

Lecture 9 (19/08/25)

ME512/ME6106: Mobile Robotics

Mathematical fundamentals for robot kinematics

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Mapping:

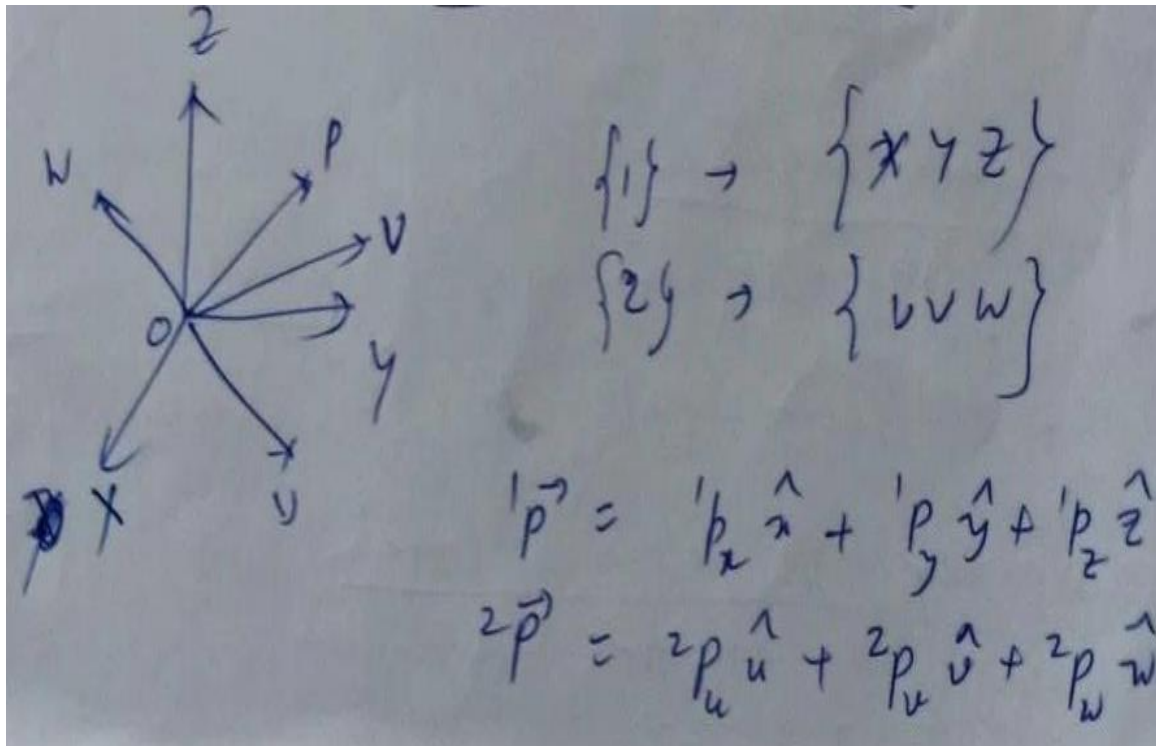
- Changing the description of a point (or value) in space from one frame to another frame.

Different cases of mapping

- (1) Second frame is rotated w.r.t the first, origin of both frames is same (changing of orientation)
- (2) Second frame is moved away from the first. Axes of both frames remain parallel (change of position)
- (3) Second frame is rotated w.r.t the first and move away from it (change of position and orientation)

Mapping:

- (1) Second frame is rotated w.r.t the first, origin of both frames is same (changing of orientation)



$${}^1p_x = \hat{x} \cdot {}^2\vec{p}$$

$$= \hat{x} \cdot ({}^2p_u \hat{u} + {}^2p_v \hat{v} + {}^2p_w \hat{w})$$

$$= \hat{x} \cdot \hat{u} \cdot {}^2p_u + \hat{x} \cdot \hat{v} \cdot {}^2p_v + \hat{x} \cdot \hat{w} \cdot {}^2p_w$$

$${}^1p_x = \hat{x} \cdot ({}^2p_u \hat{u} + {}^2p_v \hat{v} + {}^2p_w \hat{w})$$

$${}^1p_z = \hat{z} \cdot ({}^2p_u \hat{u} + {}^2p_v \hat{v} + {}^2p_w \hat{w})$$

