Differential Geometry WS 23/24

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Organization

Electronic communication only via WueCampus and by email.
Use your XYZ@stud-mail.uni-wuerzburg.de address.

- Announcements
- Changes in dates
- File up- and download
- Please check regularly...

Organization

Lectures

- Start 19.10.23
- Monday 14:15 – 15:45, Location: S1.101
  Thursday 8:30 –10:00, Location: SE30
- pdf slides offered for download (wuecampus) in advance.

Exercises

- Thursday 14:00 – 15:30, SE30
  Start 02.11.23
- Exercise sheet (pdf) for download, sketch of solutions presented by myself, possibly additional hints for self-study.
- Neither corrections nor proofreading provided.

Organization

Previous knowledge and course prerequisites:

- **Indispensable**:
  - Linear Algebra I+II
  - Calculus I+II, incl. some basic Topology
- **Recommendable**:
  - ODE
  - Calculus III+IV (Lebesgue theory, Geometric Analysis)
  - Complex Analysis (Function Theory)
  - Basic Functional Analysis
  - Differential Geometry (bachelor level)
- Basic knowledge in physics does not harm ;-)

Oral Exam (ca. 20-25 min.)

- No participation without registration (YOU have to do it).
- February 2024 or later (after the end of lecturing, end of semester).
- Venue and date by appointment via email.
- 2nd attempt ("Nachklausur") will be offered in SS 24 (Registration!).

Contents

1. Warming up: reminder, recall and repetition
   - Topological vector spaces
   - Differentiablity in $\mathbb{R}^n$
   - Implicit function theorem
   - Rank theorem
2. Topological and differentiable manifolds
   - Charts, atlasses, smooth structures
   - Smooth maps and diffeomorphisms
   - Regular submanifolds
3. The tangent structure
   - Tangent space, tangent map
   - Critical points and values
   - Tangent and cotangent bundles
   - Vector fields and 1-forms
4. Immersions and submersions
5. Lie groups, fiber bundles, tensors

Recommended Literature for Bachelor Topics

Ordinary differential equations

- Hirsch & Smale & Devaney, Differential Equations, Dynamical Systems & An Introduction to Chaos, 2nd ed., Elsevier, 2004.

Calculus (incl. Lebesgue theory and geometric analysis)

- Amann & Escher, Analysis I-III, Birkhäuser, English and German versions exist.

Complex analysis

- Ahlfors, Complex Analysis, 3rd ed., McGraw, 1979.
- Simon, Basic Complex Analysis, American Math. Soc., 2015.

Functional Analysis

- Werner, Funktionalanalysis, Springer, 8. Aufl., 2018, in German.
- Bogachev and Smolyanov, Topological Vector Spaces and Their Applications, Springer, 2017.

Literature

- Install on your computer CISCO VPN,
  for students for free, via RZ for all computer platforms available.
- Go via VPN to https://link.springer.com
- Download books on differential geometry and other topics offered by Springer Verlag.
- Additional recommendations:
  - W.M. Boothby, An introduction to differentiable manifolds and Riemannian geometry, Academic Press, 2nd ed., 1986.
  - J. Gallier and J. Quaintance, Differential geometry and Lie groups, A computational perspective, Springer, 2020.
  - J.M. Lee, Introduction to smooth manifolds, Springer, 2nd ed., 2012.
  - St. Waldmann's lecture notes *Differential Geometry* (WueCampus).
- All my "slides" from lectures will be provided on WueCampus course for download.

Time for questions...

Topological Vector Spaces

Definition

A topological vector space (TVS) is a vector space V with a Hausdorff topology s.t. the addition and scalar multiplication operations are (jointly) continuous.

The set of all neighborhoods that contain a point x in a topological vector space V is denoted $\mathcal{N}(x)$.
The families $\mathcal{N}(x)$ for various x satisfy the following neighborhood axioms:

1. Every set that contains a set from $\mathcal{N}(x)$ is also a set from $\mathcal{N}(x)$.
2. If $\{N_i\}$ is a finite family of sets from $\mathcal{N}(x)$ then $\cap_i N_i \in \mathcal{N}(x)$.
3. Every $N \in \mathcal{N}(x)$ contains x.
4. If $V \in \mathcal{N}(x)$ then there exists $W \in \mathcal{N}(x)$ s.t. for all $y \in W, V \in \mathcal{N}(y)$.

Remark

Conversely, let X be some set.

Assume for each $x \in X$ there is a family $\mathcal{N}(x)$ of subsets of X that satisfy the above neighborhood axioms.

Then there is a uniquely determined topology on X for which the families $\mathcal{N}(x)$ are the neighborhoods of the points x.

For this a subset $U \subset X$ is open iff for each $x \in U$ we have $U \in \mathcal{N}(x)$.

Lemma

A sequence $\{x_n\}$ in a TVS is a Cauchy sequence iff for every neighborhood U of 0 there is a number N_U s.t. $x_l - x_k \in U$ for all $k, l \geq N_U$.

Example

A relatively nice situation is when V has a norm that induces the topology.

Recall that a norm is a function $\|\cdot\|\colon V \to \mathbb{R}$ s.t. for all $v, w \in V$ we have

1) $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$,
2) $\|v + w\| \leq \|v\| + \|w\|$,
3) $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{R}$.

In this case we have a metric on V given by $\operatorname{dist}(v, w) := \|v - w\|$.

A seminorm is a function $\|\cdot\|\colon V \to \mathbb{R}$ s.t. 2) and 3) hold but instead of 1) we require only that $\|v\| \geq 0$.

Using a norm, we may define a metric $d(x, y) := \|x - y\|$.

A normed space V is a vector space together with a norm.

It becomes a TVS with the metric topology given by a norm.

Definition

A linear map $l\colon V \to W$ between normed spaces is called bounded iff there is a constant C s.t. for all $v \in V$ we have $\|lv\|_W \leq C\|v\|_V$.

If l is bounded then the smallest such constant C is

$$\|l\| := \sup \frac{\|lv\|_W}{\|v\|_V} = \sup\{\|lv\|_W \colon \|v\|_V \leq 1\}.$$

Definition

The set of all bounded linear maps $V \to W$ is denoted $\mathcal{B}(V, W)$.

The vector space $\mathcal{B}(V, W)$ is itself a normed space with the norm given as above.

Lemma

A linear map is bounded iff it is continuous.

Topology of Finite Dimensional TVS

The usual topology on $\mathbb{R}^n$ is given by the norm $\|x\| = \sqrt{\langle x, x\rangle}$.

Many other norms give the same topology.

In fact we have the following

Theorem

Given $\mathbb{R}^n$ the usual topology.

Then if V is any finite dimensional (Hausdorff) topological vector space of dimension n, there is a linear homeomorphism $\mathbb{R}^n \to V$.

Thus there is a unique topology on V that gives it the structure of a (Hausdorff) topological vector space.

Definition

A locally convex topological vector space V is a TVS s.t. its topology is generated by a family of seminorms $\{\|\cdot\|_\alpha\}_\alpha$.

This means that we give V the weakest topology such that all $\|\cdot\|_\alpha$ are continuous.

Since we have taken a TVS to be Hausdorff we require that the family of seminorms is sufficient in the sense that for each $x \in V$ we have $\cap\{x\colon \|x\|_\alpha = 0\} = \emptyset$.

A locally convex topological vector space is sometimes called a locally convex space (LCS).

Example

Let Ω be an open set in $\mathbb{R}^n$ (actually, or any manifold).

Let $C(\Omega)$ be the set of continuous real valued functions on Ω.

For each $x \in \Omega$ define a seminorm ρ_x on $C(\Omega)$ by $\rho_x(f) := |f(x)|$.

This family of seminorms makes $C(\Omega)$ a topological vector space.

In this topology, convergence is pointwise convergence.

Also, $C(\Omega)$ is not complete with this TVS structure.

Definition

A complete LCS (i.e., every Cauchy sequence converges) is called Fréchet space.

Definition

A complete normed space is called a Banach space.

Example

The space $C_b(\Omega)$ of bounded continuous functions on Ω is a Banach space with norm given by $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$.

Example

Consider a topological vector space V with a closed subspace S.

Then we can form the quotient V/S.

The quotient can be turned into a normed space by introducing as norm

$$\|[x]\|_{V/S} := \inf_{v \in [x]} \|v\|.$$

If S is not closed then this only defines a seminorm.

Theorem

If V is a Banach space and S a closed (linear) subspace then V/S is a Banach space with the norm $\|\cdot\|_{V/S}$ defined above.

Proof:

Let $\{x_n\}$ be a sequence in V s.t. $\{[x_n]\}$ is a Cauchy sequence in V/S.

Choose a subsequence s.t. $\|[x_n] - [x_{n+1}]\| \leq 1/2^n$ for $n = 1, 2, \ldots$.

Setting s_1 equal to zero we find $s_2 \in S$ s.t. $\|x_1 - (x_2 + s_2)\| \leq 1/2^2$ and continuing inductively define a sequence $\{s_i\}$ s.t. $\{x_n + s_n\}$ is a Cauchy sequence in V.

Thus there is an element $y \in V$ with $x_n + s_n \to y$.

But since the quotient map is norm decreasing, the sequence $\{[x_n + s_n]\} = \{[x_n]\}$ must also converge, i.e., $[x_n] \to [y]$. $\square$

Remark

It is also true that if S is closed and V/S is a Banach space then so is V.

Hilbert Spaces

Definition

A pre-Hilbert space $\mathcal{H}$ is a complex vector space with a Hermitian inner product $\langle \cdot, \cdot \rangle$.

A Hermitian inner product is a bilinear form with the following properties:

1) $\langle v, w \rangle = \overline{\langle w, v \rangle}$,
2) $\langle v, \alpha w_1 + \beta w_2 \rangle = \alpha \langle v, w_1 \rangle + \beta \langle v, w_2 \rangle$,
3) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ only if $v = 0$.

The inner product gives rise to an associate norm $\|v\| := \sqrt{\langle v, v \rangle}$ and so every pre-Hilbert space is a normed space.

If a pre-Hilbert space is complete w.r.t. this norm we call it a Hilbert space.

One of the most fundamental properties of a Hilbert space is the projection property:

Theorem

If K is a convex, closed subset of a Hilbert space $\mathcal{H}$, then for any given $x \in \mathcal{H}$ there is a unique element $P_K(x) \in \mathcal{H}$ which minimizes the distance $\operatorname{dist}(x, y) = \|x - y\|$ over $y \in K$.

That is

$$\|x - P_K(x)\| = \inf_{y \in K} \|x - y\|.$$

If K is a closed linear subspace then the map $x \mapsto P_K(x)$ is a bounded linear operator with the projection property

$$P_K^2 := P_K \circ P_K = P_K.$$

Definition

For any subset $S \in \mathcal{H}$ we have the orthogonal complement $S^\perp$ defined by

$$S^\perp := \{x \in \mathcal{H} \mid \langle x, s \rangle = 0 \text{ for all } s \in S\}.$$

$S^\perp$ is easily seen to be a linear subspace of $\mathcal{H}$.

Moreover, since $l_s \colon x \mapsto \langle x, s\rangle$ is continuous for all s and since

$$S^\perp = \cap_s l_s^{-1}(0),$$

the subspace $S^\perp$ is closed.

Since by definition

$$\|x - P_s x\|^2 \leq \|x - P_s x - \lambda s\|^2$$

for any $s \in S$ and any $\lambda \in \mathbb{R}$, we have

$$\|x - P_s x\|^2 \leq \underbrace{\|x - P_s x\|^2 - 2\lambda\langle x - P_s x, s\rangle + \lambda^2\|s\|^2}_{=:RHS}.$$

RHS is a polynomial in λ with a minimum at $\lambda = 0$.

This forces $\langle x - P_s x, s\rangle = 0$ and so $x = P_s x$.

We see that any $x \in \mathcal{H}$ can be written as $x = x - P_s x + P_s x = s + s^\perp$.

Furthermore, $S^\perp \cap S = 0$.

Thus $\mathcal{H} = S \oplus S^\perp$ for any closed linear subspace $S \subset \mathcal{H}$.

In particular the decomposition of any x as $s + s^\perp \in S \oplus S^\perp$ is unique.

Differential Calculus on Banach Spaces

- Modern differential geometry is based on the theory of differentiable manifolds - a natural extension of multivariable calculus.
- Multivariable calculus is said to be done on (or in) an n-dimensional coordinate space $\mathbb{R}^n$, Euclidean space or Cartesian space.
- The booklet Calculus on Manifolds by Michael Spivak is very suitable for modern differential geometry [22].
- Calculus easily generalizes from $\mathbb{R}^n$ to Banach spaces.

We will recall a few definitions and further facts from functional analysis and then review highlights from differential calculus while simultaneously generalizing to Banach spaces.

Definition

A topological vector space over $\mathbb{R}$ is a vector space V with a topology s.t. vector addition and scalar multiplication are continuous.

In particular, $V \times V \to V$, with $(v_1, v_2) \mapsto v_1 + v_2$ and $\mathbb{R} \times V \to V$ with $(a, v) \mapsto av$ are continuous.

Here we have given $V \times V$ and $\mathbb{R} \times V$ the product topologies.

Definition

A map between topological vector spaces which is both a continuous linear map and which has a continuous linear inverse is called a toplinear isomorphism.

Remark

A toplinear isomorphism is then just a linear isomorphism which is also a homeomorphism.

We will be interested in topological vector spaces which get their topology from a norm function.

Definition

A norm on a real vector space (an $\mathbb{R}$-vector space) V is a map

$$\|\cdot\| : V \to \mathbb{R}$$

s.t.

- $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ only if $v = 0$,
- $\|av\| = |a|\|v\|$ for all $a \in \mathbb{R}$ and all $v \in V$,
- if $v_1, v_2 \in V$, then $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ (triangle inequality).

A vector space together with a norm is called a normed vector space.

Definition

Let E and F be normed spaces.

A linear map $A : E \to F$ is said to be bounded if

$$\|A(v)\| \leq C\|v\|$$

for all $v \in E$.

For convenience, we use the same notation for the norms in both spaces.

If $\|A(v)\| = \|v\|$ for all $v \in E$ we call A an isometry.

If A is a toplinear isomorphism which is also an isometry we say that A is an isometric isomorphism.

Remark

It is a standard fact that a linear map between normed spaces is bounded iff it is continuous.

- The standard norm for $\mathbb{R}^n$ is given by $\|(x^1, \ldots, x^n)\| = \sqrt{\sum_{i=1}^n (x^i)^2}$.
- Imitating what we do in $\mathbb{R}^n$ we can define a distance function for a normed vector space by letting $\mathrm{dist}(v_1, v_2) := \|v_2 - v_1\|$.
- The distance function gives a topology in the usual way.
- The convergence of a sequence is defined w.r.t. the distance function.
- A sequence $\{v_i\}$ is said to be a Cauchy sequence if given any $\varepsilon > 0$ there is an N s.t. $\mathrm{dist}(v_n, v_m) = \|v_n - v_m\| < \varepsilon$ whenever $n, m > N$.
- In $\mathbb{R}^n$ every Cauchy sequence is a convergent sequence.
  This is a nice property with many consequences.

Definition

A normed vector space with the property that every Cauchy sequence converges is called a complete normed vector space or a Banach space.

- Up to now we have real normed vector spaces and real Banach spaces but there is also the notion of *complex* normed spaces and *complex* Banach spaces.
- In functional analysis the complex case is central but for calculus the real Banach spaces are central.
- Of course, every complex Banach space is also a real Banach space in an obvious way.
- For simplicity and definiteness all normed spaces and Banach spaces in these lectures will be real Banach spaces as defined above.

Definition

Given two normed spaces V and W with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ we can form a normed space from the Cartesian product $V \times W$ by using the norm

$$\|(v, w)\| := \max\left\{\|v\|_1, \|w\|_2\right\}.$$

The vector space structure on $V \times W$ is that of the (outer) direct sum.

Definition

Two norms on V, say $\|\cdot\|'$ and $\|\cdot\|''$ are equivalent if there exist positive constants c and C s.t.

$$c\|x\|' \leq \|x\|'' \leq C\|x\|'$$

for all $x \in V$.

Example

There are many norms for $V \times W$ equivalent to that given above including

$$\|(v,w)\|' := \sqrt{\|v\|_1^2 + \|w\|_2^2} \quad \text{and also} \quad \|(v,w)\|'' := \|v\|_1 + \|w\|_2.$$

Remark

If V and W are Banach spaces then so is $V \times W$ with either of the above norms.

The topology induced on $V \times W$ by any of these equivalent norms is exactly the product topology.

Differential Calculus on Banach Spaces, Cont'd[1]

Let W_1, W_2 be subspaces of a Banach space V.

- $W_1 + W_2$ indicates the subspace $\{v \in V \mid v = w_1 + w_2 \text{ for } w_1 \in W_1, w_2 \in W_2\}$.
- If $V = W_1 + W_2$ then any $v \in V$ can be written as $v = w_1 + w_2$ for $w_1 \in W_1, w_2 \in W_2$.
- If $W_1 \cap W_2 = 0$, then this decomposition is unique and W_1, W_2 are complementary.
- Unless a subspace is closed it will itself not be a Banach space.

  So if we are given a closed subspace W_1 of V then it is ideal if there can be found a subspace W_2 which is complementary to W_1 being *also closed*.

  In this case we write $V = W_1 \oplus W_2$.

[1] I recommend the nice monograph [1]

Theorem

If W_1, W_2 are complementary closed subspaces of a Banach space V then there is a toplinear isomorphism $W_1 \times W_2 \cong V$ given by

$$(w_1, w_2) \longleftrightarrow w_1 + w_2.$$

Idea of proof: Use the closed graph theorem. □

Remark

Let E be a Banach space and $W \subset E$ a closed subspace.

If there is a closed complementary subspace W' we say that W is a split subspace of E.

If a subspace is split we can use the isomorphism

$$W \times W' \cong W \oplus W'.$$

Definition

We will denote the set of all continuous (bounded) linear maps from a normed space E to a normed space F by $L(E, F)$.

The set of all continuous linear isomorphisms from E onto F will be denoted by $\mathrm{GL}(E, F)$.

In case $E = F$ the corresponding spaces will be denoted by $\mathfrak{gl}(E)$ and $\mathrm{GL}(E)$, respectively.

Lemma

$\mathrm{GL}(E)$ *is a group under composition and is called the general linear group.*

Definition

A multilinear map $\mu\colon V_1 \times \cdots \times V_k \to W$ is bounded iff there is a constant C s.t.

$$\|\mu(v_1, v_2, \ldots, v_k)\|_W \leq C \cdot \|v_1\|_{V_1} \cdot \|v_2\|_{V_2} \cdots \|v_k\|_{V_k}$$

for all $(v_1, \ldots, v_k) \in V_1 \times \cdots \times V_k$.

Remark

$V_1 \times \cdots \times V_k$ is a normed space in several equivalent ways just in the same way that we defined before for the case $k = 2$.

The topology is the product topology.

Proposition

A multilinear map $\mu\colon V_1 \times \cdots \times V_k \to W$ is bounded iff it is continuous.

Notation

The set of all bounded multilinear maps $V_1 \times \cdots \times V_k \to W$ *will be denoted by* $L(V_1, \ldots, V_k; W)$.

If $V_1 = \cdots = V_k = V$ *we write* $L^k(V; W)$ *instead of* $L(V, \ldots, V; W)$.

Notation

For linear maps $T \colon V \to W$ *we often write* $T \cdot v$ *instead of* $T(v)$.

Moreover, suppose for some set X, *we have map* $A \colon X \to L(V, W)$.

Then $A(x)(v)$ *makes sense but we may find ourselves in a situation where* $A|_x v$ *is even more clear.*

This latter notation suggests a family of linear maps $\{A|_x\}$ *parameterized by* $x \in X$.

Definition

A multilinear map $\mu\colon V \times \cdots \times V \to W$ is called symmetric if for any $v_1, v_2, \ldots, v_k \in V$ we have that

$$\mu\big(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)}\big) = \mu(v_1, v_2, \ldots, v_k)$$

for all permutations σ on the letters $\{1, 2, \ldots, k\}$.

Similarly, μ is called skew-symmetric or alternating if

$$\mu\big(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)}\big) = \operatorname{sgn}(\sigma)\mu(v_1, v_2, \ldots, v_k)$$

for all permutations σ.

The set of all bounded symmetric (resp. skew-symmetric) multilinear maps $V \times \cdots \times V \to W$ is denoted $L^k_{\mathrm{sym}}(V; W)$, (resp. $L^k_{\mathrm{skew}}(V; W)$ or $L^k_{\mathrm{alt}}(V; W)$).

If W is complete, i.e., if W is a Banach space then the space $L(V, W)$ is a Banach space in its own right with norm given by

$$\|A\| = \sup_{v \in V, v \neq 0} \frac{\|A(v)\|_W}{\|v\|_V} = \sup \left\{ \|A(v)\|_W \colon \|v\|_V = 1 \right\}.$$

Similarly, the spaces $L(V_1, \ldots, V_k; W)$ are also Banach spaces normed by

$$\|\mu\| := \sup \left\{ \|\mu(v_1, \ldots, v_k)\|_W \colon \|v_i\|_{V_i} = 1 \text{ for } i = 1, \ldots, k \right\}.$$

Remember:

- There is a natural linear bijection $L\big(V, L(V, W)\big) \cong L^2(V, W)$.
- Similarly, $L\Big(V, L\big(V, L(V, W)\big)\Big) \cong L^3(V; W)$.
- In general, $L\Big(V, L\big(V, L(V, ..., L(V, W))\big) \dots\Big) \cong L^k(V, W)$.
- The isomorphism above is continuous and norm preserving, i.e., an isometric isomorphism.

Definition (Differentiability)

A map $f\colon V \supset U \to W$ between normed spaces and defined on an open set $U \subset V$ is said to be differentiable at $p \in U$ iff there is a bounded linear map $A_p \in L(V, W)$ s.t.

$$\lim_{\|h\| \to 0} \frac{\|f(p+h) - f(p) - A_p \cdot h\|}{\|h\|} = 0$$

Proposition

If A_p exists for a given function f then it is unique.

Proof:
Suppose that A_p and B_p both satisfy the requirements of the definition, i.e., the limit in question equals zero. For $p + h \in U$ we have

$$A_p \cdot h - B_p \cdot h = -\big(f(p+h) - f(p) - A_p \cdot h\big) + \big(f(p+h) - f(p) - B_p \cdot h\big).$$

Taking norms, dividing by $\|h\|$ and taking the limit as $\|h\| \to 0$ we get

$$\|A_p h - B_p h\| / \|h\| \to 0.$$

Now let $h \neq 0$ be arbitrary and choose $\epsilon > 0$ small enough that $p + \epsilon h \in U$. Then we have

$$\|A_p(\varepsilon h) - B_p(\varepsilon h)\| / \|\varepsilon h\| \to 0.$$

By linearity $\|A_p(\varepsilon h) - B_p(\varepsilon h)\| / \|\varepsilon h\| = \|A_p h - B_p h\| / \|h\|$ which does not depend on ε, consequently, $\|A_p h - B_p h\| = 0$. $\square$

Definition

If a function f is differentiable at p, then the linear map A_p which exists by definition and is unique by the above result, will be denoted by $\mathrm{D}\, f(p)$.

The linear map $\mathrm{D}\, f(p)$ is called the derivative of f (differential of f) at (or evaluated at) p.

We will also use the notation $\mathrm{D}\, f|_p$ or sometimes $f'(p)$.

We often write $\mathrm{D}\, f|_p \cdot h$ instead of $\mathrm{D}\, f(p)(h)$.

Remark

It is not hard to show that the derivative of a constant map is constant and the derivative of a (bounded) linear map is the very same linear map.

If we are interested in differentiating in "one direction" then we may use the natural notion of directional derivative.

Definition

A map $f\colon V \supset U \to W$ has a directional derivative $\mathrm{D}_h f$ at p in the direction h if the following limit exists:

$$(\mathrm{D}_h f)(p) := \lim_{\varepsilon \to 0} \frac{f(p+\varepsilon h) - f(p)}{\varepsilon}.$$

In other words, $\mathrm{D}_h f(p) = \frac{\mathrm{d}}{\mathrm{d}t} f(p+th)|_{t=0}$.

Remark

$f(p)$ may have a directional derivative in every direction but yet may still **not** *be differentiable at p in the sense of the above definition.*

Notation

The directional derivative is written as $(\mathrm{D}_h f)(p)$ and, in case f is actually differentiable at p, this is equal to $\mathrm{D} f|_p h = \mathrm{D} f(p) \cdot h$.

Note that $\mathrm{D}_x f$ should not be confused with $\mathrm{D} f|_x$.

Let V, W be Banach spaces.

Consider $f\colon U \subset V \to W$ differentiable for all p throughout some open set U.

Then f is differentiable on U.

The map $\mathrm{D}\, f\colon U \subset V \to L(V, W)$ with $p \mapsto \mathrm{D}\, f(p)$ is called the derivative of f.

Whereas $\mathrm{D}\, f(p)$ is called the derivative of f evaluated at p.

If this map itself is differentiable at some $p \in V$ then its derivative at p is denoted $\mathrm{D}\,\mathrm{D}\, f(p) = \mathrm{D}^2 f(p) = \mathrm{D}^2 f|_p \in L(V, L(V, W)) \cong L^2(V; W)$, the second derivative at p.

If in turn $\mathrm{D}^2 f|_p$ exist for all p throughout U then we have a map $\mathrm{D}^2 f\colon U \to L^2(V; W)$ called the second derivative.

Similarly, we may inductively define $\mathrm{D}^k f|_p \in L^k(V; W)$ and $\mathrm{D}^k f\colon U \to L^k(V; W)$ whenever f is nice enough that the process can be iterated appropriately.

Definition

A map $f\colon U \subset V \to W$ is C^r-differentiable on U if $\mathrm{D}^r f|_p \in L^r(V, W)$ exists for all $p \in U$ and if $\mathrm{D}^r f$ is continuous as map $U \to L^r(V, W)$.

If f is C^r-differentiable on U for all $r > 0$ then f is C^∞ or smooth (on U).

Remark

Let C^0 indicate mere continuity. Clearly, a bounded multilinear map is C^∞.

Definition

A bijection f between open sets $U_\alpha \subset V$ and $U_\beta \subset W$ is called a C^r-diffeomorphism iff f and f^{-1} are both C^r-differentiable (on U_α and U_β respectively).

If $r = \infty$ then we simply call f a diffeomorphism.

Definition

Let $U \in V$ be open.

A map $f\colon U \to W$ is a local C^r-diffeomorphism iff for every $p \in U$ there is an open set $U_p \subset U$ with $p \in U_p$ s.t. $f|_{U_p}\colon U_p \to f(U_p)$ is a C^r-diffeomorphism.

Differential Calculus on Banach Spaces Cont'd[2]

Theorem (Mean Value Theorem)

Let V and W be Banach spaces.

Let $c\colon [a,b] \to V$ be a C^1-map with image contained in an open set $U \subset V$.

Let $f\colon U \to W$ be a C^1-map. Then

$$f\big(c(b)\big) - f\big(c(a)\big) = \int_a^b \mathrm{D}\, f\big(c(t)\big) \cdot c'(t)\mathrm{d}t$$

If $c(t) = (1-t)x + ty$ then

$$f(y) - f(x) = \int_0^1 \mathrm{D}\, f\big(c(t)\big)\mathrm{d}t \cdot (y-x)$$

Moreover, $\int_0^1 \mathrm{D}\, f\big(c(t)\big)\mathrm{d}t \in L(V,W)$ (integral in the sense of Cauchy-Bochner).

[2] Recommended further reading: [1],[8, 9]

Definition

A subset U of a Banach space (or any vector space) is said to be convex if it has the property that whenever $x, y \in U$ then so are all points of the line segment $l_{xy} := \{(1-t)x + ty \colon 0 \leq t \leq 1\}$.

Corollary

Let U be a convex open set in a Banach space V and $f \colon U \to W$ be C^1 into another Banach space W. Then for any $x, y \in U$ we have

$$\|f(y) - f(x)\| \leq C_{x,y} \|y - x\|$$

where $C_{x,y}$ is the supremum over all values taken by f on points of the line segment l_{xy}.

Definition

Let $f\colon U \subset E \to F$ be a map and suppose the splitting $E = E_1 \times E_2$. Consider a generic $(x, y) \in E_1 \times E_2$.

For every $(a, b) \in U \subset E_1 \times E_2$ the partial maps

$$f_{a,}\colon y \mapsto f(a, y) \text{ and } f_{,b}\colon x \mapsto f(x, b)$$

are defined in some neighborhood of b (resp. a).

(Notice the logical placement of commas in this notation).

We define the partial derivatives, when they exist, by

$$\mathrm{D}_2 f(a, b) := Df_{a,}(b) \text{ and } \mathrm{D}_1 f(a, b) := Df_{,b}(a).$$

These are, of course, linear maps

$$\mathrm{D}_1 f(a, b)\colon E_1 \to F, \qquad \mathrm{D}_2 f(a, b)\colon E_2 \to F.$$

Remark

The partial derivative can exist even in cases where f might not be differentiable in the sense we have defined.

This is a slight generalization of the point made earlier: f might be differentiable only in certain directions without being fully differentiable.

However, we have

Proposition

If f has continuous partial derivatives $\mathrm{D}_i f(x,y)\colon E_i \to F$ near $(x,y) \in E_1 \times E_2$ then $\mathrm{D} f(x,y)$ exists and is continuous.

In this case, we have for $v = (v_1, v_2)$,

$$\mathrm{D} f(x,y) \cdot (v_1, v_2) = \mathrm{D}_1 f(x,y) \cdot v_1 + \mathrm{D}_2 f(x,y) \cdot v_2.$$

Remark

Consider maps on several factors

$$f\colon E_1 \times E_2 \times \cdots \times E_n \to F,$$

we define partial derivatives $\mathrm{D}_i f\colon E_i \to F$ *in the obvious way.*

However, the meaning of $\mathrm{D}_i f$ *depends on how we factor the domain.*

For example, we have both $\mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$ *and also* $\mathbb{R}^3 \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Let U *be an open subset of* $\mathbb{R}^n$ *and let* $f\colon U \to \mathbb{R}$ *be a map.*

Then for $a \in U$ *we define*

$$(\partial_i f)(a) := (\mathrm{D}_i f)(a) \cdot e_i = \lim_{h \to 0} \frac{f(a^1, \ldots, a^i + h, \ldots, a^n) - f(a^1, \ldots, a^i, \ldots, a^n)}{h},$$

where e_i *is the* i*-th standard basis vector in* $\mathbb{R}^n$.

The function $\partial_i f$ *is defined where the above limit exists.*

With $(x^1, \ldots, x^n)$ *the standard coordinates on* $\mathbb{R}^n$ *we write* $\partial_i f$ *as* $\partial f / \partial x_i$.

- Note that in this setting, the linear map is identified $(\mathrm{D}_i f)(a) = \partial_i f(a)$.
- Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a map that is differentiable at $a = (a^1, \dots, a^n) \in \mathbb{R}^n$.

  The map f is given by m functions $f^i\colon U \to \mathbb{R}$, $1 \leq i \leq m$ s.t. $f(u) = \big(f^1(u), \dots, f^m(u)\big)$.
- The above proposition has an obvious generalization to the case where we decompose the Banach space into more than two factors as in $\mathbb{R}^m = \mathbb{R} \times \cdots \times \mathbb{R}$.

  We find that if all partials $\partial f^i / x^j$ are continuous in U then f is C^1.

Definition

With respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, the derivative is given by an $(m \times n)$-matrix called the Jacobian matrix:

$$J_a(f) := \begin{bmatrix} \frac{\partial f^1}{\partial x^1}(a) & \frac{\partial f^1}{\partial x^2}(a) & \cdots & \frac{\partial f^1}{\partial x^n}(a) \\ \frac{\partial f^2}{\partial x^1}(a) & & & \frac{\partial f^2}{\partial x^n}(a) \\ \vdots & & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1}(a) & & & \frac{\partial f^m}{\partial x^n}(a) \end{bmatrix}.$$

The rank of $J_a(f)$ is called the rank of f at a.

If $n = m$ then the Jacobian is a square matrix and $\det(J_a(f))$ is called the Jacobian determinant at a.

If f is differentiable near a then it follows from the inverse mapping theorem (see below) that if $\det\big(J_a(f)\big) \neq 0$ then there is some open set containing a on which f has a differentiable inverse.

The Jacobian of this inverse at $f(x)$ is the inverse of the Jacobian of f at x.

Theorem (Chain Rule)

Let U_1 and U_2 be open subsets of Banach spaces E_1 and E_2, respectively. Suppose we have continuous maps composing as

$$U_1 \xrightarrow{f} U_2 \xrightarrow{g} E_3$$

where E_3 is a third Banach space.

If f is differentiable at p and g is differentiable at $f(p)$ then the composition is differentiable at p and

$$\mathrm{D}(g \circ f) = \mathrm{D}\, g\big(f(p)\big) \circ \mathrm{D}\, f(p).$$

In other words, if $v \in E_1$ then

$$\mathrm{D}(g \circ f)|_p \cdot v = \mathrm{D}\, g|_{f(p)} \cdot (\mathrm{D}\, f|_p \cdot v).$$

Furthermore, if $f \in C^r(U_1)$ and $g \in C^r(U_2)$ then $g \circ f \in C^r(U_1)$.

Useful for calculations is

Lemma

Let $f\colon U \subset V \to W$ be twice differentiable at $x_0 \in U \subset V$.

Then the map $\mathrm{D}_v f\colon x \to \mathrm{D} f(x) \cdot v$ is differentiable at x_0 and its derivative at x_0 is

$$\mathrm{D}(\mathrm{D}_v f)|_{x_0} \cdot h = \mathrm{D}^2 f(x_0)(h, v).$$

Theorem

If $f\colon U \subset V \to W$ is twice differentiable on U s.t. $\mathrm{D}^2 f$ is continuous, i.e. if $f \in C^2(U)$ then $\mathrm{D}^2 f$ is symmetric:

$$\mathrm{D}^2 f(p)(w, v) = \mathrm{D}^2 f(p)(v, w).$$

More generally, if $\mathrm{D}^k f$ exists and is continuous then $\mathrm{D}^k f(p) \in L^k_{\mathrm{sym}}(V; W)$.

Idea of proof: Apply Hahn-Banach Theorem and standard 1-variable version of this theorem. Result for $\mathrm{D}^k f$ by induction. □

Theorem (Leibniz' or Product Rule)

Let $\rho \in L(F_1, F_2; W)$ be a bilinear map and let $f_1 \colon U \subset E \to F_1$ and $f_2 \colon U \subset E \to F_2$ be differentiable (resp. $C^r, r \geq 1$) maps.

Then the composition $\rho(f_1, f_2)$ is differentiable (resp. $C^r, r \geq 1$) on U where $\rho(f_1, f_2) \colon x \mapsto \rho(f_1(x), f_2(x))$. Furthermore,

$$\mathrm{D}\,\rho(f_1, f_2)|_x \cdot v = \rho\big(\mathrm{D}\,f_1|_x \cdot v, f_2(x)\big) + \rho\big(f_1(x), \mathrm{D}\,f_2|_x \cdot v\big).$$

In particular, if F is a Banach algebra with product $*$ and $f_1 \colon U \subset E \to F$ and $f_2 \colon U \subset E \to F$ then $f_1 * f_2$ is defined as a function and

$$\mathrm{D}(f_1 * f_2) \cdot v = (\mathrm{D}\,f_1 \cdot v) * f_2 + f_1 * (\mathrm{D}\,f_2 \cdot v).$$

Example

Consider

$$\begin{aligned} f_1 &\colon \mathbb{R} \to \mathbb{R}^{n\times n}, \quad t \mapsto X(t), \\ f_2 &\colon \mathbb{R} \to \mathrm{GL}_n, \quad t \mapsto Y^{-1}(t), \\ \rho(f_1, f_2) &:= \operatorname{tr}\big(X(t)Y^{-1}(t)\big). \end{aligned}$$

Then

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\rho(f_1, f_2) &= \operatorname{tr}\big(\dot{X}Y^{-1}\big) + \operatorname{tr}\Big(X\big(\mathrm{D}\,Y^{-1} \circ \dot{Y}\big)\Big) \\ &= \operatorname{tr}\big(\dot{X}Y^{-1}\big) + \operatorname{tr}\Big(X\big(-Y^{-1}\dot{Y}Y^{-1}\big)\Big). \end{aligned}$$

For a fixed x, higher derivatives $\mathrm{D}^p f|_x$ are symmetric multilinear maps.

Let $(y)^k := (y, y, \ldots, y)$.

With this notation we have the following version of Taylor's theorem.

Theorem (Taylor's Theorem)

Given Banach spaces V, W, a C^r-function $f\colon U \to W$ and a line segment $t \mapsto (1-t)x + ty \subset U$.

Then $t \mapsto \mathrm{D}^p f(x+ty)\cdot(y)^p$ is defined and continuous for $1 \leq p \leq k$ and

$$f(x+y) = f(x) + \tfrac{1}{1!}\mathrm{D}f|_x \cdot y + \tfrac{1}{2!}\mathrm{D}^2 f|_x \cdot (y)^2 + \cdots + \tfrac{1}{(k-1)!}\mathrm{D}^{k-1} f|_x \cdot (y)^{(k-1)}$$
$$+ \int_0^1 \tfrac{(1-t)^{k-1}}{(k-1)!}\, \mathrm{D}^k f(x+ty)\cdot(y)^k \mathrm{d}t.$$

Idea of Proof: By induction following the usual proof closely.

We still have an integration by parts formula coming from the product rule.

We still have the fundamental theorem of calculus. □

Example

If f is C^∞ then we may be able to extend Taylor's formula into a convergent power series.
If we can, we say f is of class C^ω, or analytic.

A standard example of a C^∞-function that is not analytic is the following

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) := \begin{cases} \mathrm{e}^{-\frac{1}{1-x^2}}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

In fact f is C^∞, and all derivatives are 0 at $x = \pm 1$. Indeed, for $|x| < 1$,

$$f^{(n)}(x) = Q_n(x) \tfrac{1}{(1-x^2)^{2n}} \, \mathrm{e}^{-\frac{1}{1-x^2}},$$

where $Q_n(x)$ are polynomials given recursively by

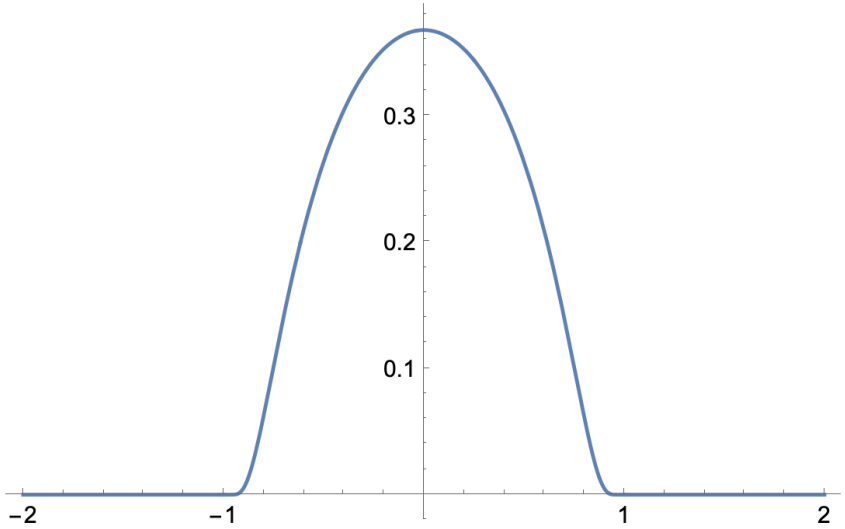
$$Q_0(x) = 1, \quad Q_{n+1}(x) = (1-x^2)^2 Q_n'(x) + 2x(2n-1-2nx^2) Q_n(x).$$

Hence all coefficients of the Taylor series around these points vanish.
But $f \not\equiv 0$ in any neighborhood of ± 1, so it cannot be analytic there.

q

In[•]:=
```
f[x_] := Piecewise[{{e^(-1/(1-x^2)), -1 < x < 1}, {0, -1 ≥ x || x ≥ 1}}];
Plot[f[x], {x, -2, 2}]
```

Out[•]=



In[•]:=
```
Limit[D[f[x], {x, 1}], x → 1, Direction → 1]
Limit[D[f[x], {x, 2}], x → 1, Direction → 1]
Limit[D[f[x], {x, 3}], x → 1, Direction → 1]
Limit[D[f[x], {x, 4}], x → 1, Direction → 1]
```

Out[•]= 0

Out[•]= 0

Out[•]= 0

Out[•]= 0

Remark

- *Our main reason for restricting our calculus to Banach spaces is that the inverse mapping theorem holds for Banach spaces.*
- *There is* **no** *simple and general inverse mapping theory on more general topological vector spaces.*
- *E.g., the Nash-Moser-Thm., cf. [15], requires special estimates and is constructed to apply only in a very controlled situation.*

Definition

Let E and F be Banach spaces.

A map will be called a C^r-diffeomorphism near p if there is some open set $U \subset \mathrm{dom}(f)$ containing p s.t. $f|_U \colon U \to f(U)$ is a C^r-diffeomorphism onto an open set $f(U)$.

If f is a C^r-diffeomorphism near p for all $p \in \mathrm{dom}(f)$ then we say that f is a local C^r-diffeomorphism.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces.

A map $f \colon X \to Y$ is said to be Lipschitz continuous (with constant k) if there is a $k > 0$ s.t.

$$d_Y\big(f(x_1), f(x_2)\big) \leq k d_X(x_1, x_2) \text{ for all } x_1, x_2 \in X.$$

If $0 < k < 1$ the map is called a contraction mapping (with constant k) and is said to be k-contractive.

Denote the n-fold composition $f^n := f \circ \cdots \circ f$.

Proposition (Contraction Map Principle or Banach's Fixed Point Thm.)

Let F be a closed subset of a complete metric space (M, d).

Let $f\colon F \to F$ be a k-contractive map s.t. $d\big(f(x), f(y)\big) \leq kd(x, y)$ for some fixed $0 \leq k < 1$.

Then

1. *there is exactly one $x_0 \in F$ s.t. $f(x_0) = x_0$ (i.e. x_0 is a fixed point for f);*
2. *for any $y \in F$ the sequence (y_n) defined by $y_n := f^n(y)$ converges to the fixed point x_0 with the error estimate*

$$d(y_n, x_0) \leq \frac{k^n}{1-k} d(y_1, x_0).$$

Idea of proof: Geometric and Cauchy series, see e.g. any advanced ODE course. □

Remark

Note that a Lipschitz map f may not satisfy the hypotheses of the last proposition even if $k < 1$ since an open U is not a complete metric space unless $U = E$.

Definition

A continuous map $f\colon U \to E$ s.t. $L_f := \mathrm{id}_U - f$ is injective has an inverse G_f (not necessarily continuous) and the invertible map $R_f := \mathrm{id}_E - G_f$ will be called the resolvent operator for f.

Theorem

Let E be a Banach space.

If $f\colon E \to E$ is a continuous map that is Lipschitz continuous with constant k where $0 \leq k < 1$, then the resolvent R_f exists and is Lipschitz continuous with constant $\frac{k}{1-k}$.

Hints for proof:
Consider $x - f(x) = y$ and show that for any $y \in E$ this equation has a unique solution as $F\colon E \to E$ with $F(x) = f(x) + y$ is k-contractive.

Then apply the contraction mapping principle. □

Lemma

The space $\mathrm{GL}(E,F)$ *of continuous linear isomorphisms is an open subset of the Banach space* $L(E,F)$.

In particular, if $\|\mathrm{id} - A\| < 1$ *for some* $A \in \mathrm{GL}(E)$ *then*

$$A^{-1} = \lim_{N\to\infty} \sum_{n=0}^{N} (\mathrm{id} - A)^n.$$

Proof:

Let $A_0 \in \mathrm{GL}(E,F)$.

The map $A \mapsto A_0^{-1} \circ A$ is continuous and onto.

We may assume $E = F$ and $A_0 = \mathrm{id}_E$.

We show that elements of $L(E,E)$ close enough to id_E are in fact elements of $\mathrm{GL}(E)$.

Now $\|\mathrm{id} - A\| < 1$ implies that $A \in \mathrm{GL}(E)$:

We use the fact that the norm on $L(E, E)$ is an algebra norm.

Thus $\|A_1 \circ A_2\| \leq \|A_1\| \cdot \|A_2\|$ for all $A_1, A_2 \in L(E, E)$.

Abbreviate id by "1" and denote $\Lambda := \mathrm{id} - A$.

Let $\Lambda^2 := \Lambda \circ \Lambda$, $\Lambda^3 := \Lambda \circ \Lambda \circ \Lambda$ and so forth.

We form a Neumann series, i.e. $\pi_k := \sum_{i=0}^{k} \Lambda^i$.

Comparing with the Neumann series of real numbers formed in the same way using $\|A\|$ instead of A we see that (π_n) is a Cauchy sequence since $\|\Lambda\| = \|\mathrm{id} - A\| < 1$.

Thus (π_n) is convergent to some element ρ.

Now we have $(1 - \Lambda)\pi_n = 1 - \Lambda^{n+1}$ and letting $n \to \infty$ we see that $(1 - \Lambda)\rho = 1$ or in other words, $A\rho = 1$. $\square$

Lemma

The map

$$\mathrm{Inv}\colon\ \mathsf{GL}(E,F) \to \mathsf{GL}(E,F)$$

given by taking inverses is a C^∞-map and the derivative of $\mathrm{Inv}\colon g \mapsto g^{-1}$ *at some* $g_0 \in \mathsf{GL}(E,F)$ *is the linear map given by the formula:*

$$\mathrm{D}\,\mathrm{Inv}|_{g_0}\colon A \mapsto -g_0^{-1} A g_0^{-1}.$$

Theorem (Inverse Mapping Theorem)

Let E and F be Banach spaces and $f\colon U \to F$ be a C^r-mapping defined on an open set $U \subset E$.

Suppose that $x_0 \in U$ and that $f'(x_0) = \mathrm{D}\, f|_{x_0}\colon E \to F$ is a continuous linear isomorphism.

Then there exists an open set $V \subset U$ with $x_0 \in V$ s.t. $f\colon V \to f(V) \subset F$ is a C^r-diffeomorphism.

Furthermore the derivative of f^{-1} at y is given by

$$\mathrm{D}\, f^{-1}|_y = \left(\mathrm{D}\, f|_{f^{-1}(y)}\right)^{-1}.$$

Corollary

Let $U \subset E$ be an open set. Suppose that $f\colon U \to F$ is differentiable with $\mathrm{D}\, f(p)\colon E \to F$ a (bounded) linear isomorphism for each $p \in U$.

Then f is a local diffeomorphism.

Example

Consider the map

$$\phi\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad \phi(x,y) := (x^2 - y^2, 2xy).$$

The derivative is given by the matrix $\begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$ which is invertible for all $(x,y) \neq (0,0)$.

Thus, by the inverse mapping theorem, the restriction of ϕ to a sufficiently small open disk centered at any point but the origin will be a diffeomorphism.

We may say that the restriction $\phi|_{\mathbb{R}^2 \setminus \{(0,0)\}}$ is a local diffeomorphism.

However, notice that $\phi(x,y) = \phi(-x,-y)$ so generically ϕ is a $2-1$ map and in particular is **not** a (global) diffeomorphism.

Theorem (Implicit Function Theorem)

Let $U \subset E$, $V \subset F$ be open and $f\colon U \times V \to G$ be C^r, $r \geq 1$, with E, F, G Banach spaces.

For some $x_0 \in U, y_0 \in V$ assume the partial derivative $\mathrm{D}_2 f(x_0, y_0)\colon F \to G$, is an isomorphism.

Then there are neighborhoods U_0 of x_0 and W_0 of $f(x_0, y_0)$ and a unique C^r-map $g\colon U_0 \times W_0 \to V$ s.t. for all $(x, w) \in U_0 \times W_0$, we have $f(x, g(x, w)) = w$.

Proof:
Define

$$\Phi\colon U \times V \to E \times G, \quad (x, y) \mapsto \big(x, f(x, y)\big).$$

Then

$$\mathrm{D}\,\Phi(x_0, y_0)(x_1, y_1) = \begin{bmatrix} I & 0 \\ \mathrm{D}_1 f(x_0, y_0) & \mathrm{D}_2 f(x_0, y_0) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

which is an isomorphism of $E \times F$ with $E \times G$.
Thus, Φ has a unique local C^r-inverse, say

$$\Phi^{-1}\colon U_0 \times W_0 \to U \times V, \quad (x, w) \mapsto \big(x, g(x, w)\big).$$

The g so defined is the desired map. □

Corollary

Derivatives of g can be computed via the chain rule:

$$\mathrm{D}_1\, g(x, w) = -\Big(\mathrm{D}_2\, f\big(x, g(x, w)\big)\Big)^{-1} \circ \mathrm{D}_1\, f\big(x, g(x, w)\big),$$

$$\mathrm{D}_2\, g(x, w) = \Big(\mathrm{D}_2\, f\big(x, g(x, w)\big)\Big)^{-1}.$$

Local Immersions and Submersions

- The following series of consequences of the inverse function theorem are important technical tools in the study of manifolds.
- The first four results give, roughly speaking, sufficient conditions to "straighten out" the range (resp., the domain) of f in a neighborhood of a point, making f look like an inclusion (resp., a projection).

Theorem (Immersion)

Let E, F be Banach spaces.

Let $U \subset E$ be open with $0 \in U$, and let $f\colon U \to E \times F$ be smooth with $f(0) = (0,0)$.
Let $\mathrm{D}\, f(0)\colon E \to E \times F$ be of the form $x \mapsto (\alpha(x), 0)$ for a continuous linear isomorphism $\alpha\colon E \to E$.

Then there exists a diffeomorphism g from an open neighborhood V of $(0,0) \in E \times F$ onto an an open neighborhood W of $(0,0) \in E \times F$ s.t.
$g \circ f\colon f^{-1}(V) \to W$ is of the form $a \mapsto (a, 0)$.

Corollary

Let U be an open neighborhood of $0 \in \mathbb{R}^k$ and $f\colon U \subset \mathbb{R}^k \to \mathbb{R}^n$ be smooth with $f(0) = 0$ s.t. $\mathrm{D}\, f(0)$ has rank k.

Then there is an open neighborhood V of $0 \in \mathbb{R}^n$, an open neighborhood W of $0 \in \mathbb{R}^n$ and a diffeomorphism $g\colon V \to W$ s.t.
$g \circ f\colon f^{-1}(V) \to W$ is of the form $(a^1, \ldots, a^k) \mapsto (a^1, \ldots, a^k, 0, \ldots, 0)$.

Theorem (Submersion)

Let E_1, E_2, F be Banach spaces and let U be an open neighborhood of a point $(a_1, a_2) \in E_1 \times E_2$.

Let $f\colon U \to F$ be a smooth map with $f(a_1, a_2) = 0$.

If the partial derivative $\mathrm{D}_1 f(a_1, a_2)\colon E_1 \to F$ is a continuous linear isomorphism then there exists a diffeomorphism $h\colon V_1 \times V_2 \to U_1$ where $U_1 \subset U$ is an open neighborhood of (a_1, a_2) and V_1, V_2 are open neighborhoods of $0 \in E_1$ and $0 \in E_2$ respectively s.t.

the composite map $f \circ h$ is of the form $(x, y) \mapsto x$.

Corollary

If U is an open neighborhood of $0 \in \mathbb{R}^n$ and $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^k$ is a smooth map with $f(0) = 0$ and

if the partial derivative $\mathrm{D}_1 f(0,0)$ is a linear isomorphism

then there exists a diffeomorphism $h\colon V \subset \mathbb{R}^n \to U_1$ where V is an open neighborhood of $0 \in \mathbb{R}^n$ and U is an open neighborhood of $0 \in \mathbb{R}^k$, respectively, s.t.

the composite map $f \circ h$ is of the form $(a^1, \dots, a^n) \mapsto (a^1, \dots, a^k)$.

Constant Rank Theorem

1. If we think about what is meant by *local* immersion and *local* submersion we will realize that in each case the derivative map $\mathrm{D}\, f(p)$ has full rank.
2. I.e., the rank of the Jacobian matrix in either case is as big as the dimensions of the spaces involved will allow.
3. But rank is only semicontinuous and this is what makes full rank extend from points out onto neighborhoods, so to speak.
4. On the other hand, we can get more general maps into the picture if we explicitly assume that the rank is *locally* constant.
5. We will state the following theorem only for the finite dimensional case.

Theorem (Rank Theorem)

Let $f\colon (\mathbb{R}^n, p) \to (\mathbb{R}^m, q)$ *be a local map s.t.* $\mathrm{D}\, f$ *has constant rank* r *in an open set containing* p.

Then there are local diffeomorphisms $g_1\colon (\mathbb{R}^n, p) \to (\mathbb{R}^n, q)$ *and* $g_2\colon (\mathbb{R}^m, q) \to (\mathbb{R}^m, 0)$ *s.t.* $g_2 \circ f \circ g_1^{-1}$ *has the form*

$$(x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^r, 0, \ldots, 0)$$

on a sufficiently small neighborhood of 0.

Recall by self-study[3] further concepts from topology such as:

- cover of a topological space X, open cover,
- refinement of a cover, locally finite cover,
- paracompactness,
- basis for a topology,
- second countability of a topology, Hausdorff property.

[3] Good read is [18]

Definition

An n-dimensional topological manifold is a paracompact Hausdorff topological space, say M, s.t. every point $p \in M$ is contained in some open set U_p that is homeomorphic to an open subset of the Euclidean space $\mathbb{R}^n$.

- We say a topological manifold is locally Euclidean.
- The integer n is referred to as the dimension of M denoted by $\dim(M)$.

Remark

At first it may seem that a locally Euclidean space must be Hausdorff, but this is not the case.

Example

1. $\mathbb{R}^n$ is trivially a topological manifold of dimension n.
2. The unit circle $S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a 1-dimensional topological manifold.
   Indeed, the map $\mathbb{R} \to S^1$, given by $\theta \mapsto (\cos\theta, \sin\theta)$ has restrictions to small open sets which are homeomorphisms.
3. The boundary of a square in the plane is a topological manifold homeomorphic to the circle, we say that it is a topological circle.
4. More generally, the n-sphere
   $$S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\}$$
   is a topological manifold.
5. If M_1 and M_2 are topological manifolds of dimensions n_1 and n_2, resp., then $M_1 \times M_2$, with the product topology, is a topological manifold of dimension $n_1 + n_2$.
   Such a manifold is called a product manifold.
   $M_1 \times M_2$ is locally Euclidean.

The product manifold construction can obviously be extended to products of a finite number of manifolds.

Example

The socalled n-torus

$$T^n := \underbrace{S^1 \times \cdots \times S^1}_{n \text{ factors}}$$

is a topological manifold of dimension n.

Definition

Let X be a topological space and $x \in X$.
The component containing x, denoted $C(x)$, is defined to be the union of all connected subsets of X that contain x.
A subset of a topological space X is a (connected) component if it is $C(x)$ for some $x \in X$.

Remark

- *A topological space is called locally connected if the topology has a basis consisting of connected sets.*
- *If a topological space is locally connected, then the connected components of each open set (considered with its subspace topology) are all open.*
- *In particular, each connected component of a locally connected space is open (and also closed).*
- *This clearly applies to manifolds.*

Definition

A continuous curve $\gamma\colon [a,b] \to M$ is said to connect a point p to a point q in M if $\gamma(a) = p$ and $\gamma(b) = q$.
We define an equivalence relation on a topological space M by declaring $p \sim q$ iff there is a continuous curve connecting p to q.
The equivalence classes are called path components, and if there is only one path component, then we say that M is path connected.

Remark

- *We shall not give many examples of topological manifolds at this time because our main concern is with smooth manifolds defined below.*
- *We give plenty of examples of smooth manifolds, and every smooth manifold is also a topological manifold.*

Charts, Atlases and Smooth Structures

Definition (Chart)

Let M be a set. A chart on M is a bijection of a subset $U \subset M$ onto an open subset of some Euclidean space $\mathbb{R}^n$.

- We say that the chart takes values in $\mathbb{R}^n$ or simply that the chart is $\mathbb{R}^n$-valued.
  A chart $\mathbf{x}\colon U \to \mathbf{x}(U) \subset \mathbb{R}^n$ is indicated by $(U, \mathbf{x})$.
- If $\mathrm{pr}_i\colon \mathbb{R}^n \to \mathbb{R}$ is the projection onto the i-th factor given by $\mathrm{pr}_i(a^1, \ldots, a^n) := a^i$, then x^i is the function defined by $x^i := \mathrm{pr}_i \circ \mathbf{x}$. It is called the i-th coordinate function for the chart $(U, \mathbf{x})$.
  We write $\mathbf{x} = (x^1, \ldots, x^n)$.
- If $p \in M$ and $(U, \mathbf{x})$ is a chart with $p \in U$ and $\mathbf{x}(p) = 0 \in \mathbb{R}^n$, then we say that the chart is centered at p.

Definition (Change of Coordinates, Atlas)

Let $A := \{(U_\alpha, \mathbf{x}_\alpha)\}_{\alpha \in I}$ be a collection of $\mathbb{R}^n$-valued charts on a set M (I some index set). We call A an $\mathbb{R}^n$ -valued atlas of class C^r if:

1. $\cup_{\alpha \in I} U_\alpha = M$.
2. The sets of the form $\mathbf{x}_\alpha(U_\alpha \cap U_\beta)$ for $\alpha, \beta \in I$ are all open in $\mathbb{R}^n$.
3. Whenever $U_\alpha \cap U_\beta$ is not empty, the map

$$\mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1} \colon \mathbf{x}_\alpha(U_\alpha \cap U_\beta) \to \mathbf{x}_\beta(U_\alpha \cap U_\beta)$$

   is a C^r-diffeomorphism.

The maps $\mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1}$ are called overlap maps or change of coordinate maps.

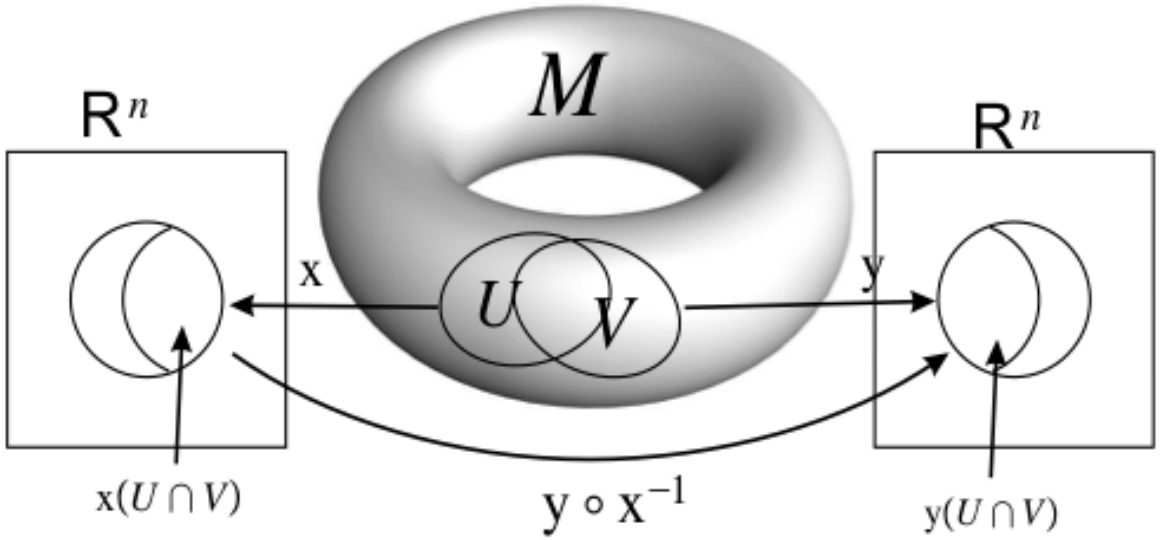
Remark

It is exactly the way we have required the overlap maps to be diffeomorphisms allowing us to have a well-defined notion for a function on M to be differentiable of class C^r.
An atlas of class C^r is also called a C^r-atlas.

Consider an arbitrary pair of charts $(U, \mathbf{x})$ and $(V, \mathbf{y})$ and the overlap map $\mathbf{y} \circ \mathbf{x}^{-1} : \mathbf{x}(U \cap V) \to \mathbf{y}(U \cap V)$.
With $\mathbf{x} := (x^1, \ldots, x^n)$ and $\mathbf{y} := (y^1, \ldots, y^n)$ we have

$$y^i(p) = y^i \circ \mathbf{x}^{-1}\big(x^1(p), \ldots, x^n(p)\big)$$

for any $p \in U \cap V$,



Definition (Differentiable Structure on Manifold)

1. Two C^r-atlases A_1, A_2 on M are said to be equivalent provided that $A_1 \cup A_2$ is also a C^r-atlas for M.
2. A C^r-differentiable structure on M is an equivalence class of C^r-atlases.
3. A C^∞-differentiable structure will also be called a smooth structure.
4. The union of all the C^r-atlases in an equivalence class, is itself an atlas that is in the equivalence class.
   Such an atlas is called a maximal C^r-atlas since it is not properly contained in any larger atlas.

Definition

1. Two $(U, \mathbf{x}), (V, \mathbf{y})$ are C^r-related if either $U \cap V = \emptyset$ or both $\mathbf{x}(U \cap V)$ and $\mathbf{y}(U \cap V)$ are open and $\mathbf{x} \circ \mathbf{y}^{-1}, \mathbf{y} \circ \mathbf{x}^{-1}$ are C^r-maps. Thus an atlas is just a family of mutually C^r-related charts whose domains form a cover of the manifold.
2. A chart $(U, \mathbf{x})$ on M is compatible with a C^r-atlas A if $A \cup \{(U, \mathbf{x})\}$ is also a C^r-atlas.

- This just means that $(U, \mathbf{x})$ is C^r-related to every chart in A.
- The maximal C^r-atlas determined by A is exactly composed of all charts compatible with A.
- We say that a chart from the maximal atlas that gives the smooth structure is admissible.
  Charts will be assumed admissible unless otherwise indicated.

Example

The space $\mathbb{R}^n$ itself has an atlas consisting of the single chart $(\mathrm{id}, \mathbb{R}^n)$, where $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.
This atlas determines a (actually the usual) differentiable structure.

Remark

- *The family of sets which are the domains of charts from a maximal atlas provide a basis for a topology on M, called the topology induced by the C^r-structure on M.*
- *I.e. the open sets are exactly the empty set plus arbitrary unions of chart domains from the maximal atlas.*
- *Since a C^r-atlas determines a C^r-structure, we will also call this the topology induced by the atlas.*
- *This topology can be characterized as follows: A subset $V \subset M$ is open iff $\mathbf{x}_\alpha(U_\alpha \cap V)$ is an open subset of Euclidean space for all charts $(U_\alpha, \mathbf{x}_\alpha)$ in any atlas $\{(U_\alpha, \mathbf{x}_\alpha)\}_{\alpha \in A}$ giving the C^r-structure.*

Definition (Differentiable Manifold)

A differentiable manifold of class C^r is a set M together with a specified C^r-structure on M s.t. the topology induced by the C^r-structure is Hausdorff and paracompact.
If the charts are $\mathbb{R}^n$-valued, then we say the manifold has dimension n, written $\dim(M) = n$.

Remark

From now on all manifolds in this lectures will be assumed to be smooth manifolds unless otherwise indicated.
Also, let us refer to an n-dimensional smooth manifold as an "n-manifold".

Example (Real Projective Plane)

The set of all lines through the origin in $\mathbb{R}^3$ is denoted $\mathbb{RP}^2$ and is called the real projective plane.
Let U_z be the set of all lines $l \in \mathbb{RP}^2$ not contained in the x, y-plane.
Every line $l \in U_z$ intersects the plane $z = 1$ at exactly one point of the form $\big(x(l), y(l), 1\big)$.
We can define a bijection $\varphi_z \colon U_z \to \mathbb{R}^2$ by letting $l \mapsto \big(x(l), y(l)\big)$.
This is a chart for $\mathbb{RP}^2$, and there are obviously two other analogous charts $(U_x, \varphi_x), (U_y, \varphi_y)$.
These charts cover $\mathbb{RP}^2$ and form a smooth atlas since they have smooth overlap maps.

Example (Real Projective n-space)

More generally, the set $\mathbb{RP}^n$ of all lines through the origin in $\mathbb{R}^{n+1}$ is called real projective n-space.
We have the surjective map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ given by letting $\pi(x)$ be the line through x and the origin.
We give $\mathbb{RP}^n$ the quotient topology, i.e. $U \subset \mathbb{RP}^n$ is open iff $\pi^{-1}(U)$ is open.
$\mathbb{RP}^n$ is given an atlas consisting of charts of the form (U_i, φ_i), where

$$U_i := \{l \in \mathbb{RP}^n \mid l \text{ is not contained in the hyperplane } x^i = 0\},$$

and $\varphi_i(l)$ is the unique $(u^1, \ldots, u^n)$ s.t. $(u^1, \ldots, u^{i-1}, 1, u^i, \ldots, u^n) \in l$.
Once again it can be checked that the overlap maps are smooth so that we have a smooth atlas.
The topology induced by the atlas is exactly the quotient topology, and it is both paracompact and Hausdorff.

Example (Real Projective n-space cont'd)

It is often useful to view $\mathbb{RP}^n$ as a quotient of the sphere S^n.
Consider the map $S^n \to \mathbb{RP}^n$ given by $x \mapsto l_x$, where l_x is the unique line through the origin in $\mathbb{R}^{n+1}$ which contains x.
Notice that if $l_x = l_y$ for $x, y \in S^n$, then $x = \pm y$.
Consequently, $\mathbb{RP}^n$ is homeomorphic to $S^n/\sim$, where $x \sim y$ iff $x = \pm y$.
We can use this homeomorphism to transfer the differentiable structure to $S^n/\sim$.
We will often identify $S^n/\sim$ with $\mathbb{RP}^n$.

Example

The set of all real $(m \times n)$-matrices $\mathbb{R}^{m \times n}$ is an mn-manifold.
We only need one chart since it is clear that $\mathbb{R}^{m \times n}$ is in one-to-one correspondence with $\mathbb{R}^{mn}$ by the map $(a_{ij}) \mapsto (a_{11}, a_{21}, \ldots, a_{mn})^\top$, i.e. by stacking matrix columns into one long column.

Example

We know already that the set of all nonsingular matrices $\mathrm{GL}(n, \mathbb{R})$ is an open submanifold of $\mathbb{R}^{n \times n} \cong \mathbb{R}^{n^2}$.

