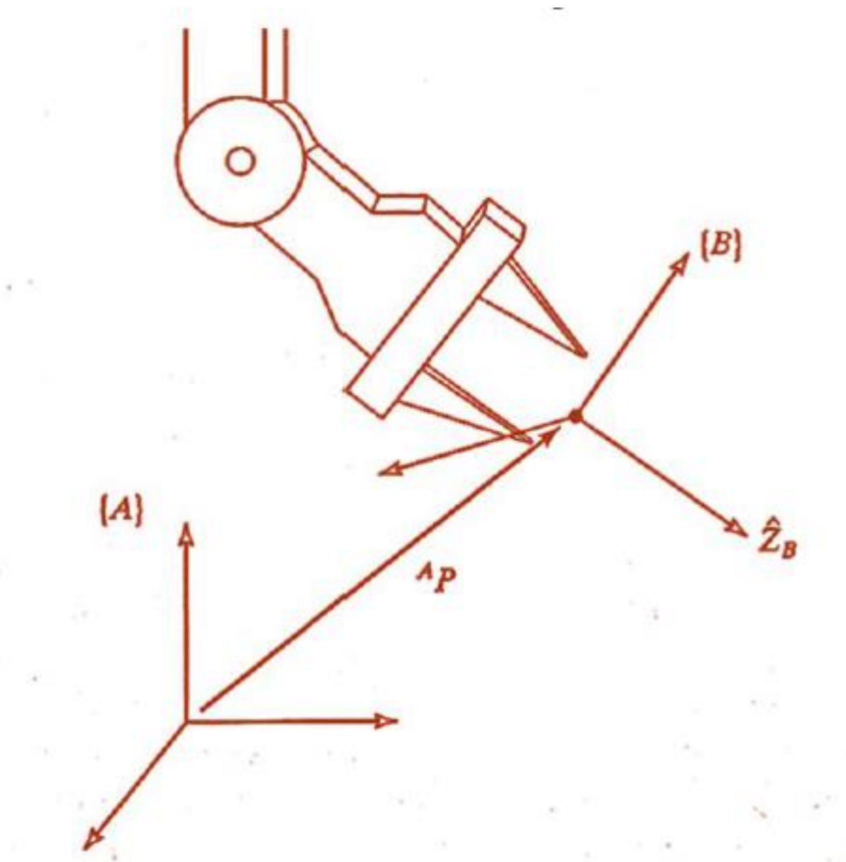
ME512/ME6106: Mobile Robotics

Geometry Fundamentals: Description of Transformations

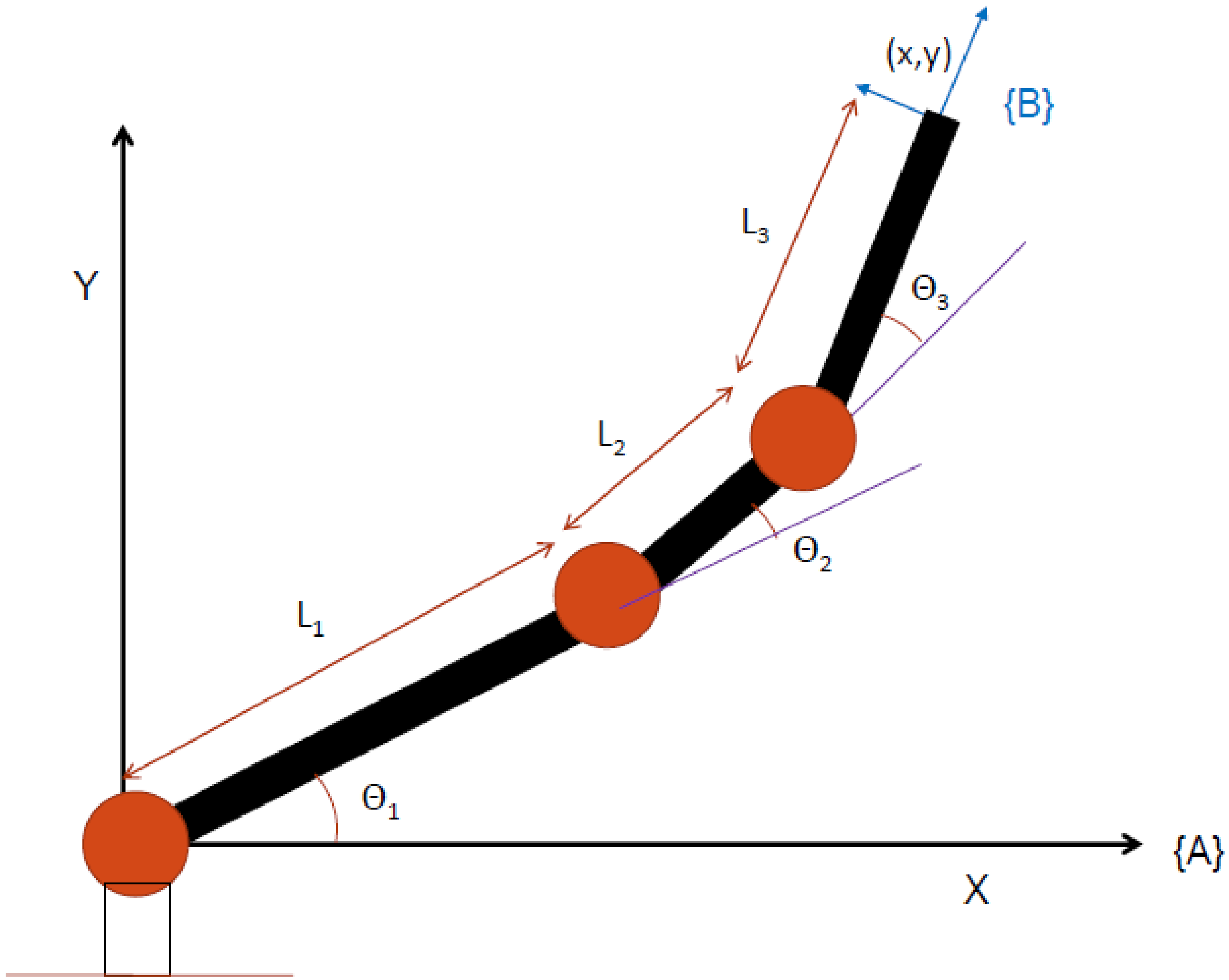
Dr. Karali Patra
Professor
Department of Mechanical Engineering
IIT Patna
Room no. 111, Block 3
Email: kpatra@iitp.ac.in
Whatsapp: 7295834306

