Quantum Mechanics Tutorial III

Engineering Physics

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How Long Are de Broglie Matter Waves?

- Calculate the de Broglie wavelength of: (a) a 0.65-kg basketball thrown at a speed of 10 m/s, (b) a nonrelativistic electron with a kinetic energy of 1.0 eV.

(a) For the basketball, the kinetic energy is

$$K = mv^2/2 = (0.65\text{kg})(10\text{m/s})^2/2 = 32.5\ \text{J}$$

and the rest mass energy is

$$E_0 = m_0c^2 = (0.65\ \text{kg})(2.998\times108\ \text{m/s})^2 = 5.84\times10^{16}\ \text{J}$$

We see that $K \ll E_0$ and so we use simply momentum

$$p = mv = (0.65\text{kg})(10\text{m/s}) = 6.5\ \text{J}\cdot\text{s/m}$$

Hence,

$$\lambda = h/p = 6.626\times10^{-34}\ \text{J.s} / 6.5\ \text{J}\cdot\text{s/m} = 1.02\times10^{-34}\text{m}.$$

(b) For non-relativistic electron , the de Broglie wavelength is

$$\lambda = h/p$$
$$= h/\sqrt{2\, m_0 K}$$
$$= (6.626\times10^{-34}\ \text{J.s})/\sqrt{2(9.1\times10^{-31}\ \text{kg})(1.0\ \text{eV})}$$
$$= 1.23\text{nm}$$

Problem based on uncertainty principle

- The average life time of an excited atomic state is 10^{-9} s. If spectral line associated with the decay of this state is 6000 Å, estimate the width of line.

We have

$$\Delta t = 10^{-9}\ \text{s}, \qquad \lambda = 6000\ \text{Å}$$

Now energy released when excited electron comes down to lower state

$$E = \frac{hc}{\lambda} \quad \text{or} \quad \Delta E = \frac{hc}{\lambda^2}\Delta\lambda$$

Multiplying both side by Δt, we get

$$\Delta E \cdot \Delta t = \frac{hc}{\lambda^2}\Delta\lambda \cdot \Delta t \cong \hbar = \frac{h}{2\pi}$$

therefore,

$$\Delta\lambda = \frac{\lambda^2}{c.\Delta t.4\pi} = \frac{36\times10^{-14}\ m^2}{\left(3\times10^8\ \frac{m}{s}\right)(10^{-9}\ s).2\pi}$$

$$= 1.91\times10^{-13}\ m$$

Problem based on particle in box

- A 10-g marble in box 10 cm across . Find its permitted energies.

Here we have m = 10 g = $1\times 10^{-2}\ kg$ and width of box L = 10 cm = $1\times 10^{-1}\ m$

So, the permitted energies of marble in this box ,

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2h^2}{8mL^2} \quad (1)$$

$$= \frac{(\mathrm{n}^2)(6.63\times 10^{-34}\ \mathrm{J.s})^2}{8(1\times 10^{-2}\mathrm{kg})(1.0\times 10^{-1}\ \mathrm{m})^2}$$

$$= 5.5\times 10^{-64}\ n^2\ \mathrm{J}$$

- The minimum energy the marble can have is 5.5×10^{-64} J, corresponding to n = 1. A marble with this kinetic energy has a speed of only $3.3\times 10^{-31}\ m/s$ and therefore can not be experimentally distinguished from a stationary marble. A reasonable speed a marble might have is 0.33 m/s which corresponds to n = 10^{30}. The permissible energy levels are so very close together that there is no way to determine whether the marble can take on only those energies predicted by equation (1). Hence, in the domain of everyday experience, quantum effects are imperceptible, which accounts for the success of Newtonian mechanics in this domain.

Normalization and extraction of expectation value

- A particle considered to move along x-axis in the domain $0 \leq x \leq L$ has a wave function $\psi(x) = \mathrm{N}\sin\left(\frac{n\pi x}{\mathrm{L}}\right)$, where n is an integer. Normalize the wave function and find the expression for N and evaluate the expectation value of its momentum.

The normalization condition gives

$$\int_{-\infty}^{\infty} \Psi^* \, \Psi \, dx = 1 \qquad \{\psi^* \text{ is complex conjugate of } \psi\}$$

For this wave function

$$|N|^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1 \quad \text{or} \quad |N|^2 \int_0^L \frac{1}{2}\left(1 - cos\,\frac{2n\pi x}{L}\right) dx = 1$$

Then,

$$|N|^2 \frac{L}{2} = 1 \quad \text{or} \quad N = \sqrt{\frac{2}{L}}$$

So, now the normalized wave function is,

$$\psi_n = \sqrt{\frac{2}{L}} \sin\,(n\pi x/L)$$

The expectation value of the momentum is obtained as:

The momentum operator is defined as $\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$

$$< \hat{p_x} > = \int_0^L \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi \, dx = -i\hbar \, \frac{2}{L} \, \frac{n\pi}{L} \int_0^L sin \frac{n\pi x}{L} \, \cos \frac{n\pi x}{L} dx = -i\hbar \, \frac{n\pi}{L^2} \int_0^L sin \frac{2n\pi x}{L} dx = 0$$