Algebra and Dynamics of Quantum Systems

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David Hilbert	John von Neumann

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Abstract

- General mathematical description of a physical system: C^*-algebras, spectra and states
- Mathematical description of quantum systems: states and representations and the GNS construction.
- Single particle quantum mechanics: Weyl algebra, Stone-von Neumann theorem, Schrödinger wave functions, minimum uncertainty states.
- Quantum dynamics: Schrödinger equation, Hamiltonian, self-adjointness, examples.
- Mathematical description of infinitely extended systems: Haag's theorem.
- Symmetries in quantum mechanics: Wigner's theorem
- Symmetry breaking: Goldstone's theorem, Higgs mechanism, superconductivity.

Contents

Vorbemerkung

Dieses Manuskript ist mein persönliches Vorlesungsmanuskript, an vielen Stellen nicht ausformuliert und kann jede Menge Fehler enthalten. Es handelt sich hoffentlich um weniger Denk- als Tippfehler, trotzdem kann ich deshalb ich keine Verantwortung für Fehler übernehmen. Zeittranslationsinvarianz ist natürlich auch nicht gegeben ...

Dennoch, oder gerade deshalb, bin ich für alle Korrekturen und Vorschläge dankbar!

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Aktuelle Informationen

WueCampus: `https://wuecampus2.uni-wuerzburg.de/moodle/course/view.php?id=54687`.

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Introduction

1.1 Literature

Lecture 01: Tue, 14. 10. 2025

1.1.1 Close to the lecture

- Franco Strocchi, *An Introduction to the Mathematical Structure of Quantum Mechanics*, World Scientific, 2nd ed., 2008.
- Franco Strocchi, *Symmetry Breaking*, Springer, 2nd ed., 2007. (used to be available as part of the University's subscription; not anymore, since 3rd ed. was published)

1.1.2 Quantum Field Theory

- Rudolf Haag, *Local Quantum Physics: Fields, Particles, Algebras*, Springer, 2nd ed., 1996.

1.1.3 C^*-Algebras

- Ola Bratteli, Derek Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vols. I, II*, Springer, 2nd ed., 2002 (these used to be available free of charge as PDFs on Ola Bratteli's homepage, but the page has dissappeared after his death).
- Jacques Dixmier, *C^*-Algebras*, North-Holland Mathematical Library, Vol. 15, 1977.
- Masamichi Takesaki, *Theory of Operator Algebra, Vol. 1*, Springer, 2001

1.1.4 Functional Analysis

- Michael Reed, Barry Simon, *Methods of modern mathematical physics, Volume 1: Functional analysis*, Academic Press, 2nd ed., 1981.

1.1.5 Historical

- John von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press 1932.

1.1.6 Mathematical Subtleties of Quantum Mechanics

- Francois Gieres, *Dirac's formalism and mathematical surprises in quantum mechanics* [Gie00]. Nice collection and discussion of mathematical pitfalls in the naive formulation of quantum mechanics.
- Gerald Teschl, *Mathematical Methods in Quantum Mechanics* [Tes14]. Textbook filling the mathematical gaps in the typical physicist's treatment of one particle quantum mechanics. Free copy on the author's web site: `https://www.mat.univie.ac.at/~gerald/ftp/book-schroe/schroe2.pdf`
- Peter Carruthers and Michael Martin Nieto, *Phase and angle variables in quantum mechanics* [CN68].

1.2 Naive Formulation of Quantum Mechanics

Schrödinger wave functions are usually the first formulation introduced in Quantum Mechanics (QM) lectures. Its ingredients are

- The state of a physical system is described by a wave function, which is assumed to be square integrable and normalized. For a single particle it is a function of the space points and time

  $$\begin{aligned} \psi : \mathbf{R}^3 \times \mathbf{R} &\to \mathbf{C} \\ (\vec{x}, t) &\mapsto \psi(\vec{x}, t) \end{aligned} \tag{1.1}$$

  and the probability of finding the particle in a domain $D \in \mathbf{R}^3$ is given by

  $$p_D(t) = \int_D \mathrm{d}^3x \, |\psi(\vec{x}, t)|^2 \,. \tag{1.2}$$

  where ψ has been normalized $\int_{\mathbf{R}^3} \mathrm{d}^3x \, |\psi(\vec{x}, t)|^2 = 1$. In other words, $|\psi(\vec{x}, t)|^2$ is interpreted as a probability density.

- Superposition principle: if $\psi_{1,2}$ are states, $\psi = c_1\psi_1 + c_2\psi_2$ with $|c_1|^2 + |c_2|^2 = 1$ is also a state.

- Other observables are constructed by a correspondence principle from a classical dynamical system

$$\text{position:} \quad \vec{x}(t) \Leftrightarrow \hat{\vec{x}} \tag{1.3}$$
$$\text{momentum:} \quad \vec{p}(t) \Leftrightarrow \hat{\vec{p}} \tag{1.4}$$

  where

$$(\hat{\vec{x}}\psi)(x,t) = \vec{x}\psi(x,t) \tag{1.5}$$
$$(\hat{\vec{p}}\psi)(x,t) = -\mathrm{i}\hbar\vec{\nabla}\psi(x,t)\,. \tag{1.6}$$

  Note:

  - The choice of $\hat{\vec{x}}$ is intuitively clear, because it yields the correct expectation values.
  - The choice of $\hat{\vec{p}}$ can be motivated by each of
    * the de Broglie relation $\vec{p} = \hbar\vec{k}$ for plain waves
    * the resulting Canonical Communation Relations (CCR) $[\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}$ reproduce the Heisenberg uncertainty relation $\Delta x \Delta p \geqslant \hbar/2$
    * as the generator of translations (cf. [Wei13, Ohl18])

    or any combination of these arguments.

- the dynamics of the system is described by the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi(\vec{x},t) = H\psi(\vec{x},t) \tag{1.7}$$

  with a Hamiltonian H that is motivated by the correspondence principle

$$H = \frac{1}{2m}\vec{p}^{\,2} + V(\vec{x}) = -\frac{1}{2m}\Delta + V(\vec{x})\,. \tag{1.8}$$

- physical interpretation:

  - possible results of measuring an observable $O = O^*$: the spectrum of the corresponding operator

– expectation value for multiple measurements of an observable $O = O^*$:

$$\langle O \rangle = \int \mathrm{d}^3x\, \bar{\psi}(\vec{x},t) O \psi(\vec{x},t) \tag{1.9}$$

and the probability of finding the particle in a domain D can be interpreted as the expectation value for the characteristic function χ_D of D

$$p(D) = \int_D \mathrm{d}^3x\, |\psi(\vec{x},t)|^2 = \int \mathrm{d}^3x\, \bar{\psi}(\vec{x},t)\chi_D(\vec{x})\psi(\vec{x},t) = \langle \chi_D \rangle\,, \tag{1.10}$$

where ψ has been normalized $\int_{\mathbf{R}^3} \mathrm{d}^3x\, |\psi(\vec{x},t)|^2 = 1$.

1.3 Problems

The spectral theorem is required to make the interpretation consistent and to define functions of operators like

$$f(A) = \int \mathrm{d}a\, f(a)\, |a\rangle\langle a|\,, \tag{1.11}$$

but

1. The CCR $[x,p] = \mathrm{i}$ can *not* be realized with bounded operators *(→ exercise)*.

2. *Hellinger-Toeplitz theorem*: an everywhere defined symmetric operator, i. e.

$$\forall \psi, \phi \in \mathcal{H} : (\psi, A\phi) = (A\psi, \phi) \tag{1.12}$$

on a Hilbert $\mathcal{H}$ space is bounded (see, e. g. [RS80], p. 84, corollary to the closed graph theorem III.12).

Therefore, we have to be careful about the domain of our operators:

1. there are square integrable $\psi : \mathbf{R} \to \mathbf{C}$ for which $(x\psi) : \mathbf{R} \to \mathbf{C}$ is not square integrable

2. the operators x, p and $[x,p]$ all have different domains.

In fact, the development of a mathematically rigorous spectral theory for unbounded operators was, to a large extent, driven by these technical difficulties of quantum mechanics.

Instead of retracing these historical steps starting from Schrödinger's and Heisenberg's heuristics, we shall go back and ask ourselves what is the *most general* framework for a mathematical description of physical systems.

1.4 Mathematical Description of Physical Systems

We will start with the intuitive example of classical mechanics and later relax some assumptions to obtain a framework that encompasses quantum mechanics.

1.4.1 Classical Mechanics

Canonical Phase Space

kinematics: the kinematical structure is determined by the *state* of the system and the *observables* that can be measured:

state: in an idealized classical mechanical system with n degrees of freedom (d.o.f.), the state is given by a point x in phase space Γ, i. e. in a chart

$$x = (q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n) \in \Gamma \tag{1.13}$$

where Γ is often a cotangent bundle T^*Q over the configuration space $Q \ni q$.

observables: we can measure all polynomials of the phase space variables q and p and this should extend to the closure of the polynomials in the $\|\cdot\| = \sup_\Gamma |\cdot|$-norm, i. e. the continuous functions on phase space: $C(\Gamma)$. Sometimes it is convenient to allow complex coefficients for the polynomials and complex valued continuous functions.

dynamics: the time evolution of the state $x = (q, p)$ of a classical mechanical system is described by a flow Φ on phase space

$$\begin{aligned} \Phi : \mathbf{R} \times \Gamma &\to \Gamma \\ (t, q, p) &\mapsto \Phi_t(q, p) = (q(t), p(t)) \end{aligned} \tag{1.14}$$

(see section 5.4.2 on page 156 of [Ohl16] for further discussions). This induces the time evolution of the observables by

$$\begin{aligned} \Phi^* : \mathbf{R} \times C(\Gamma) &\to C(\Gamma) \\ (t, f) &\mapsto \Phi_t^*(f) = f \circ \Phi_t \end{aligned} \tag{1.15}$$

i. e.

$$\Phi_t^*(f)(q, p) = f(q(t), p(t)) \,. \tag{1.16}$$

This flow is a solution of the canonical equations of motion

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \tag{1.17a}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i} \tag{1.17b}$$

with a suitable differentiable Hamiltonian function $H : \Gamma \to \mathbf{R}$. Using the antisymmetric Poisson brackets

$$\{q_i, p_j\} = \delta_{ij} \tag{1.18a}$$

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \tag{1.18b}$$

we may also write more uniformly

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i = \{q_i, H\} \tag{1.19a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} p_i = \{p_i, H\} \tag{1.19b}$$

and find with the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi_t^*(f) = \{\Phi_t^*(f), H\} \tag{1.20}$$

for all differentiable observables $f : \Gamma \to \mathbf{R}$.

Algebras of Observables

Lecture 02: Wed, 15. 10. 2025

The space $C(\Gamma)$ of all observables carries a natural *abelian $*$-algebra* structure

$$\forall \alpha, \beta \in \mathbf{C} : (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) \tag{1.21a}$$

$$(fg)(x) = f(x)g(x) = (gf)(x) \tag{1.21b}$$

$$(f^*)(x) = \overline{f(x)} \tag{1.21c}$$

with $f : x \mapsto 1$ as identity. In general we have

Definition 1.1 (algebra)**.** An *algebra* $\mathcal{A}$ over a field $\mathbf{K}$ is a vector space over $\mathbf{K}$ together with a bilinear internal binary operation

$$\begin{aligned} \cdot : \mathcal{A} \times \mathcal{A} &\to \mathcal{A} \\ (x, y) &\mapsto xy\,, \end{aligned} \tag{1.22}$$

i. e. $\forall x, y, z \in \mathcal{A}, \alpha, \beta \in \mathbf{K}$

$$z(\alpha x + \beta y) = \alpha(zx) + \beta(zy) \tag{1.23a}$$
$$(\alpha x + \beta y)z = \alpha(xz) + \beta(yz) \tag{1.23b}$$
$$(\alpha x)(\beta y) = \underbrace{(\alpha\beta)}_{\in\mathbf{K}}(xy)\,. \tag{1.23c}$$

The algebra $\mathcal{A}$ is called

- *associative*, if and only if (iff) $\forall x, y, z \in \mathcal{A} : (xy)z = x(yz)$,
- *commutative* or *abelian*, iff $\forall x, y \in \mathcal{A} : xy = yx$ and
- *unital* or with identity, iff $\exists_1 e \in \mathcal{A} : \forall x \in \mathcal{A} : ex = xe = x$.

Note that some authors reserve the term algebra for associative algebras.

In the following, we will only use algebras over $\mathbf{C}$. Most of the algebras will be associative and many unital. The more interesting cases will not be commutative, however.

Definition 1.2 (∗-algebra / involutive algebra)**.** A *∗-algebra* or *involutive algebra* $(\mathcal{A}, *)$ is an associative algebra $\mathcal{A}$ over $\mathbf{C}$ with an *antiautomorphism* or *involution*

$$\begin{aligned} * : \mathcal{A} &\to \mathcal{A} \\ x &\mapsto x^* \end{aligned} \tag{1.24}$$

i. e. $\forall x, y \in \mathcal{A}, \alpha \in \mathbf{C}$

$$(x^*)^* = x \tag{1.25a}$$
$$(x+y)^* = x^* + y^* \tag{1.25b}$$
$$(\alpha x)^* = \bar{\alpha} x^* \tag{1.25c}$$
$$(xy)^* = y^* x^* \tag{1.25d}$$

where $\bar{\alpha}$ is the usual complex conjugate of α in $\mathbf{C}$.

Note that it is in principle possible to use associative algebras over fields other than $\mathbf{C}$, but this is outside of our scope.

Definition 1.3 (normal, self-adjoint, positive, unitary)**.** An element x of a ∗-algebra is called

- *normal*, iff $x^*x = xx^*$,
- *self-adjoint*, iff $x^* = x$,

- *positive* (or $x \geqslant 0$), iff $\exists y \in \mathcal{A} : x = y^* y$,
- an *isometry*, iff $x^* x = e$,
- *unitary*, iff $x^* x = e = x x^*$.

The last two only apply to unital algebras, of course.

Lemma 1.4. *A positive element x of a $*$-algebra is self-adjoint. Furthermore, self-adjoint and unitary elements are normal.*

Proof. The first statement follows from $x^* = (y^* y)^* = y^*(y^*)^* = y^* y = x$ and the second and third statements are trivial. □

In $C(\Gamma)$, there is a natural norm

$$\|f\| = \sup_{x \in \Gamma} |f(x)| \tag{1.26}$$

and $C(\Gamma)$ is by construction complete w. r. t. to this norm, i. e. a Banach space. Since

$$\forall f, g \in C(\Gamma) : \|fg\| \leqslant \|f\| \, \|g\| \tag{1.27}$$

the product is continuous w. r. t. to the norm topology in both factors and thus we have a *Banach $*$-algebra*.

Definition 1.5 (normed algebra)**.** A *normed algebra* $(\mathcal{A}, \|\cdot\|)$ is an associative algebra $\mathcal{A}$ together with a norm $\|\cdot\|$ that satisfies

$$\forall x, y \in \mathcal{A} : \|xy\| \leqslant \|x\| \|y\|. \tag{1.28}$$

Therefore the multiplication is continuous w. r. t. to the norm topology in both factors.

Definition 1.6 (Banach algebra)**.** A *Banach algebra* is a normed algebra $(\mathcal{A}, \|\cdot\|)$ that is complete w. r. t. to the norm $\|\cdot\|$.

Definition 1.7 (normed $*$-algebra)**.** A *normed $*$-algebra* $(\mathcal{A}, *, \|\cdot\|)$ is a $*$-algebra $(\mathcal{A}, *)$ together with a norm $\|\cdot\|$ that satisfies

$$\forall x \in \mathcal{A} : \|x^*\| = \|x\| \,. \tag{1.29}$$

Definition 1.8 (Banach $*$-algebra)**.** A *Banach $*$-algebra* is a normed $*$-algebra $(\mathcal{A}, *, \|\cdot\|)$ that is complete w. r. t. to the norm $\|\cdot\|$.

Finally we find that

$$\forall f \in C(\Gamma) : \|f^* f\| = \|f^*\| \, \|f\| \tag{1.30}$$

and $C(\Gamma)$ turns out to be a unital C^*-algebra:

Definition 1.9 (C^*-algebra)**.** A C^**-algebra* is a Banach $*$-algebra $(\mathcal{A}, *, \|\cdot\|)$ that satisfies the C^**-condition*

$$\forall x \in \mathcal{A} : \|x^* x\| = \|x\|^2 \,. \tag{1.31}$$

Remark 1.10. The C^*-condition (1.31) implies (1.29), i. e. $\forall x \in \mathcal{A} : \|x^*\| = \|x\|$.

Proof. On one hand

$$\|x\|^2 \stackrel{(1.31)}{=} \|x^* x\| \leqslant \|x^*\| \|x\| \tag{1.32}$$

i. e.

$$\|x\| \leqslant \|x^*\| \tag{1.33}$$

and on the other

$$\|x^*\|^2 \stackrel{(1.31)}{=} \|(x^*)^* x^*\| = \|x x^*\| \leqslant \|x\| \|x^*\| \tag{1.34}$$

i. e.

$$\|x^*\| \leqslant \|x\| \,. \tag{1.35}$$

□

Corollary 1.11. *The C^*-condition (1.31) is equivalent to $\|x^* x\| = \|x\| \|x^*\|$.*

We will later see that the C^*-condition (1.31) is very strong and has important consequences. Therefore we should look at the example motivating the definition.

Theorem 1.12 (C^*-algebra of bounded operators)**.** *The set $L(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$ is a C^*-algebra*

Proof. $L(\mathcal{H})$ is naturally a linear space and we can define a norm via

$$\|A\| = \sup_{\psi \in \mathcal{H}, \|\psi\| = 1} \{\|A\psi\|\} \,. \tag{1.36}$$

For every $A \in L(\mathcal{H})$, we can define the adjoint operator A^*, thus turning $L(\mathcal{H})$ into a Banach $*$-algebra. Finally, using the Cauchy-Schwarz Inequality (CSI)

$$|(\psi, \phi)| \leqslant \|\psi\| \, \|\phi\| \,, \tag{1.37}$$

we can show that the C^*-condition (1.31) holds for $L(\mathcal{H})$. First

$$\|A\|^2 = \sup_{\psi\in\mathcal{H},\|\psi\|=1} \{(A\psi, A\psi)\} = \sup_{\psi\in\mathcal{H},\|\psi\|=1} \{(\psi, A^*A\psi)\}$$
$$\overset{\text{CSI}}{\leqslant} \sup_{\psi\in\mathcal{H},\|\psi\|=1} \{\|A^*A\psi\|\} = \|A^*A\| \leqslant \|A^*\|\,\|A\|\,. \tag{1.38}$$

i. e.

$$\|A\| \leqslant \|A^*\|\,. \tag{1.39}$$

Repeating (1.38) with A replaced by A^* yields

$$\|A^*\| \leqslant \|A\| \tag{1.40}$$

and therefore

$$\|A^*\| = \|A\|\,. \tag{1.41}$$

Using this in (1.38), wo obtain the C^*-condition

$$\|A\|^2 \leqslant \|A^*A\| \leqslant \|A\|^2\,. \tag{1.42}$$

□

In fact, the converse is also true

Theorem 1.13. *Every C^*-algebra is isomorphic to a norm-closed self-adjoint algebra*[1] *of bounded operators on a suitable Hilbert space.*

The proof of this theorem will not be given at this point (see section 2.3.4 of [BR02]).

A very important special case is the following characterization of *abelian* C^*-algebras. It shows that our example $C(\Gamma)$ is already the general case:

Theorem 1.14. *Every abelian C^*-algebra is isomorphic to the algebra $C_0(X)$ of continuous functions on a locally compact Hausdorff space X, vanishing at infinity.*

This theorem has the profound consequence that geometry and commutative algebra are two sides of the same coin. The maximal ideals of an algebra of functions on a space X are the functions vanishing at a point $x \in X$. Therefore the points of a space can be identified with the maximal ideals of an abelian C^*-algebra. Indeed, some aspects of the study of non-abelian C^*-algebras are called "noncommutative geometry". Again, the proof of this theorem will not be given at this point (see section 2.3.5 of [BR02]).

[1]Note that "self-adjoint algebra" does *not* mean a algebra of self-adjoint operators, but a algebra which includes the adjoint of every element.

States as Linear Functionals

Lecture 03: Tue, 21. 10. 2025

In the real world, it is *impossible* to determine the state of a system as a single point x in phase space. One rather makes measurements with finite errors and repeated measurements may or may not be correlated, depending on the experimental situation.

Given a set

$$\{\mu_n(f,\omega)\}_{n=1,\ldots,N} \tag{1.43}$$

of N measurements of an observable f of a system in the state ω, we can define an estimator

$$\langle f\rangle_{\omega,N} = \frac{1}{N}\sum_{n=1}^{N}\mu_n(f,\omega) \tag{1.44}$$

for the *expectation*

$$\omega(f) = \lim_{N\to\infty}\langle f\rangle_{\omega,N} \tag{1.45}$$

of f in the state ω. We can also determine higher moments, e. g.

$$(\Delta_\omega f)^2 = \omega\left((f-\omega(f))^2\right)\,, \tag{1.46}$$

to measure experimental uncertainties, but we can *in principle* never do better.

From the definition of the expectation $\omega(f)$, we immediately see that they are linear

$$\forall\alpha,\beta\in\mathbf{C}, f,g\in\mathcal{A}: \omega(\alpha f+\beta g) = \alpha\omega(f)+\beta\omega(g) \tag{1.47a}$$

and (ignoring systematic errors) positive

$$\forall f\in\mathcal{A}: \omega(f^*f)\geqslant 0\,. \tag{1.47b}$$

Using linearity and positivity

$$\begin{aligned}\forall\alpha,\beta\in\mathbf{C}: 0 &\leqslant \omega\left((\alpha f+\beta g)^*(\alpha f+\beta g)\right)\\ &= |\alpha|^2\omega\left(f^*f\right)+\bar\beta\alpha\omega\left(g^*f\right)+\bar\alpha\beta\omega\left(f^*g\right)+|\beta|^2\omega\left(g^*g\right)\\ &= \begin{pmatrix}\bar\alpha & \bar\beta\end{pmatrix}\begin{pmatrix}\omega\left(f^*f\right) & \omega\left(f^*g\right)\\ \omega\left(g^*f\right) & \omega\left(g^*g\right)\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\end{aligned} \tag{1.48}$$

we see that the matrix

$$M = \begin{pmatrix}\omega\left(f^*f\right) & \omega\left(f^*g\right)\\ \omega\left(g^*f\right) & \omega\left(g^*g\right)\end{pmatrix} \tag{1.49}$$

must be positive, i. e. have only real and non-negative eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\operatorname{tr} M \pm \frac{1}{2}\sqrt{\operatorname{tr}^2 M - 4\det M}\,. \tag{1.50}$$

Therefore

$$0 \leqslant \det M = \omega\left(f^*f\right)\omega\left(g^*g\right) - \omega\left(g^*f\right)\omega\left(f^*g\right) \in \mathbf{R} \tag{1.51}$$

and we have established the *CSI*

$$\omega\left(g^*f\right) = \overline{\omega\left(f^*g\right)} \tag{1.52a}$$

$$\left|\omega\left(f^*g\right)\right| \leqslant \sqrt{\omega\left(f^*f\right)}\sqrt{\omega\left(g^*g\right)}\,. \tag{1.52b}$$

From the CSI

$$|\omega(f)| \leqslant \sqrt{\omega(\mathbf{1})}\sqrt{\omega(f^*f)}\,. \tag{1.53}$$

we infer that

$$\exists f \in \mathcal{A} : \omega(f) \neq 0 \Longrightarrow \sqrt{\omega(\mathbf{1})} > 0 \tag{1.54}$$

Therefore all nontrivial states can be normalized

$$\omega_{\text{norm}}(f) = \frac{\omega(f)}{\omega(\mathbf{1})} \tag{1.55}$$

with $\omega_{\text{norm}}(\mathbf{1}) = 1$. Below, in section 4, when we have developed more tools, we will be able to prove that that every positive linear functional is continuous. This is an example, where a physical requirement (positivity) induces a functional analytic property (continuity).

Just as the flow $\Phi : \mathbf{R} \times \Gamma \to \Gamma$ on phase space induces a flow $\Phi^* : \mathbf{R} \times C(\Gamma) \to C(\Gamma)$ on the space of observables, we can also induce a flow on the states in Ω

$$\begin{aligned} \Phi_* : \mathbf{R} \times \Omega &\to \Omega \\ (t, \omega) &\mapsto \omega_t \end{aligned} \tag{1.56a}$$

with

$$\omega_t(f) = \omega(\Phi_t^*(f)) = \omega(f \circ \Phi_t)\,. \tag{1.56b}$$

Note that the idealized unphysical case of an exactly determined point in phase space can be included in this framework by considering states $\omega_{q,p}$ that correspond to *Dirac measures*

$$\begin{aligned} \omega_{q,p} : C(\Gamma) &\to \mathbf{C} \\ f &\mapsto f(q,p)\,. \end{aligned} \tag{1.57}$$

For these,

$$\begin{aligned} \left(\Delta_{\omega_{q,p}} f\right)^2 &= \omega_{q,p}\left((f - \omega_{q,p}(f))^2\right) \\ &= \omega_{q,p}\left((f - f(q,p))^2\right) = \left((f - f(q,p))^2\right)(q,p) = 0\,. \end{aligned} \tag{1.58}$$

Thus we have abstracted the kinematical structure as

- the observables A of a classical system form an abelian C^*-algebra $\mathcal{A}$ and
- the states ω of a classical system are the *normalized positive linear functionals* $\omega : \mathcal{A} \to \mathbf{C}$, i. e. a subset of the dual space $\mathcal{A}^*$.

On this level of abstraction, the flows Φ are replaced by an abelian one-parameter group $\{\alpha_t\}_{t\in\mathbf{R}}$ of C^*-automorphisms

$$\begin{aligned} \alpha : \mathbf{R} \times \mathcal{A} &\to \mathcal{A} \\ (t,x) &\mapsto \alpha_t(x) \end{aligned} \tag{1.59a}$$

with

$$\alpha_t(\alpha_{t'}(x)) = (\alpha_t \circ \alpha_{t'})(x) = \alpha_{t+t'}(x) = \alpha_{t'}(\alpha_t(x)) \,. \tag{1.59b}$$

Definition 1.15 (C^*-homomorphism)**.** A *C^*-homomorphism* $h : \mathcal{A} \to \mathcal{B}$ is a map between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ that preserves the C^*-algebra structure. It is of course linear and $\forall x, y \in \mathcal{A}$

$$(h(x))^* = h(x^*) \tag{1.60a}$$

$$h(xy) = h(x)h(y) \,. \tag{1.60b}$$

This in turn induces an abelian one-parameter group of transformations on the states:

$$\begin{aligned} \alpha^* : \mathbf{R} \times \Omega &\to \Omega \\ (t,\omega) &\mapsto \alpha^*_t(\omega) \end{aligned} \tag{1.61a}$$

with

$$\alpha^*_t(\omega)(A) = \omega(\alpha_t(A)) \,. \tag{1.61b}$$

1.4.2 General Physical Systems

Using as inspiration the algebraic structure we have just uncovered behind the description of classical systems, we will now generalize it *without* assuming the smooth phase space structure characteristic for classical systems.

Observables

For All Practical Purposes (FAPP), a physical system is nothing but a set $\mathcal{O}_0$ of *observables* $A \in \mathcal{O}_0$, which can be measured by appropriate experimental devices. This set $\mathcal{O}_0$ is not unstructured, there will relations among the observables.

Given any device measuring $A \in \mathcal{O}_0$ and any $\lambda \in \mathbf{R}$, one can construct a device measuring λA by rescaling the device. One does not even worry about finite resources, because the actual device need not be physically rescaled, just the measured value. Therefore we find

$$\forall A \in \mathcal{O}_0, \lambda \in \mathbf{R} : \lambda A \in \mathcal{O}_0 \,. \tag{1.62}$$

One can define the square A^2 of an observable A by squaring the measured value. This extends to positive powers A^n and A^0 can consistently be defined as the observable $\mathbf{1}$ that always has the value 1. Also, linear combinations $\lambda A^n + \mu A^m$ can be defined and therefore arbitrary polynomials in one observable. Note that we can not yet define functions of observables by infinite series or other limits of sequences of observables, because we have no topology and no notion of convergence.

If an observable $A \in \mathcal{O}_0$ only takes positive values, we can find another observable $B \in \mathcal{O}_0$ with $A = B^2$, by taking the square root of the measured values.

Note that we do *not* assume that we can measure multivariate polynomials in different observables, since this would require the *simultaneous* measurement of observables, which would lead us back to classical physics immediately.

States

Different *states* ω of a physical system correspond to different measured values for the observables. However, we can not say that an observable "has" a particular value, we can only estimate an *expectation* for the result of a measurement by averaging the values of repeated measurements. The expectation value for the observable $A \in \mathcal{O}_0$ when the system is in the state ω will be called $\omega(A)$.

Assuming that the set $\mathcal{O}_0$ of the observables of the system is complete, a state ω is completely determined by the expectations of all observables

$$\{\omega(A)\}_{A \in \mathcal{O}_0} \,, \tag{1.63}$$

i. e. ω is a well defined map from $\mathcal{O}_0$ to the real numbers

$$\omega : \mathcal{O}_0 \to \mathbf{R} \,. \tag{1.64}$$

Since ω is defined as an average of measurements, it is clear that the expectation of a scaled observable is the scaled expectation

$$\forall A \in \mathcal{O}_0, \lambda \in \mathbf{R} : \omega(\lambda A) = \lambda\omega(A) \tag{1.65}$$

and we also have linearity in powers

$$\forall A \in \mathcal{O}_0, \lambda, \mu \in \mathbf{R}, n, m \in \mathbf{N} : \omega(\lambda A^n + \mu A^m) = \lambda\omega(A^n) + \mu\omega(A^m)\,. \tag{1.66}$$

Above, we have said that a state is uniquely determined by measuring $\omega(A)$ for all observables $A \in \mathcal{O}_0$. The converse is also true: if two observables have the same expectation in *all* the states, they are indistinguishable, FAPP. This creates an equivalence relation among observables

$$A \sim B \Leftrightarrow \forall\omega \in \Omega : \omega(A) = \omega(B) \tag{1.67}$$

and we will in the following replace $\mathcal{O}_0$ by the equivalence classes w. r. t. this relation $\mathcal{O}_1 = \mathcal{O}_0/\sim$. This has the immediate consequence that there is at most one observable $\mathbf{1} \in \mathcal{O}_1$ with

$$\forall\omega \in \Omega : \omega(\mathbf{1}) = 1\,. \tag{1.68}$$

And we see that all ω are *normalized* functionals on $\mathcal{O}_1$. Furthermore, we can identify

$$\forall A \in \mathcal{O}_1 : A^0 = \mathbf{1} \tag{1.69}$$

and we necessarily have

$$\forall A \in \mathcal{O}_1 : \mathbf{1}A = A^0A = A^0A^1 = A^1 = A = A\mathbf{1}\,, \tag{1.70}$$

i. e. $\mathbf{1}$ is the unique identity element in $\mathcal{O}_1$.

For all positive observables $A = B^2 \in \mathcal{O}_1$, the expectation of its values must also be positive, i. e.

$$\forall\omega \in \Omega : \omega(A) = \omega(B^2) \geqslant 0 \tag{1.71}$$

and since the states characterize the observables completely

$$\forall\omega \in \Omega : \omega(A) \geqslant 0 \Leftrightarrow A \geqslant 0\,. \tag{1.72}$$

C^*-Algebra Structure

Lecture 04: Wed, 22. 10. 2025

For any physically realizable apparatus, the possible measurement values are bounded. Since ω is normalized, this allows to define a natural norm for any observable $A \in \mathcal{O}_1$

$$\|A\| = \sup_{\omega\in\Omega} |\omega(A)| < \infty\,. \tag{1.73}$$

From this definition and (1.65), we have

$$\|\lambda A\| = |\lambda|\|A\|\,. \tag{1.74}$$

Since the states characterize the observables completely, we also have

$$\|A\| = 0 \Leftrightarrow A = 0\,. \tag{1.75}$$

The first non-obvious property is

Lemma 1.16.

$$\forall A \in \mathcal{O}_1 : \|A^2\| = \|A\|^2. \tag{1.76}$$

Proof. By definition

$$\forall\omega \in \Omega : \omega(\|A\|\mathbf{1} \pm A) = \|A\| \pm \omega(A) \geqslant 0 \tag{1.77}$$

i. e.

$$\|A\|\mathbf{1} \pm A \geqslant 0\,. \tag{1.78}$$

Therefore

$$\|A\|^2\mathbf{1} - A^2 = (\|A\|\mathbf{1} + A)(\|A\|\mathbf{1} - A) \geqslant 0\,, \tag{1.79}$$

i. e.

$$\forall\omega \in \Omega : \|A\|^2 - \omega(A^2) \geqslant 0 \tag{1.80}$$

and

$$\|A\|^2 \geqslant \|A^2\|\,. \tag{1.81}$$

Also, from

$$0 \leqslant (\|A\|\mathbf{1} \pm A)^2 = \|A\|^2\mathbf{1} + A^2 \pm 2\|A\|A \tag{1.82}$$

we have

$$\forall\omega \in \Omega : 2\|A\||\omega(A)| \leqslant \|A\|^2 + \omega(A^2) \leqslant \|A\|^2 + \|A^2\|\,, \tag{1.83}$$

and

$$2\|A\|\|A\| \leqslant \|A\|^2 + \|A^2\|\,, \tag{1.84}$$

i. e.

$$\|A\|^2 \leqslant \|A^2\|\,. \tag{1.85}$$

□

Since the states characterize the observables completely, we can use them to attempt to define sums of observables

$$\forall A, B \in \mathcal{O}_1 : \exists_1 A + B : \omega(A + B) = \omega(A) + \omega(B)\,,$$

but $A + B$ might *not* be an element of the original $\mathcal{O}_1$. If this is the case, however, we will extend $\mathcal{O}_1$ to include the newly defined $A + B$, it's powers and their sums

$$\forall A, B \in \mathcal{O}_+ : \exists_1 A + B \in \mathcal{O}_+ : \forall \omega \in \Omega : \omega(A + B) = \omega(A) + \omega(B)\,. \tag{1.86}$$

Note that it is *not* necessary to measure A and B simultaneously for (1.86) to be well defined. The estimators for the expectationa $\omega(A)$ are obtained from repeated measurements of identical copies[2] of the system prepared in the state ω. This can be done for different observables as well and the sum defined from the sum of the measurements. This is *not* possible for products or other non-linear combinations.

So far, we have constructed from a set of observables $\mathcal{O}_0$ a vector space $\mathcal{O}_+$ with a linear structure defined by the set of all states ω. By the definition of the norm (1.73) on the original $\mathcal{O}_0$, we can directly extend it to the sums, satisfying the triangle inequality

$$\forall A, B \in \mathcal{O}_+ : \|A + B\| \leqslant \|A\| + \|B\|\,. \tag{1.87}$$

Therefore $\mathcal{O}_+$ is a normed vector space and by completing it in the norm $\|\cdot\|$, we can turn it into a Banach space $\mathcal{O}$. However, while the states are, by construction (1.86), well defined on the sums of observables, we must still verify that the states are well defined on the completion.

Fortunately, we have, by definition (1.73),

$$\forall \omega \in \Omega : \forall A \in \mathcal{O}_+ : |\omega(A)| \leqslant \|A\|\,, \tag{1.88}$$

i. e. all ω are continuous w. r. t. to the norm $\|\cdot\|$ and can be extended to the corresponding norm completion $\mathcal{O}$.

In order to obtain an algebra structure, we still need to define a multiplication. Note that in the case of operators on a Hilbert space, a straightforward multiplication will not work, because the set self-adjoint operators is not closed under multiplication, unless these operators commute

$$(AB)^* = B^* A^* = BA \neq AB\,.$$

[2] It is a non-trivial assumption that *identical* copies of a system can be prepared and measured without correlations. This assumption is challenged by some researchers studying the foundations of quantum mechanics and its interpretation.

Nevertheless, we can construct a symmetrical product from the sums of observables and their powers,

$$A \circ B = \frac{1}{2} \left((A+B)^2 - A^2 - B^2 \right) = B \circ A \in \mathcal{O}\,, \tag{1.89}$$

which is unfortunately neither guaranteed to be distributive or associative. Note that (1.89) implies

$$A \circ A = A^2\,. \tag{1.90}$$

To proceed, we shall now make the mild technical assumption that the product is homogeneous, i. e.

$$\forall A, B \in \mathcal{O}, \lambda \in \mathbf{R} : A \circ (\lambda B) = \lambda (A \circ B) = (\lambda A) \circ B\,. \tag{1.91}$$

The motivation for this assumption is that if $\mathcal{O}$ was an associative algebra, we would have

$$A \circ B = \frac{1}{2} (AB + BA)$$

which satisfies (1.91), of course.

