Mathematics for Deep Learning: The Value of Information Theory

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Motivating Example

Example: Time-Series Prediction

Table: BTC/USD prices $S(t)$

Date	Price(t)	
2019-01-01	3963.1	
2019-01-02	4048.8	
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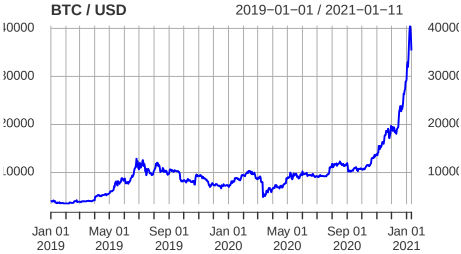

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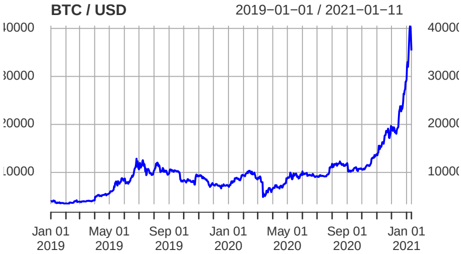

Example: Time-Series Prediction

Table: log-returns $r(t) = \log \frac{S(t+1)}{S(t)}$

Date	$r(t)$	
2019-01-02	0.021	
2019-01-03	-0.031	
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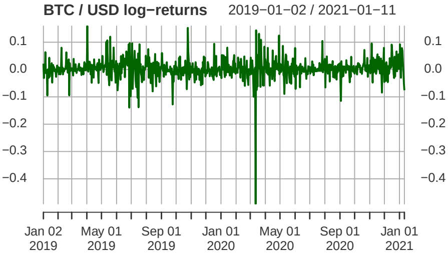

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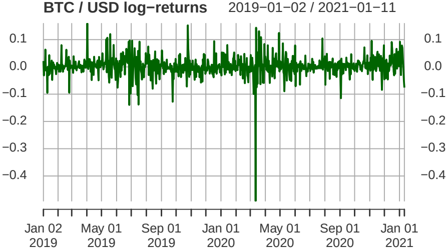

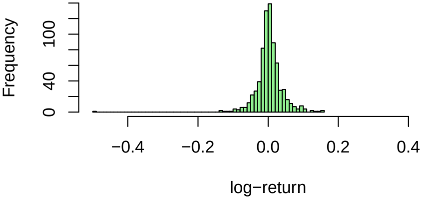

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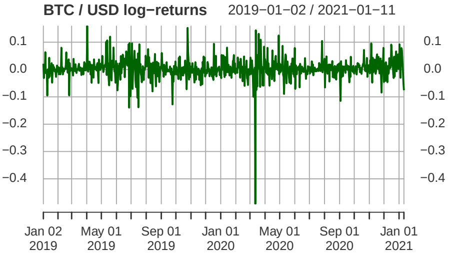

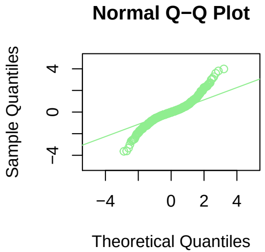

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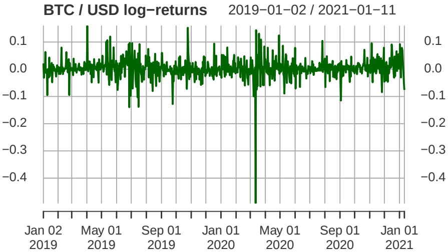

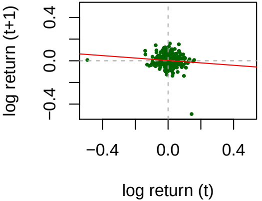

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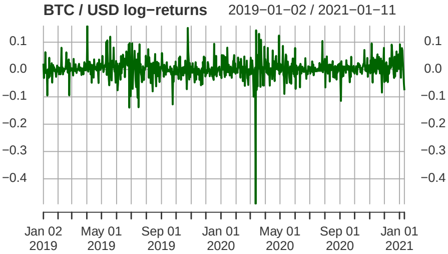

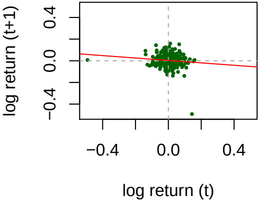

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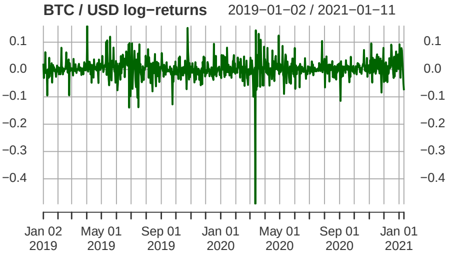
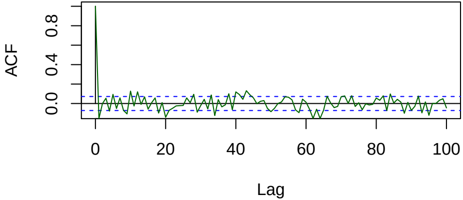


Predict $r(t+1)$ from $r(t)$:

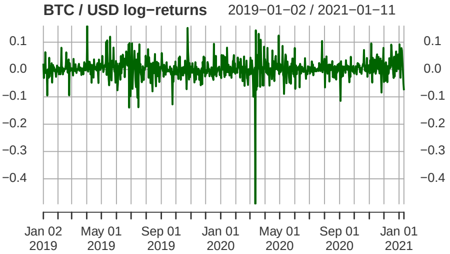
$$f\left(r(t)\right) = y \approx r(t+1)$$



Example: Time-Series Prediction

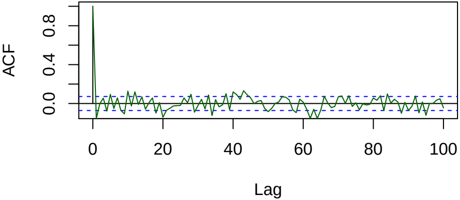
Table: log-returns $r(t) = \log \frac{S(t+1)}{S(t)}$

Date	$r(t-2)$	$r(t-1)$	$r(t)$	$r(t+1)$
2019-01-06	-0.031	0.008	-0.011	0.064
2019-01-07	0.008	-0.011	0.064	-0.013
2019-01-08	-0.011	0.064	-0.013	-0.0034
2019-01-09	0.064	-0.013	-0.0034	-0.004



Predict $r(t+1)$ from n lags of $r(t)$:

$$f\left(r(t-n), \ldots, r(t)\right) = y \approx r(t+1)$$



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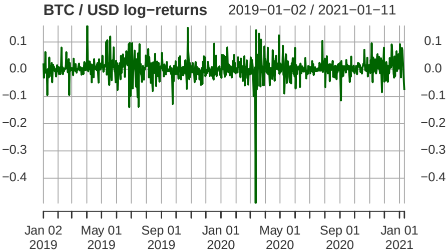
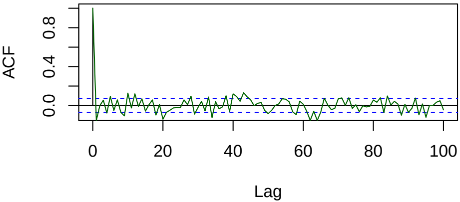


Predict $r(t+1)$ from n lags of $r(t)$ for m symbols:

$$f\begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{pmatrix} = y \approx r(t+1)$$



e.g. symbols: BTC/USD, ETH/USD, IOT/BTC, etc

Model Performance

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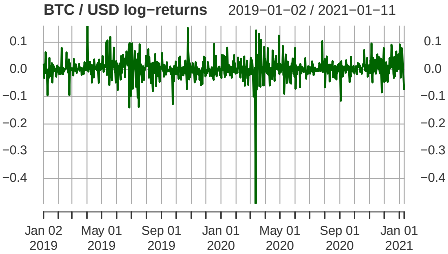
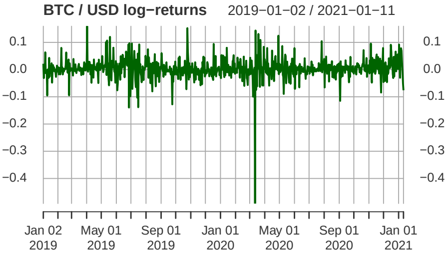


Model Performance

$$f\underbrace{\begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{m1} & \cdots & z_{mn} \end{pmatrix}}_{\text{predictors}} = y \approx \underbrace{x}_{\text{response}}$$



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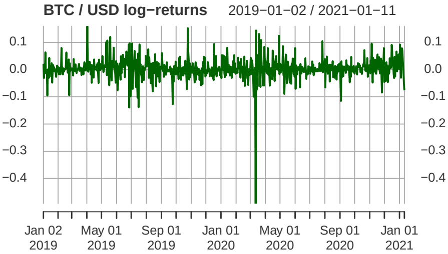

- Use $n \in [2 : 20]$ lags and $m \in [1 : 5]$ symbols (i.e. $m \times n \in [2 : 100]$).

