Einfürung in die Algebra Hausaufgaben Blatt Nr. 11

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I forgot to bring my tablet on the train so please enjoy the LaTeX solutions.

Problem 1 (Getting familiar with the Pauli spin vector). (a) Prove the relation

$$(\sigma \cdot A)(\sigma \cdot B) = A \cdot B + i\sigma \cdot (A \times B)$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ denotes the vector of the 2×2 Pauli-spin matrices. $A = (A_x, A_y, A_z)^T$ and $B = (B_x, B_y, B_z)^T$ are arbitrary vectors.

(b) Show that

$$\sigma \cdot \hat{p} = \frac{1}{r^2}(\sigma \cdot r)\left(-i\hbar r\partial_r + i\sigma \cdot \hat{L}\right)$$

where $\hat{p} = -i\hbar\nabla$ and $\hat{L} = r \times \hat{p}$.

Proof. It shall be understood here that we sum over all repeated indices.

(a)

$$\begin{aligned}
(\sigma \cdot A)(\sigma \cdot B) &= (\sigma_i A_i)(\sigma_j B_j) \\
&= A_i B_j(\delta_{ij} I + i\epsilon_{ijk}\sigma_k) \\
&= A_i B_i I + i\epsilon_{ijk} A_i B_j \sigma_k \\
&= (A \cdot B)I + i\sigma_k \epsilon_{kij} A_i B_j \\
&= (A \cdot B)I + i\sigma \cdot (A \times B)
\end{aligned}$$

(b)

$$\sigma \cdot \hat{p} = -i\hbar\sigma_i\partial_i$$

We have

$$\begin{aligned}
\frac{1}{r^2}(\sigma \cdot r)\left(-i\hbar r\partial_r + i\sigma \cdot L\right) &= \frac{1}{r^2}(\sigma_j r_j)\left(-i\hbar r\partial_r + i\epsilon_{lmn}\sigma_l r_m(-i\hbar\partial_n)\right) \\
&= \frac{-i\hbar}{r^2}(\sigma_j r_j)(r\partial_r + \epsilon_{lmn}\sigma_l r_m\partial_n) \\
&= \frac{-i\hbar}{r^2}(\sigma_j r_j)(r(\partial_r r_k)\partial_k + \epsilon_{lmn}\sigma_l r_m\partial_n) \\
&= \frac{-i\hbar}{r^2}(\sigma_j r_j)(r(\partial_r r_n) + \epsilon_{lmn}\sigma_l r_m)\partial_n \\
&= \frac{-i\hbar}{r^2}(\sigma_j r_j)(r_n\partial_n + \epsilon_{lmn}\sigma_l r_m\partial_n)
\end{aligned}$$

So now the goal is to show that

$$\frac{\sigma_j r_j}{r^2}(r(\partial_r r_n) + \epsilon_{lmn}\sigma_l r_m) = \sigma_n.$$

or

$$\frac{1}{r^2}(\sigma_j r_j)(r_n + \epsilon_{lmn}\sigma_l r_m) = \sigma_n.$$

We do this by considering

$$\begin{aligned}
&\frac{1}{r^2}(\sigma_j r_j)(r_n + \epsilon_{lmn}\sigma_l r_m)\\
=&\frac{1}{r^2}\sigma_j r_j r_n + \frac{1}{r^2}\epsilon_{lmn} r_j \sigma_j \sigma_l r_m\\
=&\frac{1}{r^2}\sigma_j r_j r_n + \frac{1}{r^2}\epsilon_{lmn} r_j r_m (\delta_{jl} I_2 + i\epsilon_{jlr}\sigma_r)\\
=&\frac{1}{r^2}\sigma_j r_j r_n + \frac{1}{r^2}\epsilon_{jmn} r_j r_m + \frac{i}{r^2}\epsilon_{lmn}\epsilon_{jlr} r_j r_m \sigma_r\\
=&\frac{1}{r^2}\sigma_j r_j r_n + \frac{1}{r^2}\epsilon_{jmn} r_j r_m + \frac{i}{r^2}\epsilon_{lmn}\epsilon_{lrj} r_j r_m \sigma_r\\
=&\frac{1}{r^2}\sigma_j r_j r_n + \frac{1}{r^2}\epsilon_{jmn} r_j r_m + \frac{i}{r^2}\left(\delta_{mr}\delta_{nj} - \delta_{mj}\delta_{nr}\right) r_j r_m \sigma_r\\
=&\frac{1}{r^2}\sigma_j r_j r_n + \frac{1}{r^2}\epsilon_{jmn} r_j r_m + \frac{i}{r^2} r_n r_r \sigma_r - \frac{i}{r^2} r_m r_m \sigma_n
\end{aligned}$$