Lemma 1.17. *The homogeneity* (1.91) *implies that* (1.89) *is distributive.*

Proof. From

$$(A+B)^2 = A^2 + B^2 + 2\,A \circ B \tag{1.92a}$$
$$(A-B)^2 = A^2 + B^2 + 2\,A \circ (-B) = A^2 + B^2 - 2\,A \circ B \tag{1.92b}$$

we find

$$A \circ B = \frac{1}{4} \left((A+B)^2 - (A-B)^2 \right) \tag{1.93a}$$
$$A^2 + B^2 = \frac{1}{2} \left((A+B)^2 + (A-B)^2 \right)\,. \tag{1.93b}$$

Then

$$\begin{aligned}
&2(A+B) \circ C - 2A \circ C - 2B \circ C \overset{(1.93a)}{=} \\
&(A+B+C)^2 - (A+B)^2 - C^2 - (A+C)^2 + A^2 + C^2 - (B+C)^2 + B^2 + C^2 \\
&= \left((A+B+C)^2 + A^2 \right) + \left(B^2 + C^2 \right) - \left((A+B)^2 + (A+C)^2 \right) - (B+C)^2 \\
&\overset{(1.93b)}{=} \frac{1}{2} \Big((2A+B+C)^2 + (B+C)^2 + (B+C)^2 + (B-C)^2 \\
&\qquad - (2A+B+C)^2 - (B-C)^2 \Big) - (B+C)^2 = 0
\end{aligned} \tag{1.94}$$

i. e.

$$(A+B)\circ C = A\circ C + B\circ C \tag{1.95a}$$

and by symmetry

$$C\circ(A+B) = C\circ A + C\circ B\,. \tag{1.95b}$$

□

From the distributivity, symmetry and homogeneity of $\circ$ and the linearity and positivity of ω follows

$$\begin{aligned}\forall\omega\in\Omega:\forall\lambda\in\mathbf{R}: 0 &\leqslant \omega((A+\lambda B)^2)\\ &= \omega((A+\lambda B)\circ(A+\lambda B)) = \omega(A^2)+\lambda^2\omega(B^2)+2\lambda\omega(A\circ B)\end{aligned} \tag{1.96}$$

and using the same argument as in the proof of the CSI (1.52b), we find

$$|\omega(A\circ B)| \leqslant \sqrt{\omega(A^2)}\sqrt{\omega(B^2)} \tag{1.97}$$

and therefore

$$\|A\circ B\| \leqslant \sqrt{\|A^2\|}\sqrt{\|B^2\|} = \|A\|\|B\|\,. \tag{1.98}$$

From the estimate[3]

$$\begin{aligned}\|A^2-B^2\| = \|(A-B)\circ(A+B)\| &\leqslant \|A+B\|\|A-B\|\\ &\leqslant (\|A-B\|+2\|B\|)\,\|A-B\|\end{aligned} \tag{1.99}$$

we see that the square $A\mapsto A^2$ is continuous in the norm topology

$$\lim_{A\to B} A^2 = B^2\,. \tag{1.100a}$$

Furthermore

$$\|A^2-B^2\| \leqslant \max\left\{\|A\|^2,\|B\|^2\right\} \tag{1.100b}$$

and we see that $\mathcal{O}$ forms a so-called *Segal system* [Seg47]. It was been shown by Segal [Seg47], that Segal systems have enough structure to recover most of quantum mechanics.

However, since the mathematical structure can become quite involved, we shall make another *technical assumption*, namely that there is a complex[4] algebra extension $(\mathcal{A},*)$ of $\mathcal{O}$ with an associative, but not necessarily commutative, product, such that $\forall A,B\in\mathcal{A}$ and $\lambda,\mu\in\mathbf{C}$

$$A\circ B = \frac{1}{2}(AB+BA) \tag{1.101a}$$

[3] NB: $4(A-B)\circ(A+B) = (2A)^2-(2B)^2$ from (1.93a).

[4] Note that it is not possible to have a non-abelian associative algebra of only self-adjoint elements, because

$$(AB)^* = B^*A^* = BA \neq AB\,.$$

$$
\begin{aligned}
(\lambda A + \mu B)^* &= \bar\lambda A^* + \bar\mu B^* && (1.101b)\\
(AB)^* &= B^* A^* && (1.101c)\\
\omega(A^*A) &\geqslant 0 && (1.101d)\\
\|AB\| = \sup_{\omega\in\Omega} |\omega(AB)| &\leqslant \|A\|\|B\| && (1.101e)\\
\|A^*A\| &= \|A^*\|\|A\| && (1.101f)
\end{aligned}
$$

where the states ω have been extended by linearity from $\mathcal{O}$ to $\mathcal{A}$ and A^*A is positive for all $A \in \mathcal{A}$.

As before in the proof of the CSI (1.52b) on page 11, we can then infer from positivity

$$\forall\lambda \in \mathbf{C} : \omega\left((\lambda A + \mathbf{1})^*(\lambda A + \mathbf{1})\right) \geqslant 0 \tag{1.102}$$

that

$$
\begin{aligned}
\omega(A^*) &= \overline{\omega(A)} && (1.103a)\\
\|A^*\| &= \|A\|\,. && (1.103b)
\end{aligned}
$$

Altogether, $\mathcal{A}$ is a C^*-algebra with identity $\mathbf{1}$, which is generated by the subset of self-adjoint elements $\mathcal{O} \subset \mathcal{A}$.

Definition 1.18 (Physical System (von Neumann))**.**

1. A physical system is defined by the unital C^*-Algebra $\mathcal{A}$ generated by its observables.

2. The states of of a system are normalized, positive linear functionals $\omega : \mathcal{A} \to \mathbf{C}$ on the observables. We assume the set $\mathcal{S} \subseteq \Omega$ of *physical* states to be *full*, i. e. to *separate* the observables. Vice versa, the observables are assumed to separate the states.

Note that we allow the set $\mathcal{S}$ of physical states to be smaller than the set Ω of all normalized, positive linear functionals on $\mathcal{A}$.

—2—
C^*-Algebras

2.1 Adjoining a Unit

In the following we will study the spectrum of elements of C^*-algebras. In order to define it, we need an identity element. We start be showing that such an element can alway be added without loss of generality.

Theorem 2.1. *Let $\mathcal{A}$ be a C^*-algebra* without *identity and $\bar{\mathcal{A}}$ denote the set of pairs*

$$\bar{\mathcal{A}} = \{(\alpha, A) : \alpha \in \mathbf{C}, A \in \mathcal{A}\} . \tag{2.1}$$

The operations

$$\mu(\alpha, A) + \lambda(\beta, B) = (\mu\alpha + \lambda\beta, \mu A + \lambda B) \tag{2.2a}$$
$$(\alpha, A)(\beta, B) = (\alpha\beta, \alpha B + \beta A + AB) \tag{2.2b}$$
$$(\alpha, A)^* = (\bar{\alpha}, A^*) \tag{2.2c}$$

turn $\bar{\mathcal{A}}$ into a $*$-algebra with identity $(1,0)$. Then

$$\|(\alpha, A)\|_{\bar{\mathcal{A}}} = \sup_{B\in\mathcal{A},\|B\|=1} \|\alpha B + AB\|_{\mathcal{A}} \tag{2.3}$$

defines a norm that turns $\bar{\mathcal{A}}$ into a C^*-algebra. $\mathcal{A}$ can be identified with the subalgebra $\{(0, A) : A \in \mathcal{A}\}$ of $\bar{\mathcal{A}}$.

Proof. The unital $*$-algebra properties are obvious. The triangle and product inequalities

$$\|(\alpha, A) + (\beta, B)\| \leqslant \|(\alpha, A)\| + \|(\beta, B)\| \tag{2.4a}$$
$$\|(\alpha, A)(\beta, B)\| \leqslant \|(\alpha, A)\|\|(\beta, B)\| \tag{2.4b}$$

are left as exercise. To show that $\|(\alpha, A)\| = 0$ implies $(\alpha, A) = (0, 0)$, we start with observing that

$$\|(0, A)\| = \sup_{B\in\mathcal{A},\|B\|=1} \|AB\| . \tag{2.5}$$

On one hand, we have

$$\sup_{B\in\mathcal{A},\|B\|=1} \|AB\| \leqslant \sup_{B\in\mathcal{A},\|B\|=1} \|A\|\|B\| = \|A\| . \tag{2.6}$$

from $\|AB\| \leqslant \|A\|\|B\|$, while on the other

$$\sup_{B\in\mathcal{A},\|B\|=1} \|AB\| \geqslant \left\| A\frac{A^*}{\|A\|} \right\| = \frac{\|AA^*\|}{\|A\|} = \|A\| \tag{2.7}$$

from the C^*-condition (1.31). Therefore

$$\|(0, A)\|_{\bar{\mathcal{A}}} = \|A\|_{\mathcal{A}} \tag{2.8}$$

and

$$\|(0, A)\| = 0 \Rightarrow \|A\| = 0 \Rightarrow A = 0 . \tag{2.9}$$

Thus we only have to study the case $\alpha \neq 0$. By linearity, we can choose Without Loss Of Generality (WLOG) $\alpha = 1$.

Lecture 05: Tue, 28. 10. 2025

From

$$\|B - AB\| = \|(1, -A)(0, B)\| \leqslant \|(1, -A)\|\|(0, B)\| \tag{2.10}$$

we can infer from $\|(1, -A)\| = 0$

$$\forall B \in \mathcal{A} : \|B - AB\| = 0 \tag{2.11}$$

or

$$\forall B \in \mathcal{A} : B = AB \tag{2.12}$$

and by involution

$$\forall B \in \mathcal{A} : B = BA^* . \tag{2.13}$$

In particular with $B = A$ and $B = A^*$

$$A = AA^* = A^* \tag{2.14}$$

and then

$$B = AB = BA = B . \tag{2.15}$$

But this means that A is an identity in $\mathcal{A}$ which is a contradiction. Thus

$$\forall 0 \neq A \in \mathcal{A} : \|(1, -A)\| > 0 . \tag{2.16}$$

Finally, to prove the C^*-property for $\bar{\mathcal{A}}$, we start from the one for $\mathcal{A}$

$$\begin{aligned}\|\alpha B + AB\|^2 &= \|(\alpha B + AB)^*(\alpha B + AB)\| \\ &= \|B^*(\bar\alpha\alpha B + (\alpha A^* + \bar\alpha A + A^*A)B)\| \\ &\leqslant \|B^*\|\|\bar\alpha\alpha B + (\alpha A^* + \bar\alpha A + A^*A)B\| \end{aligned} \tag{2.17}$$

to show that

$$\begin{aligned}\|(\alpha, A)\|^2 &= \sup_{B\in\mathcal{A},\|B\|=1} \|\alpha B + AB\|^2 \\ &\leqslant \sup_{B\in\mathcal{A},\|B\|=1} \|\bar\alpha\alpha B + (\alpha A^* + \bar\alpha A + A^*A)B\| = \|(\bar\alpha\alpha, \alpha A^* + \bar\alpha A + A^*A)\| \\ &= \|(\alpha, A)^*(\alpha, A)\| \leqslant \|(\alpha, A)^*\|\|(\alpha, A)\| \end{aligned} \tag{2.18}$$

i. e.

$$\|(\alpha, A)\| \leqslant \|(\alpha, A)^*\| \,. \tag{2.19}$$

The same argument for $(\alpha, A)^*$ instead of (α, A) yields

$$\|(\alpha, A)^*\| \leqslant \|(\alpha, A)\| \tag{2.20}$$

and we have shown

$$\|(\alpha, A)^*\| = \|(\alpha, A)\| \,. \tag{2.21}$$

Thus we find the desired C^*-property

$$\|(\alpha, A)\|^2 \leqslant \|(\alpha, A)^*(\alpha, A)\| \leqslant \|(\alpha, A)^*\|\|(\alpha, A)\| = \|(\alpha, A)\|^2 \tag{2.22}$$

i. e.

$$\|(\alpha, A)^*(\alpha, A)\| = \|(\alpha, A)\|^2 \,. \tag{2.23}$$

Finally, the completeness of $\bar{\mathcal{A}} = \mathbf{C} \times \mathcal{A}$ is obvious[1] from $\|(\alpha, A)\| \leqslant |\alpha| + \|A\|$, since both factors are complete. □

Definition 2.2 (adjoining an identity)**.** The unital algebra $\bar{\mathcal{A}}$ obtained from an algebra $\mathcal{A}$ without identity as described in theorem 2.1 will be called obtained by *adjoining an identity* **1** to $\mathcal{A}$. We will also write

$$\bar{\mathcal{A}} = \mathbf{C}\mathbf{1} + \mathcal{A} \tag{2.24}$$

and write $\alpha\mathbf{1} + A$ for the pair (α, A).

[1] See, e. g. [Put18], p. 37.

2.2 Ideals and Factors

Definition 2.3. A subspace $\mathcal{B} \subseteq \mathcal{A}$ is called a *left ideal*, if $\forall A \in \mathcal{A}, B \in \mathcal{B}: AB \in \mathcal{B}$. A subspace $\mathcal{B} \subseteq \mathcal{A}$ is called a *right ideal*, if $\forall A \in \mathcal{A}, B \in \mathcal{B}: BA \in \mathcal{B}$. If $\mathcal{B}$ is both a left and a right ideal it is called a *two sided* ideal.

Remark 2.4. Every ideal is a (sub-)algebra

Proof. *($\rightarrow$ exercise).* □

Remark 2.5. If $\mathcal{B}$ is self adjoint and a left or right ideal it is two sided.

Proof. *($\rightarrow$ exercise).* □

Remark 2.6. If $\mathcal{I}$ is a two sided ideal of an algebra $\mathcal{A}$, the factor space $\mathcal{A}/\mathcal{I}$ is also an algebra. This ist also true for $*$-algebras, iff $\mathcal{I}$ is self adjoint and Banach algebras, iff $\mathcal{I}$ is complete.

Proof. *($\rightarrow$ exercise).* □

Caveat: a two sided ideal in a $*$-algebra is *not* necessarily self adjoint. This can be seen from Stefan Waldmann's "universal counterexample" of $\mathcal{A} = C(S^2)$ with $(fg)(x) = f(x)g(x)$ and $f^*(x) = \bar{f}(-x)$. Then $\mathcal{I} = \{f \in \mathcal{A} : f(\text{north pole}) = 0\}$ is obviously a two sided ideal, but $\mathcal{I}^* = \{f \in \mathcal{A} : f(\text{south pole}) = 0\} \neq \mathcal{I}$.

2.3 Spectral Analysis

Definition 2.7 (resolvent, spectrum)**.** Let $\mathcal{A}$ be a unital algebra. The *resolvent set* $r_{\mathcal{A}}(A) \subset \mathbf{C}$ is the set of $\lambda \in \mathbf{C}$ such that $\lambda\mathbf{1} - A$ is *invertible*, i. e. has a two-sided inverse. The *spectrum* $\sigma_{\mathcal{A}}(A) = \mathbf{C} \setminus r_{\mathcal{A}}(A)$ is the complement of the resolvent set. The map

$$\begin{aligned} R : r_{\mathcal{A}}(A) &\rightarrow \mathcal{A} \\ \lambda &\mapsto R(\lambda) = (\lambda\mathbf{1} - A)^{-1} \end{aligned} \tag{2.25}$$

is called the *resolvent* of A.

Definition 2.8. Let $\bar{\mathcal{A}}$ be the algebra obtained from an algebra $\mathcal{A}$ without identity by adjoining an identity. We *define* for all $A \in \mathcal{A} \subset \bar{\mathcal{A}}$

$$r_{\mathcal{A}}(A) = r_{\bar{\mathcal{A}}}(A) \tag{2.26a}$$

$$\sigma_{\mathcal{A}}(A) = \sigma_{\bar{\mathcal{A}}}(A)\,. \tag{2.26b}$$

The most straightforward approach to spectral analysis uses series expansion and analytic continuation. Formally, we can construct the resolvent for $\lambda \neq 0$ using the geometric series

$$\frac{1}{\lambda\mathbf{1} - A} = \frac{1}{\lambda}\frac{1}{\mathbf{1} - A/\lambda} = \frac{1}{\lambda}\sum_{m=0}^{\infty}\left(\frac{A}{\lambda}\right)^m . \tag{2.27}$$

Indeed, since $\|A^n\| \leqslant \|A\|^n$, the series converges absolutely in the norm topology of the Banach algebra $\mathcal{A}$ if $|\lambda| > \|A\|$ and we may reorder the series (2.27)

$$(\lambda\mathbf{1} - A)\frac{1}{\lambda}\sum_{m=0}^{\infty}\left(\frac{A}{\lambda}\right)^m = \sum_{m=0}^{\infty}\left(\frac{A}{\lambda}\right)^m - \sum_{m=0}^{\infty}\left(\frac{A}{\lambda}\right)^{m+1} = \left(\frac{A}{\lambda}\right)^0 = \mathbf{1} . \tag{2.28}$$

Therefore the spectrum is bounded

$$\sigma_{\mathcal{A}}(A) \subseteq \{\lambda \in \mathbf{C} : |\lambda| \leqslant \|A\|\} . \tag{2.29}$$

Analogously, we can write formally for all $\lambda_0 \in r_{\mathcal{A}}(A)$

$$\begin{aligned}\frac{1}{\lambda\mathbf{1} - A} &= \frac{1}{\lambda_0\mathbf{1} - A - (\lambda_0 - \lambda)\mathbf{1}} = \frac{1}{\lambda_0\mathbf{1} - A}\frac{1}{\mathbf{1} - \left(\frac{\lambda_0 - \lambda}{\lambda_0\mathbf{1} - A}\right)} \\ &= \frac{1}{\lambda_0\mathbf{1} - A}\sum_{m=0}^{\infty}\left(\frac{\lambda_0 - \lambda}{\lambda_0\mathbf{1} - A}\right)^m = \sum_{m=0}^{\infty}(\lambda_0 - \lambda)^m (\lambda_0\mathbf{1} - A)^{-m-1} \end{aligned}\tag{2.30}$$

which converges absolutely for $|\lambda - \lambda_0| < \|(\lambda_0\mathbf{1} - A)^{-1}\|^{-1}$. This establishes that for every $\lambda_0 \in r_{\mathcal{A}}(A)$, there is an open neighborhood of λ_0 that is contained in $r_{\mathcal{A}}(A)$. Thus $r_{\mathcal{A}}(A)$ is open and consequently $\sigma_{\mathcal{A}}(A)$ is closed and therefore compact, because it is bounded. These considerations suggest the

Definition 2.9 (spectral radius)**.** The *spectral radius* of an element A of a unital Banach algebra $\mathcal{A}$ is defined as

$$\rho(A) = \rho_{\mathcal{A}}(A) = \sup_{\lambda \in \sigma_{\mathcal{A}}(A)} |\lambda| . \tag{2.31}$$

As seen above, the spectral radius $\rho(A)$ is the radius of convergence of the series (2.27) for the resolvent $1/\lambda \mapsto (\lambda\mathbf{1} - A)^{-1}$ as a function of $1/\lambda$.

In order to be able to make more detailed statements, we can prove a variation of the Cauchy-Hadamard theorem from complex analysis for power series with coefficients from a Banach algebra

Theorem 2.10. *Given a series $\{A_n\}_{n\in\mathbf{N}}$ of elements of a Banach space, the radius of convergence of the power series*

$$f(z) = \sum_{n=0}^{\infty} A_n (z-a)^n \tag{2.32}$$

is

$$r = \frac{1}{\limsup\limits_{n\to\infty} \|A_n\|^{1/n}} . \tag{2.33}$$

Proof. WLOG let $a = 0$ and define $t = \limsup_{n\to\infty} \|A_n\|^{1/n} = 1/r$.

1. $|z| < r$: For any $\epsilon > 0$, there are only a finite number of n such that $\|A_n\|^{1/n} \geqslant t + \epsilon$. Thus $\|A_n\| \leqslant (t+\epsilon)^n$ for all but a finite number of n and the series $\sum_n A_n z^n$ converges if $|z| < 1/(t+\epsilon) < 1/t = r$

2. $|z| > r$: For any $\epsilon > 0$, there are an infinite number of n such that $\|A_n\| \geqslant (t-\epsilon)^n$ and the series can not converge for $|z| \geqslant 1/(t-\epsilon) > r$, because $A_n z^n \nrightarrow 0$.

□

Theorem 2.11 (spectral radius)**.** *If $\mathcal{A}$ is a unital Banach algebra, then the spectral radius of $A \in \mathcal{A}$ is*

$$\rho(A) = \inf_{n\in\mathbf{N}} \|A^n\|^{1/n} = \lim_{n\to\infty} \|A^n\|^{1/n} \leqslant \|A\| . \tag{2.34}$$

The limit exists and $\sigma_{\mathcal{A}}(A)$ is a non-empty compact subset of $\mathbf{C}$.

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Proof. Defining

$$r = \inf_{n\in\mathbf{N}} \|A^n\|^{1/n} , \tag{2.35}$$

we have obviously

$$\forall n \in \mathbf{N} : r \leqslant \|A^n\|^{1/n} \leqslant \|A\| \tag{2.36}$$

and therefore

$$r \leqslant \liminf_{n\to\infty} \|A^n\|^{1/n} . \tag{2.37}$$

For all $\epsilon > 0$, from (2.35) there must be an m such that $\|A^m\|^{1/m} < r + \epsilon$. Using

$$\forall n \in \mathbf{N} : \exists k_n, l_n \in \mathbf{N} : n = mk_n + l_n \wedge 0 \leqslant l_n < m \tag{2.38}$$

with[2]

$$\lim_{n\to\infty} \frac{mk_n}{n} = 1 \tag{2.39a}$$

$$\lim_{n\to\infty} \frac{l_n}{n} = 0\,, \tag{2.39b}$$

we have

$$\|A^n\|^{1/n} = \|A^{k_n m} A^{l_n}\|^{1/n} \leqslant \|A^m\|^{k_n/n} \|A\|^{l_n/n} \leqslant (r+\epsilon)^{mk_n/n} \|A\|^{l_n/n} \,. \tag{2.40}$$

Therefore

$$\forall \epsilon > 0 : \limsup_{n\to\infty} \|A^n\|^{1/n} \leqslant r + \epsilon \tag{2.41}$$

and since ϵ can be arbitrarly small, the limit exists

$$\limsup_{n\to\infty} \|A^n\|^{1/n} = \liminf_{n\to\infty} \|A^n\|^{1/n} = \lim_{n\to\infty} \|A^n\|^{1/n} = r\,. \tag{2.42}$$

On The Other Hand (OTOH), using theorem 2.10, we see that r is the radius of convergence of the series (2.27) in $1/\lambda$ for the resolvent. Thus $\rho(A) = r$.

We have already seen above that $\sigma_{\mathcal{A}}(A)$ is compact. Finally, if $\sigma_{\mathcal{A}}(A)$ was empty, then $\lambda \mapsto (\lambda\mathbf{1} - A)^{-1}$ would be a function holomorphic on the whole complex plane vanishing for $|\lambda| \to \infty$. Such a function vanishes everywhere and can not be the inverse of $\lambda\mathbf{1} - A$. □

Lemma 2.12. *Let $A, B \in \mathcal{A}$ with $AB = BA$, then $(AB)^{-1} \in \mathcal{A}$ exists iff both $A^{-1} \in \mathcal{A}$ and $B^{-1} \in \mathcal{A}$ exist. Furthermore*

$$(AB)^{-1} = A^{-1}B^{-1} = B^{-1}A^{-1}\,. \tag{2.43}$$

Proof. By explicit construction:

- Assume that both A^{-1} and B^{-1} exist. Then

$$(AB)\left(A^{-1}B^{-1}\right) = BAA^{-1}B^{-1} = BB^{-1} = \mathbf{1} \tag{2.44a}$$

$$(AB)\left(B^{-1}A^{-1}\right) = AA^{-1} = \mathbf{1} \tag{2.44b}$$

$$\left(A^{-1}B^{-1}\right)(AB) = A^{-1}B^{-1}BA = A^{-1}A = \mathbf{1} \tag{2.44c}$$

$$\left(B^{-1}A^{-1}\right)(AB) = B^{-1}A^{-1}AB = B^{-1}B = \mathbf{1} \tag{2.44d}$$

- Assume that $(AB)^{-1}$ exists. Then

$$B^{-1} = (AB)^{-1}A = A(AB)^{-1} \tag{2.45a}$$

[2] Think $k_n \approx n/m$ ...

$$A^{-1} = (AB)^{-1}B = B(AB)^{-1}\,, \tag{2.45b}$$

because

$$B(AB)^{-1} = (AB)^{-1}ABB(AB)^{-1} = (AB)^{-1}BAB(AB)^{-1} = (AB)^{-1}B \tag{2.46}$$

(analogously $A(AB)^{-1} = (AB)^{-1}A$) and therefore

$$B\left((AB)^{-1}A\right) = BA(AB)^{-1} = AB(AB)^{-1} = \mathbf{1} \tag{2.47a}$$
$$\left((AB)^{-1}A\right)B = \mathbf{1} \tag{2.47b}$$
$$A\left((AB)^{-1}B\right) = AB(AB)^{-1} = \mathbf{1} \tag{2.47c}$$
$$\left((AB)^{-1}B\right)A = (AB)^{-1}AB = \mathbf{1}\,. \tag{2.47d}$$

□

Remark 2.13. Let $\mathcal{A}$ be a unital algebra. Then

$$\forall A \in \mathcal{A} : (\sigma_{\mathcal{A}}(A))^n \subseteq \sigma_{\mathcal{A}}(A^n)\,. \tag{2.48}$$

Proof. From

$$\lambda^n\mathbf{1} - A^n = (\lambda\mathbf{1} - A)\left(\lambda^{n-1}\mathbf{1} + \lambda^{n-2}A + \ldots + A^{n-1}\right) \tag{2.49}$$

and lemma 2.12, we see that if $\lambda^n\mathbf{1} - A^n$ is invertible, then $\lambda\mathbf{1} - A$ must be invertible as well. In other words, if $\lambda^n \in r_{\mathcal{A}}(A^n)$, then $\lambda \in r_{\mathcal{A}}(A)$ or $(r_{\mathcal{A}}(A))^n \supseteq r_{\mathcal{A}}(A^n)$. □

Remark 2.14. Let $\mathcal{A}$ be a unital $*$-algebra. Then for all $A, B \in \mathcal{A}$

$$\forall \mu \in \mathbf{C} : \sigma_{\mathcal{A}}(A + \mu\mathbf{1}) = \sigma_{\mathcal{A}}(A) + \mu \tag{2.50a}$$
$$\sigma_{\mathcal{A}}(A^*) = \overline{\sigma_{\mathcal{A}}(A)} \tag{2.50b}$$
$$\sigma_{\mathcal{A}}(AB) \cup \{0\} = \sigma_{\mathcal{A}}(BA) \cup \{0\}\,. \tag{2.50c}$$

If A is invertible

$$\sigma_{\mathcal{A}}(A^{-1}) = (\sigma_{\mathcal{A}}(A))^{-1}\,. \tag{2.50d}$$

Proof. The first statement is obvious from

$$(\lambda + \mu)\mathbf{1} - (A + \mu\mathbf{1}) = \lambda\mathbf{1} - A\,. \tag{2.51}$$

The second statement follows from

$$\lambda\mathbf{1} - A^* = (\bar{\lambda}\mathbf{1} - A)^* \tag{2.52}$$

and
$$\forall A \in \mathcal{A} : (A^*)^{-1} = \left(A^{-1}\right)^* \tag{2.53}$$
since $\mathbf{1} = AB = BA \Leftrightarrow \mathbf{1} = A^*B^* = B^*A^*$. If $\lambda \in r_{\mathcal{A}}(BA)$, then
$$(\lambda\mathbf{1} - AB)A\frac{1}{\lambda\mathbf{1} - BA} = A$$
from $(\lambda\mathbf{1} - AB)A = A(\lambda\mathbf{1} - BA)$. Multiplying from the right by B
$$(\lambda\mathbf{1} - AB)A\frac{1}{\lambda\mathbf{1} - BA}B = AB$$
and adding $\lambda\mathbf{1} - AB$ gives
$$(\lambda\mathbf{1} - AB) + (\lambda\mathbf{1} - AB)A\frac{1}{\lambda\mathbf{1} - BA}B = \lambda\mathbf{1}\,.$$
i. e.
$$(\lambda\mathbf{1} - AB)\left(\mathbf{1} + A\frac{1}{\lambda\mathbf{1} - BA}B\right) = \lambda\mathbf{1} \tag{2.54}$$
and therefore $\lambda \in r_{\mathcal{A}}(AB)$, as long as $\lambda \neq 0$, since we have explicitely computed an inverse for $\lambda\mathbf{1} - AB$. Including the case $\lambda = 0$, we obtain the third statement. Finally, if A is invertible, we know that $0 \notin \sigma_{\mathcal{A}}(A)$. Thus, WLOG $\lambda \neq 0$ and
$$\lambda\mathbf{1} - A = \lambda A\left(A^{-1} - \lambda^{-1}\mathbf{1}\right) \tag{2.55a}$$
$$\lambda^{-1}\mathbf{1} - A^{-1} = \lambda^{-1}A^{-1}\left(A - \lambda\mathbf{1}\right)\,. \tag{2.55b}$$
From the first equation $\lambda \in \sigma_{\mathcal{A}}(A) \Rightarrow \lambda^{-1} \in \sigma_{\mathcal{A}}(A^{-1})$ and from the second the reverse direction, i. e. the fourth statement. □

Theorem 2.15. *Let $\mathcal{A}$ be a unital C^*-algebra.*

1. *If A is normal then*
$$\rho(A) = \|A\|\,. \tag{2.56}$$
2. *If A is an isometry, i. e. $A^*A = \mathbf{1}$, but not necessarily $AA^* = \mathbf{1}$, then*
$$\rho(A) = 1\,. \tag{2.57}$$
3. *If A is unitary then*
$$\sigma_{\mathcal{A}}(A) \subseteq \{\lambda \in \mathbf{C} : |\lambda| = 1\}\,. \tag{2.58}$$

4. *If A is self-adjoint then*

$$\sigma_{\mathcal{A}}(A) \subseteq [-\|A\|, \|A\|] \subset \mathbf{R} \tag{2.59a}$$
$$\sigma_{\mathcal{A}}(A^2) \subseteq \left[0, \|A^2\|\right] \subset \mathbf{R}\,. \tag{2.59b}$$

5. *For all $A \in \mathcal{A}$ and general polynomials P*

$$\sigma_{\mathcal{A}}(P(A)) = P(\sigma_{\mathcal{A}}(A))\,. \tag{2.60}$$

Proof. 1. From

$$\begin{aligned}\|A^{2^n}\|^2 &\overset{C^*\text{-prop.}}{=} \|(A^*)^{2^n} A^{2^n}\| \overset{A^*A=AA^*}{=} \|(A^*A)^{2^n}\| \\ &\overset{C^*\text{-prop.}}{=} \|(A^*A)^{2^{n-1}}\|^2 = \ldots = \|A^*A\|^{2^n} \overset{C^*\text{-prop.}}{=} \|A\|^{2^{n+1}}\end{aligned} \tag{2.61}$$

we find

$$\rho(A) = \lim_{n\to\infty} \|A^{2^n}\|^{2^{-n}} = \|A\|\,. \tag{2.62}$$

2. From

$$\|A^n\|^2 \overset{C^*\text{-prop.}}{=} \|(A^*)^n A^n\| = \|(A^*)^{n-1} A^{n-1}\| = \ldots = \|\mathbf{1}\| = 1 \tag{2.63}$$

we find

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = 1\,. \tag{2.64}$$

3. Every unitary A is isometric. Therefore $\sigma_{\mathcal{A}}(A)$ is contained in the unit disc. Moreover

$$\sigma_{\mathcal{A}}(A) = \overline{\sigma_{\mathcal{A}}(A^*)} = \overline{\sigma_{\mathcal{A}}(A^{-1})} = \left(\overline{\sigma_{\mathcal{A}}(A)}\right)^{-1} \tag{2.65}$$

and the spectrum must lie on the boundary of the unit disk, i. e. on the unit circle.

4. Every self-adjoint A is normal and therefore $\rho(A) = \|A\|$. If $|1/\lambda| > \|A\|$, then $1/\lambda \in r_{\mathcal{A}}(A)$ and $\mathbf{1} + \mathrm{i}|\lambda|A$ is invertible. Thus

$$U = (\mathbf{1} - \mathrm{i}|\lambda|A)\,\frac{1}{\mathbf{1} + \mathrm{i}|\lambda|A} \tag{2.66}$$

exists and is unitary

$$U^*U = \frac{1}{\mathbf{1} - \mathrm{i}|\lambda|A}\,(\mathbf{1} + \mathrm{i}|\lambda|A)\,(\mathbf{1} - \mathrm{i}|\lambda|A)\,\frac{1}{\mathbf{1} + \mathrm{i}|\lambda|A}$$

$$= \frac{1}{\mathbf{1} - \mathrm{i}|\lambda|A} \left(\mathbf{1} - \mathrm{i}|\lambda|A\right) \left(\mathbf{1} + \mathrm{i}|\lambda|A\right) \frac{1}{\mathbf{1} + \mathrm{i}|\lambda|A} = \mathbf{1} \quad (2.67)$$

(the proof of $UU^* = \mathbf{1}$ is identical). From

$$\left|\frac{1 - \mathrm{i}|\lambda|\alpha}{1 + \mathrm{i}|\lambda|\alpha}\right|^2 = \frac{1 + \mathrm{i}|\lambda|\bar{\alpha}}{1 + \mathrm{i}|\lambda|\alpha} \cdot \frac{1 - \mathrm{i}|\lambda|\alpha}{1 - \mathrm{i}|\lambda|\bar{\alpha}} = 1$$
$$\Leftrightarrow 1 + |\lambda|^2|\alpha|^2 - \mathrm{i}|\lambda|(\alpha - \bar{\alpha}) = 1 + |\lambda|^2|\alpha|^2 + \mathrm{i}|\lambda|(\alpha - \bar{\alpha})$$
$$\Leftrightarrow \alpha = \bar{\alpha} \quad (2.68)$$

for $\lambda \neq 0$ and the previous statement, we infer that

$$\frac{1 - \mathrm{i}|\lambda|\alpha}{1 + \mathrm{i}|\lambda|\alpha} \in \sigma_{\mathcal{A}}(U) \Rightarrow \alpha = \bar{\alpha} \quad (2.69)$$

i. e.

$$\forall \alpha \neq \bar{\alpha} : \frac{1 - \mathrm{i}|\lambda|\alpha}{1 + \mathrm{i}|\lambda|\alpha} \in r_{\mathcal{A}}(U) \,. \quad (2.70)$$

OTOH

$$\frac{1 - \mathrm{i}|\lambda|\alpha}{1 + \mathrm{i}|\lambda|\alpha}\mathbf{1} - U = \frac{1 - \mathrm{i}|\lambda|\alpha}{1 + \mathrm{i}|\lambda|\alpha}\mathbf{1} - \left(\mathbf{1} - \mathrm{i}|\lambda|A\right) \frac{1}{\mathbf{1} + \mathrm{i}|\lambda|A} =$$
$$\frac{1}{1 + \mathrm{i}|\lambda|\alpha} \left((\mathbf{1} + \mathrm{i}|\lambda|A)(1 - \mathrm{i}|\lambda|\alpha) - (\mathbf{1} - \mathrm{i}|\lambda|A)(1 + \mathrm{i}|\lambda|\alpha)\right) \frac{1}{\mathbf{1} + \mathrm{i}|\lambda|A}$$
$$= \frac{2\mathrm{i}|\lambda|}{1 + \mathrm{i}|\lambda|\alpha} (A - \alpha\mathbf{1}) \frac{1}{\mathbf{1} + \mathrm{i}|\lambda|A} \quad (2.71)$$

and $A - \alpha\mathbf{1}$ must be invertible for all $\alpha \neq \bar{\alpha}$. Thus

$$\sigma_{\mathcal{A}}(A) \subseteq \{\lambda \in \mathbf{C} : |\lambda| \leqslant \|A\| \wedge \lambda = \bar{\lambda}\} \,. \quad (2.72)$$

The positivity of the spectrum of a positive element will follow from the following statement.

5. For commuting $\{A_i\}_{i \in I \subset \mathbf{N}}$, the invertibility of a product $A_1 A_2 \cdots A_n$ is equivalent to the invertibility of all A_i by lemma 2.12. By the fundamental theorem of algebra, for any polynomial P, we can now find $\alpha, \alpha_i \in \mathbf{C}$ such that

$$P(x) - \lambda = \alpha \prod_{i=1}^{n} (x - \alpha_i) \quad (2.73)$$

and equivalently

$$P(A) - \lambda\mathbf{1} = \alpha \prod_{i=1}^{n} (A - \alpha_i \mathbf{1}) \tag{2.74}$$

since all powers of A commute. Thus the Left Hand Side (LHS) is invertible iff all factors of the Right Hand Side (RHS) are invertible

$$\lambda \in \sigma_{\mathcal{A}}(P(A)) \Leftrightarrow \exists i : \alpha_i \in \sigma_{\mathcal{A}}(A) \,. \tag{2.75}$$

However, $\forall i : P(\alpha_i) = \lambda$ by (2.73) and therefore $\sigma_{\mathcal{A}}(P(A)) = P(\sigma_{\mathcal{A}}(A))$. □

Lecture 07: Tue, 04. 11. 2025

2.4 Uniqueness and Independence

As already alluded to above, it turns out that the C^*-property is very restrictive. In fact, a C^*-norm is unique and the spectrum of an element is independent of the (sub-)algebra it belongs to.

Lemma 2.16. *If a $*$-algebra $\mathcal{A}$ has a norm with the C^*-property, this norm is unique.*

Proof. For normal and in particular for self-adjoint elements A, we know from theorem 2.15, that the C^*-property entails

$$\|A\| = \rho(A) \tag{2.76}$$

and the latter is already determined by the algebraic structure. For general $A \in \mathcal{A}$, the C^*-property yields

$$\|A\| = \sqrt{\|A^*A\|} = \sqrt{\rho(A^*A)} \,. \tag{2.77}$$

□

Given a unital subalgebra $\mathcal{B}$ of a unital algebra $\mathcal{A}$, the spectrum of $A \in \mathcal{B}$ is in general a superset of the spectrum of $A \in \mathcal{A}$. Indeed,

$$r_{\mathcal{A}}(A) \supseteq r_{\mathcal{B}}(A) \tag{2.78}$$

because the inverse of $\lambda\mathbf{1} - A$ might be a element of $\mathcal{A}\backslash\mathcal{B}$. This is not the case in a C^*-algebra, however:

Theorem 2.17. *If $\mathcal{B}$ is a unital C^*-subalgebra of a unital C^*-algebra $\mathcal{A}$ and the unit elements of $\mathcal{A}$ and $\mathcal{B}$ coincide, then $\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A)$.*

Proof. We will show that if $\lambda\mathbf{1} - B$ is invertible in $\mathcal{A}$, then it is invertible in the C^*-subalgebra $\mathcal{C}$ generated by $\mathbf{1}$, B and B^*, i. e. $\sigma_{\mathcal{C}}(B) = \sigma_{\mathcal{A}}(B)$. Then

$$\sigma_{\mathcal{A}}(B) = \sigma_{\mathcal{C}}(B) = \sigma_{\mathcal{B}}(B)\,. \tag{2.79}$$

Thus it suffices to show that if $A = \lambda\mathbf{1} - B$ is invertible in $\mathcal{A}$, then $A^{-1} \in \mathcal{C}$.

