Classical Mechanics Tutorial III

Engineering Physics

Indian Institute of Information Technology, Allahabad

EOM using Principle of Least Action

Let us consider a Lagrangian of the following form,

$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - V(x) + \alpha\dot{x} \tag{1}$$

where $V(x)$ is the potential energy. We would determine the equation of motion (EOM) using the least action principle.

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Using, $\quad \frac{d}{dt}(m\dot{x}\delta x + \alpha\delta x) = m\ddot{x}\delta x + (m\dot{x}\delta\dot{x} + \alpha\delta\dot{x})$

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- Now let's compute the EOM for a Lagrangian which has the following form:

$$\mathcal{L}'(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - V(x) \tag{7}$$

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$$\delta S = \delta \int \mathcal{L}(x, \dot{x}, t) dt = 0 \Rightarrow m\ddot{x} + V'(x) = 0 \tag{8}$$

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- Therefore, $\mathcal{L} \rightarrow \mathcal{L}' := \mathcal{L} + \frac{dQ}{dt}$ does not change the EOM as $\frac{dQ}{dt}$ is the total derivative term in the Lagrangian.
- The same happens to $\mathcal{L}(x, \dot{x}, t) = \frac{1}{2} m\dot{x}^2 - V(x) + \dot{x}$, i.e., the last term is total derivative added to the Lagrangian. Hence, does not alter the EOM.

Calculus of Variation

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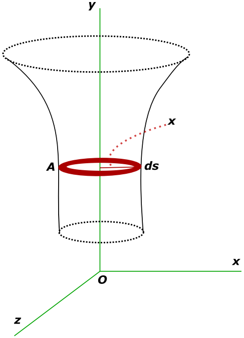
- Consider a curve between two points (x_1, y_1) and (x_2, y_2) in the xy plane whose equ is $y = y(x)$. We form a surface by revolving the curve about y-axis. We are interested in finding the nature of the curve for which the surface area is minimum.



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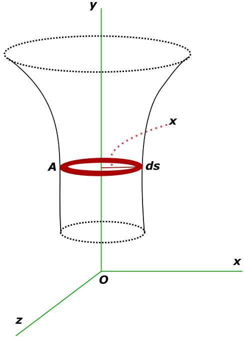

Consider a small strip at a point A formed by revolving the arc length ds about y–axis.

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If the distance of the point A on the curve from y-axis is x, then the surface area of the strip is

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The total area of the surface of revolution of the curve $y = y(x)$ about y- axis is given by

$$A = \int_{x_1}^{x_2} 2\pi x\sqrt{1 + y'^2}dx \tag{12}$$

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This surface area will be minimum iff the integrand $f = 2\pi x\sqrt{1+y'^2}dx$ satisfies Euler-Lagrange's equation, i.e.

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0. \tag{13}$$

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Integration of this will result into

$$y = a\cosh^{-1}\left(\frac{x}{a}\right) + b \;\; ; or \;\; ; \;\; x = a\cosh\left(\frac{y-b}{a}\right) \tag{16}$$

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The above equation is known as the Catenary Curve.

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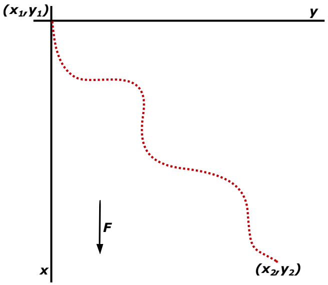
Let us now solve a classic problem from the history of physics, the brachistochrone, using the calculus of variations. Consider a particle moving in a constant force field starting from rest from some point (x_1, y_1) to some lower point (x_2, y_2). We have to find the path that allows the particle to accomplish this transit in the least possible time.



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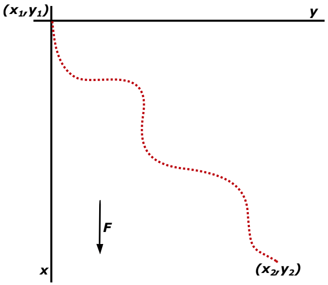
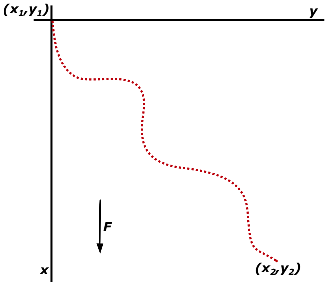


- We chose the coordinate system such that the point (x_1, y_1) is at the origin.

Calculus of Variation

The force F is along the x direction. Because the force on the particle is constant, and if we ignore the possibility of friction, the field is conservative and total energy is constant. We also consider the particle to be initially at rest s.t. $V = 0$.



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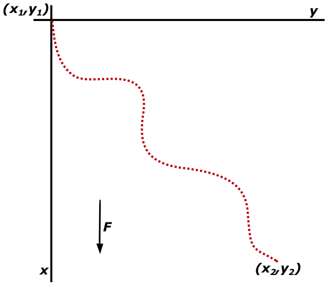

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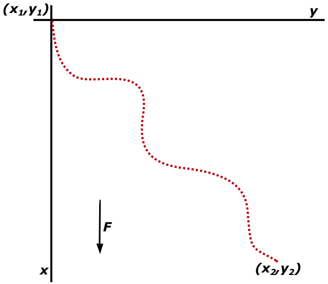

The kinetic energy is $T = 1/2mv^2$ and potential energy is $V = -mgx$, so at the origin $T + V = 0 \implies v = \sqrt{(2gx)}$

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The time of transit is to be minimized , and since the constant $(2g)^{-1/2}$ does not effect the final equation, we identify L as

$$L = \left(\frac{1 + y'^2}{x} \right)^{(1/2)} \tag{19}$$

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This gives us

$$\frac{y'^2}{x(1+y'^2)} = \frac{1}{2a} \tag{22}$$

which may be written as

$$y = \int \frac{x dx}{(2ax - x^2)^{1/2}} \tag{23}$$

Calculus of Variation

We can solve this integration

$$y = \int \frac{xdx}{(2ax - x^2)^{1/2}}$$

by introducing the following change of variable

$$x = a(1 - \cos\theta) \tag{24}$$

$$dx = a \sin\theta d\theta \tag{25}$$

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which evaluates to

$$y = a(\theta - sin\theta) + constant \tag{27}$$

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We then the get the parametric equations of the cycloid.

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- The trajectory then looks like