

Problem Sheet 10

for the tutorial on July 11th, 2025

Quantum Mechanics II

Summer term 2025

Sheet handed out on July 1st, 2025; to be handed in on July 8th, 2025 until 2 pm

Exercise 10.1: Radial component of the Pauli matrix operator

[2+2+2+2+2 P.]

The radial component of the Pauli matrix operator σ is defined as $\sigma_r := e_r \cdot \sigma$.

a) Show that

$$\sigma_r = \begin{pmatrix} \cos \theta & \sin \theta \, e^{-i\phi} \\ \sin \theta \, e^{i\phi} & -\cos \theta \end{pmatrix}.$$

- b) In the lecture it was stated that σ_r commutes with the total angular momentum operator $\hat{J} = \hat{L} + \hat{S}$. Show that explicitly for the component \hat{J}_z^1 .
- c) Show that the functions $\psi_{\kappa m}$ defined in the lecture are eigenfunctions of the parity operator \hat{P} . What are the corresponding eigenvalues²?
- d) Explain why $\sigma_r \psi_{\kappa m}$ and $\psi_{\kappa m}$ have opposite parity.
- e) According to part b) we can write that $\sigma_r \psi_{\kappa m} = a \psi_{\kappa m} + b \psi_{-\kappa m}$ with constants $a, b \in \mathbb{C}$ With the help of c) and d) show that a = 0.

Exercise 10.2: The K operator

[7 P.]

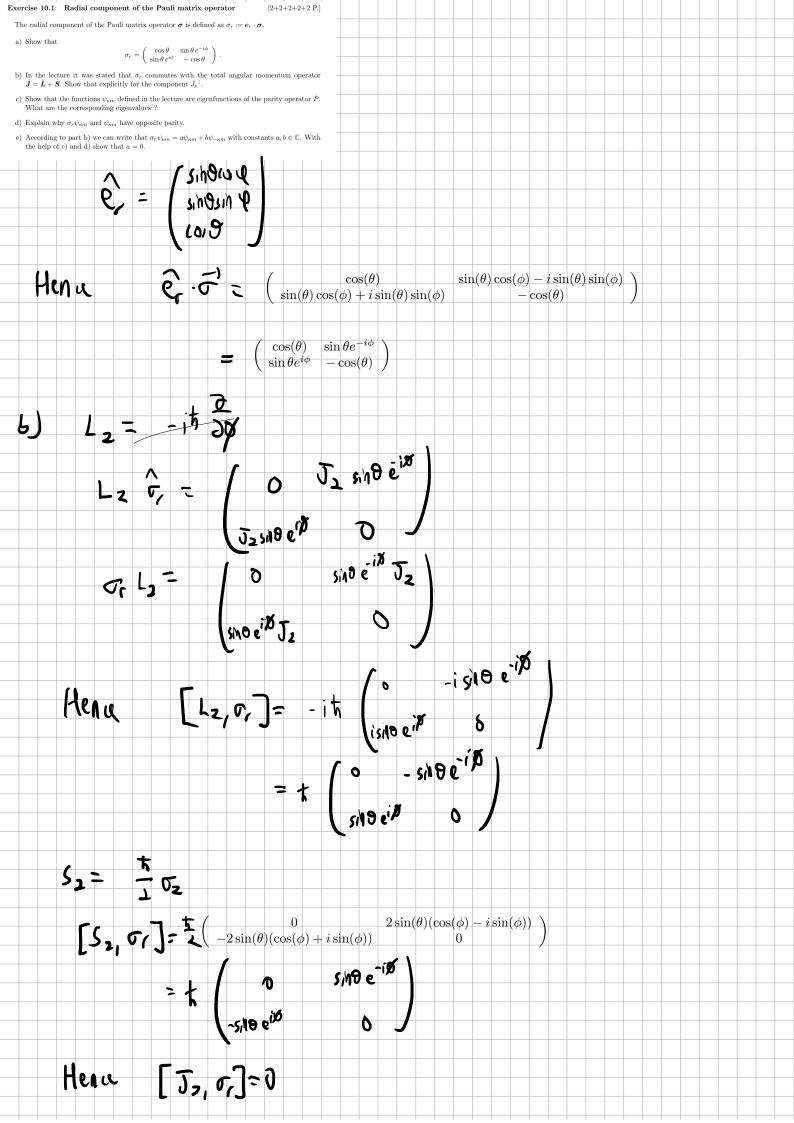
Consider the electromagnetic 4-potential $A_{\mu} = \delta_{\mu 0} \phi(r)/c$. Show that the operator introduced in the lecture

$$\hat{K} = \beta \left(\frac{2}{\hbar^2} \hat{\mathbf{S}} \cdot \hat{\mathbf{L}} + 1 \right) \tag{1}$$

does commute with the Dirac Hamiltonian $\mathcal{H}_D = \gamma^{\mu}(\hat{p}_{\mu} + eA_{\mu}) - mc$.

¹In spherical coordinates $\hat{L}_z = -i\hbar \partial/\partial \phi$.

²The parity operator P acting on a wave function $\psi(\mathbf{r})$ gives $\hat{P}\psi(\mathbf{r}) = \psi(-\mathbf{r})$. For the spherical harmonics it gives $\hat{P}Y_{lm} = (-1)^l Y_{lm}$.



$$\Psi_{jm} = \sum_{m_1 = -l}^{l} \sum_{m_2 = -1/2}^{1/2} (l \ m_1 \ \frac{1}{2} \ m_2 | j \ m) Y_{lm_1}(\theta, \varphi) \chi_{m_2}, \quad --$$

the pority operatur, (9-) (9+7) Under

Thus Yim - (-1) Yim

Eigenfunction with eigenvalue 1

d) Explain why $\sigma_r \psi_{\kappa m}$ and $\psi_{\kappa m}$ have opposite parity.