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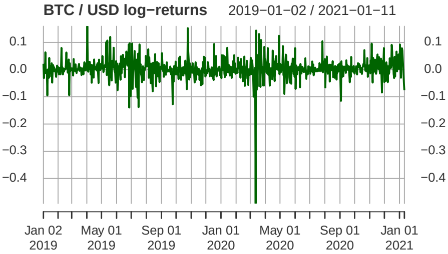

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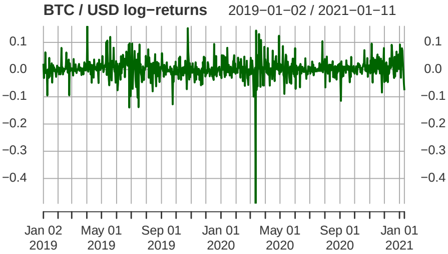

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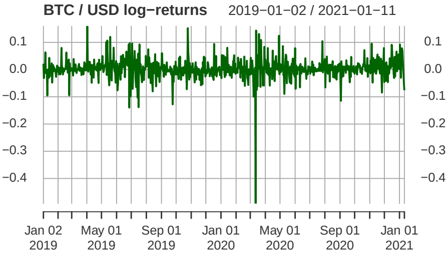

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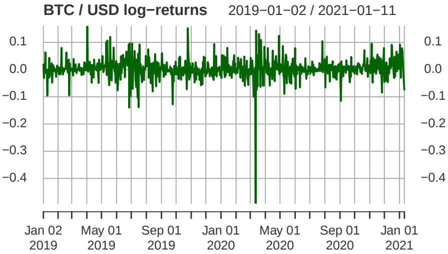

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- Is RMSE $= .035$ a good result? ($R^2 \approx .05$)

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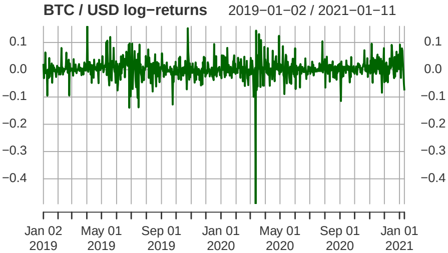

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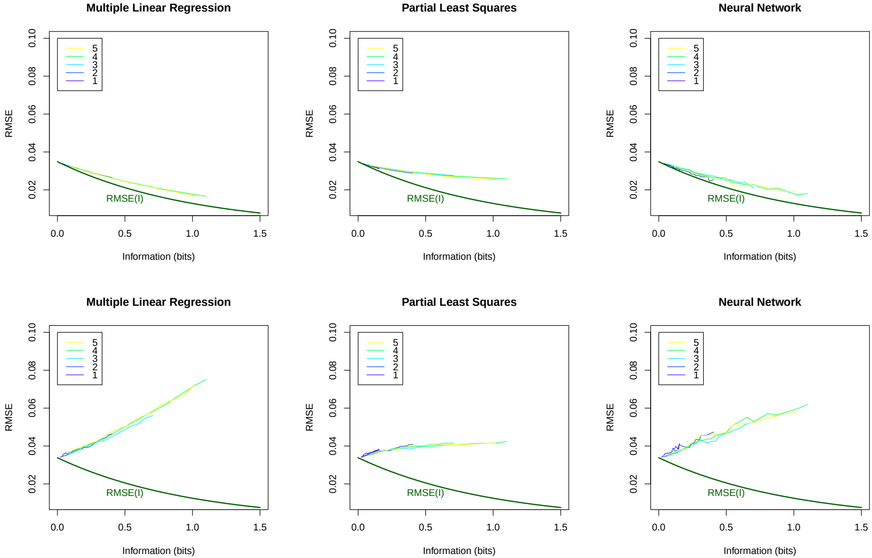
- Is RMSE $= .035$ a good result? $(R^2 \approx .05)$
- What is the smallest possible RMSE here?

Evaluation of RMSE



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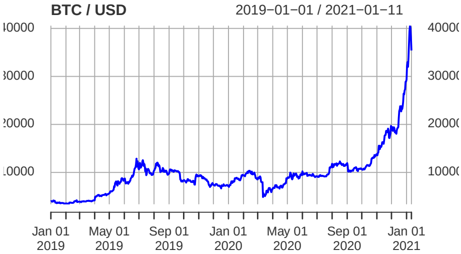

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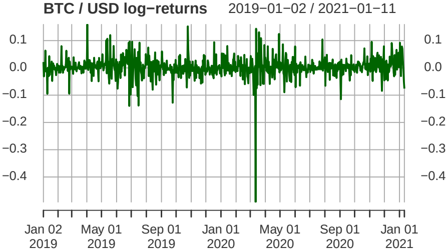

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Table: log-returns $r(t) = \log \frac{S(t+1)}{S(t)}$

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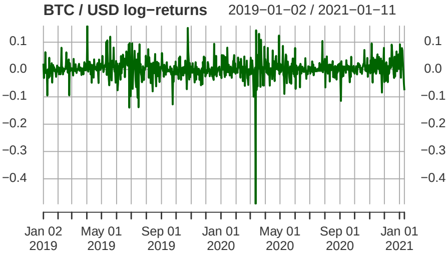

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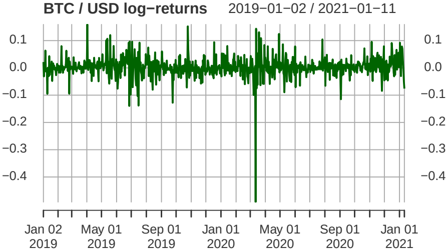

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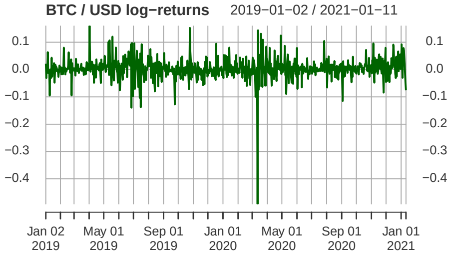

Utility $u(x, y)$ is a 2×2 matrix (confusion matrix):

$$\begin{bmatrix} u(x_1, y_1) & u(x_1, y_2) \\ u(x_2, y_1) & u(x_2, y_2) \end{bmatrix}$$

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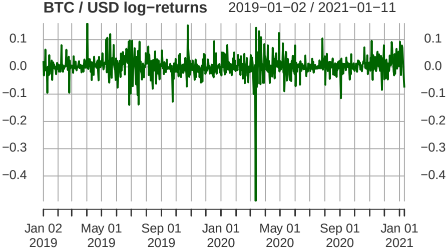

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$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

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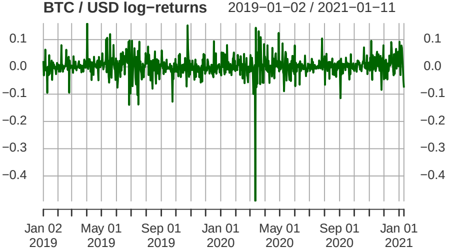

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$$\begin{bmatrix} c_1 + d_1 & c_1 - d_1 \\ c_2 - d_2 & c_2 + d_2 \end{bmatrix}$$

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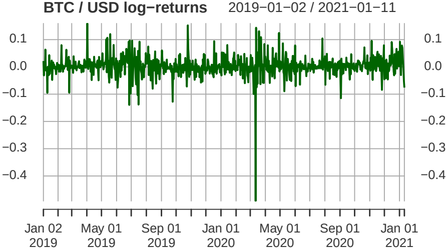

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2019-01-02	0.021	-1
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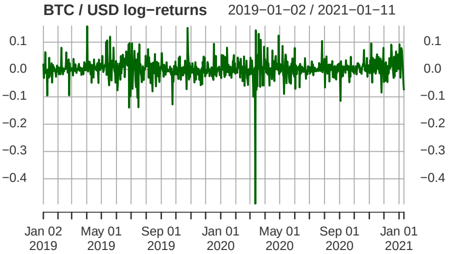
2019-01-05	-0.011	1
2019-01-06	0.064	-1
2019-01-07	-0.013	?



