4. Problemset "Quantum Algebra & Dynamics"
November 7, 2025

More Functions / Subalgebras / Positivity

4.1 Square Root and Exponential

Use again the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

to parametrize a general complex 2×2-matrix $M \in \mathcal{M}_2$ by four complex numbers $(a_0, \vec{a})$

$$M(a_0, \vec{a}) = a_0 \mathbf{1} + \vec{a}\vec{\sigma}\,. \tag{2}$$

1. Use the holomorphic functional calculus to compute $\sqrt{M}$.
2. Use the holomorphic functional calculus to compute $\mathrm{e}^{\mathrm{i}M}$ and compare the result with the corresponding power series.

4.2 Square Root Revisited

Show $B^2 = A \geq 0$ for

$$B = \int_0^\infty \frac{\mathrm{d}\lambda}{\pi} \frac{A}{\sqrt{\lambda}} \frac{1}{\lambda\mathbf{1} + A}\,. \tag{3}$$

by explicit calculation without using the holomorphic functional calculus.

Hint: the integrals can best be performed by regularizing the integrand so that we can rearrange terms and reorder integrations without having to worry about convergence.

- *Write*

$$f(\lambda) = \frac{1}{\pi} \frac{A}{\sqrt{\lambda}} \frac{1}{\lambda\mathbf{1} + A} \tag{4}$$

as shorthand for the integrand.

- *Show that B^2 can be written formally (not worrying to much about convergence for $\sigma \to 0$ and $\epsilon \to \infty$) as*

$$B^2 = 2\int_0^\infty \mathrm{d}\lambda \int_0^1 \mathrm{d}\sigma\, \lambda f(\lambda) f(\sigma\lambda) \tag{5}$$

and

$$\lambda f(\lambda) f(\sigma\lambda) = \frac{1}{\pi^2}\frac{1}{\sqrt{\sigma}}\frac{1}{1-\sigma}\left(\frac{1}{\lambda\mathbf{1}+A} - \frac{\sigma}{\sigma\lambda\mathbf{1}+A}\right) A\,. \tag{6}$$

- *Regularize the λ-integral for $\lambda \to \infty$ as*

$$f(\lambda) = \lim_{\delta\to 0+} f_\delta(\lambda) \tag{7}$$

where

$$f_\delta(\lambda) = \frac{1}{\pi}\frac{A}{\lambda^{\frac{1}{2}+\delta}}\frac{1}{\lambda\mathbf{1}+A}\,. \tag{8}$$

- *Regularize the σ-integral for $\sigma \to 0$ as*

$$\frac{1}{1-\sigma} = \lim_{\epsilon\to 0+}\frac{1}{(1-\sigma)^{1-\epsilon}}\,. \tag{9}$$

- *Show that the regularized integral can be written as*

$$B^2 = \lim_{\delta,\epsilon\to 0+}\frac{2}{\pi^2}\int_0^1 \mathrm{d}\sigma\,\frac{\sigma^{-\frac{1}{2}-\delta} - \sigma^{-\frac{1}{2}+\delta}}{(1-\sigma)^{1-\epsilon}}\int_0^\infty \mathrm{d}\lambda\,\frac{\lambda^{-2\delta}}{\lambda\mathbf{1}+A}A \tag{10}$$

- *Express the σ-integral as Euler Beta or Gamma functions and take the limit $\epsilon \to 0$ for $\delta > 0$, ignoring terms $\mathcal{O}(\delta^2)$. You should get*

$$B^2 = \lim_{\delta\to 0+} 2\delta\int_0^\infty \mathrm{d}\lambda\,\frac{\lambda^{-2\delta}}{\lambda\mathbf{1}+A}A \tag{11}$$

- *Take the limit $\delta \to 0$.*

4.3 Subalgebras

Consider a C^*-algebra $\mathcal{A}$, an element $P = P^* \in \mathcal{A}$ with

$$P^2 = P\,. \tag{12}$$

Show that the subset

$$\mathcal{A}' = P\mathcal{A}P = \{PAP : A \in \mathcal{A}\} \subseteq \mathcal{A} \tag{13}$$

is a C^*-algebra with identity $\mathbf{1}_{\mathcal{A}'} = P$.

4.4 Positivity in $*$-Algebras

Consider the vector space $\mathbf{C}^2$ as a $*$-algebra $\mathcal{A}$ with component wise addition, scalar multiplication and product

$$\alpha(x,y) + \alpha'(x',y') = (\alpha x + \alpha' x', \alpha y + \alpha' y') \tag{14a}$$

$$(x,y)(x',y') = (xx', yy') \tag{14b}$$

and involution given by

$$(x,y)^* = (\bar{y}, \bar{x})\,. \tag{14c}$$

Show that $\mathcal{A}$ can not be made into a C^*-algebra:

1. find the positive elements $a \in \mathcal{A}$.

2. Show that the sum of two strictly positive elements is not always strictly positive.