Funktionalanalysis Notizen

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CONTENTS

I. TOPOLOGY

A. Basic Concepts

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Theorem I.1 (List of Useful Identities).

(a)
$$(A \cup B)^{\mathrm{cl}} = A^{\mathrm{cl}} \cup B^{\mathrm{cl}}.$$

(b)
$$(A \cup B)^{\circ} \supseteq A^{\circ} \cup B^{\circ}.$$

(c)
$$(A \cap B)^{\mathrm{cl}} \subseteq A^{\mathrm{cl}} \cap B^{\mathrm{cl}}.$$

(d)
$$(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}.$$

(e)
$$(M \setminus A)^{\mathrm{cl}} = M \setminus A^{\circ}.$$

(f)
$$(M \setminus A)^{\circ} = M \setminus A^{\mathrm{cl}}.$$

Proof. (a) If all neighbourhoods of x intersects A, then they must certainly intersect $A \cup B$. The same thing happens if all neighbourhoods intersect B. Conversely, we suppose that x is not in A^{cl} or B^{cl}. Then we have an open neighbourhood not intersecting A, and an open neighbourhood not intersecting B. Considering the intersection of these two neighbourhoods shows that x is not in the closure of $A \cup B$ either.

(b) $A^{\circ} \cup B^{\circ}$ is an open set contained in $A \cup B$.

(c) Suppose every neighbourhood of x intersects $A \cap B$. Then every neighbourhood intersects A, and also intersects B.

(d) Clearly, $A^{\circ} \cap B^{\circ}$ is an open set contained in $A \cap B$. Conversely, it is also true that $(A \cap B)^{\circ}$ is an open set contained in A, and thus its interior, and it is also contained in B.

(e) Clearly, $M \setminus A^{\circ}$ is a closed set containing $M \setminus A$. This shows one inclusion.

Now suppose $x \in M \setminus A^\circ$. Since it is not in the interior, no open neighbourhood of x is completely contained in A; in particular, every neighbourhood must intersect $M \setminus A$. This shows the reverse inclusion.

(f) Clearly, $M \setminus A^{\mathrm{cl}}$ is an open subset of $M \setminus A$.

Conversely, suppose x is an element of $(M \setminus A)^\circ$. Then there is an open neighbourhood of x contained in $M \setminus A$, in particular not intersecting A. This shows that x is in $M \setminus A^{\mathrm{cl}}$. $\square$

Definition I.2 (Preorder)**.** A relation $\preceq$ on a set A is a preorder if the following conditions hold:

(a) $\alpha \preceq \alpha$ for all $\alpha \in A$

(b) If $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$.

Definition I.3. A *directed set* A is a set with a preorder and the following additional condition: For all $\alpha, \beta \in A$, there is some γ with $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

Remark I.4. The additional condition $\alpha \preceq \beta$ and $\beta \preceq \alpha \implies \beta = \alpha$ is known as a *partial order*. This is sometimes too restrictive, and is usually not important. The important property is the defining property of a directed set.

Definition I.5. A map from a directed set I into a set X is called a *net* (indexed by I). The elements of this net are denoted by $(x_i)_{i \in I}$.

Definition I.6 (Convergence)**.** Let X now be a topological space. We say that a net $(x_i)_{i \in I}$ converges to x if for every neighbourhood U of x there is $\alpha \in I$ such that $\beta \succeq \alpha \implies x_\beta \in U$.

Theorem I.7. The limit of a net is unique if the space X is Hausdorff.

Proof. The proof is analogous to that of a sequence. □

It is obvious that all sequences are nets. The difference is as follows: A sequence only has countably many elements. Thus, it is possible that there will be too many open sets. This leads us to our next theorem:

Theorem I.8. Let X be a topological space and $(x_i)_{i\in I}$ be a net converging to x. If X is first countable, then there is a sequence converging to x.

In general, we have the following theorem

Theorem I.9. Let A be a subset of the topological space X. $x \in A^{\mathrm{cl}}$ if and only if there is a net in A converging to x.

B. Baire Spaces

Definition I.10 (Nowhere Dense)**.** A subset A of a topological space is called nowhere dense if the interior of its closure is open, $(A^{\mathrm{cl}})^\circ = \varnothing$.

Theorem I.11. A subset A of a topological space $(M, \mathcal{M})$ is nowhere dense iff its complement contains a dense open set.

Proof. We perform the following computation:

$$\begin{aligned}
A \text{ nowhere dense} &\iff (A^{\mathrm{cl}})^\circ = \varnothing \\
&\iff M \setminus (A^{\mathrm{cl}})^\circ = M \\
&\iff (M \setminus A^{\mathrm{cl}})^{\mathrm{cl}} = M \\
&\iff ((M \setminus A)^\circ)^{\mathrm{cl}} = M \\
&\iff (M \setminus A)^\circ \text{ is dense.}
\end{aligned}$$

□

Definition I.12 (Meager). A subset is called meager if it is a countable union of nowhere dense sets.

Theorem I.13. Let $(M, \mathcal{M})$ be a topological space. Then the following are equivalent:

(a) Any countable union of closed subsets of M without inner points has no inner points.

(b) Any countable intersection of open dense subsets of M is dense.

(c) Every non-empty open subset of M is not meager

(d) The complement of every meager subset of M is dense.

Proof. The proof follows (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) $\implies$ (a).

1. Let $(U_i)_{i\in\mathbb{N}}$ be a collection of dense open subsets of M. We consider their complements, which all have empty interior. Since
$$M \setminus \left(\bigcap_{i=1}^{\infty} U_i\right) = \bigcup_{i=1}^{\infty} (M \setminus U_i)$$
and the sets in the union on the right are all closed subsets without interior points, their union has no interior points, hence the countable intersection is dense.

2. Suppose we had a meager nonempty open subset U of M, that is, we have $U = \bigcup_{i=1}^{\infty} A_i$ with A_i nowhere dense sets. Then $M \setminus A_i$ is a dense subset for all i, by (b), their intersection is still dense. Then
$$\begin{aligned}\varnothing &= M \setminus \left(\bigcap_{i=1}^{\infty} (M \setminus A_i^{\mathrm{cl}})\right)^{\mathrm{cl}} \\ &= M \setminus \left(M \setminus \bigcup_{i=1}^{\infty} A_i^{\mathrm{cl}}\right)^{\mathrm{cl}} \\ &= M \setminus \left(M \setminus \left(\bigcup_{i=1}^{\infty} A_i^{\mathrm{cl}}\right)^{\circ}\right) \\ &= \left(\bigcup_{i=1}^{\infty} A_i^{\mathrm{cl}}\right)^{\circ},\end{aligned}$$
which is a contradiction, since we had U as a nonempty open subset of the final union.

3. Suppose we have a meager subset A such that $M \setminus A$ is not dense. Then $(M \setminus A)^{\mathrm{cl}} = M \setminus A^\circ \neq M$. Then A has nonempty interior, contradicting (c).

