Universität Würzburg
Summer 2025

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4. Open quantum systems

Due date: 18.06.2025 10:00

Throughout this exercise sheet, we adopt the convention $\boxed{\hbar = 1}$.

Exercise 1 *Phase damping when* $[H_S, H_{SB}] = 0$ **3 P.**

Consider a two-level quantum system with the following Hamiltonians:

$$H_S = -\frac{1}{2}\omega_z \sigma_z, \quad H_{SB} = g\, \sigma_z \otimes B.$$

a) Identify the system operator(s) appearing in the interaction Hamiltonian and express them in the basis of H_S.

b) Using the RWA-LE (Rotating Wave Approximation in the weak coupling limit), write down the resulting master equation for the reduced density matrix $\rho(t)$ of the system.

Note: Recall that the RWA-LE in the Schrödinger picture is:

$$\frac{d\rho}{dt} = -i\left[H_S + H_{\mathrm{LS}}, \rho\right] + g^2 \sum_{\alpha\beta} \sum_{\omega} \gamma_{\alpha\beta}(\omega) \left[A_\beta(\omega)\rho A_\alpha^\dagger(\omega) - \frac{1}{2}\left\{ A_\alpha^\dagger(\omega) A_\beta(\omega), \rho \right\} \right] \tag{1}$$

Remember from the lecture that expanding

$$H_S = \sum_a \varepsilon_a |\varepsilon_a\rangle\langle\varepsilon_a|,$$

the time-evolved operator becomes

$$A_\alpha(t) = U_S^\dagger(t) A_\alpha U_S(t) = \sum_{a,b} e^{-i(\varepsilon_b - \varepsilon_a)t} \left|\varepsilon_a\right\rangle \left\langle\varepsilon_a\right| A_\alpha \left|\varepsilon_b\right\rangle \left\langle\varepsilon_b\right| = \sum_\omega A_\alpha(\omega) e^{-i\omega t},$$

with Bohr frequencies $\omega = \varepsilon_b - \varepsilon_a$, and

$$A_\alpha(\omega) = \sum_{\varepsilon_b - \varepsilon_a = \omega} \langle\varepsilon_a|A_\alpha|\varepsilon_b\rangle|\varepsilon_a\rangle\langle\varepsilon_b|, \quad A_\alpha(t) = A_\alpha^\dagger(t), \quad A_\alpha(\omega) = A_\alpha^\dagger(-\omega). \tag{2}$$

c) Derive the differential equations governing the matrix elements $\rho_{00}(t)$, $\rho_{11}(t)$, and $\rho_{01}(t)$. Solve them explicitly.

d) Determine the characteristic decoherence timescale and express it in terms of the coupling strength g and the bath correlation function $\gamma_{\alpha\beta}(\omega)$.

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$\sigma_z \propto H_s$???

b) Using the RWA-LE (Rotating Wave Approximation in the weak coupling limit), write down the resulting master equation for the reduced density matrix $\rho(t)$ of the system.

Note: Recall that the RWA-LE in the Schrödinger picture is:

$$\frac{d\rho}{dt} = -i\left[H_S + H_{LS}, \rho\right] + g^2 \sum_{\alpha\beta}\sum_{\omega} \gamma_{\alpha\beta}(\omega)\left[A_\beta(\omega)\rho A_\alpha^\dagger(\omega) - \frac{1}{2}\left\{A_\alpha^\dagger(\omega)A_\beta(\omega), \rho\right\}\right] \tag{1}$$

Remember from the lecture that expanding

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with Bohr frequencies $\omega = \varepsilon_b - \varepsilon_a$, and

$$A_\alpha(\omega) = \sum_{\varepsilon_b - \varepsilon_a = \omega} \langle\varepsilon_a|A_\alpha|\varepsilon_b\rangle |\varepsilon_a\rangle\langle\varepsilon_b|, \quad A_\alpha(t) = A_\alpha^\dagger(t), \quad A_\alpha(\omega) = A_\alpha^\dagger(-\omega). \tag{2}$$

$A = \sigma_z$

$\varepsilon_\pm = \mp \frac{1}{2}\omega_z$

$A_\alpha(\omega_z) = \langle\varepsilon_-|\sigma_z|\varepsilon_+\rangle |\varepsilon_+\rangle\langle\varepsilon_-| = 0$

