Chapter 8 TESTING OF HYPOTHESIS

8.1 POPULATION OR UNIVERSE

An aggregate of objects (animate or inanimate) under study is called **population or universe.** It is thus a collection of individuals or of their attributes (qualities) or of results of operations which can be numerically specified.

A universe containing a finite number of individuals or members is called a **finite inverse.** For example, the universe of the weights of students in a particular class.

A universe with infinite number of members is known as an **infinite universe.** For example, the universe of pressures at various points in the atmosphere.

In some cases, we may be even ignorant whether or not a particular universe is infinite, for example, the universe of stars.

The universe of concrete objects is an **existent universe.** The collection of all possible ways in which a specified event can happen is called a **hypothetical universe.** The universe of heads and tails obtained by tossing a coin an infinite number of times (provided that it does not wear out) is a hypothetical one.

8.2 SAMPLING

The statistician is often confronted with the problem of discussing a universe of which he cannot examine every member, *i.e.*, of which complete enumeration is impracticable. For example, if we want to have an idea of the average per capita income of the people of a country, enumeration of every earning individual in the country is a very difficult task. Naturally, the question arises: What can be said about a universe of which we can examine only a limited number of members? This question is the origin of the **Theory of Sampling.**

A finite subset of a universe is called a **sample.** A sample is thus a small portion of the universe. The number of individuals in a sample is called the **sample size.** The process of selecting a sample from a universe is called **sampling.**

The theory of sampling is a study of relationship existing between a population and samples drawn from the population. The fundamental object of sampling is to get as much information as possible of the whole universe by examining only a part of it. An attempt is thus made through sampling to give the maximum information about the parent universe with the minimum effort.

Sampling is quite often used in our day-to-day practical life. For example, in a shop we assess the quality of sugar, rice, or any other commodity by taking only a handful of it from the bag and then decide whether to purchase it or not. A housewife normally tests the cooked products to find if they are properly cooked and contain the proper quantity of salt or sugar, by taking a spoonful of it.

8.3 PARAMETERS OF STATISTICS

The statistical constants of the population such as mean, the variance, etc. are known as the parameters. The statistical concepts of the sample from the members of the sample to estimate the parameters of the population from which the sample has been drawn is known as *statistic*.

Population mean and variance are denoted by μ and σ^2, while those of the samples are given by $\bar{x}$, s^2.

8.4 STANDARD ERROR

The standard deviation of the sampling distribution of a statistic is known as the **standard error (S.E.).** It plays an important role in the theory of large samples and it forms a basis of the testing of hypothesis. If t is any statistic, for large sample.

$$z = \frac{t - E(t)}{S.E(t)}$$ is normally distributed with mean 0 and variance unity.

For large sample, the standard errors of some of the well known statistic are listed below:

n—sample size; σ^2—population variance; s^2—sample variance; p—population proportion ; $Q = 1 - p$; n_1, n_2—are sizes of two independent random samples.

Number	*Statistic*	*Standard error*
1.	$\overline{x}$	$\sigma/\sqrt{n}$
2.	s	$\sqrt{\sigma^2/2n}$
3.	Difference of two sample means $\overline{x}_1 - \overline{x}_2$	$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$
4.	Difference of two sample standard deviation $s_1 - s_2$	$\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}$
5.	Difference of two sample proportions $p_1 - p_2$	$\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}$
6.	Observed sample proportion p	$\sqrt{PQ/n}$

8.5 TEST OF SIGNIFICANCE

An important aspect of the sampling theory is to study the test of significance which will enable us to decide, on the basis of the results of the sample, whether

(*i*) *the deviation between the observed sample statistic and the hypothetical parameter value or*

(*ii*) *the deviation between two sample statistics is significant or might be attributed due to chance or the fluctuations of the sampling.*

For applying the tests of significance, we first set up a hypothesis which is a definite statement about the population parameter called **Null hypothesis** denoted by H_0.

Any hypothesis which is complementary to the null hypothesis (**H_0**) is called an **Alternative hypothesis** denoted by **H_1.**

For example, if we want to test the null hypothesis that the population has a specified mean μ_0, then we have

$$\mathbf{H_0}: \mu = \mu_0$$

Alternative hypothesis will be

(*i*) $H_1: \mu \neq \mu_0$ ($\mu > \mu_0$ *or* $\mu < \mu_0$) (two tailed alternative hypothesis).

(*ii*) $H_1: \mu > \mu_0$ (right tailed alternative hypothesis (*or*) single tailed).

(*iii*) $H_1: \mu < \mu_0$ (left tailed alternative hypothesis (*or*) single tailed).

Hence alternative hypothesis helps to know whether the test is two tailed test or one tailed test.

8.6 CRITICAL REGION

A region corresponding to a statistic t, in the sample space S which amounts to rejection of the null hypothesis H_0 is called as **critical region** or **region of rejection.** The region of the sample space S which amounts to the acceptance of H_0 is called *acceptance region*.

8.7 LEVEL OF SIGNIFICANCE

The probability of the value of the variate falling in the critical region is known as level of significance. The probability α that a random value of the statistic t belongs to the critical region is known as the **level of significance.**

$$P(t \in \omega \mid H_0) = \alpha$$

i.e., the level of significance is the size of the type I error or the maximum producer's risk.

8.8 ERRORS IN SAMPLING

The main aim of the sampling theory is to draw a valid conclusion about the population parameters on the basis of the sample results. In doing this we may commit the following two types of errors:

Type I Error. When H_0 is true, we may reject it.

$$P(\text{Reject } H_0 \text{ when it is true}) = P(\text{Reject } H_0/H_0) = \alpha$$

α is called the size of the type I error also referred to as **producer's risk.**

Type II Error. When H_0 is wrong we may accept it P(Accept H_0 when it is wrong) = P(Accept H_0/H_1) = β . β is called the size of the type II error, also referred to as **consumer's risk.**

NOTE *The values of the test statistic which separates the critical region and acceptance region are called the **critical values** or **significant values.** This value is dependent on (i) the level of significance used and (ii) the alternative hypothesis, whether it is one-tailed or two-tailed.*

For larger samples corresponding to the statistic t, the variable $z = \dfrac{t - E(t)}{S.E(t)}$ is normally distributed with mean 0 and variance 1. The value of z given above under the null hypothesis is known as **test statistic.**

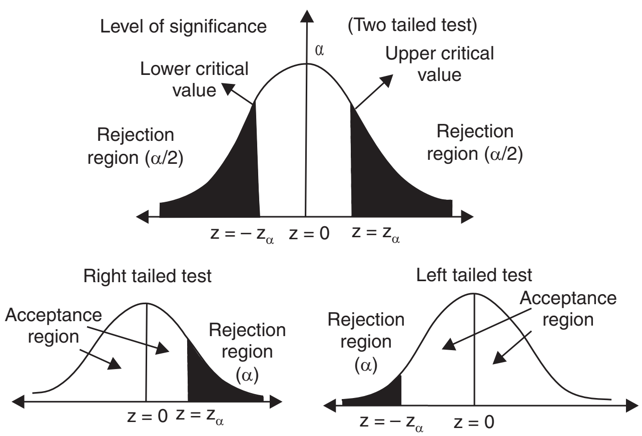
The critical value of z_α of the test statistic at level of significance α for a two-tailed test is given by

$$p(|\ z\ | > z_\alpha) = \alpha \qquad (1)$$

i.e., z_α is the value of z so that the total area of the critical region on both tails is α. Since the normal curve is symmetrical, from equation (1), we get

$$p(z > z_\alpha) + p(z < -z_\alpha) = \alpha;\ i.e.,\ 2p(z > z_\alpha) = \alpha;\ p(z > z_\alpha) = \alpha/2$$

i.e., the area of each tail is $\alpha/2$.



The critical value z_α is that value such that the area to the right of z_α is $\alpha/2$ and the area to the left of $-z_\alpha$ is $\alpha/2$.

In the case of the one-tailed test,

$$p(z > z_\alpha) = \alpha \text{ if it is right-tailed; } p(z < -z_\alpha) = \alpha \text{ if it is left-tailed.}$$

The critical value of z for a single-tailed test (right or left) at level of significance α is same as the critical value of z for two-tailed test at level of significance 2α.

Using the equation, also using the normal tables, the critical value of z at different levels of significance (α) for both single tailed and two tailed test are calculated and listed below. The equations are

$$p(|\ z\ | > z_\alpha) = \alpha;\ p(z > z_\alpha) = \alpha;\ p(z < -z_\alpha) = \alpha$$

Level of significance			
	1% (0.01)	5% (0.05)	10% (0.1)
Two tailed test	$\lvert z_\alpha \rvert = 2.58$	$\lvert z \rvert = 1.966$	$\lvert z \rvert = 0.645$
Right tailed	$z_\alpha = 2.33$	$z_\alpha = 1.645$	$z_\alpha = 1.28$
Left tailed	$z_\alpha = -2.33$	$z_\alpha = -1.645$	$z_\alpha = -1.28$

8.9 STEPS IN TESTING OF STATISTICAL HYPOTHESIS

Step 1. **Null hypothesis.** Set up H_0 in clear terms.

Step 2. **Alternative hypothesis.** Set up H_1, so that we could decide whether we should use one tailed test or two tailed test.

Step 3. **Level of significance.** Select the appropriate level of significance in advance depending on the reliability of the estimates.

Step 4. **Test statistic.** Compute the test statistic $z = \dfrac{t - E(t)}{S.E(t)}$ under the null hypothesis.

Step 5. **Conclusion.** Compare the computed value of z with the critical value z_α at level of significance (α).

If $|\ z\ | > z_\alpha$, we reject H_0 and conclude that there is significant difference. If $|\ z\ | < z_\alpha$, we accept H_0 and conclude that there is no significant difference.

8.10 TEST OF SIGNIFICANCE FOR LARGE SAMPLES

If the sample size $n > 30$, the sample is taken as large sample. For such sample we apply normal test, as Binomial, Poisson, chi square, etc. are closely approximated by normal distributions assuming the population as normal.

Under large sample test, the following are the important tests to test the significance:

1. *Testing of significance for single proportion.*
2. *Testing of significance for difference of proportions.*

3. *Testing of significance for single mean.*
4. *Testing of significance for difference of means.*
5. *Testing of significance for difference of standard deviations.*

8.10.1 Testing of Significance for Single Proportion

This test is used to find the significant difference between proportion of the sample and the population. Let X be the number of successes in n independent trials with constant probability P of success for each trial.

$$E(X) = nP;\ V(X) = nPQ;\ Q = 1 - P = \text{Probability of failure.}$$

Let $\quad p = X/n$ called the observed proportion of success.

$$E(p) = E(X/n) = \frac{1}{n}E(x) = \frac{np}{n} = p;\ E(p) = p$$

$$V(p) = V(X/n) = \frac{1}{n^2}V(X) = \frac{1(PQ)}{n} = PQ/n$$

$$S.E.(p) = \sqrt{\frac{PQ}{n}};\ z = \frac{p - E(p)}{S.E.(p)} = \frac{p - p}{\sqrt{PQ/n}} \sim N(0, 1)$$

This z is called test statistic which is used to test the significant difference of sample and population proportion.

NOTE

1. *The probable limit for the observed proportion of successes are $p \pm z_\alpha\sqrt{PQ/n}$, where z_α is the significant value at level of significance α.*

2. *If p is not known, the limits for the proportion in the population are $p \pm z_\alpha\sqrt{pq/n}$, $q = 1 - p$.*

3. *If α is not given, we can take safely 3σ limits.*

Hence, the confidence limits for observed proportion p are $p \pm 3\sqrt{\frac{PQ}{n}}$.

The confidence limits for the population proportion p are $p \pm \sqrt{\frac{pq}{n}}$.

EXAMPLES

Example 1. *A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbiased.*

Sol. H_0: The coin is unbiased *i.e.,* P = 0.5.

H_1: The coin is not unbiased (biased); P ≠ 0.5

Here $n = 400$; X = Number of success = 216

$$p = \text{proportion of success in the sample } \frac{X}{n} = \frac{216}{400} = 0.54.$$

Population proportion = 0.5 = P; Q = 1 – P = 1 – 0.5 = 0.5.

Under H_0, test statistic $z = \dfrac{p - P}{\sqrt{PQ/n}}$

$$|z| = \left| \frac{0.54 - 0.5}{\sqrt{\dfrac{0.5 \times 0.5}{400}}} \right| = 1.6$$

we use a two-tailed test.

Conclusion. Since $|z| = 1.6 < 1.96$

i.e., $|z| < z_\alpha$, z_α is the significant value of z at 5% level of significance.

i.e., the coin is unbiased is P = 0.5.

Example 2. *A manufacturer claims that only 4% of his products supplied by him are defective. A random sample of 600 products contained 36 defectives. Test the claim of the manufacturer.*

Sol. (*i*) P = observed proportion of success.

i.e., $P = \text{proportion of defective in the sample} = \dfrac{36}{600} = 0.06$

p = proportion of defectives in the population = 0.04

H_0: $p = 0.04$ is true.

i.e., the claim of the manufacturer is accepted.

