① $L = \frac{1}{2}m(\xi^2+\eta^2)(\dot{\xi}^2+\dot{\eta}^2) + \frac{1}{2}m\xi^2\eta^2\dot{\phi}^2$

ⓐ $\therefore$

$$\left.\begin{aligned} p_\xi &= \frac{\partial L}{\partial \dot{\xi}} = m(\xi^2+\eta^2)\dot{\xi} \\ p_\eta &= \frac{\partial L}{\partial \dot{\eta}} = m(\xi^2+\eta^2)\dot{\eta} \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = m\xi^2\eta^2\dot{\phi} \end{aligned}\right\}$$

$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0$

$\therefore$ p_ϕ is conserved.

ⓑ $H = p_\xi\dot{\xi} + p_\eta\dot{\eta} + p_\phi\dot{\phi} - L$

$$= \frac{p_\xi^2}{m(\xi^2+\eta^2)} + \frac{p_\eta^2}{m(\xi^2+\eta^2)} + \frac{p_\phi^2}{m\xi^2\eta^2} - \frac{p_\xi^2}{2m(\xi^2+\eta^2)} - \frac{p_\eta^2}{2m(\xi^2+\eta^2)} - \frac{p_\phi^2}{2m\xi^2\eta^2}$$

$$= \frac{p_\xi^2}{2m(\xi^2+\eta^2)} + \frac{p_\eta^2}{2m(\xi^2+\eta^2)} + \frac{p_\phi^2}{2m\xi^2\eta^2} = \frac{p_\xi^2+p_\eta^2}{2m(\xi^2+\eta^2)} + \frac{p_\phi^2}{2m\xi^2\eta^2}$$

$\Rightarrow$ Hamilton eqns of motion...

$$\dot{p}_\xi = -\frac{\partial H}{\partial \xi} = \frac{(p_\xi^2+p_\eta^2)\xi}{m(\xi^2+\eta^2)^2} + \frac{p_\phi^2}{m\xi^3\eta^2}$$

$$\dot{p}_\eta = -\frac{\partial H}{\partial \eta} = \frac{(p_\xi^2+p_\eta^2)\eta}{m(\xi^2+\eta^2)^2} + \frac{p_\phi^2}{m\xi^2\eta^3}$$

$$\dot{p}_\phi = 0$$

$$\dot{\xi} = \frac{\partial H}{\partial p_\xi} = \frac{p_\xi}{m(\xi^2+\eta^2)}$$

$$\dot{\eta} = \frac{\partial H}{\partial p_\eta} = \frac{p_\eta}{m(\xi^2+\eta^2)}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\xi^2\eta^2}$$

③ $\Psi(x,0) = Ax(x-a)$

$$\Rightarrow (a) \int_0^a A^2 x^2 (x-a)^2 \, dx = 1$$

$$\therefore A^2 \int_0^a x^2(x^2 - 2ax + a^2)\, dx = 1$$

$$\therefore A^2 \int_0^a (x^4 - 2ax^3 + x^2a^2)\, dx = 1$$

$$\therefore A^2 \left[\frac{a^5}{5} - \frac{a^5}{2} + \frac{a^5}{3}\right] = 1$$

$$\therefore \frac{A^2}{30}\left[6a^5 - 15a^5 + 10a^5\right] = 1$$

$$\therefore A^2 = \frac{30}{a^5} \Rightarrow A = \sqrt{\frac{30}{a^5}}$$

$\Rightarrow$ [redacted] $\frac{d|\Psi|^2}{dx} = 0 \Rightarrow \frac{d}{dx}\left(A^2x^2(x-a)^2\right) = 0$

$$\therefore 2x(x-a)^2 + 2x^2(x-a) = 0$$

$$\therefore 2x(x-a)\left[x - a + x\right] = 0$$

$$\therefore 2x(x-a)(2x-a) = 0$$

$$\therefore \boxed{x = \frac{a}{2}}$$

$$\Rightarrow \frac{d^2|\Psi|^2}{dx^2} = 2A^2\left[(x-a)(2x-a) + x(2x-a) + 2x(x-a)\right]$$

$$\therefore \left.\frac{d^2|\Psi|^2}{dx^2}\right|_{x=\frac{a}{2}} = 2A^2\left[-\frac{a}{2}\cancel{2}\left(\frac{a}{\cancel{2}}\right)\right]$$

$$= -\cancel{2} \times \frac{30}{a^5} \times \frac{a^2}{\cancel{8}}$$

$$= -\frac{30}{a^3} < 0$$

$\therefore |\Psi|^2$ is maximum at $x = \frac{a}{2}$.

