Mathematics for Deep Learning

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Introduction

Bayesian estimation and optimization

Functional analysis, duality, convex analysis

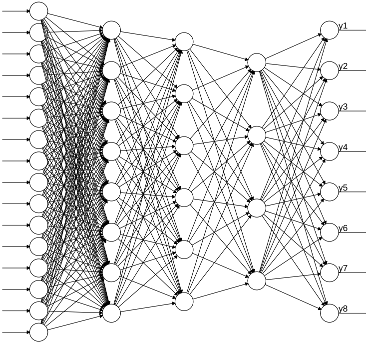
Information theory and information geometry

Probability and analysis

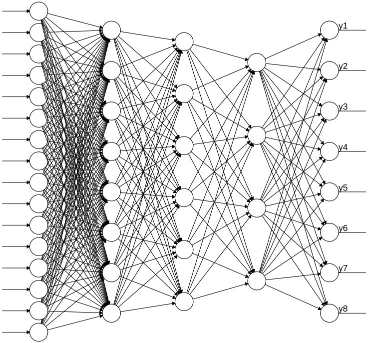
Stochastic filtering equations

Tensor algebras

Feed-forward neural networks



Feed-forward neural networks



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2. Repeat

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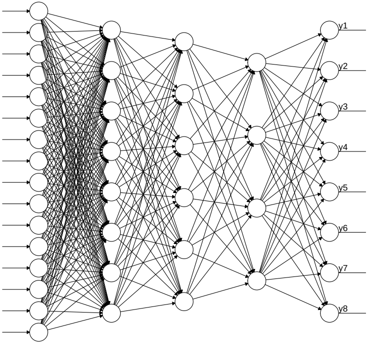

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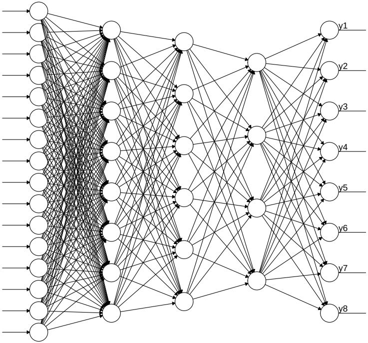

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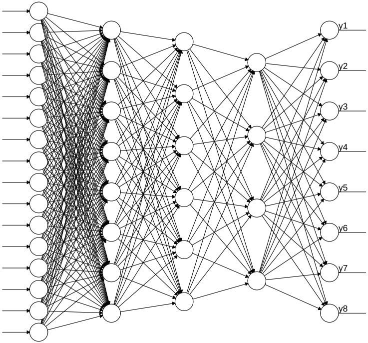

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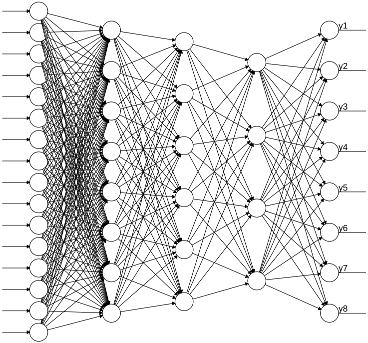

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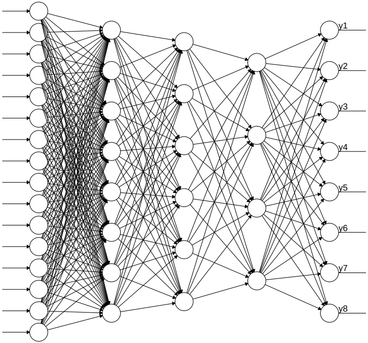

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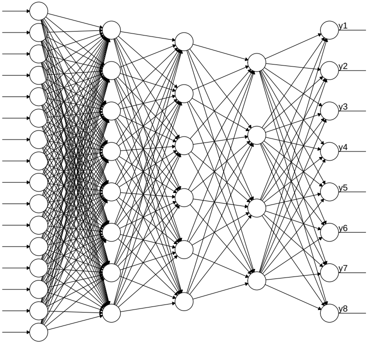

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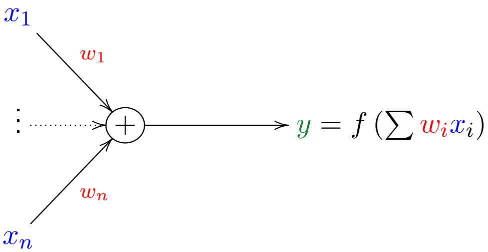
Subject to $z(x) : \ln |Z| \leq I$

