

1. Time and normal ordered exponentials of free fields

Let the operator A be linear in creation and annihilation operators. Then the normal ordered exponential is related to the time ordered exponential via

$$:e^A: = \frac{\text{Te}^A}{\langle \text{Te}^A \rangle} \equiv \frac{e^A}{\langle e^A \rangle} \quad (1)$$

where the time ordering symbol T is usually omitted by convention, as indicated. In particular, the expectation values of fields we write are always expectation values of time ordered fields.

(a) Use Wick's theorem to prove

$$\text{Te}^A = :e^A: e^{\frac{1}{2}\langle A^2 \rangle}. \quad (2)$$

$$\begin{aligned} \text{Te}^A &= T \sum_k \frac{1}{k!} A^k \\ &= \sum_k \frac{1}{k!} \text{TA}^k \end{aligned}$$

enumerate number of possible contractions in TA^k

$$\begin{aligned} \text{TA}^k &= \sum_{n=0}^{\lfloor k/2 \rfloor} :A^{k-2n}: \binom{k}{2n} \langle A^2 \rangle^n \\ &= \sum_{n=0}^{\lfloor k/2 \rfloor} \frac{k!}{j^n (2n)! (k-2n)!} :A^{k-2n}: \langle A^2 \rangle^n \\ \text{Te}^A &= \sum_{k=0}^{\infty} \sum_{n=0}^{\lfloor k/2 \rfloor} \frac{1}{j^n (2n)! (k-2n)!} :A^{k-2n}: \langle A^2 \rangle^n \end{aligned}$$

rc-index $m = k-2n$

$$\begin{aligned} \text{Te}^A &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n)! m! 2^n} :A^m: \langle A^2 \rangle^n \\ &= :e^A: e^{\frac{1}{2}\langle A^2 \rangle} \end{aligned}$$

(b) Use (2) to prove that

$$\langle e^A \rangle = e^{\frac{1}{2}\langle A^2 \rangle}. \quad (3)$$

Note that (2) and (3) imply (1).

$$\begin{aligned} \langle e^A \rangle :e^A: &= e^A \quad \dots \quad (1) \\ &= :e^A: e^{\frac{1}{2}\langle A^2 \rangle} \quad \dots \quad (2) \end{aligned}$$

$$\text{Hence } \langle e^A \rangle = e^{\frac{1}{2}\langle A^2 \rangle}$$

(c) Use (2) to derive the identity

$$T :e^{A_1}: :e^{A_2}: \dots :e^{A_N}: = :e^{A_1+A_2+\dots+A_N}: \prod_{i < j}^N e^{\langle A_i A_j \rangle}.$$

Hint: Substitute $A = \sum_{i=1}^N A_i$ in (2).

By substituting

$$\begin{aligned} \text{Te}^{\sum A_i} &= :e^{\sum A_i}: e^{\frac{1}{2}\langle (\sum A_i)^2 \rangle} \\ &= :e^{\sum A_i}: e^{\frac{1}{2}\langle \sum_{i,j} A_i A_j \rangle} \\ &=:e^{\sum A_i}: e^{\langle \sum_{i,j} A_i A_j \rangle} \\ &=:e^{\sum A_i}: \prod_{i < j} e^{\langle A_i A_j \rangle} \end{aligned}$$

At the same time,

$$T e^{\sum A_i} = T : e^{A_1} : \dots : e^{A_n} \quad (\text{I have no clue how})$$

2. Bosonization in second quantization

Consider a one-dimensional system of non-interacting fermions with anticommutation relations

$$\{c_{\alpha k}^\dagger, c_{\alpha' k'}\} = \delta_{\alpha\alpha'} \delta_{kk'} \quad (7)$$

and Hamiltonian

$$H_0 = \sum_{\alpha, k} \alpha v_F (k - \alpha k_F) c_{\alpha k}^\dagger c_{\alpha k}, \quad (8)$$

where v_F the Fermi velocity, and $\alpha = \pm$ refers to right (our $\bar{\psi}(\bar{z})$) and left (our $\psi(z)$) movers, respectively. We assume a system of length L and periodic boundary conditions (PBCs), which implies that the momenta are quantized as $k = \frac{2\pi}{L} n$, with n integer. We further assume that the single particle states at $k = \pm k_F$ (as well as all the states below) are occupied in the ground state $|0\rangle$.

