Homework for the Lecture

Functional Analysis

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Homework Sheet No 1

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(21 Points. Discussion 21. 10. 2024)

Homework 1-1: Diagonal Sequences

Let (M, d) be a metric space. Consider a sequence $(a_n)_{n\in\mathbb{N}} \subset \operatorname{Map}(\mathbb{N}, M)$ of Cauchy sequences in M, i.e. $a_n = (a_{nm})_{m\in\mathbb{N}} \subset M$ for every $n \in \mathbb{N}$.

i.) **(1 Point)** Show that the sequence $\big(d_k^{(nm)}\big)_{k\in\mathbb{N}} \subset \mathbb{R}$ defined by

$$d_k^{(nm)} := d(a_{nk}, a_{mk}) \tag{1.1}$$

is convergent.

In the following, we assume that for every $\varepsilon > 0$ there is a natural number $N \in \mathbb{N}$ such that $\lim_{k\to\infty} d_k^{(nm)} < \varepsilon$ for every $n, m \geq N$.

ii.) **(4 Points)** For a strictly monotonously increasing sequence $(m_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, we define the diagonal sequence $(D_k)_{k\in\mathbb{N}} \subset M$ as follows

$$D_k := a_{km_k}. \tag{1.2}$$

Show that there exists a diagonal Cauchy sequence $(D_k)_k$ such that $\lim_{k\to\infty} d(a_{nk}, D_k)$ converges to zero in the limit $n \to \infty$. Moreover, show that every other diagonal Cauchy sequence $(D'_k)_k$ with the same property satisfies $\lim_{k\to\infty} d(D_k, D'_k) = 0$.

iii.) **(2 Points)** Assume now that M is complete. Show that $(D_k)_k$ converges and compute its limit.

Homework 1-2: Completion of Metric Spaces

Let (M, d) be a metric space. We write $\tilde{M}$ for the set of Cauchy sequences in M.

i.) **(1 Point)** We say that two Cauchy sequences $(a_n)_n, (b_n)_n \in \tilde{M}$ are equivalent if

$$\lim_{n\to\infty} d(a_n, b_n) = 0 \tag{1.3}$$

and write $(a_n)_n \sim (b_n)_n$. Show that this defines an equivalence relation on $\tilde{M}$.

ii.) **(5 Points)** Show that there exists a metric $\hat{d}$ on the quotient space $\hat{M} := \tilde{M}/\sim$ such that $(\hat{M}, \hat{d})$ is a completion of (M, d).

iii.) **(3 Points)** Let (M', d') be another completion of (M, d). Show that M' is isometrically isomorphic to $\hat{M}$, i.e there exists a bijective isometry $\phi : \hat{M} \to M'$.

iv.) **(2 Points)** Now, assume (M', d') to be just another complete metric space and let $\Phi : M \to M'$ be an uniformly continuous map. Show that there is an unique continuous map $\phi : \hat{M} \to M'$ such that

$$\Phi = \phi \circ \iota. \tag{1.4}$$

Conclude that ϕ is even uniformly continuous.

Homework 1-3: Some Identities for the Closure and the Interior

Let $(M, \mathscr{M})$ be a topological space and $A, B \subseteq M$ be subsets. Prove the following identities.

i.) **(1 Point)**

$$(A \cup B)^{\mathrm{cl}} = A^{\mathrm{cl}} \cup B^{\mathrm{cl}} \tag{1.5}$$

and

$$(A \cup B)^{\circ} \supseteq A^{\circ} \cup B^{\circ} \tag{1.6}$$

ii.) **(1 Point)**

$$(A \cap B)^{\mathrm{cl}} \subseteq A^{\mathrm{cl}} \cap B^{\mathrm{cl}} \tag{1.7}$$

and

$$(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ} \tag{1.8}$$

iii.) **(1 Point)**

$$(M \backslash A)^{\mathrm{cl}} = M \backslash A^{\circ} \tag{1.9}$$

and

$$(M \backslash A)^{\circ} = M \backslash A^{\mathrm{cl}} \tag{1.10}$$

For (1.6) and (1.7), give examples, where one has strict subsets.