Coordinate Systems in Robots



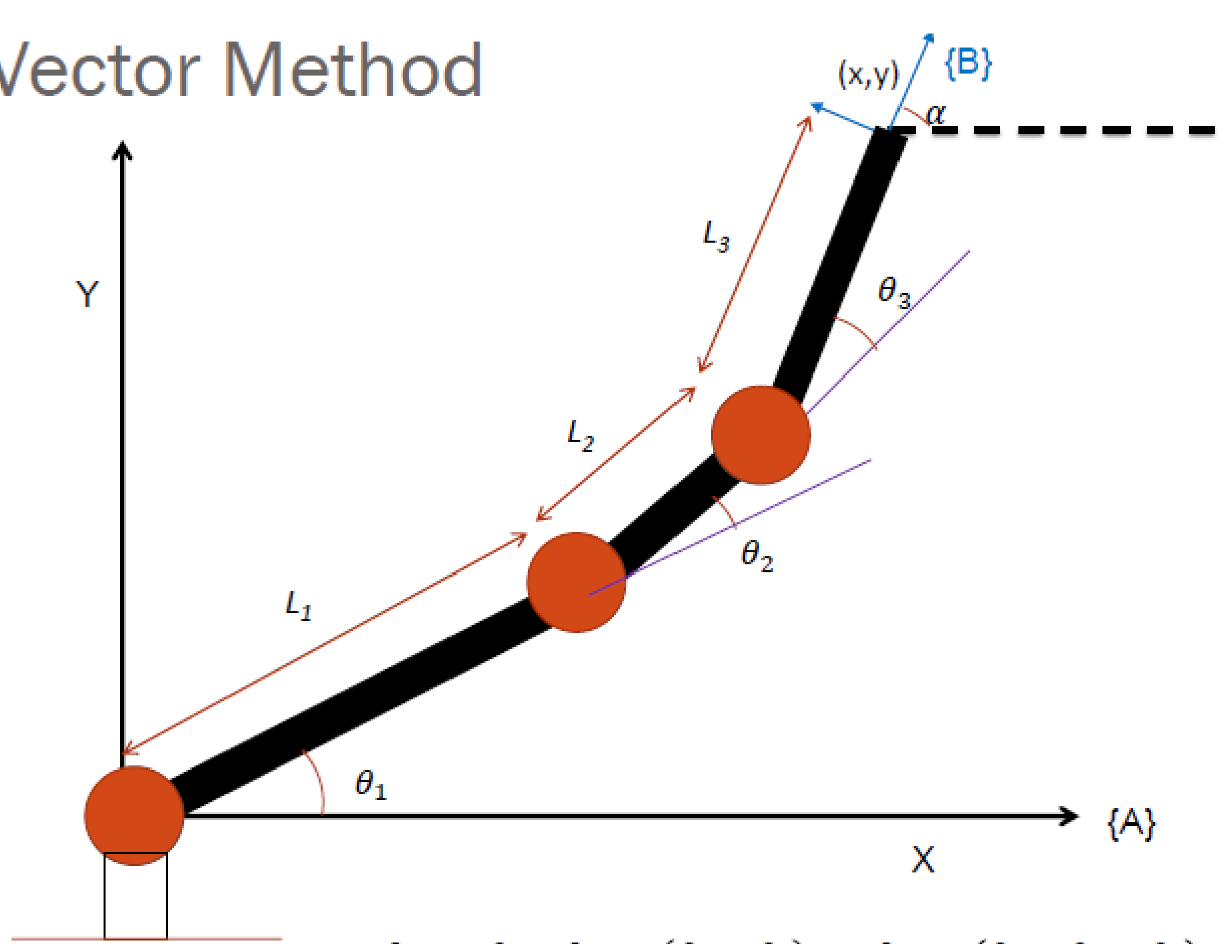
How to express frame {B} in terms of frame {A}?

Forward Kinematics Example



Forward Kinematics Example

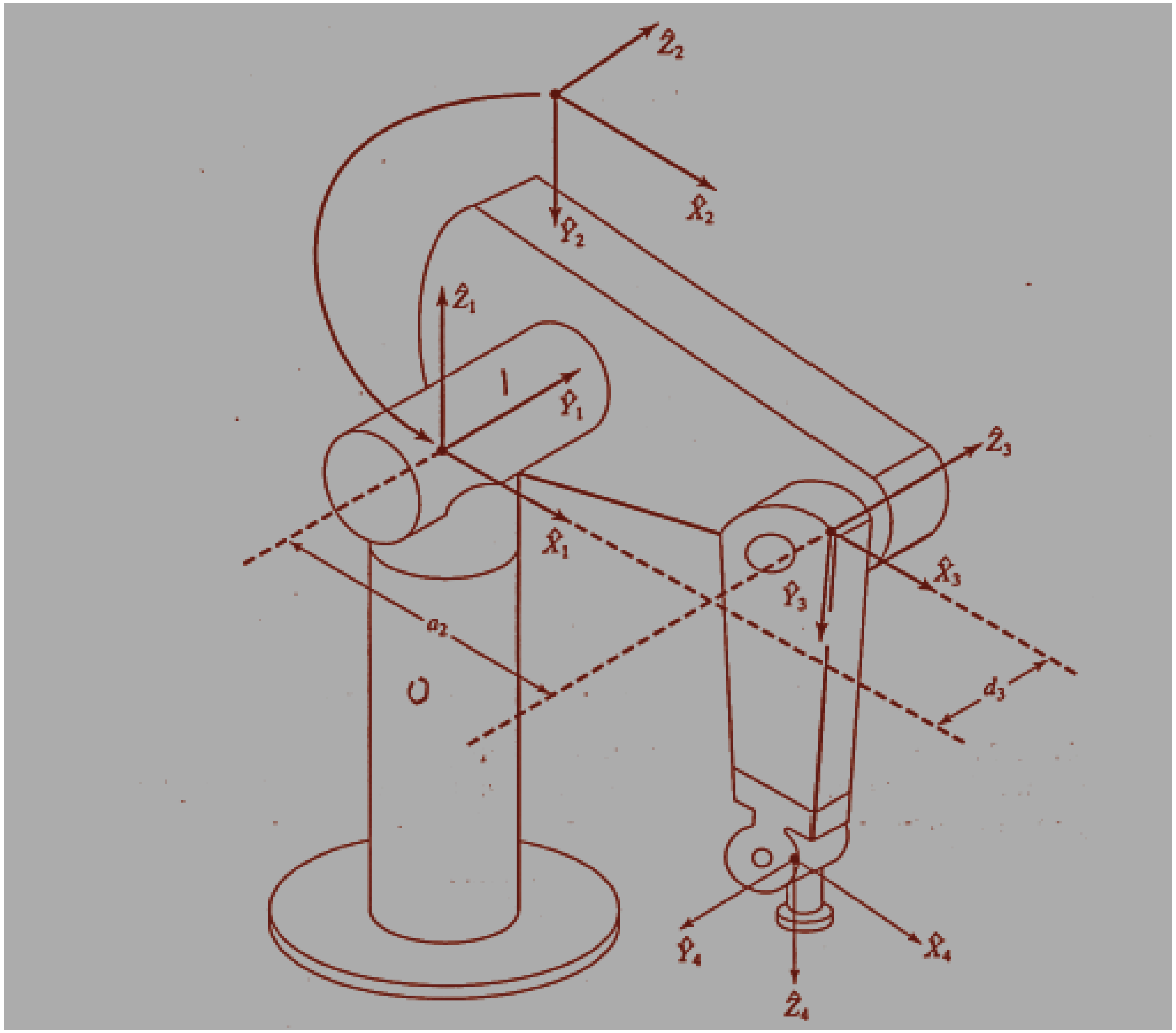
Vector Method



$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \quad + L_3 \cos(\theta_1 + \theta_2 + \theta_3)$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \quad + L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

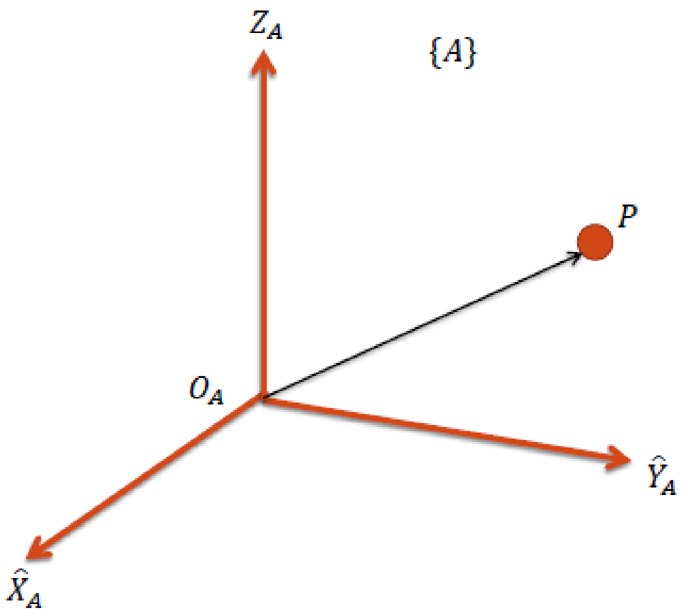
$$\alpha = \theta_1 + \theta_2 + \theta_3$$

How about this robot?