4. Finally, let us consider a sequence of closed sets $(A_n)_{n\in\mathbb{N}}$ without interior points, and we suppose that their union has a nonempty interior. Then

$$\left(M \setminus \bigcup_{n=1}^{\infty} A_n\right)^{\mathrm{cl}} = M \setminus \left(\bigcup_{n=1}^{\infty} A_n\right)^{\circ} \neq M,$$

contradicting (d), since $\bigcup_{n=1}^{\infty} A_n$ is meager and hence has a dense complement. □

Definition I.14 (Baire Spaces)**.** A topological space X is a Baire space if one (and hence all) of the conditions of Theorem I.13 hold.

Theorem I.13 is clearly unwieldy, and one must first show that a space is a Baire space before even beginning to apply this theorem. Thus, we seek easier conditions with which we can verify that a space is a Baire space. These theorems are known as Baire's Theorems.

Theorem I.15 (Baire I)**.** A complete metric space (M, d) is a Baire space.

Proof. We seek to show the condition in Theorem I.13(b). Let us consider a collection of dense open sets O_n and a point $x \in M$, as well as an open neighbourhood U of x. Since U intersects O_1, we can find an open ball $B_{\epsilon_1}(p_1)$ whose closure is contained in $U \cap O_1$. Inductively, we construct open balls $B_{\epsilon_n}(p_n)$ such that $B_{\epsilon_n}(p_n)^{\mathrm{cl}} \subseteq B_{\epsilon_{n-1}}(p_{n-1})$. Then,

$$\bigcap_{i=1}^{\infty} B_{\epsilon_n}(p_n)^{\mathrm{cl}} \subseteq U \cap \bigcap_{i=1}^{\infty} O_n$$

But since the metric space is complete, the intersection on the left side is nonempty. Thus, the intersection on the right is nonempty. Since U was an arbitrary open set, the intersection is dense. □

Theorem I.16 (Baire II)**.** A locally compact Hausdorff space $(M, \mathcal{M})$ is a Baire space.

II. TOPOLOGICAL VECTOR SPACES

Definition II.1. A topological vector space is a vector space with a topology such that addition and scalar multiplication are continuous.

Theorem II.2. Translation $T_v : x \mapsto x + v$ and multiplication $\lambda : x \mapsto \lambda x$ with $\lambda \neq 0$ are homeomorphisms.

Proof. They are invertible with continuous inverse T_{-v} and $\frac{1}{\lambda}$ respectively □

It is defined in some books that a topological vector space must be T_1 (or T_0). The usefulness of this definition comes from the fact that T_0 topological vector spaces are automatically T_2 and T_3. This is, in fact, not a statement about topological vector spaces, but about topological groups. Before we prove the result, we will need the following lemmas

Lemma II.3. Let G be a group that is also a T_1 topological space. G is a topological group iff the map $G \times G \to G, (x, y) \mapsto xy^{-1}$ is continuous.

Proof. Clearly, if G is a topological group, then the map is continuous.

Conversely, we consider $y \mapsto (e, y) \mapsto y^{-1}$, which is continuous as a composition of continuous functions. Then $(x, y) \mapsto (x, y^{-1}) \mapsto xy$ is continuous, again as a composition of continuous functions. □

Lemma II.4. A topological space X is Hausdorff iff the diagonal $X \times X$ is closed.

Proof. Suppose the diagonal is closed. Then, given (x, y), there is an open set Z and a basis element $U \times V$ such that $(x, y) \in U \times V \subseteq Z$. Since this does not intersect the diagonal, it follows that U and V must be disjoint.

Conversely, given x and y, we can find disjoint open sets U and V containing x and y, and $U \times V$ does not intersect the diagonal, since they are disjoint. □

Now, we move on to our first result about regularity. This is independent of the other separation axioms:

Theorem II.5. Let G be a topological group. It is regular. In particular, for all points $x \in G$ and closed sets A not containing x, we have a neighbourhood V of e such that $V \cdot x$ and $V \cdot A$ are disjoint.

Proof. Firstly, we show that for all neighbourhoods U of e, there is a symmetric neighbourhood V such that $V \cdot V \subseteq U$ of e.

This follows because the map $(x, y) \mapsto x \cdot y$ is continuous, hence there is a neighbourhood V such that $V \cdot V \subseteq U$. Then because $(x, y) \cdot x \cdot y^{-1}$ is continuous, there exists W with $W \cdot W^{-1} \subseteq V$. Then $W \cdot W^{-1}$ is the desired symmetric neighbourhood.

Now, since A is closed, we have $A \cdot x^{-1}$ closed (and not containing e), thus we choose an open symmetric neighbourhood V of e such that $V \cdot V \cap (A \cdot x^{-1}) = \varnothing$.

Now we show that $V \cdot x$ and $V \cdot A$ are disjoint. Suppose not. Then we have $v \cdot x = v' \cdot a$ for $v, v' \in V, a \in A$. However, this also means that $v'^{-1} \cdot v = a \cdot x^{-1}$, a contradiction. □

As a corollary we have, by a simple change of notation:

Corollary II.6. Let V be a topological vector space, $A \subseteq V$ closed and $v \in V$. Then there exists a symmetric neighbourhood U of 0 such that $(A + U) \cap (v + U) = \varnothing$.

We can also extend this to compact sets:

Corollary II.7. Let V be a topological vector space, $A \subseteq V$ closed, $K \subseteq V$ compact. Then there exists a symmetric neighbourhood U of 0 such that $(A + U) \cap (K + U)$.

Proof. As usual, we apply the previous corollary to each point $x \in K$. Then we choose a finite subcover of these, and take their intersections. □

Now, we move on to the more important result:

Theorem II.8. Let G be a T_0 topological group. Then it is T_1, T_2 and T_3.

Proof. We will prove $T_0 \implies T_1 \implies T_2$.

The goal is: We show that $\{e\}$ is closed. Pick $g \neq e$. Because G is T_0, either $G \setminus \{e\}$ is a neighbourhood of g, or $G \setminus \{g\}$ is a neighbourhood of e. Suppose the latter. Then applying the translation $T_{g^{-1}}$ shows that $G \setminus \{e\}$ is a neighbourhood of g^{-1}, and applying the inversion map shows that $G \setminus \{e\}$ is a neighbourhood of g. Thus $\{e\}$ is closed.

Then the translations show that every singleton set is closed.

Note: If the group is abelian, then the proof is significantly easier: We suppose U is a neighbourhood of x not containing y. Then $x+y-U$ is a neighbourhood of y not containing x.

Now let the identity be closed. The diagonal of $G \times G$ is the preimage of $\{e\}$ under the map $(x, y) \mapsto xy^{-1}$, hence the diagonal is closed too. This shows that G is Hausdorff.