$\Rightarrow$ (b) $\langle x\rangle = \int_0^a A^2 x^3 (x-a)^2\,dx$

$= \int_0^a \frac{30}{a^5}\left(x^5 - 2ax^4 + a^2x^3\right)dx$

$= \frac{30}{a^5}\left[\frac{a^6}{6} - \frac{2a^6}{5} + \frac{a^6}{4}\right]$

$= 30a\left(\frac{1}{6} - \frac{2}{5} + \frac{1}{4}\right)$

$= \frac{a}{2}\left(10 - 24 + 15\right)$

$= \frac{a}{2}$

$\langle p\rangle = \int_0^a A^2 x(x-a)(-i\hbar)\frac{d}{dx}\left(x(x-a)\right)dx$

$= \int_0^a \frac{30}{a^5} x(x-a)(-i\hbar)(2x-a)\,dx$

$= -i\hbar\frac{30}{a^5}\int_0^a (x^2-ax)(2x-a)\,dx$

$= -i\hbar\frac{30}{a^5}\int_0^a \left(2x^3 - 3ax^2 + a^2x\right)dx$

$= -i\hbar\frac{30}{a^5}\left[\frac{a^4}{2} - a^4 + \frac{a^4}{2}\right]$

$= -i\hbar\frac{15}{a^5}\left[a^4 - 2a^4 + a^4\right]$

$= 0$

$\langle H\rangle = \int_0^a A^2 x(x-a)\left(-\frac{\hbar^2}{2m}\right)\frac{d^2}{dx^2}\left(x(x-a)\right)dx$

$= \int_0^a \frac{30}{a^5} x(x-a)\left(\frac{-\hbar^2}{2m}\right)(2)\,dx$

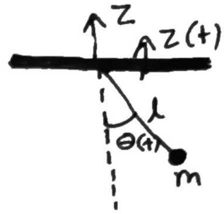
$= -\frac{30\hbar^2}{ma^5}\left[\frac{a^3}{3} - \frac{a^3}{2}\right]$

$= \frac{5\hbar^2}{ma^2}$

(2) $\sin z = a \sin \omega t$

$$\therefore \cos z \, \dot{z} = a\omega \cos \omega t$$

$$\therefore \dot{z} = \frac{a\omega \cos \omega t}{\cos z}$$



$x(t) = l \sin \theta(t)$

$y(t) = -l \cos \theta(t) + z(t)$

$$\Rightarrow \dot{x} = l \cos\theta \, \dot{\theta}$$

$$\dot{y} = l \sin\theta \, \dot{\theta} + \dot{z}$$

$$\therefore L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

$$= \frac{1}{2} m \left(l^2 \cos^2\theta \, \dot{\theta}^2 + l^2 \sin^2\theta \, \dot{\theta}^2 + \dot{z}^2 + 2l \sin\theta \, \dot{\theta} \dot{z}\right) - mg(z - l\cos\theta)$$

$$= \frac{1}{2} m \left(l^2 \dot{\theta}^2 + \frac{a^2\omega^2 \cos^2 \omega t}{\cos^2 z} + 2l \sin\theta \, \dot{\theta} \cdot \frac{a\omega \cos\omega t}{\cos z}\right) - mg(z - l\cos\theta)$$

$$= \frac{1}{2} m \left[l^2 \dot{\theta}^2 + \frac{a^2\omega^2 \cos^2\omega t}{\cos^2 z} + \frac{2la\omega \sin\theta \cos\omega t}{\cos z} \dot{\theta}\right] - mg(z - l\cos\theta)$$

