Homework for the Lecture

Functional Analysis

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Homework Sheet No 6

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(26 Points. Discussion 25. 11. 2024)

Homework 6-1: Quotients of Banach Spaces

Consider a Banach space V and a closed subspace $U \subseteq V$. The main goal is to prove that the quotient V/U is again a Banach space.

i.) **(2 Points)** Let $([v_n])_{n\in\mathbb{N}}$ be a Cauchy sequence in the quotient V/U. Show that there is a strictly monotonously increasing subsequence $(n_k)_{n\in\mathbb{N}}$ and representatives $w_k \in [v_{n_k}]$ such that we have

$$\|w_k - w_{k+1}\| < \frac{1}{2^k} \tag{6.1}$$

by recursively constructing w_{k+1} out of w_k for an arbitrary starting point w_1.

ii.) **(1 Point)** Show that the sequence $(w_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in V.

iii.) **(1 Point)** Conclude that V/U is again a Banach space.

iv.) **(2 Points)** Prove that

$$V'_U := \left\{\varphi \in V' : \varphi\big|_U \equiv 0\right\} \subseteq V' \tag{6.2}$$

is a closed subspace of V'. Moreover, show that the map

$$\phi : (V/U)' \ni \psi \mapsto \psi \circ \mathrm{pr} \in V'_U, \tag{6.3}$$

with $\mathrm{pr} : V \to V/U$ the quotient map, defines a bounded linear operator.

v.) **(4 Points)** Show that ϕ is invertible with ϕ^{-1} being continuous. Is it an isometry?

Homework 6-2: The Pull-Back

Let X, Y, Z be sets. Given two maps $\phi \in \operatorname{Map}(X, Y)$ and $f \in \operatorname{Map}(Y, Z)$, one defines the pull-back of f as the map

$$\phi^* f := f \circ \phi \in \operatorname{Map}(X, Z). \tag{6.4}$$

This induces a map

$$\phi^* : \operatorname{Map}(Y, Z) \to \operatorname{Map}(X, Z). \tag{6.5}$$

i.) **(2 Points)** Let X and Y be topological spaces, ϕ be continuous, and $Z = \mathbb{K}$. Show that $\phi^* := \phi^*\big|_{\mathscr{C}_\mathrm{b}(Y)} \in L(\mathscr{C}_\mathrm{b}(Y), \mathscr{C}_\mathrm{b}(X))$ and compute its operator norm.

ii.) **(5 Points)** Let now $X = Y = \mathbb{N}$ and $Z = \mathbb{K}$. Fix $p \in [1, \infty)$. Show that $\phi^* := \phi^*\big|_{\ell^p} \in L(\ell^p)$ iff there is a constant $C \in \mathbb{N}$ such that $|\phi^{-1}(\{n\})| \leq C$ for all $n \in \mathbb{N}$. In this case, compute its operator norm.

Homework 6-3: The Stone-Weierstraß Theorem: Part II

Here, we want to complete the proof of the Stone-Weierstraß Theorem. To this end, adopt all conditions from Homework 5-4.

i.) **(3 Points)** Assume now that $f = \overline{f} \in \mathscr{C}(X)$ is real-valued and $\epsilon > 0$ as well as $z \in X$ are given. Show that there is a real-valued function $h_z \in \mathscr{A}$ with $h_z(z) = f(z)$ as well as $h(x) \leq f(x) + \epsilon$ for all $x \in X$.

Hint: Part iii.) of Homework 5-4 gives us functions $g_y \in \mathscr{C}(X)$ with $g_y(z) = f(z)$ as well as $g_y(y) = f(y)$. By continuity, they are not too different from f in a small neighbourhood of y. Use then the compactness of X and approximate the resulting functions by means of Homework 5-4, part ii.).

ii.) **(3 Points)** Let again $f = \overline{f} \in \mathscr{C}(X, \mathbb{R})$. Prove that for every $\epsilon > 0$ there is a real-valued $g \in \mathscr{A}$ with $\|f - g\|_\infty < \epsilon$.

Hint: Let $h_z \in \mathscr{A}$ be chosen as in i.) for every $z \in X$. Use continuity to show $h_z(x) > f(x) - \epsilon$ in a small neighbourhood of z. Use then again the compactness of X and Homework 5-4, part ii.) to find a candidate for g.

iii.) **(1 Point)** Conclude the *Stone-Weierstraß Theorem*: every point-separating unital (i.e. containing the constant one-function) *-subalgebra is dense in $\mathscr{C}(X)$.

iv.) **(1 Point)** Conclude the classical *approximation Theorem of Weierstraß*: every continuous real-valued function on $[0, 1]$ is the uniform limit of polynomials $p_n \in \mathbb{R}[x]$.

v.) **(1 Point)** Show that the Fourier modes $\{f_n(x) = \mathrm{e}^{\mathrm{i}nx}\}_{n \in \mathbb{Z}}$ span a dense subspace of $\mathscr{C}(\mathbb{S}^1)$, where we interpret f_n for $x \in [0, 2\pi]$ as continuous functions on the circle.