

Time and normal ordered exponentials of free fields

Let the operator A be linear in creation and annihilation operators. Then the normal ordered exponential is related to the time ordered exponential via

$$:e^A: = \frac{T e^A}{\langle T e^A \rangle} = \frac{e^A}{\langle e^A \rangle}, \quad (1)$$

where the time ordering symbol T is usually omitted by convention, as indicated. In particular, the expectation values of fields we write are always expectation values of time ordered fields.

(a) Use Wick's theorem to prove

$$T e^A = :e^A: e^{\frac{1}{2}\langle A^2 \rangle}. \quad (2)$$

$$T e^A = T \sum_k \frac{1}{k!} A^k$$

$$= \sum_k \frac{1}{k!} T A^k$$

Enumerate number of possible contractions in $T A^k$

$$T A^k = \sum_{n=0}^{\lfloor k/2 \rfloor} :A^{k-2n}: \binom{k}{2n} \langle A^2 \rangle^n$$

$$= \sum_{n=0}^{\lfloor k/2 \rfloor} \frac{k!}{j^{(2n)} (k-2n)!} :A^{k-2n}: \langle A^2 \rangle^n$$

$$T e^A = \sum_{k=0}^{\infty} \sum_{n=0}^{\lfloor k/2 \rfloor} \frac{1}{j^{(2n)} (k-2n)!} :A^{k-2n}: \langle A^2 \rangle^n$$

re-index $m = k - 2n$

$$T e^A = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n)! m! 2^n} :A^m: \langle A^2 \rangle^n$$

$$= :e^A: e^{\frac{1}{2}\langle A^2 \rangle}$$

(b) Use (2) to prove that

$$\langle e^A \rangle = e^{\frac{1}{2}\langle A^2 \rangle}. \quad (3)$$

Note that (2) and (3) imply (1).

$$\langle e^A \rangle :e^A: = e^A \quad \text{--- (1)}$$

$$= :e^A: e^{\frac{1}{2}\langle A^2 \rangle} \quad \text{--- (2)}$$

$$\text{Hence} \quad \langle e^A \rangle = e^{\frac{1}{2}\langle A^2 \rangle}$$

(c) Use (2) to derive the identity

$$T :e^{A_1}: :e^{A_2}: \dots :e^{A_N}: = :e^{A_1+A_2+\dots+A_N}: \prod_{i < j} e^{\langle A_i A_j \rangle}.$$

Hint: Substitute $A = \sum_{i=1}^N A_i$ in (2).

By substituting

$$T e^{\sum A_i} = :e^{\sum A_i}: e^{\frac{1}{2}\langle (\sum A_i)^2 \rangle}$$

$$= :e^{\sum A_i}: e^{\frac{1}{2}\langle \sum_{i,j} A_i A_j \rangle}$$

$$= :e^{\sum A_i}: e^{\langle \sum_{i,j} A_i A_j \rangle}$$

$$= :e^{\sum A_i}: \prod_{i < j} e^{\langle A_i A_j \rangle}$$

At the same time,
 $T e^{\sum \lambda_i} = T : e^{\lambda_1} : \dots : e^{\lambda_n} : \quad (I \text{ have no clue how})$

2. Bosonization in second quantization

Consider a one-dimensional system of non-interacting fermions with anticommutation relations

$$\{c_{\alpha k}^\dagger, c_{\alpha' k'}\} = \delta_{\alpha\alpha'} \delta_{kk'} \quad (7)$$

and Hamiltonian

$$H_0 = \sum_{\alpha, k} \alpha v_F (k - \alpha k_F) c_{\alpha k}^\dagger c_{\alpha k}, \quad (8)$$

where v_F the Fermi velocity, and $\alpha = \pm$ refers to right (our $\bar{\psi}(\bar{z})$) and left (our $\psi(z)$) movers, respectively. We assume a system of length L and periodic boundary conditions (PBCs), which implies that the momenta are quantized as $k = \frac{2\pi}{L}n$, with n integer. We further assume that the single particle states at $k = \pm k_F$ (as well as all the states below) are occupied in the ground state $|0\rangle$.