$\Rightarrow$ Eqⁿs of motion---

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

$$\therefore \frac{d}{dt}\left(ml^2\dot{\theta} + \frac{mla\omega \sin\theta \cos\omega t}{\cos z}\right) = \frac{mla\omega \cos\theta \cos\omega t}{\cos z} \dot{\theta} - mgl \sin\theta$$

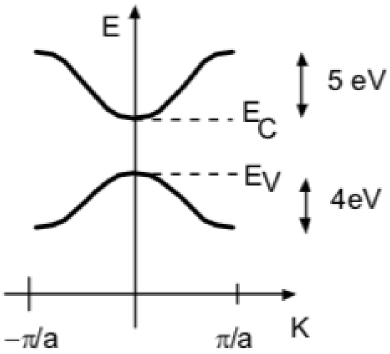
$$\therefore ml^2 \ddot{\theta} + \cancel{\frac{mla\omega \cos\theta \cos\omega t}{\cos z}} \dot{\theta} - \frac{mla\omega^2 \sin\theta \sin\omega t}{\cos z} + \frac{mla\omega \sin\theta \cos\omega t}{\cos^2 z} \sin z \, \dot{z} = \cancel{\frac{mla\omega \cos\theta \cos\omega t}{\cos z}} \dot{\theta} - mgl \sin\theta$$

$$\therefore\ ml^2\ddot{\theta} - \frac{mla\omega^2\sin\theta\sin\omega t}{\cos z} + \frac{mla\omega\sin\theta\cos\omega t}{\cos^2 z}\cdot\frac{a^2\omega\sin\omega t\cos\omega t}{\cos z} + mgl\sin\theta = 0$$

$$\therefore\ \ddot{\theta} - \frac{a\omega^2\sin\theta\sin\omega t}{l\cos z} + \frac{a\omega\sin\theta\cos\omega t}{l\cos^3 z}\cdot a^2\omega\sin\omega t\cos\omega t + \frac{g}{l}\sin\theta = 0$$

$$\therefore\ \ddot{\theta} - \frac{a\omega^2\sin\theta\sin\omega t}{l\cos z} + \frac{a^3\omega^2\sin\theta\sin\omega t\cos^2\omega t}{l\cos^3 z} + \frac{g}{l}\sin\theta = 0$$

Q4.



We use the expression for the effective mass :

$$m^* = \hbar^2 \left(\frac{d^2 E}{dK^2} \right)^{-1} \Bigg|_{K=0}$$

Taking the derivatives:

$$m^* = \hbar^2 \left[\frac{d^2}{dK^2} \left(E_C + E_1 \sin^2(Ka) \right) \right]^{-1} \Bigg|_{K=0}$$

$$= \hbar^2 \left[E_1 \frac{d}{dK} (2a \sin(Ka) \cos(Ka)) \right]^{-1} \Bigg|_{K=0}$$

$$= \hbar^2 \left[E_1 2a \left(-a \sin^2(Ka) + a \cos^2(Ka) \right) \right]^{-1} \Bigg|_{K=0}$$

$$= \frac{\hbar^2}{2 E_1 a^2}$$

$$= \frac{\left(\frac{6.63 \times 10^{-34} Js}{2\pi} \right)^2}{2 \left(5eV \cdot 1.6 \times 10^{-19} J/eV \right) \left(5 \times 10^{-10} m \right)^2}$$

$$= 2.8 \times 10^{-32} kg = 0.03 m_0$$

$$
\begin{aligned}
m^* &= \hbar^2\left[\frac{d^2}{dK^2}\left(E_V - E_2\sin^2(Ka)\right)\right]^{-1}\Bigg|_{K=0} \\
&= \hbar^2\left[E_2\frac{d}{dK}(-2a\sin(Ka)\cos(Ka))\right]^{-1}\Bigg|_{K=0} \\
&= \hbar^2\left[E_2 2a\left(+a\sin^2(Ka) - a\cos^2(Ka)\right)\right]^{-1}\Big|_{K=0} \\
&= \frac{-\hbar^2}{2E_2a^2} \\
&= \frac{-\left(\frac{6.63\times10^{-34}Js}{2\pi}\right)^2}{2\left(4eV\cdot1.6\times10^{-19}J/eV\right)\left(5\times10^{-10}m\right)^2} \\
&= -3.5\times10^{-32}kg = -0.04m_0
\end{aligned}
$$

Q5. (a) At the middle as no. of donors and acceptor ions are same.

