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Problem Sheet 12
for the tutorial on July 25th and 21st, 2025
Quantum Mechanics II
Summer term 2025
Sheet handed out on July 15th, 2025; to be handed in on July 22nd, 2025 until 2 pm

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Exercise 12.1: Annihilation and creation operators [2+7 P.]

Consider a pair of fermionic creation and annihilation operators $\hat{c}$ and $\hat{c}^\dagger$, and let $\hat{n} = \hat{c}^\dagger \hat{c}$.

a) Show that $\hat{n}^2 = \hat{n}$.

b) Using a Taylor expansion of the exponential, show that

$$e^{i\phi\hat{n}} = 1 + \left(e^{i\phi} - 1\right)\hat{n} \tag{1}$$

and from this that

$$\hat{\tilde{c}}(\phi) \equiv e^{i\phi\hat{n}}\hat{c}e^{-i\phi\hat{n}} = \hat{c}e^{-i\phi},\ \phi \in \mathbb{R} \tag{2}$$

Exercise 12.2: Electron spin operator in second quantization [6+5+5 P.]

Now let us consider fermionic creation and annihilation operators $\hat{c}_\sigma$ and $\hat{c}_\sigma^\dagger$ for electrons with spin $\sigma \in \{\uparrow, \downarrow\}$.

a) The spin operator $\hat{S}_j$ describing the electronic spin is defined as

$$\hat{S}_j = \frac{1}{2} \sum_{\sigma,\sigma'=\uparrow,\downarrow} \hat{c}_\sigma^\dagger (\tau_j)_{\sigma\sigma'} \hat{c}_{\sigma'} \tag{3}$$

with $j \in \{x, y, z\}$, the Pauli matrices

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

and the notation $(\tau_j)_{\sigma\sigma'} = \langle\sigma|\tau_j|\sigma'\rangle$. Show that the spin components can be written as

$$\begin{aligned} \hat{S}_z &= \frac{1}{2}\left(\hat{n}_\uparrow - \hat{n}_\downarrow\right) \\ \hat{S}_+ &= \hat{c}_\uparrow^\dagger \hat{c}_\downarrow \\ \hat{S}_- &= \hat{c}_\downarrow^\dagger \hat{c}_\uparrow \\ \hat{\boldsymbol{S}}^2 &= \hat{S}_z^2 + \frac{1}{2}\left(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+\right). \end{aligned} \tag{5}$$

where $\hat{n}_\sigma = \hat{c}_\sigma^\dagger \hat{c}_\sigma$ and $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$.

b) Prove that the spin components above satisfy $[\hat{S}_z, \hat{S}_\pm] = \pm\hat{S}_\pm$ and $[\hat{S}_+, \hat{S}_-] = 2\hat{S}_z$.

c) Show that

$$\hat{P} = \hat{n}_\uparrow(1 - \hat{n}_\downarrow) + \hat{n}_\downarrow(1 - \hat{n}_\uparrow) \tag{6}$$

is a projector, i.e. $\hat{P}^2 = \hat{P}$, and

$$\hat{\boldsymbol{S}}^2 = \frac{3}{4}\hat{P}. \tag{7}$$

Exercise 12.3: **Ask questions!** [P.]

Since this is the last tutorial before the exam you have the opportunity to ask some questions about the lecture or the problem sheets. In case you have some, please send them until 22nd of July to your tutor per mail.

Exercise 12.1: Annihilation and creation operators [2+7 P.]

Consider a pair of fermionic creation and annihilation operators $\hat{c}$ and $\hat{c}^\dagger$, and let $\hat{n} = \hat{c}^\dagger \hat{c}$.

a) Show that $\hat{n}^2 = \hat{n}$.

b) Using a Taylor expansion of the exponential, show that

$$e^{i\phi\hat{n}} = 1 + \left(e^{i\phi} - 1\right)\hat{n} \tag{1}$$

and from this that

$$\tilde{\hat{c}}(\phi) \equiv e^{i\phi\hat{n}}\hat{c}e^{-i\phi\hat{n}} = \hat{c}e^{-i\phi},\ \phi \in \mathbb{R}\,. \tag{2}$$

a)

$$n^2 = c^\dagger c c^\dagger c$$
$$= c^\dagger (1 - c^\dagger c) c$$
$$= c^\dagger c - \underbrace{c^\dagger c^\dagger}_{0}\underbrace{c c}_{0}$$
$$= c^\dagger c = n$$

b)

