

Funktionalanalysis Notizen

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I. MATHEMATICS

Gamma Function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

$$\Gamma(z+1) = z\Gamma(z)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(n) = (n-1)!, \quad n \in \mathbb{N}$$

Gaussian Integral

$$\int_0^\infty x^n e^{-kx^2} dx = \frac{1}{2} k^{-(n+1)/2} \Gamma\left(\frac{1+n}{2}\right)$$

Fermi-Dirac Integral

$$f_\alpha(z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{\xi^{\alpha-1}}{e^{\xi/z} + 1} d\xi$$

General Series Expansion

$$f_\alpha(z) \approx f_\alpha(z_0) + \frac{f_{\alpha-1}(z_0)}{z_0} (z - z_0) + \frac{f_{\alpha-2}(z_0) - f_{\alpha-1}(z_0)}{2z_0^2} (z - z_0)^2$$

Expansion around 0

$$f_\alpha(z) = \sum_{n=1}^\infty \frac{(-1)^{n-1} z^n}{n^\alpha}$$

Derivative

$$f_{n-1}(z) = z \frac{\partial}{\partial z} f_n(z)$$

Bose-Einstein Integral

$$g_\alpha(z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{\xi^{\alpha-1}}{e^{\xi/z} - 1} d\xi$$

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General Series Expansion

$$g_\alpha(z) \approx g_\alpha(z_0) + \frac{g_{\alpha-1}(z_0)}{z_0}(z - z_0) + \frac{g_{\alpha-2}(z_0) - g_{\alpha-1}(z_0)}{2z_0^2}(z - z_0)^2$$

Around 0

$$g_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}$$

Derivatives

$$g_{\alpha-1}(z) = z \frac{\partial}{\partial z} g_\alpha(z)$$

II. THERMODYNAMIC POTENTIALS

Energy

$$U = TS - PV + \mu N$$

$$dU = T dS - P dV + \mu dN$$

Helmholtz Free Energy

Helmholtz free energy is the maximal work in an isothermal process.

$$F = U - TS$$

$$F = -PV + \mu N$$

$$dF = -S dT - P dV + \mu dN$$

Enthalpy

$$H = U + PV$$

$$H = TS + \mu N$$

$$dH = T dS + V dP + \mu dN$$

Gibbs Free Energy

$$G = U + PV - TS$$

$$G = \mu N$$

$$dG = -T dS + V dP + \mu dN$$

Grand Canonical Potential

$$J = F - \mu N$$

$$J = -PV$$

$$dJ = -S dT - P dV - N d\mu$$

III. ENSEMBLES

General Definitions

Gibbs Entropy

$$S = -k_B \sum p_n \ln p_n = -k_B \operatorname{tr}(\hat{\rho} \ln \hat{\rho})$$

Microcanonical Ensemble

Energy fixed,

$$\mathbb{P} \propto \Omega$$

Partition function

$$\Omega = e^{S(E)}$$

Canonical Ensemble

Partition function

$$Z = \operatorname{tr} e^{-\beta \hat{H}}$$

Thermodynamic Potential

$$F = -k_B T \ln Z$$

Grand Canonical Ensemble

Partition Function

$$\begin{aligned} Z_G &= \operatorname{tr}(e^{-\beta(H-\mu N)})(\text{quantum}) \\ &= \sum_{n=0}^{\infty} z^n Z_k(n) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n (Z_k)^n (\text{classical indistinguishable particles only}) \end{aligned}$$

IV. QUANTUM STATISTICS

Determining the DOS:

1. Find the number of states with energy $E \leq \epsilon$
2. Differentiate

Momentum Space Sums

$$dk_x = \frac{2\pi}{L_x}, \quad \vec{k} = \frac{\vec{p}}{\hbar}$$

$$\sum_{\vec{p}} f(\vec{p}) = \frac{V}{h^3} \int f(\vec{p}) d\vec{p}$$

Fermi-Dirac Distribution

$$\begin{aligned} Z &= \sum_{n_1, n_1, \dots, n_N=0}^1 \langle n_1, \dots, n_N | e^{-\beta(H-\mu N)} | n_1, \dots, n_N \rangle \\ &= \sum_{n_1, n_1, \dots, n_N=0}^1 \langle n_1, \dots, n_N | \prod_{k=1}^N e^{-\beta(E_k - \mu) c_k^\dagger c_k} | n_1, \dots, n_N \rangle \\ &= \prod_{k=1}^n \sum_{n_k=0}^1 e^{-\beta(E_k - \mu) n_k} \\ &= \prod_{k=1}^n [1 + e^{-\beta(E_k - \mu)}] \end{aligned}$$