Integrate and Fire Neurons

McCulloch and Pitts (1943)



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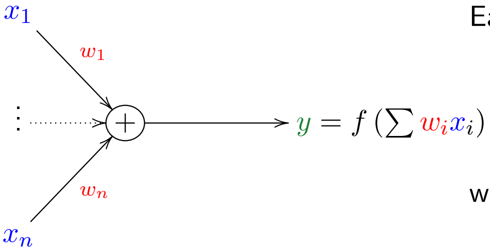

Each node computes a weighted sum:

$$v = \sum_{i=1}^{n} w_i x_i$$

which reminds us a linear model.

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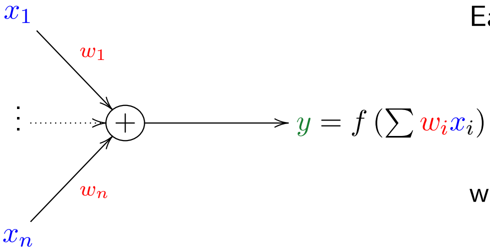

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$$f(v) = \begin{cases} 1 & \text{if } v \geq a \\ 0 & \text{otherwise} \end{cases}$$

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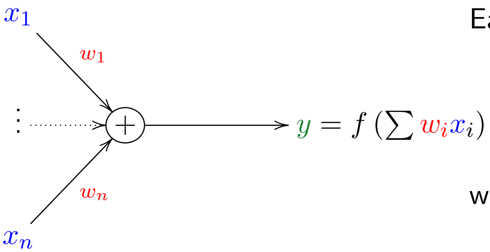

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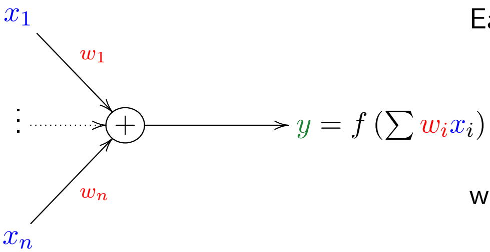

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- Each node partitions the input space into two halves.
- With several nodes the perceptron acts as a classifier.

Single layer perceptrons

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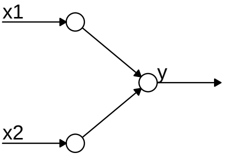

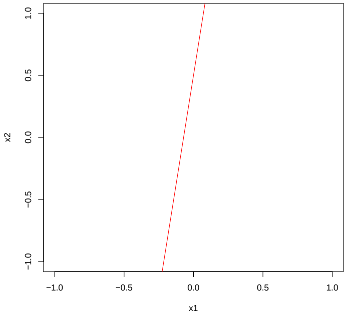

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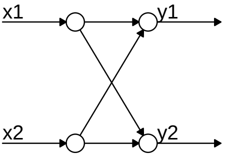

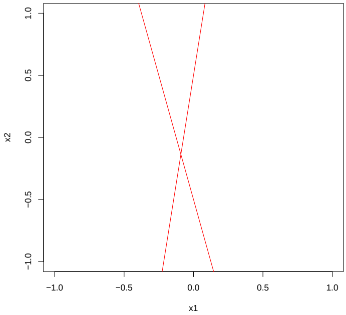

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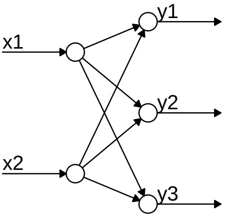

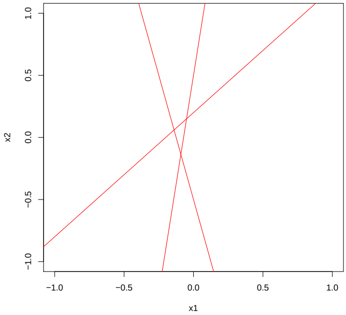

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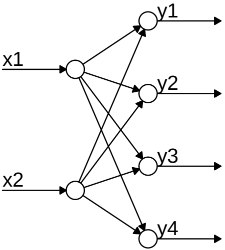