Since all topological groups are regular, this shows T_3. □

A. Continuity

Definition II.9 (Uniform Continuity). Let V, W be topological vector spaces and $\phi : V \to W$ be a map. Then ϕ is called uniformly continuous if for all neighbourhoods $U \subseteq W$ of 0 there exists a neighbourhood $Z \subseteq V$ such that $v - v' \in Z \implies \phi(v) - \phi(v') \in U$.

Theorem II.10 (Equivalence of Continuity Conditions). Let V and W be topological vector spaces and let $\phi : V \to W$ be a linear map. Then the following are equivalent

(a) The map ϕ is continuous at 0.

(b) The map ϕ is continuous at some $v \in V$.

(c) The map ϕ is continuous.

(d) The map ϕ is uniformly continuous.

Proof. 1. Assume ϕ is continuous at 0.

Recall: This means that for all neighbourhoods U of 0_W, there is a neighbourhood Z of 0_V such that $\phi(Z) \subseteq V$. Now choose $v \in V$ and a neighbourhood $U \ni \phi(v)$. Then $T_{-v}(U)$ is a neighbourhood of 0_W, and we choose a neighbourhood Z of 0 such that $\phi(Z) \subseteq T_{-v}(U)$ by continuity at 0. Then $T_v(Z) \subseteq U$ and contains v, as desired.

2. Repeating the same proof, we can get continuity at every point in V, which is equivalent to continuity.

3. We actually only need continuity at 0. Given an open neighbourhood Z of 0_W, we choose an open neighbourhood U of 0_V that maps into Z. Then we have, for $v-u \in Z$, that $\phi(v) - \phi(u) = \phi(v-u) \in W$.

4. Uniform continuity directly implies continuity at 0 if we take v' to be 0 in the definition.

□

Definition II.11 (Algebraic and Topological Dual). The algebraic dual of a vector space V is the well known space of linear maps from V to the field over which it is defined. We denote this by $V' = \mathrm{Hom}(V, \mathbb{K})$. The topological dual $V^* = L(V, \mathbb{K})$ is the space of all *continuous* linear maps. Clearly, $L(V, \mathbb{K}) \subseteq \mathrm{Hom}(V, \mathbb{K})$.

B. Subspaces

Theorem II.12. Let V be a topological vector space. If $W \subseteq V$ is a subspace, then its closure W^{cl} is also a subspace of V.

Proof. Suppose $v, w \in W^{\mathrm{cl}}$. Then there are nets $(v_i)_{i\in I}$ converging to v and $(w_i)_{i\in I}$ converging to w. Then $(v_i + w_i)_{i\in I}$ is a net converging to $v + w$, hence $v + w$ is still in W^{cl}. □

III. BANACH SPACES

A. Norms & Seminorms

Definition III.1 (Norm). A norm $\|\cdot\|$ on a vector space V is a function $V \to \mathbb{R}$ that satisfies

(a) (Homogenity) $\|sv\| = |s| \cdot \|v\|$ for all vectors v and scalars s

(b) (Triangle Inequality) $\|u + v\| \leq \|u\| + \|v\|$

(c) (Positivity) $\|v\| > 0$ for all $v \neq 0$.

Remark III.2. A norm induces a topology through the metric defined by $d(v, w) = \|v-w\|$. If a vector space has this topology, it is known as a *normed space*.

Definition III.3 (Banach Space). A Banach space is a complete normed space.

Definition III.4 (Seminorm). A seminorm p is a norm without the positivity condition. Instead, we have positive semidefiniteness, i.e.

$$p(v) \geq 0 \ \forall v \in V.$$

Note that a seminorm does not form a metric space. We define the *kernel* of the seminorm as the set $\{v \in V | p(v) = 0\}$. Note that this is a subspace. To get an actual norm, we must "divide" by the kernel.

We can define a norm on the quotient vector space $V/\ker p$ by letting the seminorm p act on any representative of this space.

Theorem III.5 (Quotient of Norms). The norm $\|\cdot\| : V/\ker p \to \mathbb{R}$, $[v] \mapsto p(v)$ is a norm.

Proof. First, we show that it is well defined. Consider $v \in V$ and let $u \in \ker p$ be arbitrary. Then we have

$$p(v + u) \leq p(v) + p(u) = p(v)$$

by the triangle inequality, and conversely

$$p(v) = p(v - u + u) \leq p(v + u) + p(u) = p(v + u)$$

which shows that the norm is independent of the choice of representative.

The triangle inequality and homogenity follow from the same properties of the seminorm. It is also true by definition that the norm is positive. $\square$

Thus, given a seminorm on a vector space, we can construct a new vector space that has a norm, and thus an induced topology.

Theorem III.6 (Operator Norm)**.** Let V, W be normed spaces and let $A : V \to W$ be a linear map. Then the following statements are equivalent:

(a) A is continuous.

(b) There exists a constant $c \geq 0$ such that

$$\|A(v)\| \leq c\|v\|$$

for all $v \in V$.

Proof. We use the fact that continuity is equivalent to continuity at 0. Then, we simply unravel the definitions. A is continuous at 0 if for every neighbourhood $Z \subseteq W$ of 0_W we have a neighbourhood $U \subseteq V$ of 0_V such that $A(U) \subseteq V$.

Equivalently, we can consider open balls in place of Z and U. This is then equivalent to the second condition. $\square$

Definition III.7 (Operator Norm)**.** The operator norm $\|A\|$ is defined as

$$\|A\| = \inf\{c | \|Av\| \leq c\|v\| \ \forall v \in V\}$$

Corollary III.8 (Equivalent Statements)**.**

(a)

$$\|A\| = \sup_{V \ni v \neq 0} \frac{\|Av\|}{\|v\|}$$

(b)

$$\|A\| = \sup_{\|v\|=1} \|Av\|$$

Theorem III.9 (Operator Composition)**.** Let V, W, Z be normed spaces, $A \in L(V, W)$, $B \in L(W, Z)$. Then

$$\|B \circ A\| \leq \|A\|\|B\|.$$

Proof. For $v \in V$, we have

$$\|BAv\| \leq \|B\|\|Av\| \leq \|B\|\|A\|\|v\|. \qquad \square$$

Note that the above definition shows once again the continuity of composition. Also note that this norm turns $L(V, W)$ into a normed space.

B. Bases

Definition III.10 (Hamel Basis). A Hamel basis is a set $B \subseteq V$ such that for all $v \in V$, we have

$$v = \sum_{k=1}^{n} a_k e_k$$

with $a_i \in \mathbb{K}$ and $e_i \in B$ for all i, and

$$\sum_{k=1}^{n} a_k e_k = 0 \implies a_i = 0 \forall i.$$

The existence of Hamel bases is equivalent to the axiom of choice, and can be proven by Zorn's Lemma. The proof is as follows: We construct minimal spanning sets and maximally linearly independent sets by Zorn's Lemma (partial order by inclusion), and show that they are the same. These sets are Hamel bases.

