

Theory and Phenomenology of Superconductivity Homework 3

Jun Wei Tan*

Julius-Maximilians-Universität Würzburg

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Problem 1 (Problem 1). Consider two reservoirs for electrons at thermal equilibrium. Refer to them as left L and right R leads which extend in the x direction. Imagine we apply small voltages to the two leads which tunes their chemical potential by eV_L and eV_R .

- Starting from the scattering matrix approach, relate the states entering and leaving each lead at the interface.
- Within the second quantization formalism, write the formula for the electrical current. **Hint:** it will be useful later to write it as an energy integration.
- Taking advantage of the elements of the scattering matrix, identify in the previous formula the transmission function.
- Identify the equilibrium conductance

$$G = \frac{e^2}{h} \text{Tr} \left[t^\dagger(E_F) t(E_F) \right], \quad (1)$$

where $t(E_F)$ is the transmission matrix of our system evaluated at the Fermi energy E_F . What is the meaning of the eigenvalues of the hermitian matrix $t^\dagger(E_F)t(E_F)$?

Proof. (a) We take each lead to have n channels, each with a quantum number \pm denoting whether a certain wave is moving right (+) or left (-). Thus, the outgoing waves for the L (R) lead are those going in the $-$ (+) direction, and the incoming waves are those going in the + (-) direction.

Then, denoting the amplitudes for each as $\vec{a}_{L/R}^\pm$, where \vec{a} is a vector denoting the n channels, the S -matrix relates the amplitudes of the ingoing and outgoing waves

* jun-wei.tan@stud-mail.uni-wuerzburg.de

as

$$\begin{pmatrix} \vec{\mathbf{a}}_L^- \\ \vec{\mathbf{a}}_R^+ \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{r} & \mathbf{t}' \\ \mathbf{t} & \mathbf{r}' \end{pmatrix}}_S \begin{pmatrix} \vec{\mathbf{a}}_L^+ \\ \vec{\mathbf{a}}_R^- \end{pmatrix}.$$

- (b) In general, the electrical current can be determined through the derivative $j^\mu = \frac{\delta S[A]}{\delta A^\mu}$, and leads to the expression for free particles coupled to an electric field

$$\begin{aligned} \vec{\mathbf{J}} &= \vec{\mathbf{J}}^\nabla + \vec{\mathbf{J}}^A \\ \vec{\mathbf{J}}^\nabla &= \frac{\hbar}{2mi} \left[\psi^\dagger \nabla \psi - (\nabla \psi^\dagger) \psi \right] \\ \vec{\mathbf{J}}^A &= -\frac{q}{m} \vec{\mathbf{A}} \psi^\dagger \psi \end{aligned}$$

In the case of a DC current, we usually have an electric field but no magnetic field. Hence, we can set $\vec{\mathbf{A}} = 0$.

□

Problem 2 (Problem 2). Consider a wide conductor along x , e.g. width in y direction is large W while the height in z direction is small. The length in the x direction is large L .

Given information: The density of states of a 2D spin degenerate system is

$$\mathcal{D}_0 = \frac{m}{\pi \hbar^2}.$$

The transmission through a wire with length L is

$$T = \frac{L_0}{L + L_0},$$

where L_0 is the mean free path. The conductivity is

$$\sigma = e^2 \mathcal{D}_0 D,$$

where

$$D = \frac{v_F L_0}{\pi}$$

is the diffusion coefficient and v_F is the Fermi velocity.

Using this information, relate the previous results for conductance G with Ohm's law. I.e., does the relationship between G and conductivity σ correspond to Ohm's law?