Predict sign of $r(t+1)$ from $r(t)$:

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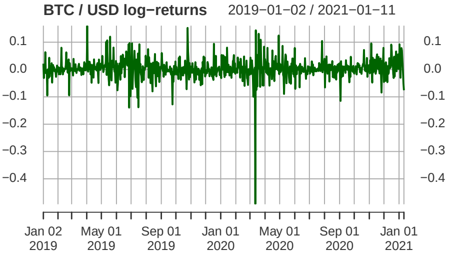

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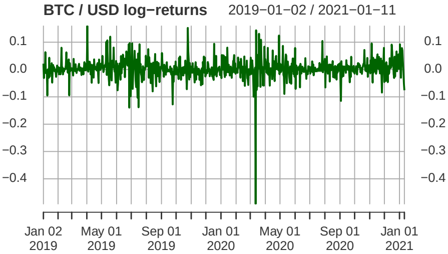

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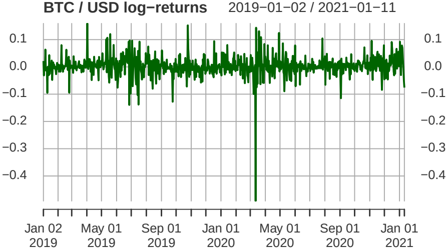
Predict sign $r(t+1)$ from n lags:

$$f\left(r(t-n), \dots, r(t)\right) = y \approx \operatorname{sign}[r(t+1)]$$



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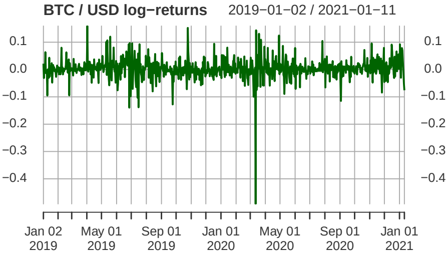

Predict sign $r(t+1)$ from n lags of m symbols (e.g. BTC/USD, ETH/USD, IOT/BTC):

$$f\begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{pmatrix} = y \approx \mathrm{sign}[r(t+1)]$$

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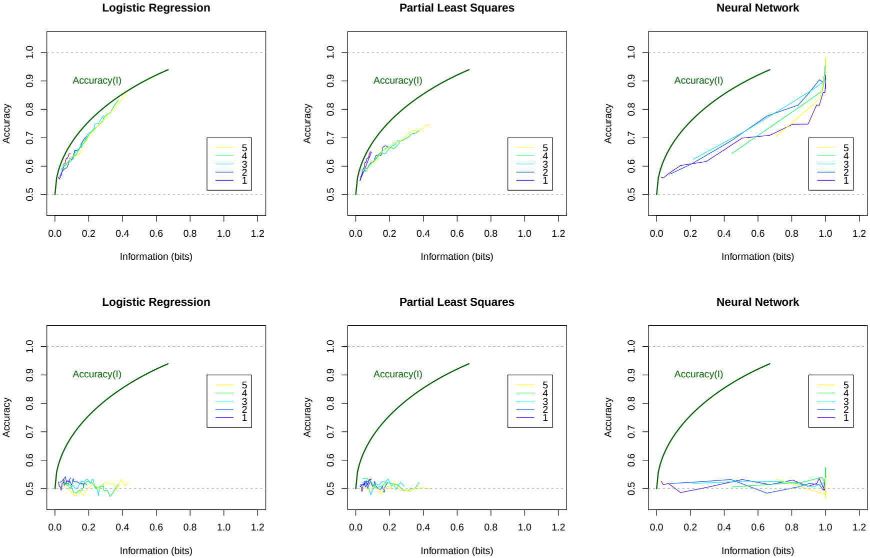
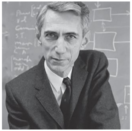
Evaluation of Accuracy



Information and its Value



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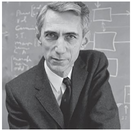

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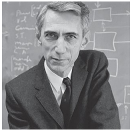

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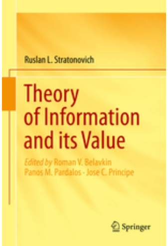

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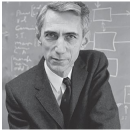

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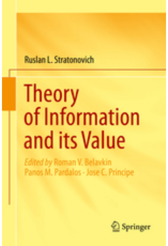

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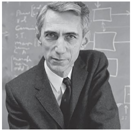

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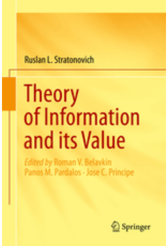

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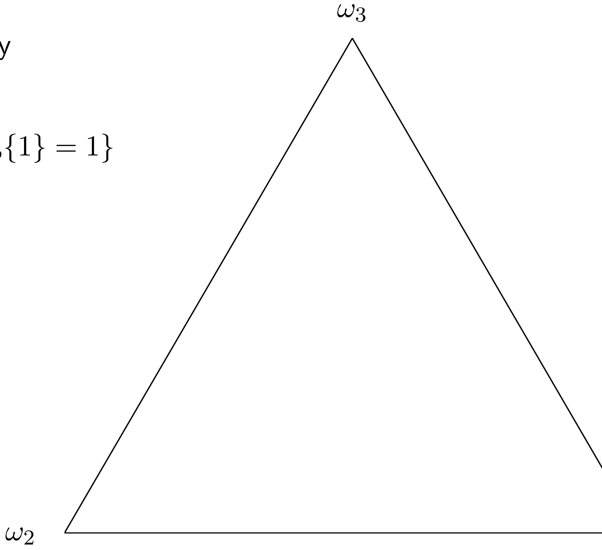

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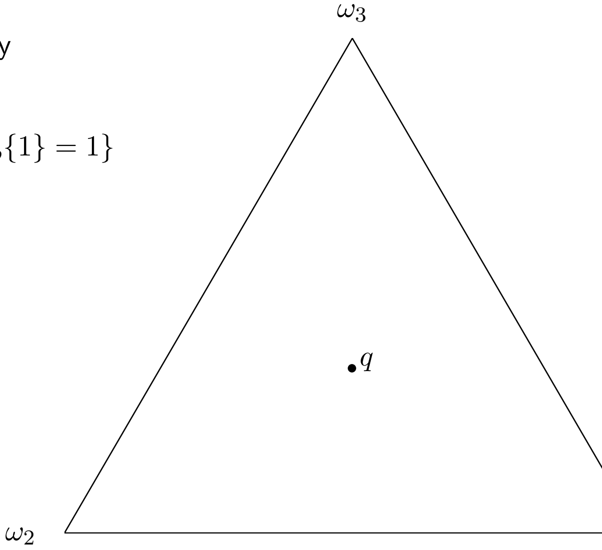

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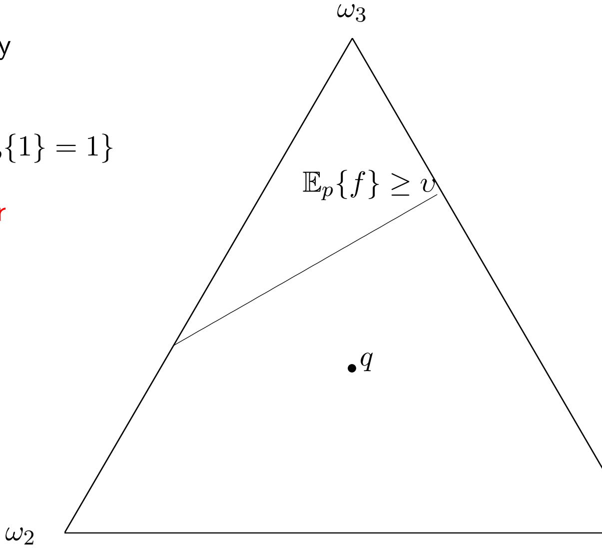

