Homework for the Lecture

Functional Analysis

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Homework Sheet No 14

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(23 Points. Discussion 03. 02. 2025)

Homework 14-1: The Lax-Milgram Theorem

Let $(\mathfrak{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $B : \mathfrak{H}^2 \to \mathbb{C}$ be a continuous sesquilinear map.

i.) **(5 Points)** Show that there is a unique bounded linear operator $T \in L(\mathfrak{H})$ such that

$$B(\psi, \phi) = \langle \psi, T\phi \rangle \tag{14.1}$$

for all $\phi, \psi \in \mathfrak{H}$.

ii.) **(4 Points)** Conclude that the set of continuous sesquilinear functionals (endowed with the operator norm topology) on $\mathfrak{H}$ is isometrically isomorphic to $L(\mathfrak{H})$.

iii.) **(6 Points)** Assume that B is coercive, i.e. there exists a constant $m > 0$ such that

$$B(\phi, \phi) \geq m\|\phi\|^2 \tag{14.2}$$

for every $\phi \in \mathfrak{H}$. Prove that in this case the corresponding operator T is invertible. Moreover, show that T^{-1} is continuous with $\|T^{-1}\| \leq \frac{1}{m}$.
Hint: Show that the orthogonal complement of the image of T does only contain the zero vector.

Homework 14-2: The Bargmann-Fock space: Part II

Consider the Bargmann-Fock space $\mathfrak{H}_{BF}$ from Homework 11-3.

i.) **(2 Points)** Prove that the orthonormal system $\{e_{k_1 \ldots k_n}(\overline{z}) : k_1, \ldots, k_n \in \mathbb{N}_0\} \subset \mathfrak{H}_{BF}$ from Homework 11-3, *iii.)* is complete.

ii.) **(2 Points)** For $i \in \{1, \ldots, n\}$, define the map

$$a_i^\dagger : \mathfrak{H}_{BF} \supset \mathbb{C}[\overline{z}_1, \ldots, \overline{z}_n] \ni p \mapsto \overline{z}_i p \in \mathbb{C}[\overline{z}_1, \ldots, \overline{z}_n]. \tag{14.3}$$

Show that $a_i^\dagger p \in \mathfrak{H}_{BF}$ for every polynomial function p.
Hint: Consider the integral $\frac{1}{(2\pi\hbar)^n} \int_{(\mathrm{B}_r(0)^{\mathrm{cl}})^n} |z_i|^2 |p(\overline{z})|^2 \mathrm{e}^{-\frac{\overline{z}z}{2\hbar}} \,\mathrm{d}z \,\mathrm{d}\overline{z}$ *in the limit* $r \to \infty$.

iii.) **(2 Points)** Determine the adjoint operator $a_i := \left(a_i^\dagger\right)^*$ of $a_i^\dagger$and compute the commutator $[a_i, a_j^\dagger]$ for $i, j \in \{1, \ldots, n\}$.

iv.) **(2 Points)** Study continuity of a_i and $a_i^\dagger$.