$$e^{i\phi\hat{n}} = 1 + i\phi\hat{n} + \frac{1}{2!}(i\phi\hat{n})^2 + \frac{1}{3!}(i\phi\hat{n})^3 + \dots$$
$$= 1 + i\phi\hat{n} + \frac{1}{2!}(i\phi)^2\hat{n} + \dots \qquad n^2 = n$$
$$= 1 - \hat{n} + \hat{n} + i\phi\hat{n} + \frac{1}{2!}(i\phi)^2\hat{n} + \dots$$
$$= 1 - \hat{n} + \left(1 + i\phi + \frac{1}{2!}(i\phi)^2 + \dots\right)\hat{n}$$
$$= 1 - \hat{n} + e^{i\phi}\hat{n}$$
$$= 1 + (e^{i\phi} - 1)\hat{n}$$

$$\hat{c}(\phi) = e^{i\phi\hat{n}}\hat{c}e^{-i\phi\hat{n}}$$
$$= \left[1 + (e^{i\phi} - 1)\hat{n}\right]\hat{c}\left[1 + (e^{-i\phi} - 1)\hat{n}\right]$$
$$= \hat{c} + (e^{i\phi} - 1)\underbrace{\hat{n}\hat{c}}_{\hat{c}^2 = 0} + \hat{c}(e^{-i\phi} - 1)\hat{n} + (e^{i\phi} - 1)(e^{-i\phi} - 1)\underbrace{\hat{n}\hat{c}}_{\hat{c}^2 = 0}\hat{n}$$
$$= \hat{c} + (e^{-i\phi} - 1)\hat{c}\hat{n}$$

$$\boxed{\begin{aligned}\hat{c}\hat{n} &= \hat{c}\hat{c}^\dagger\hat{c} \\ &= (1 - \hat{c}^\dagger\hat{c})\hat{c} = \hat{c}\end{aligned}}$$

$$= \cancel{\hat{c}} + (e^{-i\phi} \cancel{-1})\hat{c}$$
$$= \hat{c}e^{-i\phi}$$

Exercise 12.2: Electron spin operator in second quantization [6+5+5 P.]

Now let us consider fermionic creation and annihilation operators $\hat{c}_\sigma$ and $\hat{c}_\sigma^\dagger$ for electrons with spin $\sigma \in \{\uparrow, \downarrow\}$.

a) The spin operator $\hat{S}_j$ describing the electronic spin is defined as

$$\hat{S}_j = \frac{1}{2} \sum_{\sigma,\sigma'=\uparrow,\downarrow} \hat{c}_\sigma^\dagger (\tau_j)_{\sigma\sigma'} \hat{c}_{\sigma'} \tag{3}$$

with $j \in \{x, y, z\}$, the Pauli matrices

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

and the notation $(\tau_j)_{\sigma\sigma'} = \langle \sigma | \tau_j | \sigma' \rangle$. Show that the spin components can be written as

$$\begin{aligned} \hat{S}_z &= \frac{1}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow) \\ \hat{S}_+ &= \hat{c}_\uparrow^\dagger \hat{c}_\downarrow \\ \hat{S}_- &= \hat{c}_\downarrow^\dagger \hat{c}_\uparrow \\ \hat{\boldsymbol{S}}^2 &= \hat{S}_z^2 + \frac{1}{2}\left(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+\right). \end{aligned} \tag{5}$$

where $\hat{n}_\sigma = \hat{c}_\sigma^\dagger \hat{c}_\sigma$ and $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$.

$$S_j = \frac{1}{2}\begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger \end{pmatrix} \tau_j \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}$$

$$S_z = \frac{1}{2}\begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger \end{pmatrix}\begin{pmatrix} c_\uparrow \\ -c_\downarrow \end{pmatrix}$$

$$= \frac{1}{2}\left(c_\uparrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow\right) = \frac{1}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)$$

$$S_x = \frac{1}{2}\begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger \end{pmatrix}\begin{pmatrix} c_\downarrow \\ c_\uparrow \end{pmatrix}$$

$$= \frac{1}{2}\left(c_\uparrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\uparrow\right)$$

$$S_y = \frac{1}{2}\begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger \end{pmatrix}\begin{pmatrix} -ic_\downarrow \\ ic_\uparrow \end{pmatrix}$$

$$= \frac{1}{2}\left(-i\, c_\uparrow^\dagger c_\downarrow + i\, c_\downarrow^\dagger c_\uparrow\right)$$

$$S_\pm = S_x \pm iS_y = \frac{1}{2}\left(c_\uparrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\uparrow\right) \pm \frac{i}{2}\left(-i\, c_\uparrow^\dagger c_\downarrow + i\, c_\downarrow^\dagger c_\uparrow\right)$$

$$= \frac{1}{2}\left(c_\uparrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\uparrow\right) \pm \frac{1}{2}\left(c_\uparrow^\dagger c_\downarrow - c_\downarrow^\dagger c_\uparrow\right)$$

$$= \frac{1}{2}\left(c_\uparrow^\dagger c_\downarrow \pm c_\uparrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\uparrow \mp c_\downarrow^\dagger c_\uparrow\right)$$

