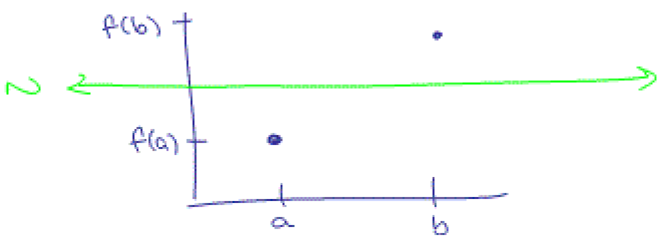
LIMITS cont…

Continuity (Section 2.4) cont…

Recall: Last day, we studied continuity. Now that we understand it, we can study a very important theorem that helps us determine if solutions exist to equations.

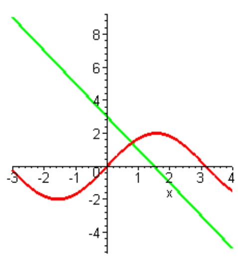
The Intermediate Value Theorem (IVT): Let f be continuous on a closed interval $[a,b]$. If z is any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, then there exists a number c in (a,b) such that $f(c) = N$.

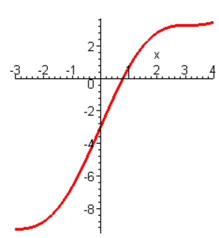


Challenge: Can you go from one point to the other without lifting your pen off the page, and without crossing the green line?

This is useful for proving the existence of roots of an equation! If your function is continuous and it's positive at one point and negative at another, then it must cross the *x*-axis somewhere between these points.

Example: Prove that $2\sin(x) = 3 - 2x$ has a root in $[0, 1]$.

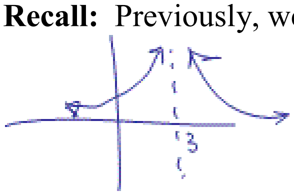




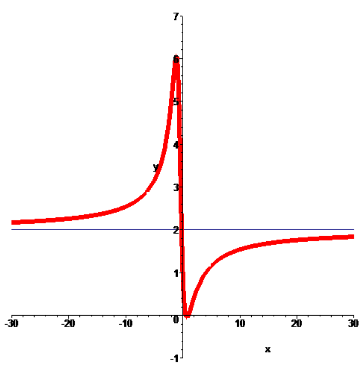
Example: Prove that there exists a positive number c such that $c^2 = 2$ (this proves the existence of $\sqrt{2}$).

Limits at Infinity and Asymptotes (Section 4.6)

Recall: Previously, we talked about infinite limits and vertical asymptotes.



Horizontal asymptotes (HA), on the contrary, are based on the behaviour as $x \to \infty$ and $x \to -\infty$.



x	f(x)
10	1.540541
100	1.950401
1000	1.995004
10000	1.9995
100000	1.99995
1000000	1.999995

$\lim_{x\to\infty} f(x) =$ _____

$\lim_{x\to-\infty} f(x) =$ _____

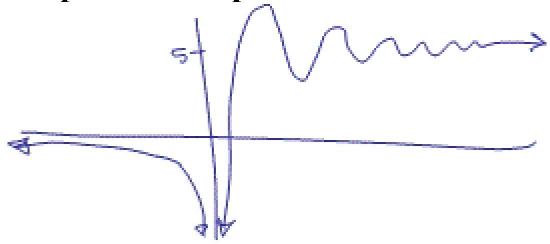
Definition: Let f be a function defined on some interval (a,∞). If the values of $f(x)$ become arbitrarily close to L as x becomes sufficiently large we say the function f has a **limit at infinity** and write

$$\lim_{x\to\infty} f(x) = L .$$

[Similarly, we can define $\lim_{x\to-\infty} f(x) = L$]

Definition: If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote (H.A.)** of f.

Graphical Example:



Now, suppose we aren't given the graph … let's try to compute the limits at infinity.

Example: $f(x) = \frac{1}{x^2}$ $\lim_{x \to \infty} \frac{1}{x^2} =$ _____ $\lim_{x \to -\infty} \frac{1}{x^2} =$ _____

Question: Does $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ always approach a particular value L?