A Hamel basis has all the beloved properties of a basis from finite dimensional linear algebra. For example, a linear map is uniquely defined through its action on the Hamel basis:

$$Av = \sum_{k=1}^{n} a_k A e_k.$$

However, a Hamel bases are usually difficult to come by, as we see with the following theorem:

Theorem III.11. A Banach space with a countable Hamel basis is finite.

Proof. Suppose we have a countable Hamel basis $(e_n)_{n \in \mathbb{N}}$ of a Banach space V. Denote $M_n = \operatorname{span}(\{e_1, \ldots, e_n\})$. This is a closed proper subspace of V. Thus it has empty interior. However, because by definition the e_ns form a Hamel basis, we have $V = \bigcup_{k=1}^{\infty} M_k$, contradicting Baire's category theorem. $\square$

Definition III.12 (Schauder Basis). A Schauder basis is a countable set $\{e_n\} \subseteq V$ such that all vectors $v \in V$ can be uniquely expressed as a sum

$$v = \sum_{k=1}^{\infty} a_k e_k,$$

where the convergence is understood to be in the topology of the vector space.

The importance of a Schauder basis is that it is countable, and we are still able to define *some* linear maps by their action on the basis. In particular, for any continuous linear map A, we have

$$Av = \sum_{k=1}^{\infty} a_k A e_k.$$

Also important to note is that this basis must be *ordered*, since the sum does not necessarily converge unconditionally.

IV. THE HAHN-BANACH THEOREMS

A. Sublinear Functionals

Definition IV.1. Let V be a vector space over $\mathbb{K}$. A functional p is called *sublinear*, if

(a) (Homogenity) $p(rx) = rp(x)$ for all real $r \geq 0$ and $v \in V$.

(b) (Subadditivity) $p(u+v) \leq p(u) + p(v)$ for all $u, v \in V$

The main property that relates sublinear functionals to linear functionals is as follows:

Theorem IV.2. If p is a sublinear functional on a real vector space V, then the following are equivalent:

(a) p is linear

(b) $p(v) + p(-v) = 0$ for all $v \in V$.

(c) $p(v) + p(-v) \leq 0$ for all $v \in V$

Proof. If p is linear, then $p(v) + p(-v) = p(v - v) = p(0) = 0$. Clearly, (b) implies (c).

Now assume (c). First, we prove (b):

$$0 = p(v - v) \leq p(v) + p(-v) \leq 0$$

or $p(v) + p(-v) = 0$, proving (b). Using (b) we have, for $r < 0$,

$$p(rv) = -p(-rv) = rp(v),$$

proving the first aspect of linearity. Then we have

$$p(v + w - w) \leq p(v + w) + p(w) = p(v + w) + p(-w),$$

implying

$$p(v) + p(w) \leq p(v + w).$$

The subadditivity yields the other inequality, completing the proof of linearity. $\square$

We can define a partial order on the sublinear functions by defining p to be less than q if p is less than q pointwise. Zorn's Lemma then yields minimal elements. We will prove eventually that these minimal elements are linear functionals. Before that, however, we need two lemmas:

Theorem IV.3 (Auxiliary Functionals)**.** If p is a sublinear functional on a real vector space V, then the auxiliary functional $q(v) = \inf\{p(v+tw) - tp(w) | t \geq 0, w \in V\}$ is a sublinear functional such that $q \leq p$.

Proof. By considering $t = 0$ in the infimum, we see that $q \leq p$. We only need to show sublinearity. Consider first $r = 0$. Then

$$q(rv) = \inf\{p(rv + tw) - tp(w) | t \geq 0, w \in V\} = \inf\{p(tw) - tp(w) | t \geq 0, w \in V\} = 0.$$

For $r > 0$, we have

$$\begin{aligned} q(rv) &= \inf\{p(rv + tw) - tp(w) | t \geq 0, w \in V\} \\ &= \inf\left\{rp\left(v + \frac{t}{a}w\right) - tp(w) | t \geq 0, w \in V\right\} \\ &= r\inf\left\{p\left(v + \frac{t}{a}w\right) - \frac{t}{a}p(w) | t \geq 0, w \in V\right\} \end{aligned}$$

$$= rq(v)$$

For subadditivity, we take the special points $w = \frac{1}{s+t}(sx + ty)$, or $(s + t)w = sx + ty$. Then

$$\begin{aligned} q(x + y) &\leq p(x + y + (s + t)w) - (s + t)p(w) \\ &\leq p(x + sw) - sp(w) + p(y + tw) - tp(w) \end{aligned}$$

which shows that $q(x + y) \leq q(x) + q(y)$. □

Now, we move on to the proof of the main result.

Theorem IV.4. A sublinear functional p on a real vector space V is linear iff it is minimal.

Proof. Suppose we have $q \leq p$, with q sublinear and p linear. Since q is sublinear, we have $0 = q(v - v) \leq q(v) + q(-v)$, which implies $-q(-v) \leq q(v)$. Since $q(-v) \leq p(-v) = -p(v)$, we have $p(v) \leq -q(-v) \leq q(v)$, suggesting that $p \leq q$. Thus $p = q$.

Conversely, suppose that p is a minimal sublinear functional. Then we must have $q = p$ where q is the auxiliary sublinear functional defined in Theorem IV.3. If we let $t = 1$ and $w = -v$ in the above definition, we get

$$p(v) \leq p(v - v) - p(-v),$$

or $p(v)+p(-v) \leq 0$, as desired. Finally, we show the boundedness which we require to apply Zorn's Lemma □

Theorem IV.5. For every sublinear functional p, we have a linear functional $f \leq p$.

Proof. We begin, as always, by considering a chain of sublinear functionals, which is also a totally ordered set of sublinear functionals P. Suppose that $q(x)$ is unbounded from below for all $q \in P$ and some $x \in V$. Then we have, for all n, a p_n such that $p_n(x) \leq -n$.

Then we construct a decreasing sequence of sublinear functionals q_n using $q_n = \min(p_1, \dots, p_n)$. Since $q_n(x) \leq -n$, we have

$$0 = q_n(x - x) \leq q_n(x) + q_n(-x) \leq -n + q_n(-x)$$

or $q_n(-x) \geq n$. Then we have, for all n, $n \leq q_n(x) \leq q_1(x)$, a contradiction. Thus we can take the infimum for all $x \in V$. This yields a functional q^*. It remains to show that q^* is sublinear.

Since $q_n(0)$ for all n, we have $q^*(0) = 0$. By homogenity, $q^*(rv) = \inf\{q_n(rv)|n \in \mathbb{N}\} = r\inf\{q_n(v)|n \in \mathbb{N}\}$. Finally, we have $q^*(u+v) = \inf\{q_n(u+v)|n \in \mathbb{N}\} \leq q^*(u) + q^*(v)$. $\square$

Theorem IV.6 (Dominated Extension)**.** Let X be a real vector space, p a sublinear functional on X, M a subspace of X and $f : M \to \mathbb{R}$ a linear functional on X such that $f \leq p$. Then there is a linear functional F on X that extends f such that $F \leq p$.

