Topological Field Theory WS 2025

Prof. Dr. Martin Greiter
M.Sc. Ludwig Bordfeldt and M.Sc. Tom Oeffner

Problem Set 7
due 19.01.2025, 10:00

---

1. Integration measure on the sphere

Consider the mapping

$$\hat{\boldsymbol{n}} : S^2 \to \mathbb{R}^3 : \ \hat{\boldsymbol{n}}(\theta, \phi) = \begin{pmatrix} \cos\varphi \, \sin\theta \\ \sin\varphi \, \sin\theta \\ \cos\theta \end{pmatrix}. \tag{1}$$

(a) Show that

$$\omega = \frac{1}{8\pi} \hat{\boldsymbol{n}}(d\hat{\boldsymbol{n}} \wedge d\hat{\boldsymbol{n}}) \equiv \frac{1}{8\pi} \varepsilon^{ijk} n_i (dn_j \wedge dn_k) = \frac{1}{4\pi} \sin\theta \, d\theta \wedge d\phi. \tag{2}$$

This is eq. (5.5) in the lectures.

(b) For $\hat{\boldsymbol{n}}(x, y) : \mathrm{M} = \mathbb{R}^2 \to \mathrm{T} = S^2$, show that

$$\int_{\mathrm{M}} \hat{\boldsymbol{n}}^* \omega = \frac{1}{4\pi} \int \hat{\boldsymbol{n}}(\partial_x \hat{\boldsymbol{n}} \times \partial_y \hat{\boldsymbol{n}}) \, dx \wedge dy. \tag{3}$$

This is eq. (5.6) in the lectures.

2. Conserved currents in the U(N) WZW model

In the Wess-Zumino-Witten model, the conserved currents are given by

$$\bar{\partial} J(z) = 0 \quad \text{with} \quad J(z) = \frac{1}{\pi} g^{-1} \partial g \tag{4}$$

$$\partial \bar{J}(\bar{z}) = 0 \quad \text{with} \quad \bar{J}(\bar{z}) = -\frac{1}{\pi} (\bar{\partial} g) g^{-1} \tag{5}$$

where $g(z, \bar{z}) \in \mathrm{U}(N)$.

(a) Show that $J^\dagger(z) = J(z)$, *i.e.*, that $J(z)$ is hermitian.

(b) Show that $\bar{\partial} J = 0$ is the equivalent to $\partial \bar{J} = 0$, *i.e.*, one implies the other.
Hint: $\partial(g^{-1} g) = \bar{\partial}(g^{-1} g) = 0$.

(c) Show that $g(z, \bar{z}) = \bar{h}(\bar{z}) h^{-1}(z)$ solves (4) and hence also (5).

(d) Show that while a conserved current $J(z)$ is not invariant under a chiral transformation

$$g(z, \bar{z}) \to \bar{\Lambda}(\bar{z}) g(z, \bar{z}) \Lambda(z), \tag{6}$$

it remains conserved, *i.e.*, $\bar{\partial} J(z) = 0$ remains valid after the transformation.