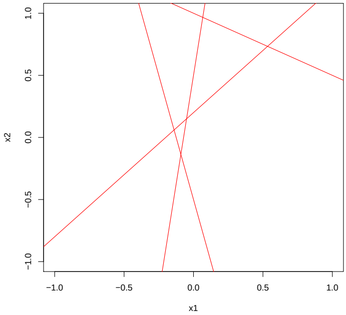

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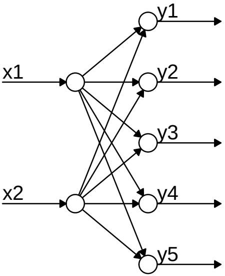

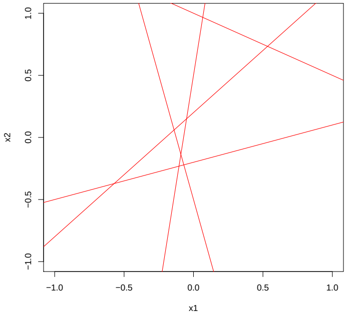

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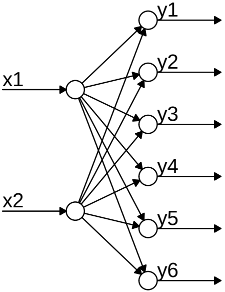

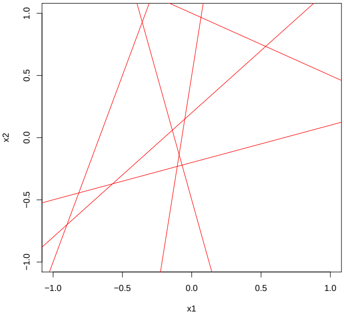

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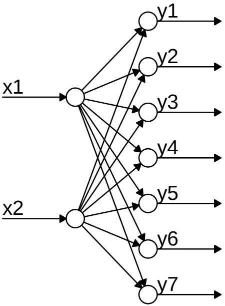

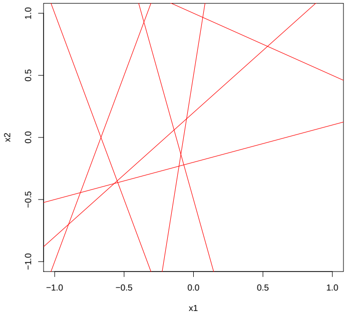

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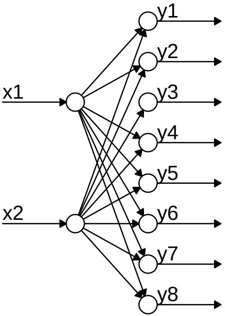

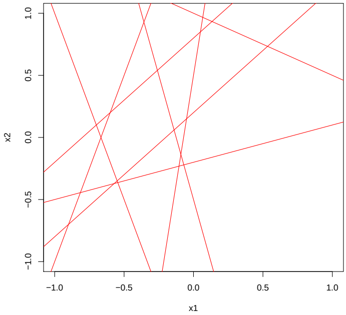

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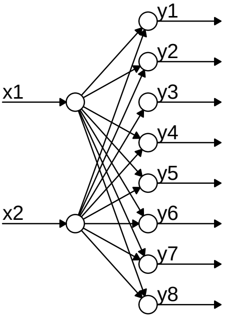

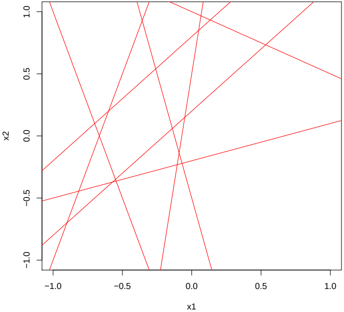

- n nodes partition the space into 2^n subsets.

Learning neural models and the value of information (VoI)

$$R(I) := \inf_{z(x)} \mathbb{E}_{P(z)} \left\{ \inf_{y(z)} \mathbb{E}_{P(x|z)} \{c(x,y) \mid z\} \right\}$$

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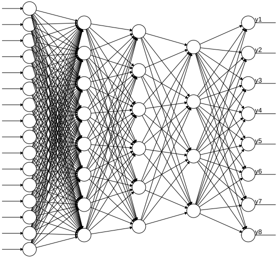

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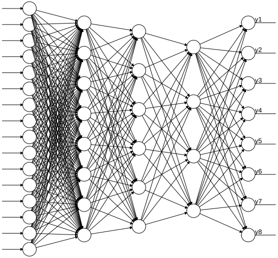

