1. Klein–Gordon theory: free real scalar field in $d+1$ dimensions

The Lagrangian for a free, massive, real, bosonic scalar field $\phi(x)$ in d space dimensions is given by

$$L = \int d^d x\, \mathcal{L}(x), \qquad \mathcal{L}(x) = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2, \tag{1}$$

where the metric has signature $(1, d)$ (*i.e.*, $g^{00} = 1$, $g^{ii} = -1$ for $i = 1, \ldots, d$).

We first consider the model (1) as a classical theory.

(a) Obtain the Euler-Lagrange equation for $\phi(x)$, which is also known as Klein–Gordon equation.

$$\mathcal{L}(x) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2$$

$$= \frac{1}{2} g^{\mu\gamma} \partial_\mu \phi \, \partial_\gamma \phi - \frac{m^2}{2}\phi^2$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)} = \frac{1}{2} g^{\mu\gamma}(\partial_\gamma \phi\, \delta_{\mu\sigma} + \partial_\mu \phi\, \delta_{\gamma\sigma})$$

$$= \frac{1}{2} g^{\sigma\gamma} \partial_\gamma \phi + \frac{1}{2} g^{\mu\sigma} \partial_\mu \phi$$

$$= g^{\sigma\mu} \partial_\mu \phi$$

$$\partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)} = g^{\sigma\mu} \partial_\sigma \partial_\mu \phi$$

$$= \partial^\mu \partial_\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\partial^\mu \partial_\mu \phi = -m^2 \phi$$

(b) Obtain the momentum field $\pi(x)$ conjugate to $\phi(x)$ and the Hamiltonian density $\mathcal{H}(x)$.

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{m^2}{2}\phi^2$$

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

$$\mathcal{H} = \pi(x)\dot{\phi} - \mathcal{L}$$

$$= \frac{1}{2}\pi(x)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2$$

(c) Obtain the energy momentum tensor $T^{\mu\nu}$, and from there the generators of translations in time H and in space P.

$$T^\mu{}_\gamma = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\gamma \phi - \delta^\mu{}_\gamma \mathcal{L}$$

$$= (g^{\mu\sigma} \partial_\sigma \phi)\partial_\gamma \phi - \delta^\mu{}_\gamma \mathcal{L}$$

$$= \partial^\mu \phi\, \partial_\gamma \phi - \delta^\mu_\gamma \mathcal{L}$$

$$T^{\mu\gamma} = \partial^\mu \phi\, \partial^\gamma \phi - \delta^{\mu\gamma} \mathcal{L},$$

conserved charges/generators are diagonal elements

We now quantize the theory by imposing the equal-time commutator

$$[\phi(\boldsymbol{x}, t), \pi(\boldsymbol{x}', t)] = i\delta^d(\boldsymbol{x} - \boldsymbol{x}'). \tag{2}$$

(d) Calculate the commutators $[H, \phi(\boldsymbol{x}, t)]$ and $[H, \pi(\boldsymbol{x}, t)]$. Are the results consistent with the Klein–Gordon equation?

Hint: Express H as a functional of $\pi(x)$, $\nabla\phi(x)$, and $\phi(x)$ rather than $\partial_\mu \phi(x)$ and $\phi(x)$.

$$H = \int \frac{1}{2}\pi(x)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2\, d^d x$$

$$[\partial_{i'}\phi, \pi(x)] = \partial_{i'}[\phi(x'), \pi(x)]$$

$$= \partial_{i'}\, i\delta^d(\vec{x}' - \vec{x})$$

$$[H, \pi] = \left[\int \frac{1}{2}\pi(x')^2 + \frac{1}{2}(\nabla\phi(x'))^2 + \frac{m^2}{2}\phi^2\, d^d x',\ \pi(x)\right]$$

$$= \int d^d x' \left[\frac{1}{2}(\nabla \phi(x'))^2, \pi(x) \right] + \frac{m^2}{2}[\phi(x')^2, \pi(x)]$$

