Quantum field theory in the solid state, Exercise sheet 9

Corrections: July 7$^{\text{th}}$

Gauge theories: Hamiltonian formulation and path integrals

Consider the Hamiltonian

$$\hat{H} = \sum_{i,\delta=a_x,a_y} \sum_{\sigma=1}^{N} \left(\hat{c}^{\dagger}_{i,\sigma} \hat{U}^{\dagger}_{i,i+\delta} \hat{c}_{i+\delta,\sigma} + \text{H.c.} \right) + h \sum_{i,\delta=a_x,a_y} \left(\hat{E}_{i,i+\delta} \right)^2 \tag{1}$$

on a square lattice. Here, $\hat{c}^{\dagger}_{i,\sigma}$ creates a fermion of spin σ at site i, the bond, $b = (i, i+\delta)$ operators are defined through the commutation rules:

$$\left[\hat{E}_b, \hat{U}_{b'} \right] = \delta_{b,b'} \hat{U}_b. \tag{2}$$

Here, $\hat{E}^{\dagger}_b = \hat{E}_b$.

1. Local symmetries.

(a) Find explicit representations of the the bond operators $\hat{U}_b$ and $\hat{E}_b$. The choice is not unique!

(b) Work in a basis where $\hat{E}_b$ is diagonal, and show that if fermion hops from site i to $i+\delta$ ($i-\delta$) the value of $\hat{E}_b$ is enhanced (reduced) by unity.

(c) Show that

$$\hat{Q}_i = \sum_{\sigma} \hat{c}^{\dagger}_{i,\sigma} \hat{c}_{i,\sigma} + \sum_{\delta=a_x,a_y} \left(\hat{E}_{i,i+\delta} - \hat{E}_{i-\delta,i} \right) \tag{3}$$

is a conserved quantity:

$$\left[\hat{H}, \hat{Q}_i \right] = 0. \tag{4}$$

(d) Consider the operator

$$\hat{Q} = \sum_i \varphi_i \hat{Q}_i \tag{5}$$

with $\varphi_i \in \mathbb{R}$, and show that

$$e^{i\hat{Q}} \hat{U}_{i,i+\delta} e^{-i\hat{Q}} = e^{i\varphi_i} \hat{U}_{i,i+\delta} e^{-i\varphi_{i+\delta}}, \quad \text{and} \quad e^{i\hat{Q}} \hat{c}_i e^{-i\hat{Q}} = e^{-i\varphi_i} \hat{c}_i \tag{6}$$

(e) Use the above results to show that

$$\langle \hat{c}_{i,\sigma} \hat{c}^{\dagger}_{j,\sigma} \rangle = \delta_{i,j} \langle \hat{c}_{i,\sigma} \hat{c}^{\dagger}_{i,\sigma} \rangle. \tag{7}$$

Show that there is no symmetry reason for

$$\langle \hat{c}^{\dagger}_{i,\sigma} \hat{U}^{\dagger}_{i,i+\delta} \hat{c}_{i+\delta,\sigma} \rangle \tag{8}$$

to be zero.

2. Strong coupling limit

Consider the strong coupling limit, $h \to \infty$, of the Hamiltonian of Eq. (1), and carry out perturbation theory up to second order in the hopping. Show that the resulting Hamiltonian is the SU(N) Heisenberg model,

$$H_J = -J \sum_b \hat{D}^{\dagger}_b \hat{D}_b + \hat{D}_b \hat{D}^{\dagger}_b \tag{9}$$

with $\hat{D}^{\dagger}_{b=(i,j)} = \sum_{\sigma=1}^{N} \hat{c}^{\dagger}_{i,\sigma} \hat{c}_{j,\sigma}$ and with $J \propto 1/h$.

Here is a hint. Follow the derivation of the SU(N) Heisenberg model starting from the Hubbard model that we discussed in class.

3. Path integral formulation

Let $\hat{U}_b$ and $\hat{E}_b$ be operators defined on $\mathbb{L}^2(\mathbb{R})$ and consider the specific form: $\hat{U}_b = e^{iA_b}$ and $\hat{E}_b = \frac{\partial}{i\partial A_b}$. Evaluate the path integral for the partition function, in the representation where $\hat{U}_b$ is diagonal and for fermion coherent states. Compare your result with the U(1) gauge theory discussed in class.

Here is a hint: For the *bosonic* part of the path integral, inspire yourself from the path integral calculation we carried out in class for a single particle in an electromagnetic field.