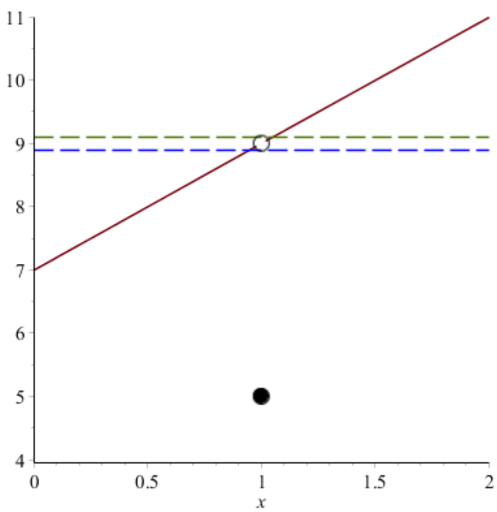
LIMITS cont…

The Precise Definition of a Limit (Section 2.5) cont…

Recall: Last day, we discussed the concept of finding the limit $\lim_{x \to a} f(x) = L$ and determining how close you'd need to be to $x = a$ in order to ensure that you could be sufficiently close to $f(x) = L$.

Example: $f(x) = \begin{cases} 2x+7, & x \neq 1 \\ 5, & x = 1 \end{cases}$

Suppose we want to be within 0.1 away from the value of the limit…how close should we get to $x = 1$?
Formally, we want to find a small number, which we'll call δ, so that
$0 < |x-1| < \delta$ guarantees that $|f(x) - 9| < 0.1$.



Now suppose we want to be within 0.01 of the limit…how close must we be to $x = 1$?

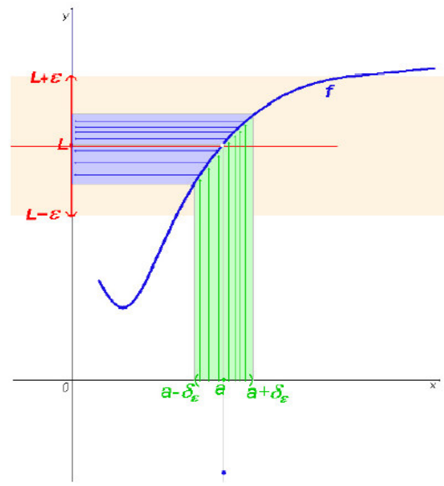
Now, we could keep playing this game, requiring $f(x)$ to be even closer to the limit, but the point is for us to be able to do it for any small distance away, say ε.

Is the choice of δ unique?

Definition: Let f be defined for all $x \neq a$ in an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.



One can also do precise definitions for sided limits in a similar way…refer to definitions 3 and 4 in your text (the only difference is the inequality for *x*). Also, infinite limits can be defined formally in a similar manner…refer to definitions 6 and 7.

Why is this formal definition helpful? Well, it can help us prove/derive the limit laws so we can calculate limits more easily.

Enrichment Example: Use the precise definition of a limit to prove the constant multiple rule, $\lim_{x \to a}[c \cdot f(x)] = c \cdot \lim_{x \to a} f(x)$

If $c=0$, this is true from constant rule (easier to prove, so let's focus on $c \neq 0$).

Let $\varepsilon > 0$

Since we're told $\lim_{x \to a} f(x)$ exists (let's call it L), there is a $\delta_1 > 0$ so that $|f(x) - L| < \frac{\varepsilon}{|c|}$ when $0 < |x-a| < \delta_1$

using $\delta = \delta_1$, if $0 < |x-a| < \delta$,

$$\begin{aligned} |cf(x) - cL| &= |c(f(x) - L)| \\ &= |c|\,\underline{|f(x) - L|} \\ &< \cancel{|c|}\,\frac{\varepsilon}{\cancel{|c|}} \\ &= \varepsilon \quad \checkmark \end{aligned}$$

Done! ☺

Application: According to Ohm's law, when a voltage of V volts is applied across a resistor with a resistance of R ohms, a current of $I = \dfrac{V}{R}$ amperes flows through the resistor. Suppose that the voltage remains constant at 3.0 volts, but temperature variations cause the resistance to vary from a value of 7.5 ohms. If the current is not allowed to vary by more than ± 0.001 ampere, answer the following questions:

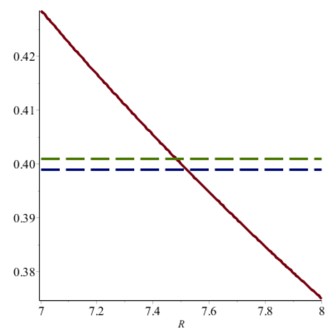
a) In the context of the delta-epsilon definition of $\lim_{x \to a} f(x) = L$, what are x, $f(x)$, a, L, ε, and δ? Complete the table below.

b) What variation of $\pm\delta$ from the value of 7.5 ohms is allowable for the resistance?

[Source: Modified from "Calculus: Early Transcendentals" 8th ed. by H. Anton, I. Bivens, and S. Davis, 2005]

a)

δ - ε definition	This application
x	
$f(x)$	
a	
L	
ε	
δ	



b)

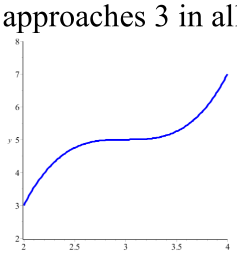
$R_{ideal} = 7.5$

$R_{\max} \approx 7.518797$

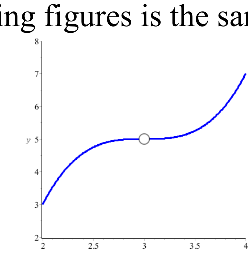
$R_{\min} \approx 7.481297$

Continuity (Section 2.4)

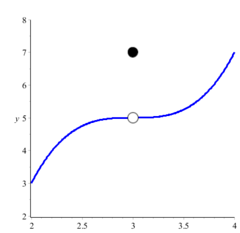
Recall: Last class, we looked at limits and found that, for example, the limit as x approaches 3 in all of the following figures is the same.



