in U$, such that

$$u_i \notin P_i,$$

but u_i is contained in all subsets

$$P_j \in \mathcal{F}' \setminus P_i.$$

Proof: Let $\mathcal{C}$ be a class of families $\mathcal{F}$ of subsets P, each subset formed by elements $u \in U$, such that the Helly number $H(\mathcal{C})$ equals h. Then each family $\mathcal{F} \in \mathcal{C}$ satisfies $H(\mathcal{F}) \leq h$. Consider a family $\mathcal{F}' \in \mathcal{C}$ whose Helly number is exactly h, and containing exactly h subsets. Such a family must exist since $\mathcal{C}$ is hereditary. Since $H(\mathcal{F}') = h$, $\mathcal{F}'$ is h-intersecting, and therefore $(h-1)$-intersecting. Furthermore, $\mathcal{F}'$ is not $(h-1)$-Helly. Applying Theorem 2, we conclude that there are h elements $U' = \{u_1, \ldots, u_h\} \subset U$, such that each set of $\mathcal{F}'$ contains at least $h-1$ elements of U'. Since $H(\mathcal{F}') > h-1$, $core(\mathcal{F}') = \emptyset$ and therefore there is no common element among the sets of $\mathcal{F}'$. In particular, since each set $P_i \in \mathcal{F}'$ contains at least $h-1$ elements of U', and $core(\mathcal{F}') = \emptyset$, we can choose h subsets P_i, in which each of them misses a distinct element $u_i \in U'$. Then for each subset $P_i \in \mathcal{F}$, there exists some element $u_i \notin P_i$, but $u_i \in P_j$, for all $P_j \in \mathcal{F}'$, $j \neq i$. □

Let $\mathcal{F}'$ be as in the previous theorem. It is simple to conclude that the removal of any subset from $\mathcal{F}'$ makes it an $(h-1)$-Helly family. Therefore we call $\mathcal{F}'$ a *minimal non-$(h-1)$-Helly family.* Moreover, the element $u_i \notin P_i$, contained in all subsets $P_j \in \mathcal{F}' \setminus P_i$, except P_i, is the *h-non-representative* of P_i.

We can apply this notion of minimal families of subsets for the B_k-EPG and B_k-VPG representations. Recall that B_k-EPG and B_k-VPG graphs are hereditary classes.

3. THE HELLY NUMBER OF B_k-EPG GRAPHS

In this section, we determine the Helly number of the classes of B_1-EPG, B_2-EPG and B_3-EPG graphs, and show that for B_k-EPG graphs, $k \geq 4$, the Helly number is unbounded. We prove the following result.

Theorem 4. *The Helly number of B_k-EPG graphs satisfy:*
(i) *$H(B_1\text{-}EPG) = 3$*
(ii) *$H(B_2\text{-}EPG) = 4$*
(iii) *$H(B_3\text{-}EPG) = 8$*
(iv) *$H(B_k\text{-}EPG)$ is unbounded, for $k \geq 4$.*

The proof consists in determining tight lower and upper bounds, as shown in the next two subsections.

3.1. Lower Bounds

We present lower bounds for the Helly number, as a function of the number k of bends.

Claim 5. *The following are lower bounds for B_k-EPG graphs.*
(i) *$H(B_1\text{-}EPG) \geq 3$*
(ii) *$H(B_2\text{-}EPG) \geq 4$*
(iii) *$H(B_3\text{-}EPG) \geq 8$*
(iv) *$H(B_k\text{-}EPG)$ is unbounded for $k \geq 4$.*

Proof. For each value of k, we exhibit a B_k-family of edge paths whose Helly number is the corresponding stated value. We refer to the pair of coordinates of grid points, to describe the paths.

