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9. Problemset "Quantum Algebra & Dynamics"

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Coherent States / Symmetries

9.1 Coherent States

Consider the family $\mathcal{U}_c$ of operators

$$U_c(z) = e^{za^* - \bar{z}a} , \tag{1}$$

where $z \in \mathbf{C}$ and

$$a^* = \frac{1}{\sqrt{2}}(x - ip) \tag{2a}$$

$$a = \frac{1}{\sqrt{2}}(x + ip) \tag{2b}$$

are the creation and annihilation operators for a harmonic oscillator with Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}x^2 = a^*a + \frac{1}{2}$.

1. Show that $U_c(z)$ creates coherent states, i.e.

$$a|z\rangle = z|z\rangle \tag{3}$$

   for $|z\rangle = U_c(z)|0\rangle$ with $a|0\rangle = 0$.

2. Show that
$$U_c(z_1)U_c(z_2) = f(z_1, z_2)U_c(g(z_1, z_2)) \tag{4}$$
   and compute $f, g : \mathbf{C}^2 \to \mathbf{C}$. Are f or g analytic?

3. Use the preceding result to find a homomorphism $\pi_c : \mathcal{A}_W \to \mathcal{U}_c$, i.e. a representation of the Weyl-algebra $\mathcal{A}_W$ on the standard Hilbert space of the harmonic oscillator.

9.2   Particle on $\mathbf{R}^3$

Consider the Weyl-Algebra $\mathcal{A}_W$ for a particle on $\mathbf{R}^3$, generated by $W(\vec{\xi}, \vec{\eta})$ with $\vec{\xi}, \vec{\eta} \in \mathbf{R}^3$ and

$$W(\vec{\xi_1}, \vec{\eta_2})W(\vec{\xi_2}, \vec{\eta_2}) = e^{\frac{i}{2}(\vec{\xi_1}\vec{\eta_2} - \vec{\xi_2}\vec{\eta_1})}W(\vec{\xi_1} + \vec{\xi_2}, \vec{\eta_1} + \vec{\eta_2}) . \tag{5}$$

1. Write down the Schrödinger representation of $\mathcal{A}_W$.

2. In classical physics we have the group of continuous symmetry transformations generated by translations $\vec{x} \mapsto \vec{x} - \vec{\alpha}$, boosts $\vec{p} \mapsto \vec{p} + \vec{\beta}$ and rotations $(\vec{x}, \vec{p}) \mapsto (R\vec{x}, R\vec{p})$. Construct the corresponding group of automorphisms of $\mathcal{A}_W$ and determine their composition laws.

3. Consider the family $\mathcal{U}_c$ of creation operators for coherent states

$$U_c(\vec{z}) = e^{\vec{z}\vec{a}^* - \vec{\bar{z}}\vec{a}} , \tag{6}$$

where $z \in \mathbf{C}^3$ and

$$\vec{a}^* = \frac{1}{\sqrt{2}}(\vec{x} - i\vec{p}) \tag{7a}$$

$$\vec{a} = \frac{1}{\sqrt{2}}(\vec{x} + i\vec{p}) \tag{7b}$$

are the creation and annihilation operators for a three dimensional harmonic oscillator and use them to realize $\mathcal{A}_W$. Construct unitary operators U that realize translations, boosts and rotations via

$$\forall A \in \mathcal{A}_W : A \mapsto U^* A U . \tag{8}$$