Only $A(0) = |\varepsilon_+\rangle\langle\varepsilon_+| - |\varepsilon_-\rangle\langle\varepsilon_-| = \sigma_z$

$$H_{LS} \equiv g^2 \sum_\omega \sum_{\alpha\beta} S_{\alpha\beta}(\omega) A_\alpha^\dagger(\omega) A_\beta(\omega)\,. = g^2 S(0)$$

$$\frac{d\rho}{dt} = -i\left[-\frac{1}{2}\omega_z\sigma_z + g^2 S(0), \rho\right] + g^2\gamma(0)\left[\sigma_z\rho\sigma_z - \rho\right]$$

$$= \frac{i\omega_z}{2}\left[\sigma_z, \rho\right] + g^2\gamma(0)\left[\sigma_z\rho\sigma_z - \rho\right]$$

$$= \begin{pmatrix} 0 & -2\gamma g^2 \rho_{12} + i\rho_{12}\omega_z \\ -2\gamma g^2 \rho_{21} - i\rho_{21}\omega_z & 0 \end{pmatrix}$$

$$\frac{d\rho_{12}}{dt} = \left(-2\gamma g^2 + i\omega_z\right)\rho_{12}$$

$$\rho_{12} = e^{i\omega_z t} e^{-2\gamma g^2 t} \rho_{12}(0)$$

$$\rho_{21} = e^{-i\omega_z t} e^{-2\gamma g^2 t} \rho_{21}(0)$$

Decoherence time $\tau = \frac{1}{2\gamma g^2}$

Exercise 2 *Phase damping when* $[H_S, H_{SB}] \neq 0$ **5 P.**

Consider a two-level quantum system with the following Hamiltonians:

$$H_S = -\frac{1}{2}\omega_x\sigma_x, \quad H_{SB} = g\,\sigma_z \otimes B.$$

a) Find the eigenstates and eigenvalues of the system Hamiltonian H_S.

b) Express the system operator σ_z in the energy eigenbasis of H_S. Using this, identify the Lindblad operators $A_\alpha(\omega)$ (see Eq.2) in the RWA-LE framework.

c) Write down the master equation using the RWA-LE framework (see Eq.1), including the Lamb shift Hamiltonian. Solve it explicitly for the components of ρ.

   Hint: It is most convenient to work in the energy eigenbasis, i.e., the basis that diagonalizes H_S, namely the $\{|+\rangle, |-\rangle\}$ basis.

d) Derive the relaxation timescale for the diagonal elements of the density matrix, $\rho_{--}(t)$ and $\rho_{++}(t)$, and compare it to the decoherence timescale of the off-diagonal element $\rho_{+-}(t)$.

Exercise 3 *Dark states in a three-level atom* **3 P.**

Consider a three-level quantum system consisting of two ground states $|1\rangle$ and $|3\rangle$, and one excited state $|2\rangle$. The system interacts with a classical field in the rotating wave approximation, with equal Rabi frequencies driving the transitions $|1\rangle \leftrightarrow |2\rangle$ and $|3\rangle \leftrightarrow |2\rangle$. The system Hamiltonian is given by:

$$H = \frac{\Omega}{2}\left(|1\rangle\langle 2| + |3\rangle\langle 2| + \text{h.c.}\right).$$

Spontaneous emission from $|2\rangle$ to both ground states occurs with equal decay rate γ, described by the jump operators:

$$L_1 = |1\rangle\langle 2|, \quad L_2 = |3\rangle\langle 2|.$$

a) Show that the antisymmetric state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle)$$

   is an eigenstate of the Hamiltonian with eigenvalue zero.

b) Verify that this state is annihilated by both jump operators: $L_1|\psi\rangle = 0$ and $L_2|\psi\rangle = 0$. What does this imply about its evolution under the Lindblad master equation?

c) Argue why there are no other pure stationary states, in particular why the symmetric state $|\psi_s\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)$ does not qualify as a stationary state of the full master equation.

Exercise 2 *Phase damping when* $[H_S, H_{SB}] \neq 0$ 5 P.

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a) Find the eigenstates and eigenvalues of the system Hamiltonian H_S.

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c) Write down the master equation using the RWA-LE framework (see Eq.1), including the Lamb shift Hamiltonian. Solve it explicitly for the components of ρ.

