Homework for the Lecture

Functional Analysis

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Winter Term 2024/2025

Homework Sheet No 10

revision: 2024-12-16 20:49:26 +0100

Last changes by christopher.rudolph@home on 2024-12-16
Git revision of funkana-ws2425: `ae251a3` (HEAD -> master)

16. 12. 2024
(29 Points. Discussion 23. 12. 2024)

Homework 10-1: ℓ^p: Now with $p < 1$

Let $p \in (0, 1)$. As usual, we define

$$\ell^p := \Big\{ x = (x_n)_{n\in\mathbb{N}} \subset \mathbb{K} : \rho_p(x) := \sum_{n=1}^{\infty} |x_n|^p < \infty \Big\}. \tag{10.1}$$

i.) **(3 Points)** Show that ℓ^p is a vector space. Moreover, prove that the map $\rho_p \colon \ell^p \to [0, \infty)$ is sublinear.
Hint: It can be helpful to recall our results about ℓ^p-spaces for $p \in [1, \infty]$.

ii.) **(2 Points)** Show that the map

$$d_p \colon \ell^p \times \ell^p \ni (x, y) \mapsto \rho_p(x - y) \tag{10.2}$$

turns ℓ^p into a complete metric space.

iii.) **(2 Points)** Show that (ℓ^p, d_p) is a topological vector space. Moreover, show that the metric d_p is invariant under translations, i.e.

$$d_p(x + z, y + z) = d_p(x, y) \tag{10.3}$$

for all $x, y, z \in \ell^p$.

iv.) **(1 Point)** Let $1 \geq q \geq p$. Show that the map

$$\iota : \ell^p \ni x \mapsto x \in \ell^q \tag{10.4}$$

is well-defined and continuous.

v.) **(3 Points)** Prove the following: Every convex subset of ℓ^p with inner points is unbounded.

vi.) **(1 Point)** Show that there is no point in ℓ^p having a basis of open convex neighborhoods.

vii.) **(6 Points)** Show that a linear functional $\varphi \in (\ell^p)^*$ is continuous iff there is a unique sequence $x_\varphi \in \ell^\infty$ such that

$$\varphi = \tau \circ m(\cdot, x_\varphi), \tag{10.5}$$

where τ and m are defined as in (the soloution of) Homework 5-3.

Homework 10-2: The δ-Functional

Consider the space $(\mathscr{C}([a,b],\mathbb{R}), \|\cdot\|_\infty)$ of continuous functions on the interval $[a,b]$ with $a < 0 < b$. Let $\rho \in \mathscr{C}(\mathbb{R})$ such that

- $\rho \geq 0$
- $\rho\big|_{\mathbb{R}\setminus[a,b]} \equiv 0$ and
- $\int_{\mathbb{R}} \rho(x)\,\mathrm{d}x = \int_{[a,b]} \rho(x)\,\mathrm{d}x = 1$.

For $1 \geq \varepsilon > 0$ and $x \in \mathbb{R}$, we define

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon}\rho\Big(\frac{x}{\varepsilon}\Big). \tag{10.6}$$

Note that this yields a family $(\rho_\varepsilon)_{1\geq\varepsilon>0} \subset \mathscr{C}(\mathbb{R})$ of continuous functions.

i.) **(3 Points)** Show that the map

$$\varphi_{\rho_\varepsilon} : \mathscr{C}([a,b],\mathbb{R}) \ni f \mapsto \int_{[a,b]} \rho_\varepsilon\big|_{[a,b]}(x) f(x)\,\mathrm{d}x \in \mathbb{R} \tag{10.7}$$

defines a continuous linear functional and compute its operator norm.
Hint: Rewrite $\varphi_{\rho_\varepsilon}(f) = \int_{\mathbb{R}} \rho_\varepsilon\big|_{[a,b]}(x)\hat{f}(x)\,\mathrm{d}x$ for a suitable integrable function $\hat{f} \in \mathrm{Map}(\mathbb{R})$.

ii.) **(2 Points)** Show that $\lim_{\varepsilon\to 0}\varphi_{\rho_\varepsilon} = \delta_0$ in the weak-*-topology.

iii.) **(4 Points)** Show that $(\varphi_{\rho_\varepsilon})_{1\geq\varepsilon>0}$ does not converge to δ_0 in the functional norm topology.
Hint: Consider the family $\Big(\int_{[a,b]\setminus \mathrm{B}_\delta(0)} \rho_\varepsilon\big|_{[a,b]}(x)\,\mathrm{d}x\Big)_{b-a\geq\delta>0}$.

Homework 10-3: A Weak Null Sequence

(2 Points) Let $(e_n)_{n\in\mathbb{N}}$ be the standard Schauder basis of ℓ^p with $p \in [1,\infty)$. Prove that $(e_n)_{n\in\mathbb{N}}$ converges to zero in the weak topology. Does the sequence converge in the norm topology?