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Ex: $W = \{(v,w,x,y,z) / \mathbb{R}^5,\ w-x-z=0,\ v=y\}$

$V = \{(v,w,x,y,z) \mid v+x-3y+z=0\}$

Find basis of $W \cap V$.

$$\underbrace{\begin{bmatrix} 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & -3 & 1 \end{bmatrix}}_{A} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

~~$b = \{(1,0,1), (0,1,0), (0,-1,1)\}$~~

~~$B = \{(1,0,0,-1,0), (0,1,0,-2,0), (0,0,1,$~~

$v - y = 0$ $\quad v = 2y$

$w - 2y = 0$ $\quad w = 2y$

$x - 2y + z = 0$

$x + z = 2y$ $\quad (y, 2y, 2y-z, y, z)$

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$$= y \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Ex: $V = \{(x,y,z) \mid x+y-z=0\}$

$$z = x+y$$

$$(x,y,z) = (x, y, x+y)$$

$$= x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Basis} = \{(1,0,1)^T, (0,1,1)^T\}$$

☆ $\mathbb{R}^n = \{(x_1, x_2 \ldots x_n) \mid x_i \in \mathbb{R}\}$

$e_1 = (1, 0, 0 \ldots 0)$

$e_2 = (0, 1, 0, \ldots 0)$

$e_n = (0, 0, 0 \ldots 1)$

$x \in \mathbb{R}^n = x_1 e_1 + x_2 e_2 + x_3 e_3 + \ldots x_n e_n$

$\underbrace{\{e_1 \ldots e_n\}}_{\text{span}(\mathbb{R}^n)}$ are lin independent.

$\downarrow$

form basis of $\mathbb{R}^n$.

★ $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 3 \end{bmatrix}_{\substack{m\times n \\ 2\times 3}}$

Row sp. $= \text{span}(\{1,2,-1\}, \{-1,0,3\})$
$=$ subsp. of $\mathbb{R}^3$ ($\mathbb{R}^n$)

Col. sp. $= \text{span}\{(1,-1), (2,0), (-1,3)\}$
$=$ subsp. of $\mathbb{R}^2$ ($\mathbb{R}^m$)

★ H.W.

If P is invertible then show that rows of P form basis of $\mathbb{F}^n$ where $\mathbb{F}^n$ is space of all row vectors $(a_1, a_2 \dots a_n)$

- P is invertible $n\times n$, $\left.\begin{matrix}\text{rows of } P \\ \text{columns of } P\end{matrix}\right\}$ Basis of $\mathbb{F}^n$

Thm: Let V be a vector space which is spanned by vectors $\{\beta_1, \beta_2 \dots \beta_m\}$ then any lin. indep. subset of V cannot have more than m vectors.

Pf: ~~To prove~~ Any set of vectors $\{\alpha_1 \dots \alpha_n\}$ where $n > m$, is lin. dep.

Let $\{\alpha_1 \dots \alpha_n\}$ be set of n vectors of V

$$x_1\alpha_1 + x_2\alpha_2 + \dots x_n\alpha_n = \sum_{j=1}^{n} x_j\alpha_j \quad \text{——} (1)$$

We know $\{\beta_1 \dots \beta_m\}$ spans V

$$\Rightarrow \alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i \quad \text{——} (2)$$

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$$\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + \dots a_{m1}\beta_m$$

$\vdots$

$$\alpha_n = a_{1n}\beta_1 + a_{2n}\beta_2 + \dots a_{mn}\beta_m$$

from ① & ②,

$$x_1\alpha_1 + \dots + x_n\alpha_n = \sum_{i,j} x_j A_{ij}\beta_i \quad \begin{matrix} i = 1 \text{ to } m \\ j = 1 \text{ to } n \end{matrix}$$

$$= \sum_{i,j} (A_{ij} x_j)\beta_i \quad \begin{matrix} i = [1,m] \\ j = [1,n] \end{matrix}$$

Now look at AX: $\sum A_{ij} x = 0$ $\quad i = [1, m]$

$n > m$, ~~therefore~~ there are m - eq^{n}s & n variables

$\Rightarrow$ There are free variables.

$\Rightarrow$ $AX = 0$ has non-trivial sol^{n}s.

