$$\theta \sim \text{Beta}(a, b)$$

$$X_1, \dots, X_n \mid \theta \overset{iid}{\sim} \text{Bernoulli}(\theta)$$

Goal: Find $p(x_{1:n}) \overset{\text{defn}}{=} \int_0^1 p(x_{1:n} \mid \theta)\, p(\theta)\, d\theta$

$$= \int_0^1 \underbrace{\theta^{\sum x_i} (1-\theta)^{n - \sum x_i}}_{\text{Binom likelih.}} \underbrace{\frac{1}{B(a,b)} \theta^{a-1} (1-\theta)^{b-1}}_{\text{Beta prior}} d\theta$$

$$= \int_0^1 \frac{1}{B(a,b)} \theta^{\sum x_i + a - 1} (1-\theta)^{n - \sum x_i + b - 1} d\theta$$ grouped like terms

$$= \frac{1}{B(a,b)} \underbrace{\int_0^1 \underbrace{\theta^{\sum x_i + a - 1} (1-\theta)^{n - \sum x_i + b - 1}}_{\text{kernel of } \text{Beta}(\theta \mid a_n = \sum x_i + a,\ b_n = n - \sum x_i + b)} d\theta \times \frac{B(a_n, b_n)}{B(a_n, b_n)}}_{1}$$

$$= \frac{1}{B(a,b)} \times B(a_n, b_n).$$

$$= \frac{B(\sum x_i + a,\ n - \sum x_i + b)}{B(a,b)}.$$

sanity check: doesn't depend on θ!