1. Topological action for a skyrmion configuration

Consider the skyrmion configuration $\hat{\boldsymbol{n}}_W\colon \mathbb{R} \to S^2$,

$$(x_1, x_2) \to \phi = W \tan^{-1}\left(\frac{x_2}{x_1}\right), \quad \theta = 2\tan^{-1}\sqrt{\frac{a^2}{x_1^2 + x_2^2}}, \tag{1}$$

where

$$\hat{\boldsymbol{n}}(\theta, \phi) = \begin{pmatrix} \cos\varphi \, \sin\theta \\ \sin\varphi \, \sin\theta \\ \cos\theta \end{pmatrix}. \tag{2}$$

Calculate

$$S_{\text{top}}[\hat{\boldsymbol{n}}] = \mathrm{i}\theta W, \quad W = -\frac{1}{4\pi}\int dx_1 dx_2 \hat{\boldsymbol{n}}(\partial_1 \hat{\boldsymbol{n}} \times \partial_2 \hat{\boldsymbol{n}}) \tag{3}$$

explicitly for (1).

Hint: Parametrize the plane (x_1, x_2) in terms of polar coordinates (r, φ) and avail yourself of the fact that topological actions are invariant of the metric.

In these coordinates,

$$\phi = W\varphi, \quad \theta = 2\tan^{-1}\left(\frac{a}{r}\right)$$

$$\sin\theta = \frac{2\left(\frac{a}{r}\right)}{1+\left(\frac{a}{r}\right)^2} = \frac{2ar}{a^2+r^2}$$

$$\cos\theta = \frac{1-\left(\frac{a}{r}\right)^2}{1+\left(\frac{a}{r}\right)^2} = \frac{r^2-a^2}{r^2+a^2}$$

Express $\hat{n}$ in terms of r & φ

$$\partial_1 \hat{n} \times \partial_2 \hat{n} = \left(\partial_1 r \, \partial_r \hat{n} + \partial_1 \varphi \, \partial_\varphi \hat{n}\right) \times \left(\partial_2 r \, \partial_r \hat{n} + \partial_2 \varphi \, \partial_\varphi \hat{n}\right)$$

$$= (\partial_1 \varphi \, \partial_2 r) \, \partial_\varphi \hat{n} \times \partial_r \hat{n} + (\partial_1 r \, \partial_2 \varphi) \, \partial_r \hat{n} \times \partial_\varphi \hat{n}$$

$$= \partial_r \hat{n} \times \partial_\varphi \hat{n} \left[\partial_1 r \, \partial_2 \varphi - \partial_1 \varphi \, \partial_2 r\right]$$

$$\partial_1 r^2 = 2r \, \partial_1 r = 2x$$

$$\partial_1 r = \frac{x}{r} = \cos\varphi$$

Similarly, $\partial_2 r = \frac{y}{r} = \sin\varphi$

$$\partial_1 \tan\varphi = \sec^2\varphi \, \partial_1 \varphi = -\frac{y}{x^2} = -\frac{r\sin\varphi}{r^2\cos^2\varphi}$$

$$\partial_1 \varphi = -\frac{\sin\varphi}{r}$$

$$\sec^2\varphi \, \partial_2 \varphi = \frac{1}{x} = \frac{1}{r\cos\varphi}$$

$$\partial_2 \varphi = \frac{\cos\varphi}{r}$$

$$\partial_1 \hat{n} \times \partial_2 \hat{n} = \partial_r \hat{n} \times \partial_\varphi \hat{n} \left[\partial_1 r \, \partial_2 \varphi - \partial_1 \varphi \, \partial_2 r\right]$$

$$= \partial_r \hat{n} \times \partial_\varphi \hat{n} \left[\cos\varphi \frac{\cos\varphi}{r} - \left(-\frac{\sin\varphi}{r}\right)\sin\varphi\right]$$

$$= \frac{1}{r} \partial_r \hat{n} \times \partial_\varphi \hat{n}$$

$$\hat{n}(\theta(r), \varphi) = \begin{pmatrix} \cos(W\varphi)\sin\theta \\ \sin(W\varphi)\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\partial_r \hat{n} = \begin{pmatrix} \cos(W\varphi)\cos\theta \\ \sin(W\varphi)\cos\theta \\ -\sin\theta \end{pmatrix} \theta'(r)$$

