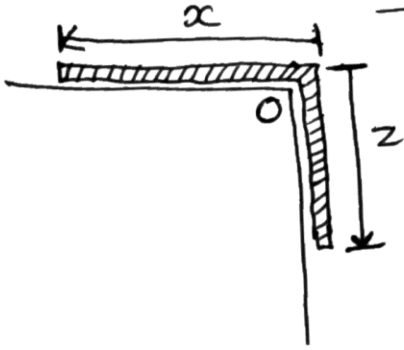
Classical Mechanics (Quiz solution)

(1)



The generalized coordinates are x and z as defined in the question.

(A) <u>Constraint</u>: The length of the rope is fixed.

$$\therefore x + z = l \Rightarrow \dot{x} = -\dot{z} \quad \text{— (1)}$$

⇒ Suppose the mass is uniformly distributed with linear mass density λ.

where, $\lambda = \frac{m}{l}$

⇒ Kinetic energy:

$$T = \frac{1}{2} m_x \dot{x}^2 + \frac{1}{2} m_z \dot{z}^2$$

where, $m_x \rightarrow$ mass of portion of rope of length x

$\therefore m_x = \lambda x$

$m_z \rightarrow$ mass of portion of rope of length z

$\therefore m_z = \lambda z$

$$= \frac{1}{2}\lambda x \dot{x}^2 + \frac{1}{2}\lambda z \dot{z}^2$$

$$= \frac{1}{2}\lambda (x+z)\dot{z}^2 \quad (\because \dot{x} = -\dot{z})$$

$$= \frac{1}{2}\lambda l \dot{z}^2 \quad (\because x + z = l)$$

$$= \frac{1}{2} m \dot{z}^2 \quad (\because m = \lambda l) \quad \text{— (2)}$$

⇒ Potential energy with $V = 0$ at $z = 0$.

$$\cancel{\int_0^V dV' = -\int_0^z m_{z'} g \, dz' \quad (\because dV' =}$$

$$dV = -(dm) g z$$

$$\therefore dV = -(\lambda dz) g z$$

$$\therefore \int dV = -\lambda g \int z \, dz$$

$$\therefore V = -\frac{1}{2}\lambda g z^2$$

$$= -\frac{1}{2}\frac{mg}{l} z^2 \quad \text{— (3)}$$

$\Rightarrow$ Lagrangian: $L = T - V$

$$\therefore L = \frac{1}{2} m \dot{z}^2 + \frac{1}{2} \frac{mg}{l} z^2 \quad \text{---(4)}$$

(B) $\Rightarrow$ Euler-Lagrange eqn: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0.$

$$\therefore \frac{d}{dt}(m\dot{z}) - \frac{mg}{l} z = 0$$

$$\therefore \ddot{z} - \frac{g}{l} z = 0$$

$$\therefore \boxed{\ddot{z} - \omega^2 z = 0} \quad \left(\because \omega^2 \equiv \frac{g}{l}\right) \quad \text{---(5)}$$

$\Rightarrow$ Solution:

$$\ddot{z} - \omega^2 z = 0$$

$$\therefore \ddot{z} + (i\omega)^2 z = 0$$

$$\therefore \frac{d^2 z}{dt^2} + (i\omega)^2 z = 0 \quad \text{---(6)}$$

This is the equation of simple Harmonic oscillator whose solution is known (only frequency is complex).

$$\therefore \boxed{z(t) = A\cos(i\omega t) + B\sin(i\omega t)} \quad \text{---(7)}$$

$\Rightarrow$ Initial conditions:
$$\left.\begin{aligned} t = 0,\ & z = z_0 \\ t = 0,\ & \dot{z} = 0 \end{aligned}\right\} \quad \text{---(8)}$$

$$\therefore z(0) = z_0 = A\cos 0 + B\sin 0$$
$$= A \quad \Rightarrow \boxed{A = z_0}$$

$$\therefore \dot{z}(t) = -Ai\omega\sin(i\omega t) + Bi\omega\cos(i\omega t)$$

$$\therefore \dot{z}(0) = 0 = -Ai\omega\sin 0 + Bi\omega\cos 0$$

$$\therefore 0 = 0 + Bi\omega$$

$$\therefore \boxed{B = 0}$$

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$\therefore z(t) = z_0 \cos(i\omega t)$