Remark

Consider two manifolds M_1, M_2 of dimensions n_1, n_2, respectively.
We form the topological Cartesian product $M_1 \times M_2$.
We may give $M_1 \times M_2$ a differentiable structure in the following way:
Let A_{M_1}, A_{M_2} be atlases for M_1, M_2.
Take as charts on $M_1 \times M_2$ the maps of the form

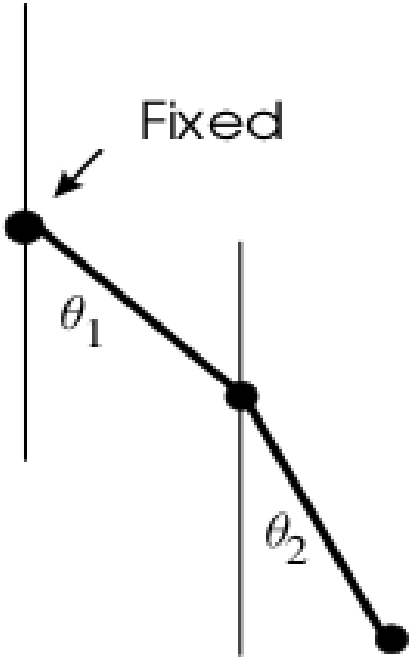
$$\mathbf{x} \times \mathbf{y} \colon U_\alpha \times V_\gamma \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2},$$

where $(U, \mathbf{x})$ is a chart from A_{M_1} and $(V, \mathbf{y})$ a chart from A_{M_2}.
This gives $M_1 \times M_2$ an atlas called the product atlas, which induces a maximal atlas and hence a differentiable structure.
With this product differentiable structure, $M_1 \times M_2$ is called a product manifold.
The product of several manifolds is also possible by an obvious iteration.
The topology induced by the product atlas is the product topology, and so the underlying topological manifold is the product topological manifold discussed earlier.

Example

The circle S^1 is a 1-manifold, and hence the product $T^2 = S^1 \times S^1$ is a 2-manifold, which is a torus (2-torus).

The set of all configurations of a double pendulum constrained to a plane and where the arms are free to swing past each other can be taken to be modeled by $T^2 = S^1 \times S^1$.



Example

For any smooth manifold M we can construct a sheet of a "generalized cylinder" $M \times (a, b)$ over M with some open interval (a, b).

E.g. think of a (closed) curve in the (x, y)-plane as M and $0 \in (a, b) \subset z$-axis.

Real Graßmann Manifolds (Graßmannians)

Let $G_{n,k}$ denote the set of k-dimensional subspaces of $\mathbb{R}^n$.

We will exhibit a natural differentiable structure on this set.

The idea is as follows:

- An alternative way of defining the points of $\mathbb{RP}^{n-1} = G_{n,1}$ is as equivalence classes of n-tuples $(v^1, \ldots, v^n) \in \mathbb{R}^n \setminus \{0\}$.
- Here $(v^1, \ldots, v^n) \sim (\lambda v^1, \ldots, \lambda v^n)$ for any nonzero λ.
- It is a way of specifying a line through the origin.

- Generalizing, we represent a k-plane as an $(n \times k)$-matrix whose column vectors span the k-plane, i.e. form a basis for the k-plane.
- Thus we put an equivalence relation on $\mathbb{R}^{n \times k}$ where $A \sim A \cdot g$ for any $g \in \mathrm{GL}_k$.

Real Graßmannians, cont'd

Let $\mathbb{R}_*^{n\times k}$ be the set of **full rank** $(n \times k)$-matrices.

Two matrices from $\mathbb{R}_*^{n\times k}$ are equivalent exactly if their columns span the same k-dimensional subspace.

Consequently, $G_{n,k} := \mathbb{R}_*^{n\times k}/\sim$ of equivalence classes is in 1-1-correspondence with the set of k-dimensional subspaces of $\mathbb{R}^n$.

Real Graßmannians, cont'd

Let U be the set of all $[A] \in G_{n,k}$, s.t. any representative A has its first k rows linearly independent.

This property is independent of the representative A of the equivalence class $[A]$, therefore U is a well-defined set.

Every $[A] \in U \subset G_{n,k}$ is an equivalence class that has a unique member A_0 of the form

$$\begin{bmatrix} I_k \\ Z \end{bmatrix},$$

obtained e.g. by Gaussian column reduction (I_k is the $(k \times k)$-identity matrix).

Thus we have a map on U defined by $\Psi\colon [A] \mapsto Z \in \mathbb{R}^{(n-k)\times k} \cong \mathbb{R}^{(n-k)k}$.

We wish to cover $G_{n,k}$ with sets similar to U and define similar maps.

Real Graßmannians, cont'd

Consider the set $U_{i_1 \dots i_k}$ of all those equivalence classes $[A] \in G_{n,k}$, s.t. any representative A has the property that the k rows indexed by $i_1, \dots, i_k$ are linearly independent.

The permutation σ putting the k rows indexed by $i_1, \dots, i_k$ into the positions $1, \dots, k$ without changing the relative order of the remaining rows induces a surjective bijection

$$\sigma_{i_1 \dots i_k} \colon U_{i_1 \dots i_k} \to U := U_{1 \dots k}.$$

We now have maps

$$\Psi_{i_1 \dots i_k} \colon U_{i_1 \dots i_k} \to \mathbb{R}^{(n-k)k}, \quad \Psi_{i_1 \dots i_k} := \Psi \circ \sigma_{i_1 \dots i_k}.$$

They form an atlas $\{(U_{i_1 \dots i_k}, \Psi_{i_1 \dots i_k})\}$ for $G_{n,k}$ providing the structure of a smooth manifold called the Graßmann manifold of real k-planes in $\mathbb{R}^n$.

The topology induced by the atlas is the same as the quotient topology.

One can check that this topology is Hausdorff and paracompact.

Definition (Smooth Map)

Let M, N be smooth manifolds with maximal atlases A_M, A_N, respectively.

A map $f\colon M \to N$ is of class C^r at $p \in M$ if there exists a chart $(V, \mathbf{y}) \in A_N$ with $f(p) \in V$, and another chart $(U, \mathbf{x}) \in A_M$ with $p \in U$, s.t. $f(U) \subset V$ and $\mathbf{y} \circ f \circ \mathbf{x}^{-1}$ is of class C^r.

If f is of class C^r at every point $p \in M$, then f is of class C^r.

Maps of class C^∞ are called smooth maps.

Remark

Let $f\colon M \to N$ be a map and suppose that $(U, \mathbf{x}), (V, \mathbf{y})$ are admissible charts for M, N.

If $f^{-1}(V) \cap U \neq \emptyset$, we have a composition

$$\mathbf{y} \circ f \circ \mathbf{x}^{-1}\colon \mathbf{x}\big(f^{-1}(V) \cap U\big) \to \mathbf{y}(V).$$

Maps of this form are called the local representative maps for f.

Definition (Notation of Partial Derivatives)

Let $(U, \mathbf{x})$ be a chart on an n-manifold M with $p \in U$.

Let $\mathbf{x} = (x^1, \ldots, x^n)$. For $f \in C^1(M, \mathbb{R}) = C^1(M)$, define a function $\frac{\partial f}{\partial x^i}$ on U by

$$\frac{\partial f}{\partial x^i}(p) := \lim_{h \to 0} \frac{f \circ \mathbf{x}^{-1}\big(a^1, \ldots, a^i + h, \ldots, a^n\big) - f \circ \mathbf{x}^{-1}\big(a^1, \ldots, a^n\big)}{h},$$

where $\mathbf{x}(p) = (a^1, \ldots, a^n)$.

Put differently,

$$\frac{\partial f}{\partial x^i}(p) := \partial_i (f \circ \mathbf{x}^{-1})\big(\mathbf{x}(p)\big) := \frac{\partial (f \circ \mathbf{x}^{-1})}{\partial u^i}\big(\mathbf{x}(p)\big),$$

where $(u^1, \ldots, u^n)$ denotes the standard coordinates on $\mathbb{R}^n$.

Definition (Diffeomorphism)

Let M, N be smooth (or C^r) manifolds.

A homeomorphism $f\colon M \to N$, s.t. both f and f^{-1} are C^r-differentiable with $r \geq 1$ is called a C^r-diffeomorphism.

For $r = \infty$, we shorten C^∞-diffeomorphism to just diffeomorphism.

The set of all C^r-diffeomorphisms of M onto itself is a group under the operation of composition.

This group is denoted $\mathrm{Diff}^r(M)$.

In the case $r = \infty$, we simply write $\mathrm{Diff}(M)$ and refer to it as the diffeomorphism group of M.

Example

The map

$$r_\theta\colon S^2 \to S^2, \quad r_\theta(x, y, z) := \big(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z\big)$$

for $x^2 + y^2 + z^2 = 1$ is a diffeomorphism.

Definition (Local Diffeomorphism)

Let N, M be smooth n-manifolds.

A map $f\colon M \to N$ is called a local diffeomorphism iff every point $p \in M$ is contained in an open subset $U \subset M$, s.t. $f|_U\colon U \to f(U)$ is a diffeomorphism onto an open subset of N.

For C^r manifolds, a C^r local diffeomorphism is defined similarly.

Example

The map $\pi\colon S^2 \to \mathbb{RP}^2$ given by taking the point (x, y, z) to the line through this point and the origin is a local diffeomorphism.

It is not a diffeomorphism since it is $2-1$ rather than $1-1$.

Cut-off Functions and Partitions of Unity

To study manifolds, it is often necessary to construct various objects (e.g. functions, vector or tensor fields), by gluing together items constructed on the domains of charts.

Partitions of unity and cut-off functions are the crucial technical tools.

Lemma (Existence of Cut-off Function)

Let M be a smooth manifold.

Let $K \subset M$ compact and $K \subset O$ open.

There exists a smooth function on M that is identically equal to 1 on K, takes values in the interval $[0, 1]$, and has compact support in O.

Remark

Cut-off functions are also called bump functions or test functions.

Example

See exercises.

Definition (Partition of Unity)

A partition of unity on a smooth manifold M is a collection $\{\varphi_\alpha\}_{\alpha\in I}$ of smooth functions on M s.t.

1. $0 \leq \varphi_\alpha \leq 1$ for all α.
2. The collection of supports $\{\mathrm{supp}(\varphi_\alpha)\}_{\alpha\in I}$ is locally finite; that is, each point p of M has a neighborhood W_p, s.t. $W_p \cap \mathrm{supp}(\varphi_\alpha) = \emptyset$ for all but a finite number of $\alpha \in I$.
3. $\sum\limits_{\alpha\in I} \varphi_\alpha(p) = 1$ for all $p \in M$.

Let $\mathcal{O} = \{O_\alpha\}_{\alpha\in I}$ be an open cover of M with $\mathrm{supp}(\varphi_\alpha) \subset O_\alpha$ for each $\alpha \in I$.

We say that $\{\varphi_\alpha\}_{\alpha\in I}$ is a partition of unity subordinate to $\mathcal{O} = \{O_\alpha\}_{\alpha\in I}$.

Theorem

Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in I}$ be a locally finite cover of M. If each U_α has compact closure, then there is a partition of unity $\{\varphi_\alpha\}_{\alpha\in I}$ subordinate to $\{U_\alpha\}_{\alpha\in I}$.

Idea of Proof:
Consider the so called shrinking lemma, e.g. lemma 2.1.4 in [20]. □

Concept of Submanifold

Definition (Regular Submanifold)

A subset S of a smooth n-manifold M is called a regular submanifold of dimension k if every point $p \in S$ is in the domain of a chart $(U, \mathbf{x})$ that has the following regular submanifold property with respect to S:

$$\mathbf{x}(U \cap S) = \mathbf{x}(U) \cap \left(\mathbb{R}^k \times \{c\}\right) \text{ for some } c \in \mathbb{R}^{n-k}.$$

The integer $n - k$ is called the codimension of S (in M), and we say that S is a regular submanifold of codimension $n - k$.

Remark

W.l.o.g $c = 0$.
If a subset $S \subset M$ is covered by charts of M of the above type, then S itself is said to have the (regular) submanifold property.

We will refer to such charts as being single-slice charts (adapted to S).

Example

The unit sphere $S^n \subset \mathbb{R}^{n+1}$ is a regular submanifold of $\mathbb{R}^{n+1}$.
Indeed, let $W_i^\pm := \{(a^1, \dots, a^{n+1}) \in \mathbb{R}^{n+1} \mid \pm a^i > 0\}$.
Then define $\psi_i^\pm \colon W_i^\pm \to \psi_i^\pm(\mathbb{R}^{n+1})$ by

$$\psi_i^\pm(a^1, \dots, a^{n+1}) = (a^1, \dots, a^{i-1}, a^{i+1}, \dots, a^{n+1}, \|a\| - 1).$$

These give charts on $\mathbb{R}^{n+1}$ smoothly related to the standard chart.
If $p \in S^2$ then p is in the domain of one of the charts $\psi_i^\pm$.
Identifying $\mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}$, we get $\psi_i^\pm(W_i^\pm \cap S^n) = \psi_i^\pm(U) \cap (\mathbb{R}^n \times \{0\})$.

I.e., these charts have the submanifold property with respect to S^n.
Let $\mathrm{pr} \colon \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$.
The resulting charts on S^n given by $\mathrm{pr} \circ \psi_i^\pm|_{W_i^\pm \cap S^n}$ have the form

$$(a^1, \dots, a^{n+1}) \mapsto (a^1, \dots, a^{i-1}, a^{i+1}, \dots, a^{n+1}).$$

These are projections from S^n onto coordinate hyperplanes in $\mathbb{R}^{n+1}$ and give the same smooth structure as the stereographic charts from the exercises.

Tangent Structures

Motivation:

- Consider the smooth curve $c\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^n$.
- We want to visualize the "velocity vector" $\dot{c}(0)$ being based at $p = c(0)$.
- It is often desirable to explicitly form a separate n-dimensional vector space for each point p, whose elements are to be thought of as being based at p.
- One way to do this is to use $\{p\} \times \mathbb{R}^n$ s.t. a tangent vector based at p is taken to be a pair (p, v) where $v \in \mathbb{R}^n$.
  Clearly $\{p\} \times \mathbb{R}^n$ inherits a vector space structure from $\mathbb{R}^n$.

Motivation cont'd:

- We *provisionally* denote $\{p\} \times \mathbb{R}^n =: T_p\mathbb{R}^n$ the tangent space at p.
- Let $c(t) = \big(x^1(t), \ldots, x^n(t)\big)$, then the velocity vector of a curve c at "time" (curve parameter) $t = 0$ is $\big(p, (\frac{\mathrm{d}x^1}{\mathrm{d}t}(0), \ldots, \frac{\mathrm{d}x^n}{\mathrm{d}t}(0))\big)$, being based at $p = c(0)$.
- Unfortunately, both $\big(p, (\frac{\mathrm{d}x^1}{\mathrm{d}t}(0), \ldots, \frac{\mathrm{d}x^n}{\mathrm{d}t}(0))\big)$ and $\big(\frac{\mathrm{d}x^1}{\mathrm{d}t}(0), \ldots, \frac{\mathrm{d}x^n}{\mathrm{d}t}(0)\big)$ are often denoted by $\dot{c}(0) = c'(0)$.
- More generally, if V is a finite-dimensional vector space, then V is a smooth manifold and the tangent space at $p \in V$ can be provisionally taken to be the set $\{p\} \times V$.
- If $v_p := (p, v)$ is a tangent vector at p, then v is called the principal part of v_p.

Remark

1. *We have a natural isomorphism between* $\mathbb{R}^n \cong T_p\mathbb{R}^n$ *given by* $v \mapsto (p, v)$, *for any* p.

   Of course we also have a natural isomorphism $T_p\mathbb{R}^n \cong T_q\mathbb{R}^n$ *for any pair of points given by* $(p, v) \mapsto (q, v)$.
2. *However, we will define in due course the notion of a tangent space at a point of an arbitrary differentiable manifold (not necessarily a vector space!).*
3. *It will be seen that there is, in general,* **no canonical** *way to identify tangent spaces at different points.*

Three Essentially Equivalent Notions of the Tangent Space

Definition (Version 1: Tangent Space via Curves or Kinematic Version)

Let p be a point in a smooth n-manifold M.

Consider smooth curves $c_1, c_2 \colon (-\varepsilon, \varepsilon) \to M$ with $c_1(0) = c_2(0) = p$.

We say that c_1 is tangent to c_2 at p if for all smooth real-valued functions f defined on an open neighborhood of p, we have

$$(f \circ c_1)'(0) = (f \circ c_2)'(0).$$

This is an equivalence relation on the set of all such curves.

Define a tangent vector at p to be an equivalence class under this relation.

If $v_p = [c]$ then we will also write $\dot{c}(0) = v_p$.

The tangent space T_pM is the set of all tangent vectors at $p \in M$.

Remark

A cut-off function argument shows that $c_1 \sim c_2$ *iff* $(f \circ c_1)'(0) = (f \circ c_2)'(0)$ *for* **all** *globally defined smooth functions* $f\colon M \to \mathbb{R}$.

Remark

Using the above mentioned cut-off argument together with the first definition (kinematic) above we can alternatively look at the following:

Definition

Given an n-manifold M, for any $p \in M$, two C^1-curves, $\gamma_1\colon (-\varepsilon_1, \varepsilon_1) \to M$ and $\gamma_2\colon (-\varepsilon_2, \varepsilon_2) \to M$, with $\gamma_1(0) = \gamma_2(0) = p$ are equivalent iff there is some chart, (U, φ), at p s.t.

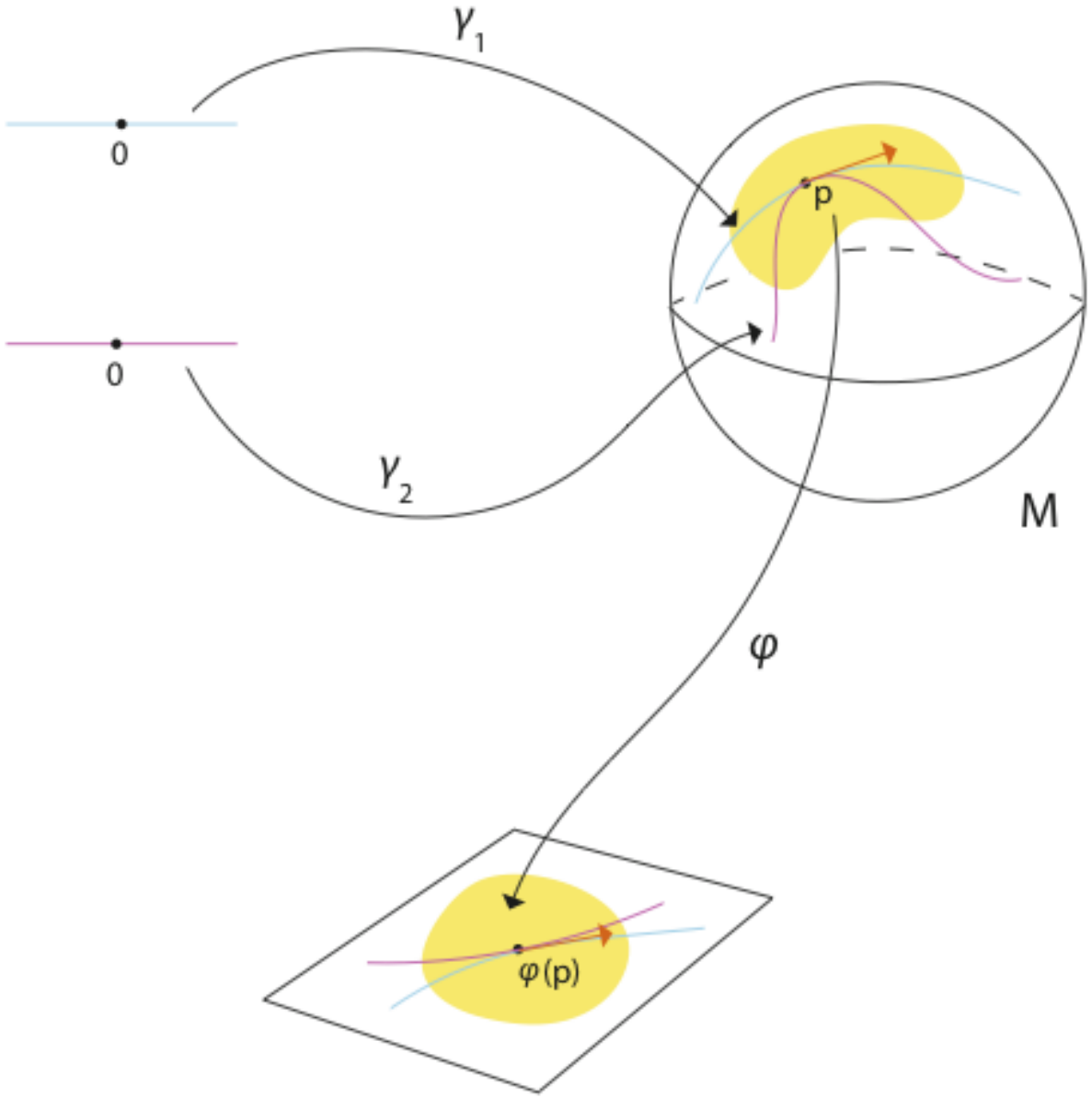
$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

However, this definition at first glance seems to depend on the choice of the chart.

Fortunately, this is not the case:
If (V, ψ) is another chart at p, we have $p \in (U \cap V)$,
so $\eta = \psi \circ \varphi^{-1}$ is C^k.

By the chain rule

$$\begin{aligned}
(\psi \circ \gamma_1)'(0) &= \big(\eta \circ \varphi \circ \gamma_1\big)'(0) \\
&= \eta'\big(\varphi(p)\big) \cdot \big((\varphi \circ \gamma_1)'(0)\big) \\
&= \eta'\big(\varphi(p)\big) \cdot \big((\varphi \circ \gamma_2)'(0)\big) \\
&= \big(\eta \circ \varphi \circ \gamma_2\big)'(0) \\
&= (\psi \circ \gamma_2)'(0).
\end{aligned}$$


γ_1
0
0
γ_2
p
M
φ
$\varphi(p)$

Three Essentially Equivalent Notions of the Tangent Space, cont'd

Remark

The above first version of a definition of the tangent space via curves is very geometric, it has one disadvantage:

It is not so obvious that T_pM is a vector space in a natural way.

The following principle is used to obtain a vector space structure:

Proposition (Consistent Transfer of Linear Structure)

Suppose that S is a set and $\{V_\alpha\}_{\alpha\in I}$ is a family of n-dimensional vector spaces.
Suppose that for each α we have a bijection $b_\alpha\colon V_\alpha \to S$.
If for every $\alpha, \beta \in I$ the map $b_\beta^{-1} \circ b_\alpha\colon V_\alpha \to V_\beta$ is a linear isomorphism, then there is a unique vector space structure on S,
s.t. each b_α is a linear isomorphism.

We use the proposition to show the natural vector space structure on T_pM. For every chart $(U_\alpha, \mathbf{x}_\alpha)$ with $p \in U$, we have a map $b_\alpha \colon \mathbb{R}^n \to T_pM$ given by $v \mapsto [\gamma_v]$, where $\gamma \colon t \mapsto \mathbf{x}_\alpha^{-1}((x_\alpha(p) + tv)$ for t in a sufficiently small but otherwise irrelevant interval containing 0.

Lemma

For each chart $(U_\alpha, \mathbf{x}_\alpha)$, the map $b_\alpha \colon \mathbb{R}^n \to T_pM$ is a bijection and $b_\beta^{-1} \circ b_\alpha = \mathrm{D}(x_\beta \circ x_\alpha^{-1})(x_\alpha(p))$.

Proof: b_α is injective. Indeed,

$$(\mathbf{x}_\alpha \circ \gamma_v)'(0) = \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_\alpha \circ \mathbf{x}_\alpha^{-1}(\mathbf{x}_\alpha(p) + tv)|_{t=0} = \tfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}_\alpha(p) + tv)|_{t=0} = v.$$

Suppose that $[\gamma_v] = [\gamma_w]$ for $v, w \in \mathbb{R}^n$. Then

$$v = (\mathbf{x}_\alpha \circ \gamma_v)'(0) = (\mathbf{x}_\alpha \circ \gamma_w)'(0) = w.$$

This means that b_α is injective.

Next we show that b_α is surjective as well.

Let $[c] \in T_pM$ be represented by $c\colon (-\varepsilon, \varepsilon) \to M$.

Let $v := (\mathbf{x}_\alpha \circ c)'(0) \in \mathbb{R}^n$.

Then we have $b_\alpha(v) = [\gamma_v]$, where $\gamma_v\colon t \mapsto \mathbf{x}_\alpha^{-1}(\mathbf{x}_\alpha(p) + tv)$.

But $[\gamma_v] = [c]$ since for any smooth f defined near p we have

$$\begin{aligned}(f \circ \gamma_v)'(0) &= \tfrac{\mathrm{d}}{\mathrm{d}t} f \circ \mathbf{x}_\alpha^{-1}\big(\mathbf{x}_\alpha(p) + tv\big)|_{t=0} \\ &= \mathrm{D}(f \circ \mathbf{x}_\alpha^{-1})\big(\mathbf{x}_\alpha(p)\big) \cdot v \\ &= \mathrm{D}(f \circ \mathbf{x}_\alpha^{-1})\big(\mathbf{x}_\alpha(p)\big) \cdot (\mathbf{x}_\alpha \circ c)'(0) \\ &= (f \circ c)'(0)\end{aligned}$$

Thus b_α is surjective.

Now $[c] \mapsto (\mathbf{x}_\alpha \circ c)'(0)$ is well-defined, and it is exactly b_α^{-1}.

Finally,

$$\begin{aligned}{b_\beta}^{-1} \circ b_\alpha(v) &= \tfrac{\mathrm{d}}{\mathrm{d}t} \mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1}\big(\mathbf{x}_\alpha(p) + tv\big)|_{t=0} \\ &= \mathrm{D}(\mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1})\big(\mathbf{x}_\alpha(p)\big) v.\end{aligned}$$

Three Essentially Equivalent Notions of the Tangent Space, cont'd

Remark

- *Temporarily, the so defined tangent space (1st version) we call kinematic tangent space* $(T_pM)_{\mathrm{kin}}$.
- *Let C_p denote the set of smooth curves c defined on some open interval containing 0, s.t. $c(0) = p$.*

  Then $(T_pM)_{\mathrm{kin}} = C_p/\sim$, where the equivalence is as described above.

Three Essentially Equivalent Notions of the Tangent Space, Cont'd:

Let A be the maximal atlas for an n-manifold M. For $p \in M$, consider

$$\Gamma_p := \Big\{ \big(p, v, (U, \mathbf{x})\big) \in \{p\} \times \mathbb{R}^n \times A, \mid p \in U \Big\}.$$

Define an equivalence relation on Γ_p by requiring

$$\big(p, v, (U, \mathbf{x})\big) \sim \big(p, w, (V, \mathbf{y})\big) \;\; :\Longleftrightarrow \;\; w = \mathrm{D}(\mathbf{y} \circ \mathbf{x}^{-1})|_{\mathbf{x}(p)} \cdot v.$$

In other words, the derivative at $\mathbf{x}(p)$ of the coordinate change $\mathbf{y} \circ \mathbf{x}^{-1}$ *identifies* v with w.

The set $\Gamma_p / \sim$ can be given a vector space structure as follows:
For each $(U, \mathbf{x})$ containing p, we have a map

$$b_{(U,\mathbf{x})} \colon \mathbb{R}^n \to \Gamma_p / \sim, \quad v \mapsto [(p, v, (U, \mathbf{x}))]$$

To see that this map is a bijection, notice that

$$\Big[\big(p, v, (U, \mathbf{x})\big)\Big] = \Big[\big(p, w, (U, \mathbf{x})\big)\Big] \implies v = \mathrm{D}(\mathbf{x} \circ \mathbf{x}^{-1})|_{\mathbf{x}(p)} \cdot v = w.$$

We obtain a vector space structure on $\Gamma_p / \sim$ via the above proposition (Consistent Transfer of Linear Structure).

Definition (Version 2: Tangent Space via Charts or Physicists' Notion)

The elements of $\Gamma_p/\sim$ are called tangent vectors.

Remark

Clearly, this is another version of the tangent space at p, and we shall (temporarily) denote this by $(T_pM)_{\mathrm{phys}}$.

The subscript "phys" refers to the fact that this version of the tangent space is based on a **"transformation law"** *and corresponds to a way of looking at things traditionally popular among physicists.*

If $v_p := \big[(p, v, (U, \mathbf{x})\big] \in (T_pM)_{\mathrm{phys}}$, then we say that $v \in \mathbb{R}^n$ represents v_p w.r.t. the chart $(U, \mathbf{x})$.

This viewpoint takes on more familiar appearance if we use **classical notation**.

Let $(U, \mathbf{x}), (V, \mathbf{y})$ be charts containing p in their domains.

If an n-tuple $(v^1, \ldots, v^n)$ represents a tangent vector at p from the point of view of $(U, \mathbf{x})$, and if the n-tuple $(w^1, \ldots, w^n)$ represents the **same** vector in the chart $(V, \mathbf{y})$, then

$$w^i = \sum_{j=1}^{n} \frac{\partial y^i}{\partial x^j}\Big|_{\mathbf{x}(p)} v^j$$

where we write the **change of coordinates** as $y^i = y^i(x^1, \ldots, x^n)$ with $1 \leq i \leq n$.

To introduce the third approach to the tangent space we abstract the notion of directional derivative.

Definition (Version 3: Tangent Space via Derivations or Algebraic Tangent Space)

A tangent vector v_p at p is a linear map $v_p \colon C^\infty(M) \to \mathbb{R}$ s.t. for $f, g \in C^\infty(M)$,

$$v_p(fg) = g(p)v_p(f) + f(p)v_p(g) \qquad (\textit{Leibniz' rule}).$$

We say that a tangent vector at p is a derivation of the algebra $C^\infty(M)$ w.r.t. to the evaluation map ev_p at p defined by $\mathrm{ev}_p(f) := f(p)$.

Alternatively, we say that v_p is a derivation at p.

The set of such derivations at p is a vector space.

We temporarily distinguish this version of the tangent space by denoting it $(T_pM)_{\mathrm{alg}}$ and referring to it as the algebraic tangent space.

Lemma

The three notions of tangent space are equivalent.

Sketch of Proof:
Let M be a n-manifold. Consider a tangent vector v_p as an equivalence class of curves represented by $c\colon I \to M$ with $c(0) = p$.
We obtain a derivation by defining

$$v_p f := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ c.$$

This gives a map $(T_pM)_{\mathrm{kin}} \to (T_pM)_{\mathrm{alg}}$ which can be shown to be an isomorphism.

We also have a natural isomorphism $(T_pM)_{\mathrm{kin}} \to (T_pM)_{\mathrm{phys}}$:

Given $[c] \in (T_pM)_{\mathrm{kin}}$, we obtain an element $v_p \in (T_pM)_{\mathrm{phys}}$ by letting v_p be the equivalence class of the triple $\big(p, v, (U, \mathbf{x})\big)$, where $v^i := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} x^i \circ c$ for a chart $(U, \mathbf{x})$ with $p \in U$.

Sketch of Proof cont'd:

If v_p is a derivation at p and $(U, \mathbf{x})$ an admissible chart with domain containing p, then v_p, as a tangent vector in the sense of the physicists' definition, is represented by the triple $\big(p, v, (U, \mathbf{x})\big)$, where $v = (v^1, \ldots, v^n)$ is given by

$$v^i = v_p x^i \quad (v_p \text{ is acting as a derivation}).$$

This gives us an isomorphism $(T_pM)_{\mathrm{alg}} \to (T_pM)_{\mathrm{phys}}$.

Next we exhibit the inverse isomorphism $(T_pM)_{\mathrm{phys}} \to (T_pM)_{\mathrm{alg}}$:

Suppose that $\big[\big(p, v, (U, x)\big)\big] \in (T_pM)_{\mathrm{phys}}$ where $v \in \mathbb{R}^n$. We obtain a derivation by defining

$$v_p f := \mathrm{D}\left(f \circ \mathbf{x}^{-1}\right)\big|_{\mathbf{x}(p)} \cdot v.$$

In other words,

$$v_p f = \sum_{i=1}^{n} v^i \tfrac{\partial}{\partial x^i}\big|_p f, \quad \text{for } v = (v^1, \ldots, v^n).$$

Moreover, v_p defined in this way is independent of the representative triple $\big(p, v, (U, x)\big)$. $\square$

Remark

However, matters become more subtle if one would consider the vector space of derivations of $C^r(M)$ at a point for $r < \infty$.

The reason is that this would not give a finite-dimensional vector space and so it is not a good candidate for the definition of the tangent space.

We do not deepen this here.

Remark

- *We now adopt the explicitly flexible attitude of interpreting a tangent vector in any of the three ways depending on the situation.*
- *Thus we effectively identify* $(T_pM)_{\mathrm{phys}} \cong (T_pM)_{\mathrm{alg}} \cong (T_pM)_{\mathrm{kin}}$.
- *Henceforth we use the* **single** *notation T_pM for the tangent space of a manifold M at a point p.*

Definition (Cotangent Space)

The dual space to a tangent space T_pM is called the cotangent space and is denoted by T_p^*M.

An element of T_p^*M is referred to as a covector.

The basis for T_p^*M that is dual to the coordinate basis $\left(\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^n}|_p\right)$ is denoted $(\mathrm{d}x^1|_p, \ldots, \mathrm{d}x^n|_p)$.

By definition $\mathrm{d}x^i|_p\left(\frac{\partial}{\partial x^j}|_p\right) = \delta^i_j$, where we have used the Kronecker symbol.

Example (Tangent space of a vector space)

The provisional definition of the tangent space at p in a vector space V was $\{p\} \times V$.
This does not immediately fit in any of the three above definitions.
It is remedied by finding a natural isomorphism $\{p\} \times V \cong T_pV$:
One may pick one version and then exhibit a natural isomorphism directly.
But we take a slightly different approach.
Namely, define a map $j_p \colon V \to T_pV$.
We think in terms of equivalence classes of curves.
For each $v \in V$ $c_{p,v} \colon \mathbb{R} \to V$ define $c_{p,v}(t) := p + tv$. Then

$$j_p(v) := [c_{p,v}] \in T_pV.$$

As a derivation, $j_p(v)$ acts according to

$$f \mapsto \tfrac{\mathrm{d}}{\mathrm{d}t} f(p + tv)|_{t=0}.$$

On the other hand, we have the obvious projection $\mathrm{pr}_2 \colon \{p\} \times V \to V$.
Then our natural isomorphism $\{p\} \times V \cong T_pV$ is just $j_p \circ \mathrm{pr}_2$.

Remark

- *Because of the identification of* $\{p\} \times V \cong T_pV$ *, we shall often denote by* pr_2 *the map* $T_pV \to V$*.*
- *Furthermore, in certain contexts,* T_pV *is identified with* V *itself,* **without any further comment***.*
- *The potential identifications introduced here are often referred to as canonical or natural and* $j_p \colon V \to T_pV$ *is often called the canonical or natural isomorphism.*
- *The inverse map* $T_pV \to V$ *is also referred to as the canonical or natural isomorphism.*
- *Context will keep things straight.*

The Tangent Map

Following our earlier approach we define three equivalent notions of the tangent map at $p \in M$ of a smooth map $f\colon M \to N$.

Given f and p define a *linear* map $T_p f\colon T_p M \to T_{f(p)} N$.

For the first definition we think of $T_p M$ as $(T_p M)_{\mathrm{kin}}$.

Definition (Tangent Map I)

Consider a smooth map between manifolds $f\colon M \to N$, with $p \in M$ and image $q = f(p) \in N$. Define the tangent map at p,

$$T_p f\colon T_p M \to T_q N,$$

in the following way: Suppose $v_p \in T_p M$ with curve c s.t. $c(0) = p$, i.e., $v_p = [c]$. Then

$$T_p f \cdot v_p := [f \circ c] \in T_q N,$$

where $[f \circ c] \in T_q N$ is the vector represented by the curve $f \circ c$.

Theorem (Chain Rule for Tangent Maps)

Let $f\colon M \to N, g\colon N \to P$ be smooth.

For each $p \in M$ we have $T_p(g \circ f) = (T_{f(p)}g) \circ T_p f$.

Proof:

Let $v \in T_pM$ be represented by the curve c, s.t. $v = [c]$.

Then $f \circ c$ represents $T_pf(v)$ and we have

$$\begin{aligned} T_p(g \circ f)(v) &= [(g \circ f) \circ c] \\ &= [g \circ (f \circ c)] \\ &= (T_{f(p)}g)\big(T_pf(v)\big) \\ &= \big((T_{f(p)}g) \circ T_pf\big)(v). \end{aligned}$$

$\square$

For the next alternative definition we consider T_pM as $(T_pM)_{\mathrm{phys}}$.

Definition (Tangent Map II)

Let $f\colon M \to N$ be a smooth map and $p \in M$ with $q = f(p) \in N$.

Choose a chart $(U, \mathbf{x})$ containing p and a chart $(V, \mathbf{y})$ containing $q = f(p)$ s.t. for $v_p \in T_pM$ we have the representative $(p, v, (U, \mathbf{x}))$.

Then the tangent map $T_pf\colon T_pM \to T_{f(p)}N$ is defined by letting the representative of $T_pf \cdot v_p$ in the chart $(V, \mathbf{y})$ be given by $(q, w, (V, \mathbf{y}))$, where

$$w = \mathrm{D}(\mathbf{y} \circ f \circ \mathbf{x}^{-1}) \cdot v.$$

This uniquely determines $T_pf \cdot v$.

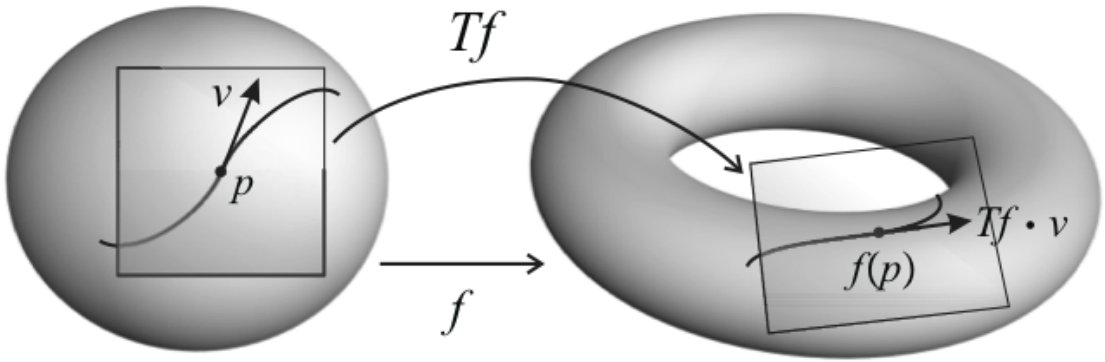
The chain rule guarantees that this is well-defined, i.e., independent of the choice of charts.

Definition (Tangent Map III)

Let M, N be smooth manifolds with smooth map $f\colon M \to N$ and $p \in M$. Define $T_p f \cdot v_p$ as a derivation by

$$(T_p f \cdot v_p) g := v_p(\underbrace{g \circ f}_{\in C^\infty(M)})$$

for each $g \in C^\infty(N)$.This map is yet another version of $T_p f$.



For examples for tangent contructions w.r.t the three approaches, see the exercises.

Definition

Let M be a smooth manifold with $p \in M$. For $f \in C^\infty(M)$ define the derivative of f at p as the linear map $\mathrm{D}\, f(p)\colon T_pM \to \mathbb{R}$ given by

$$\mathrm{D}\, f(p) \cdot v_p := v_p f \text{ for all } v_p \in T_pM.$$

Thus $\mathrm{D}\, f(p) \in T_p^*M$.

Definition

Let $I = (a, b) \subset \mathbb{R}$. If $c\colon I \to M$ is a smooth curve, the velocity at $t_0 \in I$ is the vector

$$\dot{c}(t_0) \in T_{c(t_0)}M, \quad \dot{c}(t_0) := T_{t_0}c \cdot \frac{\partial}{\partial u}\Big|_{t_0},$$

where $\frac{\partial}{\partial u}\Big|_{t_0}$ is the coordinate basis vector at $t_0 \in T_{t_0}I = T_{t_0}\mathbb{R}$ associated to the standard coordinate function on $\mathbb{R}$ (denoted here by u).
We may also occasionally write c' for $\dot{c}$.

Remark

Consider a smooth function f defined in a neighborhood of $c(t_0)$, then $\dot{c}(t_0)$ acts as a derivation as follows:

$$\dot{c}(t_0) \cdot f = \frac{\mathrm{d}}{\mathrm{d}t} f \circ c|_{t=t_0}.$$

$\dot{c}(t_0)$ is also denoted by $\frac{\mathrm{d}}{\mathrm{d}t} c|_{t_0}$.

In this notation, $\dot{c} = \frac{\mathrm{d}}{\mathrm{d}t} c$ is a map that assigns to every $t \in I$ the tangent vector $\dot{c}(t) := T_t c \cdot \frac{\partial}{\partial u}|_t$, and this is referred to as the velocity field along the curve c.

Also note that we may view $\dot{c}(t_0)$ as the equivalence class of the curve $t \mapsto c(t + t_0)$.

Definition

Let V be a vector space.

For a smooth $f\colon M \to V$ with $p \in M$ as above, the derivative $\mathrm{D}\, f(p)\colon T_pM \to V$ is the composition of the tangent map T_pf and the canonical map $T_yV \to V$ where $y = f(p)$,

$$\mathrm{D}\, f(p)\colon T_pM \overset{Tf}{\to} T_yV \to V.$$

Remark

The notational distinction between T_pf and $\mathrm{D}\, f_p$ is not universal, and $\mathrm{D}\, f_p$ is itself often used to denote T_pf!

Theorem (Inverse Mapping Theorem for Manifolds)

Let $f\colon M \to N$ be a smooth map s.t. $T_p f\colon T_p M \to T_q N$ is an isomorphism.

Then there exists an open neighborhood O of p, s.t. $f(O)$ is open and $f|_O\colon O \to f(O)$ is a diffeomorphism.

If $T_p f$ is an isomorphism for all $p \in M$, then $f\colon M \to N$ is a local diffeomorphism.

Idea of Proof:
Apply the standard inverse mapping theorem by using charts. □

Tangents of Products

Suppose that $f\colon M_1 \times M_2 \to N$ is a smooth map.
For fixed $p \in M_1, q \in M_2$, consider the insertion maps

$$\iota_p\colon y \mapsto (p, y), \quad \iota^q\colon x \mapsto (x, q).$$

Then $f \circ \iota^q, f \circ \iota_p$ are often denoted by $f(\cdot, q), f(p, \cdot)$, respectively.

Definition

Let $f\colon M_1 \times M_2 \to N$ be as above. Define the partial tangent maps $\mathrm{D}_1 f$ and $\mathrm{D}_2 f$ by

$$(\mathrm{D}_1 f)(p, q) := T_p(f \circ \iota^q)\colon T_p M_1 \to T_{f(p,q)} N,$$
$$(\mathrm{D}_2 f)(p, q) := T_q(f \circ \iota_p)\colon T_q M_2 \to T_{f(p,q)} N.$$

Lemma (Partials Lemma)

For a map $f\colon M_1 \times M_2 \to N$, we have

$$T(p, q)f \cdot (v, w) = (\mathrm{D}_1 f)(p, q) \cdot v + (\mathrm{D}_2 f)(p, q) \cdot w,$$

where we used the identification $T(p, q)(M_1 \times M_2) = T_p M_1 \times T_q M_2$. □

Critical Points and Values

Definition

Let $f\colon M \to N$ be a C^r-map and $p \in M$.

We say p is a regular point for f if $T_p f$ is a surjection.

Otherwise, p is called a critical point or singular point.

A point $q \in N$ is called a regular value of f if every point in the preimage $f^{-1}\{q\}$ is a regular point for f.
This includes the case where $f^{-1}\{q\}$ is empty.

A point of N that is not a regular value is called a critical value.

Remark

Most values of a smooth map are regular values.

In order to make this precise, we will introduce the notion of measure zero on a second countable smooth manifold.

We could define a Lebesgue measure on such a manifold, but for now the notion of measure zero is all we need.

Definition

A subset $A \subset \mathbb{R}^n$ is said to be of measure zero if for any $\varepsilon > 0$ there is a sequence of balls $(B_i)_i$ s.t. $A \subset \cup_i B_i$ and $\sum_i \mathrm{vol}(B_i) < \varepsilon$.

Here, $\mathrm{vol}(B_i)$ denotes the volume of the i-th ball.

Lemma

Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^n$ a C^1 map.

If $A \subset U$ has measure zero, then $f(A)$ has measure zero as well. □

This lemma allows us to make the following definition:

Definition

Let M be an n-manifold that is second countable.

A subset $A \subset M$ is said to be of measure zero if for every admissible chart $(U, \mathbf{x})$ the set $\mathbf{x}(A \cap U)$ has measure zero in $\mathbb{R}^n$.

In this definition we need the manifold to be second countable so that every atlas has a countable subatlas.

Theorem (Sard)

Let N be an n-manifold and M an m-manifold, both assumed second countable.

For a smooth map $f\colon N \to M$, the set of critical values has measure zero.

Proof:
See e.g. the good read [4, 5]. □

Corollary

If M, N are second countable manifolds, then the set of regular values of a smooth map $f\colon M \to N$ is dense in N. □

Tangent and Cotangent Bundles

Definition (Tangent Bundle)

The tangent bundle of a manifold M is the (disjoint) union of the tangent spaces

$$TM := \cup_{p \in M} T_p M.$$

In due course we show that TM is a smooth manifold.

Definition (Tangent Lift)

Given a smooth map $f\colon M \to N$, the tangent maps $T_p f$ on the individual tangent spaces combine to give a map

$$Tf\colon TM \to TN$$

being linear on each fiber.
The map is called tangent map or tangent lift of f.

Lemma (Chain Rule)

For smooth maps $f\colon M \to N$ and $g\colon N \to P$ we have the following version of the chain rule:

$$T(g \circ f) = Tg \circ Tf.$$

□

Definition (Derivative)

If $f\colon M \to V$, V a finite-dimensional vector space, then we have the derivative $\mathrm{D}\, f(p)\colon T_pM \to V$ of f evaluated at p.

These maps can be combined to give a single map $\mathrm{D}\, f\colon TM \to V$, called the derivative of f, which is defined by $\mathrm{D}\, f|_p(v) = \mathrm{D}\, f(p)(v)$ when $v \in T_pM$.

Definition

The map $\pi_{TM}\colon TM \to M$ defined by $\pi_{TM}(v) = p$ if $v \in T_pM$ is called the tangent bundle projection map.

Proposition

For any smooth n-manifold M, the set TM is a smooth $2n$-manifold and $\pi_{TM}\colon TM \to M$ is a smooth map.

Furthermore, for a smooth map $f\colon M \to N$, the tangent map Tf is smooth and the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Sketch of Proof: For every chart $(U, \mathbf{x})$, let $TU = \pi^{-1}(U)$ and consider the map $T\mathbf{x}\colon TU \to \mathbf{x}(U) \times \mathbb{R}^n$.
The pair $(TU, T\mathbf{x})$ actually is a chart on TM:

Suppose $(TU, T\mathbf{x}), (TV, T\mathbf{y})$ are two such charts constructed from $(U, \mathbf{x}), (V, \mathbf{y})$ with $U \cap V \neq \emptyset$.
Then $TU \cap TV \neq \emptyset$ and on the overlap we have the coordinate transitions $T\mathbf{y} \circ T\mathbf{x}^{-1}\colon (x, v) \mapsto (y, w)$ where

$$y = \mathbf{y} \circ \mathbf{x}^{-1}(x), \quad w = \mathrm{D}(\mathbf{y} \circ \mathbf{x}^{-1})|_x v.$$

Thus the overlap maps are smooth.

Moreover, $T\mathbf{x}(TU \cap TV)$ and $T\mathbf{y}(TU \cap TV)$ are open.

We obtain a smooth atlas on TM from that on M generating a topology.
It follows that TM is Hausdorff and paracompact.
To test smoothness of π, we look at maps $\mathbf{x} \circ \pi \circ (T\mathbf{x})^{-1}$. We have

$$\mathbf{x} \circ \pi \circ (T\mathbf{x})^{-1}(x, v) = \mathbf{x} \circ \pi\left(v^i \frac{\partial}{\partial x^i}\bigg|_{\mathbf{x}^{-1}(x)}\right) = \mathbf{x} \circ \mathbf{x}^{-1}(x) = x,$$

which is just a projection and so clearly smooth. $\square$

Definition (Trivialization)

A diffeomorphism $F\colon TM \to M \times V$ s.t. the map $F|_{T_pM}\colon T_pM \to \{p\} \times V$ is linear for each p s.t. the above diagram commutes is called a (global) trivialization of TM.

If a (global) trivialization exists, then we simply say TM is trivial.

For an open set $U \subset M$, a trivialization of TU is called a local trivialization of TM over U.

Remark

- *For most manifolds, there does* **not** *exist a global trivialization of the tangent bundle.*
- *On the other hand, every point p in a manifold M is contained in an open set U s.t. TM has a local trivialization over U.*
- *The existence of these local trivializations is quickly deduced from the existence of the special charts which we constructed above for a tangent bundle.*

Definition (Cotangent Bundle)

Define the cotangent bundle of a manifold M to be the set

$$T^*M := \cup_{p \in M} T_p^* M$$

and define the map $\pi_{T^*M} \colon T^*M \to M$ to be the obvious projection taking elements in each space $T_p^* M$ to the corresponding point p.

Remark

Suppose that $f \colon M \to N$ is a smooth map.

It is important to notice that even though for each $p \in M$, the map $T_p f \colon T_p M \to T_{f(p)} N$ has a dual map $(T_p f)^* \colon (T_{f(p)} N)^* \to (T_p M)^*$, these maps **do not** *generally combine to give a map from $T^*N \to T^*M$.*

In general, there is nothing like a "cotangent lift".

To see this, just consider the case where f is constant.

The situation in symplectic geometry, however, might become different.

Similar to the tangent bundle we have

Proposition

For any smooth n-manifold M, the set T^*M is a smooth $2n$-manifold and $\pi_{T^*M}\colon T^*M \to M$ is a smooth map.

Remark

- *In summary, both TM and T^*M are smooth manifolds whose smooth structure is derived from the smooth structure on M in a natural way.*
- *In both cases, the charts are derived from charts on the base M and are given by the n coordinates of the base point together with the n components of the element of TM (or T^*M) in the corresponding coordinate frame.*

Vector Fields

Roughly, a vector field is a smooth assignment of a tangent vector to each point of a manifold.

Definition

If $\pi\colon M \to N$ is a smooth map, then a (global) section of π is a map

$$\sigma\colon N \to M \quad \text{s.t.} \quad \pi \circ \sigma = \mathrm{id}\,.$$

If σ is defined only on an open subset $U \subset N$ and $\pi \circ \sigma = \mathrm{id}_U$, then we call σ a local section.

If σ is a smooth (or C^r) map, we call σ a smooth (or C^r) section.

Remark

Clearly, if $\pi\colon M \to N$ has a (global) section, then it must be surjective.

Definition

A smooth vector field on M is a smooth map $X\colon M \to TM$ s.t. $X(p) \in T_pM$ for all $p \in M$.
In other words, a vector field on M is a smooth section of the tangent bundle $\pi\colon TM \to M$. We often write $X_p = X(p)$.

Remark

A local section of TM defined on an open set U is just the same thing as a vector field on the open manifold U.
If $(U, \mathbf{x})$ is a chart on a smooth n-manifold, then writing $\mathbf{x} = (x^1, \dots, x^n)$, we have vector fields defined on U by

$$\frac{\partial}{\partial x^i}\colon p \mapsto \frac{\partial}{\partial x^i}\Big|_p.$$

The ordered set $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ is called a coordinate frame field.
If X is a smooth vector field defined on some set including this chart domain U, then for some smooth functions X^i defined on U we have

$$X(p) = \sum X^i(p) \frac{\partial}{\partial x^i}\Big|_p, \quad \text{i.e. } X|_U = \sum X^i \frac{\partial}{\partial x^i}.$$

Lemma

If $v \in T_pM$ then there exists a vector field X s.t. $X(p) = v$.

Proof:
Write $v = \sum v^i \frac{\partial}{\partial x^i}\Big|_p$.

Define a field X_U by the formula $\sum v^i \frac{\partial}{\partial x^i}$ where the v^i are taken as constant functions on U.

Let β be a cut-off function with support in U and s.t. $\beta(p) = 1$.

Finally, define $X := \beta X_U$ on U and extended to zero outside of U. $\square$

- The set of all smooth vector fields on M is denoted by $\mathfrak{X}(M)$.
- Smooth vector fields may at times be defined only on some open set $U \subset M$ so we also have the notation $\mathfrak{X}(U) = \mathfrak{X}_M(U)$ for these fields.
- We define the addition of vector fields X and Y pointwise, i.e.,

$$(X + Y)(p) := X(p) + Y(p),$$

  and scaling by real numbers, by

$$(cX)(p) := cX(p).$$

  Then the set $\mathfrak{X}(M)$ becomes a real vector space.
- Defining, moreover, multiplication of a smooth vector field X by a smooth function f by

$$(fX)(p) := f(p)X(p),$$

  makes $\mathfrak{X}(M)$ a module over the ring $C^\infty(M)$.

Definition

Let $f\colon N \to M$ be a smooth map.

A vector field along f is a smooth map $X\colon N \to TM$ s.t. $\pi_{TM} \circ X = f$.

A vector field along a regular submanifold $S \subset M$ is a vector field along the inclusion map $S \hookrightarrow M$.

We denote by $\mathfrak{X}_f$ the space of vector fields along f.

Definition (Derivation)

Let M be a smooth manifold.

A (global) derivation on $C^\infty(M)$ is a linear map $\mathcal{D}\colon C^\infty(M) \to C^\infty(M)$ s.t.

$$\mathcal{D}(fg) = \mathcal{D}(f)g + f\mathcal{D}(g).$$

We denote the set of all such derivations of $C^\infty(M)$ by $\mathrm{Der}\big(C^\infty(M)\big)$.

Definition (Lie Derivative on Functions)

To a vector field X on M, we associate the map

$$\mathcal{L}_X \colon C^\infty(M) \to C^\infty(M), \qquad (\mathcal{L}_X f)(p) := X_p f.$$

$\mathcal{L}_X$ is called the Lie derivative on functions.

Remark

Notice that

$$(\mathcal{L}_X f)(p) = X_p \cdot f = \mathrm{D}\, f(X_p) \quad \textit{for any } p,$$

and so $\mathcal{L}_X f = \mathrm{D}\, f \circ X$.

If X *is a vector field on an open set* U*, and if* f *is a function on* $V \subset U$*, then we take* $\mathcal{L}_X f$ *to be the function defined on* V *by* $p \mapsto X_p f$ *for all* $p \in V$.

It is easy to see that we have $\mathcal{L}_{aX+bY} = a\mathcal{L}_X + b\mathcal{L}_Y$ *for* $a, b \in \mathbb{R}$ *and* $X, Y \in \mathfrak{X}(M)$.

Lemma

Let $U \subset M$ be open and $X \in \mathfrak{X}(M)$.

If $\mathcal{L}_X f = 0$ for all $f \in C^\infty(U)$, then $X|_U = 0$.

Proof:

Let $p \in U$ be given.

Working locally in a chart $(V, \mathbf{x})$, let $X = \sum X^i \partial/\partial x^i$.

We may assume $p \in V \subset U$.

Using a cut-off function we may find functions f^i defined on U s.t. f^i coincides with x^i on a neighborhood of p.

Then we have

$$X^i(p) = X_p x^i = X_p f^i = (\mathcal{L}_X f^i)(p) = 0.$$

Thus $X(p) = 0$ for an arbitrary $p \in U$. $\square$

Theorem

For $X \in \mathfrak{X}(M)$, we have $\mathcal{L}_X \in \mathrm{Der}\big(C^\infty(M)\big)$.
If $\mathcal{D} \in \mathrm{Der}\big(C^\infty(M)\big)$, then $\mathcal{D} = \mathcal{L}_X$ for a **uniquely** *determined $X \in \mathfrak{X}(M)$.*

Proof: That $\mathcal{L}_X \in \mathrm{Der}\big(C^\infty(M)\big)$ follows from the Leibniz rule, i.e., from the fact that X_p is a derivation at p for each p.

If we are given a derivation $\mathcal{D}$, we define a derivation X_p at p (i.e. a tangent vector) by the rule $X_p f := (\mathcal{D}f)(p)$.

We need to show that the assignment $p \mapsto X_p$ is smooth.

Recall that any locally defined function can be extended to a global one by using a cut-off function.

It therefore suffices to show that $p \mapsto X_p f$ is smooth for any $f \in C^\infty(M)$.
But this is clear since $X_p f := (\mathcal{D}f)(p)$ and $\mathcal{D}f \in C^\infty(M)$.
Suppose now that $\mathcal{D} = \mathcal{L}_{X_1} = \mathcal{L}_{X_2}$. Notice that $\mathcal{L}_{X_1} - \mathcal{L}_{X_2} = \mathcal{L}_{X_1 - X_2}$ and so $\mathcal{L}_{X_1 - X_2}$ is the zero derivation on $C^\infty(M)$.
Consequently, we have $X_1 - X_2 = 0$. $\square$

Definition

By the above theorem, we can identify $\mathrm{Der}\big(C^\infty(M)\big)$ with $\mathfrak{X}(M)$ and we therefore often will write Xf in place of $\mathcal{L}_X f$, i.e.

$$Xf := \mathcal{L}_X f.$$

- The derivation law (i.e. the Leibniz rule) $\mathcal{L}_X(fg) = g\mathcal{L}_X f + f\mathcal{L}_X g$ becomes simply $X(fg) = gXf + fXg$.
- If we have a derivation of $C^\infty(M)$, then we know it corresponds to a vector field.

  As such, it can be restricted to any open set $U \subset M$, and thus we get a derivation of $C^\infty(U)$.

  If $f \in C^\infty(U)$ we write Xf instead of the more pedantic $X|_U f$.

Theorem

If $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Der}\big(C^\infty(M)\big)$, *then the commutator*

$$[\mathcal{D}_1, \mathcal{D}_2] := \mathcal{D}_1 \circ \mathcal{D}_2 - \mathcal{D}_2 \circ \mathcal{D}_1 \in \mathrm{Der}\big(C^\infty(M)\big).$$

Proof: We compute

$$\begin{aligned}\mathcal{D}_1\big(\mathcal{D}_2(fg)\big) &= \mathcal{D}_1\big(\mathcal{D}_2(f)g + f\mathcal{D}_2(g)\big)\\ &= (\mathcal{D}_1\mathcal{D}_2 f)g + \mathcal{D}_2 f \mathcal{D}_1 g + \mathcal{D}_1 f \mathcal{D}_2 g + f\mathcal{D}_1\mathcal{D}_2 g.\end{aligned}$$

Writing out the similar expression for $\mathcal{D}_2\big(\mathcal{D}_1(fg)\big)$ and then subtracting we obtain, after a cancellation,

$$\begin{aligned}[\mathcal{D}_1, \mathcal{D}_2](fg) &= (\mathcal{D}_1\mathcal{D}_2 f)g + f\mathcal{D}_1\mathcal{D}_2 g - \Big((\mathcal{D}_2\mathcal{D}_1 f)g + f(\mathcal{D}_2\mathcal{D}_1)g\Big)\\ &= \big([\mathcal{D}_1, \mathcal{D}_2]f\big)g + f[\mathcal{D}_1, \mathcal{D}_2]g\end{aligned}$$

Corollary

If $X, Y \in \mathfrak{X}(M)$, then there is a unique vector field $[X, Y]$ s.t.
$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X$. □

Since $\mathcal{L}_X f = Xf$, we have $[X, Y]f = X(Yf) - Y(Xf)$ or $[X, Y] = XY - YX$.

Definition

Vector field $[X, Y]$ is called the Lie bracket or commutator of X and Y.

Proposition

The map $(X, Y) \mapsto [X, Y]$ is bilinear over $\mathbb{R}$, and for $X, Y, Z \in \mathfrak{X}(M)$:

(a) $[X, Y] = -[Y, X]$ *(skew -symmetry),*

(b) $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$ *(Jacobi identity),*

(c) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ *for all $f, g \in C^\infty(M)$.*

Proof: Straight-forward calculation, however, (b) is cumbersome... □

Remark

- *The map* $(X, Y) \mapsto [X, Y]$ *is bilinear over* $\mathbb{R}$*, but by the above Proposition, it is* **not** *bilinear over* $C^\infty(M)$*.*
- *Notice that in (b),* X, Y, Z *are permuted* **cyclically***.*
- *How does the local formula for the Lie derivative looks like in conventional "index" notation?*

  Suppose we have $Y = \sum Y^i \frac{\partial}{\partial x^i}$ *and* $Z = \sum Z^i \frac{\partial}{\partial x^i}$*.*

  Then we have the local formula

$$[Y, Z] = \sum_{i.j} \left(Y^j \frac{\partial Z^i}{\partial x^j} - Z^j \frac{\partial Y^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$

Definition (Lie Algebra)

A vector space $\mathfrak{a}$ (over a field $\mathbb{F}$) is called a Lie algebra if it is equipped with a bilinear map $\mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ (a multiplication) denoted $(v, w) \mapsto [v, w]$ s.t. $[v, w] = -[w, v]$ and s.t. we have the Jacobi identity $\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0$ for all $x, y, z \in \mathfrak{a}$.

Definition

1. A Lie algebra $\mathfrak{a}$ is called Abelian (or commutative) if $[v, w] = 0$ for all $v, w \in \mathfrak{a}$.
2. A subspace $\mathfrak{h}$ of $\mathfrak{a}$ is called a Lie subalgebra if it is closed under the bracket operation.
3. A Lie subalgebra $\mathfrak{h}$ is called an ideal if $[v, w] \in \mathfrak{h}$ for any $v \in \mathfrak{a}$ and $w \in \mathfrak{h}$.

   We indicate this by writing $[\mathfrak{a}, \mathfrak{h}] \subset \mathfrak{h}$.

Remark

1. *Notice that the Jacobi identity may be restated as*

$$\big[x,[y,z]\big] = \big[[x,y],z\big] + \big[y,[x,z]\big],$$

   saying that for fixed x the map $y \mapsto [x,y]$ is a derivation of $\mathfrak{a}$.

   This is significant mathematically and also a way to remember the Jacobi identity.

2. *The Lie algebra $\mathfrak{X}(M)$ is* **infinite**-*dimensional (unless M is 0-dimensional).*

3. *In due course we will be interested in certain* **finite**-*dimensional Lie algebras which are subalgebras of $\mathfrak{X}(M)$.*

Definition (Pull-back and Push-forward for Vector Fields)

Given a diffeomorphism $\phi\colon M \to N$, the pull-back $\phi^* Y \in \mathfrak{X}(M)$ for $Y \in \mathfrak{X}(N)$ and the push-forward $\phi_* X \in \mathfrak{X}(N)$ of $X \in \mathfrak{X}(M)$ by ϕ is defined by

$$\phi^* Y := T\phi^{-1} \circ Y \circ \phi,$$
$$\phi_* X := T\phi \circ X \circ \phi^{-1}.$$

In other words,

$$(\phi^* Y)(p) = T\phi^{-1} \cdot Y_{\phi(p)}, \quad p \in M,$$
$$(\phi_* X)(q) = T\phi \cdot Y_{\phi^{-1}(q)}, \quad q \in N.$$

By construction, $\phi^* Y$ and $\phi_* X$ are both smooth vector fields.

Remark

Warning: *Sometimes f_* for the tangent map Tf is used.*
The notation $f_* X$ might then be interpreted as $Tf \circ X$, which is actually a vector field **along the map** *f rather than an element of $\mathfrak{X}(N)$.*
We will avoid to use f_* as a notation for Tf!

Remark

Summary:
If $f\colon M \to N$ is smooth, then for each $p \in M$ we have the tangent map $T_p f\colon T_p M \to T_{f(p)} N$, the tangent lift $Tf\colon TM \to TN$, i.e., a bundle map.

If f is a diffeomorphism, we have induced maps on the vector field's level

$$f_*\colon \mathfrak{X}(M) \to \mathfrak{X}(N) \quad \text{and} \quad f^*\colon \mathfrak{X}(N) \to \mathfrak{X}(M).$$

If $\phi\colon M \to N$, and $\psi\colon N \to P$ are diffeomorphisms, then we have

$$(\psi \circ \phi)_* = \psi_* \circ \phi_*\colon \mathfrak{X}(M) \to \mathfrak{X}(P),$$
$$(\psi \circ \phi)^* = \phi^* \circ \psi^*\colon \mathfrak{X}(P) \to \mathfrak{X}(M).$$

There are right and left actions of the diffeomorphism group $\mathrm{Diff}(M)$ on the space of vector fields:

The left action $\mathrm{Diff}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, given by $(\phi, X) \mapsto \phi_* X$,
the right action $\mathfrak{X}(M) \times \mathrm{Diff}(M) \to \mathfrak{X}(M)$, given by $(X, \phi) \mapsto \phi^* X$.

Definition (Pull-back and Push-forward of Functions)

On functions, the pull-back is defined by $\phi^* g := g \circ \phi$ for any smooth g.

If ϕ is even a diffeomorphism, then we can also define a push-forward by $\phi_* := (\phi^{-1})^*$.

Proposition

The Lie derivative on functions is natural w.r.t. pull-back and push-forward by diffeomorphisms.

In other words, if $\phi\colon M \to N$ is a diffeomorphism and $f \in C^\infty(M)$, $g \in C^\infty(N)$, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, then

$$\mathcal{L}_{\phi^* Y} \phi^* g = \phi^* \mathcal{L}_Y g,$$
$$\mathcal{L}_{\phi_* X} \phi_* f = \phi_* \mathcal{L}_X f.$$

Proof:
We use the algebraic definition of the tangent map.
For any p we have

$$\begin{aligned}
(\mathcal{L}_{\phi^* Y}\phi^* g)(p) &= (\phi^* Y)_p \phi^* g \\
&= (T\phi^{-1} \circ Y \circ \phi)_p [g \circ \phi] \\
&= (T\phi^{-1} \cdot Y_{\phi(p)})[g \circ \phi] \\
&= T\phi(T\phi^{-1} Y_{\phi(p)}) g \\
&= Y_{\phi(p)} g \\
&= (\mathcal{L}_Y g)\big(\phi(p)\big) \\
&= (\phi^* \mathcal{L}_Y g)(p).
\end{aligned}$$

The second statement follows from the first since $\phi_* = (\phi^{-1})^*$. $\square$

Remark

Even if $f\colon M \to N$ is **not** *a diffeomorphism, it* **may** *still be that there is a vector field $Y \in \mathfrak{X}(N)$ such that*

$$Tf \circ X = Y \circ f.$$

In other words, it may happen that $Tf \cdot X_p = Y_{f(p)}$ for all $p \in M$.

In this case, we say that Y is f-related to X and write $X \sim_f Y$.

It is not hard to check that if X_i is f-related to Y_i for $i = 1, 2$, then $aX_1 + bX_1$ is f-related to $aY_1 + bY_1$.

Example

Let M, N be smooth manifolds and consider the projections $\mathrm{pr}_1 \colon M \times N \to M$ and $\mathrm{pr}_2 \colon M \times N \to N$.
Since

$$T_{(p,q)}(M \times N) \cong T_pM \times T_qN,$$

we see that for $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ we obtain a vector field

$$X \times Y \in \mathfrak{X}(M \times N), \quad (X \times Y)(p,q) := \big(X(p), Y(p)\big).$$

Consequently, $X \times Y$ and X are pr_1-related and $X \times Y$ and Y are pr_2-related.

