

# Theory and Phenomenology of Superconductivity Homework 3

Jun Wei Tan\*

*Julius-Maximilians-Universität Würzburg*

(Dated: November 26, 2025)

**Problem 1** (Problem 1). Consider two reservoirs for electrons at thermal equilibrium. Refer to them as left  $L$  and right  $R$  leads which extend in the  $x$  direction. Imagine we apply small voltages to the two leads which tunes their chemical potential by  $eV_L$  and  $eV_R$ .

- Starting from the scattering matrix approach, relate the states entering and leaving each lead at the interface.
- Within the second quantization formalism, write the formula for the electrical current. **Hint:** it will be useful later to write it as an energy integration.
- Taking advantage of the elements of the scattering matrix, identify in the previous formula the transmission function.
- Identify the equilibrium conductance

$$G = \frac{e^2}{h} \text{Tr} \left[ t^\dagger(E_F) t(E_F) \right], \quad (1)$$

where  $t(E_F)$  is the transmission matrix of our system evaluated at the Fermi energy  $E_F$ . What is the meaning of the eigenvalues of the hermitian matrix  $t^\dagger(E_F)t(E_F)$ ?

*Proof.* (a) We take each lead to have  $n$  channels, each with a quantum number  $\pm$  denoting whether a certain wave is moving right (+) or left (−). Thus, the outgoing waves for the  $L$  ( $R$ ) lead are those going in the − (+) direction, and the incoming waves are those going in the + (−) direction.

Then, denoting the amplitudes for each as  $\vec{a}_{L/R}^\pm$ , where  $\vec{a}$  is a vector denoting the  $n$  channels, the  $S$ -matrix relates the amplitudes of the ingoing and outgoing waves

---

\* jun-wei.tan@stud-mail.uni-wuerzburg.de

as

$$\begin{pmatrix} \vec{a}_L^- \\ \vec{a}_R^+ \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{r} & \mathbf{t}' \\ \mathbf{t} & \mathbf{r}' \end{pmatrix}}_S \begin{pmatrix} \vec{a}_L^+ \\ \vec{a}_R^- \end{pmatrix}.$$

(b) In general, the electrical current can be determined through the derivative  $j^\mu = \frac{\delta S[A]}{\delta A^\mu}$ , and leads to the expression for free particles coupled to an electric field

$$\begin{aligned} \vec{J} &= \vec{J}^\nabla + \vec{J}^{\vec{A}} \\ \vec{J}^\nabla &= \frac{\hbar}{2mi} \left[ \psi^\dagger \nabla \psi - (\nabla \psi^\dagger) \psi \right] \\ \vec{J}^{\vec{A}} &= -\frac{q}{m} \vec{A} \psi^\dagger \psi \end{aligned}$$

In the case of a DC current, we usually have an electric field but no magnetic field. Hence, we can set  $\vec{A} = 0$ .

□

**Problem 2** (Problem 2). Consider a wide conductor along  $x$ , e.g. width in  $y$  direction is large  $W$  while the height in  $z$  direction is small. The length in the  $x$  direction is large  $L$ .

Given information: The density of states of a 2D spin degenerate system is

$$\mathcal{D}_0 = \frac{m}{\pi \hbar^2}.$$

The transmission through a wire with length  $L$  is

$$T = \frac{L_0}{L + L_0},$$

where  $L_0$  is the mean free path. The conductivity is

$$\sigma = e^2 \mathcal{D}_0 D,$$

where

$$D = \frac{v_F L_0}{\pi}$$

is the diffusion coefficient and  $v_F$  is the Fermi velocity.

Using this information, relate the previous results for conductance  $G$  with Ohm's law. I.e., does the relationship between  $G$  and conductivity  $\sigma$  correspond to Ohm's law?