$$\partial_\varphi \hat{n} = W \begin{pmatrix} -\sin(W\varphi)\sin\theta \\ \cos(W\varphi)\sin\theta \\ 0 \end{pmatrix}$$

$$\hat{n} \cdot (\partial_r \hat{n} \times \partial_\varphi \hat{n}) = \theta'(r) W \sin\theta$$

$$\int \hat{n} \cdot (\partial_1 \hat{n} \times \partial_2 \hat{n}) \, dx_1 dx_2 = \int \frac{1}{r} \theta'(r) W \sin\theta \, (r dr d\varphi)$$

$$= \int_0^\infty \int_0^{2\pi} W\theta'(r)\sin\theta \, d\varphi \, dr$$

$$= 2\pi W \int_0^\infty \theta'(r)\sin\theta \, dr$$

$$\theta = 2\tan^{-1}\left(\frac{a}{r}\right) \qquad \sin\theta = \frac{2\left(\frac{a}{r}\right)}{1+\left(\frac{a}{r}\right)^2} = \frac{2ar}{a^2+r^2}$$

$$\theta'(r) = \frac{2}{1+\left(\frac{a}{r}\right)^2}\left(-\frac{a}{r^2}\right) = \frac{-2a}{a^2+r^2}$$

$$\int \hat{n}\cdot(\partial_1\hat{n}\times\partial_2\hat{n})\, dx_1 dx_2 = 2\pi W \int_0^\infty \frac{-2a}{a^2+r^2}\,\frac{2ar}{a^2+r^2}\, dr$$

$$= -8\pi a^2 W \int_0^\infty \frac{r}{(a^2+r^2)^2}\, dr$$

$$= -8\pi \cancel{a^2} W \frac{1}{2\cancel{a^2}}$$

$$= -4\pi W$$

Hence $-\frac{1}{4\pi}\int \hat{n}\cdot(\partial_1\hat{n}\times\partial_2\hat{n})\,dx_1 dx_2 = W$

2. Classical equation of motion for the O(3) model

Consider the O(3) or non-linear sigma model

$$S_0[\hat{\boldsymbol{n}}] = \frac{1}{8\pi}\int d^2x\,(\partial_\mu\hat{\boldsymbol{n}})\,(\partial^\mu\hat{\boldsymbol{n}}), \quad |\hat{\boldsymbol{n}}| = 1, \tag{4}$$

with the Euclidean metric $g^{\mu\nu} = g_{\mu\nu} = \delta_{\mu\nu}$. In the lectures, we derived the inequality

$$S_0[\hat{\boldsymbol{n}}] \geq W \tag{5}$$

where W is the "winding number" as given in (3) above. The equal sign in (5) holds if and only if

$$\partial_\mu\hat{\boldsymbol{n}} - \varepsilon_{\mu\nu}\,(\hat{\boldsymbol{n}}\times\partial^\nu\hat{\boldsymbol{n}}) = 0, \tag{6}$$

where $\varepsilon_{12} = -\varepsilon_{21} = 1$. We found that if we parametrize the target space (*i.e.*, the field vector $\hat{\boldsymbol{n}}$) via a stereographic projection by a complex number w such that

$$\hat{\boldsymbol{n}}(w) = \frac{1}{w\bar{w}+1}\begin{pmatrix} w+\bar{w} \\ -\mathrm{i}(w-\bar{w}) \\ w\bar{w}-1 \end{pmatrix}, \tag{7}$$

($\bar{w}$ is just the complex conjugate of w) and our base space (x_1, x_2) by another complex number $z = x_1 + \mathrm{i}x_2$, the most general solution of (6) for $W > 0$ is given by

$$w = \prod_{i=1}^{W}\frac{a_i z + b_i}{c_i z + d_i}, \quad \text{with } a_i d_i - b_i c_i = 1 \;\forall\; i \in \{1, 2, \ldots, W\}. \tag{8}$$

(a) Find the solution of the classical equation of motion (6) of (5) for $W = 0$.