Lemma

Suppose that $f\colon M \to N$ is a smooth map, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Then X and Y are f-related iff $X(g \circ f) = (Yg) \circ f$ for all $g \in C^\infty(N)$.

Proof:
Let $p \in M$ and let $g \in C^\infty(N)$. Then

$$\begin{aligned} X(g \circ f)(p) &= X_p(g \circ f) \\ &= (T_p f \cdot X_p)g \end{aligned}$$

and

$$(Yg \circ f)(p) = Y_{f(p)}g$$

s.t. $X(g \circ f) = (Yg) \circ f$ for all such g iff $T_p f \cdot X_p = Y_{f(p)}$. $\square$

Proposition

If $f\colon M \to N$ is a smooth map and $X_i \sim_f Y_i$ for $i = 1, 2$, then $[X_1, X_2] \sim_f [Y_1, Y_2]$.

In particular, if ϕ is a diffeomorphism, then $[\phi_* X_1, \phi_* X_2] = \phi_*[X_1, X_2]$ for all $X_1, X_2 \in \mathfrak{X}(M)$.

Proof: We use the above Lemma.
Let $g \in C^\infty(N)$. Then

$$X_1 X_2(g \circ f) = X_1\big((Y_2 g) \circ f\big) = (Y_1 Y_2 g) \circ f.$$

In the same way, $X_2 X_1(g \circ f) = (Y_2 Y_1 g) \circ f$ and subtracting we obtain

$$\begin{aligned}
[X_1, X_2](g \circ f) &= X_1 X_2(g \circ f) - X_2 X_1(g \circ f) \\
&= (Y_1 Y_2 g) \circ f - (Y_2 Y_1 g) \circ f \\
&= \big([Y_1, Y_2] g\big) \circ f.
\end{aligned}$$

Using the lemma once more, we have the result. $\square$

Remark

Consider a submanifold $S \subset M$ and $X \in \mathfrak{X}(M)$.

Then the restriction $X|_S \in \mathfrak{X}(S)$ with $X|_S(p) := X(p)$ for all $p \in S$ is ι-related to X where $\iota\colon S \hookrightarrow M$ is the inclusion map.

Thus for $X, Y \in \mathfrak{X}(M)$ we always have that $[X|_S, Y|_S] \sim_\iota [X, Y]$.

This just means that $[X, Y](p) = [X|_S, Y|_S](p)$ for all p.

We immediately get

Proposition

Let $f\colon M \to N$ be a smooth map and suppose that $X \sim_f Y$.

Then we have $\mathcal{L}_X(f^*g) = f^*\mathcal{L}_Y g$ for any $g \in C^\infty(N)$. □

We now introduce the important concept of the Lie derivative of a vector field extending the previous definition.

The Lie derivative will be later extended further to tensor fields.

Definition (Lie Derivative of a Vector Field)

Given a vector field X, we define a map

$$\mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M), \quad \mathcal{L}_X Y := [X, Y].$$

This map is called the Lie derivative (with respect to X).

The Jacobi identity for the Lie bracket implies the following two identities for any $X, Y, Z \in \mathfrak{X}(M)$:

- $\mathcal{L}_X[Y, Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]$,
- $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y] \iff \mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X$.

$\mathcal{L}_X Y$ somehow measures the rate of change of Y in the direction X.

Definition (Covectors or 1-Forms)

A smooth (resp. C^r) section of the cotangent bundle is called a smooth (resp. C^r) 1-form or also a smooth (resp. C^r) covector field.

The set of all C^r 1-forms is denoted by $\mathfrak{X}^{r*}(M)$ and the set of smooth 1-forms is denoted by $\mathfrak{X}^*(M)$.

The set $\mathfrak{X}^{r*}(M)$ is a module over $C^r(M)$.
Later we will have reason to denote $\mathfrak{X}^*(M)$ also by $\Omega^1(M)$.

Definition

Let $f\colon M \to \mathbb{R}$ be a C^r function with $r \geq 1$.

The map $\mathrm{D}\,f\colon M \to T^*M$ is defined by $p \mapsto \mathrm{D}\,f(p)$ where $\mathrm{D}\,f(p)$ is the derivative at p as defined as usual.

$\mathrm{D}\,f \equiv \mathrm{d}f$ is a 1-form called the differential or derivative of f.

As a consequence of the definitions above we have

$$\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g.$$

Indeed, for $v_p \in T_pM$ we have

$$\begin{aligned}
\mathrm{d}(fg)(p) \cdot v_p &= v_p(fg) \\
&= g(p)v_p(f) + f(p)v_p(g) \\
&= g(p)(\mathrm{d}f)(p) \cdot v_p + f(p)\mathrm{d}g(p) \cdot v_p \\
&= (g\mathrm{d}f + f\mathrm{d}g)(p) \cdot v_p.
\end{aligned}$$

Three views on a 1-form:

What is the argument of a 1-form ?
One can view a 1-form in at least three different ways:

If α is a smooth 1-form, then we have the following interpretations of α:

1. View α as a map $\alpha\colon M \to T^*M$ (as in the definition) so that $\alpha(\cdot)$ takes points as arguments, i.e. $\alpha(p) \in T_p^*M$.
   We also often need to write $\alpha_p = \alpha(p)$ just as for a vector field X, where we might write $X(p)$ as X_p.
2. View α as a smooth map $\alpha\colon TM \to \mathbb{R}$ s.t. for $v_p \in T_pM$ we make sense of $\alpha(v_p)$ by $\alpha(v_p) := \alpha_p(v_p)$, i.e., as a linear functional.
3. View α as a map $\alpha\colon \mathfrak{X}(M) \to C^\infty(M)$, where for $X \in \mathfrak{X}(M)$ we interpret $\alpha(X)$ as the smooth function $p \mapsto \alpha_p(X_p)$.

The second and third interpretations obviously are dependent on the first.

The third view is very important:
in that view, $\alpha\colon \mathfrak{X}(M) \to C^\infty(M)$ is a $C^\infty(M)$-linear map.

Given a chart $(U, \mathbf{x})$ with $\mathbf{x} = (x^1, \ldots, x^n)$, the differentials $\mathrm{d}x^i \colon p \mapsto \mathrm{d}x^i|_p$ define 1-forms on U s.t. $\mathrm{d}x^1|_p, \ldots, \mathrm{d}x^n|_p$ form a basis of T_p^*M for each $p \in U$.

If α is any smooth 1-form (covector field) defined at least on U, then

$$\alpha|_U = \sum a_i \mathrm{d}x^i$$

for uniquely determined functions a_i.

In fact, $\alpha\left(\frac{\partial}{\partial x^i}\right) = a_i$ (using the third view from above).

As with vector fields we will usually just write $\alpha = \sum a_i \mathrm{d}x^i$.

In particular, $\mathrm{d}f\left(\frac{\partial}{\partial x^i}\right) = \frac{\mathrm{d}f}{\mathrm{d}x^i}$, therefore we get the familar

$$\mathrm{d}f = \sum \tfrac{\mathrm{d}f}{\mathrm{d}x^i} \mathrm{d}x^i,$$

to be interpreted to mean that at each $p \in U_\alpha$ we have

$$\mathrm{d}f(p) = \sum \tfrac{\mathrm{d}f}{\mathrm{d}x^i}\Big|_p \mathrm{d}x^i|_p.$$

Remark

The covector fields $\mathrm{d}x^i$ form what is called a coordinate coframe field over U.
Note that the component functions X^i of a vector field w.r.t. the chart above are given by $X^i = \mathrm{d}x^i(X)$, where by definition $\mathrm{d}x^i(X)$ is the function $p \mapsto \mathrm{d}x^i|_p(X_p)$. Consequently,

$$X_U = \sum \mathrm{d}x^i(X) \tfrac{\partial}{\partial x^i}.$$

If α is a 1-form on M and $p \in M$, then one can always find many functions f s.t. $\mathrm{d}f(p) = \alpha(p)$.

But there may not be a single function f s.t. this is true for all points in a neighborhood, let alone all points on M.

Definition

If $\mathrm{d}f = \alpha$ for some $f \in C^\infty(M)$, then we say the 1-form α is exact.

Definition (Pull-back of 1-forms)

If $\phi\colon M \to N$ is a C^∞ map, the pull-back of $\alpha \in \mathfrak{X}^*(N)$ by ϕ is defined by

$$(\phi^*\alpha)_p \cdot v := \alpha_{\phi(p)}(T_p\phi \cdot v) \quad \text{for} \quad v \in T_pM.$$

Remark

This extends the notion of the pull-back of a function defined earlier.

If we view a 1-form on M as a map $TM \to \mathbb{R}$, then the pull-back is given by $\phi^*\alpha = \alpha \circ T\phi$.

Lemma

The pull-back is contravariant in the sense that if $\phi_1\colon M_1 \to M_2$ and $\phi_2\colon M_2 \to N$, then for $\alpha \in \mathfrak{X}^*(N)$ we have $(\phi_2 \circ \phi_1)^*\alpha = (\phi_1^* \circ \phi_2^*)\alpha$. $\square$

Local expression for the pull-back of a 1-form:

Let $(U, \mathbf{x})$ be a chart on M and $(V, \mathbf{y})$ one on N with $\phi(U) \subset V$.
A typical 1-form has a local expression on V, i.e. $\alpha = \sum a_i \mathrm{d}y^i$ for $a_i \in C^\infty(V)$
The local expression for $\phi^*\alpha$ on U is

$$\begin{aligned}\phi^*\alpha &= \sum (a_i \circ \phi)\mathrm{d}(y^i \circ \phi) \\ &= \sum (a_i \circ \phi)\tfrac{\partial(y^i \circ \phi)}{\mathrm{d}x^j}\mathrm{d}x^j.\end{aligned}$$

Consequently, we get a local pull-back formula convenient for computations as

$$\phi^*\Big(\sum a_i \mathrm{d}y^i\Big) = \sum (a_i \circ \phi)\tfrac{\partial(y^i \circ \phi)}{\mathrm{d}x^j}\mathrm{d}x^j.$$

Remark

The pull-back of a function or 1*-form is defined whether* $\phi\colon M \to N$ *happens to be a diffeomorphism or not.*

On the other hand, the pull-back of a vector field only works in special circumstances such as where ϕ *is a diffeomorphism.*

Let $\phi\colon M \to N$ *be a* C^∞ *diffeomorphism with* $r \geq 1$*.*

Recall that the push-forward of a function $f \in C^\infty(M)$ *is denoted* $\phi_* f$ *and defined by* $\phi_* f(p) := f\left(\phi^{-1}(p)\right)$*.*

We can also define the push-forward of a 1*-form as*

$$\phi_* \alpha := \alpha \circ T\phi^{-1}.$$

Lemma

The differential is natural with respect to pull-back.
In other words, if $\phi\colon M \to N$ is a C^∞ map and $f\colon N \to \mathbb{R}$ a C^∞ function, then $\mathrm{d}(\phi^* f) = \phi^* \mathrm{d}f$.
Consequently, the differential is also natural with respect to restrictions.

Proof: We wish to show that

$$(\phi^* \mathrm{d}f)_p = \mathrm{d}(\phi^* f)_p, \quad \text{for all} \quad p \in M.$$

Let $v \in T_p M$ and write $q = \phi(p)$. Then

$$\begin{aligned}(\phi^* \mathrm{d}f)|_p v &= \mathrm{d}f|_q (T_p\phi \cdot v) \\ &= \big((T_p\phi \cdot v) f\big)(q) \\ &= v(f \circ \phi)(p) = \mathrm{d}(\phi^* f)|_p v.\end{aligned}$$

The second statement is obvious from local coordinate expressions. □

Motivation

Recall that the rank of a smooth map $f\colon M \to N$ at p is defined to be the rank of $T_p f$.
Under certain conditions on the rank of f near p, we can draw conclusions about the behavior of f near p.
The basic idea is that f behaves locally very much like $T_p f$.

Definition (Immersion)

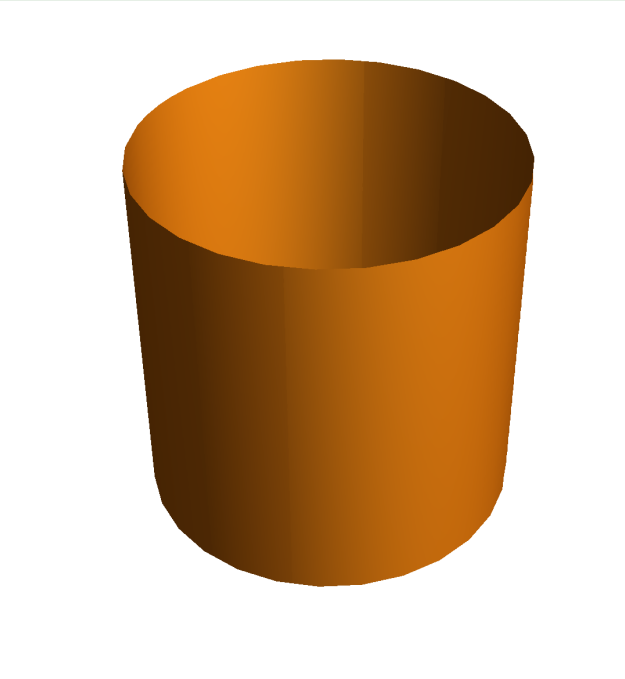
A map $f\colon M \to N$ is called an immersion at $p \in M$ if
$T_p f\colon T_p M \to T_{f(p)} N$ is an injection,
$f\colon M \to N$ is (simply) called an immersion if it is an immersion at every $p \in M$.

Remark

$T_p f\colon T_p M \to T_{f(p)} N$ is an injection iff its rank is equal to $\dim(M)$.
Thus an immersion has constant rank equal to the dimension of its domain.

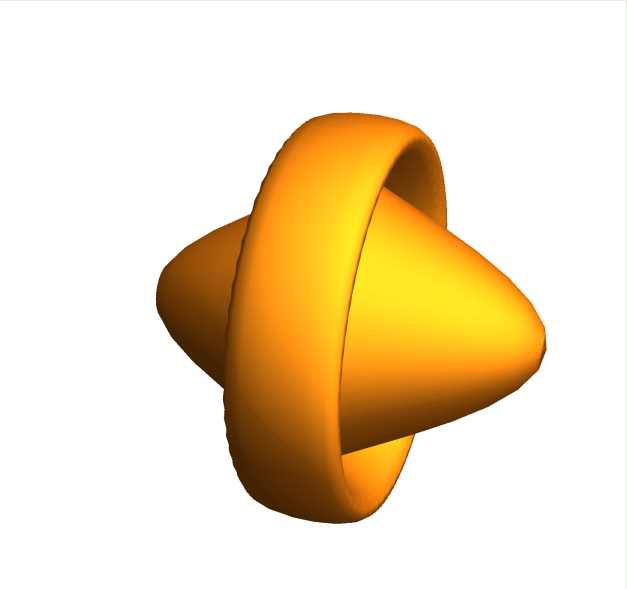
Example

Immersions of open subsets of $\mathbb{R}^2 \to \mathbb{R}^3$ appear as surfaces that may self-intersect, or periodically retrace themselves, or approach themselves in various limiting ways.
The cylinder: $\mathbb{R}^2 \to \mathbb{R}^3$, $(u, v) \mapsto (\cos u, \sin u, v)$ is an immersion.



Example

1. "2 sombreros pasted or glued together along their brims": The map $\mathbb{R}^2 \to \mathbb{R}^3$, $(u, v) \mapsto (\cos u \sin v, \sin u \sin v, (1 - 2\cos^2 v)\cos v)$ is an immersion, alternatively, i.e., parameterized differently, $S^2 \to \mathbb{R}^3$, $(x, y, z) \mapsto (x, y, z - 2z^3)$.



2. By contrast, the map $f\colon S^2 \to \mathbb{R}^3$, $(x, y, z) \mapsto (x, y, 0)$ is not an immersion at any point on the equator $S^2 \cap \{z = 0\}$.

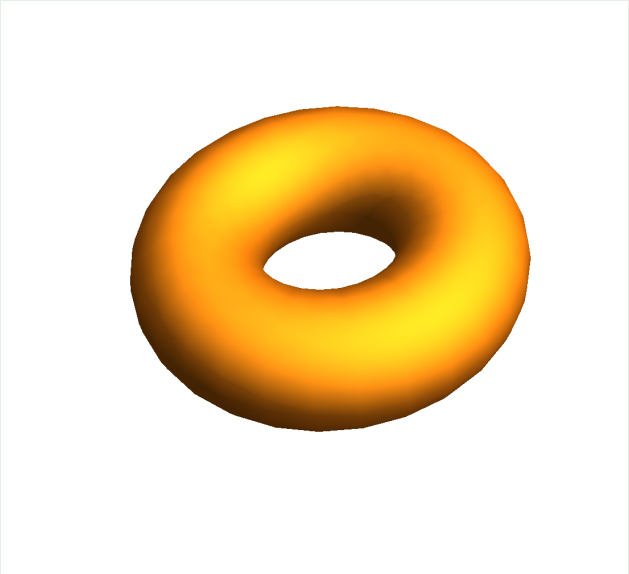
Example

An immersion of the torus $T^2 := S^1 \times S^1$ into $\mathbb{R}^3$ is as follows.

Represent points in T^2 as pairs $(\mathrm{e}^{\imath\theta_1}, \mathrm{e}^{\imath\theta_2})$.

For fixed $a, b > 0$, the following map is well-defined:

$$(\mathrm{e}^{\imath\theta_1}, \mathrm{e}^{\imath\theta_2}) \mapsto \Big((a + b\cos\theta_1)\cos\theta_2, (a + b\cos\theta_1)\sin\theta_2, b\sin\theta_1\Big).$$



Theorem

Let $f\colon M \to N$ be a smooth map that is an immersion at p.

Then for any chart $(U, \mathbf{x})$ centered at p, there is a chart $(V, \mathbf{y})$ centered at $f(p)$ s.t. $f(U) \subset V$ and s.t the corresponding coordinate expression for f is $(x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, 0, \ldots, 0) \in \mathbb{R}^n$.

Here, $\dim N = n$ and $k = \dim M$ is the rank of $T_p f$.

Idea of Proof:
This follows from the Implicit Function Theorem. □

Theorem

If $f\colon M \to N$ is an immersion and a homeomorphism onto its image $f(M)$ (using the relative topology on $f(M)$), then $f(M)$ is a regular submanifold of N.

Idea of Proof:

Let $\dim M = k, \dim N = n$.

As f is injective, let $f(p) \in f(M)$ for a unique p.

Consider charts $(U, \mathbf{x})$ with $p \in U$ and $(V, \mathbf{y})$ with $f(p) \in V$ s.t. the corresponding coordinate expression for f is $(x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, 0, \ldots, 0) \in \mathbb{R}^n$.

W.l.o.g. $f(U) \subset V$.

But $f(U)$ is open in the relative topology on $f(M)$, so there is an open set $O \subset V$ in M s.t. $f(U) = f(M) \cap O$.

Now it is clear that $(O, \mathbf{y}|_O)$ is a chart with the regular submanifold property, and so since p was arbitrary, we conclude that $f(M)$ is a regular submanifold.

Definition (Embedding)

If $f\colon M \to N$ is an immersion that is a homeomorphism onto its image (as in the above thm.), then we say that f is an embedding.

Definition (Proper Map)

A continuous map f is said to be proper if $f^{-1}(K)$ is compact whenever K is compact.

Example

A proper $1-1$ immersion is an embedding.

Definition (Immersed Submanifold)

Let S be a subset of a smooth manifold M.

If S is a smooth manifold s.t. the inclusion map $\iota\colon S \to M$ is an injective immersion,
then we call S an immersed submanifold.

Remark

- *The above S certainly need not have the subspace topology!*
- *Its topology is the one induced by its own smooth structure.*
- *If $f\colon N \to M$ is an injective immersion, then $S := f(N)$ can be given a smooth structure so that it is an immersed submanifold.*

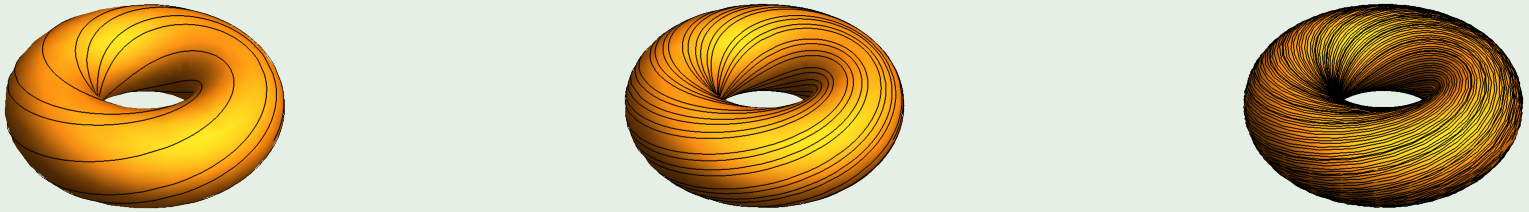
Example (Dense Windings on the Torus or "Irrational Lines")

Example of the torus T^2 immersed into $\mathbb{R}^3$ revisited.
The map

$$\gamma\colon \mathbb{R} \to T^2, \quad \gamma(t) := (\mathrm{e}^{2\pi\imath t}, \mathrm{e}^{2\pi\imath\alpha t})$$

with α any irrational number, is a smooth injective immersion.

By Dirichlet's Approximation Theorem, $\gamma(\mathbb{R})$ is actually dense in T^2.



The figures show from left to right T^2 together with $\gamma(I_1)$, $\gamma(I_2)$, $\gamma(I_3)$, where for the intervals we have $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \mathbb{R}$.

Definition (Submersion)

A map $f\colon M \to N$ is called a submersion at $p \in M$ if $T_p f\colon T_p M \to T_{f(p)} N$ is a surjection.

$f\colon M \to N$ is (simply) called a submersion if f is a submersion at every $p \in M$.

Example

The map $\mathbb{R}^3 \setminus \{0\}$ onto S^2 given by $x \mapsto x/\|x\|$ is a submersion.

Indeed, use any spherical coordinates (ρ, ϕ, θ) on $\mathbb{R}^3 \setminus \{0\}$ and the induced submanifold coordinates (ϕ, θ) on S^2.

Expressed w.r.t. these coordinates, the map becomes $(\rho, \phi, \theta) \mapsto (\phi, \theta)$ on the domain of the spherical coordinate chart.

Here we ended up locally with a projection onto a second factor $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, but this is clearly good enough to prove the statement.

Remark

As in the last example, to show that a map is a submersion at some p it is enough to find charts containing p and $f(p)$ so that the coordinate representative of the map is just a projection.

Conversely, we have

Theorem

Let M be an m-manifold and N a k-manifold and let $f\colon M \to N$ be a smooth map that is a submersion at p.
Then for any chart $(V, \mathbf{y})$ centered at $f(p)$ there is a chart $(U, \mathbf{x})$ centered at p with $f(U) \subset V$ s.t. $\mathbf{y} \circ f \circ \mathbf{x}^{-1}$ is given by
$(x^1, \ldots, x^k, \ldots, x^m) \mapsto (x^1, \ldots, x^k) \in \mathbb{R}^k$.
Here k is both the dimension of N and the rank of $T_p f$.

Sketch of proof:
Consequence of the Implicit Function Theorem, □

In practice, embedded submanifolds are most often presented as solution sets of equations or systems of equations.

Definition (Level Set)

Let M, N be smooth manifolds and consider any map $\Phi\colon M \to N$ with $c \in N$. We call $\Phi^{-1}(c)$ a level set of Φ. In the special case $N = \mathbb{R}^k$ and $c = 0$, the level set $\Phi^{-1}(0)$ is often called the zero set of Φ.

Theorem (Constant-Rank Level Set Theorem)

Let M, N be smooth manifolds, and let $\Phi\colon M \to N$ be a smooth map with constant rank r. Then each level set of Φ is an embedded submanifold of codimension r in M.

Idea of Proof:
Work in charts and apply the Rank Theorem. □

Corollary (Submersion Level Set Theorem)

If M, N are smooth manifolds and $\Phi\colon M \to N$ is a smooth submersion, then each level set of Φ is an embedded submanifold whose codimension is equal to the dimension of N. $\square$

Corollary (Regular Level Set or Regular Value Theorem)

Every regular level set of a smooth map between smooth manifolds is an embedded submanifold whose codimension is equal to the dimension of the codomain.

Proof: Let $\Phi\colon M \to N$ be a smooth map and let $c \in N$ be a regular value. The set U of points $p \in M$ where $\operatorname{rank} \mathrm{D}\Phi_p = \dim N$ is open in M and contains $\Phi^{-1}(c)$ because of the assumption that c is a regular value.
It follows that $\Phi|_U\colon U \to N$ is a smooth submersion, and the preceding corollary shows that $\Phi^{-1}(c)$ is an embedded submanifold of U.
Since the composition of smooth embeddings $\Phi^{-1}(c) \hookrightarrow U \hookrightarrow M$ is again a smooth embedding, it follows that $\Phi^{-1}(c)$ is an embedded submanifold of M; and it is closed by continuity. $\square$

Example (Spheres)

Consider $S^n \subset \mathbb{R}^{n+1}$ and the smooth function $\Phi\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $\Phi(x) = \|x\|^2$.

Now, $S^n = \Phi^{-1}(1)$ and $\mathrm{D}\,\Phi(x) = 2[x^1, \ldots, x^{n+1}]$, being surjective except at the origin.

Consequently, S^n is an n-dimensional submanifold of $\mathbb{R}^{n+1}$.

Corollary (Infinitesimal Version of Regular Value Theorem)

Let M, N be smooth manifolds, and let $\Phi\colon M \to N$ be a submersion.

Let $c \in N$ be a regular value of Φ and $P := \Phi^{-1}(c)$.

Then for any $p \in P$ we have $T_pP = \ker \mathrm{D}\,\Phi(p)$.

Example (Spheres revisited)

The tangent space at $p \in S^n \subset \mathbb{R}^{n+1}$ is computed as

$$\begin{aligned} T_p S^n &= \ker \mathrm{D}\,\Phi(p) \\ &= \ker\left(2[p^1, \dots, p^{n+1}]\right) \\ &= \{x \in \mathbb{R}^{n+1} \mid x^\top p = 0\} \\ &= (\operatorname{span} p)^\perp. \end{aligned}$$

Motivation

- We are interested in studying symmetries of smooth manifolds, Riemannian manifolds, symplectic manifolds, etc.
- The usual way to deal with symmetry in mathematics is by the use of the notion of a transformation group.
- The wonderful thing for us is that the groups that arise in the study of geometric symmetries are often themselves smooth manifolds.
- Such "group manifolds" are called Lie groups.

Definition (Lie Group)

A smooth manifold G is called Lie group if it is a group (abstract group) s.t.

$$\mu\colon G \times G \to G, \quad \mu(g,h) := gh \quad \text{(group multiplication)}$$
$$\mathrm{inv}\colon G \to G \quad \mathrm{inv}(g) := g^{-1}, \quad \text{(inversion)},$$

are C^∞ maps.

Remark

- *If the group is Abelian, we sometimes opt to use the additive notation $g + h$ for the group operation.*
- *We will usually denote the identity element of any Lie group by the same letter e. Exceptions include the case of matrix or linear groups where we use the letter I or* id.
- *The inversion map is easily seen to be a diffeomorphism.*

Example

1. $\mathbb{R}$ is a one-dimensional (Abelian) Lie group, where the group multiplication is the usual addition "$+$". Similarly, any real or complex vector space is a Lie group under vector addition.
2. The circle $S^1 := \{z \in \mathbb{C} \colon \|z\|^2 = 1\}$ is a 1-dimensional (Abelian) Lie group under complex multiplication.
3. Let $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. Then, using multiplication, $\mathbb{R}^\times$ and $\mathbb{C}^\times$ are Lie groups.
4. The group of all invertible real $(n \times n)$-matrices is a Lie group denoted $\mathrm{GL}(n, \mathbb{R})$:
   Multiplication is clearly smooth.
   For the inversion map one appeals to the usual formula for the matrix inverse, $A^{-1} = \mathrm{adj}(A)/\det(A)$. Here $\mathrm{adj}(A)$ is the classical adjoint matrix (whose entries are the cofactors).
   This shows that A^{-1} depends smoothly on the entries of A.
   Similarly, for $\mathrm{GL}(n, \mathbb{C})$.

Theorem

If G is a connected Lie group and U is a neighborhood of the identity element e, then U generates the group.
In other words, every element of g is a product of elements of U.

Proof: Note, $V := \mathrm{inv}(U) \cap U$ is an open neighborhood of e with property $\mathrm{inv}(V) = V$. We say that V is symmetric. We show that V generates G: For any open W_1 and W_2 in G, the set $W_1 W_2 := \{w_1 w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\}$ is an open set being a union of the open sets $\cup_{g \in W_1} g W_2$. Thus, in particular, the inductively defined sets

$$V^n := V \cdot V^{n-1}, \quad n = 1, 2, 3, \ldots,$$

are open. We have

$$e \in V \subset V^2 \subset \cdots \subset V^n \subset \cdots .$$

Clearly, each V^n is symmetric and so also is the union

$$V^\infty := \cup_{n=1}^\infty V^n.$$

Moreover, V^∞ is not only closed under inversion, but also obviously closed under multiplication. Thus V^∞ is an open subgroup. But V^∞ is also closed, and since G is connected, we obtain $V^\infty = G$.

Remark

In general, the connected component of a Lie group G that contains the identity is a Lie group denoted G_0, and it is generated by any open neighborhood of the identity.

We call G_0 the identity component of G.

Definition

For a Lie group G and a fixed element $g \in G$, the maps $L_g \colon G \to G$ and $R_g \colon G \to G$ are defined by

$$L_g x := gx \text{ for } x \in G,$$
$$R_g x := xg \text{ for } x \in G,$$

and are called left translation and right translation (by g) respectively.

- It is easy to see that L and R are diffeomorphisms with $L_g^{-1} = L_{g^{-1}}$ and $R_g^{-1} = R_{g^{-1}}$.
- If G, H are Lie groups, then so is the product manifold $G \times H$, where multiplication is $(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 h_2)$.

  The Lie group $G \times H$ is called the product Lie group.
- For example, the product group $S^1 \times S^1$ is called the 2-torus group.

  More generally, the higher torus groups are defined by $T^n := S^1 \times \cdots \times S^1$ (n factors).

Definition

Let H be an abstract subgroup of a Lie group G.

If H is a Lie group such that the inclusion map $H \hookrightarrow G$ is an immersion, then we say that H is a Lie subgroup of G.

Proposition

If H is an abstract subgroup of a Lie group G that is also a regular submanifold, then H is a closed Lie subgroup.

Proof: "$\cdot$", "inv" on H are the restrictions of those maps on G, and as H is a regular submanifold, we obtain the needed smoothness of these maps.

The harder part is to show that H is closed. Let $x_0 \in \overline{H}$ be arbitrary.
Let $(U, \mathbf{x})$ be a single-slice chart adapted to H whose domain contains e.
Let $\delta\colon G \times G \to G$ with $\delta(g_1, g_2) = g_1^{-1} g_2$.
Choose an open set V s.t. $e \in V \subset \overline{V} \subset U$.
By continuity of δ we find an open neighborhood O of e s.t. $O \times O \subset \delta^{-1}(V)$.
Now if (h_i) is a sequence in H converging to $x_0 \in \overline{H}$, then $x_0^{-1} h_i \to e$ and $x_0^{-1} h_i \in O$ for all sufficiently large i.
Since $h_j^{-1} h_i = (x_0^{-1} h_j)^{-1} x_0^{-1} h_i$, we have $h_j^{-1} h_i \in V$ for sufficiently large i,j.
For any sufficiently large fixed j, we have

$$\lim_{i \to \infty} h_j^{-1} h_i = h_j^{-1} x_0 \in \overline{V} \subset U.$$

As U is the domain of a single-slice chart, $U \cap H$ is closed in U . Thus since each $h_j^{-1} h_i \in U \cap H$, we see $h_j^{-1} x_0 \in U \cap H \subset H$ for all sufficiently large j.
This shows that $x_0 \in H$, and since x_0 was arbitrary, we are done. $\square$

Remark

- *By a closed Lie subgroup we shall always mean one that is a regular submanifold as in the previous theorem.*
- *It is a nontrivial fact that an abstract subgroup of a Lie group that is also a closed subset is automatically a closed Lie subgroup in this sense.*

Example

1. S^1 embedded as $S^1 \times \{1\}$ in the torus $S^1 \times S^1$ is a closed subgroup.
2. Let S^1 be considered as the set of unit modulus complex numbers. The image in the torus $T^2 = S^1 \times S^1$ of the map $\mathbb{R} \to S^1 \times S^1$ given by $t \mapsto (\mathrm{e}^{\imath 2\pi t}, \mathrm{e}^{\imath 2\pi a t})$ is a Lie subgroup.
   This map is a homomorphism. If $a \in \mathbb{Q}$, then the image is an embedded copy of S^1 wrapped around the torus several times depending on a. If a is irrational, however, the image is still a Lie subgroup but is now dense in T^2.

Linear Lie Groups

Definition

Consider an n-dimensional $\mathbb{K}$-vectorspace V with $\mathrm{GL}(V)$, the general linear Lie group.
Lie groups that are subgroups of $\mathrm{GL}(V)$ are referred to as linear Lie groups and are often realized as matrix subgroups of $\mathrm{GL}(n, \mathbb{K}) = \mathrm{GL}(V)$.

Definition

Let V be an n-dimensional vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then the group $\mathrm{SL}(V)$ defined by

$$\mathrm{SL}(V) := \mathrm{SL}(V, \mathbb{K}) := \mathrm{SL}(n, \mathbb{K}) := \{A \in \mathrm{GL}(V) \mid \det(A) = 1\}$$

is called the special linear group for V.

To find more examples of linear Lie groups we recall certain concepts from Linear Algebra:

Definition (Nondegenerate Bilinear Form)

A bilinear form $\beta\colon V \times V \to \mathbb{K}$ on a $\mathbb{K}$-vector space V is called nondegenerate if the maps $V \to V^*$ given by $\beta_R\colon v \mapsto \beta(v, \cdot)$ and $\beta_L\colon v \mapsto \beta(\cdot, v)$, are both linear isomorphisms.

If V is finite-dimensional, then β_R is an isomorphism iff β_L is an isomorphism and then β is nondegenerate provided it has the property that if $\beta(v, w) = 0$ for all $w \in V$, then $v = 0$.

Definition (Scalar Product Space)

A real scalar product on a finite-dimensional $\mathbb{R}$-vector space V is a nondegenerate symmetric bilinear form $\beta\colon V \times V \to \mathbb{R}$.

A real scalar product space is a pair (V, β) where V is an $\mathbb{R}$-vector space and β is a scalar product.
(As usual we refer to V itself as the scalar product space when β is given)

Definition (Sesquilinear and Hermitian Form)

Let V be a $\mathbb{C}$-vector space. An $\mathbb{R}$-bilinear map $\beta\colon V \times V \to \mathbb{C}$ that satisfies $\beta(av, w) = \overline{a}\beta(v, w)$ and $\beta(v, aw) = a\beta(v, w)$ for all $a \in \mathbb{C}$ and $v, w \in V$ is called a sesquilinear form.
If also $\beta(v, w) = \overline{\beta(w, v)}$, we call β a Hermitian form.
If a Hermitian form is nondegenerate, we call it a Hermitian scalar product, and then (V, β) a Hermitian scalar product space.

Remark

- *The sesquilinear conditions imposed are described by saying that β is to be conjugate linear in the first slot and linear in the second slot.*
- *Nondegeneracy for a sesquilinear form is defined as for bilinear forms.*
- *In the literature, some authors define sesquilinear and Hermitian forms to be conjugate linear in the second slot and linear in the first.*
- *Obviously, $\beta(v, v)$ is always real for a Hermitian form.*

Definition (Positive (Semi-)Definiteness)

Let β be a real scalar product or Hermitian scalar product on a vector space V. Then

(a) β is positive (resp. negative) definite if $\beta(v,v) \geq 0$ (resp. $\beta(v,v) \leq 0$) for all $v \in V$ and $\beta(v,v) = 0 \Longrightarrow v = 0$;

(b) β is positive (resp. negative) semidefinite if $\beta(v,v) \geq 0$ (resp. $\beta(v,v) \leq 0$) for all $v \in V$.

Remark

- *What is called an inner product (a term we have already used) is a positive definite scalar product (or positive definite Hermitian scalar product in the complex case).*
- **In these lectures the term inner product always implies positive definiteness**.
- *Unfortunately, in the literature one finds inner product and scalar product being sometimes defined exactly the other way round...*

Definition (ONB in Scalar Product Spaces)

A basis $(e_1, \ldots, e_n)$ for a scalar product space (V, β) is called an orthonormal basis (ONB) if $\beta(e_i, e_j) = \pm\delta_{ij}$.

Remark

- *An ONB always exists for a finite-dimensional scalar product space.*
- *The above notion of orthonormal basis for a scalar product space obviously includes indefinite scalar product spaces.*

Definition

Let V be an n-dimensional $\mathbb{K}$-vector space with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If β is a bilinear or sesquilinear form on V, then $\mathrm{Aut}(V, \beta)$ is the subgroup of $\mathrm{GL}(V)$

$$\mathrm{Aut}(V, \beta) := \big\{A \in \mathrm{GL}(V) \mid \beta(Av, Aw) = \beta(v, w) \text{ for all } v, w \in V\big\}.$$

If β is a scalar product, then the elements of $\mathrm{Aut}(V, \beta)$ are called isometries of V.

Theorem

$\mathrm{SL}(V)$ *is a closed Lie subgroup of* $\mathrm{GL}(V)$.

If β *is a bilinear or sesquilinear form as above, then* $\mathrm{Aut}(V,\beta)$ *and* $\mathrm{S\,Aut}(V,\beta) := \mathrm{Aut}(V,\beta) \cap \mathrm{SL}(V)$ *are closed Lie subgroups of* $\mathrm{GL}(V)$.
(The form β *is most often taken to be nondegenerate.)*

Sketch of Proof:
Clearly, the sets in question are subgroups. Moreover, they are also closed: E.g., $\mathrm{Aut}(V,\beta) = \cap_{v,w} F_{v,w}$, where

$$F_{v,w} := \{A \in \mathrm{GL}(V) \mid \beta(Av, Aw) = \beta(v,w)\}.$$

The fact that they are **Lie** subgroups follows from a theorem to be considered later.
However, as we shall see, most of the specific cases arising from various choices of β can be proved to be Lie groups by other means, e.g. as a consequence of the regular value theorem. □

Matrix Versions of Certain Linear Lie Groups (Faithful Matrix Representations)

- Let $\dim V = n$. After choosing a basis, $\mathrm{SL}(V)$ gives the matrix version $\mathrm{SL}(n, \mathbb{K}) := \{A \in \mathbb{K}^{n \times n} \mid \det A = 1\}$.
- Notice that even when $\mathbb{K} = \mathbb{C}$, it may be that β is only required to be $\mathbb{R}$-linear.
- Depending on whether $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and on the nature of β, the notation for the linear groups takes on special conventional forms introduced below.
- When choosing a basis in order to represent one of the groups associated to a form β in a matrix version, it is usually the case that one uses a basis under which the matrix that represents β takes on a canonical form.

Example ((Semi-) or (Pseudo-)orthogonal Groups)

Let (V, β) be a real scalar product space with $\dim V = n$.

We write $\mathrm{Aut}(V, \beta)$ as $\mathrm{O}(V, \beta)$ and refer to the semi- or pseudo-orthogonal group associated to β.

W.r.t. an appropriately ordered ONB, β is represented by the diagonal

$$\eta_{pq} := \begin{bmatrix} I_p & \\ & -I_q \end{bmatrix} \text{ with } p + q = n.$$

The group of matrices arising from $\mathrm{O}(V, \beta)$ with this basis is denoted $\mathrm{O}(p, q) = \mathrm{O}_{pq}$ and consists of the real matrices Q with $Q\eta_{pq}Q^\top = \eta_{pq}$.

These groups are called the semi- or pseudo-orthogonal matrix groups.

With this choice of ONB, the bilinear form (scalar product) is given in canonical form on $\mathbb{R}^n$:

$$\langle x, y \rangle := \sum_{i=1}^{p} x^i y^i - \sum_{i=p+1}^{p+q=n} x^i y^i.$$

An alternative description is

$$\mathrm{O}_{pq} = \{Q \in \mathrm{GL}(n) \mid \langle Qx, Qy \rangle = \langle x, y \rangle \text{ for all } x, y, \in \mathbb{R}^n\}.$$

Example

If β is positive definite, we then have $q = 0$, and $\mathrm{O}(V, \beta)$ is referred to as a real orthogonal group.

We write $\mathrm{O}(n, 0) = \mathrm{O}(n) = \mathrm{O}_n$ and refer to it as the real orthogonal matrix group;

$$Q \in O_n \Longleftrightarrow QQ^\top = I_n.$$

Example

There are also complex orthogonal groups (not to be confused with unitary groups).

In matrix representation

$$\mathrm{O}(n,\mathbb{C}) := \{Q \in \mathrm{GL}(n,\mathbb{C}) \mid QQ^\top = I_n\}.$$

Example

Let (V, β) be a Hermitian scalar product space.

We then write $\mathrm{Aut}(V, \beta, \mathbb{C})$ as $U(V, \beta)$ and refer to it as the pseudo-unitary group associated to β.

If β is positive definite, then we call it a unitary group.

We may choose a basis for V s.t. β is represented by the Hermitian form on $\mathbb{C}^n$ given by

$$\langle x, y \rangle := \sum_{i=1}^{p} \overline{x}^i y^i - \sum_{i=p+1}^{p+q=n} \overline{x}^i y^i.$$

We then obtain the pseudo-unitary matrix group

$$\mathrm{U}_{pq} := \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \langle Ax, Ay \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{C}^n\}.$$

We write $\mathrm{U}(n, 0) = \mathrm{U}(n) = \mathrm{U}_n$ and refer to it as the unitary matrix group.

In particular, $\mathrm{U}_1 = S^1 = \{z \in \mathbb{C} \colon |z| = 1\}$.

Definition

For (V, β) a real scalar product space, we have the special orthogonal group of (V, β) given by

$$\mathrm{SO}(V, \beta) := \mathrm{O}(V, \beta) \cap \mathrm{SL}(V, \mathbb{R}).$$

For (V, β) a Hermitian scalar product space, we have the special unitary group of (V, β) given by

$$\mathrm{SU}(V, \beta) := \mathrm{U}(V, \beta) \cap \mathrm{SL}(V, \mathbb{C}).$$

The group of $n \times n$ complex matrices of determinant one is the complex special linear matrix group $\mathrm{SL}(n, \mathbb{C})$.
Similarly defined is the real special linear group $\mathrm{SL}(n, \mathbb{R})$.
The special orthogonal and special pseudo-orthogonal matrix groups, $\mathrm{SO}(n)$ and $\mathrm{SO}(p, q)$, are the matrix groups defined by
$\mathrm{SO}(n) := \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SO}(p, q) := \mathrm{O}(p, q) \cap \mathrm{SL}(n, \mathbb{R})$.
The special unitary and special pseudo-unitary matrix groups $\mathrm{SU}(n)$ and $\mathrm{SU}(p, q)$ are defined similarly.

Definition (Symplectic Groups)

Let β be a nondegenerate skew-symmetric $\mathbb{C}$-bilinear (resp. $\mathbb{R}$-bilinear) form on a $2n$-dimensional complex (resp. real) vector space V.

The group $\mathrm{Aut}(V, \beta)$ is called the complex (resp. real) symplectic group and is denoted by $\mathrm{Sp}(V, \mathbb{C})$ (resp. $\mathrm{Sp}(V, \mathbb{R})$).

There exists a basis $\{f_i\}$ for V s.t. β is represented in canonical form by

$$(v, w) := \sum_{i=1}^{n} v^i w^{n+i} - \sum_{j=1}^{n} v^{n+j} w^j.$$

The symplectic matrix groups are given by

$$\mathrm{Sp}(2n, \mathbb{C}) := \{A \in \mathbb{C}^{2n \times 2n} \mid (Av, Aw) = (v, w)\},$$
$$\mathrm{Sp}(2n, \mathbb{R}) := \{A \in \mathbb{R}^{2n \times 2n} \mid (Av, Aw) = (v, w)\},$$

where (v, w) is given as above.

Lie Group Homomorphisms

Definition

Let G, H be Lie groups.

A smooth map $f\colon G \to H$ that is a group homomorphism is called a Lie group homomorphism.

A Lie group homomorphism is called a Lie group isomorphism in case it has an inverse that is also a Lie group homomorphism.

A Lie group isomorphism $G \to G$ is called a Lie group automorphism of G.

If $f\colon G \to H$ is a Lie group homomorphism, then by definition $f(g_1 g_2) = f(g_1)f(g_2)$ for all $g_1, g_2 \in G$, and it follows that $f(e) = e$ and also that $f(g^{-1}) = \big(f(g)\big)^{-1}$ for all $g \in G$.

Example

(a) The inclusion $\mathrm{SO}(n,\mathbb{R}) \hookrightarrow \mathrm{GL}(n,\mathbb{R})$ is a Lie group homomorphism.

(b) The circle $S^1 \subset \mathbb{C}$ is a Lie group under complex multiplication and the map

$$z = \mathrm{e}^{\imath\theta} \mapsto \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix}$$

is a Lie group homomorphism of S^1 into $\mathrm{SO}(n)$.

(c) The map $t \mapsto \mathrm{e}^{\imath t}$ is a Lie group homomorphism from $\mathbb{R}$ to $S^1 \subset \mathbb{C}$.

(d) The exponential map $\exp\colon \mathbb{R} \to \mathbb{R}^+$ with $x \mapsto \mathrm{e}^x$ is a Lie group homomorphism from $(\mathbb{R},+)$ to $(\mathbb{R}^+,\cdot)$.

(e) The conjugation map $C_g\colon G \to G$ given by $x \mapsto gxg^{-1}$ is a Lie group automorphism. Note that $C_g = L_g \circ R_{g^{-1}}$.

Definition

A Lie group homomorphism from the additive group $(\mathbb{R}, +)$ into a Lie group is called a one-parameter subgroup.

Despite the use of the word "subgroup", a one-parameter subgroup is actually a map.

Example

(a) The torus $S^1 \times S^1$ is a Lie group under multiplication given by $(\mathrm{e}^{\imath\tau_1}, \mathrm{e}^{\imath\theta_1})(\mathrm{e}^{\imath\tau_2}, \mathrm{e}^{\imath\theta_2}) := (\mathrm{e}^{\imath(\tau_1+\tau_2)}, \mathrm{e}^{\imath(\theta_1+\theta_2)})$.

Every homomorphism of $\mathbb{R}$ into $S^1 \times S^1$, i.e., every one-parameter subgroup of $S^1 \times S^1$, is of the form $t \mapsto (\mathrm{e}^{ta\imath}, \mathrm{e}^{tb\imath})$ for some $a, b \in \mathbb{R}$.

(b) $R\colon \mathbb{R} \to \mathrm{SO}(3)$, with $t \mapsto \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a one-parameter subgroup.

(c) Also $t \mapsto \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & \mathrm{e}^t \end{bmatrix}$ is a one-parameter subgroup of $\mathrm{GL}(3)$.

Lie Algebras and (Lie-)Exponential Maps

Definition

Let G be a Lie group.

A vector field $X \in \mathfrak{X}(G)$ is called left-invariant iff $(L_g)_* X = X$ for all $g \in G$.

A vector field $X \in \mathfrak{X}(G)$ is called right-invariant iff $(R_g)_* X = X$ for all $g \in G$.

The set of left-invariant (resp. right-invariant) vector fields is denoted $\mathfrak{X}^L(G)$ (resp. $\mathfrak{X}^R(G)$).

As

$$(L_g)_* X = TL_g \circ X \circ L_g^{-1},$$

left invariance means

$$TL_g \circ X \circ L_g^{-1} = X.$$

Equivalently, given any $x \in G$ we have $T_x L_g \cdot X(x) = X(gx)$ for all $g \in G$. Thus $X \in \mathfrak{X}(G)$ is left-invariant iff the following diagram commutes for all $g \in G$:

$$\begin{tikzcd} TG \arrow[r, "TL_g"] & TG \\ G \arrow[u, "X"] \arrow[r, "L_g"] & G \arrow[u, "X"'] \end{tikzcd}$$

There is a similar diagram for right invariance.

Lemma

$\mathfrak{X}^L(G)$ is closed under the Lie bracket operation.

Proof:
Suppose that $X, Y \in \mathfrak{X}^L(G)$.
Then we have

$$\begin{aligned}(L_g)_*[X,Y] &= \big[(L_g)_*X, (L_g)_*Y\big] \\ &= [X,Y].\end{aligned}$$

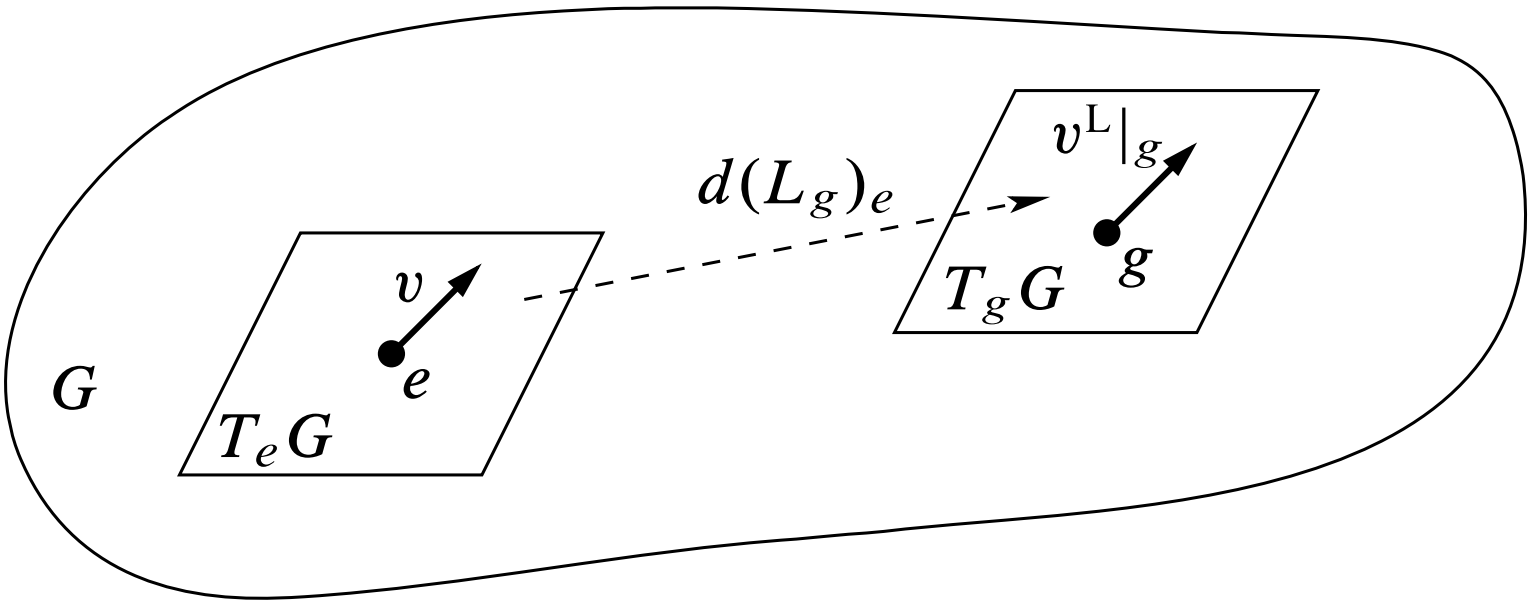
$\square$

Remark

- *Given a vector $v \in T_eG$,* **define** *a smooth left-(resp. right-)invariant vector field L^v (resp. R^v) s.t. $L^v(e) = v$ (resp. $R^v(e) = v$) by*

$$L^v(g) := TL_g \cdot v, \quad (\textit{resp.} \quad R^v(g) := TR_g \cdot v).$$

  More precisely, $L^v(g) = T_e(L_g) \cdot v$.
- *This prescription gives smooth invariant vector fields.*
- *Sometimes, we might write $L(v)$ for L^v and $R(v)$ for R^v.*
- *The map $v \mapsto L(v)$ (resp. $v \mapsto R(v)$) is a linear isomorphism from T_eG onto $\mathfrak{X}^L(G)$ (resp. $\mathfrak{X}^R(G)$).*
- *We will restrict attention to the left-invariant fields (clearly, it has analogies in the right-invariant case). We will later discover the adjoint map between the two cases.*



Defining a left-invariant vector field.

The linear isomorphism $T_eG \cong \mathfrak{X}^L(G)$ shows that $\mathfrak{X}^L(G)$ is, in fact, a vector space of finite dimension equal to $\dim G$.
We then immediately get the

Proposition

If G is a Lie group of dimension n, then $\mathfrak{X}^L(G)$ is an n-dimensional Lie algebra under the bracket of vector fields. □

We can therefore transfer the Lie algebra structure to T_eG:

Definition

For a Lie group G, define the bracket of any two elements $v, w \in T_eG$ by

$$[v, w] := [L^v, L^w](e).$$

Then T_eG becomes a Lie algebra, i.e., $\mathfrak{X}^L(G)$ and T_eG are isomorphic by construction.
By this result we denote the Lie algebra T_eG as **the** Lie algebra $\mathfrak{g}$ or $\mathfrak{L}(G)$ of the Lie group G.
Similarly, $\mathfrak{k}$ for the Lie group K, etc...

Definition (Structure Constants of a Lie Algebra)

Given an n-dimensional Lie algebra $\mathfrak{g}$, we can associate to every basis $v^1, \ldots, v^n$ for $\mathfrak{g}$, the structure constants c_{ij}^k defined by

$$[v_i, v_j] = \sum_k c_{ij}^k v_k \quad \text{for} \quad 1 \leq i, j, k \leq n.$$

Corollary

From the skew symmetry of the Lie bracket and the Jacobi identity it follows

(a) $c_{ij}^k = -c_{ji}^k$,

(b) $\sum_k c_{rs}^k c_{kt}^i + c_{st}^k c_{kr}^i + c_{tr}^k c_{ks}^i = 0$.

The structure constants characterize the Lie algebra.

Structure constants are sometimes used to actually **define** a Lie algebra once a basis is chosen.

Definition (Product of Two Lie Algebras)

Let $\mathfrak{a}, \mathfrak{b}$ be Lie algebras. For $(a_1, b_1), (a_2, b_2)$ elements of the vector space $\mathfrak{a} \times \mathfrak{b}$, define

$$\big[(a_1, b_1), (a_2, b_2)\big] := \big([a_1, a_2], [b_1, b_2]\big).$$

With this bracket, $\mathfrak{a} \times \mathfrak{b}$ is a Lie algebra called the Lie algebra product of $\mathfrak{a}$ and $\mathfrak{b}$.

Recall the definition of an ideal in a Lie algebra.
The subspaces $\mathfrak{a} \times \{0\}$ and $\{0\} \times \mathfrak{b}$ are ideals in $\mathfrak{a} \times \mathfrak{b}$ that are clearly isomorphic to $\mathfrak{a}$ and $\mathfrak{b}$, respectively.
We often identify $\mathfrak{a}$ with $\mathfrak{a} \times \{0\}$ and $\mathfrak{b}$ with $\{0\} \times \mathfrak{b}$.

Corollary

If G, H are Lie groups, then the Lie algebra $\mathfrak{g} \times \mathfrak{h}$ is (up to identifications) the Lie algebra of $G \times H$.

Definition (Lie Algebra Homomorphism)

Given two Lie algebras over a field $\mathbb{K}$, say $(\mathfrak{a}, [\ ,\ ]_{\mathfrak{a}})$ and $(\mathfrak{b}, [\ ,\ ]_{\mathfrak{b}})$, a $\mathbb{K}$-linear map σ is called a Lie algebra homomorphism iff

$$\sigma\big([v, w]_{\mathfrak{a}}\big) = [\sigma v, \sigma w]_{\mathfrak{b}}$$

for all $v, w \in \mathfrak{a}$.

A Lie algebra isomorphism is defined in the obvious way.

A Lie algebra isomorphism $\mathfrak{g} \to \mathfrak{g}$ is called an automorphism of $\mathfrak{g}$.

Remark

It is easily seen that the set of all automorphisms of $\mathfrak{g}$, denoted $\mathrm{Aut}(\mathfrak{g})$, forms a Lie group (actually a Lie subgroup of $\mathrm{GL}(\mathfrak{g})$).

Let V be a finite-dimensional $\mathbb{R}$-vector space.

Then $\mathrm{GL}(V)$ is an open subset of the linear space $L(V,V)$.

We identify

$$T\,\mathrm{GL}(V) \cong \mathrm{GL}(V) \times L(V,V).$$

Consequently,

$$T_A\,\mathrm{GL}(V) \cong \{A\} \times L(V,V).$$

The corresponding Lie algebra is $T_I\,\mathrm{GL}(V)$, and it has a Lie algebra structure derived from the Lie algebra structure on $\mathfrak{X}^L\big(\mathrm{GL}(V)\big)$.

The natural isomorphism of $T_I\,\mathrm{GL}(V) \cong L(V,V)$ puts a Lie algebra structure on $L(V,V)$.

We will now show that the resulting bracket on $L(V,V)$ is just the commutator bracket given by

$$[A,B] := A \circ B - B \circ A.$$

W.l.o.g we stay with the vector space $V = \mathbb{R}^n$.

Proposition (Lie Algebra of the General Linear Group)

The composition of the natural maps (identifications)

$$\mathfrak{gl}_n = \mathfrak{L}(\mathrm{GL}_n) \to T_{I_n}\mathrm{GL}_n \to L(\mathbb{R}^n, \mathbb{R}^n) \cong \mathbb{R}^{n\times n}$$

gives a Lie algebra isomorphism between $\mathfrak{gl}_n$ and the matrix algebra $\mathbb{R}^{n\times n}$.

Proof: Let x_{ij} be global coordinates of $X = (x_{ij})$ on $\mathrm{GL}_n \subset \mathbb{R}^{n\times n}$.
The vector space isomorphism $T_{I_n}\mathrm{GL}_n \cong \mathbb{R}^{n\times n}$ takes the form

$$\left(a_{ij}\frac{\partial}{\partial x_{ij}}|_{I_n}\right) \longleftrightarrow (a_{ij}).$$

Any $A = (a_{ij}) \in \mathfrak{gl}_n$ determines a left-invariant vector field $A^L \in \mathfrak{gl}_n$, i.e.

$$A^L|_X = \mathrm{D}(L_X)|_{I_n}(A) = \mathrm{D}(L_X)|_{I_n}\left(a_{ij}\frac{\partial}{\partial x_{ij}}|_{I_n}\right).$$

Since L_X is the restriction to GL_n of the linear map $A \mapsto AX$ its derivative is represented in coordinates by exactly the same linear map.

In other words, the left-invariant vector field A^L determined by A is the one whose value at $X \in \mathrm{GL}_n$ is

$$A^L|_X = \Big(\sum_j x_{ij} a_{jk} \tfrac{\partial}{\partial x_{ik}} |_X \Big).$$

Given $A, B \in \mathbb{R}^{n \times n}$, the Lie bracket of the corresponding left-invariant vector fields is given by

$$\begin{aligned}
[A^L, B^L] &= \Big[\Big(\sum_j x_{ij} a_{jk} \tfrac{\partial}{\partial x_{ik}} |_X \Big), \Big(\sum_q x_{pq} b_{qr} \tfrac{\partial}{\partial x_{pr}} |_X \Big) \Big] \\
&= \Big(\sum_{j,k} (x_{ij} a_{jk} b_{kr} - x_{ij} b_{jk} a_{kr}) \tfrac{\partial}{\partial x_{ir}} \Big),
\end{aligned}$$

here we used $\frac{\partial x_{pq}}{\partial x_{ik}}$ equal to 1 if $(p,q) = (i,k)$ and 0 otherwise, and A, B constant. Evaluating this last expression at $X = I_n$, we get

$$[A^L, B^L]_{I_n} = \Big(\sum_k (a_{ik} b_{kr} - b_{ik} a_{kr}) \tfrac{\partial}{\partial x_{ir}} \Big) \Big|_{I_n}.$$

This is the vector corresponding to the matrix commutator bracket $[A, B]$. Since the left-invariant vector field $[A^L, B^L]$ is determined by its value at the identity, this implies that

$$[A^L, B^L] = [A, B]^L,$$

which is precisely the statement about the Lie algebra isomorphism. $\square$

Remark

The importance of the Lie algebra of a Lie group stems from the fact that each Lie group homomorphism induces a Lie algebra homomorphism, as the next theorem shows.

Theorem (Induced Lie Algebra Homomorphisms)

Let G, H be Lie groups, and let $\mathfrak{g}, \mathfrak{h}$ be their Lie algebras.

Suppose $F\colon G \to H$ is a Lie group homomorphism.

For every $X \in \mathfrak{g}$, there is a unique vector field in $\mathfrak{h}$ that is F-related to X.

With this vector field denoted by fX, the map $f\colon \mathfrak{g} \to \mathfrak{h}$ so defined is a Lie algebra homomorphism.

Proof: If there is any vector field $Y \in \mathfrak{h}$ that is F-related to X, it must satisfy $Y_e = \mathrm{D}\,F_e(X_e)$, and thus it must be uniquely determined by

$$Y = \big(\mathrm{D}\,F_e(X_e)\big)^L.$$

To show that $Y \sim_F X$, the fact that F is a homomorphism implies

$$\begin{aligned} F(gg') = F(g)F(g') &\Rightarrow F(L_g g') = L_{F(g)}F(g') \\ &\Rightarrow F \circ L_g = L_{F(g)} \circ F \\ &\Rightarrow \mathrm{D}\,F \circ \mathrm{D}\,L_g = \mathrm{D}(L_{F(g)}) \circ \mathrm{D}\,F. \end{aligned}$$

Consequently,

$$\mathrm{D}\,F(X_g) = \mathrm{D}\,F\,\mathrm{D}\,L_g(X_e) = \mathrm{D}(L_{F(g)})\,\mathrm{D}\,F(X_e) = \mathrm{D}(L_{F(g)})(Y_e) = Y_{F(g)},$$

i.e., $X \sim_F Y$.

For each $X \in \mathfrak{g}$, let fX denote the unique vector field in $\mathfrak{h}$ that is F-related to X.

It then follows immediately that $f[X, Y] = [fX, fY]$, so f is a Lie algebra homomorphism. $\square$

Definition

The map $f\colon \mathfrak{g} \to \mathfrak{h}$ whose existence is asserted in this theorem is called the induced Lie algebra homomorphism.

Note that the theorem implies that for any left-invariant vector field $X \in \mathfrak{g}$, fX is a well-defined smooth vector field on H , even though F may **not** be a diffeomorphism.

Proposition (Properties of Induced Homomorphisms)

(a) *Let* $F := \mathrm{id}_G$ *be the identity map of* G *and* $f\colon \mathfrak{g} \to \mathfrak{g}$ *the induced Lie algebra homomorphism.*
Then f *is the identity map on* $\mathfrak{g}$.

(b) *If* $F_1\colon G \to H$ *and* $F_2\colon H \to K$ *are Lie group homomorphisms and* $F_{21} := F_2 \circ F_1$ *with induced Lie algebra homomorphism* f_1, f_2, f_{21}.
Then
$$f_{21} = f_2 \circ f_1\colon \mathfrak{g} \to \mathfrak{k}.$$

(c) *Isomorphic Lie groups have isomorphic Lie algebras.*

Proof:
Both of the relations

$$\mathrm{D}(\mathrm{id}_G)_e = \mathrm{id}_{T_eG} \quad \text{and} \quad \mathrm{D}(F_2 \circ F_1)_e = \mathrm{D}(F_2)_e \circ \mathrm{D}(F_1)_e$$

hold for derivatives.

Since the induced homomorphism is determined by the derivative at the identity, this proves (a) and (b).

Let $F\colon G \to H$ be an isomorphism with induced Lie algebra homomorphism f and corresponding $\hat{f}$ for F^{-1}.

Then (a) and (b) together imply that $f \circ \hat{f} = \hat{f} \circ f = \mathrm{id}$, so $f\colon \mathfrak{g} \to \mathfrak{h}$ is an isomorphism. $\square$

- Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $H \subset G$ a Lie subgroup.
- We might hope that the Lie algebra $\mathfrak{h}$ of H would be a Lie subalgebra of $\mathfrak{g}$.
- However, elements of $\mathfrak{h}$ are vector fields on H, not G, and so, strictly speaking, are not elements of $\mathfrak{g}$.
- Nonetheless, the next proposition gives us a way to view $\mathfrak{h}$ as a subalgebra of $\mathfrak{g}$.

Theorem (The Lie Algebra of a Lie Subgroup)

Suppose $H \subset G$ is a Lie subgroup, and $\mathrm{Inc}\colon H \hookrightarrow G$ *is the inclusion map with corresponding Lie algebra homomorphism* inc.

There is a Lie subalgebra $\hat{\mathfrak{h}} \subset \mathfrak{g}$ that is canonically isomorphic to $\mathfrak{h}$.

It is characterized by either of the following descriptions:

$$\begin{aligned}\hat{\mathfrak{h}} &= \mathrm{inc}(\mathfrak{h}) \\ &= \{X \in \mathfrak{g} \mid X_e \in T_e H\}.\end{aligned}$$

Proof:

Because Inc is a Lie group homomorphism, $\mathrm{inc}(\mathfrak{h})$ is a Lie subalgebra of $\mathfrak{g}$.

By the way we defined the induced Lie algebra homomorphism, this subalgebra is precisely the set of left-invariant vector fields on G whose values at the identity are of the form $\mathrm{D}\,\mathrm{Inc}_e(v)$ for some $v \in T_eH$.

Since the derivative $\mathrm{D}\,\mathrm{Inc}_e\colon T_eH \to T_eG$ is the inclusion of T_eH as a subspace in T_eG, the two characterizations of $\hat{\mathfrak{h}}$ are equal.

Since $\mathrm{D}\,\mathrm{Inc}_e$ is injective on T_eH, it follows that inc is injective on $\mathfrak{h}$.

Since it is surjective by definition of $\hat{\mathfrak{h}}$, it is an isomorphism between $\hat{\mathfrak{h}}$ and $\mathfrak{h}$. $\square$

Remark

Using this Theorem, whenever H is a Lie subgroup of G, we often identify $\mathfrak{h}$ as a subalgebra of $\mathfrak{g}$.

As mentioned above, elements of $\mathfrak{h}$ are not themselves left-invariant vector fields on G.

However, the preceding proposition shows that every element of $\mathfrak{h}$ corresponds to a unique element of $\mathfrak{g}$, determined by its value at the identity, and the injection of $\mathfrak{h}$ into $\mathfrak{g}$ thus determined respects Lie brackets.

