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Problem Sheet 6
for the tutorial on June 13th, 2025
Quantum Mechanics II
Summer term 2025

Sheet handed out on June 3rd, 2025; to be handed in on June 10th, 2025 until 2 pm

Exercise 6.1: Dressed states in the JC model [2+4+2+1+1+3 P.]

In this exercise we want to get familiar with the so-called *dressed state* picture of the Jaynes-Cummings (JC) model. We start with the interaction Hamiltonian expression from the lecture (written in the interaction picture), where for simplicity we have chosen the coupling constant g to be real,

$$\hat{V} = -\hbar\Delta A_{ee} + \hbar g(A_{eg}\hat{a} + A_{ge}\hat{a}^\dagger)\,. \tag{1}$$

Here, the two atomic states are denoted by $|g\rangle$ (ground) and $|e\rangle$ (excited), and the raising or lowering operators A_{ij} defined as $A_{ij} = |i\rangle\langle j|$ with $i, j \in \{g, e\}$.

a) Are $|g,\, m\, +\, 1\rangle$ and $|e,\, m\rangle$, where m denotes the number of photons, eigenstates of $\hat{V}$?

The states $|g,\, m\, +\, 1\rangle$ and $|e,\, m\rangle$ are known as the *bare states* of the system. They keep track of the atomic state and the number of photons in the field separately, by using the eigenstates of the atomic and field Hamiltonians taken separately. In strong-field scenarios, it is often more helpful to see atom and field as one combined system instead of two interacting but separate ones. Then it makes sense to directly use eigenstates of the interaction Hamiltonian instead of describing the dynamics as deviations from the eigenstates of the free system. In the next tasks we want to derive these so-called *light-dressed states of the atom* (or just *dressed states*, as opposed to the bare states) step by step.

b) First, rewrite the interaction Hamiltonian in the $\{|e,\, m\rangle\,,\, |g,\, m\, +\, 1\rangle\}$ basis and diagonalise it to obtain its eigenenergies $E_\pm$ and corresponding eigenstates $|\pm,\, m\rangle$. You can simplify your results by introducing the generalised Rabi frequency $\Omega_m = \sqrt{\Delta^2 + 4g^2(m+1)}$.

c) Defining the Stückelberg angle $\tan(2\theta) = \frac{-2g\sqrt{m+1}}{\Delta}$, show that the dressed states can be written as

$$|-, m\rangle = \cos(\theta)|e, m\rangle + \sin(\theta)|g, m+1\rangle\,,$$
$$|+, m\rangle = -\sin(\theta)|e, m\rangle + \cos(\theta)|g, m+1\rangle\,.$$

d) From now on, assume that the field mode is resonant with the atomic transition ($\Delta = 0$). Calculate the energy splitting within one dressed state manifold, $\Delta E_m = E_m^{(+)} - E_m^{(-)}$.

e) Calculate the energy splitting between dressed state manifolds, $\delta E_m = E_{m+1}^{(+)} - E_m^{(+)}$.

f) Sketch the energy level scheme of the bare states $|g, m\rangle$ and $|e, m\rangle$ for up to (and including) $m = 2$. In the same diagram, sketch the corresponding dressed states $|\pm, m\rangle$, and the energy splittings calculated in b) and c). What is non-linear in the spectrum?

$\hat{V} = -\hbar\Delta A_{ee} + \hbar g(A_{eg}\hat{a} + A_{ge}\hat{a}^\dagger)\,.$

a) Are $|g,\, m+1\rangle$ and $|e,\, m\rangle$, where m denotes the number of photons, eigenstates of $\hat{V}$?

$$\hat{V}\,|g,m+1\rangle = \hbar g\sqrt{m+1}\;|e,m\rangle$$, so no

$$\hat{V}\,|e,m\rangle = -\hbar\Delta|e,m\rangle + \hbar g\sqrt{m+1}\;|g,m+1\rangle$$, also no

b) First, rewrite the interaction Hamiltonian in the $\{|e,\, m\rangle\,,\, |g,\, m+1\rangle\}$ basis and diagonalise it to obtain its eigenenergies $E_\pm$ and corresponding eigenstates $|\pm,\, m\rangle$. You can simplify your results by introducing the generalised Rabi frequency $\Omega_m = \sqrt{\Delta^2 + 4g^2(m+1)}$.