(I have no clue how to do this I give up). □

Problem 2 (Majorana representation of the Dirac equation)**.** Multiplying the Dirac equation known from the lecture by $-\frac{i}{\hbar}$ we get

$$H_D\Psi = \left(\frac{\partial}{\partial t} + \vec{\alpha}\cdot\vec{\nabla} + im_0\beta\right)\Psi = 0$$

with

$$\beta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \vec{\alpha} = \begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix},\quad \vec{\sigma} = \left(\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right).$$

Thus, some of the matrices in H_D are imaginary. Show that the transformation

$$\Psi' = U\Psi \tag{4}$$

with

$$U = \frac{1}{\sqrt{2}}(\alpha_y + \beta)$$

results in a representation of the Dirac-equation where $H'_D = UH_DU^{-1}$ is purely real.

Proof. We begin by showing that U is unitary (which is needed to argue that this is a unitary transformation anyway):

$$\begin{aligned}
U^\dagger U &= \frac{1}{2}\begin{pmatrix}I_2 & \sigma_y^\dagger\\ \sigma_y^\dagger & -I_2\end{pmatrix}\begin{pmatrix}I_2 & \sigma_y\\ \sigma_y & -I_2\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}I_2 + \sigma_y^\dagger\sigma_y & \sigma_y - \sigma_y^\dagger\\ \sigma_y^\dagger - \sigma_y & \sigma_y^\dagger\sigma_y + I_2\end{pmatrix}\\
&= \begin{pmatrix}I_2 & 0\\ 0 & I_2\end{pmatrix}
\end{aligned}$$

Note that since U is both Hermitian and unitary, its inverse is itself. Then we simply apply this to all of the matrices:

$$\begin{aligned}
2U\beta U &= \begin{pmatrix}I_2 & \sigma_y\\ \sigma_y & -I_2\end{pmatrix}\begin{pmatrix}I_2 & 0\\ 0 & -I_2\end{pmatrix}\begin{pmatrix}I_2 & \sigma_y\\ \sigma_y & -I_2\end{pmatrix}\\
&= \begin{pmatrix}I_2 & \sigma_y\\ \sigma_y & -I_2\end{pmatrix}\begin{pmatrix}I_2 & \sigma_y\\ -\sigma_y & I_2\end{pmatrix}\\
&= \begin{pmatrix}I_2 - \sigma_y^2 & 2\sigma_y\\ 2\sigma_y & \sigma_y^2 - I_2\end{pmatrix}\\
&= \begin{pmatrix}0 & 2\sigma_y\\ 2\sigma_y & 0\end{pmatrix}
\end{aligned}$$

After the multiplication by i, this is purely real. We now proceed to the rest:

$$
\begin{aligned}
2U\alpha_i U &= \begin{pmatrix} I_2 & \sigma_y \\ \sigma_y & -I_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} I_2 & \sigma_y \\ \sigma_y & -I_2 \end{pmatrix} \\
&= \begin{pmatrix} I_2 & \sigma_y \\ \sigma_y & -I_2 \end{pmatrix} \begin{pmatrix} \sigma_i\sigma_y & -\sigma_i \\ \sigma_i & \sigma_i\sigma_y \end{pmatrix} \\
&= \begin{pmatrix} \sigma_i\sigma_y + \sigma_y\sigma_i & -\sigma_i + \sigma_y\sigma_i\sigma_y \\ \sigma_y\sigma_i\sigma_y - \sigma_i & -\sigma_y\sigma_i - \sigma_i\sigma_y \end{pmatrix}
\end{aligned}
$$

