Homework for the Lecture

Functional Analysis

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Homework Sheet No 4

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(26 Points. Discussion 11. 11. 2024)

Homework 4-1: The Space ℓ^p

(2 Points) For $p \in [1, \infty]$ we define the set

$$\ell^p := \begin{cases} \left\{ x := (x_n)_{n \in \mathbb{N}} \subset \mathbb{K} : \|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\} & p < \infty \\ \{ x := (x_n)_{n \in \mathbb{N}} \subset \mathbb{K} : \|x\|_\infty := \sup_{n \in \mathbb{N}} |x_n| < \infty \} & p = \infty. \end{cases} \tag{4.1}$$

Show that the usual operations on sequences induce a vector space structure on ℓ^p. Moreover, show that ℓ^p is a subspace of ℓ^r for $p \leq r$.

Homework 4-2: Some Inequalities

In this exercise, we consider the spaces ℓ^p for $p \in (1, \infty)$. Note that for every such p there exists a conjugate number $q \in (1, \infty)$ which satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

i.) **(1 Point)** Show that the product of two non-negative reel numbers $a, b \in [0, \infty)$ satisfies Young's inequality, that is

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{4.2}$$

Hint: Use the AM-GM inequality.

ii.) **(2 Points)** Prove that Hölder's inequality

$$\|xy\|_1 := \sum_{n=1}^{\infty} |x_n y_n| \leq \|x\|_p \|y\|_q \tag{4.3}$$

holds true for any two sequences $x \in \ell^p$ and $y \in \ell^q$.

iii.) **(3 Points)** Show Minkowski's inequality, that is

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p \tag{4.4}$$

for $x, y \in \ell^p$.

iv.) **(5 Points)** Let $\lambda := (\lambda_n)_{n\in\mathbb{N}} \subset [0,1]$ be a sequence in ℓ^1 with $\|\lambda\|_1 = 1$. Show that Jensen's inequality

$$f\Big(\sum_{n=1}^{\infty} \lambda_n x_n\Big) \leq \sum_{n=1}^{\infty} \lambda_n f(x_n) \tag{4.5}$$

holds true for every convex function $f \in \mathscr{C}(I)$ on an open inteval $I \subseteq \mathbb{R}$ and every sequence $(x_n)_{n\in\mathbb{N}} \subset I$ such that $\sum_{n=1}^{\infty} \lambda_n x_n$ and $\sum_{n=1}^{\infty} \lambda_n f(x_n)$ converge and $\sum_{n=1}^{\infty} \lambda_n x_n \in I$. Conclude that $\|x\|_r \leq \|x\|_p$ for every $x \in \ell^p$ and $p \leq r$.

Homework 4-3: A Schauder Basis for ℓ^p

(4 Points) Let $p \in [1, \infty)$. Consider the sequences $(e_n := (\delta_{nm})_{m\in\mathbb{N}})_{n\in\mathbb{N}} \subset \ell^p$. Show that for every sequence $x = (x_n)_{n\in\mathbb{N}} \in \ell^p$ the series $\sum_{n\in\mathbb{N}} x_n e_n$ converges unconditionally towards x with respect to $\|\cdot\|_p$. Does it converge absolutely? Moreover, show that a sequence $x = (x_n)_{n\in\mathbb{N}}$ lies in ℓ^p if the series $\sum_{n\in\mathbb{N}} x_n e_n$ converges unconditionally with respect to $\|\cdot\|_p$.
Hint: Having Minkowski's inequality, you can use that $(\ell^p, \|\cdot\|_p)$ is a normed space without proof.

Homework 4-4: Approximating the Square Root and the Absolute Value

In the upcoming exercise sheets, we will prove the Stone-Weierstraß theorem in several steps. Here, we do some necessary preparation we will need for the actual proof.
By recursion, define the polynomials

$$p_0(x) = 0, \quad \text{and} \quad p_{n+1}(x) = p_n(x) + \frac{1}{2}\big(x - p_n^2(x)\big). \tag{4.6}$$

i.) **(6 Points)** Show $p_n(0) = 0$ and the estimates

$$p_n(x) \geq 0, \quad \text{and} \quad 0 \leq \sqrt{x} - p_n(x) \leq \frac{2\sqrt{x}}{2 + n\sqrt{x}} \tag{4.7}$$

for $x \in [0,1]$.

Hint: First show the coarser estimates $0 \leq p_n(x) \leq 1$ for $x \in [0,1]$ by induction. Use this in a second induction to improve the estimates.

ii.) **(1 Point)** Conclude that $(p_n)_{n\in\mathbb{N}}$ converges uniformly to the square root function on the interval $[0,1]$.

iii.) **(1 Point)** Let $\alpha > 0$. Construct a sequence of polynomials that converges uniformly to the square root function on $[0, \alpha]$.

iv.) **(1 Point)** Let $\alpha > 0$. Construct a sequence of polynomials that converges uniformly to the absolute value function on $[-\alpha, \alpha]$.