1. Emergent Dirac physics

Based on the graphene exercise of the previous exercise sheet:

(a) Obtain the graphene dispersion around the K and K' points by linearizing the eigenvalues ε<sub>+</sub>(k).

**Hint:** Define  $\mathbf{k} = K + \mathbf{q}$ , with  $|\mathbf{q}| << |K|$ , and perform a linear expansion in  $\mathbf{q}$ 

$$\frac{\vec{r}_{k} = \frac{2n}{3a} (1/3)}{f(\vec{k}) = -\frac{1}{4} (e^{-\frac{1}{4}k_{x}a} + 2e^{\frac{1}{4}k_{x}a} (1/3))}$$

$$f(\vec{k}) = -\frac{1}{4} (e^{-\frac{1}{4}k_{x}a} + 2e^{\frac{1}{4}k_{x}a} (1/3))$$

$$\frac{1}{4} f(\vec{r}_{k} + \vec{q}_{k}) = (-1)^{1/3} e^{-\frac{1}{10}q_{x}} (-1+2e^{\frac{3}{10}q_{x}} \sin(\frac{\pi}{4} - \frac{1}{2}a q_{y}))$$

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$$\frac{1}{4} f(\vec{r}_{k} + \vec{q}_{k}) = -\frac{3}{4} e^{-\frac{1}{4}q_{x}} \left[ -q_{x} - \frac{1}{4}q_{y} \right]$$

$$\frac{1}{4} f(\vec{r}_{k} + \vec{q}_{k}) = \frac{3}{2} a (-1)^{1/3} q_{y} + \frac{3}{2} a (-1)^{5/4} q_{x}$$

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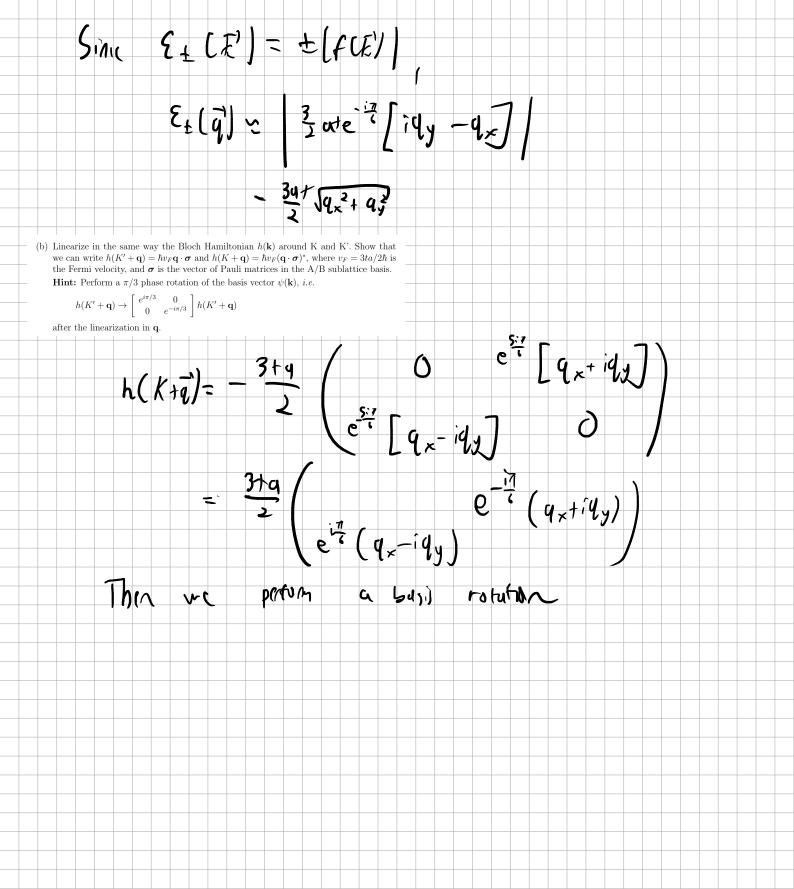
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## Points substitution

