Funktionalanalysis Hausaufgaben Blatt 2

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Problem 1. The goal of this exercise is to show that every finite dimensional vector space carries a unique Hausdorff topology. Let V be a finite dimensional topological vector space of dimension $n \in \mathbb{N}$.

(a) Use the continuity of the scalar multiplication to show that every open neighborhood U of zero contains an open balanced neighborhood U_0 of zero, that is $zU_0 \subseteq U_0$ for all $z \in \mathbb{K}$ with $|z| \leq 1$.

(b) Given a basis $(e_1, \ldots, e_n)$ of $\mathbb{K}^n$ and a basis $(v_1, \ldots, vn)$ of V , we define the map $\varphi : \mathbb{K}^n \to V$ as the K-linear extension of the map $e_i \mapsto v_i$. Recall that φ is an isomorphism of vector spaces. Show that φ is continuous if $\mathbb{K}^n$ is endowed with the standard topology.

(c) Let V be Hausdorff. Show that $0 \in \varphi(B_r(0))^\circ$ for every $r > 0$.

Hint: Consider the subset $V \setminus \varphi(\mathbb{S}^{n-1})$.

(d) Conclude that φ^{-1} is also continuous

Problem 2. Let $(M, \mathcal{M})$ be a topological space and $(f_n)_n \in \mathbb{N} \subset C(M, \mathbb{K})$ be a sequence of continuous functions that converges pointwise to a (not necessarily continuous!) function f. For $\epsilon > 0$ and $n \in \mathbb{N}$ we define

$$C_n(\epsilon) := \{p \in M : |f_n(p) - f(p)| \leq \epsilon\}$$

and set

$$C(\epsilon) := \bigcup_{n=1}^{\infty} C_n(\epsilon)^\circ$$

and

$$C := \bigcap_{n=1}^{\infty} C\left(\frac{1}{n}\right)$$

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(a) Show that f is continuous at $p \in M$ iff $p \in C$

(b) Consider the set

$$A_n(\epsilon) := \{p \in M : |f_n(p) - f_k(p)| \leq \epsilon \text{ for all } k \geq n\}.$$

Show that the boundary of $A_n(\epsilon)$ is nowhere dense.

(c) Show that the discontinuities of f form a meager set of M.

(d) Prove the following statement: There is no differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative equals the function

$$g : R \ni x \mapsto g(x) := \begin{cases} 1 & x \in (\mathbb{R} \setminus (0,1)) \cup (\mathbb{Q} \cap (0,1)) \\ 0 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap (0,1). \end{cases}$$

Proof. (a) Choose $\epsilon > 0$. Now we unravel the definitions. We know that p is in $C(1/n')$ with $1/n' < \epsilon$. This means that it is in the interior of $C_n(1/n')$ for some n. Then there is an open set containing p, such that $|f_n(x) - f(x)| \leq \epsilon$ for all x in this open set.

Using the continuity of f_n at p, we choose an smaller open set such that f_n does not vary by more than ϵ. Using this, we can show that f does not vary by more than 3ϵ within this open set, completing the proof of continuity.

(b) The set $A_n(\epsilon)$ is closed, since it is the intersection of the preimages of the closed set $[0, \epsilon]$ under the continuous map $|f_n(\cdot) - f_k(\cdot)|$.

Then we show that the boundary of a closed set A of M is nowhere dense. Since the boundary is closed, we only need to show it has empty interior. Suppose not. Then there is an open set U that lies both in A and $(M \setminus A)^{\mathrm{cl}}$. Thus there is an open set V $subseteqU$ lying in $M \setminus A$. However, since A is closed, this is a contradiction, as this subset cannot also lie in A.

(c) This is the same as showing that the complement of C is meager. The complement of C is given by

$$M \setminus C = \bigcup_{n=1}^{\infty} \left[M \setminus C\left(\frac{1}{n}\right) \right]$$

$$
\begin{aligned}
&= \bigcup_{n=1}^{\infty} \left[M \setminus \bigcup_{k=1}^{\infty} C_k \left(\frac{1}{n} \right)^{\circ} \right] \\
&= \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \left[M \setminus C_k \left(\frac{1}{n} \right)^{\circ} \right] \\
&= \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \left[M \setminus C_k \left(\frac{1}{n} \right) \right]^{\mathrm{cl}}
\end{aligned}
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$\square$