## 1. Plane curves

Normal vector $\hat{n}$ is $\frac{\pi}{2}$ anticlockwise rotation of tangent vector

$\ddot{\gamma} = \underline{\kappa}\,\hat{n}$ signed curvature

Turning angle $\dot{\gamma} = \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix}$, curvature $\dot{\theta} = \kappa$

Curve uniquely determined up to isometry by $\kappa(t)$

Regularity: $\dot{\gamma} \neq 0$ (smooth immersion)

## 2. Space Curves

Normal vector $\hat{n} = \frac{\ddot{\gamma}}{\|\ddot{\gamma}\|} = \frac{\ddot{\gamma}}{\kappa}$, binormal $\hat{b} = \dot{\gamma} \times \hat{n}$

Frenet curve $\Leftrightarrow$ $\kappa \neq 0 \ \forall\, t \in [a, b]$

Torsion $\tau_\gamma = \langle \hat{n}, \hat{b} \rangle$

Frenet equations $$\frac{d}{dt}\begin{pmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{pmatrix}$$

## 3. Parametrised Surfaces

The tangent space is spanned by $\sigma_u$ and $\sigma_v$

The parametrisation is regular if $\sigma_u$ and $\sigma_v$ are lin. indep.

## 4. Metric Quantities

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \sigma_u \cdot \sigma_u & \sigma_v \cdot \sigma_u \\ \sigma_u \cdot \sigma_v & \sigma_v \cdot \sigma_v \end{pmatrix}$$

For a parametrisation $\mathbb{R}^2 \supseteq U \to \mathbb{R}^3$ and a curve $\gamma$ in $U$

Length in $\mathbb{R}^3$ surface is $\int \sqrt{I(\dot{\gamma}, \dot{\gamma})}\, dt = \int \sqrt{\dot{\gamma}^T I \dot{\gamma}}\, dt$

Parametrisation is local isometry iff $I$ = identity matrix

Parametrisation is conformal iff $I$ = const. identity

Area of a compact subset is $\iint \sqrt{EG - F^2}\, dx\, dy$

## 5. Curvature of a Surface

The normal vector is defined by $\hat{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$

Second fundamental form $II = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$

$L = \sigma_{uu} \cdot \hat{n}$ $\quad M = \sigma_{uv} \cdot \hat{n}$ $\quad N = \sigma_{vv} \cdot \hat{n}$

Weingarten matrix $$-\hat{n}_u = W_{11}\sigma_u + W_{21}\sigma_v$$ $$-\hat{n}_v = W_{12}\sigma_u + W_{22}\sigma_v$$

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} = I^{-1} \cdot II$$

Normal curvature $\kappa_n(X) = W_p(X) \cdot X$

Note: $\|X\| = 1$ for above formula

When computing normal curvatures with a curve, the curve must be a normal section (in plane of $\hat{n}$ and $X$) and unit speed.

Principal curvatures are eigenvalues of Weingarten matrix

Curvatures $\kappa_{min}, \kappa_{max}$

If $\kappa_{min} \neq \kappa_{max}$, eigenvectors are principal directions

Gaussian curvature $K = \kappa_{min}\kappa_{max}$

Mean curvature $H = \frac{\kappa_{min} + \kappa_{max}}{2}$

Hence, $K = \det(W) = \frac{\det II}{\det I}$, $H = \frac{1}{2}\operatorname{tr}(W)$

Weingarten map depends on parametrisation up to a sign (direction of normal vector) $\Rightarrow K$ independent.

Christoffel Symbols:

$$\sigma_{uu} = \Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v + L\hat{n}$$
$$\sigma_{uv} = \Gamma^1_{12}\sigma_u + \Gamma^2_{12}\sigma_v + M\hat{n}$$
$$\sigma_{vv} = \Gamma^1_{22}\sigma_u + \Gamma^2_{22}\sigma_v + N\hat{n}$$

First & second fundamental forms determine surface uniquely

Gauss's Theorema Egregium: $K$ is determined by first fundamental form

## 5. Minimal Surfaces

Definition: A surface is minimal $\Leftrightarrow$ mean curvature $= 0$

Theorem: Minimal area $\Rightarrow$ mean curvature $= 0$

Converse is not true!