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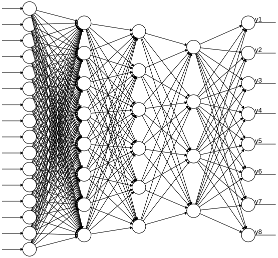

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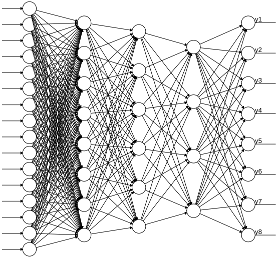

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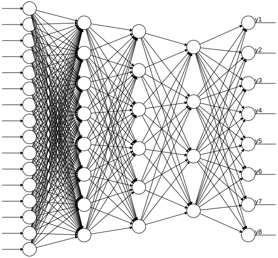

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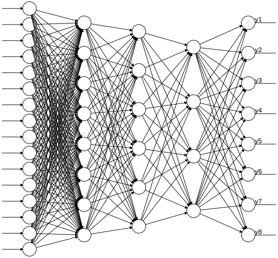

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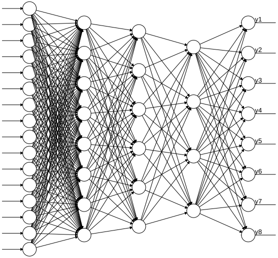

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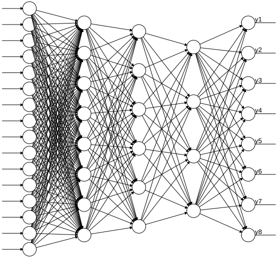

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  - Crossentropy as cost function.
- Learning weights in neural networks can be seens as a process of maximization of the value of information.

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$$U[y(z)] = \int_Z \int_X u(x, y(z)) p(x, z)\, dx\, dz$$
$$= \int_Z U[y \mid z]\, p(z)\, dz$$

where $U[y \mid z] = \int_X u(x, y)\, p(x \mid z)\, dx$ — conditional (posterior) EU.

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$$\begin{aligned} U[y(z)] &= \int_Z \int_X u(x, y(z)) p(x, z)\, dx\, dz \\ &= \int_Z U[y \mid z]\, p(z)\, dz \end{aligned}$$

where $U[y \mid z] = \int_X u(x, y)\, p(x \mid z)\, dx$ — conditional (posterior) EU.

Theorem

$$\max_{y(z)} U[y(z)] = \int_Z \max_{y(z)} U[y \mid z]\, p(z)\, dz$$

Optimal estimation / decision

- Optimal $y(z)$ is defined from the condition:

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Example (Mean-square cost)

For $u(x,y) = -\frac{1}{2}(x-y)^2$ we have $\frac{\partial}{\partial y}u(x,y) = y - x$, so that

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Example (Binary cost)

For $u(x,y) = \delta(x-y)$ we have

$$\int_X \frac{\partial}{\partial y}\delta(x-y(z))p(x \mid z)\,dx = \frac{\partial}{\partial x}p(x \mid z) = 0$$

so that $\hat{y}(z) = \arg\max p(x \mid z)$.

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For $u(x, y) = x \ln y + (1 - x) \ln(1 - y)$ for $x, y \in (0, 1)$.

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Equivalent to minimization of $KL[x, y] = \sum x[\ln x - \ln y]$

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$\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{C}$

$$\langle x, y\rangle := \sum x_i y_i \,, \qquad \langle x, y\rangle := \int x\, dy \,, \qquad \langle x, y\rangle := \operatorname{tr}\{xy\}$$

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- Y is a left (resp. right) module over $X \subseteq Y$ w.r.t. $z'y$ (resp. $yz^{*\prime *}$).

Exponents and Logarithms

- Define by the power series

$$e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}\,, \qquad \ln y := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\,(y-1)^n$$

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- Because $X \subseteq Y$, we can consider

$$\exp : X \to Y \qquad \text{and} \qquad \ln : Y \to X$$

Random variables and probability measures

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- Random variables (observables) are $f \in \mathcal{C}''_c(\Omega)$.

