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Problem Sheet 5
for the tutorial on June 6th, 2025
Quantum Mechanics II
Summer term 2025

Sheet handed out on May 27th, 2025; to be handed in on June 3rd, 2025 until 2 pm

Exercise 5.1: Perturbative calculation of the steady state [2+2+2+2 P.]

One approach to measure the properties of a quantum optical system is to probe it with a weak laser field in order to study its linear response. In this case, one is usually interested in the steady state in leading order of the probe field Rabi frequency. For simple setups, one can first calculate the full steady state, i.e., the solution of the equation $\dot{\rho} = 0$, and expand it in the probe field Rabi frequency. But often it is more efficient to directly calculate the perturbative result, and this problem discusses a method to do so.

Our model system is a laser-driven two-level atom with excited state $|2\rangle$ and ground state $|1\rangle$, spontaneous decay γ and constant real Rabi frequency Ω. The system is described by the Bloch equations

$$\frac{d}{dt}\rho_{22} = -\gamma\rho_{22} + i\Omega(\rho_{21} - \rho_{12}), \tag{1}$$

$$\frac{d}{dt}\rho_{21} = -\left(\frac{\gamma}{2} - i\Delta\right)\rho_{21} + i\Omega(\rho_{22} - \rho_{11}), \tag{2}$$

with $\rho_{12} = \rho_{21}^*$ and $\rho_{22} + \rho_{11} = 1$. We define $\vec{\rho} = (\rho_{11}, \rho_{21}, \rho_{12})^T$.

(a) The equations of motion can be written as,

$$\frac{d}{dt}\vec{\rho} = M\vec{\rho} + \vec{K}. \tag{3}$$

Determine the matrix M and the vector $\vec{K}$.

(b) Introduce the expansion $\vec{\rho} = \vec{\rho}_0 + \Omega\vec{\rho}_1 + \Omega^2\vec{\rho}_2 + \dots$ and similar for $M = M_0 + \Omega M_1 + \Omega^2 M_2 + \dots$ and $\vec{K} = \vec{K}_0 + \Omega\vec{K}_1 + \Omega^2\vec{K}_2 + \dots$. Use these expansions to show that in steady state,

$$\vec{\rho}_0 = -M_0^{-1}\vec{K}_0, \tag{4}$$

$$\vec{\rho}_1 = -M_0^{-1}\left(M_1\vec{\rho}_0 + \vec{K}_1\right), \tag{5}$$

where the superscript (-1) indicates the matrix inverse.

(c) Calculate M_0^{-1}.

(d) Use Eqs. (4) and (5) to calculate the steady state up to linear order in the Rabi frequency.

Exercise 5.2: **Coherent states** [2+1+2+2+2 P.]

We have seen in the lecture that the quantized electromagnetic field can be represented as a set of quantum mechanical harmonic oscillators. Here, we consider the case of a single field mode with basis $\{|n\rangle \mid n \in \{0, 1, 2, \dots\}\}$, where $|n\rangle$ is a Fock state with n photons in the mode. The creation and annihilation operators for our mode are denoted as $\hat{a}^\dagger$ and $\hat{a}$. A *coherent state* $|\alpha\rangle$ with complex number α as a parameter can be defined as the eigenstate of the annihilation operator given by

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,. \tag{6}$$

a) Show that

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle \tag{7}$$

fulfills Eq. (6) and is normalized ($\langle\alpha|\alpha\rangle = 1$). Thus, (7) is a representation of the coherent state in terms of the photon number state basis.

b) We define $\hat{n} = \hat{a}^\dagger\hat{a}$. Calculate $\hat{n}|n\rangle$ and interpret the meaning of the observable $\hat{n}$.

c) We define the expectation value of an operator for the coherent state as $\langle\hat{O}\rangle = \langle\alpha|\hat{O}|\alpha\rangle$. Calculate $\langle\hat{n}\rangle$, $\langle\hat{n}^2\rangle$ and $\Delta\hat{n} = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2}$. Show that $\Delta\hat{n}/\langle\hat{n}\rangle \to 0$ for $|\alpha| \to \infty$.
Hint: The coherent state is not an eigenstate of $\hat{a}^\dagger$ - but try Hermitian conjugating Eq. (6) to obtain an equation involving $\hat{a}^\dagger$.

d) Calculate the time evolution of the coherent state under the action of the Hamiltonian $\mathcal{H} = \hbar\omega\hat{n}$, i.e., calculate $|\Psi(t > 0)\rangle$ with $|\Psi(t = 0)\rangle = |\alpha_0\rangle$. Show that it can be written as $|\Psi(t)\rangle = |\alpha(t)\rangle$, i.e., as a coherent state with time dependent parameter $\alpha(t) = \alpha_0 e^{-i\omega t}$.

e) We now for simplicity assume that α_0 in the solution of the last part is a real number. Using $\hat{x} = x_0(\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = -ip_0(\hat{a} - \hat{a}^\dagger)$, calculate $\langle\hat{x}\rangle(t)$ and $\langle\hat{p}\rangle(t)$. Here, $\langle\hat{O}\rangle(t) = \langle\alpha(t)|\hat{O}|\alpha(t)\rangle$ using the solution of d).

