Homework for the Lecture

Algebra and Dynamics of Quantum Systems

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Homework Sheet No 2

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(27 Points. Submission deadline 31. 10. 2023)

Homework 2-1: States and density matrices

Consider the finite-dimensional pre-Hilbert space $\mathfrak{H} = \mathbb{C}^n$ with its canonical inner product.

i.) **(2 Points)** Show that the matrices $\mathrm{M}_n(\mathbb{C})$ act on $\mathbb{C}^n$ by adjointable operators and determine the induced *-involution. We will always endow $\mathrm{M}_n(\mathbb{C})$ with this *-involution.

ii.) **(3 Points)** Let $\omega\colon \mathrm{M}_n(\mathbb{C}) \longrightarrow \mathbb{C}$ be a positive linear functional. Prove that there exists a matrix $\varrho \in \mathrm{M}_n(\mathbb{C})$ with the property $\langle \phi, \varrho\phi \rangle \geq 0$ for all $\phi \in \mathbb{C}^n$ such that $\omega(A) = \operatorname{tr}(\varrho A)$. Show that ω is a state iff $\operatorname{tr}(\varrho) = 1$. Such a matrix ϱ is called a *density matrix*.

iii.) **(2 Points)** Conversely, show that every density matrix $\varrho \in \mathrm{M}_n(\mathbb{C})$ gives a state on $\mathrm{M}_n(\mathbb{C})$ via the definition $A \mapsto \operatorname{tr}(\varrho A)$.

iv.) **(7 Points)** Show that for a matrix $A \in \mathrm{M}_n(\mathbb{C})$ the following statements are equivalent:

(a) One has $\langle \phi, A\phi \rangle \geq 0$ for all $\phi \in \mathbb{C}^n$.

(b) One has $A = A^*$ and all eigenvalues of A are non-negative.

(c) There is a Hermitian matrix $B = B^*$ with non-negative eigenvalues and $A = B^2$.

(d) There is a Hermitian matrix $B = B^*$ with $A = B^2$.

(e) There is a matrix $B \in \mathrm{M}_n(\mathbb{C})$ with $A = B^* B$.

(f) There are matrices $B_1, \ldots, B_N \in \mathrm{M}_n(\mathbb{C})$ with $A = B_1^* B_1 + \cdots B_N^* B_N$, i.e. A is algebraically positive.

(g) One has $\omega(A) \geq 0$ for all states ω, i.e. A is a positive algebra element.

The content of this homework should be well-known (at least in parts) from linear algebra courses. One can safely skip this homework if familiar with the results. Details can be found in e.g. [1, Sect. 7.8].

Homework 2-2: Polarization identity

Let V and W be two vector spaces over $\mathbb{C}$ and $S\colon V \times V \longrightarrow W$ a *sesquilinear* map, i.e. assume that

$$S(\alpha u + \beta v, w) = \overline{\alpha} S(u, w) + \overline{\beta} S(v, w) \quad \text{and} \quad S(u, \alpha v + \beta w) = \alpha S(u, v) + \beta S(u, w) \tag{2.1}$$

hold for all $\alpha, \beta \in \mathbb{C}$ and $u, v, w \in V$.

i.) Show that the *polarization identity*

$$S(v, w) = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k \cdot S\big(v + \mathrm{i}^{-k} w, v + \mathrm{i}^{-k} w\big) \tag{2.2}$$

holds for all $v, w \in V$. Conclude that S is constant 0 iff $S(v, v) = 0$ for all $v \in V$. **(2 Points)**

ii.) Now let $W = \mathbb{C}$. A sesquilinear map $S\colon V \times V \longrightarrow \mathbb{C}$ is usually called a sesquilinear form. Such a sesquilinear form is said to be Hermitian if $\overline{S(v, w)} = S(w, v)$ holds for all $v, w \in V$. Show that a sesquilinear form S on V is Hermitian if and only if $S(v, v) \in \mathbb{R}$ holds for all $v \in V$. **(2 Points)**

iii.) Let finally $\mathscr{A}$ be a unital *-algebra over $\mathbb{C}$. Show that for every $a \in \mathscr{A}$ there exist algebraically positive elements $b_0, b_1, b_2, b_3 \in \mathscr{A}^{++}$ such that $a = \sum_{k=0}^{3} \mathrm{i}^k b_k$ holds. **(3 Points)**

Homework 2-3: Positivity in the commutative *-algebra $\mathbb{C}[x]$

Recall that $\mathbb{C}[x]$ with *-involution $\big(\sum_{n=0}^{\infty} a_n x^n\big)^* = \sum_{n=0}^{\infty} \overline{a}_n x^n$ is a commutative *-algebra. Show that for a polynomial $a \in \mathbb{C}[x]$ the following statements are equivalent:

i.) The polynomial a is an algebraically positive element of $\mathbb{C}[x]$.

ii.) The polynomial a is a positive element of $\mathbb{C}[x]$.

iii.) The polynomial a is pointwise positive, i.e. $a(y) \geq 0$ for all $y \in \mathbb{R}$.

(6 Points)

Hint: You might want to make use of the evaluation functionals at $y \in \mathbb{C}$, defined as

$$\delta_y\colon \mathbb{C}[x] \ni a \mapsto a(y) \in \mathbb{C}. \tag{2.3}$$

The fundamental theorem of algebra might also be useful.

References

[1] Waldmann, S.: *Lineare Algebra I. Die Grundlagen für Studierende der Mathematik und Physik.* Springer-Verlag, Berlin, 2. edition, 2021. 2