Then we evaluate this for $i \in \{x, y, z\}$:

$$
\begin{aligned}
2U\alpha_x U &= \begin{pmatrix} 0 & -2\sigma_x \\ -2\sigma_x & 0 \end{pmatrix} \\
2U\alpha_y U &= \begin{pmatrix} 2I_2 & 0 \\ 0 & -2I_2 \end{pmatrix} \\
2U\alpha_z U &= \begin{pmatrix} 0 & -2\sigma_z \\ -2\sigma_z & 0 \end{pmatrix}
\end{aligned}
$$

all of which are real. □

Problem 3 (Some properties of the γ matrices)**.** (a) By considering $\mu = \nu = 0$, $\mu = \nu \neq 0$ and $\mu \neq \nu$ show that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

where $\{,\}$ denotes the anti-commutator.

(b) Show that

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0.$$

Proof.

The γ matrices are defined by $$\gamma^0 = \beta, \qquad \gamma^i = \beta\alpha^i.$$

(a) We consider the different cases

(1) $\mu = \nu = 0$:

$$\beta^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = I_4 = g^{00} I_4.$$

(2) $\mu = \nu \neq 0$:

$$
\begin{aligned}
(\gamma^\mu)^2 &= \beta\alpha^\mu\beta\alpha^\mu \\
&= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \\
&= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} \\
&= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} \\
&= -I_4 = g^{\mu\mu} I_4
\end{aligned}
$$

(3) $\mu \neq \nu$: Since

$$\alpha^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix},$$

we have

$$\begin{aligned} \beta\alpha^\mu &= \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} \\ \alpha^\mu\beta &= \begin{pmatrix} 0 & -\sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \end{aligned}$$

Thus, if $\mu = 0$, we have

$$\begin{aligned} \{\beta\alpha^\mu, \beta\alpha^\nu\} &= \{\beta\beta, \beta\alpha^\nu\} \\ &= \{1, \beta\alpha^\nu\} \\ &= \begin{pmatrix} 0 & \sigma^\nu \\ -\sigma^\nu & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^\nu \\ \sigma^\nu & 0 \end{pmatrix} = 0 \end{aligned}$$

If not, we have

$$\{\beta\alpha^\mu, \beta\alpha^\nu\} = -\{\alpha^\mu, \alpha^\nu\}$$

and

$$\begin{aligned} \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu \\ \sigma^\nu & 0 \end{pmatrix} &= \begin{pmatrix} \sigma^\mu\sigma^\nu & 0 \\ 0 & \sigma^\mu\sigma^\nu \end{pmatrix} \\ \begin{pmatrix} 0 & \sigma^\nu \\ \sigma^\nu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} &= \begin{pmatrix} \sigma^\nu\sigma^\mu & 0 \\ 0 & \sigma^\nu\sigma^\mu \end{pmatrix} \end{aligned}$$

Thus, we have

$$\{\beta\alpha^\mu, \beta\alpha^\nu\} = -\begin{pmatrix} \{\sigma^\mu, \sigma^\nu\} & 0 \\ 0 & \{\sigma^\mu, \sigma^\nu\} \end{pmatrix} = 0.$$

(b) We have

$$\begin{aligned} (\gamma^\mu)^\dagger &= (\beta\alpha^\mu)^\dagger \\ &= (\alpha^\mu)^\dagger\beta^\dagger \\ &= \alpha^\mu\beta && \alpha, \beta \text{ hermitian} \\ &= \beta^{-1}\gamma^\mu\beta \\ &= \beta\gamma^\mu\beta \end{aligned}$$

where it is understood that $\alpha^0 = I_4$. □