Exercise 5.3: **Master equation for a cavity coupled to a thermal bath** [2+2+2+2 P.]

We investigate a single-mode cavity which hosts an electromagnetic field characterized by the photon creation and annihilation operators $a^\dagger$ and a. This cavity is coupled to a thermal bath with mean photon number $\bar{n}$ and has a decay rate of κ. It can be shown (in the more specialized Theoretical Quantum Optics lecture; here we just use the result) that this system is governed by the master equation

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{\kappa}{2}(\bar{n} + 1)(a^\dagger a\hat{\rho} + \hat{\rho}a^\dagger a - 2a\hat{\rho}a^\dagger) - \frac{\kappa}{2}\bar{n}(aa^\dagger\hat{\rho} + \hat{\rho}aa^\dagger - 2a^\dagger\hat{\rho}a)\,. \tag{8}$$

a) Calculate the scalar equations of motion $\dot{P}_n$ for the probabilities $P_n = \langle n|\hat{\rho}|n\rangle$ that there are n photons in the cavity.

b) The stationary or steady-state of the system can be obtained by setting $\dot{P}_n = 0$. Show that the ansatz $P_{n+1} = \bar{n}/(\bar{n} + 1)P_n$ solves the stationary state equation.

c) Calculate P_0 from the normalization condition $\sum_{n=0}^{\infty} P_n = 1$. You can use the geometric series result $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ for $|q| < 1$.

d) What is the mean photon number $\langle n\rangle$ in the cavity?

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where the superscript (-1) indicates the matrix inverse.

(c) Calculate M_0^{-1}.

(d) Use Eqs. (4) and (5) to calculate the steady state up to linear order in the Rabi frequency.

Note that $\rho_{11} = 1 - \rho_{22}$, $\frac{d\rho_{11}}{dt} = 1 - \frac{d\rho_{22}}{dt}$

$$\rho_{22} - \rho_{11} = (1-\rho_{11}) - \rho_{11} = 1 - 2\rho_{11}$$

$$\frac{d\rho_{12}}{dt} = \left(\frac{d\rho_{21}}{dt}\right)^* = -\left(\frac{\gamma}{2} + i\Delta\right)\rho_{12} - i\Omega(\rho_{22} - \rho_{11})$$

$$= -\left(\frac{\gamma}{2} + i\Delta\right)\rho_{12} - i\Omega(1 - 2\rho_{11})$$

$$\frac{d}{dt}\begin{pmatrix}\rho_{11}\\ \rho_{21}\\ \rho_{12}\end{pmatrix} = \begin{pmatrix} 1+\gamma-\gamma\rho_{11} - i\Omega(\rho_{21}-\rho_{12}) \\ -\left(\frac{\gamma}{2}-i\Delta\right)\rho_{21} + i\Omega(1-2\rho_{11}) \\ -\left(\frac{\gamma}{2}+i\Delta\right)\rho_{12} - i\Omega(1-2\rho_{11}) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -\gamma & -i\Omega & +i\Omega \\ -2i\Omega & -\left(\frac{\gamma}{2}-i\Delta\right) & 0 \\ 2i\Omega & 0 & -\left(\frac{\gamma}{2}+i\Delta\right) \end{pmatrix}}_{M} \begin{pmatrix}\rho_{11}\\ \rho_{21}\\ \rho_{12}\end{pmatrix} + \underbrace{\begin{pmatrix} 1+\gamma \\ i\Omega \\ -i\Omega \end{pmatrix}}_{\vec{K}}$$

