

Einführung in die Algebra Hausaufgaben Blatt Nr. 11

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I forgot to bring my tablet on the train so please enjoy the L^AT_EX solutions.

Problem 1 (Getting familiar with the Pauli spin vector). (a) Prove the relation

$$(\sigma \cdot A)(\sigma \cdot B) = A \cdot B + i\sigma \cdot (A \times B)$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ denotes the vector of the 2×2 Pauli-spin matrices. $A = (A_x, A_y, A_z)^T$ and $B = (B_x, B_y, B_z)^T$ are arbitrary vectors.

(b) Show that

$$\sigma \cdot \hat{p} = \frac{1}{r^2}(\sigma \cdot r) \left(-i\hbar r \partial_r + i\sigma \cdot \hat{L} \right)$$

where $\hat{p} = -i\hbar \nabla$ and $\hat{L} = r \times \hat{p}$.

Proof. (a)

$$\begin{aligned} (\sigma \cdot A)(\sigma \cdot B) &= (\sigma_i A_i)(\sigma_j B_j) \\ &= A_i B_j (\delta_{ij} I + i\epsilon_{ijk} \sigma_k) \\ &= A_i B_i I + i\epsilon_{ijk} A_i B_j \sigma_k \\ &= (A \cdot B) I + i\sigma_k \epsilon_{kij} A_i B_j \\ &= (A \cdot B) I + i\sigma \cdot (A \times B) \end{aligned}$$

(b)

$$\sigma \cdot \hat{p} = -i\hbar \sigma_i \partial_i$$

□

Problem 2 (Majorana representation of the Dirac equation). Multiplying the Dirac equation known from the lecture by $-\frac{i}{\hbar}$ we get

$$H_D \Psi = \left(\frac{\partial}{\partial t} + \vec{\alpha} \cdot \vec{\nabla} + im_0 \beta \right) \Psi = 0$$

with

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \vec{\sigma} = \left(\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right).$$

Thus, some of the matrices in H_D are imaginary. Show that the transformation

$$\Psi' = U \Psi \tag{4}$$

with

$$U = \frac{1}{\sqrt{2}}(\alpha_y + \beta)$$

results in a representation of the Dirac-equation where $H'_D = U H_D U^{-1}$ is purely real.

Proof. We begin by showing that U is unitary (which is needed to argue that this is a unitary transformation anyway):

$$\begin{aligned} U^\dagger U &= \frac{1}{2} \begin{pmatrix} I_2 & \sigma_y^\dagger \\ \sigma_y^\dagger & -I_2 \end{pmatrix} \begin{pmatrix} I_2 & \sigma_y \\ \sigma_y & -I_2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} I_2 + \sigma_y^\dagger \sigma_y & \sigma_y - \sigma_y^\dagger \\ \sigma_y^\dagger - \sigma_y & \sigma_y^\dagger \sigma_y + I_2 \end{pmatrix} \\ &= \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} \end{aligned}$$

Note that since U is both Hermitian and unitary, its inverse is itself. Then we simply apply this to all of the matrices:

$$\begin{aligned} 2U\beta U &= \begin{pmatrix} I_2 & \sigma_y \\ \sigma_y & -I_2 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} I_2 & \sigma_y \\ \sigma_y & -I_2 \end{pmatrix} \\ &= \begin{pmatrix} I_2 & \sigma_y \\ \sigma_y & -I_2 \end{pmatrix} \begin{pmatrix} I_2 & \sigma_y \\ -\sigma_y & I_2 \end{pmatrix} \\ &= \begin{pmatrix} I_2 - \sigma_y^2 & 2\sigma_y \\ 2\sigma_y & \sigma_y^2 - I_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2\sigma_y \\ 2\sigma_y & 0 \end{pmatrix} \end{aligned}$$

After the multiplication by i , this is purely real. We now proceed to the rest:

$$\begin{aligned} 2U\alpha_i U &= \begin{pmatrix} I_2 & \sigma_y \\ \sigma_y & -I_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} I_2 & \sigma_y \\ \sigma_y & -I_2 \end{pmatrix} \\ &= \begin{pmatrix} I_2 & \sigma_y \\ \sigma_y & -I_2 \end{pmatrix} \begin{pmatrix} \sigma_i \sigma_y & -\sigma_i \\ \sigma_i & \sigma_i \sigma_y \end{pmatrix} \\ &= \begin{pmatrix} \sigma_i \sigma_y + \sigma_y \sigma_i & -\sigma_i + \sigma_y \sigma_i \sigma_y \\ \sigma_y \sigma_i \sigma_y - \sigma_i & -\sigma_y \sigma_i - \sigma_i \sigma_y \end{pmatrix} \end{aligned}$$

Then we evaluate this for $i \in \{x, y, z\}$:

$$\begin{aligned} 2U\alpha_x U &= \begin{pmatrix} 0 & -2\sigma_x \\ -2\sigma_x & 0 \end{pmatrix} \\ 2U\alpha_y U &= \begin{pmatrix} 2I_2 & 0 \\ 0 & -2I_2 \end{pmatrix} \\ 2U\alpha_z U &= \begin{pmatrix} 0 & -2\sigma_z \\ -2\sigma_z & 0 \end{pmatrix} \end{aligned}$$

all of which are real. □

Problem 3 (Some properties of the γ matrices). (a) By considering $\mu = \nu = 0$, $\mu = \nu \neq 0$ and $\mu \neq \nu$ show that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

where $\{, \}$ denotes the anti-commutator.

(b) Show that

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0.$$

Proof.

The γ matrices are defined by

$$\gamma^0 = \beta, \quad \gamma^i = \beta \alpha^i.$$

(a) We consider the different cases

(1) $\mu = \nu = 0$:

$$\beta^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = I_4 = g^{00} I_4.$$

(2) $\mu = \nu \neq 0$:

$$\begin{aligned} (\gamma^\mu)^2 &= \beta \alpha^\mu \beta \alpha^\mu \\ &= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \\ &= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} \\ &= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} \\ &= -I_4 = g^{\mu\mu} I_4 \end{aligned}$$

(3) $\mu \neq \nu$: The relevant commutator is $\{\beta \alpha^\mu, \beta \alpha^\nu\}$. We then use the anticommutator identity

$$\{A, BC\} = \{A, B\}C - B[A, C]$$

as well as the commutator identity

$$[AB, C] = A[B, C] + [A, C]B.$$

to expand

$$\begin{aligned} \{\beta \alpha^\mu, \beta \alpha^\nu\} &= \{\beta \alpha^\mu, \beta\} \alpha^\nu - \beta [\beta \alpha^\mu, \alpha^\nu] \\ &= (\{\beta, \beta\} \alpha^\mu - \beta [\beta, \alpha^\mu]) \alpha^\nu - \beta (\beta [\alpha^\mu, \alpha^\nu] + [\beta, \alpha^\nu] \alpha^\mu) \end{aligned}$$

(b) We have

$$\begin{aligned} (\gamma^\mu)^\dagger &= (\beta \alpha^\mu)^\dagger \\ &= (\alpha^\mu)^\dagger \beta^\dagger \\ &= \alpha^\mu \beta && \alpha, \beta \text{ hermitian} \\ &= \beta^{-1} \gamma^\mu \beta \\ &= \beta \gamma^\mu \beta \end{aligned}$$

where it is understood that $\alpha^0 = I_4$.

□