Algebra und Dynamik von Quantensystemen Blatt Nr. 1

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Problem 1 (Norm)**.** Which of the maps $\mathcal{M}_2 \to [0, \infty)$

$$M \mapsto |\det M| \tag{1a}$$

$$M \mapsto |\operatorname{tr} M| \tag{1b}$$

$$M \mapsto \sup_{ij} |M_{ij}| \tag{1c}$$

$$M \mapsto \sup_{\substack{v \in \mathbb{C}^2, \\ \|v\|=1}} \|Mv\| \quad (\text{with } \|v\| = \sqrt{|v_1|^2 + |v_2|^2}) \tag{1d}$$

define a norm on the algebra $\mathcal{M}_2$ of complex 2×2 matrices? Which turn $\mathcal{M}_2$ into a C^*-algebra?

Proof. The first is not homogeneous, because multiplying M by a constant multiplies $\det M$ by the squared constant.

The second is not positive definite: $\operatorname{Tr}(\operatorname{diag}(1, -1)) = 0$, but $\operatorname{diag}(1, -1) \neq 0$.

The third is not a norm:

1. It is homogeneous.
2. It is positive definite
3. It satisfies the triangle inequality.
4. It is not submultiplicative: Consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

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thus

$$2 = \|A^2\| \not\leq \|A\|\|A\| = 1.$$

This is the functional norm. □

Problem 2 (Spectrum and Resolvent)**.** Use the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

to parametrize a general complex 2×2 matrix $M \in \mathcal{M}_2$ by four complex numbers $(a_0, \vec{a})$:

$$M(a_0, \vec{a}) = a_0 1 + \vec{a} \cdot \vec{\sigma} = a_0 1 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}. \tag{3}$$

1. For which $(a_0, \vec{a}) \in \mathbb{C}^4$ is $M(a_0, \vec{a})$
   (a) normal,
   (b) isometric,
   (c) unitary,
   (d) self-adjoint,
   (e) positive?
2. Determine the resolvent set $r_{\mathcal{M}_2}(M(a_0, \vec{a}))$ and spectrum $\sigma_{\mathcal{M}_2}(M(a_0, \vec{a}))$ for all $(a_0, \vec{a})$. Handle exceptional cases.
3. Test the general results for the spectrum of normal, isometric, unitary, self-adjoint, and positive matrices.
4. Compute the resolvent

$$R^{a_0, \vec{a}} : r_{\mathcal{M}_2}(M(a_0, \vec{a})) \to \mathcal{M}_2, \quad z \mapsto (z1 - M(a_0, \vec{a}))^{-1} \tag{4}$$

   as a 2×2 matrix (i.e., perform the matrix inversion explicitly!).

5. **NB:** In the lecture on *Tuesday, November 4, 2025*, it will be shown that $P_{\mathcal{C}}^{M(a_0,\vec{a})}$ is indeed a projection, if $\mathcal{C}$ encircles a part of the spectrum $\sigma(M(a_0,\vec{a}))$. You can wait until then to complete the exercise, take a peek at the script, or just do the integral choosing typical examples for $\mathcal{C}$ based on your earlier results for $\sigma(M(a_0,\vec{a}))$.

   Compute the projections

$$P_{\mathcal{C}}^{M(a_0,\vec{a})} = \int_{\mathcal{C}} \frac{dz}{2\pi i} R^{a_0,\vec{a}}(z) \tag{5}$$

   for "interesting" $\mathcal{C}$ explicitly. Are there qualitatively different cases to consider?