We wish to show that the spectrum of neutral excitations (*i.e.*, those which do not alter the number of fermions in the system) can be equally well described via the bosonic density operators

$$\rho_\alpha(q) = \sum_k c_{\alpha k+q}^\dagger c_{\alpha k}, \quad (9)$$

which create (for $\alpha q > 0$) or annihilate (for $\alpha q < 0$) electron-hole pairs.

Since k_F merely shifts the momenta in (8), we may relabel k by $p = k - \alpha k_F$ (or equivalently set $k_F = 0$).

- (a) Limiting yourself to right movers, write out all the states (via strings of operators acting on $|0\rangle$) for $k_{\text{tot}} = \frac{2\pi}{L} m$ with $m = 1, 2, 3$ and 4, once using fermion creation operators and once using bosonic density operators. Show the numbers of states for each value of m match in both descriptions.

Hint: For simplicity, label the operators via $k = \frac{2\pi}{L} n$ with the integers n , not with k or q .

The commutation relations for the boson operators are given by

$$[\rho_\alpha(q), \rho_{\alpha'}(-q')] = -\frac{\alpha q L}{2\pi} \delta_{\alpha\alpha'} \delta_{qq'}. \quad (10)$$

- (b) Verify (10) for $q = q'$ by explicit evaluation of the lhs (left hand side) using the anticommutation relations (7), and convince yourself that (10) also holds for $q \neq q'$.*

Hint: It is necessary to choose a momentum cutoff. Why?

- (c) Evaluate the commutator $[H_0, \rho_\alpha(q)]$. Use the result and (10) to write a kinetic Hamiltonian equivalent to H_0 in terms of bosonic operators.

$$\begin{aligned} b) [\rho_\alpha(q), \rho_{\alpha'}(-q)] &= \left[\sum_k c_{\alpha k+q}^\dagger, \sum_{k'} c_{\alpha' k'-q}^\dagger c_{\alpha' k'} \right] \\ &= \sum_{kk'} \left\{ c_{\alpha k+q}^\dagger \left[c_{\alpha k}, c_{\alpha' k'-q}^\dagger \right] c_{\alpha' k'} \right. \\ &\quad + c_{\alpha k+q}^\dagger c_{\alpha' k'-q}^\dagger \left[c_{\alpha k}, c_{\alpha' k'} \right] \\ &\quad + \left[c_{\alpha k+q}^\dagger, c_{\alpha' k'-q}^\dagger \right] c_{\alpha k} c_{\alpha' k'} \\ &\quad \left. + c_{\alpha' k'-q}^\dagger \left[c_{\alpha k+q}^\dagger, c_{\alpha' k'} \right] c_{\alpha k} \right\} \\ &= \sum_{kk'} \left\{ c_{\alpha' k'-q}^\dagger \left(2 c_{\alpha k+q}^\dagger (c_{\alpha' k'} - \delta_{\alpha' 0} \delta_{k', k_F}) \right) c_{\alpha k} \right. \\ &\quad \left. - c_{\alpha' k'-q}^\dagger \left(2 c_{\alpha k+q}^\dagger (c_{\alpha' k'} - \delta_{\alpha' 0} \delta_{k', k_F}) \right) c_{\alpha' k'} \right\} \end{aligned}$$

