

Stochastic Differential Equations

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Definition 1. A stochastic differential equation is a (formal) equation of the form

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)W_t, \quad (1)$$

where W_t is white noise.

This equation is to be interpreted as follows:

Definition 2. We say that the stochastic process X_t is a *solution* of the SDE (1) if

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$$

The standard method to solve a stochastic differential equation is the Itô formula

Theorem 3 (The 1-Dimensional Itô Formula). Suppose X_t is an Itô process defined by the formula

$$dX_t = u dt + v dB_t.$$

Let $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$. Then

$$Y_t = g(t, X_t)$$

is also an Itô process, and

$$dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} \cdot (dX_t)^2$$

where $(dX_t)^2$ is computed according to the rules $dt \cdot dt = dt \cdot dB_t = 0$ and $dB_t \cdot dB_t = dt$.

where an Itô process is defined as follows

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Definition 4 (Itô Process). An Itô process is a stochastic process of the form

$$X_t = X_a + \int_a^t b(s) \, ds + \int_a^t \sigma(s) \, dB(s)$$

Let us look at an example:

Example 5. A model of a population is given by the stochastic differential equation

$$\frac{dN_t}{dt} = rN_t + \alpha W_t N_t.$$

Here, r, α are constants and W_t is white noise.

This is the well known model for a population, except that we have allowed r to vary by a white noise term. The solution in the nonstochastic limit is given by a simple exponential. To solve this, we first rewrite the SDE in standard form:

$$dN_t = rN_t \, dt + \alpha N_t \, dB_t,$$

or

$$\frac{dN_t}{N_t} = r \, dt + \alpha \, dB_t.$$

Inspired by the solution in the deterministic case, we guess $g(t, x) = \ln x$ in Itô's formula, and let $Y_t = g(t, N_t)$. Then, we have

$$dY_t = \frac{dN_t}{N_t} + \frac{1}{2} \alpha^2 \, dt.$$

Substituting, we have

$$dY_t = \left(r - \frac{1}{2} \alpha^2 \right) dt + \alpha B_t$$

which integrates easily to yield

$$N_t = N_0 \exp \left(\left(r - \frac{1}{2} \alpha^2 \right) t + \alpha B_t \right).$$

Clearly, for $\alpha = 0$, this reduces to the well known exponential solution. This process is known as geometric Brownian motion, which, for example, is a solution to the Black-Scholes equation [1].

Now, we turn to the questions of existence and uniqueness. Recall the basic theorem for existence and uniqueness of a deterministic differential equation

Theorem 6 (Picard-Lindelöf). Let $\dot{x} = f(t, x)$ be a differential equation with f defined on a rectangle $[a, b] \times \mathbb{R}^n$. If f is Lipschitz continuous in x , with Lipschitz constant independent of time, and continuous in time, then the differential equation has a unique global solution on $[a, b]$.

Note that for uniqueness we do not need the continuity in time; the Lipschitz condition alone suffices. This extends to the stochastic case:

Theorem 7. Let $\sigma(t, x)$ and $f(t, x)$ be measurable functions on $[a, b] \times \mathbb{R}$ satisfying the Lipschitz condition in x , and \mathcal{F} a filtration such that the Brownian motion is adapted to it. Suppose ξ is an \mathcal{F}_a -measurable random variable satisfying $\mathbb{E}[\xi^2] < \infty$. Then the stochastic differential equation

$$\frac{dX_t}{dt} = f(t, X_t) + \sigma(t, X_t)W_t$$

has at most one continuous solution on $[a, b]$.

Proof. We will not go through the proof in detail. Instead, we will talk about the main steps. Assume we have two solutions X_t and Y_t . We seek to estimate $Z_t = X_t - Y_t$

1. We estimate the expectation value $\mathbb{E}(Z_t^2)$, using the Lipschitz condition.
2. We obtain an integral inequality

$$\mathbb{E}(Z_t^2) \leq 2K^2(1 + b - a) \int_a^t \mathbb{E}(Z_s^2) ds,$$

which, by the theory of classical differential equations, implies that Z_t is 0 almost surely for all t .