We wish to show that the spectrum of neutral excitations (i.e., those which do not alter the number of fermions in the system) can be equally well described via the bosonic density operators

$$\rho_\alpha(q) = \sum_k c_{\alpha k+q}^\dagger c_{\alpha k}, \quad (9)$$

which create (for $\alpha q > 0$) or annihilate (for $\alpha q < 0$) electron-hole pairs.

Since k_F merely shifts the momenta in (8), we may relabel k by $p = k - \alpha k_F$ (or equivalently set $k_F = 0$).

- (a) Limiting yourself to right movers, write out all the states (via strings of operators acting on $|0\rangle$) for $k_{\text{tot}} = \frac{2\pi}{L}m$ with $m = 1, 2, 3$ and 4, once using fermion creation operators and once using bosonic density operators. Show the numbers of states for each value of m match in both descriptions.

Hint: For simplicity, label the operators via $k = \frac{2\pi}{L}n$ with the integers n , not with k or q .

The commutation relations for the boson operators are given by

$$[\rho_\alpha(q), \rho_{\alpha'}(-q')] = -\frac{\alpha q L}{2\pi} \delta_{\alpha\alpha'} \delta_{qq'}. \quad (10)$$

- (b) Verify (10) for $q = q'$ by explicit evaluation of the lhs (left hand side) using the anticommutation relations (7), and convince yourself that (10) also holds for $q \neq q'$.
 Hint: It is necessary to choose a momentum cutoff. Why?
- (c) Evaluate the commutator $[H_0, \rho_\alpha(q)]$. Use the result and (10) to write a kinetic Hamiltonian equivalent to H_0 in terms of bosonic operators.

$$\begin{aligned} b) [\rho_\alpha(q), \rho_{\alpha'}(-q)] &= \left[\sum_k c_{\alpha k+q}^\dagger c_{\alpha k}, \sum_{k'} c_{\alpha' k'-q}^\dagger c_{\alpha' k'} \right] \\ &= \sum_{k, k'} \left\{ c_{\alpha k+q}^\dagger [c_{\alpha k}, c_{\alpha' k'-q}^\dagger] c_{\alpha' k'} \right. \\ &\quad + c_{\alpha k+q}^\dagger c_{\alpha' k'-q}^\dagger [c_{\alpha k}, c_{\alpha' k'}] \\ &\quad + [c_{\alpha k+q}^\dagger, c_{\alpha' k'-q}^\dagger] c_{\alpha k} c_{\alpha' k'} \\ &\quad \left. + c_{\alpha k+q}^\dagger [c_{\alpha' k'-q}, c_{\alpha' k'}] c_{\alpha k} \right\} \\ &= \sum_{k, k'} \left\{ c_{\alpha' k'-q}^\dagger (2 c_{\alpha k+q}^\dagger c_{\alpha' k'} - \delta_{\alpha, \alpha'} \delta_{k, k'-q}) c_{\alpha k} \right. \\ &\quad \left. - c_{\alpha k+q}^\dagger (2 c_{\alpha' k'-q}^\dagger c_{\alpha' k'} - \delta_{\alpha, \alpha'} \delta_{k, k'-q}) c_{\alpha' k'} \right\} \end{aligned}$$