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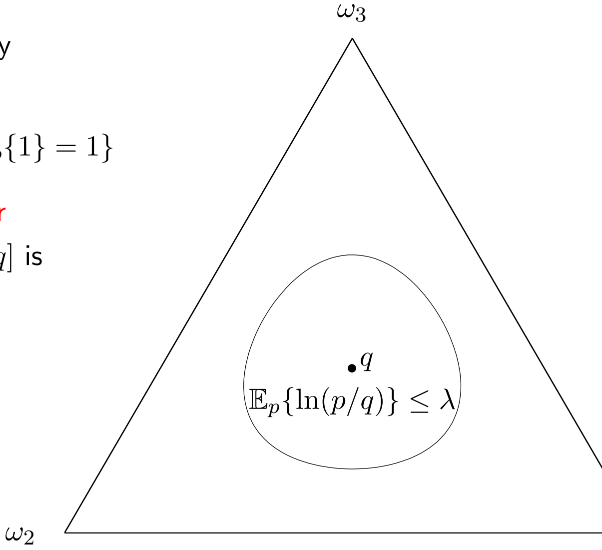

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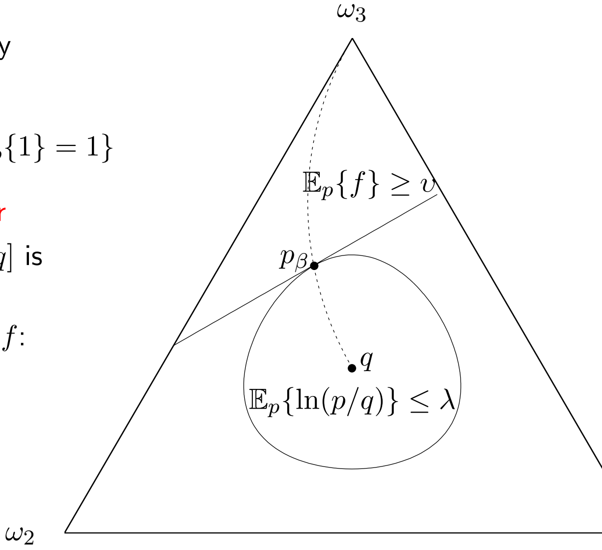

Value of Information (Hartley)

- $(\Omega, \mathcal{A}, P)$, $x, y, z : \Omega \to \mathbb{R}$
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- $u(x, y)$ — utility (or cost $c = -u$).

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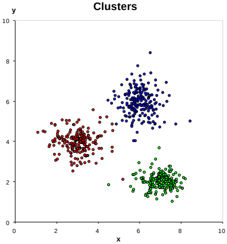

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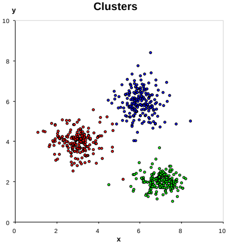

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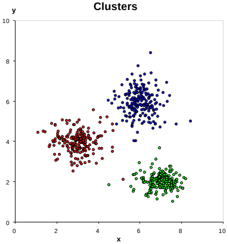

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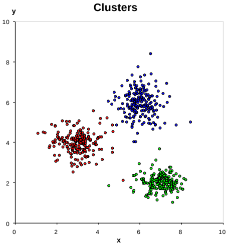

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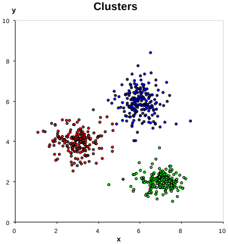

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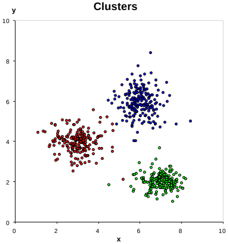

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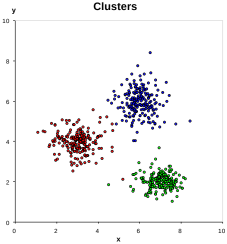

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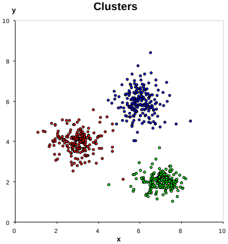

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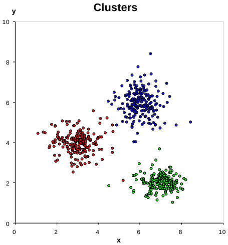

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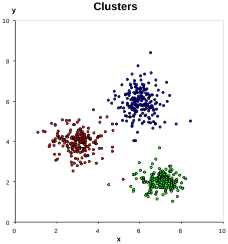

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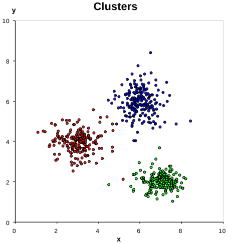

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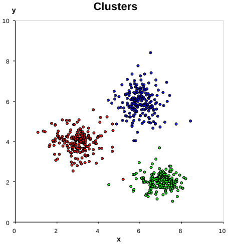

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Value of Information (Hartley)

- $(\Omega, \mathcal{A}, P)$, $x, y, z : \Omega \to \mathbb{R}$
- x — desired response (hidden), y — model response, z — data.
- $u(x, y)$ — utility (or cost $c = -u$).
-

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\begin{aligned}
x = f^{-1}(z) & \qquad \mathbb{E}_{P(x)}\{\max_{y(x)} u(x,y)\} =: U(\infty) \\
x \in f^{-1}(z) & \qquad \max_{z(x):\ln|Z|\leq I} \mathbb{E}_{P(z)}\left[\max_{y(z)} \mathbb{E}_{P(x|z)}\{u(x,y) \mid z\}\right] =: U(I) \\
P(x) & \qquad \max_{y} \mathbb{E}_{P(x)}\{u(x,y)\} =: U(0)
\end{aligned}
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Definition (Value of Information (Stratonovich, 1965))

$$V(I) := U(I) - U(0)$$

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Learning Systems

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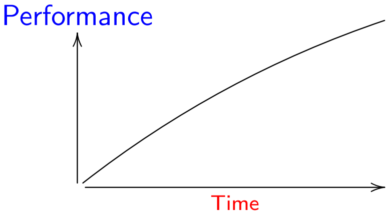

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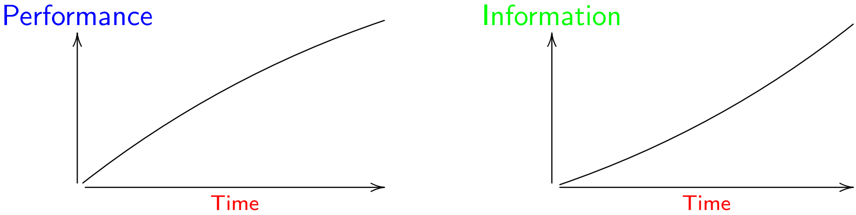

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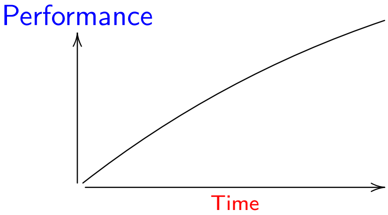
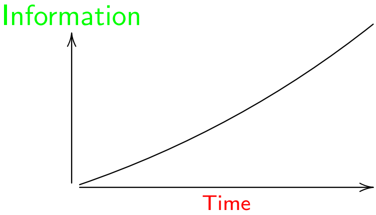
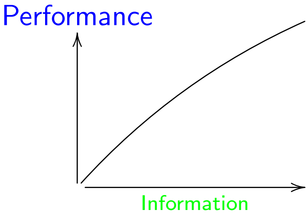




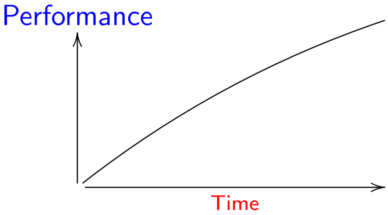
Optimal learning

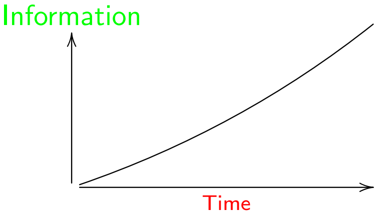


Maximize performance

s.t. information $\leq \lambda$

Learning Systems





Optimal learning

Minimize information

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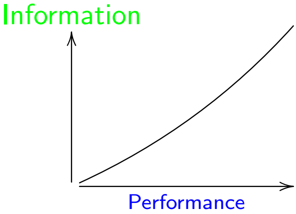
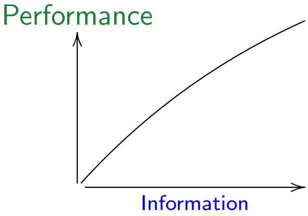


