Funktionalanalysis Hausaufgaben Blatt 2

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Problem 1. For $p \in [1, \infty]$ we define the set

$$\ell^p := \begin{cases} \{x := (x_n)_{n\in N} \subset \mathbb{K} | \|x\|_p := (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} < \infty\} & p < \infty \\ \{x := (x_n)_{n\in\mathbb{N}} \subset \mathbb{K} | \|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n| < \infty\} & p = \infty. \end{cases}$$

Show that the usual operations on sequences induce a vector space structure on ℓ^p. Moreover, show that ℓ^p is a subspace of ℓ^r for $p \leq r$.

Proof. Clearly, multiplying a vector by a constant multiplies its norm by a constant in both cases.

We show the inclusion as follows: Since the series converges, the terms (all positive) must converge to 0. Thus we can choose N such that for $|x_n| < 1$ for all $n \geq N$. For $|x| < 1$, we have $|x|^p \geq |x|^r$. This shows that the vector is also in ℓ^r. □

Problem 2. In this exercise, we consider the spaces ℓ^p for $p \in (1, \infty)$. Note that for every such p there exists a conjugate number $q \in (1, \infty)$ which satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Show that the product of two non-negative real numbers $a, b \in [0, \infty)$ satisfies Young's inequality, that is

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Hint: Use the AM-GM inequality.

(b) Prove that Hölder's inequality

$$\|xy\|_1 := \sum_{n=1}^{\infty} |x_n y_n| \leq \|x\|_p \|y\|_q$$

holds true for any two sequences $x \in \ell^p$ and $y \in \ell^q$.

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(c) Show Minkowsky's inequality, that is

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

for $x, y \in \ell^p$.

(d) Let $\lambda := (\lambda_n)_{n\in\mathbb{N}} \subset [0, 1]$ be a sequence in ℓ^1 with $\|\lambda\|_1 = 1$. Show that Jensen's inequality

$$f\left(\sum_{n=1}^{\infty} \lambda_n x_n\right) \leq \sum_{n=1}^{\infty} \lambda_n f(x_n)$$

holds true for every convex function $f \in \mathcal{C}(I)$ on an open interval $I \subseteq \mathbb{R}$ and every sequence $(x_n)_{n\in\mathbb{N}} \subset I$ such that $\sum_{n=1}^{\infty} \lambda_n x_n$ and $\sum_{n=1}^{\infty} \lambda_n f(x_n)$ converge and $\sum_{n=1}^{\infty} \lambda_n x_n \in I$. Conclude that $\|x\|_r \leq \|x\|_p$ for every $x \in \ell^p$ and $p \leq r$.

Proof. (a) Let $w_1 = \frac{1}{p}$ and $w_2 = \frac{1}{q}$The weighted AM-GM inequality yields

$$\frac{w_1 a^p + w_2 b^q}{w_1 + w_2} \geq \sqrt[w_1+w_2]{(a^p)^{w_1} + (b_q)^{w_2}} = ab.$$

(b) Suppose either norm is 0. Then that sequence must be 0 everywhere, and thus the inequality is fulfilled.

Now suppose either p or q is infinite — without loss of generality, we assume p is. Then the inequality reduces to

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sup_{n\in\mathbb{N}} x_n\right) \|y\|_1$$

which is obviously true, as we can see by replacing x_n with its supremum.

Hence, we assume that both p and q are finite, and that neither norm is 0. We can thus divide each sequence by their norm, and assume WLOG that $\|x\|_p = 1 = \|y\|_q$. Now, we apply Young's inequality

$$\begin{aligned}
\|xy\|_1 &= \sum_{n=1}^{\infty} |x_n y_n| \\
&\leq \sum_{n=1}^{\infty} \left[\frac{|x_n|^p}{p} + \frac{|y_n|^q}{q}\right] \\
&= \frac{1}{p}\|x\|_p^p + \frac{1}{q}\|y\|_q^q \\
&= \frac{1}{p} + \frac{1}{q} = 1
\end{aligned}$$

as desired.

