1. Time and normal ordered exponentials of free fields

Let the operator A be linear in creation and annihilation operators. Then the normal ordered exponential is related to the time ordered exponential via

$$:e^A: = \frac{\mathrm{T}e^A}{\langle \mathrm{T}e^A\rangle} \equiv \frac{e^A}{\langle e^A\rangle}, \tag{1}$$

where the time ordering symbol T is usually omitted by convention, as indicated. In particular, the expectation values of fields we write are always expectation values of time ordered fields.

(a) Use Wick's theorem to prove

$$\mathrm{T}e^A = :e^A: e^{\frac{1}{2}\langle A^2\rangle}. \tag{2}$$

$$Te^A = T\sum_k \frac{1}{k!}A^k$$

$$= \sum_k \frac{1}{k!} TA^k$$

enumerate number of possible contractions in TA^k

$$TA^k = \sum_{n=0}^{\lfloor k/2\rfloor} :A^{k-2n}: \binom{k}{2n}\langle A^2\rangle^n$$

$$= \sum_{n=0}^{\lfloor k/2\rfloor} \frac{k!}{2^n(2n)!(k-2n)!} :A^{k-2n}: \langle A^2\rangle^n$$

$$Te^A = \sum_{k=0}^{\infty}\sum_{n=0}^{\lfloor k/2\rfloor} \frac{1}{2^n(2n)!(k-2n)!} :A^{k-2n}: \langle A^2\rangle^n$$

re-index $m = k-2n$

$$Te^A = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{(2n)!\,m!\,2^n} :A^m: \langle A^2\rangle^n$$

$$= :e^A: e^{\frac{1}{2}\langle A^2\rangle}$$

(b) Use (2) to prove that

$$\langle e^A\rangle = e^{\frac{1}{2}\langle A^2\rangle}. \tag{3}$$

Note that (2) and (3) imply (1).

$$\langle e^A\rangle :e^A: = e^A \quad --- ①$$

$$= :e^A: e^{\frac{1}{2}\langle A^2\rangle} \quad --- ②$$

Hence $\langle e^A\rangle = e^{\frac{1}{2}\langle A^2\rangle}$

(c) Use (2) to derive the identity

$$\mathrm{T} :e^{A_1}::e^{A_2}: \ldots :e^{A_N}: = :e^{A_1+A_2+\ldots+A_N}: \prod_{i<j}^{N} e^{\langle A_iA_j\rangle}.$$

Hint: Substitue $A = \sum_{i=1}^{N} A_i$ in (2).

By substituting

$$Te^{\sum A_i} = :e^{\sum A_i}: e^{\frac{1}{2}\langle(\sum A_i)^2\rangle}$$

$$= :e^{\sum A_i}: e^{\frac{1}{2}\langle\sum_{i,j} A_iA_j\rangle}$$

$$= :e^{\sum A_i}: e^{\langle\sum_{i\le j} A_iA_j\rangle}$$

$$= :e^{\sum A_i}: \prod_{i\le j} e^{\langle A_iA_j\rangle}$$

At the same time,

$$Te^{\sum A_i} = T :e^{A_1}: \dots :e^{A_n}: \quad \text{(I have no clue how)}$$

2. Bosonization in second quantization

Consider a one-dimensional system of non-interacting fermions with anticommutation relations

$$\{c^\dagger_{\alpha k}, c_{\alpha' k'}\} = \delta_{\alpha\alpha'}\delta_{kk'} \tag{7}$$

and Hamiltonian

$$H_0 = \sum_{\alpha,k} \alpha v_F (k - \alpha k_F) c^\dagger_{\alpha k} c_{\alpha k}, \tag{8}$$

where v_F the Fermi velocity, and $\alpha = \pm$ refers to right (our $\bar{\psi}(\bar{z})$) and left (our $\psi(z)$) movers, respectively. We assume a system of length L and periodic boundary conditions (PBCs), which implies that the momenta are quantized as $k = \frac{2\pi}{L}n$, with n integer. We further assume that the single particle states at $k = \pm k_F$ (as well as all the states below) are occupied in the ground state $|0\rangle$.