VoI as Conditional Extremum

- Linear programming problem $U(I)$:

$$\text{maximize} \quad \mathbb{E}_{P(y|x)}\{u(x,y)\} \quad \text{subject to} \quad I(X,Y) \leq I$$



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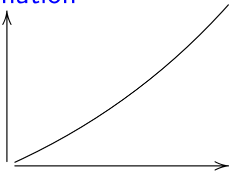
Minimize information

s.t. performance $\geq V$



Solution

- Lagrange function

$$K(p, \beta) = \mathbb{E}_p\{\ln(p/q)\} + \beta[U - \mathbb{E}_p\{u\}]$$

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- Optimal *inverse temperature* β:

$$\beta = \frac{dI(U)}{dU} \quad \text{or} \quad \beta^{-1} = \frac{dU(I)}{dI}$$

Optimal Value Functions

- Maximize $\mathbb{E}_p\{u\}$

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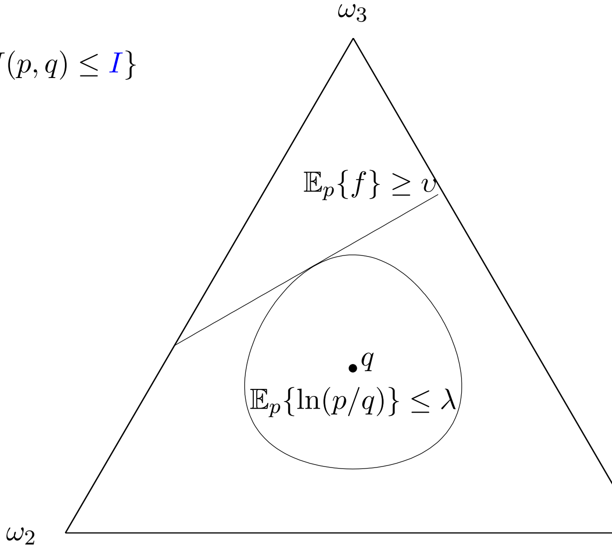

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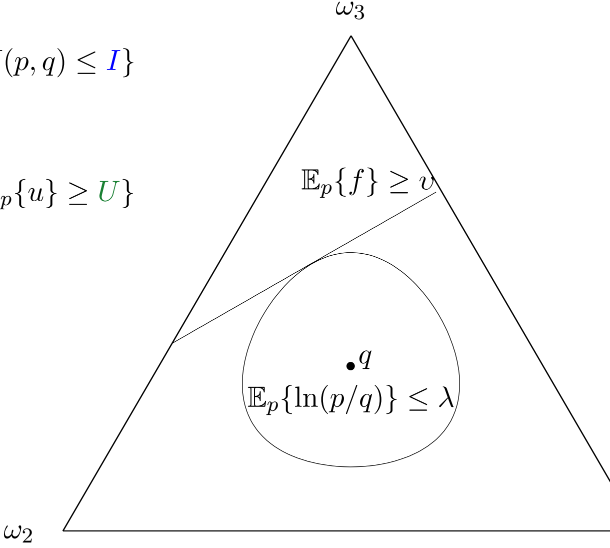

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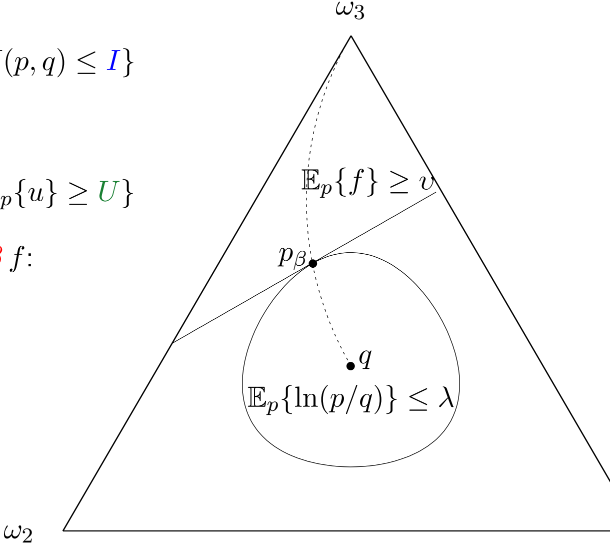

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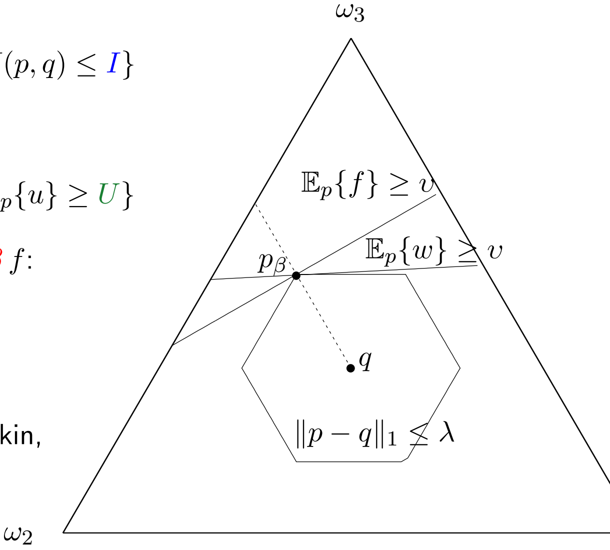

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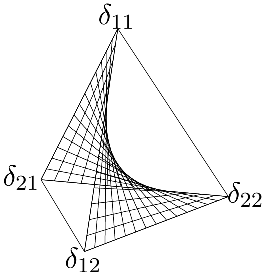

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$$\Gamma_0(\beta) = \sum_x \gamma_0(\beta,x)\, P(x) = \Gamma(\beta) + H(X)$$

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$$e^{-\gamma_0(\beta,x)} = e^{-\Gamma_0(\beta)} \iff T(1) = \sum_x e^{\beta\, u(x,y)} = e^{\Gamma_0(\beta)}$$

Computation of VoI

- Solutions to $V(I)$ are optimal joint probabilities of the form:

$$P(x \mid y) = e^{\beta u(x,y) - \Gamma_0(\beta)}$$

- If $T(\cdot) = \sum_x e^{\beta\, u(x,y)}(\cdot)$ is invertible, then

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Free Energy as the Dual of VoI

Entropy of Gibbs state

$$P(z \mid x) = \frac{e^{\beta\, u(x,z)}}{Z(\beta)}$$

β is *inverse temperature* (i.e. β^{-1} is temperature, $Z(\beta)$ is the *partition function* ($\Gamma(\beta) = \ln Z(\beta)$ — the cumulant gen. f-n).

$$H_P(X) := -\sum_X [\ln P(x)]\, P(x)$$

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Example: The Binary Case

- Let $X \times Y = \{x_1, x_2\} \times \{y_1, y_2\}$ and $u : X \times Y \to \mathbb{R}$:

$$\begin{bmatrix} u(x_1, y_1) & u(x_1, y_2) \\ u(x_2, y_1) & u(x_2, y_2) \end{bmatrix}$$

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- This gives

$$\begin{aligned} \Gamma_0(\beta) &= \beta\, c + \ln\left[2\cosh(\beta\, d)\right] \\ U(\beta) &= c + d\, \tanh(\beta\, d) \\ I(\beta) &= H(X) - [\ln[2\cosh(\beta\, d)] - \beta\, d\, \tanh(\beta\, d)] \end{aligned}$$

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- Explicit dependency

$$I(U) = H_2[p] - H_2\left[\frac{1}{2} + \frac{1}{2}\,\frac{U-c}{d}\right]$$

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$$V(I) = U(I) - U(0) = \frac{1}{4\pi\,e}e^{2H(X)}\left(1 - e^{-2I}\right)$$

Minimum RMSE

- Using $U(I)$ for $u(x,y) = -\frac{1}{2}|x-y|^2$:

$$\mathrm{RMSE}(I) = \sqrt{-2U(I)} = \frac{1}{\sqrt{2\pi\, e}}\, e^{H(X)-I}$$

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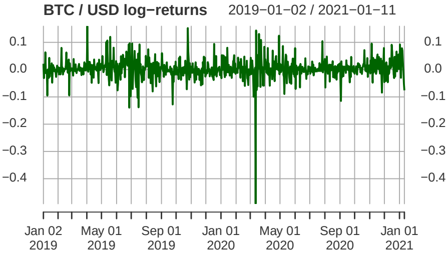
Example: RMSE in Time-Series Prediction

