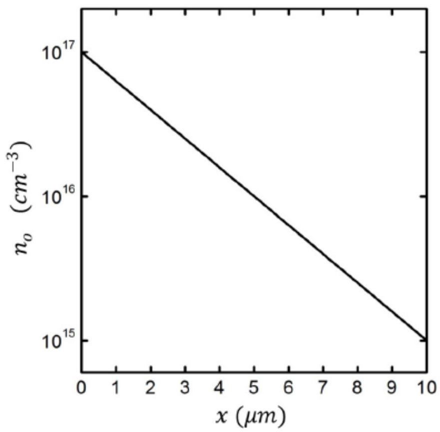
Que. 1 Consider an n-type silicon sample under thermo-dynamical equilibrium at room temperature. In a diffused region ($0 \le x \le 10$ µm) with a non-uniform doping, the electron concentration is given by $n_0(x) = 10^{17} \times 10^{-2000x}$ (cm^{-3}), (x in cm)

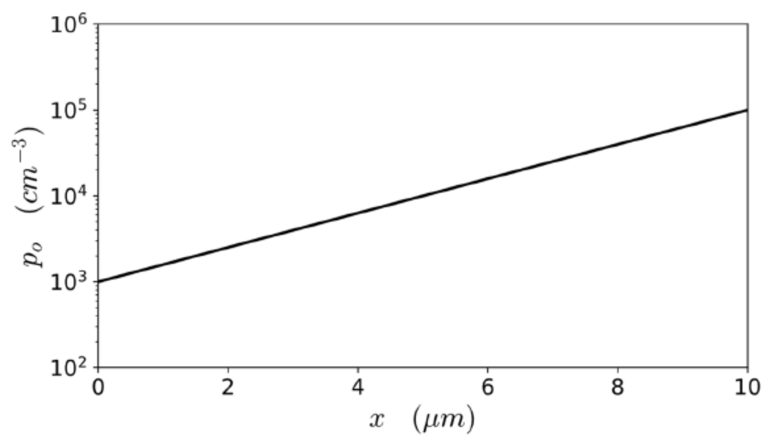


(a) Obtain an analytical expression for the hole concentration in the same region. Plot this concentration profile and give the numerical values at each extreme.

Sol. Since the semiconductor is in equilibrium, $n(x)\, p(x) = n_i^2$. Then, **(1)**

$$p(x) = \frac{n_i^2}{n(x)} = 10^3 \,.\, 10^{2000\,x}\ , (x \text{ in cm})$$ **(1)**

$$p(0) = 10^3\ cm^{-3}\ , p(10\ \mu m) = 10^5\ cm^{-3}$$ **(1)**



(1)

(b) Calculate the electric filed. Discuss the result and its sign.

Sol. Besides, in equilibrium $j_n = qn\mu_n E + qD_n \frac{dn}{dx} = 0$. Thus: **(1)**

$$E(x) = -\frac{D_n}{\mu_n}\frac{\frac{dn}{dx}}{n} = -V_T(-2000)\ln(10) = 115\ V/cm$$ **(2)**

The decreasing profile in the electron concentration causes a net flow of electrons to the right (diffusion current to the left). Then, there must be an electric field pulling the electrons to the left in order to cancel the diffusion term (drift current to the right). Yes, indeed the electric field is positive in the direction of the x axis. **(1)**

(c) Obtain the expression of the electrostatic potential in the same region. Plot it taking the reference at $x = 0$. Calculate the potential difference between both extremes of diffused region.

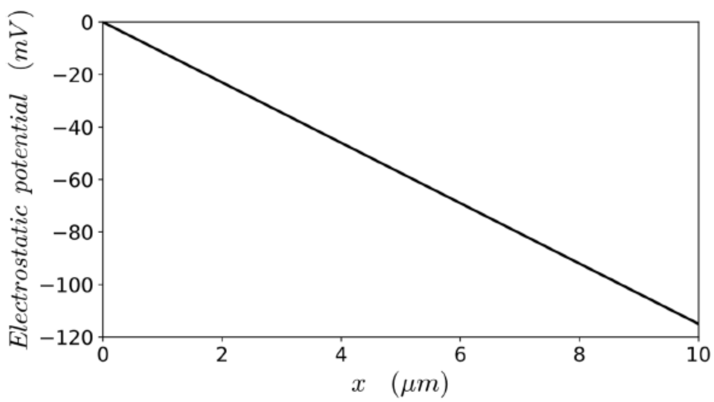
Sol. We calculate the voltage difference between both extremes of the doped region:

$$E(x) = -\frac{dV}{dx} \Rightarrow dV = -E(x)dx$$

In our case the electric field is a constant:

$$\int_0^x dV = V(x) - V(0) = -\int_0^x E\ dx = -E\ x$$

Now, taking the potential at $x = 0$ as a reference:



(2)

$$V(x) = -E\ x, \qquad (x\ in\ cm)$$ **(1)**

$$V(10\ \mu m) = -115\ mV$$ **(1)**

(d) Finally, obtain the charge density in the diffused region. Can this region be considered neutral? Would it be neutral independently of the doping profile?