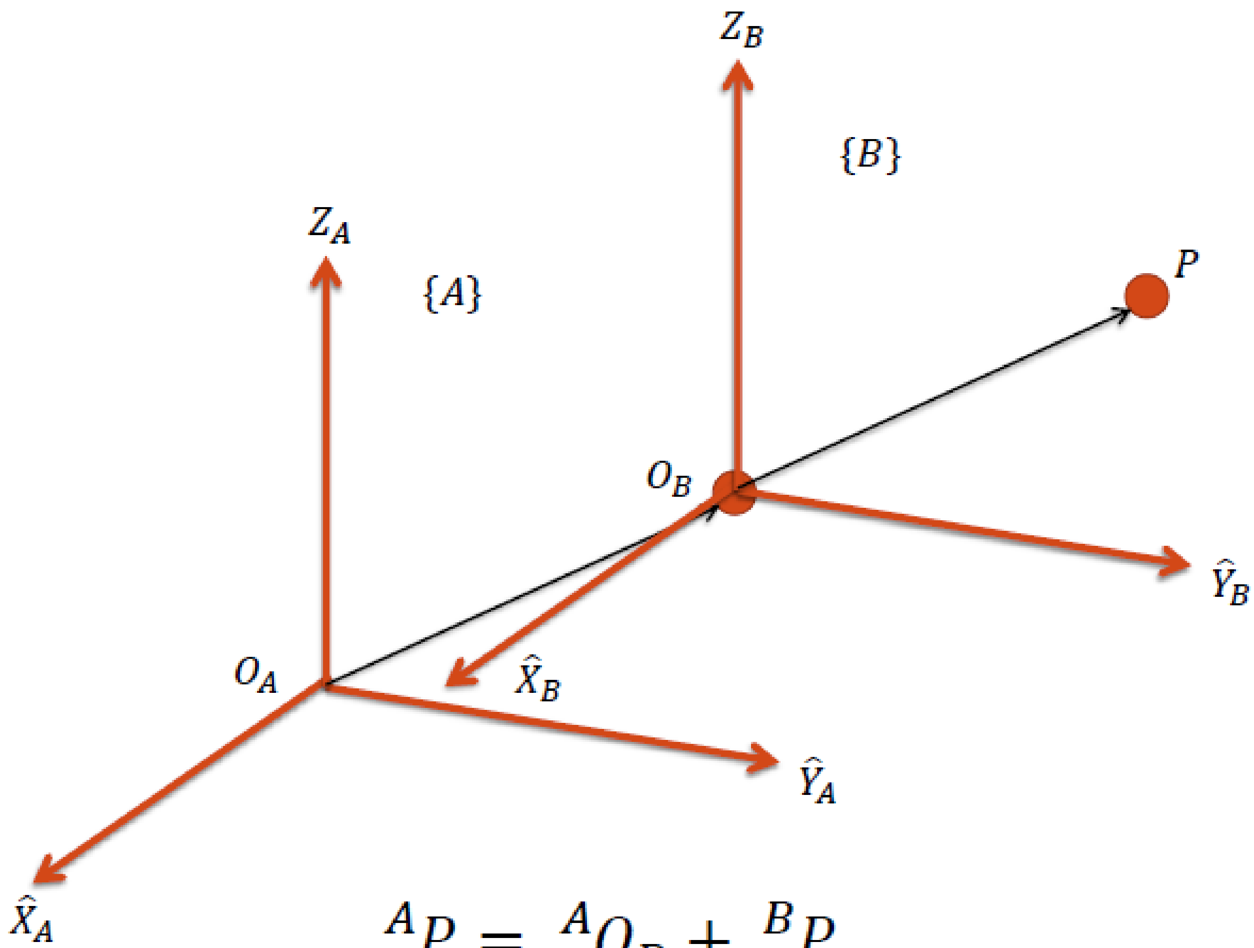
Basics of transformation

Representing Position Vectors



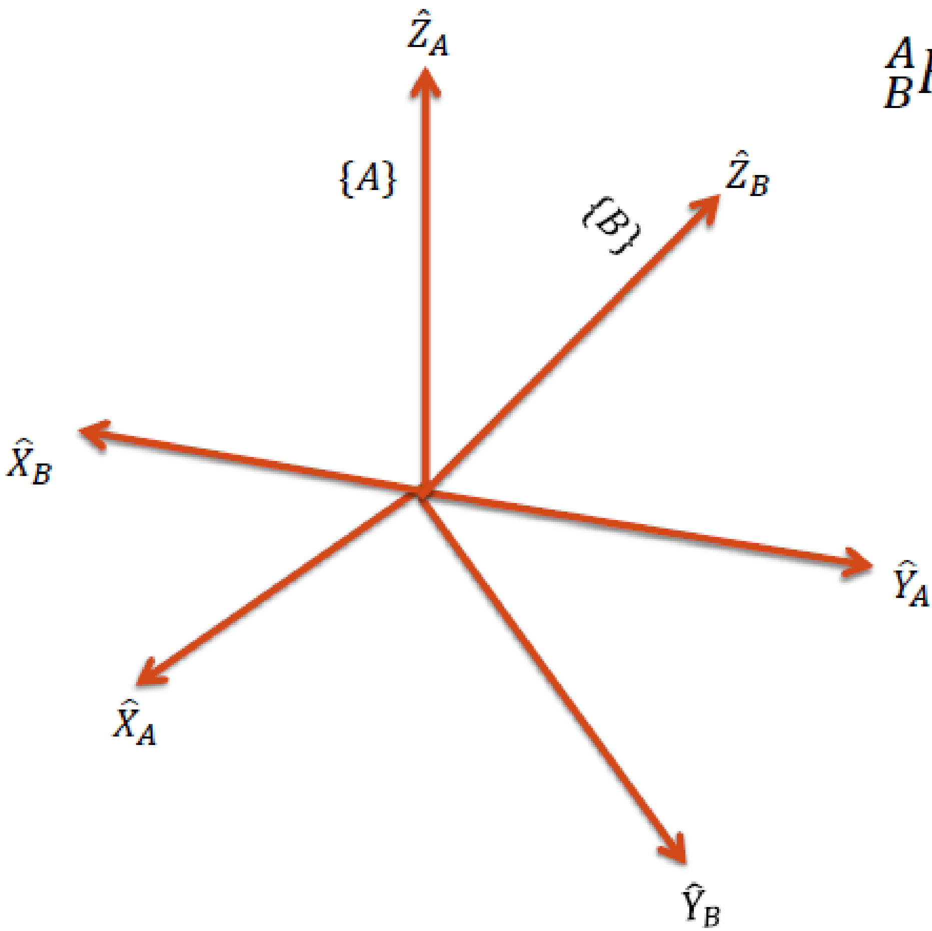
$$ {}^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} $$

