

Theory and Phenomenology of Superconductivity Homework 5

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Problem 1. Show that the Ginzburg-Landau free energy of a domain wall can be written as

$$\Delta F = A \frac{u}{4} \int dx \left[\psi_0^4 - \psi^4(x) \right],$$

where $A = L^{d-1}$ is the area of the domain wall. Using this result, show that the surface tension $\sigma = \Delta F / A$ is given by

$$\sigma = \frac{\sqrt{8}}{3} \xi u \psi_0^4,$$

with ξ the correlation length. To get the final result, you will need to recall the functional form of the soliton, i.e., the solution to the Ginzburg-Landau equation through the domain wall.

Proof. Landau theory consists of expanding the free energy as a function of the order parameter near to the phase transition, when the free energy is small:

$$F_L(\psi) = \int \frac{r}{2} \psi^2 + \frac{u}{4} \psi^4 - h\psi \, dV,$$

where h describes a conjugate field such as a magnetic field. To describe the possible effects of a spatially varying order parameter, we include a gradient term

$$F_{GL}(\psi) = \int \frac{s}{2} (\nabla \psi)^2 + \frac{r}{2} \psi^2 + \frac{u}{4} \psi^4 - h\psi \, dV.$$

We seek solutions, i.e. order parameter fields that minimise the Ginzburg-Landau functional. This is done by functional differentiation to obtain

$$\begin{aligned} \delta F_{GL}(\psi) &= \int s(\nabla \psi) \cdot \nabla \delta \psi + r\psi \delta \psi + u\psi^3 \delta \psi - h\delta \psi \, dV \\ &= \int s \nabla \cdot (\delta \psi \nabla \psi) - s(\nabla^2 \psi) \delta \psi + r\psi \delta \psi + u\psi^3 \delta \psi - h\delta \psi \, dV \\ &= \int -s(\nabla^2 \psi) \delta \psi + r\psi \delta \psi + u\psi^3 \delta \psi - h\delta \psi \, dV \end{aligned}$$

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This leads to the equation

$$-s(\nabla^2\psi) + r\psi + u\psi^3 - h = 0.$$

The domain wall we seek is a soliton like solution where the order parameter goes from one minimum to the other within a finite range of x . For $h = 0$ (no external field), these minima are symmetric, and we will call them ψ_0 . Defining the derivative of the Landau free energy as

$$f'_L[\psi] = r\psi + u\psi^3,$$

we can summarise by saying that a domain wall is a solution of the equation

$$s \frac{d^2\psi}{dx^2} = f'_L[\psi(x)]$$

with asymptotic boundary conditions

$$\psi \xrightarrow{x \rightarrow \infty} \psi_0, \quad \psi \xrightarrow{x \rightarrow -\infty} -\psi_0.$$

Due to the symmetry in all directions other than x , the free energy is the Ginzburg-Landau energy multiplied by the area A of the other dimensions., we find that

$$\begin{aligned} F[\psi] &= A \int \frac{s}{2}(\nabla\psi)^2 + \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 dV \\ &= A \int -\frac{s}{2}\psi\nabla^2\psi + \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 dV \\ &= A \int -\frac{\psi}{2}(r\psi + u\psi^3) + \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 dV \\ &= -A \int \frac{u}{4}\psi^4 dV \end{aligned}$$

We need to subtract off the energy associated with a uniform configuration ψ_0 , which is clearly given by

$$F_0[\psi_0] = A \int \frac{r}{2}\psi_0^2 + \frac{u}{4}\psi_0^4 dV.$$

Since ψ_0 must minimise the Landau free energy, we must have

$$r\psi_0 + u\psi_0^3 = 0$$

which leads to

$$F_0[\psi_0] = A \int -\frac{u}{2}\psi_0^4 + \frac{u}{4}\psi_0^4 dV = -A \int \frac{u}{4}\psi_0^4.$$