H_1: (*i*) $P \neq 0.04$ (two tailed test)

(*ii*) If we want to reject, only if $p > 0.04$ then (right tailed).

$$\text{Under } H_0,\ z = \frac{p - P}{\sqrt{PQ/n}} = \frac{0.06 - 0.04}{\sqrt{\dfrac{0.04 \times 0.96}{600}}} = 2.5.$$

Conclusion. Since $|z| = 2.5 > 1.96$, we reject the hypothesis H_0 at 5% level of significance two tailed.

If H_1 is taken as $p > 0.04$ we apply right tailed test.

$|z| = 2.5 > 1.645$ (z_α) we reject the null hypothesis here also.

In both cases, manufacturer's claim is not acceptable.

Example 3. *A machine is producing bolts a certain fraction of which are defective. A random sample of 400 is taken from a large batch and is found to contain 30 defective bolts. Does this indicate that the proportion of defectives is larger than that claimed by the manufacturer if the manufacturer claims that only 5% of his product are defective? Find 95% confidence limits of the proportion of defective bolts in batch.*

Sol. Null hypothesis H_0: The manufacturer claim is accepted *i.e.,*

$$P = \frac{5}{100} = 0.05$$

$$Q = 1 - P = 1 - 0.05 = 0.95$$

Alternative hypothesis. $p > 0.05$ (Right tailed test).

$$p = \text{observed proportion of sample} = \frac{30}{400} = 0.075$$

Under H_0, the test statistic

$$z = \frac{p - P}{\sqrt{PQ/n}} \quad \therefore \quad z = \frac{0.075 - 0.05}{\sqrt{\dfrac{0.05 \times 0.95}{400}}} = 2.2941.$$

Conclusion. The tabulated value of z at 5% level of significance for the right-tailed test is

$$z_\alpha = 1.645. \text{ Since } |\, z \,| = 2.2941 > 1.645,$$

H_0 is rejected at 5% level of significance. *i.e.,* the proportion of defective is larger than the manufacturer claim.

To find 95% confidence limits of the proportion.

It is given by $p \pm z_\alpha \sqrt{PQ/n}$

$$0.05 \pm 1.96 \sqrt{\frac{0.05 \times 0.95}{400}} = 0.05 \pm 0.02135 = 0.07136,\ 0.02865$$

Hence 95% confidence limits for the proportion of defective bolts are (0.07136, 0.02865).

Example 4. *A bag contains defective articles, the exact number of which is not known. A sample of 100 from the bag gives 10 defective articles. Find the limits for the proportion of defective articles in the bag.*

Sol. Here p = proportion of defective articles $= \dfrac{10}{100} = 0.1$;

$$q = 1 - p = 1 - 0.1 = 0.9$$

Since the confidence limit is not given, we assume it is 95%. $\therefore$ level of significance is 5% $z_\alpha = 1.96$.

Also the proportion of population P is not given. To get the confidence limit, we use P and it is given by

$$\mathrm{P} \pm z_\alpha \sqrt{pq/n} = 0.1 \pm 1.96 \sqrt{\frac{0.1 \times 0.9}{100}}$$

$$= 0.1 \pm 0.0588 = 0.1588,\ 0.0412.$$

Hence 95% confidence limits for defective articles in the bag are (0.1588, 0.0412).

ASSIGNMENT 8.1

1. A sample of 600 persons selected at random from a large city shows that the percentage of males in the sample is 53. It is believed that the ratio of males to the total population in the city is 0.5. Test whether the belief is confirmed by the observation.
2. In a city a sample of 1000 people was taken and 540 of them are vegetarian and the rest are non-vegetarian. Can we say that both habits of eating (vegetarian or non-vegetarian) are equally popular in the city at (*i*) 1% level of significance (*ii*) 5% level of significance?
3. 325 men out of 600 men chosen from a big city were found to be smokers. Does this information support the conclusion that the majority of men in the city are smokers?
4. A random sample of 500 bolts was taken from a large consignment and 65 were found to be defective. Find the percentage of defective bolts in the consignment.
5. In a hospital 475 female and 525 male babies were born in a week. Do these figures confirm the hypothesis that males and females are born in equal number?
6. 400 apples are taken at random from a large basket and 40 are found to be bad. Estimate the proportion of bad apples in the basket and assign limits within which the percentage most probably lies.

8.10.2 Testing of Significance for Difference of Proportions

Consider two samples X_1 and X_2 of sizes n_1 and n_2, respectively, taken from two different populations. Test the significance of the difference between the sample proportion p_1 and p_2. The test statistic under the null hypothesis H_0, that there is no significant difference between the two sample proportion, we have

$$z = \frac{p_1 - p_2}{\sqrt{\mathrm{PQ}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \quad \text{where} \quad \mathrm{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

and $\qquad \mathrm{Q} = 1 - \mathrm{P}.$

EXAMPLES

Example 1. *Before an increase in excise duty on tea, 800 people out of a sample of 1000 persons were found to be tea drinkers. After an increase in the duty, 800 persons were known to be tea drinkers in a sample of 1200 people. Do you think that there has been a significant decrease in the consumption of tea after the increase in the excise duty?*

Sol. Here $n_1 = 800, n_2 = 1200$

$$p_1 = \frac{X_1}{n_1} = \frac{800}{1000} = \frac{4}{5}; \; p_2 = \frac{X_2}{n_2} = \frac{800}{1200} = \frac{2}{3}$$

$$P = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{800 + 800}{1000 + 1200} = \frac{8}{11}; \; Q = \frac{3}{11}$$

Null hypothesis H_0: $p_1 = p_2$, *i.e.*, there is no significant difference in the consumption of tea before and after increase of excise duty.

H_1: $p_1 > p_2$ (right-tailed test)

The test statistic

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.8 - 0.6666}{\sqrt{\frac{8}{11} \times \frac{3}{11}\left(\frac{1}{1000} + \frac{1}{1200}\right)}} = 6.842.$$

Conclusion. Since the calculated value of $| z | > 1.645$ also $| z | > 2.33$, both the significant values of z at 5% and 1% level of significance. Hence H_0 is rejected, *i.e.,* there is a significant decrease in the consumption of tea due to increase in excise duty.

Example 2. *A machine produced 16 defective articles in a batch of 500. After overhauling it produced 3 defectives in a batch of 100. Has the machine improved?*

Sol. $p_1 = \frac{16}{500} = 0.032; \; n_1 = 500 \qquad p_2 = \frac{3}{100} = 0.03; \; n_2 = 100$

Null hypothesis H_0: The machine has not improved due to overhauling, $p_1 = p_2$.

H_1: $p_1 > p_2$ (right-tailed)

$$\therefore \qquad P = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2} = \frac{19}{600} \cong 0.032$$

Under H_0, the test statistic

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.032 - 0.03}{\sqrt{(0.032)(0.968)\left(\frac{1}{500} + \frac{1}{100}\right)}} = 0.104.$$

Conclusion. The calculated value of $|z| < 1.645$, the significant value of z at 5% level of significance, H_0 is accepted, *i.e.*, the machine has not improved due to overhauling.

Example 3. *In two large populations, there are 30% and 25%, respectively, of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900, respectively, from the two populations.*

Sol. p_1 = proportion of fair haired people in the first population = 30% = 0.3; p_2 = 25% = 0.25; $Q_1 = 0.7$; $Q_2 = 0.75$.

H_0: Sample proportions are equal, *i.e.,* the difference in population proportions is likely to be hidden in sampling.

H_1: $p_1 \neq p_2$

$$z = \frac{P_1 - P_2}{\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}} = \frac{0.3 - 0.25}{\sqrt{\frac{0.3 \times 0.7}{1200} + \frac{0.25 \times 0.75}{900}}} = 2.5376.$$

Conclusion. Since $|z| > 1.96$, the significant value of z at 5% level of significance, H_0 is rejected. However $|z| < 2.58$, the significant value of z at 1% level of significance, H_0 is accepted. At 5% level, these samples will reveal the difference in the population proportions.

Example 4. *500 articles from a factory are examined and found to be 2% defective. 800 similar articles from a second factory are found to have only 1.5% defective. Can it reasonably be concluded that the product of the first factory are inferior to those of second?*

Sol. $n_1 = 500$, $n_2 = 800$

p_1 = proportion of defective from first factory = 2% = 0.02

p_2 = proportion of defective from second factory = 1.5% = 0.015

H_0: There is no significant difference between the two products, *i.e.,* the products do not differ in quality.

H_1: $p_1 < p_2$ (one tailed test)

Under H_0, $z = \dfrac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$

$$P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{0.02(500) + (0.015)(800)}{500 + 800} = 0.01692;$$

$$Q = 1 - P = 0.9830$$

$$z = \frac{0.02 - 0.015}{\sqrt{0.01692 \times 0.983\left(\frac{1}{500} + \frac{1}{800}\right)}} = 0.68$$

Conclusion. As $|z| < 1.645$, the significant value of z at 5% level of significance, H_0 is accepted *i.e.,* the products do not differ in quality.

ASSIGNMENT 8.2

1. A random sample of 400 men and 600 women were asked whether they would like to have a school near their residence. 200 men and 325 women were in favor of the proposal. Test the hypothesis that the proportion of men and women in favor of the proposal are the same at 5% level of significance.
2. In a town A, there were 956 births, of which 52.5% were males while in towns A and B combined, this proportion in total of 1406 births was 0.496. Is there any significant difference in the proportion of male births in the two towns?
3. In a referendum submitted to the student body at a university, 850 men and 560 women voted. 500 men and 320 women voted yes. Does this indicate a significant difference of opinion between men and women on this matter at 1% level?
4. A manufacturing firm claims that its brand A product outsells its brand B product by 8%. If it is found that 42 out of a sample of 200 persons prefer brand A and 18 out of another sample of 100 persons prefer brand B. Test whether the 8% difference is a valid claim.

8.10.3 Testing of Significance for Single Mean

To test whether the difference between sample mean and population mean is significant or not.

Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample of size n from a large population $X_1, X_2, \ldots\ldots, X_N$ of size N with mean μ and variance σ^2. $\therefore$ the standard error of mean of a random sample of size n from a population with variance σ^2 is $\sigma/\sqrt{n}$.

To test whether the given sample of size n has been drawn from a population with mean μ, *i.e.* to test whether the difference between the sample mean and the population mean is significant. Under the null hypothesis, there is no difference between the sample mean and population mean.

The test statistic is $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$, where σ is the standard deviation of the population.

If σ is not known, we use the test statistic $z = \dfrac{\overline{X} - \mu}{s/\sqrt{n}}$, where s is the standard deviation of the sample.

NOTE *If the level of significance is a and z_α is the critical value*

$$-z_\alpha < |\, z \,| = \left| \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \right| < z_\alpha.$$

The limits of the population mean μ are given by

$$\bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}.$$

At 5% level of significance, 95% confidence limits are

$$\bar{x} - 1.96 \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}}$$

At 1% level of significance, 99% confidence limits are

$$\bar{x} - 2.58 \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 2.58 \frac{\sigma}{\sqrt{n}}.$$

These limits are called confidence limits or fiducial limits.

EXAMPLES

Example 1. *A normal population has a mean of 6.8 and standard deviation of 1.5. A sample of 400 members gave a mean of 6.75. Is the difference significant?*

Sol. H_0: There is no significant difference between $\bar{x}$ and μ.

H_1: There is significant difference between $\bar{x}$ and μ.

Given μ = 6.8, σ = 1.5, $\bar{x}$ = 6.75 and n = 400

$$|\, z \,| = \left| \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \right| = \left| \frac{6.75 - 6.8}{1.5/\sqrt{900}} \right| = |\, -0.67 \,| = 0.67$$

Conclusion. As the calculated value of $|\, z \,| < z_\alpha = 1.96$ at 5% level of significance, H_0 is accepted, *i.e.*, there is no significant difference between $\bar{x}$ and μ.

Example 2. *A random sample of 900 members has a mean 3.4 cms. Can it be reasonably regarded as a sample from a large population of mean 3.2 cms and standard deviation 2.3 cms?*

Sol. Here n = 900, $\bar{x}$ = 3.4, μ = 3.2, σ = 2.3

H_0: Assume that the sample is drawn from a large population with mean 3.2 and standard deviation = 2.3

H_1: $\mu \neq 3.25$ (Apply two-tailed test)

$$\text{Under } H_0;\ z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{3.4-3.2}{2.3/\sqrt{900}} = 0.261.$$

Conclusion. As the calculated value of $|z| = 0.261 < 1.96$, the significant value of z at 5% level of significance, H_0 is accepted, *i.e.,* the sample is drawn from the population with mean 3.2 and standard deviation = 2.3.

Example 3. *The mean weight obtained from a random sample of size 100 is 64 gms. The standard deviation of the weight distribution of the population is 3 gms. Test the statement that the mean weight of the population is 67 gms at 5% level of significance. Also set up 99% confidence limits of the mean weight of the population.*

Sol. Here $n = 100$, $\mu = 67$, $\bar{x} = 64$, $\sigma = 3$

H_0: There is no significant difference between sample and population mean.

i.e., $\mu = 67$, the sample is drawn from the population with $\mu = 67$.