Mapping:

$$\begin{pmatrix} {}^1p_x \\ {}^1p_y \\ {}^1p_z \end{pmatrix} = \begin{bmatrix} \hat{x} \cdot \hat{u} & \hat{x} \cdot \hat{v} & \hat{x} \cdot \hat{w} \\ \hat{y} \cdot \hat{u} & \hat{y} \cdot \hat{v} & \hat{y} \cdot \hat{w} \\ \hat{z} \cdot \hat{u} & \hat{z} \cdot \hat{v} & \hat{z} \cdot \hat{w} \end{bmatrix} \begin{pmatrix} {}^2p_u \\ {}^2p_v \\ {}^2p_w \end{pmatrix}$$

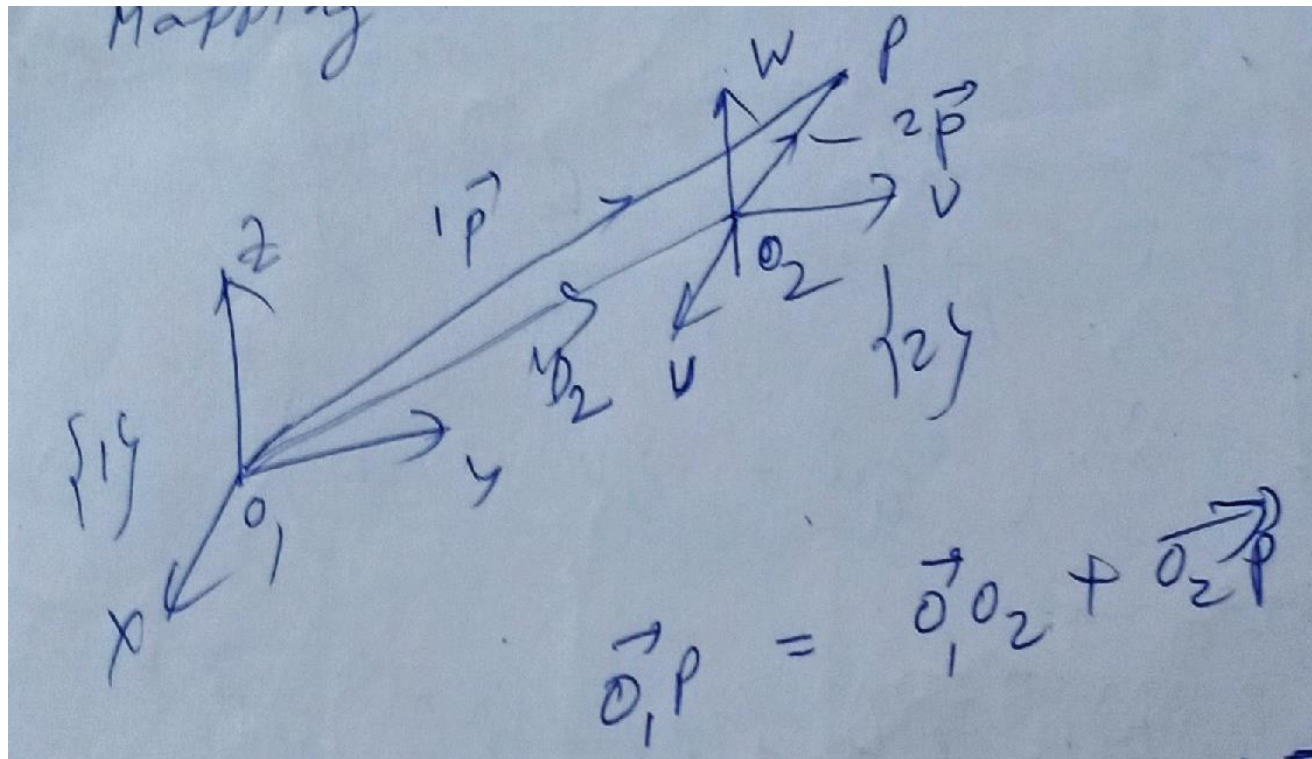
$${}^1\vec{p} = {}^1R_2 {}^2\vec{p}$$

Relation of rotation matrix

$$\begin{aligned} {}^2\vec{p} &= {}^2R_1 {}^1\vec{p} & {}^2R_1 &= ({}^1R_2)^T \\ {}^1\vec{p} &= {}^1R_2 {}^2\vec{p} \Rightarrow {}^2\vec{p} &= ({}^1R_2)^{-1} {}^1\vec{p} \\ {}^2R_1 &= ({}^1R_2)^{-1} \\ ({}^1R_2)^T &= ({}^1R_2)^{-1} \\ {}^1R_2 ({}^1R_2)^T &= I \\ \boxed{R R^T} &= \mathbf{1} \end{aligned}$$

