Homework for the Lecture

Functional Analysis

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Homework Sheet No 8

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(31 Points. Discussion 09. 12. 2024)

Homework 8-1: Vector Valued Functions

Consider the space $\mathscr{C}(X, V)$ of vector-valued continuous functions, where X is a topological space and $(V, \|\cdot\|)$ is a normed space.

i.) **(3 Points)** Show that the subset

$$\mathscr{C}_{\mathrm{b}}(X, V) := \left\{ f \in \mathscr{C}(X, V) : \|f\|_\infty := \sup_{x \in X} \|f(x)\| < \infty \right\} \subseteq \mathscr{C}(X, V)$$

of bounded (vector-valued) continuous functions together with the map $\|\cdot\|_\infty$ is a normed space, which is complete iff V is complete.

ii.) **(4 Points)** Let Y be a compact space. Show that

$$c : \mathscr{C}_{\mathrm{b}}(X \times Y, V) \ni f \mapsto (c(f) : X \ni x \mapsto (Y \in y \mapsto f(x, y))) \in \mathscr{C}_{\mathrm{b}}(X, \mathscr{C}_{\mathrm{b}}(Y, V)) \tag{8.1}$$

is a well-defined linear isometry. Here, $X \times Y$ is endowed with the product topology.
Hint: You can use that the canonical projections onto X and Y are open without proof.

iii.) **(2 Points)** Give a counterexample to prove that compactness of Y cannot be omitted in the definition of c in general.

iv.) **(3 Points)** Show that c has a continuous inverse map.

Homework 8-2: Absorbing, Balanced and Convex Sets

This exercise is devoted to the proof of Proposition 3.1.10 from the lecture. So, consider a $\mathbb{K}$-vector space V.

i.) **(1 Point)** Let $\mathrm{p} : V \to \mathbb{R}_0^+$ be a seminorm. Show that the balls

$$\mathrm{B}_{\mathrm{p},1}(0) := \{v \in V : \mathrm{p}(v) < 1\} \tag{8.2}$$

and

$$\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}} := \{v \in V : \mathrm{p}(v) \leq 1\} \tag{8.3}$$

are convex, absorbing and balanced.

ii.) **(4 Points)** Show that for every convex, absorbing and balanced subset $C \subseteq V$ the Minkowski functional

$$V \ni v \mapsto \mathrm{p}_C(v) := \inf\{\lambda > 0 : v \in \lambda C\} \tag{8.4}$$

is a seminorm on V.

iii.) **(1 Point)** Let C be a convex, absorbing and balanced subset of V. Prove the following inclusions

$$\mathrm{B}_{\mathrm{p}_C,1}(0) \subseteq C \subseteq \mathrm{B}_{\mathrm{p}_C,1}(0)^{\mathrm{cl}}. \tag{8.5}$$

iv.) **(1 Point)** Let p be a seminorm on V. Show that the equations

$$\mathrm{p}_{\mathrm{B}_{\mathrm{p},1}(0)} = \mathrm{p} = \mathrm{p}_{\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}} \tag{8.6}$$

hold true.

Homework 8-3: The (Absolute) Convex Hull

(4 Points) Let V be a $\mathbb{K}$-vector space and $X \subseteq V$ be a subset. Characterize the elements of the convex hull $\operatorname{conv}(X)$, that is the smallest convex superset of X, in terms of the elements of X. Do the same for the absolute convex hull $\operatorname{absconv}(X)$ which is defined analogously.

Homework 8-4: Infinite Dimensional Normed Spaces

Consider a (not necessarily complete) normed space $(V, \|\cdot\|)$ of infinite dimension.

i.) **(3 Points)** Prove that V has an uncountable basis if it is complete.
Hint: In case of need, one should call for the white knight.

ii.) **(1 Point)** Show that V allows for unbounded linear functionals.

iii.) **(4 Points)** Let $\varphi \in V^*$ be a linear functional. Construct a net $(\varphi_i)_{i \in I} \subset V'$ of bounded linear functionals that converges pointwise towards φ.