$$H\begin{pmatrix}|e,m\rangle\\ |g,m+1\rangle\end{pmatrix} = \begin{pmatrix}-\hbar\Delta & \hbar g\sqrt{m+1}\\ \hbar g\sqrt{m+1} & 0\end{pmatrix}\begin{pmatrix}|e,m\rangle\\ |g,m+1\rangle\end{pmatrix}$$

$$\begin{vmatrix}-\hbar\Delta - E & \hbar g\sqrt{m+1}\\ \hbar g\sqrt{m+1} & -E\end{vmatrix} = (\hbar\Delta + E)E - \hbar^2 g^2(m+1)$$

$$= E^2 + E\hbar\Delta - \hbar^2 g^2(m+1) = 0$$

$$E = \frac{-\hbar\Delta \pm\sqrt{(\hbar\Delta)^2 + 4\hbar^2 g^2(m+1)}}{2}$$

$$= \frac{\hbar}{2}\left(-\Delta \pm \Omega_m\right),$$

$\Omega_m = \sqrt{\Delta^2 + 4g^2(m+1)}.$

The eigenstates are $$|\pm,m\rangle = \begin{pmatrix}-\dfrac{\Delta \mp \Omega_m}{2g\sqrt{m+1}}\\ 1\end{pmatrix}$$

norm.: $$\langle\pm,m|\pm,m\rangle = \left(\frac{\Delta\mp\Omega_m}{2g\sqrt{m+1}}\right)^2 + 1$$

$$= \frac{\Delta^2 + \Omega_m^2 \mp 2\Delta\Omega_m}{4g^2(m+1)} + 1$$

$$= \frac{\Delta^2 + \Delta^2 + 4g^2(m+1) \mp 2\Delta\sqrt{\Delta^2+4g^2(m+1)}}{4g^2(m+1)} + 1$$

$$= 2 + 2\Delta \frac{\Delta \mp \sqrt{\Delta^2 + 4g^2(m+1)}}{4g^2(m+1)}$$

Hence
$$|\pm, m\rangle = \frac{1}{\sqrt{2}\sqrt{1 + \Delta\frac{\Delta \mp \Omega_m}{4g^2(m+1)}}} \begin{pmatrix} -\frac{\Delta \mp \Omega_m}{2g\sqrt{m+1}} \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}\sqrt{\frac{4g^2(m+1) + \Delta^2 \mp \Omega_m \Delta}{4g^2(m+1)}}} \begin{pmatrix} -\frac{\Delta \mp \Omega_m}{2g\sqrt{m+1}} \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \frac{2g\sqrt{m+1}}{\sqrt{\Omega_m^2 \mp \Omega_m \Delta}} \begin{pmatrix} -\frac{\Delta \mp \Omega_m}{2g\sqrt{m+1}} \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\Omega_m}\sqrt{\Omega_m \mp \Delta}} \begin{pmatrix} -\Delta \pm \Omega_m \\ 2g\sqrt{m+1} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\Omega_m}\sqrt{\Omega_m \mp \Delta}} \begin{pmatrix} -\Delta \pm \Omega_m \\ \sqrt{\Omega_m^2 - \Delta^2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\Omega_m}} \begin{pmatrix} \frac{-\Delta \pm \Omega_m}{\sqrt{\Omega_m \mp \Delta}} \\ \sqrt{\Omega \pm \Delta} \end{pmatrix}$$

$$\frac{-\Delta + \Omega_m}{\sqrt{\Omega_m - \Delta}} = \sqrt{\Omega_m - \Delta}$$

$$\frac{-\Delta - \Omega_m}{\sqrt{\Omega_m + \Delta}} = -\sqrt{\Omega_m + \Delta}$$

$$|+, m\rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\Omega_m}} \begin{pmatrix} -\sqrt{\Omega_m + \Delta} \\ \sqrt{\Omega_m - \Delta} \end{pmatrix}$$

$$|-, m\rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\Omega_m}} \begin{pmatrix} \sqrt{\Omega_m - \Delta} \\ \sqrt{\Omega_m + \Delta} \end{pmatrix}$$

c) Defining the Stückelberg angle $\tan(2\theta) = \frac{-2g\sqrt{m+1}}{\Delta}$, show that the dressed states can be written as

$$|-, m\rangle = \cos(\theta)|e, m\rangle + \sin(\theta)|g, m+1\rangle\,,$$
$$|+, m\rangle = -\sin(\theta)|e, m\rangle + \cos(\theta)|g, m+1\rangle\,.$$

$$\tan 2\theta = \xi =: \frac{-2g\sqrt{m+1}}{\Delta}$$

$$\sin 2\theta = \frac{\xi}{\sqrt{1+\xi^2}}$$

$$\cos 2\theta = \frac{1}{\sqrt{1+\xi^2}}$$

$$\sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$$

$$\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}$$

$$\frac{1 \mp \cos 2\theta}{2} = \frac{1}{2}\left(1 \mp \frac{1}{\sqrt{1+\xi^2}}\right)$$