Starting with self-adjoint $A = A^*$, we know that $A - \lambda\mathbf{1}$ is invertible for all λ with a non-vanishing imaginary part. For $\lambda_0 = 2\mathrm{i}\|A\|$, we know that the power series

$$\frac{1}{A - \lambda_0\mathbf{1}} = -\frac{1}{\lambda_0}\sum_{n=0}^{\infty}\left(\frac{A}{\lambda_0}\right)^n \tag{2.80}$$

converges absolutely and $(A-\lambda_0\mathbf{1})^{-1} \in \mathcal{C}$. Since $A = A^*$, the resolvent $R(\lambda) = (\lambda\mathbf{1} - A)^{-1}$ with $(R(\lambda))^* = R(\bar{\lambda})$ is normal. Furthermore $\sigma_{\mathcal{A}}((A - \lambda\mathbf{1})^{-1}) = (\sigma_{\mathcal{A}}(A) - \lambda)^{-1}$ and therefore

$$\begin{aligned}\|(A - \lambda\mathbf{1})^{-1}\| &= \rho((A - \lambda\mathbf{1})^{-1}) \\ &= \sup_{z\in\sigma_{\mathcal{A}}(A)} |(z-\lambda)^{-1}| = \left(\inf_{z\in\sigma_{\mathcal{A}}(A)} |z-\lambda|\right)^{-1} = \frac{1}{d(\lambda)}\,,\end{aligned} \tag{2.81}$$

where $d(\lambda)$ designates the closest distance from λ to $\sigma_{\mathcal{A}}(A)$. For purely imaginary λ, we have $d(\lambda) > |\lambda|$, because $0 \notin \sigma_{\mathcal{A}}(A)$ for invertible A and the compactness of the spectrum. Using this result, we can show again (as in the proof that $r_{\mathcal{A}}(A)$ is open (2.30) on page 25), that

$$\begin{aligned}\frac{1}{A - \lambda\mathbf{1}} = \frac{1}{A - \lambda_0\mathbf{1} - (\lambda - \lambda_0)\mathbf{1}} &= \frac{1}{A - \lambda_0\mathbf{1}}\frac{1}{\mathbf{1} - \frac{\lambda-\lambda_0}{A-\lambda_0\mathbf{1}}} \\ &= \frac{1}{A - \lambda_0\mathbf{1}}\sum_{n=0}^{\infty}\left(\frac{\lambda-\lambda_0}{A - \lambda_0\mathbf{1}}\right)^n\end{aligned} \tag{2.82}$$

converges for $|\lambda - \lambda_0| < d(\lambda_0) = \|(A - \lambda_0\mathbf{1})^{-1}\|^{-1}$. For purely imaginary λ_0, the series (2.82) will therefore converge for $\lambda = 0$ and

$$A^{-1} = \frac{1}{A - \lambda_0\mathbf{1}}\sum_{n=0}^{\infty}\left(\frac{-\lambda_0}{A - \lambda_0\mathbf{1}}\right)^n \in \mathcal{C}\,. \tag{2.83}$$

If A is invertible, but $A \neq A^*$, then A^*A is invertible nevertheless

$$(A^*A)^{-1} = A^{-1}\,(A^*)^{-1} = A^{-1}\left(A^{-1}\right)^* \tag{2.84}$$

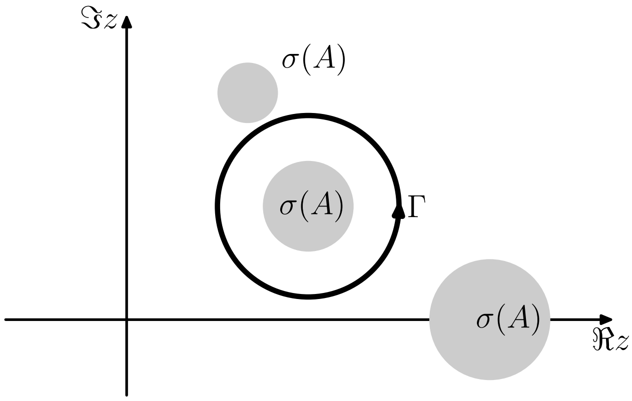

Figure 2.1: *Integration path avoiding the components of $\sigma(A)$ in (2.86).*

and $(A^*A)^{-1}$ lies in the C^*-algebra generated by $\mathbf{1}$ and A^*A. For

$$X = (A^*A)^{-1} A^* \in \mathcal{C} \tag{2.85}$$

we find $XA = \mathbf{1} = AX$, i. e. $X = A^{-1}$, i. e. $A^{-1} \in \mathcal{C}$ (cf. lemma 2.12). $\square$

Therefore, we will henceforth write $\sigma(A)$ for $\sigma_{\mathcal{A}}(A)$.

2.5 Projections

The resolvent $r(A) \ni z \mapsto (z\mathbf{1} - A)^{-1} \in \mathcal{A}$ is a very powerful object that contains more information than just the location of the spectrum.

Given $A \in \mathcal{A}$ and a closed path $\Gamma \subset r(A) \subset \mathbf{C}$, we can define an element of $\mathcal{A}$ as

$$\mathcal{A} \ni P_\Gamma^A = \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} R(z) = \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{1}{z\mathbf{1} - A}\,. \tag{2.86}$$

Note that the integral in (2.86) is well defined, because the resolvent $z \mapsto (z\mathbf{1} - A)^{-1}$ is holomorphic on $r(A)$. Such integrals can be constructed without measure theory not only for functions $\mathbf{C} \to \mathbf{C}$ but also for $\mathbf{C} \to B$, where B is an arbitrary Banach space, cf., e. g., section 9.6 of [Die68][3]

If Γ does not encircle any part of the spectrum, the integrand $z \mapsto R(z)$ is holomorphic everywhere inside of Γ and $P_\Gamma^A = 0$ by Cauchy's theorem.

In the following we will need the *resolvent identity*

[3] Or the french [Die69] and German [Die85] translations.

Remark 2.18 (Hilbert's identity).

$$\forall z_1, z_2 \in r(A) : R(z_1) - R(z_2) = (z_2 - z_1) R(z_1) R(z_2) . \tag{2.87}$$

Proof. It follows from multiplying

$$(R(z_2))^{-1} - (R(z_1))^{-1} = (z_2\mathbf{1} - A) - (z_1\mathbf{1} - A) = (z_2 - z_1)\mathbf{1} \tag{2.88}$$

by $R(z_1)$ from the left and $R(z_2)$ from the right. □

For every pair of paths $\Gamma_{1,2}$, we can compute the product

$$\begin{aligned} P^A_{\Gamma_1} P^A_{\Gamma_2} &= \int_{\Gamma_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \int_{\Gamma_2} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \frac{1}{z_1\mathbf{1} - A} \frac{1}{z_2\mathbf{1} - A} \\ &\overset{(2.87)}{=} \int_{\Gamma_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \int_{\Gamma_2} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \frac{1}{z_2 - z_1} \left(\frac{1}{z_1\mathbf{1} - A} - \frac{1}{z_2\mathbf{1} - A} \right) \\ &= \int_{\Gamma_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \frac{1}{z_1\mathbf{1} - A} \int_{\Gamma_2} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \frac{1}{z_2 - z_1} - \int_{\Gamma_2} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \frac{1}{z_2\mathbf{1} - A} \int_{\Gamma_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \frac{1}{z_2 - z_1} \\ &= \int_{\Gamma_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \frac{1}{z_1\mathbf{1} - A} w_{\Gamma_2}(z_1) + \int_{\Gamma_2} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \frac{1}{z_2\mathbf{1} - A} w_{\Gamma_1}(z_2) , \end{aligned} \tag{2.89}$$

where the *winding number* of a curve $\Gamma \subset \mathbf{C}$ relative to a point $z \in \mathbf{C}$

$$w_\Gamma(z) = \int_\Gamma \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \frac{1}{\zeta - z} \tag{2.90}$$

vanishes if z is outside of Γ. Thus, if we use only paths with winding number 1 and choose Γ_1 to lie completely inside Γ_2, or vice versa, we have either $w_{\Gamma_1}(z_2) = 1 \land w_{\Gamma_2}(z_1) = 0$ or $w_{\Gamma_2}(z_1) = 1 \land w_{\Gamma_1}(z_2) = 0$ and find

$$P^A_{\Gamma_1} P^A_{\Gamma_2} = P^A_{\Gamma_{1,2}} . \tag{2.91}$$

Since we are free to choose Γ_2 as a small deformation of Γ_1, we have derived

$$P^A_\Gamma P^A_\Gamma = P^A_\Gamma ,$$

i. e. that P^A_Γ is a projection.

OTOH, if they have relative winding number 0, then $w_{\Gamma_1}(z_2) = w_{\Gamma_2}(z_1) = 0$, i. e.

$$P^A_{\Gamma_1} P^A_{\Gamma_2} = 0 \tag{2.92}$$

and the projections are orthogonal. This includes the trivial cases when $P^A_{\Gamma_i} = 0$, because Γ does not encircle a part of the spectrum.

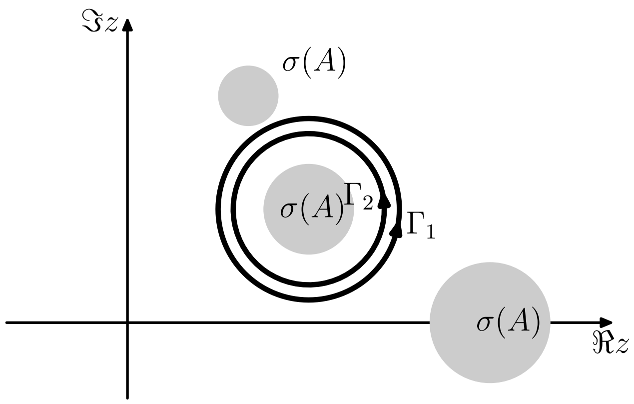

Figure 2.2: *Integration paths avoiding the components of* $\sigma(A)$ *in* (2.91).

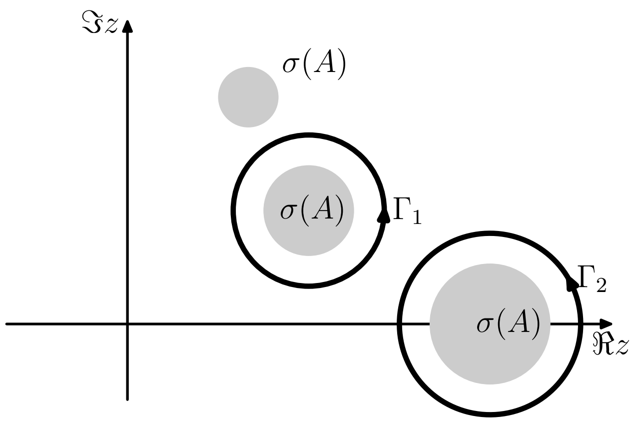

Figure 2.3: *Integration paths avoiding the components of* $\sigma(A)$ *in* (2.92).

Furthermore, if *all* of $\sigma(A)$ is enclosed by Γ, we can deform the contour and send it to infinity

$$P_\Gamma^A = \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} R(z) = \int_{|z|=r>\rho(A)} \frac{\mathrm{d}z}{2\pi\mathrm{i}} R(z) = \lim_{r\to\infty} \int_{|z|=r} \frac{\mathrm{d}z}{2\pi\mathrm{i}} R(z)$$
$$= \sum_{n=1}^{\infty} \lim_{r\to\infty} \int_{|z|=r} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{A^{n-1}}{z^n} = \sum_{n=1}^{\infty} \delta_{n,1} A^{n-1} = \mathbf{1}\,. \tag{2.93}$$

These considerations show that for every path Γ with winding number 1 the element

$$P_\Gamma^A = \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{1}{z\mathbf{1} - A} \tag{2.94}$$

can be interpreted as a projection on the part of the spectrum enclosed by Γ.

Indeed, if z_0 is an *isolated* point in the spectrum and Γ encircles only z_0, one can show that

$$(z_0\mathbf{1} - A)P_\Gamma^A = 0\,,$$

because $z_0\mathbf{1} - A$ "cancels the pole" of the resolvent at z_0. However, instead of proving this special case, we will consider a more general functional calculus now.

2.6 Holomorphic Functional Calculus

Lecture 08: Wed, 05.11.2025

For any *holomorphic* function $f : \mathbf{C} \to \mathbf{C}$, we can define a corresponding function $\hat f : \mathcal{A} \to \mathcal{A}$ via

$$\hat f(A) = \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{f(z)}{z\mathbf{1} - A} \tag{2.95}$$

where Γ encircles *all* of $\sigma(A)$ with winding number one. We can compute the product of two functions, choosing Γ' on the *outside* of Γ:

$$\begin{aligned}
\hat f_1(A)\hat f_2(A) &= \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \int_{\Gamma'} \frac{\mathrm{d}z'}{2\pi\mathrm{i}} \frac{f_1(z)}{z\mathbf{1} - A} \frac{f_2(z')}{z'\mathbf{1} - A} \\
&= \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \int_{\Gamma'} \frac{\mathrm{d}z'}{2\pi\mathrm{i}} \frac{f_1(z)f_2(z')}{z' - z} \left(\frac{1}{z\mathbf{1} - A} - \frac{1}{z'\mathbf{1} - A} \right) \\
&= \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{f_1(z)}{z\mathbf{1} - A} \underbrace{\int_{\Gamma'} \frac{\mathrm{d}z'}{2\pi\mathrm{i}} \frac{f_2(z')}{z' - z}}_{=f_2(z)} - \int_{\Gamma'} \frac{\mathrm{d}z'}{2\pi\mathrm{i}} \frac{f_2(z')}{z'\mathbf{1} - A} \underbrace{\int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{f_1(z)}{z' - z}}_{=0} \\
&= \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{f_1(z)}{z\mathbf{1} - A} f_2(z) = \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{(f_1 \cdot f_2)(z)}{z\mathbf{1} - A} = (\widehat{f_1 \cdot f_2})(A)\,.
\end{aligned} \tag{2.96}$$

i. e. $f \mapsto \hat f$ is an *homomorphism*

$$\hat f_1(A)\hat f_2(A) = (\widehat{f_1 \cdot f_2})(A)\,. \tag{2.97}$$

In physicist's notation, an analogous functional calculus for self-adjoint operators on a Hilbert space is written

$$\hat f(A) = \int \mathrm{d}\mu(a)\, f(a)\, |a\rangle\langle a| \tag{2.98}$$

with

$$\hat{f}_1(A)\hat{f}_2(A) = \int \mathrm{d}\mu(a) \int \mathrm{d}\mu(a')\, f_1(a) f_2(a')\, |a\rangle\langle a|a'\rangle\langle a'| \\ = \int \mathrm{d}\mu(a)\, (f_1 \cdot f_2)(a)\, |a\rangle\langle a| = (\widehat{f_1 \cdot f_2})(A)\,. \tag{2.99}$$

2.6.1 Examples

1. For consistency, we should have $\hat{f}(\lambda\mathbf{1}) = f(\lambda)\mathbf{1}$. Indeed

$$\hat{f}(\lambda\mathbf{1}) = \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{f(z)}{z\mathbf{1} - \lambda\mathbf{1}} = \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{f(z)}{z - \lambda}\,\mathbf{1} = f(\lambda)\,\mathbf{1}\,. \tag{2.100}$$

2. Also $\widehat{\mathrm{id}} : A \to A$ for $\mathrm{id} : z \to z$

$$\widehat{\mathrm{id}}(A) = \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \underbrace{\frac{z}{z\mathbf{1} - A}}_{= \mathbf{1} + A\frac{1}{z\mathbf{1} - A}} = \mathbf{1} \underbrace{\int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}}}_{=0} + A \underbrace{\int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{1}{z\mathbf{1} - A}}_{= P_\Gamma^A = 1} = A\,, \tag{2.101}$$

and $\widehat{\mathrm{sq}} : A \to A^2$ for $\mathrm{sq} : z \to z^2$

$$\widehat{\mathrm{sq}}(A) = \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \underbrace{\frac{z^2}{z\mathbf{1} - A}}_{= (z\mathbf{1} - A) + 2A + A^2 \frac{1}{z\mathbf{1} - A}} \\ = \underbrace{\int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} (z\mathbf{1} - A)}_{=0} + 2A \underbrace{\int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}}}_{=0} + A^2 \underbrace{\int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{1}{z\mathbf{1} - A}}_{=1} = A^2\,. \tag{2.102}$$

Obviously, this holds analogously for higher powers and all polynomials.

2.7 Holomorphic Spectral Mapping Theorem

The *spectral mapping theorem* for polynomials $\sigma(P(A)) = P(\sigma(A))$ holds for all $A \in \mathcal{A}$. For normal elements one can prove a stronger statement

$$\sigma(f(A)) = f(\sigma(A)) \tag{2.103}$$

that holds for continuous functions f. We don't have the tools yet to prove it, but the crucial ingredient ist that the $*$-algebra generated by a $A \in \mathcal{A}$ is abelian iff $AA^* = A^*A$. In this case we could have used theorem 1.14.

Here, we will prove a different spectral mapping theorem for all $A \in \mathcal{A}$, that suffices in applications where the function is analytic or can be approximated by analytic functions.

Theorem 2.19. *If $f : \mathbf{C} \to \mathbf{C}$ is holomorphic on a neighborhood of $\sigma(A)$, then $f(\sigma(A)) = \sigma(\hat{f}(A))$, with $\hat{f}$ from (2.95).*

Proof. For all $\lambda \in \mathbf{C}$, $f_\lambda : z \mapsto \lambda - f(z)$ is holomorphic on a neighborhood of $\sigma(A)$.

- If $\lambda \notin f(\sigma(A))$, then $h_1 : z \mapsto (f_\lambda(z))^{-1} = (\lambda - f(z))^{-1}$ is also holomorphic on a neighborhood of $\sigma(A)$. Using (2.97) and the function $1 : z \mapsto 1$, we find

$$\begin{aligned}\left(\lambda\mathbf{1} - \hat{f}(A)\right)\hat{h}_1(A) &= \hat{f}_\lambda(A)\hat{h}_1(A) = (\widehat{f_\lambda \cdot h_1})(A) = \hat{1}(A) = \mathbf{1} \\ &= (\widehat{h_1 \cdot f_\lambda})(A) = \hat{h}_1(A)\hat{f}_\lambda(A) = \hat{h}_1(A)\left(\lambda\mathbf{1} - \hat{f}(A)\right)\end{aligned} \tag{2.104}$$

  that an inverse of $\lambda\mathbf{1} - \hat{f}(A)$ exists, i. e. $\lambda \notin \sigma(\hat{f}(A))$ and $f(\sigma(A)) \supseteq \sigma(\hat{f}(A))$.

- If $\lambda \in f(\sigma(A))$, then $\exists z_0 \in \sigma(A)$ with $\lambda = f(z_0)$ and there is a function h_2 holomorphic on a neighborhood of $\sigma(A)$ with

$$\lambda - f(z) = f(z_0) - f(z) = (z_0 - z)h_2(z)\,. \tag{2.105}$$

  Using (2.97) again, we can compute

$$\lambda\mathbf{1} - \hat{f}(A) = (z_0\mathbf{1} - A)\hat{h}_2(A) = \hat{h}_2(A)(z_0\mathbf{1} - A)\,. \tag{2.106}$$

  Since $z_0\mathbf{1} - A$ is by assymption not invertible, $\lambda\mathbf{1} - \hat{f}(A)$ can also not be invertible, i. e. $\lambda \in \sigma(\hat{f}(A))$ and $f(\sigma(A)) \subseteq \sigma(\hat{f}(A))$.

The two inclusions are complementary, therefore $f(\sigma(A)) = \sigma(\hat{f}(A))$. □

Having established that f maps the resolvent set of A into the resolvent set of $\hat{f}(A)$, we can use the holomorphic functional calculus to compose maps:

Corollary 2.20.

$$\hat{f} \circ \hat{g} = \widehat{f \circ g} \tag{2.107}$$

Proof. With Γ_1 and Γ_2 chosen such that Γ_2 encircles $g(\Gamma_1)$ once, we have

$$\begin{aligned}\widehat{(f\circ g)}(A) &= \int_{\Gamma_1}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}\frac{(f\circ g)(z_1)}{z_1\mathbf{1}-A} = \int_{\Gamma_1}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}\overbrace{\int_{\Gamma_2}\frac{\mathrm{d}z_2}{2\pi\mathrm{i}}\frac{f(z_2)}{z_2-g(z_1)}}^{f(g(z_1))}\frac{1}{z_1\mathbf{1}-A} \\ &= \int_{\Gamma_2}\frac{\mathrm{d}z_2}{2\pi\mathrm{i}}f(z_2)\int_{\Gamma_1}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}\frac{1}{z_2-g(z_1)}\frac{1}{z_1\mathbf{1}-A} \\ &= \int_{\Gamma_2}\frac{\mathrm{d}z_2}{2\pi\mathrm{i}}f(z_2)\frac{1}{z_2\mathbf{1}-\hat{g}(A)} = \hat{f}(\hat{g}(A)) = (\hat{f}\circ\hat{g})(A)\,, \end{aligned} \tag{2.108}$$

where we have used the holomorphic functional calculus for the family of functions

$$h_\lambda : z \mapsto \frac{1}{\lambda - z} \tag{2.109}$$

that involve only addition and pointwise inverse for which we already know from (2.97) that the holomorphic functional calculus is valid.

□

2.8 Positive Elements

Earlier, we have defined the positive elements A of an algebra as those that can be written as $A = B^*B$. One can equivalently characterize them by their spectrum:

Definition 2.21. The set $\mathcal{A}_+$ of all positive elements of a $*$-algebra $\mathcal{A}$ is the set of all self-adjoint elements with the spectrum $\sigma(A)$ contained in the positive real axis.

For a positive element $A \in \mathcal{A}_+$, we can construct a *square root* with

$$B = \sqrt{A} = \int_0^\infty \frac{\mathrm{d}\lambda}{\pi}\frac{A}{\sqrt{\lambda}}\frac{1}{\lambda\mathbf{1}+A}\,. \tag{2.110}$$

For invertible A, this construction is compatible with the holomorphic functional calculus. In order to avoid a contribution from the circle at infinity, we construct the inverse and define $B = \sqrt{A} = A\sqrt{A^{-1}}$ later. This works indeed using the contour Γ from figure 2.4 which avoids the cut of $z \to \sqrt{z}$ and encircles $\sigma(A) \subset (0,\infty)$.

$$\begin{aligned}\sqrt{A^{-1}} &= \int_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{1}{\sqrt{z}}\frac{1}{z\mathbf{1}-A} \\ &= \lim_{\epsilon\to 0+}\int_0^{-\infty}\frac{\mathrm{d}x}{2\pi\mathrm{i}}\frac{1}{\sqrt{x-\mathrm{i}\epsilon}}\frac{1}{(x-\mathrm{i}\epsilon)\mathbf{1}-A} + \lim_{\epsilon\to 0+}\int_{-\infty}^0\frac{\mathrm{d}x}{2\pi\mathrm{i}}\frac{1}{\sqrt{x+\mathrm{i}\epsilon}}\frac{1}{(x+\mathrm{i}\epsilon)\mathbf{1}-A}\end{aligned}$$

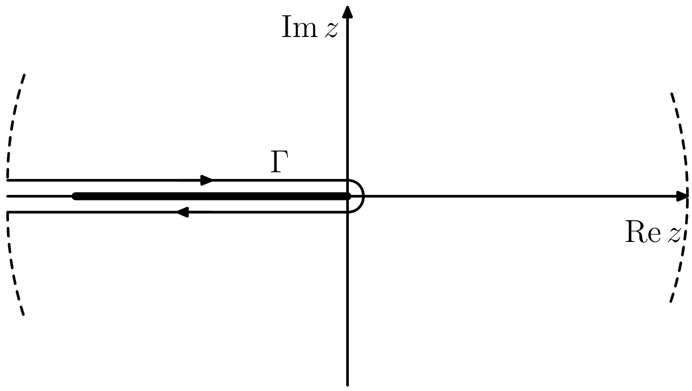

Figure 2.4: *Integration contour* Γ *used in* (2.111) *to define the square root.*

$$
\begin{aligned}
&= \lim_{\epsilon\to 0+} \int_0^{-\infty} \frac{\mathrm{d}x}{2\pi\mathrm{i}} \left(\frac{1}{\sqrt{x-\mathrm{i}\epsilon}} \frac{1}{(x-\mathrm{i}\epsilon)\mathbf{1} - A} - \frac{1}{\sqrt{x+\mathrm{i}\epsilon}} \frac{1}{(x+\mathrm{i}\epsilon)\mathbf{1} - A} \right) \\
&\stackrel{(2.113)}{=} \int_0^{-\infty} \frac{\mathrm{d}x}{\pi} \frac{1}{\sqrt{|x|}} \frac{1}{x\mathbf{1} - A} = \int_0^{\infty} \frac{\mathrm{d}x}{\pi} \frac{1}{\sqrt{x}} \frac{1}{x\mathbf{1} + A} \,, \qquad (2.111)
\end{aligned}
$$

where we have used

$$
\lim_{\epsilon\to 0+} \sqrt{-|x| \pm \mathrm{i}\epsilon} = \pm\mathrm{i}\sqrt{|x|} \qquad (2.112)
$$

and

$$
\begin{aligned}
&\frac{1}{\sqrt{x-\mathrm{i}\epsilon}} \frac{1}{(x-\mathrm{i}\epsilon)\mathbf{1} - A} - \frac{1}{\sqrt{x+\mathrm{i}\epsilon}} \frac{1}{(x+\mathrm{i}\epsilon)\mathbf{1} - A} \\
&\qquad = \frac{\mathrm{i}}{\sqrt{|x|}} \frac{1}{x\mathbf{1} - A - \mathrm{i}\epsilon\mathbf{1}} + \frac{\mathrm{i}}{\sqrt{|x|}} \frac{1}{x\mathbf{1} - A + \mathrm{i}\epsilon\mathbf{1}} \\
&\qquad = \frac{2\mathrm{i}}{\sqrt{|x|}} \frac{x\mathbf{1} - A}{(x\mathbf{1} - A)^2 + \epsilon^2\mathbf{1}} \xrightarrow{\epsilon\to 0+} \frac{2\mathrm{i}}{\sqrt{|x|}} \frac{1}{x\mathbf{1} - A} \,. \qquad (2.113)
\end{aligned}
$$

Note that (2.110) can also be constructed as a Riemannian integral. For positive A we have, from the fact that the function

$$
\begin{aligned}
f : [0,\infty) &\to [0,1) \\
z &\mapsto \frac{z}{\lambda + z}
\end{aligned} \qquad (2.114)
$$

is a bijection for $\lambda > 0$,

$$
\sigma\left(A \frac{1}{\lambda\mathbf{1} + A}\right) = f(\sigma(A)) \subseteq f([0, \|A\|]) = \left[0, \frac{\|A\|}{\lambda + \|A\|}\right] \,. \qquad (2.115)
$$

Therefore

$$\left\| A\frac{1}{\lambda\mathbf{1}+A}\right\| = \rho\left(A\frac{1}{\lambda\mathbf{1}+A}\right) \leqslant \frac{\|A\|}{\lambda+\|A\|} \tag{2.116}$$

and

$$\begin{aligned}\|B\| &= \left\|\int_0^\infty \frac{\mathrm{d}x}{\pi}\frac{A}{\sqrt{x}}\frac{1}{x\mathbf{1}+A}\right\| \leqslant \int_0^\infty \frac{\mathrm{d}x}{\pi}\left\|\frac{A}{\sqrt{x}}\frac{1}{x\mathbf{1}+A}\right\| \\ &\overset{(2.116)}{\leqslant} \int_0^\infty \frac{\mathrm{d}x}{\pi}\frac{1}{\sqrt{x}}\frac{\|A\|}{x+\|A\|} = \sqrt{\|A\|}\underbrace{\int_0^\infty \frac{\mathrm{d}y}{\pi}\frac{1}{\sqrt{y}}\frac{1}{y+1}}_{=1} = \sqrt{\|A\|}\end{aligned} \tag{2.117}$$

converges. One can show

Theorem 2.22. *A self-adjoint element A of a C^*-algebra $\mathcal{A}$ is positive iff $\exists B \in \mathcal{A} : A = B^2 \wedge B = B^*$. If A is positive, there exists a* unique *positive B with $A = B^2$ and it lies in the subalgebra of $\mathcal{A}$ generated by A.*

We will not give the proof here (see, e. g., [BR02], theorem 2.2.10), but it should be intuitively clear, that the integrand in (2.110) is positive and the corresponding Riemann sums are positive as well.

The square root plays a special role, because we can use it to define the *modulus* of an arbitrary self-adjoint $A \in \mathcal{A}$ via

$$|A| = \sqrt{A^2} \in \mathcal{A}\,. \tag{2.118}$$

Note that $\|A\| \in \mathbf{R}$, but $|A| \in \mathcal{A}$.

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One can use the modulus to write every self-adjoint $A \in \mathcal{A}$ as the difference of two positive elements:

Theorem 2.23. *The set $\mathcal{A}_+$ of all positive elements of a $*$-algebra $\mathcal{A}$ is a convex cone that is closed in the norm topology, with*

$$\mathcal{A}_+ \cap (-\mathcal{A}_+) = \{0\}\,. \tag{2.119}$$

If $A = A^* \in \mathcal{A}$, then

$$A_\pm = \frac{1}{2}\left(|A| \pm A\right)\,, \tag{2.120}$$

are the unique *elements of $\mathcal{A}$ with*

$$A_\pm \in \mathcal{A}_+ \tag{2.121a}$$
$$A = A_+ - A_- \tag{2.121b}$$
$$A_+ A_- = 0\,. \tag{2.121c}$$

Most of the proof will be skipped (see, e. g., [BR02], theorem 2.2.11). However the positivity of $A_\pm$, i. e. (2.121a), is intuitively clear, while (2.121b) is trivial and (2.121c) follows from

$$4A_+A_- = (|A| + A)(|A| - A) = |A|^2 + A|A| - |A|A - A^2 = [A, |A|] = 0 \tag{2.122}$$

because $|A|$ is an element of the abelian Algebra generated by A.

Finally, one can also show the reverse direction of theorem 2.22 (see, e. g., [BR02], theorem 2.2.12)

Theorem 2.24. *If $\mathcal{A}$ is a C^*-algebra, then*

$$\forall A \in \mathcal{A} : (A \in \mathcal{A}_+) \Leftrightarrow (\exists B \in \mathcal{A} : A = B^*B) , \tag{2.123}$$

where we don't require B to be self-adjoint.

We can use the existence of positive elements to introduce a partial order on an algebra

Definition 2.25.

$$A \geqslant B \Leftrightarrow A - B \geqslant 0 \Leftrightarrow A - B \in \mathcal{A}_+ \tag{2.124a}$$
$$A > B \Leftrightarrow A \geqslant B \wedge A \neq B \, . \tag{2.124b}$$

Obviously

$$A \geqslant 0 \wedge A \leqslant 0 \Rightarrow A = 0 \tag{2.125a}$$
$$A \geqslant B \wedge B \geqslant C \Rightarrow A \geqslant C \, , \tag{2.125b}$$

but also

Theorem 2.26. $\forall A, B, C \in \mathcal{A}$, *a C^*-algebra:*

$$A \geqslant B \geqslant 0 \Rightarrow \|A\| \geqslant \|B\| \tag{2.126a}$$
$$A \geqslant 0 \Rightarrow A\|A\| \geqslant A^2 \tag{2.126b}$$
$$A \geqslant B \geqslant 0 \Rightarrow \forall C \in \mathcal{A} : C^*AC \geqslant C^*BC \geqslant 0 \tag{2.126c}$$
$$A \geqslant B \geqslant 0 \wedge \lambda > 0 \Rightarrow \frac{1}{B + \lambda \mathbf{1}} \geqslant \frac{1}{A + \lambda \mathbf{1}} \tag{2.126d}$$

Proof. First we observe, using $\rho(A) = \|A\|$

$$\min \sigma (\|A\|\mathbf{1} - A) = \|A\| - \max \sigma (A) = \|A\| - \|A\| = 0$$
$$\Rightarrow \|A\|\mathbf{1} - A \geqslant 0 \Rightarrow B \leqslant A \leqslant \|A\|\mathbf{1} \tag{2.127}$$

and then

$$0 \leqslant \min \sigma \left(\|A\| \mathbf{1} - B \right) = \|A\| - \max \sigma \left(B \right) = \|A\| - \|B\| \\ \Rightarrow \|A\| \geqslant \|B\| \,, \tag{2.128}$$

i. e. (2.126a). Similarly

$$\sigma \left(A - \|A\| \mathbf{1}/2 \right) = \sigma \left(A \right) - \|A\|/2 \subseteq [-\|A\|/2, \|A\|/2] \tag{2.129}$$

and therefore

$$\sigma \left((A - \|A\| \mathbf{1}/2)^2 \right) \subseteq [0, \|A\|^2/4] \,. \tag{2.130}$$

This means

$$0 \leqslant (A - \|A\| \mathbf{1}/2)^2 \leqslant \|A\|^2 \mathbf{1}/4 \tag{2.131}$$

i. e. (2.126b)

$$A^2 - A\|A\| \leqslant 0 \,. \tag{2.132}$$

We obtain (2.126c) as follows

$$A \geqslant B \Rightarrow \exists D \in \mathcal{A} : A - B = D^* D \Rightarrow \\ C^* A C - C^* B C = C^* (A - B) C = C^* D^* D C = (DC)^* (DC) \geqslant 0 \,. \tag{2.133}$$

And from this (2.126d)

$$A \geqslant B \geqslant 0 \Rightarrow \forall \lambda : A + \lambda \mathbf{1} \geqslant B + \lambda \mathbf{1} \geqslant \lambda \mathbf{1} \Rightarrow \\ \forall \lambda > 0 : \frac{1}{\sqrt{B + \lambda \mathbf{1}}} (A + \lambda \mathbf{1}) \frac{1}{\sqrt{B + \lambda \mathbf{1}}} \geqslant \mathbf{1} \\ \Rightarrow \forall \lambda > 0 : \sqrt{B + \lambda \mathbf{1}} \frac{1}{A + \lambda \mathbf{1}} \sqrt{B + \lambda \mathbf{1}} \leqslant \mathbf{1} \\ \Rightarrow \forall \lambda > 0 : \frac{1}{A + \lambda \mathbf{1}} \leqslant \frac{1}{B + \lambda \mathbf{1}} \tag{2.134}$$

where we have used

$$X = X^* \geqslant \mathbf{1} \Rightarrow \sigma(X) = \sigma(X - \mathbf{1}) + 1 \subseteq [0, \infty) + 1 = [1, \infty) \\ \Rightarrow \sigma(X^{-1}) \subseteq [1, \infty)^{-1} = (0, 1] \Rightarrow X^{-1} \leqslant \mathbf{1} \tag{2.135}$$

in the next to final implication. □

—3—
Representations

In order to make contact with the Hilbert space formulation of QM, we need to realize the algebraic formulation as operators on Hilbert space. In particular, we have to find out iff there is more than one inequivalent realization. Otherwise, the algebraic formulation will not have given us a lot of new insights.

3.1 Homomorphisms

Definition 3.1 (homomorphism of $*$-algebras)**.** A $*$-*homomorphism* π is a map

$$\begin{aligned} \pi : \mathcal{A} &\to \mathcal{B} \\ A &\mapsto \pi(A) \end{aligned} \tag{3.1}$$

that preserves the $*$-algebra structure, i. e. for all $\alpha, \beta \in \mathbf{C}$ and $A, B \in \mathcal{A}$

$$\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B) \tag{3.2a}$$

$$\pi(AB) = \pi(A)\pi(B) \tag{3.2b}$$

$$\pi(A^*) = (\pi(A))^* \,. \tag{3.2c}$$

Remark 3.2. A $*$-*homomorphism* π preserves positivity, i. e.

$$\forall A \geqslant 0 : \pi(A) \geqslant 0 \,. \tag{3.3}$$

Proof. This is almost obvious:

$$\begin{aligned} &\mathcal{A} \ni A \geqslant 0 \Rightarrow \exists B \in \mathcal{A} : A = B^*B \\ &\qquad\qquad \Rightarrow \pi(A) = \pi(B^*)\pi(B) = (\pi(B))^* \pi(B) \geqslant 0 \,. \end{aligned} \tag{3.4}$$

□

Definition 3.3. The *kernel* of a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ is defined as

$$\ker \pi = \{A \in \mathcal{A} : \pi(A) = 0\} \tag{3.5}$$

Remark 3.4. $\ker \pi$ is a two-sided $*$-ideal.