Theorem: If $r > 0$ is a rational number, then $\lim_{x \to \infty} \frac{1}{x^r} = 0$ and if $r > 0$ is a rational number such that x^r is defined for all x, then $\lim_{x \to -\infty} \frac{1}{x^r} = 0$.

Application: Suppose that the diameter of an animal's pupils is given by $f(x)$ mm, where x is the intensity of light on the pupils. If $f(x) = \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15}$, find the diameter of the pupils with maximum light. NOTE: Last week, we did the same example with minimum light (i.e. $x \to 0$)

[Source: "Calculus: Concepts & Connections" by R. Smith and R. Minton, 2006]

Example: $\lim_{x \to \infty} \frac{7x^2 - 2x}{5x^2 + 9}$

Note: Our strategy for infinite limits of rational functions is to divide by the largest power in the denominator!

Shortcut: Consider the dominant powers of x in the numerator and denominator!

Example: $\lim_{x\to\infty} \frac{2-5x^2+7x^6}{8-3x^3-4x^6}$ $\qquad \lim_{x\to\infty} \frac{5x^2-1}{3x-7}$ $\qquad \lim_{x\to\infty} \frac{3x^5-6x^3+1}{4x^7-8}$

= __________ = __________ = __________

Now let's try some trickier ones…for these next 2, you'll need to remember that

$$\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Example: $\lim_{x\to\infty} \frac{5x-2}{\sqrt{x^2+4}}$

What happens if we instead consider the limit as $x \to -\infty$? i.e. $\lim_{x\to-\infty} \frac{5x-2}{\sqrt{x^2+4}}$

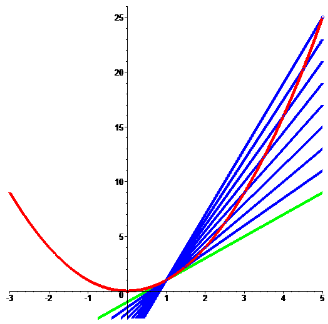
Example: $\lim_{x\to\infty} \sqrt{3x^2+6} - \sqrt{3x^2-2}$

Example: $\lim_{x\to\infty} x^8 - x^6$

DERIVATIVES

Defining the Derivative and The Derivative as a Function (Sections 3.1/3.2)

Recall: We introduced the limit concept to find slopes of tangent lines. Now that we have techniques for computing limits, let's return to this original task.



Want: Slope at a single point (green line)

Problem: We need 2 points to find slope (here, we only have 1)

Solution:

Maple Exploration: Go to Tools $\rightarrow$ Math Apps $\rightarrow$Calculus $\rightarrow$ Derivative Definition

In applications, $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ is the **average rate of change** of y with respect to x over the interval $[x_1, x_2]$, while the **instantaneous rate of change** is $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.

Recall: In high school, you learned that instantaneous rate of change is the slope of the tangent and we give this a special name: derivative.

Definition: The **derivative of a function *f***, is a new function, denoted by $f'(x)$, and is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

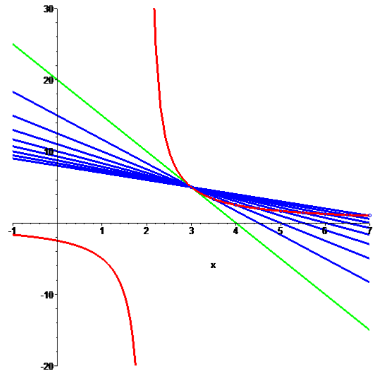
if this limit exists. This is referred to as the "**First Principles Definition**".

So to summarize, we have:

- tangent to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ with slope $f'(a)$
- the derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

Application: The temperature in an oven during a self-cleaning cycle after x hours is $T(x)$. What is the meaning of $T'(3.5) = -2$?

Example: If $f(x) = \dfrac{5}{x-2}$, find $f'(x)$. What is $f'(3)$?



Notation: If $y = f(x)$, then

$$f'(x) \quad = \quad y' \quad = \quad \frac{dy}{dx} \quad = \quad \frac{df}{dx} \quad = \quad \frac{d}{dx}f(x) \quad = \quad Df(x) \quad = \quad D_x f(x)$$