The Complexity of B_1-EPG-Helly Graph Recognition

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Abstract. Golumbic, Lipshteyn and Stern defined in 2009 the class of EPG graphs, an intersection graph class based on edge intersection of paths on a grid. An EPG graph G is a graph that admits a representation where its vertices correspond to paths in a grid Q, such that two vertices of G are adjacent if and only if their corresponding paths in Q have a common edge. If the paths in the representation have at most k changes of direction (bends), we say that this is a B_k-EPG representation. A collection C of sets satisfies the Helly property when every sub-collection of C that is pairwise intersecting has at least a common element. In this paper we show that the problem of recognizing B_k-EPG graphs $G = (V, E)$ whose edge-intersections of paths in a grid satisfy the Helly property, so-called B_k-EPG-Helly graphs, is in $\mathcal{NP}$, for every k bounded by a polynomial function of $|V(G)|$. In addition, we show that recognizing B_1-EPG-Helly graphs is NP-complete, and it remains NP-complete even when restricted to 2-apex and 3-degenerate graphs.

Keywords: Edge-intersection of paths on a grid · Helly property · Intersection graphs · NP-completeness · Single bend paths.

1 Introduction

An EPG graph G is a graph that admits a representation in which its vertices are represented by paths of a grid Q, such that two vertices of G are adjacent if and only if the corresponding paths have at least one common edge.

The study of EPG graphs has motivation related to the problem of VLSI design that combines the notion of edge intersection graphs of paths in a tree with a VLSI grid layout model [10]. The number of bends in an integrated circuit may increase the layout area, and consequently increase the cost of chip

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manufacturing. This is one of the main applications that instigate research on the EPG representations of some graph families when there are constraints on the number of bends in the paths used in the representation. Other applications and details on circuit layout problems can be found in [3, 13].

A graph is a B_k-EPG graph if it admits a representation in which each path has at most k bends. The *bend number* of a class of graphs is the smallest k for which all graphs in the class have a B_k-EPG representation. Interval graphs have bend number 0 [10], trees have bend number 1 [10] and outerplanar graphs have bend number 2 [12]. The bend number for the class of planar graphs is still open, but it is either 3 or 4 [12].

The class of EPG graphs has been studied in several papers, such as [1,2,7,10, 11, 15], among others. The investigations frequently approach characterizations with respect to the number of bends of the graph representation. Regarding the complexity of recognition of B_k-EPG graphs, only the complexity of recognizing three of these sub-classes of EPG graphs have been determined: B_0-EPG graphs can be recognized in polynomial time, since it corresponds to the class of interval graphs, see [5]. In contrast, recognizing B_1-EPG and B_2-EPG graphs are NP-complete problems, see [11, 15], and recognizing B_1-EPG graphs remains NP-complete even for L-shaped paths in grid, see [6].

This paper studies graphs that have an EPG-Helly representation. We prove that the problem of recognizing B_1-EPG-Helly graphs is NP-complete. The Helly property related to EPG representations of graphs has been studied in [9] and [10]. In particular, they have determined the strong Helly number of B_1-EPG graphs.

Next, we describe some terminology and notation used.

The term *grid* is used to denote the Euclidean space of integer orthogonal coordinates. Each pair of integer *coordinates* corresponds to a point or vertex of the grid. The term *edge of the grid*, will be used to denote a pair of vertices that are at distance one in the grid. Two edges e_1 and e_2 are *consecutive edges* when they share exactly one point of the grid. A *path in the grid* is any finite sequence of consecutive edges $e_1 = (v_1, v_2), e_2 = (v_2, v_3), \ldots, e_i = (v_i, v_{i+1}), \ldots, e_m = (v_m, v_{m+1})$, where $v_i \neq v_j$ for $i \neq j$. The first and last edges of a path are called *extremity edges*.

The *direction of an edge* is vertical when the first coordinates of its vertices are equal, and is horizontal when the second coordinates are equal. A *bend* in a path is a pair of consecutive edges e_1, e_2 of that path, such that the directions of e_1 and e_2 are different. When two edges e_1 and e_2 form a bend, they are called *bend edges*. A *segment* is a set of consecutive edges with no bends. Two paths are said to be *edge-intersecting*, or simply *intersecting*, if they share at least one edge. Throughout the paper any time we say that two paths intersect, we mean they edge-intersect.

EPG graphs are the class of intersection graphs of paths in a grid [10]. This class, consists of the class of graphs in which its vertices are represented by paths of a grid Q, such that two vertices in G are adjacent if and only if their corresponding paths intersect. If every path in a representation can be represented with at most k bends, we say that this graph G has a *B_k-EPG* representation. When $k = 1$ we say that this is a *single bend* representation.

A family of sets is *pairwise intersecting* if any two sets in the family intersect. A collection of non-empty sets C satisfies the Helly property when every pairwise intersecting sub-collection S of C has at least one element that is in every subset of S.

The Helly property can be applied to the B_k-EPG representation problem, where each path is considered to be a set of edges. A graph G has a B_k-EPG-Helly representation if there is a B_k-EPG representation of G where each path has at most k bends and this representation satisfies the Helly property.

It is easy to construct EPG representations to verify the following lemmas.

Lemma 1. *[10] Every graph is an EPG graph.*

Lemma 2. *Every graph is an EPG-Helly graph.*

Corollary 1. *Every graph G containing μ maximal cliques admits a $B_{2\mu-1}$-EPG-Helly representation.*

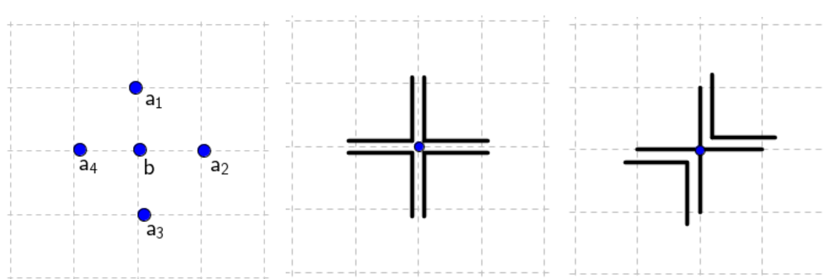
Most of the proofs of the paper are detailed in the Appendix.

2 Basic EPG representations

In this section we examine the B_1-EPG representations of a few graphs that we employ in our constructions. First, we consider EPG representations of C_4's.

