Topological Field Theory WS 2025

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PROBLEM SET 2
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1. Time and normal ordered exponentials of free fields

Let the operator A be linear in creation and annihilation operators. Then the normal ordered exponential is related to the time ordered exponential via

$$:e^A: = \frac{\mathrm{T}\, e^A}{\langle \mathrm{T}\, e^A\rangle} \equiv \frac{e^A}{\langle e^A\rangle}, \tag{1}$$

where the time ordering symbol T is usually omitted by convention, as indicated. In particular, the expectation values of fields we write are always expectation values of time ordered fields.

(a) Use Wick's theorem to prove

$$\mathrm{T}\, e^A = :e^A:\, e^{\frac{1}{2}\langle A^2\rangle}. \tag{2}$$

(b) Use (2) to prove that

$$\langle e^A\rangle = e^{\frac{1}{2}\langle A^2\rangle}. \tag{3}$$

Note that (2) and (3) imply (1).

(c) Use (2) to derive the identity

$$\mathrm{T}\, :e^{A_1}::e^{A_2}: \,\dots\, :e^{A_N}: = :e^{A_1+A_2+\dots+A_N}: \prod_{i<j}^{N} e^{\langle A_i A_j\rangle}. \tag{4}$$

Hint: Substitue $A = \sum_{i=1}^{N} A_i$ in (2).

Remarks:

(i) While Wick's theorem is a property of free field theory, it holds for all fields linear in creation and annihilation operators. Therefore the substitution in of the sum in (c) is possible.

(ii) This exercise shows that (2) is relatively straightforward to proof via Wicks theorem, and that (3) and (4) follow directly. In applications, however, the most frequently used identities may be

$$\langle :e^{A_1}::e^{A_2}: \,\dots\, :e^{A_N}:\rangle = \prod_{i<j}^{N} e^{\langle A_i A_j\rangle}, \tag{5}$$

$$\langle e^{A_1}\, e^{A_2} \,\dots\, e^{A_N}\rangle = \prod_{i<j}^{N} e^{\langle A_i A_j\rangle} \prod_{i=1}^{N} e^{\frac{1}{2}\langle A_i^2\rangle}. \tag{6}$$

2. Bosonization in second quantization

Consider a one-dimensional system of non-interacting fermions with anticommutation relations

$$\left\{c^\dagger_{\alpha k}, c_{\alpha' k'}\right\} = \delta_{\alpha\alpha'}\delta_{kk'} \tag{7}$$

and Hamiltonian

$$H_0 = \sum_{\alpha,k} \alpha v_{\mathrm{F}}(k - \alpha k_{\mathrm{F}}) c^\dagger_{\alpha k} c_{\alpha k}, \tag{8}$$

where v_{F} the Fermi velocity, and $\alpha = \pm$ refers to right (our $\bar\psi(\bar z)$) and left (our $\psi(z)$) movers, respectively. We assume a system of length L and periodic boundary conditions (PBCs), which implies that the momenta are quantized as $k = \frac{2\pi}{L}n$, with n integer. We further assume that the single particle states at $k = \pm k_{\mathrm{F}}$ (as well as all the states below) are occupied in the ground state $|0\rangle$.

We wish to show that the spectrum of neutral excitations (*i.e.*, those which do not alter the number of fermions in the system) can be equally well described via the bosonic density operators

$$\rho_\alpha(q) = \sum_k c^\dagger_{\alpha k+q} c_{\alpha k}, \tag{9}$$

which create (for $\alpha q > 0$) or annihilate (for $\alpha q < 0$) electron-hole pairs.

Since k_{F} merely shifts the momenta in (8), we may relabel k by $p = k - \alpha k_{\mathrm{F}}$ (or equivalently set $k_{\mathrm{F}} = 0$).

(a) Limiting yourself to right movers, write out all the states (via strings of operators acting on $|0\rangle$) for $k_{\mathrm{tot}} = \frac{2\pi}{L}m$ with $m = 1, 2, 3$ and 4, once using fermion creation operators and once using bosonic density operators. Show the numbers of states for each value of m match in both descriptions.

Hint: For simplicity, label the operators via $k = \frac{2\pi}{L}n$ with the integers n, not with k or q.

The commutation relations for the boson operators are given by

$$\left[\rho_\alpha(q), \rho_{\alpha'}(-q')\right] = -\frac{\alpha q L}{2\pi}\delta_{\alpha\alpha'}\delta_{qq'}. \tag{10}$$

(b) Verify (10) for $q = q'$ by explicit evaluation of the lhs (left hand side) using the anticommutation relations (7), and convince yourself that (10) also holds for $q \neq q'$.*

Hint: It is necessary to choose a momentum cutoff. Why?

(c) Evaluate the commutator $[H_0, \rho_\alpha(q)]$. Use the result and (10) to write a kinetic Hamiltonian equivalent to H_0 in terms of bosonic operators.

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*This verification is an optional exercise, and may be omitted as it not of prime importance to the topics discussed in the course.

Consider the four fermion interaction

$$H_{\text{int}} = \frac{g}{2}\sum_{\alpha,\alpha'}\int dx\, \psi^\dagger_\alpha(x)\psi_\alpha(x)\psi^\dagger_{\alpha'}(x)\psi_{\alpha'}(x). \tag{11}$$

(d) Using the Fourier transformation

$$\psi^\dagger_\alpha(x) = \frac{1}{\sqrt{L}}\sum_k e^{-\mathrm{i}kx}c^\dagger_{\alpha k}, \quad c^\dagger_{\alpha k} = \frac{1}{\sqrt{L}}\int dx e^{\mathrm{i}kx}\psi^\dagger_\alpha(x), \tag{12}$$

to express H_{int} in terms of bosonic operators.

Note: The final bosonic Hamiltonian $H_0 + H_{\text{int}}$ can be solved via a (bosonic) Bogoliubov transformation.

Bosonization Dictionary

(following Gogolin, Nersesyan, and Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge, 1998), Table 3.1. Typos are corrected where identified.)

Massless Bosons	Massless Fermions
Action	Action
$\frac{1}{2}\int d^2x(\nabla\Phi)^2$	$2\int d^2x(R^\dagger\partial_z R + L^\dagger\partial_{\bar{z}}L)$
Operators	Operators
$(\sqrt{2\pi a})^{-1}\exp\left[\pm i\sqrt{4\pi}\phi(z)\right]$	$L^\dagger, L$
$(\sqrt{2\pi a})^{-1}\exp\left[\mp i\sqrt{4\pi}\bar{\phi}(\bar{z})\right]$	$R^\dagger, R$
$(\pi a)^{-1}\cos\left[\sqrt{4\pi}\Phi(z,\bar{z})\right]$	$R^\dagger L + L^\dagger R$
$\frac{\mathrm{i}}{\sqrt{\pi}}\,\partial\Phi(z,\bar{z})$	$J(z) = :L^\dagger L:(z)$
$-\frac{\mathrm{i}}{\sqrt{\pi}}\,\bar{\partial}\Phi(z,\bar{z})$	$\bar{J}(\bar{z}) = :R^\dagger R:(\bar{z})$
with $\Phi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$	