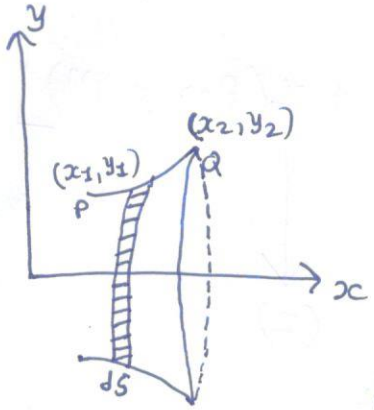
Q.1



1

$$dS = \sqrt{dx^2 + dy^2}$$

$$dA = 2\pi y \, dS$$

$$= 2\pi y \sqrt{1 + x'^2} \, dy \quad ; \quad x' = \frac{dx}{dy}$$

$$\therefore F(y, x') = 2\pi y \sqrt{1 + x'^2} \quad (2)$$

Using variational calculus, one needs to extremize the following functional.

$$I = 2\pi \int_{y_1}^{y_2} y \sqrt{1 + x'^2} \, dy$$

The Euler-Lagrange equation after setting $\delta I = 0$ yields

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0 \quad (1)$$

$$\Rightarrow \frac{\partial F}{\partial x'} = \text{constant} = c$$

$$\Rightarrow \frac{y x'}{\sqrt{1 + x'^2}} = c$$

$$\Rightarrow c^2 (1 + x'^2) = x'^2 y^2$$

$$\Rightarrow x'^2 (y^2 - c^2) = c^2$$

$$\Rightarrow \frac{dx}{dy} = \frac{c}{\sqrt{y^2 - c^2}}$$

$$\Rightarrow x = c \int \frac{dy}{\sqrt{y^2 - c^2}}$$

$$= c \int \frac{dt \, (c \sinh t)}{c \sqrt{\cosh^2 t - 1}}$$

$$y = c \cosh t$$

$$dy = c \sinh t \, dt$$

②

$$\Rightarrow x = c\int dt$$
$$= ct + D \leftarrow \text{constant} \qquad [1 + 1(\text{for sol}^n)]$$

$$\therefore x = c\cosh^{-1}\left(\frac{y}{c}\right) + D$$

$$\therefore \boxed{y = c\cosh\left(\frac{x-D}{c}\right)} \quad \text{Ans.} \qquad (2)$$

Q.2 (a) The motion is Planar as the orbit is elliptical. Kinetic energy for the Planet can be written in Plane polar coordinates (r, θ) as

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

Potential energy,

$$V = -\frac{GMm}{r}$$

$$\Rightarrow L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r} \qquad (2)$$

(b)
$$\left.\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \end{aligned}\right\} \quad 1$$

$$\Rightarrow H = p_r\dot{r} + p_\theta\dot{\theta} - L$$
$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r} \qquad 1$$

$\Rightarrow \theta$ is cyclic coordinate

$\therefore p_\theta$ is conserved. $\longrightarrow$ 1

$\therefore mr^2\dot{\theta} =$ ~~Ag~~ Angular momentum is conserved.

Q-4. (a) $\psi^*\psi = |A|^2$ for $\psi = A\, e^{\frac{i}{\hbar}(Px - Et)}$

$$\therefore \int_{-\infty}^{\infty} |A|^2\, dx = 1$$

$\Rightarrow$ The integration will diverge (No finite value for A).

So the wavefunction is not normalizable.

A can't be calculated. (1/2)

$\Rightarrow$ Since the wavefunction is not normalizable expectation value of x is not defined. $\left(\frac{1}{2}\right)$

(b) $$|c|^2 \int_0^{\infty} e^{-2ax}\, dx = 1$$

$$\Rightarrow |c|^2 \frac{1}{-2a} e^{-2ax}\Big|_0^{\infty} = 1$$

$$\Rightarrow |c|^2 \frac{1}{2a} = 1 \Rightarrow |C| = \sqrt{2a} \quad (1)$$

$$\langle x \rangle = (\sqrt{2a})^2 \int_0^{\infty} e^{-ax - ibt}\, x\, e^{-ax + ibt}\, dx$$

$$= 2a \int_0^{\infty} x\, e^{-2ax}\, dx$$

$$= 2a \left[\frac{1}{2a} \int_0^{\infty} e^{-2ax}\, dx \right]$$

$$= 2a \left(\frac{1}{4a^2} \right)$$

$$= \frac{1}{2a}$$

$$\therefore \boxed{\langle x \rangle = \frac{1}{2a}} \quad \text{Ans.} \quad (1)$$