(b)

$n_i^2 = N_cN_ve^{-E_g/KT} \sim T^3e^{-E_g/KT}$. Therefore

$$n_i(T_2) = n_i(T_1)\,(T_2/T_1)^{3/2}\exp\left(-\frac{E_g(T_2)}{2KT_2} + \frac{E_g(T_1)}{2KT_1}\right).$$

Now, $E_g(300K) = 1.12eV$, $E_g(77K) = 1.166eV$, $(T_2/T_1)^{3/2} = 0.13$, $exp(-66.14) = 1.88\times10^{-30}$. Then putting the values of n_i, we get $n_i(77K) = 2.57\times10^{20}$.

Q6.

a) Write equations for *n(x)* and *p(x)*.

At $x=0$, $n=n_i$. We will simplify by assuming it to be zero, and take the carrier concentration to be linear from zero to 10^{16}.

$$n_0 = 0 + \frac{\Delta N_D}{\Delta x} x = \frac{1\times10^{16}\,cm^{-3}}{0.5\times10^{-4}\,cm} x = 2.0\times10^{20}\,cm^{-4}\,(x)$$

$$p(x) = \frac{n_i^2}{n(x)} = \frac{n_i^2}{2.0\times10^{20}\,(x)}$$

b) Find the electron diffusion current density.

$$J_{n(diff)} = qD_n \frac{dn(x)}{dx} = qD_n\left(2.0\times10^{20}\,cm^{-4}\right)$$

For this doping range, the mobility μ_n is fairly constant, and thus so is D_n.

From Figure 3.11 $D_n = 30\ cm^2/s$.

$$J_{n(diff)} = qD_n\left(2.0\times10^{20}\,cm^{-4}\right) = \left(1.6\times10^{-19}C\right)\left(30\frac{cm^2}{V\cdot s}\right)\left(2.0\times10^{20}\,cm^{-4}\right) = 960A/cm^2$$

The hole diffusion coefficient is, from Figure 3.11 for minority carriers, about D_p =12 cm^2/V-s .

From Equation (3.41) the hole diffusion current is given by

$$J_{p(diff)} = -qD_p \frac{dp(x)}{dx} = -qD_p \left(\frac{n_i^2}{2.0\times10^{20}}\right)\left[\frac{d}{dx}\left(\frac{1}{x}\right)\right]$$

$$= -qD_p \left(\frac{n_i^2}{2.0\times10^{20}}\right)\left(-x^{-2}\right)$$

This equation is only valid once N_D becomes noticeably greater than n_i, however, which becomes clear if we try to plug in *x=0* and the equation blows up. At *x*=0, in reality the hole concentration is also n_i and not infinite It does, however, decline rapidly (as $1/x^2$) as the electron concentration increases. Clearly for small *x* the minority carriers do indeed contribute a large amount of diffusion current.

At *x*=0.5µm, the hole diffusion current is

$$J_{p(diff)} = -qD_p \left(\frac{n_i^2}{2.0\times10^{20}}\right)\left(-x^{-2}\right)$$

$$= \left(1.6\times10^{-19}C\right)\left(12cm^2/Vs\right)\left[\frac{\left(1.08\times10^{10}cm^{-3}\right)^2}{2.0\times10^{20}cm^{-3}}\right]\frac{1}{\left(0.5\times10^{-4}cm\right)^2}$$

$$= 4.5\times10^{-10}\,A/cm^2$$

$$E_C(x) - E_f = -kT\ln\left(\frac{n(x)}{N_C}\right) = -0.026eV\ln\left(\frac{2.0\times10^{20}x}{2.86\times10^{19}}\right)$$

We recognize that at *x=0*, the carrier concentration is n_i and E_C-E_f =0.56, and at *x*=0.5µm, E_C-E_f=0.2 eV. The Fermi level varies logarithmically from E_C-E_f=0.56 to 0.2 eV.

