Funktionalanalysis Hausaufgaben Blatt 2

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Problem 1. Let V be a $\mathbb{K}$-valued vector space. A seminorm on V is a homogeneous map $p : V \to [0, \infty)$ satisfying the triangle inequality, i.e.

$$p(v + w) \leq p(v) + p(w)$$

and

$$p(\lambda v) = |\lambda| p(v)$$

for any two vectors $v, w \in V$ and every scalar $\lambda \in \mathbb{K}$.

(a) Show that the kernel of a seminorm p is a subspace of V.

Given a seminorm p, we say that two vectors $v, w \in V$ are equivalent if there is a vector $u \in \ker p$ such that $w = v + u$. Make yourself clear that this yields an equivalence relation on V.

(b) Show that the quotient space $V/\ker p := V/\sim$ carries a canonical linear structure.

(c) Show that the map

$$\overline{p} : V/\ker p \ni [v] \mapsto p(v)$$

yields a well-defined norm on the quotient space.

Proof. (a) Clear by definition

(b) Also clear

$\square$

Problem 2. Let M be a topological space. Show that the space $(\mathcal{C}_b(M), \|\cdot\|_\infty)$ of continuous and bounded $\mathbb{K}$-valued functions endowed with the supremum norm is complete.

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Problem 3. In this exercise we weaken the conditions of Homework 2 by considering functions that are only essentially bounded. The goal is to find a suitable seminorm on this function space such that the corresponding quotient becomes a Banach space. But first, we shall settle the term "essentially bounded". To this end, we need the following definitions:

Let X be a set and $\mathfrak{a} \in 2^X$. We call $\mathfrak{a}$ a σ-algebra if

- $\varnothing \in \mathfrak{n}$,
- $X \setminus A \in \mathfrak{a}$ for every $A \in \mathfrak{a}$ and
- $\bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{a}$ for every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathfrak{a}$

The pair $(X, \mathfrak{a})$ is called a measurable space. One can check that for every $A \in \mathfrak{a}$ one obtains a new σ-algebra $a|_{X\setminus A} \subseteq 2^{X\setminus A}$, where $B \in \mathfrak{a}_{X\setminus A}$iff there is some $C \in \mathfrak{a}$ such that $B = C \setminus A$.

A function $f : (X, \mathfrak{a}) \to \mathbb{K}$ if $f^{-1}(B_r(z)) \subseteq \mathfrak{a}$ for every $z \in \mathbb{K}$ and $r > 0$. We denote the set of measurable $\mathbb{K}$-valued functions by $\mathcal{M}(X, a)$. Clearly, the restriction of a measurable function $f \in \mathcal{M}(X, a)$ to $X \setminus A$ yields a measurable function $f|_{X\setminus A} \in \mathcal{M}(X \setminus A, \mathfrak{a}|_{X\setminus A})$.

Finally, a subset $\mathfrak{n} \subseteq \mathfrak{a}$ is called a σ-ideal if

- $\varnothing \in \mathfrak{n}$,
- $\bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{a}$ for every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathfrak{n}$ and
- for all $A \in \mathfrak{n}$ and $B \in \mathfrak{a}$ one has the implication $B \subseteq A \implies B \in \mathfrak{n}$.

(a) For $f \in \mathcal{M}(X, \mathfrak{a})$ we define the essential range

$$\text{ess range}(f) := \{z \in \mathbb{K} : f^{-1}(B_r(z)) \notin \mathfrak{n} \text{ for all } r > 0\}$$

and the essential supremum

$$\text{ess sup}(f) := \sup\{|z| : z \in \text{ess range}(f)\}.$$

Show that $\text{ess range}(f) \subseteq \mathbb{K}$ is closed and $f^{-1}(\mathbb{K} \setminus \text{ess range}(f)) \in \mathfrak{n}$.

(b) Show that two functions $f, g \in \mathcal{M}(X, \mathfrak{a})$ have the same essential range if the essential range of $f - g$ contains only 0.

(c) The set of essentially bounded functions on X is defined as

$$\mathcal{L}^\infty(X,\mathfrak{a},\mathfrak{n}) := \{f \in \mathcal{M}(X,a) : \|f\|_{\text{ess sup}} := \text{ess sup}(f) < \infty\}.$$

Show that $\|\cdot\|_{\text{ess sup}}$ defines a seminorm on $\mathcal{L}^\infty(X,\mathfrak{a},\mathfrak{n})$ and compute its kernel. Moreover, show that the essential supremum of $f \in \mathcal{L}^\infty(X,\mathfrak{a},\mathfrak{n})$ is given by

$$\text{ess sup}(f) = C_f := \inf\{C > 0 : |f|^{-1}([C,\infty)) \in \mathfrak{n}\}.$$

Hint: You can use that $\mathcal{M}(X,\mathfrak{a})$ and $\mathcal{L}^\infty(X,\mathfrak{a},\mathfrak{n})$ are $\mathbb{K}$-vector spaces without proof.

(d) Show that $L^\infty(X,\mathfrak{a},\mathfrak{n}) := \mathcal{L}^\infty(X,\mathfrak{a},\mathfrak{n})/\ker\|\cdot\|_{esssup}$ is a Banach space, i.e. a complete normed space.

Hint: Consider the sequence $(f_n)_n$ on a suitable subset of X and copy your proof of Homework 2. You can use that a pointwise limit of a sequence of measurable functions is again measurable without proof.