CHAPTER ONE

Conventions

1.1 Spacetime Coordinates

$$
\begin{aligned}
x^0 &= x_0 = t = -i\tau \\
x^1 &= -x_1 = x \\
\mathcal{L}_E &= -\mathcal{L}_M(t \to -i\tau, \partial_t \to -i\partial_\tau) \\
x^\pm &= \frac{1}{\sqrt{2}}(x^0 \pm x^1) \\
\partial_\pm &= \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)
\end{aligned}
$$

In 1-dim,

$$A_\mu B^\mu = A^0 B_0 - A^1 B^1.$$

Complex coordinates for euclidean space:

$$z, \overline{z} = \tau \pm ix = i\sqrt{2}x^\pm,$$

$$\partial, \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial\tau} \pm i\frac{\partial}{\partial x}\right) = -\frac{i}{\sqrt{2}}(\partial_+, \partial_-).$$

and

$$\partial\overline{\partial} = \frac{1}{4}(\partial_\tau^2 + \partial_x^2) = -\frac{1}{4}(\partial_t^2 - \partial_x^2) = \frac{1}{2}\partial_+\partial_-.$$

then

$$\mathrm{d}\tau\,\mathrm{d}x = \mathrm{d}\tau \wedge \mathrm{d}x = \frac{i}{2}\,\mathrm{d}z \wedge \mathrm{d}\overline{z}\,.$$

1.1.1 Einstein Summation Convention

Replace indices $\mu, \nu \in \{0, 1\}$ with indices $\alpha, \beta \in \{z, \overline{z}\}$ s.t.

$$
\begin{aligned}
x^z &= z = \tau + ix \\
x^{\overline{z}} &= \overline{z} = \tau - ix
\end{aligned}
$$

with derivatives

$$\partial_z = \frac{\partial}{\partial x^z} = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_\tau - i\partial_x)$$

$$\partial_{\overline{z}} = \frac{\partial}{\partial x^{\overline{z}}} = \frac{1}{2}(\partial_\tau + i\partial_x)$$

We define the covariant operators

$$\begin{aligned}
x_z &= \frac{1}{2}(\tau - ix) \\
x_{\overline{z}} &= \frac{1}{2}(\tau + ix) \\
\partial^z &= \frac{\partial}{\partial x_z} = \partial_\tau + i\partial_x \\
\partial^{\overline{z}} &= \partial_\tau - i\partial_x
\end{aligned}$$

This leads us to the metric

$$g_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, g^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, {g_\alpha}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

CHAPTER TWO

Bosonisation in Field Theory

2.1 Majorana Fermions

Euclidean space action

$$S = \int \mathrm{d}\tau \, \mathrm{d}x \, \mathcal{L}_{MF},$$

where

$$\begin{aligned}
\mathcal{L}_{MF} &= \frac{1}{2\pi}(\overline{\psi}, \psi) \begin{pmatrix} \partial & 0 \\ 0 & \overline{\partial} \end{pmatrix} \begin{pmatrix} \overline{\psi} \\ \psi \end{pmatrix} \\
&= \frac{1}{2\pi}(\overline{\psi}\partial\overline{\psi} - \psi\overline{\partial}\psi).
\end{aligned}$$

The term $(\overline{\psi}, \psi)^T$ is a spinor; ψ and $\overline{\psi}$ are independent *real* fields. The fact that these are real suggests that the fermions are Majorana fermions.

Euler-Lagrange Equations

In $d+1$-dimensional Minkowski space, the action is given by

$$S = \int \mathrm{d}t \, \mathrm{d}^d x \, \mathcal{L}(\phi, \partial_\mu \phi).$$

We seek stable points in the action S and do so by performing a variation in the field

$$\begin{aligned}
0 = \delta S &= \int \mathrm{d}t \, \mathrm{d}^d x \left\{ \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \underbrace{\delta(\partial_\mu \phi)}_{\partial_\mu(\delta\phi)} - \frac{\delta \mathcal{L}}{\delta \phi} \delta\phi \right\} \\
&= \int \mathrm{d}t \, \mathrm{d}^d x \left\{ - \left(\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi} \right) \delta\phi + \text{bound. terms} \right\}.
\end{aligned}$$