Table: log-returns $r(t) = \log \frac{S(t+1)}{S(t)}$

Date	$r(t-2)$	$r(t-1)$	$r(t)$	$r(t+1)$
2019-01-06	-0.031	0.008	-0.011	0.064
2019-01-07	0.008	-0.011	0.064	-0.013
2019-01-08	-0.011	0.064	-0.013	-0.0034
2019-01-09	0.064	-0.013	-0.0034	-0.004



Predict $r(t+1)$ from n lags of $r(t)$ for m symbols (e.g. BTC/USD, ETH/USD, IOT/BTC):

$$f \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{pmatrix} = y \approx r(t+1)$$

Optimal RMSE using VoI:

$$\text{RMSE}(I) = \sigma_x \, e^{-I}$$

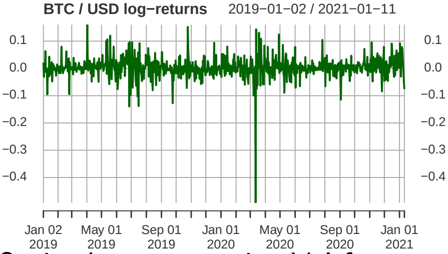
$$R^2(I) = 1 - e^{-2I}$$

Example: Accuracy in Time-Series Prediction

Table: log-returns $r(t) = \log \frac{S(t+1)}{S(t)}$

Date	$r(t-1)$	$r(t)$	sign$r(t+1)$
2019-01-06	0.008	-0.011	1
2019-01-07	-0.011	0.064	-1
2019-01-08	0.064	-0.013	-1
2019-01-09	-0.013	-0.0034	-1

Predict sign $r(t+1)$ from n lags of m symbols (e.g. BTC/USD, ETH/USD, IOT/BTC):



Optimal accuracy using VoI for binary utility as idenity matrix:

$$f \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{pmatrix} = y \approx \text{sign}[r(t+1)]$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Estimation of Mutual Information

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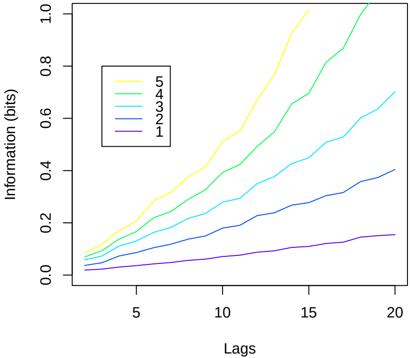
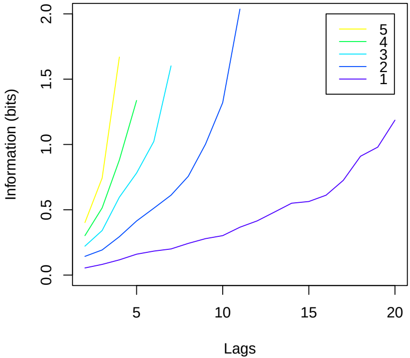
  where K_i are covariance matrices.
- This is sufficient for linear models.

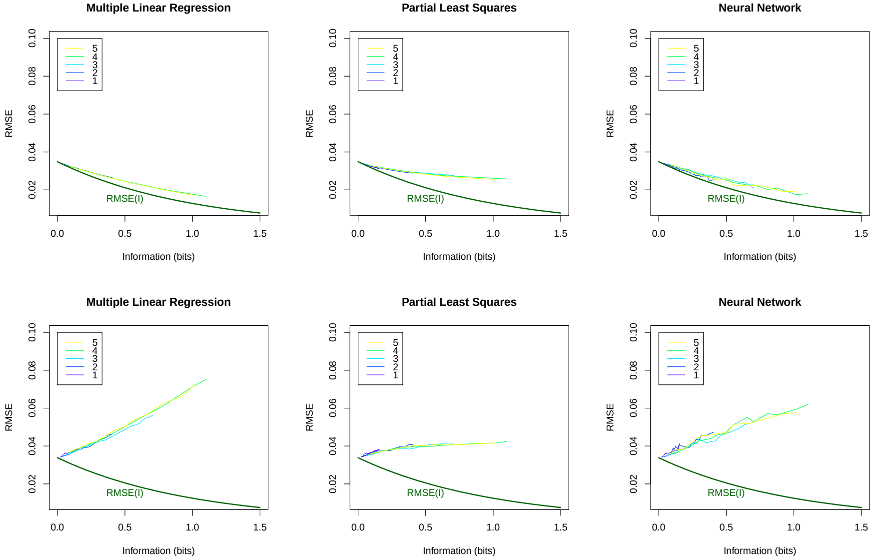
Mutual Information in Training and Testing Sets



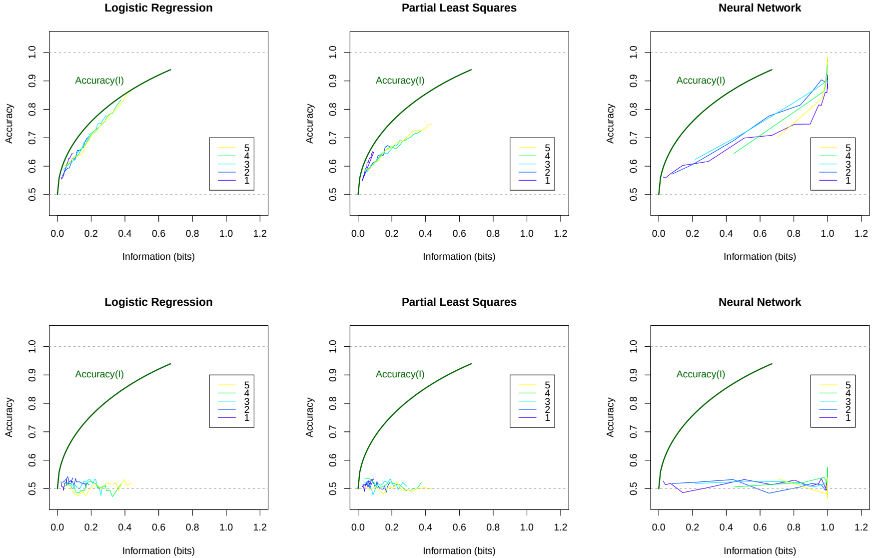


- $n \in [2:20]$ lags.
- $m \in [1:5]$ symbols (BTC/USD, ETH/USD, DAI/BTC, XRP/BTC, IOT/BTC).
- Training / testing sets 100 / 25 days.

Evaluation of RMSE



Evaluation of Accuracy



Other Measures of Model Performance

- Correlation between between prediction y and desired response x:

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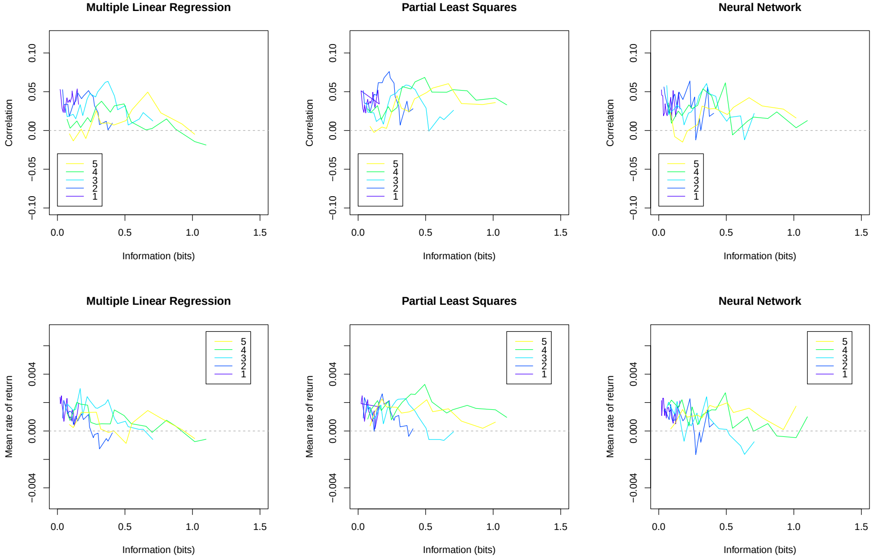
- In the context of day trading, we can estimate daily *Mean Rate of Return* (MRR):

$$\mathrm{MRR} := e^{\mathbb{E}\{\mathrm{sign}(y)\,\mathrm{sign}(x)|x|\}} - 1$$

Evaluation of Correlations and MRRs



Remark

Evolution as an Information Dynamic System

- EPSRC Sandpit '*Math of Life*' (July, 2009):