Translated Frame



$$^{A}P = {}^{A}O_B + {}^{B}P$$

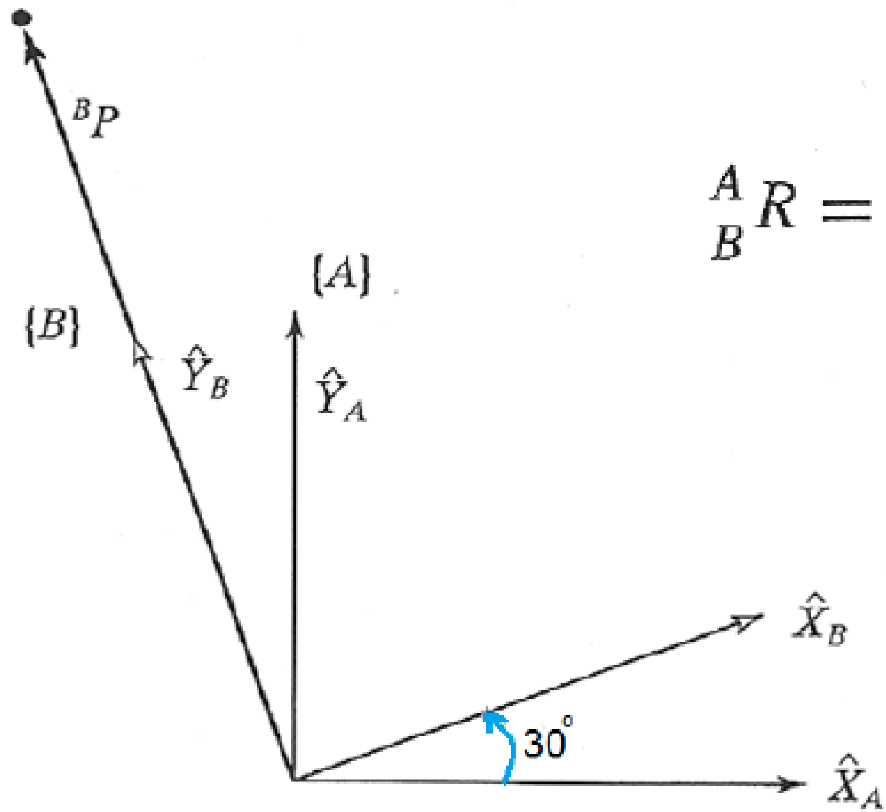
Rotated Frame



$$
{}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix}
$$

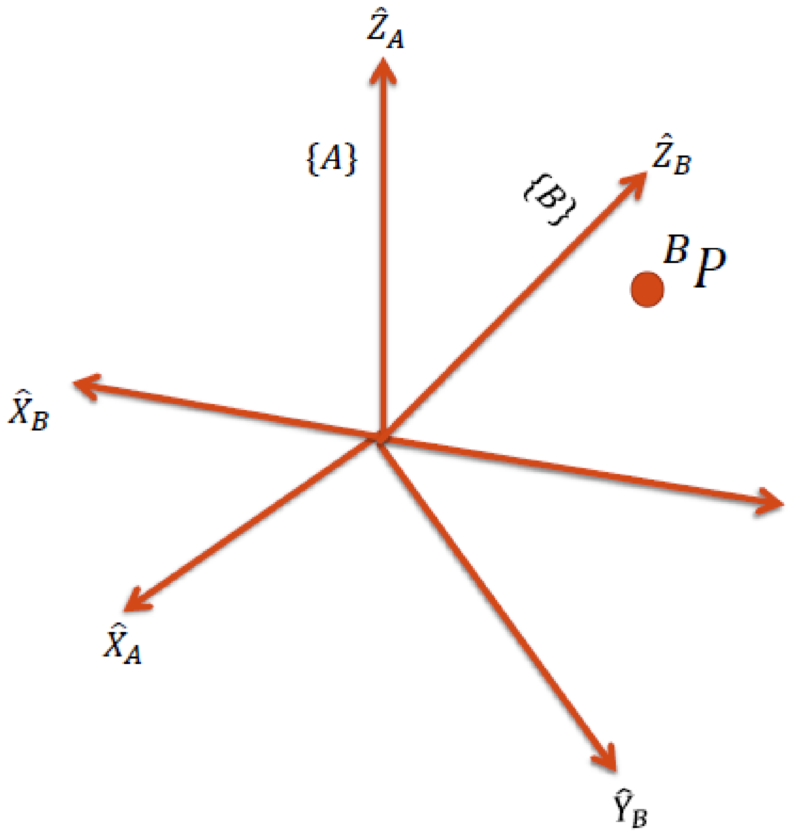
$$
= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

Rotation Matrix Example



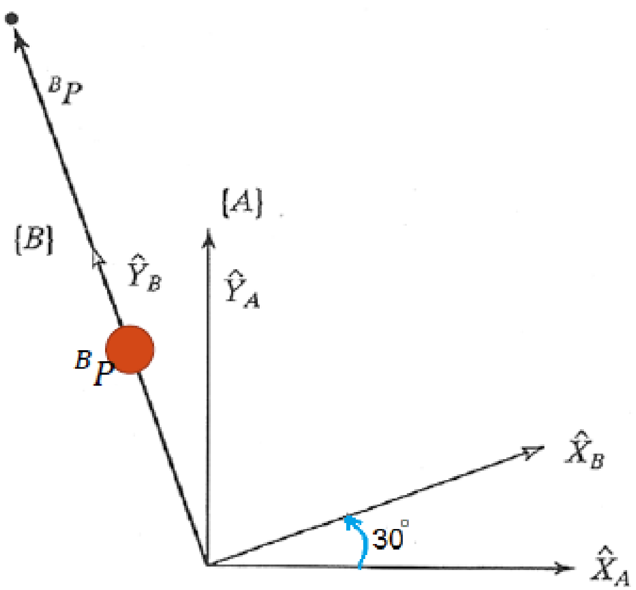
$$
{}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}.
$$

Effect of Frame Rotation



$$^AP = {}^A_BR\,{}^BP$$

Example



$$
{}^A_BR = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}
$$

$$
{}^BP = \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix},
$$

$$
{}^AP = {}^A_BR\,{}^BP = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix}.
$$

Comparing translation and rotation

Translation

$$^{A}P = {}^{A}O_{B} + {}^{B}P$$

Rotation

$$^{A}P = {}^{A}_{B}R\,{}^{B}P$$

We would like to compose both

$$^{A}P = {}^{A}_{B}T\,{}^{B}P$$

Homogenous Transformations

Translation

$$^{A}P = \begin{bmatrix} ^{A}x_p \\ ^{A}y_p \\ ^{A}z_p \\ 1 \end{bmatrix} \qquad ^{B}P = \begin{bmatrix} ^{B}x_p \\ ^{B}y_p \\ ^{B}z_p \\ 1 \end{bmatrix}$$

$$^{A}x_p = {}^{B}x_p + {}^{A}x_{OB}$$
$$^{A}y_p = {}^{B}y_p + {}^{A}y_{OB}$$
$$^{A}z_p = {}^{B}z_p + {}^{A}z_{OB}$$

$$^{A}P = \begin{bmatrix} 1 & 0 & 0 & ^{A}x_{OB} \\ 0 & 1 & 0 & ^{A}y_{OB} \\ 0 & 0 & 1 & ^{A}z_{OB} \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^{B}P$$

