Universität Würzburg
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1. Open quantum systems

Due date: 07.05.2025 10:00

Throughout this exercise sheet, we adopt the convention $\boxed{\hbar = 1}$.

Exercise 1 *Expectation value and variance* **2 P.**

Calculate the expectation value of the spin $\langle \vec{S} \rangle$ and its variance $\langle (\vec{S} - \langle \vec{S} \rangle)^2 \rangle$ for the following states of a spin-1/2 particle:

- Pure state: $|\chi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$
- Mixed state: $\hat{\rho} = \frac{1}{2}(|\uparrow\rangle \langle\uparrow| + |\downarrow\rangle \langle\downarrow|)$

Note: The first quantity is a vector, while the second is a scalar.

Exercise 2 *Bloch representation of a two-level system* **3 P.**

a) Show that the density matrix of an arbitrary two-level system in the most general form can be decomposed in terms of Pauli matrices as follows: *1 P.*

$$\hat{\rho} = \frac{1}{2}\left(\hat{\mathbb{1}} + \vec{\sigma} \cdot \vec{v}\right),$$

where $\vec{v}$ is some vector. Under which condition on $\vec{v}$ is this a valid density matrix?

b) Under which condition on $\vec{v}$ does this density matrix describe a pure state? *1 P.*

c) Calculate the expectation values $\langle \hat{\sigma}_x \rangle$, $\langle \hat{\sigma}_y \rangle$, and $\langle \hat{\sigma}_z \rangle$ for the state described by this density matrix. *1 P.*

Exercise 3 *Quantum Zeno paradox* **6 P.**

Consider a two-level system described by the following Hamiltonian

$$\hat{H} = \begin{pmatrix} E_0 & -\Delta \\ -\Delta & -E_0 \end{pmatrix}, \text{ where } \Delta \in \mathbb{R}, \Delta > 0.$$

At the initial time, the system is prepared in the state $|\psi(0)\rangle = |\uparrow\rangle$.

a) If we allow the system to evolve, it undergoes Rabi oscillations. You are asked to **derive** these Rabi oscillations and demonstrate that, at the time *3 P.*

$$T = \frac{\pi}{2\sqrt{E_0^2 + \Delta_2}},$$

the system will be found in the state $|\downarrow\rangle$ with probability $P_\downarrow(T) = \frac{\Delta^2}{E_0^2 + \Delta_2}$.

Note:
Calculate the time-evolved state $|\psi(t)\rangle$ and project this state on $|\downarrow\rangle$.

Hint:
If $\vec{v}$ is a real unit vector and θ is a real number, then

$$\exp\left(i\theta\,\vec{v}\cdot\vec{\sigma}\right) = \cos(\theta)\hat{\mathbb{1}} + i\sin(\theta)\,\vec{v}\cdot\vec{\sigma}, \quad \text{where} \quad \vec{v}\cdot\vec{\sigma} = \sum_{k=1}^{3} v_k\hat{\sigma}_k.$$

b) Now, instead of allowing the system to evolve freely, we perform measurements of the observable $\hat{s}_z = \frac{1}{2}\hat{\sigma}_z$ at intervals of time τ, where $\tau \ll T$. Determine the probability $P_\downarrow(T)$ in this case. *3 P.*

Note:
Use $\hat{M}_k = \hat{s}_z$ as a measurement operator (introduced in the lecture). Choose $\tau = \frac{T}{N}$ and send $N \to \infty$ in the final step. Calculate the state at a time τ after the measurement of $\hat{s}_z$. Project this state on $|\downarrow\rangle$. Interpret your result.

Hint:
Measurement operators $\{\hat{M}_k\}$ satisfy

$$\sum_k \hat{M}_k^\dagger \hat{M}_k = \hat{\mathbb{1}}.$$

After measuring $|\psi\rangle$, the state becomes

$$|\psi_k\rangle = \frac{\hat{M}_k|\psi\rangle}{\sqrt{p_k}}, \quad \text{where} \quad p_k = \langle\psi|\hat{M}_k^\dagger\hat{M}_k|\psi\rangle.$$