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**Problem Sheet 7**
for the tutorial on June 20th, 2025
**Quantum Mechanics II**
Summer term 2025

Sheet handed out on June 10th, 2025; to be handed in on June 17th, 2025 until 2 pm

## Exercise 7.1: Laboratory and center-of-mass systems [2 + 2 + 1 + 4 + 2 P.]

We consider a non-relativistic collision between a projectile particle A of mass $m_A$ and a target particle of mass $m_B$ like in the lecture. The laboratory system $L$ is the frame in which the target particle B is at rest before the collision. The center-of-mass system $CM$ is the coordinate system in which the center of mass of the composite system (A+B) is always at rest. In that system the projectile A and target particle B move initially with respect to the center of mass C with equal and opposite momenta, $\vec{p}_A = -\vec{p}_B = \vec{p}$, as illustrated in Fig. 1. With respect to the laboratory frame, the center of mass of the two particles moves throughout the collision with a constant velocity $\vec{v}_c$ along the direction of incidence, with $\vec{v}_c = \vec{q}_A/(m_A + m_B)$, where $\vec{q}_A$ is the momentum of particle A before the collision in the laboratory system.

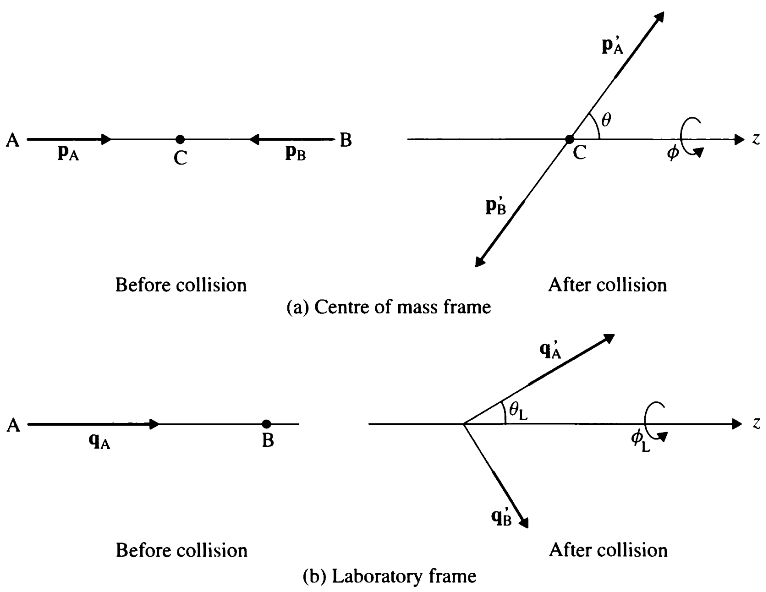



Figure 1: Elastic scattering of a projectile A by target B.

a) Using a Galilean transformation, find the relation between the momenta $\vec{q}_A$ and $\vec{p}_A$ before the collision.

b) After the collision the two particles A and B emerge with equal and opposite momenta $\vec{p'}_A = -\vec{p'_B} = \vec{p'}$ in the $CM$ frame. In the following we consider an elastic collision, such that the magnitude of the momenta of each particle remain the same $p' = p$. Relate the components of the momentum of projectile A along the direction of incidence in the two coordinate systems using the two scattering angles $\theta_L$ and $\theta$ as illustrated in Fig. 1.

c) Show that the relation between the two scattering angles is given by

$$\tan\theta_L = \frac{\sin\theta}{\cos\theta + \frac{m_A}{m_B}} \,. \tag{1}$$

d) Using the equation above, find the relation between the angular differential cross sections $\frac{d\sigma}{d\Omega}$ in the laboratory and the center-of-mass frames.

e) Let us now consider a numerical example:

   Two beams of protons intersect collinearly. If the kinetic energy of the protons is 5 keV in both beams, calculate:

   i) the magnitude of the relative velocity of a proton in one beam with respect to a proton in the other one,

   ii) the energy in the centre-of-mass system.