Steady state $\frac{d\vec{\rho}}{dt} = 0$

$$M\vec{\rho} + K = 0$$

$$\vec{\rho} = \vec{\rho}_0 + \Omega\vec{\rho}_1 + \Omega^2\vec{\rho}_2 + \ldots$$

$$M = M_0 + \Omega M_1 + \Omega^2 M_2 + \ldots$$

$$\vec{K} = \vec{K}_0 + \Omega\vec{K}_1 + \Omega^2\vec{K}_2 + \ldots$$

Expand up to 1st order

$$(M_0 + \Omega M_1)(\vec{\rho}_0 + \Omega\vec{\rho}_1) + \vec{K}_0 + \Omega\vec{K} = M_0\vec{\rho}_0 + \Omega M_0\vec{\rho}_1 + \Omega M_1\vec{\rho}_0 + \cancelto{0}{\Omega^2 M_1\vec{\rho}_1} + \vec{K}_0 + \Omega\vec{K}_1$$

$$= (M_0\vec{\rho}_0 + \vec{K}_0) + \Omega(M_0\vec{\rho}_1 + M_1\vec{\rho}_0 + \vec{K}_1)$$

b) Ω^0 coefficient: $M_0\vec{\rho}_0 + \vec{K}_0 = 0$, $\vec{\rho}_0 = -M_0^{-1}\vec{K}_0$

Ω^1 coefficient: $M_0\vec{\rho}_1 + M_1\vec{\rho}_0 + \vec{K}_1 = 0$

$$\vec{\rho}_1 = -M_0^{-1}(\vec{K}_1 + M_1\vec{\rho}_0)$$

c)
$$M = \begin{pmatrix} -\gamma & -i\Omega & +i\Omega \\ -2i\Omega & -\left(\frac{\gamma}{2}-i\Delta\right) & 0 \\ 2i\Omega & 0 & -\left(\frac{\gamma}{2}+i\Delta\right) \end{pmatrix} = \underbrace{\begin{pmatrix} -\gamma & 0 & 0 \\ 0 & -\left(\frac{\gamma}{2}-i\Delta\right) & 0 \\ 0 & 0 & -\left(\frac{\gamma}{2}+i\Delta\right) \end{pmatrix}}_{M_0} + \underbrace{\begin{pmatrix} 0 & -i & +i \\ -2i & 0 & 0 \\ 2i & 0 & 0 \end{pmatrix}\Omega}_{M_1}$$

$$M_0^{-1} = \begin{pmatrix} -\frac{1}{\gamma} & 0 & 0 \\ 0 & \frac{\gamma+i\Delta}{2\left(\frac{\gamma^2}{4}+\Delta^2\right)} & 0 \\ 0 & 0 & \frac{\gamma+i\Delta}{2\left(\frac{\gamma^2}{4}+\Delta^2\right)} \end{pmatrix}$$

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b) We define $\hat{n} = \hat{a}^\dagger\hat{a}$. Calculate $\hat{n}|n\rangle$ and interpret the meaning of the observable $\hat{n}$.

c) We define the expectation value of an operator for the coherent state as $\langle\hat{O}\rangle = \langle\alpha|\hat{O}|\alpha\rangle$. Calculate $\langle\hat{n}\rangle$, $\langle\hat{n}^2\rangle$ and $\Delta\hat{n} = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2}$. Show that $\Delta\hat{n}/\langle\hat{n}\rangle \to 0$ for $|\alpha| \to \infty$.
Hint: The coherent state is not an eigenstate of $\hat{a}^\dagger$ - but try Hermitian conjugating Eq. (6) to obtain an equation involving $\hat{a}^\dagger$.

d) Calculate the time evolution of the coherent state under the action of the Hamiltonian $\mathcal{H} = \hbar\omega\hat{n}$, i.e., calculate $|\Psi(t > 0)\rangle$ with $|\Psi(t = 0)\rangle = |\alpha_0\rangle$. Show that it can be written as $|\Psi(t)\rangle = |\alpha(t)\rangle$, i.e., as a coherent state with time dependent parameter $\alpha(t) = \alpha_0 e^{-i\omega t}$.

e) We now for simplicity assume that α_0 in the solution of the last part is a real number. Using $\hat{x} = x_0(\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = -ip_0(\hat{a} - \hat{a}^\dagger)$, calculate $\langle\hat{x}\rangle(t)$ and $\langle\hat{p}\rangle(t)$. Here, $\langle\hat{O}\rangle(t) = \langle\alpha(t)|\hat{O}|\alpha(t)\rangle$ using the solution of d).

a) $$a|\alpha\rangle = a\, e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, a|n\rangle$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}\, |n-1\rangle$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{(n-1)!}} |n-1\rangle$$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \alpha \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$= \alpha \left[e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \right]$$

$$= \alpha|\alpha\rangle$$

b) We define $\hat{n} = \hat{a}^\dagger\hat{a}$. Calculate $\hat{n}|n\rangle$ and interpret the meaning of the observable $\hat{n}$.