H_1: $\mu \neq 67$ (Two-tailed test).

$$\text{Under } H_0,\ z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{64-67}{3/\sqrt{100}} = -10 \quad \therefore \quad |z| = 10.$$

Conclusion. Since the calculated value of $|z| > 1.96$, the significant value of z at 5% level of significance, H_0 is rejected, *i.e.,* the sample is not drawn from the population with mean 67.

To find 99% confidence limits, given by

$$\bar{x} \pm 2.58\ \sigma/\sqrt{n} = 64 \pm 2.58(3/\sqrt{100}) = 64.774,\ 63.226.$$

Example 4. *The average score in mathematics of a sample of 100 students was 51 with a standard deviation of 6 points. Could this have been a random sample from a population with average scores 50?*

Sol. Here $n = 100$, $\bar{x} = 51$, $s = 6$, $\mu = 50$; σ is unknown.

H_0: The sample is drawn from a population with mean 50, $\mu = 50$

H_1: $\mu \neq 50$

$$\text{Under } H_0,\ z = \frac{\bar{x}-\mu}{s/\sqrt{n}} = \frac{51-50}{6/\sqrt{100}} = \frac{10}{6} = 1.6666.$$

Conclusion. Since $|z| = 1.666 < 1.96$, z_α the significant value of z at 5% level of significance, H_0 is accepted, *i.e.,* the sample is drawn from the population with mean 50.

ASSIGNMENT 8.3

1. A sample of 1000 students from a university was taken and their average weight was found to be 112 pounds with a standard deviation of 20 pounds. Could the mean weight of students in the population be 120 pounds?
2. A sample of 400 male students is found to have a mean height of 160 cms. Can it be reasonably regarded as a sample from a large population with mean height 162.5 cms and standard deviation 4.5 cms?
3. A random sample of 200 measurements from a large population gave a mean value of 50 and a standard deviation of 9. Determine 95% confidence interval for the mean of population.
4. The guaranteed average life of certain type of bulbs is 1000 hours with a standard deviation of 125 hours. It is decided to sample the output so as to ensure that 90% of the bulbs do not fall short of the guaranteed average by more than 2.5%. What must be the minimum size of the sample?
5. The heights of college students in a city are normally distributed with standard deviation 6 cms. A sample of 1000 students has mean height 158 cms. Test the hypothesis that the mean height of college students in the city is 160 cms.

8.10.4 Test of Significance for Difference of Means of Two Large Samples

Let $\overline{x}_1$ be the mean of a sample of size n_1 from a population with mean μ_1, and variance σ_1^2. Let $\overline{x}_2$ be the mean of an independent sample of size n_2 from another population with mean μ_2 and variance σ_2^2. The test statistic is given by $z = \dfrac{\overline{x}_1 - \overline{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$.

Under the null hypothesis that the samples are drawn from the same population where $\sigma_1 = \sigma_2 = \sigma$, *i.e.*, $\mu_1 = \mu_2$ the test statistic is given by

$$z = \frac{\overline{x}_1 - \overline{x}_2}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}.$$

NOTE **1.** *If σ_1, σ_2 are not known and $\sigma_1 \neq \sigma_2$ the test statistic in this case is*

$$z = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\dfrac{s_1^2 + s_2^2}{n_1 + n_2}}}.$$

2. *If σ is not known and $\sigma_1 = \sigma_2$. We use* $\sigma^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}$ *to calculate σ;*

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}.$$

EXAMPLES

Example 1. *The average income of persons was 210 with a standard deviation of 10 in a sample of 100 people. For another sample of 150 people, the average income was 220 with a standard deviation of 12. The standard deviation of incomes of the people of the city was 11. Test whether there is any significant difference between the average incomes of the localities.*

Sol. Here $n_1 = 100$, $n_2 = 150$, $\bar{x}_1 = 210$, $\bar{x}_2 = 220$, $s_1 = 10$, $s_2 = 12$.

Null hypothesis. The difference is not significant, *i.e.,* there is no difference between the incomes of the localities.

$$H_0: \bar{x}_1 = \bar{x}_2, \quad H_1: \bar{x}_1 \neq \bar{x}_2$$

Under H_0, $$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \frac{210 - 220}{\sqrt{\dfrac{10^2}{100} + \dfrac{12^2}{150}}} = -7.1428 \quad \therefore \quad |z| = 7.1428.$$

Conclusion. As the calculated value of $|z| > 1.96$, the significant value of z at 5% level of significance, H_0 is rejected *i.e.,* there is significant difference between the average incomes of the localities.

Example 2. *Intelligence tests were given to two groups of boys and girls.*

	Mean	*Standard deviation*	*Size*
Girls	*75*	*8*	*60*
Boys	*73*	*10*	*100*

Examine if the difference between mean scores is significant.

Sol. Null hypothesis H_0: There is no significant difference between mean scores, *i.e.,* $\bar{x}_1 = \bar{x}_2$.

$$H_1: \bar{x}_1 \neq \bar{x}_2$$

Under the null hypothesis $$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \frac{75 - 73}{\sqrt{\dfrac{8^2}{60} + \dfrac{10^2}{100}}} = 1.3912.$$

Conclusion. As the calculated value of $| z | < 1.96$, the significant value of z at 5% level of significance, H_0 is accepted *i.e.,* there is no significant difference between mean scores.

ASSIGNMENT 8.4

1. Intelligence tests on two groups of boys and girls gave the following results. Examine if the difference is significant.

	Mean	*Standard Deviation*	*Size*
Girls	70	10	70
Boys	75	11	100

2. Two random samples of 1000 and 2000 farms gave an average yield of 2000 kg and 2050 kg, respectively. The variance of wheat farms in the country may be taken as 100 kg. Examine whether the two samples differ significantly in yield.

3. A sample of heights of 6400 soldiers has a mean of 67.85 inches and a standard deviation of 2.56 inches. While another sample of heights of 1600 sailors has a mean of 68.55 inches with standard deviation of 2.52 inches. Do the data indicate that sailors are, on the average, taller than soldiers?

4. In a survey of buying habits, 400 women shoppers are chosen at random in supermarket A. Their average weekly food expenditure is 250 with a standard deviation of 40. For 500 women shoppers chosen at supermarket B, the average weekly food expenditure is 220 with a standard deviation of 45. Test at 1% level of significance whether the average food expenditures of the two groups are equal.

5. A random sample of 200 measurements from a large population gave a mean value of 50 and standard deviation of 9. Determine the 95% confidence interval for the mean of the population.

6. The means of two large samples of 1000 and 2000 members are 168.75 cms and 170 cms, respectively. Can the samples be regarded as drawn from the same population of standard deviation 6.25 cms?

8.10.5 Test of Significance for the Difference of Standard Deviations

If s_1 and s_2 are the standard deviations of two independent samples, then under the null hypothesis H_0: $\sigma_1 = \sigma_2$, *i.e.*, the sample standard deviations don't differ significantly, the statistic

$$z = \frac{s_1 - s_2}{\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}},$$ where σ_1 and σ_2 are population standard deviations.

When population standard deviations are not known, then $z = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_1} + \frac{s_2^2}{2n_2}}}$.

EXAMPLE

Example. *Random samples drawn from two countries gave the following data relating to the heights of adult males*:

	Country A	*Country B*
Mean height (in inches)	*67.42*	*67.25*
Standard deviation	*2.58*	*2.50*
Number in samples	*1000*	*1200*

(*i*) *Is the difference between the means significant?*

(*ii*) *Is the difference between the standard deviations significant?*

Sol. Given: $n_1 = 1000$, $n_2 = 1200$, $\bar{x}_1 = 67.42$; $\bar{x}_2 = 67.25$, $s_1 = 2.58$, $s_2 = 2.50$

Since the samples size are large we can take $\sigma_1 = s_1 = 2.58$; $\sigma_2 = s_2 = 2.50$.

(*i*) **Null hypothesis:** $H_0 = \mu_1 = \mu_2$ *i.e.*, sample means do not differ significantly.

Alternative hypothesis: H_1: $\mu_1 \neq \mu_2$ (two tailed test)

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{67.42 - 67.25}{\sqrt{\frac{(2.58)^2}{1000} + \frac{(2.50)^2}{1200}}} = 1.56.$$

Since $| z | < 1.96$ we accept the null hypothesis at 5% level of significance.

(*ii*) We set up the null hypothesis.

H_0: $\sigma_1 = \sigma_2$ *i.e.*, the sample standard deviations do not differ significantly.

Alternative hypothesis: $H_1 = \sigma_1 \neq \sigma_2$ (two tailed)

∴ The test statistic is given by

$$z = \frac{s_1 - s_2}{\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}} = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_1} + \frac{s_2^2}{2n_2}}}$$

($\because$ $\sigma_1 = s_1, \sigma_2 = s_2$ for large samples)

$$= \frac{2.58 - 2.50}{\sqrt{\frac{(2.58)^2}{2 \times 1000} \times \frac{(2.50)^2}{2 \times 1200}}} = \frac{0.08}{\sqrt{\frac{6.6564}{2000} + \frac{6.25}{2400}}} = 1.0387.$$

Since $|z| < 1.96$ we accept the null hypothesis at 5% level of significance.

ASSIGNMENT 8.5

1. The mean yield of two sets of plots and their variability are as given. Examine
(*i*) whether the difference in the mean yield of the two sets of plots is significant.
(*ii*) whether the difference in the variability in yields is significant.

Set of 40 plots	*Set of 60 plots*	
Mean yield per plot	1258 lb	1243 lb
Standard deviation per plot	34	28

2. The yield of wheat in a random sample of 1000 farms in a certain area has a standard deviation of 192 kg. Another random sample of 1000 farms gives a standard deviation of 224 kg. Are the standard deviation significantly different ?

8.11 TEST OF SIGNIFICANCE OF SMALL SAMPLES

When the size of the sample is less than 30, then the sample is called small sample. For such sample it will not be possible for us to assume that the random sampling distribution of a statistic is approximately normal and the values given by the sample data are sufficiently close to the population values and can be used in their place for the calculation of the standard error of the estimate.

t-TEST

8.12 STUDENT'S t-DISTRIBUTION

This t-distribution is used when sample size is ≤ 30 and the population standard deviation is unknown.

t-statistic is defined as $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}} \sim t(n-1\ d.\ f)$ d.f–degrees of freedom

where $$s = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n-1}}.$$

8.12.1 The t-Table

The t-table given at the end is the probability integral of t-distribution. The t-distribution has different values for each degrees of freedom and when the degrees of freedom are infinitely large, the t-distribution is equivalent to normal distribution and the probabilities shown in the normal distribution tables are applicable.

8.12.2 Applications of t-Distribution

Some of the applications of t-distribution are given below:

1. *To test if the sample mean* $(\bar{X})$ *differs significantly from the hypothetical value* μ *of the population mean.*
2. *To test the significance between two sample means.*
3. *To test the significance of observed partial and multiple correlation coefficients.*

8.12.3 Critical Value of t

The critical value or significant value of t at level of significance α degrees of freedom γ for two tailed test is given by

$$P[\,|\,t\,|\, > t_\gamma(\alpha)] = \alpha$$

$$P[\,|\,t\,|\, \leq t_\gamma(\alpha)] = 1 - \alpha$$

The significant value of t at level of significance α for a single tailed test can be determined from those of two-tailed test by referring to the values at 2α.

8.13 TEST I: t-TEST OF SIGNIFICANCE OF THE MEAN OF A RANDOM SAMPLE

To test whether the mean of a sample drawn from a normal population deviates significantly from a stated value when variance of the population is unknown.

H_0: There is no significant difference between the sample mean $\bar{x}$ and the population mean μ, *i.e.*, we use the statistic

$$t = \frac{\overline{X} - \mu}{s/\sqrt{n}} \qquad \text{where } \overline{X} \text{ is mean of the sample.}$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n} (X_i - \overline{X})^2 \text{ with degrees of freedom } (n - 1).$$

At a given level of significance α_1 and degrees of freedom $(n - 1)$. We refer to t-table t_α (two-tailed or one-tailed). If calculated t value is such that $| t | < t_\alpha$ the null hypothesis is accepted. $| t | > t_\alpha$, H_0 is rejected.

8.13.1 Fiducial Limits of Population Mean

If t_α is the table of t at level of significance α at $(n - 1)$ degrees of freedom.

$$\left| \frac{\overline{X} - \mu}{s/\sqrt{n}} \right| < t_\alpha \text{ for acceptance of } H_0.$$

$$\bar{x} - t_\alpha\, s/\sqrt{n} < \mu < \bar{x} + t_\alpha\, s/\sqrt{n}$$

95% confidence limits (level of significance 5%) are $\overline{X} \pm t_{0.05}\, s/\sqrt{n}$.

99% confidence limits (level of significance 1%) are $\overline{X} \pm t_{0.01}\, s/\sqrt{n}$.