So by thinking of $\mathfrak{h}$ as a subalgebra of $\mathfrak{g}$ we are not committing a "substantial" error.

This identification is especially illuminating in the case of Lie subgroups of GL_n.

Example (The Lie Algebra of O_n)

The orthogonal group O_n is a Lie subgroup of GL_n.
Using the regular value theorem (see the Exercises for details) it is equal to the level set $f^{-1}(I_n)$, where

$$f\colon \mathrm{GL}_n \to \mathrm{sym}_n, \quad f(A) := A^\top A.$$

Here $\mathrm{sym}_n \subset \mathbb{R}^{n\times n}$ is the vector space of symmetric $(n \times n)$-matrices.
It follows that $T_{I_n}\mathrm{O}_n = \ker \mathrm{D}\, f(I_n)$ where

$$\mathrm{D}\, f(I_n)\colon T_{I_n}\mathrm{GL}_n \to T_{I_n}\mathrm{sym}_n, \quad B \mapsto B^\top + B.$$

Consequently,

$$T_{I_n}\mathrm{O}_n = \{B \in \mathfrak{gl}_n \mid B^\top + B = 0\} = \mathrm{skew}_n =: \mathfrak{o}_n$$

the vector space skew_n of skew-symmetric $(n \times n)$-matrices.

The above theorems then imply that $\mathfrak{o}_n$ is a Lie subalgebra of $\mathfrak{gl}_n$ being canonically isomorphic to $\mathfrak{L}(\mathrm{O}_n)$.
Note hat we did not even have to verify directly that $\mathfrak{o}_n$ is a subalgebra.

Interlude: Lie Group Representations

- Most of the Lie groups we have seen so far can be realized as Lie subgroups of $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$.
- Are all Lie groups of this form? The key is the theory of group **representations**.
- Recall that if V is an n-dimensional $\mathbb{R}$- or $\mathbb{C}$-vector space, $\mathrm{GL}(V)$ denotes the group of invertible linear transformations of V, which is a Lie group isomorphic to $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$.

Definition (Lie Group Representation)

If G is a Lie group, a (finite-dimensional) representation of G is a Lie group homomorphism $G \to \mathrm{GL}(V)$.

(Although it is useful for many applications to consider also the case of V being infinite-dimensional, we consider only finite-dimensional representations)

Definition

If a representation $\rho\colon G \to \mathrm{GL}(V)$ is injective, it is said to be faithful.

- If ρ is faithful, it follows that $\rho(G)$ is a Lie subgroup of $\mathrm{GL}(V)$, and ρ gives a Lie group isomorphism between G and $\rho(G) \subset \mathrm{GL}(V) \cong \mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$.

  Thus, a Lie group admits a faithful representation iff it is isomorphic to a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$ for some n.
- However, not every Lie group admits such a representation.

  E.g. the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ has no faithful representation and therefore is not isomorphic to any matrix group. (The universal covering group of a Lie group G is the simply connected Lie group G^* with the same Lie algebra)

Example (Lie Group Representations)

1. Let G be a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$, the inclusion map $G \hookrightarrow \mathrm{GL}(n,\mathbb{R})$ is a faithful representation, called the defining representation of G.

   Similarly, for a Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$.

2. The inclusion map $S^1 \hookrightarrow \mathbb{C}^* \cong \mathrm{GL}(1,\mathbb{C})$ is a faithful representation of the circle group.
   More generally, the map $\rho\colon T^n \to \mathrm{GL}(n,\mathbb{C})$ given by $\rho(z_1,\ldots,z_n) := \mathrm{diag}(z_1,\ldots,z_n)$ is a faithful representation of T^n.

3. Consider
$$\sigma\colon \mathbb{R}^n \to \mathrm{GL}(n+1,\mathbb{R}) \quad \text{with} \quad \sigma(x) := \begin{bmatrix} I_n & x \\ 0 & 1 \end{bmatrix}.$$
   This is a faithful representation of the additive Lie group $\mathbb{R}^n$.

4. Another representation of $(\mathbb{R}^n,+)$ is the map $\mathbb{R}^n \to \mathrm{GL}(n,\mathbb{C})$ with $(x_1,\ldots,x_n) \mapsto \mathrm{diag}(\mathrm{e}^{2\pi\imath x_1},\ldots,\mathrm{e}^{2\pi\imath x_n})$.
   This is **not** faithful, because its kernel is the subgroup $\mathbb{Z}^n \subset \mathbb{R}^n$.

Example

Let $n \in \mathbb{N}$ and define $\beta\colon \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{R})$ by $a + \imath b \mapsto \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$ in the sense that

$$\begin{bmatrix} a_{11} + \imath b_{11} & \cdots & a_{1n} + \imath b_{1n} \\ \vdots & & \vdots \\ a_{n1} + \imath b_{n1} & \cdots & a_{nn} + \imath b_{nn} \end{bmatrix} \mapsto \begin{bmatrix} \begin{bmatrix} a_{11} & -b_{11} \\ b_{11} & a_{11} \end{bmatrix} & \cdots & \begin{bmatrix} a_{1n} & -b_{1n} \\ b_{1n} & a_{1n} \end{bmatrix} \\ \vdots & & \vdots \\ \begin{bmatrix} a_{n1} & -b_{n1} \\ b_{n1} & a_{n1} \end{bmatrix} & \cdots & \begin{bmatrix} a_{nn} & -b_{nn} \\ b_{nn} & a_{nn} \end{bmatrix} \end{bmatrix}$$

One verifies that β is an injective Lie group homomorphism whose image is a properly embedded Lie subgroup of $\mathrm{GL}(2n,\mathbb{R})$.

Thus, $\mathrm{GL}(n,\mathbb{C})$ is isomorphic to this Lie subgroup of $\mathrm{GL}(2n,\mathbb{R})$.

We showed already $\mathfrak{L}\big(\mathrm{GL}(n,\mathbb{R})\big) \cong \mathbb{R}^{n\times n}$ with notation $\mathfrak{gl}(n,\mathbb{R})$.

A similar result holds for $\mathrm{GL}(n,\mathbb{C})$.

As in the real case, the usual identification of $\mathrm{GL}(n,\mathbb{C}) \subset \mathbb{C}^{n\times n}$ as an open subset yields a sequence of vector space isomorphisms:

$$\mathfrak{L}\big(\mathrm{GL}(n,\mathbb{C})\big) \xrightarrow{\varepsilon} T_{I_n}\,\mathrm{GL}(n,\mathbb{C}) \xrightarrow{\varphi} \mathbb{C}^{n\times n}, \qquad (*)$$

where ε is the evaluation map and φ is the usual identification between the tangent space to an open subset of a vector space and the vector space itself.

Note that we are considering these as real vector spaces, not complex ones.

Proposition (The Lie Algebra of $\mathrm{GL}(n,\mathbb{C})$)

The composition of the maps in $(*)$ yields a Lie algebra isomorphism between $\mathfrak{L}\big(\mathrm{GL}(n,\mathbb{C})\big)$ and the matrix algebra $\mathbb{C}^{n\times n}$.

Proof: The Lie group homomorphism $\beta\colon\ \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{R})$ from the above example induces a Lie algebra homomorphism

$$T\beta|_{I_n}\colon \mathfrak{L}\big(\mathrm{GL}(n,\mathbb{C})\big) \to \mathfrak{L}\big(\mathrm{GL}(2n,\mathbb{R})\big) \cong \mathfrak{gl}(2n,\mathbb{R}).$$

Composing $T\beta|_{I_n}$ with our canonical isomorphisms yields a commutative diagram

$$\begin{tikzcd}
\mathfrak{L}\big(\mathrm{GL}(n,\mathbb{C})\big) \arrow[r, "\varepsilon"] \arrow[d, "T\beta|_{I_n}"] & T_{I_n}\mathrm{GL}(n,\mathbb{C}) \arrow[r, "\varphi"] \arrow[d, "\mathrm{D}\,\beta|_{I_n}"] & \mathbb{C}^{n\times n} \arrow[d, "\alpha"] \\
\mathfrak{L}\big(\mathrm{GL}(2n,\mathbb{R})\big) \arrow[r, "\varepsilon"] & T_{I_n}\mathrm{GL}(2n,\mathbb{R}) \arrow[r, "\varphi"] & \mathbb{R}^{2n\times 2n}
\end{tikzcd} \qquad (*)$$

in which $\alpha = \varphi \circ \mathrm{D}\,\beta|_{I_n} \circ \varphi^{-1}$.
We know already that the composition of the isomorphisms in the bottom row is a Lie algebra isomorphism.
We need to show the same thing for the top row.

The map β is (the restriction of) a linear map.

Therefore $\mathrm{D}\,\beta_{I_n}$ is given by the same formula as β itself, similarly for α.

Because $\beta(AB) = \beta(A)\beta(B)$, we have

$$\alpha[A, B] = \alpha(AB - BA) = \alpha(A)\alpha(B) - \alpha(B)\alpha(A) = [\alpha(A), \alpha(B)].$$

Hence, α is an injective Lie algebra homomorphism $\mathbb{C}^{n\times n} \to \mathbb{R}^{2n\times 2n}$.

Replacing the bottom row in $(*)$ by the images of the vertical maps, we obtain a commutative diagram of vector space isomorphisms

$$\begin{tikzcd}
\mathfrak{L}\big(\mathrm{GL}(n,\mathbb{C})\big) \arrow[r, "\cong"] \arrow[d, "T\beta|_{I_n}"'] & \mathfrak{gl}(n,\mathbb{C}) \arrow[d, "\alpha"] \\
T\beta|_{I_n}\mathfrak{L}\big(\mathrm{GL}(n,\mathbb{C})\big) \arrow[r, "\cong"'] & \alpha\big(\mathfrak{gl}(n,\mathbb{C})\big)
\end{tikzcd}$$

where the bottom map and the two vertical maps are Lie algebra isomorphisms.

It follows that the top map is also a Lie algebra isomorphism. $\square$

Parallel to the notion of representations of Lie groups, there is also a notion of representations of Lie algebras.

Definition (Lie Algebra Representation)

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra.

A (finite-dimensional) representation of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\varphi\colon\ \mathfrak{g} \to \mathfrak{gl}(V)$$

for some finite-dimensional vector space V.

Here $\mathfrak{gl}(V)$ denotes the Lie algebra of linear maps from V to itself.

If φ is injective, it is said to be faithful, in which case $\mathfrak{g}$ is isomorphic to the Lie subalgebra

$$\varphi(\mathfrak{g}) \subset \mathfrak{gl}(V) \cong \mathfrak{gl}(n, \mathbb{R}).$$

Theorem (Ado's Theorem, without Proof)

Every *finite-dimensional real Lie algebra admits a faithful finite-dimensional representation.*

Corollary

Every finite-dimensional real Lie algebra is isomorphic to a Lie subalgebra of some matrix algebra $\mathfrak{gl}(n, \mathbb{R})$ *with the commutator bracket.*

Before we proceed to introduce the Lie exponential map $\exp\colon \mathfrak{g} \to G$ we recall certain facts from ODE theory adapted to differentiable manifolds.

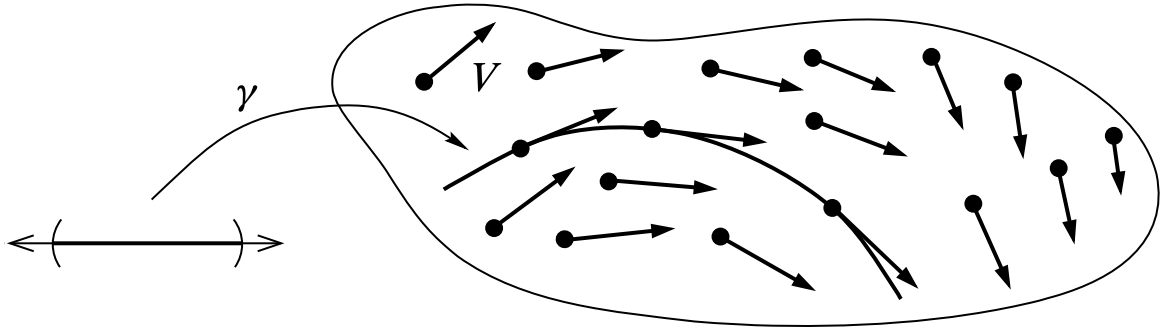
Definition

Let M be a smooth manifold and $V \in \mathfrak{X}(M)$.

An integral curve of V is a differentiable curve $\gamma\colon I \to M$ whose velocity at each point is equal to the value of V at that point:

$$\gamma'(t) = V_{\gamma(t)} \quad \text{for all} \quad t \in I.$$

If $0 \in I$, the point $\gamma(0)$ is called the starting point of γ.



Example

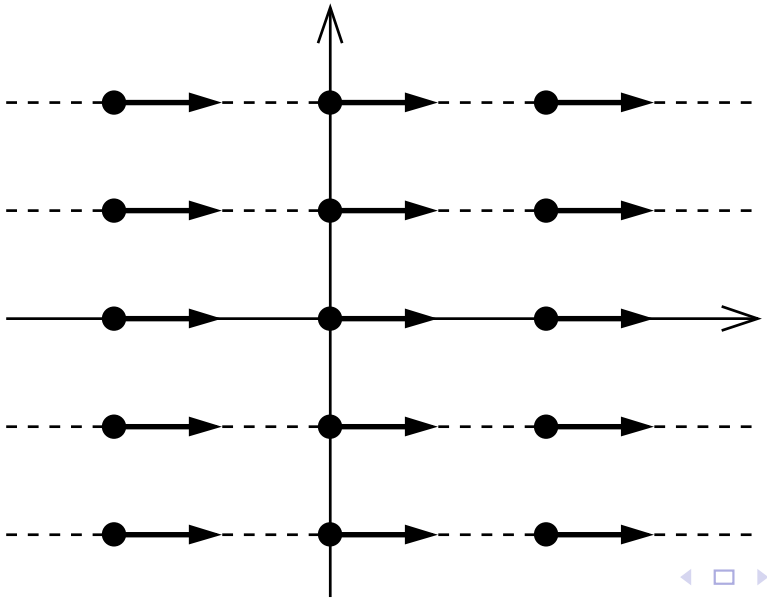
Let (x, y) be standard coordinates on $\mathbb{R}^2$, and let $V = \partial/\partial x \in \mathfrak{X}(\mathbb{R}^2)$.

Then the integral curves of V are precisely the straight lines parallel to the x-axis.

Useful parametrizations are of the form $\gamma(t) = (a + t, b)$ for constants a, b.

Thus, there is a **unique** integral curve starting at each point of the plane.

The images of different integral curves are either identical or disjoint.



Example

Let $W = x\,\partial/\partial y - y\,\partial/\partial x$ on $\mathbb{R}^2$.

Consider a smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$, written in standard coordinates as $\gamma(t) = \big(x(t), y(t)\big)$.

Then the condition $\gamma'(t) = W_{\gamma(t)}$ for γ to be an integral curve translates to

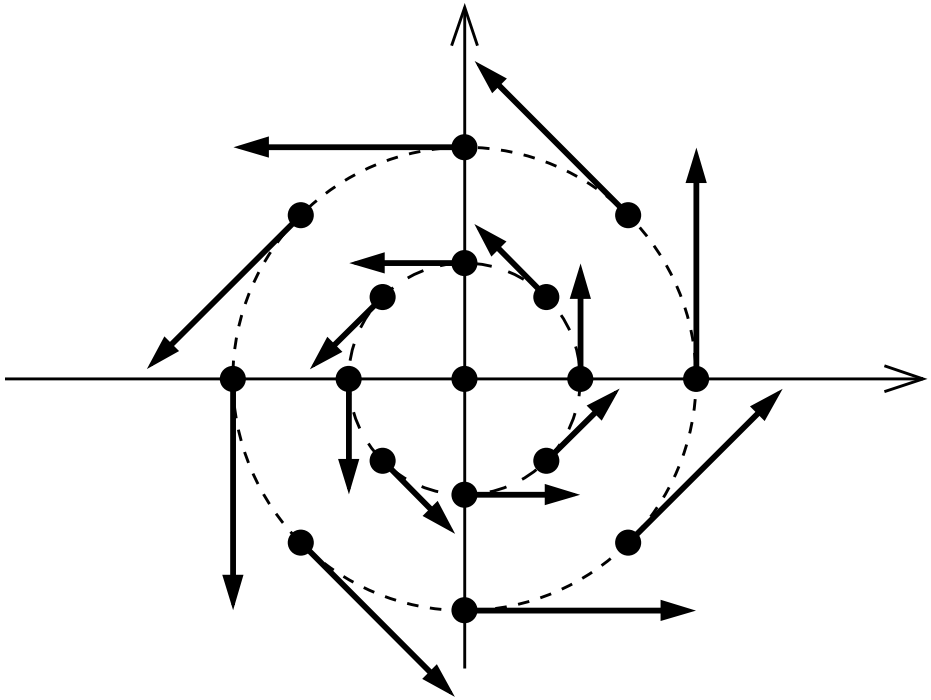
$$x'(t)\tfrac{\partial}{\partial x}\big|_{\gamma(t)} + y'(t)\tfrac{\partial}{\partial y}\big|_{\gamma(t)} = x(t)\tfrac{\partial}{\partial y}\big|_{\gamma(t)} - y(t)\tfrac{\partial}{\partial x}\big|_{\gamma(t)}.$$

Comparing the components this is equivalent to the ODE

$$\begin{aligned} x'(t) &= -y(t), \\ y'(t) &= x(t), \end{aligned}$$

with solution

$$x(t) = a\cos t - b\sin t, \quad y(t) = a\sin t + b\cos t, \quad a, b \in \mathbb{R}.$$



- As the second example illustrates, finding integral curves boils down to solving an ODE in a smooth chart.
- The fundamental fact about such systems is the existence, uniqueness, and smoothness theorem taught in ODE courses.
- Solving a system of ODEs is often referred to as integrating the system.

Proposition

Consider a smooth manifold M and let $V \in \mathfrak{X}(M)$.

For each $p \in M$ there exist $\varepsilon > 0$ and a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V starting at p.

Proof:
Apply the existence statement part of Picard-Lindelöf. □

Lemma (Rescaling Lemma)

Consider $V \in \mathfrak{X}(M)$, let $J \subset \mathbb{R}$ be an interval, and let $\gamma\colon J \to M$ be an integral curve of V.

For any $a \in \mathbb{R}$ the curve

$$\begin{aligned} \widetilde{\gamma}\colon \widetilde{J} &\to M, \\ \widetilde{\gamma}(t) &:= \gamma(at) \end{aligned}$$

is an integral curve of the vector field $a \cdot V$, where $\widetilde{J} := \{t \mid at \in J\}$.

Idea of Proof:
Application of the chain rule. □

Lemma (Translation Lemma)

Let V, M, J, γ as above.

For any $b \in \mathbb{R}$, the curve

$$\widehat{\gamma}\colon \widehat{J} \to M,$$
$$\widehat{\gamma}(t) := \gamma(t+b)$$

is also an integral curve of V, where $\widehat{J} := \{t \mid t + b \in J\}$.

Idea of Proof:
Autonomous (i.e., time-invariant) ODE... □

Proposition (Naturality of Integral Curves)

Consider smooth manifolds M, N and a smooth map $F\colon M \to N$.

Then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related iff F takes integral curves of X to integral curves of Y.

This means that for each integral curve γ of X, $F \circ \gamma$ is an integral curve of Y.

Proof:

Suppose first that $X \sim_F Y$, and $\gamma\colon J \to M$ is an integral curve of X.

If we define $\sigma\colon J \to N$ by $\sigma := F \circ \gamma$, then

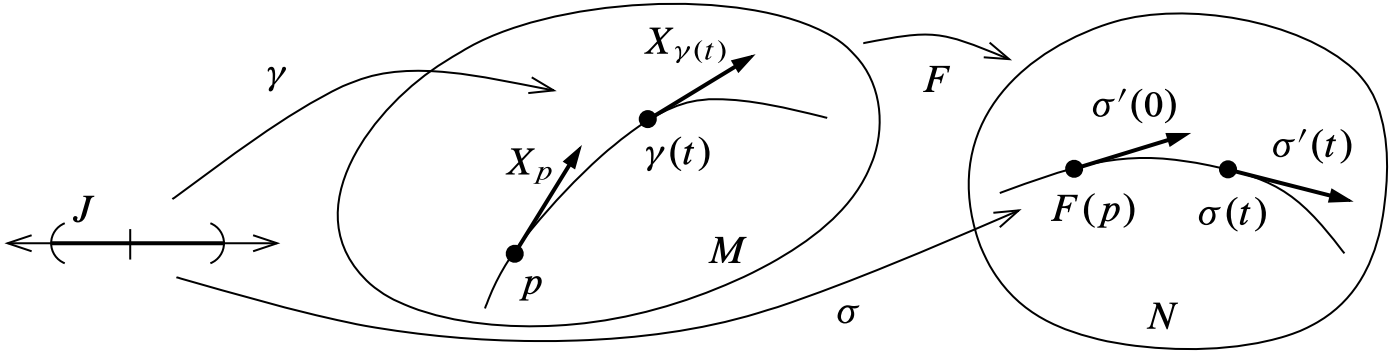
$$\begin{aligned}\sigma'(t) &= (F \circ \gamma)'(t)\\ &= \mathrm{D}\,F_{\gamma(t)}(\gamma'(t))\\ &= \mathrm{D}\,F_{\gamma(t)}(X_{\gamma(t)})\\ &= Y_{F(\gamma(t))}\\ &= Y_{\sigma(t)},\end{aligned}$$

so σ is an integral curve of Y.

Conversely, suppose F takes integral curves of X to integral curves of Y. Given $p \in M$, let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be an integral curve of X starting at p. Since $F \circ \gamma$ is an integral curve of Y starting at $F(p)$, we have

$$Y_{F(p)} = (F \circ \gamma)'(0) = \mathrm{D}\, F_p(\gamma'(0)) = \mathrm{D}\, F_p(X_p),$$

showing $X \sim_F Y$. $\square$



Flows

Motivation:

There is another way to visualize integral curves of a vector field.
Let M be a smooth manifold. Suppose for each $p \in M$, $V \in \mathfrak{X}(M)$ has an integral curve starting at p, defined **for all** $t \in \mathbb{R}$, denoted by $\theta^{(p)} \colon \mathbb{R} \to M$.

Note, however, it may not always be the case that every integral curve is defined for all t!

For each $t \in \mathbb{R}$, we can define

$$\theta_t \colon M \to M, \quad \theta_t(p) := \theta^{(p)}(t).$$

The translation lemma implies $t \mapsto \theta^{(p)}(t+s)$ is an integral curve of V starting at $q := \theta^{(p)}(s)$ (uniqueness of integral curves).

Consequently,

$$(\theta_t \circ \theta_s)(p) = \theta_{t+s}(p),$$

and with $\theta_0(p) = \theta^{(p)}(0) = p$, this implies that $\theta \colon \mathbb{R} \times M \to M$ is an action of $(\mathbb{R}, +)$ on M.

Definition (Global Flow or One-Parameter Group Action)

The continuous left $\mathbb{R}$-action, i.e., the continuous map

$$\theta\colon \mathbb{R} \times M \to M, \quad \theta\big(t, \theta(s,p)\big) := \theta(t+s, p)$$

with

$$\theta(0,p) := p, \quad \forall s,t \in \mathbb{R}, \forall p \in M$$

is called a global flow or one-parameter group action.

Some terminology:
Associated to the global flow θ are:

- For each $t \in \mathbb{R}$, define a continuous map

$$\theta_t \colon M \to M, \quad \theta_t(p) := \theta(t,p).$$

  The defining properties of θ are equivalent to the group laws

$$\theta_t \circ \theta_s = \theta_{t+s}, \quad \theta_0 = \mathrm{id}_M\,.$$

  As is the case for any continuous group action, each map $\theta_t \colon M \to M$ is a homeomorphism.
  If the flow is smooth, θ_t is even a diffeomorphism.

- For each $p \in M$ define a curve

$$\theta^{(p)} \colon \mathbb{R} \to M, \quad \theta^{(p)}(t) := \theta(t,p)$$

  The image of this curve is the orbit of p under the group action.

Definition

If $\theta\colon \mathbb{R} \times M \to M$ is a smooth global flow, for each $p \in M$ we define a tangent vector

$$V_p \in T_pM, \quad V_p := \left(\theta^{(p)}\right)'(0).$$

The assignment $p \mapsto V_p$ is a vector field on M;
the infinitesimal generator of θ.

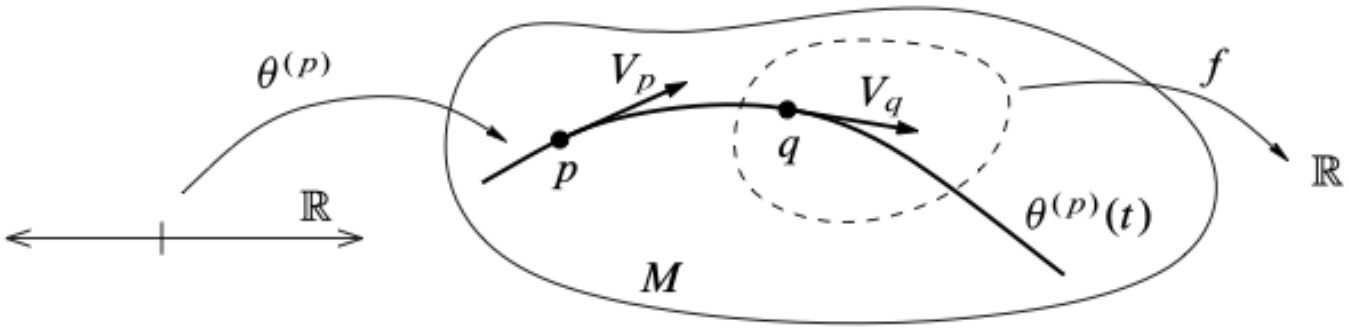
Proposition

Let $\theta\colon \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M.
The infinitesimal generator V of θ is a smooth vector field on M.
Moreover, each curve $\theta^{(p)}$ is an integral curve of V.

Proof:
To show that V is smooth, it suffices to show that Vf is smooth for every smooth real-valued function f defined on an open $U \subset M$.
For any such f and any $p \in U$

$$Vf(p) = V_p f = (\theta^{(p)})'(0)f = \frac{\mathrm{d}}{\mathrm{d}t} f\big(\theta^{(p)}(t)\big)|_{t=0} = \frac{\partial}{\partial t} f\big(\theta(t,p)\big)|_{(0,p)}.$$

Thus, V is smooth.



Next we show that $\theta^{(p)}$ is an integral curve of V, i.e., $(\theta^{(p)})'(t) = V_{\theta^{(p)}(t)}$ for all $p \in M$, $t \in \mathbb{R}$.

Let $t_0 \in \mathbb{R}$ be arbitrary, and set $q := \theta^{(p)}(t_0) = \theta_{t_0}(p)$, so what we have to show is $(\theta^{(p)})'(t_0) = V_q$, see Figure above.

By the group law, for all t

$$\theta^{(q)}(t) = \theta_t(q) = \theta_t\big(\theta_{t_0}(p)\big) = \theta_{t+t_0}(p) = \theta^{(p)}(t + t_0).$$

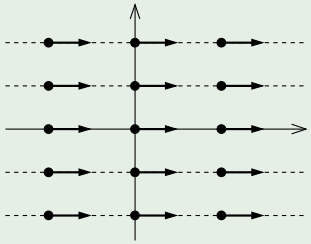
Consequently, for any smooth real-valued function f defined near q,

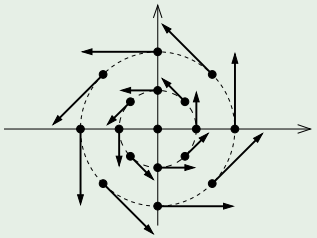
$$V_q f = (\theta^{(q)})'(0)f = \tfrac{\mathrm{d}}{\mathrm{d}t} f\big(\theta^{(q)}(t)\big)|_{t=0} = \tfrac{\mathrm{d}}{\mathrm{d}t} f\big(\theta^{(p)}(t+t_0)\big)|_{t=0} = (\theta^{(p)})'(t_0)f.$$

$\square$

Example (Global Flows)

The vector fields below both have integral curves defined for all $t \in \mathbb{R}$, so they generate global flows. We can write down the flows explicitly.





Left figure: the flow of $V = \partial/\partial x$ in $\mathbb{R}^2$ is the map

$$\tau\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad \tau_t(x, y) := (x + t, y).$$

For each $t \neq 0$, τ_t translates the plane to the right ($t > 0$) or left ($t < 0$) by a distance $|t|$.

Right figure: the flow of $W = x\partial/\partial y - y\partial/\partial x$ is the map

$$\theta\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad \theta_t(x, y) := (x \cos t - y \sin t, x \sin t + y \cos t).$$

For each $t \in \mathbb{R}$, θ_t rotates the plane through an angle t about the origin.

In general, however, the situation might be more subtle; there are smooth vector fields whose integral curves are not defined for all $t \in \mathbb{R}$:

Example

Let $M := \mathbb{R}^2 \setminus \{0\}$ with standard coordinates (x, y) and $V := \partial/\partial x \in \mathfrak{X}(M)$.

The unique integral curve of V starting at $(-1, 0)$ is $\gamma(t) = (t - 1, 0)$.

However, in this case, γ cannot be extended continuously past $t = 1$.

This is intuitively evident because of the "hole" in M at the origin.

To prove it rigorously, suppose $\widetilde{\gamma}$ is any continuous extension of γ past $t = 1$.

Then $\gamma(t) \to \widetilde{\gamma}(1) \in M$ as $t \nearrow 1$. But we can also consider γ as a map into $\mathbb{R}^2$ by composing with the inclusion $M \hookrightarrow \mathbb{R}^2$, and it is obvious from the formula that $\gamma(t) \to (0, 0)$ as $t \nearrow 1$.

Since limits are unique, this is a contradiction.

Example (Riccati ODE)

Consider $\mathbb{R}^2$ with $W := x^2\partial/\partial x \in \mathfrak{X}(\mathbb{R}^2)$.

The unique integral curve of W starting at $(1,0)$ is

$$\gamma(t) := \left(\tfrac{1}{1-t}, 0\right).$$

This curve also cannot be extended past $t = 1$, because its x-coordinate is unbounded as $t \nearrow 1$.

To deal with this kind of situations we introduce the concept of a local flow.

Definition (Local Flow)

A flow domain for M is an open $\mathscr{D} \subset \mathbb{R} \times M$ s.t. for each $p \in M$ the set $\mathscr{D}^{(p)} := \{t \in \mathbb{R} \mid (t,p) \in \mathscr{D}\}$ is an open interval containing 0.

A flow on M is a continuous map $\theta\colon \mathscr{D} \to M$ where $\mathscr{D} \subset \mathbb{R} \times M$ is a flow domain satisfying the group laws:

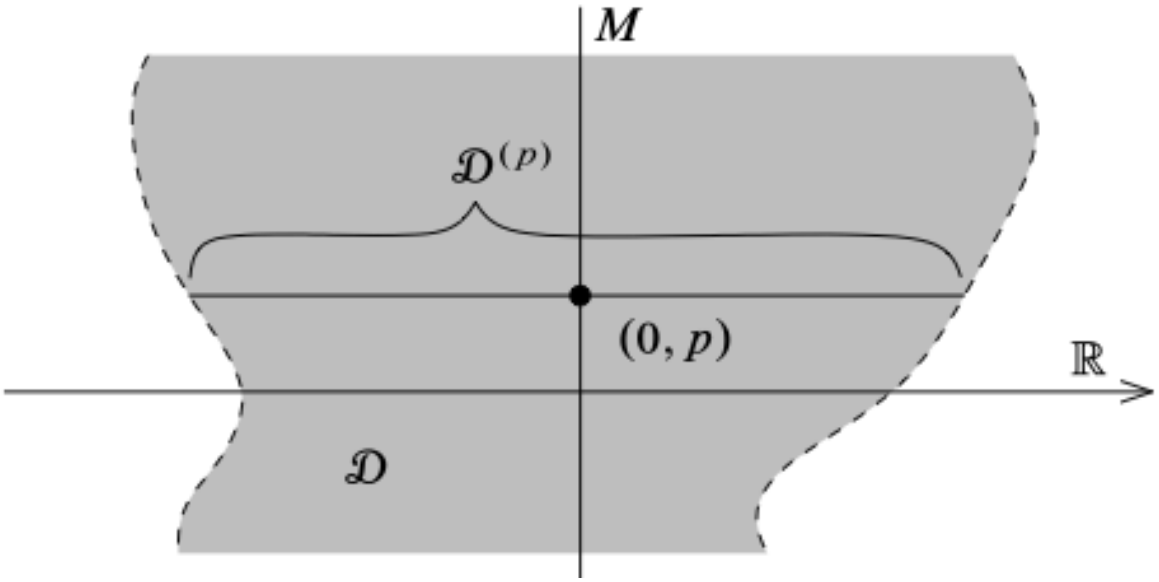
$$\theta(0,p) = p, \quad \text{for all } p \in M,$$
$$\theta(t,\theta(s,p)) = \theta(t+s,p), \quad \text{for all } s \in \mathscr{D}^{(p)}, t \in \mathscr{D}^{\theta(s,p)}, \text{s.t. } s+t \in \mathscr{D}^{(p)}.$$

θ is called local flow to distinguish it from a global one.

The term local one-parameter group action is also used.

If θ is a flow, we define $\theta_t(p) := \theta^{(p)}(t) := \theta(t,p)$ whenever $(t,p) \in \mathscr{D}$, just as for a global flow.

For smooth θ, the infinitesimal generator of θ is defined as $V_p := (\theta^{(p)})'(0)$.


M
$\mathcal{D}^{(p)}$
$(0,p)$
$\mathbb{R}$
$\mathcal{D}$

Proposition

If $\theta\colon \mathscr{D} \to M$ is a smooth flow, then the infinitesimal generator V of θ is a smooth vector field, and each curve $\theta^{(p)}$ is an integral curve of V.

Sketch of Proof:
The proof is essentially identical to the analogous proof for global flows. □

A maximal integral curve is one that cannot be extended to an integral curve on any larger open interval.

A maximal flow is a flow that admits no extension to a flow on a larger flow domain.

We proceed by recalling certain important results from ODE-theory, where usually instead of manifolds open subsets of $\mathbb{R}$-vector spaces are considered:

Theorem (Fundamental Theorem on Flows)

Let V be a smooth vector field on a smooth manifold M.

There is a unique smooth **maximal** *flow $\theta\colon \mathscr{D} \to M$ whose infinitesimal generator is V.*

This flow has the following properties:

(a) *For each $p \in M$ the curve $\theta^{(p)}\colon \mathscr{D}^{(p)} \to M$ is the unique maximal integral curve of V starting at p.*

(b) *If $s \in \mathscr{D}^{(p)}$, then $\mathscr{D}^{\theta(s,p)}$ is the interval $\mathscr{D}^{(p)} - s = \{t - s \mid t \in \mathscr{D}^{(p)}\}$.*

(c) *For each $t \in \mathbb{R}$, the set $M_t := \{p \in M \mid (t,p) \in \mathscr{D}\}$ is open in M and $\theta_t\colon M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.*

Remark

The flow whose existence and uniqueness are asserted in the fundamental theorem is called the flow generated by V, or just the flow of V.

Proposition (Naturality of Flows)

Let M, N be smooth manifolds, $F\colon M \to N$ smooth, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$.

Let θ be the flow of X and η the flow of Y.

If $X \sim_F Y$, then for each $t \in \mathbb{R}$, $F(M_t) \subset N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:

$$\begin{tikzcd} M_t \arrow[r, "F"] \arrow[d, "\theta_t"'] & N_t \arrow[d, "\eta_t"] \\ M_{-t} \arrow[r, "F"'] & N_{-t} \end{tikzcd}$$

Corollary (Diffeomorphism Invariance of Flows)

Let $F\colon M \to N$ be a diffeomorphism.

If $X \in \mathfrak{X}(M)$ and θ is the flow of X, then the flow of F_*X is $\eta_t = F \circ \theta_t \circ F^{-1}$, with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.

Definition

A smooth vector field is called complete if it generates a global flow, or equivalently if each of its maximal integral curves is defined for all $t \in \mathbb{R}$.

- Under what circumstances is a vector field complete?
- Imagine you could solve the ODE to find all of the integral curves. If they all exist for all time, then the vector field is complete.
- On the other hand, if you can find a single integral curve that cannot be extended to all of $\mathbb{R}$, then it is not complete.
- However, it is often impossible to solve the ODE explicitly, so it is useful to have some general criteria for determining when a vector field is complete.

Lemma (Uniform Time Lemma)

Let V be a smooth vector field on a smooth manifold M, and let θ be its flow.

Suppose there is a positive number ε s.t. for every $p \in M$, the domain of $\theta^{(p)}$ contains $(-\varepsilon, \varepsilon)$.

Then V is complete.

Theorem

Every compactly supported smooth vector field on a smooth manifold is complete.

Corollary

On a compact smooth manifold, every smooth vector field is complete.

Theorem

Every left-invariant vector field on a Lie group is complete.
(Similarly for right-invariant ones)

Proof:
Let G be a Lie group, let $X \in \mathfrak{g}$, and let $\theta \colon \mathscr{D} \to G$ denote the flow of X.

There is some $\varepsilon > 0$ s.t. $\theta^{(e)}$ is defined on $(-\varepsilon, \varepsilon)$.

Let $g \in G$ be arbitrary. Because X is L_g-related to itself, it follows that the curve $L_g \circ \theta^{(e)}$ is an integral curve of X starting at g and therefore is equal to $\theta^{(g)}$.

This shows that for each $g \in G$, the integral curve $\theta^{(g)}$ is defined at least on $(-\varepsilon, \varepsilon)$.

So the Uniform Time Lemma guarantees that X is complete. $\square$

Theorem (Characterization of One-Parameter Subgroups)

Let G be a Lie group.

The one-parameter subgroups of G are precisely the maximal integral curves of left-invariant vector fields starting at the identity.

Proof:

First suppose γ is the maximal integral curve of some left-invariant vector field X starting at e.

As left-invariant vector fields are complete, γ is defined on all of $\mathbb{R}$.

Left-invariance means that X is L_g-related to itself for every $g \in G$.

Therefore L_g takes integral curves of X to integral curves of X.

Applying this with $g := \gamma(s)$ for some $s \in \mathbb{R}$, we conclude that the curve $t \mapsto L_{\gamma(s)}\big(\gamma(t)\big)$ is an integral curve starting at $\gamma(s)$.

The translation lemma implies that $t \mapsto \gamma(s+t)$ is also an integral curve with the same initial point, so they are equal:

$$\gamma(s)\gamma(t) = \gamma(t+s).$$

This says precisely that $\gamma\colon \mathbb{R} \to G$ is a one-parameter subgroup.

Conversely, suppose $\gamma\colon \mathbb{R} \to G$ is a one-parameter subgroup, and let $X := \mathrm{D}\,\gamma(\mathrm{d}/\mathrm{d}t)$, treating $\mathrm{d}/\mathrm{d}t$ as a left-invariant vector field on $\mathbb{R}$.

Since $\gamma(0) = e$, we just have to show that γ is an integral curve of X.

Recall that $\mathrm{D}\,\gamma(\mathrm{d}/\mathrm{d}t)$ is defined as the unique left-invariant vector field on G that is γ-related to $\mathrm{d}/\mathrm{d}t$.

Therefore, γ is an integral curve of X as for any $t_0 \in \mathbb{R}$,

$$\gamma'(t_0) = \mathrm{D}\,\gamma_{t_0}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)|_{t_0} = X_{\gamma(t_0)}.$$

$\square$

Definition

Given $X \in \mathfrak{g}$, the one-parameter subgroup determined by X in this way is called the one-parameter subgroup generated by X.

Corollary

As left-invariant vector fields are uniquely determined by their values at e, each one-parameter subgroup is uniquely determined by its initial velocity in T_eG, and thus there are one-to-one correspondences

$$\{\textit{one-parameter subgroups of } G\} \longleftrightarrow \mathfrak{g} \longleftrightarrow T_eG.$$

The one-parameter subgroups of $\mathrm{GL}(n, \mathbb{R})$ are not hard to compute as we will see now:

Proposition

For any $A \in \mathfrak{gl}(n, \mathbb{R})$ *let*

$$\begin{aligned} \mathrm{e}^A &:= \sum_{k=0}^{\infty} \tfrac{1}{k!} A^k \\ &= I_n + A + \tfrac{1}{2} A^2 + \cdots . \end{aligned}$$

This series converges to an invertible matrix $\mathrm{e}^A \in \mathrm{GL}(n, \mathbb{R})$.

Moreover, the one-parameter subgroup of $\mathrm{GL}(n, \mathbb{R})$ *generated by* $A \in \mathfrak{gl}(n, \mathbb{R})$ *is*

$$\gamma(t) = \mathrm{e}^{tA} .$$

Proof:

We verify first convergence of the series.

The absolute convergence of the series on the right follows directly from the estimate

$$\sum_{k=0}^{\infty} \tfrac{1}{k!} \|A^k\| \leq \sum_{k=0}^{\infty} \tfrac{1}{k!} \|A\|^k = \mathrm{e}^{\|A\|},$$

exploiting the submultiplicative property $\|AB\| \leq \|A\| \cdot \|B\|$ of the Frobenius norm $\|A\| := \sqrt{\operatorname{tr} AA^\top}$ on $\mathfrak{gl}_n$, and the comparison test for absolute convergence of a series in a Banach space.

Fix an $A \in \mathfrak{gl}_n$. The matrix A corresponds to the left-invariant vector field

$$A^L|_X := \Big(\sum_j x_{ij} a_{jk} \tfrac{\partial}{\partial x_{ik}} \Big)|_X, \quad X \in \mathrm{GL}_n\,. \qquad (*)$$

Thus, the one-parameter subgroup generated by A is an integral curve of A^L on GL_n, and therefore satisfies the ODE initial value problem

$$\gamma'(t) = A^L|_{\gamma(t)}, \quad \gamma(0) = I_n.$$

Using $(*)$ the condition for γ to be an integral curve can be rewritten as

$$\dot{\gamma}_{ik}(t) = \sum_j \gamma_{ij}(t) a_{jk},$$

or in matrix notation

$$\gamma'(t) = \gamma(t)A.$$

Since $\gamma(0) = I_n$, we see that γ is the unique integral curve of A^L starting at I_n and is therefore the desired one-parameter subgroup.

Concerning differentiability:

Differentiating the exponential series term-wise yields

$$\begin{aligned} \gamma'(t) &= \sum_{k=1}^{\infty} \tfrac{k}{k!} t^{k-1} A^k \\ &= \Big(\sum_{k=1}^{\infty} \tfrac{1}{(k-1)!} t^{k-1} A^{k-1} \Big) A \\ &= \gamma(t) A. \end{aligned} \tag{1}$$

Since (1) converges uniformly on bounded sets
(apart from the additional factor A, it is the same series as before!),
the term-by-term differentiation is justified.

A similar computation shows $\gamma'(t) = A\gamma(t)$ (reordered terms on RHS!).

By smoothness of solutions to ODEs, γ is smooth.

It remains only to show that $\gamma(t)$ is invertible for all t.

Consider

$$\sigma(t) := \gamma(t)\gamma(-t) = \mathrm{e}^{tA}\,\mathrm{e}^{-tA}\,.$$

Then σ is a smooth curve in $\mathfrak{gl}_n$, and by the product rule satisfies

$$\sigma'(t) = \big(\gamma(t)A\big)\gamma(-t) - \gamma(t)\big(A\gamma(-t)\big) = 0.$$

Hence, σ is the constant curve $\sigma(t) \equiv \sigma(0) = I_n$, i.e., $\gamma(t)\gamma(-t) = I_n$.

Substituting $-t$ for t, we obtain $\gamma(-t)\gamma(t) = I_n$, which shows that $\gamma(t)$ is invertible and $\gamma(t)^{-1} = \gamma(-t)$. $\square$

Let us compute the one-parameter subgroups of GL_n, such as O_n.

To do so, we need

Proposition

Suppose G is a Lie group and $H \subset G$ a Lie subgroup.

The one-parameter subgroups of H are precisely those one-parameter subgroups of G whose initial velocities lie in T_eH.

Proof:

Let $\gamma\colon \mathbb{R} \to H$ be a one-parameter subgroup.

Then the composite map

$$\mathbb{R} \xrightarrow{\gamma} H \hookrightarrow G$$

is a Lie group homomorphism and thus a one-parameter subgroup of G, which satisfies $\gamma'(0) \in T_eH$.

Conversely, suppose $\gamma\colon \mathbb{R} \to G$ is a one-parameter subgroup with initial velocity in T_eH.

Let $\widetilde{\gamma}\colon \mathbb{R} \to H$ be the one-parameter subgroup of H with the same initial velocity as γ at 0.

Again, by composing with the inclusion map, we can also consider $\widetilde{\gamma}$ as a one-parameter subgroup of G.

Since $\gamma, \widetilde{\gamma}$ are both one-parameter subgroups of G with the same initial velocity, they must be equal. $\square$

Example

If $H \subset \mathrm{GL}_n$ is a Lie subgroup we showed that the one-parameter subgroups of H are precisely the maps of the form $\gamma(t) = \mathrm{e}^{tA}$ for $A \in \mathfrak{h}$, where $\mathfrak{h}$ is the subalgebra corresponding to the Lie algebra of H.

For example, taking $H \in \mathrm{O}_n$, this shows that the one-parameter subgroups of O_n are the maps of the form $\gamma(t) = \mathrm{e}^{tA}$ for an arbitrary **skew-symmetric** matrix A.

In particular, the exponential of any skew-symmetric matrix is orthogonal.

More explicit examples:

(a): We compute the exponential of the real skew symmetric (2×2)-matrix

$$A := \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}. \tag{2}$$

We need to find an inductive formula expressing the powers A^n. Observe

$$\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} = \theta \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{=:J} \quad \text{and} \quad \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}^2 = \theta^2 J^2 = -\theta^2 I_2,$$

by induction we get

$$A^{4n} = \theta^{4n} I_2,\ A^{4n+1} = \theta^{4n+1} J,\ A^{4n+2} = -\theta^{4n+2} I_2,\ A^{4n+3} = -\theta^{4n+3} J, \ldots$$

and consequently, after rearranging terms

$$\begin{aligned} \mathrm{e}^A &= \left(1 - \tfrac{\theta^2}{2!} + \tfrac{\theta^4}{4!} - \tfrac{\theta^6}{6!} \pm \cdots\right) I_2 + \left(\tfrac{\theta}{1!} - \tfrac{\theta^3}{3!} + \tfrac{\theta^5}{5!} \pm \cdots\right) J \\ &= I_2 \cos\theta + J \sin\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \end{aligned}$$

(b): The exponential map is not always surjective.

Let us compute the exponential of a real (2×2)-matrix with zero trace of the form

$$A := \begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

Again we need to find an inductive formula expressing the powers A^n. Observe that

$$A^2 = (a^2 + bc) I_2 = -I_2 \det A.$$

If $\det A = -(a^2 + bc) = 0$ we get the simple formula

$$\mathrm{e}^A = I_2 + A.$$

For $(-\det A) = a^2 + bc < 0$, with $\omega > 0$, s.t. $\omega^2 = -(a^2 + bc) = \det A$, we get after some induction and rearranging terms the "closed" formula

$$\mathrm{e}^A = I_2 \cosh \omega + A \tfrac{\sinh \omega}{\omega}.$$

A simple calculation gives for both cases

$$\det(\mathrm{e}^A) = 1.$$

The exponential map is a function from the set of (2×2)-matrices with zero trace, i.e., from the Lie algebra $\mathfrak{sl}_2$ to the set of (2×2)-matrices with determinant 1, i.e., the Lie group SL_2.

This function is not surjective.

Indeed,

$$\begin{aligned}
\mathrm{tr}(\mathrm{e}^A) &= 2\cos\omega \quad \text{when} \quad a^2 + bc < 0 \quad \text{(cf. Ex. (a))}, \\
\mathrm{tr}(\mathrm{e}^A) &= 2\cosh\omega \quad \text{when} \quad a^2 + bc > 0, \\
\mathrm{tr}(\mathrm{e}^A) &= 2 \quad \text{when} \quad a^2 + bc = 0.
\end{aligned}$$

As a consequence, for any matrix A with zero trace,

$$\mathrm{tr}(\mathrm{e}^A) \geq -2.$$

Moreover, any matrix B with determinant 1 and whose trace is less than -2 is **not** the exponential e^A of any matrix A with zero trace.

E.g.

$$B := \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix}$$

where $a < 0$, $a \neq -1$, is **not** the exponential of any matrix A with zero trace since

$$\begin{aligned} \tfrac{(a+1)^2}{a} &= \tfrac{a^2+1}{a} + 2 < 0 \\ &\Longrightarrow \\ \operatorname{tr} B = a + \tfrac{1}{a} &= \tfrac{a^2+1}{a} < -2. \end{aligned}$$

Definition (Exponential Map of a Lie Group)

Given a Lie group G with Lie algebra $\mathfrak{g}$, we define

$$\exp\colon \mathfrak{g} \to G, \quad \exp X := \gamma(1),$$

called the exponential map of G.

Here γ is the one-parameter subgroup generated by X,
or equivalently the integral curve of X starting at the identity.

Proposition

Let G be a Lie group.

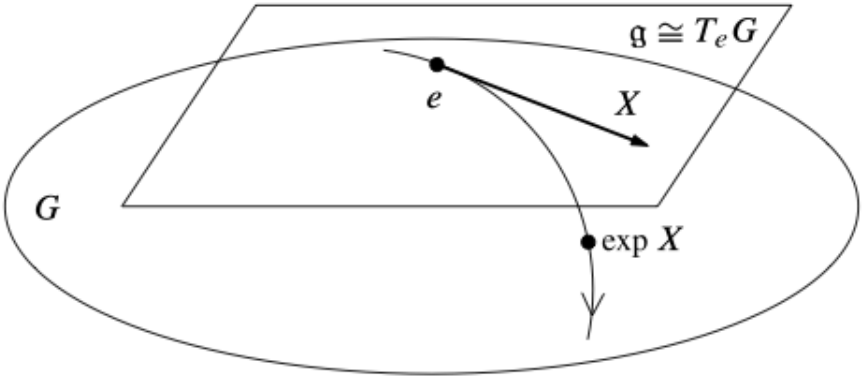
For any $X \in \mathfrak{g}$, $\gamma(s) = \exp sX$ is the one-parameter subgroup of G generated by X .

Proof:
Let $\gamma\colon \mathbb{R} \to G$ be the X-generated one-parameter subgroup, which is the integral curve of X starting at e.

For any fixed $s \in \mathbb{R}$, by the Rescaling Lemma, $\widetilde{\gamma}(t) = \gamma(st)$ is the integral curve of sX starting at e, so

$$\exp sX = \widetilde{\gamma}(1) = \gamma(s).$$

$\square$



Example

1. The exponential map of GL_n (or any Lie subgroup of it) is given by $\exp A = \mathrm{e}^A$. This is the reason for the term **exponential** map.
2. If V is a finite-dimensional $\mathbb{R}$-vector space, a choice of basis yields isomorphisms $\mathrm{GL}(V) \cong \mathrm{GL}_n$ and $\mathfrak{gl}(V) \cong \mathfrak{gl}_n$. The GL_n analysis then shows that the exponential map of GL_n can be written in the form
$$\exp A = \sum_{k=0}^{\infty} \tfrac{1}{k!} A^k,$$
where we consider $A \in \mathfrak{gl}(V)$ as a linear map $V \to V$, and $A^k := A \circ \cdots \circ A$ is the k-fold composition of A with itself.

Proposition (Properties of the Exponential Map)

Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra.

(a) *The exponential map $\mathfrak{g} \to G$ is smooth.*

(b) *For any $X \in \mathfrak{g}$ and $s, t \in \mathbb{R}$, we have* $\exp\big((s+t)X\big) = \exp(sX)\exp(tX)$.

(c) *For any $X \in \mathfrak{g}$ we have* $(\exp X)^{-1} = \exp(-X)$.

(d) *For any $X \in \mathfrak{g}$ and any $n \in \mathbb{Z}$ we have* $(\exp X)^n = \exp(nX)$.

(e) *The derivative* $(\mathrm{D}\exp)|_0 \colon T_0\,\mathfrak{g} \to T_e G$ *is the identity map, under the canonical identifications* $T_0\,\mathfrak{g} \cong T_e G \cong \mathfrak{g}$.

(f) *The exponential map restricts to a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G.*

(g) *If H is another Lie group, $\mathfrak{h}$ is its Lie algebra, and $\Phi \colon G \to H$ is a Lie group homomorphism, the diagram below commutes.*

(h) *The flow θ of a left-invariant vector field X is given by $\theta_t := R_{\exp(tX)}$ (right multiplication by $\exp(tX)$).*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{D}\,\Phi"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\Phi"'] & H \end{tikzcd}$$

Proof:
See Exercises.

Remark

Note that the above Proposition does **not** *imply* $\exp(X+Y) = \exp(X)\exp(Y)$ *for arbitrary* X, Y *in the Lie algebra.*

In fact, for connected groups, this is true only when the group is Abelian.

We demonstrate these facts by two examples:

Example

We know SO_2 is Abelian: Let $X, Y \in \mathfrak{so}_2$ with

$$X := x \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{=:J}, \quad Y := yJ$$

with $x, y \in \mathbb{R}$ but $x \neq y$ (i.e., $X \neq Y$). In particular, $[X, Y] = 0$. Then

$$\mathrm{e}^{X+Y} = I_2 \cos(x+y) + J \sin(x+y),$$

moreover, exploiting trigonometric addition theorems,

$$\begin{aligned} \mathrm{e}^X \cdot \mathrm{e}^Y &= (I_2 \cos x + J \sin x) \cdot (I_2 \cos y + J \sin y) \\ &= I_2(\cos x \cos y - \sin x \sin y) + J(\sin x \cos y + \cos x \sin y) \\ &= I_2 \cos(x+y) + J \sin(x+y). \end{aligned}$$

Example

We know SL_2 is not Abelian, it, however, has an Abelian subgroup, see above. Consider two noncommuting matrices $X, Y \in \mathfrak{sl}_2$ with

$$X := \begin{bmatrix} x & -x^2 \\ 1 & -x \end{bmatrix}, \quad Y := \begin{bmatrix} y & -y^2 \\ 1 & -y \end{bmatrix}$$

with $x, y \in \mathbb{R}$ and $x \neq y$, i.e. $X \neq Y$. Note that $\det X = \det Y = 0$. Then we compute

$$\mathrm{e}^{X+Y} = I_2 + X + Y,$$

on the other hand

$$\mathrm{e}^X \cdot \mathrm{e}^Y = (I_2 + X) \cdot (I_2 + Y) = I_2 + X + Y + XY,$$

but, because $x \neq y$ by assumption, we get

$$XY = \begin{bmatrix} xy - x^2 & -xy^2 + x^2 y \\ y - x & -y^2 + xy \end{bmatrix} \neq 0_2.$$

Proposition

Let G be a Lie group, and let $H \subset G$ be a Lie subgroup, with $\mathfrak{h} \subset \mathfrak{g}$ considered as a subalgebra in the usual way.

The exponential map of H is the restriction to $\mathfrak{h}$ of the exponential map of G, and

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp(tX) \in H \text{ for all } t \in \mathbb{R}\}.$$

Proof:
We already know that the exponential map of H is the restriction of that of G.

By the way we identified $\mathfrak{h} \subset \mathfrak{g}$, we need the following for every $X \in \mathfrak{g}$:

$$\exp tX \in H \text{ for all } t \in \mathbb{R} \quad \Longleftrightarrow \quad X_e \in T_eH.$$

Indeed, assume first that $\exp tX \in H$ for all t. Since H is embedded in G, it follows that $t \mapsto \exp tX$ is smooth into H, and thus $X_e = \gamma'(0) \in T_eH$.

Conversely, if $X_e \in T_eH$, it is implied that $\exp tX \in H$ for all t. $\square$

The Closed Subgroup Theorem

Motivation:
We already showed that a Lie subgroup is embedded iff it is closed.
In the sequel, we use the exponential map to prove a much stronger form of that theorem.
We will show that if a subgroup of a Lie group is topologically a closed subset, then it is actually an embedded Lie subgroup.

In the following sense is group multiplication in the Lie group G reflected up "to first order" in the vector space structure of its associated Lie algebra $\mathfrak{g}$:

Proposition

Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra.

For any $X, Y \in \mathfrak{g}$, there is a smooth function $Z\colon (-\varepsilon, \varepsilon) \to \mathfrak{g}$ for some $\varepsilon > 0$, such that the following identity holds for all $t \in (-\varepsilon, \varepsilon)$:

$$(\exp tX)(\exp tY) = \exp\big(t(X+Y) + t^2 Z(t)\big). \qquad (*)$$

Proof:
Since the exponential map is a diffeomorphism on some neighborhood of the origin in $\mathfrak{g}$, there is some $\varepsilon > 0$ such that

$$\begin{aligned} \varphi\colon (-\varepsilon, \varepsilon) &\to \mathfrak{g}, \\ \varphi(t) &:= \exp^{-1}\big(\exp tX \exp tY\big) \end{aligned}$$

is smooth.

Clearly, $\varphi(0) = 0$ and

$$\exp tX \exp tY = \exp \varphi(t).$$

Write φ as the composition

$$\mathbb{R} \xrightarrow{e_X \times e_Y} G \times G \xrightarrow{\text{mult}} G \xrightarrow{\exp^{-1}} \mathfrak{g};$$

where $e_X(t) := \exp tX$ and $e_Y(t) := \exp tY$.

Then for the derivative $\mathrm{D}\,\mathrm{mult}_{e,e}(X,Y) = X + Y$ for $X, Y \in T_e G$, which implies

$$\begin{aligned}\varphi'(t) &= \big((\mathrm{D}\exp)_0\big)^{-1}\big(e'_X(0) + e'_Y(0)\big)\\ &= X + Y.\end{aligned}$$

Therefore, Taylor's theorem yields

$$\varphi(t) = t(X+Y) + t^2 Z(t)$$

for some smooth function Z. $\square$

Corollary

Under the hypotheses of the proposition above,

$$\lim_{n\to\infty}\left(\exp\frac{tX}{n}\exp\frac{tY}{n}\right)^n = \exp t(X+Y)$$

Proof:
For any $t \in \mathbb{R}$ and any sufficiently large $n \in \mathbb{Z}$,

$$\exp\frac{tX}{n}\exp\frac{tY}{n} = \exp\left(\frac{t(X+Y)}{n} + \frac{t^2}{n^2}Z(\tfrac{t}{n})\right),$$

consequently,

$$\begin{aligned}\left(\exp\frac{tX}{n}\exp\frac{tY}{n}\right)^n &= \left(\exp\left(\frac{t(X+Y)}{n} + \frac{t^2}{n^2}Z(\tfrac{t}{n})\right)\right)^n \\ &= \exp\left(t(X+Y) + \frac{t^2}{n}Z(\tfrac{t}{n})\right).\end{aligned}$$

Fixing t and taking the limit as $n \to \infty$, we obtain the result. $\square$

Theorem (Closed Subgroup Theorem (proof: slide 36-42 for self study))

Suppose G is a Lie group and $H \subset G$ is a subgroup that is also a closed subset of G.

Then H is an embedded Lie subgroup.

Proof:

It suffices to show that H is an embedded submanifold of G.

We identify a subspace of $\mathfrak{g}$ that will turn out to be the Lie algebra of H.

Define

$$\mathfrak{h} := \{X \in \mathfrak{g} \mid \exp tX \in H, \text{ for all } t \in \mathbb{R}\}.$$

We start by showing that $\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$:

Indeed, $\mathfrak{h}$ is closed under scalar multiplication:

If $X \in \mathfrak{h}$, then $tX \in \mathfrak{h}$ for all $t \in \mathbb{R}$.

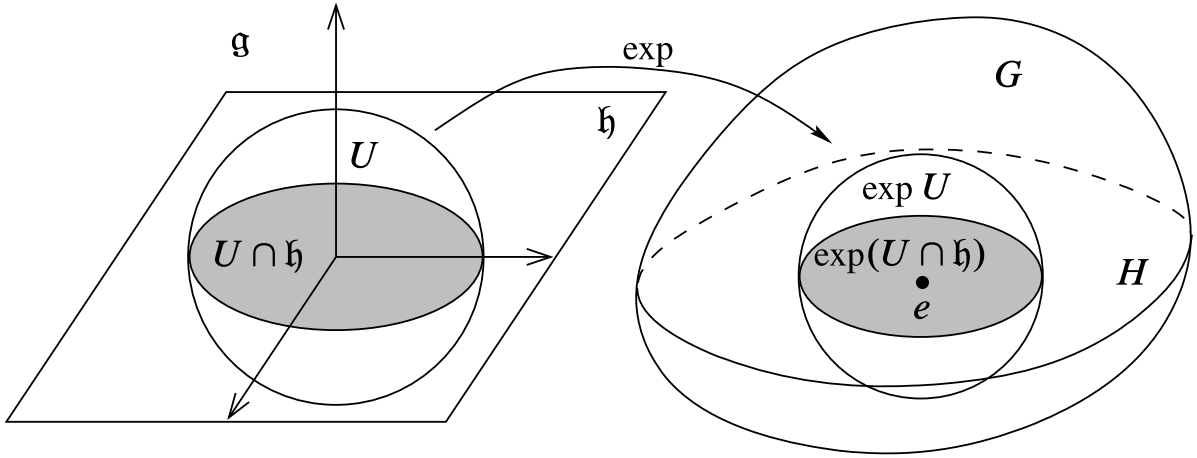
Suppose $X, Y \in \mathfrak{h}$, and let $t \in \mathbb{R}$ be arbitrary.

Then $\exp(tX/n)$ for each $n \in \mathbb{N}$, and because H is a closed subgroup of G, $\exp t(X+Y) \in H$. Thus $X + Y \in \mathfrak{h}$, so $\mathfrak{h}$ is a subspace.

We continue to show that there is a neighborhood U of $0 \in \mathfrak{g}$ on which the exponential map of G is a diffeomorphism, with property that

$$\exp(U \cap \mathfrak{h}) = (\exp U) \cap H. \qquad (*)$$

It will enable us to construct a slice chart for H in a neighborhood of e, and we then use left translation to get a slice chart in a neighborhood of any point of H.



If $U \subset \mathfrak{g}$ is any neighborhood of 0 on which exp is a diffeomorphism, then $\exp(U \cap \mathfrak{h}) \subset (\exp U) \cap H$ by definition of $\mathfrak{h}$.

So to find a neighborhood satisfying $(*)$, we need to show that U can be chosen small enough that $(\exp U) \cap H \subset \exp(U \cap \mathfrak{h})$:

Assume this is not possible.

Choose a subspace $\mathfrak{b} \subset \mathfrak{g}$ complementary to $\mathfrak{h}$.

Then $\Phi\colon \mathfrak{h} \oplus \mathfrak{b} \to G$ with $\Phi(X, Y) := \exp X \exp Y$ is a diffeomorphism in a neighborhood of $(0, 0)$.

Choose neighborhoods U_0 of $0 \in \mathfrak{g}$ and $\widetilde{U}_0$ of $(0, 0) \in \mathfrak{h} \oplus \mathfrak{b}$ s.t. both $\exp|_{U_0}$ and $\Phi|_{\widetilde{U}_0}$ are diffeomorphisms onto their images.

Let $\{U_i\}$ be a countable neighborhood basis for $\mathfrak{g}$ at 0,
e.g., a countable sequence of coordinate balls whose radii approach zero.

If we set $V_i := \exp(U_i)$ and $\widetilde{U}_i := \Phi^{-1}(V_i)$, then $\{V_i\}$ and $\{\widetilde{U}_i\}$ are neighborhood bases for G at e and $\mathfrak{h} \oplus \mathfrak{b}$ at $(0, 0)$, respectively.

By discarding finitely many terms at the beginning of the sequence, we may assume that $U_i \subset U_0$ and $\widetilde{U}_i \subset \widetilde{U}_0$ for each i.

By assumption for each i, there exists $h_i \in (\exp U_i) \cap H$ s.t. $h_i \notin \exp(U_i \cap \mathfrak{h})$.

This means $h_i = \exp Z_i$ for some $Z_i \in U_i$.

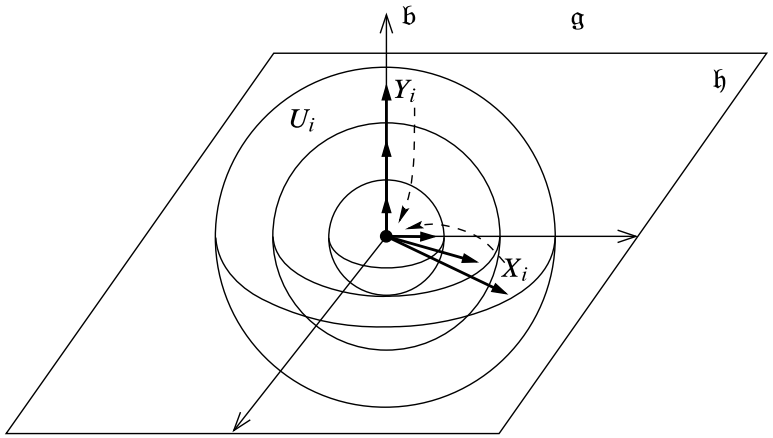
Because $\exp(U_i) = \Phi(\widetilde{U}_i)$ we write $h_i = \exp X_i \exp Y_i$ for some $(X_i, Y_i) \in \widetilde{U}_i$.

If Y_i were zero, we would have $\exp Z_i = \exp X_i \in \exp \mathfrak{h}$.

But because exp is injective on U_0, this implies $X_i = Z_i \in U_i \cap \mathfrak{h}$, which contradicts our assumption that $h_i \notin \exp(U_i \cap \mathfrak{h})$.

Since $\{\widetilde{U}_i\}$ is a neighborhood basis, $Y_i \to 0$ as $i \to \infty$.

Observe that $\exp X_i \in H$ by definition of $\mathfrak{h}$, so it follows that $\exp Y_i = (\exp X_i)^{-1} h_i \in H$ as well.



Choose an inner product on $\mathfrak{b}$ with associated norm $\|\cdot\|$.

Define the sequence $(c_i) := (\|Y_i\|)$, then $c_i \to 0$ as $i \to \infty$.

The sequence $\left(\frac{Y_i}{c_i}\right)$ lies on the unit sphere in $\mathfrak{b}$, so replacing it by a subsequence we may assume that $\frac{Y_i}{c_i} \to Y \in \mathfrak{b}$, with $\|Y\| = 1$ by continuity.

In particular, $Y \neq 0$. We will show that $\exp tY \in H$ for all $t \in \mathbb{R}$, which implies that $Y \in \mathfrak{h}$.

Since $\mathfrak{h} \cap \mathfrak{b} = \{0\}$, this is a contradiction.