Mapping:

(2) Second frame is moved away from the first. Axes of both frames remain parallel (change of position)



$${}^1\vec{p} = {}^2\vec{p} + \frac{{}^1\vec{D}_2}{{}^2D_2} {}^2\vec{p}$$

$${}^1p_x = {}^2p_x + dx$$

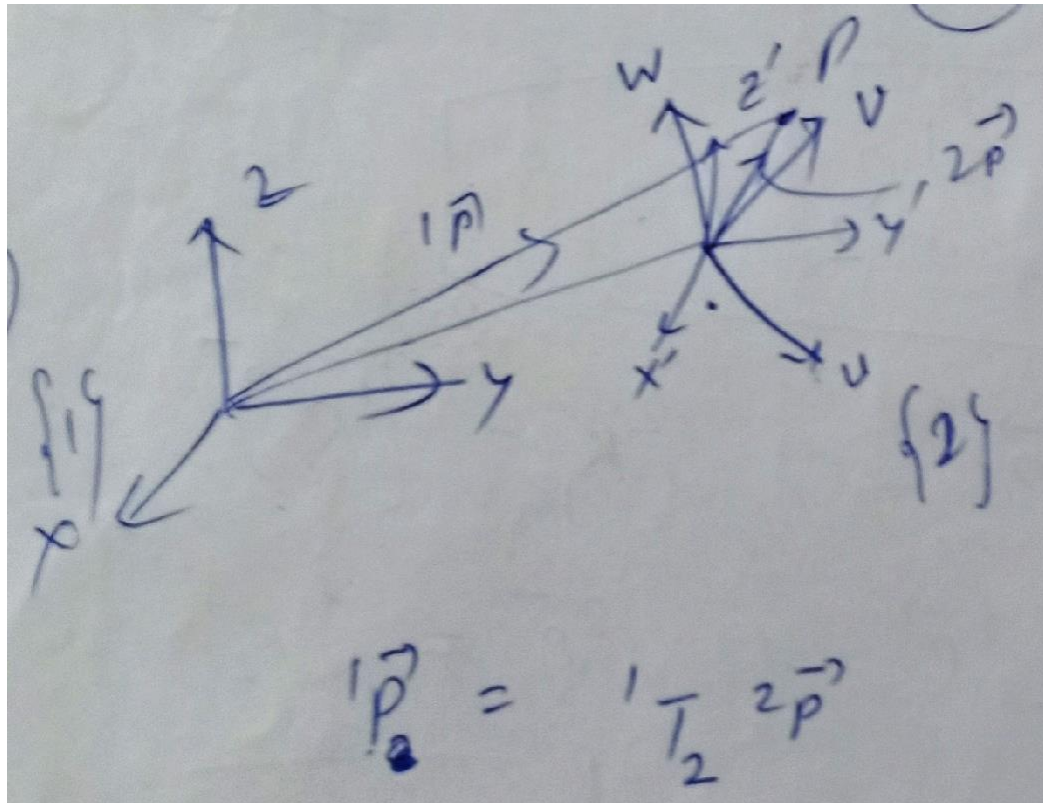
$${}^1p_y = {}^2p_y + dy$$

$${}^1p_z = {}^2p_z + dz$$

$$\begin{pmatrix} {}^1p_x \\ {}^1p_y \\ {}^1p_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^2p_x \\ {}^2p_y \\ {}^2p_z \\ 1 \end{pmatrix}$$

Mapping:

(3) Second frame is rotated w.r.t the first and move away from it (change of position and orientation)



Mapping:

(3) Second frame is rotated w.r.t the first and move away from it (change of position and orientation)