④

Q.5 (a) For 1D square well potential

$$\psi(x,t) = \sqrt{\frac{2}{L_x}} \sum_{n_x=1}^{\infty} \sin\left(\frac{n_x \pi x}{L_x}\right) e^{-\frac{i E_{n_x} t}{\hbar}}$$

or;

$$\psi_{n_x}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_x \pi x}{a}\right) \quad \Big| \quad \psi_{n_x}(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_x \pi x}{a}\right) e^{-\frac{i E_{n_x} t}{\hbar}}$$

For 3D,

$$\psi_{n_x n_y n_z}(x,y,z) = \sqrt{\frac{8}{abc}} \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{b}\right) \sin\left(\frac{n_z \pi z}{c}\right)$$

$$n_x, n_y, n_z = 1, 2, \ldots, \infty \quad (1)$$

For $L_y = b = 2a$, $L_z = c = 2a$

$$\psi_{n_x n_y n_z} = \sqrt{\frac{2}{a^3}} \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{2a}\right) \sin\left(\frac{n_z \pi z}{2a}\right)$$

$$E_{n_x} = \frac{n_x^2 \pi^2 \hbar^2}{2ma^2} \; ; \; E_{n_y} = \frac{n_y^2 \pi^2 \hbar^2}{8ma^2} \; ; \; E_{n_z} = \frac{n_z^2 \pi^2 \hbar^2}{8ma^2}$$

$$\Rightarrow E_{111} = \frac{\pi^2 \hbar^2}{2ma^2}\left(1 + \frac{1}{4} + \frac{1}{4}\right) = \frac{3\pi^2\hbar^2}{4ma^2} \quad (1)$$

$$E_{121} \text{ or } E_{112} = \frac{7\pi^2\hbar^2}{8ma^2} \quad (1)$$

two-fold degenerate

$$(b)\ \psi(x,t) = \frac{1}{\sqrt{a}}\left[\sin\left(\frac{\pi x}{a}\right)e^{-i\omega t} + \sin\left(\frac{2\pi x}{a}\right)e^{-4i\omega t}\right]$$

$$= \frac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\frac{\pi x}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)e^{-3i\omega t}\right]$$

$$\therefore |\psi|^2 = \frac{1}{a}\left[\sin\left(\frac{\pi x}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)e^{-3i\omega t}\right]\left[\sin\left(\frac{\pi x}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)e^{3i\omega t}\right]$$

$$= \frac{1}{a}\left[\sin^2\left(\frac{\pi x}{a}\right) + \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\left(e^{3i\omega t} + e^{-3i\omega t}\right) + \sin^2\left(\frac{2\pi x}{a}\right)\right]$$

$$= \frac{1}{a}\left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\right]$$

$$\Rightarrow \langle x\rangle = \frac{1}{a}\int_0^a x\left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\right]dx$$

$$= \frac{1}{a}\int_0^a x\sin^2\left(\frac{\pi x}{a}\right)dx + \frac{1}{a}\int_0^a x\sin^2\left(\frac{2\pi x}{a}\right)dx + \frac{2}{a}\cos(3\omega t)\int_0^a x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx$$

$$\Rightarrow (1)\quad I_1 = \frac{1}{a}\int_0^a x\sin^2\left(\frac{\pi x}{a}\right)dx$$

$$= \frac{1}{2a}\int_0^a x\left(1 - \cos\left(\frac{2\pi x}{a}\right)\right)dx$$

$$= \frac{1}{2a}\left[\frac{x^2}{2} - \frac{ax}{2\pi}\sin\left(\frac{2\pi x}{a}\right) + \left(\frac{a}{2\pi}\right)^2\cos\left(\frac{2\pi x}{a}\right)\right]\Big|_0^a$$

$$= \frac{1}{2a}\left[\frac{a^2}{2} - 0 - \cancel{\left(\frac{a}{2\pi}\right)^2} - 0 + 0 + \cancel{\left(\frac{a}{2\pi}\right)^2}\right]$$

$$= \frac{a}{4}$$

$$\Rightarrow (2)\quad I_2 = \frac{1}{a}\int_0^a x\sin^2\left(\frac{2\pi x}{a}\right)dx$$

$$= \frac{1}{2a}\int_0^a x\left(1 - \cos\left(\frac{4\pi x}{a}\right)\right)dx$$

$$\therefore I_2 = \frac{1}{2a}\left[\frac{a^2}{2} - 0 - \cancel{\left(\frac{a}{4\pi}\right)^2} - 0 + 0 + \cancel{\left(\frac{a}{4\pi}\right)^2}\right]$$

$$= \frac{a}{4}$$

$$\Rightarrow (3) \quad I_3 = \frac{2}{a}\cos(3\omega t)\int_0^a x \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx$$