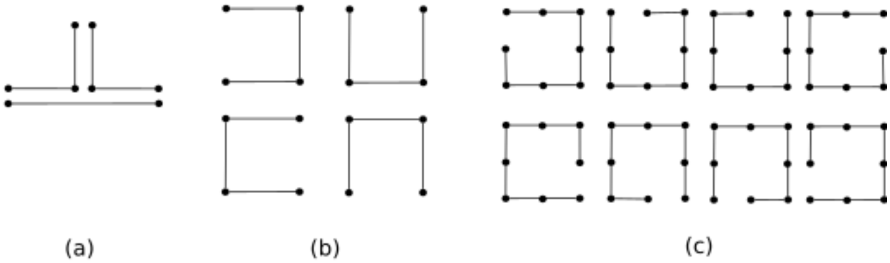

Figure 1. Minimal non-Helly sub-families for the B_1, B_2 and B_3 -families.

For $k = 1$, let $\mathcal{F}$ be a family of three 1-bend paths that pairwise intersect but which have no common edge, as depicted in Figure 1(a). Then $\mathcal{F}$ is a 2-intersecting B_1-EPG family of three paths, having an empty core. Furthermore, removing any of the paths from $\mathcal{F}$ makes its core become non-empty. Therefore $\mathcal{F}$ is a minimal non-2-Helly family and $H(B_1\text{-EPG}) \geq 3$.

Let S be the 4-cycle formed by the four edge segments, with bends at the grid points $(0,0),(0,2),(2,2),(2,0)$, respectively. For $k = 2$, consider $\mathcal{F}$ to be the family of four 2-bend paths formed when we remove exactly one of the two-edge segments that form the 4-cycle, as depicted in Figure 1(b). It follows that $\mathcal{F}$ is 3-intersecting and there is no common edge to all four paths. Hence $H(B_2$-EPG$)\geq 4$.

For $k = 3$, consider again the same cycle S as above. Note that S contains 8 grid edges. Let $\mathcal{F}$ consist of the 8 paths P_i, $1 \leq i \leq 8$, obtained by removing from S, exactly one of these distinct 8 edges, as depicted in Figure 1(c). Consequently, $\mathcal{F}$ is 7-intersecting, but $core(\mathcal{F}) = \emptyset$. Therefore, $H(B_3$-EPG$)\geq 8$.

Finally, for $k = 4$, let $\mathcal{F}$ be the family of n paths P_i, described as follows:

- P_1 is formed by the segments connecting:
  $(0,0),(0,1),(1,1),(1,0),(n,0)$;

- for $2 \leq i \leq n-1$, P_i contains the segments connecting:
  $(0,0),(0,i-1),(i-1,1),(i,1),(i,0),(n,0)$;

- P_n is formed by the segments connecting:
  $(0,0),(n-1,0),(n-1,1),(n-1,0)$.

Observe that $\mathcal{F}$ is $(n-1)$-intersecting, while $core(\mathcal{F}) = \emptyset$ (see Figure 2). Therefore $H(B_4$-EPG$)$ is unbounded. Clearly the same holds for $k > 4$. □

Next, we consider upper bounds for the Helly number B_k-EPG graphs.

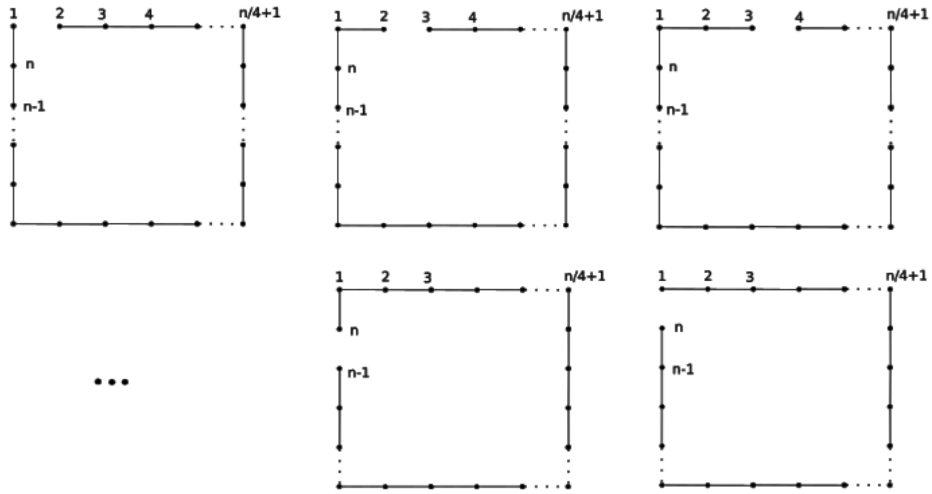

Figure 2. B_4 has an unlimited Helly number.