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- The set of all probability measures

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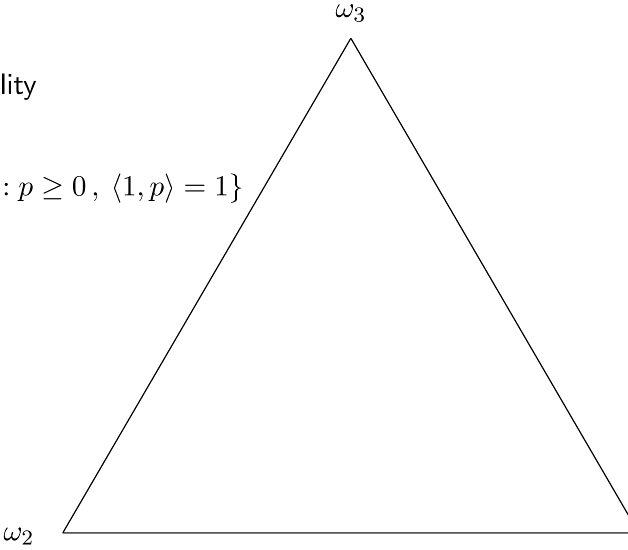

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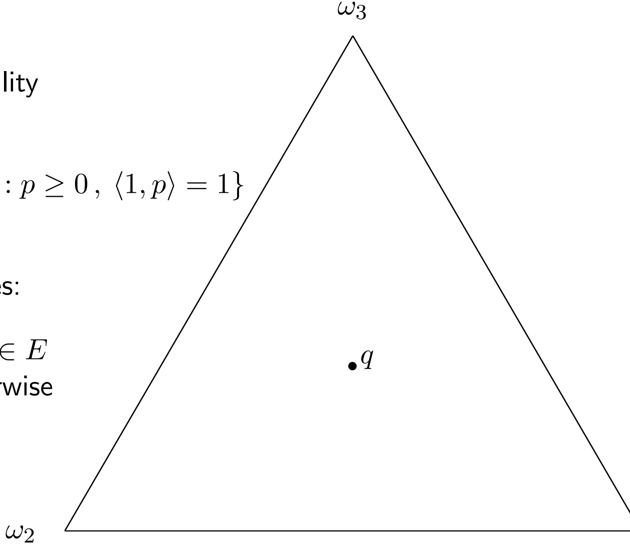
- $\delta \in \operatorname{ext} \mathcal{P}$ are Dirac (elementary) measures:

  $$\delta_\omega(E) = \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \text{otherwise} \end{cases}$$



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Expected Utility (von Neumann & Morgenstern, 1944)

$(\mathcal{P}, \lesssim)$ satisfies axioms (continuity, independence and Archimedian) if and only if there exists the expected utility representation of $(\mathcal{P}, \lesssim)$:

$$q \lesssim p \qquad \iff \qquad \mathbb{E}_q\{u\} \leq \mathbb{E}_p\{u\}$$

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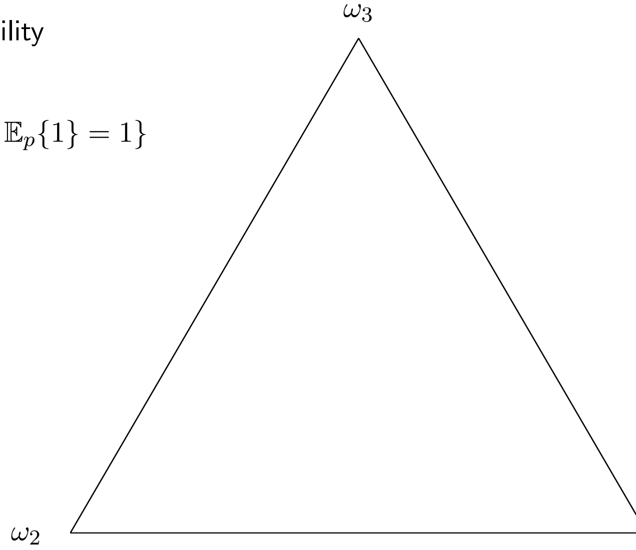