Sol. By considering the Gauss equation:

$$\frac{dE}{dx} = \frac{\rho}{\varepsilon_r \varepsilon_0} = 0$$ **(2)**

Since E is a constant, $\rho = 0$ and the sample is neutral. However, this is not a general case and it could be different for another doping profile. Nevertheless, if it does not change very fast with distance, the sample can be typically assumed to be quasi-neutral. **(1)**

Que. 2 (a) **Suppose a free particle of mass m is situated at $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ with respect to (w.r.t.) a Cartesian coordinate system installed for an inertial frame S at an instant t. Now, if another coordinate system S' is moving with a constant velocity $\mathbf{V}$ w.r.t. S, then what will be the Lagrangian for the particle in this new coordinate system? Show that Lagrange's equations of motion of the particle remain unchanged in this new system.** [4]

Sol. The Lagrangian of a free particle in the S system is

$$L = T - V = \frac{1}{2}m\dot{r}^2 = \frac{1}{2}mv^2$$

where, the potential energy V is constant as the particle is free. Thus, we can simply take V=0.

Velocity of the particle w.r.t. S$'$: **(1)**

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t$$

$$\mathbf{v}' = \mathbf{v} - \mathbf{V}$$

Lagrangian: **(1)**

$$L' = \frac{1}{2}m(\mathbf{v}-\mathbf{V})^2 = \frac{1}{2}m(\mathbf{v})^2 - m\mathbf{v}\cdot\mathbf{V} + \frac{1}{2}m\mathbf{V}^2$$

The second and third terms can be written as a total derivative

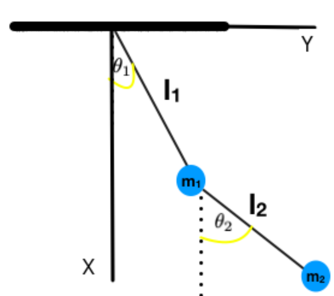
$$\frac{d}{dt}(m\mathbf{r}\cdot\mathbf{V} - \frac{1}{2}mV^2t)$$

So, the equation of motion will remain unchanged. **This also can be shown directly by obtaining Lagrange's equations of motion.** **(2)**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \implies m\ddot{x} - m\frac{d}{dt}(V_x) = 0 \implies m\ddot{x} = 0$$

As V_x is constant.

(b) **Find out the generalized velocities and generalized momenta for the double pendulum shown below.** [1+5]



Sol. Double pendulum: Generalized velocities are $\dot{\theta}_1$ and $\dot{\theta}_2$. **(1)**

Coordinates: **(1)**

$$x_1 = -l_1 \cos\theta_1 \qquad\qquad y_1 = l_1 \sin\theta_1$$
$$x_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 \qquad\qquad y_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

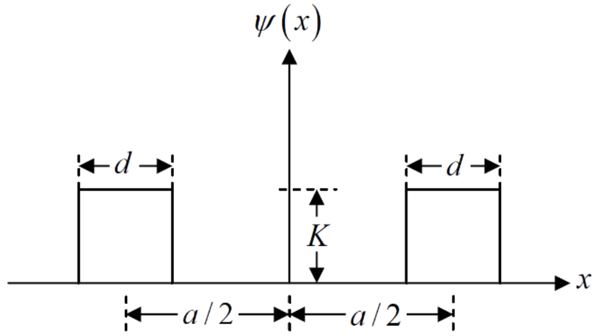
Lagrangian: **(2)**

$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - m_1 g x_1 - m_2 g x_2 \\
&= \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\right] \\
&\qquad + m_1 g l_1 \cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)
\end{aligned}$$

Generalized momenta: **(1+1)**

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

Que. 3 (a) The wave function $\psi(x)$ of a particle is as shown below



Here K is a constant, and $a > d$. Calculate the position uncertainty Δx of the particle.

Sol. $$\psi(x) = \begin{cases} k, & -\frac{a}{2}-\frac{d}{2} < x < -\frac{a}{2}+\frac{d}{2} \\ 0, & -\frac{a}{2}+\frac{d}{2} < x < \frac{a}{2}-\frac{d}{2} \\ k, & \frac{a}{2}-\frac{d}{2} < x < \frac{a}{2}+\frac{d}{2} \\ 0, & x > \frac{a}{2}+\frac{d}{2} \end{cases}$$ **(2)**