3.2. Upper Bounds

In order to obtain tight upper bounds for the Helly number, in terms of the number of bends, we introduce below more notation and lemmas.

Say that a set of edges of a grid is *co-linear* if all edges of the set belong to the same line of the grid, horizontal or vertical. The set of edges is called *parallel* if all its edges lie on parallel lines of the grid, but no two of them are co-linear.

Lemma 6. *Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of paths on a grid containing three co-linear non-representative edges. Then $\mathcal{F}$ must contain paths with at least four bends.*

Proof. Let u_i be the middle one of the three co-linear non-representative edges. It corresponds to the path P_i of $\mathcal{F}$, not containing u_i. Then P_i must go through the other two non-representative edges, but it cannot include the middle edge. Therefore path P_i must leave the common line of the grid, containing those three representatives edges, and return to that same line, thus requiring at least four bends. □

Lemma 7. *Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of paths on a grid, containing three parallel edges, and having Helly number $H(\mathcal{F}) \geq 4$. Then $\mathcal{F}$ must contain paths with at least four bends.*

Proof. Since $H(\mathcal{F}) \geq 4$ and $\mathcal{F}$ is a minimal $(h-1)$-family, it follows that $\mathcal{F}$ must contain at least four paths, P_1, P_2, P_3, P_4. Without loss of generality, let

u_1, u_2, u_3 be non-representative edges which are parallel and correspond to the paths P_1, P_2 and P_3 respectively. Then P_4 must go through all the three parallel non-representative edges u_1, u_2, u_3, thus requiring at least four bends. □

Lemma 8. *Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of paths on a grid with Helly number $H(\mathcal{F}) \geq 4$. If $\mathcal{F}$ contains three non-representative edges that lie on a common B_1-subpath P_i, then $\mathcal{F}$ must have some path with at least three bends.*

Proof. Since $\mathcal{F}$ is a minimal $(h-1)$-family having Helly number ≥ 4, it contains at least four paths. Without loss of generality, let u_1, u_2, u_3 be the three non-representative edges contained in P_4 and such that u_2 lies between u_1 and u_3 in P_4. Then path P_2 must contain u_1 and u_3, but avoid u_2, thus requiring at least three bends. □

The following are tight upper bounds for the Helly numbers of B_k-EPG paths, for $k = 1, 2, 3$.

Claim 9. $H(B_1\text{-}EPG) \leq 3$.

Proof. Assume by contradiction that the Helly number of B_1-EPG paths is $h > 3$. In this case, consider a minimal non-$(h-1)$-Helly family of $\mathcal{F}$ of B_1-EPG paths. Then $\mathcal{F}$ contains at least h paths. Any path $P_1 \in \mathcal{F}$ must contain $h-1$ non-representative edges corresponding to the $h-1$ distinct paths of $\mathcal{F}$ other than P_1. Since $h-1 \geq 3$, P_1 contains at least three distinct non-representative edges $u_2, u_3, u_4 \in P_i$, with u_3 lying between u_2 and u_4 in the path.

If u_2, u_3 and u_4 are co-linear then by Lemma 6 $P_3 \in \mathcal{F}$ must contain at least four bends. Otherwise, the edges must lie on P_1 which has a single bend. Thus, it follows from Lemma 8 that P_3 has three bends. In any situation, a contradiction arises, implying that $H(\mathcal{F}) \leq 3$. □

Claim 10. $H(B_2\text{-}EPG) \leq 4$.

Proof. Assume by contradiction that the Helly number of B_2-EPG families of paths is $h > 4$. Consider a minimal non-$(h-1)$-Helly family $\mathcal{F}$ of B_2-EPG paths. The family $\mathcal{F}$ must contain at least $h \geq 5$ distinct paths, each of them corresponding to a distinct non-representative edge. Choose arbitrarily 5 of these non-representative edges.