$$^{A}P = {}^{A}_{B}T\, {}^{B}P$$

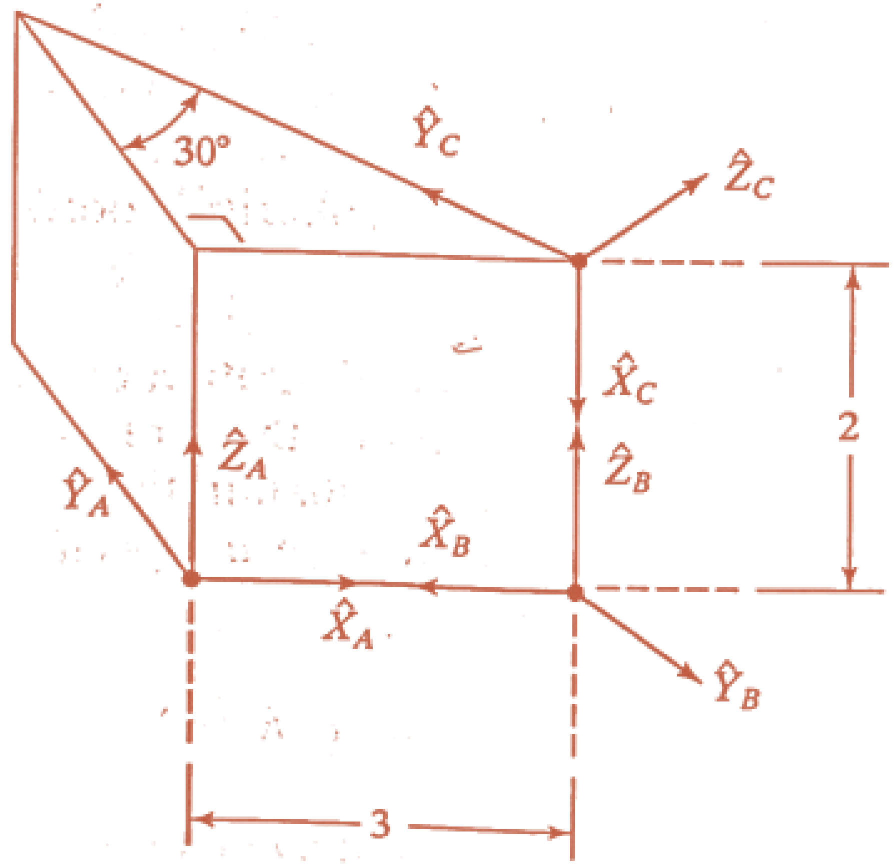
Homogenous Transformations

Rotation

$$
{}^{A}P = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] {}^{B}P
$$

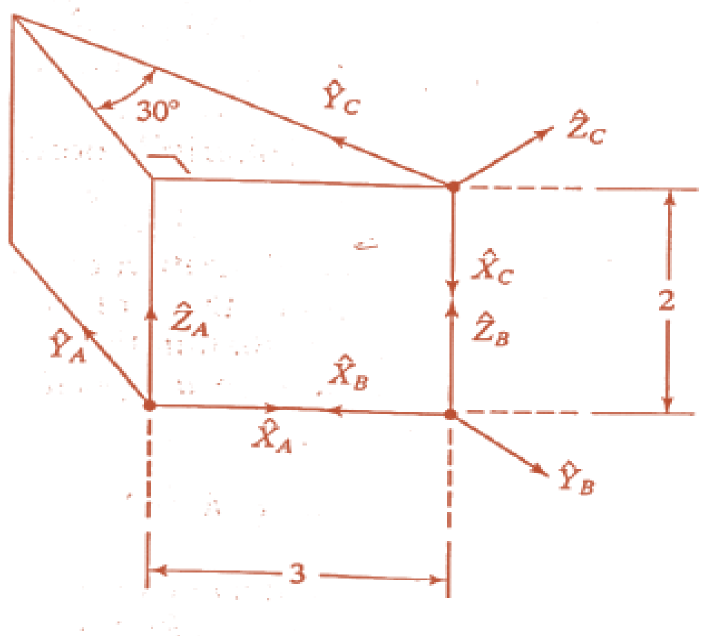
$$
\underbrace{{}^{A}P}_{4\times 1} = \underbrace{{}^{A}_{B}T}_{4\times 4}\ \underbrace{{}^{B}P}_{4\times 1}
$$

Example of ransformations



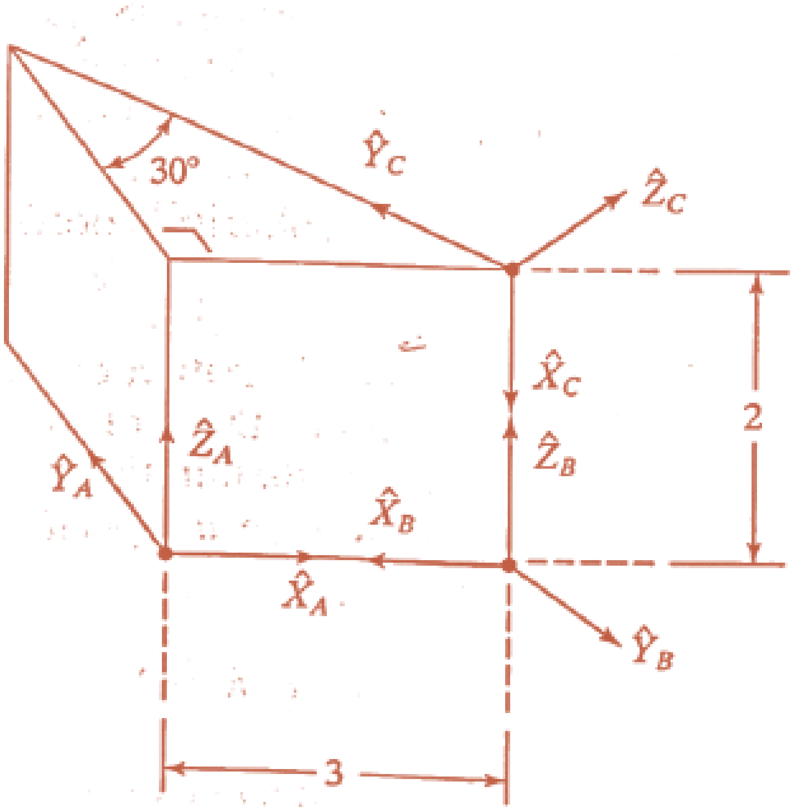
$$ {}^{A}_{B}T = ? $$

Example of transformations



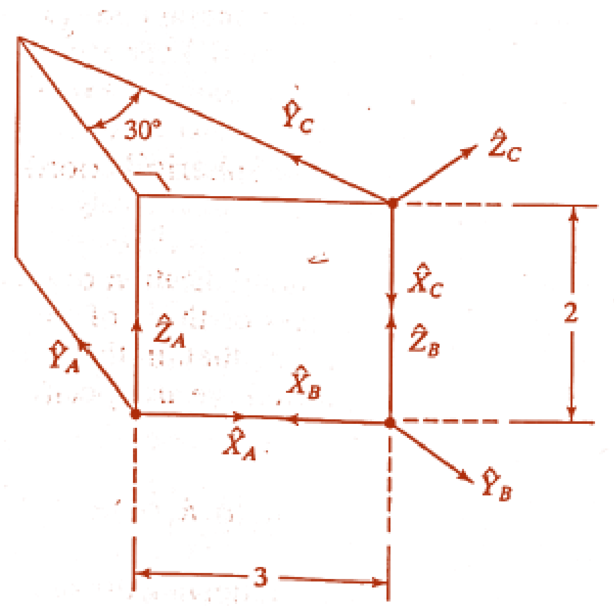
$$
{}^{A}_{B}T = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Another Example of transformations