Let $t \in \mathbb{R}$ be arbitrary, and for each i, let n_i be the greatest integer less than or equal to $\frac{t}{c_i}$. Then

$$\left|n_i - \tfrac{t}{c_i}\right| \leq 1 \quad \Longrightarrow \quad |n_i c_i - t| \leq c_i \to 0 \quad \Longrightarrow \quad n_i c_i \to t.$$

Consequently, by continuity

$$n_i Y_i = (n_i c_i)\left(\tfrac{Y_i}{c_i}\right) \to tY \quad \Longrightarrow \quad \exp n_i Y_i \to \exp tY.$$

But $\exp n_i Y_i = (\exp Y_i)^{n_i} \in H$, so the fact that H is closed implies $\exp tY \in H$.

This completes the proof of the existence of U satisfying $(*)$.

Choose any linear isomorphism $E\colon \mathfrak{g} \to \mathbb{R}^m$ that sends $\mathfrak{h}$ to $\mathbb{R}^k$.

Now $\varphi := E \circ \exp^{-1}\colon \exp U \to \mathbb{R}^m$ is a smooth chart for G, and $\varphi\big((\exp U) \cap H\big) = E(U \cap \mathfrak{h})$ is the slice obtained by setting the last $m-k$ coordinates equal to zero.

Let $h \in H$ be arbitrary, then the left translation L_h is a diffeomorphism from $\exp U$ to a neighborhood of h.

Since H is a subgroup, $L_h(H) = H$, and so $L_h\big((\exp U) \cap H\big) = L_h(\exp U) \cap H$.

Also $\varphi \circ L_h^{-1}$ is easily seen to be a slice chart for H in a neighborhood of h.

Thus, H is an embedded submanifold of G, hence a Lie subgroup. $\square$

In summary:

Corollary

Let G be a Lie group and $H \subset G$ any subgroup.

Equivalent are:

(a) *H is closed in G.*

(b) *H is an embedded submanifold of G.*

(c) *H is an embedded Lie subgroup of G.*

The Adjoint Representation of a Lie Group

Definition

Fix $g \in G$.

Recall that the map

$$C_g \colon G \to G, \quad C_g(x) := gxg^{-1}$$

is a Lie group automorphism called the conjugation map.

The tangent map

$$\mathrm{Ad}_g := T_e C_g \colon T_e G \cong \mathfrak{g} \to T_{geg^{-1}} G = T_e G \cong \mathfrak{g},$$

is called the adjoint map.

We immediately get

Proposition

The map $C_g \colon G \to G$ is a Lie group homomorphism.

Moreover, the map $C \colon g \mapsto C_g$ is a Lie group homomorphism $G \to \mathrm{Aut}(G)$ as well.

Corollary

The map $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism.

Remark

For linear Lie groups, i.e., matrix Lie groups, we have $\mathrm{Ad}_g X = gXg^{-1}$.

Lemma

Let $f\colon M \times N \to N$ be smooth and define the **partial** *map at $x \in M$ by $f_x(y) := f(x,y)$.*

Suppose, for every $x \in M$ the point y_0 is fixed by f_x, i.e., $f_x(y_0) = y_0$ for all x and, moreover, f_x is locally invertible around y_0.

Then $A_{y_0}\colon x \mapsto T_{y_0} f_x$ is a smooth map $M \to \mathrm{GL}(T_{y_0}N)$.

Idea of Proof:
It suffices to show that A_{y_0} composed with an arbitrary coordinate function from an atlas of charts on $\mathrm{GL}(T_{y_0}N)$ is smooth. $\square$

Proposition

The map $\mathrm{Ad}\colon g \to \mathrm{Ad}_g$ *is a Lie group homomorphism* $G \to \mathrm{GL}(\mathfrak{g})$*, which is called the adjoint representation of the Lie group* G*.*

Proof:
We compute

$$\begin{aligned}\mathrm{Ad}(g_1 g_2) &= \mathrm{Ad}_{g_1 g_2} \\ &= T_e C_{g_1 g_2} \\ &= T_e(C_{g_1} \circ C_{g_2}) \\ &= T_e C_{g_1} \circ T_e C_{g_2} \\ &= \mathrm{Ad}_{g_1} \circ \mathrm{Ad}_{g_2},\end{aligned}$$

which shows that Ad is a group homomorphism.

Smoothness follows immediately. □

Lemma

Let $v \in \mathfrak{g}$ with associated left invariant vectorfield L^v and associated right invariant vector field R^v.

Then $L^v(x) = R^{\mathrm{Ad}_x v}(x)$.

Proof: We compute for an arbitrary $x \in G$ for the LHS

$$L^v(x) = T_e(L_x) \cdot v,$$

and for the RHS

$$\begin{aligned} R^{\mathrm{Ad}_x v}(x) = & \; T_e(R_x) \cdot (\mathrm{Ad}_x v) = T_e(R_x) \cdot T_e(C_x) \cdot v \\ = & \underbrace{T_e(R_x) \cdot T_e(R_{x^{-1}})}_{=\mathrm{id}} \cdot T_e(L_x) \cdot v. \end{aligned}$$

□

Remark

For matrix Lie groups with $x \in G$ and $v \in \mathfrak{g}$, we have

$$\begin{aligned} L^v(x) &= xv \\ &= xvx^{-1}x \\ &= R^{\mathrm{Ad}_x v}(x). \end{aligned}$$

We now go one step further and take the derivative of Ad.

Definition

For a Lie group G with Lie algebra $\mathfrak{g}$, define the adjoint representation of the Lie algebra $\mathfrak{g}$ as the map

$$\mathrm{ad}\colon\ \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad \mathrm{ad} := T_e\,\mathrm{Ad} = \mathrm{D}\,\mathrm{Ad}(e).$$

Proposition

For all $v, w \in \mathfrak{g}$ we have $\mathrm{ad}(v)w = [v, w]$.

Proof:

In the sequel we will often write $\mathrm{ad}_v w$ instead of $\mathrm{ad}(v)w$.

Let $u^1, \ldots, u^n$ be a basis for $\mathfrak{g}$ so that $\mathrm{Ad}(x)w = \sum a_i(x)u^i$ for some functions a_i.

Let c be a curve with $\dot{c}(0) = v$.

Then we have

$$\mathrm{ad}_v w = \tfrac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ad}\,\big(c(t)\big)w|_{t=0} = \sum_i \tfrac{\mathrm{d}}{\mathrm{d}t} a_i\big(c(t)\big)u^i|_{t=0} = \sum_i (va_i)u^i. \quad (**)$$

On the other hand, by Lemma 248 we have

$$\begin{aligned} L^w(x) &= R^{\mathrm{Ad}_x w}(x) \\ &= R^{\sum_i a_i(x) u^i}(x) \\ &= \sum_i a_i(x) R^{u^i}(x). \end{aligned}$$

From the fact that the bracket of any left invariant vector field with any right invariant vector field is zero (implied by the fact that left and right translation in the group commute), we have

$$\begin{aligned} [L^v, L^w] &= \left[L^v, \sum_i a_i R^{u^i}\right] \\ &= L^v \sum_i a_i R^{u^i} - \sum_i a_i R^{u^i} L^v \\ &= \sum_i a_i \underbrace{[L^v, R^{u^i}]}_{=0} + \sum_i L^v(a_i) \cdot R^{u^i} \\ &= \sum_i L^v(a_i) \cdot R^{u^i}. \end{aligned}$$

Finally, we have

$$\begin{aligned}[v,w] &= [L^v, L^w](e) \\ &= \sum_i \underbrace{L^v(a_i)(e)}_{=va_i} \underbrace{R^{u^i}(e)}_{=u^i} \\ &= \sum_i (va_i)u^i \overset{(**)}{=} \operatorname{ad}_v w.\end{aligned}$$

□

The map $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) = \operatorname{End}(T_e G)$ is given as $T_e \operatorname{Ad}$ which is a Lie group homomorphism.

Thus we obtain for the derivative (Lie differential)

Corollary

$\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is a Lie algebra homomorphism.*

Remark

In summary, for any Lie group G we get the commutative diagrams

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{Ad}"'] & \mathrm{GL}(\mathfrak{g}) \end{tikzcd}, \qquad \begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}_g"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "C_g"'] & G. \end{tikzcd}$$

In other words,

$$\exp(t\,\mathrm{Ad}_g\, v) = g\exp(tv)g^{-1}$$

for any $g \in G$, $v \in \mathfrak{g}$ and $t \in \mathbb{R}$.

Remark

For the case of GL_n*, or any linear Lie groups accordingly, the above constructions boil down to the matrix identities:*

- $C_A(B) = ABA^{-1}$*, the* **similarity** *transformation of* B *by* A*,* $A, B \in \mathrm{GL}_n$*.*
- $\mathrm{Ad}_A X = AXA^{-1}$ *for* $A \in \mathrm{GL}_n$ *and* $X \in \mathfrak{gl}_n$*.*
- $\mathrm{ad}_X Y = [X, Y]$*, for* $X, Y \in \mathfrak{gl}_n$*.*
- *For* $X, Y \in \mathfrak{gl}_n$ *we also have*

$$\mathrm{e}^X \, Y \, \mathrm{e}^{-X} = \mathrm{e}^{\mathrm{ad}_X} \, Y := \sum_{i=0}^{\infty} \frac{(\mathrm{ad}_X)^i}{i!} Y,$$

or equivalently,

$$\mathrm{Ad} \circ \exp = \exp \circ \mathrm{ad}\,.$$

Some more results on the matrix exponential:

Theorem

The derivative of $\exp\colon \mathfrak{gl}_n \to \mathrm{GL}_n$ *with* $A \mapsto \exp A = \mathrm{e}^A$ *is*

$$\begin{aligned} \mathrm{D}\,\mathrm{e}^A\colon T_A\,\mathfrak{gl}_n \cong \mathfrak{gl}_n &\to T_{\mathrm{e}^A}\,\mathrm{GL}_n, \\ B &\mapsto \mathrm{e}^A \cdot \Big(\frac{\mathrm{id} - \mathrm{e}^{-\operatorname{ad}_A}}{\operatorname{ad}_A} \circ B \Big). \end{aligned}$$

Here, formally

$$\frac{\mathrm{id} - \mathrm{e}^{-\operatorname{ad}_A}}{\operatorname{ad}_A} := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (\operatorname{ad}_A)^k$$

denotes the convergent power series in the linear map ad_A.

Proof: See e.g. prop. 3.4.2 in [16]. □

Corollary

Consider $A, B \in \mathfrak{gl}_n$ *with* $[A, B] = 0$.

Then

$$\begin{aligned} \mathrm{D}(\mathrm{e}^A)B &= \mathrm{e}^A\, B \\ &= B\, \mathrm{e}^A\,. \end{aligned}$$

The Maurer-Cartan Form

Definition

Let G be a Lie group, and for each $g \in G$, define

$$\omega_G^{\text{left}}(g)\colon T_g G \to \mathfrak{g}, \quad \omega_G^{\text{left}}(g)(X_g) := TL_{g^{-1}} \cdot X_g,$$
$$\omega_G^{\text{right}}(g)\colon T_g G \to \mathfrak{g}, \quad \omega_G^{\text{right}}(g)(X_g) := TR_{g^{-1}} \cdot X_g.$$

The maps $\omega_G^{\text{left}} : g \mapsto \omega_G^{\text{left}}(g)$ and $\omega_G^{\text{right}} : g \mapsto \omega_G^{\text{right}}(g)$ are $\mathfrak{g}$-valued 1-forms.

They are called the left Maurer-Cartan form and right Maurer-Cartan form, respectively.

We can view ω_G^{left} and ω_G^{right} as maps $TG \to \mathfrak{g}$.

Remark

GL_n *is an open set in a vector space, and so its tangent bundle is trivial*

$$T\,\mathrm{GL}_n \cong \mathrm{GL}_n \times \mathbb{R}^{n\times n}.$$

A general abstract Lie group G, however, is not necessarily an open subset of a vector space, but we are still able to show that TG is trivial.

There are two such trivializations obtained from the Maurer-Cartan forms.

$$\begin{aligned} \mathrm{triv}_L\colon TG \to G \times \mathfrak{g}, \quad & \mathrm{triv}_L(v_g) := \big(g, \omega_G^{\mathrm{left}}(v_g)\big), \\ \mathrm{triv}_R\colon TG \to G \times \mathfrak{g}, \quad & \mathrm{triv}_R(v_g) := \big(g, \omega_G^{\mathrm{right}}(v_g)\big), \end{aligned}$$

for $v_g \in T_g G$.

Observe that $\mathrm{triv}_L^{-1}(g,v) = L^v(g)$ and similarly, $\mathrm{triv}_R^{-1}(g,v) = R^v(g)$.

Thus we have the following:

Proposition

The tangent bundle of a Lie group is trivial, i.e.

$$TG \cong G \times \mathfrak{g}\,.$$

(Lie) Group Actions

Definition

Let G be a group and M a set.

A left group action on M is a map $l\colon G \times M \to M$ such that

$$l\big(g_2, l(g_1, x)\big) = l(g_2 g_1, x), \text{ for all } g_1, g_2 \in G, \text{ and all } x \in M,$$
$$l(e, x) = x \text{ for all } x \in M.$$

Remark

Alternatively, we often write $g \cdot x = gx = l(g, x)$.
We have $g_2(g_1 x) = (g_2 g_1)x$ and $ex = x$.
Similarly, we define a right group action as an $r\colon M \times G \to M$ with $r\big(r(x, g_1), g_2\big) = r(x, g_1 g_2)$ for all $g_1, g_2 \in G$ and all $x \in M$ and $r(x, e) = x$ for all $x \in M$.
In the case of right actions, we write $r(x, g)$ as $x \cdot g$ or xg.

If $l\colon G \times M \to M$ is a left action, then for every $g \in G$ we have an associated map

$$l_g\colon M \to M, \quad l_g(x) := l(g, x).$$

Similarly, a right action gives for every g, a map $r_g\colon M \to M$.

- For every result about left actions, there is an analogous result for right actions.
- However, while $g \mapsto l_g$ is a group homomorphism, the map $g \mapsto r_g$ is a group anti-homomorphism, which means that $r_{g_1} \circ r_{g_2} = r_{g_2 g_1}$ for all $g_1, g_2 \in G$ (notice the order reversal).

Definition

Given a left action, $Gx = \{gx \mid g \in G\}$ is called the orbit of x.

Corollary

1. *Two points x, y are in the same orbit iff there is a g s.t. $gx = y$.*
2. *The orbits are equivalence classes and so they partition M.*

Let $G\backslash M$ be the set of orbits and let

$$\pi\colon M \to G\backslash M, \quad x \mapsto Gx$$

be the projection.
We give $G\backslash M$ the quotient topology:
By definition, U is open in $G\backslash M$ iff $\pi^{-1}(U)$ is open.

This makes π continuous, but in this case, it is also an open map:

Let U be open.

Then $\pi^{-1}\big(\pi(U)\big)$ is the union $\cup_{g\in G} gU$ which is open
and so $\pi(U)$ is open by the definition of quotient topology.

Definition

Consider the left action $l\colon G \times M \to M$.

1. The group G acts transitively on M if for any $x, y \in M$ there is a g with $gx = y$.

   Equivalently, the action is transitive if the action has only one orbit.
2. The action is effective provided that $l_g = \mathrm{id}_M$ implies that $g = e$.
3. If the action has the property that $gx = x$ for some $x \in M$ only when $g = e$, then G acts freely (or the action is free).

   In other words, an action is free provided that the only element of G that fixes any element of M is the identity element.

Remark

Similar statements and definitions apply for right actions except that the orbits are xG.

The quotient space will then be denoted by M/G.

Remark

The notations $G\backslash M$ and M/G are not universal in the literature!

Now specialize to Lie groups G acting **smoothly** on smooth manifolds M.

Definition

Let $l\colon G \times M \to M$ be a left action.

If l is smooth, l is a (smooth) Lie group action.

- In the sequel we use any of the notations $gp, g \cdot p, l_g(p)$ for $l(g,p)$.
- Recall that for $p \in M$, the orbit of p is denoted Gp or $G \cdot p$.
- The action is transitive if $Gp = M$.
- The action is effective if $l_g(p) = p$ for all p only if $g = e$.
- For right actions $r\colon M \times G \to M$, similar definitions apply and we write $pg = r_g(p) = r(p,g)$.
- A right action corresponds to a left action by the rule $gp := pg^{-1}$.

Definition

Let l be a Lie group action.

For a fixed $p \in M$, the isotropy group of p is defined to be

$$G_p := \{g \in G \mid gp = p\}.$$

The isotropy group of p is also called the stabilizer of p
(In German: Standgruppe oder Stabilisator von p).

Example

1. The maps $G \times G \to G$ with $(g, x) \mapsto L_g x$ and $(g, x) \mapsto R_g x$ are Lie group actions.
2. $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication. Similarly, $\mathrm{GL}_n(\mathbb{C})$ on $\mathbb{C}^n$.

   More abstractly, $\mathrm{GL}(V)$ acts on the vector space V.

   This action is smooth since Ax depends smoothly (polynomially) on the components of A and on the components of $x \in \mathbb{R}^n$ (respectively, $x \in \mathbb{C}^n$).
3. Any Lie subgroup of GL_n acts on $\mathbb{R}^n$ also by matrix multiplication.

   E.g., O_n acts on $\mathbb{R}^n$:

   For every $x \in \mathbb{R}^n$, the orbit of x is the sphere of radius $\|x\|$.

   This is trivially true if $x = 0$.

   In general, if $\|x\| \neq 0$, then $\|gx\| = \|x\|$ for any $g \in \mathrm{O}_n$.

Example

1. From the last example we can restrict the action of O_n to a transitive action on S^{n-1}:

$$\mathrm{O}_n \times S^{n-1} \to S^{n-1}, \quad (R, x) \mapsto Rx.$$

   Now SO_n also acts on S^{n-1} by restriction.

$$\mathrm{SO}_n \times S^{n-1} \to S^{n-1}, \quad (R, x) \mapsto Rx.$$

   This action on S^{n-1}, however, is only transitive as long as $n > 1$. This is easily seen as $S^0 = \mathrm{O}_1 = \{1, -1\}$ and $\mathrm{SO}_1 = \{1\}$.

2. If $H \subset G$ is a Lie subgroup, then we can consider L_h for any $h \in H$ and thereby obtain a Lie group action of H on G.

Recall that a subgroup $H \subset G$ is called a normal subgroup if $ghg^{-1} \in H$ for any $h \in H$ and all $g \in G$.

In other words, H is normal if $gHg^{-1} \subset H$ for all $g \in G$, and it is easy to see that in this case we always have $gHg^{-1} = H$.

Example

If H is a normal Lie subgroup of G, then G acts on H by conjugation

$$g \cdot h := C_g h = ghg^{-1}.$$

Notice that in this case the notation $g \cdot h$ **cannot** reasonably be abbreviated to gh.

Definition

Suppose that a Lie group G acts on smooth manifolds M and N.

For simplicity, we take both actions to be left actions, which we denote by l and λ, respectively.

A map $f\colon M \to N$ such that $f \circ l_g = \lambda_g \circ f$ for all $g \in G$, is said to be an equivariant map (equivariant with respect to the given actions).

This means that for all group elements g the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "l_g"'] & N \arrow[d, "\lambda_g"] \\ M \arrow[r, "f"'] & N \end{tikzcd} \qquad (*)$$

If f is even a diffeomorphism, then f is an equivalence of Lie group actions.

Example

Consider the torus T^n, where we identify each factor S^1 with the complex numbers of unit modulus.

Fix $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$.
Then $\mathbb{R}$ acts on $\mathbb{R}^n$ by $t \cdot x := x + tk$.

On the other hand, $\mathbb{R}$ acts on T^n by
$t \cdot (z^1, \ldots, z^n) := (\mathrm{e}^{\imath t k_1} z^1, \ldots, \mathrm{e}^{\imath t k_n} z^n)$.

The map $\mathbb{R}^n \to T^n$ given by $(x^1, \ldots, x^n) \mapsto (\mathrm{e}^{\imath x^1}, \ldots, \mathrm{e}^{\imath x^n})$ is equivariant with respect to these actions.

Theorem (Equivariant Rank Theorem)

Let $f : M \to N$ be smooth and let the Lie group G act on M and N.

The action on M is assumed to be transitive.

If f is equivariant, then it has constant rank.

In particular, each level set of f is a closed regular submanifold.

Proof:

Denote the actions on M, N by l, λ, respectively.

Pick any two points $p, q \in M$.

As G acts transitively on M, there is a g with $l_g p = q$.

By hypothesis, we have $f \circ l_g = \lambda_g \circ f$, which corresponds to the commutative diagram $(*)$.

Consequently, we get the ("derived") cummutative diagram:

$$\begin{tikzcd} T_pM \arrow[r, "T_pf"] \arrow[d, "T_pl_g"'] & T_{f(p)}N \arrow[d, "T_{f(p)}\lambda_g"] \\ T_qM \arrow[r, "T_qf"'] & T_{f(q)}N \end{tikzcd} \qquad (**)$$

Since T_pl_g, $T_{f(p)}\lambda_g$ are linear isomorphisms,
T_pf must have the same rank as T_qf.

Since p, q were arbitrary, the rank of f is constant on M. $\square$

There are several corollaries of this nice theorem:

E.g., O_n is the level set $f^{-1}(I)$, where $f\colon\ \mathrm{GL}_n \to \mathfrak{gl}_n$, $A \mapsto AA^\top$.
O_n acts on itself via left translation, and we also let O_n act on $\mathfrak{gl}(n,\mathbb{R})$ via $Q \cdot A := QAQ^\top$ (adjoint action).

f is equivariant with respect to these actions, and since the first action (left translation) is certainly transitive, we see that O_n is a closed regular submanifold of GL_n.

Consequently, O_n is a closed Lie subgroup of GL_n.

Similar arguments apply for $\mathrm{U}_n \subset \mathrm{GL}_n(\mathbb{C})$ and other linear Lie groups.

Remark

One might compare this result to the way we showed that O_n was a differentiable manifold via the regular value theorem.

More generally we have

Corollary

If $\varphi\colon G \to H$ is a Lie group homomorphism, then $\ker\varphi$ is a closed Lie subgroup of G.

Corollary

Let $l\colon G \times M \to M$ be a Lie group action, and let G_p be the isotropy subgroup of some $p \in M$.

Then G_p is a closed Lie subgroup of G.

Proper Actions and Quotients

Definition

Let $l\colon G \times M \to M$ be a smooth (or merely continuous) group action.

If

$$P\colon G \times M \to M \times M, \quad (g,p) \mapsto (l_g p, p)$$

is proper, we say that the action is a proper action.

Remark

A proper action is **not** *defined to be an action such that the defining map $l\colon G \times M \to M$ is proper.*

A useful characterization of a proper action is the following proposition.

Proposition

For any subset $K \subset M$, let $g \cdot K := \{gx \mid x \in K\}$.
Let $l\colon G \times M \to M$ be a smooth (or merely continuous) group action.
Then l is a proper action iff the set

$$G_K := \{g \in G \mid (g \cdot K) \cap K \neq \emptyset\}$$

is compact whenever K is compact.

Proof:
Suppose that l is proper so that the associated P is a proper map.
Let π_G be the first factor projection $G \times M \to G$. Then

$$\begin{aligned} G_K &= \{g \mid \text{ there exists an } x \in K \text{ s.t. } gx \in K\} \\ &= \{g \mid \text{ there exists an } x \in M \text{ s.t. } P(g,x) \in K \times K\} \\ &= \pi_G\big(P^{-1}(K \times K)\big), \end{aligned}$$

and so G_K is compact.

Next we assume that G_K is compact for all compact K.

If $C \subset M \times M$ is compact, then letting $K = \pi_1(C) \cup \pi_2(C)$, where π_1 and π_2 are the first and second factor projections $M \times M \to M$, respectively, we have

$$\begin{aligned} P^{-1}(C) &\subset P^{-1}(K \times K) \\ &\subset \big\{(g,x) \mid gx \in K \text{ and } x \in K\big\} \\ &\subset G_K \times K. \end{aligned}$$

Since $P^{-1}(C)$ is a closed subset of the compact set $G_K \times K$, it is compact.

This already means that P is proper since C was an arbitrary compact subset of $M \times M$. $\square$

Extremely useful is

Proposition

If G is compact, then any smooth action $l\colon G \times M \to M$ is proper.

Proof:
Let $B \subset M \times M$ be compact.
We find a compact $K \subset M$ such that $B \subset K \times K$ (see above Prop.).

We claim $P^{-1}(B)$ is compact. Indeed,

$$\begin{aligned} P^{-1}(B) \subset P^{-1}(K \times K) &= \cup_{k \in K} P^{-1}(K \times \{k\}) \\ &= \cup_{k \in K} \big\{(g,p) \,|\, (gp,p) \in K \times \{k\}\big\} \\ &= \cup_{k \in K} \{(g,k) \,|\, gp \in K\} \subset \cup_{k \in K} (G \times \{k\}) = G \times K. \end{aligned}$$

Thus $P^{-1}(B)$ is a closed subset of the compact set $G \times K$ and hence is compact. $\square$

Quotient Spaces

Let $l\colon G \times M \to M$ a Lie group action.

There is a natural equivalence relation on M with the equivalence classes being the orbits of the action.

The quotient space (or orbit space) is denoted $G\backslash M$, and we have the quotient map $\pi\colon M \to G\backslash M$.

We put the quotient topology on $G\backslash M$ so that $A \subset G\backslash M$ is open iff $\pi^{-1}(A)$ is open in M.

The quotient map is also open:
Indeed, let $U \subset M$ be open. We want to show that $\pi(U)$ is open, and for this it suffices to show that $\pi^{-1}\big(\pi(U)\big)$ is open.
But $\pi^{-1}\big(\pi(U)\big) = \cup_g l_g(U)$, and this is open since each $l_g(U)$ is open.

We state without proofs the following propositions
(see, however, St. Waldmann's lecture notes on our Wuecampus course):

Proposition

Let G act smoothly on M.

Then $G\backslash M$ is a Hausdorff space if the set $\Gamma := \{(gp, p) :| g \in G, p \in M\}$ is a closed subset of $M \times M$.

If M is second countable then $G\backslash M$ is also.

Proposition

If $l\colon G \times M \to M$ is a free and proper action, then $G\backslash M$ is Hausdorff.

Theorem

If $l\colon G \times M \to M$ is a free and proper Lie group action, then there is a unique smooth structure on the quotient $G\backslash M$ such that

(a) *the induced topology is the quotient topology, and $G\backslash M$ is a smooth manifold;*

(b) *the projection $\pi\colon M \to G\backslash M$ is a submersion;*

(c) $\dim(G\backslash M) = \dim M - \dim G$.

Idea of Proof:
This is essentially a consequence of the above propositions together with the choice of an appropriate atlas, s.t. the associated charts are homeomorphisms with respect to the quotient topology.

We omit the technical details. □

Homogeneous Spaces

Proposition (Quotient Lie groups)

If K is a closed normal subgroup of a Lie group G, then G/K is a Lie group and the quotient map $G \to G/K$ is a Lie group homomorphism.

Furthermore, if $f\colon G \to H$ is a surjective Lie group homomorphism, then $\ker(f)$ *is a closed normal subgroup, and the induced map* $\tilde{f}\colon G/\ker(f) \to H$ *is a Lie group isomorphism.*

Sketch of Proof: We know G/K is a smooth manifold and the quotient map is smooth.
The only thing we need to prove for the first part, is that the multiplication and inversion in the quotient are smooth. This is straight forward.

Consider a Lie group homomorphism f as in the hypothesis. Clearly, $\ker(f)$ is a normal subgroup and it is clearly closed. The induced $\tilde{f}$ is an isomorphism and therefore smooth. $\square$

Definition

If a Lie group G acts smoothly and transitively on M (on the right or left), then M is called a homogeneous space with respect to that action.

Remark

- *Of course it is possible that a single group G may act on M in more than one way and so M may be a homogeneous space in more than one way.*
- *We will give a few specific examples shortly, but we already have an abstract example on hand:*

Theorem

If H is a closed Lie subgroup of a Lie group G, then the map $G \times G/H \to G/H$, given by $l\colon (g, g_1H) \to gg_1H$, is a transitive Lie group action.
Thus G/H is a homogeneous space with respect to this action.

Proof:
The fact that l is well-defined follows since if $g_1H = g_2H$, then $g_2^{-1}g_1 \in H$, and so $gg_2H = gg_2g_2^{-1}g_1H = gg_1H$.
We already know that G/H is a smooth manifold and $\pi\colon G \to G/H$ is a surjective submersion.
We can form another submersion $\mathrm{id}_G \times \pi\colon G \times G \to G \times G/H$ making the following diagram commute:

$$\begin{tikzcd}
G \times G \arrow[r, "\mathrm{mult}"] \arrow[d, "\mathrm{id}_G \times \pi"'] \arrow[dr] & G \arrow[d, "\pi"] \\
G \times G/H \arrow[r, "l"'] & G/H
\end{tikzcd}$$

Since the diagonal map is smooth, it follows that l is smooth as well.
We see that l is transitive by observing that if $g_1H, g_2H \in G/H$, then $l_{g_2g_1^{-1}}(g_1H) = g_2H$. $\square$

Remark

- *Up to appropriate equivalence, the examples of the above type account* **for all** *homogeneous spaces.*
- *Before proving this let us look at some specific examples.*

Example (The Euclidean Group)

Let $M := \mathbb{R}^n$ and let $G := \mathrm{E}_n$ be the Euclidean group, i.e., the group of Euclidean motions (socalled rigid body transformations if $n = 3$).
We realize E_n as a matrix group

$$\mathrm{E}_n := \left\{ \begin{bmatrix} Q & v \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_{n+1} \mid v \in \mathbb{R}^n, Q \in \mathrm{O}_n \right\}.$$

The action of E_n on $\mathbb{R}^n$ is given by the rule

$$\begin{bmatrix} Q & v \\ 0 & 1 \end{bmatrix} \cdot x := Qx + v.$$

This action is **not** given by a matrix multiplication (linear map), but one can use the idea of socalled homogeneous coordinates (identifying $\mathbb{R}^n \cong \mathbb{R}^n \times \{1\}$). Rewriting as

$$\mathrm{E}_n \times \mathbb{R}^n \to \mathbb{R}^n, \quad ((Q, v), x) \mapsto \begin{bmatrix} Q & v \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Qx + v \\ 1 \end{bmatrix}.$$

The action is easily seen to be transitive.

Example (The Affine Group)

Let $M := \mathbb{R}^n$ and let $G := \mathrm{A}_n$ be the affine group. We again realize A_n as a matrix group

$$\mathrm{A}_n := \left\{ \begin{bmatrix} Q & v \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_{n+1} \mid v \in \mathbb{R}^n, Q \in \mathrm{GL}_n \right\}.$$

The action of A_n on $\mathbb{R}^n$ is given by the rule

$$\begin{bmatrix} Q & v \\ 0 & 1 \end{bmatrix} \cdot x := Qx + v.$$

Rewriting as

$$\mathrm{A}_n \times \mathbb{R}^n \to \mathbb{R}^n, \quad ((Q, v), x) \mapsto \begin{bmatrix} Q & v \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Qx + v \\ 1 \end{bmatrix}$$

again gives a transitive action.

- Comparing these first two examples, we see that we have made $\mathbb{R}^n$ into a homogeneous space in two different ways.
- It is sometimes desirable to give different names and/or notations for $\mathbb{R}^n$ to distinguish how we are acting on the space.
- In the first example we might write $\mathbb{E}^n$ (Euclidean space), and in the second case we write $\mathbb{A}^n$ and refer to it as affine space.
- Roughly speaking, the action by E_n preserves all metric properties of figures such as curves defined in $\mathbb{E}^n$.
- On the other hand, A_n always sends lines to lines, planes to planes, etc.
- There exist also the notions of the special Euclidean group SE_n (SO_n instead of O_n) and similarly, special affine group SA_n (SL_n instead of GL_n).

Example

Consider as manifold the upper half-plane $M := H := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. The group acting on H will be $\mathrm{SL}(2, \mathbb{R})$, and the action is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az + b}{cz + d}.$$

This action is transitive (cf. complex analysis course).

Example

We already know, O_n and SO_n act for $n > 1$ transitively on $S^{n-1} \subset \mathbb{R}^n$. So S^{n-1} is a homogeneous space in at least two (slightly) different ways.

Also, both SU_n and U_n act transitively on $S^{2n-1} \subset \mathbb{C}^n$.

Example (Noncompact Real Stiefel Manifold)

Let $\mathrm{ST}_{n,k}$ denote the set of all k-frames for $\mathbb{R}^n$.
This means that an n-frame is just an ordered basis for $\mathbb{R}^n$.
This set can easily be given a smooth manifold structure.
This manifold is called the (real) noncompact Stiefel manifold of k-frames.
We have the smooth Lie group action

$$\mathrm{GL}_n \times \mathrm{ST}_{n,k} \to \mathrm{ST}_{n,k}, \quad g \cdot (e_1, \dots, e_k) := (ge_1, \dots, ge_k).$$

To see that this action is transitive, let $(e_1, \dots, e_k)$ and $(f_1, \dots, f_k)$ be two k-frames.
Extend each to n-frames $(e_1, \dots, e_k, \dots, e_n)$ and $(f_1, \dots, f_k, \dots, f_n)$.
If we consider elements of $\mathbb{R}^n$ as column vectors, these two n-frames can be viewed as invertible $(n \times n)$-matrices E and F.
If we let $g := FE^{-1}$, then $gE = F$, or

$$g \cdot (e_1, \dots, e_k) = (ge_1, \dots, ge_k) = (f_1, \dots, f_k).$$

Example (Compact Real Stiefel Manifold)

Let $\mathrm{St}_{n,k}$ denote the set of all orthonormal k-frames for $\mathbb{R}^n$, $k \in \mathbb{N}$ fixed but $k \leq n$.

This manifold is called the (real) compact Stiefel manifold of k-frames.

The Lie group O_n acts transitively on $\mathrm{St}_{n,k}$ for reasons similar to those given in the last example.

Theorem

Let M be a homogeneous space via the transitive action $l\colon G \times M \to M$, and let G_p be the isotropy subgroup of $p \in M$.
Recall that G acts on G/G_p.
If G/G_p is second countable (in particular if G is second countable), then there is an equivariant diffeomorphism $\varphi\colon G/G_p \to M$ such that $\varphi(gG_p) = g \cdot p$.

Proof: We define φ by the rule $\varphi(gG_p) = g \cdot p$, but we must show that this is well-defined.
This is a standard group theory argument:

If $g_1 G_p = g_2 G_p$, then $g_1^{-1} g_2 \in G_p$, so that $(g_1^{-1} g_2) \cdot p = p$ or $g_1 \cdot p = g_2 \cdot p$.

This map is surjective by the transitivity of l.
It is also injective since if $\varphi(g_1 G_p) = \varphi(g_2 G_p)$, then $g_1 \cdot p = g_2 \cdot p$ or $(g_1^{-1} g_2) \cdot p = p$, which by definition means that $g_1^{-1} g_2 \in G_p$ and $g_1 G_p = g_2 G_p$.

Notice that the following diagram commutes:

$$\begin{tikzcd} G \arrow[d] \arrow[dr, "\theta_p"] & \\ G/G_p \arrow[r, "\varphi"'] & M \end{tikzcd},$$

with orbit map $\theta_p \colon G \to M$, $g \mapsto gp$. As the smooth θ_p is constant on the fibers of the vertical map (surjective submersion) it follows that φ smooth.

To show that φ is a diffeomorphism, it suffices to show $\operatorname{rk} \varphi = \dim M$ or in other words that φ is a submersion.
Since $\varphi(gg_1 G_p) = (gg_1) \cdot p = g\varphi(g_1 G_p)$, the map φ is equivariant and so has constant rank. Consequently, φ is a submersion and hence in the present case even a diffeomorphism. $\square$

Remark

Without the technical assumption on second countability, the proof shows that we still have that $\varphi \colon G/G_p \to M$ is a smooth equivariant bijection.

Example

Consider again the Euclidean group. The isotropy group of the origin in $\mathbb{R}^n$ is the subgroup consisting of matrices of the form

$$\begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix}, \quad Q \in \mathrm{O}_n,$$

because

$$\begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} \circ 0 = Q \cdot 0 + 0 = 0.$$

This group is clearly isomorphic to O_n and so by the above theorem we have an equivariant diffeomorphism

$$\mathbb{R}^n \cong \mathrm{E}_n \,/\, \mathrm{O}_n \,.$$

Similarly, for the affine group

$$\mathbb{R}^n \cong \mathrm{A}_n \,/\, \mathrm{GL}_n \,.$$

Example

Consider again the action of $\mathrm{SL}(2,\mathbb{R})$ on the complex upper half-plane $H := \mathbb{C}_+$.
We determine the isotropy subgroup for the point $\imath = \sqrt{-1}$. A matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is in this subgroup iff

$$\frac{a\imath + b}{c\imath + d} = \imath.$$

This is true exactly if $a = d$ and $b = -c$ (together with $ad - bc = 1$), and so the isotropy subgroup is $\mathrm{SO}(2,\mathbb{R})$ $\big(\cong S^1 = \mathrm{U}(1,\mathbb{C})\big)$.
Thus we have an equivariant diffeomorphism

$$H = \mathbb{C}_+ \cong \mathrm{SL}_2 \,/\, \mathrm{SO}_2\,.$$

Example

Moreover, we derive for $n > 1$

$$S^{n-1} \cong \mathrm{O}_n \,/\, \mathrm{O}_{n-1} \cong \mathrm{SO}_n \,/\, \mathrm{SO}_{n-1},$$
$$S^{2n-1} \cong \mathrm{U}_n \,/\, \mathrm{U}_{n-1} \cong \mathrm{SU}_n \,/\, \mathrm{SU}_{n-1}\,.$$

Example ($\mathrm{ST}_{n,k}$ and $\mathrm{St}_{n,k}$ revisited)

Let $(e_1, \ldots, e_n)$ be the standard basis for $\mathbb{R}^n$. Under the GL_n-action on $\mathrm{ST}_{n,k}$ $(1 \leq k < n)$, the isotropy group of the k-frame $(e_1, \ldots, e_k)$ is the subgroup of GL_n of the form:

$$\begin{bmatrix} I_k & 0 \\ 0 & A \end{bmatrix}, \quad A \in \mathrm{GL}_{n-k},$$

because

$$\begin{bmatrix} I_k & 0 \\ 0 & A \end{bmatrix} \cdot \underbrace{[e_1 | \cdots | e_k]}_{=\left[\begin{smallmatrix} I_k \\ 0 \end{smallmatrix}\right]} = \begin{bmatrix} I_k \\ 0 \end{bmatrix}.$$

We identify this group with GL_{n-k} and then we obtain

$$\mathrm{ST}_{n,k} \cong \mathrm{GL}_n \,/\, \mathrm{GL}_{n-k}\,.$$

Similarly, for $1 \leq k < n$

$$\mathrm{St}_{n,k} \cong \mathrm{O}_n \,/\, \mathrm{O}_{n-k}\,.$$

In particular, $\mathrm{St}_{n,1} \cong S^{n-1}$.
However, $\mathrm{St}_{n,n} \cong \mathrm{O}_n$, as O_n acts transitively on itself from the left.

Remark

The situation for the SO_n *action on* $\mathrm{St}_{n,k}$ *is for arbitrary* k *slightly more subtle:*

$$\mathrm{St}_{n,k} \cong \mathrm{SO}_n \,/\, \mathrm{SO}_{n-k} \quad \textit{for} \quad 1 \leq k < n.$$

In particular, $\mathrm{St}_{n,n-1} \cong \mathrm{SO}_n$*, as an orthonormal* $n-1$*-frame in* $\mathbb{R}^n$ *can be uniquely extended to a* **positively** *oriented* n*-frame, (provided we have chosen the positive orientation as the one with determinant equal to* $+1$*).*

However, there cannot exist an SO_n*-based group model (quotient model) for* $\mathrm{St}_{n,n} \cong \mathrm{O}_n$*, as clearly,* SO_n *does not act transitively on* O_n *from the left.*

Proposition

Let G be a Lie group acting smoothly on M.
Let the action be a left (resp. right) action. If both G and $M\backslash G$ (resp. M/G) are connected, then M is connected.

Proof: Assume for convenience that the action is a left action and that G and $M\backslash G$ are connected.
Suppose by way of contradiction that M is not connected.
Then there are disjoint open sets U, V with $U \cup V = M$.
Each orbit $G \cdot p$ is the image of the connected space G under the orbit map $g \mapsto g \cdot p$ and so is connected.
This means that each orbit must be contained in one and only one of U and V.
Since the quotient map π is an open map, $\pi(U)$ and $\pi(V)$ are open, and from what we have just observed they must be disjoint and $\pi(U) \cup \pi(V) = M\backslash G$.
This contradicts the assumption that $M\backslash G$ is connected. $\square$

An immediate consequence is

Corollary

Let H be a closed Lie subgroup of G.

If both H and G/H are connected, then G is connected.

Corollary

For each $n \geq 1$, the groups $\mathrm{SO}_n, \mathrm{SU}_n$ and U_n are connected.

However, the group O_n has exactly two components:
SO_n (the rotations) and the subset of O_n consisting of elements with determinant equal to -1 (the reflections).

Proof:
SO_1 and SU_1 are both connected since they each contain only one element, namely 1.

$\mathrm{U}_1 \cong S^1$, and so it too is connected.

To proceed we use induction:

Suppose that SO_k and SU_k are connected for $1 \leq k \leq n-1$.
We show that this implies that $\mathrm{SO}_n, \mathrm{SU}_n$ and U_n are connected:

We already know that $S^{n-1} = \mathrm{SO}_n / \mathrm{SO}_{n-1}$.

Since S^{n-1} and SO_{n-1} are connected (the second by the induction hypothesis), we see that SO_n is connected.

The same argument works for SU_n and U_n.

Every element of O_n has determinant either 1 or -1.

The subset $\mathrm{SO}_n \subset \mathrm{O}_n$ is closed since it is exactly $\{g \in \mathrm{O}_n \mid \det g = 1\}$.

Fix an element a_0 with $\det a_0 = -1$.

Now $a_0 \,\mathrm{SO}_n$ is exactly the set of elements of O_n with determinant -1 so that $\mathrm{SO}_n \cup a_0 \,\mathrm{SO}_n = \mathrm{O}_n$ and $\mathrm{SO}_n \cap a_0 \,\mathrm{SO}_n = \emptyset$.

Indeed, by the multiplicative property of determinants, each element of $a_0 \,\mathrm{SO}_n$ has determinant -1.

But $a_0 \,\mathrm{SO}_n$ also contains **every** element of determinant -1 since for any such g we have $g = a_0(a_0^{-1} g)$ and $a_0^{-1} g \in \mathrm{SO}_n$.

Since SO_n and $a_0\,\mathrm{SO}_n$ are complements, they are both open.

Both sets are connected, since $g \mapsto a_0 g$ is a diffeomorphism which maps the first to the second.

Hence, SO_n and $a_0\,\mathrm{SO}_n$ are the connected components of O_n. $\square$

Remark

An alternative argument to show that O_n is not connected is as follows:

The function

$$\det\colon\ \mathrm{O}_n \to \mathbb{R}, \quad X \mapsto \det X$$

takes values exactly in the discrete and disconnected set $\{+1, -1\}$.

The determinantal function, however, is continuous.

It follows that the preimage $\det^{-1}(\{+1, -1\})$ cannot be connected.

Example

The Lorentz group

$$\mathrm{O}_{1,3} := \left\{ \Lambda \in \mathrm{GL}_4 \mid \Lambda^\top I_{1,3} \Lambda = I_{1,3} \right\}$$

with

$$I_{1,3} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

has exactly four components.
For a proof see e.g. prop. I.2 in A. Jaffe's notes (on wuecampus for download).
For a more elegant but less down-to-earth proof using the socalled polar decomposition and the singular value decomposition (SVD) of a matrix $\Lambda \in \mathrm{O}_{1,3}$, see section 6.1 in [12].

- We now relate the notion of a Lie group action with that of a Lie group representation.
- We give just a few basic definitions, some of which will be used later.

Definition

A linear action of a Lie group G on a finite-dimensional vector space V is a left Lie group action $\lambda\colon G \times V \to V$ such that for each $g \in G$ the map $\lambda_g\colon v \mapsto \lambda(g, v)$ is linear.

Remark

The map $G \to \mathrm{GL}(V)$, $g \mapsto \lambda(g) := \lambda_g$ is a Lie group homomorphism and will be denoted by the same letter λ as the action so that $\lambda(g)v := \lambda(g, v)$.

Recall: A Lie group homomorphism $\lambda\colon G \to \mathrm{GL}(V)$ is called a representation of G.

- Given such a representation, we obtain a linear action by letting $\lambda(g, v) := \lambda(g)v$.
- Thus a linear action of a Lie group is basically the same as a Lie group representation.
- The kernel of the action is the kernel of the associated homomorphism (representation).

Definition

An effective linear action is one s.t. the associated homomorphism has trivial kernel, which is the same as saying that the representation is faithful.

Definition

Two representations $\lambda\colon G \to \mathrm{GL}(V)$ and $\lambda'\colon G \to \mathrm{GL}(V')$ are equivalent if there exists a linear isomorphism $T\colon V \to V'$ such that $T \circ \lambda_g = \lambda'_g \circ T$ for all g.

- We have already seen one important example of a Lie group representation, namely, the adjoint representation.
- The adjoint representation came from first considering the action of G on itself given by conjugation which leaves the identity element fixed.
- The idea can be generalized:

Theorem

Let $l\colon G \times M \to M$ be a (left) Lie group action.

Suppose $p_0 \in M$ is a fixed point of the action, i.e., $l_g(p_0) = p_0$ for all g.

The map

$$l^{(p_0)}\colon G \to \mathrm{GL}(T_{p_0}M),$$
$$l^{(p_0)}(g) := T_{p_0} l_g$$

is a Lie group representation.

Proof: Since

$$\begin{aligned} I^{(p_0)}(g_1 g_2) &= T_{p_0}(l_{g_1 g_2}) \\ &= T_{p_0}(l_{g_1} \circ l_{g_2}) \\ &= T_{p_0} l_{g_1} \circ T_{p_0} l_{g_2} \\ &= I^{(p_0)}(g_1) I^{(p_0)}(g_2), \end{aligned}$$

we see that $I^{(p_0)}$ is a homomorphism.

We must show that $I^{(p_0)}$ is smooth:
It will be enough to show that $g \mapsto \alpha(T_{p_0} l_g \cdot v)$ is smooth for any $v \in T_{p_0} M$ and any $\alpha \in T^*_{p_0} M$.
This will follow if we can show that for fixed $v_0 \in T_{p_0} M$, the map $G \to TM$ given by $g \mapsto T_{p_0} l_g \cdot v_0$ is smooth. However, this map is a composition

$$G \to TG \times TM \cong T(G \times M) \xrightarrow{Tl} TM,$$

where the first map is $g \mapsto (0_g, v_0)$, being smooth. But this implies that the map $G \times T_{p_0} M \to T_{p_0} M$ given by $(g, v) \mapsto T_{p_0} l_g \cdot v$ is smooth.

Definition

For a Lie group action $l\colon G \times M \to M$ with fixed point p_0, the representation $l^{(p_0)}$ from the last theorem is called the isotropy representation for the fixed point.

More terminology and very basic concepts from representation theory:

Definition

Given a representation λ of G in a vector space V, we have a dual representation λ^* of G in the dual space V^* by defining

$$\lambda^*(g) := \left(\lambda(g^{-1})\right)^*\colon V^* \to V^*.$$

Recall that if $L\colon V \to V$ is linear, then $L^*\colon V^* \to V^*$ is defined by $L^*(\alpha)(v) := \alpha(Lv)$ for $\alpha \in V^*$ and $v \in V$. This dual representation is also sometimes called the contragredient representation (especially when V is an $\mathbb{R}$-vector space).

Definition

Now let λ^V and λ^W be representations of a Lie group G in $\mathbb{F}$-vector spaces V and W, respectively.
We can then form the direct sum representation $\lambda^V \oplus \lambda^W$ by

$$(\lambda^V \oplus \lambda^W)_g := \lambda^V_g \oplus \lambda^W_g, \text{ for } g \in G,$$
$$(v, w) \mapsto (\lambda^V_g v, \lambda^W_g w).$$

Definition

A representation $\lambda\colon G \to \mathrm{GL}(V)$ is said to be irreducible if there is no nonzero proper subspace $W \subset V$ s.t. $\lambda_g(W) \subset W$ for all g.

Remark

A large class of Lie groups known as semisimple Lie groups have the property that their representations break into direct sums of irreducible representations.

Motivation:

- In view of (pseudo-)Riemannian and symplectic or even Poisson geometry we introduce the notion of vector bundles.
- We recall certain facts from multilinear algebra.
- Moreover, we generalize the concept of tensors on vector spaces to tensor fields on smooth manifolds.

Vector Bundles

Recall, the tangent bundle TM of M is the disjoint union of the tangent spaces at all points of M.
TM can be thought of both as a union of vector spaces and as a manifold in its own right.
This kind of structure, called a vector bundle, is extremely common in differential geometry. We discuss the definitions and basic properties of vector bundles.
In the sequel we will restrict our considerations to the smooth category.

Definition (Vector Bundle)

For any $k \in \mathbb{N}$, a smooth vector bundle of rank k is a pair of smooth manifolds E, M together with a smooth surjective map $\pi\colon E \to M$ satisfying the following conditions:

(i) For each $p \in M$, the set $E_p := \pi^{-1}(p)$ is endowed with the structure of a k-dimensional real vector space.

(ii) For each $p \in M$, there exist a neighborhood U of p and a diffeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ s.t. $\pi_U \circ \Phi := \pi$ where $\pi_U\colon U \times \mathbb{R}^k \to U$ is the projection onto the first factor.
For each $q \in U$, the diffeomorphism Φ restricts to a linear isomorphism $E_q \to \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

M is the base of the bundle, E its total space, and π its projection. Each set E_p is the fiber of E over p, and each diffeomorphism Φ described above is called a smooth local trivialization.

A tangent bundle TM admits a smooth structure via an appropriate chosen atlas induced from M. We similarly can check:

Lemma (Vector Bundle Chart Lemma)

Let M be a smooth manifold and suppose that for each $p \in M$ we are given a real vector space E_p of some fixed dimension k.
Let $E := \cup_{p \in M} E_p$ (the disjoint union of all the spaces E_p), and let

$$\pi \colon E \to M, \quad E_p \mapsto p.$$

Suppose furthermore that we are given

(i) *an indexed open cover $\{U_\alpha\}_{\alpha \in A}$ of M;*

(ii) *for each $\alpha \in A$, a bijective map $\Phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a linear isomorphism $E_p \to \{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$;*

(iii) *for each $\alpha, \beta \in A$ s.t. $U_\alpha \cap U_\beta \neq \emptyset$ a smooth map $\tau_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ s.t. the composite map $\Phi_\alpha \circ \Phi_\beta^{-1} \colon (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ has the form*

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = \big(p, \tau_{\alpha\beta}(p) v\big).$$

Then E has a unique smooth manifold structure making it a smooth vector bundle of rank k over M, with π as projection and the maps Φ_α as smooth local trivializations.

Definition

The smooth $\mathrm{GL}_k(\mathbb{R})$-valued maps $\tau_{\alpha\beta}$ are called transition functions for E.

If $\pi\colon E \to M$ is a smooth vector bundle over M, then a section of E is a smooth map $\sigma\colon M \to E$, $\pi \circ \sigma = \mathrm{id}_M$, or equivalently, $\sigma(p) \in E_p$ for all p.

If $U \subset M$ is open, then a local section of E over U is a smooth map $\sigma\colon U \to E$ with $\pi \circ \sigma = \mathrm{id}_U$.

A local frame for E is an ordered k-tuple $(\sigma_1, \ldots, \sigma_k)$ of local sections over an open set U whose values at each $p \in U$ constitute a basis for E_p.

It is called a global frame if $U = M$.

If τ is a (local or global) section of E and $(\sigma_1, \ldots, \sigma_k)$ is a local frame for E over $U \subset M$, then the value of τ at each $p \in U$ can be written

$$\tau(p) = \sum_i \tau^i(p)\sigma_i(p)$$

for $\tau^1, \ldots, \tau^k\colon U \to \mathbb{R}$, called component functions of σ w.r.t. the frame.

Lemma (Local Frame Criterion for Smoothness)

Let $\pi\colon E \to M$ be a smooth vector bundle, and let $\tau\colon M \to E$ be a section. If (σ_i) is a smooth local frame for E over an open subset $U \subset M$, then τ is smooth on U iff its component functions with respect to (σ_i) are smooth.

Definition

If $E \to M$ is a smooth vector bundle, then the set of smooth sections $\Gamma(E)$ of E is a vector space (usually infinite-dimensional) under pointwise addition and multiplication by constants.

Its zero element is the zero section ζ defined by $\zeta(p) = 0$ for all $p \in M$.

For every section σ of E, the support of σ is the closure of the set $\{p \in M \mid \sigma(p) \neq 0\}$.

Review of Tensors

Motivation:

- Of all the constructions in smooth manifold theory, the ones that play the most fundamental roles in (pseudo-)Riemannian , symplectic and Poisson geometry are tensors and tensor fields.
- Most of the technical machinery of these types of geometry is built up using tensors; indeed, (pseudo-)Riemannian metrics, symplectic and Poisson forms themselves are tensor fields.

Remark (Einstein Summation Convention)

From now on, we write coordinates with upper indices, as in x^i, and expressions with indices are interpreted using the Einstein summation convention:
If in some monomial term the same index appears exactly twice, once as upper and once as lower index, then that term is understood to be summed over all possible values of that index, i.e., usually from 1 to the dimension of the space.
E.g., the expression $a^i v_i$ is to be read as $\sum_i a^i v_i$.

Recall:

Definition (Covectors)

The dual space V^* of an $\mathbb{R}$-vector space V is the space of linear maps from $V \to \mathbb{R}$.
The elements are called covectors or linear functionals on V.
Under the operations of pointwise addition and multiplication by constants, V^* is a vector space.

Definition (Dual Basis)

Suppose $(b_1, \ldots, b_n)$ is an ordered basis for the finite-dimensional $\mathbb{R}$-vector space V. For each $i = 1, \ldots, n$, define a covector $\beta^i \in V^*$ by

$$\beta^i(b_j) = \delta^i_j,$$

where δ^i_j is the Kronecker delta symbol, defined by

$$\delta^i_j := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then $(\beta^1, \ldots, \beta^n)$ is a basis for V^*, called the dual basis to (b_i).

Corollary

Clearly, V^* is finite-dimensional as well, and its dimension is the same as that of V.
Every covector $\omega \in V^*$ can thus be written in terms of the dual basis as

$$\omega = \omega_j \beta^j,$$

where the components ω_j are defined by $\omega_j := \omega(b_j)$.
The action of ω on an arbitrary vector $v = v^i b_i \in V$ is then given by

$$\omega(v) = \omega_j v^j.$$

Remark

- *Every $v \in V$ uniquely determines a linear functional on V^*.*
- *Because V is finite-dimensional, there is a canonical (basis-independent) isomorphism between V and V^{**}.*
- *Given $\omega \in V^*$ and $v \in V$ we can denote the natural action of ω on v either by $\omega(v)$, or by either of the more symmetric notations $\langle \omega, v \rangle$ or $\langle v, \omega \rangle$.*
- *By abuse of notation, this socalled duality pairing makes sense without any choice of an inner product on V.*

We now generalize from linear functionals to multilinear ones:

Definition

If $V_1, \ldots, V_k, W$ are vector spaces, a map $F\colon V_1 \times \cdots \times V_k \to W$ is said to be multilinear if it is linear as a function of each variable separately, when all the others are held fixed, i.e.,

$$F(v_1, \ldots, av_i + a'v_i', \ldots, v_k) = aF(v_1, \ldots, v_i, \ldots, v_k) + a'F(v_1, \ldots, v_i', \ldots, v_k).$$

Given a finite-dimensional vector space V, a covariant k-tensor on V is a multilinear map

$$F\colon \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}.$$

Similarly, a contravariant k-tensor on V is a multilinear map

$$F\colon \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \to \mathbb{R}.$$

A mixed tensor of type (k, l), also called a k-contravariant, l-covariant tensor, is a multilinear map

$$F\colon \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}} \to \mathbb{R}.$$

Remark

- *Often it is necessary to consider real-valued multilinear functions whose arguments consist of k covectors and l vectors, but not necessarily in the order implied by the definition above.*
- *Such an object is still called a tensor of type (k, l).*
- *For any given tensor, we will make it clear which arguments are vectors and which are covectors.*

Definition

The spaces of tensors on V of various types are denoted by

$$\begin{aligned} T^k(V^*) &:= \{\text{covariant } k\text{-tensors on } V\}, \\ T^k(V) &:= \{\text{contravariant } k\text{-tensors on } V\}, \\ T^{(k,l)}(V) &:= \{\text{mixed } (k,l)\text{-tensors on } V\}. \end{aligned}$$

Definition (Rank of Tensor)

The rank of a tensor is the number of arguments (vectors and/or covectors) it takes.
By convention, a 0-tensor is just a real number.

Remark

However notice:
The notation conventions for describing the spaces of covariant, contravariant, and mixed tensors are not universally agreed upon! Historically, it changed at least once, also it might nowadays still differ between communities, say physics, engineering, mathematics....

Definition (Tensor Product)

If $F \in T^{(k,l)}(V)$ and $G \in T^{(p,q)}(V)$, the tensor product

$$F \otimes G \in T^{(k+p,l+q)}(V)$$

is defined by

$$F \otimes G\ (\omega^1, \dots, \omega^{k+p}, v_1, \dots, v_{l+q}) :=$$
$$F(\omega^1, \dots, \omega^k, v_1, \dots, v_l) G(\omega^{k+1}, \dots, \omega^{k+p}, v_{l+1}, \dots, v_{l+q}).$$

The tensor product is associative, so we can unambiguously form tensor products of three or more tensors on V.
If (b_i) is a basis for V and (β^j) is the associated dual basis, then a basis for $T^{(k,l)}(V)$ is given by the set of all tensors of the form

$$b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l},$$

as the indices i_p, j_q range from 1 to n.

These tensors act on basis elements by

$$b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l}(\beta^{s_1}, \ldots, \beta^{s_k}, b_{r_1}, \ldots, b_{r_l}) = \delta^{s_1}_{i_1} \cdots \delta^{s_k}_{i_k} \delta^{j_1}_{r_1} \cdots \delta^{j_l}_{r_l}.$$

Consequently, $T^{(k,l)}(V)$ has dimension n^{k+l}, where $n = \dim V$.
Every tensor $F \in T^{(k,l)}(V)$ can be written in terms of this basis as

$$F = F^{i_1 \ldots i_k}_{j_1 \ldots j_l} b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l},$$

where

$$F^{i_1 \ldots i_k}_{j_1 \ldots j_l} := F(\beta^{i_1}, \ldots, \beta^{i_k}, b_{j_1}, \ldots, b_{j_l}).$$

However, if arguments of mixed tensor F occur in nonstandard order, then horizontal as well as vertical positions of the indices are significant and reflect which arguments are vectors and which are covectors.

Example

Let A be a $(1,2)$-tensor whose first argument is a vector, second is a covector, and third is again a vector, its basis expression would be written

$$A = A_i{}^j{}_k \beta^i \otimes b_j \otimes \beta^k, \text{ where } A_i{}^j{}_k := A(b_i, \beta^j, b_k).$$

Corollary

There are obvious identifications among some of these tensor spaces:

$$\begin{aligned}
T^{(0,0)}(V) &= T^0(V) = T^0(V^*) = \mathbb{R}, \\
T^{(1,0)}(V) &= T^1(V) = V, \\
T^{(0,1)}(V) &= T^1(V^*) = V^*, \\
T^{(k,0)}(V) &= T^k(V), \\
T^{(0,k)}(V) &= T^k(V^*).
\end{aligned}$$

A less obvious, but extremely important, identification is the following:

$$T^{(1,1)}(V) \cong \mathrm{End}(V).$$

It is a special case of the following proposition.

Proposition

Let V be a finite-dimensional vector space. There is a natural (basis-independent) isomorphism between $T^{(k+1,l)}(V)$ and the space of multilinear maps

$$\underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}} \to V.$$

Idea of Proof: In the special case $k = 0, l = 1$, consider the map $\Phi\colon \mathrm{End}(V) \to T^{(1,1)}(V)$ defined by letting ΦA be the $(1,1)$-tensor defined by $\Phi A(\omega, v) := \omega(Av)$. The general case is similar. $\square$

Definition (Tensor Contraction)

By the trace or contraction operation, the rank of a tensor is lowered by 2: We define

$$\operatorname{tr}\colon T^{(k+1,l+1)}(V) \to T^{(k,l)}(V)$$

by letting $(\operatorname{tr} F)\big(\omega^1, \dots, \omega^k, v_1, \dots, v_l\big)$ be the trace of the $(1,1)$-tensor

$$F\big(\omega^1, \dots, \omega^k, (\cdot), v_1, \dots, v_l, (\cdot)\big) \in T^{(1,1)}(V).$$

Example

$\operatorname{tr}\colon T^{(1,1)}(V) \to \mathbb{R}$ is just the (ordinary) trace of F when regarded as an endomorphism of V.

In other words the sum of the diagonal entries of any matrix representation of F.

Since the trace of a linear endomorphism is basis-independent, this is well defined.

Remark

- *In terms of a basis, the components of the tensor* $\operatorname{tr} F$ *are*

$$(\operatorname{tr} F)^{i_1 \dots i_k}_{j_1 \dots j_l} = F^{i_1 \dots i_k m}_{j_1 \dots j_l m}.$$

- *In other words, just set the last upper and lower indices equal and sum.*
- *Even more generally, we can contract a given tensor on any pair of indices as long as one is contravariant and one is covariant.*
- *There is no general notation for this operation, so we just describe it in words each time it arises!*

Symmetric and Skew Symmetric (or Alternating) Tensors

- There are two classes of tensors that play particularly important roles in differential geometry:
- The symmetric and alternating tensors.
- Here we discuss the symmetric ones.
- Alternating tensors are discussed thoroughly in Geometric Mechanics (i.e., Symplectic and Poisson Geometry).

Definition

If V is a finite-dimensional vector space, a covariant tensor $F \in T^k(V^*)$ is said to be symmetric if its value is unchanged by interchanging any pair of arguments:

$$F(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = F(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

whenever $1 \leq i < j \leq k$.

- Clearly, the value is unchanged under **every** rearrangement of the arguments.
- If we denote the components of F with respect to a basis by $F_{i_1 \ldots i_k}$, then F is symmetric iff its components are unchanged by every permutation of the indices $i_1, \ldots, i_k$.
- 0-tensors and 1-tensors are symmetric trivially.
- The set of symmetric k-tensors on V is a linear subspace of $T^k(V^*)$, which we denote by $\Sigma^k(V^*)$.

- Certainly, we can build tensor products also from symmetric tensors.
- This does not necessarily lead us to a new *symmetric* tensor. However, we have:

Definition

There is a product of symmetric tensors that does yield symmetric tensors. Given a covariant k-tensor F, the symmetrization of F is the k-tensor $\operatorname{Sym} F$ defined by

$$(\operatorname{Sym} F)(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} F(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

with the symmetric group on k letters, S_k, i.e., the group of all permutations of $\{1, \ldots, k\}$.

- It is easy to check that $\operatorname{Sym} F$ is symmetric, and $\operatorname{Sym} F = F$ iff F is symmetric.
- Then if F, G are symmetric tensors on V, of ranks k and l, respectively, their symmetric product is defined to be the $(k+l)$-tensor FG given by

$$FG := \operatorname{Sym}(F \otimes G),$$

denoted by juxtaposition without an explicit product symbol.

Example

An important special case is the symmetric product of two 1-tensors, which can be characterized as follows: if ω, η are covectors, then

$$\omega\eta := \tfrac{1}{2}(\omega \otimes \eta + \eta \otimes \omega).$$

To prove this, evaluate both sides on an arbitrary pair of vectors (v, w) and use the definition of $\omega\eta$.
If ω is any 1-tensor, then $\omega^2 := \omega\omega = \omega \otimes \omega$.

Remark

In linear algebra the symbol for the tensor product $\otimes$ is sometimes called symbol of the Kronecker product.

Example

Having fixed a basis for $\mathbb{R}^n$ we explicitly compute the tensor (Kronecker) product of two elements of $T^{(1,1)}(\mathbb{R}^n) \cong \mathrm{End}(\mathbb{R}^n)$ in terms of their representing matrices $A, B \in \mathbb{R}^{n \times n}$ as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}.$$

On M, we can perform the same linear-algebraic constructions on each T_pM that we perform on any vector space, yielding tensors at p.

Definition

The disjoint union of tensor spaces of a particular type at all points of the manifold yields a vector bundle, called a tensor bundle.
The cotangent bundle is defined as

$$T^*M := \cup_{p\in M} T_p^*M.$$

More generally, the bundle of (k,l)-tensors on M is defined as

$$T^{(k,l)}TM := \cup_{p\in M} T^{(k,l)}(T_pM).$$

Example

- Bundle of covariant k-tensors is denoted by $T^kT^*M = T^{(0,k)}TM$,
- bundle of contravariant k-tensors is denoted by $T^kTM = T^{(k,0)}TM$,
- bundle of symmetric k-tensors is $\Sigma^kT^*M = \cup_{p\in M}\Sigma^k(T_p^*M)$.

There are the usual identifications among these bundles:

$$T^1 TM = T^{(1,0)} TM = TM,$$
$$T^1 T^*M = T^{(0,1)} TM = \Sigma^1 T^*M = T^*M.$$

Definition

A tensor field on M is a section of some tensor bundle over M.
A section of $T^1 T^*M = T^{(0,1)} TM$ (a covariant 1-tensor field) is also called a covector field.

Remark

As we do with vector fields, we write the value of a tensor field F at $p \in M$ as F_p or $F|_p$.
Because covariant tensor fields are the most common and important tensor fields we work with, we use the following shorthand notation for the space of all smooth covariant k-tensor fields:

$$\mathcal{T}^k M := \Gamma(T^k T^*M).$$

The space of smooth 0-tensor fields is then just $C^\infty(M)$.

Let $(E_i) = (E_1, \dots, E_n)$ be any smooth local frame for TM over an open $U \subset M$.

Associated with such a frame is the dual coframe, denoted by $(\varepsilon^1, \dots, \varepsilon^n)$.

These are smooth covector fields satisfying $\varepsilon^i(E_j) = \delta^i_j$.

For example, given a coordinate frame $(\partial/\partial x^1, \dots, \partial/\partial x^n)$ over some open $U \subset M$, the dual coframe is $(\mathrm{d}x^1, \dots, \mathrm{d}x^n)$, where $\mathrm{d}x^i$ is the differential of the coordinate function x^i.

In terms of any smooth local frame (E_i) and its dual coframe (ε^i) the tensor fields $E_{i_1} \otimes \cdots E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l}$ form a smooth local frame for $T^{(k,l)}T^*M$.

In particular, in local coordinates (x^i), a (k,l)-tensor field F has a coordinate expression of the form

$$F = F^{i_1 \dots i_k}_{j_1 \dots j_l} \partial_{i_1} \otimes \cdots \otimes \partial_{i_k} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_l},$$

with each coefficient $F^{i_1 \dots i_k}_{j_1 \dots j_l} \in C^\infty(U)$.

