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Problem Sheet 12
for the tutorial on July 25th and 21st, 2025
Quantum Mechanics II
Summer term 2025

Sheet handed out on July 15th, 2025; to be handed in on July 22nd, 2025 until 2 pm

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Exercise 12.1: Annihilation and creation operators [2+7 P.]

Consider a pair of fermionic creation and annihilation operators $\hat{c}$ and $\hat{c}^\dagger$, and let $\hat{n} = \hat{c}^\dagger \hat{c}$.

a) Show that $\hat{n}^2 = \hat{n}$.

b) Using a Taylor expansion of the exponential, show that

$$e^{i\phi\hat{n}} = 1 + \left(e^{i\phi} - 1\right)\hat{n} \tag{1}$$

and from this that

$$\hat{\tilde{c}}(\phi) \equiv e^{i\phi\hat{n}}\hat{c}e^{-i\phi\hat{n}} = \hat{c}e^{-i\phi},\ \phi \in \mathbb{R}\,. \tag{2}$$

Exercise 12.2: Electron spin operator in second quantization [6+5+5 P.]

Now let us consider fermionic creation and annihilation operators $\hat{c}_\sigma$ and $\hat{c}_\sigma^\dagger$ for electrons with spin $\sigma \in \{\uparrow, \downarrow\}$.

a) The spin operator $\hat{S}_j$ describing the electronic spin is defined as

$$\hat{S}_j = \frac{1}{2} \sum_{\sigma,\sigma'=\uparrow,\downarrow} \hat{c}_\sigma^\dagger (\tau_j)_{\sigma\sigma'} \hat{c}_{\sigma'} \tag{3}$$

with $j \in \{x, y, z\}$, the Pauli matrices

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

and the notation $(\tau_j)_{\sigma\sigma'} = \langle\sigma|\tau_j|\sigma'\rangle$. Show that the spin components can be written as

$$\begin{aligned} \hat{S}_z &= \frac{1}{2}\left(\hat{n}_\uparrow - \hat{n}_\downarrow\right) \\ \hat{S}_+ &= \hat{c}_\uparrow^\dagger \hat{c}_\downarrow \\ \hat{S}_- &= \hat{c}_\downarrow^\dagger \hat{c}_\uparrow \\ \hat{\boldsymbol{S}}^2 &= \hat{S}_z^2 + \frac{1}{2}\left(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+\right)\,. \end{aligned} \tag{5}$$

where $\hat{n}_\sigma = \hat{c}_\sigma^\dagger \hat{c}_\sigma$ and $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$.

b) Prove that the spin components above satisfy $[\hat{S}_z, \hat{S}_\pm] = \pm\hat{S}_\pm$ and $[\hat{S}_+, \hat{S}_-] = 2\hat{S}_z$.

c) Show that

$$\hat{P} = \hat{n}_\uparrow(1 - \hat{n}_\downarrow) + \hat{n}_\downarrow(1 - \hat{n}_\uparrow) \tag{6}$$

is a projector, i.e. $\hat{P}^2 = \hat{P}$, and

$$\hat{\boldsymbol{S}}^2 = \frac{3}{4}\hat{P}. \tag{7}$$

Exercise 12.3: **Ask questions!** [P.]

Since this is the last tutorial before the exam you have the opportunity to ask some questions about the lecture or the problem sheets. In case you have some, please send them until 22nd of July to your tutor per mail.