Quantum field theory in the solid state, Exercise sheet 1

Corrections: Monday 5^{th} of May

Ising model in a transverse field

The Ising model in a transverse field reads:

$$\hat{H} = -J \sum_{\langle i,j \rangle} \hat{S}_i^z \hat{S}_j^z - h \sum_i \hat{S}_i^x, \text{ with } J > 0 \text{ and } h \geq 0. \tag{1}$$

Here $\hat{S}^\alpha = \frac{1}{2}\sigma^\alpha$ with $\sigma^x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma^y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, and $\sigma^z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Here, we will consider a d-dimensional hypercubic lattice and $\langle i,j \rangle$ denotes nearest neighbor sites. Note that this Hamiltonian operator acts in the tensor product Hilbertspace $\mathcal{H} = \otimes_{i=1}^N \mathbb{C}^2$ with N corresponding to the number of sites on the hypercubic lattice. The operators $\hat{S}_i^z$ and $\hat{S}_i^x$ act on the i^{th} $\mathbb{C}^2$ Hilbert space in above tensor product. We will also consider periodic boundary conditions.

a) Find an operator $\hat{T}$ that satisfies:

$$\hat{T}^{-1}\hat{S}_i^z\hat{T} = -\hat{S}_i^z, \text{ and } \hat{T}^{-1}\hat{S}_i^x\hat{T} = \hat{S}_i^x \tag{2}$$

and show that

$$\left[\hat{T}, \hat{H}\right] = 0. \tag{3}$$

b) Discuss the ground state of the system at $h = \infty$ (or equivalently $J = 0$) and at $h = 0$. Do these ground states have the same symmetry as the Hamiltonian? Is there a phase transition as a function of h?

c) We would now like to compute the propagator ($\hbar = 1$):

$$K(\boldsymbol{s}, \boldsymbol{s}', t) = \langle \boldsymbol{s} | e^{-it\hat{H}} | \boldsymbol{s}' \rangle \text{ with } |\boldsymbol{s}'\rangle = |s_1'\rangle \otimes |s_2'\rangle \otimes \cdots \otimes |s_N'\rangle \text{ and } \hat{S}_i^z |\boldsymbol{s}'\rangle = s_i' |\boldsymbol{s}'\rangle \tag{4}$$

First show that for an infinitesimal time propagation you can write

$$K(\boldsymbol{s}, \boldsymbol{s}', t) = \langle \boldsymbol{s} | e^{-i\epsilon\hat{H}} | \boldsymbol{s}' \rangle = C e^{+i\epsilon J \sum_{\langle i,j \rangle} s_i s_j + iK \sum_i s_i s_i'} + \mathcal{O}(\epsilon^2) \tag{5}$$

and find the values for K and C. Then compute the propagator for a finite time interval t.

d) Repeat the same calculation as above, but now for imaginary time:

$$\tilde{K}(\boldsymbol{s}, \boldsymbol{s}', \beta) = \langle \boldsymbol{s} | e^{-\beta\hat{H}} | \boldsymbol{s}' \rangle. \tag{6}$$

Using the above path integral formulation write an expression for the partition function:

$$Z = \sum_{\boldsymbol{s}} \tilde{K}(\boldsymbol{s}, \boldsymbol{s}, \beta). \tag{7}$$

The important point that you should realize in this calculation is that the partition function of the d-dimensional Ising model in a transverse magnetic field is equivalent to the partition function of the (anisotropic) Ising model in d+1 dimensions. This has very important consequences for the understanding of the phase transition.

e) Both for real and imaginary time, discuss the limit where the transverse field vanishes (i.e. $h \to 0$ in Eq. 1).