An important property of tensor fields is that they are multilinear over the space of smooth functions.

Lemma

Suppose $F \in \Gamma(T^{(k,l)}TM)$ is a smooth tensor field.
Given smooth $\omega^1, \ldots, \omega^k \in \mathcal{T}^1(M)$ and smooth $X_1, \ldots, X_l \in \mathfrak{X}(M)$, then the function $F(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l)$ is smooth, and thus F induces a map

$$\widetilde{F}\colon \underbrace{\mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M)}_{k \text{ factors}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l \text{ factors}} \to C^\infty(M).$$

Moreover, this map is multilinear over $C^\infty(M)$ that is, for all functions $u, v \in C^\infty(M)$ and smooth vector or covector fields α, β,

$$\widetilde{F}(\ldots, u\alpha + v\beta, \ldots) = u\widetilde{F}(\ldots, \alpha, \ldots) + v\widetilde{F}(\ldots, \beta, \ldots).$$

The converse holds true as well:
Every such map that is multilinear over $C^\infty(M)$ defines a tensor field.
We have

Lemma (Tensor Characterization Lemma)

A map

$$\mathcal{F}\colon \underbrace{\mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M)}_{k \text{ factors}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l \text{ factors}} \to C^\infty(M)$$

is induced by a smooth (k,l)-tensor field as above iff it is multilinear over $C^\infty(M)$. Similarly, a map

$$\mathcal{F}\colon \underbrace{\mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M)}_{k \text{ factors}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l \text{ factors}} \to \mathfrak{X}(M)$$

is induced by a smooth $(k+1,l)$-tensor field iff it is multilinear over $C^\infty(M)$.

Remark

- *It is common to use the same symbol for both a tensor field and the multilinear map on sections that it defines.*
- *We refer to either of these objects as a tensor field.*

Definition

Suppose $F\colon M \to N$ is a smooth and A is a covariant k-tensor field on N. For every $p \in M$, we define a tensor $\mathrm{D}\,F_p^*(A) \in T^k(T_p^*M)$; called the pointwise pullback of A by F at p, by

$$\mathrm{D}\,F_p^*(A)(v_1, \dots, v_k) := A(\mathrm{D}\,F_p(v_1), \dots, \mathrm{D}\,F_p(v_k))$$

for $v_1, \dots, v_k \in T_pM$.
We define the pullback of A by F to be the tensor field F^*A on M defined by

$$(F^*A)_p := \mathrm{D}\,F_p^*(A_{F(p)}).$$

Proposition (Properties of Tensor Pullbacks)

Suppose $F\colon M \to N$ and $G\colon P \to M$ are smooth maps, A, B are covariant tensor fields on N; and f is a real-valued function on N. Then

(a) $F^*(fB) = (f \circ F)F^*B$,

(b) $F^*(A \otimes B) = F^*A \otimes F^*B$,

(c) $F^*(A + B) = F^*A + F^*B$,

(d) *F^*B is a tensor field, and it is smooth if B is smooth,*

(e) $(F \circ G)^*B = G^*(F^*B)$,

(f) $(\mathrm{id}_N)^*B = B$.

If f is a 0-tensor field, then $F^*f = f \circ F$, i.e. (a) as special case of (b).

Remark

We omit for the time being:

- *How pullbacks are computed in local coordinates,*
- *the generalization of Lie derivatives applied to tensor fields.*

Introduction to Riemannian Geometry

Definition

Let M be a smooth manifold.

A Riemannian metric on M is a smooth covariant 2-tensor field $g \in \mathcal{T}^2(M)$ whose value g_p at each $p \in M$ is an inner product on T_pM.

Thus g is a symmetric 2-tensor field that is positive definite in the sense that $g_p(v, v) \geq 0$ for each $p \in M$ and each $v \in T_pM$, with equality iff $v = 0$.

A Riemannian manifold is a pair (M, g), where M is a smooth manifold and g is a specific choice of Riemannian metric on M.

Proposition

Every smooth manifold admits a Riemannian metric.

Proof: Consider an atlas $\{(x_\alpha, U_\alpha) | \alpha \in \mathcal{A}\}$ for the n-dimensional manifold M, together with a partition of unity $(\varphi_\alpha)_{\alpha\in\mathcal{A}}$ subordinate to $(U_\alpha)_{\alpha\in\mathcal{A}}$.

For simplicity of notation, we use the same index set for (φ_α) and (U_α), being justified by replacing, if necessary, the original covering (U_α) by a locally finite refinement.

For $v, w \in T_pM$ and $\alpha \in \mathcal{A}$ with $p \in U_\alpha$ let the coordinate representations be $(v_\alpha^1, \ldots, v_\alpha^n)$ and $(w_\alpha^1, \ldots, w_\alpha^n)$.

Then we put

$$\langle v, w\rangle := \sum_{\alpha\in\mathcal{A}, p\in U_\alpha} \varphi_\alpha(p) \sum_{i=1}^n v_\alpha^i w_\alpha^i.$$

This defines a Riemannian metric, i.e., the metric is simply constructed by piecing the Euclidean metrics of the coordinate images together by means of a partion of unity. □

Let (M, g) be a Riemannian manifold.

Because g_p is an inner product on T_pM for each $p \in M$, we use the angle-bracket notation for $v, w \in T_pM$:

$$\langle v, w \rangle_g := g_p(v, w).$$

We can then define

- lengths of tangent vectors,
- angles between nonzero tangent vectors,
- orthogonality of tangent vectors

as usual.

The length of a vector $v \in T_pM$ is denoted by

$$\|v\|_g := \langle v, v \rangle_g^{1/2}.$$

Example (The Euclidean Metric)

The Euclidean metric is the Riemannian metric $\bar{g}$ on $\mathbb{R}^n$ whose value at each $x \in \mathbb{R}^n$ is just the usual inner product on $T_x\mathbb{R}^n$ under the natural identification $T_x\mathbb{R}^n \cong \mathbb{R}^n$.

This means that for $v, w \in T_x\mathbb{R}^n$ written in standard coordinates $(x^1, \ldots, x^n)$ as $v = v^i \partial_i|_x$ and $w = w^j \partial_j|_x$ we have

$$\langle v, w \rangle_{\bar{g}} = \sum_{i=1}^{n} v^i w^i.$$

When we work with $\mathbb{R}^n$ as Riemannian manifold from now on we always think of $(\mathbb{R}, \bar{g})$.

Definition (Riemannian Isometry)

Consider two Riemannian manifolds $(M, g), (\widetilde{M}, \tilde{g})$.

An isometry from (M, g) to $(\widetilde{M}, \tilde{g})$ is a diffeomorphism $\varphi\colon M \to \widetilde{M}$ s.t.

$$\varphi^* \tilde{g} = g.$$

In other words φ is a smooth bijection and each derivative

$$\mathrm{D}\,\varphi_p\colon T_pM \to T_{\varphi(p)}\widetilde{M}$$

is a linear isometry.

$(M, g), (\widetilde{M}, \tilde{g})$ are isometric if there exists an isometry between them.

Remark

A composition of isometries and the inverse of an isometry are again isometries, so being isometric is an equivalence relation on the class of Riemannian manifolds.

Definition

If $(M, g), (\widetilde{M}, \tilde{g})$ are Riemannian manifolds, a map $\varphi\colon M \to \widetilde{M}$ is a local isometry if each point $p \in M$ has a neighborhood U s.t. $\varphi|_U$ is an isometry onto an open subset of $\widetilde{M}$.

Definition

A Riemannian n-manifold is said to be flat if it is locally isometric to a Euclidean space, i.e., if every point has a neighborhood that is isometric to an open set in $(\mathbb{R}^n, \bar{g})$

Definition

An isometry from (M, g) to itself is called an isometry of (M, g).
The set of all isometries of (M, g) is a group under composition, called the isometry group of (M, g).
It is denoted by $\mathrm{Iso}(M, g)$ or sometimes just $\mathrm{Iso}(M)$ if the metric is understood.

- In many applications or just for concrete calculations we often need local representations for metrics.
- In other words, we need to be able to work in charts.

Suppose (M, g) is a Riemannian manifold. If $(x^1, \ldots, x^n)$ are smooth local coordinates on an open $U \subset M$, then g can be written locally in U as

$$g = g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$$

for n^2 smooth functions g_{ij} with $i, j = 1, \ldots, n$.

The component functions of this tensor field constitute a matrix-valued function (g_{ij}) characterized by

$$g_{ij}(p) = \langle \partial_i|_p, \partial_j|_p \rangle,$$

where $\partial_i := \partial/\partial x^i$ is the i -th coordinate vector field.

We have $(g_{ij}) = (g_{ij})^\top$ and it depends smoothly on $p \in U$.

Let $v = v^i \partial_i|_p \in T_pM$ s.t. $g_{ij}(p)v^j = 0$, then $\langle v, v\rangle = g_{ij}(p)v^i v^j = 0$, which implies $v = 0$.

Thus the matrix $\big(g_{ij}(p)\big)$ is always nonsingular.

The notation for g can be shortened by expressing it in terms of the symmetric tensor product. We compute

$$\begin{aligned} g &= g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j = \tfrac{1}{2}\big(g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ji}\mathrm{d}x^i \otimes \mathrm{d}x^j\big) \\ &= \tfrac{1}{2}\big(g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ij}\mathrm{d}x^j \otimes \mathrm{d}x^i\big) = g_{ij}\mathrm{d}x^i\mathrm{d}x^j. \end{aligned}$$

Example

For $\bar{g}$ of $\mathbb{R}^n$ we have $\bar{g} = \sum_i \mathrm{d}x^i\mathrm{d}x^i = \sum_i (\mathrm{d}x^i)^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$, or just $\bar{g}_{ij} = \delta_{ij}$ (in these standard coordinates).

More generally, if $(E_1, \dots, E_n)$ is any smooth local frame for TM on an open $U \subset M$ and $(\varepsilon^1, \dots, \varepsilon^n)$ is its dual coframe, we can write g locally in U as $g = g_{ij}\varepsilon^i\varepsilon^j$, where $g_{ij}(p) = \langle E_i|_p, E_j|_p\rangle$, and the matrix-valued function (g_{ij}) is symmetric and smooth as before.

Frames and the Riemannian Metric

- As a covariant 2-tensor field a Riemannian metric g acts on smooth vector fields X, Y to yield a real-valued function $\langle X, Y\rangle$.
- In terms of any smooth local frame, this function is expressed locally by $\langle X, Y\rangle = g_{ij}X^iY^j$ and therefore is smooth.
- Similarly, we obtain a nonnegative real-valued function $\|X\| := \langle X, X\rangle^{1/2}$, which is continuous everywhere and smooth on the open subset where $X \neq 0$.

Definition

A local frame (E_i) on open $U \subset M$ is an orthonormal frame if $E_1|_p, \ldots, E_n|_p$ form an orthonormal basis for T_pM at each $p \in U$.

Equivalently, (E_i) is an orthonormal frame iff

$$\langle E_i, E_j\rangle = \delta_{ij},$$

in which case g has the local expression

$$g = \sum_{i=1}^{n}(\varepsilon^i)^2 = \sum_{i=1}^{n}\varepsilon^i \otimes \varepsilon^i.$$

Proposition (Existence of Orthonormal Frames)

Let (M, g) be a Riemannian n-manifold.

Let (X_j) be any smooth local frame for TM over an open $U \subset M$.

Then there is a smooth orthonormal frame (E_j) over U s.t.

$$\operatorname{span}(E_1|_p, \ldots, E_k|_p) = \operatorname{span}(X_1|_p, \ldots, X_k|_p)$$

for each $k = 1, \ldots n$ and each $p \in U$.

In particular, for every $p \in M$, there is a smooth orthonormal frame (E_j) defined on some neighborhood of p.

Idea of Proof:
Apply the Gram-Schmidt procedure to the vectors $X_1|_p, \ldots, X_n|_p$ at each $p \in U$.

We then obtain an ordered n-tuple of orthonormal vector fields $(E_1, \ldots, E_n)$ over U satisfying the span conditions.

Smoothness follows from the fact that the GS-procedure consists of smooth calculations only, in particular denominators never vanish. □

Remark

As a spiteful aside, usual numerical implementations of the GS-procedure (e.g. Matlab) do **not** *rely on smooth formulas, as they are based on socalled Householder transformations which are essentially based on nonsmooth reflections.*

Remark

- *Proposition 41 does* **not** *say that one can find coordinates in a whole* **neighborhood** *of p s.t. the coordinate frame (∂_i) is orthonormal.*
- *We will learn later that this is possible only when the metric is flat, that is, locally isometric to the Euclidean metric.*
- *The situation is, however, different when we deal with symplectic manifolds instead of Riemannian ones (Darboux's Theorem).*

Methods for Constructing Riemannian Metrics

Motivation:
Many examples of Riemannian manifolds arise naturally as

- submanifolds,
- products,
- quotients

of other Riemannian manifolds.

Lemma

Suppose $(\widetilde{M}, \tilde{g})$ is a Riemannian manifold, M is a smooth manifold, and $F\colon M \to \widetilde{M}$ is a smooth map.
The smooth 2-tensor field

$$g := F^*\widetilde{g}$$

is a Riemannian metric on M iff F is an immersion.

Sketch of Proof:
Symmetry and bilinearity are clearly ensured by the corresponding properties of $\widetilde{g}$.

However, positive definiteness of g requires the injectivity of $\mathrm{D}\,F$:

Indeed, e.g. consider a non-zero element v of $\ker \mathrm{D}\,F_p$ then $(F^*g)_p(v, v) = 0$ but $v \neq 0$.

Injectivity of $\mathrm{D}\,F$ eliminates this possibility. $\square$

Definition

Suppose $(\widetilde{M}, \tilde{g})$ is a Riemannian manifold.

Given a smooth immersion $F\colon M \to \widetilde{M}$, the metric $g := F^*\tilde{g}$ is called the metric induced by F.

On the other hand, if M is already endowed with a *given* Riemannian metric g, an immersion or embedding

$$F\colon M \to \widetilde{M} \text{ satisfying } F^*\tilde{g} = g$$

is called an isometric immersion or isometric embedding, respectively.

Example

Suppose $M \subset \widetilde{M}$ is an (immersed or embedded) submanifold.

The induced metric on M is the metric $g := \iota^*\tilde{g}$ induced by the inclusion map $\iota\colon M \hookrightarrow \widetilde{M}$.

With this metric, M is called a Riemannian submanifold of $\widetilde{M}$.

We always consider submanifolds of Riemannian manifolds to be endowed with the induced metrics unless otherwise specified.

Remark

If (M, g) is a Riemannian submanifold of $(\widetilde{M}, \widetilde{g})$, then for every $p \in M$ and $v, w \in T_pM$, the definition of the induced metric reads

$$g_p(v, w) = \widetilde{g}_p\big(\mathrm{D}\,\iota_p(v), \mathrm{D}\,\iota_p(w)\big).$$

Because we usually identify T_pM with its image in $T_p\widetilde{M}$ under $\mathrm{D}\,\iota_p$, and think of $\mathrm{D}\,\iota_p$ as an inclusion map, what this really amounts to is

$$g_p(v, w) = \widetilde{g}_p(v, w) \quad \text{for} \quad v, w \in T_pM.$$

In other words, the induced metric g is just the restriction $g = \widetilde{g}|_{TM}$.

Example (Spheres)

For each $n \in \mathbb{N}$, the unit n-sphere $S^n \subset \mathbb{R}^{n+1}$ is an embedded n-dimensional submanifold. The Riemannian metric induced on S^n by the Euclidean metric is known as the round metric or standard metric $\mathring{g}$ on S^n.

Definition

If $(\widetilde{M}, \tilde{g})$ is an m-dimensional smooth Riemannian manifold and $M \subset \widetilde{M}$ is an n-dimensional submanifold.

A local frame $(E_1, \ldots, E_m)$ for $\widetilde{M}$ on an open subset $\widetilde{U} \subset \widetilde{M}$ is said to be adapted to M if the first n vector fields $(E_1, \ldots, E_n)$ are tangent to M.

Proposition (Existence of Adapted Orthonormal Frames)

Let $(\widetilde{M}, \tilde{g})$ be a Riemannian manifold, and let $M \subset \widetilde{M}$ be an embedded smooth submanifold.

Given $p \in M$, there exist a neighborhood $\widetilde{U}$ of p in $\widetilde{M}$ and a smooth orthonormal frame for $\widetilde{M}$ on $\widetilde{U}$ that is adapted to M.

Idea of Proof:
Application of GS-procedure to a coordinate frame in slice coordinates. □

Definition

Suppose $(\widetilde{M}, \tilde{g})$ is a Riemannian manifold and $M \subset \widetilde{M}$ is a smooth submanifold.

Given $p \in M$, a vector $v \in T_p\widetilde{M}$ is said to be normal to M if $\langle v, w \rangle = 0$ for every $w \in T_pM$.

The space of all vectors normal to M at p is a subspace of $T_p\widetilde{M}$ called the normal space at p and denoted by $N_pM = (T_pM)^\perp$.

At each $p \in M$, the ambient tangent space $T_p\widetilde{M}$ splits as an orthogonal direct sum $T_p\widetilde{M} = T_pM \oplus N_pM$.

A section N of the ambient tangent bundle $T\widetilde{M}|_M$ is called a normal vector field along M if $N_p \in N_pM$ for each $p \in M$.

The set

$$NM := \cup_{p \in M} N_pM$$

is called the normal bundle of M.

Proposition (Normal Bundle)

If $\widetilde{M}$ is a Riemannian m-manifold and $M \subset \widetilde{M}$ is an immersed or embedded n-dimensional submanifold, then NM is a smooth rank-$m-n$ vector subbundle of the ambient tangent bundle $T\widetilde{M}|_M$.

There are smooth bundle homomorphisms

$$\pi^\top \colon T\widetilde{M}|_M \to TM,$$
$$\pi^\perp \colon T\widetilde{M}|_M \to NM$$

called the tangential and normal projections, that for each $p \in M$ restrict to orthogonal projections from $T_p\widetilde{M}$ to T_pM and N_pM, respectively.

Sketch of Proof: Because every smooth immersion is locally an embedding there is a neighborhood U of p in M that is embedded in $\widetilde{M}$.

Now Prop. 42 ensures the existence of a smooth orthonormal frame $(E_1, \ldots, E_m)$ being adapted to U on some neighborhood $\widetilde{U}$ of p in $\widetilde{M}$. This means that $(E_1, \ldots, E_n)|_{\widetilde{U} \cap U}$ form a local orthonormal frame for M.

Given such an adapted frame, the $m - n$ vector fields $(E_{n+1}, \ldots, E_m)|_M$ form a smooth local frame for NM, so NM is a smooth subbundle.

$\pi^\top$ and $\pi^\perp$ are defined pointwise as orthogonal projections onto the tangent and normal spaces, respectively, which shows that they are uniquely defined.

In terms of an adapted orthonormal frame, they can be written

$$\pi^\top \sum_{i=1}^{m} X^i E_i = \sum_{j=1}^{n} X^j E_j,$$

$$\pi^\perp \sum_{i=1}^{m} X^i E_i = \sum_{j=n+1}^{m} X^j E_j,$$

which shows that they are smooth.

Example

Consider S^{n-1} as Riemannian submanifold of Euclidean $\mathbb{R}^n$.

Then the projections are explicitly

$$\begin{aligned} \pi^\top \colon T_p\mathbb{R}^n \cong \mathbb{R}^n &\to T_pS^{n-1}, \\ v &\mapsto (I - pp^\top)v \end{aligned}$$

and

$$\begin{aligned} \pi^\perp \colon T_p\mathbb{R}^n \cong \mathbb{R}^n &\to N_pS^{n-1}, \\ v &\mapsto pp^\top v. \end{aligned}$$

Example (Metrics in Graph Coordinates)

Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}$ a smooth function.

The graph of f, i.e.,

$$\Gamma(f) := \Big\{ \big(x, f(x)\big) \mid x \in U \Big\} \subset \mathbb{R}^{n+1},$$

is an embedded submanifold of dimension n.

It has a **global** parametrization $X\colon U \to \mathbb{R}^{n+1}$ called a graph parametrization, defined by $X(u) := \big(u, f(u)\big)$.

The corresponding coordinates $(u^1, \dots, u^n)$ on $\Gamma(f)$ are called graph coordinates.

In graph coordinates, the induced metric on $\Gamma(f)$ is

$$X^*\bar{g} = X^*\Big((\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^{n+1})^2\Big) = (\mathrm{d}u^1)^2 + \cdots + (\mathrm{d}u^n)^2 + (\mathrm{D}\,f)^2.$$

Example

We apply Ex. 335 to the upper hemisphere of $S^2 \subset \mathbb{R}^3$.

Consider the open unit ball $B^2 := \{x \in \mathbb{R}^3 \colon \|x\| < 1\}$. The upper hemisphere of S^2 can be globally parameterized by

$$X \colon B^2 \to \mathbb{R}^3, \quad X(u,v) := \big(u, v, \sqrt{1-u^2-v^2}\big).$$

Hence, the round metric on S^2 can be written locally as

$$\begin{aligned} \mathring{g} &= X^* \bar{g} \\ &= \mathrm{d}u^2 + dv^2 + \big(\tfrac{u\mathrm{d}u + v\mathrm{d}v}{\sqrt{1-u^2-v^2}}\big)^2 \\ &= \frac{(1-v^2)\mathrm{d}u^2 + (1-u^2)\mathrm{d}v^2 + 2uv\ \mathrm{d}u\ \mathrm{d}v}{1-u^2-v^2}. \end{aligned}$$

Definition (Riemannian Product Metric)

If $(M_1, g_1), (M_2, g_2)$ are Riemannian manifolds, $M_1 \times M_2$ has a natural Riemannian metric $g = g_1 \oplus g_2$, called the product metric, defined by

$$g_{(p_1,p_2)}\big((v_1, v_2), (w_1, w_2)\big) := g_1|_{p_1}(v_1, w_1) + g_2|_{p_2}(v_2, w_2),$$
$$(v_1, v_2), (w_1, w_2) \in T_{p_1}M_1 \oplus T_{p_2}M_2 \cong T_{(p_1,p_2)}(M_1 \times M_2).$$

Local coordinates $(x^1, \ldots, x^n)$ for M_1 and $(x^{n+1}, \ldots, x^{n+m})$ for M_2 give coordinates $(x^1, \ldots, x^{n+m})$ for $M_1 \times M_2$.
In these coordinates, the product metric has the local expression $g = \sum g_{ij} \mathrm{d}x^i \mathrm{d}x^j$, where (g_{ij}) is the block diagonal matrix

$$(g_{ij}) = \begin{bmatrix} (g_1)_{ab} & 0 \\ 0 & (g_2)_{cd} \end{bmatrix},$$

here $a, b \in \{1, \ldots, n\}$, and $c, d \in \{n+1, \ldots, n+m\}$.
Product metrics on products of three or more Riemannian manifolds are defined similarly.

Definition (Warped Products)

Suppose $(B, g_B), (F, g_F)$ are Riemannian manifolds.
Let $f\colon B \to \mathbb{R}^+$ be a strictly positive smooth function.

The warped product $M := B \times_f F$ is the product manifold $B \times F$ endowed with the Riemannian metric $g = g_B \oplus f^2 g_F$, defined by

$$g_{(p_B,p_F)}\big((v_B, v_F), (w_B, w_F)\big) := g_B|_{p_B}(v_B, w_B) + f^2(p_B) g_F|_{p_F}(v_F, w_F),$$
$$(v_B, v_F), (w_B, w_F) \in T_{p_B}B \oplus T_{p_F}F.$$

The function f is called the warping function of the warped product M, whereas B is called base of M and F the fiber.

Fibers $\{p_B\} \times F$ and leaves $B \times \{p_F\}$ are Riemannian submanifolds of $B \times_f F$.

With $f \equiv 1$, $B \times_f F$ is just $B \times F$ with the product metric.

Despite the similarity with the notation for product metrics, $g_B \oplus f^2 g_F$ is generally **not** a product metric unless f is constant.

Example ($\mathbb{R}^3 \setminus \{0\}$ as a warped product)

Write the "line element" $\mathrm{d}s$ of $\mathbb{R}^3 \setminus \{0\}$ in spherical coordinates. With the parametrization $r \in \mathbb{R}^+, \theta \in (0, \pi), \varphi \in (0, 2\pi)$ associated to spherical coordinates

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r \cos\theta,$$

we get

$$\begin{aligned}(\mathrm{d}s)^2 &= (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 \\ &= \big(\mathrm{d}(r \sin\theta \cos\varphi)\big)^2 + \big(\mathrm{d}(r \sin\theta \sin\varphi)\big)^2 + \big(\mathrm{d}(r \cos\theta)\big)^2 \\ &= (\mathrm{d}r)^2 + r^2\big((\mathrm{d}\theta)^2 + \sin^2\theta(\mathrm{d}\varphi)^2\big).\end{aligned}$$

Moreover, we have the diffeomorphism

$$\mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^+ \times S^2, \quad \big(\|v\|, \tfrac{v}{\|v\|}\big),$$

and since for $r = 1$, $\mathrm{d}s$ furnishes the line element on S^2, we have, with the notation as in Def. 338,

$$\mathbb{R}^3 \setminus \{0\} \cong \mathbb{R}^+ \times_r S^2,$$

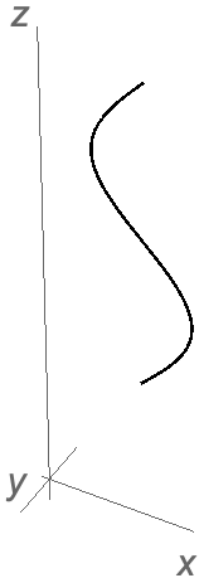
here $\cong$, means "isometric to".

Example (Surface of Revolution as Warped Product)

Every surface of revolution (not crossing the axis of revolution) can be expressed as a warped product, as follows.

The leaves are the different positions of the rotated curve and the fibers are the circles of revolution.

More explicitly, if M is gotten by revolving a plane curve C about an axis in $\mathbb{R}^3$ and $f\colon C \to \mathbb{R}^+$ gives distance to the axis, then $M = C \times_f S^1$.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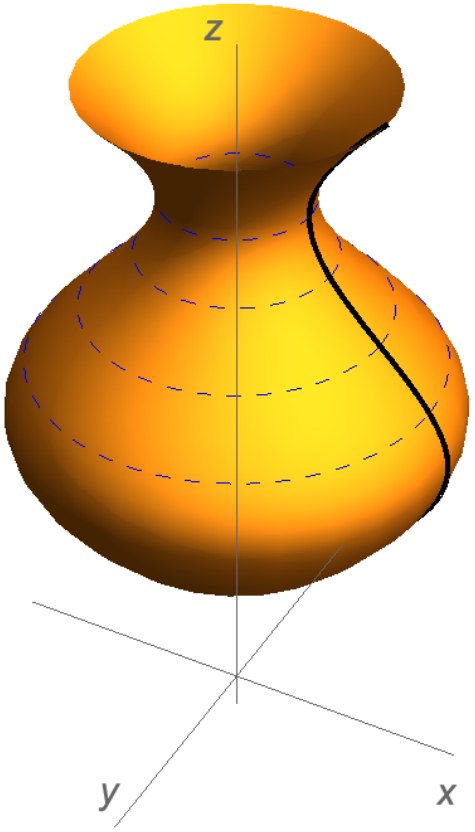
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Riemannian Submersions

- Submanifolds and products of Riemannian manifolds, automatically inherit Riemannian metrics of their own.
- Quotients of Riemannian manifolds inherit Riemannian metrics only under very special circumstances.
- We will see what those circumstances are.

Suppose $\widetilde{M}$ and M are smooth manifolds $\pi\colon \widetilde{M} \to M$ is a smooth submersion, and $\tilde{g}$ is a Riemannian metric on $\widetilde{M}$.

By the regular value theorem, each fiber $\widetilde{M}_y := \pi^{-1}(y)$, i.e., the preimage of $y \in M$ under π, is a properly embedded smooth submanifold of $\widetilde{M}$.

At each point $x \in \widetilde{M}$, we define two subspaces of the tangent space $T_x\widetilde{M}$.

Definition

The vertical space at x is

$$V_x := \ker \mathrm{D}\,\pi_x = T_x\big(\widetilde{M}_{\pi(x)}\big),$$

i.e., the tangent space to the fiber containing x.
The horizontal space at x is its orthogonal complement

$$H_x := (V_x)^\perp.$$

Then the tangent space $T_x\widetilde{M}$ decomposes as an orthogonal direct sum $T_x\widetilde{M} = H_x \oplus V_x$.

Remark

Note that the vertical space is well defined for **every** *submersion, because it does not refer to the metric.*

However, the horizontal space depends on the metric.

Definition

A vector field on $\widetilde{M}$ is said to be a horizontal vector field if its value at each point lies in the horizontal space at that point.

A vertical vector field is defined similarly.

Given a vector field X on M, a vector field $\widetilde{X}$ on $\widetilde{M}$ is called a horizontal lift of X if $\widetilde{X}$ is horizontal and π-related to X.

The latter property means that $\mathrm{D}\,\pi_x\big(\widetilde{X}_x\big) = X_{\pi(x)}$ for each $x \in \widetilde{M}$.

As the principal tool for doing computations on Riemannian submersion we have

Proposition (Properties of Horizontal Vector Fields)

Let $\widetilde{M}$ and M be smooth manifolds, let $\pi\colon \widetilde{M} \to M$ be a smooth submersion, and let $\tilde{g}$ be a Riemannian metric on $\widetilde{M}$.

(a) *Every smooth vector field W on $\widetilde{M}$ can be expressed uniquely in the form $W = W^H + W^V$, where W^H is horizontal, W^V is vertical, and both W^H and W^V are smooth.*

(b) *Every smooth vector field on M has a unique smooth horizontal lift to $\widetilde{M}$.*

(c) *For every $x \in \widetilde{M}$ and $v \in H_x$, there is a vector field $X \in \mathfrak{X}(M)$ whose horizontal lift $\widetilde{X}$ satisfies $\widetilde{X}_x = v$.*

Proof:

(a): Let $p \in \widetilde{M}$ be arbitrary. As π is a smooth submersion, the rank theorem ensures that there exist smooth coordinate charts $\big(\widetilde{U}, (x^i)\big)$ centered at p and $\big(U, (u^j)\big)$ centered at $\pi(p)$ in which π has the coordinate representation

$$\pi(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots, x^n),$$

where $m = \dim \widetilde{M}$ and $n = \dim M$.

It follows that at each point $q \in \widetilde{U}$, the vertical space V_q is spanned by the vectors $\partial_{n+1}|_q, \ldots, \partial_m|_q$.

It probably will not be the case, however, that the *horizontal* space is spanned by the other n basis vectors.

Now apply Gram–Schmidt to the ordered frame $(\partial_{n+1}, \ldots, \partial_m, \partial_1, \ldots, \partial_n)$ to obtain a smooth orthonormal frame $(E_1, \ldots, E_m)$ on $\widetilde{U}$ s.t. V_q is spanned by $(E_1|_q, \ldots, E_{m-n}|_q)$ at each $q \in \widetilde{U}$.

It follows that H_q is spanned by $(E_{m-n+1}|_q, \ldots, E_m|_q)$.

Now let $W \in \mathfrak{X}(\widetilde{M})$ be arbitrary.

At each $q \in \widetilde{M}$, W_q can be written uniquely as a sum of a vertical vector plus a horizontal vector.

This defines a decomposition $W = W^V + W^H$ into vertical and horizontal vector fields.

To see that they are smooth, just note that in a neighborhood of each point we can express W in terms of a frame $(E_1, \ldots, E_m)$ of the type constructed above as $W = \sum_{i=1}^m W^i E_i$ with smooth coefficients (W^i).

Then it follows that $W^V = \sum_{i=1}^{m-n} W^i E_i$ and $W^H = \sum_{i=m-n+1}^m W^i E_i$, both of which are smooth.

Sketch of proof for (b) and (c) (The arguments are the same as for product manifolds):

Obviously, for any $p \in \widetilde{M}$ the restriction $\mathrm{D}\,\pi_p|_{H_p} \colon H_p \to T_{\pi(p)}M$ is an isomorphism.

Consequently, a vector field on M can be lifted to a unique horizontal vector field on $\widetilde{M}$, and we only need to show smoothness.

Let $\widetilde{U} \subset \widetilde{M}$ an open subset, and let $(E_{m-n+1}, \ldots, E_m)$ be a smooth orthonormal frame of the horizontal bundle along $\widetilde{U}$.

Now, $\mathrm{D}\,\pi|_{H_p}$ is smooth and maps each of the (E_i) to a smooth vector field on M.

Since taking the inverse is a smooth function on the space of invertible linear maps, $(\mathrm{D}\,\pi|_{H_p})^{-1}$ is smooth and maps a smooth vector field on M to a smooth vector field on $\widetilde{M}$. $\square$

Definition (Riemannian Submersion)

Suppose $(\widetilde{M}, \tilde{g})$ and (M, g) are Riemannian manifolds, and $\pi\colon \widetilde{M} \to M$ is a smooth submersion.

Then π is said to be a Riemannian submersion if for each $x \in \widetilde{M}$, the derivative $\mathrm{D}\,\pi_x$ restricts to a linear isometry from H_x onto $T_{\pi(x)}M$.

In other words, whenever $v, w \in H_x$, we have

$$\tilde{g}_x(v, w) = g_{\pi(x)}\big(\mathrm{D}\,\pi_x(v), \mathrm{D}\,\pi_x(w)\big).$$

Example

1. The projection $\pi\colon \mathbb{R}^{n+k} \to \mathbb{R}^n$ onto the first n coordinates is a Riemannian submersion, if both $\mathbb{R}^{n+k}$ and $\mathbb{R}^n$ are endowed with their Euclidean metrics.
2. If M, N are Riemannian manifolds and $M \times N$ is endowed with the product metric, then both projections $\pi_M\colon M \times N \to M$ and $\pi_N\colon M \times N \to N$ are Riemannian submersions.
3. If $B \times_f F$ is a warped product manifold, the projection $\pi_B\colon B \times_f F \to B$ is a Riemannian submersion, but π_F typically is not.

- Given $(\widetilde{M}, \tilde{g})$ and a surjective submersion $\pi\colon \widetilde{M} \to M$, it is rarely the case that there exists a metric on M that makes π into a Riemannian submersion.
- For this to be the case, whenever $p_1, p_2 \in \widetilde{M}$ are two points in the same fiber $\pi^{-1}(y)$, both linear maps $(\mathrm{D}\,\pi_{p_i}|_{H_{p_i}})^{-1}\colon T_yM \to H_{p_i}$ have to pull $\tilde{g}$ back to the **same** inner product on T_yM.
- There is, however, an important special case in which there is such a metric:

Definition

Given a smooth surjective submersion $\pi\colon \widetilde{M} \to M$ and a Lie group G acting on $\widetilde{M}$.

The action is vertical if every element $g \in G$ takes each fiber to itself, meaning that $\pi(g \cdot p) = \pi(p)$ for all $p \in \widetilde{M}$.

The action is transitive on fibers if for each $p, q \in \widetilde{M}$ s.t. $\pi(p) = \pi(q)$, there exists $g \in G$ such that $g \cdot p = q$.

If in addition $\widetilde{M}$ is endowed with a Riemannian metric, the action is said to be an isometric action or an action by isometries.

And the metric is said to be invariant under G, if the map $x \mapsto g \cdot x$ is an isometry for each $g \in G$.

Theorem

Consider $(\widetilde{M}, \tilde{g})$ and $\pi\colon \widetilde{M} \to M$ a surjective smooth submersion.

Let G be a group acting on $\widetilde{M}$ with $\theta\colon G \times \widetilde{M} \to \widetilde{M}$, and $(g, \tilde{p}) \mapsto g \cdot \tilde{p}$ and associated $\theta_g\colon \widetilde{M} \to \widetilde{M}$, with $\tilde{p} \mapsto g \cdot \tilde{p}$.

If the action is isometric, vertical, and transitive on fibers, then there is a unique Riemannian metric on M such that π is a Riemannian submersion.

Proof: Let $p \in M$ and $u, v \in T_pM$.
For $\tilde{p} \in \pi^{-1}(p)$, there exist unique vectors $\tilde{u}, \tilde{v} \in H_{\tilde{p}}$ such that $\mathrm{D}\,\pi\tilde{u} = u$ and $\mathrm{D}\,\pi\tilde{v} = v$.

To make $\mathrm{D}\,\pi$ be an isometry between $H_{\tilde{p}}$ and T_pM, we must set $g_p(u, v) = \tilde{g}_{\tilde{p}}(\tilde{u}, \tilde{v})$.

The metric g_p does not depend on the choice of $\tilde{p}$ in the fiber.

Indeed, if $\pi(\tilde{p}) = \pi(\tilde{p}') = p$, there exists $g \in G$ such that $\theta_g(\tilde{p}) = g \cdot \tilde{p} = \tilde{p}'$.

As $\mathrm{D}\,\theta_g$ is an isometry between $H_{\tilde{p}}$ and $H_{\tilde{p}'}$, we have $\tilde{g}_{\tilde{p}}(\tilde{u}, \tilde{v}) = \tilde{g}_{\tilde{p}'}\big(\mathrm{D}\,\theta_g\tilde{u}, \mathrm{D}\,\theta_g\tilde{v}\big)$.

The next corollary describes a situation to which the above thm. applies.

Corollary

Consider $(\widetilde{M}, \tilde{g})$ and G is a Lie group acting smoothly, freely, properly, and isometrically on $\widetilde{M}$.

Then the orbit space $M = G\backslash\widetilde{M}$ has a unique smooth manifold structure and Riemannian metric such that π is a Riemannian submersion.

Proof:
The quotient manifold theorem shows that M has a unique smooth manifold structure such that the quotient map $\pi\colon \widetilde{M} \to M$ is a smooth submersion.

It follows easily from the definitions in that case that the given action of G on $\widetilde{M}$ is vertical and transitive on fibers.

Since the action is also isometric, the thm. shows that M inherits a unique Riemannian metric making π into a Riemannian submersion. $\square$

Raising and Lowering Indices

- An elementary important property of Riemannian metrics is that they allow us to convert vectors to covectors and vice versa.
- Given a Riemannian metric g on M, we define a bundle homomorphism $\hat{g}\colon TM \to T^*M$ by setting for all $p \in M$ and $v, w \in T_pM$
$$\hat{g}(v)(w) := g_p(v, w).$$
- If X and Y are smooth vector fields on M, this yields
$$\hat{g}(X)(Y) := g_p(X, Y),$$
which implies:
  (a) $\hat{g}(X)(Y)$ is linear over $C^\infty(M)$ in Y and thus $\hat{g}(X)$ is a smooth *covector* field,
  (b) the covector field $\hat{g}(X)$ is linear over $C^\infty(M)$ as a function of X,
  (c) $\hat{g}$ is a smooth bundle homomorphism.

Given a smooth local frame (E_i) with dual coframe (ε^i).

Let $g = g_{ij}\varepsilon^i\varepsilon^j$ be the local expression for g.

If $X = X^i E_i$ is a smooth vector field, the covector field $\hat{g}(X)$ has the coordinate expression

$$\hat{g}(X) = (g_{ij}X^i)\varepsilon^j.$$

Thus the matrix of $\hat{g}$ in any local frame is the same as the matrix of g itself.

Definition (Index Gymnastics)

Given a vector field X, we denote the components of the covector field $\hat{g}(X)$ by

$$X_j := g_{ij} X^i, \quad \text{s.t.} \quad \hat{g}(X) = X_j \varepsilon^j.$$

I.e., $\hat{g}(X)$ is obtained from X by lowering an index.

$\hat{g}(X)$ is denoted $X^\flat$ and called X flat.

As $\hat{g}$ is invertible the matrix of $\hat{g}^{-1}$ is just the inverse matrix of (g_{ij}), denoted by (g^{ij}), so that $g^{ij} g_{jk} = g_{kj} g^{ji} = \delta^i_k$.

The symmetry of (g_{ij}) implies symmetry of (g^{ij}).

In terms of a local frame, the inverse map $\hat{g}^{-1}$ is given by

$$\hat{g}^{-1} = \omega^i E_i, \quad \text{where} \quad \omega^i = g^{ij} \omega_j.$$

If ω is a covector field, the vector field $\hat{g}^{-1}(\omega)$ is called ω sharp and denoted by $\omega^\sharp$, it is obtained from ω by raising an index.

Remark

Borrowed from the musical notation for lowering or raising a tone, the two inverse isomorphisms $\flat$ and $\sharp$ are known as the musical isomorphisms.

Definition (Gradient)

Consider the Riemannian manifold (M, g) together with $f \in C^\infty(M)$.

The unique vector field grad_f, defined by

$$g\big(\operatorname{grad}_f, u\big)_p = \mathrm{D}\, f(p) \cdot u, \quad \text{for all } p \in M \quad \text{and } u \in T_pM,$$

is called the (Riemannian) gradient of f.

Clearly, by this definition the gradient is uniquely defined, as the metric is nondegenerate and, moreover, Riesz' representation theorem applies.

The Riemannian gradient of the Rayleigh quotient as an example:

For a given $A = A^\top \in \mathbb{R}^{n\times n}$ consider $S^{n-1} \subset \mathbb{R}^n$ and the smooth function

$$\begin{aligned} f\colon S^{n-1} &\to \mathbb{R}, \\ x &\mapsto x^\top A x. \end{aligned}$$

We consider $(S^{n-1}, \overline{g}|_{TS^{n-1}})$, i.e., the sphere as Riemannian manifold with induced metric from the embedding Euclidean $(\mathbb{R}^n, \overline{g})$.

Let $\gamma\colon \mathbb{R} \to S^{n-1}$ be a smooth curve with $\gamma(0) = p$ and $\dot\gamma(0) = \xi$.

The directional derivative of f evaluated at p in direction $\xi \in T_pS^{n-1}$ is

$$\begin{aligned} \mathrm{D}\, f(p)\xi &= \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\big(f\circ\gamma\big)(\varepsilon)|_{\varepsilon=0} \\ &= \mathrm{D}\, f\big(\gamma(0)\big)\cdot\dot\gamma(0) \\ &= 2p^\top A\xi \\ &= 2\overline{g}(Ap,\xi). \end{aligned}$$

One might be tempted to identify the vector Ap above already as the Riemannian gradient of $f\colon S^{n-1} \to \mathbb{R}$.

But such a conclusion is **wrong** as in general $Ap \notin T_pS^{n-1}$.

Therefore the expression $\overline{g}|_{T_pS^{n-1}}(Ap, \xi)$ is not defined.

However, this situation is easily handled:

We project the vector $Ap \in \mathbb{R}^n \cong T_p\mathbb{R}^n$ orthogonally into T_pS^{n-1}.

As $\xi \in T_pS^{n-1}$ we get

$$\begin{aligned} \mathrm{D}\, f(p)\xi &= 2\overline{g}(Ap, \xi) \\ &= 2\overline{g}\big(\pi_p^\top(Ap), \xi\big) \\ &= 2\overline{g}|_{T_pS^{n-1}}\big(\pi_p^\top(Ap), \xi\big), \end{aligned}$$

with orthogonal projection operator

$$\begin{aligned} \pi_p^\top\colon T_p\mathbb{R}^n \cong \mathbb{R}^n &\to T_pS^{n-1}, \\ x &\mapsto x - \overline{g}(p, x) \cdot p = \big(I - pp^\top\big)x. \end{aligned}$$

We conclude that

$$\begin{aligned}\operatorname{grad}_f(p) &= 2\big(I - pp^\top\big)Ap \\ &= 2\big(A - f(p)I\big)p \in T_pS^{n-1}.\end{aligned}$$

A necessary and sufficient condition for $p \in S^{n-1}$ to be critical is that

$$\mathrm{D}\, f(p) = 0 \quad \Longleftrightarrow \quad \operatorname{grad}_f(p) = 0.$$

Consequently, we have the critical point condition

$$2\big(A - f(p)I\big)p = 0 \quad \Longleftrightarrow \quad Ap = f(p)p.$$

This means, p is critical iff p is a unit norm eigenvector of A with corresponding critical value being the corresponding eigenvalue $f(p)$.

If, moreover, the spectrum of A was simple, then all the critical points were isolated.

To show the latter we need, however, second order derivatives, to be discussed in the next semester.

Remark

With the terminology of raising and lowering indices and the notion of musical isomorphisms we can revisit the Riemannian gradient.

On a Riemannian manifold (M, g) *with* $f \in C^\infty(M)$ *we have*

$$\operatorname{grad}_f = (\mathrm{D}\, f)^\sharp,$$

obtained from the derivative by raising an index.

The local expression then is

$$\operatorname{grad}_f = (g^{ij} E_i f) E_j.$$

Thus if (E_i) *is an* **orthonormal** *frame, then* grad_f *is the vector field whose components are the same as the components of the covector field* $\mathrm{D}\, f$*.*

However, in other frames, this will not be the case.

Length, Energy and Distance

Motivation:
A Riemannian metric offers us to measures, lengths of curves and distances between points.

Some terminology:

- A curve in M always means parameterized curve, i.e., a continuous map $\gamma\colon I \to M$ with some interval $I \subset \mathbb{R}$.
- To say that γ is a smooth curve means that it is smooth as a map from the manifold I (possibly with boundary) to M.

  If I has one or two endpoints, then γ is smooth iff it extends to a smooth curve defined on some open interval containing I.
- A curve segment is a curve whose domain is a compact interval.
- γ is a regular curve if the velocity $\gamma'(t) \neq 0$ for all $t \in I$.

  It implies that γ is an immersion, in particular its image does neither have corners nor kinks.

- $\gamma\colon [a,b] \to M$ is called piecewise regular if there exists a partition $(a_0, \dots, a_k)$ of $[a,b]$ such that $\gamma|_{[a_{i-1},a_i]}$ with $i = 1, \dots, k$ is a regular segment.
- A piecewise regular curve segment is called admissible curve.
  Any partition $(a_0, \dots, a_k)$ such that $\gamma|_{[a_{i-1},a_i]}$ for all i is smooth is called an admissible partition for γ.
- If $\gamma\colon I \to M$ is smooth, a reparametrization of γ is a curve of the form $\tilde{\gamma} = \gamma \circ \varphi\colon I' \to M$, where $I' \subset \mathbb{R}$ is another interval and $\varphi\colon I' \to I$ is a diffeomorphism.
- Because intervals are connected, φ is either strictly increasing or strictly decreasing on I'.
  We say that $\tilde{\gamma}$ is a forward reparametrization if φ is increasing, and a backward reparametrization if it is decreasing.
- For an admissible curve $\gamma\colon [a,b] \to M$, we define a reparametrization of γ a little more broadly, as a curve of the form $\tilde{\gamma} = \gamma \circ \varphi$, where $\varphi\colon [c,d] \to [a,b]$ is a homeomorphism for which there is a partition $(c_0, \dots, c_k)$ of $[c,d]$ such that the restriction of φ to each subinterval $[c_{i-1}, c_i]$ is a diffeomorphism onto its image.

Definition

If $\gamma\colon [a,b] \to M$ is an admissible curve, we define the length of γ to be

$$L_g(\gamma) := \int_a^b \|\gamma'(t)\|_g \mathrm{d}t.$$

Proposition (Properties of Lengths)

Let (M,g) be a Riemannian manifold and $\gamma\colon [a,c] \to M$ an admissible curve. Then we have

(a) *additivity of length: if $a < b < c$, then $L_g(\gamma) = L_g(\gamma|_{[a,b]}) + L_g(\gamma|_{[b,c]})$;*

(b) *parameter independence: if $\tilde{\gamma}$ is a reparameterization of γ, then $L_g(\gamma) = L_g(\tilde{\gamma})$;*

(c) *isometry invariance: if (M,g) and $(\widetilde{M},\tilde{g})$ are Riemannian manifolds and $\varphi\colon M \to \widetilde{M}$ is a local isometry, then $L_g(\gamma) = L_{\tilde{g}}(\varphi \circ \gamma)$.*

Proof:
(a): Additivity follows from the definition and properties of the integral.

(b): Parameter independence follows from the chain rule together with the bilinearity of g.

(c): Invariance by local isometries follows from the definition of a local isometry together with the rules for integration by substitution. □

Definition

Suppose $\gamma\colon [a,b] \to M$ is an admissible curve.

The arc length function of γ is the function $s\colon [a,b] \to \mathbb{R}$ defined by

$$\begin{aligned} s(t) &:= L_g(\gamma|_{[a,t]}) \\ &= \int_a^t \|\gamma'(u)\|_g \mathrm{d}u. \end{aligned}$$

Corollary

Clearly, s is continuous and it is smooth wherever γ is smooth. Moreover, $s'(t) = \|\gamma'(t)\|_g$.

Definition

By the speed of γ at any $t \in I$ we mean $\|\gamma'(t)\|_g$.[4]

Remark

If $\gamma\colon [a,b] \to M$ is a unit-speed admissible curve, then its arc length function has the simple form $s(t) = t - a$.

If, in addition, its parameter interval is of the form $[0,b]$ for some $b > 0$, then the arc length function is $s(t) = t$.

For this reason, a unit-speed admissible curve whose parameter interval is of the form $[0,b]$ is said to be parameterized by arc length.

[4] Unfortunately, in German the two words speed and velocity have the same translation, namely *Geschwindigkeit*. Consequently, in a mathematical or physical context one says *Geschwindigkeitsvektor* if γ' is meant.

Proposition

Suppose (M, g) is a Riemannian manifold.

(a) *Every regular curve in M has a unit-speed forward reparameterization.*

(b) *Every admissible curve in M has a unique forward reparameterization by arc length.*

Sketch of Proof: The proof is similar to the Euclidean case:

Consider an admissible $c\colon [a, b] \to M$ of length $L = \int_a^b \|c'(t)\|_g \mathrm{d}t$.

Introduce the arc length parameter s by

$$s(t) := \psi(t) = \int_a^t \|c'(\tau)\|_g \mathrm{d}\tau,$$

which defines a map $\psi\colon [a, b] \to [0, L]$.

Now $s'(t) = \psi'(t) = c'(t) \neq 0$ and consequently, there is an inverse function $\varphi := \psi^{-1}$ s.t. $c \circ \varphi = c \circ \psi^{-1}$ is parameterized by arc length.

This parameterization is unique up to translation $s \mapsto s \pm s_0$. $\square$

Definition

If $\gamma\colon [a,b] \to M$ is a smooth curve, we define the energy of γ to be

$$E_g(\gamma) := \tfrac{1}{2}\int_a^b \|\gamma'(t)\|_g^2 \mathrm{d}t.$$

Lemma

For each smooth curve $\gamma\colon [a,b] \to M$ we have

$$L_g(\gamma)^2 \leq 2(b-a)E_g(\gamma),$$

and equality holds if and only if the speed of γ is constant.

Proof.

By Hölder's inequality

$$\int_a^b \|\gamma'(t)\|_g \mathrm{d}t \leq (b-a)^{\frac{1}{2}} \Big(\int_a^b \|\gamma'(t)\|_g^2 \mathrm{d}t \Big)^{\frac{1}{2}}$$

with equality precisely if $\|\gamma'(t)\| \equiv$ constant. □

On Riemannian manifolds we also have the important concept of distance between two points.

Definition (Riemannian distance)

Suppose (M, g) is a connected Riemannian manifold.

For $p, q \in M$ we define the Riemannian distance from p to q, denoted by $d_g(p, q)$, to be the infimum of the lengths of all admissible curves connecting p with q.

That $d_g(p, q) \geq 0$ is well defined ensures the following proposition.

Proposition

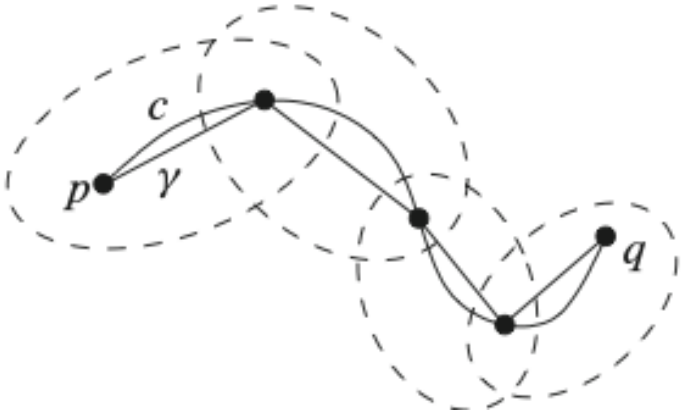
If M is a connected smooth manifold, then any two points of M can be joined by an admissible curve.

Proof: Given arbitrary $p, q \in M$.

As a connected manifold is path-connected as well, we even can connect p with q by a continuous path $c\colon [a, b] \to M$.

By compactness, there is a partition of $[a, b]$ such that the image $c([a_{i-1}, a_i])$ is contained in a single smooth coordinate ball for each i.

We may then replace each such curve segment by a straight-line path in coordinates, yielding ultimately an admissible curve γ between the two points. $\square$



Proposition

Let φ be an isometry between two Riemannian manifolds (M, g) and $(\widetilde{M}, \tilde{g})$.
Then $d_{\tilde{g}}\big(\varphi(x), \varphi(y)\big) = d_g(x, y)$ for all $x, y \in M$.

Proof:
As φ and φ^{-1} preserve the length of curves they are both distance preserving as well. $\square$

The Riemannian distance function turns M into a metric space, whose topology is the same as its original manifold topology.

Lemma

Consider the n-dimensional Riemannian manifold (M, g).

Then the distance function satifies the usual axioms:

(a) $d_g(p, q) \geq 0$ *for all* p, q, *and* $d_g(p, q) > 0$ *for all* $p \neq q$;

(b) $d_g(p, q) = d_g(q, p)$;

(c) $d_g(p, q) \leq d_g(p, r) + d_g(r, q)$ *for all* $p, q, r \in M$ *(triangle inequality).*

Proof: (b),(c) are clear. (a): we only have to show $d_g(p,q) > 0 \,\forall\, p \neq q$.
We work with a chart $\varphi\colon U \to \mathbb{R}^n$ with $p \in U$. There exists an $\varepsilon > 0$ with

$$D_\varepsilon(\varphi(p)) := \left\{ y \in \mathbb{R}^n \;\middle|\; \|y - \varphi(p)\|_{\overline{g}} \leq \varepsilon \right\} \subset \varphi(U)$$

and

$$q \notin \varphi^{-1}\big(D_\varepsilon(\varphi(p))\big). \tag{3}$$

By a slight abuse of notation let the metric be represented by $\big(g_{ij}(x)\big)$ in our chart around p with $\varphi(p) = x$.
Since $\big(g_{ij}(x)\big)$ is positive definite and smooth, hence continuous around x and $D_\varepsilon(x)$ is compact, there exists $\lambda > 0$ with

$$g_{ij}(y)\xi^i\xi^j \geq \lambda^2 \big(\|\xi\|_{\overline{g}}\big)^2 \tag{4}$$

for all $y \in D_\varepsilon(x)$ and $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n$.
Therefore, for any $\gamma\colon [a,b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$

$$L(\gamma) \geq L\Big(\gamma \cap \varphi^{-1}\big(D_\varepsilon(x)\big)\Big) \geq \lambda\varepsilon > 0,$$

because $\varphi(\gamma)$ by (3) has to contain a point $z \in \partial D_\varepsilon(x)$, i.e., a point whose Euclidean distance from x is ε.
By (4), z then has distance from x at least $\lambda\varepsilon$ w.r.t. the metric (g_{ij}). $\square$

Corollary

The topology on (M, g) induced by the distance function d_g coincides with the original manifold topology of M.

Proof: It suffices to show that in each chart the topology induced by d_g coincides with the one of $\mathbb{R}^n$, i.e., by the Euclidean distance function. Now for every x in some chart φ, there exists $\varepsilon > 0$ for which $D_\varepsilon(x)$ is contained in the same chart, and positive constants λ, μ with

$$\lambda^2 \|\xi\|_{\overline{g}}^2 \leq g_{ij}(y)\xi^i\xi^j \leq \mu^2 \|\xi\|_{\overline{g}}^2 \quad \text{for all} \quad y \in D_\varepsilon(x), \xi \in \mathbb{R}^n.$$

Thus

$$\lambda \|y - x\|_{\overline{g}} \leq d_g\big(\varphi^{-1}(y), \varphi^{-1}(x)\big) \leq \mu \|y - x\|_{\overline{g}} \quad \forall\, y \in D_\varepsilon(x),$$

and each Euclidean ball contains a distance ball for d_g, and vice versa. In other words we have with

$$B\big(\varphi^{-1}(z), \delta\big) := \big\{\varphi^{-1}(y) \in M \mid d_g\big(\varphi^{-1}(z), \varphi^{-1}(y)\big) \leq \delta\big\}$$

that

$$\mathring{D}_{\lambda\delta}(x) \subset \mathring{B}\big(\varphi^{-1}(x), \delta\big) \subset \mathring{D}_{\mu\delta}(x). \qquad \square$$

Concepts from metric spaces apply to connected Riemannian manifolds:

Definition

The connected Riemannian manifold (M, g) is called (metrically) complete if every Cauchy sequence in M converges.

A subset $A \subset M$ is bounded if there is a constant C such that $d_g(p, q) \leq C$ for all $p, q \in A$.

In this case the diameter of A is the smallest such constant, i.e.

$$\operatorname{diam}(A) := \sup \left\{ d_g(p, q) \mid p, q \in A \right\}.$$

Example

The sphere $S^2 \subset \mathbb{R}^3$ with **induced** metric has (Riemannian) diameter π (not 2), as the **Riemannian** distance between antipodal points is π: Antipodal points on S^2 are at the same time antipodal points on a great circle on S^2 with diameter 2. The latter measured in the Euclidean metric of the embedding $\mathbb{R}^3$ but a half circle of radius 1 has length π.

Pseudo-Riemannian Metrics

- There is a generalization of Riemannian metrics that has become especially important because of its application not only to physics.
- Before defining this, we recall some linear algebraic preliminaries.

- Suppose V is a finite dimensional vector space, and q is a symmetric covariant 2-tensor on V, i.e., a **symmetric bilinear form**.
- Just as for an inner product, there is an associated linear map

$$\begin{aligned} \hat{q}\colon V &\to V^*, \\ \hat{q}(v)(w) &:= q(v,w) \quad \text{for all} \quad v,w \in V. \end{aligned}$$

- We say that q is nondegenerate if $\hat{q}$ is an isomorphism.

From linear algebra courses we (should) know:

Lemma

Suppose q is a symmetric bilinear form on a finite dimensional vector space V.

The following are equivalent:

1. *q is nondegenerate;*
2. *for every non-zero $v \in V$, there is some $w \in V$ such that $q(v, w) \neq 0$;*
3. *if $q = q_{ij}\varepsilon^i\varepsilon^j$ in terms of basis (ε^j) for V^*, then the matrix (q_{ij}) is invertible.* □

- We review well-known concepts and results from finite dimensional scalar product spaces, not necessarily being inner product ones.
- Although the following results are not necessarily part of standard linear algebra courses we omit proofs and refer to [2], [3], [13].

Lemma

Suppose (V, q) is a finite dimensional scalar product space, and $S \subset V$ is a linear subspace. Then

1. $\dim S + \dim S^{\perp} = \dim V$,
2. $(S^{\perp})^{\perp} = S$. □

Definition

An ordered k-tuple $(v_1, \ldots, v_k)$ of elements of V is said to be **orthonormal** if $\langle v_i, v_j \rangle = \pm\delta_{ij}$ for each i, j.

To prove that every scalar product space has an ONB, note that the usual Gram–Schmidt algorithm does not always work in this situation, because the vectors that appear in the denominators during the procedure might have vanishing norms.

The situation, however, can be easily handled.

Definition

If (V, q) is a finite dimensional scalar product space, a subspace $S \subset V$ is said to be nondegenerate if the restriction $q|_{S \times S}$ is nondegenerate.

An ordered k-tuple of vectors $(v_1, \ldots, v_k)$ in V is said to be nondegenerate if for each $j = 1, \ldots, k$ the vectors $v_1, \ldots, v_j$ span a nondegenerate j-dimensional subspace of V.

Obviously, every orthonormal basis is nondegenerate.

Lemma

Suppose (V, q) is a finite dimensional scalar product space, and $S \subset V$ is a linear subspace.

The following are equivalent:

1. *S is nondegenerate;*
2. *$S^\perp$ is nondegenerate;*
3. *$S \cap S^\perp = \{0\}$;*
4. *$V = S \oplus S^\perp$.* $\square$

Lemma (Completion of Nondegenerate Bases)

Suppose (V, q) is an n-dimensional scalar product space, and $(v_1, \ldots, v_k)$ is a nondegenerate k-tuple in V with $0 \leq k < n$.

Then there exist vectors $v_{k+1}, \ldots, v_n$ such that $(v_1, \ldots, v_n)$ is a nondegenerate basis for V.

Proposition (Gram-Schmidt Procedure for Scalar Product Spaces)

Suppose (V, q) is an n-dimensional scalar product space.

If (v_i) is a nondegenerate basis for V, then there is an orthonormal basis (b_i) with the property that $\operatorname{span}(b_1, \ldots, b_k) = \operatorname{span}(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$.

Proof:

The constructive proof is exactly as in the positive definite case.

However, one has to start with a **nondegenerate** basis, ensuring that the induction will work. □

Corollary

Suppose (V, q) is an n-dimensional scalar product space.

There is a basis (β^i) for V^* with respect to which q has the expression

$$q = (\beta^1)^2 + \cdots + (\beta^r)^2 - (\beta^{r+1})^2 - \cdots - (\beta^{r+s})^2, \tag{5}$$

for some nonnegative integers r, s with $r + s = n$.

Remark

The integers r, s in (5) are independent of the choice of basis.

The key to proving this is the following classical result from linear algebra.

Proposition (Sylvester's Law of Inertia)

Suppose (V, q) is an n-dimensional scalar product space.

If q has the representation (5) in some basis, then the number r is the maximum dimension among all subspaces on which the restriction of q is positive definite.

Thus r and s are independent of the choice of basis.

Definition

The integer s in the expression (5) (the number of negative terms) is called the index of q, and the ordered pair (r, s) is called the signature of q.

Definition (Pseudo-Riemannian Metric)

Suppose M is a smooth manifold.

A pseudo-Riemannian metric on M is a smooth covariant symmetric 2-tensor field g that is nondegenerate at each point of M, and with the same signature everywhere,

(by some authors called semi-Riemannian metric).

Example

Clearly, every Riemannian manifold is pseudo-Riemannian as well.

Example

For nonnegative integers r and s, we define the pseudo-Euclidean space of signature (r, s) denoted by $\mathbb{R}^{r,s}$ to be the manifold $\mathbb{R}^{r+s}$, with standard coordinates denoted by $(\xi^1, \dots, \xi^r, \tau^1, \dots, \tau^s)$, and with the pseudo-Riemannian metric $\overline{q}^{(r,s)}$ defined by

$$\overline{q}^{(r,s)} := (\mathrm{d}\xi^1)^2 + \cdots + (\mathrm{d}\xi^r)^2 - (\mathrm{d}\tau^1)^2 - \cdots - (\mathrm{d}\tau^s)^2.$$

The pseudo-Riemannian metrics of index 1 are called Lorentz metrics.
The pseudo-Euclidean metric $\overline{q}^{(r,1)}$ is a Lorentz metric called the Minkowski metric.
The Lorentz manifold $\mathbb{R}^{r,1}$ is called $(r+1)$-dimensional Minkowski space.
In special relativity $\mathbb{R}^{3,1}$ is important.

Outlook

- Some basic theorems of Riemannian geometry can be generalized to the pseudo-Riemannian case.
- In particular, the socalled **fundamental theorem of Riemannian geometry**, is true for pseudo-Riemannian manifolds as well.
- This allows one to speak of the **Levi-Civita connection** on a pseudo-Riemannian manifold along with the associated **curvature tensor**.
- On the other hand, there are many theorems in Riemannian geometry which do not hold in the generalized case.
- E.g., it is not true that every smooth manifold admits a pseudo-Riemannian metric of a given signature; there are certain topological obstructions.

Pseudo-Riemannian Submanifolds

The theory of submanifolds is only slightly more complicated in the pseudo-Riemannian case.

Definition

If $\left(\widetilde{M}, \tilde{g}\right)$ is a pseudo-Riemannian manifold, a smooth submanifold $\iota\colon M \hookrightarrow \widetilde{M}$ is called a pseudo-Riemannian submanifold of $\widetilde{M}$ if $\iota^*\tilde{g}$ is nondegenerate with constant signature.

If this is the case, we always consider M to be endowed with the induced pseudo-Riemannian metric $\iota^*\tilde{g}$.

In the special case in which $\iota^*\tilde{g}$ is positive definite, M is called a Riemannian submanifold of $\widetilde{M}$.

Example

The nondegeneracy hypothesis is not automatically satisfied:

If $M \subset \mathbb{R}^{1,1}$ is the submanifold $\{(\xi, \tau) \mid \xi = \tau\}$ and $\iota\colon M \to \mathbb{R}^{1,1}$ the inclusion, then the pullback tensor $\iota^* \overline{g}^{1,1}$ is identically zero on M.

More explicitly, M is equal to the one-dimensional subspace

$$\operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \mathbb{R} \cdot \begin{bmatrix}1\\1\end{bmatrix} \cong T_pM$$

for any $p \in M$.

Indeed, for any $v, w \in T_pM$ with $v = \alpha \begin{bmatrix}1\\1\end{bmatrix}$ and $w = \beta \begin{bmatrix}1\\1\end{bmatrix}$, we get for all $\alpha, \beta \in \mathbb{R}$

$$\langle v, w \rangle_{1,1} = \alpha\beta \begin{bmatrix}1 & 1\end{bmatrix} \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix} = 0.$$

Remark

- *Similar as in the Riemannian case we can consider adapted orthonormal frames to a pseudo-Riemannian manifold.*
- *Moreover, the normal bundle to a pseudo-Riemannian submanifold is defined exactly as in the Riemannian case.*

Further Generalizations of Riemannian Metrics

- Relax the requirement that the metric be defined on the whole tangent space:
  A sub-Riemannian metric (also sometimes known as a singular Riemannian metric or Carnot–Carathéodory metric) on a smooth manifold M is a smooth fiber metric on a smooth tangent distribution $S \subset TM$, i.e., on a vector subbundle of TM.
- Relax the requirement that norms of vectors be defined in terms of an inner product on each tangent space:
  Suppose M is a smooth manifold. A Finsler metric on M is a continuous function $F : TM \to \mathbb{R}$, smooth on the set of nonzero vectors, whose restriction to each tangent space T_pM is a norm. Again, the norm function associated with a Riemannian metric is a special case.
- Sub-Riemmanian geometry and Finsler geometry are both areas of active research.

Motivation

- Certain model Riemannian manifolds help to motivate the general theory.
- These manifolds are distinguished by having a high degree of **symmetry**.

Definition

A Riemannian manifold (M, g) is called homogeneous Riemannian manifold if its isometry group $\mathrm{Iso}(M, g)$ acts transitively on M.