Let ~~us~~ us denote it as $(c_1 \dots c_n)$

$$c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = \sum_{i,j} (A_{ij} c_j)\beta_i$$

$$= 0\beta_1 + 0\beta_2 + \dots 0\beta_m$$

$$= 0$$

$\Rightarrow$ $\alpha_1, \alpha_2 \dots \alpha_n$ are linearly dependent.

Thm: V vector sp. of dimension n then any 2 basis of V have same no. of elements.

Pf: Let B_1 & B_2 are basis of V containing m & n elements resp.
W. l.o.g. let $m > n$.

$B_1 = \{u_1, u_2 \ldots u_m\}$ $m > n$
$B_2 = \{v_1, v_2 \ldots v_n\}$ both are basis

By prev. result, $\{v_1, v_2 \ldots v_n\}$ span V

$\therefore$ Any set containing more than n vectors are linearly dependent.
$\therefore$ $\{u_1 \ldots u_m\}$ are not linearly & indep.
$\therefore$ B_1 with m $(> n)$ ~~no~~ elements will not be basis.

$\therefore$ By contradiction, no. of elements (B_1) = no. of elements (B_2)

☆ No. of elements in basis is called DIMENSION.

☆ Corrollaries:
① If dim. $(V) = n$, any set containing $> n$ elements is lin. dep. set.
② If dim. $(V) = n$ then no set containing $< n$ elements can span V.

$\dim(V) = n$

$\{v_1, v_2\}$ lin. indep. set

$v_3 \notin \text{span}\{v_1, v_2\} \rightarrow$ You cannot write v_3 as lin. combination of v_1, v_2

$\Rightarrow \{v_1, v_2, v_3\}$ is lin. indep. set

||ly $\{v_1, v_2, v_3, v_4\}$ " " " "

⋮

$\{v_1, v_2, v_3, \ldots\ldots, v_n\}$ " " " "

$\dim = n$ then any $(n+1)$ vectors are lin. dep.

In other words, if $\dim(V) = n$ then any linearly independent set containing n vectors forms basis of V.

Computation with basis.

B_1, B_2 two sets of basis.

How are they connected?

Ordered Basis :

$B_1 = \{(0,1), (1,0)\}$

$B_2 = \{(1,0), (0,1)\}$

$B = \{v_1, v_2\}$

$B_1 \neq B_2$

★ Co-ordinate :

$V = \mathbb{R}^3$

$B = \{ \underset{v_1}{(1,1,1)}, \underset{v_2}{(1,1,0)}, \underset{v_3}{(1,0,0)} \}$

$v = (4, -3, 2) \in V$

as B spans V, $v = v_1 x_1 + v_2 x_2 + v_3 x_3$

$(4,-3,2) = x_1(1,1,1) + x_2(1,1,0) + x_3(1,0,0)$

$\Rightarrow \quad 4 = x_1 + x_2 + x_3$

$\Rightarrow \quad -3 = x_3 + x_2$

$\Rightarrow \quad 2 = x_3$

$\Rightarrow \quad x_3 = 2, \quad x_2 = -5, \quad x_1 = 7$

$\begin{bmatrix} 7 \\ -5 \\ 2 \end{bmatrix}$ is called co-ordinate of v wrt B

$v = x_1 v_1 + x_2 v_2 + \dots x_n v_n$

then $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is called co-ordinate vector of v ~~over~~ w.r.t $\{v_1 \dots v_n\}$

x_1 is called 1st component

$\vdots$

x_n is called n^{th} component.

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$$(v_1 \quad v_2 \quad v_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = v_1x_1 + v_2x_2 + v_3x_3$$

$$B \qquad X = V$$

★ $\mathbb{P}_2(\mathbb{R})$ = Vector sp. of all poly. of degree ≤ 2 over $\mathbb{R}$

$$B' = \{(1-x, 1+x, x^2)\}$$

$$a_0 + a_1x + a_2x^2 = \qquad + a_2x^2$$

$$\alpha - \alpha x + \beta + \beta x$$

$$\alpha + \beta = a_0$$

$$\beta - \alpha = a_1$$

$$\beta = \frac{a_1 + a_0}{2}$$

$$\alpha = \frac{a_0 - a_1}{2}$$

$$X = \begin{pmatrix} \frac{a_0 - a_1}{2} \\ \frac{a_0 + a_1}{2} \\ a_2 \end{pmatrix}$$