Definition 1. *Let Q be a grid and let (a_1, b), (a_2, b), (a_3, b), (a_4, b) be a 4-star as depicted in Figure 1(a). Let $\mathcal{P} = \{P_1, \ldots, P_4\}$ be a collection of paths each containing exactly two edges of the 4-star:*

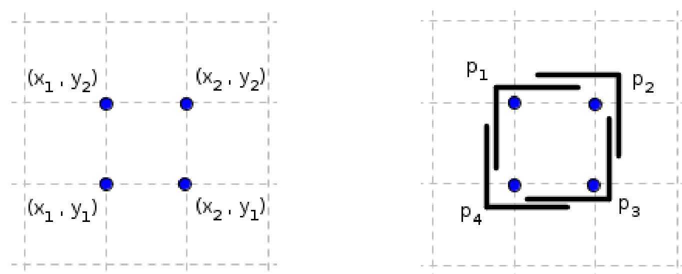
- *A* true pie *is a representation where each P_i of $\mathcal{P}$ forms a bend in b.*
- *A* false pie *is a representation where two of the paths P_i do not contain bends, while the remaining two do not share an edge.*



(a) 4-star in grid (b) True pie (c) False pie

Fig. 1. B_1-EPG representation of the induced cycle of size 4 as pies with emphasis in center b

Definition 2. *Consider a rectangle of any size with 4 corners at vertices* (x_1, y_1); (x_2, y_1); (x_2, y_2); (x_1, y_2)*, positioned as in Figure 2. A* frame *is a representation containing 4 paths* $\mathcal{P} = \{P_1, \ldots, P_4\}$*, each having a bend in a different corner of a rectangle, and such that the sub-paths* $P_1 \cap P_2, P_2 \cap P_3, P_3 \cap P_4, P_4 \cap P_1$*, share at least one edge. While the sub-paths* $P_2 \cap P_4$ *and* $P_1 \cap P_3$ *do not share edges.*



(a) Points of the coordinates of bends of a frame (c) Paths of a frame

Fig. 2. B_1-EPG representation of the induced cycle of size 4 as frame

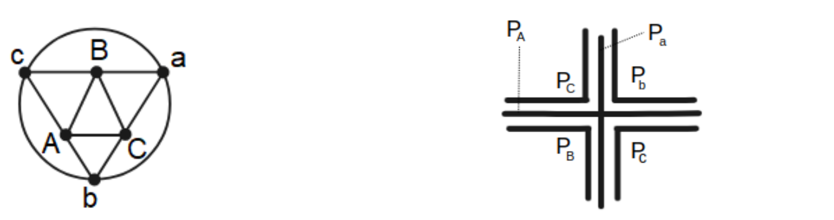
Lemma 3. *[10] Every* C_4 *that is an induced subgraph of a graph* G *corresponds, in any representation, to a true pie, a false pie, or a frame.*

Definition 3. *A* B_k*-EPG representation is* minimal *when its set of edges does not properly contain another* B_k*-EPG representation.*

The *octahedral* graph is the graph containing 6 vertices and 12 edges, depicted in Figure 3(a). Next, we consider representations of the octahedral.

Next lemma follows directly from the discussion presented in [11].

Lemma 4. *The octahedral graph* O_3 *has a unique minimal* B_1*-EPG representation.*



(a) The octahedral O_3 graph (b) B_1-EPG representation of the graph O_3

Fig. 3. The octahedral O_3 graph and its B_1-EPG representation

By Lemma 4, O_3 has a unique minimal B_1-EPG representation, up to isomorphisms, as depicted in Figure 3(b). The paths P_a, P_b and P_c do not satisfy the Helly property. Therefore $O_3 \notin B_1$-EPG-Helly.

3 Membership in $\mathcal{NP}$

B_k-EPG-Helly recognition problem can be formally described as follows:

B_k-EPG-Helly Recognition	
Input: A graph G, and an integer k.	
Goal:	Determine if there is a set of k-bend paths $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ in a grid Q such that: • $i, j \in V(G)$ are adjacent if only if P_i, P_j share an edge in Q; • $\mathcal{P}$ satisfies the Helly property.

In this section, we show that B_k-EPG-Helly Recognition, where k is bounded by a polynomial function of $|V(G)|$, belongs to $\mathcal{NP}$.

A (positive) certificate for the B_k-EPG-Helly recognition consists of a grid Q, a set $\mathcal{P}$ of k-bend paths of Q, which have a one-to-one correspondence with the vertex set $V(G)$ of G, such that, for each pair of distinct paths $P_i, P_j \in \mathcal{P}, P_i \cap P_j \neq \emptyset$ if and only if the corresponding vertices are adjacent in G. Furthermore, $\mathcal{P}$ satisfies the Helly property.

The following are key concepts that make it easier to control the size of an EPG representation. A relevant edge of a path in a B_k-EPG representation is one which is either an extremity edge or a bend edge of the path. Therefore each path with at most k bends can have up to $2(k+1)$ relevant edges, and any B_k-EPG representation contains at most $2|\mathcal{P}|(k+1)$ distinct relevant edges.

To show that there is a non-deterministic polynomial-time algorithm for B_k-EPG-Helly recognition, it is enough to consider as certificate a B_k-EPG representation R containing a collection $\mathcal{P}$ of paths, $|\mathcal{P}| = |V(G)|$, such that each path $P_i \in P$ is given by its set of relevant edges along with the relevant edges of each path that intersects P_i. The relevant edges for each path are given in the order that they appear in the path, so as to make checking that the edges correspond to a unique path with at most k bends straightforward. This representation is also handy for checking that the paths form an intersection model for G.

In order to verify in polynomial time that the input is in fact is a positive certificate for the problem, we have to assert the following:

(i) The sequence of relevant edges of a path $P_i \in \mathcal{P}$ determines P_i in polynomial time;
(ii) Two paths $P_i, P_j \in \mathcal{P}$ intersect if and only if they intersect in some relevant edge;

(iii) The set $\mathcal{P}$ of relevant edges satisfies the Helly property.

The following lemma states that condition (i) holds.

Lemma 5. *Each path P_i can be determined uniquely in polynomial time by the sequence of its relevant edges.*

Next we assert property (ii).

Lemma 6. *Let $\mathcal{P}$ be the paths in a B_k-EPG representation of G, and let $P_1, P_2 \in \mathcal{P}$. Then P_1, P_2 are intersecting paths if and only if their intersection contains at least one relevant edge.*

The two previous lemmas let us check that a certificate is an actual B_k-EPG representation of a given graph G. The next lemma says we can also verify in polynomial time that the representation encoded in the certificate is a Helly representation. Fortunately we do not need to check every subset of intersecting paths of the representation to make sure they have a common intersection.

