1. Integration measure on the sphere

Consider the mapping

$$\hat{\boldsymbol{n}} : S^2 \to \mathbb{R}^3 : \ \hat{\boldsymbol{n}}(\theta, \phi) = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}. \tag{1}$$

(a) Show that

$$\omega = \frac{1}{8\pi}\hat{\boldsymbol{n}}(d\hat{\boldsymbol{n}} \wedge d\hat{\boldsymbol{n}}) \equiv \frac{1}{8\pi}\varepsilon^{ijk} n_i (dn_j \wedge dn_k) = \frac{1}{4\pi}\sin\theta\, d\theta \wedge d\phi. \tag{2}$$

This is eq. (5.5) in the lectures.

(b) For $\hat{\boldsymbol{n}}(x,y) : \mathrm{M} = \mathbb{R}^2 \to \mathrm{T} = S^2$, show that

$$\int_{\mathrm{M}} \hat{\boldsymbol{n}}^* \omega = \frac{1}{4\pi}\int \hat{\boldsymbol{n}}(\partial_x \hat{\boldsymbol{n}} \times \partial_y \hat{\boldsymbol{n}})\, dx \wedge dy. \tag{3}$$

This is eq. (5.6) in the lectures.

$$dn_0 = -\sin\varphi\sin\theta\, d\varphi + \cos\varphi\cos\theta\, d\theta$$
$$dn_1 = \cos\varphi\sin\theta\, d\varphi + \sin\varphi\cos\theta\, d\theta$$
$$dn_2 = -\sin\theta\, d\theta$$

Then we sum up the possibilities:

$$(i,j,k) = (0,1,2): \ \cos\varphi\sin\theta\left[\left(\cos\varphi\sin\theta\, d\varphi + \sin\varphi\cos\theta\, d\theta\right) \wedge \left(-\sin\theta\, d\theta\right)\right]$$
$$= \cos\varphi\sin\theta\left[\cos\varphi\sin^2\theta\right] d\theta \wedge d\varphi$$
$$= \cos^2\varphi\sin^3\theta \ d\theta\wedge d\varphi$$

$$(1,2,0): \ \sin\varphi\sin\theta\left[-\sin\theta\, d\theta \wedge \left(-\sin\varphi\sin\theta\, d\varphi + \cos\varphi\cos\theta\, d\theta\right)\right]$$
$$= \sin\varphi\sin\theta\left[\sin^2\theta\sin\varphi\right] d\theta\wedge d\varphi$$

$$(2,0,1): \ \cos\theta\left[\left(-\sin\varphi\sin\theta\, d\varphi + \cos\varphi\cos\theta\, d\theta\right) \wedge \left(\cos\varphi\sin\theta\, d\varphi + \sin\varphi\cos\theta\, d\theta\right)\right]$$
$$= \cos\theta\left[\sin^2\varphi\cos\theta\sin\theta + \cos^2\varphi\sin\theta\cos\theta\right] d\theta\wedge d\varphi$$
$$= \sin\theta\cos^2\theta\ d\theta\wedge d\varphi$$

$$\omega = \frac{1}{4\pi}\left[\cos^2\varphi\sin^3\theta + \sin^2\varphi\sin^3\theta + \sin\theta\cos^2\theta\right] d\theta\wedge d\varphi$$
$$= \frac{1}{4\pi}\left[\sin^3\theta + \sin\theta\cos^2\theta\right] d\theta\wedge d\varphi$$
$$= \frac{1}{4\pi}\left[\sin\theta\left(1-\cos^2\theta\right) + \sin\theta\cos^2\theta\right] d\theta\wedge d\varphi$$
$$= \frac{1}{4\pi}\sin\theta\ d\theta\wedge d\varphi$$

Note the extra factor of 2 due to combinations like (0,2,1) which are the same as (0,1,2)

(b)
$$\omega = \frac{1}{8\pi}\varepsilon^{ijk} n_i\left(dn_j \wedge dn_k\right)$$
$$= \frac{1}{8\pi}\varepsilon^{ijk} n_i\left(\frac{\partial n_j}{\partial x^l}dx^l \wedge \frac{\partial n_k}{\partial x^r}dx^r\right)$$
$$= \frac{1}{8\pi}\varepsilon^{ijk} n_i(\partial_l n_j)(\partial_r n_k)\, dx^l\wedge dx^r$$

$$= \frac{1}{4\pi} \varepsilon^{ijk} n_i (\partial_x n_j)(\partial_y n_k)\, dx \wedge dy$$
$$= \frac{1}{4\pi} \hat{n} \cdot (\partial_x \hat{n} \times \partial_y \hat{n})\, dx \wedge dy$$

2. Conserved currents in the U(N) WZW model

In the Wess-Zumino-Witten model, the conserved currents are given by

$$\bar{\partial} J(z) = 0 \quad \text{with} \quad J(z) = \frac{1}{\pi} g^{-1} \partial g \tag{4}$$

$$\partial \bar{J}(\bar{z}) = 0 \quad \text{with} \quad \bar{J}(\bar{z}) = -\frac{1}{\pi} (\bar{\partial} g) g^{-1} \tag{5}$$

where $g(z, \bar{z}) \in \mathrm{U}(N)$.