where g is an integer. Because of the symmetry of the ring we look for eigenfunctions ψ such that

$$\psi(x + a) = C\psi(x), \tag{13.3}$$

where C is a constant. Then

$$\psi(x + ga) = C^g\psi(x); \tag{13.4}$$

and, if the eigenfunction is to be single-valued,

$$\psi(x + Na) = \psi(x) = C^N\psi(x), \tag{13.5}$$

so that C is one of the N roots of unity, or

$$C = e^{i2\pi g/N}; \qquad g = 0, 1, 2, \cdots, N - 1. \tag{13.6}$$

We have then

$$\psi(x) = e^{i2\pi xg/Na}u_g(x) \tag{13.7}$$

as a satisfactory solution, where $u_g(x)$ has periodicity a. Letting

$$k = 2\pi g/Na, \tag{13.8}$$

we have

$$\psi = e^{ikx}u_k(x), \tag{13.9}$$

which is the Bloch result.

KRONIG-PENNEY MODEL

We demonstrate some of the characteristic features of electron propagation in crystals by considering the periodic square-well struc-

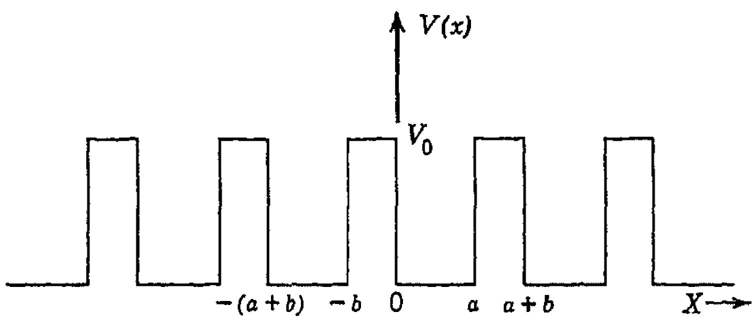


Fig. 13.3. Kronig and Penney one-dimensional periodic potential.

ture[2] in one dimension (Fig. 13.3). The wave equation of the problem is

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(W - V)\psi = 0. \tag{13.10}$$

[2] R. de L. Kronig and W. G. Penney, Proc. Roy. Soc., (London) **A130**, 499 (1930); see also D. S. Saxon and R. A. Hutner, Philips Research Repts. **4**, 81 (1949); J. M. Luttinger, Philips Research Repts. **6**, 303 (1951).

The running wave solutions will be of the form of a plane wave modulated with the periodicity of the lattice. Using (12.4) and (12.5) for plane waves, we obtain solutions of the form

$$\psi = u_k(x)e^{ikx}, \tag{13.11}$$

where $u(x)$ is a periodic function in x with the period $(a + b)$ and is determined by substituting (13.11) into (13.10):

$$\frac{d^2u}{dx^2} + 2ik\frac{du}{dx} + \frac{2m}{\hbar^2}(W - W_k - V)u = 0, \tag{13.12}$$

where $W_k = \hbar^2k^2/2m$.

In the region $0 < x < a$ the equation has the solution

$$u = Ae^{i(\alpha-k)x} + Be^{-i(\alpha+k)x}, \tag{13.13}$$

provided that

$$\alpha = (2mW/\hbar^2)^{1/2}. \tag{13.14}$$

In the region $a < x < a + b$ the solution is

$$u = Ce^{(\beta-ik)x} + De^{-(\beta+ik)x}, \tag{13.15}$$

provided that

$$\beta = [2m(V_0 - W)/\hbar^2]^{1/2}. \tag{13.16}$$

The constants A, B, C, D are to be chosen so that u and du/dx are continuous at $x = 0$ and $x = a$, and by the periodicity required of $u(x)$ the values at $x = a$ must equal those at $x = -b$. Thus we have the four linear homogeneous equations:

$$A + B = C + D;$$

$$i(\alpha - k)A - i(\alpha + k)B = (\beta - ik)C - (\beta + ik)D;$$

$$Ae^{i(\alpha-k)a} + Be^{-i(\alpha+k)a} = Ce^{-(\beta-ik)b} + De^{(\beta+ik)b};$$

$$i(\alpha - k)Ae^{i(\alpha-k)a} - i(\alpha + k)Be^{-i(\alpha+k)a} = (\beta - ik)Ce^{-(\beta-ik)b} - (\beta + ik)De^{(\beta+ik)b}.$$

These have a solution only if the determinant of the coefficients vanishes, or[3]

$$\frac{\beta^2 - \alpha^2}{2\alpha\beta}\sinh\beta b\sin\alpha a + \cosh\beta b\cos\alpha a = \cos k(a + b). \tag{13.17}$$

[3] Before verifying this for himself the reader should refer to the alternative derivation in the following section.

