FUNCTIONS AND GRAPHS

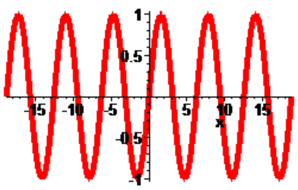
NOTE: Remember to do the posted reading on sections 1.1-1.5…it's important high school review, and you WILL be tested on it…for more practice, check out the pre-calc resources at www.nool.ca and do the homework posted in the pre-calc checklist.

Inverse Functions (Section 1.4)

Inverse trigonometric functions:

Recall: $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

So, what if you're asked to solve $\sin(x) = \frac{1}{2}$?

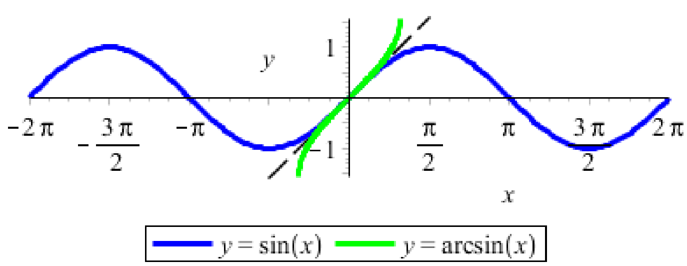


Notation: In the above example, we want $x = \sin^{-1}\left(\frac{1}{2}\right)$, also written as $x = \arcsin\left(\frac{1}{2}\right)$

Caution: It is important to understand that $\sin^{-1}(x)$ is NOT the same as $(\sin(x))^{-1}$!!

To find an inverse, we must restrict the domain to ensure that the function is one-to-one.

$$\boxed{\sin^{-1}(x) = y \Leftrightarrow \sin y = x \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}}$$

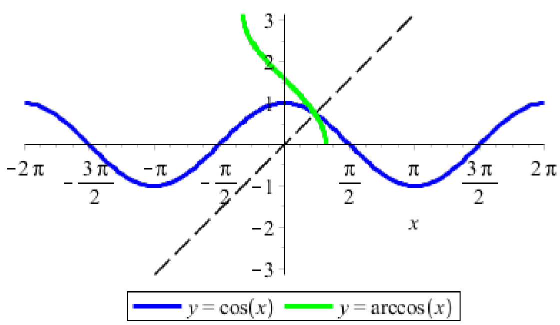


Application: A compound lens is made from crown glass (n = 1.52) bonded to flint glass (n = 1.89). What is the critical angle for light incident on the flint-crown interface? [Some physics background: Refraction is the bending of light at an interface between transparent mediums. Total internal reflection occurs when light propagating in a medium with refractive index n_1 is incident on an interface at an angle greater than the critical angle θ_c given by $\sin(\theta_c) = \frac{n_2}{n_1}$.]

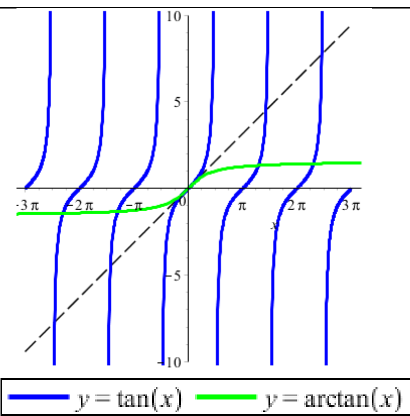
[Source: "Physics with Modern Physics for Scientists and Engineers" by R. Wolfson and J. Pasachoff, 1995]

Similarly, we restrict the domains of $\cos(x)$ and $\tan(x)$ as follows:

$\boxed{\cos^{-1}(x) = y \Leftrightarrow \cos y = x, \quad 0 \le y \le \pi}$



$\boxed{\tan^{-1}(x) = y \Leftrightarrow \tan y = x, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}}$



It is also possible to find the inverse functions of csc(x), sec(x), and cot(x) by restricting their domains...refer to the text if you're interested.

Note: Cancellation laws apply when f and f^{-1} are applied in succession; these are often very useful when working with inverse trigonometric functions.

For example, we have the following cancellation laws for $\sin(x)$ and $\sin^{-1}(x)$

$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\sin(\sin^{-1} x) = x \quad \text{for} \quad -1 \le x \le 1$$

Example: $\sin^{-1}\left(\sin\left(\frac{\pi}{12}\right)\right)$

Example: $\arccos(\cos(2\pi))$

Now let's try a few examples which are more complicated.