$= z_0 \cosh(\omega t)$ — 9

$\left(\because \cos(i\omega t) = \frac{e^{i(i\omega t)} + e^{-i(i\omega t)}}{2} = \frac{e^{-\omega t} + e^{\omega t}}{2} = \frac{e^{\omega t} + e^{-\omega t}}{2} = \cosh(\omega t)\right)$

$\uparrow$ Hyperbolic funcn

(c) Velocity at time t:

$\dot{z} = z_0 \omega \sinh(\omega t)$

$= z_0 \omega \sqrt{\cosh^2 \omega t - 1}$

$(\because \cosh^2 \omega t - \sinh^2 \omega t = 1)$

$\therefore \dot{z} = z_0 \omega \sqrt{\frac{z^2}{z_0^2} - 1}$ $\quad (\because z = z_0 \cosh \omega t)$

$\therefore \boxed{\dot{z} = \omega \sqrt{z^2 - z_0^2}}$ — 10

when the rope leaves the surface of the table, $z = l$

$\therefore \boxed{\dot{z} = \omega \sqrt{l^2 - z_0^2}}$ — 11

(d) Now, $\frac{z}{z_0} = \cosh \omega t = \frac{e^{\omega t} + e^{-\omega t}}{2}$

$\therefore e^{\omega t} + e^{-\omega t} = \frac{2z}{z_0}$

$\therefore (e^{\omega t})^2 + 1 = \frac{2z}{z_0} e^{\omega t}$

$\therefore (e^{\omega t})^2 - \frac{2z}{z_0} e^{\omega t} + 1 = 0$ — 12

$\therefore e^{\omega t} = \frac{\frac{2z}{z_0} \pm \sqrt{\frac{4z^2}{z_0^2} - 4}}{2}$

$= \frac{z}{z_0} \pm \frac{1}{z_0} \sqrt{z^2 - z_0^2}$

$$\therefore \omega t = \ln\left[\frac{z}{z_0} + \frac{1}{z_0}\sqrt{z^2 - z_0^2}\right]$$

$$\therefore t = \frac{1}{\omega}\ln\left\{\frac{z}{z_0} + \sqrt{\frac{z^2}{z_0^2} - 1}\right\}$$

$\Rightarrow$ At $t = \tau$, $z = l$.

$$\therefore \tau = \frac{1}{\omega}\ln\left\{\frac{l}{z_0} + \sqrt{\frac{l^2}{z_0^2} - 1}\right\}$$

$$\therefore \boxed{\tau = \sqrt{\frac{l}{g}}\ln\left\{\frac{l}{z_0} + \sqrt{\frac{l^2}{z_0^2} - 1}\right\}} \quad \text{— (13)}$$

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(2) (A) Let us consider any arbitrary curve between two points $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ in plane-polar coordinates.

The small arc length of the curve is

$$ds = \sqrt{dr^2 + r^2 d\theta^2} \quad \text{— (14)}$$

Therefore, the length of the curve is

$$l = \int_C ds = \int_C \sqrt{dr^2 + r^2 d\theta^2}$$

$$\therefore l = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

$$= \int_{r_1}^{r_2} \sqrt{1 + r^2\theta'^2}\, dr \quad \text{where, } \theta' \equiv \frac{d\theta}{dr}$$

(15)

$\Rightarrow$ Euler-Lagrange eqn: $\frac{d}{dr}\left(\frac{\partial F(\theta, \theta', r)}{\partial \theta'}\right) - \frac{\partial F(\theta, \theta', r)}{\partial \theta} = 0$

where, $F(\theta, \theta', r) = \sqrt{1 + r^2\theta'^2}$

$$\therefore \frac{d}{dr}\left(\frac{r^2\theta'}{\sqrt{1 + r^2\theta'^2}}\right) - 0 = 0 \Rightarrow \frac{r^2\theta'}{\sqrt{1 + r^2\theta'^2}} = C \quad \text{— (16)}$$