This implies $K \leq 0$

## 6. Geodesics

Let $\tilde{\gamma}: \mathbb{R} \supseteq I \to U \subseteq \mathbb{R}^2$ be smooth curve

$\sigma: U \to \mathbb{R}^3$ smooth, regular and $\hat{n}$ field of unit vectors,

$\gamma = \sigma \circ \tilde{\gamma}$ constant speed

$$\ddot{\gamma} = \underset{\substack{\uparrow \\ \text{normal curvature}}}{\kappa_n(t)}\, \hat{n}(\tilde{\gamma}(t)) + \underset{\substack{\uparrow \\ \text{geodesic curvature}}}{\kappa_g(t)}\, \hat{n}(\tilde{\gamma}(t)) \times \dot{\gamma}(t)$$

Geodesics have zero geodesic curvature

Geodesics have constant speed, because $\frac{d}{dt}\|\dot{\gamma}\| = 0$

Normal section is intersection of manifold with a plane orthogonal to tangent space


Theorem: Constant speed parametrised normal sections are geodesics

Proof idea: $\gamma = \sigma \circ \tilde{\gamma}$ unit speed

$$0 = \frac{d}{dt}\underbrace{\langle \dot{\gamma}, \hat{n} \circ \tilde{\gamma} \rangle}_{0} = \langle \ddot{\gamma}, \hat{n} \circ \tilde{\gamma} \rangle + \langle \dot{\gamma}, \tfrac{d}{dt}(\hat{n} \circ \tilde{\gamma}) \rangle$$

$$\kappa_n(\dot{\gamma}) = \langle W(\dot{\gamma}), \dot{\gamma} \rangle = -\langle \tfrac{d}{dt}(n \circ \tilde{\gamma}), \dot{\gamma} \rangle = \langle \ddot{\gamma}, \hat{n} \circ \tilde{\gamma} \rangle$$

Then show using the plane that $\ddot{\gamma}$ and $\hat{n} \circ \tilde{\gamma}$ are lin. dep. "all curvature is normal"

Existence & uniqueness: Let $p \in M$ and $X \in T_pM$. Then there exists a unique maximal geodesic passing through $p$ with tangent vector $X$

Geodesics are preserved under local isometries

Geodesics are length minimisers

Geodesic Equation: $\gamma: \mathbb{R} \supseteq I \to U$ smooth curve, $\sigma \circ \gamma$ unit speed. $\gamma: t \mapsto (x(t), y(t))$

$\sigma \circ \gamma$ geodesic iff

$$x'' + (x')^2 \cdot (\Gamma^1_{11} \circ \gamma) + 2x'y'(\Gamma^1_{12} \circ \gamma) + (y')^2 \cdot (\Gamma^1_{22} \circ \gamma) = 0$$
$$y'' + (x')^2 \cdot (\Gamma^2_{11} \circ \gamma) + 2x'y'(\Gamma^2_{12} \circ \gamma) + (y')^2 \cdot (\Gamma^2_{22} \circ \gamma) = 0$$

GR language: $\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}$

$$\Gamma^\beta_{\alpha\gamma} = \frac{1}{2}g^{\mu\beta}(g_{\mu\gamma,\alpha} + g_{\mu\alpha,\gamma} - g_{\alpha\gamma,\mu})$$

$g^{\mu\gamma}g_{\gamma\beta} = \delta^\mu_\beta$ (contravariant tensor is inverse)

## 7. Gauß-Bonnet

$\gamma$ closed curve in $U \subseteq \mathbb{R}^2$, nullhomologous, positively oriented (normal $J \circ \dot{\gamma}$ points towards interior), unit speed

$$\int_0^T \kappa_g\, dt + \int_{int(\gamma)} K\, dA_\sigma = 2\pi$$

To prove this, we first need a local frame s.t. $\{e', e'', N\}$ is a right handed system

