Goal: Find the Bayes rule.

- assume $\hat{\theta}$ estimates θ (r.v., unknown)
- assume data $x_1, \dots, x_n \to x_{1:n}$
- assume $\ell(\theta, \hat{\theta}) = (\theta - \hat{\theta})$ sq. error loss

To find the Bayes rule

$\to$ $\boxed{\text{minimize the posterior risk}}$

① Find the posterior risk ← plug in my loss!

$$\rho(\hat{\theta}, x_{1:n}) \overset{\text{defn}}{=} E[\ell(\theta, \hat{\theta}) \mid x_{1:n}]$$

$$= E[(\theta - \hat{\theta})^2 \mid x_{1:n}] \quad \text{expand the square}$$

$$= E[\theta^2 - 2\theta\hat{\theta} + \hat{\theta}^2 \mid x_{1:n}]$$

$\hat{\theta}$ is est (known)

θ is random

$$= \underline{E[\theta^2 \mid x_{1:n}] - 2\hat{\theta}\, E[\theta \mid x_{1:n}] + \hat{\theta}^2}$$

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② Find the Bayes rule

$$\frac{\partial \rho(\hat{\theta}, x_{1:n})}{\partial \hat{\theta}} = \frac{\partial}{\partial \hat{\theta}} \left\{ \underline{E[\theta^2 \mid x_{1:n}]} - 2\hat{\theta}\, E[\theta \mid x_{1:n}] + \hat{\theta}^2 \right]$$

$$= 0 - 2E[\theta \mid x_{1:n}] + 2\hat{\theta} \overset{\text{set}}{=} 0$$

(solve for $\hat{\theta}$)

$$\hat{\theta} = E[\theta \mid X_{1:n}]$$

$$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$\mu \sim N(\theta, \tau^2).$$

$$X \mid \theta \sim \mathrm{Bin}(\theta), \quad \boxed{0 < \theta < 1}$$

$$p(x \mid \theta) = \binom{n}{x} \underline{\theta^{x} (1-\theta)^{n-x}}$$

$$\propto \theta^{x} (1-\theta)^{n-x}$$