$\uparrow$ constant

$$\therefore r^4\theta'^2 = c^2(1 + r^2\theta'^2)$$

$$\therefore r^4\theta'^2 - c^2r^2\theta'^2 = c^2$$

$$\therefore r^2\theta'^2(r^2 - c^2) = c^2$$

$$\therefore \theta' = \frac{c}{r\sqrt{r^2-c^2}}$$

$$\therefore \frac{d\theta}{dr} = \frac{c}{r\sqrt{r^2-c^2}} \quad \text{— (17)}$$

suppose $u = \sqrt{r^2-c^2}$

$$\therefore du = \frac{r\,dr}{\sqrt{r^2-c^2}}$$

$$\therefore d\theta = \frac{c}{r^2}\cdot\frac{r\,dr}{\sqrt{r^2-c^2}}$$

$$\therefore d\theta = \frac{c\,du}{u^2+c^2}$$

$$\therefore \int d\theta = \int \frac{c\,du}{u^2+c^2}$$

$$\therefore \theta = \frac{c}{c}\tan^{-1}\left(\frac{u}{c}\right) + c_1$$

↑ constant

$$\therefore (\theta - c_1) = \tan^{-1}\left(\frac{\sqrt{r^2-c^2}}{c}\right)$$

$$\therefore \tan(\theta - c_1) = \frac{\sqrt{r^2-c^2}}{c}$$

$$\therefore r^2 - c^2 = c^2\tan^2(\theta - c_1)$$

$$\therefore r^2 = c^2\{1 + \tan^2(\theta - c_1)\}$$

$$\therefore r^2 = c^2\sec^2(\theta - c_1)$$

$$\therefore r^2\cos^2(\theta - c_1) = c^2$$

$$\therefore r\cos(\theta - c_1) = c$$

$$\therefore r\{\cos\theta\cos c_1 + \sin\theta\sin c_1\} = c$$

$$\therefore \boxed{r\cos\theta\cos c_1 + r\sin\theta\sin c_1 = c}$$

Which is a straight line equation.

$$\because r\cos\theta = x$$
$$r\sin\theta = y$$

$$\therefore \boxed{x\cos c_1 + y\sin c_1 = c}$$

(b) $L = \frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} - x\dot{y}$

$\Rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x}$

$p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} - x \Rightarrow \dot{y} = p_y + x$

$$\Rightarrow H = p_x \dot{x} + p_y \dot{y} - L$$

$$= p_x \cdot p_x + p_y(p_y + x) - \left\{ \frac{p_x^2}{2} + \frac{(p_y + x)^2}{2} - x(p_y + x) \right\}$$

$$= p_x^2 + p_y^2 + xp_y - \frac{p_x^2}{2} - \frac{(p_y + x)^2}{2} + xp_y + x^2$$

$$= \frac{p_x^2}{2} + p_y^2 + 2xp_y - \frac{p_y^2}{2} - xp_y - \frac{x^2}{2} + x^2$$

$$= \frac{p_x^2}{2} + \frac{p_y^2}{2} + xp_y + \frac{x^2}{2}$$

$$= \frac{p_x^2}{2} + \frac{(p_y + x)^2}{2}$$

$\Rightarrow$ Lagrangian is independent of y. Thus, $p_y = \frac{\partial L}{\partial \dot{y}} \Rightarrow \dot{p}_y = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right.$

$\therefore \dot{p}_y = \frac{\partial L}{\partial y} = 0$

$\Rightarrow p_y$ is conserved.

$\Rightarrow$ since L does not ~~depend~~ explicitly depend on time, $\frac{\partial L}{\partial t} = 0$

$\therefore \frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0 \Rightarrow H$ is conserved.