We wish to show that the spectrum of neutral excitations (*i.e.*, those which do not alter the number of fermions in the system) can be equally well described via the bosonic density operators

$$\rho_\alpha(q) = \sum_k c^\dagger_{\alpha k+q} c_{\alpha k}, \tag{9}$$

which create (for $\alpha q > 0$) or annihilate (for $\alpha q < 0$) electron-hole pairs.

Since k_F merely shifts the momenta in (8), we may relabel k by $p = k - \alpha k_F$ (or equivalently set $k_F = 0$).

(a) Limiting yourself to right movers, write out all the states (via strings of operators acting on $|0\rangle$) for $k_{tot} = \frac{2\pi}{L}m$ with $m = 1, 2, 3$ and 4, once using fermion creation operators and once using bosonic density operators. Show the numbers of states for each value of m match in both descriptions.

Hint: For simplicity, label the operators via $k = \frac{2\pi}{L}n$ with the integers n, not with k or q.

Right movers: $\alpha = +1$

$$H_0 = \sum_k v_F k\, c_k^\dagger c_k$$

$$\rho(q) = \sum_k c^\dagger_{k+q} c_k$$

Since the number of fermions is constant, for every fermion we create we need to annihilate one fermion. To avoid annihilating the state, the fermion annihilated needs to be with k <= 0, and the state created needs to be with k >0

$k_{tot} = 1$: $c_1^\dagger c_0|0\rangle = \rho(1)|0\rangle$

$k_{tot} = 2$: $c_2^\dagger c_0|0\rangle,\ c_1^\dagger c_{-1}|0\rangle$

Bosonic: $\rho(1)^2|0\rangle,\ \rho(2)|0\rangle$

$k_{tot} = 3$: $c_3^\dagger c_0|0\rangle,\ c_2^\dagger c_{-1}|0\rangle,\ c_1^\dagger c_{-2}|0\rangle$

$\rho(1)^3|0\rangle,\ \rho(1)\rho(2)|0\rangle,\ \rho(3)|0\rangle$

$k_{tot} = 4$: $c_4^\dagger c_0|0\rangle,\ c_3^\dagger c_{-1}|0\rangle,\ c_2^\dagger c_{-2}|0\rangle,\ c_1^\dagger c_{-3}|0\rangle,\ c_2^\dagger c_1^\dagger c_0 c_{-1}|0\rangle$

Bosonic: $\rho(1)^4|0\rangle,\ \rho(2)^2|0\rangle,\ \rho(1)\rho(3)|0\rangle,\ \rho(4)|0\rangle,\ \rho(1)^2\rho(2)|0\rangle$

The commutation relations for the boson operators are given by

$$[\rho_\alpha(q), \rho_{\alpha'}(-q')] = -\frac{\alpha q L}{2\pi}\delta_{\alpha\alpha'}\delta_{qq'}. \tag{10}$$

(b) Verify (10) for $q = q'$ by explicit evaluation of the lhs (left hand side) using the anticommutation relations (7), and convince yourself that (10) also holds for $q \neq q'$.*

Hint: It is necessary to choose a momentum cutoff. Why?

(c) Evaluate the commutator $[H_0, \rho_\alpha(q)]$. Use the result and (10) to write a kinetic Hamiltonian equivalent to H_0 in terms of bosonic operators.

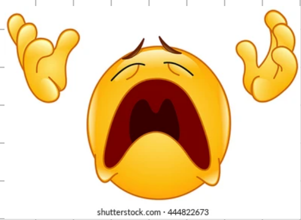
b) $$[\rho_\alpha(q), \rho_{\alpha'}(-q)] = \Big[\sum_k c^\dagger_{\alpha k+q} c_{\alpha k}, \sum_{k'} c^\dagger_{\alpha' k'-q} c_{\alpha' k'}\Big]$$