Proof. 1. (a) We compute $AA^\dagger - A^\dagger A$ explicitly: We have

$$\begin{aligned}
AA^\dagger &= \left(\sum_{i=0}^{3} a_i\sigma_i\right)\left(\sum_{j=0}^{3} a_j^*\sigma_j\right)\\
&= \sum_{i,j=0}^{3} a_i a_j^* \sigma_i\sigma_j\\
&= |a_0|^2 + \sum_{i=1}^{3} a_i a_0^*\sigma_i + \sum_{i=1}^{3} a_0 a_i^*\sigma_i + \sum_{i,j=1}^{3} a_i a_j^*\sigma_i\sigma_j\\
&= |a_0|^2 + \sum_{i=1}^{3}(a_i a_0^* + a_i^* a_0)\sigma_i + \sum_{i,j=1}^{3} a_i a_j^*\left(\delta_{ij}I + i\sum_k \epsilon_{ijk}\sigma_k\right)\\
&= \sum_i |a_i|^2 + \sum_{i=1}^{3}(a_i a_0^* + a_i^* a_0)\sigma_i + i\sum_{i,j,k=1}^{3}\epsilon_{ijk}a_i a_j^*\sigma_k
\end{aligned}$$

By the symmetry, we get that

$$A^\dagger A = \sum_i |a_i|^2 + \sum_{i=1}^{3}(a_i a_0^* + a_i^* a_0)\sigma_i + i\sum_{i,j,k=1}^{3}\epsilon_{ijk}a_i^* a_j\sigma_k.$$

Thus, we subtract them and demand that their difference vanishes, which leads to

$$0 = \sum_{i,j,k=1}^{3}[\epsilon_{ijk}a_i a_j^* - \epsilon_{ijk}a_i^* a_j]\sigma_k.$$

Using the orthogonality of the σ_k s, this is equivalent to the expression that

$$\vec{a}\times\vec{a}^* = 0.$$

(b) As in the last part, we have

$$A^\dagger A = \sum_i |a_i|^2 + \sum_{k=1}^{3}\left[(a_k a_0^* + a_k^* a_0) + i\sum_{i,j=1}^{3}\epsilon_{ijk}a_i^* a_j\right]\sigma_k.$$

Since this must be the identity, we must have $\sum_{i=0}^{3} |a_i|^2$ and

$$a_k a_0^* + a_k^* a_0 + i(\vec{a}^* \times \vec{a})_k = 0.$$

(c) It is unitary if it is isometric and normal. By substituting, we get

$$\vec{a}^* \times \vec{a} = 0$$
$$\mathrm{Re}(a_k a_0^*) = 0$$

(d) It is self adjoint if all as are real.

(e) It is positive if

$$a_0 \geq \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

2. The eigenvalues are

$$a_0 \pm \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

3. As example unitary matrix, we choose

$$M = `\frac{1}{9\sqrt{2}} \begin{pmatrix} 2+3i & 10+7i \\ -10+7i & 2-3i \end{pmatrix}$$

which corresponds to vector

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{9} \\ \frac{7i}{9\sqrt{2}} \\ \frac{5i\sqrt{2}}{9} \\ \frac{i}{3\sqrt{2}} \end{pmatrix}.$$

We can see clearly that it satisfies the equations for a unitary matrix.

4.

$$\frac{1}{z - M(a_0, \vec{a})} = \begin{pmatrix} -\frac{-a_0+a_3+z}{2a_0z-a_0^2+a_1^2+a_2^2+a_3^2-z^2} & -\frac{a_1-ia_2}{2a_0z-a_0^2+a_1^2+a_2^2+a_3^2-z^2} \\ -\frac{a_1+ia_2}{2a_0z-a_0^2+a_1^2+a_2^2+a_3^2-z^2} & -\frac{-a_0-a_3+z}{2a_0z-a_0^2+a_1^2+a_2^2+a_3^2-z^2} \end{pmatrix}.$$

5. We expect qualitatively that there should be 4 classes of contours $\mathcal{C}$: Two, that enclose either point of the spectrum, one that encloses both and one that encloses neither.

We write the integral explicitly:

$$\mathcal{P}_{\mathcal{C}}^{M(a_0,\vec{a})} = -\int_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{1}{2a_0 z - a_0^2 + |\vec{a}|^2 - z^2} \begin{pmatrix} z - a_0 + a_3 & a_1 - i a_2 \\ a_1 + i a_2 & z - a_0 - a_3 \end{pmatrix} .$$

Factorising the denominator yields

$$z^2 - 2a_0 z + a_0^2 - |\vec{a}|^2 = .$$

□