10. Problemset "Quantum Algebra & Dynamics"
December 19, 2025

Fock Representation(s)

10.1 Bogoliubov Transformations

Even in our favorite units with $\hbar = c = 1$, the definition

$$a = \frac{1}{\sqrt{2}}\left(x + \mathrm{i}p\right)$$
$$a^* = \frac{1}{\sqrt{2}}\left(x - \mathrm{i}p\right)$$

is not well defined, because x and p have different dimensions and neither is dimensionless. A more suitable definition introduces a mass m and frequency ω to make

$$a = \sqrt{\frac{m\omega}{2}}\left(x + \frac{\mathrm{i}}{m\omega}p\right) \tag{1a}$$
$$a^* = \sqrt{\frac{m\omega}{2}}\left(x - \frac{\mathrm{i}}{m\omega}p\right) \tag{1b}$$

dimensionless. It is well known that the harmonic oscillator

$$H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2 \tag{2}$$

is diagonal in the Fock representation $(\mathcal{H}, \pi, \Omega)$ with

$$a\Omega = 0 \tag{3a}$$

and $\mathcal{H}$ the completion of the linear combinations of

$$\Psi_n = \frac{1}{\sqrt{n!}}\left(a^*\right)^n \Omega\,. \tag{3b}$$

Consider now a harmonic oscillator with the same mass, but different frequency $\omega_\alpha = \alpha^2\omega$

$$H_\alpha = \frac{1}{2m}p^2 + \frac{m\omega_\alpha^2}{2}x^2 \tag{4}$$

with $\mathbf{R} \ni \alpha \neq 0$.

1. Find annihilation and creation operators b_α and b_α^* such that
$$H_\alpha = \omega_\alpha \left(b_\alpha^* b_\alpha + \frac{1}{2} \right) . \tag{5}$$
2. Express b_α, b_α^* and H_α as functions of a and a^*.
3. Compute the matrix elements
$$(\Psi_k, a\Psi_l) \tag{6a}$$
$$(\Psi_k, a^*\Psi_l) \tag{6b}$$
$$(\Psi_k, H\Psi_l) \tag{6c}$$
and
$$(\Psi_k, b_\alpha\Psi_l) \tag{7a}$$
$$(\Psi_k, b_\alpha^*\Psi_l) \tag{7b}$$
$$(\Psi_k, H_\alpha\Psi_l) \tag{7c}$$
4. Show that
$$b_\alpha = \left(U_s(\zeta(\alpha))\right)^* a U_s(\zeta(\alpha)) \tag{8a}$$
$$b_\alpha^* = \left(U_s(\zeta(\alpha))\right)^* a^* U_s(\zeta(\alpha)) \tag{8b}$$
with the *squeeze operators*
$$U_s(\zeta) = \mathrm{e}^{\frac{1}{2}\left(\zeta a^* a^* - \bar\zeta a a\right)} \tag{9}$$
and compute $\zeta(\alpha)$.
5. Express the Fock state Ω_α with $b_\alpha\Omega_\alpha = 0$ in terms of the $\{\Psi_n\}_{n\in\mathbf{N}_0}$. Here the formula
$$U_s(\zeta) = \mathrm{e}^{-\frac{\zeta}{2}\frac{\tanh|\zeta|}{|\zeta|}a^*a^*}\mathrm{e}^{-\ln(\cosh|\zeta|)\left(a^*a+\frac{1}{2}\right)}\mathrm{e}^{\frac{\bar\zeta}{2}\frac{\tanh|\zeta|}{|\zeta|}aa} \tag{10}$$
is helpful.
6. Can you prove (10)? *Hint: use the fact that* $A = aa/2$, $N = a^*a$ *and* $A^* = a^*a^*/2$ *form a closed Lie algebra that can be realized by* 2×2-*matrices. The coefficients in* (10) *can then be obtained from the explicit product of matrix exponentials.*