$\dot{\gamma} = \cos\theta\, e' + \sin\theta\, e''$

$\hat{N} \times \dot{\gamma} = -\sin\theta\, e' + \cos\theta\, e''$

$\kappa_g = (\hat{N} \times \dot{\gamma}) \cdot \ddot{\gamma} = \ldots = \dot{\theta} - e' \cdot \dot{e}''$

Green's theorem for double integrals

$$\int_0^{\ell(\gamma)} e' \cdot \dot{e}''\, ds = \int_0^{\ell(\gamma)} e' \cdot (\dot{u} e''_u + \dot{v} e''_v)\, ds$$
$$= \int_\pi (e' \cdot e''_u)\, du + (e' \cdot e''_v)\, dv$$
$$= \int_{int(\pi)} [(e' \cdot e''_v)_u - (e' \cdot e''_u)_v]\, du\, dv$$
$$= \int_{int(\pi)} [(e'_u \cdot e''_v) - (e'_v \cdot e''_u)]\, du\, dv$$
$$= \int_{int(\pi)} \frac{LN - M}{(EG - F^2)^{1/2}}\, du\, dv$$
$$= \int_{int(\pi)} \frac{LN - M}{EG - F^2}\sqrt{EG - F^2}\, du\, dv$$
$$= \int_{int(\pi)} K\, dA$$

$\int \dot{\theta}\, dt = 2\pi$ (Umlaufsatz, nullhomotopy required here, must be integer multiple, deform to 0)

## Curvilinear Polygons

A curvilinear polygon is a $T$-periodic map, smooth on $(t_0, t_1), \dots (t_{n-1}, t_n)$, one-sided derivatives

$$\dot{\rho}^+(t_i) = \lim_{t \downarrow t_i} \frac{\rho(t) - \rho(t_i)}{t - t_i}$$
$$\dot{\rho}^-(t_i) = \lim_{t \uparrow t_i} \frac{\rho(t) - \rho(t_i)}{t - t_i}$$

exist and are nonparallel. We define the angle of rotation as the turning angle.

Thm: For any unit speed parametrised positively oriented curvilinear polygon with $n$ edges on a surface $\sigma$,

$$\iint_{int(\gamma \circ \rho)} K\,dA + \int_0^T \kappa_g\,ds + \sum_{i=1}^n \beta_i = 2\pi$$

For any geodesic triangle

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + \int K\,dA \quad (\text{cor. } 10.2.3)$$

## Euler characteristic

We define a triangulation to be a set of triangles that meets only at a (one only) common edge/vertex, or not at all.

$$\chi(S) = v - e + f$$

We say that a surface is compact if it can be covered by finitely many triangles.

$$\iint_S K\,dA = 2\pi \chi(S)$$

## 8. Possibly useful formulae

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}$$

$$\Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}$$

$$\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)}$$

$$\Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)}$$

$$\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}$$

$$\Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}$$

If $\gamma$ is a plane curve, then

$$\kappa(t) = \frac{\ddot{\gamma} \cdot J(\dot{\gamma}(t))}{\|\dot{\gamma}\|^3} = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}$$

If $\gamma$ is a smooth regular space curve,

$$\kappa = \frac{\sqrt{\|\ddot{\gamma}\| - \langle \ddot{\gamma}, \hat{t} \rangle^2}}{\|\dot{\gamma}\|^2}$$

If $\gamma$ is frenet, then

$$\hat{n}_\gamma = \frac{\ddot{\gamma} - \langle \ddot{\gamma}, \hat{t} \rangle \hat{t}}{\|\ddot{\gamma} - \langle \ddot{\gamma}, \hat{t} \rangle \hat{t}\|}$$

$$\hat{b}_\gamma = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma}\| \, \|\ddot{\gamma} - \langle \ddot{\gamma}, \hat{t} \rangle \hat{t}\|}$$

$$\tau_\gamma = \frac{\det(\dot{\gamma}, \ddot{\gamma}, \dddot{\gamma})}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}$$

