Topological Field Theory WS 2025

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Problem Set 1
due 27.10.25, 10:00

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1. Gaussian integral for bosonic fields

Verify the functional integral

$$\int \mathcal{D}\phi \, \exp\left(-\frac{1}{2}\phi^\top M\phi + J^\top\phi\right) = (2\pi)^{\frac{N}{2}}(\det M)^{-\frac{1}{2}} \exp\left(\frac{1}{2}J^\top M^{-1}J\right) \tag{1}$$

where $\phi^\top = (\phi_1, \ldots, \phi_N)$ and $J^\top = (J_1, \ldots, J_N)$ are real vectors, M a symmetric $N \times N$ matrix, and the integration is carried out over all the fields ϕ_i, $i = 1, \ldots, N$,

$$\int \mathcal{D}\phi \equiv \int d\phi_1 \ldots \int d\phi_N. \tag{2}$$

2. Green's function of Laplacian in two dimensions

Verify

$$\bar\partial\partial \ln(z\bar z) = \pi\delta(\tau)\delta(x), \tag{3}$$

where $z = \tau + \mathrm{i}x$, $\bar z = \tau - \mathrm{i}x$.

3. Dirac Lagrangian in two dimensions

Consider the 2D Dirac Lagrangian in Minkowski space,

$$\mathcal{L}_{\mathrm{D,M}} = \frac{1}{\pi}\bar\Psi_{\mathrm{D}}\mathrm{i}\partial\!\!\!/\Psi_{\mathrm{D}}, \tag{4}$$

where $\bar\Psi_{\mathrm{D}} = \Psi_{\mathrm{D}}^\dagger\gamma^0$, $\Psi_{\mathrm{D}}^\dagger = (\bar\psi^*, \psi^*)$, $\partial\!\!\!/ = \gamma^\mu\partial_\mu = \gamma^0\partial_0 + \gamma^1\partial_1$,

$$\Psi_{\mathrm{D}} = \begin{pmatrix}\bar\psi\\ \psi\end{pmatrix}, \ \gamma^0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \ \gamma^1 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \ \text{and } \gamma_5 = \gamma^0\gamma^1 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

(a) Show that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, where $g^{00} = -g^{11} = 1$ is the Minkowski metric.

(b) Obtain the equation of motion from $\mathcal{L}_{\mathrm{D,M}}$.

(c) Verify that $\mathcal{L}_{\mathrm{D,M}}$ is invariant under the U(1) vector symmetry $\Psi_{\mathrm{D}} \to e^{i\lambda}\Psi_{\mathrm{D}}$ and obtain the associated conserved current, the vector current J^μ_{V}.

(d) Verify that $\mathcal{L}_{\mathrm{D,M}}$ is invariant under the U(1) axial symmetry $\Psi_{\mathrm{D}} \to e^{i\lambda\gamma_5}\Psi_{\mathrm{D}}$ and obtain the associated conserved current, the axial current J^μ_{A}.

(e) Compare the Lagrangian, as well as the results from (b)-(d), to the results we obtained in Euclidean space with complex space time coordinates $z = \tau + \mathrm{i}x$, $\bar z = \tau - \mathrm{i}x$ in class.

4. Partial integration in the complex plane

Determine the coefficients a and b in the formula

$$\frac{1}{\pi}\int d\tau dx \left(\bar\partial f(z) + \partial\bar f(\bar z)\right) = a\oint dz f(z) + b\oint d\bar z \bar f(\bar z), \tag{5}$$

where $f(z)$ and $\bar f(\bar z)$ are independent functions, the $d\tau dx$ integration extends over the entire plane and the contour integrals are taken counter-clockwise around the entire z or $\bar z$ planes in the respective terms.

Spherical coordinates: Vectors and vector fields are given by

$$\boldsymbol{r} = r\boldsymbol{e}_{\mathrm{r}} \quad \text{and} \quad \boldsymbol{v}(\boldsymbol{r}) = v_{\mathrm{r}}\boldsymbol{e}_{\mathrm{r}} + v_\theta\boldsymbol{e}_\theta + v_\varphi\boldsymbol{e}_\varphi, \tag{6}$$

with

$$\boldsymbol{e}_{\mathrm{r}} = \begin{pmatrix}\cos\varphi\,\sin\theta\\ \sin\varphi\,\sin\theta\\ \cos\theta\end{pmatrix}, \quad \boldsymbol{e}_\theta = \begin{pmatrix}\cos\varphi\,\cos\theta\\ \sin\varphi\,\cos\theta\\ -\sin\theta\end{pmatrix}, \quad \boldsymbol{e}_\varphi = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}, \tag{7}$$

where $\varphi \in [0, 2\pi[$ and $\theta \in [0, \pi]$. This implies

$$\boldsymbol{e}_{\mathrm{r}} \times \boldsymbol{e}_\theta = \boldsymbol{e}_\varphi, \quad \boldsymbol{e}_\theta \times \boldsymbol{e}_\varphi = \boldsymbol{e}_{\mathrm{r}}, \quad \boldsymbol{e}_\varphi \times \boldsymbol{e}_{\mathrm{r}} = \boldsymbol{e}_\theta, \tag{8}$$

and

$$\begin{aligned}
&\frac{\partial\boldsymbol{e}_{\mathrm{r}}}{\partial\theta} = \boldsymbol{e}_\theta, && \frac{\partial\boldsymbol{e}_\theta}{\partial\theta} = -\boldsymbol{e}_{\mathrm{r}}, && \frac{\partial\boldsymbol{e}_\varphi}{\partial\theta} = 0,\\
&\frac{\partial\boldsymbol{e}_{\mathrm{r}}}{\partial\varphi} = \sin\theta\,\boldsymbol{e}_\varphi, && \frac{\partial\boldsymbol{e}_\theta}{\partial\varphi} = \cos\theta\,\boldsymbol{e}_\varphi, && \frac{\partial\boldsymbol{e}_\varphi}{\partial\varphi} = -\sin\theta\,\boldsymbol{e}_{\mathrm{r}} - \cos\theta\,\boldsymbol{e}_\theta.
\end{aligned} \tag{9}$$

With

$$\nabla = \boldsymbol{e}_{\mathrm{r}}\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \tag{10}$$

we obtain

$$\nabla\boldsymbol{v} = \frac{1}{r^2}\frac{\partial(r^2 v_{\mathrm{r}})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}, \tag{11}$$

$$\begin{aligned}
\nabla\times\boldsymbol{v} &= \boldsymbol{e}_{\mathrm{r}}\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\, v_\varphi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\varphi}\right)\\
&+ \boldsymbol{e}_\theta\left(\frac{1}{r\sin\theta}\frac{\partial v_{\mathrm{r}}}{\partial\varphi} - \frac{1}{r}\frac{\partial(r v_\varphi)}{\partial r}\right)\\
&+ \boldsymbol{e}_\varphi\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_{\mathrm{r}}}{\partial\theta}\right),
\end{aligned} \tag{12}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{13}$$