For the MF, we use the $z, \overline{z}$convention to get the EL equation

$$\underbrace{\partial \frac{\delta \mathcal{L}}{\delta \partial \overline{\psi}}}_{-\frac{1}{2\pi}\overline{\psi}} + \underbrace{\overline{\partial} \frac{\delta \mathcal{L}}{\delta \left(\overline{\partial \psi}\right)}}_{0} - \underbrace{\frac{\delta \mathcal{L}}{\delta \overline{\psi}}}_{\frac{1}{2\pi}\partial\overline{\psi}} = 0.$$

This leads to

$$\partial\psi = 0, \qquad \overline{\psi} = \overline{\psi}(\overline{z}).$$

This tells us that $\overline{\psi}$ is the right mover R, because it depends on $\tau - ix = -i(t-x)$. By performing something similar with the other field, we get that

$$\overline{\partial}\psi = 0, \qquad \psi = \psi(z)$$

which tells us that the field ψ is the left mover L. Here, it is important to note that this does not mean that the two fields do not depend on the two different complex variables from the start; this is the *equation of motion*. We started out with a Lagrangian that depended on two fields, and the equations of motion tell us that the fields are constant on the other variable.

Quantum Propagator For a feld theory defined by the action

$$S = \frac{1}{2}\int \mathrm{d}\tau\, \mathrm{d}^d x\, \phi M \phi$$

where M is some kind of matrix, for e.g. $-\partial_\mu\partial^\mu$, the propagator is given by the time ordered correlation function

$$\langle \mathcal{T}\phi(x)\phi(y)\rangle.$$

In this course, we will omit the time ordering as it is understood in all correlation functions. By direct evaluation of the functional integral, we arrive at the expression

$$\langle \mathcal{T}\phi(x)\phi(y)\rangle = (M^{-1})_{xy}.$$

The above expression holds for both bosonic and fermionic variables.

Example 2.1. Compute the inverse of $M = \frac{1}{\pi}\overline{\partial}$.

Proof. We use the identity $\overline{\partial}\partial \ln(z\overline{z}) = \pi\delta^{(2)}(z) = \pi\delta(\tau)\delta(x)$. Thus

$$\langle \psi(z)\psi(w)\rangle = \frac{1}{z-w}$$

and

$$\langle \overline{\psi}(\overline{z})\overline{\psi}(\overline{w})\rangle = \frac{1}{\overline{z}-\overline{w}}.$$

□

2.2 Dirac Fermions

We assemble two real majorana fermions $(\overline{\psi}_1, \psi_1), (\overline{\psi}_2, \psi_2)$ into 1 complex dirac fermion

$$\begin{pmatrix}\overline{\psi}_D \\ \psi_D\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\overline{\psi}_1 - i\overline{\psi}_2 \\ \psi_1 + i\psi_2\end{pmatrix}$$

$$\mathcal{L}_D = \frac{1}{\pi}\begin{pmatrix}\overline{\psi}^*_D & \psi^*_D\end{pmatrix}\begin{pmatrix}\partial & 0 \\ 0 & \overline{\partial}\end{pmatrix}\begin{pmatrix}\overline{\psi}_D \\ \psi_D\end{pmatrix} = \mathcal{L}_{MF_1} + \mathcal{L}_{MF_2}$$

which leads us to the propagator

$$\langle \psi^*_D(z)\psi_D(w)\rangle = \frac{1}{z-w}$$

$$\langle \overline{\psi}^*_D(\overline{z})\overline{\psi}(\overline{z})\rangle = \frac{1}{\overline{z}-\overline{w}}$$

2.3 Noether's Theorem

Suppose we have a continuous symmetry of a field $\phi(x) \to \phi(x, \lambda)$ such that

$$\mathcal{D}\mathcal{L} = \frac{\mathrm{d}\mathcal{L}(x,\lambda)}{\mathrm{d}\lambda}|_{\lambda=0} = \partial_\mu F^\mu(x).$$

Then

$$J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\mathcal{D}\phi - F^\mu(x)$$

is conserved.