**Exercise 7.2: Partial waves and phase shifts** [6 + 8 P.]

We consider in the following the scattering by a central potential $V(r)$ such that the system is completely symmetrical about the direction of incidence, which we choose to be the $z$-axis. In this case both the wave function $\psi_{\boldsymbol{k}}$ and the scattering amplitude $f$ do not depend on the azimuthal angle $\varphi$. We then expand them in a series of Legendre polynomials, which form a complete set in the interval $-1 \leq \cos\theta \leq 1$,

$$\psi_{\boldsymbol{k}}(r,\theta) = \sum_{l=0}^{\infty} R_l(k,r) P_l(\cos\theta)\,, \tag{2}$$

$$f(k,\theta) = \sum_{l=0}^{\infty} f_l(k) P_l(\cos\theta)\,. \tag{3}$$

Each term in the series is known as a partial wave and is a simultaneous eigenfunction of the operators $\vec{L}^2$ and $L_z$ belonging to eigenvalues $l(l+1)\hbar^2$ and zero, respectively. The radial wave function for the far region where the potential can be neglected is given by a linear combination of Bessel and Neumann functions $j_l(kr)$ and $n_l(kr)$

$$R_l(k,r) = B_l(k) j_l(kr) + C_l(k) n_l(kr) \tag{4}$$

with coefficients $B(k)$ and $C(k)$. Using the asymptotic expressions for the Bessel and Neumann functions given in the lecture, this leads to

$$R_l(k,r) \overset{r\to\infty}{\approx} \frac{1}{kr}\left[B_l(k)\sin\left(kr - \frac{l\pi}{2}\right) - C_l(k)\cos\left(kr - \frac{l\pi}{2}\right)\right]. \tag{5}$$

a) In the lecture, we have considered a particular choice for the coefficients $B_l$ and $C_l$. Here we consider a more general case. To this end, it is convenient to rewrite the expression above as

$$R_l(k,r) \overset{r\to\infty}{\approx} A_l(k)\frac{1}{kr}\sin\left(kr - \frac{l\pi}{2} + \delta_l(k)\right). \tag{6}$$

Determine the expressions of the amplitudes $A_l(k)$ and the phase shifts $\delta_l(k)$ introduced above as a function of $B_l$ and $C_l$. The phase shifts $\delta_l(k)$ are real quantities and characterize the strength of the scattering in the $l$th partial wave by the potential $V(r)$ at the energy $E = \hbar^2k^2/(2m)$.

b) We would like now to relate the phase shifts $\delta_l(k)$ to the partial wave amplitudes $f_l(k)$ and to the scattering amplitude $f(k,\theta)$ in Eq. (3). Use the radial wave function determined above and the general relation between wave function and scattering amplitude

$$\psi_{\boldsymbol{k}}(r) \overset{r\to\infty}{\approx} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(k,\theta)\frac{e^{ikr}}{r} \tag{7}$$

to determine the partial wave amplitudes $f_l(k)$. Write then the expression of the scattering amplitude $f(k,\theta)$ as a function of the phase shifts.

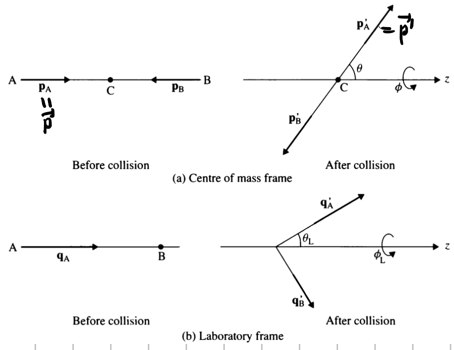
**Hint:** Use the plane wave expansion

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)\,. \tag{8}$$

a) Using a Galilean transformation, find the relation between the momenta $\vec{q}_A$ and $\vec{p}_A$ before the collision.

$$\frac{\vec{p}_A}{m_A} = \frac{\vec{q}_A}{m_A} - \vec{v}_c = \frac{\vec{q}_A}{m_A} - \frac{\vec{q}_A}{m_A+m_B} = \frac{m_B}{m_A}\frac{\vec{q}_A}{m_A+m_B}$$

$$\vec{q}_A = \frac{m_A+m_B}{m_B}\vec{p}_A$$




(a) Centre of mass frame

(b) Laboratory frame

b) After the collision the two particles A and B emerge with equal and opposite momenta $\vec{p'}_A = -\vec{p'_B} = \vec{p'}$ in the $CM$ frame. In the following we consider an elastic collision, such that the magnitude of the momenta of each particle remain the same $p' = p$. Relate the components of the momentum of projectile A along the direction of incidence in the two coordinate systems using the two scattering angles $\theta_L$ and $\theta$ as illustrated in Fig. 1.