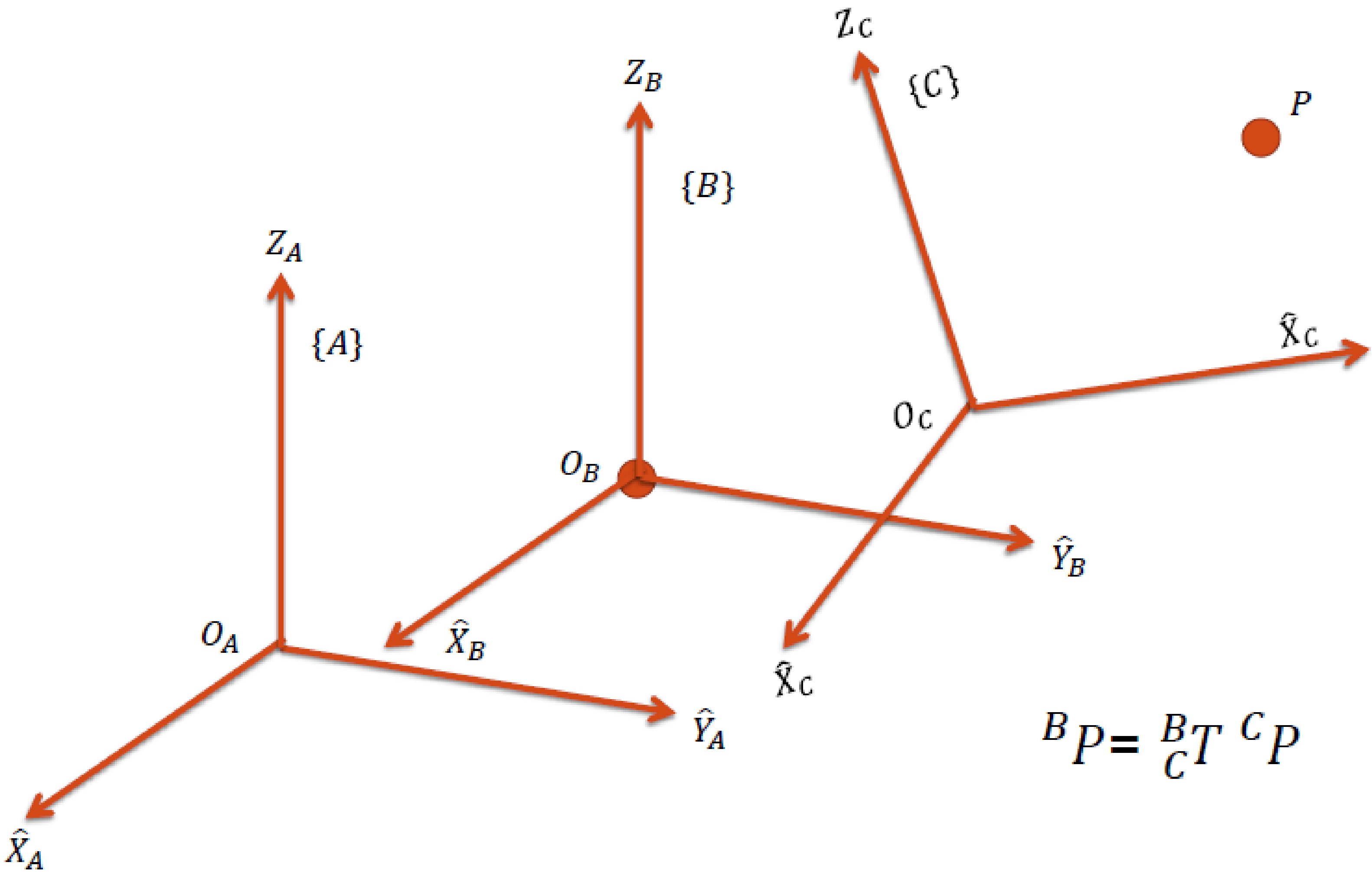
$$ {}^{A}_{C}T = ? $$

Another Example of transformations



$$
{}^{A}_{C}T = \begin{bmatrix} 0 & -0.5 & 0.866 & 3 \\ 0 & 0.866 & 0.5 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Composition of Transformations

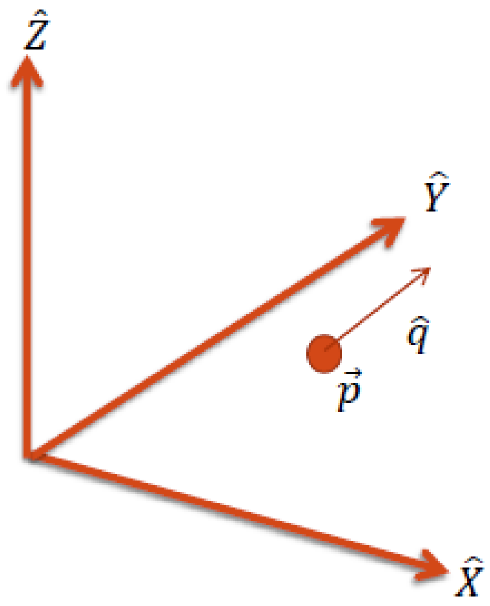


$$^{B}P = {}^{B}_{C}T\,{}^{C}P$$

$$^{A}P = {}^{A}_{B}T\,{}^{B}P = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P$$

Translation of a Position Vector

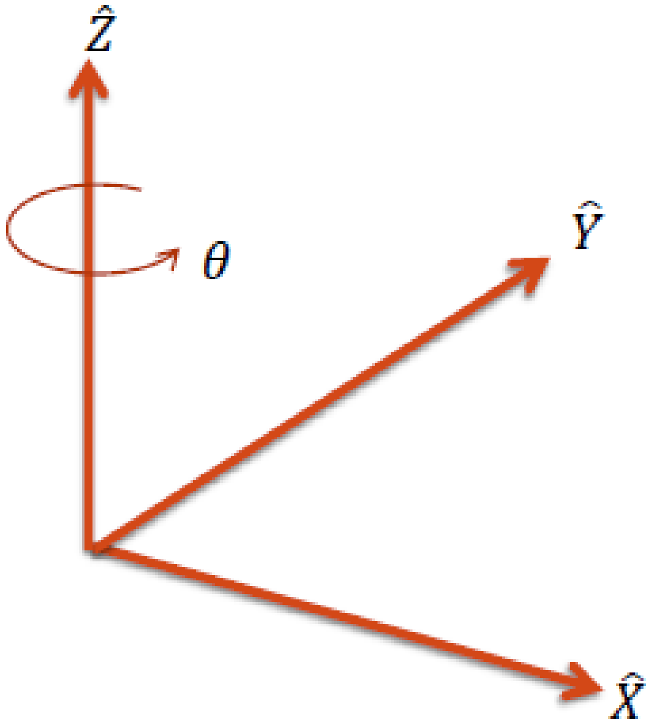
Translation by vector $q = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$



$$T(q) = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Fundamental Rotation matrices

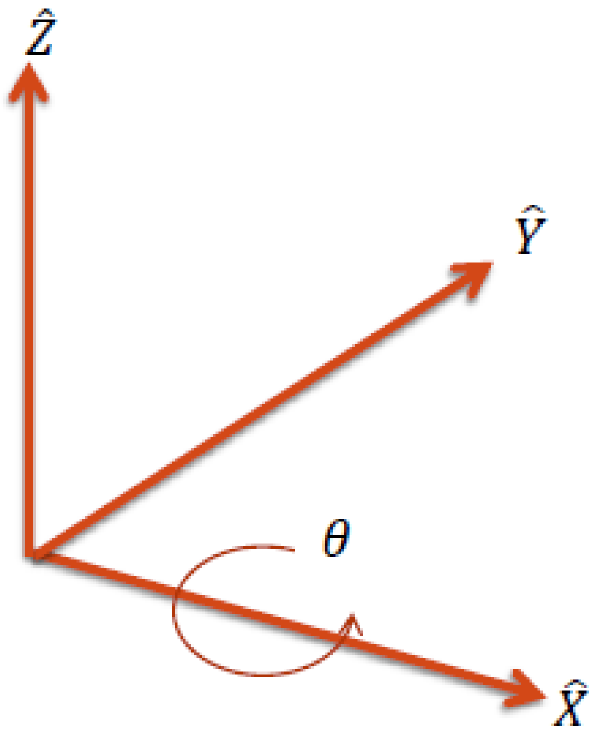
Rotation about Z axis by angle θ



$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Fundamental Rotation matrices