(b) For which values of W (and a) are the skyrmion configurations in (1) solutions of (6)?

a) $W=0$ means that solution is homotopic to constant solution

We see that the constant vector is a solution.

b) By comparison between the n vectors

$$\hat{n} = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix} = \frac{1}{w\bar{w}+1}\begin{pmatrix} w+\bar{w} \\ -i(w-\bar{w}) \\ w\bar{w}-1 \end{pmatrix}$$

$w = \cot\frac{\theta}{2}\, e^{i\phi}$ (Stereographic Projection)

Substituting, we get $w = \frac{r}{a} e^{iW\varphi}$

$$z = re^{i\varphi}$$

This is w

$$w = \frac{1}{a} r^W r^{-(W-1)} e^{iW\varphi}$$

$$= \frac{1}{a} r^{1-W} z^W$$

That's W

Hence w is holomorphic iff $W=1$

(contribution of $r = \sqrt{z\bar{z}}$)

If $W=-1$, w is antiholomorphic which is also a solution (see next part)

(c) What is the general solution of (6) for $W < 0$?

Original argument

$$0 \leq \frac{1}{2}\int d^2x\,(\partial_\mu \mathbf{n} + \epsilon_{\mu\nu}\mathbf{n}\times\partial_\nu \mathbf{n})\cdot(\partial_\mu \mathbf{n} + \epsilon_{\mu\nu'}\mathbf{n}\times\partial_{\nu'} \mathbf{n})$$
$$= \int d^2x\,(\partial_\mu \mathbf{n}\cdot\partial_\mu \mathbf{n} + \epsilon_{\nu\mu}\mathbf{n}\cdot(\partial_\mu \mathbf{n}\times\partial_\nu \mathbf{n}))\,.$$

Modify to

$$0 \leq \frac{1}{2}\int d^2x\,(\partial_\mu \hat{n} - \epsilon_{\mu\gamma}\hat{n}\times\partial_\gamma \hat{n})\cdot(\partial_\mu \hat{n} - \epsilon_{\mu\gamma'}\hat{n}\times\partial_{\gamma'} \hat{n})$$
$$= \int d^2x\left[\partial_\mu \hat{n}\cdot\partial_\mu \hat{n} - \epsilon_{\gamma\mu}\hat{n}\cdot(\partial_\mu \hat{n}\times\partial_\gamma \hat{n})\right]$$

Hence equation flips sign

$$\partial_\mu \hat{n} - \epsilon_{\mu\gamma}\hat{n}\times\partial_\gamma \hat{n} = 0$$

We had the complex equation

$$\partial \hat{n} - i\hat{n}\times\partial\hat{n} = 0$$
$$\bar{\partial}\hat{n} + i\hat{n}\times\bar{\partial}\hat{n} = 0$$

Now we just flip the signs to get

$$\partial \hat{n} + i\hat{n}\times\partial\hat{n} = 0$$
$$\bar{\partial}\hat{n} - i\hat{n}\times\bar{\partial}\hat{n} = 0$$

Claim: This is satisfied by antiholomorphic functions

$$\hat{n} = \frac{1}{w\bar{w}+1}\begin{pmatrix} w+\bar{w} \\ -i(w-\bar{w}) \\ w\bar{w}-1 \end{pmatrix}$$

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In[1]:= n[x1_, x2_] := 1/(x1 x2 + 1) {x1 + x2, -I (x1 - x2), x1 x2 - 1}
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This is to confirm a correct implementation of n
I verify that the eom with positive W is actually solved by holomorphic functions *w(z)*, wbar(zbar)

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In[3]:= D[n[w[z], wbar[zbar]], z] - I n[w[z], wbar[zbar]] × D[n[w[z], wbar[zbar]], z] // Simplify
Out[3]= {0, 0, 0}

In[4]:= D[n[w[z], wbar[zbar]], zbar] + I n[w[z], wbar[zbar]] × D[n[w[z], wbar[zbar]], zbar] // Simplify
Out[4]= {0, 0, 0}
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Now I show that the claim is actually true

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In[5]:= D[n[w[zbar], wbar[z]], z] + I n[w[zbar], wbar[z]] × D[n[w[zbar], wbar[z]], z] // Simplify
Out[5]= {0, 0, 0}

In[6]:= D[n[w[zbar], wbar[z]], zbar] - I n[w[zbar], wbar[z]] × D[n[w[zbar], wbar[z]], zbar] // Simplify
Out[6]= {0, 0, 0}
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General solution:

$$w = \prod_{i=1}^{|W|}\frac{a_i\bar{z}+b_i}{c_i\bar{z}+d_i}$$