A Bloch state of a one-band Hamiltonian with a periodic potential  $U(\mathbf{r} + \mathbf{R}) = U(\mathbf{r})$  (where  $\mathbf{R}$  is a linear combination of the lattice basis vectors with integer coefficients) is given by

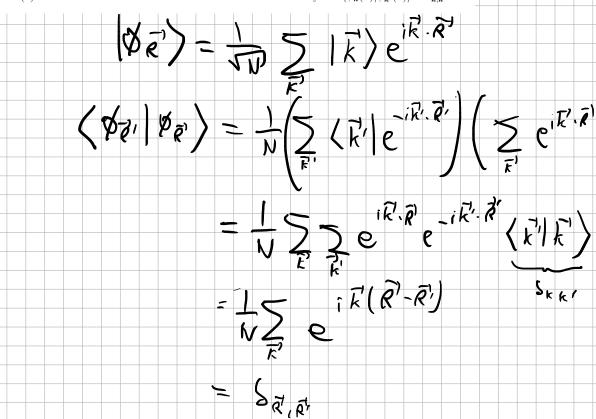
$$\psi_{k\sigma}(\mathbf{r}) = u_{k\sigma}(\mathbf{r})e^{i\mathbf{k}\mathbf{r}}\chi_{\sigma},\tag{1}$$

where  $u_{\mathbf{k}\sigma}(\mathbf{r})$  has the same periodicity as the crystal. For the following calculation we will neglect the spin index  $\sigma$  and its respective wave function  $\chi_{\sigma}$ . Then the Wannier functions are defined by

$$\phi_{\mathbf{R}}(\mathbf{r}) = \frac{1}{\sqrt{N}} \sum_{k} \psi_{\mathbf{k}}(\mathbf{r}) e^{i\mathbf{k}\mathbf{R}}, \tag{2}$$

where N is the number of unit cells.

(a) Show that two Wannier states are orthonormal. You may use  $\langle \psi_{k}(r) | \psi_{k'}(r) \rangle = \delta_{k,k'}$ .



We now introduce a magnetic field with vector potential  $\boldsymbol{A}(\boldsymbol{r})$ , leading to the modified Hamiltonian  $H = \frac{1}{2m} \left(\boldsymbol{p} - \frac{e}{c} \boldsymbol{A}(\boldsymbol{r})\right)^2 + U(\boldsymbol{r})$ . Ultimately, we are interested in the hopping amplitude between the lattice sites

$$t_{lm} = \int d\mathbf{r} \phi_{\mathbf{R}_l}(\mathbf{r})^* H(\mathbf{r}) \phi_{\mathbf{R}_m}(\mathbf{r}). \tag{3}$$

(b) Show that the vector potential  $\mathbf{A}(\mathbf{r})$  does only lead to a phase factor  $\exp\left(i\frac{e}{\hbar c}\int_{\mathbf{R}}^{\mathbf{r}}d\mathbf{r}'\mathbf{A}(\mathbf{r}')\right)$  in the Wannier states, where the integral is understood to be a line integral from  $\mathbf{R}$  to  $\mathbf{r}$ . Why do we choose the lower boundary to be  $\mathbf{R}$ ? Hint: Consider

$$\phi_{\mathbf{R}}(\mathbf{r}) = \exp\left(i\frac{e}{\hbar c} \int_{\mathbf{R}}^{\mathbf{r}} d\mathbf{r}' \mathbf{A}(\mathbf{r}')\right) \tilde{\phi}_{\mathbf{R}}(\mathbf{r})$$
(4)

and show that  $\exp\left(i\frac{e}{\hbar c}\int_{\pmb{R}}^{\pmb{r}}d\pmb{r}'\pmb{A}(\pmb{r}')\right)H(\pmb{A}\to 0)\tilde{\phi}_{\pmb{R}}(\pmb{r})=H(\pmb{A}\neq 0)\phi_{\pmb{R}}(\pmb{r}).$