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★ $\{v_1, v_2, v_3\}$ of vector sp. V of dim^n n.

when w_1 & w_2 $\in$ span $\{v_1, v_2, v_3\}$ = B

$$w_1 = Bx_1$$
$$w_2 = Bx_2$$

$$[v_1, v_2, v_3]\,[x_1\ x_2] = (w_1\ w_2)$$

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} v_1x_1 + v_2x_2 + v_3x_3 \\ v_1y_1 + v_2y_2 + v_3y_1 \end{bmatrix}$$

Row matrix

$$B \quad X_{3\times 2} = (w_1\ w_2)$$

★ $\{(1,0,0), (0,1,0), (1,0,0)\} = B$

$w_1 = (4, -3, 2)$ $w_2 = (1, -1, 0)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -3 & -1 \\ 2 & 0 \end{bmatrix} = (w_1, w_2)$$

★ $B_1 = \{u_1, u_2 \ldots u_n\}$

$B_2 = \{v_1, v_2 \ldots v_n\}$

$B_1 . [\quad]_{n\times n} = B_2$

★ $BX = Y$

Y is vector of V

If B is invertible, $X = B^{-1} Y$.

★ Basis(B) = $\{ v_1, \dots, v_n \}$

Co-ordinate of v w.r.t B = $[v]_B$

★ $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ basis of $\mathbb{R}^2$

$$B = [v_1 \ v_2] = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 + 5x_2 \end{bmatrix}$$

Let $x_1 = x_2 = 1$; $= \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

$$[(4,7)]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} 4 \\ 7 \end{pmatrix} \in \text{span}(v_1, v_2)$$

★ ~~$V \to \dim(n)$, $B = \{v_1 \dots v_n\}$ (basis)~~

~~then $v \in$ span~~

★ $V = \mathbb{F}^n$

$[v_1 \dots v_n]$, $y \in V$ arbitrary vector

we say $y \in \text{span}(v_1 \dots v_n)$ if $\exists$ column

vector x s.t. $(v_1 \dots v_n) x = \begin{bmatrix} y_1 \\ y_n \end{bmatrix} = y$

We say $y_1, y_2 \in \text{span}(v_1, \ldots, v_n)$,

if $\exists\, x_1$ & x_2 s.t. $Bx_1 = y_1$ & $Bx_2 = y_2$

In other words, if we can get matrix $A = [x_1 \;\; x_2]$ s.t.

$$BA = Y \quad , \quad Y = (y_1, y_2)$$

Eg. $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$; vectors of $\mathbb{R}^2$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ -1 & -3 & 7 \end{bmatrix} = \begin{bmatrix} -2 & -7 & 26 \\ -3 & -11 & 45 \end{bmatrix}$$

$w_1 \quad w_2 \quad w_3$

$$[w_1]_B = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \quad [w_2]_B = \begin{bmatrix} -7 \\ -11 \end{bmatrix} \quad [w_3]_B = \begin{bmatrix} 26 \\ 45 \end{bmatrix}$$

Thm: V be vector space of dim n & let $\{v_1 \ldots v_m\}$ by any vectors of V, then any subset of V: $\{w_1 \ldots w_k\}$ is in span $(v_1, \ldots, v_m)$ if $\exists A$ s.t. $[v_1 \ldots v_m] A = (w_1, w_2 \ldots w_k)$

// $V = \mathbb{F}^n$

Eg: $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ - invertible matrix

$$B^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Thm

Pf:

$$X = B^{-1}Y$$
$$= \begin{bmatrix} -5 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}_B = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

Thm: Let A be $n \times n$ matrix with entries in $\mathbb{F}$. The columns of A forms basis of $\mathbb{F}^n$ iff A is invertible.

Pf: Dim^n of $\mathbb{F}^n$ is n over $\mathbb{F}$

A is $n \times n$ matrix so:

Case 1: Suppose columns of A form basis of $\mathbb{F}^n$, then they are lin. indep. they are n of them so $AX = 0$ has only trivial sol^n $\Rightarrow$ A is inv.