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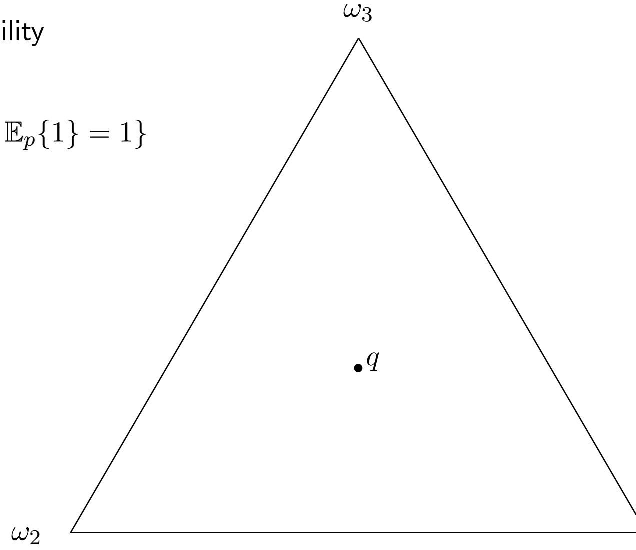

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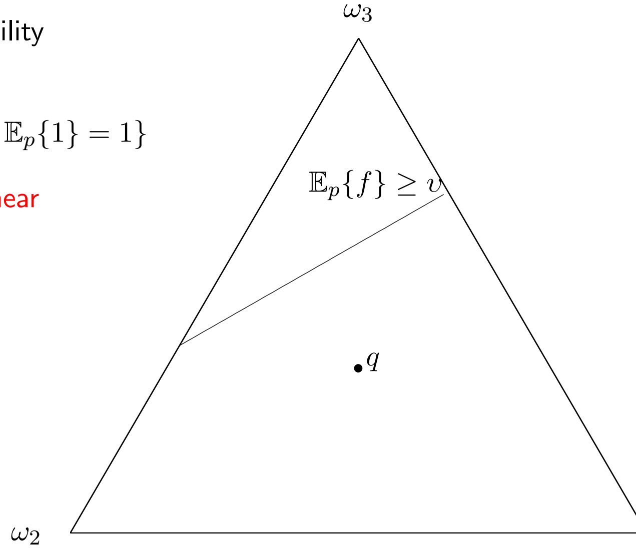

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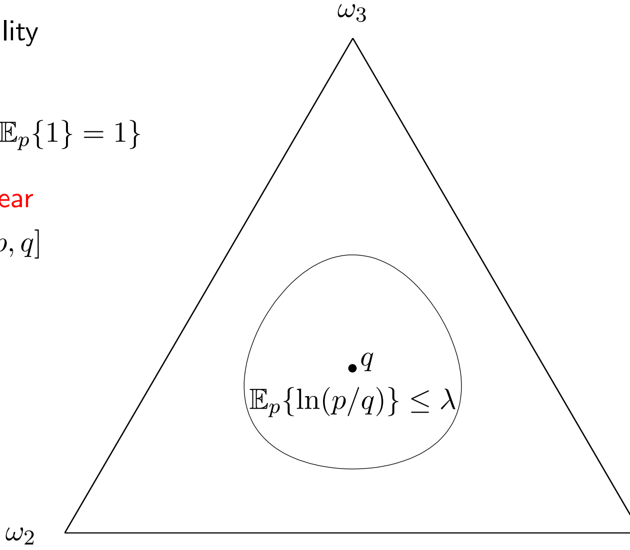

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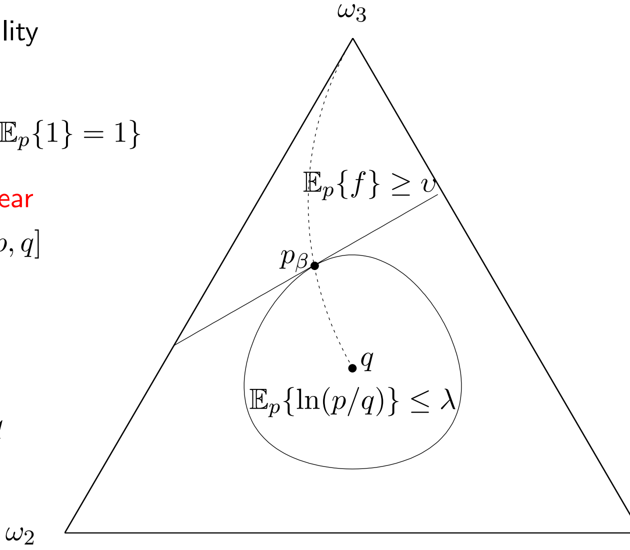