Cauchy-Schwarz

$$|\langle x, y \rangle| \le \|x\| \cdot \|y\|$$

Exercise: Let $\gamma$ be a unit-speed curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Let $\hat{n}$ be the principal normal of $\gamma$, viewed as a curve on $S^2$, and let $s$ be the arc length of $\hat{n}$. Show that the geodesic curvature of $\hat{n}$ is, up to a sign, $\frac{d}{ds}\tan^{-1}\left(\frac{\tau}{\kappa}\right)$ where $\kappa$ and $\tau$ are the curvature and torsion of $\gamma$. Show also that if $\hat{n}$ is a simple closed curve on $S^2$, then interior and exterior are of equal areas.

Solution: Since the unit normal of $S^2$ is also $\pm\hat{n}$, the geodesic curvature is $\kappa_g = n'' \cdot (n \times n')$. Let $t$ be the arc length of $\gamma$ and denote $\frac{d}{dt}$ by a dot. $\frac{ds}{dt} = \|\dot{n}\| = \|-\kappa\hat{t} + \tau\hat{b}\| = \sqrt{\kappa^2 + \tau^2} = R$ (defn of $R$). Then $n' = (-\kappa\hat{t} + \tau\hat{b})/R$, $\hat{n} \times \hat{n}' = (\kappa\hat{b} + \tau\hat{t})/R$, and $\hat{n}'' = \frac{1}{R}\frac{d}{dt}\left(\frac{-\kappa\hat{t} + \tau\hat{b}}{R}\right) = -R^{-1}\frac{d}{dt}\left(\frac{\kappa}{R}\right)\hat{t} + R^{-1}\frac{d}{dt}\left(\frac{\tau}{R}\right)\hat{b} - R^{-1}(\kappa^2 + \tau^2)\hat{n}$

Hence $n'' \cdot (n \times n') = -R^{-2}\tau\frac{d}{dt}\left(\frac{\kappa}{R}\right) + R^{-2}\kappa\frac{d}{dt}\left(\frac{\tau}{R}\right) = \frac{\kappa\dot{\tau} - \tau\dot{\kappa}}{R^3}$. Since $\dot{\kappa} = R\kappa'$, etc.

$$\kappa_g = \pm\frac{\kappa\tau' - \tau\kappa'}{\kappa^2 + \tau^2} = \pm\frac{d}{ds}\tan^{-1}\left(\frac{\tau}{\kappa}\right)$$

Then we note that $K = 1$ on $S^2$ and $\int_0^{\ell(\hat{n})} \kappa_g\,dt = 0$ because $\kappa_g$ is the derivative of an $\ell(\hat{n})$-periodic function. Thus the area inside $\hat{n}$ is $2\pi$.

Exercise: Suppose $Im(\gamma) \subseteq \overline{B_R(0)}$ and $\gamma(t_0) \in \partial B_R(0)$. Then $\kappa_\gamma \geq \frac{1}{R}$.

Solution: WLOG $\gamma$ is unit speed. $f(t) := \|\gamma(t)\|^2$

$$f''(t_0) = 2\|\dot{\gamma}(t_0)\|^2 + 2\gamma(t_0) \cdot \ddot{\gamma}(t_0) \le 0$$
$$\gamma(t_0) \cdot \ddot{\gamma}(t_0) \le -1$$

Cauchy-Schwarz yields $-1 \geq \gamma(t_0) \cdot \ddot{\gamma}(t_0) \geq -\|\gamma(t_0)\| \cdot \|\ddot{\gamma}(t_0)\| = -R\,\kappa_\gamma(t_0)$

$$\kappa_\gamma(t_0) \geq \frac{1}{R}$$