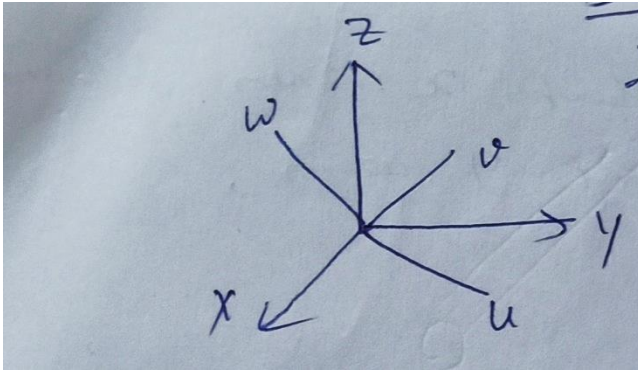
Handwritten representation of the Denavit-Hartenberg (DH) transformation matrix T_2 :

$$T_2 = \begin{bmatrix} \hat{x}, \hat{u} & \hat{x}, \hat{v} & \hat{x}, \hat{w} & d_x \\ \hat{y}, \hat{u} & \hat{y}, \hat{v} & \hat{y}, \hat{w} & d_y \\ \hat{z}, \hat{u} & \hat{z}, \hat{v} & \hat{z}, \hat{w} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Annotations in the image:

- Rotation matrix (3x3)**: Points to the top-left 3x3 submatrix of the DH matrix.
- Translation vector**: Points to the column vector $[d_x, d_y, d_z]^T$.
- Representative transform (1x3)**: Points to the bottom row $[0, 0, 0, 1]$.
- Scale factor**: Points to the bottom-right element 1 .

Rotation matrix:

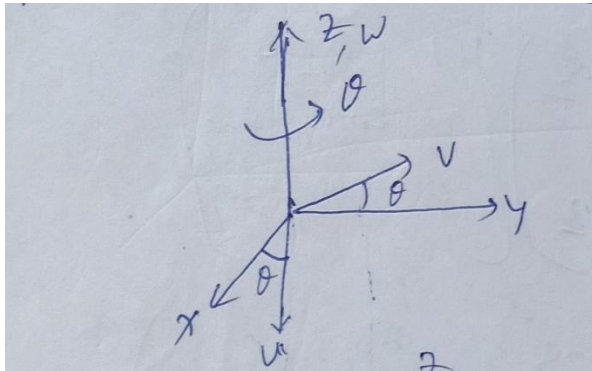


$$\begin{aligned} \{x, y, z\} &\rightarrow \{1\} \\ \{u, v, w\} &\rightarrow \{2\} \end{aligned}$$

$${}^1R_2 = \begin{bmatrix} \hat{x} \cdot \hat{u} & \hat{x} \cdot \hat{v} & \hat{x} \cdot \hat{w} \\ \hat{y} \cdot \hat{u} & \hat{y} \cdot \hat{v} & \hat{y} \cdot \hat{w} \\ \hat{z} \cdot \hat{u} & \hat{z} \cdot \hat{v} & \hat{z} \cdot \hat{w} \end{bmatrix}$$

Fundamental rotation matrices

(1) Rotation about z axis



$$\begin{bmatrix} \hat{x} \cdot \hat{u} & \hat{x} \cdot \hat{v} & \hat{x} \cdot \hat{w} \\ \hat{y} \cdot \hat{u} & \hat{y} \cdot \hat{v} & \hat{y} \cdot \hat{w} \\ \hat{z} \cdot \hat{u} & \hat{z} \cdot \hat{v} & \hat{z} \cdot \hat{w} \end{bmatrix}$$

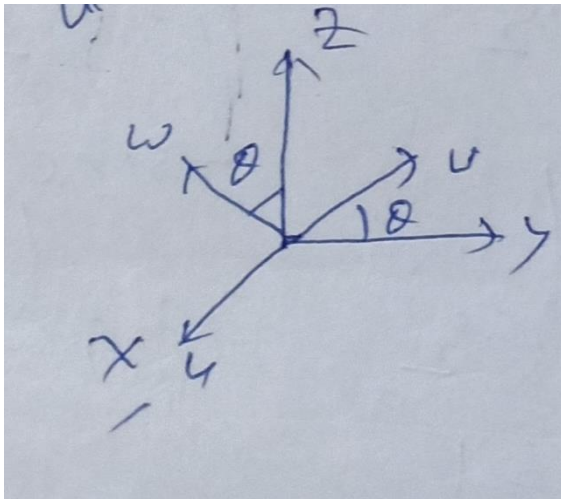
$$R_z(\theta) = \begin{bmatrix} \cos \theta & \cos(\theta + 90^\circ) & \cos 90^\circ \\ \cos(90^\circ - \theta) & \cos(\theta) & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Fundamental rotation matrices