This, of course, could also have been derived by substituting ψ_0 for ψ in the earlier expression for $F[\psi]$. Subtracting, we find that the energy of the domain wall is given by

$$\Delta F = \frac{u}{4} \int dx [\psi_0^4 - \psi^4(x)].$$

Before we proceed further, we will need to solve the equation for the order parameter field explicitly. We can solve this by analogy to Newton's Second Law with f'_L as the potential. Defining $v = \frac{d\psi}{dx}$, we can rewrite the equation using the chain rule as

$$sv \frac{dv}{d\psi} = f'_L[\psi] = r\psi + u\psi^3.$$

We note that the initial conditions are that at the start, $v = 0$ and $\psi = -\psi_0$. Thus, separating variables yields

$$\begin{aligned} \int_0^v sv dv &= \int_{-\psi_0}^{\psi} r\psi + u\psi^3 d\psi \\ \frac{1}{2}sv^2 &= \left[\frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 \right]_{-\psi_0}^{\psi} \\ &= \frac{r}{2}(\psi^2 - \psi_0^2) + \frac{u}{4}(\psi^4 - \psi_0^4) \end{aligned}$$

Again, we simplify this by noting that ψ_0 was the minimum of the Landau energy and hence satisfied the equation

$$r + u\psi_0^2 = 0.$$

Substituting, we find that

$$\begin{aligned} \frac{1}{2}sv^2 &= -\frac{u\psi_0^2}{2}(\psi^2 - \psi_0^2) + \frac{u}{4}(\psi^4 - \psi_0^4) \\ &= u \left[-\frac{1}{2}\psi^2\psi_0^2 + \frac{1}{2}\psi_0^4 + \frac{1}{4}\psi^4 - \frac{1}{4}\psi_0^4 \right] \\ &= \frac{u}{4} \left[\psi_0^4 + \psi^4 - 2\psi^2\psi_0^2 \right] \\ &= \frac{u}{4}(\psi_0^2 - \psi^2)^2 \end{aligned}$$

Since $v = \psi'$ must always be of the same sign, we choose the positive sign to allow us to travel from $-\psi_0$ to ψ_0 . Thus, this simplifies to

$$\begin{aligned} v &= \sqrt{\frac{u}{2s}}(\psi_0^2 - \psi^2) \\ &= \frac{\psi_0}{\sqrt{2\xi}} \left(1 - \frac{\psi^2}{\psi_0^2} \right) \end{aligned}$$

where ξ is defined as the correlation length $\xi = \sqrt{s/|r|}$. We can then integrate this differential equation to get

$$\frac{\psi_0}{\sqrt{2}\xi} dx = \frac{1}{1 - \frac{\psi^2}{\psi_0^2}} d\xi.$$

This yields the desired solution

$$\psi(x) = \psi_0 \tanh\left(\frac{x - x_0}{\sqrt{2}\xi}\right).$$

□

Problem 2. Consider a two-component Dirac electron moving in one dimension through a domain wall, described by the wave equation

$$(-i\sigma_1 \nabla_x - m(x)\sigma_3)\psi = E\psi,$$

where the mass field forms a domain wall, changing sign at the origin according to

$$m(x) = m_0 \tanh\left(\frac{x}{\sqrt{2}\xi}\right).$$

Asymptotically, the energy of the excitation is gapped, with an excitation spectrum $E(k) = \sqrt{k^2 + m_0^2}$. Show that the domain wall gives rise to a zero energy bound state and derive the form of its wave function.

Proof. Since it is given that the bound state has 0 energy, we seek such states directly:

$$(-i\sigma_1 \nabla_x - m(x)\sigma_3)\psi = 0.$$

This corresponds to, in matrix form, if $\psi = (\psi_1, \psi_2)^T$

$$\begin{pmatrix} i\partial_x \psi_2 + m(x)\psi_1 \\ i\partial_x \psi_1 + m(x)\psi_2 \end{pmatrix} = 0.$$

We differentiate the first equation one more time to get

$$\begin{aligned} 0 &= i\partial_x^2 \psi_2 + m(x)\partial_x \psi_1 \\ &= i\partial_x^2 \psi_2 + m(x)[im(x)\psi_2] \end{aligned}$$

Thus the equation for ψ_2 is

$$\partial_x^2 \psi_2 + m(x)^2 \psi_2 = 0.$$

This is a Schrödinger like equation in the position basis. We seek a solution inspired by the WKB method. However, since no perturbative corrections are demanded, we will simply write down the solution:

$$\psi_2(x) = A e^{\int_0^x i m(x) x dx} \psi_2(0).$$

This can be proven by simply differentiating: □