$$\hat{n}|n\rangle = a^\dagger a |n\rangle$$

$$= a^\dagger \sqrt{n}\, |n-1\rangle$$

$$= n\, |n\rangle$$

Hence $\hat{n}$ is the number operator

c) We define the expectation value of an operator for the coherent state as $\langle\hat{O}\rangle = \langle\alpha|\hat{O}|\alpha\rangle$. Calculate $\langle\hat{n}\rangle$, $\langle\hat{n}^2\rangle$ and $\Delta\hat{n} = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2}$. Show that $\Delta\hat{n}/\langle\hat{n}\rangle \to 0$ for $|\alpha| \to \infty$.
Hint: The coherent state is not an eigenstate of $\hat{a}^\dagger$ - but try Hermitian conjugating Eq. (6) to obtain an equation involving $\hat{a}^\dagger$.

$$a|\alpha\rangle = \alpha\,|\alpha\rangle$$

$$\langle\alpha|\, a^\dagger = \alpha^*\langle\alpha|$$

$$\langle\alpha|a^\dagger a|\alpha\rangle = \alpha^* \alpha\langle\alpha|\alpha\rangle$$
$$= |\alpha|^2$$

$$\langle\alpha|n^2|\alpha\rangle = \langle\alpha|a^\dagger a\, a^\dagger a|\alpha\rangle$$
$$= |\alpha|^2\langle\alpha|a a^\dagger|\alpha\rangle$$
$$= |\alpha|^2\langle\alpha|(a^\dagger a+1)|\alpha\rangle$$
$$= |\alpha|^4 + |\alpha|^2$$
$$\langle\hat{n}^2\rangle - \langle n\rangle^2 = |\alpha|^2$$

d) $\mathcal{H} = \hbar\omega\hat{n}$

$$U = e^{-\frac{i\mathcal{H}t}{\hbar}} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\mathcal{H}t}{\hbar}\right)^n$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}(i\omega t\,\hat{n})^n$$
$$= \sum_{n=0}^{\infty}\frac{(-i\omega t)^n}{n!}\hat{n}^n$$

$$U(t)|\alpha_0\rangle = \left[\sum_{n=0}^{\infty}\frac{(-i\omega t)^n}{n!}\hat{n}^n\right]\left[e^{-\frac{|\alpha_0|^2}{2}}\sum_{k=0}^{\infty}\frac{\alpha_0^k}{\sqrt{k!}}|k\rangle\right]$$
$$= e^{-\frac{|\alpha|^2}{2}}\sum_{n,k=0}^{\infty}\frac{(-i\omega t)^n}{n!}\frac{\alpha_0^k}{\sqrt{k!}}\hat{n}^n|k\rangle$$
$$= e^{-\frac{|\alpha|^2}{2}}\sum_{n,k=0}^{\infty}\frac{(-i\omega t)^n}{n!}\frac{\alpha_0^k}{\sqrt{k!}}k^n|k\rangle$$
$$= e^{-\frac{|\alpha|^2}{2}}\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{(-i\omega t)^n}{n!}k^n\right)\frac{\alpha_0^k}{\sqrt{k!}}|k\rangle$$
$$= e^{-\frac{|\alpha|^2}{2}}\sum_{k=0}^{\infty}e^{-i\omega t k}\frac{\alpha_0^k}{\sqrt{k!}}|k\rangle$$
$$= e^{-\frac{|\alpha_0|^2}{2}}\sum_{k=0}^{\infty}\frac{(\alpha_0 e^{-i\omega t})^k}{\sqrt{k!}}|k\rangle$$
$$= |\alpha(t)\rangle$$
$$\alpha(t) = \alpha_0 e^{-i\omega t}$$

e) We now for simplicity assume that α_0 in the solution of the last part is a real number. Using $\hat{x} = x_0(\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = -ip_0(\hat{a} - \hat{a}^\dagger)$, calculate $\langle \hat{x} \rangle(t)$ and $\langle \hat{p} \rangle(t)$. Here, $\langle \hat{O} \rangle(t) = \langle \alpha(t)|\hat{O}|\alpha(t)\rangle$ using the solution of d).

$$\langle x \rangle = x_0 \langle \alpha | a + a^\dagger | \alpha \rangle$$

$$= x_0 (\alpha + \alpha^\dagger) \langle \alpha | \alpha \rangle$$

$$= x_0 (\alpha + \alpha^\dagger)$$

$$\langle p \rangle = -ip_0 \langle \alpha | a - a^\dagger | \alpha \rangle$$

$$= -ip_0 (\alpha - \alpha^\dagger)$$

Exercise 5.3: Master equation for a cavity coupled to a thermal bath [2+2+2+2 P.]

