Funktionalanalysis Hausaufgaben Blatt 1

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Problem 1. Let (M,d) be a metric space. Consider a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathrm{Map}(N,M)$ of Cauchy sequences in M i.e. $a_n = (a_{mn})_{m\in\mathbb{N}} \subset M$ for every $n \in \mathbb{N}$.

(a) Show that the sequence $(d_k^{(mn)})_{k\in N} \subset \mathbb{R}$ defined by.

$$d_k^{(mn)} := d(a_{nk}, a_{mk})$$

is convergent. In the following, we assume that for every $\epsilon > 0$ there is a natural number $N \in \mathbb{N}$ such that $\lim_{k\to\infty} d_k^{(nm)} < \epsilon$ for every $n, m \geq N$.

(b) For a strictly monotonously increasing sequence $(m_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, we define the diagonal sequence $(D_k)_{k\in\mathbb{N}}$ as follows

$$D_k := a_{km_k}.$$

Show that there exists a diagonal Cauchy sequence $(D_k)_k$ such that $\lim_{k\to\infty} d(a_{nk}, D_k)$ converges to zero in the limit $n \to \infty$. Moreover, show that every other diagonal Cauchy sequence $(D'_k)_k$ with the same property satisfies $\lim_{k\to\infty} d(D_k, D'_k) = 0$.

(c) Assume now that M is complete. Show that $(D_k)_k$ converges and compute its limit.

Proof. (a) We show that the sequence is Cauchy. Choose $N \in \mathbb{N}$ such that for all $k_1, k_2 \geq N$, we have $d(a_{nk_1}, a_{nk_2}) < \epsilon$ and $d(a_{mk_1}, a_{mk_2}) < \epsilon$. Then we apply the triangle inequality

$$\begin{aligned} d(a_{nk_1}, a_{m,k_1}) &\leq d(a_{nk_1}, a_{nk_2}) + d(a_{nk_2}, a_{mk_2}) + d(a_{mk_2}, a_{mk_1}) \\ d(a_{nk_1}, a_{m,k_1}) - d(a_{nk_2}, a_{mk_2}) &\leq d(a_{nk_1}, a_{nk_2}) + d(a_{mk_2}, a_{mk_1}) \end{aligned}$$

Thus the sequence is Cauchy. Since $\mathbb{R}$ is complete, it is convergent.

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(b) We construct this sequence as follows: Let us fix $k \geq 0$ and choose

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Problem 2. Let (M, d) be a metric space. We write $\tilde{M}$ for the set of Cauchy sequences in M.

1. We say that two Cauchy sequences $(a_n)_n, (b_n)_n \in \tilde{M}$ are equivalent if

$$\lim_{n \to \infty} d(a_n, b_n) = 0$$

   and write $(a_n)_n \sim (b_n)_n$. Show that this defines an equivalence relation on $\tilde{M}$

2. Show that there exists a metric $\hat{d}$ on the quotient space $\hat{M} := \tilde{M}/\sim$ such that $(\hat{M}, \hat{d})$ is a completion of (M, d).

3. Let (M', D') be another completion of (M, d). Show that M' is isometrically isomorphic to $\hat{M}$, i.e. there exists a bijective isometry $\phi : \hat{M} \to M'$.

4. Now, assume (M', d') to be another complete metric space and let $\Phi : M \to M'$ be a uniformly continuous map. Show that there is a unique continuous map $\phi : \hat{M} \to M'$ such that

$$\Phi = \phi \circ \iota.$$

   Conclude that ϕ is even uniformly continuous.

Proof. (a) Clear.

(b) We define the metric between two Cauchy sequences $(a_n)_n$ and $(b_n)_n$ by $\hat{d}(a, b) = \lim_{n \to \infty} d(a_n, b_n)$ (cf Pr. 1).

Note that this is actually not a metric on $\tilde{M}$, since it is not positive. Two applications of the triangle inequality show that this is well defined on the quotient space.

We map each element of M to the (equivalence class containing the) constant sequence at that element. Note that these sequences are convergent, and this map is an isometry.

It remains to show that (the image of) M is dense, and that $\hat{M}$ is complete.

Let us consider a Cauchy sequence $(a_n)_n$ and an $\epsilon > 0$. Since the sequence is Cauchy, we have $N \in \mathbb{N}$ such that for all $n, m \geq N$, $d(a_n, a_m) < \epsilon$. In particular, $d(a_N, a_m) < \epsilon$.

Thus $\hat{d}(a, a_N) < \epsilon$ (Note: a_N is the constant Cauchy sequence at a_N). Thus, the open ball with radius ϵ intersects M. M is therefore dense.

Now we show completeness. Consider a Cauchy sequence of elements of $\hat{M}$. Since this is Cauchy, it satisfies the conditions of Pr. 1(b), and converges to the diagonal sequence mentioned there.

Comment: Pr 1(c) shows that this construction would yield M if M were complete, because the diagonal sequence would converge to a limit.

(c) Consider a map from $\hat{M}$ to M' that sends all elements of M to elements of M. Clearly, when we restrict the map to these elements, it is an isometry.

The question is: Can we now define

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Problem 3. Let $(M, \mathcal{M})$ be a topological space and $A, B \subseteq M$ be subsets. Prove the following identities.

(a)

$$(A \cup B)^{\mathrm{cl}} = A^{\mathrm{cl}} \cup B^{\mathrm{cl}}$$

and

$$(A \cup B)^{\circ} \supseteq A \circ \cup B^{\circ}.$$

(b)

$$(A \cap B)^{\mathrm{cl}} \subseteq A^{\mathrm{cl}} \cap B^{\mathrm{cl}}$$

and

$$(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}.$$

(c)

$$(M \setminus A)^{\mathrm{cl}} = M \setminus A^{\circ}$$

and

$$(M \setminus A)^{\circ} = M \setminus A^{\mathrm{cl}}.$$

For the identities with inequalities, give examples where one has strict subsets.

Proof. (a) The reverse inclusion is obvious, because if all neighbourhoods of a point p intersect A or B, then they intersect $A \cup B$. Conversely, suppose a point is neither in the closure of A nor in the closure of B. Then it has a neighbourhood that does not intersect A, and a neighbourhood that does not intersect B. The intersection of these two neighbourhoods does not intersect $A \cup B$.

For the second inclusion, we simply note that A° and B° are open; therefore, their union is an open set contained in $A \cup B$.

An example where equality fails is the Cantor set and its complement.

(b) The closures of A and B are closed sets containing $A \cap B$; therefore, their intersection is closed and contains $(A \cap B)^{\mathrm{cl}}$.

Equality fails

Suppose x is in the interior of A and of B. Then it has a neighbourhood entierly contained in A and B, and thus in $A \cap B$. The same thing holds in reverse.

(c)

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