Proof. $\forall A \in \mathcal{A}, B \in \ker \pi$:

$$\pi(AB) = \pi(A)\pi(B) = \pi(A)\,0 = 0 \tag{3.6a}$$
$$\pi(BA) = \pi(B)\pi(A) = 0\,\pi(A) = 0 \tag{3.6b}$$
$$\pi(B^*) = (\pi(B))^* = 0^* = 0 \tag{3.6c}$$

□

Therefore, $\mathcal{A}/\ker \pi$ is a $*$-algebra and the map

Definition 3.5.

$$\begin{aligned} \hat\pi : \mathcal{A}_\pi = \mathcal{A}/\ker \pi \to \mathcal{B}_\pi = \operatorname{Im} \pi = \pi(\mathcal{A}) \\ \hat A \mapsto \pi(A \in \hat A) \end{aligned} \tag{3.7}$$

is well defined

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\pi"] \arrow[d, "\hat{}"'] & \mathcal{B}_\pi \subseteq \mathcal{B} \\ \mathcal{A}_\pi \arrow[ur, "\hat\pi"'] & \end{tikzcd} \tag{3.8}$$

because

$$\pi(\hat A) = \pi(A + I) = \pi(A) + \pi(I) = \pi(A) + 0 = \pi(A)\,, \tag{3.9}$$

and an isomorphism, because $\ker \hat\pi = \hat 0$, by construction.

3.2 Approximate Identities and Factor Algebras

While we have shown in theorem 2.1 that one can always adjoin an identity, it is sometimes relevant that the original algebra does not contain one. However, we want to show that certain factor algebras that appear naturally, i. e. $\mathcal{A}/\ker \pi$, carry a C^*-algebra structure. In order to prove this, we need to introduce approximate identities of the ideal $\ker \pi$. In some way, approximate identities resemble the representations of Dirac δ-distributions as the limit of a sequence of functions in a given function space, while the Dirac δ-distributions themselves are not in this function space.

Definition 3.6. If $\mathcal{I}$ is a right ideal of a C^*-algebra, an *approximate identity* of $\mathcal{I}$ is a family[1] $\{E_\alpha\}_{\alpha\in U}$ of *positive* elements $E_\alpha \in \mathcal{I}$, indexed by a *directed set*[2] U with the properties

$$\|E_\alpha\| \leqslant 1 \tag{3.10a}$$
$$\forall \alpha \leqslant \beta \in U : E_\alpha \leqslant E_\beta \tag{3.10b}$$
$$\forall I \in \mathcal{I} : \|E_\alpha I - I\| \to 0 \,. \tag{3.10c}$$

An approximate identity of a left ideal is defined analogously with $\|IE_\alpha - I\| \to 0$.

This definition is useful because we can show

Theorem 3.7. *Every right ideal $\mathcal{I}$ of a C^*-algebra $\mathcal{A}$ possesses an approximate identity.*

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Proof. If $\mathcal{A}$ is not unital, we can adjoin an identity as in definition 2.2 for the purpose of the proof and use (2.8), i. e. $\|(0, A)\|_{\bar{\mathcal{A}}} = \|A\|_{\mathcal{A}}$ where needed. We observe that $\mathcal{I}$ is also a right ideal in $\bar{\mathcal{A}}$:

$$\forall I \in \mathcal{I}, (\alpha, A) \in \bar{\mathcal{A}} : (0, I)(\alpha, A) = (0, \alpha I + IA) \in \mathcal{I} \tag{3.11}$$

if we identify $\mathcal{I}$ with the canonical injection $\mathcal{A} \supseteq \mathcal{I} \cong \{(0, I) : I \in \mathcal{I}\} \subseteq \bar{\mathcal{A}}$.

We can partially order the set $\mathcal{U}$ of all *finite* families $\alpha = \{A_1, A_2, \ldots, A_{|\alpha|}\}$ with $A_i \in \mathcal{I}$ by inclusion, i. e. $\alpha \geqslant \beta$ iff β is a subfamily of α. We can define for every family α the positive algebra elements

$$\mathcal{I} \ni F_\alpha = \sum_{i=1}^{|\alpha|} A_i A_i^* \geqslant 0 \tag{3.12}$$

and

$$\mathcal{I} \ni E_\alpha = |\alpha| F_\alpha \frac{1}{\mathbf{1} + |\alpha| F_\alpha} = \mathbf{1} - \frac{1}{\mathbf{1} + |\alpha| F_\alpha} \leqslant \mathbf{1} \tag{3.13}$$

since $(\mathbf{1} + |\alpha| F_\alpha)^{-1} \in \bar{\mathcal{A}}$ and $\mathcal{I}$ is a right ideal in $\bar{\mathcal{A}}$. Obviously $\|E_\alpha\| \leqslant 1$, but also

[1] Such a family $\{E_\alpha\}_{\alpha\in U}$ is called a *net*.

[2] A directed set is a nonempty set with a preorder $\leqslant$ such that every pair of elements $a, b \in U$ has an *upper bound* $c \in U$: $a \leqslant c \wedge b \leqslant c$.

$$\begin{aligned}
\forall A_i \in \alpha : {}& (E_\alpha A_i - A_i)(E_\alpha A_i - A_i)^* \\
&= (E_\alpha - \mathbf{1}) A_i A_i^* (E_\alpha - \mathbf{1}) = (\mathbf{1} - E_\alpha) A_i A_i^* (\mathbf{1} - E_\alpha) \\
&\leqslant \sum_{i=1}^{|\alpha|} (\mathbf{1} - E_\alpha) A_i A_i^* (\mathbf{1} - E_\alpha) = \frac{1}{\mathbf{1} + |\alpha| F_\alpha} F_\alpha \frac{1}{\mathbf{1} + |\alpha| F_\alpha} \\
&= \sqrt{F_\alpha} \left(\frac{1}{\mathbf{1} + |\alpha| F_\alpha} \right)^2 \sqrt{F_\alpha} \overset{(2.126c)}{\leqslant} \sqrt{F_\alpha} \frac{1}{\mathbf{1} + |\alpha| F_\alpha} \sqrt{F_\alpha} \\
&= \frac{1}{|\alpha|} \left(\mathbf{1} - \frac{1}{\mathbf{1} + |\alpha| F_\alpha} \right) \leqslant \frac{1}{|\alpha|} \mathbf{1} \,,
\end{aligned} \tag{3.14}$$

i. e.

$$\| E_\alpha A_i - A_i \|^2 \leqslant \frac{1}{|\alpha|} \tag{3.15}$$

and since every $A \in \mathcal{I}$ belongs to some family

$$\forall A \in \mathcal{I} : \| E_\alpha A - A \| \to 0 \,. \tag{3.16}$$

Finally

$$E_\alpha - E_\beta = \frac{1}{\mathbf{1} + |\beta| F_\beta} - \frac{1}{\mathbf{1} + |\alpha| F_\alpha} \tag{3.17}$$

and $\alpha \geqslant \beta$ implies $|\alpha| F_\alpha \geqslant |\beta| F_\beta$ and $E_\alpha \geqslant E_\beta$. □

Theorem 3.8. *Every closed two sided ideal $\mathcal{I}$ of a C^*-algebra $\mathcal{A}$ is self-adjoint and the factor algebra $\mathcal{A}/\mathcal{I}$ with the norm*

$$\| \hat{A} \|_{\mathcal{A}/\mathcal{I}} = \inf_{I \in \mathcal{I}} \| A + I \|_{\mathcal{A}} \tag{3.18}$$

is a C^*-algebra.

Proof. If $\{E_\alpha\}_{\alpha \in U}$ is an approximate identity of $\mathcal{I}$, then

$$\forall A \in \mathcal{I} : \| A^* E_\alpha - A^* \|_{\mathcal{A}} = \| E_\alpha A - A \|_{\mathcal{A}} \to 0 \,. \tag{3.19}$$

Since $A^* E_\alpha \in \mathcal{I}$ and $\mathcal{I}$ is closed, we have $A^* \in \mathcal{I}$. Again, if $\mathcal{A}$ is not unital, we can adjoin an identity as in definition 2.2 to be able to write

$$A - E_\alpha A + I - E_\alpha I = (\mathbf{1} - E_\alpha)(A + I) \tag{3.20}$$

and find

$$\begin{aligned}
\limsup_\alpha \| A - E_\alpha A \|_{\mathcal{A}} &= \limsup_\alpha \| A - E_\alpha A \|_{\bar{\mathcal{A}}} \\
&= \limsup_\alpha \| A - E_\alpha A + I - E_\alpha I \|_{\bar{\mathcal{A}}} = \limsup_\alpha \| (\mathbf{1} - E_\alpha)(A + I) \|_{\bar{\mathcal{A}}}
\end{aligned}$$

$$\leqslant \limsup_{\alpha} \|\mathbf{1} - E_\alpha\|_{\bar{\mathcal{A}}} \|A + I\|_{\bar{\mathcal{A}}} \leqslant \|A + I\|_{\bar{\mathcal{A}}} = \|A + I\|_{\mathcal{A}} \,, \tag{3.21}$$

because $0 \leqslant E_\alpha \leqslant \mathbf{1}$, or $\sigma(\mathbf{1} - E_\alpha) \subseteq [0, 1]$. Therefore

$$\begin{aligned} \|\hat{A}\|_{\mathcal{A}/\mathcal{I}} &= \inf_{I \in \mathcal{I}} \|A + I\|_{\mathcal{A}} \geqslant \limsup_{\alpha} \|A - E_\alpha A\|_{\mathcal{A}} \\ &\geqslant \liminf_{\alpha} \|A - \underbrace{E_\alpha A}_{\in \mathcal{I}}\|_{\mathcal{A}} \geqslant \inf_{I \in \mathcal{I}} \|A + I\|_{\mathcal{A}} = \|\hat{A}\|_{\mathcal{A}/\mathcal{I}} \,, \end{aligned} \tag{3.22}$$

i. e.

$$\|\hat{A}\|_{\mathcal{A}/\mathcal{I}} = \lim_{\alpha} \|A - E_\alpha A\|_{\mathcal{A}} \tag{3.23}$$

and we get the C^*-property on $\mathcal{A}/\mathcal{I}$ from the C^*-property on $\mathcal{A}$

$$\begin{aligned} \|\hat{A}\|^2_{\mathcal{A}/\mathcal{I}} &= \lim_{\alpha} \|A - E_\alpha A\|^2_{\mathcal{A}} = \lim_{\alpha} \|A - E_\alpha A\|^2_{\bar{\mathcal{A}}} \\ &= \lim_{\alpha} \|(A - E_\alpha A)(A - E_\alpha A)^*\|_{\bar{\mathcal{A}}} = \lim_{\alpha} \|(\mathbf{1} - E_\alpha) A A^* (\mathbf{1} - E_\alpha)\|_{\bar{\mathcal{A}}} \\ &= \lim_{\alpha} \|(\mathbf{1} - E_\alpha) A A^* (\mathbf{1} - E_\alpha) + (\mathbf{1} - E_\alpha) I (\mathbf{1} - E_\alpha)\|_{\bar{\mathcal{A}}} \\ &= \lim_{\alpha} \|(\mathbf{1} - E_\alpha)(A A^* + I)(\mathbf{1} - E_\alpha)\|_{\bar{\mathcal{A}}} \\ &\leqslant \|A A^* + I\|_{\bar{\mathcal{A}}} = \|A A^* + I\|_{\mathcal{A}} \end{aligned} \tag{3.24}$$

for arbitrary $I \in \mathcal{I}$. Thus

$$\|\hat{A}\|^2_{\mathcal{A}/\mathcal{I}} \leqslant \inf_{I \in \mathcal{I}} \|A A^* + I\|_{\mathcal{A}} = \|\hat{A}\hat{A}^*\|_{\mathcal{A}/\mathcal{I}} \leqslant \|\hat{A}\|_{\mathcal{A}/\mathcal{I}} \|\hat{A}^*\|_{\mathcal{A}/\mathcal{I}} \tag{3.25}$$

i. e.

$$\|\hat{A}\|_{\mathcal{A}/\mathcal{I}} \leqslant \|\hat{A}^*\|_{\mathcal{A}/\mathcal{I}} \tag{3.26}$$

and

$$\|\hat{A}^*\|_{\mathcal{A}/\mathcal{I}} \leqslant \|\hat{A}\|_{\mathcal{A}/\mathcal{I}} \tag{3.27}$$

from replacing $\hat{A}$ by $\hat{A}^*$ in the above argument. This implies the C^*-property

$$\|\hat{A}\|^2_{\mathcal{A}/\mathcal{I}} = \|\hat{A}\hat{A}^*\|_{\mathcal{A}/\mathcal{I}} \,. \tag{3.28}$$

$\square$

3.3 Continuity

Using this theorem, we can prove a non-obvious result, with profound consequences:

Theorem 3.9. *Let $\mathcal{A}$ be a unital Banach $*$-algebra and $\mathcal{B}$ a C^*-algebra and $\pi : \mathcal{A} \to \mathcal{B}$ a $*$-homomorphism. Then π is continuous and even*

$$\forall A \in \mathcal{A} : \|\pi(A)\| \leqslant \|A\| . \tag{3.29}$$

In addition, if $\mathcal{A}$ is a C^*-algebra, then $\mathcal{B}_\pi = \pi(\mathcal{A}) \subseteq \mathcal{B}$ is a C^*-subalgebra of $\mathcal{B}$.

Proof. For self-adjoint A, we know that

$$\|\pi(A)\| = \rho(\pi(A)) = \sup_{\lambda \in \sigma(\pi(A))} |\lambda| . \tag{3.30}$$

Now $P = \pi(\mathbf{1}_\mathcal{A})$ is a projection

$$P^2 = \pi(\mathbf{1}_\mathcal{A})\pi(\mathbf{1}_\mathcal{A}) = \pi(\mathbf{1}_\mathcal{A}^2) = \pi(\mathbf{1}_\mathcal{A}) = P . \tag{3.31}$$

and can be used to define the C^*-algebra *($\to$ exercise)*

$$\mathcal{B}' = P\mathcal{B}P = \{PBP : B \in \mathcal{B}\} \subseteq \mathcal{B} \tag{3.32}$$

with identity $\mathbf{1}_{\mathcal{B}'} = P$. Since

$$\pi(A) = \pi(\mathbf{1}_\mathcal{A} A \mathbf{1}_\mathcal{A}) = \pi(\mathbf{1}_\mathcal{A})\pi(A)\pi(\mathbf{1}_\mathcal{A}) = P\pi(A)P \in \mathcal{B}' , \tag{3.33}$$

we know that $\pi(\mathcal{A}) \subseteq \mathcal{B}'$ and we can compare the respective spectra: if $(\lambda\mathbf{1}_\mathcal{A} - A)^{-1} \in \mathcal{A}$, then $(\lambda P - \pi(A))^{-1} = \pi((\lambda\mathbf{1}_\mathcal{A} - A)^{-1}) \in \mathcal{B}'$, i. e. $r_{\mathcal{B}'}(\pi(A)) \supseteq r_\mathcal{A}(A)$ or

$$\sigma_{\mathcal{B}'}(\pi(A)) \subseteq \sigma_\mathcal{A}(A). \tag{3.34}$$

Note that we have indicated the respective algebras, because $\mathcal{B}'$ and $\mathcal{A}$ are not subsets of each other and theorem 2.17 does *not* apply. Thus

$$\|\pi(A)\| = \sup_{\lambda \in \sigma(\pi(A))} |\lambda| \leqslant \sup_{\lambda \in \sigma(A)} |\lambda| \leqslant \|A\| . \tag{3.35}$$

and for non self-adjoint A we can use

$$\|\pi(A)\|^2 = \|\pi(A^*A)\| \leqslant \|A^*A\| \leqslant \|A\|^2 \tag{3.36}$$

to prove $\|\pi(A)\| \leqslant \|A\|$.

Having shown that π is continuous, we infer that its kernel $\ker\pi$ is a closed two-sided ideal. By theorem 3.8, $\mathcal{A}_\pi = \mathcal{A}/\ker\pi$ is then a C^*-algebra. Therefore $\hat\pi : \mathcal{A}_\pi \to \mathcal{B}_\pi = \pi(\mathcal{A})$ with $\hat\pi(\hat A) = \pi(A \in \hat A)$ is an isomorphism and $\hat\pi^{-1} : \mathcal{B}_\pi \to \mathcal{A}_\pi$ with

$$\hat\pi^{-1}(\hat\pi(\hat A)) = \hat A \tag{3.37}$$

is well defined. Then from (3.29) applied to $\hat\pi^{-1}$ and $\hat\pi$, we have

$$\|\hat A\| = \|\hat\pi^{-1}(\hat\pi(\hat A))\| \leqslant \|\hat\pi(\hat A)\| \leqslant \|\hat A\| \tag{3.38}$$

and find that the norms agree

$$\|\hat A\|_{\mathcal{A}/\ker\pi} = \|\hat\pi(\hat A)\|_{\mathcal{B}} = \|\pi(A \in \hat A)\|_{\mathcal{B}}\,. \tag{3.39}$$

Therefore, whenever a series $\{\pi(A_n)\}_{n\in\mathbf{N}}$ converges to a $B \in \mathcal{B}$, the series $\{\hat A_n\}_{n\in\mathbf{N}}$ must converge to a $\hat A \in \mathcal{A}_\pi$, because $\mathcal{A}_\pi$ is complete. By continuity of π and $\hat\pi$, $B = \hat\pi(\hat A) = \pi(A \in \hat A)$ and $B \in \mathcal{B}_\pi = \pi(\mathcal{A})$, which is therefore closed. $\square$

Definition 3.10 (isomorphism of C^*-algebras)**.** A C^*-*isomorphism* $\pi : \mathcal{A} \to \mathcal{B}$ is a C^*-homomorphism that is *one-to-one* and *onto*, i. e. $\pi(\mathcal{A}) = \mathcal{B}$ and the inverse map $\pi^{-1} : \mathcal{B} \to \mathcal{A}$ exists.

From linear algebra, we know

Remark 3.11. A C^*-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ is a C^*-isomorphism, iff its kernel is trivial $\ker\pi = \{0\}$ and $\pi(\mathcal{A}) = \mathcal{B}$.

3.4 Representations

Theorem 1.13 stated that every C^*-algebra is isomorphic to an algebra of operators on a suitable Hilbert space. Thus C^*-homomorphisms that map into the C^*-algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ are particularly interesting:

Definition 3.12 (representation of a C^*-algebra)**.** A *representation* of a C^*-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$ consisting of a complex Hilbert space $\mathcal{H}$ and a C^*-homomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$. The representation $(\mathcal{H}, \pi)$ is called *faithful* iff $\pi : \mathcal{A} \to \pi(\mathcal{A})$ is a C^*-isomorphism.

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Theorem 3.13. *Let $(\mathcal{H},\pi)$ be a representation of the C^*-algebra $\mathcal{A}$. It is faithful, iff it satisfies the following* equivalent *conditions*

$$\ker\pi = \{0\} \tag{3.40a}$$
$$\forall A \in \mathcal{A} : \|\pi(A)\| = \|A\| \tag{3.40b}$$
$$\forall A > 0 : \pi(A) > 0\,. \tag{3.40c}$$

Proof. We already know (3.40a) from linear algebra. OTOH, if (3.40a) holds, there is a well defined map $\pi^{-1} : \pi(\mathcal{A}) \to \mathcal{A}$ with $\pi^{-1}(\pi(A)) = A$ for all A. Using theorem 3.9 twice, i. e.

$$\|A\| = \|\pi^{-1}(\pi(A))\| \leqslant \|\pi(A)\| \leqslant \|A\|\,, \tag{3.41}$$

we obtain (3.40b). Since $A > 0$ implies $\|A\| > 0$, this yields

$$A > 0 \Rightarrow 0 < \|A\| = \|\pi(A)\| \Rightarrow \pi(A) \neq 0\,. \tag{3.42}$$

Using remark 3.2, we obtain $\pi(A) \geqslant 0$, i. e. (3.40c). Finally, if $\ker\pi \neq \{0\}$, then $\exists B \in \ker\pi \subseteq \mathcal{A} : B \neq 0 \wedge \pi(B^*B) = \pi(B^*)\pi(B) = 0$. OTOH, $0 < \|B\|^2 = \|B^*B\|$, i. e. $B^*B > 0$ and (3.40c) is false. □

Definition 3.14 (automorphism of a C^*-algebra)**.** A C^*-isomorphism $\pi : \mathcal{A} \to \mathcal{A}$ is called a C^*-*automorphism.*

Corollary 3.15. *If π is a C^*-isomorphism of $\mathcal{A}$, it is norm preserving*

$$\forall A \in \mathcal{A} : \|\pi(A)\| = \|A\|\,. \tag{3.43}$$

Proof. This is a direct consequence of theorem 3.13, in particular the equivalence of (3.40a) and (3.40b). □

Definition 3.16. A subspace $\mathcal{H}_1 \subseteq \mathcal{H}$ is called an *invariant subspace*, also known as (a.k.a.) a *stable subspace*, of a representation $(\mathcal{H},\pi)$ of a C^*-algebra, iff

$$\forall A \in \mathcal{A} : \pi(A)\mathcal{H}_1 \subseteq \mathcal{H}_1\,. \tag{3.44}$$

If $\mathcal{H}_1$ is a closed subspace of $\mathcal{H}$ and $P_1 = P_1^*$ is the corresponding *orthogonal* projector with $P_1\mathcal{H} = \mathcal{H}_1$, then

$$\forall A \in \mathcal{A} : P_1\pi(A)P_1 = \pi(A)P_1\,. \tag{3.45}$$

As a consequence

$$\pi(A)P_1 = P_1\pi(A)P_1 = (P_1^*\pi(A)^*P_1^*)^* = (P_1\pi(A)^*P_1)^* \\ = (\pi(A)^*P_1)^* = P_1^*\pi(A) = P_1\pi(A)\,, \quad (3.46)$$

i. e.

$$\pi(A)P_1 = P_1\pi(A)\,. \quad (3.47)$$

And indeed this is the necessary *and* sufficient condition for $\mathcal{H}_1$ to be stable under π.

Remark 3.17. If $\mathcal{H}_1$ is stable under π and P_1 is the corresponding orthogonal projection, then $(\mathcal{H}_1, \pi_1)$ with

$$\pi_1(A) = P_1\pi(A)P_1 \quad (3.48)$$

is also a representation. It is called a *subrepresentation*.

Proof.

$$\pi_1(A)\pi_1(B) = P_1\pi(A)P_1P_1\pi(B)P_1 \\ = P_1\pi(A)\pi(B)P_1 = P_1\pi(AB)P_1 = \pi_1(AB)\,. \quad (3.49)$$

□

If P_1 is an orthogonal projection, then $\mathbf{1} - P_1 = P_2$ is one as well and projects on the orthogonal subspace $\mathcal{H}_2$. Since

$$P_2\pi(A) = (\mathbf{1} - P_1)\pi(A) = \pi(A)(\mathbf{1} - P_1) = \pi(A)P_2\,, \quad (3.50)$$

$\mathcal{H}_2$ is also stable under π and $(\mathcal{H}_2, \pi_2)$ with

$$\pi_2(A) = P_2\pi(A)P_2 \quad (3.51)$$

is again a subrepresentation. In this way, we obtain a *decomposition* of a representation into a direct sum

$$(\mathcal{H}, \pi) = (\mathcal{H}_1, \pi_1) \oplus (\mathcal{H}_2, \pi_2) = (\mathcal{H}_1 \oplus \mathcal{H}_2, \pi_1 \oplus \pi_2) \quad (3.52)$$

where

$$\forall A \in \mathcal{A}, \psi_{1,2} \in \mathcal{H}_{1,2} : \pi(A)(\psi_1 \oplus \psi_2) = \pi_1(A)\psi_1 \oplus \pi_2(A)\psi_2\,. \quad (3.53)$$

As always, the definition of a representation allows for *trivial representations* with

$$\pi : \mathcal{A} \to \{0\} \subseteq \mathcal{L}(\mathcal{H}) \quad (3.54)$$

that are not particularly interesting. Moreover, there are nontrivial representations with parts that are not interesting either. The set

$$\mathcal{H}_0 = \bigcap_{A\in\mathcal{A}} \ker(\pi(A)) = \{\psi \in \mathcal{H} : \pi(A)\psi = 0, \forall A \in \mathcal{A}\} \tag{3.55}$$

is obviously a linear subspace and invariant under $\pi(\mathcal{A})$. The corresponding subrepresentation $(\mathcal{H}_0, \pi_0)$ has

$$\pi_0(A) = P_0\pi(A)P_0 = 0\,. \tag{3.56}$$

Representations with a trivial subrepresentation are called degenerate. OTOH, the interesting cases are covered by

Definition 3.18 (*nondegenerate representation*)**.** A representation $(\mathcal{H}, \pi)$ is called nondegenerate, iff the subspace annihilated by $\pi(\mathcal{A})$ is trivial: $\mathcal{H}_0 = \{0\}$.

Particularly interesting is the case when there is

Definition 3.19 (*cyclic vector*)**.** A vector $\Omega \in \mathcal{H}$ is called cyclic in $\mathcal{H}$ for a set $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$, if the linear span of $\{B\Omega : B \in \mathcal{B}\}$ is dense in $\mathcal{H}$.

Then we have

Definition 3.20. A *cyclic representation* of a C^*-algebra $\mathcal{A}$ is a triple $(\mathcal{H}, \pi, \Omega)$ with $(\mathcal{H}, \pi)$ a representation of $\mathcal{A}$ and Ω a cyclic vector for $\pi(\mathcal{A})$ in $\mathcal{H}$.

It is obvious that every cyclic representation is nondegenerate. Indeed, if this were not the case, then

$$\exists B \in \mathcal{A} : \pi(B)\Omega \neq 0 \wedge \forall A \in \mathcal{A} : 0 = \pi(A)\pi(B)\Omega = \pi(AB)\Omega\,, \tag{3.57}$$

which implies $\pi(B)\Omega = 0$ for $A = \mathbf{1}$ in a unital algebra or A an approximate identity, which exists by theorem 3.7.

It is less obvious that

Theorem 3.21. *If $(\mathcal{H}, \pi)$ is a nondegenerate representation of a C^*-algebra, it is a direct sum of a family of* cyclic *subrepresentations.*

Before we can prove the theorem, we need a precise

Definition 3.22 (*direct sum of representations*)**.** Let $\{(\mathcal{H}_i, \pi_i)\}_{i\in I}$ be a family[3] of representations of a C^*-algebra $\mathcal{A}$. Then the direct sum of representation spaces

$$\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i \tag{3.58}$$

[3]The index set I is not necessarily countable.

is well defined[4] and the action of the direct sum of representations

$$\pi = \bigoplus_{i\in I} \pi_i \tag{3.59}$$

is defined as a bounded operator

$$\pi(A)\left(\bigoplus_{i\in I} \psi_i\right) = \bigoplus_{i\in I} (\pi_i(A)\psi_i) \tag{3.60}$$

since all $\pi_i(A)$ are bounded operators with $\|\pi_i(A)\| \leqslant \|A\|$.

Proof. (of theorem 3.21) Choose[5] a maximal family $\{\Omega_i\}_{i\in I}$ of non-zero $\Omega_i \in \mathcal{H}$ with

$$\forall A, B \in \mathcal{A}, \forall i \neq j \in I : (\pi(A)\Omega_i, \pi(B)\Omega_j) = 0\,. \tag{3.61}$$

We can then define

$$\forall i \in I : \mathcal{H}_i = \overline{\pi(\mathcal{A})\Omega_i}\,, \tag{3.62}$$

which are, by construction, invariant subspaces and we obtain subrepresentations

$$\pi_i(A) = P_i\pi(A)P_i\,, \tag{3.63}$$

and a family $\{(\mathcal{H}_i, \pi_i, \Omega_i)\}_{i\in I}$ of cyclic representations. Since $(\mathcal{H}, \pi)$ is non-degenerate and the family $\{\Omega_i\}_{i\in I}$ is maximal, there is no non-zero $\psi \in \mathcal{H}$ with $(\psi, \phi) = 0, \forall\phi \in \oplus_{i\in I}\mathcal{H}_i$. □

This theorem shows that we esentially know *all* representations, when we know all cyclic representations. This is important, because we will soon learn that the cyclic representations are intimately related the the physical states, i. e. the normalized postive linear functionals on the C^*-algebra of (complexified) observables.

Close relatives of the cyclic representations are

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[4] As above in the proof of theorem 3.7, we can order the finite subsets of I by inclusion to build a directed set. The elements of $\mathcal{H}$ are then families $\{\psi_i\}_{i\in\alpha}$ such that

$$\lim_{|\alpha|\to\infty} \sum_{i\in\alpha} \|\psi_i\|_i^2 < \infty\,.$$

This way we obtain a norm and corresponding inner product and can complete the direct sum with respect to (wrt) the topology induced by this norm.

[5] For the existence of such a family, in the case of a not countable I, we have to assume the *axiom of choice* or, equivalently, *Zorn's lemma*.

Definition 3.23 (*irreducible representation*)**.** A self-adjoint set $\mathcal{B}$ of bounded operators on a Hilbert space $\mathcal{H}$ is called irreducible on $\mathcal{H}$, if there are no closed subspaces of $\mathcal{H}$ invariant under $\mathcal{B}$ other than the trivial $\{0\}$ and $\mathcal{H}$ itself. Analogously, a representation $(\mathcal{H}, \pi)$ of a C^*-algebra $\mathcal{A}$ is called irreducible iff $\pi(\mathcal{A})$ is irreducible on $\mathcal{H}$.

Theorem 3.24. *Let $\mathcal{B}$ be a self-adjoint set of bounded operators on a Hilbert space $\mathcal{H}$. The following are equivalent:*

1. *$\mathcal{B}$ is irreducible.*
2. *only multiples of $\mathbf{1} \in \mathcal{L}(\mathcal{H})$ commute with all $B \in \mathcal{B}$, i. e. the* commutant *$\mathcal{B}'$ of $\mathcal{B}$ is $\mathcal{B}' = \{\lambda\mathbf{1} : \lambda \in \mathbf{C}\}$.*
3. *every nonzero $\psi \in \mathcal{H}$ is cyclic for $\mathcal{B}$ in $\mathcal{H}$, unless $\mathcal{B} = \{0\}$ or $\mathcal{H} = \mathbf{C}$.*

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Proof. We prove the cyclic dependence:

1. (1)→(3): If there is a nonzero $\psi \in \mathcal{H}$ that is not cyclic, then the orthogonal complement of $\mathcal{B}\psi$ is a non-empty invariant[6] subspace and $\mathcal{B}$ is reducible on $\mathcal{H}$ (unless $\mathcal{B} = \{0\}$ or $\mathcal{H} = \mathbf{C}$).
2. (3)→(2): If $B \in \mathcal{B}'$, then $B^* \in \mathcal{B}'$, since $\mathcal{B}$ is self-adjoint. Also $B + B^* \in \mathcal{B}'$ and $(B - B^*)/\mathrm{i} \in \mathcal{B}'$ and if $\mathcal{B}' \neq \{\lambda\mathbf{1}\}_{\lambda\in\mathbf{C}}$, there are $C = C^* \in \mathcal{B}'$ with $C \neq \lambda\mathbf{1}, \forall\lambda \in \mathbf{C}$. The corresponding spectral projections P of C are then also in $\mathcal{B}'$ and a $\psi \in \mathcal{H}$ with $(1 - P)\psi = 0$ cannot be cyclic.
3. (2)→(1): If $\mathcal{B}$ is not irreducible on $\mathcal{H}$, then there is a closed subspace $\mathcal{K} \subseteq \mathcal{H}$ that is stable under $\mathcal{B}$ and the corresponding non-trivial projector $P_{\mathcal{K}} \in \mathcal{B}'$.

□

Note that *not* all cyclic representations are irreducible: we have just proven that all non-zero vectors of an irreducible representation are cyclic. But there are cyclic representations where *not all* non-zero vectors are cyclic.

[6] $(\mathcal{B}\psi, B(\mathcal{B}\psi)^\perp) = (B^*\mathcal{B}\psi, (\mathcal{B}\psi)^\perp) = (\mathcal{B}\psi, (\mathcal{B}\psi)^\perp) = 0$, i.e. $B(\mathcal{B}\psi)^\perp \subseteq (\mathcal{B}\psi)^\perp$.

Definition 3.25 (*unitary equivalence*)**.** Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are called unitarily equivalent, iff there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\forall A \in \mathcal{A} : \pi_2(A) = U\pi_1(A)U^* \,. \tag{3.64}$$

i. e. the diagram

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\pi_1(A)} & \mathcal{H}_1 \\ U\downarrow & & \downarrow U \\ \mathcal{H}_2 & \xrightarrow[\pi_2(A)]{} & \mathcal{H}_2 \end{array} \tag{3.65}$$

commutes for all $A \in \mathcal{A}$.

Remark 3.26. Obviously, given a representation $(\mathcal{H}_1, \pi_1)$ and a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$, we obtain another representation $(\mathcal{H}_2, \pi_2)$ from (3.64):

$$\begin{aligned} \pi_2(A)\pi_2(B) = U\pi_1(A)UU^*\pi_1(B)U^* &= U\pi_1(A)\pi_1(B)U^* \\ &= U\pi_1(AB)U^* = \pi_2(AB) \end{aligned} \tag{3.66a}$$

$$\pi_2(A)^* = U_1^\pi(A)^*U^* = U\pi_1(A)^*U^* = U\pi_1(A^*)U^* = \pi_2(A^*) \,. \tag{3.66b}$$

—4— States

As before, the positive linear functionals on a C^*-algebra of observables will play the role of physical states.

Definition 4.1 (*dual space*)**.** The space of continuous linear functionals $\omega : \mathcal{A} \to \mathbf{C}$ on the C^*-algebra $\mathcal{A}$ is denoted $\mathcal{A}^*$.

We can define a natural norm on $\mathcal{A}^*$ by

$$\|\omega\| = \sup_{A\in\mathcal{A},\|A\|=1} |\omega(A)| \,. \tag{4.1}$$

Definition 4.2 (*positive linear functional, state*)**.** A linear functional $\omega : \mathcal{A} \to \mathbf{C}$ on the C^*-algebra $\mathcal{A}$ is called positive, iff

$$\forall A \in \mathcal{A} : \omega(A^*A) \geqslant 0 \,. \tag{4.2}$$

A positive $\omega : \mathcal{A} \to \mathbf{C}$ with $\|\omega\| = 1$ is called a state.

Typical examples that should be familiar from the QM lecture

- *pure states*, a.k.a. *vector states*:

$$\omega(A) = (\Omega, \pi(A)\Omega) \tag{4.3}$$

  with $\Omega \in \mathcal{H}$ normalized, i. e. $(\Omega, \Omega) = 1$. Positivity is obvious

$$\begin{aligned}\omega(A^*A) &= (\Omega, \pi(A^*A)\Omega) = (\Omega, \pi(A^*)\pi(A)\Omega) \\ &= (\Omega, \pi(A)^*\pi(A)\Omega) = (\pi(A)\Omega, \pi(A)\Omega) \geqslant 0\end{aligned} \tag{4.4}$$

  and the normalization $\|\omega\| = 1$ will be a consequence of theorem 4.4 below.

- statistical mixtures
$$\omega(A) = \sum_i p_i \omega_i(A) \tag{4.5}$$
with $p_i \geqslant 0$ and $\sum_i p_i = 1$.
- *density matrices*
$$\omega(A) = \operatorname{tr}(\rho\pi(A)) \tag{4.6}$$
with $\rho = \rho^* > 0$ and $\operatorname{tr}\rho = 1$.

These will be discussed in more detail below.

In the following we will frequently use the CSI (1.52b), which we have already derived on page 11 when we discussed algebras of observables, before introducing C^*-algebras. The identical arguments will also establish

Lemma 4.3 (*Cauchy-Schwarz Inequality*)**.** *For every postive linear functional ω on a C^*-algebra $\mathcal{A}$, we have $\forall A, B \in \mathcal{A}$*

$$\omega(B^*A) = \overline{\omega(A^*B)} \tag{4.7a}$$
$$|\omega(A^*B)| \leqslant \sqrt{\omega(A^*A)}\sqrt{\omega(B^*B)}\,. \tag{4.7b}$$

We will now see that every positive $\omega : \mathcal{A} \to \mathbf{C}$ is continuous. ,

Theorem 4.4 (continuity of postive functionals)**.** *For all $\omega : \mathcal{A} \to \mathbf{C}$, the following are equivalent*

1. *ω is positive*
2. *ω is continuous and*
$$\|\omega\| = \lim_\alpha \omega(E_\alpha^2) \tag{4.8}$$
for some *approximate identity $\{E_\alpha\}$ of $\mathcal{A}$.*

Furthermore, if ω is positive, then $\forall A, B \in \mathcal{A}$

$$\omega(A^*) = \overline{\omega(A)} \tag{4.9a}$$
$$|\omega(A)|^2 \leqslant \omega(A^*A)\|\omega\| \tag{4.9b}$$
$$|\omega(A^*BA)| \leqslant \omega(A^*A)\|B\| \tag{4.9c}$$
$$\|\omega\| = \sup_{A\in\mathcal{A}, \|A\|=1} \omega(A^*A) \tag{4.9d}$$

and
$$\|\omega\| = \lim_\alpha \omega(E_\alpha) \tag{4.10}$$
for any *approximate identity $\{E_\alpha\}$ of $\mathcal{A}$.