(a) Show that $J^\dagger(z) = J(z)$, *i.e.*, that $J(z)$ is hermitian.

$$J^\dagger(z) = \frac{1}{\pi}\left[(\partial g)^\dagger (g^{-1})^\dagger\right]$$
$$= \frac{1}{\pi}\left[-\partial(g^\dagger)\, g\right]$$
$$= \frac{1}{\pi}\left[-\partial(g^{-1})\, g\right].$$
$$= \frac{1}{\pi}\left[g^{-1}(\partial g) g^{-1} g\right]$$
$$= J(z)$$

Lemma: $\partial(g^{-1}) = g^{-1}(\partial g) g^{-1}$

Proof: $0 = \partial(\mathbb{1})$
$$= \partial(g g^{-1})$$
$$= g\,\partial(g^{-1}) + (\partial g) g^{-1}$$

Note also $\partial^\dagger = -\partial$ because z is imaginary

(b) Show that $\bar{\partial} J = 0$ is the equivalent to $\partial \bar{J} = 0$, *i.e.*, one implies the other.
Hint: $\partial(g^{-1} g) = \bar{\partial}(g^{-1} g) = 0$.

$$\bar{\partial} J = \frac{1}{\pi} \bar{\partial}\left[g^{-1} \partial g\right]$$
$$= \frac{1}{\pi}\left[\bar{\partial} g^{-1} \partial g + g^{-1} \bar{\partial} \partial g\right]$$
$$= \frac{1}{\pi}\left[g^{-1}(\bar{\partial} g) g^{-1} \partial g + g^{-1} \bar{\partial} \partial g\right]$$
$$g \bar{\partial} J g^{-1} = \frac{1}{\pi}\left[(\bar{\partial} g) g^{-1} (\partial g) g^{-1} + (\bar{\partial} \partial g) g^{-1}\right]$$
$$\partial \bar{J} = -\frac{1}{\pi} \partial\left[(\bar{\partial} g) g^{-1}\right]$$
$$= -\frac{1}{\pi}\left[(\partial \bar{\partial} g) g^{-1} + (\bar{\partial} g)(\partial g^{-1})\right]$$
$$= -\frac{1}{\pi}\left[(\partial \bar{\partial} g) g^{-1} + (\bar{\partial} g) g^{-1} \partial g\, g^{-1}\right]$$
$$= -g \bar{\partial} J g^{-1}$$

Hence they are equivalent

(c) Show that $g(z,\bar z) = \bar h(\bar z)h^{-1}(z)$ solves (4) and hence also (5).

$$g^{-1} = h(z)\,\bar h^{-1}(\bar z)$$

$$\bar\partial J(z) \propto \bar\partial\Big[h(z)\bar h^{-1}(\bar z)\Big\{\big(\cancelto{0}{\partial \bar h(\bar z)}\big)h^{-1}(z) + \bar h(\bar z)\,\partial h^{-1}(z)\Big\}\Big]$$

$$= \bar\partial\big[h(z)\bar h^{-1}(\bar z)\bar h(\bar z)\,h^{-1}(z)\big]$$

$$= \bar\partial\,[\mathbb{1}] = 0$$

(d) Show that while a conserved current $J(z)$ is not invariant under a chiral transformation

$$g(z,\bar z) \to \bar\Lambda(\bar z)g(z,\bar z)\Lambda(z), \tag{6}$$

it remains conserved, *i.e.*, $\bar\partial J(z) = 0$ remains valid after the transformation.

$$g^{-1} \to \Lambda^{-1}g^{-1}\bar\Lambda^{-1}$$

$$J(z) \to \frac{1}{\pi}\Lambda^{-1}g^{-1}\bar\Lambda^{-1}\partial(\bar\Lambda g\Lambda)$$

$$\bar\partial J(z) \to \frac{1}{\pi}\bar\partial\Big[\Lambda^{-1}(z)g^{-1}(z,\bar z)\cancel{\bar\Lambda^{-1}(\bar z)}\Big[\cancel{\bar\Lambda(\bar z)}\big(\partial g(z,\bar z)\big)\Lambda(z) + \cancel{\bar\Lambda(\bar z)}\,g(z,\bar z)\,\partial\Lambda(z)\Big]\Big]$$

$$= \frac{1}{\pi}\Lambda^{-1}(z)\,\bar\partial\Big[g^{-1}(z,\bar z)\big(\partial g(z,\bar z)\big)\Lambda(z) + \partial\Lambda(z)\Big]$$

0 by $\bar\partial J = 0$

0 because antiholomorphic

$$= 0$$