2.4 Bosons & Bosonisation

We have

$$\begin{aligned} J(z) &= i\partial\Phi \\ \overline{J}(\overline{z}) &= -i\overline{\partial}\Phi \end{aligned}$$

The conservation law

$$\overline{\partial}J(z) = \partial\overline{J}(\overline{z}) = 0$$

is equivalent to the equation of motion. These currents can be derived from a "symmetry" of the bosonic theory: If we translate the fields, we get

$$\Phi(\lambda) = \Phi + \lambda,$$

the Lagrangian transforms as

$$\mathcal{L}_B = \frac{1}{2\pi}\partial\Phi\overline{\partial}\Phi$$

and thus $D\mathcal{L}_B = 0$. Thus, we get the conserved currents

$$\begin{aligned} J^z &= \frac{\delta\mathcal{L}}{\delta\partial_z\Phi}1 = \frac{i}{2\pi}\overline{J}(\overline{z}) \\ J^{\overline{z}} &= \frac{\delta\mathcal{L}}{\delta(\partial_{\overline{z}}\Phi)} = -\frac{i}{2\pi}J(z) \end{aligned}$$

The current conservation tells us that

$$\partial_z J^z + \partial_{\overline{z}} J^{\overline{z}} = 0$$

or

$$\frac{1}{\pi}\partial\overline{\partial}\Phi = 0.$$

This is not a problem since $\Phi \to \Phi + \lambda$ since Φ is not a physical field!

If $\Phi \to \Phi + \lambda$ were a physical symmetry, it could be spontaneously broken, contradicting the Mermin-Wagner Theorem (no goldstone modes in $1 + 1$ dim!).

2.5 Why non-abelian bosonization?

Consider N Dirac fermions $\begin{pmatrix}\overline{\psi}_r \\ \psi_r\end{pmatrix}$, $r = 1, 2, \ldots, N$. The Lagrangian is

$$\mathcal{L}_F = \frac{1}{\pi}\sum_{r=1}^{N}(\overline{\psi}_r^*\partial\overline{\psi}_r + \psi_r^*\overline{\partial}\psi_r.$$

Abelian bosonisation in Sec 1.3 suggests that this is physically equivalent to

$$\mathcal{L}_B = \frac{1}{2\pi}\sum_{r=1}^{N}\partial\Phi_r\overline{\partial}\Phi_r.$$

Note: While $\mathcal{L}_F$ has a $U(N)\times U(N)$ symmetry with $\psi(z) = \begin{pmatrix}\psi_1 \\ \vdots \\ \psi_N\end{pmatrix}$, $\mathcal{L}_F$ is also invariant under

$$\psi(z) \to \Lambda\psi(z), \qquad \psi^\dagger(z) \to \psi^\dagger(z)\Lambda^\dagger$$

where $\Lambda \in U(N)_L$ is a unitary $N\times N$ matrix such that $\Lambda^\dagger\Lambda = 1$. The conserved currents generate

$$J_{rr'}(z) = \frac{1}{\pi} : \psi_r^*\psi_{r'} : (z)$$

and

$$\overline{J}_{rr'}(\overline{z}) = \frac{1}{\pi} : \overline{\psi}_r^*\overline{\psi}_{r'} : (\overline{z}).$$

We note that $J_{rr'}^\dagger = J_{r'r}$ so that the off diagonal maps map into each other under hermitian conjugation. Only $J_{rr'}^\dagger + J_{rr'}$ is hermitian. Thus, there are a total of $2\cdot\frac{1}{2}N(N+1)$, where the first 2 comes from the left and right movers.

$$\begin{aligned}\overline{\partial}J_{rr'}(z) &= 0\\ \partial\overline{J}_{rr'}(z) &= 0\end{aligned}$$

for all $r, r \in 1, \ldots, N$. However, $\mathcal{L}_B$ only has $2N$ conserved currents $\partial\Phi_r$, $\overline{\partial}\Phi_r$. Note that this problem shows up already when particles have $SU(2)$ spin. The solution is non-abelian bosonization.