Proof. First, assume that ω is positive, i. e. (1), and let $\{A_n\}_{n\in\mathbf{N}}$ be a sequence with $A_n \geqslant 0$ and $\|A_n\| \leqslant 1$. Then for *all* sequences $\{\lambda_n\}_{n\in\mathbf{N}}$ with $\lambda_n \geqslant 0$ and $\sum_{n\in\mathbf{N}} \lambda_n < \infty$

$$\sum_{n=1}^{N} \lambda_n A_n \overset{N\to\infty}{\to} A \geqslant 0 \tag{4.11}$$

converges monotonically from below. Thus, by positivity and linearity of ω

$$\sum_{n=1}^{N} \lambda_n \omega(A_n) \leqslant \omega(A) < \infty\,. \tag{4.12}$$

This can only be true for *all* such sequences $\{\lambda_n\}_{n\in\mathbf{N}}$ *independently* of $\{A_n\}_{n\in\mathbf{N}}$, if the $\{\omega(A_n)\}_{n\in\mathbf{N}}$ are *uniformly* bounded

$$M_+ = \sup_{A\geqslant 0, \|A\|=1} \omega(A) < \infty\,. \tag{4.13}$$

We can decompose each $A \in \mathcal{A}$ with $\|A\| = 1$ first into two self-adjoint and then into four positive $\{A_k\}$ with $\|A_k\| \leqslant 1$

$$A = \sum_{k=1}^{4} \mathrm{i}^k A_k \tag{4.14}$$

and therefore

$$\|\omega\| = \sup_{\|A\|=1} |\omega(A)| \leqslant 4M_+\,. \tag{4.15}$$

Since this entails

$$|\omega(A)| \leqslant 4M_+\|A\|\,, \tag{4.16}$$

we have established that $\omega : \mathcal{A} \to \mathbf{C}$ is continuous. For any $E_\alpha = E_\alpha^* \in \mathcal{A}$ we can use the CSI

$$|\omega(AE_\alpha)|^2 \overset{(1.52b)}{\leqslant} \omega(A^*A)\omega(E_\alpha^*E_\alpha) \leqslant M_+\|A\|^2\omega(E_\alpha^2) \tag{4.17}$$

and if $\{E_\alpha\}$ is an approximate identity, we can take the limit $|\alpha| \to \infty$

$$|\omega(A)|^2 \leqslant M_+M\|A\|^2 \tag{4.18}$$

where

$$M = \sup_\alpha \omega(E_\alpha^2)\,. \tag{4.19}$$

Taking the supremum over $\|A\| = 1$, we obtain

$$\|\omega\|^2 \leqslant M_+M\,. \tag{4.20}$$

However, $\|E_\alpha\| \leqslant 1$ and $E_\alpha \geqslant 0$ by definition and therefore

$$M = \sup_\alpha \omega(E_\alpha^2) \leqslant M_+ = \sup_{A\geqslant 0, \|A\|=1} \omega(A) \leqslant \sup_{\|A\|=1} \omega(A) = \|\omega\| . \tag{4.21}$$

This entails

$$\|\omega\|^2 \leqslant M_+ M \leqslant M_+ \|\omega\| \leqslant \|\omega\|^2 \tag{4.22a}$$
$$\|\omega\|^2 \leqslant M_+ M \leqslant M \|\omega\| \leqslant \|\omega\|^2 \tag{4.22b}$$

and the inequalities must be equalities

$$\|\omega\| = M_+ = M = \lim_\alpha \omega(E_\alpha^2) , \tag{4.23}$$

i. e. we have established (2). While we're at it, we observe that since $E_\alpha^2 \leqslant E_\alpha$

$$\|\omega\| = \lim_\alpha \omega(E_\alpha^2) \leqslant \lim_\alpha \omega(E_\alpha) \leqslant \|\omega\| \tag{4.24}$$

i. e. (4.10). The remaining implications of positivity are straightforward: (4.9a) follows from (4.7a) with $B = E_\alpha$ in the limit $\alpha \to \infty$ and (4.9b) from (4.7b) in a similar way. Also (4.9c) follows from using (4.7b), this time in the form

$$|\omega(A^*BA)|^2 \leqslant \omega(A^*A)\omega(A^*B^*BA) \tag{4.25}$$

and noting that

$$A^*B^*BA \leqslant A^*\|B\|^2 A . \tag{4.26}$$

Finally, the estimate (4.9b) implies (4.9d).

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Conversely, assume that (2) holds and WLOG $\|\omega\| = 1$. If $\mathcal{A}$ is unital, then

$$\begin{aligned} \|\mathbf{1} - E_\alpha^2\| &\leqslant \|\mathbf{1} - E_\alpha\| + \|E_\alpha - E_\alpha^2\| \\ &\leqslant \|\mathbf{1} - E_\alpha\| + \|\mathbf{1} - E_\alpha\|\|E_\alpha\| = \|\mathbf{1} - E_\alpha\| \left(1 + \|E_\alpha\|\right) \end{aligned} \tag{4.27}$$

therefore

$$\lim_\alpha E_\alpha^2 = \mathbf{1} \tag{4.28}$$

and

$$\omega(\mathbf{1}) = \lim_\alpha \omega(E_\alpha^2) = \|\omega\| = 1 \tag{4.29}$$

from (2). OTOH, if $\mathcal{A}$ is not unital, we can adjoin a unit and extend $\omega : \mathcal{A} \to \mathbf{C}$ to

$$\begin{aligned} \tilde\omega : \mathbf{C1} + \mathcal{A} &\to \mathbf{C} \\ (\lambda, A) &\mapsto \lambda + \omega(A) . \end{aligned} \tag{4.30}$$

Then

$$A - AE_\alpha^2 = (A - AE_\alpha) + (A - AE_\alpha)E_\alpha\,, \tag{4.31}$$

i. e.

$$\lim_\alpha AE_\alpha^2 = A\,. \tag{4.32}$$

Furthermore

$$|\omega(A)| \leqslant \|\omega\|\|A\| \tag{4.33}$$

i. e.

$$|\omega(A)| \leqslant \|A\| \tag{4.34}$$

and

$$\begin{aligned} |\tilde\omega(\lambda\mathbf{1} + A)| &= |\lambda + \omega(A)| = \lim_\alpha |\lambda\omega(E_\alpha^2) + \omega(AE_\alpha^2)| \\ &\overset{(4.34)}{\leqslant} \lim_\alpha \sup \|\lambda E_\alpha^2 + AE_\alpha^2\|_{\mathcal{A}} \leqslant \sup_{\|B\|=1} \|\lambda B + AB\|_{\mathcal{A}} = \|\lambda\mathbf{1} + A\|_{\bar{\mathcal{A}}}\,, \end{aligned} \tag{4.35}$$

i. e.

$$\|\tilde\omega\| \leqslant 1\,. \tag{4.36}$$

OTOH, $\tilde\omega(\mathbf{1}) = 1$ from the definition and therefore $\|\tilde\omega\| = 1$. Thus we have in any case

$$\omega(\mathbf{1}) = \|\mathbf{1}\| = 1 = \|\omega\|\,. \tag{4.37}$$

In order to show that $\omega(A)$ is real for self adjoint A, we observe that for

$$\omega(A) = \alpha + \mathrm{i}\beta \qquad (\alpha, \beta \in \mathbf{R}) \tag{4.38}$$

we have

$$\omega(A + \mathrm{i}\gamma\mathbf{1}) = \alpha + \mathrm{i}(\beta + \gamma)\,. \tag{4.39}$$

For any self-adjoint A, the spectrum satisfies

$$\sigma(A) + \mathrm{i}\gamma \subseteq [-\|A\|, \|A\|] + \mathrm{i}\gamma \tag{4.40}$$

and since $A + \mathrm{i}\gamma\mathbf{1}$ is normal, we have

$$\|A + \mathrm{i}\gamma\mathbf{1}\| = \rho(A + \mathrm{i}\gamma\mathbf{1}) = \sqrt{\|A\|^2 + \gamma^2}\,. \tag{4.41}$$

OTOH

$$|\omega(A + \mathrm{i}\gamma\mathbf{1})| = |\alpha + \mathrm{i}(\beta + \gamma)| = \sqrt{\alpha^2 + (\beta + \gamma)^2} \geqslant |\beta + \gamma| \tag{4.42}$$

and therefore

$$\forall\gamma \in \mathbf{R} : |\beta + \gamma| \leqslant \sqrt{\|A\|^2 + \gamma^2}\,. \tag{4.43}$$

Choosing $\gamma = \delta \|A\|^2\beta/|\beta|$, we find

$$\forall\delta \in \mathbf{R} : \beta^2 + 2\delta|\beta|\|A\|^2 \leqslant \|A\|^2 , \tag{4.44}$$

which can hold for $\delta \to \infty$ only if $\beta = 0$, i. e. $\omega(A)$ must be real if $A = A^*$. However, since $A^*A \geqslant 0$

$$\left\| \mathbf{1} - \frac{A^*A}{\|A^*A\|} \right\| = \left\| \mathbf{1} - \frac{A^*A}{\|A\|^2} \right\| \leqslant 1 \tag{4.45}$$

and therefore, using (4.33),

$$\left| \omega(\mathbf{1}) - \frac{\omega(A^*A)}{\|A\|^2} \right| \leqslant 1 . \tag{4.46}$$

With $\omega(\mathbf{1}) = 1$ and $\omega(A^*A) \in \mathbf{R}$ this entails the positivity

$$\omega(A^*A) \geqslant 0 . \tag{4.47}$$

□

It is obvious that the sum of two positive linear functionals is again positive and the foregoing theorem tells us that the norm is additive.

Corollary 4.5. *Given two positive linear functionals $\omega_1, \omega_2 \in \mathcal{A}^*$ on a C^*-algebra $\mathcal{A}$, their sum is positive and*

$$\|\omega_1 + \omega_2\| = \|\omega_1\| + \|\omega_2\| \tag{4.48}$$

and the states form a convex subset of $\mathcal{A}^*$.

Proof. Using

$$\begin{aligned}\|\omega_1 + \omega_2\| &= \lim_\alpha \left(\omega_1(E_\alpha^2) + \omega_2(E_\alpha^2)\right) \\ &= \lim_\alpha \omega_1(E_\alpha^2) + \lim_\alpha \omega_2(E_\alpha^2) = \|\omega_1\| + \|\omega_2\| \end{aligned} \tag{4.49}$$

for a suitable approximate identity $\{E_\alpha\}$ we obtain the additivity and with a similar argument

$$\|\omega\| = \|\lambda\omega_1 + (1-\lambda)\omega_2\| = \lambda\|\omega_1\| + (1-\lambda)\|\omega_2\| = 1 \tag{4.50}$$

the correct normalization of the conical sum. □

We have seen in theorem 2.1 that given a C^*-algebra $\mathcal{A}$ without identity, we can always adjoin an identity to obtain a unital C^*-algebra $\tilde{\mathcal{A}} = \mathbf{C1} + \mathcal{A}$. We can extend every $\omega \in \mathcal{A}^*$ to $\tilde\omega \in \tilde{\mathcal{A}}^*$

$$\tilde\omega\,(\lambda\mathbf{1} + A) = \lambda\|\omega\| + \omega(A) . \tag{4.51}$$

It can be shown easily that $\tilde\omega$ inherits the positivity, normalization and additivity from ω.

4.1 Pure and Impure States

Just as we used positivity of algebra elements to define an order on the algebra by $A \leqslant B \Leftrightarrow B - A \geqslant 0$, we can order the dual space by

$$\forall \omega_1, \omega_2 \in \mathcal{A}^* : \omega_1 \leqslant \omega_2 \Leftrightarrow \omega_2 - \omega_1 \geqslant 0 \,. \tag{4.52}$$

We say that ω_2 *majorizes* ω_1, iff $\omega_2 \geqslant \omega_1$. If a state ω is a *conical sum* of the states ω_1 and ω_2, i. e.

$$\omega = \lambda\omega_1 + (1-\lambda)\omega_2 \tag{4.53}$$

with $0 < \lambda < 1$, then

$$\omega - \lambda\omega_1 = (1-\lambda)\omega_2 \geqslant 0 \tag{4.54a}$$

$$\omega - (1-\lambda)\omega_2 = \lambda\omega_1 \geqslant 0 \tag{4.54b}$$

i. e. ω majorizes both $\lambda\omega_1$ and $(1-\lambda)\omega_2$. The normalization of the states means that there must be *smallest* states wrt to the ordering just defined. Therefore one cannot write all of them as conical sums of majorized states. Thus one can distinguish states ω that can be written as a conical sum of two other states from those that can not.

Definition 4.6 (*pure states*)**.** A state $\omega \in \mathcal{A}^*$ over a C^*-algebra $\mathcal{A}$ is called *pure*, if all $0 \leqslant \omega' \in \mathcal{A}^*$ majorized by ω are of the form $\omega' = \lambda\omega$ with $0 \leqslant \lambda \leqslant 1$.

We will denote the convex cone of all states by $\Omega_{\mathcal{A}} \subset \mathcal{A}^*$ and the subset of all pure states by $\Pi_{\mathcal{A}} \subset \Omega_{\mathcal{A}}$.

4.2 The GNS Construction

We have seen that for any representation $(\mathcal{H}, \pi)$ of a C^*-algebra $\mathcal{A}$ and for all $\Omega \in \mathcal{H}$ with $\|\Omega\| = 1$, we obtain a *vector state* $\omega_\Omega : A \mapsto (\Omega, \pi(A)\Omega)$. In this section, we will show *by construction* that *every* state ω on $\mathcal{A}$ is a vector state for a suitable cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ and that this representation will be irreducible, iff ω is pure.

Since every C^*-algebra $\mathcal{A}$ is, by definition, a Banach space, we can recycle its vector space structure to construct a Hilbert space, iff we can find a suitable nondegenerate *sesquilinear form*

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle : \mathcal{A} \times \mathcal{A} &\to \mathbf{C} \\ (A, B) &\mapsto \langle A, B\rangle \end{aligned} \tag{4.55}$$

with $\forall A, B, C \in \mathcal{A}$ and $\forall \beta, \gamma \in \mathbf{C}$

$$\langle A, B\rangle = \overline{\langle B, A\rangle} \tag{4.56a}$$
$$\langle A, \beta B + \gamma C\rangle = \beta\langle A, B\rangle + \gamma\langle A, C\rangle \tag{4.56b}$$
$$\langle A, A\rangle \geqslant 0 \tag{4.56c}$$
$$\langle A, A\rangle = 0 \Leftrightarrow A = 0\,. \tag{4.56d}$$

In fact, we can construct a sesquilinear form from any positive linear functional ω on an algebra

$$\langle A, B\rangle_\omega = \omega(A^*B)\,. \tag{4.57}$$

Indeed, all properties (4.56) except (4.56d) can be verified trivially, where (4.56a) follows from the positivity of ω via (4.7a). However, there will in general be $0 \neq A \in \mathcal{A}$ with $\omega(A^*A) = 0$.

Fortunately, these elements form a left ideal

$$\mathcal{I}_\omega = \{A \in \mathcal{A} : \omega(A^*A) = 0\}\,, \tag{4.58}$$

because

$$\begin{aligned}\forall I \in \mathcal{I}_\omega, A \in \mathcal{A} : 0 \leqslant \omega((AI)^*AI) &= \omega(I^*A^*AI)\\ &\leqslant \omega(I^*I)\|A^*A\| = \omega(I^*I)\|A\|^2 = 0\,,\end{aligned} \tag{4.59}$$

by result (4.9c) of theorem 4.4. Thus we have shown $\forall I \in \mathcal{I}_\omega, A \in \mathcal{A} : AI \in \mathcal{I}_\omega$. Having identified this left ideal, we can form the factor space $\mathcal{H}^0_\omega = \mathcal{A}/\mathcal{I}_\omega$ consisting of the equivalence classes

$$\psi^A_\omega = \{\hat{A} : \hat{A} = A + I, I \in \mathcal{I}_\omega\} \in \mathcal{H}^0_\omega\,. \tag{4.60}$$

By construction, $\mathcal{H}^0_\omega$ is a linear space and we get again a sesquilinear form

$$\begin{aligned}\langle\,\cdot\,,\,\cdot\,\rangle : \mathcal{H}^0_\omega \times \mathcal{H}^0_\omega &\to \mathbf{C}\\ (\psi^A_\omega, \psi^B_\omega) &\mapsto \langle\psi^A_\omega, \psi^B_\omega\rangle = \omega(A^*B)\,,\end{aligned} \tag{4.61}$$

since we can use the CSI to show that it does not depend on which representative of each equivalence class is used

$$\forall I \in \mathcal{I}_\omega, A \in \mathcal{A} : |\omega(A^*I)| \leqslant \sqrt{\omega(A^*A)}\sqrt{\omega(I^*I)} = 0\,, \tag{4.62}$$

i. e.

$$\forall I \in \mathcal{I}_\omega, A, B \in \mathcal{A} : \omega(A^*(B + I)) = \omega(A^*B) + \omega(A^*I) = \omega(A^*B)\,. \tag{4.63}$$

Conversely

$$\forall I \in \mathcal{I}_\omega, \mathcal{A} \ni A \notin \mathcal{I}_\omega : \omega((A+I)^*(A+I)) = \omega(A^*A) > 0 \tag{4.64}$$

by the definition of $\mathcal{I}_\omega$. Thus $\langle \cdot , \cdot \rangle$ turns $\mathcal{H}^0_\omega$ into a proper pre-Hilbert space that can be completed to a Hilbert space $\mathcal{H}_\omega = \overline{\mathcal{H}^0_\omega}$ and $\langle \cdot , \cdot \rangle$ can be extended to $(\cdot , \cdot)$ on $\mathcal{H}_\omega$ by the continuity of ω.

Having constructed a Hilbert space is not yet enough for a representation, we still need to find a homomorphism $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$. Fortunately, the algebra structure of $\mathcal{A}$ suffices to allow a *canonical* construction

$$\begin{aligned} \pi_\omega(A) : \mathcal{H}_\omega &\to \mathcal{H}_\omega \\ \psi^B_\omega &\mapsto \pi_\omega(A)\psi^B_\omega = \psi^{AB}_\omega \,. \end{aligned} \tag{4.65}$$

This definition has *all* the required properties:

- it is well defined, because it's independent of the representative

$$\forall I \in \mathcal{I}_\omega : A(B+I) = AB + \underbrace{AI}_{\in \mathcal{I}_\omega} \tag{4.66}$$

- it's linear due to the linear structure of the factor space

$$\begin{aligned} \pi_\omega(\alpha_1 A_1 + \alpha_2 A_2)\psi^B_\omega = \psi^{(\alpha_1 A_1 + \alpha_2 A_2)B}_\omega &= \alpha_1 \psi^{A_1 B}_\omega + \alpha_2 \psi^{A_2 B}_\omega \\ &= \alpha_1 \pi_\omega(A_1)\psi^B_\omega + \alpha_2 \pi_\omega(A_2)\psi^B_\omega \end{aligned} \tag{4.67}$$

- it's an algebra homomorphism due to the associativity of the algebra

$$\begin{aligned} \pi_\omega(A_1)\pi_\omega(A_2)\psi^B_\omega &= \pi_\omega(A_1)\psi^{A_2 B}_\omega \\ &= \psi^{A_1(A_2 B)}_\omega = \psi^{(A_1 A_2)B}_\omega = \pi_\omega(A_1 A_2)\psi^B_\omega \end{aligned} \tag{4.68}$$

- the involution in the algebra is mapped to the adjoint of the operators

$$\begin{aligned} \left(\psi^A_\omega, \pi_\omega(B)^*\psi^C_\omega\right) &= \left(\pi_\omega(B)\psi^A_\omega, \psi^C_\omega\right) = \left(\psi^{BA}_\omega, \psi^C_\omega\right) \\ &= \omega((BA)^*C) = \omega(A^*B^*C) = \omega(A^*(B^*C)) \\ &= \left(\psi^A_\omega, \psi^{B^*C}_\omega\right) = \left(\psi^A_\omega, \pi_\omega(B^*)\psi^C_\omega\right) \end{aligned} \tag{4.69}$$

- $\pi_\omega(A)$ is bounded

$$\begin{aligned} \|\pi_\omega(A)\psi^B_\omega\|^2 = \|\psi^{AB}_\omega\|^2 &= \left(\psi^{AB}_\omega, \psi^{AB}_\omega\right) = \omega\left((AB)^*AB\right) \\ &= \omega\left(B^*A^*AB\right) \leqslant \|A^*A\|\omega\left(B^*B\right) = \|A\|^2\omega\left(B^*B\right) \\ &= \|A\|^2\|\psi^B_\omega\|^2 \end{aligned} \tag{4.70}$$

  i. e.

$$\|\pi_\omega(A)\| \leqslant \|A\| \,. \tag{4.71}$$

If $\mathcal{A}$ is unital, the cyclic vector Ω_ω is given simply by

$$\Omega_\omega = \psi_\omega^{\mathbf{1}} \tag{4.72}$$

recovering the vector state

$$(\Omega_\omega, \pi_\omega(A)\Omega_\omega) = \left(\psi_\omega^{\mathbf{1}}, \psi_\omega^A\right) = \omega(\mathbf{1}^* A) = \omega(A)\,, \tag{4.73}$$

since

$$\pi_\omega(A)\Omega_\omega = \pi_\omega(A)\psi_\omega^{\mathbf{1}} = \psi_\omega^A\,. \tag{4.74}$$

We verify that

$$\{\pi_\omega(A)\Omega_\omega : A \in \mathcal{A}\} = \left\{\psi_\omega^A : A \in \mathcal{A}\right\} = \mathcal{H}_\omega \tag{4.75}$$

i. e. that Ω_ω is cyclic for $(\mathcal{H}_\omega, \pi_\omega)$.

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OTOH, if $\mathcal{A}$ is not unital, we can again adjoin an identity $\mathbf{1}$ to get $\tilde{\mathcal{A}}$ and proceed as above[1] This time, however, while it is again obvious that $\Omega_\omega = \psi_\omega^{\mathbf{1}}$ is cyclic for $\pi_\omega(\tilde{\mathcal{A}})$, we must also show that it is cyclic for $\pi_\omega(\mathcal{A})$. Since only $\pi_\omega(\mathbf{1})$ is missing from $\pi_\omega(\mathcal{A})$, it suffices to show that $\Omega_\omega = \pi_\omega(\mathbf{1})\Omega_\omega$ itself is already in $\mathcal{H}_\omega = \pi_\omega(\mathcal{A})\Omega_\omega$.

Again, the existence of approximate identities, as ensured by theorem 3.7, comes to the rescue. Let $\{E_\alpha\}$ be an approximate identity for $\mathcal{A}$, then

$$\begin{aligned}\|\pi_\omega(E_\alpha)\Omega_\omega - \Omega_\omega\|^2 &= \underbrace{\|\Omega_\omega\|^2}_{\omega(\mathbf{11})} + \underbrace{\|\pi_\omega(E_\alpha)\Omega_\omega\|^2}_{\omega(E_\alpha E_\alpha)} - 2\,\underbrace{(\Omega_\omega, \pi_\omega(E_\alpha)\Omega_\omega)}_{\omega(\mathbf{1}E_\alpha)} \\ &= 1 + \omega(E_\alpha^2) - 2\omega(E_\alpha) \xrightarrow{\alpha} 1 + 1 - 2 = 0\end{aligned} \tag{4.77}$$

i. e.

$$\lim_\alpha \|\pi_\omega(E_\alpha)\Omega_\omega - \Omega_\omega\| = 0 \tag{4.78}$$

and Ω_ω lies in the closure of $\bigcup_\alpha \pi_\omega(E_\alpha)\Omega_\omega \subset \pi_\omega(\mathcal{A})\Omega_\omega$.

With this construction, we have already almost proved the

[1] Note that one can not circumvent $\tilde{\mathcal{A}}$ by directly using a *right* approximate identity to construct Ω_ω as $\lim_\alpha \psi_\omega^{E_\alpha}$, because, in general, only the limit $\lim_\alpha E_\alpha A = A \in \mathcal{A}$ exists and not the limit $\lim_\alpha E_\alpha \in \mathcal{A}$. Actually, iff $\{E_\alpha\}_\alpha$ was a *left* approximate identity, the limit $\lim_\alpha \psi_\omega^{E_\alpha} \in \mathcal{H}_\omega$ exists because $\mathcal{H}_\omega$ is complete and then one can argue

$$\pi_\omega(A) \lim_\alpha \psi_\omega^{E_\alpha} = \lim_\alpha \psi_\omega^{AE_\alpha} = \psi_\omega^A\,. \tag{4.76}$$

Unfortunately, we have only proven the existence of *right* approximate identities in theorem 3.7.

Theorem 4.7 (*GNS representation*)**.** *If ω is a state over the C^*-algebra $\mathcal{A}$, then there exists a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathcal{A}$ with*

$$\forall A \in \mathcal{A} : \omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega) \tag{4.79}$$

and

$$\|\Omega_\omega\|^2 = (\Omega_\omega, \Omega_\omega) = \omega(\mathbf{1}) = \|\omega\| \,. \tag{4.80}$$

This representation is unique up to unitary equivalence *and called the* GNS representation *associated to ω.*

Proof. After the explicit description of the construction above, it only remains to prove the uniqueness up to unitary equivalence, i. e. that if there is a second cyclic representation $(\mathcal{H}'_\omega, \pi'_\omega, \Omega'_\omega)$ with $\omega(A) = (\Omega'_\omega, \pi'_\omega(A)\Omega'_\omega)$, there is a unitary map $U : \mathcal{H}_\omega \to \mathcal{H}'_\omega$ such that

$$\begin{tikzcd}
\mathcal{H}_\omega \arrow[r, "\pi_\omega(A)"] \arrow[d, "U"] & \mathcal{H}_\omega \arrow[d, "U"] \\
\mathcal{H}'_\omega \arrow[r, "\pi'_\omega(A)"] & \mathcal{H}'_\omega
\end{tikzcd} \tag{4.81a}$$

commutes for all $A \in \mathcal{A}$ and that

$$U\Omega_\omega = \Omega'_\omega \,. \tag{4.81b}$$

Due to the cyclicity of the representations, we can construct such a U easily by demanding (4.81b), defining

$$\forall A \in \mathcal{A} : U\psi_\omega^A = U\pi_\omega(A)\Omega_\omega = \pi'_\omega(A)U\Omega_\omega = \pi'_\omega(A)\Omega'_\omega = \psi'^{A}_\omega \tag{4.82}$$

and extending it to all of $\mathcal{H}_\omega$. This map respects the inner products

$$\begin{aligned}(U\psi_\omega^A, U\psi_\omega^B) &= (U\pi_\omega(A)\Omega_\omega, U\pi_\omega(B)\Omega_\omega) = (\pi'_\omega(A)\Omega'_\omega, \pi'_\omega(B)\Omega'_\omega) \\ &= \omega(A^*B) = (\pi_\omega(A)\Omega_\omega, \pi_\omega(B)\Omega_\omega) = (\psi_\omega^A, \psi_\omega^B)\end{aligned} \tag{4.83}$$

and the existence of the inverse map $U^{-1} : \mathcal{H}'_\omega \to \mathcal{H}_\omega$ can be verified by explicit construction with $U^{-1}\Omega'_\omega = \Omega_\omega$ and

$$\forall A \in \mathcal{A} : U^{-1}\psi'^{A}_\omega = U^{-1}\pi'_\omega(A)\Omega'_\omega = \pi_\omega(A)U^{-1}\Omega'_\omega = \pi_\omega(A)\Omega_\omega = \psi_\omega^A \,. \tag{4.84}$$

□

The unitary equivalence of all representations constructed from a state ω means that the result of the GNS construction is essentially unique. Furthermore, it implies that symmetries, i. e. automorphisms, of a C^*-algebra are realized in the GNS representations by unitary operators.

Corollary 4.8. *If $\phi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of C^*-algebras, then we have for the GNS representations derived from the states $\omega : \mathcal{B} \to \mathbf{C}$ and $\omega \circ \phi : \mathcal{A} \to \mathbf{C}$*

$$\pi_{\omega\circ\phi} = \pi_\omega \circ \phi \,, \tag{4.85}$$

upto unitary equivalence.

Proof. The situation is clarified by the commutative diagram

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "\omega\circ\phi"'] \arrow[r, "\pi_{\omega\circ\phi}"] \arrow[dd, "\phi"] & \mathcal{L}(\mathcal{H}_{\omega\circ\phi}) \arrow[dd, dash, "\cong"] \\ \mathbf{C} & & \\ & \mathcal{B} \arrow[ul, "\omega"] \arrow[r, "\pi_\omega"] & \mathcal{L}(\mathcal{H}_\omega) \end{tikzcd} \tag{4.86}$$

Given the representations $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ and $(\mathcal{H}_{\omega\circ\phi}, \pi_{\omega\circ\phi}, \Omega_{\omega\circ\phi})$, we can compute

$$(\Omega_{\omega\circ\phi}, \pi_{\omega\circ\phi}(A)\Omega_{\omega\circ\phi}) = (\omega\circ\phi)(A) = \omega(\phi(A)) = (\Omega_\omega, \pi_\omega(\phi(A))\Omega_\omega) \tag{4.87}$$

and by theorem 4.7 we infer that, up to unitary equivalence,

$$\pi_{\omega\circ\phi} = \pi_\omega \circ \phi \,. \tag{4.88}$$

□

Corollary 4.9 (unitary realization of automorphisms)**.** *If $\tau : \mathcal{A} \to \mathcal{A}$ is an automorphism of a C^*-algebra that leaves a state $\omega : \mathcal{A} \to \mathbf{C}$ invariant, i. e. $\omega \circ \tau = \omega$ or*

$$\forall A \in \mathcal{A} : \omega(\tau(A)) = \omega(A) \,, \tag{4.89}$$

then there exists a unique *unitary $U_\omega^\tau : \mathcal{H}_\omega \to \mathcal{H}_\omega$ with*

$$U_\omega^\tau \pi_\omega(A) \left(U_\omega^\tau\right)^{-1} = \pi_\omega(\tau(A)) \tag{4.90a}$$

$$U_\omega^\tau \Omega_\omega = \Omega_\omega \,. \tag{4.90b}$$

Proof. This is simple consequence of corollary 4.8 and

$$\omega \circ \tau = \omega \,. \tag{4.91}$$

□

An important result concerns the relation of the purity of states and irreducibility of the corresponding GNS representations:

Theorem 4.10 (purity vs. irreducibility)**.** *If ω is a state over a C^*-algebra $\mathcal{A}$ and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the associated GNS representations of $\mathcal{A}$, then the following statements are equivalent*

1. *the representation $(\mathcal{H}_\omega, \pi_\omega)$ is irreducible*
2. *ω is pure.*

In addition, there is a one-to-one correspondence between the positive functionals $\omega_T \leqslant \omega$ and positive operators T with $T \leqslant 1_{\mathcal{H}_\omega}$ and commuting with all operators in $\pi_\omega(\mathcal{A})$. This correspondence is given by

$$\omega_T(A) = (T\Omega_\omega, \pi_\omega(A)\Omega_\omega) \;. \tag{4.92}$$

Proof. In order to prove that 2) follows from 1), assume that ω is not pure. Then there is a positive $\rho \neq \lambda\omega, \forall\lambda \in \mathbf{R}_+$ majorized by ω, i.e. $\forall A \in \mathcal{A} : 0 \leqslant \rho(A^*A) \leqslant \omega(A^*A)$. Using once more the CSI

$$\begin{aligned} |\rho(B^*A)|^2 \leqslant \rho(B^*B)\rho(A^*A) \leqslant \omega(B^*B)\omega(A^*A) \\ = \|\pi_\omega(B)\Omega_\omega\|^2 \|\pi_\omega(A)\Omega_\omega\|^2 = \|\psi_\omega^B\|^2 \|\psi_\omega^A\|^2 \,, \end{aligned} \tag{4.93}$$

we see that

$$\begin{aligned} \hat\rho : \mathcal{H}_\omega \times \mathcal{H}_\omega &\to \mathbf{C} \\ (\psi_\omega^B, \psi_\omega^A) &\mapsto \rho(B^*A) \end{aligned} \tag{4.94}$$

is a densly defined, bounded sesquilinear functional. By the Riesz representation theorem, there is a bounded operator $T \in \mathcal{L}(\mathcal{H}_\omega)$, such that

$$\rho(B^*A) = \left(\psi_\omega^B, T\psi_\omega^A\right) = (\pi_\omega(B)\Omega_\omega, T\pi_\omega(A)\Omega_\omega) \;. \tag{4.95}$$

Since ρ is not a multiple of ω, this T must not be a multiple of $\mathbf{1}_{\mathcal{H}_\omega}$. From

$$\begin{aligned} 0 \leqslant (\pi_\omega(A)\Omega_\omega, T\pi_\omega(A)\Omega_\omega) = \rho(A^*A) \\ \leqslant \omega(A^*A) = (\pi_\omega(A)\Omega_\omega, \pi_\omega(A)\Omega_\omega) \end{aligned} \tag{4.96}$$

we conclude

$$0 < T < \mathbf{1}_{\mathcal{H}_\omega} \,. \tag{4.97}$$

Furthermore

$$\forall C \in \mathcal{A} : (\pi_\omega(B)\Omega_\omega, T\pi_\omega(C)\pi_\omega(A)\Omega_\omega) = \rho(B^*CA) = \rho((C^*B)^*A)$$

$$= (\pi_\omega(C)^*\pi_\omega(B)\Omega_\omega, T\pi_\omega(A)\Omega_\omega) = (\pi_\omega(B)\Omega_\omega, \pi_\omega(C)T\pi_\omega(A)\Omega_\omega) \tag{4.98}$$

i. e.

$$\forall C \in \mathcal{A} : T\pi_\omega(C) = \pi_\omega(C)T \tag{4.99}$$

or $T \in \pi_\omega(\mathcal{A})'$ and $(\mathcal{H}_\omega, \pi_\omega)$ is reducible.

OTOH, assume that $(\mathcal{H}_\omega, \pi_\omega)$ is reducible. Then there is a $T \in \pi_\omega(\mathcal{A})'$ with $T \neq \lambda \mathbf{1}_{\mathcal{H}_\omega}$. Since $(\pi_\omega(\mathcal{A}))^* = \pi_\omega(\mathcal{A})$, also T^*, $T + T^*$ and $(T - T^*)/\mathrm{i}$ lie in the commutant $\pi_\omega(\mathcal{A})'$. Consequently, there is a self-adjoint $S \in \pi_\omega(\mathcal{A})'$ with $S \neq \lambda \mathbf{1}_{\mathcal{H}_\omega}$ and a corresponding spectral projection $P \in \pi_\omega(\mathcal{A})'$ with $0 < P < \mathbf{1}_{\mathcal{H}_\omega}$. This allows to define a $\rho : \mathcal{A} \to \mathbf{C}$ via

$$\rho(A) = (P\Omega_\omega, \pi_\omega(A)\Omega_\omega)\ . \tag{4.100}$$

This ρ is positive, because

$$\begin{aligned}\rho(A^*A) &= (P\Omega_\omega, \pi_\omega(A^*A)\Omega_\omega) \\ &= (\pi_\omega(A)P\Omega_\omega, \pi_\omega(A)\Omega_\omega) = (P\pi_\omega(A)\Omega_\omega, P\pi_\omega(A)\Omega_\omega) \geqslant 0\,.\end{aligned} \tag{4.101}$$

Also

$$\omega(A^*A) - \rho(A^*A) = (\pi_\omega(A)\Omega_\omega, (\mathbf{1} - P)\pi_\omega(A)\Omega_\omega) \geqslant 0 \tag{4.102}$$

and ω majorizes ρ. But ρ can not be a multiple of ω, because P is not a multiple of $\mathbf{1}_{\mathcal{H}_\omega}$ and therefore ω is not pure. This shows that 1) follows from 2).

The correspondence between postive functionals and postive operators has been shown by construction *en passant*. □

Note that this relation of purity and irreducibility applies only to the GNS representation. One can realize an impure state with a density matrix in an irreducible representation, but this is *not* the GNS representation associated with this state.

—5— Quantum Mechanics

5.1 Weyl Systems

Definition 5.1. A *symplectic vector* space (V, Θ) is a vector space with V a real vector space and $\Theta : V \times V \to \mathbf{R}$ a non-degenerate antisymmetric bilinear map, i. e. $\forall v, u, w \in V, \alpha, \beta \in \mathbf{R}$

$$\Theta(v, u) = -\Theta(u, v) \tag{5.1a}$$

$$\Theta(v, \alpha u + \beta w) = \alpha\Theta(v, u) + \beta\Theta(v, w) \tag{5.1b}$$

$$(\forall v \in V : \Theta(v, u) = 0) \Rightarrow u = 0\,. \tag{5.1c}$$

If the dimension of V is finite, it is necessarily even.

