Quantum Mechanics Tutorial II

Engineering Physics

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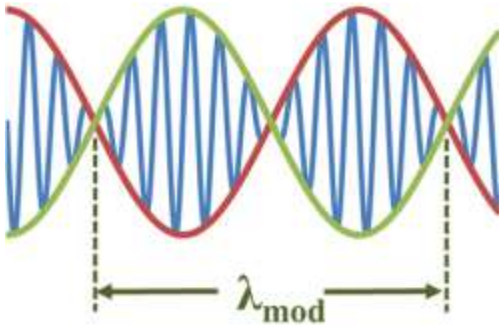
- **Phase velocity and Group Velocity of Quantum Particles:**

To simply understand this, we can say that **phase velocity** is the speed of a monochromatic wave moving with some constant amplitude A, an angular frequency ω and a wave vector k. For example, $y(x,t) = A\cos(kx - \omega t)$ where the phase velocity is

$$v_{ph} = \frac{\omega}{k}$$

Now, if the similar waves with same amplitude but different frequency and wave vector are superposed, the wave with higher frequency envelopes within the wave with lower frequency. The velocity of this envelope is the **group velocity**.

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}$$



For example, consider two sinusoidal waves:

$$y_1 = A\sin(k_1 x - \omega_1 t)$$
$$y_2 = A\sin(k_2 x - \omega_2 t)$$

Their superposition is

$$\left.\begin{aligned} y &= y_1 + y_2 \\ &= A\{\sin(k_1 x - \omega_1 t) + \sin(k_2 x - \omega_2 t)\} \\ &= 2A\cos(\Delta k\, x - \Delta\omega\, t)\sin(kx - \omega t) \end{aligned}\right\}$$

Group velocity: $v_g = \frac{\Delta\omega}{\Delta k}$

Phase velocity: $v_{ph} = \frac{\omega}{k}$

Where, $\Delta k = \frac{k_1 - k_2}{2}$ $\quad \Delta\omega = \frac{\omega_1 - \omega_2}{2}$ $\quad k = \frac{k_1 + k_2}{2}$ $\quad \omega = \frac{\omega_1 + \omega_2}{2}$

For the relativistic particle, the total energy is $E = \sqrt{m_0^2 c^4 + p^2 c^2}$

For the matter waves, $\omega \to E$ and $k \to p$.

Therefore, the phase velocity becomes $v_{ph} = \frac{E}{p} = \frac{\gamma m_0 c^2}{\gamma m_0 v} = \frac{c^2}{v}$

And the group velocity becomes

$$v_g = \frac{dE}{dp} = \frac{d}{dp}\sqrt{m_0^2 c^4 + p^2 c^2}$$

$$= \frac{pc^2}{\sqrt{m_0^2 c^4 + p^2 c^2}}$$

$$= \frac{pc^2}{E} \quad \Rightarrow \quad v_g = \frac{c^2}{\left(\frac{E}{p}\right)} = \frac{c^2}{v_{ph}} \quad \Rightarrow \quad v_g \cdot v_{ph} = c^2$$

$$= \frac{v\, c^2}{c^2}$$

$$= v$$

The group velocity of the matter waves is equal to the velocity of the particle.

Problem based on uncertainty principle

- **The radius of the hydrogen atom is 5.3 $\times$ 10^{-11} m. Use the uncertainty principle to estimate the minimum energy an electron can have in this atom.**

The uncertainty relation is given mathematically as:

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}$$

Here, we find that Δx = 5.3$\times$ 10^{-11} m .

$$\therefore\ \Delta p \geq \frac{\hbar}{2\Delta x} = \frac{1.053\ \times\ 10^{-34}\, J.s}{2 \times (5.3\ \times\ 10^{-11}\ m)} = 9.9 \times 10^{-25}\ kg\,.m/s$$

Since pc $\ll$ rest mass energy of the electron (0.51 MeV)
So, an electron whose momentum is of this order behaves like a classical particle, and its kinetic energy is:

$$\text{KE} = \frac{p^2}{2m} \geq \frac{\left(9.9\times10^{-25}\ kg.\frac{m}{s}\right)^2}{2(9.1\times10^{-31}\ kg)} = 5.4\times\ 10^{-19}\ J$$

Or KE $\geq$ 3.4 eV

- **The lifetime of a nucleus in an excited state is 10^{-12} s. Calculate the probable uncertainty in the energy and frequency of γ-ray photon emitted by it.**

The energy-time uncertainty relation is $\Delta E\, \Delta t \approx \frac{\hbar}{2}$

$$\therefore \Delta E \approx \frac{\hbar}{2\,\Delta t} = \frac{1.054 \times 10^{-34}\, J.s}{2 \times 10^{-12}\, s} = 0.527 \times 10^{-22}\, J$$

The uncertainty in frequency is

$$\Delta \nu = \frac{\Delta E}{h} = \frac{0.527 \times 10^{-22}\, J}{6.3 \times 10^{-34}\, J.s} = 0.836 \times 10^{11}\, Hz$$

