6. Problemset "Quantum Algebra & Dynamics"
November 21, 2025

Direct Sum, Tensor Product & Hilbert Space(s)

6.1 Sum and Product

Consider the direct sum $\mathcal{A}_1 \oplus \mathcal{A}_2$ and tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ of two C^*-algebras $\mathcal{A}_{1,2}$.

1. Show that both are $*$-algebras with the natural definitions of the respective products.
2. Show that $\mathcal{A}_1 \oplus \mathcal{A}_2$ can be made into a C^*-algebra with the natural definition of a norm.
3. Can you find a norm that turns $\mathcal{A}_1 \otimes \mathcal{A}_2$ into a C^*-algebra?

6.2 Spin Chain

Consider a chain of N spin-1/2 systems in the Hilbert space

$$\mathcal{H}_N = \bigotimes_{i=1}^{N} \mathcal{H}^{(i)} \tag{1}$$

with

$$\forall i \in \{1, 2, \ldots, N\} : \mathcal{H}^{(i)} = \{c_\uparrow \Psi_\uparrow + c_\downarrow \Psi_\downarrow : c_\uparrow, c_\downarrow \in \mathbf{C}\} \cong \mathbf{C}^2 \,, \tag{2}$$

in which the C^*-Algebra $\mathcal{A}_N$ of observables generated by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \; \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

is represented by

$$\Sigma_k^{(i)} = \bigotimes_{k=1}^{i-1} \mathbf{1} \otimes \sigma_k \otimes \bigotimes_{k=i+1}^{N} \mathbf{1} \,. \tag{4}$$

1. Which dimension has $\mathcal{H}_N$?
2. Which dimension has $\mathcal{A}_N$?

3. Compute the commutation relations

$$\left[\Sigma_k^{(i)}, \Sigma_l^{(j)}\right]_- = \Sigma_k^{(i)}\Sigma_l^{(j)} - \Sigma_l^{(j)}\Sigma_k^{(i)} . \tag{5}$$

4. Construct the states $\Psi_{\vec{a}}^N \in \mathcal{H}_N$ with the property

$$\forall i \in \{1, 2, \ldots, N\} : \left(\vec{a}\vec{\Sigma}^{(i)}\right) \Psi_{\vec{a}}^N = \Psi_{\vec{a}}^N \tag{6}$$

   for all $\vec{a} \in \mathbf{R}^3$ with $\|\vec{a}\| = 1$.

5. Find a unitary operator $U_N(\vec{a}, \vec{b})$ with

$$U_N(\vec{a}, \vec{b})\Psi_{\vec{b}}^N = \Psi_{\vec{a}}^N . \tag{7}$$

6. In the limit $N \to \infty$, we can study the Hilbert spaces

$$\mathcal{H}_{\vec{a}} = \overline{\lim_{N\to\infty} \mathcal{A}_N \Psi_{\vec{a}}^N} \ni \Psi_{\vec{a}} = \lim_{N\to\infty} \Psi_{\vec{a}}^N \tag{8}$$

   that are obtained by completing the spaces of states obtained from applying elements of $\lim_{N\to\infty} \mathcal{A}_N$ to $\Psi_{\vec{a}}$.

   (a) Do we have $\Psi_{\vec{a}} \in \mathcal{H}_{\vec{b}}$?

   (b) Does $U = \lim_{N\to\infty} U_N$ exist in the operator topology?