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Problem Sheet 8
for the tutorial on June 27th, 2025
Quantum Mechanics II
Summer term 2025

Sheet handed out on June 17th, 2025; to be handed in on June 24th, 2025 until 2 pm

Exercise 8.1: **Scattering phase shifts in first Born approximation** [13 P.]

In first Born approximation for spherically symmetric potentials, we found for the scattering amplitude (see lecture notes)

$$f^{(1B)} = -\frac{1}{\Delta}\int_0^\infty dr\, r\, U(r)\sin(\Delta r)\,, \qquad \Delta = 2k\sin(\theta/2)\,, \tag{1}$$

for the central potential $U(r)$. Using the expansion

$$\frac{\sin(\Delta r)}{\Delta r} = \frac{\pi}{2kr}\sum_{l=0}^{\infty}(2l+1)[J_{l+1/2}(kr)]^2\, P_l(\cos\theta)\,, \tag{2}$$

show that the scattering phases introduced in HW 7.2 last week are in the first Born approximation given by

$$\delta_l^B(k) = -\frac{\pi}{2}\int_0^\infty U(r)[J_{l+1/2}(kr)]^2\, r\, dr\,. \tag{3}$$

In the equations above, $J_{l+1/2}(kr)$ denotes the Bessel functions of the first kind (these are siblings of the spherical Bessel functions $j_n(kr)$ already used in the lecture) and $P_l(\cos\theta)$ the Legendre polynomials, respectively.

Exercise 8.2: **Gaussian potential** [12 P.]

Show that the differential cross section in the first Born approximation for the Gaussian potential $U(r) = \frac{2m}{\hbar^2}V_0\, e^{-r^2/r_0^2}$ is given by

$$\frac{d\sigma}{d\Omega} = \frac{\pi r_0^2}{4}\left(\frac{mV_0 r_0^2}{\hbar^2}\right)^2 e^{-\Delta^2 r_0^2/2} \tag{4}$$

where $\Delta^2 = 2k^2(1-\cos(\theta)) = 4k^2\sin^2(\theta/2)$ is the modulus square of the momentum transfer $\vec{\Delta} = \vec{k} - \vec{k}'$, as introduced in the lecture.
How does $\mathrm{d}\sigma/\mathrm{d}\Omega$ change as a function of θ with increasing energy of the incident particles?
Hint: Don't hesitate to use Mathematica or similar tools to solve the complicated integral!

Exercise 8.1: Scattering phase shifts in first Born approximation [13 P.]

In first Born approximation for spherically symmetric potentials, we found for the scattering amplitude (see lecture notes)

$$f^{(1B)} = -\frac{1}{\Delta}\int_0^\infty dr\, r\, U(r)\sin(\Delta r)\,, \qquad \Delta = 2k\sin(\theta/2)\,, \tag{1}$$

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In the equations above, $J_{l+1/2}(kr)$ denotes the Bessel functions of the first kind (these are siblings of the spherical Bessel functions $j_n(kr)$ already used in the lecture) and $P_l(\cos\theta)$ the Legendre polynomials, respectively.

$$f(k,\theta) = \sum_{l=0}^{\infty} f_l(k)P_l(\cos\theta)\,.$$

$$f^{(1B)} = -\frac{1}{\Delta}\int_0^\infty dr\, r\, U(r)\sin(\Delta r)$$

$$= -\int_0^\infty U(r)\,\frac{\sin\Delta r}{\Delta r}\, r^2\, dr$$

$$= -\int_0^\infty U(r)r^2\,\frac{\pi}{2kr}\sum_{l=0}^{\infty}(2l+1)\left[J_{l+\frac{1}{2}}(kr)\right]^2 P_l(\cos\theta)\,dr$$

$$= \sum_{l=0}^{\infty}\left[-\int_0^\infty \frac{\pi r}{2k}U(r)\left(J_{l+\frac{1}{2}}(kr)\right)^2 dr\right](2l+1)P_l(\cos\theta)$$

From previous tutorial sheet:

$$f(k,\theta) = \sum_{l=0}^{\infty} f_l(k)P_l(\cos\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left[e^{2i\delta_l(k)}-1\right]P_l(\cos\theta)$$

By identification, $\forall l$, $-\int_0^\infty \frac{\pi}{\cancel{2k}}U(r)\left(J_{l+\frac{1}{2}}(kr)\right)^2 r\,dr = \frac{1}{\cancel{2}i\cancel{k}}\left[e^{2i\delta_l(k)}-1\right]$

Born approximation $\Rightarrow$ weak potential $\Rightarrow$ $\delta_l(k) \ll 1$

$e^{2i\delta_l(k)}-1 \approx 2i\delta_l(k)$

Hence,

$$\delta_l(k) = -\frac{\pi}{2}\int_0^\infty U(r)\left(J_{l+\frac{1}{2}}(kr)\right)^2 r\,dr$$

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$$\frac{d\sigma}{d\Omega} = \frac{\pi r_0^2}{4} \left(\frac{m V_0 r_0^2}{\hbar^2} \right)^2 e^{-\Delta^2 r_0^2/2} \tag{4}$$

where $\Delta^2 = 2k^2(1 - \cos(\theta)) = 4k^2 \sin^2(\theta/2)$ is the modulus square of the momentum transfer $\vec{\Delta} = \vec{k} - \vec{k}'$, as introduced in the lecture.

How does $d\sigma/d\Omega$ change as a function of θ with increasing energy of the incident particles?

Hint: Don't hesitate to use Mathematica or similar tools to solve the complicated integral!

$$f^{(1B)} = -\frac{1}{\Delta} \int_0^\infty dr \, r \, U(r) \sin(\Delta r)$$

$$= -\frac{1}{\Delta} \int_0^\infty r \frac{2m}{\hbar^2} V_0 e^{-\frac{r^2}{r_0^2}} \sin(\Delta r) \, dr$$

The principle of this should just be to write the sine as a sum of exponentials, complete the square and then substitute u = (r + \alpha)^2, but since the tutorial says that we can use mathematica I will just throw it in

$$f^{(1B)} = -\frac{2mV_0}{\hbar^2 \cancel{\Delta}} \left(\frac{1}{4} \sqrt{\pi} \cancel{\Delta} r_0^3 e^{-\frac{1}{4}\Delta^2 r_0^2} \right)$$

$$= -\frac{m V_0 \sqrt{\pi}}{2\hbar^2} r_0^3 e^{-\frac{1}{4}\Delta^2 r_0^2}$$

$$\frac{d\sigma}{d\Omega} = \left| f^{1B}(k,\theta) \right|^2$$

$$= \frac{m^2 V_0^2 \pi}{4\hbar^4} r_0^6 e^{-\frac{1}{2}\Delta^2 r_0^2}$$

$$= \frac{\pi r_0^2}{4} \left(\frac{m V_0 r_0^2}{\hbar^2} \right)^2 e^{-\frac{\Delta^2 r_0^2}{2}}$$

The θ dependence is entirely in Δ^2 which is only in the exponential

$e^{-k^2 \sin^2\frac{\theta}{2}}$ just goes down with θ, the k controls the sharpness