$$1+\xi^2 = 1 + \frac{4g^2(m+1)}{\Delta^2} = \frac{\Delta^2 + 4g^2(m+1)}{\Delta^2} = \frac{\Omega_m^2}{\Delta^2}$$

$$\cos\theta = \pm\frac{1}{\sqrt{2}}\sqrt{1+\frac{\Delta}{\Omega_m}}$$

$$\sin\theta = \pm\frac{1}{\sqrt{2}}\sqrt{1-\frac{\Delta}{\Omega_m}}$$

$$|+,m\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{\Omega_m}}\begin{pmatrix} -\sqrt{\Omega_m+\Delta} \\ \sqrt{\Omega_m-\Delta} \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$|-,m\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{\Omega_m}}\begin{pmatrix} \sqrt{\Omega_m-\Delta} \\ \sqrt{\Omega_m+\Delta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

d) From now on, assume that the field mode is resonant with the atomic transition ($\Delta = 0$). Calculate the energy splitting within one dressed state manifold, $\Delta E_m = E_m^{(+)} - E_m^{(-)}$.

$$E_m^{\pm} = \frac{\hbar}{2}\left(-\Delta \pm \Omega_m\right)$$

$$\Delta E_m = \hbar\Omega_m$$

e) Calculate the energy splitting between dressed state manifolds, $\delta E_m = E_{m+1}^{(+)} - E_m^{(+)}$.

$$\delta E_m = \frac{\hbar}{2}\left(\Omega_{m+1} - \Omega_m\right)$$

f) Sketch the energy level scheme of the bare states $|g,m\rangle$ and $|e,m\rangle$ for up to (and including) $m = 2$. In the same diagram, sketch the corresponding dressed states $|\pm,m\rangle$, and the energy splittings calculated in b) and c). What is non-linear in the spectrum?

what

Exercise 6.2: Dispersive JC model and quantum non-demolition measurements [2+3+3+4 P.]

We consider the Jaynes-Cummings (JC) model to describe an atom flying through a cavity that hosts a single field mode. In this exercise we focus on the dispersive limit, i.e., the detuning $\Delta \to \infty$. We use the dressed state picture introduced in the previous exercise, with the mixing angle $\tan(2\theta) = -2g\sqrt{n+1}/\Delta$ with $-\pi/2 \leq \theta \leq 0$.

(a) Show that in the dispersive limit $|-, m\rangle \to |e, m\rangle$ and $|+, m\rangle \to |g, m+1\rangle$, where e and g stand for the excited and the ground states of the atom, respectively.

(b) Calculate the dressed state energies $E_m^{\pm}$ in the limit $\Delta \to \infty$ up to and including the term of order $1/\Delta$.

(c) Using the result of (b), calculate the time evolution of the states $|e, m\rangle$ and $|g, m\rangle$.

(d) Let us assume that the atom enters the cavity in the state $|\Psi(t = 0)\rangle = c|g\rangle + d|e\rangle$ and that the cavity field is initially in the Fock state $|m\rangle$. The interaction between atom and cavity is assumed constant throughout the flight of the atom through the cavity. Denoting with τ the flight time of the atom through the cavity, determine the state $|\Psi(\tau)\rangle$ of the atom when it leaves the cavity.

Show that the relative phase between the states $|g\rangle$ and $|e\rangle$ at time τ depends on the photon number in the cavity, and that the cavity field remains unchanged. As a consequence, this method enables the measurement of the cavity photon number without destroying the cavity field. This is called a quantum non-demolition measurement. A nice example can be found in the work of the Haroche group (S. Haroche was awarded the Nobel Prize in Physics in 2012), see Gleyzes *et al.*, Nature 446, 297 (2007).

(a) Show that in the dispersive limit $|-,m\rangle \to |e,m\rangle$ and $|+,m\rangle \to |g,m+1\rangle$, where e and g stand for the excited and the ground states of the atom, respectively.

$$\Delta \to \infty, \quad \tan(2\theta) \to 0$$

$$\cos 2\theta \to 1$$

$$\sin\theta \to 0$$

$$\cos\theta \to 1$$

$$|-,m\rangle = \cos(\theta)|e,m\rangle + \sin(\theta)|g,m+1\rangle\,, \to |e,m\rangle$$

$$|+,m\rangle = -\sin(\theta)|e,m\rangle + \cos(\theta)|g,m+1\rangle\,. \to |g,m+1\rangle$$

(b) Calculate the dressed state energies $E_m^{\pm}$ in the limit $\Delta \to \infty$ up to and including the term of order $1/\Delta$.