Example: Evaluate $\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

Example: Evaluate $\cos\left(\sin^{-1}(x)\right)$

Example: Evaluate $\csc\left(\tan^{-1}(2x)\right)$

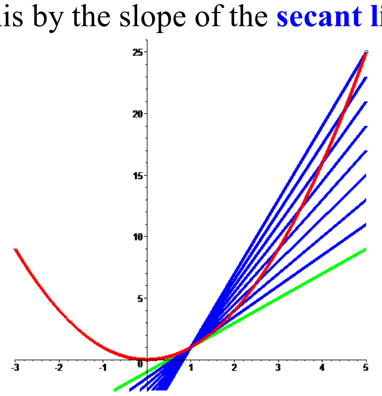
Example: Evaluate $\cot\left(\sec^{-1}(x^2)\right)$

LIMITS

Now that we have an understanding of functions, we will move on to study limits of functions, which is the foundation of our future work with derivatives and integrals.

A Preview of Calculus (Section 2.1)

Motivation: Often, we are interested in finding the **tangent** to a curve. For example, we may want to find the instantaneous speed of a moving object, whose position is $s(t) = t^2$. To find the slope of the tangent line (shown here at $x = 1$, in green), we can approximate this by the slope of the **secant line**, PQ, that goes through the point Q.



If we try moving Q closer to P, first from the left, and then from the right, we obtain:

x coord of Q	mpq
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

and

x coord of Q	mpq
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

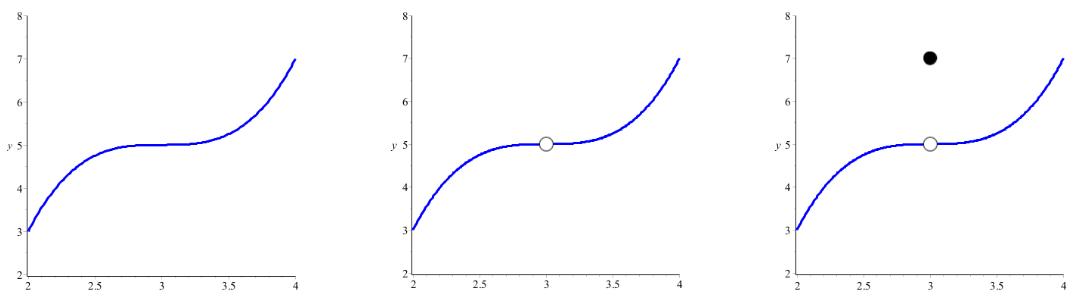
We say that $\lim_{Q \to P} m_{pq} = m$

The Limit of a Function (Section 2.2)

Definition: We write

$$\lim_{x \to a} f(x) = L$$

and say that "the **limit** of $f(x)$, as x approaches a, equals L"
if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (on either side of a), but not equal to a.



Before we go on to develop techniques for finding limits, let us consider some examples where the limit may not exist.

One-Sided Limits

Definition: We write

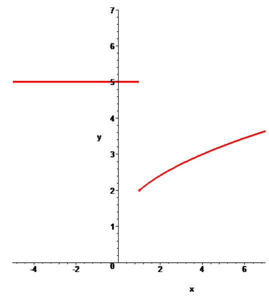
$$\lim_{x \to a^-} f(x) = L$$

and say that "the limit of $f(x)$, as x approaches a from the left, equals L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a. We define $\lim_{x \to a^+} f(x) = L$ in a similar manner.

Example:

Consider $f(x) = \begin{cases} 5, & x < 1 \\ \sqrt{x} + 1, & x \geq 1 \end{cases}$

Find $\lim_{x \to 1} f(x)$



NOTE: $\boxed{\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a+} f(x) = L}$

Infinite Limits

Definition: If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = \infty .$$

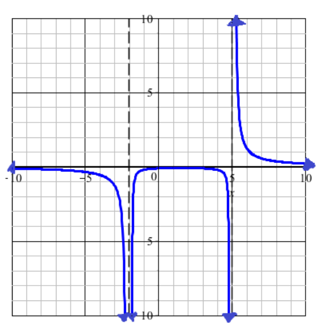
If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty .$$

We have similar definitions for the corresponding one-sided limits.

Caution: We're NOT saying the limit exists here; it doesn't (∞ is not a real number)…we're just specifying in more detail how/why the limit DNE).

Graphical Example:



Example: $\lim_{x \to 2} \frac{-5x}{(x-2)^4}$

Definition: Let f be a function. The line $x = a$ is called a **vertical asymptote** of the function $y = f(x)$ if any of the following conditions hold

$\lim_{x \to a^-} f(x) = \pm\infty$ $\quad$ $\lim_{x \to a^+} f(x) = \pm\infty$ $\quad$ or $\quad$ $\lim_{x \to a} f(x) = \pm\infty$.