Example 5.2 (*canonical phase space*). Consider a flat n-dimensional configuration space $q \in \mathbf{R}^n$ and the corresponding phase space

$$z = (z_1, \ldots, z_{2n}) = (q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbf{R}^{2n} = V\,. \tag{5.2}$$

The Poisson brackets $(i, j = 1, \ldots, n)$

$$\{q_i, p_j\} = \delta^{ij} \tag{5.3a}$$

$$\{q_i, q_j\} = 0 \tag{5.3b}$$

$$\{p_i, p_j\} = 0 \tag{5.3c}$$

define a symplectic structure $(\alpha, \beta = 1, \ldots, 2n)$

$$\{z_\alpha, z_\beta\} = \theta^{\alpha\beta} \tag{5.4}$$

with the antisymmetric $2n \times 2n$-matrix

$$\theta = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}\,. \tag{5.5}$$

This corresponds to a symplectic form

$$
\begin{aligned}
\Theta : \mathbf{R}^{2n} \times \mathbf{R}^{2n} &\to \mathbf{R} \\
\big(\underbrace{(q,p)}_{z}, \underbrace{(q',p')}_{z'}\big) &\mapsto \sum_{\alpha,\beta=1}^{2n} z_\alpha \theta^{\alpha\beta} z'_\beta = \sum_{i=1}^{n} (q_i p'_i - p_i q'_i) \,.
\end{aligned}
\tag{5.6}
$$

Example 5.3 (*complex vector space*). Using the canonical identification

$$\mathbf{C}^n \ni (x_1 + \mathrm{i}y_1, \ldots, x_n + \mathrm{i}y_n) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbf{R}^{2n} \tag{5.7}$$

the preceding example is equivalent to

$$
\begin{aligned}
\Theta : \mathbf{C}^n \times \mathbf{C}^n &\to \mathbf{R} \\
(z, z') &\mapsto \sum_{i=1}^{n} \operatorname{Im} \bar z_i z'_i \,.
\end{aligned}
\tag{5.8}
$$

Lecture 15: Wed, 03. 12. 2025

Example 5.4 (*Peierls brackets*). Consider the vector space $\mathcal{S}_m(\mathbf{R}^{n+1})$ of (smooth) solutions $\phi : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ of the *Klein-Gordon equation* with mass m

$$\left(\frac{\partial^2}{\partial x_0^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + m^2 \right) \phi(x) = \left(\Box + m^2\right) \phi(x) = 0 \,. \tag{5.9}$$

that fall off "sufficiently fast" at "spatial infinity", i. e. $|x_{1,\ldots,n}| \to \infty$. The *Peierls bracket*

$$
\begin{aligned}
\langle\,\cdot\,,\,\cdot\,\rangle : \mathcal{S}_m(\mathbf{R}^{n+1}) \times \mathcal{S}_m(\mathbf{R}^{n+1}) &\to \mathbf{R} \\
(\phi, \psi) &\mapsto \langle \phi, \psi \rangle = \langle \phi, \psi \rangle_t
\end{aligned}
\tag{5.10}
$$

with

$$\langle \phi, \psi \rangle_t = \int_{x_0 = t} \mathrm{d}^n x \, \left(\phi(x) \frac{\partial \psi}{\partial x_0}(x) - \frac{\partial \phi}{\partial x_0}(x) \psi(x) \right) \tag{5.11}$$

is, due to the Klein-Gordon equation and falloff at spatial infinity, independent of t and can be used to turn $\mathcal{S}_m(\mathbf{R}^{n+1})$ into the symplectic vector space $(\mathcal{S}_m(\mathbf{R}^{n+1}), \langle\,\cdot\,,\,\cdot\,\rangle)$. The nondegeneracy can shown by considering the initial conditions for the Klein-Gordon equation *(→ exercise)*.

Definition 5.5. A *Weyl system* $(\mathcal{A}, W)$ of a symplectic vector space (V, Θ) is a C^*-algebra $\mathcal{A}$ together with a map $W : V \to \mathcal{A}$, such that $\forall \phi, \psi \in V$

$$W(0) = \mathbf{1} \tag{5.12a}$$
$$(W(\phi))^* = W(-\phi) \tag{5.12b}$$
$$W(\phi)W(\psi) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\Theta(\phi,\psi)} W(\phi + \psi)\,. \tag{5.12c}$$

Note that we don't require W to be continuous. In fact, we don't even assume V to have a topology.

Example 5.6 (one particle *quantum mechanics* in one dimension). Consider $V = \mathbf{R}^2$ with

$$\begin{aligned} \Theta : V \times V &\to \mathbf{R} \\ ((\xi_1, \eta_1), (\xi_2, \eta_2)) &\mapsto \eta_1 \xi_2 - \xi_1 \eta_2\,. \end{aligned} \tag{5.13}$$

Then the composition law in the corresponding Weyl system is

$$W(\xi_1, \eta_1) W(\xi_2, \eta_2) = \mathrm{e}^{\frac{\mathrm{i}}{2}(\xi_1\eta_2 - \eta_1\xi_2)} W(\xi_1 + \xi_2, \eta_1 + \eta_2)\,. \tag{5.14}$$

The connection with quantum mechanics is to *formally* write

$$W(\xi, \eta) = \mathrm{e}^{\mathrm{i}\xi p + \mathrm{i}\eta x} \tag{5.15}$$

with the *unbounded* hermitian operators

$$x = x^* \tag{5.16a}$$
$$p = p^* \tag{5.16b}$$
$$[x, p] = \mathrm{i} \tag{5.16c}$$
$$[x, x] = [p, p] = 0 \tag{5.16d}$$

and to use

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]} = \mathrm{e}^{A+B} \mathrm{e}^{\frac{1}{2}[A,B]} \tag{5.17}$$

for $[A, [A, B]] = [[A, B], B] = 0$. Then we obtain

$$\begin{aligned} W(\xi_1, \eta_1) W(\xi_2, \eta_2) &= \mathrm{e}^{\mathrm{i}\xi_1 p + \mathrm{i}\eta_1 x} \mathrm{e}^{\mathrm{i}\xi_2 p + \mathrm{i}\eta_2 x} \\ &= \mathrm{e}^{\mathrm{i}\xi_1 p + \mathrm{i}\eta_1 x + \mathrm{i}\xi_2 p + \mathrm{i}\eta_2 x} \mathrm{e}^{-\frac{1}{2}[\xi_1 p + \eta_1 x, \xi_2 p + \eta_2 x]} = \mathrm{e}^{\mathrm{i}(\xi_1+\xi_2)p + \mathrm{i}(\eta_1+\eta_2)x} \mathrm{e}^{\frac{\mathrm{i}}{2}(\xi_1\eta_2 - \eta_1\xi_2)} \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}(\xi_1\eta_2 - \eta_1\xi_2)} W(\xi_1 + \xi_2, \eta_1 + \eta_2) \end{aligned} \tag{5.18}$$

i. e., up to domain issues for unbounded operators, the *canonical commutation relations* (5.16) are equivalent to the *Weyl form of the canonical commutation relations* (5.14).

Definition 5.7. The algebra $\mathcal{A}_W$ generated by the $W(\xi,\eta)$, i. e. the completion of the linear span of the $W(\xi,\eta)$ is called the *Weyl algebra*.

Example 5.8 (n particle quantum mechanics). Consider $V = \mathbf{R}^{2n}$ with

$$\begin{aligned}\Theta : V \times V &\to \mathbf{R} \\ ((\xi_1,\eta_1),(\xi_2,\eta_2)) &\mapsto \sum_{i=1}^{n} (\eta_{1,i}\xi_{2,i} - \xi_{1,i}\eta_{2,i}) \,.\end{aligned} \tag{5.19}$$

Then the composition law in the corresponding Weyl system is

$$W(\xi_1,\eta_1)W(\xi_2,\eta_2) = \mathrm{e}^{\frac{\mathrm{i}}{2}\sum_{i=1}^{n}(\xi_{1,i}\eta_{2,i}-\eta_{1,i}\xi_{2,i})} W(\xi_1+\xi_2,\eta_1+\eta_2)\,. \tag{5.20}$$

The connection with quantum mechanics is again to *formally* write

$$W(\xi,\eta) = \exp\left(\mathrm{i}\sum_{i=1}^{n}(\xi_i p_i + \eta_i x_i)\right) \,. \tag{5.21}$$

Example 5.9 (*general Weyl system*). Let $\mathcal{H} = L^2(V,\mathbf{C})$ be the Hilbert space of all functions $f : V \to \mathbf{C}$ that are square integrable wrt the *counting measure*[1], i. e. $f \in \mathcal{H}$, iff f vanishes everywhere except for *countably* many points $\phi \in V$ and

$$\|f\|_{\mathcal{H}}^2 = \sum_{\phi\in V} |f(\phi)|^2 < \infty \,. \tag{5.22}$$

The inner product on $\mathcal{H}$ is then accordingly

$$\langle f, g\rangle_{\mathcal{H}} = \sum_{\phi\in V} \overline{f(\phi)} g(\phi)\,. \tag{5.23}$$

Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be the C^*-algebra of bounded operators on $\mathcal{H}$ and define

$$\begin{aligned} W : V &\to \mathcal{A} \\ \phi &\mapsto W(\phi)\end{aligned} \tag{5.24a}$$

where the action of $W(\phi) : \mathcal{H} \to \mathcal{H}$ is given by

$$(W(\phi)f)(\psi) = \mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\phi,\psi)} f(\phi+\psi)\,. \tag{5.24b}$$

[1] Note that everything works for $L^2(V,\mathbf{C})$ with more general measures, including the Lebesgue measure, except for statement 4) of theorem 5.10, whose proof uses an induction argument.

It's easy to check (5.12): each $W(\phi)$ is obviously bounded

$$\forall \phi \in V, f \in \mathcal{H} : \|W(\phi)f\| = \|f\| \tag{5.25}$$

and (5.12a) is also obvious

$$\forall f \in \mathcal{H} : (W(0)f)(\psi) = f(\psi) . \tag{5.26}$$

Verifying (5.12b) requires a short calculation

$$\begin{aligned}
\forall \phi \in V, f, g \in \mathcal{H} : \left\langle f, (W(\phi))^* g\right\rangle_{\mathcal{H}} &= \langle W(\phi)f, g\rangle_{\mathcal{H}} \\
&= \sum_{\psi\in V} \overline{(W(\phi)f)(\psi)} g(\psi) = \sum_{\psi\in V} \overline{\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\phi,\psi)} f(\phi+\psi)} g(\psi) \\
&\stackrel{\chi=\phi+\psi}{=} \sum_{\chi\in V} \overline{\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\phi,\chi-\phi)} f(\chi)} g(\chi-\phi) = \sum_{\chi\in V} \overline{f(\chi)} \mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(-\phi,\chi)} g(\chi-\phi) \\
&= \sum_{\chi\in V} \overline{f(\chi)} \left(W(-\phi)g\right)(\chi) = \langle f, W(-\phi)g\rangle_{\mathcal{H}}
\end{aligned} \tag{5.27}$$

similarly for (5.12c)

$$\begin{aligned}
\forall \phi, \psi, \chi \in V, f \in \mathcal{H} : \left(\left(W(\phi)W(\psi)\right) f\right)(\chi) &= \left(W(\phi) \left(W(\psi)f\right)\right)(\chi) \\
&= \mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\phi,\chi)} \left(W(\psi)f\right)(\phi+\chi) = \mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\phi,\chi)} \mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\psi,\phi+\chi)} f(\psi+\phi+\chi) \\
&= \mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\psi,\phi)} \mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\phi+\psi,\chi)} f(\psi+\phi+\chi) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\Theta(\phi,\psi)} \left(W(\phi+\psi)f\right)(\chi) \\
&= \left(\mathrm{e}^{-\frac{\mathrm{i}}{2}\Theta(\phi,\psi)} W(\phi+\psi) f\right)(\chi) .
\end{aligned} \tag{5.28}$$

Theorem 5.10. *Let $(\mathcal{A}, W)$ be a Weyl system of a symplectic vector space (V, Θ), then*

1. *$W(\phi)$ is unitary for all $\phi \in V$,*
2. *$\|W(\phi) - W(\psi)\| = 2$ for all $\phi \neq \psi \in V$,*
3. *$\mathcal{A}$ is* not *separable, unless $V = \{0\}$,*
4. *the family $\{W(\phi)\}_{\phi\in V}$ is linearly independent.*

Proof.

1. Obviously

$$(W(\phi))^* W(\phi) = W(-\phi)W(\phi) = \mathbf{1} \tag{5.29a}$$

$$W(\phi) (W(\phi))^* = W(\phi)W(-\phi) = \mathbf{1} . \tag{5.29b}$$

2. $\forall\phi,\chi\in V$:

$$W(\chi)W(\phi)\left(W(\chi)\right)^{-1} = W(\chi)W(\phi)W(-\chi)$$
$$= \mathrm{e}^{-\frac{\mathrm{i}}{2}\Theta(\chi,\phi)}W(\chi+\phi)W(-\chi) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\Theta(\chi,\phi)}\mathrm{e}^{-\frac{\mathrm{i}}{2}\Theta(\phi,-\chi)}W(\phi)$$
$$= \mathrm{e}^{-\mathrm{i}\Theta(\chi,\phi)}W(\phi)\,. \quad (5.30)$$

Note that $z\mathbf{1}-A$ is invertible, iff $U(z\mathbf{1}-A)U^{-1} = z\mathbf{1}-UAU^{-1}$ is invertible for invertible U. Therefore the spectrum is invariant under similarity transformations

$$\sigma(A) = \sigma\left(UAU^{-1}\right) \quad (5.31)$$

and (5.30) implies

$$\sigma\left(W(\phi)\right) = \sigma\left(W(\chi)W(\phi)\left(W(\chi)\right)^{-1}\right) = \mathrm{e}^{-\mathrm{i}\Theta(\chi,\phi)}\sigma\left(W(\phi)\right)\,. \quad (5.32)$$

If $\phi\neq 0$ and Θ is non degenerate, there is a χ such that $V\ni\phi\mapsto\Theta(\chi,\phi)$ is *onto* $\mathbf{R}$ and $V\ni\phi\mapsto\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\chi,\phi)}$ is onto $S^1=\{z\in\mathbf{C}:|z|=1\}$. Thus, for $\phi\neq 0$, the spectrum of $W(\phi)$ is invariant under *arbitrary* rotations $z\mapsto\mathrm{e}^{\mathrm{i}\alpha}z$. Since $\sigma\left(W(\phi)\right)\neq\emptyset$ by theorem 2.11 and, by unitarity, $\sigma\left(W(\phi)\right)\subseteq S^1$ we conclude that

$$\forall\phi\neq 0:\sigma\left(W(\phi)\right) = S^1\,. \quad (5.33)$$

Therefore

$$\forall\phi\neq\psi:\sigma\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\psi,\phi)}W(\phi-\psi)-\mathbf{1}\right) = S^1-1 \quad (5.34)$$

and since $\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\psi,\phi)}W(\phi-\psi)-\mathbf{1}$ is normal, the norm is given by the spectral radius

$$\forall\phi\neq\psi:\left\|\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\psi,\phi)}W(\phi-\psi)-\mathbf{1}\right\| = \rho\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\psi,\phi)}W(\phi-\psi)-\mathbf{1}\right) = 2\,. \quad (5.35)$$

Using

$$W(\phi)-W(\psi) = W(\psi)\left(W(-\psi)W(\phi)-\mathbf{1}\right)$$
$$= W(\psi)\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\psi,\phi)}W(\phi-\psi)-\mathbf{1}\right)\,, \quad (5.36)$$

we obtain the desired result

$$\left\|W(\phi)-W(\psi)\right\|^2 = \left\|\left(W(\phi)-W(\psi)\right)^*\left(W(\phi)-W(\psi)\right)\right\|$$

$$= \left\| \left(\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\psi,\phi)} W(\phi-\psi) - \mathbf{1}\right)^* (W(\psi))^* W(\psi) \left(\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\psi,\phi)} W(\phi-\psi) - \mathbf{1}\right) \right\|$$
$$= \left\| \left(\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\psi,\phi)} W(\phi-\psi) - \mathbf{1}\right)^* \left(\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\psi,\phi)} W(\phi-\psi) - \mathbf{1}\right) \right\|$$
$$= \left\| \mathrm{e}^{\frac{\mathrm{i}}{2}\Theta(\psi,\phi)} W(\phi-\psi) - \mathbf{1} \right\|^2 = 4\,. \tag{5.37}$$

3. Since $\|W(\phi) - W(\psi)\| = 2$, the open balls of radius 1 that are centered at $\phi \in V$ are a collection of disjoint open sets that is *not* countable.

Lecture 16: Tue, 09. 12. 2025

4. Assume that there is a $\{\phi_i\}_{i=1,\ldots,n}$ such that $\phi_i \neq \phi_j$ for $i \neq j$ and
$$\sum_{i=1}^n \alpha_i W(\phi_i) = 0\,. \tag{5.38}$$
We will show by induction on n that $\alpha_i = 0$. The case $n = 1$ is trivial, because all $W(\phi)$ are unitary and thus never null. For the induction step, assume WLOG that the $\{W(\phi_i)\}_{i=1,n-1}$ are linearly independent and $\alpha_n \neq 0$. Then
$$W(\phi_n) = -\sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} W(\phi_i) \tag{5.39}$$
and for all $\psi \in V$
$$\mathrm{e}^{-\mathrm{i}\Theta(\psi,\phi_n)} \left(-\sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} W(\phi_i)\right) \overset{(5.39)}{=} \mathrm{e}^{-\mathrm{i}\Theta(\psi,\phi_n)} W(\phi_n)$$
$$\overset{(5.30)}{=} W(\psi) W(\phi_n) W(-\psi)$$
$$\overset{(5.39)}{=} -\sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} W(\psi) W(\phi_i) W(-\psi) \overset{(5.30)}{=} -\sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} \mathrm{e}^{-\mathrm{i}\Theta(\psi,\phi_i)} W(\phi_i) \tag{5.40}$$
i. e.
$$\sum_{i=1}^{n-1} \alpha_i W(\phi_i) = \sum_{i=1}^{n-1} \alpha_i \mathrm{e}^{\mathrm{i}\Theta(\psi,\phi_n-\phi_i)} W(\phi_i) \tag{5.41}$$
or, by the induction hypothesis that the $\{W(\phi_i)\}_{i=1,\ldots,n-1}$ are independent
$$\forall i = 1,\ldots,n-1 : \alpha_i = \alpha_i \mathrm{e}^{\mathrm{i}\Theta(\psi,\phi_n-\phi_i)}\,. \tag{5.42}$$

If $\alpha_i \neq 0$ for $i < n$, then

$$\forall\psi \in V : \Theta(\psi, \phi_n - \phi_i) = 0 \mod 2\pi\,. \tag{5.43}$$

By linearity

$$\forall\psi \in V : \Theta(\psi, \phi_n - \phi_i) = 0 \tag{5.44}$$

and since Θ is non-degenerate we find the contradiction

$$\phi_n = \phi_i\,. \tag{5.45}$$

Thus $\forall i = 1, \ldots, n : \alpha_i = 0$.

□

5.2 *The Stone-von Neumann Theorem*

In the following, it will sometimes be convenient to use in place of the $W(\xi, \eta)$ from example 5.6 Weyl's

$$U(\eta) = W(0, \eta) \tag{5.46a}$$
$$V(\xi) = W(\xi, 0)\,, \tag{5.46b}$$

which form two abelian one-parameter groups of unitary elements of $\mathcal{A}_W$

$$U(\eta)U(\eta') = U(\eta + \eta') \tag{5.47a}$$
$$V(\xi)V(\xi') = V(\xi + \xi')\,. \tag{5.47b}$$

The general $W(\xi, \eta)$ can of course be expressed by $U(\eta)$ and $V(\xi)$ since

$$U(\eta)V(\xi) = W(0, \eta)W(\xi, 0) = W(\xi, \eta)\mathrm{e}^{-\frac{\mathrm{i}}{2}\eta\xi} \tag{5.48a}$$
$$V(\xi)U(\eta) = W(\xi, 0)W(0, \eta) = W(\xi, \eta)\mathrm{e}^{\frac{\mathrm{i}}{2}\xi\eta}\,, \tag{5.48b}$$

i. e.

$$W(\xi, \eta) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\eta\xi}V(\xi)U(\eta) = \mathrm{e}^{\frac{\mathrm{i}}{2}\eta\xi}U(\eta)V(\xi)\,. \tag{5.49}$$

The analogous notation can also be used for the n degress of freedom in example 5.8.

Definition 5.11 (*regular representation*)**.** A representation $(\mathcal{H}, \pi)$ of the Weyl algebra $\mathcal{A}_W$ on a separable Hilbert space $\mathcal{H}$ is called *regular*, if $\mathbf{R} \ni \xi \mapsto \pi(V(\xi)) = \pi(W(\xi, 0))$ and $\mathbf{R} \ni \eta \mapsto \pi(U(\eta)) = \pi(W(0, \eta))$ are strongly continuous in ξ and η, respectively.

Theorem 5.12 (*Stone-von Neumann*). *All regular irreducible representations of the Weyl algebra $\mathcal{A}_W$ of a finite dimensional symplectic vector space are unitarily equivalent.*

Proof. Starting from the *Fock state* $\omega_F : \mathcal{A}_W \to \mathbf{C}$, defined by

$$\omega_F(W(\xi,\eta)) = \exp\left(-\frac{\xi^2+\eta^2}{4}\right) \tag{5.50}$$

and linearity, we construct the corresponding GNS representation $(\mathcal{H}_F, \pi_F, \Omega_F)$ with

$$(\Omega_F, \pi_F(W(\xi,\eta))\Omega_F) = \omega_F(W(\xi,\eta)) \,. \tag{5.51}$$

Now consider another regular representation π and compute the integral

$$P_\pi = \int \frac{\mathrm{d}\xi \mathrm{d}\eta}{2\pi} \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}} \pi(W(\xi,\eta)) \,. \tag{5.52}$$

This integral exists as a limit of Riemann sums in the strong operator topology, because $(\xi,\eta) \mapsto \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}} \in L^1(\mathbf{R}^2)$ and $\pi(W(\xi,\eta))$ is both bounded and continuous in ξ and η. P_π is obviously self-adjoint

$$P_\pi^* = \int \frac{\mathrm{d}\xi \mathrm{d}\eta}{2\pi} \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}} \pi(W(-\xi,-\eta)) = P_\pi \,. \tag{5.53}$$

It is not obvious that P_π doesn't vanish. However, if this was the case, then using

$$\begin{aligned} W(-\xi',-\eta')W(\xi,\eta)W(\xi',\eta') &= \mathrm{e}^{-\frac{\mathrm{i}}{2}(\xi'\eta-\eta'\xi)} W(\xi-\xi',\eta-\eta')W(\xi',\eta') \\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2}(\xi'\eta-\eta'\xi)} \mathrm{e}^{\frac{\mathrm{i}}{2}(\xi\eta'-\eta\xi')} W(\xi,\eta) = \mathrm{e}^{\mathrm{i}\xi\eta'-\mathrm{i}\eta\xi'} W(\xi,\eta) \end{aligned} \tag{5.54}$$

(that's (5.30) again), we find

$$\begin{aligned} 0 &= \pi(W(-\xi',-\eta'))P_\pi \pi(W(\xi',\eta')) \\ &= \int \frac{\mathrm{d}\xi \mathrm{d}\eta}{2\pi} \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}} \pi(W(-\xi',-\eta')W(\xi,\eta)W(\xi',\eta')) \\ &= \int \frac{\mathrm{d}\xi \mathrm{d}\eta}{2\pi} \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}} \pi(W(\xi,\eta)) \mathrm{e}^{\mathrm{i}\xi\eta'-\mathrm{i}\eta\xi'} \,, \end{aligned} \tag{5.55}$$

i. e. that the Fourier transform of all matrix elements of

$$\mathrm{e}^{-\frac{\xi^2+\eta^2}{4}} \pi(W(\xi,\eta)) \tag{5.56}$$

would vanish, which would imply $\pi(W(\xi,\eta)) = 0$. Also, using

$$\begin{aligned}W(\xi_1,\eta_1)W(\xi,\eta)W(\xi_2,\eta_2) &= \mathrm{e}^{\frac{\mathrm{i}}{2}(\xi_1\eta-\eta_1\xi)}W(\xi+\xi_1,\eta+\eta_1)W(\xi_2,\eta_2) \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}(\xi_1\eta-\eta_1\xi)}\mathrm{e}^{\frac{\mathrm{i}}{2}((\xi+\xi_1)\eta_2-(\eta+\eta_1)\xi_2)}W(\xi+\xi_1+\xi_2,\eta+\eta_1+\eta_2) \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}(\xi_1\eta+\xi\eta_2+\xi_1\eta_2-\eta_1\xi-\eta\xi_2-\eta_1\xi_2)}W(\xi+\xi_1+\xi_2,\eta+\eta_1+\eta_2) \end{aligned} \tag{5.57}$$

we find by a straightforward computation of two Gaussian integrals *(→ exercise)*

$$P_\pi\pi(W(\xi,\eta))P_\pi = \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}}P_\pi\,. \tag{5.58}$$

In particular from

$$P_\pi^2 = P_\pi\pi(W(0,0))P_\pi = P_\pi\,, \tag{5.59}$$

we see that $P_\pi \neq 0$ is an orthogonal projection. Thus there must be a $\Omega_0 \in \mathcal{H}_\pi$ with

$$P_\pi\Omega_0 = \Omega_0 \tag{5.60a}$$

$$\|\Omega_0\| = 1 \tag{5.60b}$$

and we can compute

$$\begin{aligned}(\Omega_0,\pi(W(\xi,\eta))\Omega_0) &= (\Omega_0,P_\pi\pi(W(\xi,\eta))P_\pi\Omega_0) \\ &\overset{(5.58)}{=} \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}}\underbrace{(\Omega_0,P_\pi\Omega_0)}_{\|\Omega_0\|^2} = \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}} = \omega_F(W(\xi,\eta))\,. \end{aligned} \tag{5.61}$$

This shows that $\Omega_0 = \Omega_\pi$ (up to unitary equivalence) and we have already shown in theorem 4.7 that all GNS representations corresponding to the same state are unitarily equivalent. □

We have proven the theorem for one degree of freedom or $V = \mathbf{R}\times\mathbf{R}$, but the same argument works for any *finite* number of degrees of freedom. However, it breaks down for an infinite number of degrees of freedom, because then the integral in (5.52) doesn't exist straightforwardly.

Remark 5.13. P_π projects on a one-dimensional subspace.

Proof. Assume

$$\exists\Psi\in\mathcal{H}_\pi : (\Psi,\Omega_0) = 0 \wedge P_\pi\Psi = \Psi \neq 0, \tag{5.62}$$

then

$$\begin{aligned}\forall\xi,\eta\in\mathbf{R} : (\Psi,\pi(W(\xi,\eta))\Omega_0) &= (P_\pi\Psi,\pi(W(\xi,\eta))P_\pi\Omega_0) \\ &= \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}}(\Psi,P_\pi\Omega_0) = \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}}(\Psi,\Omega_0) = 0 \end{aligned} \tag{5.63}$$

and thus
$$\forall A \in \mathcal{A}_W : (\Psi, \pi(A)\Omega_0) = 0 \,. \tag{5.64}$$
Since every vector in $\mathcal{H}_\pi$ is cyclic in an irreducible representation, we conclude that $\Psi = 0$. □

Note that we needed the strong continuity of the representation only to show that the integral (5.52) exists. We could have relaxed this condition to measurability of

$$\xi \mapsto (\Psi, \pi(V(\xi))\Phi) \tag{5.65a}$$
$$\eta \mapsto (\Psi, \pi(U(\eta))\Phi) \tag{5.65b}$$

for all $\Psi, \Phi \in \mathcal{H}$. However there is a theorem by von Neumann (see, e. g., [RS80], theorem VIII.9, p. 268), that weak measurability implies strong continuity. Thus we may demand strong continuity, WLOG.

Using (5.48), we can compute the Fock state of a product of $U(\eta)$ and $V(\xi)$:

$$\omega_F(U(\eta)V(\xi)) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\xi\eta}\omega_F(W(\xi,\eta)) = \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}-\frac{\mathrm{i}}{2}\xi\eta} \,. \tag{5.66}$$

Computing the partial derivatives from this explicit expression

$$\frac{\partial}{\partial\xi}\omega_F(U(\eta)V(\xi)) = -\frac{\mathrm{i}\eta+\xi}{2}\omega_F(U(\eta)V(\xi)) \tag{5.67a}$$
$$\frac{\partial}{\partial\eta}\omega_F(U(\eta)V(\xi)) = -\frac{\mathrm{i}\xi+\eta}{2}\omega_F(U(\eta)V(\xi)) \,, \tag{5.67b}$$

we can derive the differental equation

$$\left(-\mathrm{i}\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta} + \eta\right)\omega_F(U(\eta)V(\xi)) = 0 \,. \tag{5.68}$$

Since $\eta \mapsto \pi_F(U(\eta))$ and $\xi \mapsto \pi_F(V(\xi))$ are strongly continuous, Stone's Theorem (see, e. g., [RS80], theorem VIII.8, p. 266), guarantees the existence of self-adjoint operators X and P that are in general unbounded, but defined on dense subdomains of $D_X, D_Y \subseteq \mathcal{H}_F$

$$X : D_X \to \mathcal{H}_F \tag{5.69a}$$
$$P : D_P \to \mathcal{H}_F \tag{5.69b}$$

with

$$\pi_F(U(\eta)) = \mathrm{e}^{\mathrm{i}\eta X} \tag{5.70a}$$

$$\pi_F(V(\xi)) = \mathrm{e}^{\mathrm{i}\xi P} \,. \tag{5.70b}$$

Note that I could have written X_F and P_F or X_{π_F} and P_{π_F} to stress the fact that the X and P constructed this way depend on the representation $(\mathcal{H}, \pi)$.

Thus we can write

$$-\mathrm{i}\frac{\partial}{\partial\xi}\pi_F(V(\xi)) = P\pi_F(V(\xi)) = \pi_F(V(\xi))P \tag{5.71a}$$

$$\frac{\partial}{\partial\eta}\pi_F(U(\eta)) = \mathrm{i}X\pi_F(U(\eta)) = \mathrm{i}\pi_F(U(\eta))X \tag{5.71b}$$

and find

$$\left(-\mathrm{i}\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta} + \eta\right)\pi_F(U(\eta))\pi_F(V(\xi)) = \pi_F(U(\eta))\left(P + \mathrm{i}X + \eta\right)\pi_F(V(\xi)) \tag{5.72}$$

i. e., from (5.68),

$$\begin{aligned} 0 = \left(-\mathrm{i}\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta} + \eta\right)&\left(\Omega_F, \pi_F(U(\eta)V(\xi))\Omega_F\right) \\ &= \left(\Omega_F, \pi_F(U(\eta))\left(P + \mathrm{i}X + \eta\right)\pi_F(V(\xi))\Omega_F\right) \,. \end{aligned} \tag{5.73}$$

Since

$$X = -\mathrm{i}\lim_{\eta\to 0}\frac{\partial}{\partial\eta}\pi_F(U(\eta)) \tag{5.74a}$$

$$P = -\mathrm{i}\lim_{\xi\to 0}\frac{\partial}{\partial\xi}\pi_F(V(\xi)) \,. \tag{5.74b}$$

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we can compute

$$\begin{aligned} &X\pi_F(W(\xi,\eta)) - \pi_F(W(\xi,\eta))X \\ &\quad = -\mathrm{i}\lim_{\eta'\to 0}\frac{\partial}{\partial\eta'}\left(\pi_F(W(0,\eta'))\pi_F(W(\xi,\eta)) - \pi_F(W(\xi,\eta))\pi_F(W(0,\eta'))\right) \\ &\quad = -\mathrm{i}\lim_{\eta'\to 0}\frac{\partial}{\partial\eta'}\left(\pi_F(W(\xi,\eta+\eta'))\mathrm{e}^{-\frac{\mathrm{i}}{2}\xi\eta'} - \pi_F(W(\xi,\eta+\eta'))\mathrm{e}^{\frac{\mathrm{i}}{2}\xi\eta'}\right) \\ &\quad = -2\lim_{\eta'\to 0}\frac{\partial}{\partial\eta'}\left(\pi_F(W(\xi,\eta+\eta'))\sin\frac{\xi\eta'}{2}\right) \\ &\quad = -2\lim_{\eta'\to 0}\left(\sin\frac{\xi\eta'}{2}\frac{\partial}{\partial\eta'}\pi_F(W(\xi,\eta+\eta')) + \pi_F(W(\xi,\eta+\eta'))\frac{\xi}{2}\cos\frac{\xi\eta'}{2}\right) \end{aligned}$$

$$= -\xi\pi_F(W(\xi,\eta)) \quad (5.75)$$

and analogously

$$\begin{aligned}
&P\pi_F(W(\xi,\eta)) - \pi_F(W(\xi,\eta))P \\
&\quad= -\mathrm{i}\lim_{\xi'\to 0}\frac{\partial}{\partial\xi'}\left(\pi_F(W(\xi',0))\pi_F(W(\xi,\eta)) - \pi_F(W(\xi,\eta))\pi_F(W(\xi',0))\right) \\
&\quad= -\mathrm{i}\lim_{\xi'\to 0}\frac{\partial}{\partial\xi'}\left(\pi_F(W(\xi+\xi',\eta))\mathrm{e}^{\frac{\mathrm{i}}{2}\xi'\eta} - \pi_F(W(\xi+\xi',\eta))\mathrm{e}^{-\frac{\mathrm{i}}{2}\xi'\eta}\right) \\
&\quad= 2\lim_{\xi'\to 0}\frac{\partial}{\partial\xi'}\left(\pi_F(W(\xi+\xi',\eta))\sin\frac{\xi'\eta}{2}\right) \\
&\quad= 2\lim_{\xi'\to 0}\left(\sin\frac{\xi'\eta}{2}\frac{\partial}{\partial\xi'}\pi_F(W(\xi+\xi',\eta)) + \pi_F(W(\xi+\xi',\eta))\frac{\eta}{2}\cos\frac{\xi'\eta}{2}\right) \\
&\quad= \eta\pi_F(W(\xi,\eta))\,, \qquad (5.76)
\end{aligned}$$

i. e.

$$X\pi_F(W(\xi,\eta)) = \pi_F(W(\xi,\eta))(X-\xi) \quad (5.77a)$$
$$P\pi_F(W(\xi,\eta)) = \pi_F(W(\xi,\eta))(P+\eta)\,. \quad (5.77b)$$

Using this, we can infer from (5.73)

$$\begin{aligned}
0 &= (\Omega_F, \pi_F(U(\eta))(X - \mathrm{i}P - \mathrm{i}\eta)\pi_F(V(\xi))\Omega_F) \\
&= (\Omega_F, (X - \mathrm{i}P)\pi_F(U(\eta))\pi_F(V(\xi))\Omega_F) \\
&= ((X + \mathrm{i}P)\Omega_F, \pi_F(U(\eta))\pi_F(V(\xi))\Omega_F) \qquad (5.78)
\end{aligned}$$

i. e.

$$\forall\Psi\in\mathcal{H}_F : ((X+\mathrm{i}P)\Omega_F, \Psi) = 0 \quad (5.79)$$

or

$$(X+\mathrm{i}P)\Omega_F = 0\,, \quad (5.80)$$

which should be familiar from the algebraic solution of the harmonic oscillator

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 = a^*a + \frac{1}{2} \quad (5.81)$$

with

$$a = \frac{1}{\sqrt{2}}(x + \mathrm{i}p) \quad (5.82)$$

in elementary quantum mechanics.

5.3 Schrödinger Representation & Wave Functions

In the following, we will construct a particular representation $(\mathcal{H}_S, \pi_S)$ of $\mathcal{A}_W$. The Hilbert space is $\mathcal{H}_S = L^2(\mathbf{R},\mathbf{C})$ with the standard Lebesgue measure on $\mathbf{R}$. And $\pi_S : \mathcal{A}_W \to \mathcal{L}(\mathcal{H}_S)$ is uniquely defined by the action of the generating elements $\{\pi_S(W(\xi,\eta))\}_{\xi,\eta\in\mathbf{R}}$ on the elements of $\mathcal{H}_S$:

$$(\pi_S(U(\eta))\psi)(x) = \mathrm{e}^{\mathrm{i}\eta x}\psi(x) \tag{5.83a}$$

$$(\pi_S(V(\xi))\psi)(x) = \psi(x+\xi) \tag{5.83b}$$

therefore

$$\begin{aligned}(\pi_S(W(\xi,\eta))\psi)(x) &= \left(\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\eta\xi}\pi_S(U(\eta))\pi_S(V(\xi))\right)\psi\right)(x)\\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}\eta\xi}\left(\pi_S(U(\eta))\left(\pi_S(V(\xi))\psi\right)\right)(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}\eta\xi}\mathrm{e}^{\mathrm{i}\eta x}\left(\pi_S(V(\xi))\psi\right)(x)\\ &= \mathrm{e}^{\mathrm{i}\eta(x+\xi/2)}\psi(x+\xi)\,. \end{aligned} \tag{5.84}$$

This definition satisfies the Weyl relations (5.14) *(→ exercise)*. Note that is different from the general Weyl system described in example 5.9, where the $\pi(W(\xi,\eta))$ act on functions on $\mathbf{R}^2$. Here we have functions on $\mathbf{R}$.

Theorem 5.14. *The Schrödinger representation $(\mathcal{H}_S,\pi_S)$ with (5.83) is irreducible.*

Proof. If $(\mathcal{H}_S,\pi_S)$ were reducible, there would be a $\mathcal{H}'\subset\mathcal{H}_S$ invariant under $\pi_S(\mathcal{A}_W)$ and a $\phi\in\mathcal{H}_S$ with

$$\begin{aligned}\forall\psi\in\mathcal{H}',\xi,\eta\in\mathbf{R} : 0 &= (\phi,\pi_S(U(\eta)V(\xi))\psi)\\ &= (\pi_S(U(-\eta))\phi,\pi_S(V(\xi))\psi) = \int dx\,\mathrm{e}^{\mathrm{i}\eta x}\overline{\phi(x)}\psi(x+\xi)\,, \end{aligned} \tag{5.85}$$

i. e., the Fourier transform of

$$x\mapsto\overline{\phi(x)}\psi(x+\xi) \tag{5.86}$$

vanishes for all $\xi\in\mathbf{R}$. Therefore the intersection of the support of ϕ and the support of ψ shifted by ξ vanishes. Since ξ is arbitrary, and the support of ψ is not empty, the support of ϕ must be empty, i. e. $\phi = 0$. Thus $(\mathcal{H}_S,\pi_S)$ can't be reducible. □

Theorem 5.15. *The Schrödinger representation $(\mathcal{H}_S,\pi_S)$ is strongly continuous as maps $\mathbf{R}\to\mathcal{L}(L^2(\mathbf{R},\mathbf{C}))$*

$$\xi\mapsto\pi_S(V(\xi)) \tag{5.87a}$$

$$\eta\mapsto\pi_S(U(\eta)) \tag{5.87b}$$

with (5.83).