Since, $\langle\psi|\psi\rangle = 1$

$$\Rightarrow k^2 \int_{-\frac{a}{2}-\frac{d}{2}}^{-\frac{a}{2}+\frac{d}{2}} dx + k^2 \int_{\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}} dx = 1$$

$$\Rightarrow k^2\left[\left(-\frac{a}{2}+\frac{d}{2}\right)-\left(-\frac{a}{2}-\frac{d}{2}\right)\right] + k^2\left[\left(\frac{a}{2}+\frac{d}{2}\right)-\left(\frac{a}{2}-\frac{d}{2}\right)\right] = 1$$

$$\Rightarrow k^2\left[\frac{d}{2}+\frac{d}{2}+\frac{d}{2}+\frac{d}{2}\right] = 1$$

$$\Rightarrow k = \frac{1}{\sqrt{2d}}$$ **(2)**

Since the wavefunction is symmetric about $x = 0$, so $\langle x\rangle = 0$. **(1)**

Now,

$$\langle x^2\rangle = k^2 \int_{-\frac{a}{2}-\frac{d}{2}}^{-\frac{a}{2}+\frac{d}{2}} x^2\,dx + k^2 \int_{\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}} x^2\,dx$$

$$= \frac{k^2}{3}\left[[x^3]_{-\frac{a}{2}-\frac{d}{2}}^{-\frac{a}{2}+\frac{d}{2}} + [x^3]_{\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}}\right]$$

$$= \frac{k^2}{3\times 8}[(-a+d)^3-(-a-d)^3+(a+d)^3-(a-d)^3]$$
$$= \frac{k^2}{24}[(-a^3+d^3+3a^2d-3ad^2)+(a^3+d^3+3a^2d+3ad^2)$$
$$+(a^3+d^3+3a^2d+3ad^2)-(a^3-d^3-3a^2d+3ad^2)]$$
$$\Rightarrow \langle x^2\rangle = \frac{k^2}{24}[4d^3+12a^2d]$$
$$= \frac{4d(d^2+3a^2)}{24\times 2d}$$
$$= \frac{3a^2+d^2}{12} \qquad \textbf{(2)}$$

Now,

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$
$$= \sqrt{\frac{3a^2+d^2}{12}} \qquad \textbf{(1)}$$

(b) Consider a 1-D particle which is confined within the region $0 \le x \le a$ and whose wave function is $\psi(x,t) = sin\left(\frac{\pi x}{a}\right)e^{-i\omega t}$. Calculate the probability of finding the particle in the interval $\frac{a}{4} \le x \le \frac{3a}{4}$.

Sol. The probability of finding the particle in the interval $\frac{a}{4} \le x \le \frac{3a}{4}$ is given by:

$$P = \frac{\int_{a/4}^{3a/4}|\psi(x)|^2\,dx}{\int_0^a|\psi(x)|^2\,dx} \qquad \textbf{(1)}$$
$$= \frac{\int_{a/4}^{3a/4}\sin^2\left(\frac{\pi x}{a}\right)dx}{\int_0^a\sin^2\left(\frac{\pi x}{a}\right)dx}$$
$$= \frac{2+\pi}{2\pi}$$
$$= 0.82 \qquad \textbf{(2)}$$

(c) Consider the motion of a particle along the x-axis in a potential $V(x) = F|x|$. Its ground state energy E_o is estimated using the uncertainty principle and is estimated as $E_o \propto F^{\alpha}$. Find the value of α.

Sol. Given $V = F|x|$. Then,

$$E = \frac{p^2}{2m} + F|x|$$

$$\Rightarrow E = \frac{p^2}{2m} + Fx \quad for \quad x > 0$$

$$; \ E = \frac{p^2}{2m} - Fx \quad for \quad x < 0 \qquad \textbf{(1)}$$

From the uncertainty theory we have,

$$\Delta x \Delta p \approx \hbar$$
$$\Rightarrow \Delta p = \frac{\hbar}{\Delta x}$$

Then,

$$E = \frac{(\Delta p)^2}{2m} + F(\Delta x)$$
$$= \frac{\hbar^2}{2m(\Delta x)^2} + F\Delta x \qquad \textbf{(1)}$$

For minimum energy,

$$\frac{dE}{d\Delta x} = 0$$

$$\Rightarrow \ -\frac{\hbar^2}{m(\Delta x)^3} + F = 0$$

$$\Rightarrow (\Delta x)^3 = \frac{\hbar^2}{mF} \Rightarrow \Delta x = \left[\frac{\hbar^2}{mF}\right]^{\frac{1}{3}} \qquad \textbf{(1)}$$

So,

$$E_o = \frac{\hbar^2}{2m}\left(\frac{mF}{\hbar}\right)^{\frac{2}{3}} + \left(\frac{\hbar^2}{m}\right)^{\frac{1}{3}} F^{\frac{2}{3}}$$

$$\Rightarrow E_o \propto F^{\frac{2}{3}}$$

So,

$$\Rightarrow \alpha = \frac{2}{3} \qquad \textbf{(1)}$$