By Lemmas 6 and 7 any three of these chosen edges can neither be co-linear nor parallel. Therefore, at least one of the five chosen non-representative edges must be in a different direction from the majority of the chosen edges. Call the direction of this edge vertical and the direction of the majority of the chosen edges horizontal. Consider a path P_1 from the family $\mathcal{F}$ that goes through this vertical edge. The path P_1 contains at least four of the chosen non-representative edges, at least one of which is vertical. Since P_1 has at most two bends, then it must have at most three segments. Since we have three segments and four non-representative edges which P_1 must contain, by the pigeon hole principle, one of these segments must have two non-representative edges. If this pair of edges are in a horizontal segment of P_1, then such pair of edges, along with the vertical edge are in two consecutive path segments, forming a B_1-subpath in $\mathcal{F}$. Then Lemma 8 implies that some path of $\mathcal{F}$ must have at least three bends. Otherwise, the two edges are vertical. But the others must be horizontal, and again we have at least three edges in a pair of consecutive segments forming a subpath in $\mathcal{F}$ having one bend. Again, Lemma 8 implies that some path has at least three bends. □

Claim 11. $H(B_3\text{-}EPG) \leq 8$.

Proof. Assume by contradiction that the Helly number of B_3-EPG paths is $h > 8$. In this case, consider a minimal non-$(h-1)$-Helly family $\mathcal{F}$ of B_3-EPG paths. Then $\mathcal{F}$ contains at least h distinct non-representative edges, corresponding to h distinct paths. By Lemma 7, since we can have at most three bends in any path, then these h non-representative edges must lie in at most two vertical and two horizontal lines of the grid. Therefore one of these four possible lines must contain at least three distinct non-representative edges. By Lemma 6, that would imply the existence of a path with four bends. □

This completes the proof of Theorem 4.

4. HELLY NUMBER OF B_k-VPG GRAPHS

In this section, we determine the Helly number of B_k-VPG graphs. We prove the following results.

Theorem 12. *The Helly numbers for B_k-VPG graphs satisfy:*

1. *$H(B_1\text{-}VPG) = 4$*
2. *$H(B_2\text{-}VPG) = 6$*
3. *$H(B_3\text{-}VPG) = 12$*
4. *$H(B_4\text{-}VPG)$ is unbounded.*

Again, we prove the theorem by showing tight lower and upper bounds.

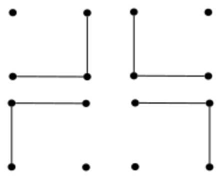

Figure 3. Lower bound for B_1-VPG graphs

4.1. Lower Bounds

We start by describing some sets of paths that achieve our lower bounds. Figure 3 shows a set of 4 B_1-paths of a graph G, in a 2×2 grid, such that each path covers three vertices of the grid, and avoids exactly one of the vertices.

Figure 4 shows a set of 6 B_2-paths of a graph G, in a 2×3 grid, such that each path covers five vertices of the grid, and avoids exactly one.

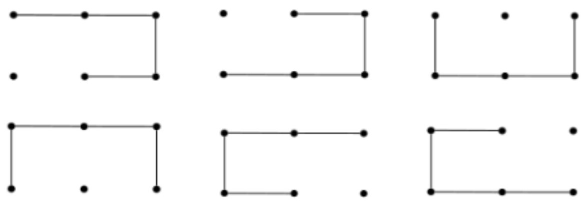

Figure 4. Lower bound for B_2-VPG graphs

Figure 5 shows 12 B_3-paths of a graph G, in a grid, of perimeter 12, such that each path covers 11 vertices of the grid, avoiding one of them.

Figure 6 shows a set of n B_4-paths of a n-vertex graph G, in a grid having perimeter n, such that each path covers $n-1$ vertices of G, avoiding one of them.

Applying Theorem 3, we can then conclude that the number of vertices of each of the above-described graphs is lower bound for the corresponding class. Then, we can claim the following bounds.