$$\frac{q'_A\cos\theta_L}{m_A} = \frac{p'\cos\theta}{m_A} - v_c$$

$$= \frac{p\cos\theta}{m_A} - v_c$$

c) Show that the relation between the two scattering angles is given by

$$\tan\theta_L = \frac{\sin\theta}{\cos\theta + \frac{m_A}{m_B}}.$$

no ' needed because elastic

$$\tan\theta_L = \frac{q'_{A,y}}{q'_{A,x}} = \frac{p_y}{p_x + m_A v_c} = \frac{p\sin\theta}{p\cos\theta + \frac{m_A}{m_A+m_B}q_A}$$

$$= \frac{p\sin\theta}{p\cos\theta + \frac{m_A}{m_B}p} \quad \text{(see (a))}$$

$$= \frac{\sin\theta}{\cos\theta + \frac{m_A}{m_B}}$$

d) Using the equation above, find the relation between the angular differential cross sections $\frac{d\sigma}{d\Omega}$ in the laboratory and the center-of-mass frames.

$$d\Omega = \sin\theta\, d\theta\, d\phi = -d(\cos\theta)\, d\phi$$

$$1 + \tan^2\theta_L = \sec^2\theta_L = \frac{\sin^2\theta}{\cos^2\theta + \left(\frac{m_A}{m_B}\right)^2 + 2\frac{m_A}{m_B}\cos\theta} + 1$$

$$= \frac{1 + \left(\frac{m_A}{m_B}\right)^2 + 2\frac{m_A}{m_B}\cos\theta}{\cos^2\theta + \left(\frac{m_A}{m_B}\right)^2 + 2\frac{m_A}{m_B}\cos\theta}$$

$$\cos^2\theta_L = \frac{\left(\frac{m_A}{m_B}\right)^2 + \cos^2\theta + 2\frac{m_A}{m_B}\cos\theta}{1 + \left(\frac{m_A}{m_B}\right)^2 + 2\frac{m_A}{m_B}\cos\theta}$$

$$\cos\theta_L = \frac{\cos\theta + \frac{m_A}{m_B}}{\left[1 + \left(\frac{m_A}{m_B}\right)^2 + 2\left(\frac{m_A}{m_B}\right)\cos\theta\right]^{1/2}}$$

$$d(\cos\theta_L) = \frac{1 + \left(\frac{m_A}{m_B}\right)\cos\theta}{\left(1 + 2\left(\frac{m_A}{m_B}\right)\cos\theta + \left(\frac{m_A}{m_B}\right)^2\right)^{3/2}} (d\cos\theta)$$

Since nothing happens to $\phi$,

$$\frac{d\sigma}{d\Omega_L} = \frac{\left(1 + 2\left(\frac{m_A}{m_B}\right)\cos\theta + \left(\frac{m_A}{m_B}\right)^2\right)^{3/2}}{1 + \left(\frac{m_A}{m_B}\right)\cos\theta} \frac{d\sigma}{d\Omega}$$

e) Let us now consider a numerical example:

Two beams of protons intersect collinearly. If the kinetic energy of the protons is 5 keV in both beams, calculate:

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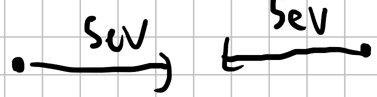


$$p = \sqrt{2mE}$$

$$v = \sqrt{\frac{2E}{m}}$$

$$= 9.79 \times 10^5 \text{ ms}^{-1}$$

$$v_{rel} = 2v = 1.96 \times 10^6 \text{ ms}^{-1}$$

$$E = 2(5\text{ keV}) = 10\text{ keV}$$

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a) In the lecture, we have considered a particular choice for the coefficients $B_l$ and $C_l$. Here we consider a more general case. To this end, it is convenient to rewrite the expression above as

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Determine the expressions of the amplitudes $A_l(k)$ and the phase shifts $\delta_l(k)$ introduced above as a function of $B_l$ and $C_l$. The phase shifts $\delta_l(k)$ are real quantities and characterize the strength of the scattering in the $l$th partial wave by the potential $V(r)$ at the energy $E = \hbar^2k^2/(2m)$.