Proof. First compute

$$\|\pi_S(U(\eta))\psi - \psi\|^2 = \int \mathrm{d}x \, \underbrace{|\psi(x)|^2 \left|\mathrm{e}^{\mathrm{i}\eta x} - 1\right|^2}_{\leqslant 4|\psi(x)|^2} \tag{5.88}$$

and observe that both

$$x \mapsto 4|\psi(x)|^2 \in L^1(\mathbf{R},\mathbf{C}) \tag{5.89}$$

and

$$\lim_{\eta\to 0} |\psi(x)|^2 \left|\mathrm{e}^{\mathrm{i}\eta x} - 1\right|^2 = 0 \tag{5.90}$$

pointwise. Therefore, by the *Lebesgue dominated convergence theorem*[2], we have convergence of the integrand in the $L^1(\mathbf{R},\mathbf{C})$ topology and therefore

$$\lim_{\eta\to 0} \|\pi_S(U(\eta))\psi - \psi\|^2 = 0 \tag{5.91}$$

in the $L^2(\mathbf{R},\mathbf{C})$ topology. In the case of $\pi_S(V(\xi))$, we can use the same argument again for the Fourier transform $k \mapsto \tilde\psi(k)$ of $x \mapsto \psi(x)$ to find[3]

$$\lim_{\xi\to 0} \|\pi_S(V(\xi))\psi - \psi\|^2 = 0\,. \tag{5.92}$$

□

[2]E. g. stated as Theorem I.11 on p. 17 of [RS80]:

Theorem 5.16 (*Lebesgue dominated convergence theorem*)**.** *If* $\forall x \in \mathbf{R} : f_n(x) \to f(x)$ *pointwise and* $\exists g \in L^1(\mathbf{R}) : \forall x \in \mathbf{R}, \forall n : |f_n(x)| \leqslant g(x)$*, then* $f \in L^1(\mathbf{R})$ *and* $\int \mathrm{d}x\, |f_n(x) - f(x)| \to 0$*.*

[3]NB: in a direct proof, without using the Fourier transform, the estimate

$$\|\pi_S(V(\xi))\psi - \psi\|^2 = \int \mathrm{d}x \, \underbrace{|\psi(x+\xi) - \psi(x)|^2}_{\leqslant (|\psi(x+\xi)| + |\psi(x)|)^2}$$

is *not* sufficient for an application of the *Lebesgue dominated convergence theorem*, because it is not uniform in ξ.

However, as already suggested during the lecture, the argument can be salvaged by starting with continuous functions of compact support, for which

$$\forall\ \xi_0 : \exists \epsilon(\xi_0) : \forall 0 \leqslant \xi < \xi_0 : \forall x : |\psi(x+\xi) - \psi(x)| \leqslant \epsilon(\xi_0)\,.$$

We can approximate any $\psi \in \mathcal{H}_S$ by a sequence $\{\psi_k\}_{k\in\mathbf{N}}$ of continuous functions of compact support and estimate

$$\|\pi_S(V(\xi))\psi - \psi\| \leqslant \underbrace{\|\pi_S(V(\xi))\psi - \pi_S(V(\xi))\psi_k\|}_{=\|\psi-\psi_k\|\to 0} + \underbrace{\|\pi_S(V(\xi))\psi_k - \psi_k\|}_{\to 0} + \underbrace{\|\psi_k - \psi\|}_{\to 0}\,,$$

using the unitarity of $\pi_S(V(\xi))$.

Having established the strong continuity, we can use Stone's theorem to construct the self-adjoint operators

$$X = -\mathrm{i} \lim_{\eta\to 0} \frac{\partial}{\partial\eta} \pi_S(U(\eta)) \tag{5.93a}$$

$$P = -\mathrm{i} \lim_{\xi\to 0} \frac{\partial}{\partial\xi} \pi_S(V(\xi)) \,. \tag{5.93b}$$

On one hand

$$XP = -\lim_{\xi,\eta\to 0} \frac{\partial}{\partial\eta}\frac{\partial}{\partial\xi} \pi_S(U(\eta)V(\xi)) \tag{5.94}$$

and on the other

$$\begin{aligned} PX &= -\lim_{\xi,\eta\to 0} \frac{\partial}{\partial\eta}\frac{\partial}{\partial\xi} \pi_S(V(\xi)U(\eta)) \\ &= -\lim_{\xi,\eta\to 0} \frac{\partial}{\partial\eta}\frac{\partial}{\partial\xi} \left(\pi_S(U(\eta)V(\xi))\mathrm{e}^{\mathrm{i}\eta\xi}\right) = XP - \mathrm{i}\mathbf{1} \,. \end{aligned} \tag{5.95}$$

Thus

$$XP - PX = [X, P] = \mathrm{i}\mathbf{1} \tag{5.96}$$

and X and P can play the role of position and momentum operators on a common dense domain $\mathcal{S}(\mathbf{R}) \subset \mathcal{H}_S$, i. e. the *Schwartz space* of smooth functions that fall off faster than any inverse power. We can therefore use $f(X)$ and $g(P)$ for the observables representing functions of position and momentum, respectively.

5.3.1 N Degrees of Freedoms

We can easily extend the Schrödinger representation $(\mathcal{H}_S, \pi_S)$ to a $\mathcal{A}_W$ with N degrees of freedom. The Hilbert space is now $\mathcal{H}_S = L^2(\mathbf{R}^N, \mathbf{C})$ with the standard Lebesgue measure on $\mathbf{R}^N$. The homomorphism $\pi_S : \mathcal{A}_W \to \mathcal{L}(\mathcal{H}_s)$ is uniquely defined by the action of $\{\pi_S(W(\vec{\xi},\vec{\eta}))\}_{\vec{\xi},\vec{\eta}\in\mathbf{R}^N}$ on the elements of $\mathcal{H}_S$ by

$$(\pi_S(W(\vec{\xi},\vec{\eta}))\psi)(\vec{x}) = \mathrm{e}^{\mathrm{i}\vec{\eta}(\vec{x}+\vec{\xi}/2)}\psi(\vec{x}+\vec{\xi}) \,. \tag{5.97}$$

All regular representations of the Weyl algebra for N d.o.f. are again unitarily equivalent, since the proof given above goes through.

5.3.2 Equivalent Representations

We can generalize the above representation $\pi = \pi_S$ to a two-parameter family of representations $(\mathcal{H}_{\alpha,\beta}, \pi_{\alpha,\beta})$ with $\alpha, \beta \in \mathbf{R}$

$$\pi_{\alpha,\beta}(W(\xi,\eta)) = \mathrm{e}^{\mathrm{i}(\xi\beta-\eta\alpha)}\pi(W(\xi,\eta)) \tag{5.98}$$

where obviously $\mathcal{H} = \mathcal{H}_{0,0}$ and $\pi = \pi_{0,0}$. In particular

$$(\pi_{\alpha,\beta}(U(\eta))\psi)(x) = \mathrm{e}^{\mathrm{i}\eta(x-\alpha)}\psi(x) \tag{5.99a}$$

$$(\pi_{\alpha,\beta}(V(\xi))\psi)(x) = \mathrm{e}^{\mathrm{i}\xi\beta}\psi(x+\xi) \tag{5.99b}$$

or

$$(\pi_{\alpha,\beta}(W(\xi,\eta))\psi)(x) = \mathrm{e}^{\mathrm{i}\eta(x+\xi/2)+\mathrm{i}(\xi\beta-\eta\alpha)}\psi(x+\xi)\,. \tag{5.100}$$

According to the Stone-von Neumann theorem, all these representations must be unitarily equivalent. This is indeed the case and we can use $\mathcal{H}_{\alpha,\beta} = \mathcal{H}_{0,0} = \mathcal{H}_{0,0}$ and $\pi_{\alpha,\beta} = \pi_{0,0} \circ U_{\alpha,\beta} = \pi \circ U_{\alpha,\beta}$ with the intertwiners

$$U_{\alpha,\beta} = \pi(W(\alpha,\beta)) \in \mathcal{L}(\mathcal{H})\,, \tag{5.101}$$

because from (5.57)

$$W(-\alpha,-\beta)W(\xi,\eta)W(\alpha,\beta) = \mathrm{e}^{\mathrm{i}(\beta\xi-\eta\alpha)}W(\xi,\eta) \tag{5.102}$$

and therefore

$$U^{-1}_{\alpha,\beta}\pi(W(\xi,\eta))U_{\alpha,\beta} = \mathrm{e}^{\mathrm{i}(\beta\xi-\eta\alpha)}\pi(W(\xi,\eta)) = \pi_{\alpha,\beta}(W(\xi,\eta))\,. \tag{5.103}$$

5.3.3 Pitfalls on the Circle

The proof of the Stone-von Neumann theorem presented above works equally well for any finite number of degrees of freedom, but fails for an infinite number of d.o.f., because the infinite product of Gaussian integrals in (5.52) is not well defined. However, there are simpler systems, where the Stone-von Neumann theorem doesn't apply.

Consider a particle on the circle $S^1 = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$. A more convenient parametrization is

$$S^1 = \{(\cos\phi, \sin\phi) \in \mathbf{R}^2 : 0 \leqslant \phi < 2\pi\}\,, \tag{5.104}$$

but we must impose periodic boundary conditions for wavefunctions at $\phi \to 2\pi \cong 0$.

In particular, we must have

$$\forall\psi\in\mathcal{H}_\pi : (\pi(W(\xi,\eta))\psi)(x) = (\pi(W(\xi,\eta))\psi)(x+2\pi) \tag{5.105}$$

because otherwise $\pi(W(\xi,\eta))$ would destroy periodicity. Thus

$$\mathrm{e}^{\mathrm{i}\eta(x+\xi/2)}\psi(x+\xi) \stackrel{!}{=} \mathrm{e}^{\mathrm{i}\eta(x+2\pi+\xi/2)}\psi(x+2\pi+\xi) = \mathrm{e}^{\mathrm{i}\eta(x+2\pi+\xi/2)}\psi(x+\xi)\,, \tag{5.106}$$

or

$$\mathrm{e}^{\mathrm{i}\eta(x+\xi/2)} \stackrel{!}{=} \mathrm{e}^{\mathrm{i}\eta(x+2\pi+\xi/2)} \tag{5.107}$$

i. e. $\eta\in\mathbf{Z}$. Going back from the representation to the algebra, we see that the Weyl-algebra for S^1 is generated by

$$\left\{\widetilde{W}_\nu(\xi) = W(\xi,\nu) : \xi\in\mathbf{R}, \nu\in\mathbf{Z}\right\}\,. \tag{5.108}$$

We can again study the representations $(\mathcal{H}_{\alpha,\beta},\pi_{\alpha,\beta})$, but this time, there are fewer intertwiners available and we can only construct the representations with $\beta\in\mathbf{Z}$ using

$$U_{\alpha,\beta} = \pi\left(\widetilde{W}_\beta(\alpha)\right)\,, \tag{5.109}$$

while the representations $(\mathcal{H}_{\alpha,\beta},\pi_{\alpha,\beta})$ with $0<\beta<1$ *could* be inequivalent. The Stone-von Neumann theorem is also of little help, because the conditions are not met and the proof breaks down, since we can't use Gaussian integrals over η to construct a projector. We can however obtain a very simple necessary condition for two representations to be unitarily equivalent from the observation

Lemma 5.17. *The spectrum $\sigma(A)$ of an operator $A\in\mathcal{L}(\mathcal{H})$ is invariant under similarity transformations $A\to SAS^{-1}\in\mathcal{L}(\mathcal{H}')$ with invertible $S:\mathcal{H}\to\mathcal{H}'$.*

$$\forall A\in\mathcal{L}(\mathcal{H}) : \sigma_{\mathcal{L}(\mathcal{H})}(A) = \sigma_{\mathcal{L}(\mathcal{H}')}(SAS^{-1})\,. \tag{5.110}$$

Proof. $S^{-1}:\mathcal{H}'\to\mathcal{H}$ exists and $B=SAS^{-1}\in\mathcal{L}(\mathcal{H}')$ is well defined. The operator

$$z\mathbf{1}-B = z\mathbf{1}-SAS^{-1} = S\,(z\mathbf{1}-A)\,S^{-1} \tag{5.111}$$

is invertible iff $z\mathbf{1}-A$ is invertible. Thus the resolvent set and the spectrum agree. □

Corollary 5.18. *The spectrum of the representatives is the same in two unitarily equivalent representations $(\mathcal{H}_1,\pi_1)$ and $(\mathcal{H}_2,\pi_2)$ of a C^*-algebra $\mathcal{A}$:*

$$\forall A\in\mathcal{A} : \sigma_{\mathcal{L}(\mathcal{H}_1)}(\pi_1(A)) = \sigma_{\mathcal{L}(\mathcal{H}_2)}(\pi_2(A))\,. \tag{5.112}$$

Proof. The unitary intertwiner $U:\mathcal{H}_1\to\mathcal{H}_2$ defines a similarity transformation $\pi_1(A)\to\pi_2(A)=U\pi_1(A)U^{-1}$. □

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Of course, $z\mathbf{1} - \pi(W(\xi,\eta))$ is invertible, i. e. $z \notin \sigma(\pi(W(\xi,\eta)))$, iff

$$\mathrm{e}^{\mathrm{i}(\beta\xi-\eta\alpha)} z\mathbf{1} - \pi_{\alpha,\beta}(W(\xi,\eta)) = \mathrm{e}^{\mathrm{i}(\beta\xi-\eta\alpha)} \left(z\mathbf{1} - \pi(W(\xi,\eta))\right) \tag{5.113}$$

is invertible, i. e. $\mathrm{e}^{\mathrm{i}(\beta\xi-\eta\alpha)} z \notin \sigma(\pi_{\alpha,\beta}(W(\xi,\eta)))$. Thus we have shown that

$$\sigma(\pi_{\alpha,\beta}(W(\xi,\eta))) = \mathrm{e}^{\mathrm{i}(\beta\xi-\eta\alpha)} \sigma(\pi(W(\xi,\eta))) \tag{5.114}$$

and in particular

$$\sigma(\pi_{\alpha,\beta}(W(\xi,0))) = \mathrm{e}^{\mathrm{i}\beta\xi} \sigma(\pi(W(\xi,0)))\,. \tag{5.115}$$

If we can compute $\sigma(\pi(W(\xi,0)))$ and show that is a *strict* subset of S^1 *not* invariant under arbitrary rotations, we have proven the existence of inequivalent representations. Indeed,

$$\sigma(\pi(\widetilde{W}_0(\xi))) = \sigma(\pi(W(\xi,0))) = \left\{\mathrm{e}^{\mathrm{i}n\xi}\right\}_{n\in\mathbf{Z}} \tag{5.116}$$

since[4] for $\psi_n : x \mapsto \mathrm{e}^{\mathrm{i}nx}$

$$(\pi(W(\xi,0))\psi_n)(x) = \psi_n(x+\xi) = \mathrm{e}^{\mathrm{i}n\xi}\psi_n(x)\,. \tag{5.117}$$

Even for irrational ξ, $\left\{\mathrm{e}^{\mathrm{i}n\xi}\right\}_{n\in\mathbf{Z}}$, while dense in $S^1 \subset \mathbf{C}$, remains always strictly smaller. In any case, given a ξ, we can find a $\beta \in (0,1)$ such that

$$\left\{\mathrm{e}^{\mathrm{i}n\xi}\right\}_{n\in\mathbf{Z}} \neq \mathrm{e}^{\mathrm{i}\beta\xi} \left\{\mathrm{e}^{\mathrm{i}n\xi}\right\}_{n\in\mathbf{Z}}\,. \tag{5.118}$$

The example of a particle on S^1 introduces a new feature, a non trivial center of a C^*-algebra.

---

[4] Using the ansatz

$$(\pi(W(\xi,0))\psi_n)(x) = \psi_n(x+\xi) = \alpha_n(\xi)\psi_n(x)$$

we find from multiple applications and the group property

$$\forall \xi \in \mathbf{R}, k \in \mathbf{Z} : (\alpha_n(\xi))^k = \alpha_n(k\xi) \wedge \alpha_n(2\pi) = 1$$

and therefore exponentials

$$\alpha_n(\xi) = \mathrm{e}^{\mathrm{i}m\xi}$$

with $m \in \mathbf{Z}$. WLOG we can choose $m = n$.

Definition 5.19 (*commutant, center*)**.** The *commutant* $\mathcal{B}'$ of a subset $\mathcal{B}$ of a C^*-algebra $\mathcal{A}$ is the set of all elements of $\mathcal{A}$ that commute with all elements of $\mathcal{B}$, i. e.

$$\mathcal{B}' = \{A \in \mathcal{A} : \forall B \in \mathcal{B} : AB = BA\} . \tag{5.119}$$

The *center* $Z(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ is the set of all elements of $\mathcal{A}$ that commute with all other elements, i. e.

$$Z(\mathcal{A}) = \mathcal{A}' \cap \mathcal{A} . \tag{5.120}$$

Indeed, we can compute

$$\forall n, \nu \in \mathbf{Z} : V(2n\pi)U(\nu) = U(\nu)V(2n\pi)\mathrm{e}^{\mathrm{i}2n\nu\pi} = U(\nu)V(2n\pi) \tag{5.121}$$

and, of course,

$$V(2n\pi)V(\xi) = V(\xi)V(2n\pi) \tag{5.122}$$

or more generally

$$\forall n \in \mathbf{Z} : \widetilde{W}_0(2n\pi)\widetilde{W}_\nu(\xi) = \widetilde{W}_\nu(\xi)\widetilde{W}_0(2n\pi)\mathrm{e}^{\mathrm{i}2n\pi\nu} = \widetilde{W}_\nu(\xi)\widetilde{W}_0(2n\pi) . \tag{5.123}$$

Therefore

$$\forall n \in \mathbf{Z} : \widetilde{W}_0(2n\pi) \in Z(\mathcal{A}_W^{S^1}) \tag{5.124}$$

but from

$$\widetilde{W}_0(2n\pi)\widetilde{W}_\nu(\xi) = \mathrm{e}^{\mathrm{i}n\pi\nu}\widetilde{W}_\nu(\xi + 2n\pi) \tag{5.125}$$

and the fact that there is a priori *no* 2π-periodicity in the family of algebra elements $\widetilde{W}_\nu : \mathbf{R} \to \mathcal{A}_W^{S^1}$, we can expect that $\widetilde{W}_0(2n\pi) \neq \mathbf{1}$, i. e.

$$Z(\mathcal{A}_W^{S^1}) \neq \{z\mathbf{1} : z \in \mathbf{C}\} . \tag{5.126}$$

Indeed, computing the action of $\widetilde{W}_0(2n\pi)$ in the representations $(\mathcal{H}_{\alpha,\beta}, \pi_{\alpha,\beta})$, we find from (5.100)

$$(\pi_{\alpha,\beta}(\widetilde{W}_0(2n\pi))\psi)(x) = \mathrm{e}^{\mathrm{i}2n\pi\beta}\psi(x + 2n\pi) = \mathrm{e}^{\mathrm{i}2n\pi\beta}\psi(x) , \tag{5.127}$$

i. e.

$$\pi_{\alpha,\beta}(\widetilde{W}_0(2n\pi)) = \mathrm{e}^{\mathrm{i}2n\pi\beta}\mathbf{1} . \tag{5.128}$$

For $0 < \beta < 1$, this is not compatible with $\widetilde{W}_0(2n\pi) = z\mathbf{1}$, because for all representations

$$\pi_{\alpha,\beta}(z\mathbf{1}) = z\pi_{\alpha,\beta}(\mathbf{1}) = z\mathbf{1} , \tag{5.129}$$

independently of β.

As mentioned in section 4.2 on page 69, symmetries are realized as automorphisms of the C^*-algebra of observables. For example the translations $x \mapsto x - \alpha$

$$\begin{aligned} \tau_\alpha : \mathcal{A}_W &\to \mathcal{A}_W \\ W(\xi,\eta) &\mapsto \mathrm{e}^{-\mathrm{i}\eta\alpha} W(\xi,\eta) \end{aligned} \tag{5.130}$$

and *Galileo boosts* $p \mapsto p + \beta$

$$\begin{aligned} \sigma_\beta : \mathcal{A}_W &\to \mathcal{A}_W \\ W(\xi,\eta) &\mapsto \mathrm{e}^{\mathrm{i}\xi\beta} W(\xi,\eta) \end{aligned} \tag{5.131}$$

form an abelian[5] two-parameter group of automorphisms $\Gamma_{\alpha,\beta} : \mathcal{A}_W \to \mathcal{A}_W$, with $\alpha, \beta \in \mathbf{R}$,

$$\Gamma_{\alpha,\beta} = \sigma_\beta \circ \tau_\alpha = \tau_\alpha \circ \sigma_\beta \tag{5.132}$$

and[6]

$$\begin{aligned} \Gamma_{\alpha,\beta}(W(\xi,\eta)) &= \tau_\alpha(\sigma_\beta(W(\xi,\eta)) \\ &= \mathrm{e}^{\mathrm{i}(\xi\beta-\eta\alpha)} W(\xi,\eta) = W(-\alpha,-\beta)W(\xi,\eta)W(\alpha,\beta)\,. \end{aligned} \tag{5.135}$$

That τ_α and σ_β correspond to translations in position and momenta is obvious from

$$X \mapsto -\mathrm{i} \lim_{\eta\to 0} \frac{\partial}{\partial\eta} \pi_S(\tau_\alpha(W(0,\eta))) = X - \alpha \tag{5.136a}$$

$$P \mapsto -\mathrm{i} \lim_{\xi\to 0} \frac{\partial}{\partial\xi} \pi_S(\sigma_\beta(W(\xi,0))) = P + \beta\,. \tag{5.136b}$$

As we have seen above (5.101), in a representation $(\mathcal{H},\pi)$, the $\Gamma_{\alpha,\beta}$ are represented unitarily

$$\pi\left(\Gamma_{\alpha,\beta}(W(\xi,\eta)\right) = \pi\left(W(-\alpha,-\beta)W(\xi,\eta)W(\alpha,\beta)\right) = U^*_{\alpha,\beta} W(\xi,\eta) U_{\alpha,\beta} \tag{5.137}$$

by

$$U_{\alpha,\beta} = \pi\left(W(\alpha,\beta)\right)\,. \tag{5.138}$$

[5] Cf. section 3.8.4 on page 59 of [Ohl16].

[6] Note that

$$W(\alpha,0)W(0,\beta) \neq W(0,\beta)W(\alpha,0) \neq W(\alpha,\beta)\,, \tag{5.133}$$

but

$$\begin{aligned} W(0,-\beta)\left(W(-\alpha,0)AW(\alpha,0)\right)W(0,\beta) &= W(-\alpha,-\beta)AW(\alpha,\beta) \\ &= W(-\alpha,0)\left(W(0,-\beta)AW(0,\beta)\right)W(\alpha,0)\,, \end{aligned} \tag{5.134}$$

because the phases cancel.

However, while $\Gamma_{\alpha,\beta} : \mathcal{A}_W^{S^1} \to \mathcal{A}_W^{S^1}$ remains well defined on the circle for all $\alpha, \beta \in \mathbf{R}$, only $\Gamma_{\alpha,\nu}$ with $\alpha \in \mathbf{R}$ and $\nu \in \mathbf{Z}$ can be realized by unitary operators on $\mathcal{H}$. We may say that we have a case of *Spontaneous Symmetry Breaking (SSB)*: the symmetry is an automorphism of the microscopic algebra of observables, but not of the representation corresponding to the macroscopic state. In the present case, all translations $\tau_\alpha = \Gamma_{\alpha,0}$ remain unbroken, but the Galileo boosts $\sigma_\nu = \Gamma_{0,\nu}$ are broken by the quantization of momenta, except for integer amounts.

In this example, the breaking of the symmetry is of course not fully spontaneous, because it is brought about by a enforcing the non-trivial topology of the configuration space S^1. Another way to describe the situation is that the classical boost symmetry $p \mapsto p + \beta$ is *anomalous* in the quantum theory or that we have an example of an *anomaly*, where a classical symmetry can not (fully) be realized in the quantum theory.

Nevertheless, there remains a weaker form of the Stone-von Neumann theorem, namely that all representations $(\mathcal{H}, \pi)$ with $\pi(\widetilde{W}_0(2n\pi)) = \mathbf{1}$ are unitarily equivalent. In fact, we can show that there is a cyclic vector Ω_0 with

$$\forall \nu \in \mathbf{Z}, \xi \in \mathbf{R} : \left(\Omega_0, \pi_{\alpha,\beta}(\widetilde{W}_\nu(\xi))\Omega_0\right) = \delta_{\nu,0} \tag{5.139}$$

corresponding to the state

$$\omega(\widetilde{W}_\nu(\xi)) = \delta_{\nu,0} \,. \tag{5.140}$$

Indeed from

$$\widetilde{W}_0(\xi)\widetilde{W}_0(\xi') = \widetilde{W}_0(\xi + \xi') \tag{5.141}$$

we infer

$$\widetilde{W}_0(\xi + 2\pi) = \widetilde{W}_0(2\pi)\widetilde{W}_0(\xi) \tag{5.142}$$

and

$$\pi(\widetilde{W}_0(\xi + 2\pi)) = \pi(\widetilde{W}_0(2\pi))\pi(\widetilde{W}_0(\xi)) = \pi(\widetilde{W}_0(\xi)) \,. \tag{5.143}$$

Thus $\pi(\widetilde{W}_0(\xi))$ is periodic with period 2π and we are let to define

$$P_\pi = \int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi} \pi(\widetilde{W}_0(\xi)) \,. \tag{5.144}$$

Again, this is self-adjoint

$$\begin{aligned} P_\pi^* = \int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi} \pi(\widetilde{W}_0(-\xi)) &= -\int_0^{-2\pi} \frac{\mathrm{d}\xi}{2\pi} \pi(\widetilde{W}_0(\xi)) \\ &= \int_{-2\pi}^{0} \frac{\mathrm{d}\xi}{2\pi} \pi(\widetilde{W}_0(\xi)) = \int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi} \pi(\widetilde{W}_0(\xi)) = P_\pi \end{aligned} \tag{5.145}$$

and $P_\pi \neq 0$ since

$$\begin{aligned}\pi(\widetilde{W}_{-\nu}(-\xi))P_\pi\pi(\widetilde{W}_\nu(\xi)) &= \int_0^{2\pi}\frac{\mathrm{d}\xi'}{2\pi}\pi(\widetilde{W}_{-\nu}(-\xi)\widetilde{W}_0(\xi')\widetilde{W}_\nu(\xi)) \\ &= \int_0^{2\pi}\frac{\mathrm{d}\xi'}{2\pi}\mathrm{e}^{\mathrm{i}\nu\xi'}\pi(\widetilde{W}_0(\xi'))\end{aligned} \tag{5.146}$$

can not vanish, because it's the νth Fourier coefficient of the periodic unitary operator $\pi(\widetilde{W}_0(\xi))$. Finally, similarly to the proof of the Stone-von Neumann

$$P_\pi\pi(\widetilde{W}_\nu(\xi))P_\pi = \delta_{\nu,0}P_\pi\,. \tag{5.147}$$

Thus we see that P_π is an orthogonal projection and can construct Ω_0 as a normalized eigenvector of P_π.

5.4 Reducible Representations

A representation $(\mathcal{H},\pi)$ of a C^*-algebra $\mathcal{A}$ corresponding to an impure state

$$\omega = \sum_{i=1}^N p_i\omega_i \tag{5.148}$$

with each ω_i pure, $\mathbf{R} \ni p_i > 0$ and $\sum_{i=1}^N p_i = 1$, will not be irreducible and can be written as a direct sum (or integral) of N irreducible representations

$$\mathcal{H} = \bigoplus_{i=1}^N \mathcal{H}_i \tag{5.149a}$$

$$\pi = \bigoplus_{i=1}^N \pi_i\,, \tag{5.149b}$$

where we must allow that not all $(\mathcal{H}_i,\pi_i)$ are different. Since all $(\mathcal{H}_i,\pi_i)$ in (5.149) are, by definition, irreducible, they correspond to the pure states

$$\begin{aligned}\omega_i : \mathcal{A} &\to \mathbf{C} \\ A &\mapsto \omega_i(A) = (\Omega_i, \pi_i(A)\Omega_i)\end{aligned} \tag{5.150}$$

with $\Omega_i \in \mathcal{H}_i$. Therefore the state ω has the decomposition

$$\begin{aligned}\omega : \mathcal{A} &\to \mathbf{C} \\ A &\mapsto \omega(A) = \sum_{i=1}^N p_i\,(\Omega_i, \pi_i(A)\Omega_i)\,.\end{aligned} \tag{5.151}$$

Introducing the operator

$$\begin{aligned}\rho : \mathcal{H} &\to \mathcal{H} \\ \Psi &\mapsto \sum_{i=1}^{N} \Omega_i \left(\Omega_i, \Psi\right) p_i \,,\end{aligned} \tag{5.152}$$

we can express the state ω as a trace

$$\omega(A) = \mathrm{tr}_{\mathcal{H}}(\rho\pi(A))\,. \tag{5.153}$$

This suggests the pair of definitions

Definition 5.20. A bounded operator $A \in \mathcal{L}(\mathcal{H})$ in a separable Hilbert space $\mathcal{H}$ is called *trace class*, iff the sum for any orthonormal basis $\{\psi_i\}_{i\in I\subseteq\mathbf{N}}$ satisfies

$$\mathrm{tr}_{\mathcal{H}}(|A|) = \mathrm{tr}_{\mathcal{H}}(\sqrt{A^*A}) = \sum_{i\in I} \left(\psi_i, \sqrt{A^*A}\psi_i\right) < \infty\,. \tag{5.154}$$

Then the *trace* of A

$$\mathrm{tr}_{\mathcal{H}}(A) = \sum_{i\in I} (\psi_i, A\psi_i) \tag{5.155}$$

is absolutely convergent and *independent* of the basis.

Definition 5.21. A positive trace class operator $\rho \in \mathcal{L}(\mathcal{H})$ with

$$\mathrm{tr}_{\mathcal{H}}(\rho) = 1 \tag{5.156}$$

is called a *density matrix.*

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Theorem 5.22. *Given a representation $(\mathcal{H}, \pi)$ of a C^*-algebra $\mathcal{A}$, each density matrix $\rho : \mathcal{H} \to \mathcal{H}$ defines a state*

$$\begin{aligned}\omega_\rho : \mathcal{A} &\to \mathbf{C} \\ A &\mapsto \mathrm{tr}_{\mathcal{H}}(\rho\,\pi(A))\,.\end{aligned} \tag{5.157}$$

Proof. ω_ρ is normalized by definition

$$\omega_\rho(\mathbf{1}) = \mathrm{tr}_{\mathcal{H}}\,\rho = 1 \tag{5.158}$$

and since positivity implies self-adjointness, we can diagonalize ρ

$$\rho : \phi \mapsto \sum_{i\in I} p_i\psi_i(\psi_i, \phi) \tag{5.159}$$

with $\mathbf{R} \ni p_i > 0$ and $\sum_{i\in I} p_i = 1$. Thus

$$\begin{aligned}\omega_\rho(A^*A) &= \mathrm{tr}_{\mathcal{H}}(\rho\,\pi(A^*A)) = \sum_{i\in I} (\psi_i, \rho\,\pi(A^*A)\psi_i) \\ &= \sum_{i,j\in I} p_j \underbrace{(\psi_i,\psi_j)}_{=\delta_{ij}} (\psi_j, \pi(A^*A)\psi_i) = \sum_{i\in I} p_j\,(\psi_i, \pi(A^*A)\psi_i) \\ &= \sum_{i\in I} \underbrace{p_j}_{\geqslant 0} \underbrace{(\pi(A)\psi_i, \pi(A)\psi_i)}_{\geqslant 0} \geqslant 0\,. \end{aligned} \tag{5.160}$$

□

Remark 5.23. The pure states correspond to density matrices that are one-dimensional projections

$$\rho : \Phi \mapsto \Psi_\rho\,(\Psi_\rho, \Phi) \tag{5.161}$$

with $\|\Psi_\rho\| = 1$.

Proof.

$$\begin{aligned}\omega_\rho(A) &= \sum_{i\in I} (\psi_i, \rho\,\pi(A)\psi_i) = \sum_{i\in I} (\psi_i, \Psi_\rho)\,(\Psi_\rho, \pi(A)\psi_i) \\ &= \sum_{i\in I} (\Psi_\rho, \pi(A)\psi_i)\,(\psi_i, \Psi_\rho) = (\Psi_\rho, \pi(A)\Psi_\rho)\ , \end{aligned} \tag{5.162}$$

using the completeness of the basis $\{\psi_i\}_{i\in I\subseteq\mathbf{N}}$:

$$\forall\phi\in\mathcal{H} : \sum_{i\in I} \psi_i\,(\psi_i, \phi) = \phi\,. \tag{5.163}$$

□

Note that density matrices appear here in two very different ways. First we have decomposed a reducible representation and written the state as a sum of pure vector states

$$\omega : A \mapsto \sum_{i=1}^{N} p_i\,(\Omega_i, \pi_i(A)\Omega_i)\ , \tag{5.164}$$

which we have rewritten as a trace of a density matrix in the big Hilbert space. But after introducing the notion of a density matrix, we have noticed that *every* density matrix defines a state

$$\omega_\rho : A \mapsto \mathrm{tr}_{\mathcal{H}}(\rho\,\pi(A))\,, \tag{5.165}$$

which can be pure or not.

Example 5.24. Consider the C^*-algebra $\mathcal{M}_2$ of 2×2-Matrices, which furnish their own representation $(\mathcal{H}, \pi) = (\mathbf{C}^2, \mathrm{id})$. As we have also seen in the exercises, the states on $\mathcal{M}_2$ can be parametrized by three real numbers $\vec{\alpha}$ with $|\vec{\alpha}| \leqslant 1$ and

$$\begin{aligned} \omega_{\vec{\alpha}} : \mathcal{M}_2 &\to \mathbf{C} \\ M &\mapsto \mathrm{tr}(M\rho(\vec{\alpha}))\,, \end{aligned} \tag{5.166}$$

where

$$\rho(\vec{\alpha}) = \frac{1}{2}\left(\mathbf{1} + \vec{\alpha}\vec{\sigma}\right)\,. \tag{5.167}$$

The pure states are those with $|\vec{\alpha}| = 1$. It is important to realize that the representation $(\mathbf{C}^2, \mathrm{id})$ is in general *not* the GNS-representation associated to the state $\omega_{\vec{\alpha}}$ or even unitarily equivalent to it.

In fact, the dimension of the representation space $\mathcal{H}_{\vec{\alpha}}$ of the GNS-representation $(\mathcal{H}_{\vec{\alpha}}, \pi_{\vec{\alpha}})$ constructed from $\omega_{\vec{\alpha}}$ depends on the dimension of the Gel'fand ideal

$$\mathcal{I}_{\vec{\alpha}} = \{M \in \mathcal{M}_2 : \omega_{\vec{\alpha}}(M^*M) = 0\} \tag{5.168}$$

via

$$\dim \mathcal{H}_{\vec{\alpha}} = \dim\left(\mathcal{M}_2/\mathcal{I}_{\vec{\alpha}}\right) = \underbrace{\dim \mathcal{M}_2}_{=4} - \dim \mathcal{I}_{\vec{\alpha}}\,. \tag{5.169}$$

Iff $\omega_{\vec{\alpha}}$ is not a pure state, $\rho(\vec{\alpha})$ has rank 2, the Gel'fand ideal is trivial and $\dim \mathcal{H}_{\vec{\alpha}} = 4$. OTOH, iff $\omega_{\vec{\alpha}}$ is pure, $\rho(\vec{\alpha})$ is a projector with rank 1, the Gel'fand ideal has dimension 2 (all matrices with vanishing first column) and $\dim \mathcal{H}_{\vec{\alpha}} = 2$.

5.4.1 Fell's Theorem

In order to organize the representations, we make the following

Definition 5.25 (*vector states*)**.** Given a state ω over a C^*-algebra $\mathcal{A}$ and the corresponding representation $(\mathcal{H}_\omega, \pi_\omega)$, we call the members of the set

$$\{\omega_\Psi : \Psi \in \mathcal{H}_\omega, \|\Psi\| = 1\} \tag{5.170}$$

with

$$\begin{aligned} \omega_\Psi : \mathcal{A} &\to \mathbf{C} \\ A &\mapsto (\Psi, \pi_\omega(A)\Psi)\,, \end{aligned} \tag{5.171}$$

the *vector states* of this representation.

Definition 5.26 (*folium*)**.** Given a state ω over a C^*-algebra $\mathcal{A}$ and the corresponding representation $(\mathcal{H}_\omega, \pi_\omega)$, we call the set

$$\{\omega_\rho : \rho \in \mathcal{L}(\mathcal{H}_\omega),\ 0 \leqslant \rho \text{ trace class},\ \mathrm{tr}_{\mathcal{H}_\omega}\rho = 1\} \tag{5.172}$$

with

$$\begin{aligned} \omega_\rho : \mathcal{A} &\to \mathbf{C} \\ A &\mapsto \mathrm{tr}_{\mathcal{H}_\omega}(\rho\, \pi_\omega(A))\,, \end{aligned} \tag{5.173}$$

the *folium* of this representation.

Besides the norm topology on the space $\mathcal{A}^*$ of all linear functionals $\omega : \mathcal{A} \to \mathbf{C}$,

$$\|\omega\| = \sup_{A\in\mathcal{A}, \|A\|=1} |\omega(A)| \,, \tag{4.1}$$

introduced earlier, we can define a weaker topology, that is probably physically better motivated. We start from a family of seminorms

$$\begin{aligned} \|\cdot\|_{A_1,\ldots,A_n} : \mathcal{A}^* &\to \mathbf{R}_+ \cup \{0\} \\ \omega &\mapsto \sup_{k=1,\ldots,n} |\omega(A_k)| \end{aligned} \tag{5.174}$$

indexed by finite subsets of $\mathcal{A}$ with $\|A_k\| = 1$ and define neighborhoods of the origin

$$\begin{aligned} N(\epsilon; A_1,\ldots,A_n) &= \{\omega \in \mathcal{A}^* : \|\omega\|_{A_1,\ldots,A_n} < \epsilon\} \\ &= \{\omega \in \mathcal{A}^* : \forall k = 1,\ldots,n : |\omega(A_k)| < \epsilon\}\,. \end{aligned} \tag{5.175}$$

The set of all such neighborhoods define the *weak topology* on $\mathcal{A}^*$. Because we use only a *finite* number of elements of $\mathcal{A}$ to measure distances, the neighborhoods in the weak topology are cylinders, while the neighborhoods in the norm topology are balls. Therefore, the open sets in the weak topology can be generated by *infinite unions* of open sets in the norm topology, but the open sets in the norm topology can not be generated as *finite intersections* of open sets in the weak topology. Thus every open set in the weak topology is also an open set in the norm topology, but not vice-versa (unless $\mathcal{A}$ is finite-dimensional). Therefore, the weak topology is coarser than the norm topology.