3. Then, we extend the solution to show that Z_t is 0 almost surely, using sample path continuity. □

The existence theorem is as follows:

Theorem 8. The stochastic differential equation (1) has a unique solution with initial condition ξ , where ξ^2 has finite expectation, σ and b are Lipschitz in x , with Lipschitz constant independent of t , and continuous in t .

Proof. The proof follows by Picard Iteration much as it does in the deterministic case. We define $X_0 = \xi$ and

$$X_t^{(n+1)} = X_0 + \int_0^t b(s, X_s^{(n)}) ds + \int_0^t \sigma(s, X_s^{(n)}) dB_s.$$

Then, the proof proceeds in two steps. First, we show that this sequence converges, If it does, and converges in a dominated manner, it satisfies the integral equation by the dominated convergence theorem. \square

It is a known property of initial value problems that the future solution is not dependent on the past. In particular, we can imagine that we have some initial state $x(0)$ and let it evolve a time t to $x(t)$. Then we can let it evolve further. Alternatively, we can consider an initial value problem that has the value $x(t)$ at time t . We expect that these two solutions are identical. In a stochastic differential equation, this “memory” property is known as the Markov property.

Definition 9. A stochastic process X_t , with $a \leq t \leq b$, is said to have the *Markov property* if for all sequences $a < t_1 < \dots < t_n < t < b$ and corresponding x_1, \dots, x_n , we have

$$\mathbb{P}(X_t \leq x | X_{t_1} = x_1, \dots, X_{t_n} = x_n) = \mathbb{P}(X_t \leq x | X_{t_n} = x_n).$$

As an example, all processes with independent increments have the Markov property. The theorem we seek is thus

Theorem 10. The solution to (1) is a Markov process.

The final property that is of interest to us is time translation invariance. For a deterministic differential equation $\dot{x} = f(x)$, we know that the solution exhibits time translation invariance. In the deterministic case, this is quite easy to see, and follows from the fact that $\frac{d}{dt}\varphi(t - t_0) = \varphi'(t - t_0)$.

In this case, the relevant property is called the *stationary Markov property*

Definition 11. A stochastic process X is called stationary if the moments are time translation invariant:

$$\langle X_{t_1+\tau} X_{t_2+\tau} \dots X_{t_n+\tau} \rangle = \langle X_{t_1} X_{t_2} \dots X_{t_n} \rangle$$

for all n, τ and t_1, \dots, t_n .

Thus, we have our final theorem

Theorem 12. Suppose that $b(x)$ and $\sigma(x)$ are functions satisfying the Lipschitz condition. Then the solution to

$$\frac{dX_t}{dt} = b(X_t) + \sigma(X_t)W_t, \quad (2)$$

is a stationary Markov process.

As an example of this, we solve the Langevin equation

Example 13. The Langevin equation is the SDE given by

$$dX_t = \mu X_t dt + \sigma dB_t.$$

It has solutions

$$X_t = e^{\mu t} X_0 + \int_0^t \sigma e^{\mu(t-s)} dB_s$$

Proof. We multiply by the “integrating factor” $e^{-\mu t}$ and consider

$$Y_t = e^{-\mu t} X_t.$$

By Itô’s formula, we have

$$\begin{aligned} dY_t &= -\mu e^{-\mu t} X_t dt + e^{-\mu t} dX_t \\ &= -\mu e^{-\mu t} X_t dt + e^{-\mu t} (\mu X_t dt + \sigma dB_t) \\ &= e^{-\mu t} \sigma dB_t \end{aligned}$$

implying that

$$X_t = e^{\mu t} X_0 + \int_0^t \sigma e^{\mu(t-s)} dB_s.$$

□

Finally, we note that some stochastic processes can be described through density functions. Where such a density function is available, it satisfies the *Fokker-Planck Equation*

Theorem 14. The probability density of the solution to Eq. (1) $p(x, t)$ satisfies the equation

$$\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x}[b(x, t)p(x, t)] + \frac{\partial^2}{\partial x^2}[D(x, t)p(x, t)]$$

where $D(x, t) = \frac{\sigma^2(X_t, t)}{2}$ is the diffusion coefficient.

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