$\mathbb{K}$-linear extension of $e_i \mapsto v_i$

Need only show continuity at 0!

Suppose $\phi^{-1}(U)$ is not continuous, at 0

contains $0|_V$

Every $\varepsilon$-ball about 0 contains sth in $\mathbb{K}^n \setminus \phi^{-1}(U)$

Sequence $(a_n)_n \to 0|_{\mathbb{K}^n}$

$(x_n^i e_i)_n \xrightarrow{n\to\infty} 0 \quad \forall i$

$(x_n^i)_n \xrightarrow{n\to\infty} 0$

Cont. scalar. mult. $\quad x_n^i v_i \to 0 \quad \forall i$

Fin. dim. hypothesis $\quad \sum_{i=1}^{k} x_n^i v_i \to 0$

Possible alternative solution!

$\phi_i : \sum_{n=1}^{N} x^n e_n \mapsto x^n \mapsto x^n v_n$

iii) $\quad S^{n-1} \subset \mathbb{K}^n$ is compact closed

$\phi(S^{n-1})$ is compact, + $V$ hausdorff = closed too

$V \setminus \phi(S^{n-1})$ open nbhd of 0

$\exists$ balanced open nbhd $U_0 \subseteq V \setminus \phi(S^{n-1})$

Want! $U_0 \subseteq \phi(B_1(0))$

Assume $\quad \exists u \in U_0 \cap V \setminus \phi(B_1(0)^c) \to \|\phi^{-1}(u)\| > 1$

$\frac{1}{\|\phi^{-1}(u)\|} u \in U_0 \cap \phi(S^{n-1}) \to$ ↯

Use continuity of scalar multiplication to extend to $B_r(0)$ for any $r$

1. $A$ is closed, complement open

$p \in M \setminus A_n(\varepsilon)$ and let $k \geq n$

$|f_n(p) - f_k(p)| = \varepsilon + \delta, \quad \delta \geq 0.$

Reverse $\Delta$ inequality $|f_n(q) - f_k(q)| \geq |f_n(p) - f_k(p)| - |f_n(q) - f_n(p) + f_k(p) - f_k(q)|$

$$\geq |f_n(p) - f_k(p)| - |f_n(q) - f_n(p)| - |f_k(p) - f_k(q)|$$

$$\geq \varepsilon + \delta'$$

$$A_n(\varepsilon) = A_n(\varepsilon)^o \cup \partial A_n(\varepsilon)$$

$$\underbrace{(A_n(\varepsilon)^o \cup \partial A_n(\varepsilon))^o}_{A_n(\varepsilon)^o} \supseteq A_n(\varepsilon)^o \cup \underbrace{\partial A_n(\varepsilon)^o}_{\text{empty!}}$$

(ii) Pointwise convergence implies $M = \bigcup_{n \in \mathbb{N}} A_n(\varepsilon)$

and $A_n(\varepsilon)^o \subseteq C_n(\varepsilon)^o$

$$M \setminus C = \bigcup_{m \in \mathbb{N}} M \setminus C(\tfrac{1}{m}) \qquad \lim_{k \to \infty} |f_n(q) - f_k(q)| \leq \varepsilon$$

$$\subseteq \bigcup_{m \in \mathbb{N}} \Big(\underbrace{M}_{= \bigcup_{n \in \mathbb{N}} A_n(\frac{1}{m})} \setminus \bigcup_{n \in \mathbb{N}} A_n(\tfrac{1}{m})^o\Big) \subseteq \bigcup_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \left(A_n(\tfrac{1}{m}) \setminus A_n(\tfrac{1}{m})^o\right) \qquad \square$$

ii) Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable

$$f_n'(x) := n\left(f(x + \tfrac{1}{n}) - f(x)\right) = \frac{f(x+\frac{1}{n}) - f(x)}{x + \frac{1}{n} - x}$$

$f_n'$ converges pointwise to $f'$