$$= \frac{1}{a}\cos(3\omega t)\int_0^a x\left(\cos\left(\frac{\pi x}{a}\right) - \cos\left(\frac{3\pi x}{a}\right)\right)dx$$

$$= \frac{1}{a}\cos(3\omega t)\int_0^a \left(x\cos\left(\frac{\pi x}{a}\right) - x\cos\left(\frac{3\pi x}{a}\right)\right)dx$$

$$= \frac{1}{a}\cos(3\omega t)\left[\frac{ax}{\pi}\sin\left(\frac{\pi x}{a}\right) + \left(\frac{a}{\pi}\right)^2\cos\left(\frac{\pi x}{a}\right) - \frac{xa}{3\pi}\sin\left(\frac{3\pi x}{a}\right) - \left(\frac{a}{3\pi}\right)^2\cos\left(\frac{3\pi x}{a}\right)\right]\Bigg|_0^a$$

$$= \frac{1}{a}\cos(3\omega t)\left[-\left(\frac{a}{\pi}\right)^2 + \left(\frac{a}{3\pi}\right)^2 - \left(\frac{a}{\pi}\right)^2 + \left(\frac{a}{3\pi}\right)^2\right]$$

$$= \frac{2}{\cancel{a}}\cos(3\omega t)\left[\frac{a^{\cancel{2}}}{9\pi^2} - \frac{a^{\cancel{2}}}{\pi^2}\right]$$

$$= \frac{2a}{\pi^2}\cos(3\omega t)\left(-\frac{8}{9}\right)$$

$$= -\frac{16a}{9\pi^2}\cos(3\omega t)$$

$$\Rightarrow \langle x \rangle = \frac{a}{4} + \frac{a}{4} - \frac{16a}{9\pi^2}\cos(3\omega t)$$

$$= \frac{a}{2} - \frac{16a}{9\pi^2}\cos(3\omega t)$$

$$= \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right] \quad \underline{\text{Ans.}}$$

It oscillates with an angular frequency 3ω.

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(a). For no dispersion, $v_g = v$, i.e., $\frac{dv}{d\lambda} = 0$. 1

Differentiating the expression $v^2 = \frac{g\lambda}{2\pi} + \frac{2\pi\sigma}{\rho\lambda}$, with respect to λ and setting

$\frac{dv}{d\lambda} = 0$ and $\lambda = \lambda_0$ gives $\lambda_0 = 2\pi\sqrt{\sigma / \rho g}$

Substituting the values of g, σ, ρ and λ_0 yields,

$$\lambda_0 = 2\pi\sqrt{\sigma / \rho g} = 2\pi\sqrt{\frac{7.2\times10^{-2}}{1000\times9.8}} = 0.017$$

1

$$\lambda_0 = 1.7\times10^{-2} m = 1.7 cm$$

Thus, waves of average wavelength of 1.7cm do not disperse in water. The group and

phase velocities at λ_0 are equal to each other.

$$v = v_g = \left(\frac{g\lambda_0}{2\pi} + \frac{2\pi\sigma}{\rho\lambda_0}\right)^{1/2}$$ ½

Substituting the values of g, σ, ρ and λ_0 yields,

1

$$v = v_g = \left(\frac{9.8\times0.017}{2\pi} + \frac{2\pi\times7.2\times10^{-2}}{1000\times0.017}\right)^{1/2} = 0.23 m/s$$ ½

Related

$$v = v_g = 23 cms^{-1}$$

(b) For $\lambda << \lambda_0$ the surface tension term dominates, so that

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$$v^2 = \frac{2\pi\sigma}{\rho\lambda} \Rightarrow v = \sqrt{\frac{2\pi\sigma}{\rho\lambda}}$$

Now,

$$v_g = v - \lambda\frac{dv}{d\lambda} = v - \lambda\left(\frac{2\pi\sigma}{\rho}\right)^{1/2}\left(-\frac{1}{2}\right)\lambda^{-3/2}$$

1½

$$= v + \frac{1}{2}\left(\frac{2\pi\sigma}{\rho\lambda}\right)^{1/2} = v + \frac{1}{2}v = 1.5v$$

This is the speed with which ripples (short-wavelength waves) propagate in water.

For

such waves $v_g > v$, indicating that they show anomalous dispersion.

And if $\lambda >> \lambda_0$ the gravity term dominates. The phase velocity of these long wavelength waves is, therefore, given by $v = \left(\frac{g\lambda}{2\pi}\right)^{1/2} \Rightarrow v_g = v - \lambda\frac{dv}{d\lambda} = \frac{v}{2}$ 1½

These waves show normal dispersion, i.e., their phase velocity decrease with decrease in wavelength.

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