Proof. The overarching idea of this proof is as follows: We seek a sublinear functional q such that $q \leq p$ on X, and $q \leq f$ on M. By the previous theorem, we will then get a linear functional $F \leq q \leq p$ on X and $F \leq f$ on M. Since f is minimal on M, we then have $f \leq F$, or $f = F$ on M.

We choose as our candidate

$$q(x) = \inf\{p(x+m) + f(m)|m \in M\}.$$

It is clear that $q \leq f$ on M. We first show that q is real valued. We do this by showing two different estimates:

First, we have $f(-m) \leq p(-m)$. Thus, for all $x \in X$, we have $-p(-x) + f(-m) \leq -p(-x) + p(m)$, or

$$-p(-x) \leq p(m) - p(-x) - f(-m).$$

Then, we estimate $p(-m) \leq p(x-m) + p(-x)$ and thus

$$p(-m) - p(-x) \leq p(x-m).$$

Combining the two equalities, we get

$$-p(-x) \leq p(x-m) + f(m)$$

which shows that q is real valued. Now we show sublinearity.

1. Since $f(-m) \leq p(-m)$, we have $p(-m) + f(m) \geq 0$. This shows that $q(0) \geq 0$. Conversely, we have $p(0) + f(0) = 0$, which shows that $q(0) \leq 0$.

2. For $a \in \mathbb{R}_0^+$, we have $q(ax) = \inf\{p(ax-m)+f(ax)\} = \inf a\{p(x-m/a)+af(m/a)\} = aq(x)$.

3. For sublinearity, we choose $x, y \in X$ and $r > 0$. There exists $m, n \in M$ such that

$$q(x) > p(x-m) + f(m) + \frac{r}{2}, \qquad q(y) > p(y-n) + f(n) + \frac{r}{2}.$$

Then

$$q(x) + q(y) \geq p(x+y-(m+n)) - f(m+n) - r \geq q(x+y) - r.$$

□

Theorem IV.7 (Hahn-Banach (Normed)). Let V be a normed space, $W \subseteq V$ a subspace, and f a continuous linear functional on W. Then there is a continuous extension of f to V.

Proof. We consider $g : V \to \mathbb{R}$, $v \mapsto \|f\|\|v\|$. This map is clearly sublinear and $f \leq g$. Then, by the Dominated Extension Theorem, we get an extension to all of V. Because this map is bounded, it is continuous. □

Corollary IV.8. Let V be a normed space and $v \in V$. Then there exists a linear functional $\varphi \in V'$ such that $\varphi(v) = \|v\|$.

Proof. We let $W = \text{span}\{v\}$ and $\varphi(x) = k$ if $x = kv$ in Theorem IV.7 □

B. Geometric Hahn-Banach

Definition IV.9. Let V be a normed space. A subset A of V is called

(a) *Absorbing*, if for every $v \in V$ there is $\lambda > 0$ with $v \in \lambda A$.

(b) *Balanced*, if for all $|z| \leq 1$ we have $zA \subseteq A$.

(c) *Convex*, if for all $v, w \in A$ and $\lambda \in [0, 1]$ we have $\lambda v + (1-\lambda)w \in A$.

(d) *Absolutely convex*, if it is balanced and convex.

Theorem IV.10. Let V be a vector space over $\mathbb{K}$.

(a) If $p : V \to \mathbb{R}_0^+$ is a seminorm then

$$B_{p,1}(0) = \{v \in V | p(v) < 1\}$$

and

$$B_{p,1}(0)^{\mathrm{cl}} = \{v \in V | p(v) \leq 1\}$$

are absorbing and absolutely convex.

(b) If $C \subseteq V$ is convex, balanced and absorbing then

$$p_c(V) = \inf\{\lambda | \lambda > 0, v \in \lambda C\}$$

is a seminorm.

(c) For C absolutely convex and absorbing

$$B_{p_c,1}(0) \subseteq C \subseteq B_{p_c,1}(0)^{\mathrm{cl}}.$$

Proof. (a) It is clear from the definition of a seminorm that both of these are absolutely convex.

If $v \in V$ and $p(v) = \alpha$, then $p(v/2\alpha) = 1/2$, which tells us that $v/2\alpha$ is an element of the ball, and thus the ball is absorbing.

(b) First, we show absolute homogenity. We note that $v \in \lambda C$ if and only if $|\alpha| v \in |\alpha| \lambda C$. Since C is balanced, so is $|\alpha|\lambda C$, and thus this happens iff $\alpha v \in |\alpha|\lambda C$. Since the infimum is absolutely homogeneous, this shows that the p_c is too.

Suppose $v \in \lambda_v C$ and $u \in \lambda_u C$. If we prove that $u + v \in (\lambda_u + \lambda_v)C$, we will be done. In fact, we know more - we know that $aC + bC = (a + b)C$. Let us prove this.

Suppose $v \in aC + bC$. Then $v = au + bw$ for $u, w \in C$. Since C is convex, we have

$$\frac{v}{a+b} = \frac{a}{a+b}u + \frac{b}{a+b}w \in C$$

suggesting that $v \in (a + b)C$.

(c) Clearly, for all $v \in C$, we have $p(v) \leq 1$.

If $p(v) < 1$, then $v \in \lambda C$ for some $\lambda < 1$. Since C is balanced, this shows that $v \in C$. □

Definition IV.11 (Minkowski Functional)**.** Let $C \subseteq V$ be an absorbing subset in a vector space V over a $\mathbb{K}$. Then $p_c : V \to \mathbb{R}_0^+$ defined by

$$p_c(v) = \inf\{\lambda > 0 | v \in \lambda C\}$$

is called the Minkowski functional of C.

(b) of the previous theorem shows that this is a seminorm if C is absolutely convex too.

Theorem IV.12 (Separation I)**.** Let V be a normed space with two nonempty disjoint convex subsets $A, B \subseteq V$.

(a) Suppose A is open. Then there exists $\Phi \in V'$ and $\alpha \in \mathbb{R}$ such that

$$\mathrm{Re}(\Phi(v)) < \alpha \leq \mathrm{Re}(\Phi(u))$$

for $v \in A < u \in B$

(b) Suppose both A and B are open. Then there exists $\Phi \in V'$ and $\alpha \in \mathbb{R}$ with

$$\mathrm{Re}(\Phi(v)) < \alpha < \mathrm{Re}(\Phi(v))$$

for $v \in A, u \in B$.

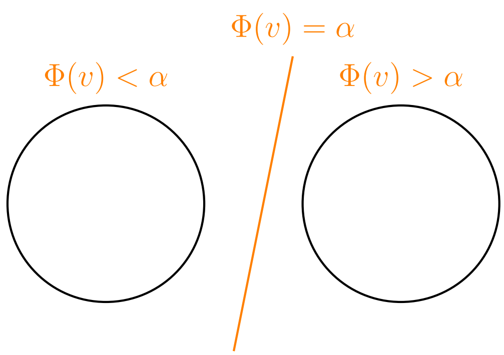
Proof. Fix $a_0 \in A,\ b_0 \in B$. We define $x_0 = b_0 - a_0$, and let $C = A - B + x_0$. Then C is a convex neighbourhood of 0. Let p be its Minkowski functional. Since C is a neighbourhood of 0, it is absorbing. Thus, its Minkowski functional is a sublinear functional on X, with $p(x_0) \geq 1$.