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University of Manchester : Chris Knight, Rok Krašovec & Danna Gifford

Optimal Mutation Operator

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- The temperature β^{-1} is the slope of $V(I)$:

$$\beta^{-1} = \frac{dV(I)}{dI}$$

Special Case: Hamming Space

Example (Hamming metric)

DNA sequences of length l and alphabet $\{1,\dots,\alpha\}$ are elements of Hamming space $\mathcal{H}^l_\alpha := \{1,\dots,\alpha\}^l$ with Hamming metric

$$d_H(a,b) = \|a-b\|_H = l - \sum_{i=1}^{l} \delta_{a_i}(b_i)$$

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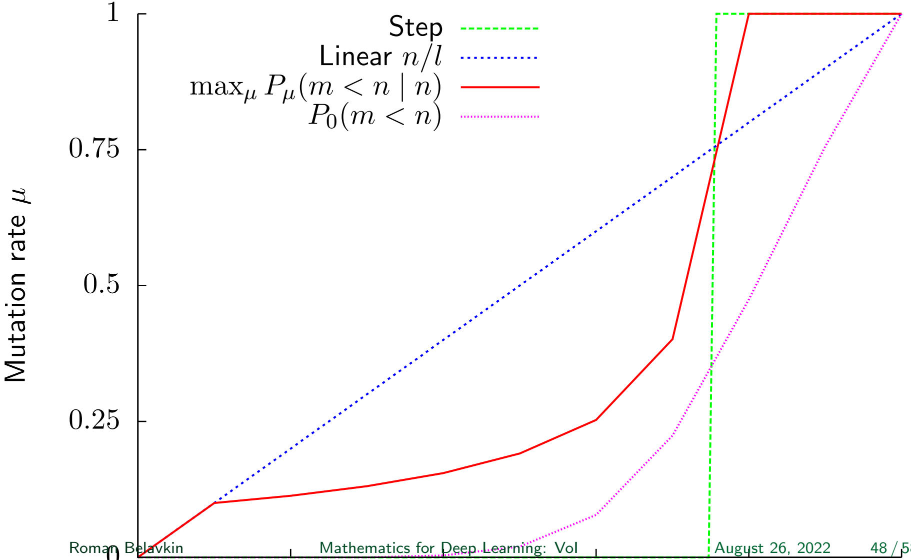
Solution

$$P_\beta(b \mid a) = \frac{e^{-\beta\,\|a-b\|_H}}{[1+(\alpha-1)e^{-\beta}]^l} = \prod_{i=1}^{l} \frac{e^{-\beta\,(1-\delta_{a_i}(b_i))}}{1+(\alpha-1)e^{-\beta}}$$

The constraint $\mathbb{E}\{r\} \leq v$ on $r = \|a-b\|_H$ defines $\beta = \ln\left(\mu^{-1}-1\right) + \ln(\alpha-1)$, where $\mu = v/l$ is the mutation rate.

Optimal mutation rate control functions in $\mathcal{H}_4^{10}$



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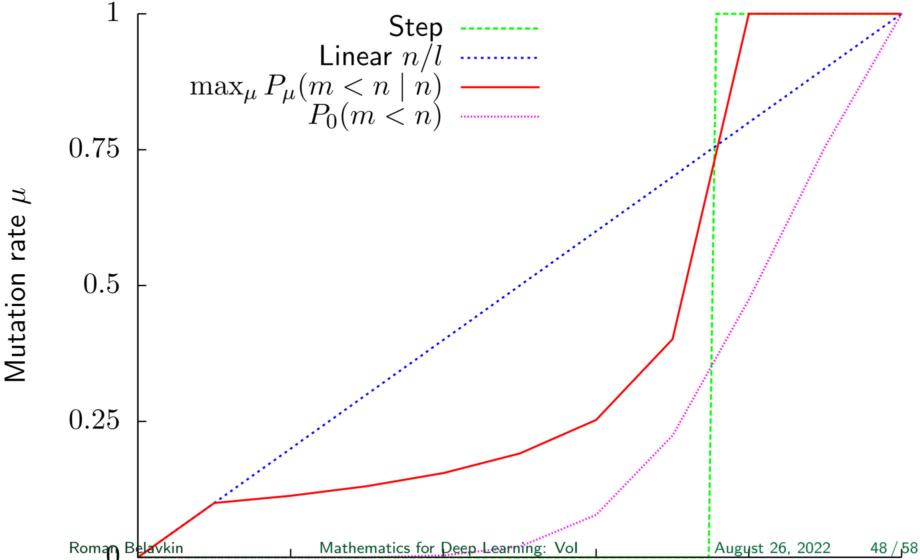
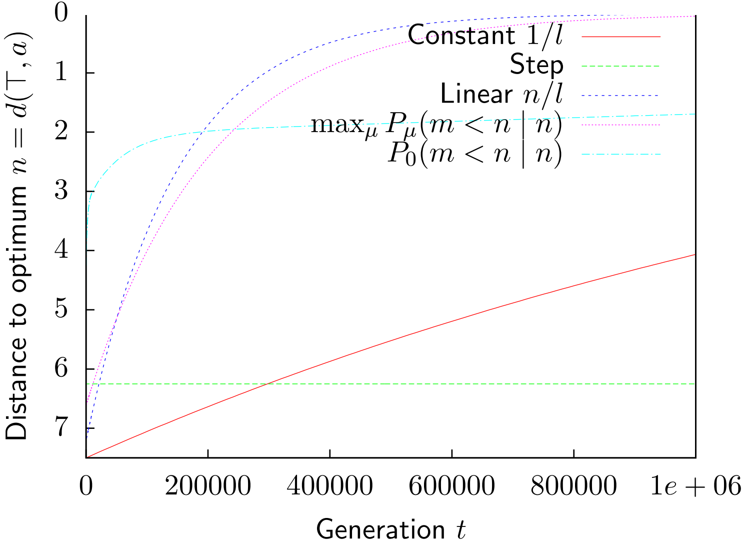
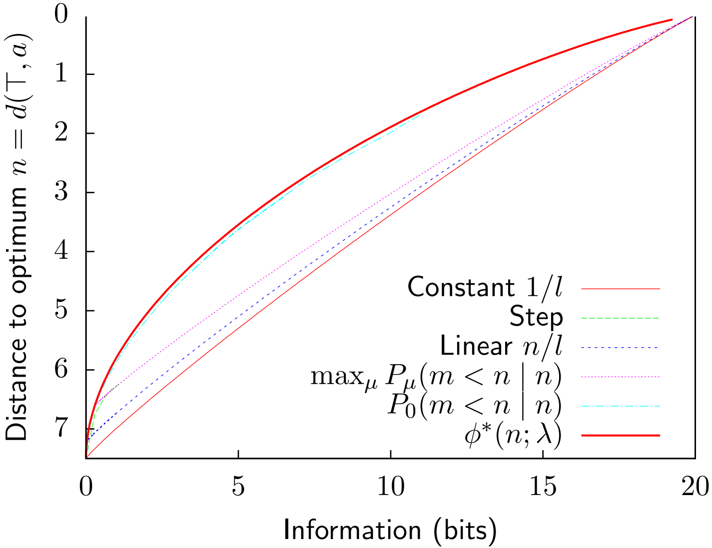