Hint: It is most convenient to work in the energy eigenbasis, i.e., the basis that diagonalizes H_S, namely the $\{|+\rangle, |-\rangle\}$ basis.

d) Derive the relaxation timescale for the diagonal elements of the density matrix, $\rho_{--}(t)$ and $\rho_{++}(t)$, and compare it to the decoherence timescale of the off-diagonal element $\rho_{+-}(t)$.

a) $|\varepsilon_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\quad \varepsilon_- = -\frac{1}{2}\omega_x$

$|\varepsilon_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ $\quad \varepsilon_+ = \frac{1}{2}\omega_x$

b) $\sigma_z = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ in eigenbasis of σ_x

$= |\varepsilon_-\rangle\langle\varepsilon_+| + |\varepsilon_+\rangle\langle\varepsilon_-| =: A$

in $\varepsilon_+, \varepsilon_-$ basis

$$A_\alpha(\omega) = \sum_{\varepsilon_b - \varepsilon_a = \omega} \langle\varepsilon_a|A_\alpha|\varepsilon_b\rangle |\varepsilon_a\rangle\langle\varepsilon_b|,$$

$A(\omega_x) = |\varepsilon_-\rangle\langle\varepsilon_+| = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$

$A(-\omega_x) = |\varepsilon_+\rangle\langle\varepsilon_-| = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$

$$\frac{d\rho}{dt} = -i\,[H_S + H_{\mathrm{LS}}, \rho] + g^2 \sum_{\alpha\beta}\sum_{\omega} \gamma_{\alpha\beta}(\omega)\left[A_\beta(\omega)\rho A_\alpha^\dagger(\omega) - \frac{1}{2}\left\{A_\alpha^\dagger(\omega)A_\beta(\omega), \rho\right\}\right]$$

$$H_{\mathrm{LS}} \equiv g^2 \sum_\omega \sum_{\alpha\beta} S_{\alpha\beta}(\omega) A_\alpha^\dagger(\omega) A_\beta(\omega) = g^2 \sum_{\omega = \pm\omega_x} S(\omega) A^\dagger(\omega) A(\omega)$$

$$= g^2\left[S(\omega_x)\,|\varepsilon_+\rangle\langle\varepsilon_+| + S(-\omega_x)\,|\varepsilon_-\rangle\langle\varepsilon_-|\right]$$

$$\frac{d\rho}{dt} = -i\begin{pmatrix} 0 & (\varepsilon_+ + S(\omega_x))\rho_{12} - (\varepsilon_- + S(-\omega_x))\rho_{12} \\ -(\varepsilon_+ + S(\omega_x))\rho_{21} + (\varepsilon_- + S(-\omega_x))\rho_{21} & 0 \end{pmatrix}$$

$$+ g^2\gamma(\omega_x)\begin{pmatrix} -\rho_{11} & -\frac{\rho_{12}}{2} \\ -\frac{\rho_{21}}{2} & \rho_{11}\end{pmatrix} + g^2\gamma(-\omega_x)\begin{pmatrix} \rho_{22} & -\frac{\rho_{12}}{2} \\ -\frac{\rho_{21}}{2} & -\rho_{22}\end{pmatrix}$$

$$\frac{d\rho_{11}}{dt} = -g^2 \gamma(\omega_x)\rho_{11} + g^2\gamma(-\omega_x)\rho_{22}$$

$$\frac{d\rho_{22}}{dt} = g^2\gamma(\omega_x)\rho_{11} - g^2\gamma(-\omega_x)\rho_{22}$$

$$\frac{d\rho_{12}}{dt} = \left[-i\left(\varepsilon_+ - \varepsilon_- + S(\omega_x) - S(-\omega_x)\right) - \frac{g^2}{2}\left[\gamma(\omega_x) + \gamma(-\omega_x)\right]\right]\rho_{12}$$

$$\frac{d\rho_{21}}{dt} = \left[i\left(\varepsilon_+ - \varepsilon_- + S(\omega_x) - S(-\omega_x)\right) - \frac{g^2}{2}\left[\gamma(\omega_x) + \gamma(-\omega_x)\right]\right]\rho_{12}$$

Decoherence time $\tau_{decoherence} = \dfrac{2}{g^2(\gamma(\omega_x) + \gamma(-\omega_x))}$

Relaxation time $T_1 = \dfrac{1}{g^2(\gamma(\omega_x) + \gamma(-\omega_x))}$

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c) Argue why there are no other pure stationary states, in particular why the symmetric state $|\psi_s\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)$ does not qualify as a stationary state of the full master equation.

a) $H = \frac{\Omega}{2}\left(|1\rangle\langle 2| + |3\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 3|\right)$

$H|\psi\rangle = 0$ clear

b) Clear

$$\dot{\rho}(t) = \mathcal{L}\rho(t) = -i\underbrace{[H, \rho(t)]}_{\to 0} + \underbrace{\sum_{\alpha \geq 1}\left(L_\alpha \rho(t) L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho(t)\}\right)}_{\to 0}$$

stationary !!!

c) $H|\psi_s\rangle \neq 0$

$L_1|\psi\rangle, L_2|\psi\rangle = 0 \Rightarrow |\psi\rangle \in \text{span}\{|1\rangle, |3\rangle\}$