Universität Würzburg
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5. Open quantum systems

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Throughout this exercise sheet, we adopt the convention $\boxed{\hbar = 1}$.

Exercise 1 *Gain/loss dynamics and encircling exceptional points* **7 P.**

Consider a two-level quantum system described by the non-Hermitian Hamiltonian:

$$H_{\mathrm{NHH}} = \begin{pmatrix} E_1 - i\gamma_1 & H_{12} \\ H_{12} & E_2 - i\gamma_2 \end{pmatrix},$$

where $E_1, E_2, \gamma_1, \gamma_2 \in \mathbb{R}$, and $H_{12} \in \mathbb{C}$.

Define the following quantities:

$$E = \frac{E_1 + E_2 - i(\gamma_1 + \gamma_2)}{2},$$
$$e = \frac{E_1 - E_2}{2}, \quad \gamma = \frac{\gamma_1 - \gamma_2}{2}, \quad B = \frac{e - i\gamma}{H_{12}}.$$

a) Compute the eigenvalues of H_{NHH}, and determine the condition under which the two eigenvalues coalesce to form an exceptional point (EP).

b) Interpret the physical meaning of γ_1 and γ_2 in this system. What happens when $\gamma_2 > \gamma_1$? Which level experiences relative loss? Can this be interpreted as a relative gain for the other level?

   Hint: Rewrite H_{NHH} in the form:

$$H_{\mathrm{NHH}} = \begin{pmatrix} * & H_{12} \\ H_{12} & * \end{pmatrix} - i\frac{\gamma_1 + \gamma_2}{2}\mathbb{1},$$

   where you should compute the entries denoted by * yourself. Identify which term corresponds to relative gain/loss and which to overall decay.

c) Show that the EPs occur when $B = \pm i$.

d) Explain how encircling an EP in the complex B-plane leads to a transformation of the eigenvectors due to the square-root branch cut. Assume the normalized right eigenvectors of H_{NHH} are given by:

$$|r_1\rangle = \begin{pmatrix} \cos u \\ \sin u \end{pmatrix}, \quad |r_2\rangle = \begin{pmatrix} -\sin u \\ \cos u \end{pmatrix},$$

where

$$\tan u = -B + \sqrt{B^2 + 1}.$$

Show that after one encirclement of the EP at $B = i$, the eigenvectors transform as:

$$u \to u + \frac{\pi}{2}, \quad \Rightarrow \quad |r_1\rangle \to |r_2\rangle\,, \quad |r_2\rangle \to -\,|r_1\rangle\,,$$

or

$$u \to u - \frac{\pi}{2}, \quad \Rightarrow \quad |r_1\rangle \to -\,|r_2\rangle\,, \quad |r_2\rangle \to |r_1\rangle\,.$$

e) How many full turns around the EP are required to return the eigenvectors to their original form?

Exercise 2 *Lindbladian Dynamics and Exceptional Points (EPs)* **4 P.**

Consider a two-level quantum system with the system Hamiltonian:

$$H_S = \omega\sigma_x,$$

and assume the environment induces both spontaneous excitation and decay, modeled by the Lindblad jump operators:

$$L_1 = \sqrt{\gamma}\,\sigma_-, \quad L_2 = \sqrt{\gamma}\,\sigma_+$$

where $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ are the lowering and raising operators, respectively.

a) Write the full Lindblad master equation for the density matrix $\rho(t)$.

b) Vectorize the density matrix using the procedure:

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \quad \Rightarrow \quad |\rho\rangle = \begin{pmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{10} \\ \rho_{11} \end{pmatrix}.$$

Construct the corresponding 4×4 Lindbladian superoperator $\mathcal{L}$ such that:

$$\frac{d}{dt}\,|\rho\rangle = \mathcal{L}\,|\rho\rangle\,.$$

Hint: Recall from the lecture that the Lindbladian superoperator $\mathcal{L}$, acting on the vectorized density matrix $|\rho\rangle$, can be expressed as:

$$\mathcal{L} = -\,i\left(H \otimes \mathbb{1} - \mathbb{1} \otimes H^{\mathrm{T}}\right) + \sum_j \left(L_j \otimes L_j^* - \frac{1}{2}\left(L_j^\dagger L_j \otimes \mathbb{1} + \mathbb{1} \otimes L_j^{\mathrm{T}} L_j^*\right)\right),$$

where H is the system Hamiltonian, L_j are the Lindblad operators, and $\otimes$ denotes the Kronecker product.

c) Determine the conditions under which the Lindbladian $\mathcal{L}$ exhibits an exceptional point (EP); that is, when two or more of its eigenvalues and corresponding eigenvectors coalesce.