Claim 13. *The following are lower bounds for the Helly numbers of B_k-VPG graphs.*

1. $H(B_1\text{-}VPG) \geq 4$
2. $H(B_2\text{-}VPG) \geq 6$
3. $H(B_3\text{-}VPG) \geq 12$
4. $H(B_4\text{-}VPG)$ *is unbounded.*

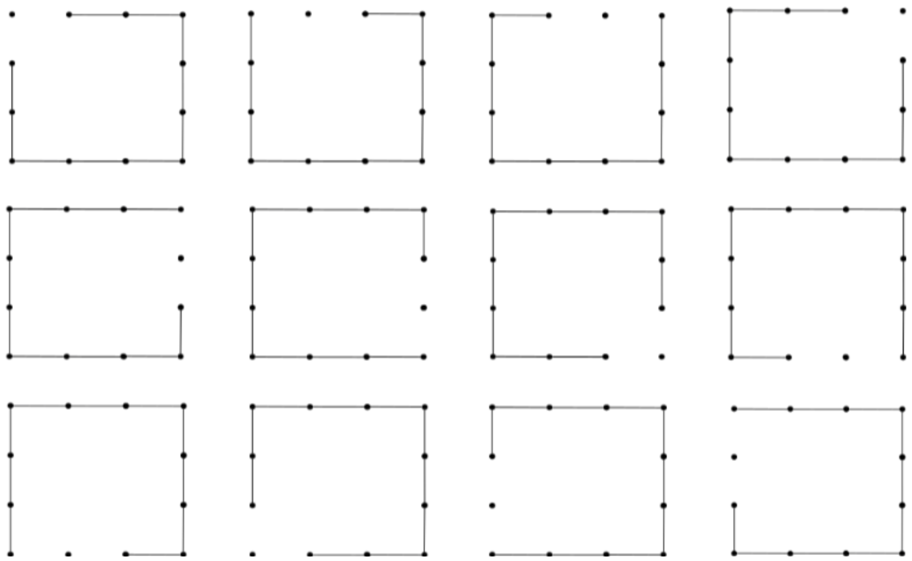

Figure 5. Lower bound for B_3-VPG graphs

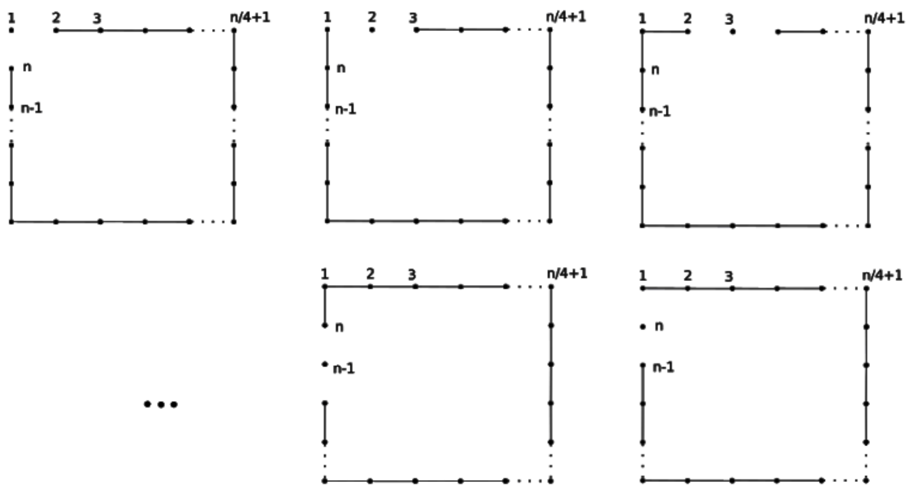

Figure 6. Lower bound for B_4-VPG graphs

4.2. Upper Bounds

Next, we provide upper bounds for the Helly number of B_k-VPG graphs. The following lemmas are employed.

Lemma 14. *Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of paths, for some h, containing $k \in \{3, 4, 5\}$ distinct co-linear non-representative points of the grid.*

Then $\mathcal{F}$ contains a path having at least $k-1$ bends.