Conversely: suppose A is invertible.
$\Rightarrow$ columns of A are lin. indep. & they are n of them which is dim^n of $\mathbb{F}^n$.
$\Rightarrow$ Colums of A form basis of $\mathbb{F}^n$.

map between them which is injective & surjective.

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Thm: If V is vector space over $\mathbb{F}$ of dimension n, then V is isomorphic to $\mathbb{F}^n$.

space of all column vectors $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

Pf: $\dim(V) = n$ ... fix some basis

$B = (v_1, v_2 \ldots v_n)$ of V. Then any vector $v \in V$ will have unique representation $v = x_1 v_1 + x_2 v_2 + \ldots x_n v_n$

Define function $\psi : F^n \to V$ s.t.

$$\psi \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto x_1 v_1 + x_2 v_2 + \ldots v_n x_n$$

ψ is surjective: Because for every $v \in V$ we have unique $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

s.t. $v = BX$.

~~ψ is surjective: Because for every $v \in V$ we have~~

$$\psi\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = v$$

$\therefore \psi$ is surjective.

ψ is injective: Let $\psi(x_1) = \psi(x_2)$

$$X_1 = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \quad X_2 = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

$\Rightarrow \alpha_1 v_1 + \ldots \alpha_n v_n = \beta_1 v_1 + \ldots \beta_n v_n$

$\Rightarrow \alpha_i = \beta_i \quad i = 1$ to n → every element has unique representation.

$\Rightarrow X_1 = X_2$

Rmk: All ...

Change ...

Let ... dim ...

Let $B = $

can ... of ...

Im ...

Matrix

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Rmk: All finite dimensional vectors spaces are classified

if dim. $(V) = n$ then $V \cong \mathbb{F}^n$,
↓
isomorphism.

Change of Basis :

Let V be finite diml vector space of dim^n n,

Let $B = (v_1, v_2, \ldots, v_n)$

$B' = (w_1, w_2, \ldots, w_n)$ are 2 basis of V.

As $w_1, w_2, \ldots w_n$ are basis of V, we can write each v_i as linear comb. of $w_1, w_2, \ldots, w_n$.

In other words, $(v_1, v_2 \ldots v_n) \in \text{span}(w_1 \ldots w_n)$

$\Rightarrow \exists$ matrix P s.t.

$$(w_1, w_2 \ldots w_n) P = (v_1, v_2 \ldots v_n)$$

$$[w_1, w_2 \ldots w_n][x_1 \ldots x_n] = [v_1 \ldots v_n]$$

$$B' P = B$$

$$x_1 = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$[v_1]_{B'} = x_1$

IIIly $[v_i]_{B'} = x_i$ $\quad i = 1$ to n

Matrix P is called matrix of change of basis from B' to B.

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Claim: P is invertible.

Pf: Now we can $(w_1, \dots w_n)$ in terms of $(v_1, \dots v_n)$ because $(v_1 \dots v_n)$ are also basis.

$\Rightarrow \exists P'$ s.t.

$$BP' = B'$$

Also, $B'P = B$

$\therefore$ ~~$\boxed{P^{-1} = P'}$~~

$$\therefore B'PP' = B'$$

$$\Rightarrow PP' = I$$

$$\Rightarrow \boxed{P^{-1} = P'}$$

Because B' is basis (every element has unique representation)

Rmk: For any vector $v \in V$.

$$[v]_B = x \quad \& \quad [v]_{B'} = x'$$

Connection between x & x'?

$\Rightarrow Bx = v \qquad \Rightarrow B'x' = v$

$\Rightarrow B'Px = v \qquad \Rightarrow B'x' = v$

~~$\Rightarrow Bx = B'x'$~~

~~$\Rightarrow B'Px = B'x'$~~

~~$\Rightarrow Px = x'$~~

Px & x' are 2 co-ordinate vectors of v wrt B'.

$$\Rightarrow \boxed{Px = x'}$$

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$\dim(V) = n$

$B = (v_1 \; v_2 \; \ldots \; v_n)$ basis of V

Let $(w_1, w_2, \ldots w_n)$ be ANY n vectors of V s.t. $(w_1, w_2, \ldots w_n) \in$ span B i.e. each w_i = lin. combination of $(v_1, \ldots, v_n)$

Not necessary

∴ ~~i.e. each w_i =~~

$\exists$ matrix A s.t. $BA = (w_1, \ldots w_n)$

claim: $(w_1, \ldots w_n)$ forms basis of V iff A is invertible.