Generating Functions

- $P(\omega)$ gives us all moments of $x : \mathcal{A} \subseteq 2^{\Omega} \to \mathbb{R}$:

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The KL-divergence and its Dual $\Gamma[u] = \ln \Theta(u)$

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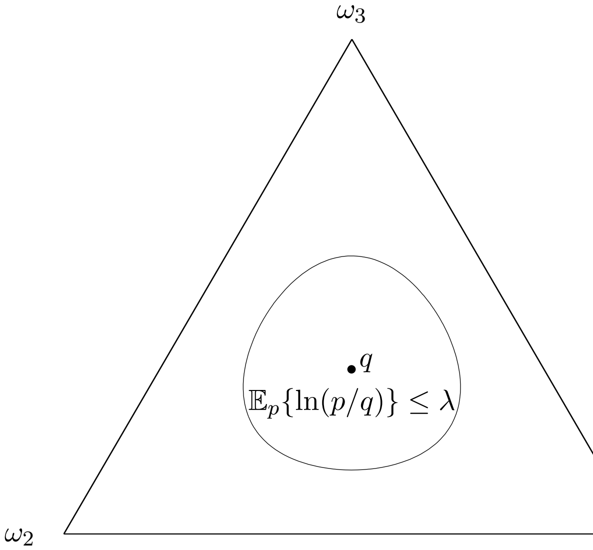

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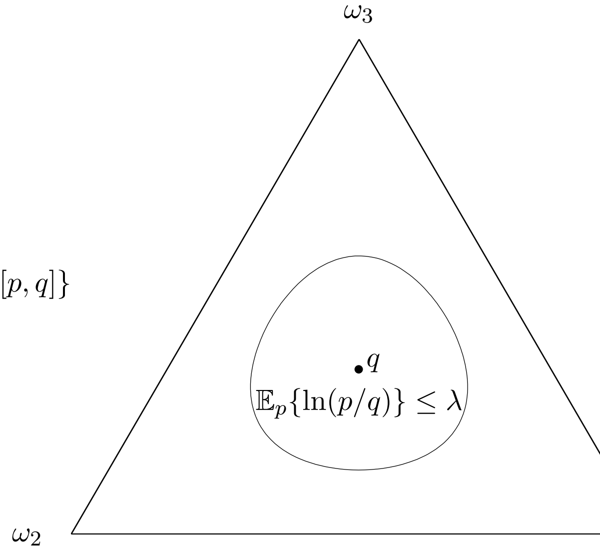

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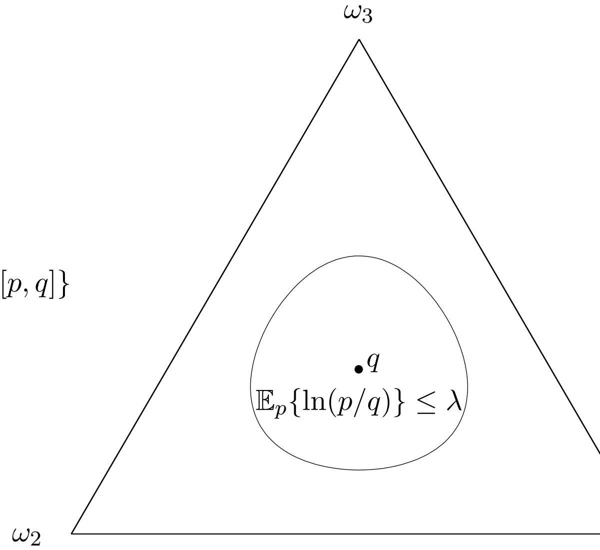

Markov Processes and PDE

Consider a transformation of $p(x)$ into $p_\tau(x_\tau)$ after time τ:

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- Thus, p_τ turns out to be ‘expanded’ at p:

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- In the limit $\tau \to 0$, we obtain the kinetic equation:

$$\dot{p}(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x}\right)^n [K_n(x)\, p(x)]$$

where $K_n(x) := \lim_{\tau \to 0} \frac{m_n(x)}{\tau}$

The Fokker-Planck equation

Definition (Continuous Markov process)

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Example (Diffusion equation)

If the drift $K_1 = 0$ and diffusion $K_2 = 1$, then

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}$$

This corresponds to the Wiener process $\{x(t)\}_{t\in(0,T)}$.

