Funktionalanalysis Notizen

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I. BAIRE SPACES

Theorem 1 (List of Useful Identities).

(a)

$$(A \cup B)^{\mathrm{cl}} = A^{\mathrm{cl}} \cup B^{\mathrm{cl}}.$$

(b)

$$(A \cup B)^{\circ} \supseteq A^{\circ} \cup B^{\circ}.$$

(c)

$$(A \cap B)^{\mathrm{cl}} \subseteq A^{\mathrm{cl}} \cap B^{\mathrm{cl}}.$$

(d)

$$(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}.$$

(e)

$$(M \setminus A)^{\mathrm{cl}} = M \setminus A^{\circ}.$$

(f)

$$(M \setminus A)^{\circ} = M \setminus A^{\mathrm{cl}}.$$

Proof. (a) If all neighbourhoods of x intersects A, then they must certainly intersect $A \cup B$. The same thing happens if all neighbourhoods intersect B. Conversely, we suppose that x is not in A^{cl} or B^{cl}. Then we have an open neighbourhood not intersecting A, and an open neighbourhood not intersecting B. Considering the intersection of these two neighbourhoods shows that x is not in the closure of $A \cup B$ either.

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(b) $A^\circ \cup B^\circ$ is an open set contained in $A \cup B$.

(c) Suppose every neighbourhood of x intersects $A \cap B$. Then every neighbourhood intersects A, and also intersects B.

(d) Clearly, $A^\circ \cap B^\circ$ is an open set contained in $A \cap B$. Conversely, it is also true that $(A \cap B)^\circ$ is an open set contained in A, and thus its interior, and it is also contained in B.

(e) Clearly, $M \setminus A^\circ$ is a closed set containing $M \setminus A$. This shows one inclusion.

Now suppose $x \in M \setminus A^\circ$. Since it is not in the interior, no open neighbourhood of x is completely contained in A; in particular, every neighbourhood must intersect $M \setminus A$. This shows the reverse inclusion.

(f) Clearly, $M \setminus A^{\mathrm{cl}}$ is an open subset of $M \setminus A$.

Conversely, suppose x is an element of $(M \setminus A)^\circ$. Then there is an open neighbourhood of x contained in $M \setminus A$, in particular not intersecting A. This shows that x is in $M \setminus A^{\mathrm{cl}}$. □

Definition 2 (Nowhere Dense)**.** A subset A of a topological space is called nowhere dense if the interior of its closure is open, $(A^{\mathrm{cl}})^\circ = \varnothing$.

Theorem 3. A subset A of a topological space $(M, \mathcal{M})$ is nowhere dense iff its complement contains a dense open set.

Proof. We perform the following computation:

$$\begin{aligned} A \text{ nowhere dense} &\iff (A^{\mathrm{cl}})^\circ = \varnothing \\ &\iff M \setminus (A^{\mathrm{cl}})^\circ = M \\ &\iff (M \setminus A^{\mathrm{cl}})^{\mathrm{cl}} = M \\ &\iff ((M \setminus A)^\circ)^{\mathrm{cl}} = M \\ &\iff (M \setminus A)^\circ \text{ is dense.} \end{aligned}$$

□

Definition 4 (Meager)**.** A subset is called meager if it is a countable union of nowhere dense sets.

Theorem 5. Let $(M, \mathcal{M})$ be a topological space. Then the following are equivalent:

(a) Any countable union of closed subsets of M without inner points has no inner points.

(b) Any countable intersection of open dense subsets of M is dense.

(c) Every non-empty open subset of M is not meager

(d) The complement of every meager subset of M is dense.

Proof. The proof follows (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) $\implies$ (a).

1. Let $(U_i)_{i\in\mathbb{N}}$ be a collection of dense open subsets of M. We consider their complements, which all have empty interior. Since
$$M \setminus \left(\bigcap_{i=1}^{\infty} U_i\right) = \bigcup_{i=1}^{\infty} (M \setminus U_i)$$
and the sets in the union on the right are all closed subsets without interior points, their union has no interior points, hence the countable intersection is dense.

2. Suppose we had a meager nonempty open subset U of M, that is, we have $U = \bigcup_{i=1}^{\infty} A_i$ with A_i nowhere dense sets. Then $M \setminus A_i$ is a dense subset for all i, by (b), their intersection is still dense. Then
$$\begin{aligned}
\varnothing &= M \setminus \left(\bigcap_{i=1}^{\infty}(M \setminus A_i^{\mathrm{cl}})\right)^{\mathrm{cl}} \\
&= M \setminus \left(M \setminus \bigcup_{i=1}^{\infty} A_i^{\mathrm{cl}}\right)^{\mathrm{cl}} \\
&= M \setminus \left(M \setminus \left(\bigcup_{i=1}^{\infty} A_i^{\mathrm{cl}}\right)^{\circ}\right) \\
&= \left(\bigcup_{i=1}^{\infty} A_i^{\mathrm{cl}}\right)^{\circ},
\end{aligned}$$
which is a contradiction, since we had U as a nonempty open subset of the final union.

3. Suppose we have a meager subset

□

Definition 6 (Baire Spaces). ...

Theorem 7 (Baire I). ...

Theorem 8 (Baire II). ...

II. TOPOLOGICAL VECTOR SPACES

Definition 9. A topological vector space is a vector space with a topology such that addition and scalar multiplication are continuous.

Theorem 10. Translation $T_v : x \mapsto x + v$ and multiplication $\lambda : x \mapsto \lambda x$ with $\lambda \neq 0$ are homeomorphisms.

Proof. They are invertible with continuous inverse T_{-v} and $\frac{1}{\lambda}$ respectively □

Definition 11 (Uniform Continuity).

Theorem 12 (Equivalence of Completeness Conditions).

Theorem 13 (Equivalence of Continuity Conditions).

Definition 14 (Algebraic and Topological Dual).

III. BANACH SPACES

A. Basic Properties

Definition 15 (Banach Space).

Definition 16 (Norm).

Definition 17 (Seminorm).

Theorem 18 (Quotient of Norms).

Definition 19 (Operator Norm).

Corollary 20 (Equivalent Statements).

Theorem 21 (Continuity & The Operator Norm).

Theorem 22 (Operator Composition).

Definition 23 (Schauder Basis).

B. Quotients of Banach Spaces

Definition 24 (Quotient (Semi-)Norm).

Theorem 25. This is actually a seminorm

Theorem 26 (Quotient Norm).

C. Various Examples