$$E_m^{\pm} = \frac{\hbar}{2}\left(-\Delta \pm \Omega_m\right), \qquad \Omega_m = \sqrt{\Delta^2 + 4g^2(m+1)}.$$

$$= \frac{\hbar}{2}\left(-\Delta \pm \sqrt{\Delta^2 + 4g^2(m+1)}\right)$$

$$= \frac{\hbar\Delta}{2}\left(-1 \pm \sqrt{1 + \frac{4g^2(m+1)}{\Delta^2}}\right)$$

$$= \frac{\hbar\Delta}{2}\left(-1 \pm \left(1 + \frac{2g^2(m+1)}{\Delta^2} + O\left(\frac{1}{\Delta^4}\right)\right)\right)$$

$$= \frac{\hbar\Delta}{2}\left(-1 \pm 1 \pm \frac{2g^2(m+1)}{\Delta^2}\right)$$

$$E_m^{+} = \frac{\hbar g^2(m+1)}{2\Delta}, \qquad E_m^{-} = \hbar\Delta\left(-1 - \frac{g^2(m+1)}{\Delta^2}\right)$$

(c) Using the result of (b), calculate the time evolution of the states $|e,m\rangle$ and $|g,m\rangle$.

$$|-,m\rangle = \cos(\theta)|e,m\rangle + \sin(\theta)|g,m+1\rangle\,,$$

$$|+,m\rangle = -\sin(\theta)|e,m\rangle + \cos(\theta)|g,m+1\rangle\,.$$

$$|g,m+1\rangle = \sin\theta\,|-,m\rangle + \cos\theta\,|+,m\rangle$$

$$|g,m\rangle = \sin\theta\,|-,m-1\rangle + \cos\theta\,|+,m-1\rangle$$

$$|g,m(t)\rangle = e^{-\frac{iE_m^- t}{\hbar}} \sin\theta\,|-,m-1\rangle + e^{-\frac{iE_{m-1}^+ t}{\hbar}} \cos\theta\,|+,m-1\rangle$$

$$= \exp\left(i\Delta\left(1+\frac{g^2 m}{\Delta^2}\right)t\right)\sin\theta\,|-,m-1\rangle$$

$$+\exp\left(-i\frac{\hbar g^2 m}{2\Delta}t\right)\cos\theta\,|+,m-1\rangle$$

$$|-,m\rangle = \cos(\theta)|e,m\rangle + \sin(\theta)|g,m+1\rangle\,,$$
$$|+,m\rangle = -\sin(\theta)|e,m\rangle + \cos(\theta)|g,m+1\rangle\,.$$

$$|e,m\rangle = \cos\theta\,|-,m\rangle - \sin\theta\,|+,m\rangle$$

$$|e,m(t)\rangle = e^{-\frac{iE_n^- t}{\hbar}}\cos\theta\,|-,m\rangle + e^{-\frac{iE_n^+ t}{\hbar}}\sin\theta\,|+,m\rangle$$

$$= \exp\left(i\Delta\left(1+\frac{g^2(m+1)}{\Delta^2}\right)t\right)\cos\theta\,|-,m\rangle + \exp\left(-i\frac{g^2(m+1)}{2\Delta}t\right)\sin\theta\,|+,m\rangle$$

(d) Let us assume that the atom enters the cavity in the state $|\Psi(t=0)\rangle = c|g\rangle + d|e\rangle$ and that the cavity field is initially in the Fock state $|m\rangle$. The interaction between atom and cavity is assumed constant throughout the flight of the atom through the cavity. Denoting with τ the flight time of the atom through the cavity, determine the state $|\Psi(\tau)\rangle$ of the atom when it leaves the cavity.

Show that the relative phase between the states $|g\rangle$ and $|e\rangle$ at time τ depends on the photon number in the cavity, and that the cavity field remains unchanged. As a consequence, this method enables the measurement of the cavity photon number without destroying the cavity field. This is called a quantum non-demolition measurement. A nice example can be found in the work of the Haroche group (S. Haroche was awarded the Nobel Prize in Physics in 2012), see Gleyzes *et al.*, Nature 446, 297 (2007).

$$|\Psi(\tau)\rangle = \exp\left(i\Delta\left(1+\frac{g^2 m}{\Delta^2}\right)\tau\right)\sin\theta\,|-,m-1\rangle$$

$$+\exp\left(-i\frac{\hbar g^2 m}{2\Delta}\tau\right)\cos\theta\,|+,m-1\rangle$$

$$+\exp\left(i\Delta\left(1+\frac{g^2(m+1)}{\Delta^2}\right)\tau\right)\cos\theta\,|-,m\rangle$$

$$+\exp\left(-i\frac{g^2(m+1)}{2\Delta}\tau\right)\sin\theta\,|+,m\rangle$$