NOTE *Instead of calculating s, we calculate S for the sample.*

Since $$s^2 = \frac{1}{n-1}\sum_{i=1}^{n} (X_i - \overline{X})^2$$

∴ $$S^2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - \overline{X})^2. \qquad | \because \quad (n-1)s^2 = nS^2$$

EXAMPLES

Example 1. *A random sample of size 16 has 53 as mean. The sum of squares of the derivation from mean is 135. Can this sample be regarded as taken from the population having 56 as mean? Obtain 95% and 99% confidence limits of the mean of the population.*

Sol. H_0: There is no significant difference between the sample mean and hypothetical population mean.

$$H_0: \mu = 56; \quad H_1: \mu \neq 56 \quad \text{(Two-tailed test)}$$

$$t: \frac{\overline{X} - \mu}{s/\sqrt{n}} \sim t(n-1 \text{ difference})$$

Given: $\overline{X} = 53, \mu = 56, n = 16, \Sigma(X - \overline{X})^2 = 135$

$$s = \sqrt{\frac{\Sigma(X-\overline{X})^2}{n-1}} = \sqrt{\frac{135}{15}} = 3; t = \frac{53-56}{3/\sqrt{16}} = -4$$

$| t | = 4$. *d.fv.* $= 16 - 1 = 15$.

Conclusion. $t_{0.05} = 1.753$.

Since $| t | = 4 > t_{0.05} = 1.753$, the calculated value of t is more than the table value. The hypothesis is rejected. Hence, the sample mean has not come from a population having 56 as mean.

95% confidence limits of the population mean

$$= \overline{X} \pm \frac{s}{\sqrt{n}} t_{0.05} = 53 \pm \frac{3}{\sqrt{16}} (1.725) = 51.706; 54.293$$

99% confidence limits of the population mean

$$= \overline{X} \pm \frac{s}{\sqrt{n}} t_{0.01,} = 53 \pm \frac{3}{\sqrt{16}} (2.602) = 51.048; 54.951.$$

Example 2. *The lifetime of electric bulbs for a random sample of 10 from a large consignment gave the following data:*

Item	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Life in '000' hrs.	*4.2*	*4.6*	*3.9*	*4.1*	*5.2*	*3.8*	*3.9*	*4.3*	*4.4*	*5.6*

Can we accept the hypothesis that the average lifetime of a bulb is 4000 hrs?

Sol. H_0: There is no significant difference in the sample mean and population mean. *i.e.,* $\mu = 4000$ hrs.

Applying the t-test: $t = \dfrac{\overline{X} - \mu}{s/\sqrt{n}} \sim t(10 - 1 \text{ difference})$

X	4.2	4.6	3.9	4.1	5.2	3.8	3.9	4.3	4.4	5.6
$X - \overline{X}$	– 0.2	0.2	– 0.5	– 0.3	0.8	– 0.6	– 0.5	– 0.1	0	1.2
$(X - \overline{X})^2$	0.04	0.04	0.25	0.09	0.64	0.36	0.25	0.01	0	1.44

$$\overline{X} = \frac{\Sigma X}{n} = \frac{44}{10} = 4.4 \qquad \Sigma(X - \overline{X})^2 = 3.12$$

$$s = \sqrt{\frac{\Sigma(X - \overline{X})^2}{n-1}} = \sqrt{\frac{3.12}{9}} = 0.589; \; t = \frac{4.4 - 4}{\frac{0.589}{\sqrt{10}}} = 2.123$$

For $\gamma = 9$, $t_{0.05} = 2.26$.

Conclusion. Since the calculated value of t is less than table $t_{0.05}$. $\therefore$ The hypothesis $\mu = 4000$ hrs is accepted, *i.e.*, the average lifetime of bulbs could be 4000 hrs.

Example 3. *A sample of 20 items has mean 42 units and standard deviation 5 units. Test the hypothesis that it is a random sample from a normal population with mean 45 units.*

Sol. H_0: There is no significant difference between the sample mean and the population mean. *i.e.,* $\mu = 45$ units

H_1: $\mu \neq 45$ (Two tailed test)

Given: $n = 20$, $\overline{X} = 42$, $S = 5$; $\gamma = 19$ difference

$$s^2 = \frac{n}{n-1} S^2 = \left[\frac{20}{20-1}\right](5)^2 = 26.31 \qquad \therefore s = 5.129$$

Applying t-test $\quad t = \dfrac{\overline{X} - \mu}{s/\sqrt{n}} = \dfrac{42 - 45}{5.129/\sqrt{20}} = -2.615; \; |t| = 2.615$

The tabulated value of t at 5% level for 19 d.f. is $t_{0.05} = 2.09$.

Conclusion. Since $|t| > t_{0.05}$, the hypothesis H_0 is rejected, *i.e.,* there is significant difference between the sample mean and population mean.

i.e., the sample could not have come from this population.

Example 4. *The 9 items of a sample have the following values 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these values differ significantly from the assumed mean 47.5?*

Sol. H_0: $\mu = 47.5$

i.e., there is no significant difference between the sample and population mean.

H_1: $\mu \neq 47.5$ (two tailed test); Given: $n = 9$, $\mu = 47.5$

X	45	47	50	52	48	47	49	53	51
$X - \overline{X}$	– 4.1	– 2.1	0.9	2.9	– 1.1	– 2.1	– 0.1	3.9	1.9
$(X - \overline{X})^2$	16.81	4.41	0.81	8.41	1.21	4.41	0.01	15.21	3.61

$$\overline{X} = \frac{\Sigma x}{n} = \frac{442}{9} = 49.11; \ \Sigma(X - \overline{X})^2 = 54.89; \ s^2 = \frac{\Sigma(X - \overline{X})^2}{(n-1)} = 6.86$$

$\therefore \qquad s = 2.619$

Applying t-test $\qquad t = \dfrac{\overline{X} - \mu}{s/\sqrt{n}} = \dfrac{49.1 - 47.5}{2.619/\sqrt{8}} = \dfrac{(1.6)\sqrt{8}}{2.619} = 1.7279$

$$t_{0.05} = 2.31 \text{ for } \gamma = 8.$$

Conclusion. Since $|\ t\ | < t_{0.05}$, the hypothesis is accepted *i.e.,* there is no significant difference between their mean.

ASSIGNMENT 8.6

1. Ten individuals are chosen at random from a normal population of students and their scores found to be 63, 63, 66, 67, 68, 69, 70, 70, 71, 71. In the light of these data discuss the suggestion that mean score of the population of students is 66.
2. The following values gives the lengths of 12 samples of Egyptian cotton taken from a consignment: 48, 46, 49, 46, 52, 45, 43, 47, 47, 46, 45, 50. Test if the mean length of the consignment can be taken as 46.
3. A sample of 18 items has a mean 24 units and standard deviation 3 units. Test the hypothesis that it is a random sample from a normal population with mean 27 units.
4. A filling machine is expected to fill 5 kg of powder into bags. A sample of 10 bags gave the following weights: 4.7, 4.9, 5.0, 5.1, 5.4, 5.2, 4.6, 5.1, 4.6, and 4.7. Test whether the machine is working properly.

8.14 TEST II: t-TEST FOR DIFFERENCE OF MEANS OF TWO SMALL SAMPLES (FROM A NORMAL POPULATION)

This test is used to test whether the two samples $x_1, x_2, \ldots\ldots, x_{n_1}, y_1, y_2, \ldots\ldots, y_{n_2}$ of sizes n_1, n_2 have been drawn from two normal populations with mean μ_1

and μ_2 respectively, under the assumption that the population variance are equal. ($\sigma_1 = \sigma_2 = \sigma$).

H_0: The samples have been drawn from the normal population with means μ_1 and μ_2, *i.e.*, H_0: $\mu_1 \neq \mu_2$.

Let $\overline{X}$, $\overline{Y}$ be their means of the two samples.

Under this H_0 the test of statistic t is given by

$$t = \frac{(\overline{X} - \overline{Y})}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim t(n_1 + n_2 - 2 \text{ difference})$$

NOTE **1.** *If the two sample's standard deviations s_1, s_2 are given, then we have*

$$s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}.$$

2. *If $n_1 = n_2 = n$, $t = \dfrac{\overline{X} - \overline{Y}}{\sqrt{\dfrac{s_1^2 + s_2^2}{n-1}}}$ can be used as a test statistic.*

3. *If the pairs of values are in some way associated (correlated) we can't use the test statistic as given in Note 2. In this case, we find the differences of the associated pairs of values and apply for single mean i.e., $t = \dfrac{\overline{X} - \mu}{s/\sqrt{n}}$ with degrees of freedom $n - 1$.*

The test statistic is

$$t = \frac{\overline{d}}{s/\sqrt{n}}$$

or $$t = \frac{\overline{d}}{s/\sqrt{n-1}},$$ where $\overline{d}$ is the mean of paired difference.

i.e., $$d_i = x_i - y_i$$

$\overline{d_i} = \overline{X} - \overline{Y}$, where (x_i, y_i) are the paired data $i = 1, 2, \ldots\ldots, n$.

EXAMPLES

Example 1. *Two samples of sodium vapor bulbs were tested for length of life and the following results were obtained:*

	Size	*Sample mean*	*Sample S.D.*
Type I	*8*	*1234 hrs*	*36 hrs*
Type II	*7*	*1036 hrs*	*40 hrs*

Is the difference in the means significant to generalize that Type I is superior to Type II regarding length of life?

Sol. $H_0: \mu_1 = \mu_2$ *i.e.,* two types of bulbs have same lifetime.

$H_1: \mu_1 > \mu_2$ *i.e.,* type I is superior to Type II.

$$s^2 = \frac{n_1 s_1{}^2 + n_2 s_2{}^2}{n_1 + n_2 - 2} = \frac{8(36)^2 + 7(40)^2}{8+7-2} = 1659.076$$

$$\therefore \qquad s = 40.7317$$

The t-statistic
$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{1234 - 1036}{40.7317\sqrt{\frac{1}{8} + \frac{1}{7}}}$$

$$= 18.1480 \sim t(n_1 + n_2 - 2 \text{ difference})$$

$t_{0.05}$ at difference 13 is 1.77 (one tailed test).

Conclusion. Since calculated $|\, t \,| > t_{0.05}$, H_0 is rejected, *i.e.* H_1 is accepted.

$\therefore$ Type I is definitely superior to Type II.

where $$\overline{X} = \sum_{i=1}^{n_1} \frac{X_i}{n_i}, \quad \overline{Y} = \sum_{j=1}^{n_2} \frac{Y_j}{n_2}; \; s^2 = \frac{1}{n_1 + n_2 - 2}[\Sigma(X_i - \overline{X})^2 + (Y_j - \overline{Y})^2]$$

is an unbiased estimate of the population variance σ^2.

t follows t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

Example 2. *Samples of sizes 10 and 14 were taken from two normal populations with standard deviation 3.5 and 5.2. The sample means were found to be 20.3 and 18.6. Test whether the means of the two populations are the same at 5% level.*

Sol. $H_0: \mu_1 = \mu_2$ *i.e.,* the means of the two populations are the same.

$H_1 : \mu_1 \neq \mu_2$.

Given $\overline{X} = 20.3, \overline{X}_2 = 18.6; n_1 = 10, n_2 = 14, s_1 = 3.5, s_2 = 5.2$

$$s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{10(3.5)^2 + 14(5.2)^2}{10 + 14 - 2} = 22.775 \quad \therefore \quad s = 4.772$$

$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{20.3 - 18.6}{\left(\sqrt{\frac{1}{10} + \frac{1}{14}}\right)4.772} = 0.8604$$

The value of t at 5% level for 22 difference is $t_{0.05} = 2.0739$.

Conclusion. Since $|\ t\ | = 0.8604 < t_{0.05}$ the hypothesis is accepted, *i.e.,* there is no significant difference between their means.

Example 3. *The height of 6 randomly chosen sailors in inches is 63, 65, 68, 69, 71, and 72. Those of 9 randomly chosen soldiers are 61, 62, 65, 66, 69, 70, 71, 72, and 73. Test whether the sailors are, on average, taller than soldiers.*

Sol. Let X_1 and X_2 be the two samples denoting the heights of sailors and soldiers.

Given the sample size $n_1 = 6$, $n_2 = 9$, H_0: $\mu_1 = \mu_2$,

i.e., the means of both populations are the same.