Lemma 7. *Let $\mathcal{P}$ be a collection of paths encoded as a sequence of relevant edges that constitute a B_k-EPG representation of a graph G. We can verify in polynomial time if $\mathcal{P}$ has the Helly property.*

Proof. Let T be the set of relevant edges of $\mathcal{P}$. Consider each triple T_i of edges of T . Let P_i be the set of paths of $\mathcal{P}$ containing at least two of the edges in the triple T_i. By Gilmore's Theorem [4], $\mathcal{P}$ has the Helly property if an only if the subset of paths P_i corresponding to each triple T_i has a non-empty intersection. By Lemma 6, it suffices to examine the intersections on relevant edges. Therefore a polynomial algorithm for checking if $\mathcal{P}$ has the Helly property could examine each of the subsets P_i, and for each relevant edge e of a path in P_i, to compute the number of paths in P_i that contain e. $\mathcal{P}$ has the Helly property if and only if for every P_i there exists some relevant edge that is present in all paths in P_i, yielding a non-empty intersection. □

Corollary 2. *Let $\mathcal{P}'$ be a set a pairwise intersecting paths in a B_k-EPG-Helly representation of a graph G. Then the intersection of all paths of $\mathcal{P}'$ contains at least one relevant edge.*

Note that the property described in Corollary 2 is due to Gilmore's Theorem [4], and it applies only to representations that satisfy Helly's property.

Lemma 8. *Let G be a B_k-EPG-Helly graph. Then G admits a B_k-EPG-Helly representation on a grid of size at most $4n(k+1) \times 4n(k+1)$.*

Proof. Let R be a B_k-EPG representation of a graph G on a grid Q with the smallest possible size. Let $\mathcal{P}$ be the set of paths of R. Note that $|\mathcal{P}| = n$. A counting argument shows that there are at most $2|\mathcal{P}|(k+1)$ relevant edges in R. If Q has a pair of consecutive columns c_i, c_{i+1} neither of which contains relevant edges of R, and such that there is no relevant edge crossing from c_i to c_{i+1}, then we can contract each edge crossing from c_i to c_{i+1} into single vertices so as to obtain a new B_k-EPG representation of G on a smaller grid, which is a contradiction. An analogous argument can be applied to pairs of consecutive rows of the grid. Therefore the grid Q is such that each pair of consecutive columns and of consecutive rows of Q has at least one relevant edge of R or contains a relevant edge crossing it. Since Q is the smallest possible grid for representing G then the first row and the first column of Q must both contain at least one point belonging to some relevant edge of R. Thus, if G is B_k-EPG then it admits a B_k-EPG representation on a grid of size at most $4|\mathcal{P}|(k+1) \times 4|\mathcal{P}|(k+1)$. In addition, by Corollary 2, it holds that the contraction operation previously described preserves the Helly property, if any. Hence, letting R be a B_k-EPG-Helly representation of a graph G on a grid Q with the smallest possible size it holds that Q has size at most $4|\mathcal{P}|(k+1) \times 4|\mathcal{P}|(k+1)$. □

Theorem 1. B_k-EPG-HELLY RECOGNITION *is in $\mathcal{NP}$, whenever k is bounded by a polynomial function of $|V(G)|$.*

Proof. By Lemma 8 and the fact that k is bounded by a polynomial function of $|V(G)|$, it follows that the collection $\mathcal{P}$ can be encoded through its relevant edges with $n^{\mathcal{O}(1)}$ bits.

Finally, by Lemmata 5, 6 and 7, it follows that one can verify in polynomial-time on the size of G whether $\mathcal{P}$ is a family of paths encoded as a sequence of relevant edges that constitute a B_k-EPG-Helly representation of a graph G. □

4 NP-hardness

Now we will prove that B_1-EPG-Helly graph recognition is NP-complete. For this proof we follow the roadmap set out in the prior hardness proof of [11]. We set up a reduction from POSITIVE (1 IN 3)-3SAT defined as follows:

POSITIVE (1 IN 3)-3SAT	
Input:	A set X of positive variables; a collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses on X such that for each $C_i \in \mathcal{C}$, $|C_i| = 3$.
Goal:	Determine if there is an assignment of values to the variables in X so that every clause in $\mathcal{C}$ has exactly one true literal.

POSITIVE (1 IN 3)-3SAT is a well known NP-complete problem (see [8], problem [L04], page 259). POSITIVE (1 IN 3)-3SAT remains NP-complete when the incidence graph of the input CNF (Conjunctive Normal Form) formula is a planar graph [14].

Given a formula F that is an instance of POSITIVE (1 IN 3)-3SAT we will present a polynomial-time construction of a graph G_F such that G_F is B_1-EPG-Helly if and only if F is satisfiable. This graph will contain an induced subgraph

G_{C_i} with 12 vertices (called *clause gadget*) for every clause $C_i \in \mathcal{C}$, and an induced subgraph (*variable gadget*) for each variable x_j, containing a special vertex v_j, plus a *base gadget* with 55 additional vertices.

We will use a graph H isomorphic to the graph presented in Figure 4, as a gadget to perform the proof. For each clause C_i of F of the target problem, we will have a *clause gadget* isomorphic to H, denoted by G_{c_i}.

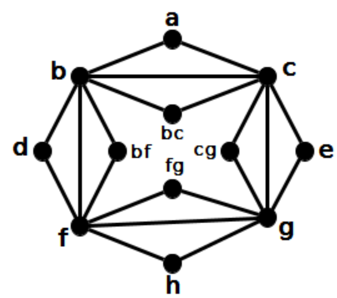


Fig. 4. The partial gadget graph H

The reduction of a formula F from POSITIVE (1 IN 3)-3SAT to a particular graph G_F such that G_F has a B_1-EPG-Helly representation if only if F is satisfiable, is given below.

Definition 4. *Let F be a CNF-formula with no negative literals, in which every clause has exactly three literals. The graph G_F is constructed as follows:*

1. *For each clause $C_i \in \mathcal{C}$ create a clause gadget G_{C_i}, isomorphic to graph H;*
2. *For each variable x_j create a* variable vertex *v_j that is adjacent to the vertex a, e, or h of G_{C_i}, when x_j is the first, second or third variable in C_i, respectively;*
3. *For each variable vertex v_j, construct a* variable gadget *formed by adding two copies of H, H_1 and H_2, and making v_j adjacent to the vertices of the 2 triangles (a, b, c) in H_1 and H_2.*
4. *Create a vertex V, that will be used as vertical reference of the construction, and add an edge from V to each vertex d of a clause gadget;*
5. *Create a bipartite graph $K_{2,4}$ with a particular vertex T that is in the largest stable set. This vertex is nominated* true vertex. *T is adjacent to all v_j and also to V;*
6. *Create two graphs isomorphic to H, G_{B1} and G_{B2}. The vertex T is connected to each vertex of the triangle (a,b,c) in G_{B1} and G_{B2};*
7. *Create two graphs isomorphic to H, G_{B3} and G_{B4}. The vertex V is connected to each vertex of the triangle (a,b,c) in G_{B3} and G_{B4};*
8. *The subgraph induced by the set of vertices $\{V(K_{2,4}) \cup \{T, V\} \cup V(G_{B1}) \cup V(G_{B2}) \cup V(G_{B3}) \cup V(G_{B4})\}$ will be referred to as the* base gadget.

Figure 5 illustrates how this construction works on a small formula.