$f(x)$



$g(x)$



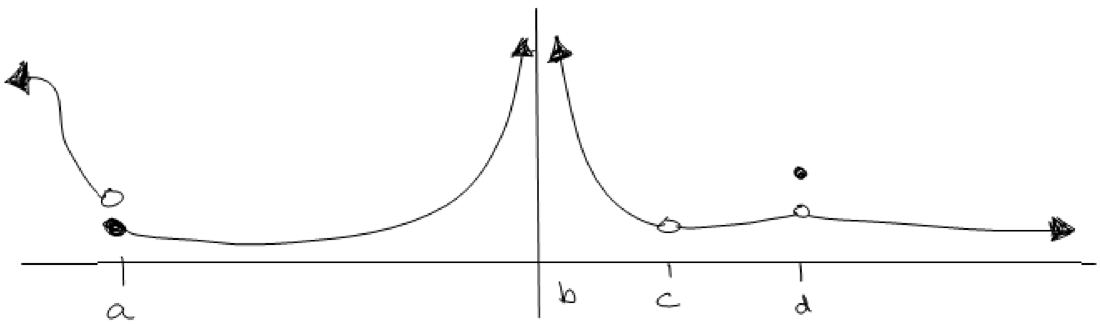
$h(x)$

Clearly, however, the 3 graphs do NOT represent the same function…so what's the difference?

Definition: A function f is **continuous** at a number a if $\lim_{x \to a} f(x) = f(a)$

So, what actually has to hold?

Graphical Example:



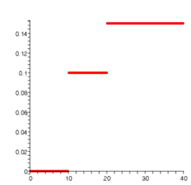
What happens if we are given a formula for the function?…how do we determine where the function is continuous?

Example: $f(x) = \dfrac{x^2 + 5x + 4}{x + 1}$

Example: $f(x) = \begin{cases} \dfrac{1}{x-3} & x \neq 3 \\ 6 & x = 3 \end{cases}$

Application: Suppose that a certain country has the following income tax rates, where x is income in thousands of dollars.

$$f(x) = \begin{cases} 0 & x < 10 \\ 0.1 & 10 \leq x \leq 20 \\ 0.15 & x > 20 \end{cases}$$



Definition: A function f is **continuous from the right** at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left** at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Graphical Example:

Example: $f(x) = \begin{cases} 4x+1 & x \le 0 \\ -2x+7 & x > 0 \end{cases}$

So far, we've only talked about continuity at a point, but what do we mean when we say a function is continuous on e.g. [0, 6]?

Definition: A function f is **continuous on an interval** if it is continuous at every number in the interval. (If it is defined only on one side of an endpoint of the interval, "continuous" at the endpoint means "continuous from the right" or "continuous from the left")

To help us determine if a function is continuous, the following two theorems will help:

Theorem: The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions
- Exponential Functions
- Logarithmic Functions

Theorem: If f and g are continuous at a and c is a constant, then the following are also continuous at a:

1. $f+g$
2. $f-g$
3. $c \cdot f$
4. $f \cdot g$
5. $\dfrac{f}{g}$ if $g(a) \ne 0$
6. $f(g(x))$

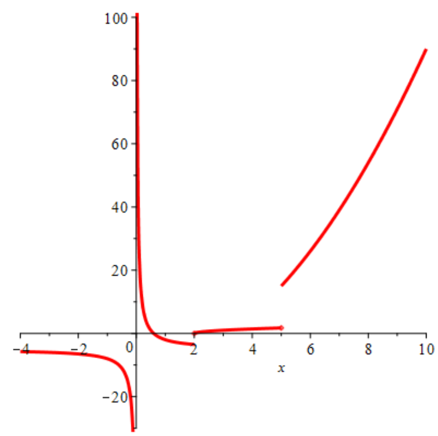
Example: On what interval is the function $h(x)=\dfrac{4}{x-6}+\sqrt{x-2}-\dfrac{1}{x^2+1}+\sin x$ continuous?

Application: Suppose a neuron has the following response to inputs: if it receives a voltage input V greater than or equal to a threshold of V_o, it outputs a voltage of kV for some constant k. If it receives an input less than the threshold value of V_o, it outputs a fixed voltage V^*. What would k have to be to make the function continuous?

$$f(V)=\begin{cases} V^* & V<V_0 \\ kV & V\geq V_0 \end{cases}$$

[Source: Modified from "Modeling the Dynamics of Life: Calculus and Probability for Life Scientists", Frederick R. Adler, 1998]

Example: Given $f(x)=\begin{cases} \dfrac{3}{x}-5 & x<2 \\ \sqrt{x-2} & 2\leq x\leq 5 \\ x^2-10 & x>5 \end{cases}$, which of the following are true?



Exercise: Explain your answers.

- ☐ $f(x)$ is continuous when $x<2$.
- ☐ $f(x)$ is continuous at $x=2$.
- ☐ $f(x)$ is continuous from the left at $x=5$.
- ☐ $f(x)$ is continuous when $x>5$.