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- This will allow us to derive a (stochastic) differential equation for $w(x, t)$.

Recursive posterior densities

Theorem (Stratonovich (1959b, 1959a, 1960))

If the likelihood function is multiplicative $L_m(x_m,\dots,x_1)=p(z_m,\dots,z_1\mid x_m,\dots,x_1)=\prod_{t=1}^m p(z_t\mid x_t)$, *then*

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Proof.

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The Filtering Equation

Theorem (Stratonovich-Kushner-Zakai)

Let $\{x(t)\}_{t\in[0,T]}$ *be a continuous Markov process, and* $\{z(t)\}_{t\in[0,T]}$ *be the observed process given by*

$$dx(t) = a(x,t)dt + b(x,t)\,dw(t)\,, \qquad dz(t) = s(x,t)dt + \sqrt{N_0/2}\,dv(t)$$

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where $\mathcal{L}[\cdot] := -\frac{\partial}{\partial x}[a(x,t)\cdot] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[b(x,t)\cdot]$ is the Kolmogorov-Fokker-Planck operator.

The Filtering Equation

Theorem (Stratonovich-Kushner-Zakai)

Let $\{x(t)\}_{t\in[0,T]}$ be a continuous Markov process, and $\{z(t)\}_{t\in[0,T]}$ be the observed process given by

$$dx(t) = a(x,t)dt + b(x,t)\,dw(t)\,, \qquad dz(t) = s(x,t)dt + \sqrt{N_0/2}\,dv(t)$$

where v is the standard Gaussian white noise.
Then the (normalized) posterior density $w(x,t) := p(x_t \mid z_t, \ldots, z_1)$ and the unnormalized measure $u(x,t) = w(x,t)/C(t)$ satisfy

$$dw(x,t) = \mathcal{L}[w(x,t)]\,dt + \frac{2}{N_0}w(x,t)[s(x,t) - \langle s(x,t)\rangle]\,\underbrace{[dz - \langle s(x,t)\rangle\,dt]}_{\textit{Innovation}}$$

$$du(x,t) = \mathcal{L}[u(x,t)]\,dt + \frac{2}{N_0}u(x,t)s(x,t)\,dz$$

where $\mathcal{L}[\cdot] := -\frac{\partial}{\partial x}[a(x,t)\cdot] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[b(x,t)\cdot]$ is the Kolmogorov-Fokker-Planck operator.

Proof sketch.

Consider variations $\delta z_m := \int_{\Delta t_m} z(x,t)\,dt = s_m \Delta t + \sqrt{N_0/2}\delta v_m$ during small Δt and use recursion:

$$\frac{w_m(x_m)}{C_m} = L_m(x_m) \int \pi(x_m \mid x_{m-1}) w_{m-1}(x_{m-1})\, dx_{m-1}$$

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&= L_m(x_m)\int \{\delta(x_m - x_{m-1}) + \Delta t\mathcal{L}[\delta(x_m - x_{m-1})]\}\, w_{m-1}(x_{m-1}) \\
&= e^{-\frac{1}{N_0\Delta t}[\delta z_m - s_m\Delta t]^2}\,\{w_{m-1}(x_m) + \Delta t\mathcal{L}[w_{m-1}(x_m)]\}
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&= L_m(x_m) \int \{\delta(x_m - x_{m-1}) + \Delta t \mathcal{L}[\delta(x_m - x_{m-1})]\}\, w_{m-1}(x_{m-1}) \\
&= \left[1 + \frac{2}{N_0}\delta z_m s_m + o(\Delta t)\right] \{w_{m-1}(x_m) + \Delta t \mathcal{L}[w_{m-1}(x_m)]\}
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where we used the Levy's rule $\mathbb{E}\{\delta z^2\} = (N_0/2)\Delta t + o(\Delta t)$.

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\end{aligned}$$

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$$C_m^{-1} = 1 + \frac{2}{N_0}\delta z_m\langle s_m\rangle + o(\Delta t)$$

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where we used the Levy's rule $\mathbb{E}\{\delta z^2\} = (N_0/2)\Delta t + o(\Delta t)$.

$$C_m = 1 - \frac{2}{N_0}\delta z_m \langle s_m \rangle + \frac{2}{N_0}\langle s_m \rangle^2 + o(\Delta t)$$

□

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- The main result proving optimality of such decompositions.

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