In-class exercise

1. Consider $X_1, \ldots, X_n | \mu, \lambda \overset{iid}{\sim} \mathcal{N}(\mu, \lambda^{-1})$. Then independently consider

$$\boldsymbol{\mu} \sim \mathcal{N}(\mu_0, \lambda_0^{-1})$$
$$\boldsymbol{\lambda} \sim \text{Gamma}(a, b).$$

   (a) Derive the conditional distribution of $\mu \mid \lambda, x_{1:n}$.
   (b) Derive the conditional distribution of $\lambda \mid x_{1:n}$.
   (c) Explain how you can use both conditional distributions to approximate the distribution $p(\mu, \lambda \mid x_{1:n})$.
   (d) Explain how you could calculate $P(\mu \leq 5 \mid x_1, \ldots, x_n)$?

Solution:

(a) We know that for the Normal–Normal model, we know that for any fixed value of λ,

$$\boldsymbol{\mu} | \lambda, x_{1:n} \sim \mathcal{N}(M_\lambda, L_\lambda^{-1})$$

where

$$L_\lambda = \lambda_0 + n\lambda \;\text{ and }\; M_\lambda = \frac{\lambda_0 \mu_0 + \lambda \sum_{i=1}^n x_i}{\lambda_0 + n\lambda}.$$

(b) For any fixed value of μ, it is straightforward to derive[1] that

$$\boldsymbol{\lambda} | \mu, x_{1:n} \sim \text{Gamma}(A_\mu, B_\mu) \tag{0.1}$$

where $A_\mu = a + n/2$ and

$$B_\mu = b + \tfrac{1}{2} \sum (x_i - \mu)^2 = n\hat{\sigma}^2 + n(\bar{x} - \mu)^2$$

where $\hat{\sigma}^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$.

(c) Initialize $(\mu, \lambda) = (\mu_o, \lambda_o)$. To update μ, we draw from its conditional distribution. That is, we find μ_1 by sampling

$$\mu \sim p(\mu \mid \lambda = \lambda_o, x_{1:n}).$$

[1] do this on your own

This gives an intermediate output of (μ_1, λ_o). Next, we update λ from its conditional distribution. That is, we find λ_1 by sampling

$$\lambda \sim p(\lambda \mid x_{1:n}).$$

This gives an update of (μ_1, λ_1). We repeat this M times until we have the following samples:

$$(\mu_0, \lambda_0), (\mu_1, \lambda_1), \ldots, (\mu_M, \lambda_M).$$

(d) Given the answer in part c, we approximate

$$P(\mu \leq 5 \mid x_1, \ldots, x_n) \approx \frac{1}{M} \sum_{i=1}^{M} I(\mu_i \leq 5).$$