For example, the WZW action

$$S = S_0 + K\Gamma,$$

where

$$S_0 = \frac{1}{16\pi}\int_{S^2}\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{Tr}\big(\partial_\mu g\partial^\mu g^{-1}\big),\ \mu = 0, 1$$

where $g(z, \overline{z}) \in U(N)$ are matrix-valued fields. Γ is defined by

$$\Gamma = \frac{1}{24\pi}\int_{B^3}\tilde{g}^* W,$$

where $\tilde{g}^*$ is the pullback of 3-form W onto B^3,

$$W = \mathrm{Tr}\big(\tilde{g}^{-1}\,\mathrm{d}\tilde{g}\wedge\tilde{g}^{-1}\,\mathrm{d}\tilde{g}\wedge\tilde{g}^{-1}\,\mathrm{d}\tilde{g}\big).$$

The conserved currents in this case would be given by

$$\begin{aligned}J(z) &= \frac{1}{\pi}g^{-1}\partial g, &\qquad \overline{\partial}J(z) = 0\\ \overline{J}(\overline{z}) &= -\frac{1}{\pi}(\overline{\partial}g)g^{-1}, &\qquad \partial\overline{J}(\overline{z}) = 0\end{aligned}$$

The WZW model is far more general than the problem of non-Abelian bosonization! Instead of $U(N) \sim SU(N)$ "spin" $\times\, U(1)$ "charge". we can take any lie group G.

CHAPTER THREE

θ Terms

3.1 Example: Particle on a Ring

Consider a particle on a ring, parametrised by coordinate ϕ with a flux Φ through the ring. The Hamiltonian is given by

$$H = \frac{1}{2}(-i\partial_\phi - A)^2$$

where $A = \frac{\Phi}{\Phi_0}$ and $\Phi_0 = 2\pi$. The wavefunction is naturally periodic $\psi(2\pi) = \psi(0)$. The solutions are

$$\psi_n = \frac{1}{\sqrt{2\pi}} e^{in\phi}, \qquad \epsilon_n = \frac{1}{2}\left(n - \frac{\Phi}{\Phi_0}\right)^2.$$

We reformulate the problem as a path integral. First, we express the Hamiltonian as a function of p and x:

$$H = \frac{1}{2}(p_\varphi - A)^2.$$

The Hamiltonian equations of motion are

$$\begin{aligned} \frac{\mathrm{d}\varphi}{\mathrm{d}t} &= \frac{\partial H}{\partial p_\varphi} = p_\varphi - A \\ \frac{\mathrm{d}p_\varphi}{\mathrm{d}t} &= -\frac{\partial H}{\partial \varphi} = 0 \end{aligned}$$

The Lagrangian is

$$L = p_\varphi \dot{\phi} - H = (\dot{\varphi} + A)\dot{\varphi} - \frac{1}{2}\dot{\varphi}^2 = \frac{1}{2}\dot{\varphi}^2 + A\dot{\varphi}.$$

In euclidean space, this is

$$L_E = -L(t \to -i\tau, \partial_t \to i\partial_\tau) = \frac{1}{2}(\partial_\tau \varphi)^2 - iA\partial_\tau \varphi.$$

The equation of motion is

$$\partial_\tau \frac{\delta L_E}{\delta(\partial_\tau \varphi)} - \frac{\delta L_E}{\delta \varphi} = 0$$

which is

$$\partial_\tau^2 \varphi = 0.$$

We generate the path integral using the standard formula

$$Z = \int_{\varphi(\beta) - \varphi(0) \in 2\pi\mathbb{Z}} \mathcal{D}\varphi e^{-S_E}, \qquad S_E = \int \mathrm{d}\tau\, L_E.$$