We investigate a single-mode cavity which hosts an electromagnetic field characterized by the photon creation and annihilation operators $a^\dagger$ and a. This cavity is coupled to a thermal bath with mean photon number $\bar{n}$ and has a decay rate of κ. It can be shown (in the more specialized Theoretical Quantum Optics lecture; here we just use the result) that this system is governed by the master equation

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a) Calculate the scalar equations of motion $\dot{P}_n$ for the probabilities $P_n = \langle n|\hat{\rho}|n\rangle$ that there are n photons in the cavity.

b) The stationary or steady-state of the system can be obtained by setting $\dot{P}_n = 0$. Show that the ansatz $P_{n+1} = \bar{n}/(\bar{n}+1)P_n$ solves the stationary state equation.

c) Calculate P_0 from the normalization condition $\sum_{n=0}^{\infty} P_n = 1$. You can use the geometric series result $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ for $|q| < 1$.

d) What is the mean photon number $\langle n\rangle$ in the cavity?

$$\langle n|\frac{d\rho}{dt}|n\rangle = \frac{d}{dt}P_n$$

$$= -\frac{\kappa}{2}(\bar{n}+1)\langle n|(\underbrace{a^\dagger a\rho}_{nP_n} + \underbrace{\rho a^\dagger a}_{nP_n} \underbrace{-2a\rho a^\dagger}_{-2(n+1)P_{n+1}})|n\rangle - \frac{\kappa}{2}\bar{n}\langle n|(\underbrace{aa^\dagger\rho}_{(n+1)P_n} + \underbrace{\rho aa^\dagger}_{(n+1)P_n} \underbrace{-2a^\dagger\rho a}_{-2nP_{n-1}})|n\rangle$$

$$= -\frac{\kappa}{\cancel{2}}(\bar{n}+1)\cancel{2}(nP_n - (n+1)P_{n+1}) - \frac{\kappa}{\cancel{2}}\bar{n}\cdot\cancel{2}((n+1)P_n - nP_{n-1})$$

$$= -\kappa(\bar{n}+1)(nP_n - (n+1)P_{n+1}) - \kappa\bar{n}((n+1)P_n - nP_{n-1})$$

b) $$0 = -\kappa(\bar{n}+1)(nP_n - (n+1)P_{n+1}) - \kappa\bar{n}((n+1)P_n - nP_{n-1})$$

Substituting ansatz $P_{n+1} = \frac{\bar{n}}{\bar{n}+1}P_n$, we get

$$-\kappa(\bar{n}+1)(nP_n - (n+1)P_{n+1}) - \kappa\bar{n}((n+1)P_n - nP_{n-1})$$

$$= -\kappa(\bar{n}+1)\left[nP_n - (n+1)\frac{\bar{n}}{\bar{n}+1}P_n\right] - \kappa\bar{n}\left[(n+1)P_n - n\frac{\bar{n}+1}{\bar{n}}P_n\right]$$

$$= -\kappa(\cancel{\bar{n}}+1)nP_n + \kappa(\cancel{n}+1)\bar{n}P_n - \kappa\bar{n}(n+1)P_n + \kappa n(\bar{n}+1)P_n$$

$$= -\cancel{\kappa nP_n} + \cancel{\kappa\bar{n}P_n} - \cancel{\kappa\bar{n}nP_n} - \cancel{\kappa\bar{n}P_n} + \kappa n(\cancel{\bar{n}}+\cancel{1})P_n$$

$$= 0$$

c) $$P_{n+1} = \frac{\bar{n}}{\bar{n}+1}P_n$$

$$P_n = \left(\frac{\bar{n}}{\bar{n}+1}\right)^n P_0$$

$$\sum_{n=0}^{\infty} P_n = P_0\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n$$

$$= P_0\frac{1}{1-\left(\frac{\bar{n}}{\bar{n}+1}\right)}$$

$$= P_0\frac{1}{\frac{1}{\bar{n}+1}} = (\bar{n}+1)P_0 = 1$$

$$P_0 = \frac{1}{\bar{n}+1}$$

d) $$\langle n \rangle = \sum_{n=0}^{\infty} n P_n$$

$$= P_0 \sum_{n=0}^{\infty} n \left(\frac{\bar{n}}{\bar{n}+1}\right)^n$$

$$= P_0 \frac{\left(\frac{\bar{n}}{\bar{n}+1}\right)}{\left(1-\frac{\bar{n}}{\bar{n}+1}\right)^2}$$

$$= \frac{1}{\bar{n}+1} \quad \frac{\frac{\bar{n}}{\bar{n}+1}}{\left(\frac{1}{\bar{n}+1}\right)^2} = \bar{n}$$