Proof. For $k \in \{3,5\}$, the path avoiding the middle point has at least $k-1$ bends, while for $k=4$, the path avoiding one of the middle points also has this same property. □

Lemma 15. *Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of paths, on a grid containing $k < h$ distinct pairwise non-co-linear non-representative points. Then $\mathcal{F}$ must contain a path with at least $k-1$ bends.*

Proof. Since $k < h$, $\mathcal{F}$ must contain a path that visits all such k pairwise non-co-linear points. Such a path requires at least one bend, between two consecutive non-co-linear points. Therefore $\mathcal{F}$ contains a path with at least $k-1$ bends. □

We also employ some additional concepts and notation, below described.

Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of B_{k-1}-paths on a grid Q. By Theorem 3, we can choose h paths $P_i \in \mathcal{F}$, each of them associated to a distinct non-representative grid point p_i, such that P_i avoids p_i, but contains all the other $h-1$ distinct non-representative points $p_j \in P_J$, for each $j \neq i$. Denote by P_N, $|P_N| = h$, the subset of grid points of Q, restricted to the chosen set of distinct non-representative points p_i. By Lemmas 14 and 15, the grid points of P_N are contained in at most k columns (lines), and each column (line) contains at most k points of P_N. Consequently, the cardinalities of the points of P_N, contained in the columns (lines) of Q, form a partition of the integer h, into at most k parts, such that each part is at most k. Call such a partition as a *feasible partition of h, relative to P_N*. Therefore, each non-representative point $p_i \in P_N$ contributes with one unit to some part of the partition, which is then referred to, as the part of the partition *corresponding* to p_i.

The following lemma describes sufficient conditions for an integer h to be an upper bound for the Helly number.

Lemma 16. *Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of B_{k-1}-paths on a grid Q, and P_N the set of non-representative points of Q. Let k, h be integers, $1 \leq k \leq 3$ and $k < h$. The following conditions imply $H(B_k\text{-}VPG) \leq h$*

(i) there is no feasible partition of $h+1$, relative to P_N, or

(ii) for any possible feasible partition, and for any arrangement of the grid points of P_N in Q, there is some non-representative point $p_i \in P_N$, such that no path exists in Q, having at most k bends, containing all points of P_N, except p_i.

Proof: The proof of (i) follows from Lemmas 14 and 15, while the proof of (ii) is a consequence of Theorem 3. □

The following are upper bounds for the Helly number of B_k-VPG graphs, for each k, $1 \leq k \leq 3$, obtained by applying Lemma 16.

Claim 17. *$H(B_1$-VPG$) \leq 4$.*

Proof. There is no partition of the integer 5, into two parts, in which each part is at most 2. Consequently, the result follows from Lemma 16 (i). □

Claim 18. *$H(B_2$-VPG$) \leq 6$.*

Proof. Assume the contrary. Then $H(B_2\text{-VPG}) \geq 7$, let $\mathcal{F}$ be a minimal non-6-Helly family of B_2-paths, and P_N be the set of non-representative points of $\mathcal{F}$ in Q. There are two possible partitions of the integer 7, in three parts, each of them of size at most 3, namely $(3, 3, 1)$ and $(3, 2, 2)$. In any of these cases, it is always possible to choose some point $p_i \in P_N$, belonging to a part of the partition of size 3, such that a path in $\mathcal{F}$ which avoids p_i and covers the other six non-representative points, must contain at least three bends. Then by Lemma 16, indeed $H(B_2\text{-VPG}) \leq 6$. □

Claim 19. *$H(B_3$-VPG$) \leq 12$.*

Proof. Assume the contrary, $H(B_3\text{-VPG}) \geq 12$. Let $\mathcal{F}$ be a minimal non-12-Helly family of B_3-paths, and P_N be the set of non-representative points of $\mathcal{F}$ in Q. There are three possible partitions of the integer 13, into four parts, each of them of size at most 4, namely $(4, 4, 4, 1)$, $(4, 4, 3, 2)$ and $(4, 3, 3, 3)$. In this case, choose $p_i \in P_N$ to be a non-representative point, corresponding to a part of size 4 of the partition. The path of $\mathcal{F}$, which avoids p_i, must cover the other 12 non-representative points. These points are located in 4 distinct columns, of cardinalities 4,4,3,1, 4,3,3,2, or 3,3,3,3, considering the three possible partitions, respectively. Such a path must contain at least four bends, a contradiction. Then by Lemma 16, $H(B_3\text{-VPG}) \leq 12$. □

From the lower and upper bounds described in the previous subsections, we obtain the results for the Helly numbers of B_k-VPG graphs, completing the proof of Theorem 12.

