Einfürung in die Algebra Hausaufgaben Blatt Nr. 11

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(Dated: October 21, 2025)

TOPOLOGICAL STUFF FOR WARMING UP...

Problem 1. Give an example that in general a family of open sets is not closed under infinite intersections.

Proof.

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0, 1].$$

□

Problem 2. Give an example that a bijective continuous function $f : E \to F$ between topological spaces is not necessarily a homeomorphism.

Proof. Consider $f : [0, 2\pi) \to S^1$, $f(x) = e^{ix}$. This is bijective and continuous, but its inverse is not, because two points that are near $(1, 0)$ map to different ends of the interval. □

Problem 3. Give an example of a connected topological space which is not path (or arcwise) connected.

Proof. Topologists' sine curve (see Ana2). □

Problem 4. Give an example of a connected topological space which is not locally connected.

Proof. Topologists' sine curve □

Problem 5. Recall the **Definition.** A topological space X is *totally disconnected* if the connected components of X are single points.

Give an example of a totally disconnected topological space.

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Proof. Discrete space. $\square$

Problem 6. Recall the **Definition.** Let X be a topological space. A subset $S \subset X$ is said to be *dense* in X if, for each $x \in X$ and each neighborhood U of x there is a point $s \in S$ s.t. $s \in U$.

Give an example of a dense subset of $\mathbb{R}$ (with usual topology).

Proof.

$$\mathbb{Q}$$

$\square$

Problem 7. Let X be the space of continuous functions on the interval $I := [0, 1]$. Show that $d(f, g) := \max_{x \in I} |f(x) - g(x)|$ defines a metric on X.

Proof. We check 3 properties

1. $d(f, f) = 0$: Clear
2. $d(f, g) = d(g, f)$: Also clear
3. Triangle inequality:

$$\begin{aligned} d(f,g) &= \max_{x \in I} |f(x) - g(x)| \\ &= \max_{x \in I} |f(x) - h(x) + h(x) - g(x)| \\ &\leq \max_{x \in I} [|f(x) - h(x)| + |h(x) - g(x)|] \\ &\leq \max_{x \in I} |f(x) - h(x)| + \max_{x \in I} |h(x) - g(x)| \\ &= d(f,h) + d(h,g). \end{aligned}$$

$\square$

DIFFERENTIAL CALCULUS

Problem 8. Recall from elementary linear algebra that the dual space E^* of a finite dimensional vector space E of dimension n also has dimension n and so the space and its dual are isomorphic. For general Banach spaces this is no longer true. However, it is true for Hilbert spaces.

Prove the following

Theorem. (Riesz Representation Theorem) Let E be a real (resp., complex) Hilbert space. The map $e \mapsto \langle \cdot, e \rangle$ is a linear (resp., antilinear) norm-preserving isomorphism of E with E^*; for short, $E \cong E^*$.

Recall that a map $A : E \to F$ between complex vector spaces is called antilinear if we have the identities $A(e + e') = Ae + Ae'$, and $A(\alpha e) = \bar{\alpha} Ae$.

Proof. It is clearly antilinear. The proof idea is that a continuous 1-form is uniquely defined up to scaling by its kernel. $\square$

Problem 9. Add to your personal mathematical tool box the following

Definition.

(a) Let E be a normed space and $f : U \subset E \to \mathbb{R}$ be differentiable so that $Df(u) \in L(E, \mathbb{R}) = E^*$. In this case we sometimes write $df(u)$ for $Df(u)$ and call df the differential of f. Thus $df : U \to E^*$.

(b) Let $(E, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $f : U \subset E \to \mathbb{R}$ be differentiable. The *gradient* of f is the map $\nabla f := \nabla f : U \to E$ defined (implicitly) by $\langle \nabla f(u), e \rangle := df(u) \cdot e$, meaning the linear map $df(u)$ applied to the vector e (directional derivative).