In other words, for each pair of points $p, q \in M$ there is an isometry

$$\begin{aligned} \varphi\colon M &\to M, \\ p &\mapsto \varphi(p) = q. \end{aligned}$$

Clearly, $\mathrm{Iso}(M,g)$ does more than just act on M itself:

For every $\varphi \in \mathrm{Iso}(M,g)$, the derivative $\mathrm{D}\,\varphi$ maps TM to itself and restricts to a linear isometry

$$\mathrm{D}\,\varphi(p)\colon T_pM \to T_{\varphi(p)}M, \quad p \in M.$$

Given $p \in M$, let $\mathrm{Iso}_p(M,g)$ denote the isotropy subgroup (or stabilizer) at p, that is, the subgroup of $\mathrm{Iso}(M,g)$ consisting of those isometries that fix p.

Definition

For each $\varphi \in \mathrm{Iso}_p(M,g)$, the linear map $\mathrm{D}\,\varphi(p)$ takes T_pM to itself.

Moreover,

$$I_p\colon \mathrm{Iso}_p(M,g) \to \mathrm{GL}(T_pM), \quad I_p(\varphi) = \mathrm{D}\,\varphi(p)$$

is a (group) representation of $\mathrm{Iso}_p(M,g)$, called the isotropy representation.

Remark

A deep theorem by S.B. Myers and N.E. Steenrod, cf. [17], shows that if M has finitely many components,

then $\mathrm{Iso}(M, g)$ has a topology and smooth structure making it into a finite dimensional Lie group acting smoothly on M.

We will, however, not prove nor use the Myers–Steenrod theorem.

Example (n-dimensional Euclidean space $(\mathbb{R}^n, \overline{g})$)

More generally, if $\big(V, \langle\cdot,\cdot\rangle\big)$ is any real n-dimensional inner product space, set $g(v,w) := \langle v, w\rangle$ for any $p \in V$ and any $v, w \in T_pV \cong V$.

Choosing an ONB $(b_1, \dots, b_n)$ defines a basis isomorphism

$$\mathbb{R}^n \to V, \quad (x^1, \dots, x^n) \mapsto x^i b_i.$$

I.e., an isometry of $(V, g) \cong (\mathbb{R}^n, \overline{g})$, i.e., n-dimensional inner product spaces are isometric to each other as Riemannian manifolds.

Constructions of isometries of $(\mathbb{R}^n, \overline{g})$ are as follows:
The **Euclidean group** $\mathrm{E}_n \cong \mathrm{O}_n \ltimes \mathbb{R}^n$, with multiplication given by $(A, b) \circ (A', b') := (AA', Ab' + b)$ has a faithful representation

$$\rho(\mathrm{E}_n) \to \mathrm{GL}_{n+1}, \quad \rho(A, b) := \left[\begin{smallmatrix} A & b \\ 0 & 1 \end{smallmatrix}\right].$$

E_n acts on $\mathbb{R}^n$ transitively via

$$(A, b) \cdot x := Ax + b.$$

E_n is in fact the full isometry group of $(\mathbb{R}^n, \overline{g})$,
e.g., see chapter 1, section 4.3 of [11] for a proof for the case of $n = 3$.

Example (Spheres)

For $R > 0$ consider the n-dimensional spheres of radius R, denoted by $S^n(R)$, seen as Riemannian submanifolds of $(\mathbb{R}^{n+1}, \overline{g})$.

Clearly, $S^n = S^n(1)$. These spaces are highly symmetric.

We get a large group of isometries of $S^n(R)$ by observing that the linear action of the orthogonal group O_{n+1} on $\mathbb{R}^{n+1}$ preserves $S^n(R)$ and the Euclidean metric as well.

Consequently, its restriction to $S^n(R)$ acts isometrically on the sphere.

From linear algebra it is well-known that this action is tansitive and furthermore O_{n+1} is the full isometry group of $S^n(R)$.

Example (Hyperbolic Spaces)

There is a third family of highly symmetric spaces, the socalled hyperbolic spaces of radius R.

One prominent example is the Poincaré half space together with several mutually isometric siblings, introduced sometimes in a basic course on complex analysis.

We omit here the details, but come back to them in connection with specific examples of pseudo-Riemannian manifolds together with (pseudo-)Riemannian submanifolds.

Invariant Metrics on Lie Groups

Let G be a Lie group.
Recall:
Left translation

$$L_h \colon G \to G,$$
$$L_h(k) := hk,$$

is a diffeomorphism with inverse $L_{h^{-1}}$.

Similarly, right translation, i.e.,

$$R_h \colon G \to G,$$
$$R_h(k) := kh,$$

with inverse $R_{h^{-1}}$.

Definition

A Riemannian metric g on G is said to be left-invariant if it is invariant under all left translations, i.e., $L_h^* g = g$ for all $h \in G$.

Pointwise this means:

$$g(u,v)_k = g\big(\mathrm{D}\,L_h(k)u, \mathrm{D}\,L_h(k)v\big)_{hk},\ \forall h,k \in G,\ \forall u,v \in T_kG.$$

Similarly, g is right-invariant if it is invariant under all right translations, and bi-invariant if it is both left- and right-invariant.

In G, since L_h, R_h are diffeomorphisms for all $h \in G$, it is natural to consider the metrics for which these maps are isometries.

Proposition

There is a bijective correspondence between left-invariant (resp. right-invariant) metrics on a Lie group G, and inner products on the Lie algebra $\mathfrak{g}$ of G.

Proof:
If g is left-invariant, then for all $k \in G$ and all $u, v \in T_k G$, we have

$$\begin{aligned} g(u,v)_k &= \langle u, v \rangle_k \\ &= \left\langle \mathrm{D}\, L_{k^{-1}}(k)u, \mathrm{D}\, L_{k^{-1}}(k)v \right\rangle_e, \end{aligned}$$

i.e., g is completely determined by its restriction to $\mathfrak{g} = T_e G$.
Conversely, let $\langle\langle \cdot, \cdot \rangle\rangle$ be an inner product on $\mathfrak{g}$ and set

$$\begin{aligned} g(u,v)_k &= \langle u, v \rangle_k \\ &:= \left\langle\left\langle \mathrm{D}\, L_{k^{-1}}(k)u, \mathrm{D}\, L_{k^{-1}}(k)v \right\rangle\right\rangle, \forall u, v \in T_k G, k \in G. \end{aligned} \tag{6}$$

Clearly, $\langle \cdot, \cdot \rangle_k$, yields a Riemannian metric on G.

Left-invariance follows by the chain rule and left translations being group isomorphisms:

$$\begin{aligned}
\left\langle \mathrm{D}\, L_h(k)u, \mathrm{D}\, L_h(k)v \right\rangle_{hk} &= \left\langle\!\!\left\langle \mathrm{D}L_{(hk)^{-1}}(hk)\big(\mathrm{D}L_h(k)u\big), \mathrm{D}L_{(hk)^{-1}}(hk)\big(\mathrm{D}L_h(k)v\big) \right\rangle\!\!\right\rangle \\
&= \left\langle\!\!\left\langle \mathrm{D}\left(L_{(hk)^{-1}} \circ L_h\right)(k)u, \mathrm{D}\left(L_{(hk)^{-1}} \circ L_h\right)(k)v \right\rangle\!\!\right\rangle \\
&= \left\langle\!\!\left\langle \mathrm{D}\left(L_{k^{-1}h^{-1}} \circ L_h\right)(k)u, \mathrm{D}\left(L_{k^{-1}h^{-1}} \circ L_h\right)(k)v \right\rangle\!\!\right\rangle \\
&= \left\langle\!\!\left\langle \mathrm{D}\, L_{k^{-1}}(k)u, \mathrm{D}\, L_{k^{-1}}(k)v \right\rangle\!\!\right\rangle = \langle u, v \rangle_k .
\end{aligned}$$

For right-invariance, instead of (6) use

$$g(u,v)_k = \langle u, v \rangle_k := \left\langle\!\!\left\langle \mathrm{D}\, R_{k^{-1}}(k)u, \mathrm{D}\, R_{k^{-1}}(k)v \right\rangle\!\!\right\rangle. \qquad \square$$

Remark

If the Lie group G has dimension n, then since inner products on $\mathfrak{g}$ are in one-to-one correspondence with $n \times n$-positive definite matrices, we see that G possesses a family of left-invariant metrics of dimension $n(n+1)/2$.

Proposition

Every Lie group G equipped with a left-invariant (resp. right-invariant) metric is complete.

Proof:
As G is locally compact, we can pick some $\varepsilon > 0$ small enough so that the closed ε-ball around the identity is compact.

By translation, every ε-ball is compact, hence every Cauchy sequence eventually lies within a compact set, and thus, converges. $\square$

Recall the notion of adjoint representations.

Definition

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

For every $g \in G$, conjugation by g gives a Lie group automorphism $\mathrm{conj}_g \colon G \to G$, called an inner automorphism, by $\mathrm{conj}_g(h) := ghg^{-1}$.

Let

$$\mathrm{Ad}(g) := \mathrm{Ad}_g := (\mathrm{conj}_g)_* \colon \mathfrak{g} \to \mathfrak{g}$$

be the induced Lie algebra automorphism.

More explicitly, with $\mathrm{D}\,\mathrm{conj}_g(h) \colon T_h G \to T_{ghg^{-1}} G$ we get

$$(\mathrm{conj}_g)_* := \mathrm{D}\,\mathrm{conj}_g(e) \colon T_e G = \mathfrak{g} \to T_{geg^{-1}} G = T_e G = \mathfrak{g}\,.$$

It follows from the definition that $\mathrm{conj}_g \circ \mathrm{conj}_h = \mathrm{conj}_{gh}$, and therefore $\mathrm{Ad}_g \circ \mathrm{Ad}_h = \mathrm{Ad}_{gh}$.

In other words, $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$ is a representation, called the adjoint representation of G.

Example (Adjoint Representations of Lie Groups)

1. If G is an Abelian Lie group, then conj_g is the identity map for every $g \in G$.

   So the adjoint representation of G is trivial, i.e., $\mathrm{Ad}_g = \mathrm{id}\,|_{\mathfrak{g}}$ for all $g \in G$.

2. Suppose G is a real matrix Lie group, i.e., a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$.

   Then for each $A \in G$, the conjugation map $\mathrm{conj}_A(B) = ABA^{-1}$ is the restriction to G of a linear map $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$.

   Its derivative, therefore, is given by the same formula, i.e., $\mathrm{Ad}_A(X) = AXA^{-1}$ for every $A \in G$ and $X \in \mathfrak{gl}_n$.

There is also an adjoint representation for Lie algebras:

Definition

If $\mathfrak{g}$ is any Lie algebra, a representation of $\mathfrak{g}$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(V)$, the Lie algebra of all linear endomorphisms of some vector space V.

For each $X \in \mathfrak{g}$, define a map

$$\begin{aligned} \mathrm{ad}_X \colon\ & \mathfrak{g} \to \mathfrak{g}, \\ & Y \mapsto \mathrm{ad}_X(Y) := \mathrm{ad}_X\, Y := [X, Y]. \end{aligned}$$

This defines ad as a map from $\mathfrak{g}$ to $\mathfrak{gl}(\mathfrak{g})$, and a straightforward computation shows that it is a representation of $\mathfrak{g}$.
It is called the adjoint representation of $\mathfrak{g}$.

The two adjoint representations are actually related as the next proposition shows.

Proposition

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra, and let $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ *and* $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *be their respective adjoint representations.*

The induced Lie algebra representation $\mathrm{Ad}_*\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is given by* $\mathrm{Ad}_* = \mathrm{ad}$.

Proof: See Exercises.

Proposition

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

Suppose g is a left-invariant Riemannian metric on G, and let $\langle \cdot, \cdot \rangle$ denote the corresponding inner product on $\mathfrak{g}$.

Then g is bi-invariant iff $\langle \cdot, \cdot \rangle$ is invariant under the action of $\mathrm{Ad}(G) \subset \mathrm{GL}(\mathfrak{g})$,

in the sense that

$$\langle \mathrm{Ad}_h X, \mathrm{Ad}_h Y \rangle = \langle X, Y \rangle \quad \forall \quad X, Y \in \mathfrak{g}, \quad \forall h \in G.$$

Proof:

Suppose g is left-invariant and $\langle \cdot, \cdot \rangle$ is the associated inner product on $\mathfrak{g}$.

Let $h \in G$ be arbitrary, and recall that $\mathrm{conj}_h = L_h \circ R_{h^{-1}}$.

Thus for every $X \in \mathfrak{g}$, left-invariance implies

$$\begin{aligned}(R_{h^{-1}})_* X &= (R_{h^{-1}})_* (L_h)_* X \\ &= (\mathrm{conj}_h)_* X \\ &= \mathrm{Ad}_h X.\end{aligned}$$

Therefore, for all $k \in G$ and $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned}\left((R_{h^{-1}})^* g\right)_k (X_k, Y_k) &= g_{kh^{-1}}\left(\left((R_{h^{-1}})_* X\right)_{kh^{-1}}, \left((R_{h^{-1}})_* Y\right)_{kh^{-1}}\right) \\ &= g_{kh^{-1}}\left(\left(\mathrm{Ad}_h X\right)_{kh^{-1}}, \left(\mathrm{Ad}_h Y\right)_{kh^{-1}}\right) \\ &= \left\langle \mathrm{Ad}_h X, \mathrm{Ad}_h Y \right\rangle.\end{aligned} \tag{7}$$

Now assume $\langle \cdot, \cdot \rangle$ is invariant under $\mathrm{Ad}(G)$.

Then the expression on the last line of (7) is equal to $\langle X, Y \rangle = g_k(X_k, Y_k)$, showing that $(R_{h^{-1}})^* g = g$.

Since this is true for all $h \in G$, it follows that g is bi-invariant.

Conversely, assuming that g is bi-invariant, we have $(R_{h^{-1}})^* g = g$ for each $h \in G$, so the above computation yields

$$\begin{aligned} \langle X, Y \rangle &= g_k(X_k, Y_k) \\ &= \left((R_{h^{-1}})^* g \right)_k (X_k, Y_k) \\ &= \langle \mathrm{Ad}_h X, \mathrm{Ad}_h Y \rangle, \end{aligned}$$

which shows that $\langle \cdot, \cdot \rangle$ is $\mathrm{Ad}(G)$-invariant. $\square$

We mention without proof two existence results:

Theorem (Existence of Bi-invariant Metrics I)

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

Then G admits a bi-invariant metric iff $\mathrm{Ad}(G)$ has compact closure in $\mathrm{GL}(\mathfrak{g})$.

Theorem (Existence of Bi-invariant Metrics II)

A connected Lie group G admits a bi-invariant metric iff it is isomorphic to the cartesian product of a compact group and an additive vector group $\mathbb{R}^n$.

Corollary

Every compact Lie group admits a bi-invariant Riemannian metric. □

For additional discussions of the above results, see the Exercises.

Example

Every left-invariant metric on an Abelian Lie group is bi-invariant, because its adjoint representation is trivial.

Thus e.g. the Euclidean metric on $\mathbb{R}^n$ is bi-invariant.

Example

If a metric g on a Lie group G is left-invariant, then the induced metric on every Lie subgroup $H \subset G$ is left-invariant, as well.

Similarly, if g is bi-invariant, then the induced metric on H is bi-invariant.

Example

The special linear group

$$\mathrm{SL}_2(\mathbb{R}) := \{X \in \mathbb{R}^{2\times 2} \mid \det X = 1\}$$

admits many left-invariant metrics, but **no** bi-invariant ones:

Recall

$$\mathfrak{sl}_2(\mathbb{R}) \cong \{X \in \mathbb{R}^{2\times 2} \mid \operatorname{tr} X = 0\}$$

with adjoint representation given by $\mathrm{Ad}_A(X) = AXA^{-1}$. Consider

$$X := \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in \mathfrak{sl}_2(\mathbb{R}) \text{ and } A_c := \begin{bmatrix} c & 0 \\ 0 & 1/c \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) \text{ with } c > 0$$

then $\mathrm{Ad}_{A_c}(X) = \left[\begin{smallmatrix} 0 & c^2 \\ 0 & 0 \end{smallmatrix}\right]$, which is unbounded as $c \to \infty$.

Thus the A_c-orbit of X is not contained in any compact subset, which by the above theorem implies that there is no bi-invariant metric on $\mathrm{SL}_2(\mathbb{R})$.

Similarly, $\mathrm{SL}_n(\mathbb{R})$ admits no bi-invariant metric for any $n \geq 2$.

In view of our exercises, $\mathrm{GL}_n(\mathbb{R})$ admits no bi-invariant metric for $n \geq 2$.

However, $\mathrm{GL}_1(\mathbb{R})$ does admit bi-invariant metrics because it is Abelian.

Example

Let $\mathfrak{so}_n$ denote the Lie algebra of SO_n identified with the algebra of real skew-symmetric $(n \times n)$-matrices.

Define a bilinear form (so-called trace form or Frobenius inner product) on $\mathfrak{so}_n$ by

$$\langle A, B \rangle := \mathrm{tr}(A^\top B).$$

This is an Ad-invariant inner product, and thus determines a bi-invariant Riemannian metric on SO_n.

Definition (Killing Form)

Given any Lie algebra $\mathfrak{g}$ over $\mathbb{K}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

The Killing form B of $\mathfrak{g}$ is the symmetric $\mathbb{K}$-bilinear form

$$\begin{aligned} B\colon\ & \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \\ & B(u,v) := \operatorname{tr}\big(\operatorname{ad}(u) \circ \operatorname{ad}(v)\big), \quad \text{for all } u, v \in \mathfrak{g}\,. \end{aligned}$$

If $\mathfrak{g}$ is the Lie algebra of a Lie group G, we also refer to B as the Killing form of the group G.

SU_2 as Example I

Consider the special unitary group SU_2 with Lie algebra

$$\mathfrak{su}_2 := \left\{ X \in \mathbb{C}^{2\times 2} \mid -X = \overline{X}^\top, \operatorname{tr} X = 0 \right\}.$$

Any element of this **real** 3-dimensional vector space can be uniquely parameterized as

$$X = \begin{bmatrix} a\imath & b+\imath c \\ -b+\imath c & -a\imath \end{bmatrix}, \quad a, b, c \in \mathbb{R}, \quad \imath := \sqrt{-1}.$$

Consider another element of $\mathfrak{su}_2$

$$Y = \begin{bmatrix} d\imath & e+\imath f \\ -e+\imath f & -d\imath \end{bmatrix}, \quad d, e, f \in \mathbb{R}.$$

We choose a suitable basis of $\mathfrak{su}_2$, namely

$$e_1 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad e_2 := \begin{bmatrix} 0 & \imath \\ \imath & 0 \end{bmatrix}, \quad e_3 := \begin{bmatrix} \imath & 0 \\ 0 & -\imath \end{bmatrix}.$$

SU_2 as Example II

Straight-forward computations give us

$$\begin{aligned} \mathrm{ad}_X(e_1) &= -2ce_3 + 2ae_2, \\ \mathrm{ad}_X(e_2) &= -2ae_1 + 2be_3, \\ \mathrm{ad}_X(e_3) &= 2ce_1 - 2be_2, \end{aligned}$$

implying the representation matrices

$$\begin{aligned} \mathrm{ad}_X &= \begin{bmatrix} 0 & -2a & 2c \\ 2a & 0 & -2b \\ -2c & 2b & 0 \end{bmatrix}, \\ \mathrm{ad}_Y &= \begin{bmatrix} 0 & -2d & 2f \\ 2d & 0 & -2e \\ -2f & 2e & 0 \end{bmatrix}. \end{aligned}$$

SU_2 as Example III

Thus

$$\begin{aligned} B(X,Y) &= \operatorname{tr}\left(\operatorname{ad}_X \circ \operatorname{ad}_Y\right) \\ &= \operatorname{tr}\begin{bmatrix} -4ad-4cf & 4ce & 4ae \\ 4bf & -4ad-4be & 4af \\ 4bd & 4cd & -4be-4cf \end{bmatrix} \\ &= -8ad - 8be - 8cf\,. \end{aligned}$$

However,

$$\operatorname{tr}(XY) = -2ad - 2be - 2cf\,.$$

Hence

$$B(X,Y) = 4\operatorname{tr}(XY).$$

U_2 as Example

Now consider the unitary group U_2, its Lie algebra $\mathfrak{u}_2$ is the 4-dimensional real Lie algebra consisting of all skew-Hermitian (2×2)-matrices; that is, (the $\mathbb{R}$-vector space of) matrices of the form

$$\begin{bmatrix} a\imath & b+\imath c \\ -b+\imath c & \imath d \end{bmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

Choosing the basis

$$e_1 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad e_2 := \begin{bmatrix} 0 & \imath \\ \imath & 0 \end{bmatrix}, \quad e_3 := \begin{bmatrix} \imath & 0 \\ 0 & 0 \end{bmatrix}, \quad e_4 := \begin{bmatrix} 0 & 0 \\ 0 & \imath \end{bmatrix},$$

another straight-forward computation shows

$$B(X, Y) := 4\,\mathrm{tr}(XY) - 2\,\mathrm{tr}(X)\,\mathrm{tr}(Y).$$

Proposition

The following identities hold:

$$\begin{aligned} \mathrm{GL}_n(\mathbb{R}), \mathrm{U}_n: &\quad B(X,Y) = 2n\,\mathrm{tr}(XY) - 2\,\mathrm{tr}(X)\,\mathrm{tr}(Y), \\ \mathrm{SL}_n(\mathbb{R}), \mathrm{SU}_n: &\quad B(X,Y) = 2n\,\mathrm{tr}(XY), \\ \mathrm{SO}_n: &\quad B(X,Y) = (n-2)\,\mathrm{tr}(XY). \end{aligned}$$

Sketch of Proof:
It suffices to compute the quadratic form $B(X,X)$, because $B(X,Y)$ is symmetric bilinear so it can be recovered from the polarization identity, i.e.,

$$B(X,Y) = \tfrac{1}{2}\big(B(X+Y,X+Y) - B(X,X) - B(Y,Y)\big).$$

□

Proposition

The Killing form B of a Lie algebra $\mathfrak{g}$ has the following properties.

(a) *It is a symmetric bilinear form invariant under all automorphisms of $\mathfrak{g}$. In particular, if $\mathfrak{g}$ is the Lie algebra of a Lie group G, then B is Ad_g-invariant, for all $g \in G$; that is*

$$B\big(\mathrm{Ad}_g(u), \mathrm{Ad}_g(v)\big) = B(u,v), \quad \text{for all } u, v \in \mathfrak{g} \text{ and all } g \in G.$$

(b) *The linear map ad_u is skew-adjoint w.r.t B for all $u \in \mathfrak{g}$; that is,*

$$\begin{aligned} B\big(\mathrm{ad}_u v, w\big) &= B\big([u,v],w\big) \\ &= -B\big(v,[u,w]\big) \\ &= -B\big(v, \mathrm{ad}_u w\big), \text{ for all } v, w \in \mathfrak{g}. \end{aligned}$$

Proof:
To prove a), note that the form B is bilinear, and as $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, it is symmetric.

If φ is an automorphism of $\mathfrak{g}$, the preservation of the Lie bracket implies

$$\operatorname{ad}_{\varphi(u)} \circ \varphi = \varphi \circ \operatorname{ad}_u \quad \Longrightarrow \quad \operatorname{ad}_{\varphi(u)} = \varphi \circ \operatorname{ad}_u \circ \varphi^{-1}.$$

From $\operatorname{tr}(XY) = \operatorname{tr}(YX)$, we get $\operatorname{tr}(A) = \operatorname{tr}(BAB^{-1})$, therefore

$$\begin{aligned} B\big(\varphi(u), \varphi(v)\big) &= \operatorname{tr}\left(\operatorname{ad}_{\varphi(u)} \circ \operatorname{ad}_{\varphi(v)}\right) \\ &= \operatorname{tr}\left(\varphi \circ \operatorname{ad}_u \circ \varphi^{-1} \circ \varphi \circ \operatorname{ad}_v \circ \varphi^{-1}\right) \\ &= \operatorname{tr}\left(\operatorname{ad}_u \circ \operatorname{ad}_v\right) \\ &= B(u, v). \end{aligned}$$

Since Ad_g is an automorphism of $\mathfrak{g}$ for all $g \in G$,
B is indeed Ad_g-invariant.

To prove b), we have

$$\begin{aligned} B(\operatorname{ad}_u v, w) &= B\big([u,v], w\big) \\ &= \operatorname{tr}\big(\operatorname{ad}_{[u,v]} \circ \operatorname{ad}_w\big), \\ B(v, \operatorname{ad}_u w) &= B\big(v, [u,w]\big) \\ &= \operatorname{tr}\big(\operatorname{ad}_v \circ \operatorname{ad}_{[u,w]}\big). \end{aligned}$$

However, the Jacobi identity is equivalent to

$$\operatorname{ad}_{[u,v]} = \operatorname{ad}_u \circ \operatorname{ad}_v - \operatorname{ad}_v \circ \operatorname{ad}_u .$$

Consequently,

$$\begin{aligned}
\operatorname{tr}\left(\operatorname{ad}_{[u,v]} \circ \operatorname{ad}_w\right) &= \operatorname{tr}\left((\operatorname{ad}_u \circ \operatorname{ad}_v - \operatorname{ad}_v \circ \operatorname{ad}_u) \circ \operatorname{ad}_w\right) \\
&= \operatorname{tr}(\operatorname{ad}_u \circ \operatorname{ad}_v \circ \operatorname{ad}_w) - \operatorname{tr}(\operatorname{ad}_v \circ \operatorname{ad}_u \circ \operatorname{ad}_w), \\
\operatorname{tr}\left(\operatorname{ad}_v \circ \operatorname{ad}_{[u,w]}\right) &= \operatorname{tr}\left((\operatorname{ad}_v \circ (\operatorname{ad}_u \circ \operatorname{ad}_w - \operatorname{ad}_w \circ \operatorname{ad}_u)\right) \\
&= \operatorname{tr}(\operatorname{ad}_v \circ \operatorname{ad}_u \circ \operatorname{ad}_w) - \operatorname{tr}(\operatorname{ad}_v \circ \operatorname{ad}_w \circ \operatorname{ad}_u).
\end{aligned}$$

As

$$\operatorname{tr}(\operatorname{ad}_u \circ \operatorname{ad}_v \circ \operatorname{ad}_w) = \operatorname{tr}(\operatorname{ad}_v \circ \operatorname{ad}_w \circ \operatorname{ad}_u),$$

we deduce that

$$\begin{aligned}
B(\operatorname{ad}_u v, w) &= \operatorname{tr}(\operatorname{ad}_{[u,v]} \circ \operatorname{ad}_w) \\
&= -\operatorname{tr}(\operatorname{ad}_v \circ \operatorname{ad}_{[u,w]}) \\
&= -B(v, \operatorname{ad}_u w),
\end{aligned}$$

as claimed. □

There are many homogeneous Riemannian manifolds besides the Lie groups with invariant metrics.

To identify more examples, it is natural to ask the following question:

If M is a smooth manifold endowed with a smooth, transitive action by a Lie group G (called a homogeneous G-space or just a homogeneous space), is there a Riemannian metric on M that is invariant under the group action?

The next theorem gives a necessary and sufficient condition for existence of an invariant Riemannian metric that is usually easy to check.

Theorem (Existence of Invariant Metrics on Homogeneous Spaces)

Suppose G is a Lie group and M is a homogeneous G-space.

Let p_0 be a point in M, and let $I_{p_0} \colon G_{p_0} \to \mathrm{GL}(T_{p_0}M)$ denote the isotropy representation at p_0.

There exists a G-invariant Riemannian metric on M iff $I_{p_0}(G_{p_0})$ has compact closure in $\mathrm{GL}(T_{p_0}M)$.

Sketch of Proof:
Assume first that g is a G-invariant metric on M.

Then the inner product g_{p_0} on T_pM is invariant under the isotropy representation.

So it essentially follows from lemma 3.13 in [19] that $I_{p_0}(G_{p_0})$ has compact closure in $\mathrm{GL}(T_{p_0}M)$.

For the converse one applies the socalled equivariant rank theorem which was strived earlier in these lectures. □

Corollary

If a Lie group G acts smoothly and transitively on a smooth manifold M with compact isotropy groups, then there exists a G-invariant Riemannian metric on M. □

Some Outlook

There is a certain class of Riemannian manifolds with abundant symmetry, called symmetric spaces.

They were completely classified by E. Cartan [6, 7].

Definition

If (M, g) is a Riemannian manifold and $p \in M$, a point reflection at p is an isometry $\varphi\colon M \to M$ that fixes p and satisfies

$$\mathrm{D}\,\varphi_p = -\,\mathrm{id}\colon T_pM \to T_pM.$$

A Riemannian manifold (M, g) is called a (Riemannian) symmetric space if it is connected and for each $p \in M$ there exists a point reflection at p.

- The modifier "Riemannian" is included to distinguish such spaces from other kinds of symmetric spaces that can be defined, such as e.g. pseudo-Riemannian symmetric spaces.
- Since we will be concerned currently only with Riemannian symmetric spaces, we will sometimes refer to them simply as "symmetric spaces" for brevity.
- But note that pseudo-Riemannian symmetric spaces have not been completely classified yet.
- Lateron, we will learn about tools to prove, that every Riemannian symmetric space is homogeneous.
- More generally, (M, g) is called a (Riemannian) locally symmetric space if each $p \in M$ has a neighborhood U on which there exists an isometry $\varphi\colon U \to U$ that is a point reflection at p.
- Clearly every Riemannian symmetric space is locally symmetric.

The next lemma can be used to facilitate the verification that a given Riemannian manifold is symmetric.

Lemma

If (M, g) is a connected homogeneous Riemannian manifold that possesses a point reflection at one point, then it is symmetric.

Proof:
Suppose (M, g) satisfies the hypothesis, and let $\varphi\colon M \to M$ be a point reflection at $p \in M$.

Given any other point $q \in M$, by homogeneity there is an isometry $\psi\colon M \to M$ satisfying $\psi(p) = q$.

Then $\widetilde{\varphi} := \psi \circ \varphi \circ \psi^{-1}$ is an isometry that fixes q.

Because $\mathrm{D}\,\psi_p$ is linear, it commutes with multiplication by -1, so

$$\begin{aligned}
\mathrm{D}\,\widetilde{\varphi}_q &= \mathrm{D}\,\psi_p \circ (-\operatorname{id}_{T_pM}) \circ \mathrm{D}\left(\psi^{-1}\right)_q \\
&= (-\operatorname{id}_{T_qM}) \circ \mathrm{D}\,\psi_p \circ \mathrm{D}\left(\psi^{-1}\right)_q \\
&= -\operatorname{id}_{T_qM}\,.
\end{aligned}$$

Thus $\widetilde{\varphi}$ is a point reflection at q. $\square$

Example (Lie groups as symmetric spaces)

Let (G, g) be a connected Lie group with bi-invariant Riemannian metric.

Define $\varphi\colon G \to G$ by $\varphi(x) := x^{-1}$, then $\mathrm{D}\,\varphi_e(v) = -v$ for every $v \in T_eG$. Consequently, for the pointwise pullback of the metric tensor g by φ at e we have $\mathrm{D}\,\varphi_e^*(g_e) = g_e$, or more explicitly for all $v, w \in \mathfrak{g}$:

$$\mathrm{D}\,\varphi_e^*(g_e)(v, w) = g_e(\mathrm{D}\,\varphi_e \cdot v, \mathrm{D}\,\varphi_e \cdot w) = g_e(-v, -w) = g_e(v, w),$$

To see that φ is an isometry, let $p \in G$ be arbitrary.

The identity $q^{-1} = (p^{-1}q)^{-1}p^{-1}$ for all $q \in G$ implies $\varphi = R_{p^{-1}} \circ \varphi \circ L_{p^{-1}}$, and therefore it follows from bi-invariance of g that

$$(\varphi^* g)_p = \mathrm{D}\,\varphi_p^* g_{p^{-1}} = \mathrm{D}(L_{p^{-1}})_p^* \circ \mathrm{D}\,\varphi_e^* \circ \mathrm{D}(R_{p^{-1}})_e^* g_{p^{-1}} = g_p.$$

Therefore φ is an isometry of g and hence a point reflection at e.

The above lemma then implies that (G, g) is a symmetric space.

Example (Spheres as symmetric spaces)

Consider S^{n-1} as Riemannian submanifold of the Euclidean space $(\mathbb{R}^n, \overline{g})$.
Let $p \in S^{n-1}$ and define

$$\varphi\colon S^{n-1} \to S^{n-1}, \quad \varphi(q) := (2pp^\top - I)q.$$

Obviously,

$$\varphi(p) = p \quad \text{and} \quad 2pp^\top - I \in \mathrm{O}_n\,.$$

Therefore φ is an isometry of $\overline{g}|_{TS^{n-1}}$.
Moreover, as we have

$$\mathrm{D}\,\varphi_p\colon T_pS^{n-1} \to T_{\varphi(p)}S^{n-1}, \quad v \mapsto (2pp^\top - I)v = -v$$

we get $\mathrm{D}\,\varphi_p = -\,\mathrm{id} = -I$, so φ is a point reflection as well.
Note that $p \in S^{n-1}$ also lies in the normal space $(T_pS^{n-1})^\perp$, i.e., $p^\top v = \overline{g}(p, v) = 0$.
Consequently, S^{n-1} is a symmetric space.

Example

Every product of Riemannian symmetric spaces is easily seen to be a symmetric space when endowed with the product metric.

Definition

A symmetric space is said to be irreducible if it is not isometric to a product of positive-dimensional symmetric spaces.

Motivations for Connections

- To define notion of **curvature** for arbitrary (Pseudo-)Riemannian manifolds.
- To relate to other geometric and topological properties.
- To study **geodesics**, generalizations to Riemannian manifolds of **straight lines** in Euclidean space.
- Two properties of straight lines in $\mathbb{R}^n$, either of which serves to characterize them uniquely:
  (1) Every segment of a straight line is the unique shortest path between its endpoints (sounds geometric),
  (2) straight lines are the only curves that have parametrizations with zero acceleration (needs new concept).
- To make sense of acceleration on a manifold, introduce new object called a **connection**.
- Essentially coordinate-independent set of rules, for taking directional derivatives of vector fields.

Motivation, cont'd

Consider smooth curve

$$\gamma\colon I \to \mathbb{R}^n,$$

in standard coordinates as

$$\gamma(t) = \big(\gamma^1(t), \ldots, \gamma^n(t)\big).$$

$\gamma(t)$ has well-defined velocity $\gamma'(t)$ and acceleration $\gamma''(t)$ at each $t \in I$, computed by differentiating the components:

$$\begin{aligned}\gamma'(t) &= \dot\gamma^1(t)\tfrac{\partial}{\partial x^1}\big|_{\gamma(t)} + \cdots + \dot\gamma^n(t)\tfrac{\partial}{\partial x^n}\big|_{\gamma(t)},\\ \gamma''(t) &= \ddot\gamma^1(t)\tfrac{\partial}{\partial x^1}\big|_{\gamma(t)} + \cdots + \ddot\gamma^n(t)\tfrac{\partial}{\partial x^n}\big|_{\gamma(t)}.\end{aligned}$$

We use dots to denote ordinary derivatives w.r.t. t when there are superscripts that would make primes hard to read.

Fact: $\gamma\colon I \to \mathbb{R}^n$ is a straight line iff it has a parametrization with $\gamma''(t) \equiv 0$.

Directional derivatives of vector fields on $\mathbb{R}^n$ we get by computing ordinary directional derivatives of the component functions in standard coordinates:

Definition

Given $Y \in \mathfrak{X}(\mathbb{R}^n)$ and $v \in T_p\mathbb{R}^n$, define the Euclidean directional derivative of Y in the direction v by

$$\overline{\nabla}_v Y := v(Y^1)\tfrac{\partial}{\partial x^1}\big|_p + \cdots + v(Y^n)\tfrac{\partial}{\partial x^n}\big|_p.$$

For each i, $v(Y^i)$ is the result of applying v to function Y^i:

$$v(Y^i) := v^1 \frac{\partial Y^i(p)}{\partial x^1} + \cdots + v^n \frac{\partial Y^i(p)}{\partial x^n}.$$

Consider another $X \in \mathfrak{X}(\mathbb{R}^n)$.

We obtain a new vector field $\overline{\nabla}_X Y$ by evaluating $\overline{\nabla}_{X_p} Y$ at each point:

$$\overline{\nabla}_X Y := X(Y^1)\tfrac{\partial}{\partial x^1} + \cdots + X(Y^n)\tfrac{\partial}{\partial x^n}. \tag{8}$$

- More generally, we can play the same game with curves and vector fields on a **submanifold** of $\mathbb{R}^n$.
- Suppose $M \subset \mathbb{R}^n$ is an embedded submanifold, and consider a smooth $\gamma\colon I \to M$.
- Think of a **geodesic** in M as a curve in M that is "as straight as possible".
- Of course, if M itself is curved, then $\gamma'(t)$, thought of as a vector in $\mathbb{R}^n$, will probably have to vary, or else the curve will leave M.
- But we can try to insist that the velocity not change any more than necessary for the curve to stay in M.
- One way to do this is to compute the Euclidean acceleration $\gamma''(t)$, and apply the **tangential projection** $\pi^\top\colon T_{\gamma(t)}\mathbb{R}^n \to T_{\gamma(t)}M$.

Definition

The vector

$$\gamma''(t)^\top := \pi^\top\big(\gamma''(t)\big)$$

tangent to M is called the tangential acceleration of γ.

Remark

We say that γ is as straight as it is possible for a curve in M to be if its tangential acceleration is zero.

Question:
For $Y \in \mathfrak{X}(U)$ with $U \subset M$ open, how much is Y varying *in* M in the direction of a vector $v \in T_pM$?

We look at this problem from the point of view of the embedding $\mathbb{R}^n$:

- The vector field Y might be forced to vary just so that it can remain tangent to M.
- A plausible way to answer the question is
  - **extend** Y to a smooth vector field $\widetilde{Y}$ on an open subset of $\mathbb{R}^n$,
  - compute the Euclidean directional derivative of $\widetilde{Y}$ in the direction v,
  - project orthogonally onto T_pM.

Definition

Define the tangential directional derivative of Y in the direction v to be

$$\nabla_v^\top Y := \pi^\top(\overline{\nabla}_v \widetilde{Y}).$$

Remark

- *Clearly, the tangential directional derivative is well defined and preserved by rigid motions of* $\mathbb{R}^n$.
- *Is there any reason why the tangential directional derivative is an intrinsic invariant of* M*, i.e., one that depends only on the Riemannian geometry of* M *with its induced metric?*
- *On an abstract Riemannian manifold, for which there is a priori no ambient Euclidean space in which to differentiate, this technique is not available!*

Bad news:

- The velocity of γ is a well-defined vector $\gamma'(t) \in T_{\gamma(t)}M$, whose representation in any coordinates is given just as in Euclidean space.
- However, unlike velocity, acceleration has **no** such coordinate-independent interpretation.

Example

Consider $S^1 \subset \mathbb{R}^2$ by $\gamma(t) := \big(x(t), y(t)\big) = \big(\cos(t), \sin(t)\big)$.

As a smooth curve in $\mathbb{R}^2$, it has an acceleration vector at time t given by

$$\gamma''(t) = x''(t)\tfrac{\partial}{\partial x}\big|_{\gamma(t)} + y''(t)\tfrac{\partial}{\partial y}\big|_{\gamma(t)} = -\cos(t)\tfrac{\partial}{\partial x}\big|_{\gamma(t)} - \sin(t)\tfrac{\partial}{\partial y}\big|_{\gamma(t)}.$$

In polar coordinates γ is described by $\big(r(t), \theta(t)\big) = (1, t)$.

In these coordinates, if we try to compute the acceleration vector by the analogous formula, we get

$$\gamma''(t) = r''(t)\tfrac{\partial}{\partial r}\big|_{\gamma(t)} + \theta''(t)\tfrac{\partial}{\partial \theta}\big|_{\gamma(t)} = (0, 0).$$

The problem is this:

- To define $\gamma''(t)$ by differentiating $\gamma'(t)$ with respect to t, we have to take a limit of a difference quotient involving the vectors $\gamma'(t+h)$ and $\gamma'(t)$.
- But these live in **different** vector spaces, $T_{\gamma(t+h)}M$ and $T_{\gamma(t)}M$, respectively.
- It does not make sense to subtract them.
- The definition of acceleration works in the special case of smooth curves in $\mathbb{R}^n$ expressed in standard coordinates, because

$$T_{\gamma(t)}\mathbb{R}^n \cong T_{\gamma(t+h)}\mathbb{R}^n \cong \mathbb{R}^n.$$

- On a general smooth manifold, there is no such natural identification.

Remark

The velocity vector $\gamma'(t)$ is an example of a **vector field along a curve**, *a concept for which we will give a rigorous definition in due course.*

Definition (Connection)

A connection in TM (connection in M) is a map

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (X, Y) \mapsto \nabla_X Y, \tag{9}$$

satisfying:

(a) $\nabla_X Y$ is linear over $C^\infty(M)$ in X: for $f_1, f_2 \in C^\infty(M)$ and $X_1, X_2 \in \mathfrak{X}(M)$,

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y; \tag{10}$$

(b) $\nabla_X Y$ is linear over $\mathbb{R}$ in Y: for $a_1, a_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \mathfrak{X}(M)$,

$$\nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2; \tag{11}$$

(c) ∇ satisfies the following product rule: for $f \in C^\infty(M)$,

$$\nabla_X (fY) = f \nabla_X (Y) + (Xf) Y. \tag{12}$$

The symbol $\nabla_X Y$ ("∇" is called "nabla") is called the covariant derivative of Y in the direction X.

A connection is defined by its action on *global* vector fields.

However, a connection is actually a *local* operator:

Lemma (Locality)

Suppose ∇ is a connection in M.

For every $X, Y \in \mathfrak{X}(M)$ and $p \in M$, the covariant derivative $\nabla_X Y|_p$ depends only on the values of X and Y in an arbitrarily small neighborhood of p.

More precisely, if $X = \widetilde{X}$ and $Y = \widetilde{Y}$ on a neighborhood of p, then $\nabla_X Y|_p = \nabla_{\widetilde{X}} \widetilde{Y}|_p$.

Proof: First consider Y. Replacing Y by $Y - \widetilde{Y}$ shows that it suffices to prove $\nabla_X Y|_p = 0$ if Y vanishes on a neighborhood of p.

Thus suppose Y is a smooth vector field that is identically zero on a neighborhood U of p.

Choose a bump function $\varphi \in C^\infty(M)$ with support in U s.t. $\varphi(p) = 1$.

The hypothesis that Y vanishes on U implies that $\varphi Y \equiv 0$ on all of M, so for every $X \in \mathfrak{X}(M)$ we have

$$\nabla_X(\varphi Y) = \nabla_X(0 \cdot \varphi Y) = 0\nabla_X(\varphi Y) = 0.$$

By the product rule:

$$0 = \nabla_X(\varphi Y) = (X\varphi)Y + \varphi(\nabla_X Y). \qquad (\dagger)$$

Now $Y \equiv 0$ on the support of φ, so the first term on the right is identically zero.

Evaluating $(\dagger)$ at p shows that $\nabla_X Y|_p = 0$.

The argument for X is similar. $\square$

Proposition (Restriction of a Connection)

Suppose ∇ is a connection in M.

For every open $U \subset M$, there is a unique connection ∇^U on the restricted tangent bundle TU satisfying for every $X, Y \in \mathfrak{X}(M)$:

$$\nabla^U_{(X|_U)}(Y|_U) = (\nabla_X Y)|_U. \tag{13}$$

Proof:
Uniqueness: Let ∇^U any such connection and $X, Y \in \mathfrak{X}(U)$ are arbitrary.

Given $p \in U$, use a bump function to construct smooth $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(M)$, s.t. $\widetilde{X}|_U$ agrees with X and $\widetilde{Y}|_U$ agrees with Y on some neighborhood of p.

Then the above Lemma together with (13) implies

$$\nabla^U_X Y|_p = \nabla^U_{(\widetilde{X}|_U)}(\widetilde{Y}|_U)|_p = (\nabla_{\widetilde{X}}\widetilde{Y})|_p. \tag{14}$$

Since R.H.S. is completely determined by ∇, ∇^U is uniquely defined if it exists.
Existence: Given $X, Y \in \mathfrak{X}(U)$, for every $p \in U$ we just construct $\widetilde{X}, \widetilde{Y}$ as above, and define $\nabla^U_X Y|_p$ by (14).

This is independent of the choices of $\widetilde{X}, \widetilde{Y}$ by the above Lemma, and it is smooth because the same formula holds on some neighborhood of p.

The see that it satisfies the connection properties is straight-forward. $\square$

We can strengthen the above Proposition:

Proposition

Under the hypotheses of Lemma 404, $\nabla_X Y|_p$ depends only on the values of Y in a neighborhood of p and the value of X at p.

Proof: The claim about Y was proved in the above Lemma.

To prove the claim about X, it suffices by linearity to assume that $X_p = 0$ and show that $\nabla_X Y|_p = 0$.

Choose a coordinate neighborhood U of p, and write $X = X^i \partial_i$ in coordinates on U , with $X^i(p) = 0$.

Thanks to the Proposition, it suffices to work with the restricted connection on U, which we also denote by ∇.

For every $Y \in \mathfrak{X}(U)$, we have

$$\nabla_X Y|_p = \nabla_{X^i \partial_i} Y|_p = X^i(p) \nabla_{\partial_i} Y|_p = 0.$$

For computations, we need to examine how a connection appears in terms of a local frame.

Definition

Let (E_i) be a smooth local frame for TM on an open subset $U \subset M$.

For every choice of the indices i, j, we can expand the vector field $\nabla_{E_i} E_j$ in terms of this same frame:

$$\nabla_{E_i} E_j = \Gamma_{ij}^k E_k.$$

As $i, j, k = 1, \ldots, n$ with $n = \dim M$, this defines n^3 smooth functions $\Gamma_{ij}^k \colon U \to \mathbb{R}$ called the connection coefficients of ∇ w.r.t. the given frame.

Proposition

Let M be a smooth manifold, and let ∇ be a connection in TM.

Suppose (E_i) is a smooth local frame over an open $U \subset M$.

Let $\{\Gamma_{ij}^k\}$ be the connection coefficients of ∇ w.r.t. this frame.

For smooth vector fields $X, Y \in \mathfrak{X}(U)$, written in terms of the frame as $X = X^i E_i$, $Y = Y^j E_j$, respectively, one has

$$\nabla_X Y = \Big(X(Y^k) + X^i Y^j \Gamma_{ij}^k \Big) E_k .$$

Proof:
We use the defining properties of a connection and compute:

$$\begin{aligned}
\nabla_X Y &= \nabla_X (Y^j E_j) \\
&= X(Y^j) E_j + Y^j \nabla_{X^i E_i} E_j \\
&= X(Y^j) E_j + X^i Y^j \nabla_{E_i} E_j \\
&= X(Y^j) E_j + X^i Y^j \Gamma_{ij}^k E_k \\
&= X(Y^k) E_k + X^i Y^j \Gamma_{ij}^k E_k .
\end{aligned}$$

$\square$

Existence of Connections

Example (The Euclidean Connection)

In $T\mathbb{R}^n$, define the Euclidean connection $\overline{\nabla}$ by

$$\overline{\nabla}_X Y := X(Y^1)\frac{\partial}{\partial x^1} + \cdots + X(Y^n)\frac{\partial}{\partial x^n}. \tag{15}$$

This satisfies the required properties for a connection, and its connection coefficients in the standard coordinate frame are all zero.

Example (The Tangential Connection on a Submanifold $M \subset \mathbb{R}^n$)

Define the tangential connection $\nabla^\top$ on TM

$$\nabla^\top_X Y := \pi^\top\Big(\overline{\nabla}_{\widetilde{X}} \widetilde{Y}|_M\Big). \tag{16}$$

With $\pi^\top$ the orthogonal projection onto TM, $\overline{\nabla}$ the Euclidean connection on $\mathbb{R}^n$, and $\widetilde{X}, \widetilde{Y}$ smooth extensions of X, Y to an open set in $\mathbb{R}^n$.

Since the value of $\overline{\nabla}_{\widetilde{X}} \widetilde{Y}$ at $p \in M$ depends only on $\widetilde{X}_p = X_p$, it boils down to defining $(\nabla^\top_X Y)_p := \nabla^\top_{X_p} Y$, defined before.

It is independent of the choice of extension $\widetilde{Y}$, so $\nabla^\top$ is well defined.

To see smoothness express $\overline{\nabla}_{\widetilde{X}} \widetilde{Y}$ by an adapted orthonormal frame.

$\nabla^\top_X Y$ is linear over $C^\infty(M)$ in X and over $\mathbb{R}$ in Y. The product rule still needs to be checked. Let $f \in C^\infty(M)$, and let $\tilde{f}$ be an extension of f to a neighborhood of M in $\mathbb{R}^n$. Then $\tilde{f}\widetilde{Y}$ is a smooth extension of fY to a neighborhood of M, so

$$\begin{aligned}\nabla^\top_X(fY) &= \pi^\top\Big(\overline{\nabla}_{\widetilde{X}}(\tilde{f}\widetilde{Y})\big|_M\Big) = \pi^\top\Big((\widetilde{X}\tilde{f})\,\widetilde{Y}\big|_M\Big) + \pi^\top\Big(\tilde{f}\overline{\nabla}_{\widetilde{X}}\widetilde{Y}\big|_M\Big)\\ &= (Xf)Y + f\nabla^\top_X Y.\end{aligned}$$

There are many connections on $\mathbb{R}^n$, in fact on every smooth manifold that admits a global frame, e.g., if a manifold is covered by a single chart.

Lemma

Suppose M is a smooth n-manifold, and M admits a global frame (E_i).

$$\nabla_X Y = \left(X(Y^k) + X^i Y^j \Gamma_{ij}^k\right) E_k \tag{17}$$

already gave a one-to-one correspondence between connections in TM and choices of n^3 smooth real-valued functions $\{\Gamma_{ij}^k\}$ on M.

Sketch of Proof:
Every connection determines functions $\{\Gamma_{ij}^k\}$, and they satisfy (17).
On the other hand, given $\{\Gamma_{ij}^k\}$, we can define $\nabla_X Y$ by (17).
The resulting expression is smooth if X and Y are smooth, linear over $\mathbb{R}$ in Y, and linear over $C^\infty(M)$ in X.
To prove that it is a connection, the product rule requires checking. □

Remark

By a partition of unity argument one can show that the tangent bundle of every smooth manifold admits a connection.

Covariant Derivatives of Tensor Fields

- By definition, a connection in TM is a rule for computing covariant derivatives of vector fields.
- We will show that every connection in TM automatically induces connections in all tensor bundles over M.
- Thus it gives us a way to compute covariant derivatives of tensor fields of any type, e.g. of the Riemannian metric.
- We mention here several results without proof but refer to [19] for more details, and in particular to appendix B therein for a review of the concept of a tensor (field).

Proposition

Let M be a smooth manifold and let ∇ be a connection in TM.

Then ∇ uniquely determines a connection in each tensor bundle $T^{(k,l)}TM$, also denoted by ∇, such that the following four conditions are satisfied.

1. *In $T^{(1,0)}TM = TM$, ∇ agrees with the given connection;*
2. *in $T^{(0,0)}TM = M \times \mathbb{R}$, ∇ is given by ordinary differentiation of functions, i.e., $\nabla_X f = Xf$;*
3. *∇ obeys the following product rule with respect to tensor products:*

$$\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G); \tag{18}$$

4. *∇ commutes with all contractions: if tr denotes a trace on any pair of indices, one covariant and one contravariant, then*

$$\nabla_X(\operatorname{tr} F) = \operatorname{tr}(\nabla_X F). \tag{19}$$

Moreover, a connection as in our Prop. satisfies the following properties:

- ∇ obeys the following product rule with respect to the natural pairing[5] between a covector field ω and a vector field Y:

$$\nabla_X \langle \omega, Y \rangle = \langle \nabla_X \omega, Y \rangle + \langle \omega, \nabla_X Y \rangle;$$

- consider the smooth sections $\Gamma(T^{(k,l)}TM)$. Then for all $F \in \Gamma(T^{(k,l)}TM)$, smmoth 1-forms $\omega^1 \ldots, \omega^k$ and smooth vector fields $Y_1, \ldots, Y_l$,

$$\begin{aligned}(\nabla_X F)(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l) &= X(F(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l)) \\ &\quad - \sum_{i=1}^{k} F(\omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^k, Y_1, \ldots, Y_l) \\ &\quad - \sum_{j=1}^{l} F(\omega^1, \ldots, \omega^k, Y_1, \ldots, \nabla_X Y_j, \ldots, Y_l).\end{aligned}$$

[5] By abuse of notation, the natural pairing between a covector field and a vector field, i.e., $T^*M \ni \omega \colon TM \to \mathbb{R}$, given by $Y \mapsto \omega(Y)$ is often rewritten as $\omega(Y) =: \langle \omega, Y \rangle$. A misunderstanding due to confusion with the notation of the metric is unlikely.

- The Proposition is useful for ensuring existence and uniqueness of the connections in tensor bundles
- It is not too practical for computation, because computing the value of $\nabla_X F$ at a point requires extending all of its arguments to vector fields and covector fields in an open set, and computing a great number of derivatives.
- For computing the components of a covariant derivative in terms of a local frame, the formulas in the following proposition are more useful.

Proposition

Let ∇ be a connection in TM, (E_i) a local frame, (ε^j) its dual, and $\{\Gamma^k_{ij}\}$ the connection coefficients.

Let $X \in \mathfrak{X}(M)$ and $X^i E_i$ its local expression.

1. *The covariant derivative of a 1-form $\omega = \omega_i \varepsilon^i$ is given locally by*
$$\nabla_X(\omega) = \big(X(\omega_k) - X^j \omega_i \Gamma^i_{jk}\big)\varepsilon^k.$$

2. *If $F \in \Gamma(T^{(k,l)}TM)$ is a smooth mixed tensor field of any rank, expressed locally as*
$$F = F^{i_1 \dots i_k}_{j_1 \dots j_l} E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l},$$
then the covariant derivative of F is given locally by
$$\nabla_X F = \Big(X\big(F^{i_1 \dots i_k}_{j_1 \dots j_l}\big) + \sum_{s=1}^{k} X^m F^{i_1 \dots p \dots i_k}_{j_1 \dots j_l} \Gamma^{i_s}_{mp} - \sum_{s=1}^{l} X^m F^{i_1 \dots i_k}_{j_1 \dots p \dots j_l} \Gamma^{p}_{m i_s}\Big) \times$$
$$E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l}.$$

$\nabla_X F$ of a tensor field F (or, as a special case, a vector field) is linear over $C^\infty(M)$ in X, the covariant derivatives of F in all directions can be handily encoded in a single tensor field whose rank is one more than the rank of F, as follows.

Proposition (The Total Covariant Derivative)

Let M be a smooth manifold and let ∇ be a connection in TM.

For every $F \in \Gamma(T^{(k,l)}TM)$, the map

$$\nabla F\colon \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k \text{ copies}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l+1 \text{ copies}} \to C^\infty(M)$$

given by

$$(\nabla F)(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l, X) := (\nabla_X F)(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l)$$

defines a smooth $(k, l+1)$-tensor field on M called the total covariant derivative of F.

When we express the components of a total covariant derivative in terms of a local frame, by standard practice we use a semicolon to separate indices resulting from differentiation from the preceding indices.

Thus, e.g., if $Y = Y^i E_i$ is a vector field, the components of the $(1,1)$-tensor field ∇Y are written $Y^i_{;j}$, so that

$$\nabla Y = Y^i_{;j} E_i \otimes \varepsilon^j,$$

with

$$Y^i_{;j} = E_j Y^i + Y^k \Gamma^i_{jk}.$$

For a 1-form ω, the formulas read

$$\nabla \omega = \omega_{i;j} \varepsilon^i \otimes \varepsilon^j, \quad \text{with } \omega_{i;j} = E_j \omega_i - \omega_k \Gamma^k_{ji}.$$

More generally, the next result gives a formula for the components of total covariant derivatives of arbitrary tensor fields.

Proposition

Let ∇ be a connection in TM, (E_i) a smooth local frame for TM and $\{\Gamma_{ij}^k\}$ the corresponding connection coefficients.

The components of the total covariant derivative of a (k,l)-tensor field F with respect to this frame are given by

$$F_{j_1\dots j_l;m}^{i_1\dots i_k} = E_m\big(F_{j_1\dots j_l}^{i_1\dots i_k}\big) + \sum_{s=1}^{k} F_{j_1\dots j_l}^{i_1\dots p\dots i_k}\Gamma_{mp}^{i_s} - \sum_{s=1}^{l} F_{j_1\dots p\dots j_l}^{i_1\dots i_k}\Gamma_{mj_s}^{p}.$$

Second Covariant Derivatives

- Having defined the tensor field ∇F for a (k, l)-tensor field F, we can in turn take its total covariant derivative and obtain a $(k, l+2)$-tensor field
$$\nabla^2 F := \nabla(\nabla F).$$
- Given $X, Y \in \mathfrak{X}(M)$, introduce $\nabla^2_{X,Y} F$ for the (k, l)-tensor field obtained by inserting X, Y in the last two slots of $\nabla^2 F$:
$$\nabla^2_{X,Y} F(\ldots) := \nabla^2 F(\ldots, Y, X).$$
- Note the reversal of order of X and Y: necessitated by our convention that the *last* index position in ∇F is the one resulting from differentiation, while it is conventional to let $\nabla^2_{X,Y}$ stand for differentiating first in the Y direction, then in the X direction.
- For this reason, some textbooks adopt the convention that the new index position introduced by differentiation is the first instead of the last. As usual, be sure to check author's conventions when you read.

- It is important to be aware that $\nabla^2_{X,Y}F$ is **not** the same as $\nabla_X(\nabla_Y F)$.
- The main reason is that the former is linear over $C^\infty(M)$ in Y, while the latter is not.
- The relationship between the two expressions is given in the following proposition.

Proposition

Let M be a smooth manifold and let ∇ be a connection in TM.

For every smooth vector field or tensor field F,

$$\nabla^2_{X,Y}F = \nabla_X(\nabla_Y F) - \nabla_{(\nabla_X Y)}F.$$

Sketch of Proof: A covariant derivative $\nabla_Y F$ can be expressed as the trace of $\nabla F \otimes Y$ on its last two indices

$$\nabla_Y F = \operatorname{tr}(\nabla F \otimes Y).$$

This statement is correct as both expressions have the same formula in components, namely $F^{i_1 \dots i_k}_{j_1 \dots j_l;m} Y^m$.
Similarly, $\nabla^2_{X,Y} F$ can be expressed as an iterated trace as

$$\nabla^2_{X,Y} F = \operatorname{tr}\big(\operatorname{tr}(\nabla^2 F \otimes X) \otimes Y\big).$$

Indeed, first trace the last index of $\nabla^2 F$ with that of X, then trace the last remaining free index (originally the second-to-last in $\nabla^2 F$) with that of Y.
Consequently, as ∇_X commutes with contractions and satisfies the product rule (w.r.t. tensor products) we get

$$\begin{aligned}
\nabla_X(\nabla_Y F) &= \nabla_X\big(\operatorname{tr}(\nabla F \otimes Y)\big) = \operatorname{tr}\big(\nabla_X(\nabla F \otimes Y)\big) \\
&= \operatorname{tr}\big(\nabla_X(\nabla F) \otimes Y + \nabla F \otimes \nabla_X Y\big) \\
&= \operatorname{tr}\big(\operatorname{tr}(\nabla^2 F \otimes X) \otimes Y\big) + \operatorname{tr}(\nabla F \otimes \nabla_X Y) \\
&= \nabla^2_{X,Y} F + \nabla_{(\nabla_X Y)} F. \qquad \square
\end{aligned}$$

Example (The Covariant Hessian)

Let $f \in C^\infty(M)$. Then

$$\Gamma\big(T^{(0,1)}TM\big) = \Omega^1(M) \ni \nabla f = \mathrm{d}f = \mathrm{D}\, f,$$

because both tensors have the same action on vectors:

$$\nabla f(X) = \nabla_X f = Xf = \mathrm{D}\, f(X).$$

The covariant 2-tensor

$$\nabla^2 f := \nabla(\mathrm{D}\, f)$$

is called the covariant Hessian of the function f. Its action on smooth vector fields X, Y can be computed by the following formula:

$$\nabla^2 f(Y, X) = \nabla^2_{X,Y} f = \nabla_X(\nabla_Y f) - \nabla_{(\nabla_X Y)} f = X(Yf) - (\nabla_X Y)f.$$

Moreover, in local coordinates we get

$$\nabla^2 f = f_{;ij}\mathrm{d}x^i \otimes \mathrm{d}x^j, \quad \text{with } f_{;ij} = \partial_j\partial_i f - \Gamma^k_{ji}\partial_k f.$$

We discuss more specific examples after having introduced geodesics.

Vector and Tensor Fields Along Curves

Back to the former question:

How can we make sense of the derivative of a vector field along a curve?

Definition

Let M be a smooth manifold and consider a smooth curve $\gamma\colon I \to M$.

A vector field along γ is a continuous map $V\colon I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for every $t \in I$.

It is a smooth vector field along γ if it is smooth as a map from I to TM.

Remark

We let $\mathfrak{X}(\gamma)$ denote the set of all smooth vector fields along γ.
It is a real vector space under pointwise vector addition and multiplication by constants.
It is a module over $C^\infty(I)$ with multiplication defined pointwise:

$$(fX)(t) := f(t)X(t).$$

Example

An example of a vector field along a smooth curve γ is its velocity $\gamma'(t) \in T_{\gamma(t)}M$ for each t. Its coordinate expression shows smoothness.

Example

If γ is a curve in $\mathbb{R}^2$, let $N(t) = R\gamma'(t)$, where R is counterclockwise rotation by $\pi/2$, so $N(t)$ is normal to $\gamma'(t)$. In standard coordinates, $N(t) = \big(-\dot\gamma^2(t), \dot\gamma^1(t)\big)$, so N is a smooth vector field along γ.

More examples are provided by the following construction:

Suppose $\gamma\colon I \to M$ is a smooth curve and $\widetilde{V} \in \mathfrak{X}(U)$ with open $U \subset M$ and $\gamma(I) \subset U$.
Define

$$V\colon I \to TM, \quad V(t) := \widetilde{V}|_{\gamma(t)}, \qquad \text{for each } t \in I.$$

Since V is equal to the composition $\widetilde{V} \circ \gamma$, it is smooth.

Definition

A smooth vector field along γ is said to be extendible if there exists a smooth vector field $\widetilde{V}$ on a neighborhood of the image of γ that is related to V in this way.

Not every vector field along a curve need be extendible:

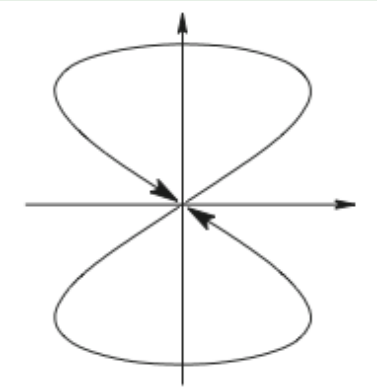
Consider a self intersecting curve, where at the intersection point we have $\gamma(t_1) = \gamma(t_2)$ but $\gamma'(t_1) \neq \gamma'(t_2)$, then γ' is not extendible.

Even if γ is injective, $\gamma'(t)$ need not be extendible, see:

Example

Consider the figure eight curve $\gamma\colon (-\pi, \pi) \to \mathbb{R}^2, \gamma(t) := (\sin 2t, \sin t)$. Clearly, γ is an injective smooth immersion, but γ' is not extendible. Although γ is neither self intersecting nor closed (the domain is open), in the limit we have

$$\lim_{t\to\pm\pi} \gamma(t) = \gamma(0) = (0,0) \quad \text{but} \quad \lim_{t\to\pm\pi} \gamma'(t) = (2,-1) \neq (2,1) = \gamma'(0).$$



Definition

More generally, a tensor field along γ is a continuous map

$$\sigma\colon I \to T^{(k,l)}TM, \quad \text{s.t.} \quad \sigma(t) \in T^{(k,l)}(T_{\gamma(t)}M), \text{ for each } t \in I.$$

It is a smooth tensor field along γ if it is smooth as a map from I to $T^{(k,l)}TM$.

It is extendible if there is a smooth tensor field $\widetilde{\sigma}$ on a neighborhood of $\gamma(I)$ such that $\sigma = \widetilde{\sigma} \circ \gamma$.

A connection offers a way to take derivatives of vector fields along curves:

Theorem (Covariant Derivative Along a Curve)

Let M be a smooth manifold and let ∇ be a connection in TM. For each smooth curve $\gamma\colon I \to M$, the connection determines a unique operator

$$\mathrm{D}_t\colon \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma),$$

called the covariant derivative along γ, satisfying:

(a) *Linearity over $\mathbb{R}$:*

$$\mathrm{D}_t(aV + bW) = a\,\mathrm{D}_t\,V + b\,\mathrm{D}_t\,W \quad \text{for all } a, b \in \mathbb{R}. \tag{20}$$

(b) *Product rule:*

$$\mathrm{D}_t(fV) = f'V + f\,\mathrm{D}_t\,V \quad \text{for } f \in C^\infty(I). \tag{21}$$

(c) *If $V \in \mathfrak{X}(\gamma)$ is extendible, then for every extension $\widetilde{V}$ of V,*

$$\mathrm{D}_t\,V(t) = \nabla_{\gamma'(t)}\widetilde{V}. \tag{22}$$

An analogous operator on the space of smooth tensor fields of any type exists along γ.

Proof: (Only for vector fields along γ) Uniqueness: Suppose D_t is such an operator, and let $t_0 \in I$ be arbitrary. Then the value of $\mathrm{D}_t\, V(t_0)$ depends only on the values of V in any interval $(t_0 - \varepsilon, t_0 + \varepsilon)$.
In coordinates (x^i) and for t near t_0

$$V(t) = V^j(t)\partial_j|_{\gamma(t)}. \tag{23}$$

By the properties of D_t, since each ∂_j is extendible,

$$\begin{aligned} \mathrm{D}_t\, V(t) &= \dot{V}^j(t)\partial_j\big|_{\gamma(t)} + V^j(t)\nabla_{\gamma'(t)}\partial_j\big|_{\gamma(t)} \\ &= \Big(\dot{V}^k(t) + \dot{\gamma}^i(t)V^j(t)\Gamma^k_{ij}\big(\gamma(t)\big)\Big)\partial_k\Big|_{\gamma(t)}. \end{aligned} \tag{24}$$

This shows that such an operator is unique if it exists.
Existence: If $\gamma(I)$ is contained in one chart, we can *define* $\mathrm{D}_t\, V$ by (24). It satisfies the requisite properties. In general, we can cover $\gamma(I)$ with charts and define $\mathrm{D}_t\, V$ by this formula in each chart. Uniqueness implies that the various definitions agree whenever charts overlap. □

Remark

In physics, the covariant derivative along a curve is often called absolute derivative.

We can now define acceleration and geodesics:

Definition

Let M be a smooth manifold and let ∇ be a connection in TM.

