Universität Würzburg
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4. Open quantum systems

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Throughout this exercise sheet, we adopt the convention $\boxed{\hbar = 1}$.

Exercise 1 *Phase damping when* $[H_S, H_{SB}] = 0$ **3 P.**

Consider a two-level quantum system with the following Hamiltonians:

$$H_S = -\frac{1}{2}\omega_z \sigma_z, \quad H_{SB} = g\, \sigma_z \otimes B.$$

a) Identify the system operator(s) appearing in the interaction Hamiltonian and express them in the basis of H_S.

b) Using the RWA-LE (Rotating Wave Approximation in the weak coupling limit), write down the resulting master equation for the reduced density matrix $\rho(t)$ of the system.

Note: Recall that the RWA-LE in the Schrödinger picture is:

$$\frac{d\rho}{dt} = -i\left[H_S + H_{\mathrm{LS}}, \rho\right] + g^2 \sum_{\alpha\beta} \sum_{\omega} \gamma_{\alpha\beta}(\omega) \left[A_\beta(\omega)\rho A_\alpha^\dagger(\omega) - \frac{1}{2}\left\{A_\alpha^\dagger(\omega)A_\beta(\omega), \rho\right\}\right] \tag{1}$$

Remember from the lecture that expanding

$$H_S = \sum_a \varepsilon_a |\varepsilon_a\rangle\langle\varepsilon_a|,$$

the time-evolved operator becomes

$$A_\alpha(t) = U_S^\dagger(t) A_\alpha U_S(t) = \sum_{a,b} e^{-i(\varepsilon_b - \varepsilon_a)t} \left|\varepsilon_a\right\rangle \left\langle\varepsilon_a\right| A_\alpha \left|\varepsilon_b\right\rangle \left\langle\varepsilon_b\right| = \sum_\omega A_\alpha(\omega) e^{-i\omega t},$$

with Bohr frequencies $\omega = \varepsilon_b - \varepsilon_a$, and

$$A_\alpha(\omega) = \sum_{\varepsilon_b - \varepsilon_a = \omega} \langle\varepsilon_a|A_\alpha|\varepsilon_b\rangle |\varepsilon_a\rangle\langle\varepsilon_b|, \quad A_\alpha(t) = A_\alpha^\dagger(t), \quad A_\alpha(\omega) = A_\alpha^\dagger(-\omega). \tag{2}$$

c) Derive the differential equations governing the matrix elements $\rho_{00}(t)$, $\rho_{11}(t)$, and $\rho_{01}(t)$. Solve them explicitly.

d) Determine the characteristic decoherence timescale and express it in terms of the coupling strength g and the bath correlation function $\gamma_{\alpha\beta}(\omega)$.

Exercise 2 *Phase damping when* $[H_S, H_{SB}] \neq 0$ **5 P.**

Consider a two-level quantum system with the following Hamiltonians:

$$H_S = -\frac{1}{2}\omega_x\sigma_x, \quad H_{SB} = g\,\sigma_z \otimes B.$$

a) Find the eigenstates and eigenvalues of the system Hamiltonian H_S.

b) Express the system operator σ_z in the energy eigenbasis of H_S. Using this, identify the Lindblad operators $A_\alpha(\omega)$ (see Eq.2) in the RWA-LE framework.

c) Write down the master equation using the RWA-LE framework (see Eq.1), including the Lamb shift Hamiltonian. Solve it explicitly for the components of ρ.

   Hint: It is most convenient to work in the energy eigenbasis, i.e., the basis that diagonalizes H_S, namely the $\{|+\rangle, |-\rangle\}$ basis.

d) Derive the relaxation timescale for the diagonal elements of the density matrix, $\rho_{--}(t)$ and $\rho_{++}(t)$, and compare it to the decoherence timescale of the off-diagonal element $\rho_{+-}(t)$.

Exercise 3 *Dark states in a three-level atom* **3 P.**

Consider a three-level quantum system consisting of two ground states $|1\rangle$ and $|3\rangle$, and one excited state $|2\rangle$. The system interacts with a classical field in the rotating wave approximation, with equal Rabi frequencies driving the transitions $|1\rangle \leftrightarrow |2\rangle$ and $|3\rangle \leftrightarrow |2\rangle$. The system Hamiltonian is given by:

$$H = \frac{\Omega}{2}\left(|1\rangle\langle 2| + |3\rangle\langle 2| + \text{h.c.}\right).$$

Spontaneous emission from $|2\rangle$ to both ground states occurs with equal decay rate γ, described by the jump operators:

$$L_1 = |1\rangle\langle 2|, \quad L_2 = |3\rangle\langle 2|.$$

a) Show that the antisymmetric state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle)$$

   is an eigenstate of the Hamiltonian with eigenvalue zero.

b) Verify that this state is annihilated by both jump operators: $L_1|\psi\rangle = 0$ and $L_2|\psi\rangle = 0$. What does this imply about its evolution under the Lindblad master equation?

c) Argue why there are no other pure stationary states, in particular why the symmetric state $|\psi_s\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)$ does not qualify as a stationary state of the full master equation.