Right movers: $\alpha = +1$

$$H_0 = \sum_k V_F k (c_k^\dagger c_k)$$

$$\rho(q) = \sum_k c_{k+q}^\dagger c_k$$

Since the number of fermions is constant, for every fermion we create we need to annihilate one fermion. To avoid annihilating the state, the fermion annihilated needs to be with $k <= 0$, and the state created needs to be with $k > 0$

$$k_{\text{tot}} = 1: |c_1^\dagger c_1|_0\rangle = |\rho(1)|_0\rangle$$

$$k_{\text{tot}} = 2: |c_2^\dagger c_2|_0\rangle, |c_1^\dagger c_1|_0\rangle$$

Bosonic: $|\rho(1)^2|_0\rangle, |\rho(2)|_0\rangle$

$$k_{\text{tot}} = 3: |c_3^\dagger c_3|_0\rangle, |c_2^\dagger c_2|_0\rangle, |c_1^\dagger c_1|_0\rangle$$

$$|\rho(1)^3|_0\rangle, |\rho(1)\rho(2)|_0\rangle, |\rho(3)|_0\rangle$$

$$k_{\text{tot}} = 4: |c_4^\dagger c_4|_0\rangle, |c_3^\dagger c_3|_0\rangle, |c_2^\dagger c_2|_0\rangle$$

$$|c_1^\dagger c_1|_0\rangle, |c_2^\dagger c_1 c_1|_0\rangle$$

Bosonic: $|\rho(1)^4|_0\rangle, |\rho(2)^2|_0\rangle, |\rho(1)\rho(3)|_0\rangle$

$$|\rho(4)|_0\rangle, |\rho(1)^2 \rho(2)|_0\rangle$$

$$= \sum_{kk'} \left[C_{\alpha' k - q}^+ C_{\alpha k} \delta_{k', k+q} - C_{\alpha k+q}^+ C_{\alpha' k} \delta_{k', k-q} \right] \delta_{\alpha \alpha'}$$

$$= \delta_{\alpha \alpha'} \left[\sum_k C_{\alpha' k}^+ C_{\alpha k} - \sum_{k'} C_{\alpha k'}^+ C_{\alpha' k'} \right]$$



But anyway we need cutoffs so that we can rewrite delta functions

$$\delta_{dk+q, d'k'+q'} = \delta_{\alpha \alpha'} \delta_{k'q, k'q'}$$

We demand that q is small so that excitations do not have momentum large enough to turn left movers into right movers.

(c) Evaluate the commutator $[H_0, \rho_\alpha(q)]$. Use the result and (10) to write a kinetic Hamiltonian equivalent to H_0 in terms of bosonic operators.

$$[H_0, \rho_\alpha(q)] = \left[\sum_{\alpha' k'} \alpha' V_F k' C_{\alpha k'}^+ C_{\alpha k'}, \sum_k C_{\alpha k+q}^+ C_{\alpha k} \right]$$

$$= \sum_{kk'} \alpha' V_F k' \left[C_{\alpha k'}^+ C_{\alpha k'}, C_{\alpha k+q}^+ C_{\alpha k} \right]$$

$$= \sum_{kk'} \alpha' V_F k' \left(C_{\alpha k'}^+ C_{\alpha k} C_{\alpha k+q}^+ C_{\alpha k} - C_{\alpha k+q}^+ C_{\alpha k} C_{\alpha k'}^+ C_{\alpha k'} \right)$$

$$= \sum_{kk'} \alpha' V_F k' \left[C_{\alpha k'}^+ \left(\{ C_{\alpha k}, C_{\alpha k'}^+ \} - C_{\alpha k+q}^+ C_{\alpha k'} \right) C_{\alpha k} \right. \\ \left. - C_{\alpha k+q}^+ \left(\{ C_{\alpha k}, C_{\alpha k'}^+ \} - C_{\alpha k'}^+ C_{\alpha k} \right) C_{\alpha k'} \right]$$

$$= \sum_{kk'} \alpha' V_F k' \left[C_{\alpha k'}^+ C_{\alpha k} \delta_{k', k+q} - C_{\alpha k+q}^+ C_{\alpha k'} \delta_{k', k-q} \right]$$

$$= \sum_{k'} \alpha' V_F k' \left[C_{\alpha k'}^+ C_{\alpha(k'-q)} - C_{\alpha k+q}^+ C_{\alpha k'} \right]$$