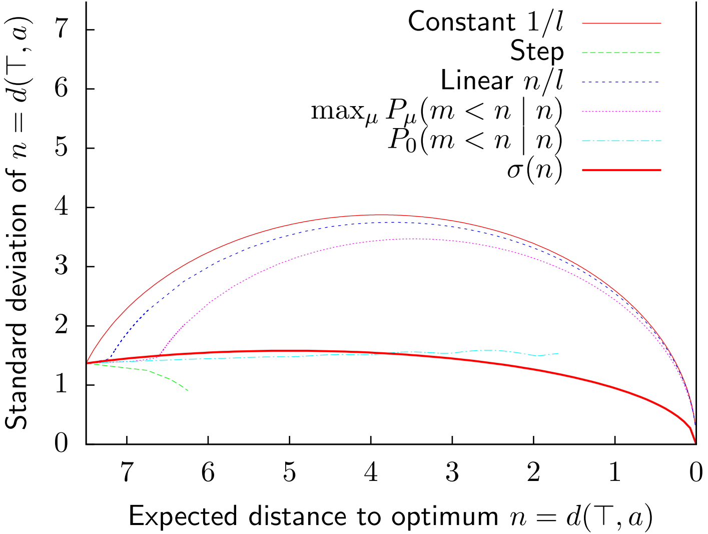
Expected Fitness in Time



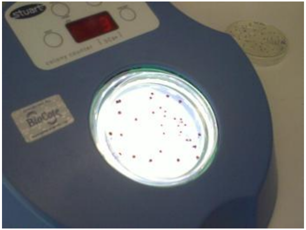
Evolution of Fitness in Information



Fitness Variance and Expectation



Mutation Rate Control in *E. coli*



- Used strains of *Escherichia coli* K-12 MG1665

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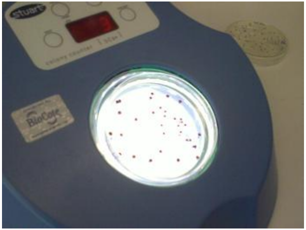

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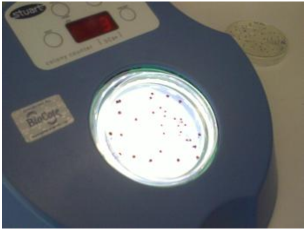

- Used strains of *Escherichia coli* K-12 MG1665
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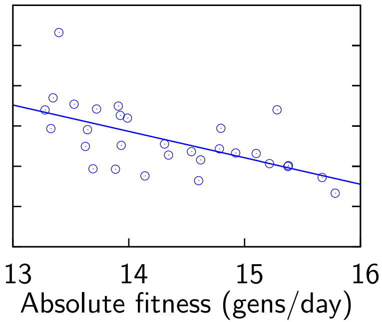
- Estimated mutation rates μ in *E.coli* strains grown in Davis minimal medium with different amount of glucose.

Experimental Results (Krašovec et al., 2014)



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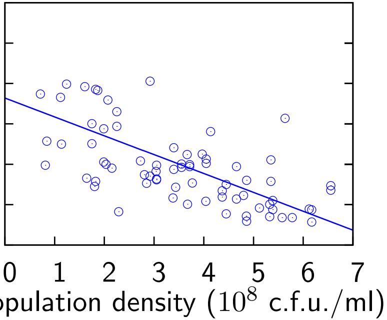

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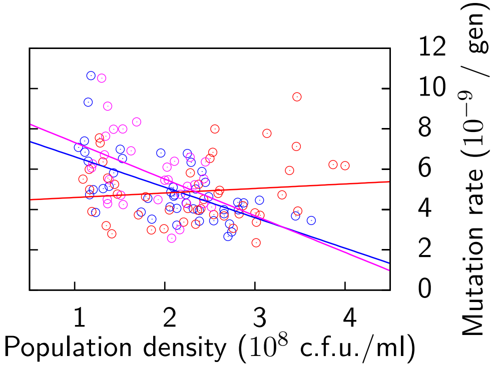

- Strong relationship between μ and density of cells ($p < .0001$).

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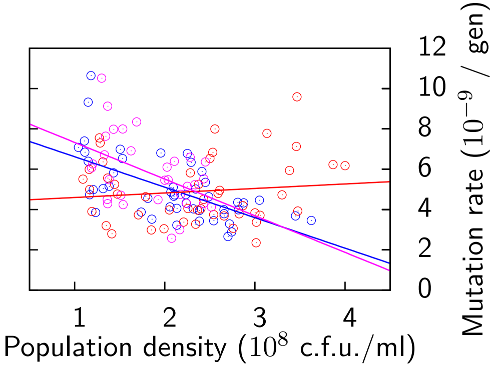

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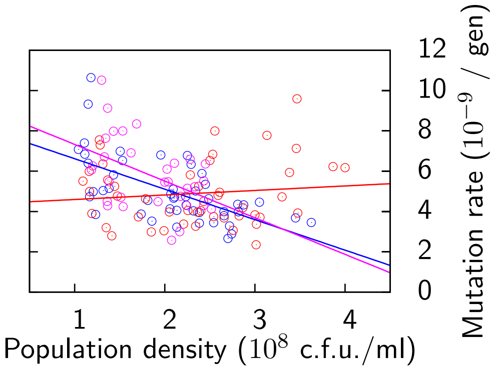

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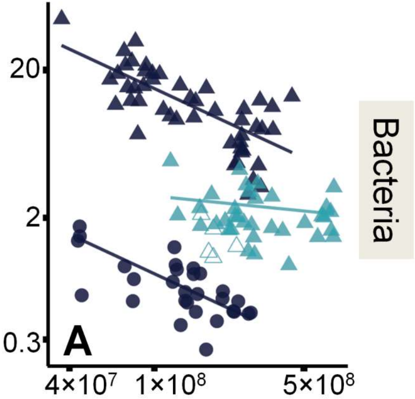
Plastic mutation rates in bacteria (Krašovec et al., 2017)



- *rifampicin* (triangles)

Krašovec, R., Richards, H., Gifford, D. R., Hatcher, C., Faulkner, K. J., Belavkin, R. V., Channon, A., Aston, E., McBain, A. J., Knight, C. G. (2017). Spontaneous mutation rate is a plastic trait associated with population density across domains of life. *PLoS Biology*, 15:8.

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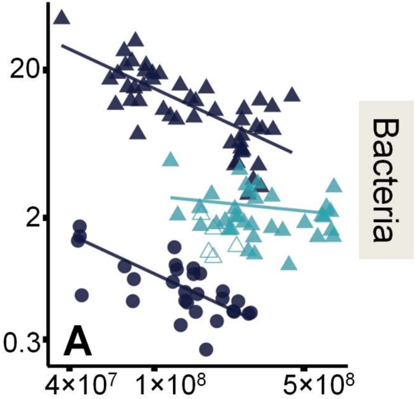

- *rifampicin* (triangles)
- *nalidixic acid* in *E.coli* (dark circles) and in *P. aeruginosa* (light circles)

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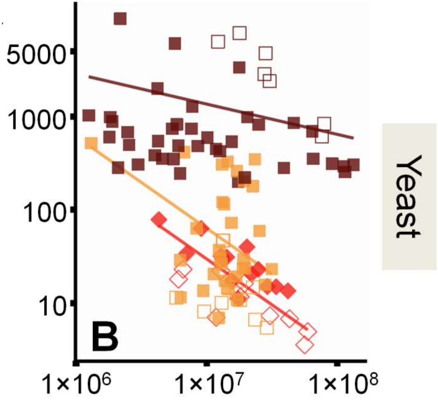

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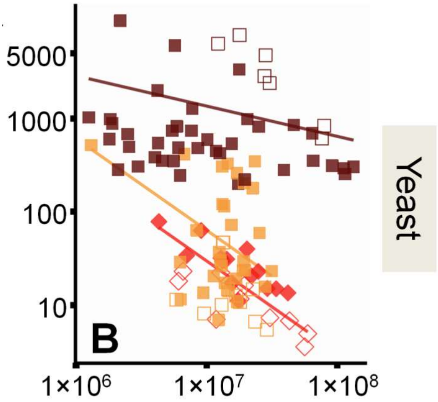

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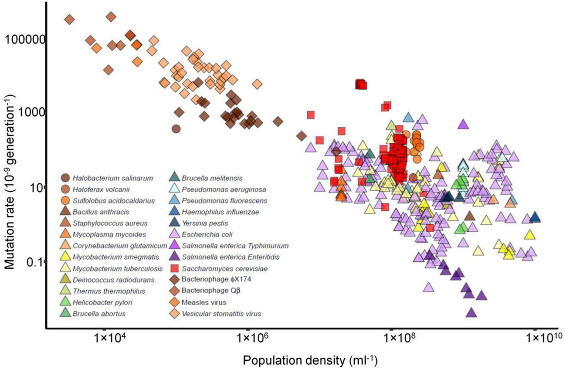
Plastic rates in all domains of life (Krašovec et al., 2017)



>70 years of published data (1943–2016), 67 studies, 26 species.

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- Control of parameters (mutation rates, learning rates, annealing schedule, exploration-exploitation balance, etc).

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