Then, by Corollary IV.8, we define a linear functional on the subspace spanned by x_0 such that $f(x_0) = 1$. Since this is bounded by p, f extends to a linear functional Λ on X. Since $\Lambda(C) \leq 1$, we have $\Lambda(-C) \geq -1$ and Λ is bounded on a neighbourhood $C \cap -C$ of 0 and is hence continuous. Then, we let $a \in A, b \in B$, and note that $a - b + x_0 \in C$, thus

$$\Lambda(a) - \Lambda(b) + 1 = \Lambda(a - b + x_0) < 1,$$

or $\Lambda(a) < \Lambda(b)$. Thus the images of A and B under Λ are convex subsets of the real line, and the image of A is an open subset, since linear functionals are always open maps. However, the only convex open subsets of the real line are the intervals, and choosing the right endpoint of $\Lambda(A)$ yields (a). (b) follows almost identically. □

The picture for this theorem is as follows: A linear functional Φ defines a hyperplane for each fixed value of α, then splitting the space into two parts, one where the functional is less than, and one where the functional is greater than α. In $\mathbb{R}^2$, the picture looks as follows:



Theorem IV.13 (Separation II). Let V be a normed vector space, $K \subseteq V$ compact, convex, nonempty, $C \subseteq V$ closed, convex, nonempty, with $K \cap C = \varnothing$. Then there exists $\Phi \in V'$ and $\alpha, \beta \in \mathbb{R}$ with $\operatorname{Re}\Phi(v) < \alpha < \beta < \operatorname{Re}\Phi(u)$ for all $v \in K$ and $u \in C$.

Proof. There is a convex neighbourhood V of 0 in V such that $K + V \cap C = \varnothing$ (Corollary II.7). Then, we apply Theorem IV.12 to $K + V$ and C to get a linear functional that satisfies all the required properties, since K is a compact subset of $K + V$.

Note that we used that V is normed in the first line, as Corollary II.7 only yields a not necessarily convex neighbourhood. Local convexity would also have sufficed. □

Theorem IV.14 (Separation III). Let V be a normed space, $K \subseteq V$ compact convex, $C \subseteq V$ closed convex nonempty, balanced, such that $K \cap C = \varnothing$. Then there exists

$\Phi \in V'$ such that $\sup_{u\in C} |\Phi(u)| < \inf_{v\in K} |\Phi(v)|$.

Proof. Follows directly from Theorem IV.14. □

Corollary IV.15. Let $C \subseteq V$ be a closed balanced convex subset and $v \in V \setminus C$. Then there exists $\Phi \in V'$ with $\Phi(v) > 1$, $\Phi(u) \leq 1$ for all $u \in C$.

Proof. Apply Theorem IV.14 with $K = \{v\}$ and rescale as necessary. □

V. DUALITY

A. The Weak Topology

In this subsection, we will want to work with components of vectors. However, due to the difficulty of picking a basis, we cannot define the components the same way we do in finite dimensional linear algebra. Thus, we define the components to be the outputs of linear functionals applied to vectors. This notion leads us to the weak topology.

Definition V.1. Let V be normed.

(a) For $\varphi \in V'$ we define the seminorm $p_\varphi : V \to \mathbb{R}_0^+$ by $p_\varphi(v) = |\varphi(v)|$ and one calls $B_{p_\varphi,r}(v) = \{w \in V | p_\varphi(v - w) < r\}$ the open ball.

(b) We define the weak topology to be the topology on V generated by all open balls $B_{p_\varphi,r}(v)$.

Equivalently, we could also have defined the weak topology as the weakest topology such that all the seminorms p_φ are continuous. Note that the weak topology is locally convex as the subbasis elements are convex due to Theorem IV.10.

Theorem V.2. Let V be a normed space.

(a) The weak topology turns V into a topological vector space.

(b) The weak topology is coarser than the norm topology.

(c) $\varphi \in V^*$ is weakly continuous iff φ is norm continuous.

(d) The weak topology is Hausdorff

(e) p_φ for $\varphi \in V'$ is weakly continuous.

Proof. (a) We only check the continuity condition for a subbasis:

$$p_\varphi(u + u' - (v + v')) \leq p_\varphi(u - v) + p_\varphi(u' - v') < r$$

which implies that $u + u' \subseteq B_{p_\varphi, r}(v + v')$. Thus

$$B_{p_{\varphi,\frac{r}{2}}}(v) \times B_{p_{\varphi,\frac{r}{2}}}(v') \subseteq +^{-1} B_{p_{\varphi,r}}(v + v').$$

Thus, $+$ is continuous at (v, v'). Continuity of multiplication follows by the homogenity of seminorms.

(b) If $p_\varphi(v - v') < \epsilon$, we also have $p_\varphi(v - v') = |\varphi(v - v')| \leq \|\varphi\| \|v - v'\|$, thus showing that we can choose δ such that $B_\delta(v) \subseteq B_{p_{\varphi,\epsilon}}(v)$.

(c) If φ is weakly continuous it is norm continuous because of (b). Conversely, we see that $\varphi^{-1}(B_r(0)) = B_{p_{\varphi,r}}(0)$, which is an element of the subbasis. Thus, φ is weakly continuous.

(d) Suppose $v \neq 0$. Then, we define a continuous linear functional φ such that $r = \varphi(v) > 0$ by Hahn-Banach. Then $B_{p_{\varphi,r/2}}(0) \cap B) p_{\varphi,r/2}(v) = \varnothing$.

(e) Follows by definition. □

Theorem V.3. Let V be a topological vector space.

(a) A net $(v_i)_{i \in I}$ in V is weakly convergent to $v \in V$ if and only if for every $\varphi \in V'$ one has $\lim_{i \in I} \varphi(v_i) = \varphi(v)$.

(b) A net $(v_i)_{i \in I}$ in V is a weak Cauchy net if and only if for every $\varphi \in V'$ the net $(\varphi(v_i))_{i \in I}$ is a Cauchy net in $\mathbb{K}$.

Proof. (a) Since φ is (weakly) continuous, we have the desired equality.

Conversely, assume that the equality is satisfied. What we need to show is that the net v_i is eventually in every neighbourhood of v. Since V is locally convex, it suffices to consider absolutely convex neighbourhoods K. Consider the Minkowski functional of K, p_K. Since p_K also generates the weak topology, we can find $\alpha \in I$ such that $\beta \succeq \alpha \implies p_K(v_i - v) < 1$, or $v_i \in K$.