$A = [x_1 \; \ldots \; x_n]$

$x_i = [w_i]_B$

Pf: Let $B' = (w_1, \ldots, w_n)$.

Suppose B' is basis of V, then,

$BA = B' \Rightarrow A$ is matrix of change of basis

$\Rightarrow A$ is invertible.

Suppose A is invertible.

$\Rightarrow B = B'A^{-1}$ as $BA = B'$

$\Rightarrow B \subseteq$ span of B'

$\Rightarrow V = \text{span}(B) \subseteq \text{span}(B')$

$\Rightarrow V \subseteq \text{span}(B')$

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B' has n vectors & $\dim(V) = n$.

$\therefore$ All vectors in B' are lin. indep.

$\therefore$ B' is basis of V.

Rmk: $\dim(V) = n$

If we start with some basis $B = \{v_1 \ldots v_n\}$ then all other basis can be obtained by multiplying any invertible matrix P to B.

$\therefore$ BP is new basis.

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Assignment 3

A1 -

$$A = \begin{bmatrix} 5 & 1 & 1 & 1 & \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}$$

A2 -

$$a u_1 + b u_2 + c u_3 = w$$

$$a + 4b + c = 2$$
$$a + 2c = 1$$
$$2b + c = 5$$
$$2a + b + 3c = 0$$

$$4b - c = 1$$
$$c - b = 2$$
$$3b = 3$$
$$b = 1$$
$$c = 3$$
$$a = -5$$

$$w = -5u_1 + u_2 + 3u_3$$

A3 -

$$a + b + c = 4$$
$$2a + 3b + c = 9 \qquad 2a + 3b = 8$$
$$a + 2b + 3c = 8 \qquad a + 2b = 5$$
$$a + b + 2c = 5 \qquad a + b = 3$$
$$\Rightarrow c = 1 \qquad b = 2$$
$$a = 1$$

$\boxed{\text{Yes}}$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 8 \\ 1 & 1 & 2 & 5 \end{bmatrix}$$

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$S = L(1-x^3+x^2, -x^3+x, 1)$

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A4 - $f = ax^3+bx^2+cx+d$

$f(0) = d$

$f(1) = a+b+c+d$

$\boxed{a+b+c=0}$

$S = \{ax^3+bx^2+cx+d \mid a,b,c,d \in \mathbb{R};\ a+b+c=0\}$

$v_1 = (a, b, -(a+b), d)$

$v_2 = (a', b', -(a'+b'), d')$

$\alpha v_1 + \beta v_2 = (\alpha a+\beta a')x^3 + (\alpha b+\beta b')x^2 + (\alpha c+\beta c')x + d$

$= \alpha(a+b+c) + \beta(a'+b'+c') + d+d'$

$v_1+v_2 = 0 + 0 + d+d' \in S$

$\therefore$ It is subspace of $\mathbb{P}_3$.

A5 - (a) $\alpha A = \alpha A^T$

$\beta B = \beta B^T$

$(\alpha A+\beta B)^T = \alpha A^T + \beta B^T$

$= \alpha A + \beta B$ ✓

(b) $A = I$

$B = -I$

$A+B = 0 \longrightarrow$ not invertible ✗

(c) ✓

(d) $\alpha A = -\alpha A^T$

$\beta B = -\beta B^T$

$(\alpha A+\beta B)^T = -\alpha A - \beta B$ ✓

A6

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A6 (a) ×

(b) $\alpha a_1 = 0$

$\alpha a_1 + \beta a_2 = a_3$

$\alpha (a_1, a_2, a_3) + \beta (a_1', a_2', a_3')$

$(\alpha a_1 + \beta a_1'), (\alpha a_2 + \beta a_2'), (\alpha a_3 + \beta a_3')$

