**1. Dispersion relation of magnons in ferromagnetic spin chains**

In the lectures, we derived the effective action for a ferromagnetic spin chain with Hamiltonian

$$H = -J_{\mathrm{F}} \sum_i \boldsymbol{S}_i \boldsymbol{S}_{i+1} \tag{1}$$

and obtained

$$S[\hat{\boldsymbol{n}}] = \frac{1}{a} \int d\tau dx \left\{ \frac{J_{\mathrm{F}} s^2 a^2}{2} (\partial_x \hat{\boldsymbol{n}})^2 + \mathrm{i} s \mathcal{L}_{\mathrm{WZ}}(\hat{\boldsymbol{n}}, \partial_\tau \hat{\boldsymbol{n}}) \right\}, \tag{2}$$

with

$$\mathcal{L}_{\mathrm{WZ}}(\hat{\boldsymbol{n}}, \partial_\tau \hat{\boldsymbol{n}}) = (1 - \cos\theta) \partial_\tau \phi. \tag{3}$$

(a) Write the Lagrangian density for the action in (2) and (3) in terms of $\theta$, $\phi$, as well as their spacial ($\partial_x$) and temporal ($\partial_\tau$) derivatives.

$$\hat{\boldsymbol{n}}(\theta, \phi) = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}.$$

$$(\partial_x \hat{n})^2 = \theta'(x)^2 + \sin^2(\theta(x))\phi'(x)^2$$

$$\mathcal{L} = \frac{J_F s^2 a^2}{2}\left[(\partial_x \theta)^2 + \sin^2\theta\, (\partial_x \phi)^2\right] + is(1-\cos\theta)\partial_\tau \phi$$

(b) Obtain the equation of motion and show that the dispersion for spin wave excitations (magnons) is quadratic, *i.e.*, $\omega \propto q^2$.

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \xi)} = \frac{\partial \mathcal{L}}{\partial \xi}$$

For $\phi$: $\partial_\tau\left[is(1-\cos\theta)\right] + \partial_x\left[J_F s^2 a^2 \sin^2\theta\, \partial_x \phi\right] = 0$

$$is \sin\theta\, \partial_\tau \theta + J_F s^2 a^2\left[(2\sin\theta\cos\theta)\partial_x\theta\, \partial_x \phi + \sin^2\theta\, \partial_x^2 \phi\right] = 0$$

For $\theta$: $J_F s^2 a^2\, \partial_x^2 \theta = \frac{\partial}{\partial\theta}\left[\frac{J_F s^2 a^2}{2}\sin^2\theta\, (\partial_x\phi)^2 + is(1-\cos\theta)\partial_\tau\phi\right]$

$$= J_F s^2 a^2 \sin\theta\cos\theta\, (\partial_x \phi)^2 + is\sin\theta\, \partial_\tau \phi$$

Dispersion relation: Substitute $\phi = \hat{A} e^{i(kx-\omega t)}$, $\theta = \hat{B} e^{i(kx-\omega t)}$

From the time derivative, we get one factor of $\omega$, but since the spatial derivatives come in pairs we get $k^2$