$$S_+ = c_\uparrow^\dagger c_\downarrow$$

$$S_- = c_\downarrow^\dagger c_\uparrow$$

$$S_+S_- = (S_x + iS_y)(S_x - iS_y)$$

$$= S_x^2 + iS_yS_x - iS_xS_y + S_y^2$$

$$S_-S_+ = (S_x - iS_y)(S_x + iS_y)$$

$$= S_x^2 - iS_yS_x + iS_xS_y + S_y^2$$

$$S_x^2 + S_y^2 = \frac{1}{2}(S_+S_- + S_-S_+)$$

$$S^2 = S_z^2 + S_x^2 + S_y^2$$

$$= S_z^2 + \frac{1}{2}(S_+S_- + S_-S_+)$$

b) Prove that the spin components above satisfy $[\hat{S}_z, \hat{S}_\pm] = \pm\hat{S}_\pm$ and $[\hat{S}_+, \hat{S}_-] = 2\hat{S}_z$.

Here we will make copious use of the facts that $[n_\uparrow, c_\downarrow] = [n_\uparrow, c_\downarrow^\dagger] = 0$ and analogously when swapping $\downarrow$ and $\uparrow$; this is proven by anticommuting the creation/annihilation operators twice

$$
\begin{aligned}
[S_+, S_-] &= [c_\uparrow^\dagger c_\downarrow, c_\downarrow^\dagger c_\uparrow] \\
&= c_\uparrow^\dagger c_\downarrow c_\downarrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\uparrow c_\uparrow^\dagger c_\downarrow \\
&= c_\downarrow c_\downarrow^\dagger c_\uparrow^\dagger c_\uparrow - c_\uparrow c_\uparrow^\dagger c_\downarrow^\dagger c_\downarrow \\
&= (1 - \cancel{c_\downarrow^\dagger c_\downarrow}) c_\uparrow^\dagger c_\uparrow - (1 - \cancel{c_\uparrow^\dagger c_\uparrow}) c_\downarrow^\dagger c_\downarrow \\
&= c_\uparrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow \\
&= 2S_z
\end{aligned}
$$

$$[S_z, S_\pm] = \frac{1}{2}[\hat{n}_\uparrow - \hat{n}_\downarrow, S_\pm]$$

$$
\begin{aligned}
[S_z, S_+] &= \frac{1}{2}\left\{ [n_\uparrow, c_\uparrow^\dagger c_\downarrow] - [n_\downarrow, c_\uparrow^\dagger c_\downarrow] \right\} \\
&= \frac{1}{2}\left\{ [n_\uparrow, c_\uparrow^\dagger] c_\downarrow + c_\uparrow^\dagger \cancelto{0}{[n_\uparrow, c_\downarrow]} - \cancelto{0}{[n_\downarrow, c_\uparrow^\dagger]} c_\downarrow - c_\uparrow^\dagger [n_\downarrow, c_\downarrow] \right\} \\
&= \frac{1}{2}\left\{ (c_\uparrow^\dagger c_\uparrow c_\uparrow^\dagger - \cancelto{0}{c_\uparrow^\dagger c_\uparrow^\dagger} c_\uparrow) c_\downarrow - c_\uparrow^\dagger (\cancelto{0}{c_\downarrow^\dagger c_\downarrow c_\downarrow} - c_\downarrow c_\downarrow^\dagger c_\downarrow) \right\} \\
&= \frac{1}{2}\left\{ c_\uparrow^\dagger (1 - \underbrace{c_\uparrow^\dagger c_\uparrow}_{\text{dies}}) c_\downarrow + c_\uparrow^\dagger (1 - \underbrace{c_\downarrow^\dagger c_\downarrow}_{\text{dies}}) c_\downarrow \right\} \\
&= \frac{1}{2}\left\{ c_\uparrow^\dagger c_\downarrow + c_\uparrow^\dagger c_\downarrow \right\} = S_\pm
\end{aligned}
$$