$\alpha (a_1 + 3a_2) + \beta (a_1' + 3a_2') = \alpha a_3 + \beta a_3$

✓

(c) $a_2 = a_1^2$

$\alpha (a_1, a_1^2, \ldots)$

$+ \beta (a_2, a_2^2, \ldots)$

$(\alpha a_1 + \beta a_2, \alpha a_1^2 + \beta a_2^2, \ldots)$

×

(d) $a_1 a_2 = 0$

$\alpha (a_1, a_2, \ldots)$

$+ \beta (a_1', a_2' \ldots)$

$(\alpha a_1 + \beta a_1') \cdot (\alpha a_2 + \beta a_2') = \cancel{\alpha^2 a_1 a_2} + \cancel{\beta^2 a_1' a_2'}$

$+ \alpha\beta\, a_1 a_2' + a_2 a_1'$

×

(e) ✓

A7 - (0), Good
(a)

A8 - $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} a \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ b \end{pmatrix}$

$\begin{pmatrix} a \\ b \end{pmatrix}$

A9 - $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$ $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$ $\begin{pmatrix} a \\ 0 \\ c \end{pmatrix}$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\begin{pmatrix} 0 \\ b \\ c \end{pmatrix}$

A10 - a) $\omega_1 = u + v$
$= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

b) $\omega_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

Ex

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Ex: $B = \{(1,1,1)\ (1,-1,1)\ (1,1,0)\}$

$B' = \{(1,0,0)\ (1,1,0)\ (1,1,1)\}$

$V = \mathbb{R}^3$

$\underbrace{B'P}_{\text{matrix of change of basis}} = B$

$[v]_B = x$

$[v]_{B'} = x'$ } $Px = x'$

Sol^n: $(1,1,1) = a_1(1,0,0) + a_2(1,1,0) + a_3(1,1,1)$

$[(1,1,1)]_{B'} = [0\ 0\ 1]^T$

$(1,-1,1) =$ " " "

$1 = a_1 + a_2 + a_3$

$-1 = a_2 + a_3$

$1 = a_3$

$a_2 = -2$

$a_1 = 2$

$[(1,-1,1)]_{B'} = (2\ -2\ 1)^T$

$|||^{ly}$ ~~$a_1 + a_2 + a_3$~~ $[(1,1,0)]_{B'} = (0\ 1\ 0)^T$

$\therefore P = \begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

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$(x\ y\ z)_{B'} = a_1(1\ 0\ 0) + a_2(1\ 1\ 0) + a_3(1\ 1\ 1)$

Px

$x = a_1 + a_2 + a_3$
$y = a_2 + a_3$
$z = a_3$

$a_3 = z \quad , \quad a_2 = y - z \quad , \quad a_1 = x - y$

$$x' = [(x, y, z)_{B'}] = \begin{bmatrix} x-y \\ y-z \\ z \end{bmatrix}$$

$(x, y, z)_B = a_1(1, 1, 1) + a_2(1, -1, 1) + a_3(1, 1, 0)$

$x = a_1 + a_2 + a_3$
$y = a_1 - a_2 + a_3$
$z = a_1 + a_2$

$\frac{x-y}{2} = a_2$

$a_1 = z - \frac{x-y}{2}$

$a_1 = \frac{2z - x + y}{2}$

$\cancel{a_1 0} \; x - a_3 = x - z$

$$x = [(x, y, z)]_B = \begin{bmatrix} \frac{2z - x + y}{2} \\ \frac{x-y}{2} \\ x - z \end{bmatrix}$$

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$$Px = \begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} (2z - x+y)/2 \\ (x-y)/2 \\ x-z \end{bmatrix}$$

$$= \begin{bmatrix} x-y \\ y \cancel{-x} \cancel{+x} -z \\ z \end{bmatrix}$$

$$Px = x'$$

Linear Transformation

Let V & W are 2 vector spaces over $\mathbb{F}$

A map $T: V \to W$ is called linear transformation, If it satisfies following:

① $\forall u, v \in V$ $T(u+v) = T(u) + T(v)$
② $\forall c \in \mathbb{F}$ $T(cu) = cT(u)$

Rmk: ① $T(0_V) = 0_W$

$0_V = 0_V + 0_V$
$T(0_V) = T(0_V) + T(0_V)$
$\Rightarrow T(0_V) = 0_W$

Ex: ① $T: \mathbb{R}^2 \to \mathbb{R}^2$
$T(x,y) = (x+1, y+3)$ $\longrightarrow$ not linear trans
$T(0,0) = (1,3) \neq (0,0)$

② $T: \mathbb{R}^2 \to \mathbb{R}^2$
$T(v) = v$ Id map
It's linear trans.

