Topological Field Theory WS 2025

Prof. Dr. Martin Greiter
M.Sc. Ludwig Bordfeldt and M.Sc. Tom Oeffner

Problem Set 3
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1. Klein–Gordon theory: free real scalar field in $d+1$ dimensions

The Lagrangian for a free, massive, real, bosonic scalar field $\phi(x)$ in d space dimensions is given by

$$L = \int d^d x\, \mathcal{L}(x), \qquad \mathcal{L}(x) = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{m^2}{2}\phi^2, \tag{1}$$

where the metric has signature $(1,d)$ (*i.e.*, $g^{00}=1$, $g^{ii}=-1$ for $i=1,\ldots,d$).

We first consider the model (1) as a classical theory.

(a) Obtain the Euler-Lagrange equation for $\phi(x)$, which is also known as Klein–Gordon equation.

(b) Obtain the momentum field $\pi(x)$ conjugate to $\phi(x)$ and the Hamiltonian density $\mathcal{H}(x)$.

(c) Obtain the energy momentum tensor $T^{\mu\nu}$, and from there the generators of translations in time H and in space P.

We now quantize the theory by imposing the equal-time commutator

$$\left[\phi(\boldsymbol{x},t), \pi(\boldsymbol{x'},t)\right] = \mathrm{i}\delta^d(\boldsymbol{x}-\boldsymbol{x'}). \tag{2}$$

(d) Calculate the commutators $[H, \phi(\boldsymbol{x},t)]$ and $[H, \pi(\boldsymbol{x},t)]$. Are the results consistent with the Klein–Gordon equation?
Hint: Express H as a functional of $\pi(x)$, $\nabla\phi(x)$, and $\phi(x)$ rather than $\partial_\mu\phi(x)$ and $\phi(x)$.

(e) Calculate the commutators $[P, \phi(\boldsymbol{x},t)]$ and $[P, \pi(\boldsymbol{x},t)]$. What does the result confirm?

2. Quantisation of Klein–Gordon theory in terms of linear harmonic oscillators

To begin with, consider a single linear harmonic oscillator with Hamiltonian

$$H_0 = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2. \tag{3}$$

To quantise the oscillator, we introduce ladder operators a and $a^\dagger$ obeying $\left[a, a^\dagger\right] = 1$, and write

$$x = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right), \quad p = -\mathrm{i}\sqrt{\frac{\omega}{2}}\left(a - a^\dagger\right). \tag{4}$$

(a) Calculate the commutator $[x,p]$.

(b) Is the quantisation procedure we have chosen physically different from taking $p \to -\mathrm{i}\partial_x$?

(c) Write H_0 in terms of a and $a^\dagger$.

(d) Write down the eigenstates $|n\rangle$ in terms of the ladder operators and obtain their energies $E_{0,n}$.

We now expand the Klein–Gordon field $\phi(\boldsymbol{x},t)$ from Problem 1 in terms of Fourier modes,

$$\phi(\boldsymbol{x},t) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{p}} e^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\phi_{\boldsymbol{p}}(t), \tag{5}$$

where $\phi^*_{\boldsymbol{p}}(t) = \phi_{-\boldsymbol{p}}(t)$ assures that $\phi(\boldsymbol{x},t)$ is real.

(e) Use the Euler-Lagrange equation for $\phi(\boldsymbol{x},t)$ to show that $\phi_{\boldsymbol{p}}(t)$ obeys the equation of motion of a linear harmonic oscillator for each mode $\boldsymbol{p}$,

$$\left(\partial_t^2 + \omega_{\boldsymbol{p}}^2\right)\phi_{\boldsymbol{p}}(t) = 0, \tag{6}$$

and determine $\omega_{\boldsymbol{p}}$.

To quantise the theory, we introduce ladder operators $a_{\boldsymbol{p}}$ and $a^\dagger_{\boldsymbol{p}}$ which obey

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p'}}\right] = \delta_{\boldsymbol{pp'}}, \tag{7}$$

and write at $t=0$:

$$\phi(\boldsymbol{x}) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{p}} e^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right), \tag{8}$$

$$\pi(\boldsymbol{x}) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{p}} e^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}(-\mathrm{i})\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}}\right). \tag{9}$$

(f) Are the fields $\phi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ hermitian?

(g) Calculate the equal time commutator $\left[\phi(\boldsymbol{x}), \pi(\boldsymbol{x'})\right]$ using (8)and (9) and compare the result to (2).

(h) Obtain the Klein–Gordon Hamiltonian H in terms of $a_{\boldsymbol{p}}$ and $a^\dagger_{\boldsymbol{p}}$.

Now consider a massless ($m=0$) scalar field in one dimension ($d=1$) with periodic boundary conditions with a system length L.

(j) Obtain the zero point energy E_0 as a sum over modes n, where n is integer and $p = \frac{2\pi}{L}n$. Does the sum converge?