$$\boxed{[A^2, B] = A[A, B] + [A, B]A}$$

$$[\phi(x')^2, \pi(x)] = \phi(x')[\phi(x'), \pi(x)] + [\phi(x'), \pi(x)]\phi(x')$$

$$= 2i\phi(x')\delta^d(x'-x)$$

$$\int d^d x' [\phi(x')^2, \pi(x)] = 2i\phi(x)$$

$$[(\partial_{i'}\phi)^2, \pi(x)] = \partial_{i'}\phi[\partial_{i'}\phi, \pi(x)] + [\partial_{i'}\phi, \pi(x)]\partial_{i'}\phi$$

$$= 2i(\partial_{i'}\phi)\partial_{i'}\delta(x'-x)$$

$$= 2i\,\partial^2_{i'}\phi(x)\,\delta(x'-x) \quad \text{(integration by parts)}$$

$$[H, \pi(x)] = i(\nabla^2\phi)(x) + im^2\phi(x)$$

$$[H, \phi(x)] = \int d^d x' \frac{1}{2}[\pi(x')^2, \phi(x)]$$

$$= \frac{1}{2}\int d^d x' \; \pi(x')[\pi(x'), \phi(x)] + [\pi(x'), \phi(x)]\pi(x')$$

$$= \frac{1}{2}\int d^d x' \left(2\pi(x')\delta(x-x') \right)$$

$$= -i\pi(x)$$

idk is it consistent :(((

(e) Calculate the commutators $[P, \phi(\boldsymbol{x}, t)]$ and $[P, \pi(\boldsymbol{x}, t)]$. What does the result confirm?

2. Quantisation of Klein–Gordon theory in terms of linear harmonic oscillators

To begin with, consider a single linear harmonic oscillator with Hamiltonian

$$H_0 = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2. \tag{3}$$

To quantise the oscillator, we introduce ladder operators a and $a^\dagger$ obeying $[a, a^\dagger] = 1$, and write

$$x = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right), \quad p = -\mathrm{i}\sqrt{\frac{\omega}{2}}\left(a - a^\dagger\right). \tag{4}$$

(a) Calculate the commutator $[x, p]$.

$$[x,p] = -i\frac{1}{\sqrt{2\omega}}\sqrt{\frac{\omega}{2}}\,[a+a^\dagger, a-a^\dagger]$$

$$= -\frac{i}{2}\left\{ \cancel{[a,a]}^{\,0} + [a,-a^\dagger] + [a^\dagger,a] + \cancel{[a^\dagger,-a^\dagger]}^{\,0} \right\}$$

$$= -\frac{i}{2}\left\{ -2\,[a,a^\dagger] \right\} = i$$

(b) Is the quantisation procedure we have chosen physically different from taking $p \to -\mathrm{i}\partial_x$?

No, because the canonical commutation relations are kept, and hence the transformation is canonical.

(c) Write H_0 in terms of a and $a^\dagger$.

$$x = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right), \quad p = -\mathrm{i}\sqrt{\frac{\omega}{2}}\left(a - a^\dagger\right).$$

$$H_0 = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 = -\frac{1}{2}\frac{\omega}{2}(a-a^\dagger)^2 + \frac{1}{2}\omega^2\frac{1}{2\omega}(a+a^\dagger)^2$$

$$= -\frac{\omega}{4}\left(\cancel{a^2} - aa^\dagger - a^\dagger a + \cancel{a^{\dagger 2}}\right) + \frac{\omega}{4}\left(\cancel{a^2} + aa^\dagger + a^\dagger a + \cancel{a^{\dagger 2}}\right)$$

$$= \frac{\omega}{2}\left(aa^\dagger + a^\dagger a\right)$$

$$= \frac{\omega}{2}\left(2a^\dagger a + 1\right) = \omega\left(a^\dagger a + \frac{1}{2}\right)$$

(d) Write down the eigenstates $|n\rangle$ in terms of the ladder operators and obtain their energies $E_{0,n}$.