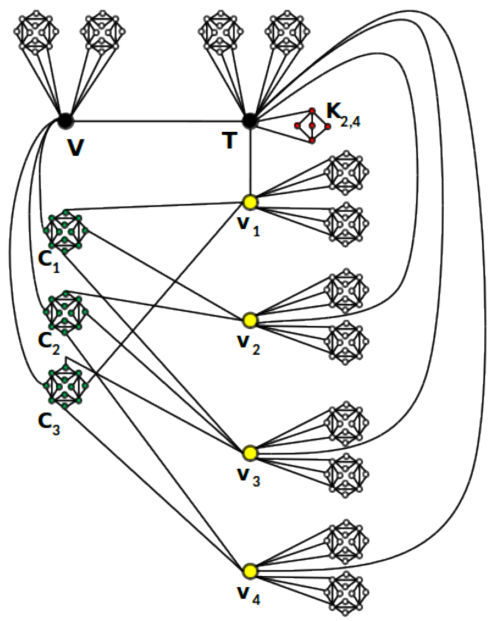


Fig. 5. The G_F graph corresponding to formula $F = (x_1 + x_2 + x_3) \wedge (x_2 + x_3 + x_4) \wedge (x_3 + x_1 + x_4)$

Lemma 9. *Given a satisfiable instance F of* Positive (1 in 3)-3SAT*, the graph G_F constructed from F according to Definition 4 admits a B_1-EPG-Helly representation.*

Next, we consider the converse. Let a B_1-EPG-Helly representation R of G_F.

Definition 5. *Let H be the graph shown in Figure 4, such that the 4-cycle $H[\{b, c, f, g\}]$ corresponds in R to a false pie or true pie, then:*

- *the* center *is the unique grid-point of this representation which is contained in every path representing 4-cycle $\{b, c, f, g\}$;*
- *a* central ray *is an edge-intersection between two of the paths corresponding to vertices b, c, f, g, respectively.*

Note that every B_1-EPG representation of a C_4 satisfies the Helly property, see Lemma 3, and triangles have B_1-EPG representations that satisfy the Helly property, e.g. the one shown in Figure 8(b). The graph H is composed by a 4-cycle $C_4^H = H[b, c, f, g]$ and eight cycles of size 3.

As C_4^H has well known representations (see in Lemma 3), then we can start drawing the B_1-EPG-Helly representation of H from these structures. Figure 6 shows possible representations for H.

If C_4^H is represented by pie, then the paths P_b, P_c, P_f, P_g share a central point of the representation. On the other hand, if C_4^H is represented by a frame then the bends of the 4 paths corresponding to the four distinct corners of a rectangle, i.e. all path representing the vertices of C_4^H have distinct bend points, see [10].

Next we examine the use of the frame structure.

Proposition 1. *In a B_1-EPG representation of a C_4 isomorphic to a frame, every path P_i that represents a vertex of the C_4 intersects exactly two other paths P_{i-1} and P_{i+1} of the frame, so that one of the intersections is horizontal and the other is vertical.*

Proposition 2. *Given a B_1-EPG-Helly representation of a graph G that has an induced C_4 whose representation is isomorphic to a frame. If there is a vertex v of G, outside this C_4, that is adjacent to exactly two consecutive vertices of this C_4, then the path representing v shares at least one common edge-intersection with the paths representing both of these vertices.*

By Proposition 1 and Proposition 2 we can conclude that for every vertex $v_i \in V(H)$ such that $v_i \neq V(C_4^H)$, when we use a frame to represent the C_4^H, P_{v_i} will have at least one common edge-intersection to a pair of paths representing its neighbor vertices of H. Figure 6(c) presents a possible B_1-EPG-Helly representation of H. Note that we can apply rotations and mirroring operations, while maintaining it as a B_1-EPG-Helly representation of H.

Definition 6. *In a single bend representation of a graph C_4 isomorphic to a frame, the paths that represent consecutive vertices in the C_4 are called* consecutive paths *and the segment that corresponds to the intersection between two consecutive paths is called* side intersection.

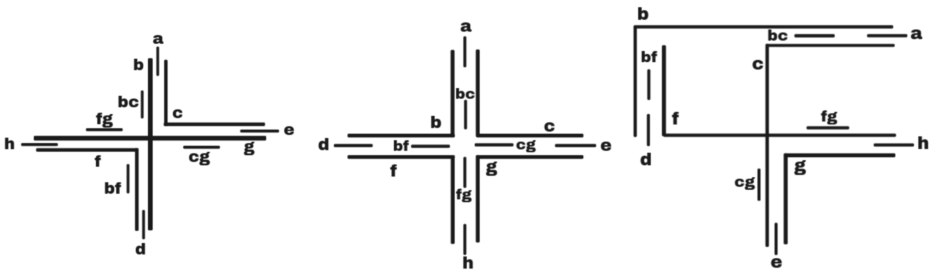
Lemma 10. *In any single bend minimal representation of a graph isomorphic to H, there are two paths in $\{P_a, P_e, P_d, P_h\}$ that have horizontal directions and the other two paths have vertical directions.*

Corollary 3. *In any single bend minimal representation of a graph isomorphic to H, the following paths are on the same central ray or side intersection: P_a and P_{bc}; P_e and P_{cg}; P_h and P_{fg}; P_d and P_{bf}.*

Definition 7. *Consider a graph G and a vertex $v \in V(G)$. If in a B_1-EPG representation of G the two bend edges (or one extremity edge) of the path P_v intersects other paths, then we say that P_v has an* obstructed bend (or extremity). *In addition, given a B_1-EPG representation of G where P_v has an obstructed bend (or extremity), we say that a subset of paths* obstructs *a bend edge (or an extremity edge) of P_v if it intersects such an edge.*

Remark 1. In every single bend representation of a $K_{2,4}$, the path representing each vertex of the larger part has its bend in a false pie (claim in [12] and reasoning in [2]).

Definition 8. *We say that a segment s is* internally contained *in a path P_x if s is contained in P_x, and it does not intersect a relevant edge of P_x.*



(a) Based in false pie (b) Based in true pie (c) Based in frame

Fig. 6. Different single bend representations of the graph H using a false pie (a), a true pie (b) and a frame (c) for representing C_4^H

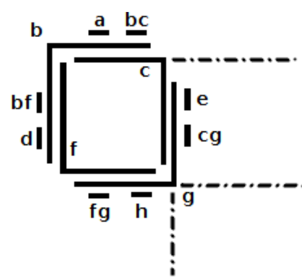


Fig. 7. A frame representation where the bend of dashed paths change directions

Lemma 11. *If a graph G_F, constructed according to Definition 4, admits a B_1-EPG-Helly representation, then the associated CNF-formula F is a yes-instance of* POSITIVE (1 IN 3)-3SAT.

Note that a B_1-EPG representation is Helly if and only if each clique is represented by a edge clique (and not by a claw-clique). More details on edge clique and claw-clique can be found in [10]. Thus, an alternative way to check that a representation is Helly is to note that all cliques are represented as edge clique.