③ $T: \mathbb{R}^3 \to \mathbb{R}^3$
$T(x,y,z) \to (x,y,0)$ (✓)

④ Let $A_{m \times n}$ matrix, then A gives linear trans
$T: \mathbb{F}^n \to \mathbb{F}^m$
$T: v \to Av$

$\mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| x_i \in \mathbb{F} \right\}$

$v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

Ex:

Ex:

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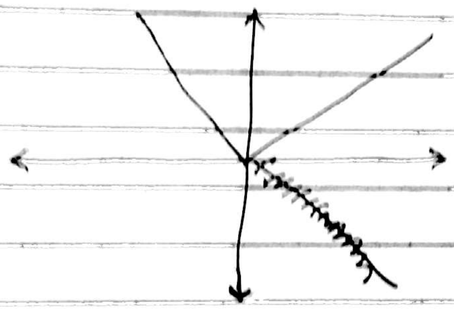
$$Av = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix} \quad \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} \in \mathbb{R}^m$$

Ex: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$A : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} -y \\ x \end{bmatrix}$$



Ex: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$x = (r, \alpha) \longrightarrow$ polar co ordinates

$v = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$

$$Av = \begin{bmatrix} \cancel{r\cos\theta\cos\alpha} - r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$$

A gives rotation by angle θ.

$(r, \alpha) \longrightarrow (r, \theta+\alpha)$

$f : X \to Y$

set of all all $x \in X$ s.t. $f(x) = 0$
is called kernel of f.

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$V, W \to 2$ vector spaces

$T : V \to W$ be lin. trans (LT)

$Ax = b$ $\quad R(T) = \{ T(x) / x \in V \} = \text{Im } T$ ~~range space~~ image space

$Ax = 0$ $\quad N(T) = \{ x \in V / T(x) = 0 \} = \ker T$ null space
kernel

$N(T) \subseteq V$

$R(T) \subseteq W$

Thm: $T : V \to W$ be a LT. ~~from~~
Suppose $\{ v_1, v_2, \ldots, v_n \}$ have property that
$\{ T(v_1), T(v_2) \ldots, T(v_n) \}$ are lin. indep.,
then $\{ v_1, v_2, \ldots, v_n \}$ are lin. indep.

Pf: We have to show $\{ v_1 \ldots v_n \}$ are lin. indep.

Let $c_1 v_1 + c_2 v_2 \ldots + c_n v_n = 0$

$T(\quad \text{"} \quad) = T(0) = 0$

$c_1 T(v_1) + c_2 T(v_2) + \ldots c_n T(v_n) = 0$

But, $\{ T(v_1), \ldots T(v_n) \}$ are lin. indep.

$\Rightarrow \quad c_i = 0 \quad \forall i \in [1, n]$

$\Rightarrow \quad \{ v_1, \ldots, v_n \}$ are lin. indep.

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Rmk: $T: V \to W$

$\{v_1, v_2 \ldots v_m\}$ are basis of V.

It is enough to know how T acts on $\{v_1, \ldots v_m\}$

$v = a_1 v_1 + \ldots a_m v_m$

$T(v) = a_1 T(v_1) + \ldots a_m T(v_m)$

Thm: $T: V \to W$ be a LT, then,

① T is one-one iff $\ker T = \{0\}$

② " " onto " $\operatorname{Im} T = W$

③ T is bijective " $\operatorname{Im} T = W$ & $\ker T = \{0\}$

Pf: ① T is one-one. $T(x_1) = T(x_2) \Leftrightarrow x_1 = x_2$

Let T be one-one. we know $T(0) = 0$.
Let $v \in \ker T$ be any aribitrary vector of $\ker T$

$\Rightarrow T(v) = 0$

$\because T$ is one-one, $\Rightarrow v = 0 \Rightarrow \ker T = \{0\}$

Conversely, Let $\ker T = \{0\}$. Show T is one-one.