$$[a^\dagger a, a] = a^\dagger[a,a] + [a^\dagger,a]\,a = -a$$

$$[a^\dagger a, a^\dagger] = a^\dagger[a,a^\dagger] + [a^\dagger,a^\dagger]\,a = a^\dagger$$

Hence: If $|n\rangle$ is eigenvector of $\hat{n}$ with eigenvalue n

$$\hat{n}a^\dagger|n\rangle = \left(a^\dagger\hat{n} + [\hat{n},a^\dagger]\right)|n\rangle = a^\dagger\hat{n}|n\rangle + a^\dagger|n\rangle = (n+1)a^\dagger|n\rangle$$

$a^\dagger|n\rangle$ is eigenvector with eigenvalue $n+1$. Similarly,

$$\hat{n}a|n\rangle = \left(a\hat{n} + [\hat{n},a]\right)|n\rangle = a\hat{n}|n\rangle - a|n\rangle = (n-1)a|n\rangle$$

$$\tilde{E}_{0,n} = \left(n+\tfrac{1}{2}\right)\omega$$

Normalisation:

$$\| a^\dagger |n\rangle \|^2 = \langle n | a a^\dagger | n \rangle$$
$$= \langle n | a^\dagger a + 1 | n \rangle$$
$$= n+1$$
$$a^\dagger |n\rangle = \frac{1}{\sqrt{n+1}} |n+1\rangle$$
$$\| a |n\rangle \|^2 = \langle n | a^\dagger a | n \rangle = n$$
$$a|n\rangle = \frac{1}{\sqrt{n}} |n-1\rangle$$

We now expand the Klein–Gordon field $\phi(\boldsymbol{x},t)$ from Problem 1 in terms of Fourier modes,

$$\phi(\boldsymbol{x},t) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi_{\boldsymbol{p}}(t), \tag{5}$$

where $\phi^*_{\boldsymbol{p}}(t) = \phi_{-\boldsymbol{p}}(t)$ assures that $\phi(\boldsymbol{x},t)$ is real.

(e) Use the Euler-Lagrange equation for $\phi(\boldsymbol{x},t)$ to show that $\phi_{\boldsymbol{p}}(t)$ obeys the equation of motion of a linear harmonic oscillator for each mode $\boldsymbol{p}$,

$$\left(\partial_t^2 + \omega_{\boldsymbol{p}}^2\right)\phi_{\boldsymbol{p}}(t) = 0, \tag{6}$$

and determine $\omega_{\boldsymbol{p}}$.

$$\partial_\mu \partial^\mu \phi = \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = -m^2 \phi$$
$$\nabla^2 \phi(\vec{x},t) = \nabla^2 \frac{1}{\sqrt{V}} \sum_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \phi_p(t)$$
$$= -\frac{1}{\sqrt{V}} \sum_{\vec{p}} |\vec{p}|^2 e^{i\vec{p}\cdot\vec{x}} \phi_p(t)$$
$$\frac{1}{\cancel{\sqrt{V}}} \sum_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \left(\partial_t^2 \phi_p(t) - |\vec{p}|^2 \phi_p(t)\right) = \frac{-m^2}{\cancel{\sqrt{V}}} \sum_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \phi_p(t)$$
$$\left[\partial_t^2 - \underbrace{\left(|\vec{p}|^2 + m^2\right)}_{\omega_p^2}\right] \phi_p(t) = 0$$

To quantise the theory, we introduce ladder operators $a_{\boldsymbol{p}}$ and $a^\dagger_{\boldsymbol{p}}$ which obey

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}\right] = \delta_{\boldsymbol{p}\boldsymbol{p}'}, \tag{7}$$

and write at $t=0$:

$$\phi(\boldsymbol{x}) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right), \tag{8}$$

$$\pi(\boldsymbol{x}) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} (-i) \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} \left(a_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}}\right). \tag{9}$$

(f) Are the fields $\phi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ hermitian?

