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Problem Sheet 7
for the tutorial on June 20th, 2025
Quantum Mechanics II
Summer term 2025

Sheet handed out on June 10th, 2025; to be handed in on June 17th, 2025 until 2 pm

Exercise 7.1: Laboratory and center-of-mass systems [2 + 2 + 1 + 4 + 2 P.]

We consider a non-relativistic collision between a projectile particle A of mass m_A and a target particle of mass m_B like in the lecture. The laboratory system L is the frame in which the target particle B is at rest before the collision. The center-of-mass system CM is the coordinate system in which the center of mass of the composite system (A+B) is always at rest. In that system the projectile A and target particle B move initially with respect to the center of mass C with equal and opposite momenta, $\vec{p}_A = -\vec{p}_B = \vec{p}$, as illustrated in Fig. 1. With respect to the laboratory frame, the center of mass of the two particles moves throughout the collision with a constant velocity $\vec{v}_c$ along the direction of incidence, with $\vec{v}_c = \vec{q}_A/(m_A + m_B)$, where $\vec{q}_A$ is the momentum of particle A before the collision in the laboratory system.

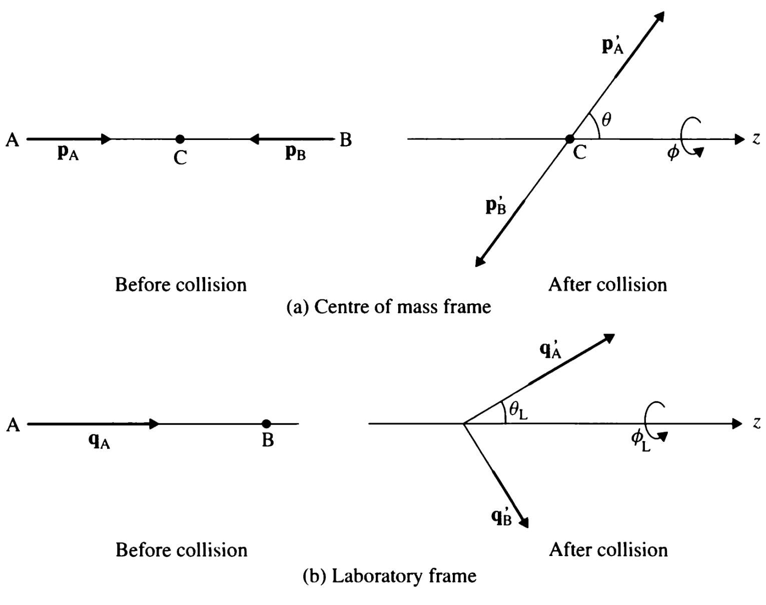


Figure 1: Elastic scattering of a projectile A by target B.

a) Using a Galilean transformation, find the relation between the momenta $\vec{q}_A$ and $\vec{p}_A$ before the collision.

b) After the collision the two particles A and B emerge with equal and opposite momenta $\vec{p'}_A = -\vec{p'_B} = \vec{p'}$ in the CM frame. In the following we consider an elastic collision, such that the magnitude of the momenta of each particle remain the same $p' = p$. Relate the components of the momentum of projectile A along the direction of incidence in the two coordinate systems using the two scattering angles θ_L and θ as illustrated in Fig. 1.

c) Show that the relation between the two scattering angles is given by

$$\tan\theta_L = \frac{\sin\theta}{\cos\theta + \frac{m_A}{m_B}} \,. \tag{1}$$

d) Using the equation above, find the relation between the angular differential cross sections $\frac{d\sigma}{d\Omega}$ in the laboratory and the center-of-mass frames.

e) Let us now consider a numerical example:

Two beams of protons intersect collinearly. If the kinetic energy of the protons is 5 keV in both beams, calculate:

i) the magnitude of the relative velocity of a proton in one beam with respect to a proton in the other one,

ii) the energy in the centre-of-mass system.

Exercise 7.2: **Partial waves and phase shifts** [6 + 8 P.]

We consider in the following the scattering by a central potential $V(r)$ such that the system is completely symmetrical about the direction of incidence, which we choose to be the z-axis. In this case both the wave function $\psi_{\boldsymbol{k}}$ and the scattering amplitude f do not depend on the azimuthal angle φ. We then expand them in a series of Legendre polynomials, which form a complete set in the interval $-1 \le \cos\theta \le 1$,

$$\psi_{\boldsymbol{k}}(r,\theta) = \sum_{l=0}^{\infty} R_l(k,r) P_l(\cos\theta) \,, \tag{2}$$

$$f(k,\theta) = \sum_{l=0}^{\infty} f_l(k) P_l(\cos\theta) \,. \tag{3}$$

Each term in the series is known as a partial wave and is a simultaneous eigenfunction of the operators $\vec{L}^2$ and L_z belonging to eigenvalues $l(l+1)\hbar^2$ and zero, respectively. The radial wave function for the far region where the potential can be neglected is given by a linear combination of Bessel and Neumann functions $j_l(kr)$ and $n_l(kr)$