This gives a family of classical solutions

$$\varphi_W(\tau) = 2\pi W \frac{\tau}{\beta}.$$

We note that the solutions are mappings

$$\varphi : S^1 \to S^1, \qquad \tau \mapsto \varphi(\tau).$$

The former must be periodic because of the periodic boundary conditions in the partition function, while $\varphi(\tau)$ lives on a circle. There is an integer winding number W associated with each path that cannot be changed by a continuous deformation. We can perform the path integral separately over each of these sectors, and there will be a classical solution in each of them:

$$Z = \sum_W \int_{\varphi(\beta)-\varphi(0)=2\pi W} \mathcal{D}\varphi e^{-\mathrm{d}\tau L_E(\varphi,\partial_\tau\varphi)}.$$

The integral over the second term of the euclidean space action can be performed directly

$$\int_{\varphi(\beta)-\varphi(0)=2\pi W} (-iA\partial_\tau\varphi) = -iA\varphi(\tau)|_0^\beta = -2\pi i A W$$

leading to the euclidean space action

$$Z = \sum_W e^{2\pi i A W} \int_{\varphi(\beta)-\varphi(0)=2\pi W} \mathcal{D}\varphi e^{-\int \mathrm{d}\tau (\partial_\tau\varphi)^2}.$$

The first term is an example of a topological term: It depends on the boundary conditions of the path, in the same sense that a gauge field can only be measured when one moves along a complete path. The kinetic information remains in the quadratic term.

Some notes

1. $S_{\text{top}} = -2\pi i W A$ is the simplest example of a topological term and belongs to the class of θ terms.
2. S_{top} is sensitive only to the topological sector, which is a global property, and does not depend on the local properties of the path. Thus, it cannot affect the equation of motion.
3. S_{top} is invariant to changes of the metric of the base manifold (e.g. scaling $\tau \to \alpha_\tau$)
4. $e^{-S_{\text{top}}}$ (or $e^{iS^M_{\text{top}}}$ in Minkowski space) is always a pure phase.

Note that $e^{-S_{\text{top}}}$ is invariant under $A \to A + \mathbb{Z}$.

3.2 Homotopy

Fields are mappings on a manifold $\phi : M \to T$. M is known as the base manifold, and T is known as the target space. Usually, the base manifold is $\mathbb{R}^d$. For low energy physics, it is sensible to consider field configurations which approach a constant value at the boundary of M. Thus, we can consider $M \cong S^d$.

We say that two field configurations $\phi_1,\ \phi_2$ are topologically equivalent if they can be deformed continuously into each other, meaning there exists a homotopy

$$\hat{\phi} : S^d \times [0,1] \to T,\ (x,\lambda) \to \hat{\phi}(x,\lambda)$$

suuch that ϕ is continuous as a function from $S^d \times [0,1]$ in the usual product topology. Clearly, this is an equivalence relation; we call this the homotopy class. In general, we have $\pi_n(S^d) = \{0\}$ if $n < d$ and $\pi_n(S^n) = \mathbb{Z}$.

Some examples:

1. $\pi_n(S^d) = \{0\}$ if $n < d$, $\pi_n(S^n) = \mathbb{Z}$.
2. $\pi_1(T^d) = \pi_1(S^1 \times S^1 \cdots \times S^1) = \mathbb{Z} \times \mathbb{Z} \cdots \times Z$

3.2.1 θ-terms in general

Recall that each $\phi : M \to T$ belongs to a certain homotopy class $[\phi]$. Thus, we can organise Z.

$$Z = \sum_{w \in G} \int \mathrm{D}\phi_W \, e^{-S[\phi]}.$$

where W labels the homotopy class, G is the homotopy group and we integrate over all ϕs in the homotopy class. This leads to an action

$$S[\phi] = S_0[\phi] + \underbrace{S_{\text{top}}[\phi]}_{=F(W)}$$

and

$$Z = \sum_{w \in G} e^{-F(w)} \int \mathrm{D}\phi_W \, e^{-S_0[\phi]}.$$

Now, we can ask how $F(W)$ depends on W. In general, we will see that this is linear - $F(w_1 + w_2) = F(w_1) + F(w_2)$.