(b) Follows analogously to (a). □

Theorem V.4. Let V be a normed space and let $C \subseteq V$ be convex. Then the weak closure of C coincides with the norm closure.

Proof. Since the weak topology is weaker, $C^{\mathrm{cl}} \subseteq C^{\mathrm{wcl}}$.

Conversely, we choose $v_0 \in V$, $v_0 \notin C^{\mathrm{cl}}$. Then Theorem IV.13 yields $\gamma \in \mathbb{R}$ and $\Lambda \in V'$ such that

$$\operatorname{Re} \Lambda x_0 < \gamma < \operatorname{Re} \Lambda x.$$

Thus, the set $\{x | \operatorname{Re} \Lambda x < \gamma\}$ is a weak neighbourhood of x_0 that does not intersect C, showing that $x \notin C^{\mathrm{wcl}}$. □

Corollary V.5. Let $U \subseteq V$ be a subspace of a normed space. Then $U^{\mathrm{cl}} = U^{\mathrm{wcl}}$.

Proof. Subspaces are automatically convex. □

Theorem V.6. Let V be a normed space. Then the canonical map $i : v \to V''$ is norm preserving and hence injective.

Proof. We define this injection by

$$i(v)\varphi = \varphi(v).$$

We have $i \in V''$ and $\|i(v)\| \leq \|v\|$ for a given $v \in V$. THis implies that there is $\varphi \in V'$ with $\|\varphi\| = 1$ and $\varphi(v) = \|v\|$. Thus,

$$|i(v)\varphi| = |\varphi(v) = \|v\| = \|v\|\|\varphi\|.$$

Thus, $\|i(v)\| \geq \|v\|$. Together, we have $\|i(v)\| \geq \|v\|$. □

Theorem V.7 (Completion of normed space)**.** Every normed space V can be completed to a Banach space. More precisely $i(v)^{\mathrm{cl}} \subseteq V''$ is a completion.

Proof. Note that the dual space of *any* TVS is always complete. As a closed subspace of a complete space, we have a complete space. □

The canonical map i allows us to define a weak topology on V' generated by the functionals $i(V) \in V''$. We call this the weak *-topology. Convergence in the weak *-topology is defined by the same conditions as in the weak topology. In particular, a net $(v_i)_{i \in I} \subseteq V'$ converges to $v \in V'$ if and only if $v_i(x) \to v(x)$ for all $x \in V$ - that is, the linear functionals converge pointwise.

Theorem V.8. Let V, W be normed spaces with $A : V \to W$ continuous and linear. Then

$$\|A\| = \sup_{\varphi \neq 0, w \neq 0} \frac{|\varphi(Av)|}{\|\varphi\|_{W'} \|v\|_V} = \sup_{\|\varphi\| = \|v\| = 1} |\varphi(Av)| = \|A'\|.$$

B. Polars & Banach-Aloglu Theorems

Definition V.9. Let V be a topological vector space with topological dual V'.

(a) The polar $A^{\circledast}$ of a subset $A \subseteq V$ is defined by

$$A^{\circledast} = \{\varphi \in V' | |\varphi(v)| \leq 1 \text{ for all } v \in A\}.$$

(b) The polar $B_{\circledast}$ of a subset $B \subseteq V'$

$$B_{\circledast} = \{v \in V | |\varphi(v)| \leq 1 \text{ for all } \varphi \in B\}.$$

The polar $A^{\circledast}$ of a set $A \subseteq V$ is the set of linear functionals which are small on A. Note that the 1 in the definition comes without loss of generality, as

$$\varphi(v) < r \iff \frac{\varphi}{r}(v) < 1.$$

Theorem V.10. Let V be a topological vector space and $B \subseteq V'$. Then $B_{\circledast} \subseteq V$ is absolutely convex and weakly closed.

Proof. For $z \in \mathbb{K}$ with $|z| \leq 1$ and $v \in B_X$, then for $\varphi \in B$ we have $|\varphi(zv)| = |z||\varphi(v)| \leq 1$, which shows that $B_{\circledast}$ is balanced.

Then we choose $\lambda \in [0,1]$ and $v, w \in B_{\circledast}$. Then for $\varphi \in B$ we have

$$\varphi(\lambda v + (1-\lambda)w) \leq \lambda|\varphi(v) + (1-\lambda)|\varphi(w)| \leq 1.$$

This implies that $B_{\circledast}$ is convex.

To show that it is closed, we have

$$\begin{aligned} B_{\circledast} &= \{v \in V | |\varphi(v)| \leq 1 \text{ for all } \varphi \in B\} \\ &= \{v \in V | p_\varphi(v) \leq 1 \text{ for all } \varphi \in B\} \\ &= \bigcap_{\varphi \in B} B_{p_\varphi,1}(0)^{\text{cl}} \\ &= \bigcap_{\varphi \in B} p_\varphi^{-1}([0,1]) \end{aligned}$$

which is an intersection of closed sets and hence closed. □

We also have the following facts

Theorem V.11. Let V be a topological vector space.

(a) If $A \subseteq B \subseteq V$, then $A^{\circledast} \supseteq B^{\circledast}$.

(b) For $a \neq 0$, we have $(aA)^{\circledast} = a^{-1}A^{\circledast} = |a|^{-1}A^{\circledast}$.

(c) $A \subseteq A^{\circledast\circledast}$ and $A^{\circledast} = A^{\circledast\circledast\circledast}$.

Proof. (a) If $|\varphi(v)| < 1$ for all $v \in B$, then it holds true for all $v \in A$.

(b) We have $\varphi(av) = a\varphi(v)$, as well as $|\varphi(av)| = |a||\varphi(v)|$.

(c) We have $A \subseteq A^{\circledast\circledast}$ by definition. Thus, $A^{\circledast} \subseteq (A^{\circledast})^{\circledast\circledast}$. However, by (a), $A \subseteq A^{\circledast\circledast}$ implies that $A^{\circledast} \supseteq A^{\circledast\circledast\circledast}$. □

Next, we will consider what happens to polar sets of hulls. However, before we continue, we first need to define what hulls are.

Definition V.12. The convex hull C_c of a subset $C \subseteq V$ is the intersection of all convex sets containing C.

The most useful characterisation of convex hulls is as follows:

Lemma V.13. The convex hull C_c of a set C is the set consisting of all convex combinations of vectors of C.

Proof. Let K be the set of all convex combinations of elements of C. Since K is convex, we have $C_c \subseteq K$. Conversely, all linear combinations of elements of C must be in C_c, and thus $K \subseteq C_c$. □

We also have balanced hulls:

Definition V.14. The balanced hull C_b of a subset $C \subseteq V$ is the intersection of all balanced sets containing C.

Again, this is characterised by the following Lemma:

Lemma V.15. The balanced hull C_b of a subset $C \subseteq V$ is the union of all sets zC, with $|z| \leq 1$.