5. STRONG HELLY NUMBER

In this section, we first consider determining the strong Helly number of B_k-EPG graphs.

We start by describing a theorem similar to Theorem 3.

Theorem 20. *Let $\mathcal{C}$ be a hereditary class of families $\mathcal{F}$ of subsets of the universal set U, whose strong Helly number $sH(\mathcal{C})$ equals h. Then there exists a family $\mathcal{F}' \in \mathcal{C}$ with exactly h subsets satisfying the following condition:*

For each subset $P_i \in \mathcal{F}'$, there is exactly one distinct element $u_i \in U$, such that

$$u_i \notin P_i,$$

but u_i is contained in all subsets

$$P_j \in \mathcal{F}' \setminus P_i.$$

Proof: The strong Helly number of $\mathcal{C}$ is h and not $h-1$, so that there must exist some family $\mathcal{F} \in \mathcal{C}$ whose strong Helly number is exactly h, i.e., $\mathcal{F}$ contains h subsets P_i whose intersection equals core$(\mathcal{F}')$ but is such that no $h-1$ of its subsets have the same intersection. In particular, let $\mathcal{F}'$ be the family containing exactly the h subsets P_i described above. Such a family must exist, since $\mathcal{C}$ is hereditary. Then each P_i does not contain at least one element u_i in the intersection of the remaining $h-1$ subsets P_j, $j \neq i$, since the intersection of these $h-1$ subsets must not be equal to the core$(\mathcal{F}')$. $\square$

Again, if we consider the family $\mathcal{F}'$ described in the theorem above it is simple to conclude that the removal of any subset from $\mathcal{F}'$ turns it $(h-1)$-strong Helly. Then call $\mathcal{F}'$ a *minimal* non-$(h-1)$-strong Helly family. Moreover, the element $u_i \notin P_i$, contained in all subsets $P_j \in \mathcal{F}' \setminus P_i$, except P_i, is the h *non-representative* of P_i.

As before, we employ the above minimal families of subsets, applied to paths in a grid.

We prove that the strong Helly number of B_k-EPG graphs coincide with the Helly number, for each corresponding value of k. Similarly, for B_k-VPG graphs. For $k = 0$, it is simple to show that if a set of intervals $\mathcal{I}$ in a line pairwise intersect, then there exist two intervals of $\mathcal{I}$, whose intersection equals the intersection of all intervals of $\mathcal{I}$. Consequently, the k-strong Helly number of B_0-EPG graphs equals 2. Similarly, for B_0-VPG graphs. Recall that the strong Helly number is at least equal to the Helly number of a family so that the lower bounds presented in Claim 5 also hold for the strong Helly number. The proofs for the strong Helly numbers for $k \geq 1$ are similar to those described in Section 3.

6. Concluding Remarks

We have determined the Helly number and strong Helly number of B_k-EPG graphs and B_k-VPG graphs, for $k \geq 0$.

Table 1 summarizes the results obtained.

We leave two questions to be investigated concerning the presented results.

k	B_k-EPG	B_k-VPG
0	2	2
1	3	4
2	4	6
3	8	12
≥ 4	unbounded	unbounded

Table 1. Helly and Strong Helly Numbers for B_k-EPG and B_k-VPG Graphs

1. Given a *specific* EPG or VPG graph, the question is to formulate an algorithm to determine its Helly and strong Helly numbers. See [5], for instance, for such algorithms, applied to general graphs.
2. The values of the Helly and strong Helly numbers, which were determined in this paper, coincided in all cases. Clearly, in general, this is not the case. We leave as an open question, to find the conditions for such equality to occur.

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