$T(x_1) = T(x_2)$

$\Rightarrow T(x_1) - T(x_2) = 0$

$\Rightarrow T(x_1 - x_2) = 0$

$\Rightarrow x_1 - x_2 \in \ker(T)$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

$\therefore T$ is one-one.

② Directly def^n

③ from ① & ②

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Thm: $T: V \to W$ form LT, V, W are vector sp.

Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V, then

(1) If $\{T(v_1), T(v_2) \dots T(v_n)\}$ are lin. indep.
$\Rightarrow$ T is one-one.

(2) If $T(v_1), \dots T(v_n)$ span W, then
T is onto

(3) If $\{T(v_1) \dots T(v_n)\}$ is basis of W, then
T is bijective.

Pf: (1) Any $v \in \ker(T)$
$\Rightarrow v \in V$ s.t. $T(v) = 0$

$$v = a_1 v_1 + \dots a_n v_n \quad \text{— (1)}$$

and

$$0 = a_1 T(v_1) + \dots a_n T(v_n) \longrightarrow \text{lin. indep.}$$

$$\Rightarrow a_i = 0 \quad \forall i \in [1, n] \quad \text{— (2)}$$

$$\Rightarrow v = 0 \quad \text{— from (1) \& (2)}$$

(2) We need to show $\text{Im}(T) = W$

We know $\text{Im} \subseteq W$ — (1)

Let $w \in W$,
but $(T(v_1) + T(v_2) \dots T(v_n))$ spans W.

$$w = a_1 T(v_1) + \dots a_n T(v_n)$$
$$= T(a_1 v_1 + \dots a_n v_n)$$
$$= T(v)$$

$\Rightarrow w \in \text{Im}(T)$

$\Rightarrow W \subseteq \text{Im}(T)$ — (2)

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from ① & ②, $Im(T) = W$

$\therefore$ onto

③ T is one one → ~~T(v_i)~~ $T(v_i)$ are lin indep.

T is onto → ~~T(v_i)~~ $T(v_i)$ are onto

T is bijective → ① + ②

→ ~~Basis~~ $T(v_i)$ are basis

$i \in [1, n]$

Thm: $T: V \to W$ be a linear transformation. Suppose $\{v_1 \ldots v_n\}$ are basis of V

ⓐ
① $R(T)$ is subspace of W.
② $R(T) = span\{T(v_1) \ldots T(v_n)\}$
③ $dim(R(T)) \leq dim. W$

ⓑ ① $N(T)$ is subsp. of V
② $dim(N(T)) \leq dim. V$

ⓒ T is one-one $\iff N(T) = \{0\}$

$\iff \{T(v_i)\}$ $i = 1$ to n form basis of $R(T)$

ⓓ $dim.(R(T)) = dim. V$ iff $N(T) = \{0\}$

Pf: ⓐ ① Let $T(v_1), T(v_2) \in R(T)$

$T(v_1) + T(v_2) = T(v_1 + v_2)$

$\in R(T) \because v_1 + v_2 \in V$

Let $T(v_1) \in R(T)$

$\alpha T(v_1) = T(\alpha v_1) \in R(T)$

$\because \alpha v_1 \in V$

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(c) T is one-one $\Rightarrow$ $T(v_1) \ldots T(v_n)$ are lin. indep. — ①

By ① (b) $T(v_1) \ldots T(v_n)$ ...... span W

$W = \text{span}\{T(v_i)\}$ — ②

from ① & ②, $T(v_i)$ from basis of W

(d) dim ~~dim~~ $V = n$

By (c), if $\ker = \{0\}$ $\Rightarrow$ $T(v_1) \ldots T(v_n)$ is ~~a~~ basis of W

$\Rightarrow \text{Dim. } V = \text{Dim}(R(T))$

☆ Rank nullity theorem :

Let V & W be 2 vector spaces & let $T: V \to W$ be a LT then,

$\dim(V) = \dim.(R(T)) + \dim.(N(T))$
$= \text{rank} + \text{nullity}$