$$= \sum_{kk'} \Big\{ c^\dagger_{\alpha k+q}[c_{\alpha k}, c^\dagger_{\alpha' k'-q}] c_{\alpha' k'} + c^\dagger_{\alpha k+q} c^\dagger_{\alpha' k'-q} \cancel{[c_{\alpha k}, c_{\alpha' k'}]} \to 0 + \cancel{[c^\dagger_{\alpha k+q}, c^\dagger_{\alpha' k'-q}]} \to 0\ c_{\alpha k} c_{\alpha' k'} + c^\dagger_{\alpha' k'-q}[c^\dagger_{\alpha k+q}, c_{\alpha' k'}] c_{\alpha k} \Big\}$$

$$= \sum_{kk'} \Big\{ c^\dagger_{\alpha' k'-q}\big(\cancel{2c^\dagger_{\alpha k+q} c_{\alpha' k'}} - \delta_{\alpha\alpha'}\delta_{k',k+q}\big) c_{\alpha k} - c^\dagger_{\alpha k+q}\big(\cancel{2c^\dagger_{\alpha' k'-q} c_{\alpha k}} - \delta_{\alpha\alpha'}\delta_{k,k'-q}\big) c_{\alpha' k'} \Big\}$$

$$= \sum_{kk'} \left[c^\dagger_{\alpha' k'-q} c_{\alpha k}\, \delta_{k',k+q} - c^\dagger_{\alpha k+q} c_{\alpha' k'}\, \delta_{k,k'-q} \right] \delta_{\alpha\alpha'}$$

$$= \delta_{\alpha\alpha'} \left[\sum_k c^\dagger_{\alpha' k} c_{\alpha k} - \sum_{k'} c^\dagger_{\alpha k'} c_{\alpha' k'} \right]$$



But anyway we need cutoffs so that we can rewrite delta functions $\delta_{\alpha k+q,\alpha' k'+q'} = \delta_{\alpha\alpha'}\delta_{k+q,k'+q'}$

We demand that q is small so that excitations do not have momentum large enough to turn left movers into right movers.

(c) Evaluate the commutator $[H_0, \rho_\alpha(q)]$. Use the result and (10) to write a kinetic Hamiltonian equivalent to H_0 in terms of bosonic operators.

$$[H_0, \rho_\alpha(q)] = \left[\sum_{\alpha',k'} \alpha' v_F k' c^\dagger_{\alpha' k'} c_{\alpha' k'} , \sum_k c^\dagger_{\alpha k+q} c_{\alpha k} \right]$$

$$= \sum_{kk'} \alpha v_F k' \left[c^\dagger_{\alpha k'} c_{\alpha k'} , c^\dagger_{\alpha k+q} c_{\alpha k} \right]$$

$$= \sum_{kk'} \alpha v_F k' \left(c^\dagger_{\alpha k'} c_{\alpha k'} c^\dagger_{\alpha k+q} c_{\alpha k} - c^\dagger_{\alpha k+q} c_{\alpha k} c^\dagger_{\alpha k'} c_{\alpha k'} \right)$$

$$= \sum_{kk'} \alpha v_F k' \Big[c^\dagger_{\alpha k'} \left(\{ c_{\alpha k'}, c^\dagger_{\alpha k+q} \} - \cancel{c^\dagger_{\alpha k+q} c_{\alpha k'}} \right) c_{\alpha k} - c^\dagger_{\alpha k+q} \left(\{ c_{\alpha k}, c^\dagger_{\alpha k'} \} - \cancel{c^\dagger_{\alpha k'} c_{\alpha k}} \right) c_{\alpha k'} \Big]$$

$$= \sum_{kk'} \alpha v_F k' \left[c^\dagger_{\alpha k'} c_{\alpha k}\, \delta_{k',k+q} - c^\dagger_{\alpha k+q} c_{\alpha k'}\, \delta_{kk'} \right]$$

$$= \sum_{k'} \alpha v_F k' \left[c^\dagger_{\alpha k'} c_{\alpha (k'-q)} - c^\dagger_{\alpha k'+q} c_{\alpha k'} \right]$$