Under the parity transformation,

(0) 0 -) (0) (7-0) = - (0) 9

sing - sin (7-9) = sin 9

e' p = - e'p

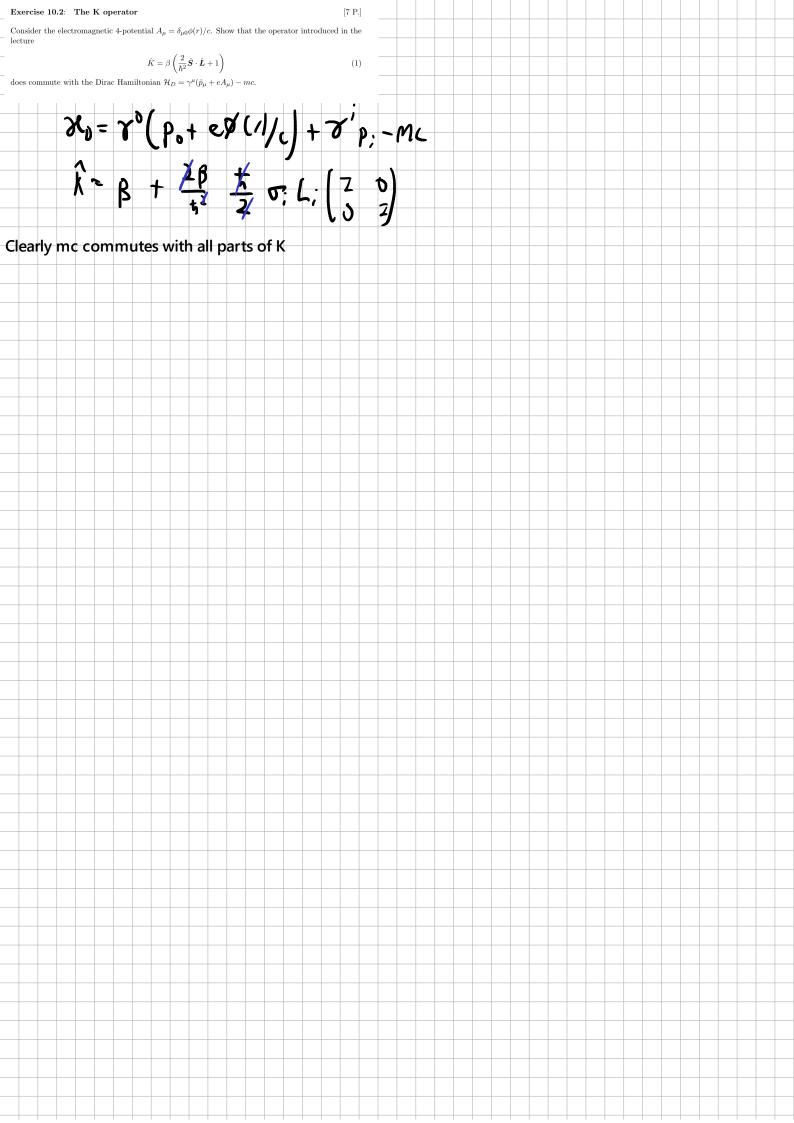
 $\begin{pmatrix} \cos(\theta) & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos(\theta) \end{pmatrix} \rightarrow \begin{pmatrix} -\sin\theta & -\sin\theta e^{-i\phi} \\ -\sin\theta & -\sin\theta \end{pmatrix} = -\nabla$

Hence when applying the parity operator to both, we pick up a negative sign when pulling the or, Alpany the punty parity operator through

e) According to part b) we can write that $\sigma_r \psi_{\kappa m} = a \psi_{\kappa m} + b \psi_{-\kappa m}$ with constants $a, b \in \mathbb{C}$. With the help of c) and d) show that a = 0.

By cancelling out all the powers of -1

adding the two equations, we get



In this task the operator $\hat{\mathbf{S}} \cdot \hat{\mathbf{p}}$ is to be examined where the spin operator reads

$$\hat{\mathbf{S}} = \frac{\hbar}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} . \tag{2}$$

a) The wave function of a free relativistic particle with momentum $p = \hbar k$ is given by

$$\psi_{\pm 1/2} = \begin{pmatrix} \chi_{\pm 1/2} \\ \frac{c\hbar \vec{k}\vec{\sigma}}{E + mc^2} \chi_{\pm 1/2} \end{pmatrix} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$
(3)

where $\chi_{\pm 1/2}$ is the eigenspinor of σ_z with eigenvalue ± 1 . Show that the wave functions $\psi_{\pm 1/2}$ for $\mathbf{k} = k\mathbf{e}_z$ are eigenfunctions of the operator $\hat{\mathbf{S}} \cdot \hat{\mathbf{p}}$. What are the corresponding eigenvalues?

- b) Dividing the eigenvalues from a) by $\hbar^2 k$ gives us the so called helicities of the wave functions. What is the physical meaning of the helicity in this context?
- c) Does $\hat{\mathbf{S}} \cdot \hat{\mathbf{p}}$ commute with the free Dirac Hamiltonian?