Right movers: $\alpha = +1$

$$H_0 = \sum_k v_F k (c_k^\dagger c_k)$$

$$\rho(q) = \sum_k c_{k+q}^\dagger c_k$$

Since the number of fermions is constant, for every fermion we create we need to annihilate one fermion. To avoid annihilating the state, the fermion annihilated needs to be with $k \leq 0$, and the state created needs to be with $k > 0$

$$k_{\text{tot}} = 1: c_1^\dagger |0\rangle = \rho(1)|0\rangle$$

$$k_{\text{tot}} = 2: c_2^\dagger |0\rangle, c_1^\dagger c_1 |0\rangle$$

$$\text{Bosonic: } \rho(1)^2 |0\rangle, \rho(2)|0\rangle$$

$$k_{\text{tot}} = 3: c_3^\dagger |0\rangle, c_2^\dagger c_1 |0\rangle, c_1^\dagger c_2 |0\rangle$$

$$\rho(1)^3 |0\rangle, \rho(1)\rho(2)|0\rangle, \rho(3)|0\rangle$$

$$k_{\text{tot}} = 4: c_4^\dagger |0\rangle, c_3^\dagger c_1 |0\rangle, c_2^\dagger c_2 |0\rangle$$

$$c_1^\dagger c_3 |0\rangle, c_2^\dagger c_1 c_2 |0\rangle$$

$$\text{Bosonic: } \rho(1)^4 |0\rangle, \rho(2)^2 |0\rangle, \rho(1)\rho(3)|0\rangle$$

$$\rho(4)|0\rangle, \rho(1)^2 \rho(2)|0\rangle$$

$$= \sum_{kk'} \left[c_{\alpha'k+q}^\dagger c_{\alpha k} \delta_{k',k+q} - c_{\alpha k+q}^\dagger c_{\alpha'k'} \delta_{k,k'+q} \right] \delta_{\alpha\alpha'}$$

$$= \delta_{\alpha\alpha'} \left[\sum_k c_{\alpha'k}^\dagger c_{\alpha k} - \sum_{k'} c_{\alpha k'}^\dagger c_{\alpha'k'} \right]$$



But anyway we need cutoffs so that we can rewrite delta functions

$$\delta_{\alpha k+q, \alpha' k'+q} = \delta_{\alpha\alpha'} \delta_{k+q, k'+q}$$

We demand that q is small so that excitations do not have momentum large enough to turn left movers into right movers.

(c) Evaluate the commutator $[H_0, \rho_\alpha(q)]$. Use the result and (10) to write a kinetic Hamiltonian equivalent to H_0 in terms of bosonic operators.

$$[H_0, \rho_\alpha(q)] = \left[\sum_{\alpha', k'} \alpha V_F k' c_{\alpha'k'}^\dagger c_{\alpha'k'}, \sum_k c_{\alpha k+q}^\dagger c_{\alpha k} \right]$$

$$= \sum_{kk'} \alpha V_F k' \left[c_{\alpha'k'}^\dagger c_{\alpha k'}, c_{\alpha k+q}^\dagger c_{\alpha k} \right]$$

$$= \sum_{kk'} \alpha V_F k' \left(c_{\alpha k}^\dagger c_{\alpha k'} c_{\alpha k+q}^\dagger c_{\alpha k} - c_{\alpha k+q}^\dagger c_{\alpha k} c_{\alpha'k'}^\dagger c_{\alpha'k'} \right)$$

$$= \sum_{kk'} \alpha V_F k' \left[c_{\alpha'k'}^\dagger \left(\{c_{\alpha k'}, c_{\alpha k+q}^\dagger\} - c_{\alpha k+q}^\dagger c_{\alpha k'} \right) c_{\alpha k} \right. \\ \left. - c_{\alpha k+q}^\dagger \left(\{c_{\alpha k}, c_{\alpha'k'}^\dagger\} - c_{\alpha'k'}^\dagger c_{\alpha k} \right) c_{\alpha k'} \right]$$

$$= \sum_{kk'} \alpha V_F k' \left[c_{\alpha'k'}^\dagger c_{\alpha k} \delta_{k',k+q} - c_{\alpha k+q}^\dagger c_{\alpha'k'} \delta_{k,k'+q} \right]$$

$$= \sum_{k'} \alpha V_F k' \left[c_{\alpha'k'}^\dagger c_{\alpha(k'-q)} - c_{\alpha k+q}^\dagger c_{\alpha k'} \right]$$