(1) Rotation about x axis

$R_x =$



$$\begin{bmatrix} \cos 90^\circ & \cos 90^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos \theta & \cos (90^\circ + \theta) \\ \cos 90^\circ & \cos (90^\circ - \theta) & \cos(\theta) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

Problem#1

Frame $\{2\}$ is rotated w.r.t frame $\{1\}$ about the x axis by angle of 60° . The position of the origin of frame $\{2\}$ as seen from frame $\{1\}$ is ${}^1D_2 = \{7.0, 5.0, 7.0\}^T$

Obtain the transformation matrix 1T_2 which describes frame $\{1\}$ w.r.t $\{2\}$.

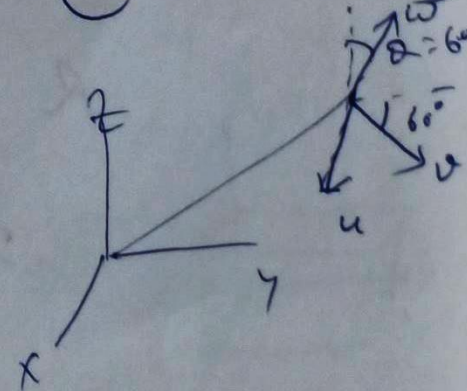
Also find the description of point P in frame $\{1\}$ if ${}^2P = [2.0, 4.0, 6.0]^T$

solution

$${}^1T_2 = \begin{bmatrix} {}^1R_2 & {}^1D_2 \\ 0 & 1 \end{bmatrix}$$

$${}^1R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60^\circ & -\sin 60^\circ \\ 0 & \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -0.866 \\ 0 & 0.866 & 0.5 \end{bmatrix}$$

$${}^1T_2 = \begin{bmatrix} 1 & 0 & 0 & 7.0 \\ 0 & 0.5 & -0.866 & 5.0 \\ 0 & 0.866 & 0.5 & 7.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$${}^1\vec{p} = {}^1T_2 {}^2\vec{p}$$