(c)

$$
\begin{aligned}
\|x+y\|_p &= \left[\sum_{n=1}^{\infty} |x_n + y_n|^p\right]^{1/p} \\
&= \left[\sum_{n=1}^{\infty} |x_n + y_n||x_n + y_n|^{p-1}\right]^{1/p} \\
&\leq \left[\sum_{n=1}^{\infty} \left(|x_n||x_n + y_n|^{p-1} + |y_n||x_n + y_n|^{p-1}\right)\right]^{1/p}
\end{aligned}
$$

□

Problem 3. Let $p \in [1, \infty)$. Consider the sequences $(e_n := (\delta_{nm})_{m \in \mathbb{N}})_{n \in \mathbb{N}} \subset \ell^p$. Show that for every sequence $x = (x_n)_{n \in \mathbb{N}} \in \ell^p$ the series $\sum_{n \in \mathbb{N}} x_n e_n$ converges unconditionally towards x with respect to $\|x\|_p$. Does it converge absolutely? Moreover, show that a sequence $x = (x_n)_{n \in \mathbb{N}}$ lies in ℓ^p if the series $\sum_{n \in \mathbb{N}} x_n e_n$ converges unconditionally with respect to $\| \cdot \|_p$.

Hint: Having Minkowski's inequality, you can use that $(\ell^p, \| \cdot \|_p)$ *is a normed space without proof*

Proof. Since the sequence x is an element of ℓ^p, we can find N such that the sum

$$
(\sum_{n=N}^{\infty} |x_n|^p)^{\frac{1}{p}} < \epsilon.
$$

Now we consider a reordering of the sum a_n, $a : \mathbb{N} \to \mathbb{N}$. Since the reordering contains all natural numbers, we can find N' such that $a_1, \ldots, a_{N'}$ contains all $1, \ldots, N$. Since the terms in the sum are all positive, we have not put ourselves in a worse situation. Thus the sum converges unconditionally. □

Problem 4. In the upcoming exercise sheets, we will prove the Stone-Weierstraß theorem in several steps. Here, we do some necessary preparation we will need for the actual proof.

By recursion, define the polynomials

$$
p_0(x) = 0, \qquad \text{and} \qquad p_{n+1}(x) = p_n(x) + \frac{1}{2}(x - p_n^2(x)).
$$

(a) Show $p_n(0) = 0$ and the estimates

$$
p_n(x) \geq 0, \qquad \text{and} \qquad 0 \leq \sqrt{x} - p_n(x) \leq \frac{2\sqrt{x}}{2 + n\sqrt{x}}
$$

for $x \in [0,1]$.

Hint: First show the coarser estimates $0 \leq p_n(x) \leq 1$ *for* $x \in [0,1]$ *by induction. Use this in a second induction to improve the estimates.*

(b) Conclude that $(p_n)_{n\in\mathbb{N}}$ converges uniformly to the square root function on the interval $[0,1]$.

(c) Let $\alpha > 0$. Construct a sequence of polynomials that converges uniformly to the square root function on $[0,\alpha]$.

(d) Let $\alpha > 0$. Construct a sequence of polynomials that converges uniformly to the absolute value function on $[-\alpha,\alpha]$.

Proof. (a) We begin by showing the coarser estimates as suggested. We rewrite the expression as

$$p_{n+1}(x) = p_n(x)\left(1 - \frac{p_n(x)}{2}\right) + \frac{x}{2}.$$

The former expression is a quadratic in $p_n(x)$, which we can show never exceeds $1/2$. Thus $p_{n+1}(x)$ is between 0 and 1. The proof follows by induction. Now we suppose the other inequality holds for p_n, and we consider p_{n+1}:

$$\begin{aligned}\sqrt{x} - p_{n+1}(x) &= \sqrt{x} - p_n(x) - \frac{1}{2}(x - p_n^2(x)) \\ &= \sqrt{x} - p_n(x) - \frac{1}{2}(\sqrt{x} - p_n(x))(\sqrt{x} + p_n(x))\end{aligned}$$

at which point it is already clearly positive. We proceed further:

$$= (\sqrt{x} - p_n(x))\left[1 - \frac{\sqrt{x}}{2} - \frac{p_n(x)}{2}\right]$$

(b) For $x = 0$ the upper bound is 0. For $x \geq 0$ we divide by $\sqrt{x}$:

$$\sqrt{x} - p_n(x) \leq \frac{2}{\frac{2}{\sqrt{x}} + n} \leq \frac{2}{n}.$$

Thus the sequence converges in the supremum norm and therefore uniformly.

(c) We use the identity $\sqrt{x} = \sqrt{\alpha}\sqrt{x/\alpha}$. Since $x/\alpha \in [0,1]$, the sequence of polynomials $\sqrt{\alpha}p_n(x/\alpha)$ converges to the square root function on $[0,\alpha]$.

(d)

$\square$