H_1: $\mu_1 > \mu_2$ (one tailed test)

Calculation of two sample means:

X_1	63	65	68	69	71	72
$X_1 - \overline{X}_1$	– 5	– 3	0	1	3	4
$(X_1 - \overline{X}_1)^2$	25	9	0	1	9	16

$$\overline{X}_1 = \frac{\Sigma X_1}{n_1} = 68; \Sigma(X_1 - \overline{X}_1)^2 = 60$$

X_2	61	62	65	66	69	70	71	72	73
$X_2 - \overline{X}_2$	– 6.66	– 5.66	– 2.66	1.66	1.34	2.34	3.34	4.34	5.34
$(X_2 - \overline{X}_2)^2$	44.36	32.035	7.0756	2.7556	1.7956	5.4756	11.1556	18.8356	28.5156

$$\overline{X}_2 = \frac{\Sigma X_2}{n_2} = 67.66; \Sigma(X_2 - \overline{X}_2)^2 = 152.0002$$

$$s^2 = \frac{1}{n_1 + n_2 - 2}[\Sigma(X_1 - \overline{X}_1)^2 + \Sigma(X_2 - \overline{X}_2)^2]$$

$$= \frac{1}{6+9-2}[60 + 152.0002] = 16.3077 \quad \therefore \quad s = 4.038$$

Under H_0, $t = \dfrac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \dfrac{68 - 67.666}{4.0382\sqrt{\dfrac{1}{6} + \dfrac{1}{9}}}$

$$= 0.3031 \sim t(n_1 + n_2 - 2 \text{ difference})$$

The value of t at 10% level of significance ($\because$ the test is one-tailed) for 13 difference is 1.77.

Conclusion. Since $|\,t\,| = 0.3031 < t_{0.05} = 1.77$, the hypothesis H_0 is accepted.

There is no significan difference between their average.

The sailors are not, on average, taller than the soldiers.

Example 4. *A certain stimulus administered to each of 12 patients resulted in the following increase in blood pressure: 5, 2, 8, – 1, 3, 0, – 2, 1, 5, 0, 4, 6. Can it be concluded that the stimulus will in general be accompanied by an increase in blood pressure?*

Sol. To test whether the mean increase in blood pressure of all patients to whom the stimulus is administered will be positive, we have to assume that this population is normal with mean μ and standard deviation σ which are unknown.

$$H_0: \mu = 0; H_1: \mu_1 > 0$$

The test statistic under H_0

$$t = \frac{\overline{d}}{s/\sqrt{n-1}} \sim t(n - 1 \text{ degrees of freedom})$$

$$\overline{d} = \frac{5 + 2 + 8 + (-1) + 3 + 0 + 6 + (-2) + 1 + 5 + 0 + 4}{12} = 2.583$$

$$s^2 = \frac{\Sigma d^2}{n} - \overline{d}^2 = \frac{1}{12}[5^2 + 2^2 + 8^2 + (-1)^2 + 3^2 + 0^2 + 6^2 + (-2)^2 + 1^2 + 5^2 + 0^2 + 4^2] - (2.583)^2$$

$$= 8.744 \quad \therefore \quad s = 2.9571$$

$$t = \frac{\overline{d}}{s/\sqrt{n-1}} = \frac{2.583}{2.9571/\sqrt{12-1}} = \frac{2.583\sqrt{11}}{2.9571}$$

$$= 2.897 \sim t(n - 1 \text{ difference})$$

Conclusion. The tabulated value of $t_{0.05}$ at 11 difference is 2.2.

$\because \quad | t | > t_{0.05}$, H_0 is rejected.

i.e., the stimulus does not increase the blood pressure. The stimulus in general will be accompanied by an increase in blood pressure.

Example 5. *Memory capacity of 9 students was tested before and after a course of meditation for a month. State whether the course was effective or not from the data below (in same units):*

Before	*10*	*15*	*9*	*3*	*7*	*12*	*16*	*17*	*4*
After	*12*	*17*	*8*	*5*	*6*	*11*	*18*	*20*	*3*

Sol. Since the data are correlated and concerned with the same set of students we use paired t-test.

H_0: Training was not effective $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$ (Two-tailed test).

Before training (X)	*After training (Y)*	$d = X - Y$	d^2
10	12	– 2	4
15	17	– 2	4
9	8	1	1
3	5	– 2	4
7	6	1	1
12	11	1	1
16	18	– 2	4
17	20	– 3	9
4	3	1	1
		$\Sigma d = -7$	$\Sigma d^2 = 29$

$$\bar{d} = \frac{\Sigma d}{n} = \frac{-7}{9} = -0.7778; \; s^2 = \frac{\Sigma d^2}{n} - (\bar{d})^2 = \frac{29}{9} - (-0.7778)^2 = 2.617$$

$$t = \frac{\bar{d}}{s/\sqrt{n-1}} = \frac{-0.7778}{\sqrt{2.6172/\sqrt{8}}} = \frac{-0.7778 \times \sqrt{8}}{1.6177} = -1.359$$

The tabulated value of $t_{0.05}$ at 8 difference is 2.31.

Conclusion. Since $| t | = 1.359 < t_{0.05}$, H_0 is accepted, training was not effective in improving performance.

Example 6. *The following figures refer to observations in live independent samples:*

Sample I	*25*	*30*	*28*	*34*	*24*	*20*	*13*	*32*	*22*	*38*
Sample II	*40*	*34*	*22*	*20*	*31*	*40*	*30*	*23*	*36*	*17*

Analyse whether the samples have been drawn from the populations of equal means.

Sol. H_0: The two samples have been drawn from the population of equal means, *i.e.*, there is no significant difference between their means

i.e., $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$ (Two tailed test)

Given n_1 = Sample I size = 10 ; n_2 = Sample II size = 10

To calculate the two sample mean and sum of squares of deviation from mean. Let X_1 be the Sample I and X_2 be the Sample II.

X_1	25	30	28	34	24	20	13	32	22	38
$X_1 - \overline{X}_1$	– 1.6	3.4	1.4	7.4	– 2.6	– 6.6	– 13.6	5.4	4.6	11.4
$(X_1 - \overline{X}_1)^2$	2.56	11.56	1.96	54.76	6.76	43.56	184.96	29.16	21.16	129.96
X_2	40	34	22	20	31	40	30	23	36	17
$X_2 - \overline{X}_2$	10.7	4.7	– 7.3	– 9.3	1.7	10.7	0.7	– 6.3	6.7	– 12.3
$(X_2 - \overline{X}_2)^2$	114.49	22.09	53.29	86.49	2.89	114.49	0.49	39.67	44.89	151.29

$$\overline{X}_1 = \sum_{i=1}^{10} \frac{X_1}{n_1} = 26.6 \qquad \overline{X}_2 = \sum_{i=1}^{10} \frac{X_2}{n_2} = \frac{293}{10} = 29.3$$

$$\Sigma(X_1 - \overline{X}_1)^2 = 486.4 \qquad \Sigma(X_2 - \overline{X}_2)^2 = 630.08$$

$$s^2 = \frac{1}{n_1 + n_2 - 2}[\Sigma(X_1 - \overline{X}_1)^2 + \Sigma(X_2 - \overline{X}_2)^2]$$

$$= \frac{1}{10 + 10 - 2}[486.4 + 630.08] = 62.026$$

$$\therefore \qquad s = 7.875$$

Under H_0 the test statistic is given by

$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{26.6 - 29.3}{7.875\sqrt{\frac{1}{10} + \frac{1}{10}}} = -0.7666 \sim t(n_1 + n_2 - 2 \text{ difference})$$

$| t | = 0.7666.$

Conclusion. The tabulated value of t at 5% level of significance for 18 difference is 2.1. Since the calculated value $| t | = 0.7666 < t_{0.05}$, H_0 is accepted.

There is no significant difference between their means.

The two samples have been drawn from the populations of equal means.

ASSIGNMENT 8.7

1. The mean life of 10 electric motors was found to be 1450 hrs with a standard deviation of 423 hrs. A second sample of 17 motors chosen from a different batch showed a mean life of 1280 hrs with a standard deviation of 398 hrs. Is there a significant difference between means of the two samples ?

2. The scores obtained by a group of 9 regular course students and another group of 11 part time course students in a test are given below:

Regular:	56	62	63	54	60	51	67	69	58		
Part time:	62	70	71	62	60	56	75	64	72	68	66

Examine whether the scores obtained by regular students and part time students differ significantly at 5% and 1% level of significance.

3. A group of 10 boys fed on diet A and another group of 8 boys fed on a different diet B recorded the following increase in weight (kgs):

Diet A:	5	6	8	1	12	4	3	9	6	10
Diet B:	2	3	6	8	10	1	2	8		

Does it show the superiority of diet A over the diet B?

4. Two independent samples of sizes 7 and 9 have the following values:

Sample A:	10	12	10	13	14	11	10		
Sample B:	10	13	15	12	10	14	11	12	11

Test whether the difference between the means is significant.

5. To compare the prices of a certain product in two cities, 10 shops were visited at random in each city. The price was noted below:

City 1:	61	63	56	63	56	63	59	56	44	61
City 2:	55	54	47	59	51	61	57	54	64	58

Test whether the average prices can be said to be the same in two cities.

6. The average number of articles produced by two machines per day are 200 and 250 with standard deviation 20 and 25 respectively on the basis of records of 25 days production. Are both machines equally efficient at 5% level of significance?

8.15 SNEDECOR'S VARIANCE RATIO TEST OR F-TEST

In testing the significance of the difference of two means of two samples, we assumed that the two samples came from the same population or populations with equal variance. The object of the F-test is to discover whether two independent estimates of population variance differ significantly or whether the two samples may be regarded as drawn from the normal populations having the same variance. Hence before applying the t-test for the significance of the difference of two means, we have to test for the equality of population variance by using the F-test.

Let n_1 and n_2 be the sizes of two samples with variance s_1^2 and s_2^2. The estimate of the population variance based on these samples is $s_1^2 = \dfrac{n_1 s_1^2}{n_1 - 1}$ and $s_2^2 = \dfrac{n_2 s_2^2}{n_2 - 1}$. The degrees of freedom of these estimates are $\nu_1 = n_1 - 1$, $\nu_2 = n_2 - 1$.

To test whether these estimates, s_1^2 and s_2^2, are significantly different or if the samples may be regarded as drawn from the same population or from two populations with same variance σ^2, we set-up the null hypothesis

$$H_0: \sigma_1^2 = \sigma_2^2 = \sigma^2,$$

i.e., the independent estimates of the common population do not differ significantly.

To carry out the test of significance of the difference of the variances we calculate the test statistic $F = \dfrac{s_1^2}{s_2^2}$, the Numerator is greater than the Denominator, *i.e.*, $s_1^2 > s_2^2$.

Conclusion. If the calculated value of F exceeds $F_{0.05}$ for $(n_1 - 1)$, $(n_2 - 1)$ degrees of freedom given in the table, we conclude that the ratio is significant at 5% level.

We conclude that the sample could have come from two normal population with same variance.

The assumptions on which the F-test is based are:

1. *The populations for each sample must be normally distributed.*
2. *The samples must be random and independent.*

3. *The ratio of σ_1^2 to σ_2^2 should be equal to 1 or greater than 1. That is why we take the larger variance in the Numerator of the ratio.*

Applications. *F-test is used to test*

(*i*) *whether two independent samples have been drawn from the normal populations with the same variance σ^2.*

(*ii*) *Whether the two independent estimates of the population variance are homogeneous or not.*

EXAMPLES

Example 1. *Two random samples drawn from 2 normal populations are as follows:*

A	*17*	*27*	*18*	*25*	*27*	*29*	*13*	*17*
B	*16*	*16*	*20*	*27*	*26*	*25*	*21*	

Test whether the samples are drawn from the same normal population.

Sol. To test if two independent samples have been drawn from the same population we have to test (*i*) equality of the means by applying the *t*-test and (*ii*) equality of population variance by applying F-test.

Since the *t*-test assumes that the sample variances are equal, we shall first apply the F-test.

F-test. 1. **Null hypothesis H_0:** $\sigma_1^2 = \sigma_2^2$ *i.e.*, the population variance do not differ significantly.

Alternative hypothesis. $H_1: \sigma_1^2 \neq \sigma_2^2$

Test statistic: $F = \dfrac{s_1^2}{s_2^2}$, (if $s_1^2 > s_2^2$)

Computations for s_1^2 and s_2^2

X_1	$X_1 - \overline{X}_1$	$(X_1 - \overline{X}_1)^2$	X_2	$X_2 - \overline{X}_2$	$(X_2 - \overline{X}_2)^2$
17	– 4.625	21.39	16	– 2.714	7.365
27	5.735	28.89	16	– 2.714	7.365
18	– 3.625	13.14	20	1.286	1.653
25	3.375	11.39	27	8.286	68.657
27	5.735	28.89	26	7.286	53.085
29	7.735	54.39	25	6.286	39.513
13	– 8.625	74.39	21	2.286	5.226
17	– 4.625	21.39			

$$\bar{X}_1 = 21.625;\ n_1 = 8;\ \Sigma(X_1 - \bar{X}_1)^2 = 253.87$$

$$\bar{X}_2 = 18.714;\ n_2 = 7;\ \Sigma(X_2 - \bar{X}_2)^2 = 182.859$$

$$s_1^2 = \frac{\Sigma(X_1 - \bar{X}_1)^2}{n_1 - 1} = \frac{253.87}{7} = 36.267;$$

$$s_2^2 = \frac{\Sigma(X_2 - \bar{X}_2)^2}{n_2 - 1} = \frac{182.859}{6} = 30.47$$

$$F = \frac{s_1^2}{s_2^2} = \frac{36.267}{30.47} = 1.190.$$

Conclusion. The table value of F for $\nu_1 = 7$ and $\nu_2 = 6$ degrees of freedom at 5% level is 4.21. The calculated value of F is less than the tabulated value of F. $\therefore$ H_0 is accepted. Hence we conclude that the variability in two populations is same.

t-test: Null hypothesis. H_0: $\mu_1 = \mu_2$ *i.e.,* the population means are equal.