$$= \begin{bmatrix} 9.0 & 1.84 & 13.46 \end{bmatrix}^T$$

Where,

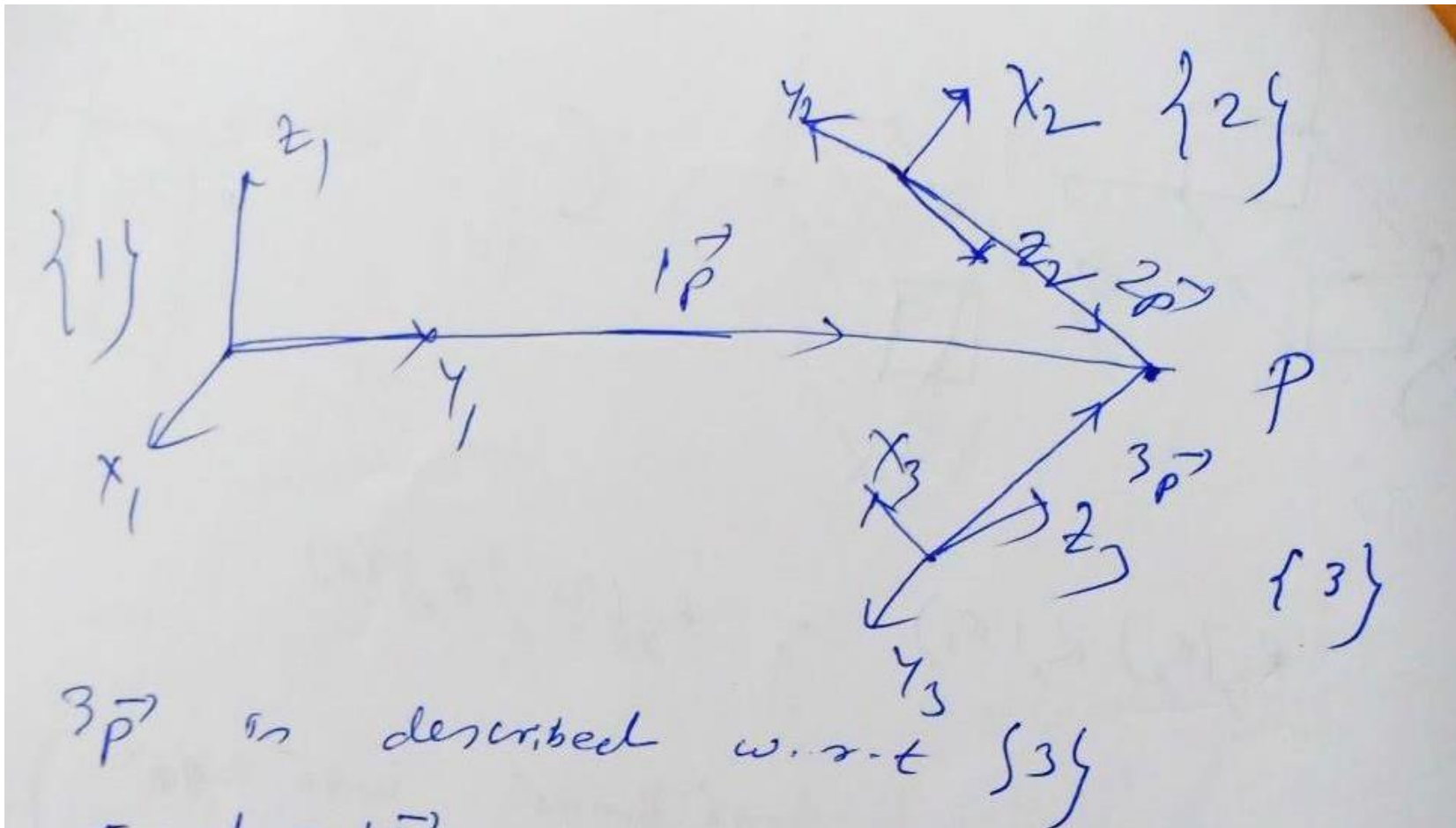
$1T_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 7.0 \\ 0 & 0.5 & -0.866 & 5 \\ 0 & 0.866 & 0.5 & 7.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2\vec{p} = \begin{bmatrix} 2.0 & 4.0 & 6.0 \end{bmatrix}^T$$

Compound or composite transformation

❖ The point P is defined w.r.t frame 3, obtain the transformation matrix between frame 1 and 3.



Compound or composite transformation

❖ The point P is defined w.r.t frame 3, obtain the transformation matrix between frame 1 and 3.

$$\begin{aligned}
 {}^2\vec{p} &= {}^2T_3 {}^3\vec{p} & {}^1\vec{p} &= {}^1T_2 {}^2\vec{p} \\
 {}^1\vec{p} &= ({}^1T_3) {}^3\vec{p} & &= ({}^1T_2 {}^2T_3) {}^3\vec{p}
 \end{aligned}$$