$$R_l(k,r) = B_l(k) j_l(kr) + C_l(k) n_l(kr) \tag{4}$$

with coefficients $B(k)$ and $C(k)$. Using the asymptotic expressions for the Bessel and Neumann functions given in the lecture, this leads to

$$R_l(k,r) \overset{r\to\infty}{\approx} \frac{1}{kr}\left[B_l(k)\sin\left(kr - \frac{l\pi}{2}\right) - C_l(k)\cos\left(kr - \frac{l\pi}{2}\right)\right] . \tag{5}$$

a) In the lecture, we have considered a particular choice for the coefficients B_l and C_l. Here we consider a more general case. To this end, it is convenient to rewrite the expression above as

$$R_l(k,r) \overset{r\to\infty}{\approx} A_l(k)\frac{1}{kr}\sin\left(kr - \frac{l\pi}{2} + \delta_l(k)\right). \tag{6}$$

Determine the expressions of the amplitudes $A_l(k)$ and the phase shifts $\delta_l(k)$ introduced above as a function of B_l and C_l. The phase shifts $\delta_l(k)$ are real quantities and characterize the strength of the scattering in the lth partial wave by the potential $V(r)$ at the energy $E = \hbar^2k^2/(2m)$.

b) We would like now to relate the phase shifts $\delta_l(k)$ to the partial wave amplitudes $f_l(k)$ and to the scattering amplitude $f(k,\theta)$ in Eq. (3). Use the radial wave function determined above and the general relation between wave function and scattering amplitude

$$\psi_{\boldsymbol{k}}(r) \overset{r\to\infty}{\approx} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(k,\theta)\frac{e^{ikr}}{r} \tag{7}$$

to determine the partial wave amplitudes $f_l(k)$. Write then the expression of the scattering amplitude $f(k,\theta)$ as a function of the phase shifts.

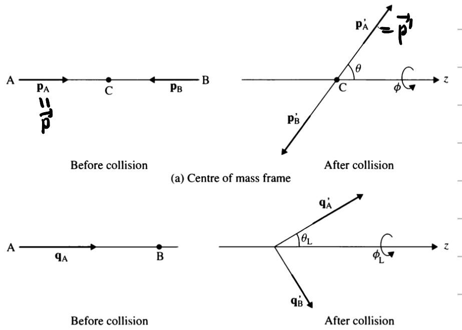
Hint: Use the plane wave expansion

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)\,. \tag{8}$$

a) Using a Galilean transformation, find the relation between the momenta $\vec{q}_A$ and $\vec{p}_A$ before the collision.

$$\frac{\vec{p}_A}{m_A} = \frac{\vec{q}_A}{m_A} - \vec{V}_C = \frac{\vec{q}_A}{m_A} - \frac{\vec{q}_A}{m_A+m_B} = \frac{m_B}{m_A}\frac{\vec{q}_A}{m_A+m_B}$$

$$\vec{q}_A = \frac{m_A+m_B}{m_B}\vec{p}_A$$



(a) Centre of mass frame

(b) Laboratory frame

b) After the collision the two particles A and B emerge with equal and opposite momenta $\vec{p'}_A = -\vec{p'_B} = \vec{p'}$ in the CM frame. In the following we consider an elastic collision, such that the magnitude of the momenta of each particle remain the same $p' = p$. Relate the components of the momentum of projectile A along the direction of incidence in the two coordinate systems using the two scattering angles θ_L and θ as illustrated in Fig. 1.

$$\frac{q'_A \cos\theta_L}{m_A} = \frac{p'\cos\theta}{m_A} - V_c$$

$$= \frac{p\cos\theta}{m_A} - V_c$$

c) Show that the relation between the two scattering angles is given by

$$\tan\theta_L = \frac{\sin\theta}{\cos\theta + \frac{m_A}{m_B}}.$$

no ' needed because elastic

$$\tan\theta_L = \frac{q'_{A,y}}{q'_{A,x}} = \frac{p_y}{p_x + m_A V_c} = \frac{p\sin\theta}{p\cos\theta + \frac{m_A}{m_A+m_B} q_A}$$

$$= \frac{p\sin\theta}{p\cos\theta + \frac{m_A}{m_B} p} \quad \text{(see (a))}$$

$$= \frac{\sin\theta}{\cos\theta + \frac{m_A}{m_B}}$$

d) Using the equation above, find the relation between the angular differential cross sections $\frac{d\sigma}{d\Omega}$ in the laboratory and the center-of-mass frames.

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

$$\tan\theta_L = \frac{\sin\theta}{\cos\theta + \frac{m_A}{m_B}}$$

$$(\tan\theta_L)\left(\cos\theta + \frac{m_A}{m_B}\right) = \sin\theta$$

$$\sec^2\theta_L\left(\cos\theta+\frac{m_A}{m_B}\right)d\theta_L-\tan\theta_L\sin\theta\,d\theta=\cos\theta\,d\theta$$