Some of the vertices of G_F have highly constrained B_1-EPG representations. Vertex T has its bend and both extremities obstructed by its neighbors in G_{B1}, G_{B2} and in the $K_{2,4}$ subgraphs. Vertex V and each variable vertex v_i must have one of its segments internally contained in T, and also have its extremities and bend obstructed. Therefore, vertex V and each variable vertex have only one segment each that can be used in an EPG representation to make them adjacent to the clause gadget. The direction of this segment, being either horizontal or vertical, can be used to represent the true or false value for the variable. The clause gadgets, on the other hand, are such that exactly two of its adjacencies to the variable vertices and to V can be realized with a horizontal intersection whereas the other two must be realized with a vertical intersection. If we consider the direction used by V as a truth assignment, we get that exactly one of the variables in each clause will be true in any possible representation of G_F. Conversely, it is fairly straightforward to obtain a B_1-EPG representation for G_F given a truth assignment for the formula F.

Theorem 2. *B_1-EPG-Helly graph recognition is NP-complete.*

Proof. By Theorem 1 and Lemmas 9 and 11. □

We say that a k-apex graph is a graph that can be made planar by the removal of k vertices. In addition, a d-degenerate graph is a graph in which every subgraph has a vertex of degree at most d. Remind that Positive (1 in 3)-3SAT remains NP-complete when the incidence graph of the input formula is planar [14]. Thus, the following corollary holds.

Corollary 4. *B_1-EPG-Helly graph recognition is NP-complete on 2-apex and 3-degenerate graphs.*

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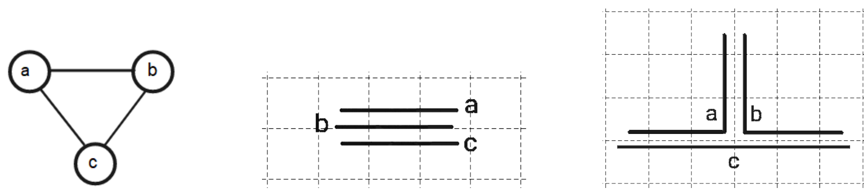
Appendix

The following are some examples to illustrate definitions that were cited in the text.

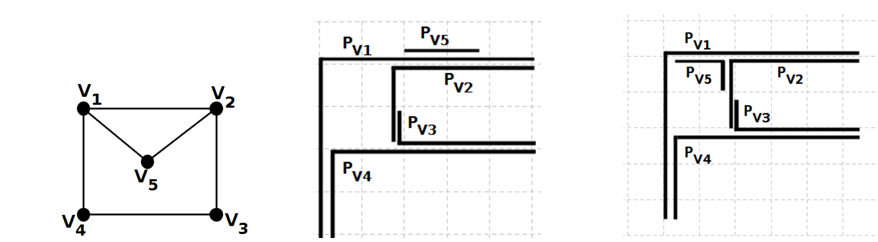
A graph is a B_k-EPG graph if it admits a representation in which each path has at most k bends. As an example, Figure 8(a) shows a C_3, Figure 8(b) shows an EPG representation where the paths have no bends and Figure 8(c) shows a representation with 1 bend. Consequently, C_3 is a B_0-EPG graph. More generally, B_0-EPG graphs coincide with interval graphs.



(a) The graph C_3 (b) B_0-EPG representation of C_3 (c) B_1-EPG representation of C_3

Fig. 8. The graph C_3 and representations without bends and with 1 bend

The Helly property can be applied to the B_k-EPG representation problem, where each path is considered to be a set of edges. A graph G has a B_k-EPG-Helly representation if there is a B_k-EPG representation of G where each path has at most k bends and this representation satisfies the Helly property. Figure 9(a) presents two B_1-EPG representations of a graph with five vertices. Figure 9(b) presents 3 pairwise intersecting paths $(P_{v_1}, P_{v_2}, P_{v_5})$, containing a common edge, so it is a B_1-EPG-Helly representation. In Figure 9(c), although the three paths are pairwise intersecting, there is no common edge in all 3 paths, and therefore they do not satisfy the Helly property.



(a) A graph with 5 vertices (b) B_1-EPG representation that satisfies the Helly property (c) B_1-EPG representation that does not satisfy the Helly property

Fig. 9. A graph with 5 vertices in (a) and some single bend representations: Helly in (b) and not Helly in (c)

Now, we describe the proofs that have been omitted in the text, as well as some additional comments.

Lemma 2: Every graph is an EPG-Helly graph.

Proof. 2: Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$ and μ maximal cliques $C_1, C_2, \ldots, C_\mu$. We construct an EPG-Helly representation of G, using a grid Q of size $(\mu \times 2n)$. The rows correspond to the maximal cliques and are numbered $1, 2, \ldots, \mu$. Each vertex v_j corresponds to the pair of columns $j, j+n$. Each maximal clique C_i is mapped into an edge $(i, n), (i, n+1)$. Each path P_j contains all edges $(i, n), (i, n+1)$ corresponding to the maximal cliques C_i containing v_j.

Moreover, two distinct paths P_j, P_k intersect at the edges corresponding to the maximal cliques containing v_j, v_k.

Let $v_j \in V(G)$, consider the maximal cliques containing v_j in ascending order of their indices. The path P_j representing v_j, start at vertex $(1, j)$ of Q and descends column j until (i, j), where C_i is the first clique containing v_j then P_j bends at vertex (i, j) and proceeds to the right, travessing edge $(i, n), (i, n+1)$ representing C_i. Then P_j proceeds further on row i until reaching vertex $(i, j+n)$, where it bends again, descending column $j+n$ until reaching vertex $(l, j), l > i$, where C_l is the next maximal clique containing v_j where it bends again. It proceeds in row l, travessing edge $(l, n), (l, n+1)$, and so on, until all edges of Q corresponding to the maximal cliques containing v_j have been travessed by P_j.

It follows that two distinct paths, P_j and P_q, intersect exactly at the rows corresponding to the maximal cliques containing both v_j and v_q. □

Figure 10 shows the grid Q and the path P_2 corresponding to the vertex $v_2 \in V(G)$, contained in maximal cliques C_2, C_4 and C_5 of G.

This proof follows directly from the discussion presented in [11].

Lemma 4: The octahedral graph O_3 has a unique minimal B_1-EPG representation.

Proof. 4: The octahedral graph O_3 has in its constitution induced cycles of size 4 (C_4's). Take an induced C_4 subgraph of the octahedral O_3. The pairs of non-adjacent vertices of the induced C_4 are false twins whose neighborhoods are the remaining vertices of the induced C_4. Each of the vertices outside the C_4 are adjacent to all vertices of the C_4. Thus, if in a B_1-EPG representation of C_4, the C_4 is represented as a frame, no single bend path can simultaneously intersect the 4 paths representing the vertices of the induced C_4. Therefore, we conclude that the frame structure can not be part of a O_3 representation.

