Homework 1

Due 01/31/25

January 22, 2025

Definition 1. *$a \mid b$ ("a divides b") if and only if there exists some integer k such that $b = ak$. Equivalently, $a \mid b$ if and only if b has a remainder of 0 when divided by a (see question 2).*

1. Using the formal definition of divisibility above, prove that there exist positive integers a, b, and c such that $a \mid bc$, but $a \nmid b$ and $a \nmid c$.

   Theorem 1. *The Division Algorithm. For any integers a and b where $b \neq 0$, there exist a unique pair of integers q and r such that $a = qb + r$ and $0 \leq r < b$. The integers q and r are known as the quotient and remainder of $a \div b$, respectively.*

2. Using the formal definition of the remainder above, prove that if n and m are positive integers such that n has a remainder of r when divided by m and $r < \sqrt{m}$, n^2 has a remainder of r^2 when divided by m.
3. Use the formal definition of Big-Oh to prove that if $f(n) = n^x + an^y$, where a, x, and y are positive integers such that $x > y$, $f(n) = O(n^x)$.
4. Use the formal definition of Big-Omega to prove that if $f_1(n)$, $f_2(n)$, $g_1(n)$, and $g_2(n)$ are functions such that $f_1(n) = \Omega(g_1(n))$ and $f_2(n) = \Omega(g_2(n))$, then $f_1(n) + f_2(n) = \Omega(g_1(n) + g_2(n))$.

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Ans.1

Integers a, b and c that bc is multiple of a, but neither b nor c

$a = 6, b = 2, c = 3$

$bc = 2 * 3 = 6$ (multiple of a i.e. 6)

2 is not multiple of 6

3 is not multiple of 6

a, b and c satisfy the conditions.

Ans.2

Let n = qm + r (integers q and r, 0 <= r < m)

Now, n^2 = (qm + r)^2

n^2 = q^2m^2 + 2qmr + r^2

Since, r < $\sqrt{m}$ and r^2 < m

Therefore, n^2 has a remainder of r^2 when divided by m.

Ans.3

Finding a constant C > 0 such that for all n ≥ 1, f(n) ≤ C * n^x

Since x > y, we have n^x ≥ n^y for all n ≥ 1

Therefore, f(n) = n^x + an^y ≤ n^x + an^x = (1 + a)n^x

Putting C = 1 + a, we have f(n) ≤ C * n^x for all n ≥ 1

Thus, f(n) = O(n^x)

Ans.4

Finding a constant C > 0 and an integer N ≥ 1 such that for all n ≥ N, f1(n) ≥ C * g1(n) and f2(n) ≥ C * g2(n)

Adding both inequalities, then f1(n) + f2(n) ≥ C * (g1(n) + g2(n))

So, f1(n) + f2(n) = Ω(g1(n) + g2(n)).