In order to obtain a handier equation we represent the potential by a periodic delta function, passing to the limit where $b = 0$ and $V_0 = \infty$ in such a way that $\beta^2 b$ stays finite. We set

$$\lim_{\substack{b \to 0 \\ \beta \to \infty}} \frac{\beta^2 ab}{2} = P, \tag{13.18}$$

so that the condition (13.17) becomes

$$P \frac{\sin \alpha a}{\alpha a} + \cos \alpha a = \cos ka. \tag{13.19}$$

This transcendental equation must have a solution for α in order that wave functions of the form (13.11) should exist.

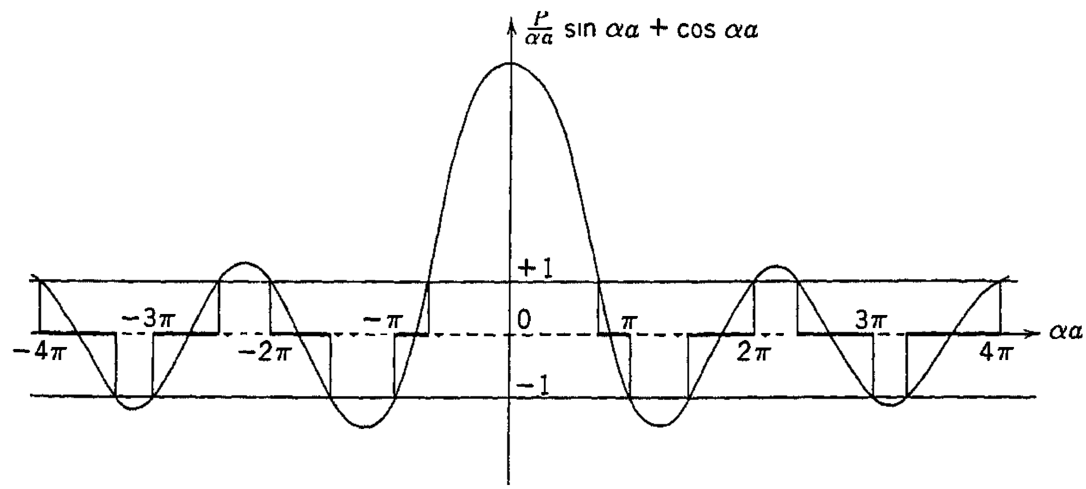


Fig. 13.4. Plot of the function $P \dfrac{\sin \alpha a}{\alpha a} + \cos \alpha a$, for $P = 3\pi/2$. The allowed values of the energy W are given by those ranges of $\alpha = [2mW/\hbar^2]^{1/2}$ for which the function lies between +1 and −1. (After Kronig and Penney.)

In Fig. 13.4 we have plotted the left side of (13.19) as a function of αa, for the arbitrary value $P = 3\pi/2$. As the cosine term on the right side can have values only between +1 and −1, only those values of αa are allowed for which the left side falls in this range. The allowed ranges of αa are drawn heavily in the figure, and through the relation $\alpha = [2mW/\hbar^2]^{1/2}$ they correspond to allowed ranges of the energy W. The boundaries of the allowed ranges of αa correspond to the values $n\pi/a$ for k. In Fig. 13.5 W vs. k is plotted.

If P is small, the forbidden ranges disappear. If $P \to \infty$, the allowed ranges of αa reduce to the points $n\pi$ ($n = \pm 1, \pm 2, \cdots$). The energy spectrum becomes discrete, and the eigenvalues

$$W = n^2 h^2 / 8ma^2$$

are those of an electron in a box of length a.