Quantum field theory in the solid state, Exercise sheet 6

Corrections: Week of June 16th

The Holstein model

We will consider a model describing identical fermions on a lattice. Each lattice site is subject to harmonic vibrations (phonons) that alter the on-site energy. The model reads:

$$\hat{H} = \sum_{\boldsymbol{i},\boldsymbol{j},\sigma} \hat{c}^{\dagger}_{\boldsymbol{i},\sigma} T_{\boldsymbol{i},\boldsymbol{j}} \hat{c}_{\boldsymbol{j},\sigma} + g \sum_{\boldsymbol{i}} \hat{Q}_{\boldsymbol{i}} \hat{n}_{\boldsymbol{i}} + \sum_{\boldsymbol{i}} \left(\frac{\hat{P}^2_{\boldsymbol{i}}}{2M} + \frac{k}{2} \hat{Q}^2_{\boldsymbol{i}} \right) \tag{1}$$

. Here,

$$\left\{ \hat{c}_{\boldsymbol{i},\sigma}, \hat{c}^{\dagger}_{\boldsymbol{j},\sigma'} \right\} = \delta_{\boldsymbol{i},\boldsymbol{j}} \delta_{\sigma,\sigma'} \quad \left\{ \hat{c}^{\dagger}_{\boldsymbol{i},\sigma}, \hat{c}^{\dagger}_{\boldsymbol{j},\sigma'} \right\} = 0 \tag{2}$$

satisfy fermion commutation rules,

$$\hat{n}_{\boldsymbol{i}} = \sum_{\sigma} \hat{c}^{\dagger}_{\boldsymbol{i},\sigma} \hat{c}_{\boldsymbol{i},\sigma} \tag{3}$$

corresponds to the fermion particle number on Wannier state centered around unit cell $\boldsymbol{i}$,

$$\left[\hat{Q}_{\boldsymbol{i}}, \hat{P}_{\boldsymbol{j}} \right] = i\hbar \delta_{\boldsymbol{i},\boldsymbol{j}} \tag{4}$$

correspond the the position and momentum operator of the lattice vibration, and finally

$$\left[\hat{Q}_{\boldsymbol{i}}, \hat{c}_{\boldsymbol{j}} \right] = \left[\hat{P}_{\boldsymbol{i}}, \hat{c}_{\boldsymbol{j}} \right] = 0. \tag{5}$$

a)

Write the path integral in the coherent state representation for the fermions and in real space for the phonons.

b)

Integrate out the phonon degrees of freedom. The result you should obtain for the imaginary path integral at inverse temperature β is:

$$Z = \int \prod_{\boldsymbol{i},\sigma,\tau} d\xi^{\dagger}_{\boldsymbol{i},\sigma}(\tau) d\xi_{\boldsymbol{i},\sigma}(\tau) e^{-S\left(\{\xi^{\dagger},\xi\}\right)} \tag{6}$$

with

$$S(\{\xi^{\dagger},\xi\}) = S_0(\{\xi^{\dagger},\xi\}) - \int_0^{\beta} d\tau d\tau' \sum_{\boldsymbol{i},\boldsymbol{j},\sigma,\sigma'} \xi^{\dagger}_{\boldsymbol{i},\sigma}(\tau) \xi_{\boldsymbol{i},\sigma}(\tau) D_{\boldsymbol{i},\boldsymbol{j}} \left(\tau - \tau'\right) \xi^{\dagger}_{\boldsymbol{j},\sigma'}(\tau') \xi_{\boldsymbol{j},\sigma'}(\tau'). \tag{7}$$

Here the phonon propagator reads:

$$D_{\boldsymbol{i},\boldsymbol{j}}(\tau-\tau') = \delta_{\boldsymbol{i},\boldsymbol{j}} \frac{g^2}{2k} P(\tau-\tau') \tag{8}$$

with

$$P(\tau) = \frac{\omega_0}{2(1-e^{-\beta\omega_0})}\left[e^{-|\tau|\omega_0} + e^{-(\beta-|\tau|)\omega_0}\right], \quad \omega_0 = \sqrt{\frac{k}{M}}. \tag{9}$$

c)

Show that in the anti-adiabatic limit $\omega_0 \to \infty$ the interaction becomes instantaneous and the resulting action corresponds to the attractive Hubbard model:

$$\hat{H} = \sum_{\boldsymbol{i},\boldsymbol{j},\sigma} \hat{c}^{\dagger}_{\boldsymbol{i},\sigma} T_{\boldsymbol{i},\boldsymbol{j}} \hat{c}_{\boldsymbol{j},\sigma} - \frac{g^2}{2k}\sum_{i} \hat{n}^2_{\boldsymbol{i}}. \tag{10}$$

Here you can use the following. Let $g_n(x) = n g_1(nx)$ with $\int dx g_1(x) = 1$, then $\lim_{n\to\infty} g_n(x) = \delta(x)$ where $\delta(x)$ is the Dirac δ-function.

d)

Give a physical picture of why the interaction is attractive.

Comment. This calculation explicitly shows that the electron-phonon interaction leads to a retarded attractive interaction. In the anti-adiabatic limit, the attractive Hubbard model is know to have a superconducting ground state.