Topological Field Theory WS 2025

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PROBLEM SET 6
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1. Exterior multiplication in phase space

Consider the 2-form

$$\omega^2 = \sum_{i=1}^{n} p_i \wedge q_i. \tag{1}$$

Evaluate the $2k$ form

$$\underbrace{\omega^2 \wedge \omega^2 \wedge \ldots \wedge \omega^2}_{k \text{ factors}} \tag{2}$$

in terms of the p_i's and the q_i's.

2. Example of a pullback of a differential form

Consider the mapping

$$f : M = \mathbb{R}^3 \to N = \mathbb{R}^2 : \; y_1 = x_1 + x_2^2 + x_3^3, \; y_2 = x_1 x_2 x_3, \tag{3}$$

and the 2-form $\omega = dy_1 \wedge dy_2$.

Evaluate the form $f^*\omega$ on M. Is $f^*\omega$ a 2-form or a 3-form?

3. Derivation of the Jacobian determinant formula

Consider the mapping

$$f : M = \mathbb{R}^k \to N = \mathbb{R}^k : \; \boldsymbol{x} \in \mathrm{D} \to \boldsymbol{y} \in f(\mathrm{D}), \tag{4}$$

where D is a convex polyhedra in M.

Use the identity (4.13) from the lectures,

$$\int_{\mathrm{D}} f^*\omega = \int_{f(\mathrm{D})} \omega \tag{5}$$

to derive the Jacobian for a transformation of integration variables from $dy_1 dy_2 \ldots dy_k$ to $dx_1 dx_2 \ldots dx_k$.

Hint: The most general k-form on N is

$$\omega^k = \phi(\boldsymbol{y}) dy_1 \wedge dy_2 \wedge \ldots \wedge dy_k. \tag{6}$$

4. Exterior differentiation

(a) Show $d(\omega^k \wedge \omega^l) = d\omega^k \wedge \omega^l + (-1)^k \omega^k \wedge d\omega^l$.

(b) Show $dd\omega = 0 \; \forall \omega$.

(c) Let $f : M \to N$ be a differentiable mapping, and ω^k a k-form on N. Show $f^*(d\omega) = d(f^*\omega)$.

5. Exterior derivatives in $\mathbb{R}^3$

Recall the following definitions from Sect. 4.3 of the lectures:

$$\omega^1_{\boldsymbol{A}}(\boldsymbol{\xi}) = (\boldsymbol{A}, \boldsymbol{\xi}) \equiv \boldsymbol{A} \cdot \boldsymbol{\xi}, \tag{7}$$

$$\omega^2_{\boldsymbol{A}}(\boldsymbol{\xi_1}, \boldsymbol{\xi_2}) = (\boldsymbol{A}, \boldsymbol{\xi_1}, \boldsymbol{\xi_2}) \equiv \boldsymbol{A} \cdot (\boldsymbol{\xi_1} \times \boldsymbol{\xi_2}), \tag{8}$$

where $\boldsymbol{\xi} = (dx_1, dx_2, dx_3)$, as well as eqs. (4.5a) and (4.5b):

$$\omega^1_{\boldsymbol{A}} \wedge \omega^1_{\boldsymbol{B}} = \omega^2_{\boldsymbol{A} \times \boldsymbol{B}} \tag{9}$$

$$\omega^1_{\boldsymbol{A}} \wedge \omega^2_{\boldsymbol{B}} = (\boldsymbol{A}, \boldsymbol{B})\, dx_1 \wedge dx_2 \wedge dx_3 \tag{10}$$

Show

(a) $df = \omega^1_{\boldsymbol{\nabla} f}$.

(b) $d\omega^1_{\boldsymbol{A}} = \omega^2_{\boldsymbol{\nabla} \times \boldsymbol{A}}$.

(c) $d\omega^2_{\boldsymbol{A}} = (\boldsymbol{\nabla} \boldsymbol{A})\, dx_1 \wedge dx_2 \wedge dx_3$

6. Vector analysis in $\mathbb{R}^3$

Show

(a) $\boldsymbol{\nabla}(\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{\nabla} \times \boldsymbol{A})\boldsymbol{B} - \boldsymbol{A}(\boldsymbol{\nabla} \times \boldsymbol{B})$.
Hint: Use (9), (10), (b), and (c) from Problem 5 above.

(b) $\boldsymbol{\nabla} \times (a\boldsymbol{A}) = (\boldsymbol{\nabla} a) \times \boldsymbol{A} + a(\boldsymbol{\nabla} \times \boldsymbol{A})$.

(c) $\boldsymbol{\nabla}(a\boldsymbol{A}) = (\boldsymbol{\nabla} a)\boldsymbol{A} + a(\boldsymbol{\nabla} \boldsymbol{A})$.

(d) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0$.

(e) $\boldsymbol{\nabla}(\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$.