With the same reasoning, take a B_1-EPG representation of C_4 where the induced C_4 subgraph is represented as true pie or false pie. When adding the false twin vertices, which are neighbors of all vertices of C_4 taken from O_3, both representations converge to the structure represented in Figure 3(b). □

Lemma 5: Each path P_i can be determined in polynomial time, from the considered sequence of edge.

Proof. 5: It is easy to verify that (i) is true, consider the sequence of relevant edges of some path $P_i \in \mathcal{P}$. Start from an extremity edge of P_i. Let t be the row

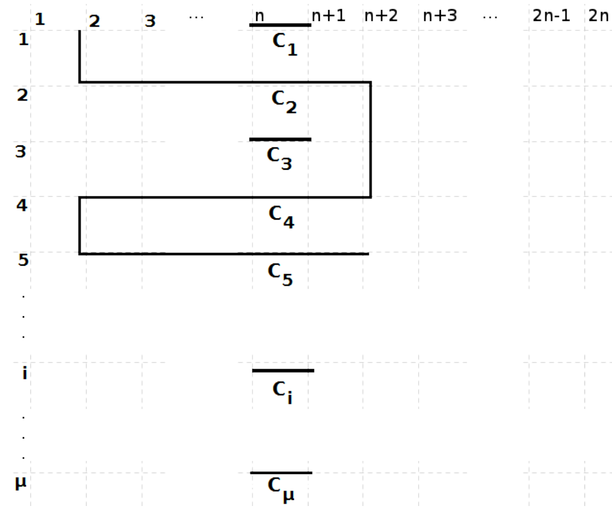


Fig. 10. Representation of the path P_2 corresponding to vertex v_2 contained in cliques C_2, C_4 and C_5

(column) containing the last considered relevant edge. The next relevant edge e' in the sequence, must be also contained in row (column) t. If e' is an extremity edge, the process is finished and the path has been determined. It contains all edges between the considered relevant edges in the sequence. Otherwise, if e' is a bend edge , the next relevant edge is the second bend edge e'' of this same bend, which is contained in some column (row) t'. The process continues until the second extremity edge of P_i is located.

With the above procedure, we can determine in $\mathcal{O}(k+|V(G)|)$ time, whether path P_i contains any given edge of the grid Q. Therefore, the sequence of relevant edges of P_i uniquely determines P_i. □

Lemma 6: Let R be a B_k-EPG-Helly representation of G, and $P_1, P_2 \in \mathcal{P}$ paths of R. Then P_1, P_2 are intersecting paths if and only if they contain a common relevant edge.

Proof. 6: If P_1, P_2 contain a common relevant edge there is nothing to prove. Otherwise, assume that P_1, P_2 are intersecting and we show they contain a common relevant edge. Without loss of generality, suppose P_1, P_2 intersect at row i of the grid, in the B_k-EPG representation R. The following are the possible cases that may occur:

- **Case 1:** Neither P_1 nor P_2 contain bends in row i.
  Then P_1 and P_2 are entirely contained in row i. Since they intersect, either P_1, P_2 overlap, or one of the paths contains the other. In any these situations, they intersect in a common extremity edge, that is a relevant edge.
- **Case 2:** P_1 does not contain bends in i, but P_2 does.

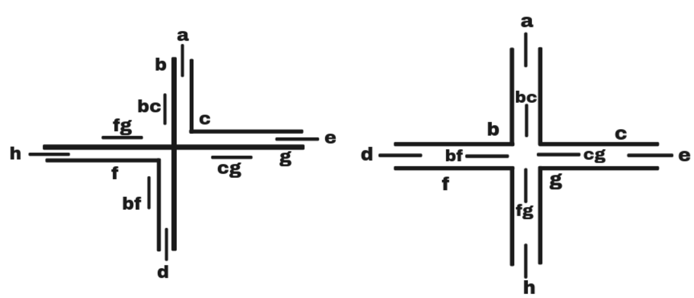
If some bend edge of P_2 also belongs to P_1, then P_1, P_2 intersect in a relevant edge. Otherwise, since P_1, P_2 intersect, the only possibility is that the intersection contains an extremity edge of P_1 or P_2. Hence the paths intersect in relevant edge.

– **Case 3:** Both P_1, P_2 contain bends in i
Again, if the intersection occurs in some bend edge of P_1 or P_2, the lemma follows. Otherwise, the same situation as above must occur, that is P_1, P_2 must intersect in same extremity edge.

In any of the cases, P_1 and P_2 intersect in some relevant edge. □

Lemma 9: Given a satisfiable instance F of POSITIVE (1 IN 3)-3SAT, the graph G_F constructed from F according to Definition 4 admits a B_1-EPG-Helly representation.

Proof. 9: We will use the true pie and false pie structures to represent the *clause gadgets* G_C, but the construction could also be done with the frame structure without loss of generality, see Figure 11.



(a) Based in false pie (b) Based in true pie

Fig. 11. Single bend representations of a clause gadget isomorph to graph H

The *variable gadgets* will be represented by structures as of Figure 12.

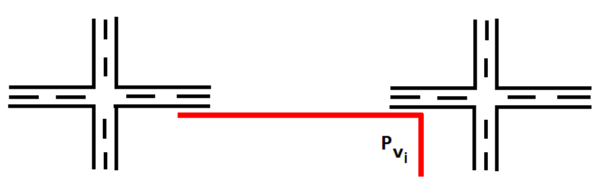


Fig. 12. Single bend representation of a variable gadget

The *base gadget* will be represented by the structure of Figure 13.

