Homework for the Lecture

Algebra and Dynamics of Quantum Systems

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Homework Sheet No 3

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(27 Points. Submission deadline 7. 11. 2023)

Homework 3-1: Sums of squares and the Motzkin polynomial

We consider the *-algebra $\mathscr{C}^\infty(\mathbb{R}^2)$ of complex-valued smooth functions on the plane.

i.) **(4 Points)** Show that the polynomial $p \in \mathbb{C}[x, y] \subseteq \mathscr{C}^\infty(\mathbb{R}^2)$

$$p(x, y) = 1 + x^4y^2 + x^2y^4 - 3x^2y^2 \tag{3.1}$$

is non-negative on $\mathbb{R}^2$ but not a sum of squares of polynomials. This an explicit example to the 17th Hilbert problem due to Motzkin [1].

Hint: Use the AM-GM inequality to check that p is actually pointwise non-negative.

ii.) **(2 Points)** Show that p is not even a sum of squares inside the bigger algebra $\mathscr{C}^\infty(\mathbb{R}^2)$.

Hint: Write $p = \sum_{i=1}^n \overline{f}_i f_i$ with smooth functions $f_i \in \mathscr{C}^\infty(\mathbb{R}^2)$ and use Taylor expansions of the f_i around 0.

Homework 3-2: Uncertainty, characters and eigenvectors

Let $\mathscr{A}$ be a unital *-algebra and let $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a state.

i.) **(2 Points)** Let $a, b \in \mathscr{A}$ be given. Show that

$$\left|\omega(a^*b) - \omega(a^*)\omega(b)\right|^2 \le \operatorname{Var}_\omega(a) \operatorname{Var}_\omega(b) \tag{3.2}$$

holds.

ii.) **(2 Points)** Show the *uncertainty relation*

$$4\,\mathrm{Var}_\omega(a)\,\mathrm{Var}_\omega(b) \geq \big|\omega([a,b])\big|^2. \tag{3.3}$$

for Hermitian $a, b \in \mathscr{A}$.

iii.) **(2 Points)** Show that $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ is a unital *-homomorphism if and only if ω is a state that fulfills $\mathrm{Var}_\omega(a) = 0$ for all $a \in \mathscr{A}$.

iv.) **(2 Points)** Finally, let $\mathcal{H}$ be a pre-Hilbert space and $\mathscr{A} \subseteq \mathscr{B}(\mathcal{H})$ a unital *-subalgebra of all adjointable endomorphisms of $\mathcal{H}$. Given $A \in \mathscr{A}$ and $\phi \in \mathcal{H}$ with $\langle \phi, \phi \rangle = 1$, then show that the positive linear functional

$$\omega_\phi\colon \mathscr{A} \ni B \;\mapsto\; \omega_\phi(B) = \langle \phi, B\phi \rangle \in \mathbb{C} \tag{3.4}$$

fulfills $\mathrm{Var}_{\omega_\phi}(A) = 0$ if and only if ϕ is an eigenvector of A to the eigenvalue $\omega_\phi(A)$.

Homework 3-3: Functorial properties of the GNS construction

Let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a morphism of *-algebras $\mathscr{A}$ and $\mathscr{B}$ as well as $\omega\colon \mathscr{B} \longrightarrow \mathbb{C}$ be a positive linear functional.

i.) **(1 Point)** Verify that the pull-back $\Phi^*\omega = \omega \circ \Phi$ is a positive linear functional on $\mathscr{A}$.

ii.) **(2 Points)** Prove that for the Gel'fand ideals we have

$$\Phi(\mathscr{J}_{\Phi^*\omega}) \subseteq \mathscr{J}_\omega. \tag{3.5}$$

Conclude that Φ descends to a well-defined linear map $U_\phi\colon \mathcal{H}_{\Phi^*\omega} \longrightarrow \mathcal{H}_\omega$.

iii.) **(1 Point)** Show that the map U_Φ is isometric. Note that it may well happen that U_Φ is *not* adjointable.

iv.) **(1 Point)** Prove that the isometry U_Φ is a *intertwiner* along Φ, i.e. for all $a \in A$ we have

$$\pi_\omega(\Phi(a))U_\Phi = U_\Phi \pi_{\Phi^*\omega}(a) \tag{3.6}$$

as maps between the corresponding GNS representations of $\mathscr{A}$ and $\mathscr{B}$.

v.) **(1 Point)** Suppose now in addition that Φ is surjective. Show that in this case the intertwiner U_Φ is unitary.

vi.) **(2 Points)** If $\Psi\colon \mathscr{C} \longrightarrow \mathscr{A}$ is yet another *-homomorphism, what can we say about the relations of U_Ψ, U_Φ and $U_{\Psi\circ\Phi}$? What is $U_{\mathrm{id}_\mathscr{A}}$?

Homework 3-4: Unitary group representations

Let G be a group and let $\mathscr{A}$ be a *-algebra over $\mathbb{C}$. Moreover, assume that $\Phi\colon G \longrightarrow {}^*\text{-}\mathrm{Aut}(\mathscr{A})$ is a group morphism, i.e. G acts on $\mathscr{A}$ by *-automorphisms.

i.) **(1 Point)** Show that the properties of Φ are equivalent to a map $\Phi\colon G \ni g \mapsto \Phi_g \in {}^*\text{-}\mathrm{Aut}(\mathscr{A})$ with $\Phi_e = \mathrm{id}_\mathscr{A}$ and $\Phi_g \circ \Phi_h = \Phi_{gh}$ for all $g, h \in G$.

ii.) **(3 Points)** Suppose now that $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ is a positive functional on $\mathscr{A}$ which is G-invariant, i.e. $\Phi_g^*\omega = \omega$ for all $g \in G$. Show that the construction of $U_g = U_{\Phi_g}$ form Homework 3-3 yields a unitary representation of G on the GNS pre-Hilbert space $\mathcal{H}_\omega$.

iii.) **(1 Point)** Show that the GNS representation π_ω of $\mathscr{A}$ is G-covariant in the sense that

$$\pi_\omega(\Phi_g(a)) = U_g \pi_\omega(a) U_g^* \tag{3.7}$$

holds for all $g \in G$ and $a \in \mathscr{A}$.

References

[1] Motzkin, T. S.: *The arithmetic-geometric inequality.* In: *Inequalities (Proc. Sympos. Wright-Patterson Air Force Base, Ohio, 1965)*, 205–224. Academic Press, New York, 1967. 1