Quantum field theory in the solid state, Exercise sheet 8

Corrections: June 30$^{\text{th}}$

Grassmann numbers and path integral for non-interacting fermions

1. Normal ordering. Consider a product of fermion creation and annihilation operators,

$$\hat{A} = \hat{c}^{\#_1}_{\alpha_1} \cdots \hat{c}^{\#_N}_{\alpha_N}. \tag{1}$$

Here $\# = \dagger, .$ such that $\hat{c}^{\#}_{\alpha_1} = \hat{c}^{\dagger}_{\alpha_1}$ if $\# = \dagger$ and $\hat{c}^{\#}_{\alpha_1} = \hat{c}_{\alpha_1}$ if $\# = .$. The normal ordering of the operator $\hat{A}$ is denoted by $: \hat{A} :$ and defined as:

$$: \hat{A} := (-1)^{\pi}\, \hat{c}^{\#_{\pi(1)}}_{\alpha_{\pi(1)}} \cdots \hat{c}^{\#_{\pi(N)}}_{\alpha_{\pi(N)}}. \tag{2}$$

where π is an permutation of N numbers chosen such that all the destruction operators are on the right.

a) For fermion coherent states $|\xi\rangle$ and $|\xi'\rangle$, show that:

$$\begin{aligned} &\langle \xi' | : \hat{A} : |\xi\rangle = \xi^{\#_1}_{\alpha_1} \cdots \xi^{\#_N}_{\alpha_N} \; \langle \xi' | \xi \rangle \\ &\text{with } \; \xi^{\#}_{\alpha} = \begin{cases} \xi'^{\dagger}_{\alpha} & \text{if } \; \# = \dagger \\ \xi_{\alpha} & \text{if } \; \# = . \end{cases} \end{aligned} \tag{3}$$

b) Show that normal ordering is not a linear operation.

c) Show that

$$\langle \xi | e^{\hat{\mathbf{c}}^{\dagger} A \hat{\mathbf{c}}} | \xi' \rangle = e^{\boldsymbol{\xi}^{\dagger} e^{A} \boldsymbol{\xi}'}. \tag{4}$$

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a)
$$\langle \xi' | :A: | \xi \rangle = (-1)^{\pi} \langle \xi' | c^{\#_{\pi(1)}}_{\alpha_{\pi(1)}} \cdots c^{\#_{\pi(N)}}_{\alpha_{\pi(N)}} | \xi \rangle$$
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b)
$$:-c c^{\dagger}: = :c^{\dagger} c:$$
$$= c^{\dagger} c$$
$$\neq -:c c^{\dagger}:$$
$$= -c^{\dagger} c$$

c)
$$\langle \xi | e^{\vec{c}^{\dagger} A \vec{c}} | \xi' \rangle$$
$$= \langle \xi | \sum_{n=0}^{\infty} (\vec{c}^{\dagger} A \vec{c})^n | \xi' \rangle$$

$$\langle \xi | (c^{\dagger} A c)^n | \xi' \rangle$$
$$= \langle \xi | \vec{c}^{\dagger} A \vec{c} \ldots \vec{c}^{\dagger} A \vec{c} | \xi' \rangle$$
$$= \langle \xi | \vec{\xi}^{\dagger} A \vec{c} \ldots \vec{c}^{\dagger} A \vec{\xi}' | \xi' \rangle$$

2. Coherent state path integrals for Gaussian systems

Consider the quadratic Hamiltonian:

$$\hat{H} = \sum_{\boldsymbol{i},\boldsymbol{j}} \hat{c}^{\dagger}_{\boldsymbol{i}} T(\boldsymbol{i}-\boldsymbol{j}) \hat{c}_{\boldsymbol{j}}. \tag{5}$$

