Electrodynamics of superconductor

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The following exercises are intended to understand the London-Pippard phenomenological theory. We are going to derive them with a simple model. For that we start considering the supercurrent can be thought as a nonviscous charged fluid with field velocity $\vec{v}(\vec{x},t)$, that obeys

$$\vec{j}(\vec{x},t) = -n_S e\vec{v}(\vec{x},t),$$

where n_S is the density of the super-electron number density and $-e$ is the electronic charge. The continuity equation together with the Newton's second law give

$$\vec{\nabla}\cdot\vec{j} = \vec{\nabla}\cdot\vec{v} = 0,$$
$$\frac{d\vec{v}}{dt} = -\frac{e}{m}\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{h}\right),$$

with $\vec{h}$ is the *microscopic field* which is larger compared to the atomic size but smaller compared to the penetration depth. Notice that this also involves the total derivative with respect time of the velocity and not just its partial (explicit) dependence.

Problem 1

- Define the effective field
$$\vec{Q} = \vec{\nabla}\times\vec{v} - \frac{e\vec{h}}{mc}$$
Together with the Maxwell's equations, derive the dynamical equation for $\vec{v}$ for a superconducting bulk under the constrain $\vec{Q} = 0$. **Hint**: Notice that in Newton's second law equation it was written d/dt and not $\partial/\partial t$. Recall for a field vector which is the relation between those two and use it to get the equation $\partial v/\partial t$.
- How can we interpret $\vec{Q}$?

Problem 2

Now consider a semi-infinite slab in the 3D space that occupies $z > 0$. One of the equations of the previous item can be rewritten as $\vec{h} = -\frac{mc}{n_S e^2}\vec{\nabla}\times\vec{j}$.

- Rewrite the previous equation using the Maxwell's equations for static fields to write $\vec{h}$ in terms of spatial derivatives of $\vec{h}$.
- Consider an applied field $\vec{H} = H_0\hat{x}$ parallel to the surface. Recalling that inside of the superconductor the field has to be zero, find an acceptable solution for the microscopic field. What is the penetration depth? Identify the London penetration depth λ_L.
- If the slab has a 2d thickness, under the same external field, can you find an appropriate solution for $h(z)$? Check that it is a solution of the differential equation that you found before. Consider the mean magnetic field in the superconductor as the spatial average of $h(z)$. What can you say about the Meissner effect in the two limits $d \ll \lambda_L$ and $d \gg \lambda_L$?

Problem 3

Now we can find a conservation law. For simplicity we start with linearized equations

$$-\frac{1}{n_S e}\frac{\partial \vec{j}}{\partial t} = \frac{\partial \vec{v}}{\partial t} = -\frac{e\vec{E}}{m}.$$

Consider a surface S bounded by a fixed closed curve C that lies wholly in the superconducting material. S can be wherever placed. Integrating Maxwell's equations one has

$$\int d\vec{S}\cdot\frac{\partial \vec{h}}{\partial t} = -c\int d\vec{S}\cdot(\vec{\nabla}\times\vec{E}) = -c\oint_C d\vec{l}\cdot\vec{E}.$$

- Find the conserved quantity.
- Can you add more assumptions to get more specific results? For instance
  * consider that your curve C is far from the boundaries
  * if the interior of C is wholly superconducting
- It is interesting to rewrite the conserved quantity in terms of the vector potential $\vec{A}$ and momenta p. Rederive the conserved quantity. What does it remind you of?

Problem 4

Choosing the London gauge $\vec{\nabla}\cdot\vec{A} = 0$ and $\vec{A}\cdot\hat{n}$ ($\hat{n}$ surface normal) for the vector potential we have

$$\vec{j} = -\frac{n_S e^2}{mc}\vec{A}.$$

We are going to consider a generalization of this equation where $\vec{j}$ is determined as a spatial average of $\vec{A}$ throughout some neighboring region of dimension r_0. The motivation behind this is to take into account non-local superconducting effects. In heavily doped alloys, r_0 is comparable with the electronic mean free path l in the normal metal. We introduce the Pippard's coherence length ξ_0

$$\frac{1}{r_0} = \frac{1}{\xi_0} + \frac{1}{l}.$$

Pippard then rewrote the London equation as an average over space for the field $\vec{A}$

$$\vec{j}(\vec{x}) = -\frac{n_S e^2}{mc}\frac{3}{4\pi\xi_0}\int d^3x' \frac{\vec{X}\left(\vec{X}\cdot\vec{A}(\vec{x}')\right)}{X^4} e^{-X/r_0}.$$

where $\vec{X} = \vec{x} - \vec{x}'$.

- That equation has to be solved together with its corresponding Maxwell's equation, making it quite cumbersome to solve. Nevertheless we can extract physical features. Consider $r_0 \ll \lambda_L$ with λ_L the London penetration length and get a relation for $\vec{j}$ and $\vec{A}$.
- The previous result was obtained under the assumption $r_0 \ll \lambda_L$. Consider the other limit $r_0 \gg \lambda_L$ and consider a current sheet $j_0\delta(z)\hat{y}$ in the xy-plane. The current sheet generates a magnetic field which then induces a supercurrent $\vec{j}$. Write the Maxwell's equation for $\vec{h}$ that has to be solved together with the Pippard's equation.

  * Use Fourier transformation to find the (linear) equation that relates $\vec{j}$ to $\vec{A}$ in the momenta space.

  * Write the final result for $\vec{A}$ and h. Note that it is enough to have them expressed as integrals

  * Can you find the proper limit to recover the Meissner effect?