Alternative hypothesis. H_1: $\mu_1 \neq \mu_2$

Test of statistic

$$s^2 = \frac{\Sigma(X_1 - \bar{X}_1)^2 + \Sigma(X_2 - \bar{X}_2)^2}{n_1 + n_2 - 2} = \frac{253.87 + 182.859}{8 + 7 - 2} = 33.594$$

$\therefore \quad s = 5.796$

$$t = \frac{\bar{X}_1 - \bar{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{21.625 - 18.714}{5.796\sqrt{\frac{1}{8} + \frac{1}{7}}} = 0.9704 \sim t(n_1 + n_2 - 2) \text{ difference}$$

Conclusion. The tabulated value of t at 5% level of significance for 13 difference is 2.16.

The calculated value of t is less than the tabulated value. H_0 is accepted, *i.e.*, there is no significant difference between the population mean. *i.e.,* $\mu_1 = \mu_2$. $\therefore$ We conclude that the two samples have been drawn from the same normal population.

Example 2. *Two independent sample of sizes 7 and 6 had the following values:*

Sample A	*28*	*30*	*32*	*33*	*31*	*29*	*34*
Sample B	*29*	*30*	*30*	*24*	*27*	*28*	

Examine whether the samples have been drawn from normal populations having the same variance.

Sol. H_0: The variance are equal. *i.e.,* $\sigma_1^2 = \sigma_2^2$

i.e., the samples have been drawn from normal populations with same variance.

$$H_1: \sigma_1^2 \neq \sigma_2^2$$

Under null hypothesis, the test statistic $F = \dfrac{s_1^2}{s_2^2} (s_1^2 > s_2^2)$

Computations for s_1^2 and s_2^2

X_1	$X_1 - \overline{X}_1$	$(X_1 - \overline{X}_1)^2$	X_2	$X_2 - \overline{X}_2$	$(X_2 - \overline{X}_2)^2$
28	– 3	9	29	1	1
30	– 1	1	30	2	4
32	1	1	30	2	4
33	2	4	24	– 4	16
31	0	0	27	– 1	1
29	– 2	4	28	0	0
34	3	9			
		28			26

$$\overline{X}_1 = 31, \quad n_1 = 7; \quad \Sigma(X_1 - \overline{X}_1)^2 = 28$$

$$\overline{X}_2 = 28, \quad n_2 = 6; \quad \Sigma(X_2 - \overline{X}_2)^2 = 26$$

$$s_1^2 = \frac{\Sigma(X_1 - \overline{X}_1)^2}{n_1 - 1} = \frac{28}{6} = 4.666; \quad s_2^2 = \frac{\Sigma(X_2 - \overline{X}_2)^2}{n_2 - 1} = \frac{26}{5} = 5.2$$

$$F = \frac{s_2^2}{s_1^2} = \frac{5.2}{4.666} = 1.1158. \qquad (\because \ s_2^2 > s_1^2)$$

Conclusion. The tabulated value of F at $\nu_1 = 6 - 1$ and $\nu_2 = 7 - 1$ difference for 5% level of significance is 4.39. Since the tabulated value of F is less than the calculated value, H_0 is accepted, *i.e.,* there is no significant difference between the variance. The samples have been drawn from the normal population with same variance.

Example 3. *The two random samples reveal the following data:*

Sample number	*Size*	*Mean*	*Variance*
I	*16*	*440*	*40*
II	*25*	*460*	*42*

Test whether the samples come from the same normal population.

Sol. A normal population has two parameters namely the mean μ and the variance σ^2. To test whether the two independent samples have been drawn from the same normal population, we have to test

(*i*) the equality of means (*ii*) the equality of variance.

Since the *t*-test assumes that the sample variance are equal, we first apply F-test.

F-test: Null hypothesis. $\sigma_1^2 = \sigma_2^2$

The population variance do not differ significantly.

Alternative hypothesis. $\sigma_1^2 \neq \sigma_2^2$

Under the null hypothesis the test statistic is given by $F = \frac{s_1^2}{s_2^2}$, $(s_1^2 > s_2^2)$

Given: $n_1 = 16$, $n_2 = 25$; $s_1^2 = 40$, $s_2^2 = 42$

$$\therefore \qquad F = \frac{s_1^2}{s_2^2} = \frac{\dfrac{n_1 s_1^2}{n_1 - 1}}{\dfrac{n_2 s_2^2}{n_2 - 1}} = \frac{16 \times 40}{15} \times \frac{24}{25 \times 42} = 0.9752.$$

Conclusion. The calculated value of F is 0.9752. The tabulated value of F at 16 – 1, 25 – 1 difference for 5% level of significance is 2.11.

Since the calculated value is less than that of the tabulated value, H_0 is accepted, the population variance are equal.

t-test: Null hypothesis. H_0: $\mu_1 = \mu_2$ *i.e.,* the population means are equal.

Alternative hypothesis. H_1: $\mu_1 \neq \mu_2$

Given: $n_1 = 16$, $n_2 = 25$, $\overline{X}_1 = 440$, $\overline{X}_2 = 460$

$$s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{16 \times 40 + 25 \times 42}{16 + 25 - 2} = 43.333$$

$$\therefore \qquad s = 6.582$$

$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \frac{440 - 460}{6.582\sqrt{\dfrac{1}{16} + \dfrac{1}{25}}}$$

$$= -9.490 \text{ for } (n_1 + n_2 - 2) \text{ difference}$$

Conclusion. The calculated value of $|\ t\ |$ is 9.490. The tabulated value of t at 39 difference for 5% level of significance is 1.96.

Since the calculated value is greater than the tabulated value, H_0 is rejected, *i.e.,* there is significant difference between means. *i.e.,* $\mu_1 \neq \mu_2$.

Since there is significant difference between means, and no significant difference between variance, we conclude that the samples do not come from the same normal population.

ASSIGNMENT 8.8

1. From the following two sample values, find out whether they have come from the same population:

Sample 1	17	27	18	25	27	29	27	23	17
Sample 2	16	16	20	16	20	17	15	21	

2. The daily wages in Rupees of skilled workers in two cities are as follows:

	Size of sample of workers	*Standard deviation of wages in the sample*
City A	16	25
City B	13	32

3. The standard deviation calculated from two random samples of sizes 9 and 13 are 2.1 and 1.8 respectively. Can the samples be regarded as drawn from normal populations with the same standard deviation?

4. Two independent samples of size 8 and 9 had the following values of the variables:

Sample I	20	30	23	25	21	22	23	24	
Sample II	30	31	32	34	35	29	28	27	26

Do the estimates of the population variance differ significantly?

8.16 CHI-SQUARE (χ^2) TEST

When a coin is tossed 200 times, the theoretical considerations lead us to expect 100 heads and 100 tails. But in practice, these results are rarely achieved. The quantity χ^2 (a Greek letter, pronounced as chi-square) describes the magnitude of discrepancy between theory and observation. If $\chi = 0$, the observed and expected frequencies completely coincide. The greater the discrepancy between the observed and expected frequencies, the greater is the value of χ^2. Thus χ^2 **affords a measure of the correspondence between theory and observation.**

If O_i ($i = 1, 2, ..., n$) is a set of observed (experimental) frequencies and E_i ($i = 1, 2, ..., n$) is the corresponding set of expected (theoretical or hypothetical) frequencies, then, χ^2 **is defined as**

$$\chi^2 = \sum_{i=1}^{n} \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

where $\Sigma O_i = \Sigma E_i = N$ (total frequency) and degrees of freedom (difference)

$= (n - 1)$.

NOTE *(i) If $\chi^2 = 0$, the observed and theoretical frequencies agree exactly.*

(ii) If $\chi^2 > 0$ they do not agree exactly.

8.16.1 Degrees of Freedom

While comparing the calculated value of χ^2 with the table value, we have to determine the degrees of freedom.

If we have to choose any four numbers whose sum is 50, we can exercise our independent choice for any three numbers only, the fourth being 50 minus the total of the three numbers selected. Thus, though we were to choose any four numbers, our choice was reduced to three because of one condition imposed. There was only one restraint on our freedom and our degrees of freedom were $4 - 1 = 3$. If two restrictions are imposed, our freedom to choose will be further curtailed and degrees of freedom will be $4 - 2 = 2$.

In general, the number of degrees of freedom is the total number of observations less the number of independent constraints imposed on the observations. Degrees of freedom (difference) are usually denoted by ν (the letter 'nu' of the Greek alphabet).

Thus, $\nu = n - k$, where k is the number of independent constraints in a set of data of n observations.

NOTE *(i) For a $p \times q$ contingency table (p columns and q rows), $\nu = (p - 1)(q - 1)$*

(ii) In the case of a contingency table, the expected frequency of any class

$$= \frac{\textit{Total of rows in which it occurs} \times \textit{Total of columns in which it occurs}}{\textit{Total number of observations}}$$

8.16.2 Applications

χ^2 test is one of the simplest and the most general test known. It is applicable to a very large number of problems in practice which can be summed up under the following heads:

(*i*) as a test of goodness of fit.
(*ii*) as a test of independence of attributes.
(*iii*) as a test of homogeneity of independent estimates of the population variance.
(*iv*) as a test of the hypothetical value of the population variance s^2.
(*v*) as a list to the homogeneity of independent estimates of the population correlation coefficient.

8.16.3 Conditions for Applying χ^2 Test

Following are the conditions which should be satisfied before χ^2 test can be applied:

(*a*) N, the total number of frequencies should be large. It is difficult to say what constitutes largeness, but as an arbitrary figure, we may say that **N should be atleast 50,** however small the number of cells.

(*b*) No theoretical cell-frequency should be small. Here again, it is difficult to say what constitutes smallness, but 5 should be regarded as the very minimum and **10 is better.** If small theoretical frequencies occur (*i.e.*, < 10), the difficulty is overcome by grouping two or more classes together before calculating (O – E). **It is important to remember that the number of degrees of freedom is determined with the number of classes after regrouping.**

(*c*) The constraints on the cell frequencies, if any, should be linear.

NOTE *If any one of the theoretical frequency is less than 5, then we apply a corrected given by F Yates, which is usually known as 'Yates correction for continuity', we add 0.5 to the cell frequency which is less than 5 and adjust the remaining cell frequency suitably so that the marginal total is not changed.*

8.17 THE χ^2 DISTRIBUTION

For large sample sizes, the sampling distribution of χ^2 can be closely approximated by a continuous curve known as the chi-square distribution. The probability function of χ^2 distribution is given by

$$f(\chi^2) = c(\chi^2)^{(\nu/2-1)}\, e^{-x^2/2}$$

where e = 2.71828, ν = number of degrees of freedom; c = a constant depending only on ν.

Symbolically, the degrees of freedom are denoted by the symbol ν or by difference and are obtained by the rule $\nu = n - k$, where k refers to the number of independent constraints.

In general, when we fit a binomial distribution the number of degrees of freedom is one less than the number of classes; when we fit a Poisson distribution the degrees of freedom are 2 less than the number of classes, because we use the total frequency and the arithmetic mean to get the parameter of the Poisson distribution. When we fit a normal curve the number of degrees of freedom are 3 less than the number of classes, because in this fitting we use the total frequency, mean and standard deviation.

If the data is given in a series of "n" numbers then degrees of freedom

$= n - 1.$

In the case of Binomial distribution difference $= n - 1$

In the case of Poisson distribution difference $= n - 2$

In the case of Normal distribution difference $= n - 3.$

8.18 χ^2 TEST AS A TEST OF GOODNESS OF FIT

χ^2 test enables us to ascertain how well the theoretical distributions such as Binomial, Poisson or Normal etc. fit empirical distributions, *i.e.*, distributions obtained from sample data. If the **calculated value of χ^2 is less than the table value** at a specified level (generally 5%) of significance, the **fit is considered to be good,** *i.e.*, the divergence between actual and expected frequencies is attributed to fluctuations of simple sampling. If the calculated value of χ^2 is greater than the table value, the fit is considered to be poor.

EXAMPLES

Example 1. *The following table gives the number of accidents that took place in an industry during various days of the week. Test if accidents are uniformly distributed over the week.*

Day	*Mon*	*Tue*	*Wed*	*Thu*	*Fri*	*Sat*
Number of accidents	*14*	*18*	*12*	*11*	*15*	*14*

Sol. Null hypothesis H_0: The accidents are uniformly distributed over the week.

Under this H_0, the expected frequencies of the accidents on each of these days $= \dfrac{84}{6} = 14$

Observed frequency O_i	14	18	12	11	15	14
Expected frequency E_i	14	14	14	14	14	14
$(O_i - E_i)^2$	0	16	4	9	1	0

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{30}{14} = 2.1428.$$

Conclusion. Table value of χ^2 at 5% level for (6 – 1 = 5 d.f.) is 11.09.

