**Quantum field theory in the solid state, Exercise sheet 11**

**Corrections: July 21rst**

Consider the Hamiltonian:

$$\hat{H} = \sum_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \hat{\sigma}^z_{\langle \boldsymbol{i},\boldsymbol{j}\ \rangle} \hat{f}^\dagger_{\boldsymbol{i},\sigma} \hat{f}_{\boldsymbol{j},\sigma} + h \sum_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \hat{\sigma}^x_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \tag{1}$$

The Hamiltonian is defined on a square lattice, $\langle \boldsymbol{i}, \boldsymbol{j} \rangle$ denotes the nearest neighbor sites, $\hat{f}^\dagger_{\boldsymbol{i},\sigma}$ creates a fermion of spin $\sigma$ at site $\boldsymbol{i}$, and $\hat{\sigma}^{z,x}_{\langle \boldsymbol{i},\boldsymbol{j}\ \rangle}$ are the Pauli matrix acting on the nearest neighbor sites $\boldsymbol{i}$ and $\boldsymbol{j}$.

**1. Symmetries**

**(a)** Let

$$\hat{c}^\dagger_{\boldsymbol{i},\sigma} = \frac{1}{2}(\hat{\gamma}_{\boldsymbol{i},\sigma,1} - i\hat{\gamma}_{\boldsymbol{i},\sigma,2}) \tag{2}$$

on one sub-lattice and

$$\hat{c}^\dagger_{\boldsymbol{i},\sigma} = \frac{i}{2}(\hat{\gamma}_{\boldsymbol{i},\sigma,1} - i\hat{\gamma}_{\boldsymbol{i},\sigma,2}) \tag{3}$$

on the other sub-lattice. Note that hopping only occurs between different sub-lattices. Compute $\hat{\gamma}^\dagger_{\boldsymbol{i},\sigma,n}$ and the anti-commutation relations of the $\hat{\gamma}$-operators.

**(b)** Show that the Hamiltonian can be rewritten as:

$$\hat{H} = \frac{i}{2} \sum_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \sum_{\sigma,n} \hat{\sigma}^z_{\langle \boldsymbol{i},\boldsymbol{j}\ \rangle} \hat{\gamma}_{\boldsymbol{i},\sigma,n} \hat{\gamma}_{\boldsymbol{j},\sigma,n} + h \sum_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \hat{\sigma}^x_{\langle \boldsymbol{i},\boldsymbol{j} \rangle}. \tag{4}$$

Show that the Hamiltonian is invariant under global O(4) rotations:

$$\tilde{\gamma}_{i,\eta} = O_{\eta,\eta'} \hat{\gamma}_{i,\eta'} \tag{5}$$

with $OO^T = 1$, and $\eta = 1, 2, 3, 4$ the four components of the $\hat{\gamma}_{\boldsymbol{i}}$-operators.

**(c)** Show that this O(4) symmetry leads to the equality:

$$\langle \hat{S}^+_{\boldsymbol{i}} \hat{S}^-_{\boldsymbol{j}} \rangle = \langle \hat{\Delta}^\dagger_{\boldsymbol{i}} \hat{\Delta}_{\boldsymbol{j}} \rangle \tag{6}$$

with $\hat{S}^+_{\boldsymbol{i}} = \hat{f}^\dagger_{\boldsymbol{i},\uparrow} \hat{f}_{\boldsymbol{i},\downarrow}$, $\hat{S}^-_{\boldsymbol{i}} = \hat{f}^\dagger_{\boldsymbol{i},\downarrow} \hat{f}_{\boldsymbol{i},\uparrow}$, and $\hat{\Delta}^\dagger_{\boldsymbol{i}} = \hat{f}^\dagger_{\boldsymbol{i},\uparrow} \hat{f}^\dagger_{\boldsymbol{i},\downarrow}$.

**(d)** Show that a Hubbard U-term: $\hat{H}_U = U \sum_{\boldsymbol{i}} (\hat{n}_{\boldsymbol{i},\uparrow} - 1/2)(\hat{n}_{\boldsymbol{i},\downarrow} - 1/2)$, with $\hat{n}_{\boldsymbol{i},\sigma} = \hat{f}^\dagger_{\boldsymbol{i},\sigma} \hat{f}_{\boldsymbol{i},\sigma}$, reduces the $O(4)$ symmetry to $SO(4)$. Here is a hint: Show that the Hubbart term can be written as $\sum_i \gamma_{\boldsymbol{i},1} \gamma_{\boldsymbol{i},2} \gamma_{\boldsymbol{i},3} \gamma_{\boldsymbol{i},4}$ and that under an O(4) transformation $\gamma_{\boldsymbol{i},1} \gamma_{\boldsymbol{i},2} \gamma_{\boldsymbol{i},3} \gamma_{\boldsymbol{i},4} \rightarrow$

$\det(O)\,\gamma_{\boldsymbol{i},1}\gamma_{\boldsymbol{i},2}\gamma_{\boldsymbol{i},3}\gamma_{\boldsymbol{i},4}$.

**(e)** Show that

$$\hat{D}_{\boldsymbol{i}} = (-1)^{\sum_\sigma \hat{f}^\dagger_{i,\sigma}\hat{f}^\dagger_{i,\sigma}} \hat{\sigma}^x_{\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_x\rangle} \hat{\sigma}^x_{\langle \boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_x\rangle} \hat{\sigma}^x_{\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_y\rangle} \hat{\sigma}^x_{\langle \boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_y\rangle} \tag{7}$$

is a locally conserved quantity, with $\hat{D}^\dagger_{\boldsymbol{i}} = \hat{D}_{\boldsymbol{i}}$, and $\hat{D}^2_{\boldsymbol{i}} = 1$. Here is a hint: work in a basis where $\sigma^x$ is diagonal.

**(f)** Check that the $\hat{c}^\dagger_{\boldsymbol{i},\sigma}$ operators as well as the bond $\hat{\sigma}^z_{\langle \boldsymbol{i},\boldsymbol{j}\ \rangle}$ operators carry $Z_2$ charge. That is, they anti-commute with $\hat{D}_{\boldsymbol{i}}$.

## 2. Fluxes

Consider the flux

$$\hat{W}_p = \hat{\sigma}^z_{\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_x\rangle} \hat{\sigma}^z_{\langle \boldsymbol{i}+\boldsymbol{a}_x,\boldsymbol{i}+\boldsymbol{a}_x+\boldsymbol{a}_y\rangle} \hat{\sigma}^z_{\langle \boldsymbol{i}+\boldsymbol{a}_y,\boldsymbol{i}+\boldsymbol{a}_x+\boldsymbol{a}_y\rangle} \hat{\sigma}^z_{\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_y\rangle} \tag{8}$$

**(a)** Does the flux operator carry $Z_2$ charge.

**(b)** Show that the flux is a conserved quantity only in the limit $h \to 0$

**(c)** At $h = 0$ show that the energy eigenstates on open manifolds only depend upon the flux quantum numbers, and not on the specific choice of the bond operators.

**(d)** Does the above hold for periodic boundary conditions?