(a) Show that the partition function reads:

$$Z = \mathrm{Tr} e^{-\beta \hat{H}} = \int \left\{ \prod_{\boldsymbol{i},\tau=1}^{L} d\xi^{\dagger}_{\boldsymbol{i},\tau} d\xi_{\boldsymbol{i},\tau} \right\} \exp\left[-\int_0^{\beta} d\tau \sum_{\boldsymbol{i},\boldsymbol{j}} \xi^{\dagger}_{\boldsymbol{i}}(\tau) \left(\delta_{\boldsymbol{i},\boldsymbol{j}} \frac{\partial}{\partial \tau} + T(\boldsymbol{i}-\boldsymbol{j}) \right) \xi_{\boldsymbol{j}}(\tau) \right] \tag{6}$$

(b) Show that

$$\langle \mathcal{T} \hat{c}_{\boldsymbol{i}}(\tau) \hat{c}^{\dagger}_{\boldsymbol{j}}(\tau') \rangle = \tag{7}$$
$$\frac{1}{Z} \int \left\{ \prod_{\boldsymbol{i},\tau=1}^{L} d\xi^{\dagger}_{\boldsymbol{i},\tau} d\xi_{\boldsymbol{i},\tau} \right\} \exp\left[-\int_0^{\beta} d\tau \sum_{\boldsymbol{i},\boldsymbol{j}} \xi^{\dagger}_{\boldsymbol{i}}(\tau) \left(\delta_{\boldsymbol{i},\boldsymbol{j}} \frac{\partial}{\partial \tau} + T(\boldsymbol{i}-\boldsymbol{j}) \right) \xi_{\boldsymbol{j}}(\tau) \right] \xi_{\boldsymbol{i}}(\tau) \xi^{\dagger}_{\boldsymbol{j}}(\tau')$$

In the above equation the time ordering is defined as:

$$\mathcal{T} \hat{c}_{\boldsymbol{i}}(\tau) \hat{c}^{\dagger}_{\boldsymbol{j}}(\tau') = \begin{cases} \hat{c}_{\boldsymbol{i}}(\tau) \hat{c}^{\dagger}_{\boldsymbol{j}}(\tau') & \text{if } \tau \geq \tau' \\ -\hat{c}^{\dagger}_{\boldsymbol{j}}(\tau') \hat{c}_{\boldsymbol{i}}(\tau) & \text{if } \tau < \tau' \end{cases} \tag{8}$$

(c) Show that:

$$\langle \mathcal{T} \hat{c}_{\boldsymbol{i}}(\tau) \hat{c}^{\dagger}_{\boldsymbol{j}}(\tau') \rangle = M^{-1}_{(\boldsymbol{i},\tau),(\boldsymbol{j},\tau')} \tag{9}$$

In the above we have discretized the imaginary time so as to obtain

$$\int_0^{\beta} d\tau \sum_{\boldsymbol{i},\boldsymbol{j}} \xi^{\dagger}_{\boldsymbol{i}}(\tau) \left(\delta_{\boldsymbol{i},\boldsymbol{j}} \frac{\partial}{\partial \tau} + T(\boldsymbol{i}-\boldsymbol{j}) \right) \xi_{\boldsymbol{j}}(\tau) = \sum_{(\boldsymbol{i},\tau),(\boldsymbol{j},\tau')} \xi^{\dagger}_{\boldsymbol{i}}(\tau) M_{(\boldsymbol{i},\tau),(\boldsymbol{j},\tau')} \xi_{\boldsymbol{j}}(\tau') \tag{10}$$

(d) Consider now a hypercubic lattice of linear length L with periodic boundary conditions and consider the transformed Grassmann variables:

$$\eta_{\boldsymbol{k},i\omega_m} = \frac{1}{\sqrt{\beta N}} \int_0^{\beta} d\tau \sum_{\boldsymbol{j}} e^{i\omega_m \tau - i\boldsymbol{k}\cdot\boldsymbol{j}} \xi_{\boldsymbol{j}}(\tau) \tag{11}$$

with $N = L^d$ and d the dimension of the hyper-cubic lattice. Determine the quantization of the crystal momenta, $\boldsymbol{k}$, and of the so called fermionic Matsubara frequencies ω_m. Show that the above transformation diagonalizes M.

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$$Z = \mathrm{Tr}\, e^{-\beta H}$$

$$= \langle -\xi | e^{-\beta H} | \xi \rangle$$