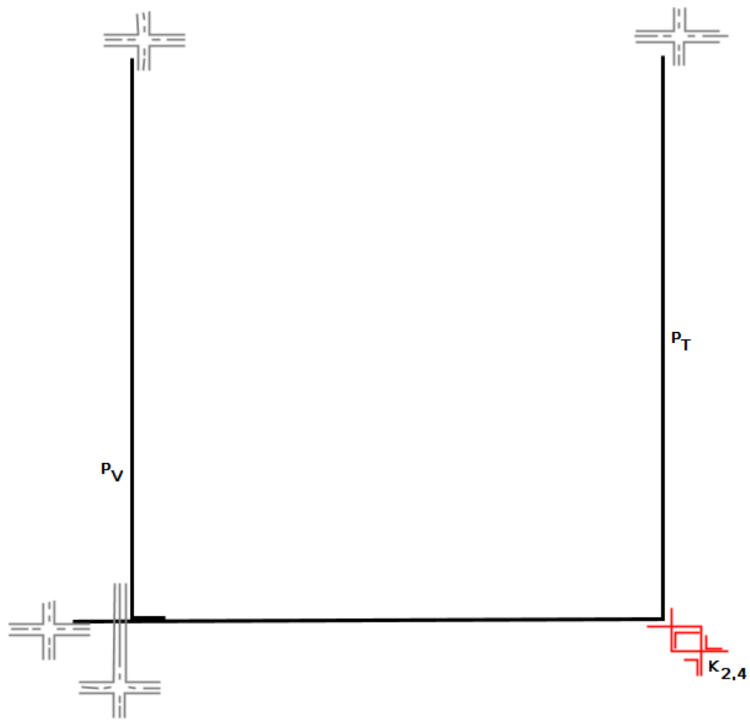
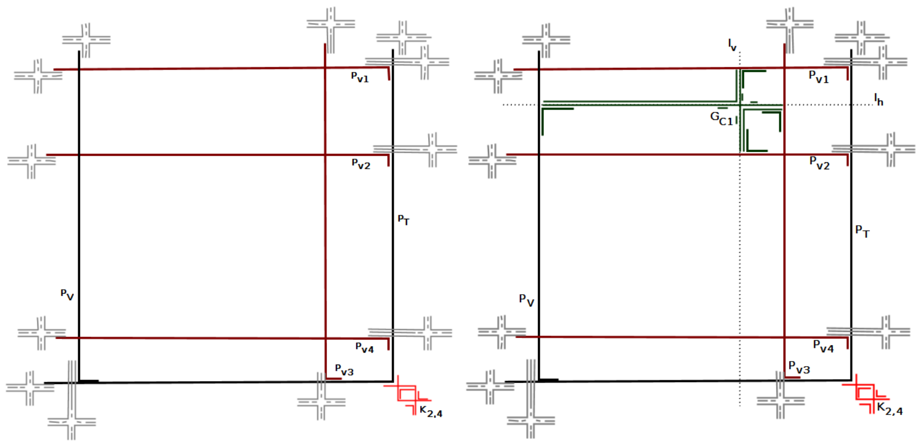


Fig. 13. Single bend representation of the base gadget

It is easy to see that the representations of the clause gadgets, variable gadgets, and base gadgets are all B_1-EPG-Helly. Now we need to describe how these representations can be combined in order to construct a single bend representation R_{G_F}.

Given an assignment A that satisfies F, we can construct a B_1-EPG-Helly representation R_{G_F}. First we will fix the representation structure of the base gadget in the grid to guide the single bend representation, see Figure 13. Next we will insert the variable gadgets with the following rule: if the variable x_i related to the path P_{v_i} had assignment *True*, then the adjacency between the path P_{v_i} with P_T is horizontal, and vertical otherwise. For example, for an assignment $A = \{x_1 = False; x_2 = False; x_3 = True; x_4 = False\}$ to variables of the formula F that generated the gadget G_F of Figure 5, it will give us a single bend representation (base gadget + variables gadget) according to the Figure 14(a).

When a formula F of Positive (1-in-3)-3sat has clauses whose format of assignment is $(False, True, False)$ or $(False, False, True)$ then we will use false pie to represent these clauses, but when the clause has format $(True, False, False)$ we will use true pie to represent this clause. To insert a *clause gadget* G_C, we introduce a horizontal line l_h in the grid between the horizontal rows used by the paths for the two false variables in C. Then we connect the path $P_{d_{c_i}}$ of G_{C_i} to P_V vertically using the bend of $P_{d_{c_i}}$. However, we introduce a vertical line l_v in the grid, between the vertical line of the grid used by P_V and the path to the true variable in C_i, i.e. between P_V and the path of the true variable $x_j \in C_i$. Where l_h and l_v cross, to insert the center of the *clause gadget* as can be seen in Figure 14(b). A complete construction of this single bend representation for the G_F can be verified in Figure 15.



(a) Representation with omitted clause gadgets

(b) Representation with G_{C_1} associated with the clause $(x_1+x_2+x_3)$ in highlighted

Fig. 14. Single bend representation of the base and variables gadgets associated with the assignment $x_1 = False, x_2 = False, x_3 = True, x_4 = False$

Note that when we join all these representations of gadgets that form R_{G_F} we do not insert more bends in paths that had already bend then the representation is necessarily B_1-EPG. Let us show that it satisfies the Helly property.

A simple way to check that R_{G_F} satisfies the Helly property is to note that the particular graph G_F never forms triangles between variable, clause, and base gadgets. Thus, any triangle of G_F is inside a variable, clause or base gadget. As we only use B_1-EPG-Helly representations of such gadgets, R_{G_F} is a B_1-EPG-Helly representation of G_F. $\square$

Lemma 10: In any single bend minimal representation of a graph isomorphic to H, there are two paths in $\{P_a, P_e, P_d, P_h\}$ that have horizontal direction and the other two paths have vertical direction.

Proof. 10: If the $C_4^H = [b, c, f, g]$ is represented by a true pie or false then each path of C_4^H share two central rays with two other paths of C_4^H, where each central ray corresponds to one pair of consecutive vertices in C_4^H.

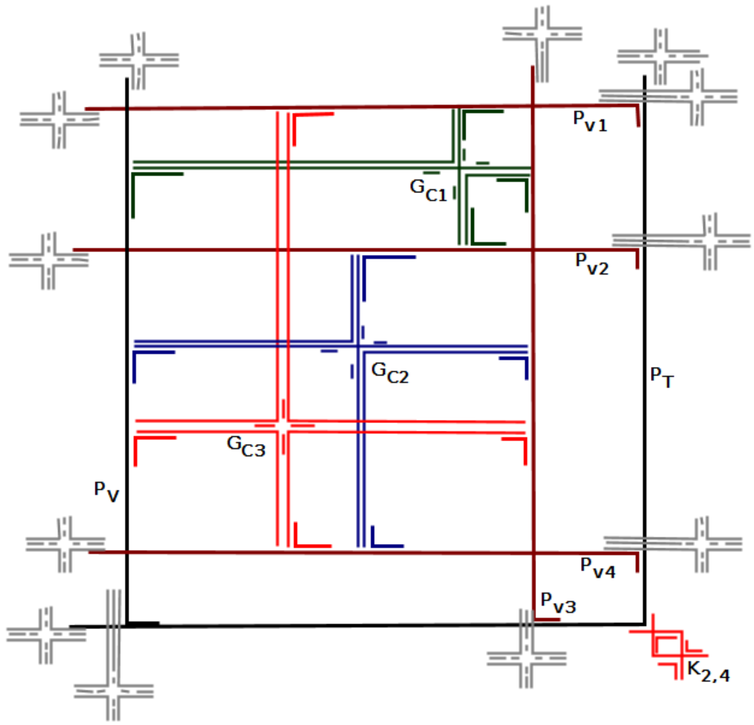


Fig. 15. Single bend representation of G_F

As the vertices a, e, d and h are adjacent to pairs of consecutive vertices in C_4^H so the paths P_a, P_e, P_d and P_h have to be positioned in each one of the different central rays, 2 are horizontal and 2 are vertical.

