Green's function formalism, origin of attractive interaction, and Cooper instability

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In the following exercises, we are going to review some properties of the Green functions for both electrons and phonons

Problem 1

Let's consider an infinite multimoded wire, i.e.: 2D, that is infinite in the $x-$ direction but finite on $y-$ direction. The Hamiltonian for the wire can be written as

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + U(y) \tag{1}$$

with $U(y)$ representing the confining potential in the $y-$ direction. Derive the following Green's function for a semi infinite wire using eigenfunction expansion (Lehman representation)

$$G^r(\vec{x}, \vec{x}') = -\frac{i}{\hbar\nu_m}\sum_m \chi_m(y)\chi_m(y')e^{ik_m|x-x'|}; \tag{2}$$

with $k_m = \sqrt{2m(E-\varepsilon_{m,0})}/\hbar$, $\nu_m = \hbar k_m/m$ and $\chi_m(y)$ is the eigenfunction for the problem in the y direction.

Problem 2

The scattering matrix is related to the Green function in the following way

$$s_{nm} = -\delta_{nm} + \frac{i\hbar\sqrt{\nu_n\nu_m}}{a}\int dy_p dy_q \chi(y_q) G^r(y_q, y_p')\chi(y_p'), \tag{3}$$

where $G^r(y_q, y_p')$ is the retarded Green function that connects a point from the lead q with a one from the lead p. The previous formula is the so-called Fisher-Lee relation which expresses the elements of the scattering matrix in terms of the Green function. Derive the expression for the transmission probability of going from lead p to lead q:

$$T_{pq} = \mathrm{Tr}\left[\Gamma_p G^r \Gamma_q G^a\right] \tag{4}$$

Compare the result with the one obtained in the exercise sheet of Normal metals.

Problem 3

Consider the following Hamiltonian

$$H = H_{\mathrm{el-gas}} + H_{\mathrm{ph}} + \sum_{\boldsymbol{kq}} g_{\boldsymbol{q}}(a_{\boldsymbol{q}} + a^\dagger_{-\boldsymbol{q}})c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma}c_{\boldsymbol{k}\sigma}, \tag{5}$$

where $H_{\text{el-gas}}$ describes a free electronic gas, H_{ph} describes the free phonon part. The last term describes an interaction between the electrons and phonon with a coupling constant $g_{\boldsymbol{q}}$, and $c_{\boldsymbol{k}\sigma}$ ($a_{\boldsymbol{q}}$) describes an electronic (phononic) destruction operator. Since the phonons operators appear in the combination $a_{\boldsymbol{q}} + a^{\dagger}_{-\boldsymbol{q}}$ it is often convenient to define the phonon Green's function in terms of the combined operator $A_{\boldsymbol{q}} = a_{\boldsymbol{q}} + a^{\dagger}_{-\boldsymbol{q}}$.

- a) Find the bare phonon Green's function from the definition

$$D_0(\boldsymbol{q}, t) = -i \langle \Phi_0 | T[A_{\boldsymbol{q}}(t) A^{\dagger}_{\boldsymbol{q}}(0)] | \Phi_0 \rangle . \tag{6}$$

  Here, $|\Phi_0\rangle$ is the phonon vacuum/ground state and T is the time-ordering operator. Use the interaction picture where $A_{\boldsymbol{q}}(t) = e^{iH_0 t} A_{\boldsymbol{q}} e^{-iH_0 t}$, where $H_0 = H_{\text{ph}} = \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} a^{\dagger}_{\boldsymbol{q}} a_{\boldsymbol{q}}$.

- b) Apply a Fourier transform to frequency space to get

$$D_0(\boldsymbol{q}, \omega) = \frac{2\omega_{\boldsymbol{q}}}{\omega^2 - \omega^2_{\boldsymbol{q}} + i\eta}, \tag{7}$$

  where $\eta = 0^+$. Note: during the Fourier transform you will need to introduce factors $i\delta$, with $\delta = 0^+$ as appropriate to make sure the integrals converge.

- c) Draw a Feynman diagram depicting an effective electon-electron interaction mediated by the phonon propagator D_0. I.e., glue two phonon lines together from the diagram depicting an electron-phonon coupling. Also assume the two incoming electrons have opposite momentum $\boldsymbol{k}, -\boldsymbol{k}$ and opposite spin. From this you can read off the interaction strength as ($\boldsymbol{k}' = \boldsymbol{k} + \boldsymbol{q}$)

$$V_{\boldsymbol{k},\boldsymbol{k}'} = |g_{\boldsymbol{k}'-\boldsymbol{k}}|^2 D_0(\boldsymbol{k}' - \boldsymbol{k}, \omega). \tag{8}$$

  We often take the static limit $\omega = 0$ and assume the phonon-mediated attraction is only active for electrons with an energy smaller that the Debye frequency (maximum phonon energy) relative to the Fermi level. Show that the interaction becomes independent of momentum if we take the simple jellium model acoustic phonons $g_{\boldsymbol{q}} = g\sqrt{q}, \omega_{\boldsymbol{q}} = uq$.

Problem 4

Generalize Cooper's calculation to a pair with finite momentum. In particular

- a) Show that the operator that creates a Cooper pair at a finite momentum $\vec{p}$,

$$\Lambda^{\dagger}_{\vec{p}} = \int d^3x d^3x' \phi(\vec{x} - \vec{x}') \psi^{\dagger}_{\uparrow}(\vec{x}) \psi^{\dagger}_{\downarrow}(\vec{x}') e^{i\vec{p}(\vec{x}+\vec{x}')/2}, \tag{9}$$

  can be rewritten in the form

$$\Lambda^{\dagger}(p) = \sum_k \phi(\vec{k}) c^{\dagger}_{\vec{k}+\vec{p}/2,\uparrow} c^{\dagger}_{-\vec{k}+\vec{p}/2,\downarrow}. \tag{10}$$

- b) Show that the energy E_p of the pair state $\Lambda^{\dagger}(\vec{p})\,|\text{FS}\rangle$ is given by the (complex) roots $z = E_p$ of the equation

$$1 + \frac{g_0}{V} \sum_{0<\varepsilon_{k\pm p/2}<\omega_D} \frac{1}{z - \left(\varepsilon_{k+p/2} + \varepsilon_{k-p/2}\right)} = 0, \tag{11}$$

  where $|\text{FS}\rangle = \Pi_{k<k_f} c^{\dagger}_{\vec{k}\uparrow} c^{\dagger}_{-\vec{k}\downarrow} |0\rangle$ defines the filled Fermi sea. ε_k is the energy of an electron with momentum k, g_0 is the strength of the phonon-mediated electron-electron interaction, V is a volume factor, and ω_D is the Debye frequency.