Since the calculated value of χ^2 is less than the tabulated value, H_0 is accepted, the accidents are uniformly distributed over the week.

Example 2. *A die is thrown 270 times and the results of these throws are given below*:

Number appeared on the die	*1*	*2*	*3*	*4*	*5*	*6*
Frequency	*40*	*32*	*29*	*59*	*57*	*59*

Test whether the die is biased or not.

Sol. Null hypothesis H_0: Die is unbiased.

Under this H_0, the expected frequencies for each digit is $\frac{276}{6} = 46$.

To find the value of χ^2

O_i	40	32	29	59	57	59
E_i	46	46	46	46	46	46
$(O_i - E_i)^2$	36	196	289	169	121	169

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{980}{46} = 21.30.$$

Conclusion. Tabulated value of χ^2 at 5% level of significance for (6 – 1 = 5) d.f. is 11.09. Since the calculated value of χ^2 = 21.30 > 11.07 the tabulated value, H_0 is rejected.

i.e., die is not unbiased or die is biased.

Example 3. *The following table shows the distribution of digits in numbers chosen at random from a telephone directory*:

Digits	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
Frequency	*1026*	*1107*	*997*	*966*	*1075*	*933*	*1107*	*972*	*964*	*853*

Test whether the digits may be taken to occur equally frequently in the directory.

Sol. Null hypothesis H_0: The digits taken in the directory occur equally frequently.

i.e., there is no significant difference between the observed and expected frequency.

Under H_0, the expected frequency is given by $= \dfrac{10{,}000}{10} = 1000$

To find the value of χ^2

O_i	1026	1107	997	996	1075	1107	933	972	964	853
E_i	1000	1000	1000	1000	1000	1000	1107	1000	1000	1000
$(O_i - E_i)^2$	676	11449	9	1156	5625	11449	4489	784	1296	21609

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{58542}{1000} = 58.542.$$

Conclusion. The tabulated value of χ^2 at 5% level of significance for 9 difference is 16.919. Since the calculated value of χ^2 is greater than the tabulated value, H_0 is rejected.

There is significant difference between the observed and theoretical frequency.

The digits taken in the directory do not occur equally frequently.

Example 4. *Records taken of the number of male and female births in 800 families having four children are as follows:*

Number of male births	*0*	*1*	*2*	*3*	*4*
Number of female births	*4*	*3*	*2*	*1*	*0*
Number of families	*32*	*178*	*290*	*236*	*94*

Test whether the data are consistent with the hypothesis that the Binomial law holds and the chance of male birth is equal to that of female birth, namely $p = q = 1/2$.

Sol. H_0: The data are consistent with the hypothesis of equal probability for male and female births, *i.e.*, $p = q = 1/2$.

We use Binomial distribution to calculate theoretical frequency given by:

$$N(r) = N \times P(X = r)$$

where N is the total frequency. $N(r)$ is the number of families with r male children:

$$P(X = r) = {}^nC_r p^r q^{n-r}$$

where p and q are probability of male and female births, n is the number of children.

$$N(0) = \text{Number of families with 0 male children} = 800 \times {}^4C_0 \left(\frac{1}{2}\right)^4$$

$$= 800 \times 1 \times \frac{1}{2^4} = 50$$

$$N(1) = 800 \times {}^4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 200; \; N(2) = 800 \times {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$$

$$= 300$$

$$N(3) = 800 \times {}^4C_3 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 200; \; N(4) = 800 \times {}^4C_4 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4$$

$$= 50$$

Observed frequency O_i	32	178	290	236	94
Expected frequency E_i	50	200	300	200	50
$(O_i - E_i)^2$	324	484	100	1296	1936
$\frac{(O_i - E_i)^2}{E_i}$	6.48	2.42	0.333	6.48	38.72

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = 54.433.$$

Conclusion. Table value of χ^2 at 5% level of significance for $5 - 1 = 4$ difference is 9.49.

Since the calculated value of χ^2 is greater than the tabulated value, H_0 is rejected.

The data are not consistent with the hypothesis that the Binomial law holds and that the chance of a male birth is not equal to that of a female birth.

NOTE *Since the fitting is Binomial, the degrees of freedom*

$$\nu = n - 1 \text{ i.e., } \nu = 5 - 1 = 4.$$

Example 5. *Verify whether Poisson distribution can be assumed from the data given below:*

Number of defects	*0*	*1*	*2*	*3*	*4*	*5*
Frequency	*6*	*13*	*13*	*8*	*4*	*3*

Sol. H_0: Poisson fit is a good fit to the data.

Mean of the given distribution $= \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{94}{47} = 2$

To fit a Poisson distribution we require m. Parameter $m = \overline{x} = 2$.

By Poisson distribution the frequency of r success is

$$N(r) = N \times e^{-m} \cdot \frac{m^r}{r!}, \text{ N is the total frequency.}$$

$$N(0) = 47 \times e^{-2} \cdot \frac{(2)^0}{0!} = 6.36 \approx 6; \; N(1) = 47 \times e^{-2} \cdot \frac{(2)^1}{1!} = 12.72 \approx 13$$

$$N(2) = 47 \times e^{-2} \cdot \frac{(2)^2}{2!} = 12.72 \approx 13; \; N(3) = 47 \times e^{-2} \cdot \frac{(2)^3}{3!} = 8.48 \approx 9$$

$$N(4) = 47 \times e^{-2} \cdot \frac{(2)^4}{4!} = 4.24 \approx 4; \; N(5) = 47 \times e^{-2} \cdot \frac{(2)^5}{5!} = 1.696 \approx 2.$$

X	0	1	2	3	4	5
O_i	6	13	13	8	4	3
E_i	6.36	12.72	12.72	8.48	4.24	1.696
$\dfrac{(O_i - E_i)^2}{E_i}$	0.2037	0.00616	0.00616	0.02716	0.0135	1.0026

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = 1.2864.$$

Conclusion. The calculated value of χ^2 is 1.2864. Tabulated value of χ^2 at 5% level of significance for $\gamma = 6 - 2 = 4$ d.f. is 9.49. Since the calculated value of χ^2 is less than that of tabulated value. H_0 is accepted *i.e.,* Poisson distribution provides a good fit to the data.

Example 6. *The theory predicts the proportion of beans in the four groups, G_1, G_2, G_3, G_4 should be in the ratio 9: 3: 3: 1. In an experiment with 1600 beans the numbers in the four groups were 882, 313, 287 and 118. Does the experimental result support the theory.*

Sol. H_0: The experimental result support the theory. *i.e.,* there is no significant difference between the observed and theoretical frequency under H_0, the theoretical frequency can be calculated as follows:

$$E(G_1) = \frac{1600 \times 9}{16} = 900; \ E(G_2) = \frac{1600 \times 3}{16} = 300;$$

$$E(G_3) = \frac{1600 \times 3}{16} = 300; \ E(G_4) = \frac{1600 \times 1}{16} = 100$$

To calculate the value of χ^2.

Observed frequency O_i	882	313	287	118
Expected frequency E_i	900	300	300	100
$\frac{(O_i - E_i)^2}{E_i}$	0.36	0.5633	0.5633	3.24

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = 4.7266.$$

Conclusion. The table value of χ^2 at 5% level of significance for 3 difference is 7.815. Since the calculated value of χ^2 is less than that of the tabulated value. Hence H_0 is accepted and the experimental results support the theory.

ASSIGNMENT 8.9

1. The following table gives the frequency of occupance of the digits 0, 1, ..., 9 in the last place in four logarithm of numbers 10–99. Examine if there is any peculiarity.

Digits:	0	1	2	3	4	5	6	7	8	9
Frequency:	6	16	15	10	12	12	3	2	9	5

2. The sales in a supermarket during a week are given below. Test the hypothesis that the sales do not depend on the day of the week, using a significant level of 0.05.

Days:	Mon	Tues	Wed	Thurs	Fri	Sat
Sales:	65	54	60	56	71	84

3. A survey of 320 families with 5 children each revealed the following information:

Number of boys:	5	4	3	2	1	0
Number of girls:	0	1	2	3	4	5
Number of families:	14	56	110	88	40	12

Is this result consistent with the hypothesis that male and female births are equally probable?

4. 4 coins were tossed at a time and this operation is repeated 160 times. It is found that 4 heads occur 6 times, 3 heads occur 43 times, 2 heads occur 69 times, one head occurs 34 times. Discuss whether the coin may be regarded as unbiased?

5. Fit a Poisson distribution to the following data and best the goodness of fit:

x:	0	1	2	3	4
f:	109	65	22	3	1

6. In the accounting department of bank, 100 accounts are selected at random and estimated for errors. The following results were obtained:

Number of errors:	0	1	2	3	4	5	6
Number of accounts:	35	40	19	2	0	2	2

Does this information verify that the errors are distributed according to the Poisson probability law?

7. In a sample analysis of examination results of 500 students, it was found that 180 students failed, 170 secured a third class, 90 secured a second class and the rest, a first class. Do these figures support the general belief that the above categories are in the ratio 4:3:2:1, respectively?

8. What is χ^2–test?

A die is thrown 90 times with the following results:

Face:	1	2	3	4	5	6	Total
Frequency:	10	12	16	14	18	20	90

Use χ^2-test to test whether these data are consistent with the hypothesis that die is unbiased.

Given $\chi^2_{0.05} = 11.07$ for 5 degrees of freedom.

9. A survey of 320 families with 5 children shows the following distribution:

Number of boys & girls:	5 boys & 0 girl	4 boys & 1 girl	3 boys & 2 girls	2 boys & 3 girls	1 boy & 4 girls	0 boy & 5 girls	Total
Number of families:	18	56	110	88	40	8	320

Given that values of χ^2 for 5 degrees of freedom are 11.1 and 15.1 at 0.05 and 0.01 significance level respectively, test the hypothesis that male and female births are equally probable.

8.19 χ^2 TEST AS A TEST OF INDEPENDENCE

With the help of χ^2 test, we can find whether or not two attributes are associated. We take the null hypothesis that there is no association between the attributes under study, *i.e.*, **we assume that the two attributes are independent. If the calculated value of χ^2 is less than the table value** at a specified level (generally 5%) of significance, the hypothesis holds good, *i.e.*, **the attributes are independent** and do not bear any association. On the other hand, if the calculated value of χ^2 is greater than the table value at a specified level of significance, we say that the results of the experiment do not support the hypothesis. In other words, the attributes are associated. Thus a very useful application of χ^2 test is to investigate the relationship between trials or attributes which can be classified into two or more categories.

The sample data set out into two-way table, called **contingency table.**

Let us consider two attributes A and B divided into r classes $A_1, A_2, A_3, \ldots\ldots, A_r$, and B divided into s classes $B_1, B_2, B_3, \ldots\ldots, B_s$. If (A_i), (B_j) represents the number of persons possessing the attributes A_i, B_j respectively, ($i = 1, 2, \ldots\ldots, r$, $j = 1, 2, \ldots\ldots, s$) and $(A_i B_j)$ represent the number of persons possessing attributes A_i and B_j. Also we have $\sum_{i=1}^{r} A_i = \sum_{j=1}^{s} B_j = N$ where N is the total frequency. The contingency table for $r \times s$ is given below:

B \ *A*	A_1	A_2	A_3	$\ldots A_r$	*Total*
B_1	(A_1B_1)	(A_2B_1)	(A_3B_1)	$\ldots\ldots(A_rB_1)$	B_1
B_2	(A_1B_2)	(A_2B_2)	(A_3B_2)	$\ldots\ldots(A_rB_2)$	B_2
B_3	(A_1B_3)	(A_2B_3)	(A_3B_3)	$\ldots\ldots(A_rB_3)$	B_3
......	......	......	......	......	......
......	......	......	......	......	......
B_s	(A_1B_s)	(A_2B_s)	(A_3B_s)	$\ldots\ldots(A_rB_s)$	(B_s)
Total	(A_1)	(A_2)	(A_3)	$\ldots\ldots(A_r)$	N

H_0: Both the attributes are independent. *i.e.,* A and B are independent under the null hypothesis, we calculate the expected frequency as follows:

$P(A_i)$ = Probability that a person possesses the attribute

$$A_i = \frac{(A_i)}{N} \quad i = 1, 2, \ldots\ldots, r$$

$P(B_j)$ = Probability that a person possesses the attribute $B_j = \dfrac{(B_j)}{N}$

$P(A_iB_j)$ = Probability that a person possesses both attributes A_i and B_j

$$= \frac{(A_iB_j)}{N}$$

If $(A_iB_j)_0$ is the expected number of persons possessing both the attributes A_i and B_j

$$(A_iB_j)_0 = NP(A_iB_j) = NP(A_i)(B_j)$$

$$= N\frac{(A_i)}{N}\frac{(B_j)}{N} = \frac{(A_i)(B_j)}{N} \qquad (\because \text{ A and B are independent})$$

Hence $$\chi^2 = \sum_{i=1}^{r}\sum_{j=1}^{s}\left[\frac{[(A_iB_j)-(A_iB_j)_0]^2}{(A_iB_j)_0}\right]$$

which is distributed as a χ^2 variate with $(r-1)(s-1)$ degrees of freedom.