Since only a finite number of measurements can be performed in real life, the weak topology appears to be more appropriate than the norm topology, which can only be realized when *all* observables are measured.

The importance of the notion of folium is explained by

Theorem 5.27 (Fell [Fel60])**.** *The folium of a faithful representation of a C^*-algebra $\mathcal{A}$ is weakly dense in the set of all states on $\mathcal{A}$.*

This theorem is however somewhat disappointing, because it means that one can not distinguish folia by physically realizable measurements, which necessarily involve only a finite number of observables. In particular, if we have two states ω and ω' corresponding to unitarily inequivalent representations, states from one folium can nevertheless be approximated arbitrarily well by states from the other folium. Thus one might conclude that “one folium fits all” or “one Hilbert space fits them all”.

This conclusion is misleading, however, because the qualitative description of important physical phenomena like *SSB* and *phase transitions* depend on idealizations like the *infinite volume limit*, which corresponds to choosing values for an infinite number of measurements. While a sample of finite size will never undergo a phase transition or SSB, we get a better description of its properties by making this idealization.

—6— Infinite Number of Degrees of Freedom

6.1 Fock Representation

Coming back to example 5.8, i. e. the Weyl algebra $\mathcal{A}_W$ for quantum mechanics with n d.o.f.

$$W(\xi_1,\eta_1)W(\xi_2,\eta_2) = \mathrm{e}^{\frac{\mathrm{i}}{2}\sum_{i=1}^n(\xi_{1,i}\eta_{2,i}-\eta_{1,i}\xi_{2,i})}W(\xi_1+\xi_2,\eta_1+\eta_2) \tag{6.1}$$

we will now allow for $n\to\infty$. Of course, in order to have a well defined multiplication law (6.1), we must assume that $\xi,\eta\in l_2$.

Since we know from Stone's theorem that

$$\pi(x_i) = -\mathrm{i}\lim_{\eta\to0}\frac{\partial}{\partial\eta_i}\pi(W(0,\eta)) \tag{6.2a}$$

$$\pi(p_i) = -\mathrm{i}\lim_{\xi\to0}\frac{\partial}{\partial\xi_i}\pi(W(\xi,0)) \tag{6.2b}$$

are well defined symmetric operators on a common dense domain in any regular representation $(\mathcal{H},\pi)$, we can suppress the π and write

$$W(\xi,\eta) = \exp\left(\mathrm{i}\sum_{i=1}^n(\xi_i p_i+\eta_i x_i)\right). \tag{6.3}$$

For convenience, we will in the following also use representations of the *Heisenberg algebra* $\mathcal{A}_H$, generated by the symmetric x_i, p_i with

$$[x_i,p_j] = \mathrm{i}\delta_{ij} \tag{6.4a}$$

$$[x_i,x_j] = [p_i,p_j] = 0\,. \tag{6.4b}$$

Since x_i and p_i can not be simultaneously bounded, $\mathcal{A}_H$ is *not* a C^*-algebra and we have to specify a representation $(\mathcal{H},\pi)$ and a common dense domain $D\subset\mathcal{H}$. Nevertheless will will in the following often write x_i for $\pi(x_i)$ and p_i for $\pi(p_i)$ and leave the representation implicit.

As suggested by the proof of the Stone-von Neumann theorem 5.12 and also discussed in the exercises, one can introduce

$$a_i = \frac{1}{\sqrt{2}}(x_i + \mathrm{i}p_i) \tag{6.5a}$$
$$a_i^* = \frac{1}{\sqrt{2}}(x_i - \mathrm{i}p_i)\ , \tag{6.5b}$$

the (unbounded) *annihilation and creation operators*, together with the *number operators*

$$N_i = a_i^* a_i \tag{6.5c}$$

for all $i \in \mathbf{N}$. Obviously, one has

$$[a_i, a_j^*] = \delta_{ij} \tag{6.6a}$$
$$[a_i, a_j] = [a_i^*, a_j^*] = 0 \tag{6.6b}$$

and

$$[N_i, a_j] = -\delta_{ij} a_i \tag{6.6c}$$
$$[N_i, a_j^*] = \delta_{ij} a_i^*\,. \tag{6.6d}$$

Theorem 6.1. *In an irreducible representation $(\mathcal{H}, \pi)$ of the Heisenberg algebra $\mathcal{A}_H$ with common dense domain $D \subset \mathcal{H}$, the following are equivalent (suppressing the π):*

1. *The total number operator*

$$N = \sum_{i=1}^{\infty} N_i \tag{6.7}$$

exists in the sense that the strong limit

$$\forall \alpha \in \mathbf{R} : \operatorname*{s-lim}_{n\to\infty} \mathrm{e}^{\mathrm{i}\alpha \sum_{i=1}^{n} a_i^* a_i} = \mathrm{e}^{\mathrm{i}\alpha N} = T(\alpha) \tag{6.8}$$

exists on D and defines a one-parameter group of unitary operators $T(\alpha) : D \to D$, that is strongly continuous in α, such that N exists as its generator.

2. *There exists a non-zero vector $\Omega \in \mathcal{H}$, called the* Fock state, *or* Fock vacuum state, *such that*

$$\forall i : a_i \Omega = 0\,. \tag{6.9}$$

Such a representation is called a Fock representation $(\mathcal{H}, \pi, \Omega)$.

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Proof. From 1) and the commutation relations, we see, using the *Hausdorff formula*[1]

$$e^{A} B e^{-A} = e^{[A,\cdot]} B \tag{6.16}$$

that

$$T(\alpha) a_i \left(T(\alpha)\right)^{-1} = e^{-i\alpha} a_i \,. \tag{6.17}$$

Therefore

$$T(2\pi) a_i = a_i T(2\pi) \tag{6.18}$$

or

$$[T(2\pi), a_i] = 0 \,. \tag{6.19}$$

Analogously

$$[T(2\pi), a_i^*] = 0 \,, \tag{6.20}$$

i. e.

$$[T(2\pi), \mathcal{A}_H] = 0 \,. \tag{6.21}$$

Since the representation is irreducible, this implies that $T(2\pi)$ is proportional to $\mathbf{1}_{\mathcal{H}}$ and since it is unitary

$$\exists \theta \in \mathbf{R} : T(2\pi) = e^{i\theta} \mathbf{1}_{\mathcal{H}} \,. \tag{6.22}$$

[1] A simple proof introduces linear operators for multiplication from left and right

$$L_A B = AB \tag{6.10a}$$
$$R_A B = BA \tag{6.10b}$$

with

$$[A, B] = L_A B - R_A B = (L_A - R_A) B \tag{6.11}$$

and

$$L_A R_B = R_B L_A \,. \tag{6.12}$$

Therefore

$$e^{L_A} e^{R_B} = e^{L_A + R_B} = e^{R_B} e^{L_A} \,, \tag{6.13}$$

which lead to

$$e^{A} B = e^{L_A} B \tag{6.14a}$$
$$B e^{-A} = e^{R_{-A}} B = e^{-R_A} B \tag{6.14b}$$

and finally

$$e^{A} B e^{-A} = e^{-R_A} e^{L_A} B = e^{L_A - R_A} B = e^{[A,\cdot]} B \,. \tag{6.15}$$

This suggests to define

$$T'(\alpha) = \mathrm{e}^{-\mathrm{i}\alpha\theta/2\pi} T(\alpha) \tag{6.23}$$

with

$$T'(2\pi) = \mathbf{1}_{\mathcal{H}} . \tag{6.24}$$

In the spectral decomposition

$$T'(\alpha) = \int_{\sigma(N')} \mathrm{d}P(\lambda)\, \mathrm{e}^{\mathrm{i}\alpha\lambda} , \tag{6.25}$$

where

$$N' = N - \frac{\theta}{2\pi} , \tag{6.26}$$

this means that

$$\forall\lambda \in \sigma(N') : \mathrm{e}^{\mathrm{i}2\pi\lambda} = 1 . \tag{6.27}$$

Thus $\sigma(N') \subseteq \mathbf{Z}$ and $\sigma(N)$ is also discrete. If $\sigma(N) \ni \lambda > 0$ with Ψ_λ one of the corresponding eigenvectors[2], we have

$$0 < \lambda \|\Psi_\lambda\|^2 = (\Psi_\lambda, N\Psi_\lambda) = \sum_i \|a_i\Psi_\lambda\|^2 \tag{6.28}$$

and therefore

$$\exists i \in \mathbf{N} : a_i\Psi_\lambda \neq 0 . \tag{6.29}$$

Using the commutation relations

$$T(\alpha)a_i\Psi_\lambda = a_i\mathrm{e}^{-\mathrm{i}\alpha}T(\alpha)\Psi_\lambda = a_i\mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\alpha N}\Psi_\lambda = a_i\mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\alpha\lambda}\Psi_\lambda = \mathrm{e}^{\mathrm{i}(\lambda-1)\alpha}a_i\Psi_\lambda \tag{6.30}$$

we conclude that a_i acts as a lowering operator and $\lambda - 1 \in \sigma(N)$ as well. OTOH $N \geqslant 0$, because it is a sum of positive operators and therefore $\sigma(N) \subseteq \mathbf{R}_+ \cup \{0\}$. Thus there must be a $\lambda \in \sigma(N)$ where the lowering terminates for all a_i. This λ must be 0, i. e.

$$\forall i : a_i\Psi_0 = 0 , \tag{6.31}$$

and we have shown 2) with $\Omega = \Psi_0$.

The other direction follows from

$$\mathcal{A}_H\Omega = \mathcal{P}(a^*)\Omega \subset \mathcal{H} , \tag{6.32}$$

where $\mathcal{P}(a^*)$ are the polynomials in $\{a_i^*\}$. $\mathcal{P}(a^*)\Omega$ is dense in $\mathcal{H}$, since the representation is irreducible. N exists on $\mathcal{P}(a^*)\Omega$ by construction and the exponential series for $T(\alpha)$ converges strongly to a one parameter group of unitary operators. □

[2] λ might be a degenerate eigenvalue.

Remark 6.2. The eigenvalue 0 of the total number operator N is not degenerate in a Fock representation $(\mathcal{H}, \pi, \Omega)$, i. e. Ω is unique.

Proof. Assume that there is a second $\mathcal{H} \ni \Omega' \neq \Omega$ with $N\Omega' = 0$. Then, since $N \geqslant 0$,

$$\forall i : a_i \Omega' = 0 \tag{6.33}$$

and for every polynomial P

$$(\Omega', P(a^*)\Omega) = (P(a)\Omega', \Omega) = 0 \tag{6.34}$$

and $\Omega' = 0$, because Ω is cyclic. □

We can rescue a part of the Stone-von Neumann to the case of an infinite number of d.o.f.

Theorem 6.3. *All irreducible Fock representations $(\mathcal{H}, \pi, \Omega)$ of a Heisenberg algebra $\mathcal{A}_H$ are unitarily equivalent.*

Therefore, it makes sense to speak of "the" Fock representation.

Proof. Given two irreducible Fock representations $\{(\mathcal{H}_i, \pi_i, \Omega_i)\}_{i=1,2}$ of $\mathcal{A}_H$, we can use the cyclic vectors Ω_1 and Ω_2 to define a map $U : \mathcal{H}_1 \supseteq \pi_1(\mathcal{A}_H)\Omega_1 \to \pi_2(\mathcal{A}_H)\Omega_2 \subseteq \mathcal{H}_2$ and its inverse $U^{-1} : \pi_2(\mathcal{A}_H)\Omega \to \pi_1(\mathcal{A}_H)\Omega$ on dense subsets by

$$\forall A \in \mathcal{A}_H : U\pi_1(A)\Omega_1 = \pi_2(A)\Omega_2 \tag{6.35a}$$
$$\forall A \in \mathcal{A}_H : U^{-1}\pi_2(A)\Omega_2 = \pi_1(A)\Omega_1 \,, \tag{6.35b}$$

including $U\Omega_1 = \Omega_2$ and $U^{-1}\Omega_2 = \Omega_1$. Since we can compute the matrix elements

$$(\pi_i(A)\Omega_i, \pi_i(B)\Omega_i) \tag{6.36}$$

for all $A, B \in \mathcal{P}(a^*)$ from the commutation relations and the condition $a_j\Omega_i = 0$, they can't depend on the representation and U must be unitary. □

The polynomials in the creation operators $\{\pi(a_i^*)\}$ applied to Ω form a dense subset of $\mathcal{H}$, in which all vectors are simultaneous eigenvectors of all the $\{N_i\}$. Therefore the Fock representation is also called the *occupation number representation.*

Note that we have not yet made any reference to the dynamics of the system. The Fock representation exists independently of the Hamiltonian. In the case of decoupled harmonic oscillators it provides a diagonalization of the Hamiltonian

$$H_0 = \sum_i \omega_i a_i^* a_i + \text{const.} \,, \tag{6.37}$$

but it exists also for free particles and other potentials. However *only* in the case (6.37) is Ω the ground state of the system. By the Stone-von Neumann theorem, the Fock representation is as good as any other representation for a finite number of d.o.f., even though there might be calculationally more convenient choices.

In the case of an infinite number of d.o.f., this is no longer true. As we shall see below, one can show with simple arguments that interacting relativistic quantum fields and many body systems with non-zero density can *not* be described by a Fock representation or the folium of a Fock representation.

The Fock representation a good representation, iff the total occupation number N is a good quantum number for the description of the system. Assume that the system described by (6.37) has a *mass gap* or *energy gap* m

$$\forall i : \omega_i \geqslant m > 0\,. \tag{6.38}$$

Then the series defining H_0 dominates the series defining N term by term and

$$N \leqslant \frac{1}{m} H_0 \tag{6.39}$$

and the existence of H_0 as a self adjoint operator implies the existence of the number operator N.

6.2 Non-Fock Representations

Consider a many particle system with non-zero density n in the thermodynamic limit

$$\left.\begin{matrix} N \to \infty \\ V \to \infty \end{matrix}\right\} \qquad n = \frac{N}{V} = \text{const.} > 0\,. \tag{6.40}$$

If N_V denotes the number operator in the volume N_V, we have

$$N_V \leqslant N \tag{6.41}$$

and therefore

$$\forall \Psi \in D(N) : \|n\Psi\| = \lim_{V\to\infty} \frac{1}{V} \|N_V \psi\| \leqslant \lim_{V\to\infty} \frac{1}{V} \underbrace{\|N\psi\|}_{<\infty} = 0\,. \tag{6.42}$$

Thus if $n \neq 0$, the total number operator N *must not* be a well defined operator with a dense domain. Thus we cannot use the Fock representation to describe such a system.

6.2.1 Haag's Theorem

A typical description of a physical system starts from a Hamiltonian for "free" or "non-interacting" states that can be described by decoupled harmonic oscillators (6.37) in the bosonic case (e. g. photons, phonons and Cooper pairs) and by anti-commuting annihilation and creation operators in the fermionic case (e. g. electrons). These states can be simple plane waves for photons in the vacuum or or complicated wave functions for electrons in condensed matter.

The interactions among these states is then taken into account by adding a polynomial in the annihilation and creation operators

$$H_g = \underbrace{\sum_i \omega_i a_i^* a_i}_{= H_0} + g H_{\text{int.}}(a, a^*) \,. \tag{6.43}$$

If we could diagonalize the Hamiltonian (6.43) explicitely, we could introduce annihilation and creation operators A_i and A_i^* for the interacting modes with energies E_i and write the Hamiltonian in a quadratic form

$$H_g = \sum_i E_i A_i^* A_i + \text{const.} \,. \tag{6.44}$$

A more realistic way to solve the dynamics of (6.43) is to first solve the linear Heisenberg picture equations corresponding to H_0

$$\frac{\mathrm{d}}{\mathrm{d}t} a_i^{(0)}(t) = \mathrm{i}[H_0, a_i^{(0)}(t)] = -\mathrm{i}\omega_i a_i^{(0)}(t) \tag{6.45}$$

as

$$a_i^{(0)}(t) = \mathrm{e}^{\mathrm{i}H_0 t} a_i(0) \mathrm{e}^{-\mathrm{i}H_0 t} = \mathrm{e}^{-\mathrm{i}\omega_i t} a_i(0) \,. \tag{6.46}$$

OTOH, the full Heisenberg equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t} a_i(t) = \mathrm{i}[H_g, a_i(t)] \tag{6.47}$$

for $g \neq 0$ are not linear and we can not derive a closed expression for

$$a_i(t) = \mathrm{e}^{\mathrm{i}H_g t} a_i(0) \mathrm{e}^{-\mathrm{i}H_g t} \,. \tag{6.48}$$

The compatibility of matrix elements in the Heisenberg and Schrödinger picture

$$(\Psi_H, a_H(t)\Phi_H) = \left(\Psi, \mathrm{e}^{\mathrm{i}H_g t} a \mathrm{e}^{-\mathrm{i}H_g t} \Phi\right)$$

$$= \left(\mathrm{e}^{-\mathrm{i}H_g t}\Psi, a\mathrm{e}^{-\mathrm{i}H_g t}\Phi\right) = (\Psi_S(t), a_S\Phi_S(t)) \tag{6.49}$$

can be extended to the interaction picture

$$\begin{aligned}(\Psi_H, a_H(t)\Phi_H) &= \left(\mathrm{e}^{\mathrm{i}H_0 t}\mathrm{e}^{-\mathrm{i}H_g t}\Psi, \mathrm{e}^{\mathrm{i}H_0 t}a\mathrm{e}^{-\mathrm{i}H_0 t}\mathrm{e}^{\mathrm{i}H_0 t}\mathrm{e}^{-\mathrm{i}H_g t}\Phi\right) \\ &= \left(U(t)\Psi, a^{(0)}(t)U(t)\Phi\right) = \left(\Psi, U^*(t)a^{(0)}(t)U(t)\Phi\right) \end{aligned}\tag{6.50}$$

where

$$a^{(0)}(t) = \mathrm{e}^{\mathrm{i}H_0 t}a\mathrm{e}^{-\mathrm{i}H_0 t} \tag{6.51a}$$

$$U(t) = \mathrm{e}^{\mathrm{i}H_0 t}\mathrm{e}^{-\mathrm{i}H_g t} = 1 + \mathcal{O}(g)\,. \tag{6.51b}$$

Since

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}U(t) &= \mathrm{e}^{\mathrm{i}H_0 t}\left(H_g - H_0\right)\mathrm{e}^{-\mathrm{i}H_g t} = g\mathrm{e}^{\mathrm{i}H_0 t}H_{\text{int.}}\mathrm{e}^{-\mathrm{i}H_g t} \\ &= g\mathrm{e}^{\mathrm{i}H_0 t}H_{\text{int.}}\mathrm{e}^{-\mathrm{i}H_0 t}\mathrm{e}^{\mathrm{i}H_0 t}\mathrm{e}^{-\mathrm{i}H_g t} = gH_{\text{int.}}(t)U(t)\,,\end{aligned}\tag{6.52}$$

the *formally* unitary operator $U(t) = (U^*(t))^{-1}$ satisfies

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}U(t) = gH_{\text{int.}}(t)U(t) \tag{6.53a}$$

with a *time dependent* interaction

$$H_{\text{int.}}(t) = \mathrm{e}^{\mathrm{i}H_0 t}H_{\text{int.}}(a,a^*)\mathrm{e}^{-\mathrm{i}H_0 t} = H_{\text{int.}}\left(a^{(0)}(t), \left(a^{(0)}(t)\right)^*\right)\,. \tag{6.53b}$$

The Schrödinger equation (6.53a) is solved formally by *Dyson's series*

$$U(t) = \mathrm{T}\mathrm{e}^{-\mathrm{i}g\int_0^t \mathrm{d}t'\,H_{\text{int.}}(t')}\,. \tag{6.54}$$

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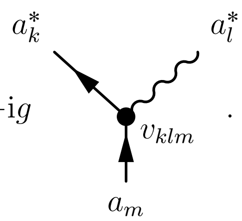
Matrix elements involving Dyson's series (6.54) are evaluated in perturbation theory using *Feynman diagrams*, for example,

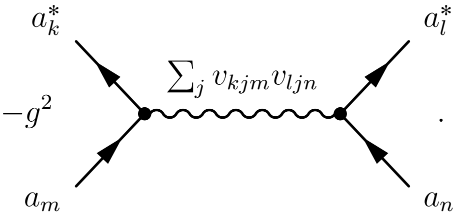
$$gH_{\text{int.}}(a,a^*) = g\sum_{klm} v_{klm}a_k^*a_l^*a_m + \mathrm{h.\,c.} \equiv -\mathrm{i}g$$



. (6.55)

A typical second order contribution is then

$$-g^2$$



. (6.56)

While perturbation theory has been spectacularly successful for predicting scattering cross sections in weakly interacting theories, it can fail spectacularly if the couplings become large or if the energy denominators appearing in the perturbation series become small.

This brings us to the question under which circumstances the operator $U(t)$ in (6.53a) is a well defined unitary operator relating the free and interacting modes. The surprising and somewhat disappointing answer is provided by

Theorem 6.4 (Haag (1955))**.** *Let $\mathcal{A}_H$ be the Heisenberg algebra of a system with local degrees of freedom that is invariant under translations. Let $(\mathcal{H}_0, \pi_0, \Omega_0)$ and $(\mathcal{H}_g, \pi_g, \Omega_g)$ be the Fock representations of $\mathcal{A}_H$ in which the free and interacting Hamiltonians H_0 and H_g, respectively, are well defined. These representations are* not *unitarily equivalent for $g \neq 0$.*

Thus the interaction picture does *not* exist for translation invariant systems, such as quantum field theories in the vacuum.

Before we can prove this theorem, we must define the systems under consideration more precisely. By "local degrees of freedom", we mean that the annihilation and creation operators spanning the corresponding $\mathcal{A}_H$ can be written in terms of operator valued distributions

$$a_i = \psi(f_i) = \int \mathrm{d}^n x\, \bar{f}_i(x)\psi(x) \tag{6.57}$$

with

$$[\psi(x), \psi^*(x')] = \delta^n(x - x') \tag{6.58}$$

or

$$\left[a_i, a_j^*\right] = \int \mathrm{d}^n x\, \bar{f}_i(x) f_j(x) = \langle f_i, f_j \rangle = \delta_{ij} \tag{6.59}$$

and $\{f_i\}$ an orthonormal set of square integrable smooth functions. Conversely, if $\{f_i\}$ is complete

$$\psi(x) = \sum_i f_i(x) a_i\,. \tag{6.60}$$

Note that x corresponds to a point in space, *not* an event in space time. The dynamics is given by the temporal evolution of the a_i or $\psi(x)$. If we have defined the annihilation and creation operators as in (6.57), we can implement space translations straighforwardly

$$\begin{aligned}\alpha : \mathbf{R}^n \times \mathcal{A}_H &\to \mathcal{A}_H \\ (\xi, \psi(f)) &\mapsto \alpha_\xi(\psi(f)) = \psi(\tilde\alpha_\xi f)\end{aligned} \tag{6.61a}$$

where

$$(\tilde\alpha_\xi f)(x) = f(x-\xi)\,. \tag{6.61b}$$

One could also write formally

$$(\alpha_\xi \psi)(x) = \psi(x+\xi)\,, \tag{6.62}$$

but this is only to be understood as a shorthand, because the ψ are distributions and $\psi(x)$ is *not* an element of $\mathcal{A}_H$.

Obviously, each α_ξ defines a $*$-automorphism of $\mathcal{A}_H$ and α therefore a n-parameter family of $*$-automorphisms.

Lemma 6.5. *In an irreducible Fock representation $(\mathcal{H}, \pi, \Omega)$ of a $\mathcal{A}_H$ that is invariant under translations (6.61), the Fock state Ω is the unique translation invariant state and space translations are implemented by strongly continuous unitary operators $U(\xi)$, i. e.*

$$U(\xi)\psi(x)U^*(\xi) = \psi(x+\xi)\,. \tag{6.63}$$

Proof. In a Fock representation, the number operator

$$\begin{aligned}N = \sum_i a_i^* a_i = \sum_i \int \mathrm{d}^n x \mathrm{d}^n y\, f_i(x) \bar f_i(y) \psi^*(x)\psi(y) \\ = \int \mathrm{d}^n x \mathrm{d}^n y\, \delta^n(x-y) \psi^*(x)\psi(y) = \int \mathrm{d}^n x \psi^*(x)\psi(x)\end{aligned} \tag{6.64}$$

is well defined. It is obviously invariant under translations $\alpha_\xi(N) = N$ and the spacial translations commute with it $[U(\xi), N] = 0$. Thus is suffices to study the sectors of $\mathcal{H}$ with a fixed number of modes, WLOG m. These correspond to wavefunctions for m “particles”, i. e. square integrable functions $\mathbf{R}^{nm} \to \mathbf{C}$. But square integrable functions must fall of in every direction for $nm > 0$ and can not be invariant under the action of $\tilde\alpha_\xi$, i. e. constant shift $\forall i = 1, \ldots, m : x_i \mapsto x_i + \xi$. Thus only the 0-“particle” state Ω is translation invariant. □

Proof of Haag's Theorem 6.4. By lemma 6.5, Ω_g is the unique translation invariant state with $U_g(\xi)\Omega_g = \Omega_g$ for all members of the family of translation operators $U_g(\xi)$. OTOH, assume that $(\mathcal{H}_0, \pi_0, \Omega_0)$ is another, unitarily equivalent Fock representation, we can use the operators a_i and a_i^* with the well defined transformation properties under translation in both of them. Then we can use *both* $U_g(\xi)$ and $U_0(\xi)$ to translate them

$$U_g(\xi) a_i U_g^*(\xi) = \alpha_\xi(a_i) \tag{6.65a}$$
$$U_0(\xi) a_i U_0^*(\xi) = \alpha_\xi(a_i) \tag{6.65b}$$

and find

$$U_g(\xi) U_0^*(\xi) a_i U_0(\xi) U_g^*(\xi) = \alpha_\xi(\alpha_{-\xi}(a_i)) = a_i\,. \tag{6.66}$$

Therefore

$$\left[U_g U_0^{-1}, a_i\right] = \left[U_g U_0^{-1}, a_i^*\right] = 0 \tag{6.67}$$

or

$$\left[U_g U_0^{-1}, \mathcal{A}_H\right] = 0 \tag{6.68}$$

for all representations π. In an irreducible representation, we must have

$$U_g(\xi) U_0^{-1}(\xi) = \mathbf{1} \mathrm{e}^{\mathrm{i}\theta_g(\xi)} = \mathbf{1} \mathrm{e}^{\mathrm{i}\theta_g \cdot \xi}\,, \tag{6.69}$$

where the second equation is a result of the group property. We can therefore absorb the phase $\mathrm{e}^{\mathrm{i}\theta_g \cdot \xi}$

$$U_g(\xi) \to U_g(\xi) \mathrm{e}^{-\mathrm{i}\theta_g \cdot \xi} \tag{6.70}$$

to get

$$U_g(\xi) = U_0(\xi)\,. \tag{6.71}$$

Finally, if the translation operators are the same, the unique states left invariant by them must also be the same

$$\Omega_g = \Omega_0\,. \tag{6.72}$$

This implies that the Fock state, which is the lowest energy state in both representations agrees, which can not be, since the interaction changes the ground state. □

In fact, this phenomenon can be demonstrated already by very simple examples

Example 6.6. While this caricature

$$H = \sum_i \omega_i a_i^* a_i + g \sum_i \left(\bar{j}_i a_i + j_i a_i^*\right) \tag{6.73}$$

of a scalar field $\psi \propto a + a^*$ coupled to a classical "source" $\{j_i\}$ can be diagonalized exactly for any $\{j_i\}$ and the corresponding unitary operators intertwining between $g = 0$ and $g \neq 0$ can be constructed explicitely for each mode *seperately*, the product exists only if $\sum_i |j_i|^2/\omega_i^2 \leqslant \infty$. In a local theory, we will have $\lim_{i\to\infty} j_i \not\to 0$, however.

Example 6.7. A quadratic perturbation

$$H = \sum_i \left(\frac{1}{2m} p_i^2 + \frac{m\omega_i^2}{2} x_i^2 \right) + g \sum_i \frac{m\omega_i^2}{2} x_i^2 \tag{6.74}$$

can also be diagonalized exactly by *Bogolyubov tranformations*

$$\begin{pmatrix} a_i \\ a_i^* \end{pmatrix} \mapsto \begin{pmatrix} u & v \\ \bar v & \bar u \end{pmatrix} \begin{pmatrix} a_i \\ a_i^* \end{pmatrix} \tag{6.75}$$

with $|u|^2 - |v|^2 = 1$ and the intertwining unitary operators exist for each mode, but the product does not exist.

These examples show how important it is, to solve the linear and quadratic interactions *exactly* before attempting perturbation theory for the remaining nonlinear interactions. Of course, the perturbation series for the remaining terms can still diverge and the intertwining operators can fail to exist even if the series converges.

6.3 Local Quantum Theory

While locality in concert with translation invariance provided the ingredients for Haag's theorem, it is nevertheless an important physical principle, realized in most models of microphysics.

A systematic approach considers the C^*-algebras $\mathcal{A}(\mathcal{V})$ and $\mathcal{A}(\mathcal{O})$ generated by the observables localized in a volume $\mathcal{V} \subseteq \mathbf{R}^n$ or a space-time region $\mathcal{O} \subseteq \mathbf{R} \times \mathbf{R}^n$. The algebras obviously must satisfy

$$\forall \mathcal{O}_1 \subseteq \mathcal{O}_2 : \mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)\,. \tag{6.76}$$

In relativistic physics, the measurements of observables in regions $\mathcal{O}_1$ and $\mathcal{O}_1$ that are separated by a space-like distance, as in figure 6.1, *must not* influence each other, i. e. the corresponding algebras must commute

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0\,. \tag{6.77}$$

This usually called *Einstein locality.*

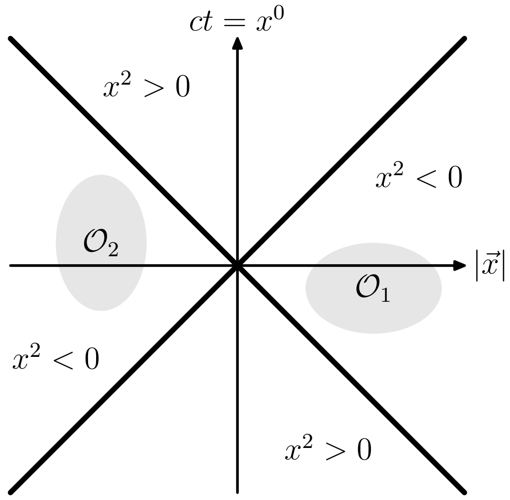


Figure 6.1: *The light cone $x^2 = c^2t^2 - \vec{x}^2 = 0$ separates time-like $x^2 > 0$ from space-like $x^2 < 0$ distances in $\mathbf{R} \times \mathbf{R}^n$. The regions $\mathcal{O}_1$ and $\mathcal{O}_2$ are separated by a space-like distance.*

In any case, the physically relevant algebra is the *local algebra*

$$\mathcal{A}_L = \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O}) \tag{6.78}$$

or $\bigcup_{\mathcal{V}} \mathcal{A}(\mathcal{V})$, respectively. While the multiplication of two elements from different algebras $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ is *a priori* not defined, this is never a problem in practice when the elements of the algebras are generated by smearings of local operators, e. g.

$$\exp\left(\mathrm{i}\int \mathrm{d}x \psi(x)\bar{f}(x)\right) \tag{6.79}$$

with $\mathrm{supp}\,(f) \in \mathcal{O}_i$. But note that $\mathcal{A}_L$ is *not* complete by construction and one needs to complete it in an appropriate topology to the *local algebra*

$$\mathcal{A} = \overline{\mathcal{A}_L}\,, \tag{6.80}$$

In relativistic physics, the finite propagation speed causes the completion in the norm topology to be invariant under the time evolution described by a one parameter family of automorphisms

$$\begin{aligned} \alpha : \mathbf{R} \times \mathcal{A} &\to \mathcal{A} \\ (t, A) &\mapsto \alpha_t(A)\,, \end{aligned} \tag{6.81}$$

but in the non-relativistic approximation with instantaneous interactions, it can happen that only a larger algebra, obtained by completing in a weaker topology, is stable under the time evolution α.

6.3.1 Asymptotic Abelianness

In general, even in the non-relativistic case, we should be able to *require* that two observables commute asymptotically, if we move them apart

$$\forall A, B \in \mathcal{A} : \lim_{|x|\to\infty} [\alpha_x(A), B] = 0\,, \tag{6.82}$$

where α_x are the automorphisms (6.61) of $\mathcal{A}$ that realize spacial translations. Indeed, if *asymptotic abelianness* (6.82) was not satisfied, observables would be influenced by measurements at spacial infinity, which would be unphysical, of course.

However, one might relax (6.82) to *weak asymptotic abelianness*

$$\forall A, B \in \mathcal{A} : \operatorname*{w-lim}_{|x|\to\infty} [\pi(\alpha_x(A)), \pi(B)] = 0\,, \tag{6.83}$$

for any "relevant" representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$.

6.3.2 Physically Relevant Representations

The local algebra $\mathcal{A}$ necessarily corresponds to a Weyl or Heisenberg algebra with an infinite number of d.o.f., since the creation and annihilation operators are convolutions of field operators with functions

$$a(f) = \int \mathrm{d}x\, \bar{f}(x)\psi(x)\,. \tag{6.84}$$

Thus we cannot rely on the Stone-von Neumann theorem to make sure that predictions are independent of the representation chosen.

Therefore we must develop criteria to choose the correct representation(s) $(\mathcal{H}, \pi)$ describing a given system in a given state:

1. *Existence of energy and momentum:* the space translation $\mathbf{R}^n \times \mathcal{A} \to \mathcal{A}$ and time evolution $\mathbf{R}^n \times \mathcal{A} \to \mathcal{A}$ automorphisms are realized by strongly continuous abelian groups of unitary operators

$$\forall x \in \mathbf{R}^n, A \in \mathcal{A} : U(x)\pi(A)U^*(x) = \pi(\alpha_x(A)) \tag{6.85a}$$

$$\forall t \in \mathbf{R}, A \in \mathcal{A} : U(t)\pi(A)U^*(t) = \pi(\alpha_t(A))\,. \tag{6.85b}$$

By Stone's theorem, this guarantees that there are self-adjoint generators P and H with a common dense domain $D \subset \mathcal{H}$ and corresponding to momentum and energy, respectively. Note that in condensed matter theory, the continuous group of translations might have to be replaced by the discrete subgroup of lattice translations.

2. *Stability:* the spectrum $\sigma(H)$ of the Hamiltonian is bounded from below
$$\sigma(H) \subseteq [E_{\text{min}}, \infty) \tag{6.86}$$
and by the redefinition $U(t) \mapsto U(t)\mathrm{e}^{-\mathrm{i}E_{\text{min}}t}$, we may choose $E_{\text{min}} = 0$. The relativistically invariant form of the spectral condition is
$$\sigma(H) \subseteq [0, \infty) \tag{6.87a}$$
$$H^2 \geqslant P^2 \,. \tag{6.87b}$$

3. *Existence of a ground state:* The infimum of $\sigma(H)$ is a non degenerate eigenvalue of H with associated eigenvector $\Omega \in \mathcal{H}$, which is called the *ground state* and has the properties:
   (a) Ω is cyclic wrt to the local algebra $\mathcal{A}_L$, i. e.
$$\mathcal{H} = \overline{\pi(\mathcal{A}_L)\Omega} \,. \tag{6.88}$$
   and
   (b) Ω is the unique translation invariant state.

   Note that the non-degeracy of the lowest eigenvalue is refers to the chosen representation $(\mathcal{H}, \pi)$ *only*. There is in general more than one representation with the same local eigenvalue and corresponding ground state.

The cyclicity of Ω wrt $\mathcal{A}_L$ is equivalent to the requirement that it must be possible, at least in principle, to prepare *all* states by local operations. The translation invariance of the ground state means that the large distance behaviour of the system is encoded in the ground state.

In fact, the uniqueness of the translation invariant state follows from asymptotic abelianness

Theorem 6.8. *In any irreducible cyclic representation $(\mathcal{H}, \pi, \Omega)$ of the algebra $\mathcal{A}$, that satisfies weak asymptotic abelianness (6.83), the ground state is the unique translation invariant state.*

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This can be shown using

Theorem 6.9 (Von Neumann bicommutant theorem)**.** *For any unital $*$-subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$, the following are equivalent*

1. $\mathcal{A} = \mathcal{A}'' = (\mathcal{A}')'$
2. $\mathcal{A}$ *is closed in the weak topology*
3. $\mathcal{A}$ *is closed in the strong topology,*

which we state without proof.

—A— Acronyms

a.k.a. also known as

CCR Canonical Communation Relations

CSI Cauchy-Schwarz Inequality

d.o.f. degrees of freedom

FAPP For All Practical Purposes

iff if and only if

ITP Infinite Tensor Product

LHS Left Hand Side

OTOH On The Other Hand

QM Quantum Mechanics

RHS Right Hand Side

SSB Spontaneous Symmetry Breaking

WLOG Without Loss Of Generality

wrt with respect to

—B— Bibliography

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