Proof. We define

$$K := \bigcup_{|z|\leq 1} zC.$$

Clearly, K is balanced, and thus $C_b \subseteq K$. Conversely, we must have $K \subseteq C_b$. □

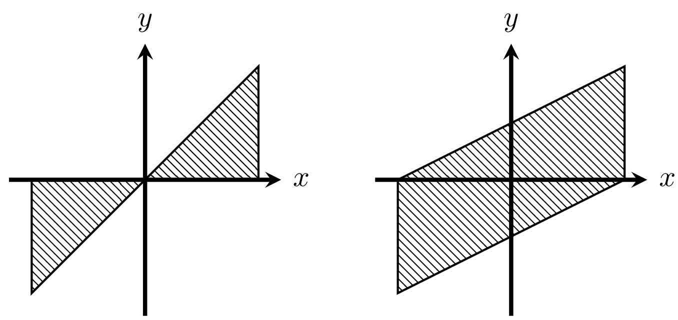
We also have the absolutely convex hull of a set

Definition V.16. The absolutely convex hull A_{bc} of a subset $A \subseteq V$ is defined by $(A_b)_c$.

We note that this is convex by definition, and we can easily verify that it is also balanced. It is also important to note that this is not the same as $(A_c)_b$, as $(A_c)_b$ may not be convex.

Example V.17. Let A be the set in $\mathbb{R}^2$ consisting of $(0,0)$, $(1,0)$ and $(1,1)$. Its convex hull is the triangle consisting of these 3 points, and its balanced hull is its convex hull plus its mirror image, which is not convex.

(a) $(A_c)_b$ (b) $(A_b)_c$



Now we are ready to discuss their polar sets:

Theorem V.18 (Polars of Hulls)**.** Let V be a topological vector space, $A \subseteq V$.

(a) $A^{\circledast} = (A_b)^{\circledast}$.

(b) $A^{\circledast} = (A_c)^{\circledast}$.

(c) $A^{\circledast} = (A^{\mathrm{wcl}})^{\circledast}$.

(d) $A^{\circledast} = ((A_{bc})^{\mathrm{wcl}})^{\circledast}$.

Proof. For all of these, the reverse inclusion follows due to A being a subset of the other set.

(a) Now suppose $|\varphi(v)| \leq 1$ for all $v \in A$. Then consider $u \in A_b$. We have $u = zv$ for some $v \in A$, and thus $\varphi(u) = \varphi(zv) = z\varphi(v)$.

(b) Consider $u \in A_c$, meaning that $u = \sum_{k=1}^n \alpha_k v_k$ for $\sum_{k=1}^n \alpha_k = 1$, $v_k \in A$ for all k. We have
$$\varphi(u) = \sum_{k=1}^n \alpha_k \varphi(v_k) \leq \sum_{k=1}^n \alpha_k = 1.$$

(c) Suppose $|\varphi(v)| \leq 1$ for all $v \in A$. Then, if $w \in A^{\mathrm{wcl}}$, it means that there is a net $(v_i)_{i\in I} \subseteq A$ such that $v_i \to w$. In the weak topology, this means that $\varphi(v_i) \to \varphi(w) \leq$

1.

(d) We note that $A \subseteq (A_{bc})^{\mathrm{wcl}}$. Since $A^{\circledast\circledast}$ is absolutely convex and weakly closed by Theorem V.10, we have $(A_{bc})^{\mathrm{wcl}} \subseteq A^{\circledast\circledast}$. This implies that

$$A^{\circledast} \supseteq (A_{bc})^{\mathrm{wcl}} \supseteq A^{\circledast\circledast\circledast} = A^{\circledast}. \qquad \square$$

Theorem V.19 (Bipolar Theorem). Let A be a subset of a topological vector space V. Then $A^{\circledast\circledast} = (A_{bc})^{\mathrm{wcl}}$.

Proof. As in V.18 we have $A \subseteq (A_{bc})^{\mathrm{wcl}} \subseteq A^{\circledast\circledast}$. Choose $w \notin (A_{bc})^{\mathrm{wcl}}$. Then, by Theorem IV.14, there is a weakly continuous linear functional Φ such that $\sup_{v\in(A_{bc})^{\mathrm{wcl}}} |\Phi(v)| < \Phi(w)$. Since $(A_{bc})^{\mathrm{wcl}}$ is balanced, we have $0 = \Phi(0) < a$. Hence, we may rescale the functional to get a linear functional Ψ with $\sup_{v\in(A_{bc})^{\mathrm{wcl}}} |\Psi(v)| = 1$. This implies that $\Psi \in A^{\circledast}$. Since $\Psi(w) > 1$, we have $w \notin A^{\circledast\circledast}$, completing the proof. $\square$

Theorem V.20 (Banach-Alaoglu). If U is a balanced neighbourhood of 0 in the topological vector space V then $U^{\circledast}$ is weak *-compact.

Proof. Since neighbourhoods of 0 are absorbing, we choose $\gamma(v)$ for each $v \in V$ such that $v \in \gamma(v)U$. Then we define a set P by $P = \prod_{v\in V} K_{\gamma(v)}(0)^{\mathrm{cl}}$. Since the disks are compact, P is compact by Tychonoff's Theorem. P consists of all functions $f : V \to \mathbb{K}$ such that $|f(x)| \le \gamma(x)$. Thus, we have $U^{\circledast} \subseteq V' \cap P$.

We now have two topologies on $U^{\circledast}$, induced by V' and P respectively. We show that these topologies coincide, and $U^{\circledast}$ is closed in the topology induced by P. Since P is compact, this will imply that $U^{\circledast}$ is compact.

Fix $\Lambda_0 \in A^{\circledast}$, and choose $v_i \in V$ for $i = 1, \ldots, n$ as well as $\delta > 0$

$$W_1 = \{\Lambda \in V' | |\Lambda v_i - \Lambda_0 v_i| < \delta, i = 1, \ldots, n\}$$
$$W_2 = \{f \in P | |f v_i - \Lambda_0 v_i| < \delta, i = 1, \ldots, n\}$$

The Ws form a local basis for the topology. Since $A^{\circledast} \cap W_1 = A^{\circledast} \cap W_2$, the topologies coincide.

Now suppose that we have f_0 in the P-closure of $A^{\circledast}$. Then we choose scalars α, β, and $u, v \in V$. The set of all f such that $|f - f_0| < \epsilon$ at x, y and $\alpha x + \beta y$ is a P-neighbourhood of f_0. Thus, $A^{\circledast}$ contains such an f. Since f is linear, we have

$$\begin{aligned} & f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y) \\ =& (f_0 - f)(\alpha x + \beta y) + \alpha(f - f_0)(x) + \beta(f - f_0)(y). \end{aligned}$$

Approximating each of these by ϵ, we have'

$$|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| = \epsilon(1 + |\alpha| + |\beta|)$$

showing that f_0 is linear. Similarly, we conclude that $|f_0(v)| \leq 1$ for all $v \in A$. Thus, we have $f_0 \in A^{\circledast}$, completing the proof. $\square$