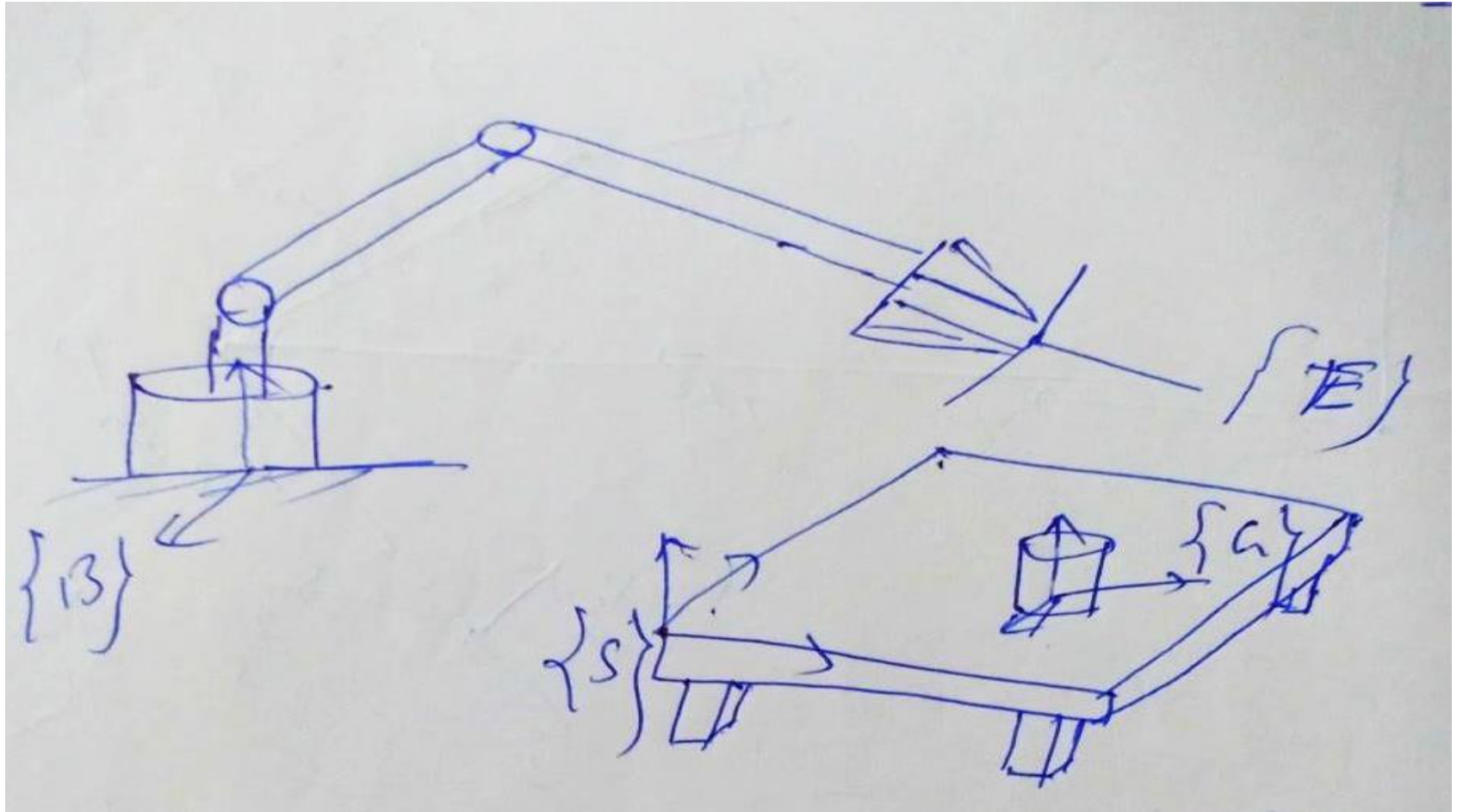
$${}^1T_3 = {}^1T_2 {}^2T_3$$

So ${}^1T_i = {}^1T_2 {}^2T_3 \dots {}^jT_{j+1} \dots {}^{i-1}T_i$

or in general from frame $\{i\}$ to frame $\{j\}$

$${}^jT_i = {}^jT_{j+1} {}^{j+1}T_{j+2} \dots {}^{i-1}T_i \quad (j > i)$$

- ❖ To find the position and orientation of manipulator end effector w.r.t. object.



Composite transformation in manipulator

❖ To find the position and orientation of manipulator end effector w.r.t. object.

$${}^B T_E = {}^B T_1 {}^1 T_2 \dots {}^n T_E \quad \begin{array}{l} n \rightarrow \text{no. of} \\ \text{joint} \\ \text{(D.O.F)} \end{array}$$

\rightarrow Known.

$${}^B T_S \rightarrow \text{Known} \left(\phi \text{ between } \{B\} \Delta \{S\} \right)$$

$$\textcircled{B} S T_w = \text{Known} \left(\text{between } \{S\} \Delta \{G\} \right)$$

- ❖ To find the position and orientation of manipulator end effector w.r.t. object.

Find the position and orientation of ~~last frame~~ end effector
w.r.t $\{E\}$ w.r.t $\{A\}$

$$E_{T_A}$$

$$E_{T_A} = E_{T_B} \cdot B_{T_S} \cdot S_{T_A}$$

$$E_{T_A} = (B_{T_E})^{-1} \cdot B_{T_S} \cdot S_{T_A}$$