NOTE

1. *For a 2 × 2 contingency table where the frequencies are $\frac{a/b}{c/d}$, χ^2 can be calculated from independent frequencies as*

$$\chi^2 = \frac{(a+b+c+d)(ad-bc)^2}{(a+b)(c+d)(b+d)(a+c)}.$$

2. *If the contingency table is not 2 × 2, then the formula for calculating χ^2 as given in Note 1, can't be used. Hence, we have another formula for calculating the expected frequency $(A_iB_j)_0 = \frac{(A_i)(B_j)}{N}$ i.e., expected frequency in each cell is = $\frac{\text{Product of column total and row total}}{\text{whole total}}$.*

3. *If $\frac{a|b}{c|d}$ is the 2 × 2 contingency table with two attributes, $Q = \frac{ad-bc}{ad+bc}$ is called the coefficient of association. If the attributes are independent then $\frac{a}{b} = \frac{c}{d}$.*

4. ***Yates's Correction.*** *In a 2 × 2 table, if the frequencies of a cell is small, we make Yates's correction to make χ^2 continuous.*

Decrease by $\frac{1}{2}$ those cell frequencies which are greater than expected frequencies, and increase by $\frac{1}{2}$ those which are less than expectation. This will not affect the marginal columns. This correction is known as Yates's correction to continuity.

$$\text{After Yates's correction } \chi^2 = \frac{N\left(bc - ad - \frac{1}{2}N\right)^2}{(a+c)(b+d)(c+d)(a+b)} \quad \text{when } ad - bc < 0$$

$$\chi^2 = \frac{N\left(ad - bc - \frac{1}{2}N\right)^2}{(a+c)(b+d)(c+d)(a+b)} \quad \text{when} \quad ad - bc > 0.$$

EXAMPLES

Example 1. *What are the expected frequencies of 2 × 2 contingency tables given below:*

(*i*)

a	*b*
c	*d*

(*ii*)

2	10
6	6

Sol. Observed frequencies → Expected frequencies

(*i*)

a	*b*	*a* + *b*
c	*d*	*c* + *d*
a + *c*	*b* + *d*	*a* + *b* + *c* + *d* = N

→

$\frac{(a+c)(a+b)}{a+b+c+d}$	$\frac{(b+d)(a+b)}{a+b+c+d}$
$\frac{(a+c)(c+d)}{a+b+c+d}$	$\frac{(b+d)(c+d)}{a+b+c+d}$

Observed frequencies Expected frequencies

(*ii*)

2	10	12
6	6	12
8	16	24

$\frac{8 \times 12}{24} = 4$	$\frac{16 \times 12}{24} = 8$
$\frac{8 \times 12}{24} = 4$	$\frac{16 \times 12}{24} = 8$

Example 2. *From the following table regarding the color of eyes of father and son test if the color of son's eye is associated with that of the father.*

Eye color of son

Eye color of father		*Light*	*Not light*
	Light	471	51
	Not light	148	230

Sol. Null hypothesis H_0: The color of son's eye is not associated with that of the father, *i.e.,* they are independent.

Under H_0, we calculate the expected frequency in each cell as

$$= \frac{\text{Product of column total and row total}}{\text{Whole total}}$$

Expected frequencies are:

Eye color of son / *Eye color of father*	*Light*	*Not light*	*Total*
Light	$\frac{619 \times 522}{900} = 359.02$	$\frac{289 \times 522}{900} = 167.62$	522
Not light	$\frac{619 \times 378}{900} = 259.98$	$\frac{289 \times 378}{900} = 121.38$	378
Total	619	289	900

$$\chi^2 = \frac{(471-359.02)^2}{359.02} + \frac{(51-167.62)^2}{167.62} + \frac{(148-259.98)^2}{259.98} + \frac{(230-121.38)^2}{121.38}$$

$= 261.498.$

Conclusion. Tabulated value of χ^2 at 5% level for 1 difference is 3.841.

Since the calculated value of χ^2 > tabulated value of χ^2, H_0 is rejected. They are dependent, *i.e.,* the color of son's eye is associated with that of the father.

Example 3. *The following table gives the number of good and bad parts produced by each of the three shifts in a factory:*

	Good parts	*Bad parts*	*Total*
Day shift	*960*	*40*	*1000*
Evening shift	*940*	*50*	*990*
Night shift	*950*	*45*	*995*
Total	*2850*	*135*	*2985*

Test whether or not the production of bad parts is independent of the shift on which they were produced.

Sol. Null hypothesis H_0: The production of bad parts is independent of the shift on which they were produced.

The two attributes, production and shifts are independent.

Under H_0, $$\chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{3} \left[\frac{[(A_iB_j)_0 - (A_iB_j)]^2}{(A_iB_j)_0} \right]$$

Calculation of expected frequencies

Let A and B be the two attributes namely production and shifts. A is divided into two classes A_1, A_2 and B is divided into three classes B_1, B_2, B_3.

$$(A_1B_1)_0 = \frac{(A_1)(B_2)}{N} = \frac{(2850) \times (1000)}{2985} = 954.77;$$

$$(A_1B_2)_0 = \frac{(A_1)(B_2)}{N} = \frac{(2850) \times (990)}{2985} = 945.226$$

$$(A_1B_3)_0 = \frac{(A_1)(B_3)}{N} = \frac{(2850) \times (995)}{2985} = 950;$$

$$(A_2B_1)_0 = \frac{(A_2)(B_1)}{N} = \frac{(135) \times (1000)}{2985} = 45.27$$

$$(A_2B_2)_0 = \frac{(A_2)(B_2)}{N} = \frac{(135) \times (990)}{2985} = 44.773;$$

$$(A_2B_3)_0 = \frac{(A_2)(B_3)}{N} = \frac{(135) \times (995)}{2985} = 45.$$

To calculate the value of χ^2

Class	O_i	E_i	$(O_i - E_i)^2$	$(O_i - E_i)^2/E_i$
(A_1B_1)	960	954.77	27.3529	0.02864
(A_1B_2)	940	945.226	27.3110	0.02889
(A_1B_3)	950	950	0	0
(A_2B_1)	40	45.27	27.7729	0.61349
(A_2B_2)	50	44.773	27.3215	0.61022
(A_2B_3)	45	45	0	0
				1.28126

Conclusion. The tabulated value of χ^2 at 5% level of significance for 2 degrees of freedom $(r - 1)(s - 1)$ is 5.991. Since the calculated value of χ^2 is less than the tabulated value, we accept H_0, *i.e.,* the production of bad parts is independent of the shift on which they were produced.

ASSIGNMENT 8.10

1. In a locality 100 persons were randomly selected and asked about their educational achievements. The results are given below:

Education				
		Middle	*High school*	*College*
Sex	Male	10	15	25
	Female	25	10	15

Based on this information can you say the education depends on sex.

2. The following data is collected on two characters:

	Smokers	*Non smokers*
Literate	83	57
Illiterate	45	68

Based on this information can you say that there is no relation between habit of smoking and literacy.

3. In an experiment on the immunisation of goats from anthrax, the following results were obtained. Derive your inferences on the efficiency of the vaccine.

	Died anthrax	*Survived*
Inoculated with vaccine	2	10
Not inoculated	6	6

TABLE 1: Significant values t_v (α) of t-distribution (Two Tail Areas)
$[|t| > t_v(\alpha)] = \alpha$

difference	*Probability (Level of significance)*					
(ν)	*0.50*	*0.10*	*0.05*	*0.02*	*0.01*	*0.001*
1	1.00	6.31	12.71	31.82	63.66	636.62
2	0.82	0.92	4.30	6.97	6.93	31.60
3	0.77	2.32	3.18	4.54	5.84	12.94
4	0.74	2.13	2.78	3.75	4.60	8.61
5	0.73	2.02	2.57	3.37	4.03	6.86
6	0.72	1.94	2.45	3.14	3.71	5.96
7	0.71	1.90	2.37	3.00	3.50	5.41
8	0.71	1.80	2.31	2.90	3.36	5.04
9	0.70	1.83	2.26	2.82	3.25	4.78
10	0.70	1.81	2.23	2.76	3.17	4.59
11	0.70	1.80	2.20	2.72	3.11	4.44
12	0.70	1.78	2.18	2.68	3.06	4.32
13	0.69	1.77	2.16	2.65	3.01	4.22
14	0.69	1.76	2.15	2.62	2.98	4.14
15	0.69	1.75	2.13	2.60	2.95	4.07
16	0.69	1.75	2.12	2.58	2.92	4.02
17	0.69	1.74	2.11	2.57	2.90	3.97
18	0.69	1.73	2.10	2.55	2.88	3.92
19	0.69	1.73	2.09	2.54	2.86	3.88
20	0.69	1.73	2.09	2.53	2.85	3.85
21	0.69	1.72	2.08	2.52	2.83	3.83
22	0.69	1.72	2.07	2.51	2.82	3.79
23	0.69	1.71	2.07	2.50	2.81	3.77
24	0.69	1.71	2.06	2.49	2.80	3.75
25	0.68	1.71	2.06	2.49	2.79	3.73
26	0.68	1.71	2.06	2.48	2.78	3.71
27	0.68	1.70	2.05	2.47	2.77	3.69
28	0.68	1.70	2.05	2.47	2.76	3.67
29	0.68	1.70	2.05	2.46	2.76	3.66
30	0.68	1.70	2.04	2.46	2.75	3.65
∞	0.67	1.65	1.96	2.33	2.58	3.29

TABLE 2: F-Distribution

Values of F for F-Distributions with 0.05 of the Area in The Right Tall

	Degrees of freedom for numerator																		
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*12*	*15*	*20*	*24*	*20*	*40*	*60*	*120*	*∞*
1	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5
3	10.1	9.55	9.28	9.12	9.01	9.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	6.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	3.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	3.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92

19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.17	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.98	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.94	2.40	2.34	2.28	2.24	2.16	2.29	2.01	1.96	1.92	1.87	1.82	1.77	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.64	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

TABLE 3: CHI-SQUARE DISTRIBUTION

Significant Values χ^2 (α) of Chi-Square Distribution Right Tail Areas for Given Probability α, P = P_r ($\chi^2 > \chi^2$ (α)) = α And ν Degrees of Freedom (difference)

Degrees of freedom (ν)	*Probability (Level of significance)*						
	0 = .99	*0.95*	*0.50*	*0.10*	*0.05*	*0.02*	*0.01*
1	.000157	.00393	.455	2.706	3.841	5.214	6.635
2	.0201	.103	1.386	4.605	5.991	7.824	9.210
3	.115	.352	2.366	6.251	7.815	9.837	11.341
4	.297	.711	3.357	7.779	9.488	11.668	13.277
5	.554	1.145	4.351	9.236	11.070	13.388	15.086
6	.872	2.635	5.348	10.645	12.592	15.033	16.812
7	.1.239	2.167	6.346	12.017	14.067	16.622	18.475
8	3.646	2.733	7.344	13.362	15.507	18.168	20.090
9	2.088	3.325	8.343	14.684	16.919	19.679	21.669
10	2.558	3.940	9.340	15.987	18.307	21.161	23.209
11	3.053	4.575	10.341	17.275	19.675	22.618	24.725
12	3.571	5.226	11.340	18.549	21.026	24.054	26.217
13	4.107	5.892	12.340	19.812	22.362	25.472	27.688
14	4.660	6.571	13.339	21.064	23.685	26.873	29.141
15	4.229	7.261	14.339	22.307	24.996	28.259	30.578
16	5.812	7.962	15.338	23.542	26.296	29.633	32.000
17	6.408	8.672	15.338	24.769	27.587	30.995	33.409
18	7.015	9.390	17.338	25.989	28.869	32.346	34.805
19	7.633	10.117	18.338	27.204	30.144	33.687	36.191
20	8.260	10.851	19.337	28.412	31.410	35.020	37.566
21	8.897	11.591	20.337	29.615	32.671	36.343	38.932
22	9.542	12.338	21.337	30.813	33.924	37.659	40.289
23	10.196	13.091	22.337	32.007	35.172	38.968	41.638
24	10.856	13.848	23.337	32.196	36.415	40.270	42.980
25	11.524	14.611	24.337	34.382	37.65	41.566	44.314
26	12.198	15.379	25.336	35.363	38.885	41.856	45.642
27	12.879	16.151	26.336	36.741	40.113	41.140	46.963
28	13.565	16.928	27.336	37.916	41.337	45.419	48.278
29	14.256	17.708	28.336	39.087	42.557	46.693	49.588
30	14.933	18.493	29.336	40.256	43.773	47.962	50.892

NOTE *For degrees of freedom (ν) greater than 30, the quantity* $\sqrt{2\chi^2} - \sqrt{2\nu - 1}$ *may be used as a normal variate with unit variance.*