- **The average lifetime of an excited atomic state is 10^{-8}s. If the wavelength of the spectral line associated with the transition from this state to the ground state is 6000 Å, estimate the width of this line.**

Since $E = h\nu = \frac{hc}{\lambda}$, we have $\Delta E = -\frac{hc}{\lambda^2}\, \Delta\lambda$

According to the uncertainty principle, $\Delta E\, \Delta t = \frac{\hbar}{2}$

$$\therefore -\frac{hc}{\lambda^2}\, \Delta\lambda\, \Delta t = \frac{\hbar}{2}$$

$$\therefore |\Delta\lambda| = \frac{\lambda^2}{4\pi c\, \Delta t} = \frac{(6 \times 10^{-7})^2}{2 \times 3.14 \times 3 \times 10^{8} \times 10^{-8}} = 1.9 \times 10^{-14}\, m$$

Expectation value: How to extract information from a wave function

The expectation value of the position of the single particle is:

$$< x > = \frac{\int_{-\infty}^{\infty} x|\Psi|^2 \, dx}{\int_{-\infty}^{\infty} |\Psi|^2 \, dx}$$

If Ψ is a normalized wave function, the denominator of above equation equals the probability that the particle exists somewhere between x = -∞ and x = ∞ and therefore has the value 1.

- A particle limited to the x axis has wave function Ψ = a x between x = 0 and x = 1; Ψ = 0 elsewhere.
  Find (a) probability that the particle can be found between x = 0.45 and x = 0.55
  (b) Expectation value $< x >$ of the particle's position.

(a) The probability is

$\int_{x_1}^{x_2} |\Psi|^2 \, dx = a^2 \int_{0.45}^{0.55} x^2 \, dx = 0.0251a^2$

(b) The expectation value is

$< x > = \int_0^1 x|\Psi|^2 \, dx = a^2 \int_0^1 x^3 \, dx = a^2/4$

- Consider a particle whose normalized wave function is

$$\Psi(x) = 2\alpha\sqrt{\alpha}\, x\, e^{-\alpha x} \quad ; \quad x > 0$$
$$= 0 \quad ; \quad \text{otherwise}$$

(a) For what value of x does the probability $|\Psi|^2$ be maximum?
(b) Calculate $< x >$, $< x^2 >$, $< p >$, $< p^2 >$.
(c) Verify uncertainty principle.

Use the following integral:

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx \qquad \text{for n} > 0$$
$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n!$$
$$\Gamma(1) = 1 \quad \& \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

(a) The probability be maximum when $\frac{d|\Psi|^2}{dx} = 0$

$$\therefore \frac{d}{dx}\left(x^2\, e^{-2\alpha x}\right) = 0$$
$$\therefore 2x\, e^{-2\alpha x}(1 - \alpha x) = 0$$
$$\therefore x = \frac{1}{\alpha}$$

(b) $$\langle x\rangle = \int_0^\infty dx\, x(4\alpha^3 x^2 e^{-2\alpha x}) = \frac{1}{4\alpha}\int_0^\infty dy\, y^3 e^{-y} = \frac{3!}{4\alpha} = \frac{3}{2\alpha}$$

$$\langle x^2\rangle = \int_0^\infty dx\, x^2(4\alpha^3 x^2 e^{-2\alpha x}) = \frac{4!}{8\alpha^2} = \frac{3}{\alpha^2}$$

$$\langle p\rangle = -i\hbar\left(4\alpha^3\right)\int_0^\infty xe^{-\alpha x}\frac{\partial}{\partial x}\left(xe^{-\alpha x}\right)dx = 0$$

$$\langle p^2\rangle = -\hbar^2\left(4\alpha^3\right)\int_0^\infty xe^{-\alpha x}\frac{\partial^2}{\partial x^2}\left(xe^{-\alpha x}\right)dx = -\hbar^2\left(4\alpha^3\right)\left[-2\alpha\int_0^\infty xe^{-2\alpha x}dx + \alpha^2\int_0^\infty x^2 e^{-2\alpha x}dx\right]$$

$$= -\hbar^2\left[-\frac{2\alpha\left(4\alpha^3\right)}{\left(2\alpha\right)^2}\int_0^\infty ye^{-y}dy + \alpha^2\left(4\alpha^3\int_0^\infty x^2 e^{-2\alpha x}dx\right)\right] = -\hbar^2\left[-2\alpha^2+\alpha^2\right] = \alpha^2\hbar^2$$

(c) $$\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{3}{\alpha^2} - \left(\frac{3}{2\alpha}\right)^2 = \frac{3}{4\alpha^2};\ \Delta x = \frac{\sqrt{3}}{2\alpha}$$

$$\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2 = \alpha^2\hbar^2;\ \Delta p = \alpha\hbar$$

$$\Delta x\Delta p = \frac{\sqrt{3}}{2\alpha}\alpha\hbar = \frac{\sqrt{3}}{2}\hbar > \frac{\hbar}{2}$$