$$R_\ell(k,r) \xrightarrow{r\to\infty} \frac{1}{kr}\left[B_\ell(k)\sin\left(kr - \frac{\ell\pi}{2}\right) - C_\ell(k)\cos\left(kr - \frac{\ell\pi}{2}\right)\right]$$

R formula: $a\sin\theta + b\cos\theta = R\sin(\theta+\alpha)$

$$R = \sqrt{a^2+b^2}\,,\quad \alpha = \tan^{-1}\left(\frac{b}{a}\right)$$

Proof: $R\sin(\theta+\alpha) = R(\sin\theta\cos\alpha + \sin\alpha\cos\theta)$

$$= a\sin\theta + b\cos\theta$$

Comparing coefficients, $a = R\cos\alpha$, $b = R\sin\alpha$

Inverting, $R = \sqrt{a^2+b^2}$, $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

$$\frac{1}{kr}\left[B_\ell(k)\sin\left(kr - \frac{\ell\pi}{2}\right) - C_\ell(k)\cos\left(kr - \frac{\ell\pi}{2}\right)\right] = \frac{1}{kr}A_\ell(k)\sin\left(kr - \frac{\ell\pi}{2} + \delta(k)\right)$$

$$A_\ell(k) = \sqrt{B_\ell(k)^2 + C_\ell(k)^2}$$

$$\delta(k) = -\tan^{-1}\left(\frac{C_\ell(k)}{B_\ell(k)}\right)$$

b) We would like now to relate the phase shifts $\delta_l(k)$ to the partial wave amplitudes $f_l(k)$ and to the scattering amplitude $f(k,\theta)$ in Eq. (3). Use the radial wave function determined above and the general relation between wave function and scattering amplitude

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$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr) P_l(\cos\theta)\,. \tag{8}$$

$$\psi_K(r,\theta) = \sum_{\ell=0}^{\infty} R_\ell(k,r) P_\ell(\cos\theta)$$

$$\xrightarrow{r\to\infty} e^{i\vec{k}\cdot\vec{r}} + f(k,\theta)\frac{e^{ikr}}{r}$$

$$= \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}j_{\ell}(kr)P_{\ell}(\cos\theta) + \frac{e^{ikr}}{r}\sum_{\ell=0}^{\infty}f_{\ell}(k)P_{\ell}(\cos\theta)$$

$$\sum_{\ell=0}^{\infty}\left[(2\ell+1)i^{\ell}j_{\ell}(kr) + \frac{e^{ikr}}{r}f_{\ell}(k)\right]P_{\ell}(\cos\theta) \overset{r\to\infty}{=} \sum_{\ell=0}^{\infty} R_{\ell}(k,r)P_{\ell}(\cos\theta)$$

$$(2\ell+1)i^{\ell}j_{\ell}(kr) + \frac{e^{ikr}}{r}f_{\ell}(k) \overset{r\to\infty}{=} R_{\ell}(k,r)$$

$$\underbrace{(2\ell+1)i^{\ell}j_{\ell}(kr)}_{} + \frac{e^{ikr}}{r}f_{\ell}(k) = \frac{A_{\ell}(k)}{kr}\sin\left(kr-\frac{\ell\pi}{2}+\delta_{\ell}(k)\right)$$

$$\to (2\ell+1)i^{\ell}\frac{1}{kr}\sin\left(kr-\frac{\ell\pi}{2}\right)$$

$$f_{\ell}(k) = e^{-ikr}\left[\frac{A_{\ell}(k)}{k}\sin\left(kr-\frac{\ell\pi}{2}+\delta_{\ell}(k)\right) - \frac{(2\ell+1)i^{\ell}}{k}\sin\left(kr-\frac{\ell\pi}{2}\right)\right]$$

$$f(k,\theta) = \sum_{\ell=0}^{\infty}e^{-ikr}\left[\frac{A_{\ell}(k)}{k}\sin\left(kr-\frac{\ell\pi}{2}+\delta_{\ell}(k)\right) - \frac{(2\ell+1)i^{\ell}}{k}\sin\left(kr-\frac{\ell\pi}{2}\right)\right]P_{\ell}(\cos\theta)$$