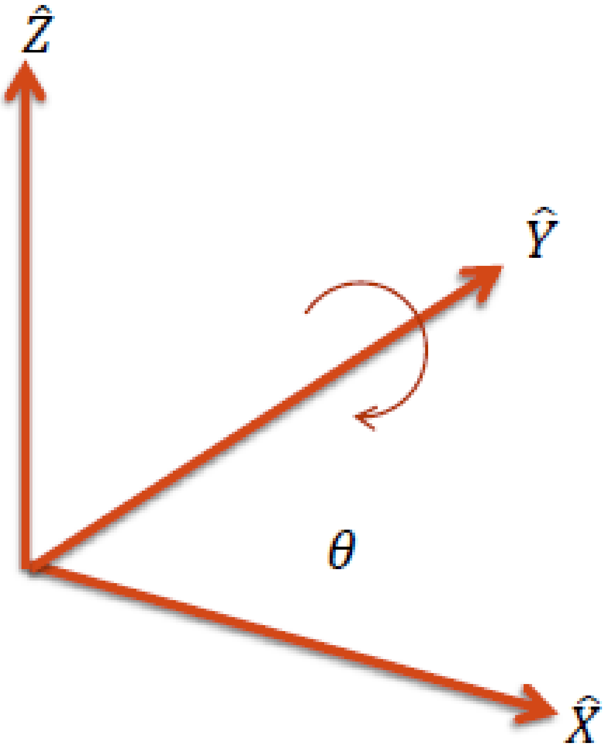
Rotation about X axis by angle θ



$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

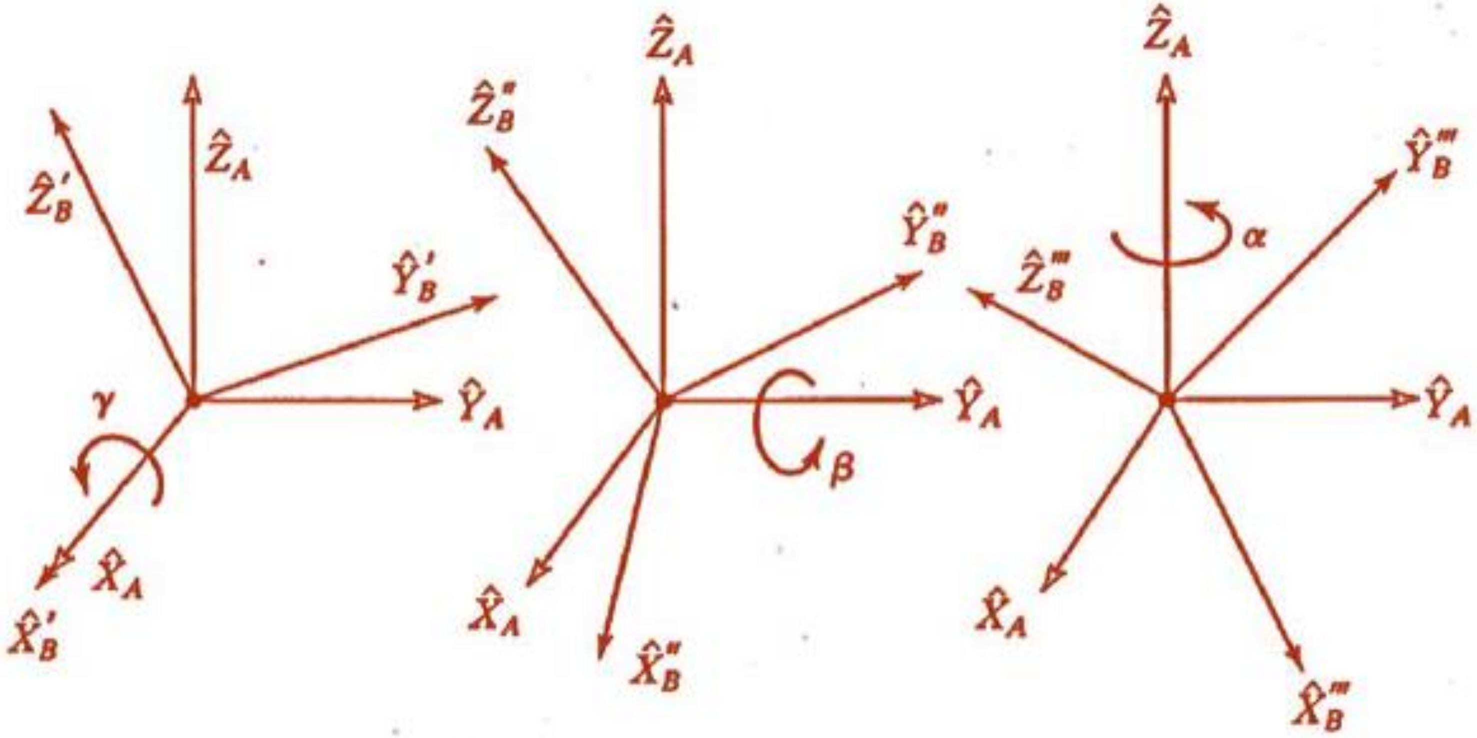
Fundamental Rotation matrices

Rotation about Y axis by angle θ



$$R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

X-Y-Z Fixed Angle Rotation



X-Y-Z Fixed Angle Rotation

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma)
$$

$$
= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix},
$$

here $c\alpha$ is shorthand for $\cos\alpha$, $s\alpha$ for $\sin\alpha$, and so on.

