Universität Würzburg
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Prof. Dr. Björn Trauzettel
Aisel Shiralieva

3. Open quantum systems

Due date: 04.06.2025 10:00

Throughout this exercise sheet, we adopt the convention $\boxed{\hbar = 1}$.

Exercise 1 *Properties of the Lindblad equation* **6 P.**

Let $\rho(t)$ be the density matrix of a quantum system evolving according to the Lindblad master equation:

$$\frac{d\rho}{dt} = -i[H, \rho] + \sum_\alpha \gamma_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\} \right),$$

where H is the system Hamiltonian, and L_α are the Lindblad operators describing the interaction with the environment.

- Show that the trace of $\rho(t)$ is preserved under this evolution, i.e.,

$$\frac{d}{dt}\,\mathrm{Tr}[\rho(t)] = 0.$$

- Prove that the purity of the state, defined as $\mathrm{Tr}[\rho^2(t)]$, is a non-increasing function of time. Specifically, show that

$$\frac{d}{dt}\,\mathrm{Tr}[\rho^2(t)] \leq 0.$$

  In this case, assume for simplicity that all the jump operators L_α are *Hermitian.*

  Hint: Since the density matrix ρ is Hermitian, it can be diagonalized as

$$\tilde{\rho} = \sum_i \Lambda_i \left|\Lambda_i\right\rangle \left\langle\Lambda_i\right|,$$

  with real eigenvalues Λ_i and corresponding eigenvectors $|\Lambda_i\rangle$. The ordering $\Lambda_0 \geq \Lambda_1 \geq \cdots \geq \Lambda_d$ can be assumed. This property holds at any time since the inequality must be satisfied for all t. The jump operators in this basis are denoted $\tilde{L}_\alpha$.

Exercise 2 *Lindblad equation for bit-flip noise* **5 P.**

Consider a single-qubit system with Hamiltonian

$$H = -\frac{1}{2}\omega\,\sigma_z,$$

and a single Lindblad (jump) operator

$$L_1 = \sigma_x,$$

which corresponds to bit-flip noise. Let the decay rate be $\gamma_1 = \gamma$, and assume all other rates vanish: $\gamma_{\alpha \geq 2} = 0$.

- Write the Lindblad master equation for this system.
- Express the density matrix in Bloch form:

$$\rho = \frac{1}{2}\left(\mathbb{1} + \vec{v} \cdot \vec{\sigma}\right),$$

  where $\vec{v}(t) = (v_x(t), v_y(t), v_z(t))$ is the Bloch vector. Using this form, derive the differential equations that govern the time evolution of $v_x(t)$, $v_y(t)$, and $v_z(t)$.
- Solve these differential equations to obtain the explicit time dependence of the Bloch vector components.
- How does the Bloch vector behave as $t \to \infty$? Interpret your results.