(g) Calculate the equal time commutator $\left[\phi(\boldsymbol{x}), \pi(\boldsymbol{x}')\right]$ using (8)and (9) and compare the result to (2).

$$\phi^\dagger = \frac{1}{\sqrt{V}} \sum_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a^\dagger_{\vec{p}} + a_{-\vec{p}}\right)$$

Substitute $\vec{p} \to -\vec{p}$ and use $\omega_{-\vec{p}} = \omega_{\vec{p}} \Rightarrow \phi^\dagger(\vec{x}) = \phi(\vec{x})$ ✓

$$\pi^\dagger = \frac{1}{\sqrt{V}} \sum_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} (i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a^\dagger_{\vec{p}} - a_{-\vec{p}}\right)$$
$$= \frac{1}{\sqrt{V}} \sum_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{-\vec{p}} - a^\dagger_{\vec{p}}\right)$$

Substitute $\vec{p} \to -\vec{p}$, same argument ✓

$$\left[\phi(\vec{x}), \pi(\vec{x}')\right] = \frac{1}{V} \sum_{\vec{p},\vec{p}'} \left[e^{i\vec{p}\cdot\vec{x}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} + a^\dagger_{-\vec{p}}\right), e^{i\vec{p}'\cdot\vec{x}'} (-i) \sqrt{\frac{\omega_{\vec{p}'}}{2}} \left(a_{\vec{p}'} - a^\dagger_{-\vec{p}'}\right)\right]$$
$$= \frac{-i}{2V} \sum_{\vec{p},\vec{p}'} e^{i\vec{p}\cdot\vec{x}} e^{i\vec{p}'\cdot\vec{x}'} \sqrt{\frac{\omega_{\vec{p}'}}{\omega_{\vec{p}}}} \left[a_{\vec{p}} + a^\dagger_{-\vec{p}}, a_{\vec{p}'} - a^\dagger_{-\vec{p}'}\right]$$

Clearly, if p is not equal to -p', the creation and annihilation operators have nothing to do with each other and the commutator vanishes (if p = p', then a creation operator with momentum p is paired with an annihilation operator of momentum -p and it vanishes anyway)

$$= \frac{-i}{2V} \sum_{\vec{p}} [a_{\vec{p}} + a^\dagger_{-\vec{p}}, a_{-\vec{p}} - a^\dagger_{\vec{p}}]$$

$$= \frac{-i}{2V} \sum_{\vec{p}} \left\{ [a_{\vec{p}}, a_{-\vec{p}}] + [a^\dagger_{-\vec{p}}, a_{-\vec{p}}] - [a_{\vec{p}}, a^\dagger_{\vec{p}}] - [a^\dagger_{-\vec{p}}, a^\dagger_{-\vec{p}}] \right\}$$

$$= \frac{i}{2V} \sum_{\vec{p}} (2) = i$$

(h) Obtain the Klein–Gordon Hamiltonian H in terms of a_p and $a_p^\dagger$.

$$H = \int \frac{1}{2} \pi(x)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 \, d^d x$$

$$\int \pi(x)^2 d^d x = \frac{-1}{V} \int d^d x \left(\sum_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \sqrt{\frac{w_{\vec{p}}}{2}} (a_{\vec{p}} - a^\dagger_{-\vec{p}}) \right) \left(\sum_{\vec{p}'} e^{i\vec{p}'\cdot\vec{x}} \sqrt{\frac{w_{\vec{p}'}}{2}} (a_{\vec{p}'} - a^\dagger_{-\vec{p}'}) \right)$$

Again $\vec{p} \neq -\vec{p}'$ terms vanish under the integral

$$\int \pi(x)^2 d^d x = -\frac{1}{2V} \int d^d x \sum_{\vec{p}} w_{\vec{p}} (a_{\vec{p}} - a^\dagger_{-\vec{p}})(a_{-\vec{p}} - a^\dagger_{\vec{p}})$$

$$= -\frac{1}{2} \sum_{\vec{p}} w_{\vec{p}} (a_{\vec{p}} - a^\dagger_{-\vec{p}})(a_{-\vec{p}} - a^\dagger_{\vec{p}})$$

can the sheets be a bit shorter :((