$$
\begin{aligned}
[S_z, S_-] &= \frac{1}{2}\left\{ [n_\uparrow, c_\downarrow^\dagger c_\uparrow] - [n_\downarrow, c_\downarrow^\dagger c_\uparrow] \right\} \\
&= \frac{1}{2}\left\{ \cancelto{0}{[n_\uparrow, c_\downarrow^\dagger]} c_\uparrow + c_\downarrow^\dagger [n_\uparrow, c_\uparrow] - [n_\downarrow, c_\downarrow^\dagger] c_\uparrow - c_\downarrow^\dagger \cancelto{0}{[n_\downarrow, c_\uparrow]} \right\} \\
&= \frac{1}{2}\left\{ c_\downarrow^\dagger (\cancelto{0}{c_\uparrow^\dagger c_\uparrow c_\uparrow} - c_\uparrow c_\uparrow^\dagger c_\uparrow) - (c_\downarrow^\dagger c_\downarrow c_\downarrow^\dagger - \cancelto{0}{c_\downarrow^\dagger c_\downarrow^\dagger} c_\downarrow) c_\uparrow \right\} \\
&= -\frac{1}{2}\left\{ c_\downarrow^\dagger c_\uparrow c_\uparrow^\dagger c_\uparrow + c_\downarrow^\dagger c_\downarrow c_\downarrow^\dagger c_\uparrow \right\} \\
&= -\frac{1}{2}\left\{ c_\downarrow^\dagger (1 - \underbrace{c_\uparrow^\dagger c_\uparrow}_{\text{dies}}) c_\uparrow + c_\downarrow^\dagger \underbrace{(1 - c_\downarrow^\dagger c_\downarrow)}_{\text{dies}} c_\uparrow \right\}
\end{aligned}
$$

$$= -\frac{1}{2}\left\{ c_\downarrow^\dagger c_\uparrow + c_\downarrow^\dagger c_\uparrow \right\} = -S_-$$

c) Show that

$$\hat{P} = \hat{n}_\uparrow(1-\hat{n}_\downarrow) + \hat{n}_\downarrow(1-\hat{n}_\uparrow) \qquad (6)$$

is a projector, i.e. $\hat{P}^2 = \hat{P}$, and

$$\hat{S}^2 = \frac{3}{4}\hat{P}. \qquad (7)$$

$$P^2 = \left[n_\uparrow - n_\uparrow n_\downarrow + n_\downarrow - n_\downarrow n_\uparrow\right]^2$$

$$= \left[n_\uparrow - n_\uparrow n_\downarrow + n_\downarrow - n_\downarrow n_\uparrow\right]\left[n_\uparrow - n_\uparrow n_\downarrow + n_\downarrow - n_\downarrow n_\uparrow\right]$$

$$= n_\uparrow^2 - n_\uparrow^2 n_\downarrow + n_\uparrow n_\downarrow - n_\uparrow n_\downarrow n_\uparrow$$
$$- n_\uparrow n_\downarrow n_\uparrow + n_\uparrow n_\downarrow n_\uparrow n_\downarrow - n_\uparrow n_\downarrow^2 + n_\uparrow n_\downarrow^2 n_\uparrow$$
$$+ n_\downarrow n_\uparrow - n_\downarrow n_\uparrow n_\downarrow + n_\downarrow^2 - n_\downarrow^2 n_\uparrow$$
$$- n_\downarrow n_\uparrow^2 + n_\downarrow n_\uparrow n_\uparrow n_\downarrow - n_\downarrow n_\uparrow n_\downarrow + n_\downarrow n_\uparrow n_\downarrow n_\uparrow$$

(Because the number operator is idempotent and most of the operators here commute, there are really only four terms: One with each number operator, and the two ways they can be paired together)

$$= n_\uparrow - n_\uparrow n_\downarrow + n_\downarrow - n_\downarrow n_\uparrow$$
$$= P$$

$$S^2 = S_z^2 + \frac{1}{2}(S_+S_- + S_-S_+)$$

$$= \frac{1}{4}(n_\uparrow - n_\downarrow)(n_\uparrow - n_\downarrow) + \frac{1}{2}\left(c_\uparrow^\dagger c_\downarrow c_\downarrow^\dagger c_\uparrow + c_\downarrow^\dagger c_\uparrow c_\uparrow^\dagger c_\downarrow\right)$$

$$= \frac{1}{4}\left(n_\uparrow^2 - n_\uparrow n_\downarrow - n_\downarrow n_\uparrow + n_\downarrow^2\right) + \frac{1}{2}\left(c_\uparrow^\dagger c_\downarrow c_\downarrow^\dagger c_\uparrow + c_\downarrow^\dagger c_\uparrow c_\uparrow^\dagger c_\downarrow\right)$$

$$= \frac{1}{4}\left(n_\uparrow + n_\downarrow - 2n_\uparrow n_\downarrow\right) + \frac{1}{2}\left[c_\uparrow^\dagger\left(1 - c_\downarrow^\dagger c_\downarrow\right)c_\uparrow + c_\downarrow^\dagger\left(1 - c_\uparrow^\dagger c_\uparrow\right)c_\downarrow\right]$$

$$= \frac{1}{4}\left(n_\uparrow + n_\downarrow - 2n_\uparrow n_\downarrow\right) + \frac{1}{2}\left[n_\uparrow - n_\uparrow n_\downarrow + n_\downarrow - n_\downarrow n_\uparrow\right]$$

$$= \frac{3}{4}\left(n_\uparrow + n_\downarrow - 2n_\uparrow n_\downarrow\right)$$

$$= \frac{3}{4}\hat{P}$$