Algebra und Dynamik von Quantensystemen Blatt Nr. 1

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Problem 1 (CCRs vs. Boundedness)**.** Consider two bounded operators A and B on a Hilbert space $\mathcal{H}$, i.e.

$$\exists a \in \mathbb{R} : \forall \psi \in \mathcal{H} : \|A\psi\| \leq a\|\psi\|, \tag{1a}$$

$$\exists b \in \mathbb{R} : \forall \psi \in \mathcal{H} : \|B\psi\| \leq b\|\psi\|. \tag{1b}$$

Show that the canonical commutation relations

$$[A, B] = AB - BA = i \tag{2}$$

are inconsistent with the assumption of boundedness for the operators A and B.

NB: It is not necessary to find an original proof. It suffices to find, understand and present a proof from the literature.

Proof. The proof comes from rudin: Let A be a normed algebra, and $x, y \in A$. We assume that

$$xy - yx = 1$$

The first step is to prove that $xy^n - y^n x = ny^{n-1}$ for all $n \in \mathbb{N}$. This is true for $n = 1$. Then, by induction, if

$$xy^n - y^n x = ny^{n-1},$$

it follows that

$$\begin{aligned} xy^{n+1} - y^{n+1}x &= (xy^n - y^n x)y + y^n(xy - yx) \\ &= ny^n + y^n \\ &= (n+1)y^n \end{aligned}$$

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Additionally, we note that $y^n \neq 0$ for all n. Otherwise, we would choose the minimum n, and get

$$0 = xy^n - y^n x = ny^{n-1},$$

a contradiction to the minimality of n. Now, we have

$$n\|y^{n-1}\| = \|xy^n - y^n x\| \leq 2\|x\|\|y\|\|y\|^{n-1}$$

and

$$2\|x\|\|y\| \geq n,$$

a contradiction. □

Problem 2 (Classical Dynamics on the 2-Torus)**.** Consider a classical dynamical system with the 2-Torus $T^2 = S^1 \times S^1$ as phase space Γ (this is not a cotangent bundle, but it has the technical advantage of being compact).

Using standard coordinates $(\theta_1, \theta_2) \in [0, 2\pi)^2$, a consistent Poisson bracket is given by

$$\{f, g\} = \frac{\partial f}{\partial \theta_1}\frac{\partial g}{\partial \theta_2} - \frac{\partial f}{\partial \theta_2}\frac{\partial g}{\partial \theta_1}. \tag{3}$$

Assume that the Hamiltonian is

$$\begin{aligned} H &: \Gamma \to \mathbb{R} \\ (\theta_1, \theta_2) &\mapsto H(\theta_1, \theta_2) = c\cos\theta_1. \end{aligned} \tag{4}$$

In order to be well defined globally, the Hamiltonian must be periodic in θ_1 and θ_2. This is the simplest choice.

1. Derive the equations of motion.

2. Determine the flow Φ of a phase space point $(\theta_1, \theta_2) \in \Gamma$.

3. Determine the time evolution of the state ω, where

$$\omega(f) = \int_\Gamma \mathrm{d}^2\theta\, f(\theta)\,\omega(\theta) \tag{5}$$

with

$$\begin{aligned} \omega &: \Gamma \to \mathbb{R} \\ (\theta_1, \theta_2) &\mapsto \omega(\theta_1, \theta_2) = \frac{1}{\pi^2}\sin^2\theta_1 \sin^2\theta_2. \end{aligned} \tag{6}$$

HAMILTONIAN DYNAMICS

The Hamiltonian is a function on the phase space

$$H = H(q_1, \ldots, q_n, p_1, \ldots, p_n).$$

The flow solves the canonical equations of motion

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}$$
$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}$$

We can also define the Poisson bracket

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

It is immediately clear that the Poisson bracket is antisymmetric; additionally, we have the canonical commutation relations

$$\{q, p\} = 1, \{q, q\} = \{p, p\} = 0$$

Clearly, because $\frac{\partial q}{\partial p} = \frac{\partial p}{\partial q} = 0$, we can rewrite the canonical commutation relations as

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \{q_i, H\}$$
$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \{p_i, H\}$$

STATES AS FUNCTIONALS

Proof. 1. The equations of motion are

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \frac{\partial H}{\partial \theta_2} = 0$$

and

$$\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = -\frac{\partial H}{\partial \theta_1} = c \sin \theta_1$$

2. Clearly, the first equation tells us that θ_1 is a constant; since the right hand side of

equation 2 is now a constant, θ_2 varies linearly with time.

$$\Phi^t \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 + ct\cos\theta_1 \end{pmatrix}$$

3. The flow is defined by

$$\omega_t(f) = \omega(\Phi_t(f)) = \omega(f \circ \Phi_t).$$

On the right hand side, we desire

$$\begin{aligned} \omega_t(f) &= \frac{1}{\pi^2}\int_\Gamma \mathrm{d}^2\theta\, f(\theta_1, \theta_2 + ct\cos\theta_1)\omega(\theta_1, \theta_2) \\ &= \frac{1}{\pi^2}\int_\Gamma \mathrm{d}^2\theta\, f(\theta_1, \theta_2)\omega(\theta_1, \theta_2 - ct\cos\theta_1). \end{aligned}$$ □

Problem 3 (Classical Dynamics on the 2-Sphere)**.** Consider a classical dynamical system with the 2-Sphere S^2 as phase space Γ (this is again not a cotangent bundle, but the technical advantage of being compact and is highly symmetric).

Using standard spherical coordinates $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$, a consistent Poisson bracket is given by

$$\{f, g\} = \frac{1}{\sin\theta}\left(\frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \phi} - \frac{\partial f}{\partial \phi}\frac{\partial g}{\partial \theta}\right). \tag{7}$$

Assume that the Hamiltonian is

$$\begin{aligned} &H : \Gamma \to \mathbb{R} \\ &(\phi, \theta) \mapsto H(\phi, \theta) = c\cos\theta. \end{aligned} \tag{8}$$

In order to be well defined globally, the Hamiltonian must be periodic in θ and ϕ. This is one of the simplest choices.

1. Show that the Poisson bracket satisfies all requirements.

2. Determine the flow Φ of a phase space point $(\theta, \phi) \in \Gamma$.

3. Determine the time evolution of the state ω, where

$$\omega(f) = \int_\Gamma \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, f(\theta, \phi)\, \omega(\theta, \phi) \tag{9}$$

with

$$\begin{aligned} &\omega : \Gamma \to \mathbb{R} \\ &(\theta, \phi) \mapsto \omega(\theta, \phi) = \frac{2}{\pi^2}\sin\theta\cos^2\phi. \end{aligned} \tag{10}$$

Proof. 1. It is clearly antisymmetric

2. We have the equations of motion

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \{\theta, H\} = 0$$
$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \{\phi, H\} = -\frac{1}{\sin\theta}(-c\sin\theta) = c$$

and solutions

$$\theta(t) = \theta(0)$$
$$\phi(t) = \phi(0) + ct$$

3. Again

$$\begin{aligned}\omega_t(f) &= \omega(f \circ \Phi_t) \\ &= \frac{2}{\pi^2}\int_\Gamma \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, f(\theta, \phi + ct)\, \omega(\theta, \phi) \\ &= \frac{2}{\pi^2}\int_\Gamma \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, f(\theta, \phi)\, \omega(\theta, \phi - ct)\end{aligned}$$

□