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}.
$$

X-Y-Z Fixed Angle Rotation

Final matrix

$$
{}^{A}_{B}R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.
$$

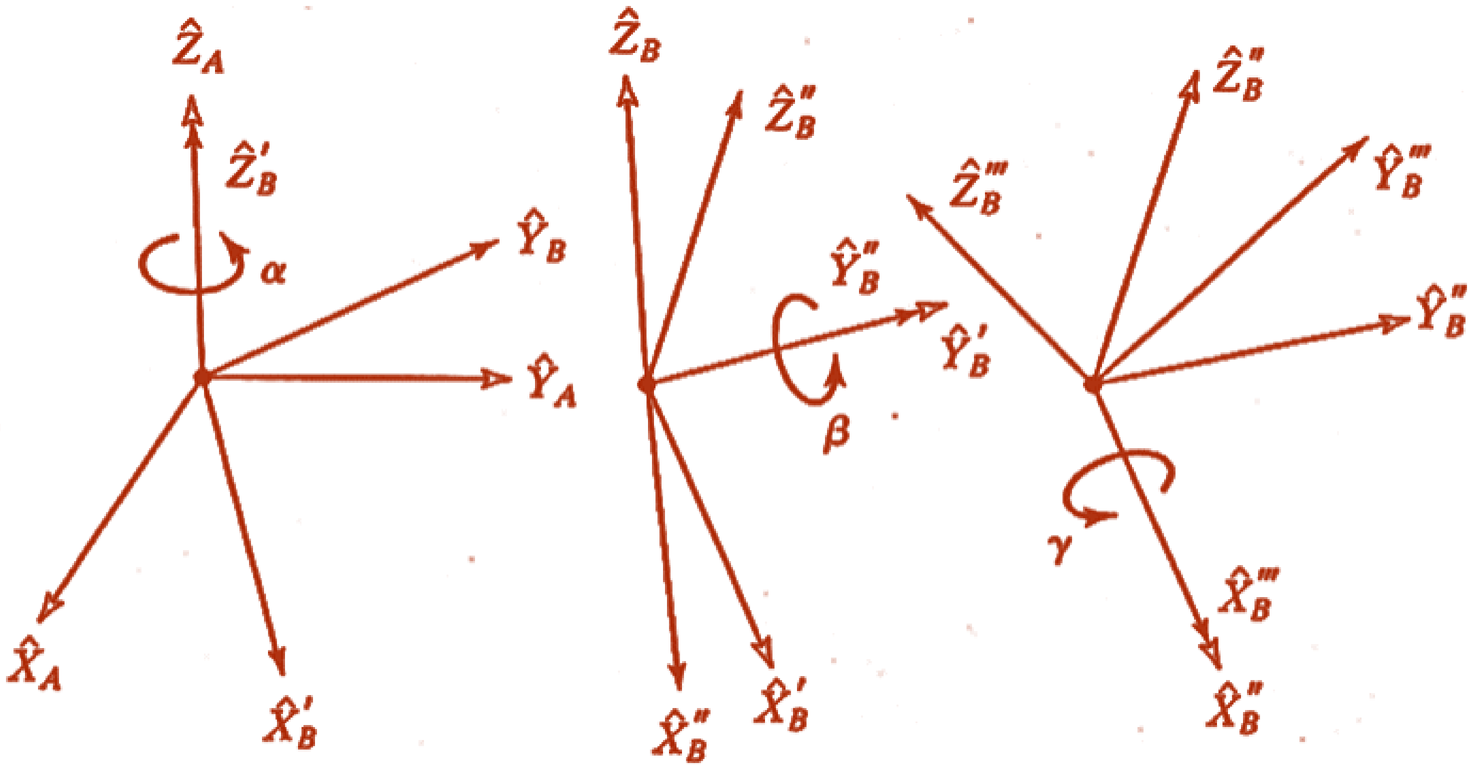
Recovering angles from final matrix

$$\beta = \text{Atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right),$$

$$\alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta),$$

$$\gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta),$$

Z-Y-X Euler Angles



Z-Y-X Euler Angles

$$ {}^{A}_{B}R = {}^{A}_{B'}R \; {}^{B'}_{B''}R \; {}^{B''}_{B}R, $$

$$ {}^{A}_{B}R_{Z'Y'X'} = R_Z(\alpha) R_Y(\beta) R_X(\gamma) $$

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}, $$

Z-Y-X Euler Angles

$$
{}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}.
$$

Determination of Euler angles

Combined rotation matrix:

$$^{1}R_{2} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Equivalent rotation matrix Z-Y-X $(\theta_3, \theta_2, \theta_1)$:

$$^{1}R_{2} = \begin{bmatrix} c_2 c_3 & s_1 s_2 c_3 - c_1 s_3 & s_1 s_3 + c_1 s_2 c_3 \\ c_2 s_3 & s_1 s_2 s_3 + c_1 c_3 & -s_1 c_3 + c_1 s_2 s_3 \\ -s_2 & s_1 c_2 & c_1 c_2 \end{bmatrix}$$

Determination of Euler angles

Equation element (11) and (21) of both matrices:

$$c_2 c_3 = r_{11} \qquad c_2 s_3 = r_{21}$$

$$r_{11}^2 + r_{21}^2 = c_2^2 c_3^2 + c_2^2 s_3^2$$

$$= c_2^2 (c_3^2 + s_3^2) = c_2^2$$

$$c_2 = \pm\sqrt{r_{11}^2 + r_{21}^2}$$

Again equating element (31) of both matrices

$$-s_2 = r_{31}$$

Determination of Euler angles

$$\theta_2 = \text{Atan2}\left(-r_{31}, \pm\sqrt{r_{11}^2 + r_{21}^2}\right)$$

Atan2 (a, b) → two argument arc tangent

$$\text{Atan2}(-2, -2) = -135^\circ$$

$$\text{Atan2}(2, 2) = 45^\circ$$

A distinction which would be lost with ~~single-~~ single-argument arc tangent function.

Determination of Euler angles

Case 1

$$\theta_2 \neq 90^\circ$$

$$\tan\theta_3 = \frac{s_3}{c_3}$$

$$r_{11} = c_2 c_3 \Rightarrow c_3 = \frac{r_{11}}{c_2}$$

$$r_{21} = c_2 s_3 \Rightarrow s_3 = \frac{r_{21}}{c_2}$$

$$\boxed{\theta_3 = \text{Atan2}\left(\frac{r_{21}}{c_2}, \frac{r_{11}}{c_2}\right)}$$

$$\boxed{\theta_1 = \text{Atan2}\left(\frac{r_{32}}{c_2}, \frac{r_{33}}{c_2}\right)}$$

Determination of Euler angles

Case 2 $\quad \theta_2 = \pm 90^\circ$

comparing element (1,2) & (2,2)

$$r_{12} = s_1 s_2 c_3 - c_1 s_3$$

$$r_{22} = s_1 s_2 s_3 + c_1 c_3$$

If $\theta_2 = +90^\circ$, $\quad r_{12} = s_1 c_3 - c_1 s_3$

$$= \sin\theta_1 \cos\theta_3 - \cos\theta_1 \sin\theta_3$$

$$= \sin(\theta_1 - \theta_3)$$

$$r_{22} = \sin\theta_1 \sin\theta_3 + \cos\theta_1 \cos\theta_3$$

$$r_{22} = \cos(\theta_1 - \theta_3)$$

Determination of Euler angles

$$(\theta_1 - \theta_3) = \text{Atan2}(r_{12}, r_{22})$$

Now choosing $\theta_3 = 0$, $\theta_2 = 90^\circ$

$$\boxed{\theta_1 = \text{Atan2}(r_{12}, r_{22})}$$

When $\theta_2 = -90^\circ$

$r_{12} = -\sin(\theta_1 + \theta_3)$ & $r_{22} = \cos(\theta_1 + \theta_3)$

$$\theta_1 + \theta_3 = \text{Atan2}(-r_{12}, r_{22})$$

If $\theta_3 = 0$

$$\boxed{\theta_1 = \text{Atan2}(-r_{12}, r_{22})}$$

Determination of R-P-Y angles

Problem# 1

The concept of roll, pitch and yaw angles has been used to represent rotation of frame {2} with respect to the reference frame {1}.
The above rotation can be expressed as

$$ {}^{1}R_{2} = \begin{bmatrix} -0.250 & 0.433 & -0.866 \\ 0.433 & -0.750 & -0.500 \\ -0.866 & -0.500 & 0.000 \end{bmatrix} $$

Determine the angles of rolling, pitching and yawing.

Determination of R-P-Y angles

Solution# 1

Consider Rolling angle $= \theta_1$, pitching $= \theta_2$, yaw $= \theta_3$

From the given matrix, first find pitching angle

$$\theta_2 = A\tan 2\left(-r_{31}, \pm\sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$= A\tan 2\left(0.866, \pm\sqrt{(0.250)^2 + (0.433)^2}\right)$$

$$\theta_2 = A\tan 2\left(0.866, \pm 0.499\right)$$

$$= 60^\circ \ \& \ 120^\circ$$

Determination of R-P-Y angles

Solution# 1 (contd)

Now

$\theta_2 \neq 90^\circ$

when $\theta_2 = 60^\circ$

$$\theta_3 = \text{Atan2}\left(\frac{r_{21}}{c_2}, \frac{r_{11}}{c_2}\right)$$

$$= \text{Atan2}\left(\frac{0.433}{\cos 60^\circ}, \frac{-0.250}{\cos 60^\circ}\right)$$

$$= \text{Atan2}(0.866, -0.500)$$

$$= 120^\circ$$

Determination of R-P-Y angles

Solution# 1 (contd)

$$\theta_1 = A\tan 2\left(\frac{r_{32}}{c_2}, \frac{r_{33}}{c_2}\right)$$

$$= A\tan 2\left(\frac{-0.500}{\cos 60^\circ}, \frac{0.000}{\cos 60^\circ}\right)$$

$$= A\tan 2\left(-1, 0\right)$$

$$= 315^\circ$$

Determine the other set of angles
Also show you will get same rotation matrix for both sets.

Determination of Euler angles

Problem#2

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

(a) Show that it is a rotation matrix
(b) Determine Euler angles of rotation considering Z-Y-X rotation sequence
(c) Determine R-P-Y angles considering fixed angle representation.
(d) Show both representations are equivalent geometrically with the help of any position vector attached to the moving frame.