If the C_4^H is represented by a frame then each path of the C_4^H has a bend positioned in the corners of the frame. In the frame, the adjacency relationship of pairs of consecutive vertices in the C_4^H is represented by the edge-intersection of the paths that constitute the frame. Thus, since a frame has two parts in the vertical direction and two parts in the horizontal direction, then there are two paths in $\{P_a, P_e, P_d, P_h\}$ that have horizontal direction and two that have vertical direction.

Note that, by minimality of the representation, no additional edge is needed on the different paths. □

Proposition 2: Given a B_1-EPG-Helly representation of a graph G that has an induced C_4 whose representation is isomorphic to a frame, if there is a vertex v of G, outside this C_4, that is adjacent to exactly two consecutive vertices of this C_4, then the path representing v shares at least one common edge-intersection with the paths representing both vertices.

Proof. 2: By assumption, G has a triangle containing v and two vertices of a C_4. Therefore the path representing v shares at least one common edge intersecting with the paths representing these neighbors, otherwise the representation does not satisfy the Helly property. □

The following proposition helps us in understanding of the NP-hardness proof.

Proposition 3. *In any single bend Helly representation of the graph G' presented in Figure 16(a), the path P_x has obstructed extremities and bend.*

Proof. Consider G' consisting of a vertex x, two graphs isomorphic to H, H_1 and H_2, and a bipartite graph $K_{2,4}$, such that: x is a vertex of the largest stable set of the $K_{2,4}$; x is adjacent to an induced cycle of size 3 of H_1, $C_3^{H_1}$ and to an induced cycle of size 3 of H_2, $C_3^{H_2}$, see Figure 16(a).

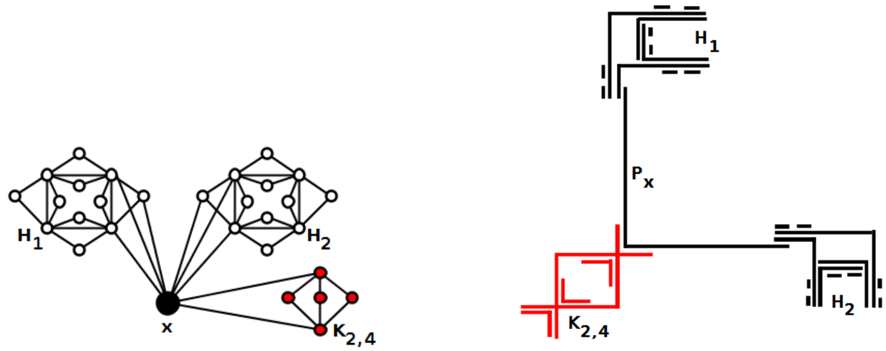
We know that the paths belonging to the largest stable set of a $K_{2,4}$ always will bend into a false pie, see Fact 1. Since P_x is part of the largest stable set of the $K_{2,4}$, then P_x has an *obstructed bend*, see Figure 16(b).

The vertex x is adjacent to $C_3^{H_1}$ and $C_3^{H_2}$, so that its path P_x intersects the paths representing them. But in a single bend representations of a graph isomorphic to H there are pairs of paths that always are on some segment of a central ray or a side intersection, see Corollary 3, and the representation of $C_3^{H_1}$ (similarly $C_3^{H_2}$) has one these paths. Therefore, there is an edge in the set paths that represent H_1 (similarly in H_2) that has a intersection of 3 paths representing $C_3^{H_1}$ (and $C_3^{H_2}$), otherwise the representation would not be Helly, and there is one other different edge in the same central ray or side intersection that contains three other paths and one of them is not in set paths $C_3^{H_1}$ (similarly $C_3^{H_2}$). Thus in a single bend representation of G', the paths that represent $C_3^{H_1}$ (similarly $C_3^{H_2}$) must intersect in a bend edge or an extremity edge of P_x, because P_x intersects only one of set paths that are on some central ray or side intersection where $C_3^{H_1}$ (similarly $C_3^{H_2}$) is. As the bend of G' is already obstructed by structure of $K_{2,4}$, then H_1 (similarly in H_2) must be positioned at an extremity edge of P_x. This implies that P_x has a condition of *obstructed extremities*, see Figure 16(b). □

Remark 2. In a single bend representation of a graph, if a path P_y has an obstructed bend and obstructed extremities, and some path P_x intersects P_y, but does not intersect paths that obstruct the extremities and the bend edges of P_y, then the intersection between P_x and P_y is internally contained into P_y.

Lemma 11: If a graph G_F, constructed according to Definition 4, admits a B_1-EPG-Helly representation, then the associated CNF-formula F is a yes-instance of Positive (1 in 3)-3sat.

Proof. 11: Suppose that G_F has a B_1-EPG-Helly representation, R_{G_F}. From R_{G_F} we will construct an assignment that satisfies F.



(a) The graph G' (b)A B_1-EPG representation of G'

Fig. 16. The sample of obstructed extremities and bend.

First, note that in every single bend representation of a $K_{2,4}$, the path of each vertex of the greater stable set, in particular P_T (in R_{G_F}), has bends contained in a false pie (see Remark 1).

The vertex T is adjacent to the vertices of a triangle of G_{B1} and G_{B2}. As the $K_{2,4}$ is positioned in the bend of P_T, then in R_{G_F} the representation of G_{B1} and G_{B2} are positioned at the extremities of P_T, see Proposition 4.3.

Without loss of generality assume that $P_V \cap P_T$ is a horizontal segment in R_{G_F}.

We can note in R_{G_F} that: the number of paths P_d with segment internally contained in P_V is the number of clauses in F; the intersection between each P_a, P_e, P_h in the gadget clause and each path P_{v_j} indicates the variables composing the clause. Thus, we can assign to each variable x_j the value *True* if the edge intersecting P_{v_j} and P_T is horizontal, and *False* otherwise.

In Lemma 10 it was shown that any minimal B_1-EPG representation of a clause gadget has two paths in $\{P_a, P_d, P_e, P_h\}$ with vertical direction and the other two paths have horizontal direction. Since P_d intersects P_V, it follows that in a single bend representation of G_F we must connect two of these in order to represent a false assignment, and exactly one will represent a true assignment. Thus, from R_{G_F} we construct an assignment to F such that every clause has exactly one variable with a true value. □

Corollary 4: B_1-EPG-HELLY GRAPH RECOGNITION remains NP-complete on 2-apex and 3-degenerate graphs.

Proof. 4: It is easy to see that the graphs constructed according to Definition 4 are 3-degenerate. As POSITIVE (1 IN 3)-3SAT remains NP-complete when the incidence graph of F is planar [14], from an instance F of PLANAR POSITIVE (1 IN 3)-3SAT, by using a planar embedding of the incidence graph of F, one can observe that by removing V and T from G_F we obtain a planar graph. Thus, G_F is 2-apex. □