For every smooth curve $\gamma\colon I \to M$, we define the acceleration of γ to be the vector field $\mathrm{D}_t\,\gamma'$ along γ.

A smooth curve γ is called a geodesic (w.r.t. ∇) if its acceleration is zero:

$$\mathrm{D}_t\,\gamma' \equiv 0.$$

Corollary

In terms of coordinates (x^i), if we write the component functions of γ as $\gamma(t) = \big(x^1(t), \ldots, x^n(t)\big)$, then it follows from (24) that γ is a geodesic iff its component functions satisfy the following geodesic equation:

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}\big(x(t)\big) = 0. \tag{25}$$

□

This is a system of second order ordinary differential equations (ODEs) for the real-valued functions $x^1, \ldots, x^n$.

The next theorem follows from ODE theory to ensure existence and uniqueness of geodesics with suitable initial conditions.

Theorem (Existence and Uniqueness of Geodesics)

Let M be a smooth manifold and ∇ a connection in TM.

For every $p \in M$, $w \in T_pM$, and $t_0 \in \mathbb{R}$, there exist an open interval $I \subset \mathbb{R}$ containing t_0 and a geodesic $\gamma\colon I \to M$ satisfying $\gamma(t_0) = p$ and $\gamma'(t_0) = w$.

Any two such geodesics agree on their common domain. □

Definition

A geodesic $\gamma\colon I \to M$ is said to be maximal if it cannot be extended to a geodesic on a larger interval.

In other words, if there does not exist a geodesic $\widetilde{\gamma}\colon \widetilde{I} \to M$ defined on an interval $\widetilde{I}$ properly containing I and satisfying $\widetilde{\gamma}|_I = \gamma$.

A geodesic segment is a geodesic whose domain is a compact interval.

Corollary

Let M be a smooth manifold and let ∇ be a connection in TM.

For each $p \in M$ and $v \in T_pM$, there is a unique maximal geodesic $\gamma\colon I \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$, defined on some open interval I containing 0.

The Concept of Parallel Transport

Definition

Let M be a smooth manifold and let ∇ be a connection in TM.

A smooth vector field V along a smooth curve γ is said to be parallel along γ (w.r.t. ∇), if $\mathrm{D}_t\, V \equiv 0$.

Thus a geodesic can be characterized as a curve whose velocity vector field is parallel along the curve.

Example

Let $\gamma\colon I \to \mathbb{R}^n$ be a smooth curve, and let V be a smooth vector field along γ.

Then V is parallel along γ w.r.t. the Euclidean connection iff its component functions w.r.t. the standard basis are constants.

The fundamental fact about parallel vector and tensor fields along curves is that every tangent vector or tensor at any point on a curve can be uniquely extended to a parallel field along the entire curve. (Proved in due course.)

First examine what the equation of parallelism looks like in coordinates.

Given a smooth curve $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$ in coordinates, we know (last lecture) that a vector field V is parallel along γ iff

$$\dot{V}^k(t) = -\dot{\gamma}^i(t) V^j(t) \Gamma^k_{ij}\big(\gamma(t)\big), \quad k = 1, \ldots, n. \tag{26}$$

This is a system of **first-order linear ordinary differential equations** for the unknown coefficients of the vector or tensor field.

ODE theory tells us that existence and uniqueness results for the solution of (26) apply for the entire parameter interval.

Theorem (Existence and Uniqueness of Parallel Transport)

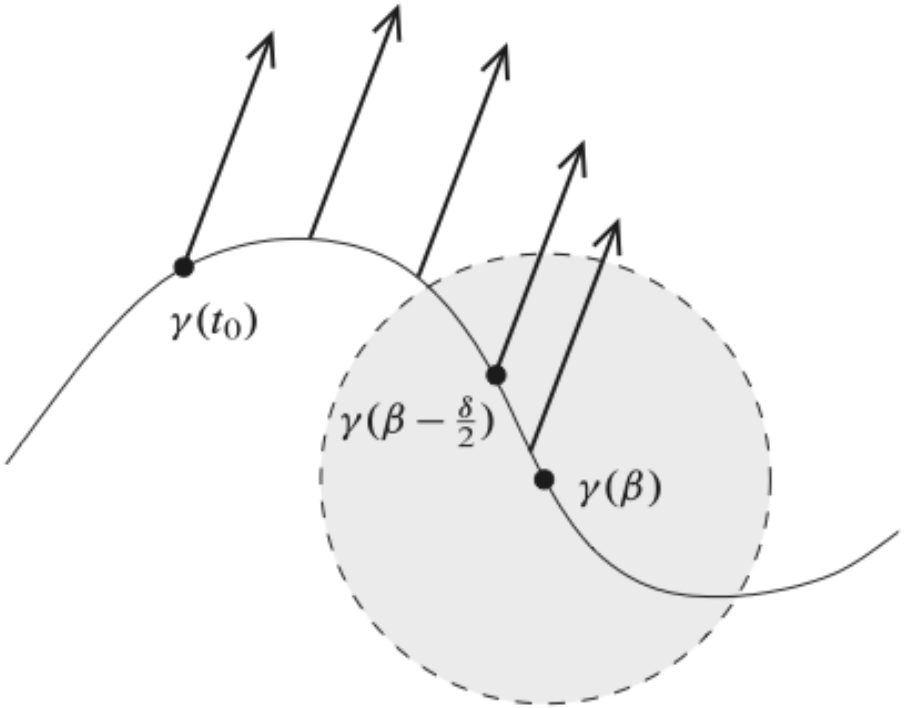
Suppose M is a smooth manifold and ∇ is a connection in TM.

Given a smooth curve $\gamma\colon I \to M$, and a vector $v \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field V along γ s.t. $V(t_0) = v$.

Proof: First suppose $\gamma(I)$ is contained in a *single* chart.
Then it follows that V is parallel along γ iff its components satisfy (26).
Existence and uniqueness of a solution for (26) with $V(t_0) = v$ on all of I is ensured by linear ODE theory.

Now suppose $\gamma(I)$ is *not* covered by a single chart.
Let β denote the supremum of all $b > t_0$ for which a unique parallel transport exists on $[t_0, b]$.
(Similar for $t < t_0$.)
We know that $\beta > t_0$, since for b close enough to t_0, $\gamma([t_0, b])$ is contained in a single chart and the above argument applies.
Then a unique parallel transport V exists on $[t_0, \beta)$.
If β is equal to $\sup I$, we are done.

If not, choose smooth coordinates on an open set containing $\gamma\big((\beta-\delta,\beta+\delta)\big)$ for some positive δ.
Then there exists a unique parallel vector field $\widetilde{V}$ on $(\beta-\delta,\beta+\delta)$ satisfying the initial condition $\widetilde{V}(\beta-\delta/2)=V(\beta-\delta/2)$.
By uniqueness, $V=\widetilde{V}$ on their common domain, and therefore $\widetilde{V}$ would be a parallel extension of V past β, which is a contradiction. $\square$



Definition

The vector field whose existence and uniqueness is proved in Theorem 424 is called the parallel transport of v along γ.

For each $t_0, t_1 \in I$, we define a map

$$P^{\gamma}_{t_0 t_1} \colon T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M, \tag{27}$$

called the parallel transport map, by setting $P^{\gamma}_{t_0 t_1}(v) = V(t_1)$ for each $v \in T_{\gamma(t_0)}M$, where V is the parallel transport of v along γ.

- Obviously, this map is linear, because the equation of parallelism is linear.
- It is in fact an isomorphism, because $P^{\gamma}_{t_1 t_0}$ is an inverse for it.
- It is also useful to extend the parallel transport operation to curves that are merely piecewise smooth.
- The generalization is straight-forward. We then talk about piecewise smooth vector fields along γ.

Corollary (Parallel Transport Along Piecewise Smooth Curves)

Suppose M is a smooth manifold and ∇ is a connection in TM.

Given an admissible curve $\gamma\colon [a,b] \to M$ and a vector $v \in T_{\gamma(a)}M$, there exists a unique piecewise smooth parallel vector field V along γ such that $V(a) = v$, and V is smooth wherever γ is.

Proof: Let $(a_0, \ldots, a_k)$ be an admissible partition for γ.

First define $V|_{[a_0,a_1]}$ to be the parallel transport of v along the first smooth segment $\gamma|_{[a_0,a_1]}$;

then define $V|_{[a_1,a_2]}$ to be the parallel transport of $V(a_1)$ along the next smooth segment $\gamma|_{[a_1,a_2]}$;

and continue by induction. $\square$

A useful tool for working with parallel transport:

- Given any basis $(b_1, \ldots, b_n)$ for $T_{\gamma(t_0)}M$, we parallel transport the b_i along γ, obtaining parallel vector fields $(E_1, \ldots, E_n)$ along γ.
- Because each parallel transport map is an isomorphism, the vectors $\big(E_i(t)\big)$ form a basis for $T_{\gamma(t)}M$ at each point $\gamma(t)$.
- Such an n-tuple of vector fields along γ is called a parallel frame along γ.
- Every smooth (or piecewise smooth) vector field along γ can be expressed in terms of such a frame as $V(t) = V^i(t)E_i(t)$, and then the properties of covariant derivatives along curves, together with the fact that the E_i's are parallel, imply

$$\mathrm{D}_t\, V(t) = \dot{V}^i(t)E_i(t), \tag{28}$$

  wherever V and γ are smooth.
- This means that a vector field is parallel along γ iff its component functions w.r.t. the frame (E_i) are constants.

- The parallel transport map is the means by which a connection "connects" nearby tangent spaces.
- In other words, it gives a rule how to identify nearby tangent spaces.
- The next theorem and its corollary show that parallel transport determines covariant differentiation along curves, and thereby the connection itself.

Theorem (Parallel Transport Determines Covariant Differentiation)

Let M be a smooth manifold and let ∇ be a connection in TM.

Let $\gamma\colon I \to M$ be a smooth curve and V a smooth vector field along γ.

For each $t_0 \in I$,

$$\mathrm{D}_t\, V(t_0) = \lim_{t_1 \to t_0} \frac{P^{\gamma}_{t_1 t_0} V(t_1) - V(t_0)}{t_1 - t_0}. \tag{29}$$

Proof:
Let (E_i) be a parallel frame along γ, and expand

$$V(t) = V^i(t)E_i(t) \quad \text{for} \quad t \in I.$$

On the one hand, (28) shows that $\mathrm{D}_t\, V(t_0) = \dot{V}^i(t_0)E_i(t_0)$.
On the other hand, for every fixed $t_1 \in I$, the parallel transport of the vector $V(t_1)$ along γ is the constant coefficient vector field $W(t) = V^i(t_1)E_i(t)$ along γ, so

$$P^{\gamma}_{t_1 t_0} V(t_1) = V^i(t_1)E_i(t_0).$$

Inserting these formulas into (29) and taking the limit $t_1 \to t_0$, we conclude that the right-hand side is also equal to $\dot{V}^i(t_0)E_i(t_0)$. $\square$

Corollary (Parallel Transport Determines the Connection)

Let M be a smooth manifold and let ∇ be a connection in TM . Suppose X and Y are smooth vector fields on M. For every $p \in M$,

$$\nabla_X Y|_p = \lim_{h \to 0} \frac{P^{\gamma}_{h0} Y_{\gamma(h)} - Y_p}{h}, \tag{30}$$

where $\gamma \colon I \to M$ is any smooth curve such that $\gamma(0) = p$ and $\gamma'(0) = X_p$.

Proof:
Given $p \in M$ and a smooth curve γ s.t. that $\gamma(0) = p$ and $\gamma'(0) = X_p$.

Let $V(t)$ denote the vector field along γ determined by Y, so $V(t) = Y_{\gamma(t)}$.

Then $\nabla_X Y|_p$ is equal to $\mathrm{D}_t\, V(0)$, so the result follows from the above theorem. □

Definition

A smooth vector field on M is said to be parallel (w.r.t. ∇) if it is parallel along every smooth curve in M .

Example

Every constant coefficient vector field on $\mathbb{R}^n$ is parallel.

Corollary

Suppose M is a smooth manifold, ∇ is a connection in TM, and A is a smooth vector field on M.

Then A is parallel on M iff $\nabla A \equiv 0$. $\square$

Remark

Although we showed above that it is always possible to extend a vector at a point to a parallel vector field along any given curve,
it may not be possible in general to extend it to a parallel vector field on an open subset of the manifold.

The impossibility of finding such extensions is intimately connected with the phenomenon of curvature, which will occupy a major portion of our attention in the sequel.

Pullback Connections

Suppose $M, \widetilde{M}$ are smooth manifolds and $\varphi\colon M \to \widetilde{M}$ is a diffeomorphism. For $X \in \mathfrak{X}(M)$ recall that the **pushforward** of X is the unique vector field $\varphi_* X \in \mathfrak{X}(\widetilde{M})$ that satisfies for all $p \in M$

$$\mathrm{D}\,\varphi_p(X_p) = (\varphi_* X)_{\varphi(p)}.$$

Lemma (Pullback Connection)

Suppose $M, \widetilde{M}$ are smooth manifolds.

If $\widetilde{\nabla}$ is a connection in $T\widetilde{M}$ and $\varphi\colon M \to \widetilde{M}$ is a diffeomorphism, then

$$\begin{aligned} \varphi^*\widetilde{\nabla}\colon \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M), \\ (X, Y) &\mapsto (\varphi^*\widetilde{\nabla})_X Y := \left(\varphi^{-1}\right)_*\left(\widetilde{\nabla}_{\varphi_* X}(\varphi_* Y)\right) \end{aligned}$$

is a connection in TM, called the pullback of $\widetilde{\nabla}$ by φ.

Proof:

(i) Obviously, $(\varphi^*\widetilde{\nabla})_X Y$ is linear over $\mathbb{R}$ in Y .

(ii) To see that it is linear over $C^\infty(M)$ in X, consider $f \in C^\infty(M)$. Let $\widetilde{f} := f \circ \varphi^{-1}$, so $\varphi_*(fX) = \widetilde{f}\varphi_* X$. Then

$$\begin{aligned}(\varphi^*\widetilde{\nabla})_{fX} Y &= (\varphi^{-1})_*\Big(\widetilde{\nabla}_{\widetilde{f}\varphi_* X}(\varphi_* Y)\Big)\\ &= (\varphi^{-1})_*\Big(\widetilde{f}\widetilde{\nabla}_{\varphi_* X}(\varphi_* Y)\Big) = f(\varphi^*\widetilde{\nabla})_X Y.\end{aligned}$$

(iii) To prove the product rule in Y, let f and $\widetilde{f}$ be as above, and note that $(\varphi_* X)(\widetilde{f}) = (Xf) \circ \varphi^{-1}$ (because X and $\varphi_* X$ are "φ-related"). Thus

$$\begin{aligned}(\varphi^*\widetilde{\nabla})_X (fY) &= (\varphi^{-1})_*\Big(\widetilde{\nabla}_{\varphi_* X}(\widetilde{f}\varphi_* Y)\Big)\\ &= (\varphi^{-1})_*\Big(\widetilde{f}\widetilde{\nabla}_{\varphi_* X}(\varphi_* Y) + (\varphi_* X)(\widetilde{f})\varphi_* Y\Big)\\ &= f(\varphi^*\widetilde{\nabla})_X Y + (Xf)Y. \qquad \square\end{aligned}$$

Proposition (Properties of Pullback Connections)

Suppose $M, \widetilde{M}$ are smooth manifolds and $\varphi\colon M \to \widetilde{M}$ is a diffeomorphism. Let $\widetilde{\nabla}$ be a connection in $T\widetilde{M}$ and let $\nabla := \varphi^*\widetilde{\nabla}$ be the pullback connection in TM. Suppose $\gamma\colon I \to M$ is a smooth curve.

1. *φ takes covariant derivatives along curves to covariant derivatives along curves: if V is a smooth vector field along γ, then*

$$\mathrm{D}\,\varphi \circ \mathrm{D}_t\, V = \widetilde{\mathrm{D}}_t(\mathrm{D}\,\varphi \circ V),$$

   where D_t is covariant differentiation along γ w.r.t. ∇, and $\widetilde{\mathrm{D}}_t$ is covariant differentiation along $\varphi \circ \gamma$ w.r.t. $\widetilde{\nabla}$.

2. *φ takes geodesics to geodesics: if γ is a ∇-geodesic in M, then $\varphi \circ \gamma$ is a $\widetilde{\nabla}$-geodesic in $\widetilde{M}$.*

3. *φ takes parallel transport to parallel transport: for every $t_0, t_1 \in I$,*

$$\mathrm{D}\,\varphi_{\gamma(t_1)} \circ P^{\gamma}_{t_0 t_1} = P^{\varphi\circ\gamma}_{t_0 t_1} \circ \mathrm{D}\,\varphi_{\gamma(t_0)}.$$

The Levi-Civita Connection

Motivation:

- We need/want a way to single out a particular connection on a Riemannian manifold that reflects the properties of the metric.
- We describe two properties that together determine a **unique** connection on every Riemannian manifold.
- **Compatibility** with the metric, is easy to motivate and understand.
- **Symmetry**, is a bit more mysterious; but it is motivated by the fact that it is invariantly defined, and is always satisfied by the tangential connection.
- These two conditions are enough to determine a unique connection associated with any Riemannian or pseudo-Riemannian metric, called the **Levi-Civita connection**.
- **Note**: Except where noted otherwise, the results and proofs of the following do not use positivity of the metric, so they apply equally well to Riemannian **and** pseudo-Riemannian manifolds.

The Tangential Connection Revisited

- The tangential connection defined earlier computes covariant derivatives on $M \subset \mathbb{R}^n$ by taking ordinary derivatives in $\mathbb{R}^n$ and projecting them orthogonally to TM.
- It is easy to compute covariant derivatives along curves in M with respect to the tangential connection.
- Suppose $\gamma\colon I \to M$ is a smooth curve. Then γ can be regarded as either a smooth curve in M or a smooth curve in $\mathbb{R}^n$.
- A smooth vector field V along γ that takes its values in TM can be regarded as either a vector field along γ in M or a vector field along γ in $\mathbb{R}^n$.
- Let $\overline{\mathrm{D}}_t V$ denote the covariant derivative of V along γ (as a curve in $\mathbb{R}^n$) w.r.t. the Euclidean connection $\overline{\nabla}$, and let $\mathrm{D}_t^\top V$ denote its covariant derivative along γ (as a curve in M) w.r.t. the tangential connection $\nabla^\top$.

Proposition

Let $M \subset \mathbb{R}^n$ be an embedded submanifold.

Let $\gamma\colon I \to M$ a smooth curve in M.

Let V a smooth vector field along γ that takes its values in TM.

Then for each $t \in I$,

$$\mathrm{D}_t^\top V(t) = \pi^\top\big(\overline{\mathrm{D}}_t V(t)\big).$$

Proof:

Let $t_0 \in I$ be arbitrary. On a neighborhood U of $\gamma(t_0) \in \mathbb{R}^n$ there is an adapted orthonormal frame for TM, i.e., a local orthonormal frame $(E_1, \dots, E_n)$ for $T\mathbb{R}^n$ s.t. $(E_1, \dots, E_k)$ restricts to an orthonormal frame for TM at points of $M \cap U$ (where $k = \dim M$).

If $\varepsilon > 0$ is small enough that $\gamma\big((t_0 - \varepsilon, t_0 + \varepsilon)\big) \subset U$ then for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ we can expand

$$V(t) = V^1(t)E_1(t)|_{\gamma(t)} + \dots + V^k(t)E_k(t)|_{\gamma(t)},$$

for smooth $V^1, \dots, V^k \colon (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R}$. Now

$$\begin{aligned}
\pi^\top\big(\overline{\mathrm{D}}_t V(t)\big) &= \pi^\top\Big(\sum_{i=1}^k \Big(\dot V^i(t) E_i\big|_{\gamma(t)} + V^i(t)\overline{\nabla}_{\gamma'(t)} E_i\big|_{\gamma(t)}\Big)\Big) \\
&= \sum_{i=1}^k \Big(\dot V^i(t) E_i\big|_{\gamma(t)} + V^i(t)\pi^\top\Big(\overline{\nabla}_{\gamma'(t)} E_i\big|_{\gamma(t)}\Big)\Big) \\
&= \sum_{i=1}^k \Big(\dot V^i(t) E_i\big|_{\gamma(t)} + V^i(t)\nabla^\top_{\gamma'(t)} E_i\big|_{\gamma(t)}\Big) = \mathrm{D}_t^\top V(t). \qquad \square
\end{aligned}$$

Corollary

Suppose $M \subset \mathbb{R}^n$ is an embedded submanifold.

A smooth curve $\gamma\colon I \to M$ is a geodesic w.r.t. the tangential connection on M iff its ordinary acceleration $\gamma''(t)$ is orthogonal to $T_{\gamma(t)}M$ for all $t \in I$.

Proof:

In an earlier example we showed that the connection coefficients of the Euclidean connection on $\mathbb{R}^n$ are all zero.

Consequently, the Euclidean covariant derivative of γ' along γ is just its ordinary acceleration:

$$\overline{\mathrm{D}}_t \gamma'(t) = \gamma''(t).$$

The corollary then follows from Proposition 66. $\square$

- These considerations can be extended to pseudo-Riemannian manifolds as well.
- Let $\left(\mathbb{R}^{r,s}, \overline{q}^{(r,s)}\right)$ be the pseudo-Euclidean space of signature (r,s).
- If $M \subset \mathbb{R}^{r,s}$ is an embedded Riemannian or pseudo-Riemannian submanifold, then for each $p \in M$, the tangent space decomposes as a direct sum
$$T_p\mathbb{R}^{r,s} = T_pM \oplus N_pM.$$
Here $N_pM := (T_pM)^\perp$ is the orthogonal complement of T_pM w.r.t. $\overline{q}^{(r,s)}$.

Definition

We let $\pi^\top \colon T_p\mathbb{R}^{r,s} \to T_pM$ be the $\overline{q}^{(r,s)}$-orthogonal projection.

Define the tangential connection $\nabla^\top$ on M by

$$\nabla^\top_X Y := \pi^\top\big(\overline{\nabla}^\top_{\widetilde{X}} \widetilde{Y}\big).$$

Here $\widetilde{X}, \widetilde{Y}$ are smooth extensions of X, Y to a neighborhood of M.

$\overline{\nabla}$ is the ordinary Euclidean connection on $\mathbb{R}^{r,s}$.

- The tangential connection is a well-defined connection on M by the same argument as in the Euclidean case.
- The next proposition is proved in exactly the same way as Corollary 433.

Proposition

Suppose M is an embedded Riemannian or pseudo-Riemannian submanifold of the pseudo-Euclidean space $\mathbb{R}^{r,s}$.

A smooth curve $\gamma\colon I \to M$ is a geodesic w.r.t. $\nabla^\top$ iff $\gamma''(t)$ is $\overline{q}^{(r,s)}$-orthogonal to $T_{\gamma(t)}M$ for all $t \in I$. □

Connections on Abstract Riemannian Manifolds

- A celebrated theorem by John Nash [21] says that every Riemannian metric on a smooth manifold can be realized as the induced metric of some embedding in a Euclidean space.
- Later generalized independently by Robert Greene [14] and Chris J. S. Clarke [10] to pseudo-Riemannian metrics.
- So we do not lose generality by studying only submanifolds of (pseudo-)Euclidean spaces with their induced metrics, for which the tangential connection would suffice.
- However, when we are trying to understand intrinsic properties of a Riemannian manifold, an embedding introduces a great deal of extraneous information, and in some cases actually makes it harder to discern which geometric properties depend only on the metric.
- We will distinguish some important properties of the tangential connection that make sense for connections on an abstract (pseudo-)Riemannian manifold.
- We use them to single out a unique connection in the abstract case.

The Concept of a Metric Connection

- The Euclidean connection on $\mathbb{R}^n$ has one very nice property w.r.t. the Euclidean metric: it satisfies the product rule

$$\overline{\nabla}_X \langle Y, Z \rangle = \langle \overline{\nabla}_X Y, Z \rangle + \langle Y, \overline{\nabla}_X Z \rangle.$$

- In this formula, the l.h.s. represents the covariant derivative of the real-valued function $\langle Y, Z \rangle$, which is really just $X \langle Y, Z \rangle$.
- The Euclidean connection has the same property w.r.t. the pseudo-Euclidean metric on $\mathbb{R}^{r,s}$.
- It is almost immediate that the tangential connection on a (pseudo-)Riemannian submanifold satisfies the same product rule, if we now interpret all the vector fields as being tangent to M and interpret the inner products as being taken w.r.t. the induced metric on M.
- This property makes sense on an abstract Riemannian or pseudo-Riemannian manifold.

Definition

Let g be a Riemannian or pseudo-Riemannian metric on a smooth manifold M.

A connection ∇ on TM is said to be compatible with g, or to be a metric connection, if it satisfies the following product rule for all $X, Y, Z \in \mathfrak{X}(M)$:

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle. \tag{31}$$

The next proposition gives several alternative characterizations of compatibility with a metric, any one of which could be used as the definition.

Proposition (Characterizations of Metric Connections)

Let (M, g) be a Riemannian or pseudo-Riemannian manifold and let ∇ be a connection on TM. The following conditions are equivalent:

(a) *∇ is comapitble with g, i.e., $\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$;*

(b) *g is parallel with respect to ∇, i.e., $\nabla g \equiv 0$;*

(c) *in terms of a smooth loca frame (E_i), the connection coefficients of ∇ satisfy*
$$\Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il} = E_k(g_{ij}); \tag{32}$$

(d) *if V, W are smooth vector fields along any smooth curve γ, then*
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\langle V, W\rangle = \langle \mathrm{D}_t V, W\rangle + \langle V, \mathrm{D}_t W\rangle; \tag{33}$$

(e) *if V, W are parallel vector fields along a smooth curve γ in M , then $\langle V, W\rangle$ is constant along γ;*

(f) *given any smooth curve γ in M, every parallel transport map along γ is a linear isometry;*

(g) *given any smooth curve γ in M, every orthonormal basis at a point of γ can be extended to a parallel orthonormal frame along γ.*

Proof:
We prove first a) $\Leftrightarrow$ b). The total derivative of the metric is given by

$$\begin{aligned}(\nabla g)(Y,Z,X) &= (\nabla_X g)(Y,Z)\\ &= X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z),\end{aligned}$$

which is zero for all X, Y, Z iff (31) is satisfied for X, Y, Z.

To prove b) $\Leftrightarrow$ c), note that the components of ∇g in terms of a smooth local frame (E_i) are given as

$$g_{ij;k} = E_k(g_{ij}) - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il}. \tag{34}$$

They are all equal to zero iff (32) is satified.

Now we prove a) $\Leftrightarrow$ d). Assume a), and let V, W be smooth vector fields along a smooth curve $\gamma\colon I \to M$.
Given $t_0 \in I$, in a neighborhood of $\gamma(t_0)$ we may choose coordinates (x^i) and expand

$$V = V^i \partial_i, \quad W = W^j \partial_j, \quad \text{with} \quad V^i, W^j \in C^\infty(I).$$

Applying (32) to the extendible vector fields ∂_i, ∂_j, we obtain

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\langle V, W\rangle &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big(V^i W^j \langle \partial_i, \partial_j\rangle\big) \\
&= \big(\dot V^i W^j + V^i \dot W^j\big)\langle \partial_i, \partial_j\rangle + V^i W^j \big(\langle \nabla_{\gamma'(t)}\partial_i, \partial_j\rangle + \langle \partial_i, \nabla_{\gamma'(t)}\partial_j\rangle\big) \\
&= \langle \mathrm{D}_t V, W\rangle + \langle V, \mathrm{D}_t W\rangle,
\end{aligned}$$

which implies d).
Conversely, if d) holds, then in particular it holds for extendible vector fields along γ, and then a) follows from the notion/definition of covariant derivatives along curves.

Now we will prove d) $\Rightarrow$ e) $\Rightarrow$ f) $\Rightarrow$ g) $\Rightarrow$ d).

Assume first that d) holds. If V and W are parallel along γ, then (33) shows that $\langle V, W\rangle$ has zero derivative w.r.t. t, so it is constant along γ.

Now assume e).

Let $v_0, w_0 \in T_{\gamma(t_0)}M$ be arbitrary.

Let V, W be their parallel transports along γ, s.t.

$$V(t_0) = v_0,\ W(t_0) = w_0,\ P^{\gamma}_{t_0 t_1} v_0 = V(t_1),\ P^{\gamma}_{t_0 t_1} w_0 = W(t_1).$$

Because $\langle V, W\rangle$ is constant along γ, it follows that

$$\begin{aligned}\langle P^{\gamma}_{t_0 t_1} v_0, P^{\gamma}_{t_0 t_1} w_0\rangle &= \langle V(t_1), W(t_1)\rangle \\ &= \langle V(t_0), W(t_0)\rangle \\ &= \langle v_0, w_0\rangle,\end{aligned}$$

so $P^{\gamma}_{t_0 t_1}$ is a linear isometry.

Next, assuming f), we suppose $\gamma\colon I \to M$ is a smooth curve and (b_i) is an orthonormal basis for $T_{\gamma(t_0)}M$, for some $t_0 \in I$.

We can extend each b_i by parallel transport to obtain a smooth parallel vector field E_i along γ, and the assumption that parallel transport is a linear isometry guarantees that the resulting n-tuple (E_i) is an orthonormal frame at all points of γ.

Finally, assume that g) holds, and let (E_i) be a parallel orthonormal frame along γ.

Given smooth vector fields V and W along γ, we can expand them as $V = V^i E_i$ and $W = W^j E_j$.

The fact that the frame is orthonormal means that the metric coefficients $g_{ij} = \langle E_i, E_j \rangle$ are constants along γ, i.e., they are equal to ± 1 or 0.

The fact that it is parallel means that $\mathrm{D}_t\, V = \dot{V}^i E_i$ and $\mathrm{D}_t\, W = \dot{W}^i E_i$.

Thus both sides of (33) reduce to the following expression:

$$g_{ij}(\dot{V}^i W^j + V^i \dot{W}^j).$$

This proves d). $\square$

As an immediate consequence we get:

Corollary

Suppose (M, g) is a Riemannian or pseudo-Riemannian manifold, ∇ is a metric connection on M, and $\gamma\colon I \to M$ is a smooth curve.

1. $\|\gamma'(t)\|$ *is constant iff* $\mathrm{D}_t\,\gamma'(t)$ *is orthogonal to* $\gamma'(t)$ *for all* $t \in I$.
2. *If* γ *is a geodesic, then* $\|\gamma'(t)\|$ *is constant.* □

Proposition

If M is an embedded Riemannian or pseudo-Riemannian submanifold of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$, the tangential connection on M is compatible with the induced Riemannian or pseudo-Riemannian metric.

Proof:
We will show that $\nabla^\top$ satisfies (31).

Suppose $X, Y, Z \in \mathfrak{X}(M)$, and let $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ be smooth extensions of them to an open subset of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$.

At points of M, we have

$$\begin{aligned}
\nabla^\top_X \langle Y, Z\rangle &= X\langle Y, Z\rangle \\
&= \widetilde{X}\langle \widetilde{Y}, \widetilde{Z}\rangle \\
&= \overline{\nabla}_{\widetilde{X}}\langle \widetilde{Y}, \widetilde{Z}\rangle \\
&= \langle \overline{\nabla}_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}\rangle + \langle \widetilde{Y}, \overline{\nabla}_{\widetilde{X}}\widetilde{Z}\rangle \\
&= \langle \pi^\top(\overline{\nabla}_{\widetilde{X}}\widetilde{Y}), \widetilde{Z}\rangle + \langle \widetilde{Y}, \pi^\top(\overline{\nabla}_{\widetilde{X}}\widetilde{Z})\rangle \\
&= \langle \nabla^\top_X Y, Z\rangle + \langle Y, \nabla^\top_X Z\rangle,
\end{aligned}$$

where the next-to-last equality follows from the fact that $\widetilde{Z}$ and $\widetilde{Y}$ are tangent to M. $\square$

Symmetric Connections

Motivation:

- Every abstract (pseudo-)Riemannian manifold admits many different metric connections.
- Requiring compatibility with the metric is not sufficient to pin down a unique connection on such a manifold.
- We need another key property of the tangential connection.
- Recall the expression on the r.h.s. of the Euclidean connection being reminiscent of part of the coordinate expression for the Lie bracket:
$$[X, Y] = X(Y^i)\frac{\partial}{\partial x^i} - Y(X^i)\frac{\partial}{\partial x^i}. \tag{35}$$
In fact, the two terms in the Lie bracket formula are exactly the coordinate expressions for $\overline{\nabla}_X Y$ and $\overline{\nabla}_Y X$.
- Therefore, the Euclidean connection satisfies for all X, Y:
$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y]. \tag{36}$$
- This expression is coordinate-independent and makes sense for every connection on the tangent bundle.

Definition

We say that a connection ∇ on the tangent bundle of a smooth manifold M is symmetric if

$$\nabla_X Y - \nabla_Y X \equiv [X, Y] \quad \text{for all} \quad X, Y \in \mathfrak{X}(M). \tag{37}$$

Definition

The symmetry condition can also be expressed in terms of the torsion tensor of the connection.

This is the smooth $(1, 2)$-tensor field

$$\begin{aligned} \tau \colon \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M), \\ (X, Y) &\mapsto \tau(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]. \end{aligned} \tag{38}$$

Corollary

Thus a connection ∇ is symmetric iff its torsion vanishes identically. □

Remark

Look at the torsion tensor in a coordinate frame:
A connection is symmetric iff its connection coefficients in every coordinate frame satisfy

$$\Gamma_{ij}^k = \Gamma_{ji}^k.$$

This is the origin of the term "symmetric".

Proposition

If M is an embedded (pseudo-)Riemannian submanifold of a (pseudo)-Euclidean space,
then the tangential connection on M is symmetric.

Proof:
Let M be an embedded (pseudo-)Riemannian submanifold of $\mathbb{R}^n$, where $\mathbb{R}^n$ is endowed with the (pseudo-)Euclidean metric $\overline{q}^{(r,s)}$ with $r + s = n$.

Let $X, Y \in \mathfrak{X}(M)$, and let $\tilde{X}, \tilde{Y}$ be smooth extensions of them to an open subset of the ambient space.

If $\iota\colon M \hookrightarrow \mathbb{R}^n$ represents the inclusion map, it follows that X and Y are ι-related to $\tilde{X}$ and $\tilde{Y}$, respectively, and consequently $[X, Y]$ is ι-related to $[\tilde{X}, \tilde{Y}]$.

In particular, $[\tilde{X}, \tilde{Y}]$ is tangent to M, and its restriction to M is equal to $[X, Y]$.

Therefore,

$$\begin{aligned}\nabla^\top_X Y - \nabla^\top_Y X &= \pi^\top\left(\overline{\nabla}_{\tilde{X}}\tilde{Y}|_M - \overline{\nabla}_{\tilde{Y}}\tilde{X}|_M\right)\\ &= \pi^\top\left([\tilde{X},\tilde{Y}]|_M\right)\\ &= [\tilde{X},\tilde{Y}]|_M\\ &= [X,Y],\end{aligned}$$

as required. $\square$

- We wish to single out a connection on each (pseudo-)Riemannian manifold in such a way that it matches the tangential connection when the manifold is presented as an embedded submanifold of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$ with the induced metric.
- Then we must require at least that the connection be compatible with the metric and symmetric.
- It is a pleasant fact that these two conditions are enough to determine a unique connection.

One of the most important basic statements about (pseudo)-Riemannian geometry is

Theorem (Fundamental Theorem of Riemannian Geometry)

Let (M, g) be a Riemannian or pseudo-Riemannian manifold.

There exists a unique connection ∇ on TM that is compatible with g and symmetric.

It is called the Levi-Civita connection of g.

Or also, when g is positive definite, the Riemannian connection.

Proof:

We prove uniqueness first, by deriving a formula for ∇.

Suppose, ∇ is such a connection, and let $X, Y, Z \in \mathfrak{X}(M)$.

Write the compatibility equation 3 times with X,Y,Z cyclically permuted:

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \\ Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \end{aligned}$$

Using the symmetry condition on the last term in each line, and rewrite

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X, Z]\rangle, \\ Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle + \langle Z, [Y, X]\rangle, \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X, [Z, Y]\rangle. \end{aligned}$$

Adding the first two of these equations and subtracting the third:

$$X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle = 2\langle \nabla_X Y, Z\rangle + \langle Y, [X, Z]\rangle + \langle Z, [Y, X]\rangle - \langle X, [Z, Y]\rangle.$$

Finally, solving for $\langle \nabla_X Y, Z \rangle$, we get

$$\begin{aligned} \langle \nabla_X Y, Z \rangle = \tfrac{1}{2}\Big(& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle \Big). \end{aligned} \tag{39}$$

Now suppose ∇^1 and ∇^2 are two connections on TM that are symmetric and compatible with g.

Since the r.h.s. of (39) does not depend on the connection, it follows that $\langle \nabla^1_X Y - \nabla^2_X Y, Z \rangle = 0$ for all X, Y, Z.

This can happen only if $\nabla^1_X Y = \nabla^2_X Y$ for all X and Y, so $\nabla^1 = \nabla^2$.

To prove existence, we use (39), or rather a coordinate version of it.

It suffices to prove that such a connection exists in each coordinate chart, for then uniqueness ensures that the connections in different charts agree where they overlap.

Let $\big(U,(x^i)\big)$ be any smooth local coordinate chart.

Applying (39) to the coordinate vector fields, whose Lie brackets are zero, we obtain

$$\langle \nabla_{\partial_i}\partial_j, \partial_l\rangle = \tfrac{1}{2}\Big(\partial_i\langle\partial_j,\partial_l\rangle + \partial_j\langle\partial_l,\partial_i\rangle - \partial_l\langle\partial_i,\partial_j\rangle\Big). \tag{40}$$

Recall the definitions of the metric coefficients and the connection coefficients

$$g_{ij} = \langle\partial_i,\partial_j\rangle, \quad \nabla_{\partial_i}\partial_j = \Gamma_{ij}^m\partial_m. \tag{41}$$

Inserting these into (40) yields

$$\Gamma_{ij}^m g_{ml} = \tfrac{1}{2}\Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\Big). \tag{42}$$

Finally, multiplying both sides by the inverse matrix g^{kl} and noting that $g_{ml}g^{kl} = \delta_m^k$, we get

$$\Gamma_{ij}^k = \tfrac{1}{2}g^{kl}\Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\Big). \tag{43}$$

This formula certainly defines a connection in each chart, and it is evident from this formula that $\Gamma_{ij}^k = \Gamma_{ji}^k$, i.e., the connection is symmetric.

It remains to show the compatibility with the metric.

Using (42) twice, we get

$$\begin{aligned}\Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{il} &= \tfrac{1}{2}\Big(\partial_k g_{ij} + \partial_i g_{kj} - \partial_j g_{ki}\Big) + \tfrac{1}{2}\Big(\partial_k g_{ji} + \partial_j g_{ki} - \partial_i g_{kj}\Big) \\ &= \partial_k g_{ij}.\end{aligned}$$

This shows that ∇ is compatible with g. $\square$

To summarize:

Corollary (Formulas for the Levi-Civita Connection)

Let (M, g) be a (pseudo-)Riemannian manifold, and let ∇ be its Levi-Civita connection.

(a) *For $X, Y, Z \in \mathfrak{X}(M)$ we get Koszul's formula*

$$\langle \nabla_X Y, Z\rangle = \tfrac{1}{2}\big(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle\big).$$

(b) *In a smooth chart for M, the coefficients ∇ are given by*

$$\Gamma_{ij}^k = \tfrac{1}{2} g^{kl}\Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\Big).$$

(c) *Let (E_i) be a smooth local frame on an open $U \subset M$, and let $c_{ij}^k \colon U \to \mathbb{R}$ be the n^3 functions defined by $[E_i, E_j] = c_{ij}^k E_k$. Then*

$$\Gamma_{ij}^k = \tfrac{1}{2} g^{kl}\Big(E_i g_{jl} + E_j g_{il} - E_l g_{ij} - g_{jm} c_{il}^m - g_{lm} c_{ji}^m + g_{im} c_{lj}^m\Big). \tag{44}$$

(d) *If g is Riemannian, (E_i) a local orthonormal frame, the c_{ij}^k as above, then*

$$\Gamma_{ij}^k = \tfrac{1}{2}\Big(c_{ij}^k - c_{ik}^j - c_{jk}^i\Big). \tag{45}$$

Proof:
(a) and (b) were already shown. To prove (44) apply Koszul's formula with $X = E_i$, $Y = E_j$, and $Z = E_l$, to obtain

$$\begin{aligned}\Gamma_{ij}^q g_{ql} &= \langle \nabla_{E_i} E_j, E_l \rangle \\ &= \tfrac{1}{2}\Big(E_i g_{jl} + E_j g_{il} - E_l g_{ij} - g_{jm} c_{il}^m - g_{lm} c_{ji}^m + g_{im} c_{lj}^m \Big).\end{aligned}$$

Multiplying both sides by g^{kl} yields (44).

Finally, under the hypotheses of d), we have $g_{ij} = \delta_{ij}$, so (44) reduces to (45) after rearranging and using the fact that c_{ij}^k is skewsymmetric in i, j. $\square$

- In the sequel, on every Riemannian or pseudo-Riemannian manifold, we will always use the Levi-Civita connection without further comment.
- Geodesics with respect to this connection are called Riemannian (or pseudo-Riemannian) geodesics, or simply "geodesics" as long as there is no risk of confusion.
- The connection coefficients Γ_{ij}^k of the Levi-Civita connection in coordinates, given by (43), are called the Christoffel symbols of g.
- The next proposition shows that these connections are familiar ones in the case of embedded submanifolds of Euclidean or pseudo-Euclidean spaces.

Proposition

(a) *The Levi-Civita connection on a (pseudo-)Euclidean space is equal to the Euclidean connection.*

(b) *Suppose M is an embedded (pseudo-)Riemannian submanifold of a (pseudo-) Euclidean space.*
Then the Levi-Civita connection on M is equal to the tangential connection $\nabla^\top$.

Proof:
We observed earlier that the Euclidean connection is symmetric and compatible with both the Euclidean metric $\overline{g}$ and the pseudo-Euclidean metrics $\overline{q}^{(r,s)}$, which implies (a).

Part (b) then follows from the Propositions above. $\square$

Proposition (Naturality of the Levi-Civita Connection)

Suppose $(M, g), (\widetilde{M}, \tilde{g})$ are Riemannian or pseudo-Riemannian manifolds. Let $\nabla, \widetilde{\nabla}$ denote the corresponding Levi-Civita connections. If $\varphi\colon M \to \widetilde{M}$ is an isometry, then $\varphi^*\widetilde{\nabla} = \nabla$.

Proof: See Exercises.

Corollary (Naturality of Geodesics)

Suppose $(M, g), (\widetilde{M}, \tilde{g})$ are Riemannian or pseudo-Riemannian manifolds and $\varphi\colon M \to \widetilde{M}$ is a local isometry.

If γ is a geodesic in M, then $\varphi \circ \gamma$ is a geodesic in $\widetilde{M}$.

Proof:
This is an immediate consequence of the above Proposition, together with the fact that being a geodesic is a local property. □

The Exponential Map

- Let (M, g) be a Riemannian or pseudo-Riemannian n-manifold, endowed with its Levi-Civita connection.
- We showed that each initial point $p \in M$ and each initial velocity vector $v \in T_pM$ determine a unique maximal geodesic γ_v.
- To deepen our understanding of geodesics, we need to study their collective behavior, and in particular, to address the following question:
  How do geodesics change if we vary p or v?
- The dependence of geodesics on the initial data is encoded in a map from the tangent bundle into the manifold, called the exponential map.
- Its properties are fundamental to the further study of Riemannian geometry.

Remark

Note that the existence of the exponential map and the basic properties expressed in the Proposition below hold for **every** *connection in TM, not just for the Levi-Civita connection.*

For simplicity, we restrict attention here to the latter case, because that is all we need.

Geodesics with proportional initial velocities are related in a simple way:

Lemma (Rescaling Lemma)

Consider $\gamma\colon I \to M$ a geodesic.

For every $p \in M$, $v \in T_pM$, and $c, t \in \mathbb{R}$,

$$\gamma_{cv}(t) = \gamma_v(ct), \tag{46}$$

whenever either side is defined.

Here we use the agreement that $\gamma_v\colon I \to M$ denotes a geodesic with $\gamma_v(0) = p \in M$ and $\dot\gamma_v(0) = v \in T_pM$.

That is, γ is a geodesic on M starting for $t = 0$ in p and emanating at 0 in tangent direction v.

Proof:
If $c = 0$, then both sides of (46) are equal to p for all $t \in \mathbb{R}$, so we may assume that $c \neq 0$.

It suffices to show that $\gamma_{cv}(t)$ exists and (46) holds whenever the r.h.s. is defined.

Certainly, the same argument with the substitutions $v = c'v'$, $t = c't'$, and $c = 1/c'$ then implies that the conclusion holds when only the l.h.s. is known to be defined.

Suppose the maximal domain of γ_v is the open interval $I \subset \mathbb{R}$.

For simplicity, write $\gamma = \gamma_v$, and define a new curve $\tilde{\gamma} \colon c^{-1}I \to M$ by $\tilde{\gamma}(t) := \gamma(ct)$, where $c^{-1}I := \{c^{-1}t \mid t \in I\}$.

We will show, $\tilde{\gamma}$ is a geodesic with initial point p and initial velocity cv.

It then follows by uniqueness and maximality of solutions of ODEs that it must be equal to γ_{cv}.

It is immediate from the definition that $\tilde{\gamma}(0) = \gamma(0) = p$.

Choose smooth local coordinates on M and write $\gamma = \big(\gamma^1(t), \dots, \gamma^n(t)\big)$.

The chain rule gives

$$\dot{\tilde{\gamma}}^i(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}\gamma^i(ct) = c\dot{\gamma}^i(ct).$$

In particular, it follows that $\tilde{\gamma}'(0) = c\gamma'(0) = cv$.

Now let $\mathrm{D}_t, \widetilde{\mathrm{D}}_t$ denote the covariant differentiation operators along $\gamma, \tilde{\gamma}$, respectively. Using the chain rule again in coordinates yields

$$\begin{aligned}
\widetilde{\mathrm{D}}_t\tilde{\gamma}'(t) &= \Big(\tfrac{\mathrm{d}}{\mathrm{d}t}\dot{\tilde{\gamma}}^k(t) + \Gamma^k_{ij}\big(\tilde{\gamma}(t)\big)\dot{\tilde{\gamma}}^i(t)\dot{\tilde{\gamma}}^j(t)\Big)\partial_k \\
&= \Big(c^2\ddot{\gamma}^k(ct) + c^2\Gamma^k_{ij}\big(\gamma(ct)\big)\dot{\gamma}^i(ct)\dot{\gamma}^j(ct)\Big)\partial_k \\
&= c^2\,\mathrm{D}_t\,\gamma'(ct) = 0.
\end{aligned}$$

Thus $\tilde{\gamma}$ is a geodesic, so $\tilde{\gamma} = \gamma_{cv}$, as claimed. $\square$

- We continue by the following idea.
  The assignment $v \mapsto \gamma_v$ defines a map from TM to the set of geodesics in M.
- More importantly, by virtue of the rescaling lemma 443, it allows us to define a map from (a subset of) the tangent bundle to M itself.
- This sends each line through the origin in T_pM to a geodesic.

Definition

Define a subset $\mathcal{E} \subset TM$, the domain of the (Riemannian) exponential map, by

$$\mathcal{E} := \{v \in TM \mid \gamma_v \text{ is defined on an interval containing } [0,1]\},$$

and then define the (Riemannian) exponential map $\exp\colon \mathcal{E} \to M$ by

$$\exp(v) := \gamma_v(1).$$

For each $p \in M$, the restricted (Riemannian) exponential map at p, denoted by $\exp_p$, is the restriction of exp to the set $\mathcal{E}_p := \mathcal{E} \cap T_pM$.

The exponential map of a Riemannian manifold should **not** be confused with the exponential map of a Lie group.

The two are closely related for bi-invariant metrics, but in general they need not be.

The Exponential Map, cont'd

- The next proposition describes some essential properties of the exponential map.
- Recall: a subset S of a vector space is said to be star-shaped w.r.t. a point $x \in S$ if for every $y \in S$, the line segment from x to y is contained in S.

Proposition (Properties of the Exponential Map)

Let (M, g) be a Riemannian or pseudo-Riemannian manifold, and let $\exp\colon \mathcal{E} \to M$ be its exponential map.

a) *The set $\mathcal{E}$ is an open subset of TM containing the image of the zero section (i.e., the zero vector field $(p, 0)$ for all $p \in M$), and each $\mathcal{E}_p := (\mathcal{E} \cap T_pM) \subset T_pM$ is star-shaped w.r.t. 0.*

b) *For each $v \in T_pM$, the geodesic γ_v is given by*

$$\gamma_v(t) = \exp(tv) \tag{47}$$

for all t such that either side is defined.

c) *The exponential map is smooth.*

d) *For each point $p \in M$, the derivative $\mathrm{D}(\exp_p)_0\colon T_0(T_pM) \cong T_pM \to T_pM$ is the identity map of T_pM, under the usual identification $T_0(T_pM) \cong T_pM$.*

Proof:
Write $n = \dim M$. The rescaling lemma with $t = 1$ says precisely that $\exp(cv) = \gamma_{cv}(1) = \gamma_v(c)$ whenever either side is defined; this shows b).

Moreover, if $v \in \mathcal{E}_p$, then by definition γ_v is defined at least on $[0, 1]$. Thus for $0 \leq t \leq 1$, the rescaling lemma says that

$$\begin{aligned} \exp_p(tv) &= \gamma_{tv}(1) \\ &= \gamma_v(t) \end{aligned} \tag{48}$$

is defined. This shows that $\mathcal{E}_p$ is star-shaped with respect to 0, showing a).

Next we will prove that $\mathcal{E}$ is open and exp is smooth.
We first reformulate the proof for existence and uniqueness for geodesics in a more invariant way.

Let (x^i) be local coordinates on $U \subset M$ open,
$\pi\colon TM \to M$ is the projection, and
(x^i, v^i) denote associated coordinates for $\pi^{-1}(U) \subset TM$.

A standard trick in ODE theory exploits the fact that the second-order ODE (here the geodesic equation on M), is equivalent to the system of first-order ODEs

$$\begin{aligned} \dot{x}^k(t) &= v^k(t), \\ \dot{v}^k(t) &= -v^i(t)v^j(t)\Gamma^k_{ij}\big(x(t)\big), \end{aligned} \tag{49}$$

under the substitution $v^k = \dot{x}^k$.
Recognize (49) as the equations for the flow of $G \in \mathfrak{X}(U \times \mathbb{R}^n)$ given by

$$G_{(x,v)} = v^k \frac{\partial}{\partial x^k}\Big|_{(x,v)} - v^i v^j \Gamma^k_{ij}(x) \frac{\partial}{\partial v^k}\Big|_{(x,v)}. \tag{50}$$

Stated somewhat more invariantly:
Every integral curve of G on $\pi^{-1}(U)$ **projects** to a geodesic under $\pi\colon TM \to M$, which in these coordinates is just $\pi(x, v) = x$.

Conversely, every geodesic $\gamma(t) = \big(x^1(t), \dots, x^n(t)\big)$ in U **lifts** to an integral curve of G in $\pi^{-1}(U)$ by setting $v^i(t) = \dot{x}^i(t)$.

The importance of G stems from the fact that it actually defines a **global** vector field on the total space of TM, called the geodesic vector field.

A key observation, to be proved below, is that G acts on $f \in C^\infty(TM)$ by

$$Gf(p,v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f\big(\gamma_v(t), \gamma_v'(t)\big). \tag{51}$$

Since this formula is independent of coordinates, it shows that the various definitions of G given by (50) in different coordinate systems agree.

To prove (51), write the components of $\gamma_v(t)$ as $x^i(t)$ and $v^i(t) = \dot{x}^i(t)$.

Using the chain rule and the geodesic eq. (49), the r.h.s. of (51) is as

$$\begin{aligned}\Big(\frac{\partial f}{\partial x^k}\big(x(t), v(t)\big)\dot{x}^k(t) + \frac{\partial f}{\partial v^k}\big(x(t), v(t)\big)\dot{v}^k(t)\Big)\Big|_{t=0} \\ &= \frac{\partial f}{\partial x^k}(p,v)v^k + \frac{\partial f}{\partial v^k}(p,v)v^i v^j \Gamma^k_{ij}(p) \\ &= Gf(p,v).\end{aligned}$$

By ODE theory, there exist an open set $\mathcal{D} \subset \mathbb{R} \times TM$ containing $\{0\} \times TM$ and a smooth map $\theta\colon \mathcal{D} \to TM$,
s.t. each $\theta^{(p,v)}(t) = \theta\big(t,(p,v)\big)$ is the unique maximal integral curve of G starting at (p,v), defined on an open interval containing 0.

Now suppose $(p,v) \in \mathcal{E}$.
This means that γ_v is defined at least on $[0,1]$.

Therefore so is the integral curve of G starting at $(p,v) \in TM$.

Since $\big(1,(p,v)\big) \in \mathcal{D}$, there is a neighborhood of $\big(1,(p,v)\big)$ in $\mathbb{R} \times TM$ on which the flow of G is defined.
In particular, this means that there is a neighborhood of (p,v) on which the flow exists for $t \in [0,1]$, and therefore on which the exponential map is defined.

This shows that $\mathcal{E}$ is open.

Geodesics are projections of integral curves of G, so exp can be expressed as

$$\exp_p(v) = \gamma_v(1) = \pi \circ \theta\big(1, (p, v)\big),$$

wherever it is defined, and therefore $(p, v) \mapsto \exp_p(v)$ is smooth in (p, v).

To compute $\mathrm{D}(\exp_p)_0(v) = \mathrm{D}\exp_p(0) \cdot v$ for an arbitrary $v \in T_pM$, we choose a curve τ in T_pM with $\tau(0) = p$ and $\dot\tau(0) = v$.
Then we compute the initial velocity of $\exp_p \circ \tau$.

Certainly, already the straight line $\tau(t) := tv$ does the job, yielding

$$\begin{aligned}
\mathrm{D}(\exp_p)_0(v) &= \tfrac{\mathrm{d}}{\mathrm{d}t}(\exp_p \circ \tau)(t)\big|_{t=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t} \exp_p(tv)\big|_{t=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}\gamma_v(t)\big|_{t=0} = \dot\gamma_v(0) \\
&= v.
\end{aligned}$$

Thus $\mathrm{D}(\exp_p)_0$ is the identity map. $\square$

Our corollary on the naturality of geodesics translates into the following important property of the exponential map:

Proposition (Naturality of the Exponential Map)

Suppose $(M, g), (\widetilde{M}, \tilde{g})$ are Riemannian or pseudo-Riemannian manifolds and $\varphi\colon M \to \widetilde{M}$ is a local isometry.
Then for every $p \in M$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{E}_p \arrow[r, "\mathrm{D}\,\varphi_p"] \arrow[d, "\exp_p"'] & \widetilde{\mathcal{E}}_{\varphi(p)} \arrow[d, "\exp_{\varphi(p)}"] \\ M \arrow[r, "\varphi"] & \widetilde{M} \end{tikzcd}$$

where $\mathcal{E}_p \subset T_pM$ and $\widetilde{\mathcal{E}}_{\varphi(p)} \subset T_{\varphi(p)}\widetilde{M}$ are the domains of the restricted exponential maps $\exp_p$ (w.r.t. g) and $\exp_{\varphi(p)}$ (w.r.t. $\tilde{g}$), respectively.

An important consequence of the naturality of exp is the following:

Local isometries of connected manifolds are completely determined by their values and differentials at a single point. More precisely:

Corollary

Let $(M, g), (\widetilde{M}, \tilde{g})$ be Riemannian or pseudo-Riemannian manifolds, with M connected.

Suppose $\varphi, \psi \colon M \to \widetilde{M}$ are local isometries, s.t. for some $p \in M$, we have

$$\varphi(p) = \psi(p), \quad \mathrm{D}\,\varphi(p) = \mathrm{D}\,\psi(p).$$

Then $\varphi \equiv \psi$. □

Definition

A Riemannian or pseudo-Riemannian manifold M is geodesically complete if $\mathcal{E}_p = T_pM$ for all $p \in M$.

I.e., the exponential $\exp_p(v)$ is defined for all $p \in M$ and for all $v \in T_pM$, or equivalently, if every maximal geodesic is defined for all $t \in \mathbb{R}$.

Remark

Some examples of manifolds that are **not** *geodesically complete:*

- *In every proper open subset of $\mathbb{R}^n$ with its (pseudo-)Euclidean metric, there are geodesics that reach the boundary in finite time. E.g. think of any secant in the open unit disc of $\mathbb{R}^2$.*
- *Similarly, on $\mathbb{R}^n$ with the pull back metric $(\sigma^{-1})^*\mathring{g}$, obtained from the sphere with round metric $\mathring{g} = \bar{g}|_{S^{n-1}}$, by stereographic projection σ. There are geodesics that escape to infinity in finite time. E.g., think of any meridian, i.e., any great semicircle connecting south pole with north pole on S^2. A meridian has length π, it is however mapped to a half-line.*

Normal Neighborhoods and Normal Coordinates

- Let (M, g) be a (pseudo-)Riemannian manifold of dimension n.
- For every $p \in M$, the restricted exponential map $\exp_p$ maps the open subset $\mathcal{E}_p \subset T_pM$ smoothly into M.
- As $\mathrm{D}(\exp_p)_0$ is invertible, the inverse function theorem guarantees that there exist a neighborhood V of 0 in T_pM and a neighborhood U of p in M s.t. $\exp_p \colon V \to U$ is a diffeomorphism.

Definition

A neighborhood U of $p \in M$ that is the diffeomorphic image under $\exp_p$ of a *star-shaped* neighborhood of $0 \in T_pM$ is called a normal neighborhood of p.

Definition

Every ONB (b_i) for T_pM determines a basis isomorphism

$$B\colon \mathbb{R}^n \to T_pM, \quad B(x^1,\dots,x^n) = x^i b_i.$$

If $U := \exp_p(V)$ is a normal neighborhood of p,
we can combine this isomorphism with the exponential map to get a smooth coordinate map

$$\varphi = B^{-1} \circ (\exp_p|_V)^{-1}\colon U \to \mathbb{R}^n :$$

$$\begin{tikzcd} T_pM \arrow[r, "B^{-1}"] & \mathbb{R}^n \\ U \arrow[u, "(\exp_p|_V)^{-1}"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Such coordinates are called pseudo-Riemannian normal coordinates centered at p or simply Riemannian normal coordinates if the metric is clear.

Proposition (Properties of Normal Coordinates)

Let (M, g) be a (pseudo-)Riemannian n-manifold, let $\big(U, (x^i)\big)$ be any normal coordinate chart centered at $p \in M$.

a) *The coordinates of p are $(0, \dots, 0)$.*

b) *The components of the metric at p are $g_{ij} = \delta_{ij}$ if g is Riemannian, and $g_{ij} = \pm\delta_{ij}$ otherwise.*

c) *For every $v = v^i \partial_i|_p \in T_pM$, the geodesic γ_v starting at p with initial velocity v is represented in normal coordinates by the line*

$$\gamma_v(t) = \big(tv^1, \dots, tv^n\big), \tag{52}$$

as long as t is in some interval I containing 0 such that $\gamma_v(I) \subset U$.

d) *The Christoffel symbols in these coordinates vanish at p.*

e) *All of the first partial derivatives of g_{ij} in these coordinates vanish at p.*

Proof:
Parts a), b) and c) follow more or less directly from the definition of normal coordinates and from a proposition of the last lecture (properties of exp).

To prove d), let $v = v^i \partial_i|_p \in T_pM$ be arbitrary. Then in coordinates the geodesic equation for $\gamma_v(t) = \left(tv^1, \ldots, tv^n\right)$ simplifies to

$$\Gamma^k_{ij}(tv)v^i v^j = 0.$$

Evaluating at $t = 0$ shows that $\Gamma^k_{ij}(0)v^i v^j = 0$ for every k and every v.

In particular, with $v = \partial_a$ for some fixed a, this shows that $\Gamma^k_{aa} = 0$ for each a and k (no summation!).

Substituting $v = \partial_a + \partial_b$ and $v = \partial_b - \partial_a$ for any fixed pair of indices a and b and subtracting, we conclude also that $\Gamma^k_{ab} = 0$ at p for all a, b, k.

Finally, e) follows from d) together with the symmetry condition of the connection coefficents in the case $E_k = \partial_k$. $\square$

Definition

Because they are given by the simple formula (52), the geodesics starting at p and lying in a normal neighborhood of p are called radial geodesics.

Remark

1. *Be warned, that geodesics lying in a normal neigborhood of p, but that do* **not** *pass through p, do* **not** *in general have a* **simple** *form in normal coordinates.*
2. *Be careful also with the statements in the above proposition.*

   The statements b), d) and e) tell you certain properties **at** *p.*

   They are in general wrong in a neighborhood of p.

Example

Consider $S^{n-1} \subset \mathbb{R}^n$ as Riemannian submanifold of the Euclidean $(\mathbb{R}^n, \overline{g})$.

1. A geodesic starting in $\gamma(0) = p \in S^{n-1}$ emanating in direction $\dot{\gamma}(0) = v \in T_p S^{n-1}$ is given by

$$\gamma_v \colon \mathbb{R} \to S^{n-1}, \quad \gamma_v(t) = \mathrm{e}^{t(vp^\top - pv^\top)} \cdot p. \tag{53}$$

2. The restricted Riemannian exponential map for $\left(S^{n-1}, \overline{g}|_{TS^{n-1}}\right)$ is given by

$$\exp_p \colon \mathcal{E}_p = T_p S^{n-1} \to S^{n-1}, \quad \exp_p(v) = \gamma_v(1). \tag{54}$$

For a proof see the Exercises.

Restricted exp, $\exp^{\mathrm{Lie}}$, and Geodesics on SO_n

Consider $\mathrm{SO}_n \subset \mathbb{R}^{n\times n}$ as Riemannian submanifold of Euclidean $(\mathbb{R}^{n\times n}, g)$.
Here g denotes the metric defined by the Frobenius inner product on $\mathbb{R}^{n\times n}$, i.e., for any $(X, A), (X, B) \in T_X\mathbb{R}^{n\times n}$ we get

$$g\big((X, A), (X, B)\big) := \mathrm{tr}(A^\top B).$$

By restriction this equals a multiple of the bi-invariant Killing form of SO_n.
We compute the (restricted) Riemannian exponential map.
The ODE, a curve on SO_n has to satisfy to be a geodesic, is

$$\begin{aligned} R'' &= (\mathrm{id} - \pi^\top) \circ R'' \\ &= R'' - \tfrac{1}{2}(R'' - RR''^\top R) = \tfrac{1}{2}(R'' + RR''^\top R). \end{aligned} \tag{55}$$

Here $\pi^\top$ denotes the orthogonal projection operator $\mathbb{R}^{n\times n} \to T_R\,\mathrm{SO}_n$ introduced in the exercise sheet. Eq. (55) means, that R'' has only a normal component.

Rewriting (55) exploiting orthogonality of R gives

$$R'' = RR''^{\top}R \quad \Longleftrightarrow \quad R^{\top}R'' = R''^{\top}R. \tag{56}$$

In particular we see that the product $R''^{\top}R$ is a **symmetric** matrix.
The ODEs (55) and (56) still look very ugly, being nonlinear and implicit.
However, we can rewrite them as **linear time-invariant ODE**.

We proceed as follows:
The defining equation for orthogonal matrices is

$$R^{\top}R = RR^{\top} = I. \tag{57}$$

Taking derivatives in (57), thinking of smooth curves $I \to \mathrm{SO}_n$, $t \mapsto R(t)$, gives us four more relations,

$$\begin{aligned} R'^{\top}R + R^{\top}R' &= 0, \\ R''^{\top}R + 2R'^{\top}R' + R^{\top}R'' &= 0, \\ R'R^{\top} + RR'^{\top} &= 0, \\ R''R^{\top} + 2R'R'^{\top} + RR''^{\top} &= 0. \end{aligned} \tag{58}$$

They imply, because of the second equality in (56), that

$$-R'^\top R' = R''^\top R, \quad -R'R'^\top = R''R^\top. \tag{59}$$

Consequently, using (59), our geodesic equation can be rewritten as

$$\begin{aligned} R'' &= RR''^\top R \\ &= -RR'^\top R' \\ &= R'R^\top R'. \end{aligned} \tag{60}$$

We claim that the skew-symmetric matrix

$$\Omega(t) := R'(t)R^\top(t) \tag{61}$$

is actually **time-independent**.
Skew-symmetry of Ω follows from the third equality in (58).

Indeed, taking the derivative of (61) gives

$$\Omega' = \underbrace{R''}_{\overset{\text{by (56)}}{=} RR''^\top R} \cdot R^\top + R'R'^\top$$

$$= R \cdot \underbrace{R''^\top R}_{\overset{\text{by (59)}}{=} -R'^\top R'} \cdot R^\top + R'R'^\top = \underbrace{-\overbrace{RR'^\top}^{=-R'R^\top} \cdot \overbrace{R'R^\top}^{=-RR'^\top}}_{=-R'R'^\top} + R'R'^\top = 0. \tag{62}$$

Hence,

$$\Omega(t) = \Omega(0) = R'(0)R^\top(0) =: \Omega_0.$$

Now, the geodesic equation is equivalent to

$$R'' = \Omega_0 R', \tag{63}$$

and after integration we get the linear time-invariant ODE

$$R'(t) = \Omega_0 R(t) + C, \tag{64}$$

but the integration constant C is equal to zero (evaluate (64) at zero).

To summarize, the unique solution to the initial value problem

$$R'' = RR''^{\top}R, \quad R(0) = R_0 \in \mathrm{SO}_n, \quad R'(0) = R_0' \in T_{R_0}\mathrm{SO}_n$$

is as

$$R(t) = \mathrm{e}^{tR_0'R_0^{\top}}\, R_0 = \mathrm{e}^{t\Omega_0}\, R_0.$$

In this case we therefore have the remarkable result at the identity I:

$$\big(\exp_I\colon T_I\,\mathrm{SO}_n \to \mathrm{SO}_n\big) = \big(\exp^{\mathrm{Lie}}\colon \mathfrak{so}_n \to \mathrm{SO}_n\big) = \big(\mathrm{e}^{(\cdot)}\colon \mathfrak{so}_n \to \mathrm{SO}_n\big).$$

The reason for this comes from the fact that the metric was bi-invariant.

Normal coordinates we get by mapping a suitable neighborhood V of the origin (i.e., the zero matrix) in the tangent space $T_R\,\mathrm{SO}_n \cong \mathfrak{so}_n \cdot R$ to its image in the group, i.e., $U := \mathrm{e}^V \cdot R \subset \mathrm{SO}_n$, with U forming a neighborhood of R.

Explicitly, we have for the restricted exponential map

$$\exp_R\colon T_R\,\mathrm{SO}_n \to \mathrm{SO}_n, \quad \Omega R \mapsto \mathrm{e}^{\Omega}\, R, \quad \Omega \in \mathfrak{so}_n\,.$$

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