Collected Homeworks and Exercises for the Lecture

Algebra and Dynamics of Quantum Systems

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Homeworks for Algebra and Dynamics of Quantum Systems

Homework 1-1: The commutator

Let $\mathscr{A}$ be an associative algebra over $\mathbb{C}$. We define the commutator

$$[a, b] = ab - ba \tag{1.1}$$

for $a, b \in \mathscr{A}$ as usual. Furthermore, we write $\mathrm{ad}(a)\colon b \mapsto [a, b]$.

i.) Prove that $[\,\cdot\,,\,\cdot\,]$ turns $\mathscr{A}$ into a Lie algebra.

ii.) Let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be an algebra morphism into another associative algebra $\mathscr{B}$. Show that Φ is also a Lie algebra morphism with respect to the commutator Lie brackets. Conclude that this yields a functor from the category of associative algebras $\mathsf{alg}_{\mathbb{C}}$ into the category of Lie algebras over $\mathbb{C}$.

iii.) Consider the left and right multiplications

$$\mathsf{L}_a, \mathsf{R}_b\colon \mathscr{A} \longrightarrow \mathscr{A} \tag{1.2}$$

for a fixed algebra element $a \in \mathscr{A}$, i.e. $\mathsf{L}_a(b) = ab$ as well as $\mathsf{R}_a(b) = ba$. Show $[\mathsf{L}_a, \mathsf{R}_b] = 0$ as well as $\mathrm{ad}(a) = \mathsf{L}_a - \mathsf{R}_a$ for all $a, b \in \mathscr{A}$.

iv.) Let $\mathfrak{g}$ be a Lie algebra. Prove that $\mathrm{ad}\colon \mathfrak{g} \ni \xi \mapsto (\eta \mapsto \mathrm{ad}(\xi)\eta = [\xi, \eta]) \in \mathrm{End}(\mathfrak{g})$ yields a homomorphism of Lie algebras

$$\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathrm{End}\big(\mathfrak{g}\big), \tag{1.3}$$

where we equip $\mathrm{End}(\mathfrak{g})$ with the commutator as Lie bracket.

v.) Prove that the map $\mathrm{ad}(a)$ is a derivation of the associative product for $a \in \mathscr{A}$. Furthermore show that the set of derivations of $\mathscr{A}$ constitutes a Lie subalgebra $\mathrm{Der}(\mathscr{A}) \subseteq \mathrm{End}(\mathscr{A})$ of all endomorphisms of $\mathscr{A}$. Finally, prove that $\mathrm{ad}\colon \mathscr{A} \longrightarrow \mathrm{Der}(\mathscr{A})$ is a Lie algebra homomorphism.

vi.) Derivations of the form $\mathrm{ad}(a)$ are called *inner derivations*, whose set we denote by $\mathrm{InnDer}(\mathscr{A})$. Show first that $\mathrm{InnDer}(\mathscr{A})$ is a subspace of $\mathrm{End}(\mathscr{A})$. Furthermore prove

$$[D, \mathrm{ad}(a)] = \mathrm{ad}(Da) \tag{1.4}$$

for every derivation $D \in \mathrm{Der}(\mathscr{A})$ and every algebra element $a \in \mathscr{A}$. Conclude that the quotient $\mathrm{OutDer}(\mathscr{A}) = \mathrm{Der}(\mathscr{A}) \big/ \mathrm{InnDer}(\mathscr{A})$ carries a Lie algebra structure. The elements of $\mathrm{OutDer}(\mathscr{A})$ are called *outer derivations* of $\mathscr{A}$.

vii.) Let now $\mathscr{A}$ be a *-algebra. Compute $[a, b]^*$ for $a, b \in \mathscr{A}$. Use this to characterize the elements $a \in \mathscr{A}$, for which $\mathrm{ad}(a)$ is a *-derivation.

Homework 1-2: A positive quadratic polynomial

Consider complex numbers $a, b, b', c \in \mathbb{C}$ with

$$p(z, w) = a\overline{z}z + bz\overline{w} + b'\overline{z}w + cw\overline{w} \geq 0 \tag{1.5}$$

for all $z, w \in \mathbb{C}$. Show that this implies $a \geq 0$, $c \geq 0$, $\overline{b} = b'$ and $ac \geq b\overline{b}$.

Homework 1-3: The polynomial calculus I

Let $\mathscr{A}$ be a unital associative algebra over some field $\Bbbk$ and let $a \in \mathscr{A}$ be a fixed element. For a polynomial $p \in \Bbbk[x]$ one defines $p(a) \in \mathscr{A}$ as usual by substituting the variable x by the algebra element a. If $\mathscr{A}$ is not unital, then this is only possible for polynomials $p \in x\Bbbk[x]$ with vanishing constant part.

i.) Show that the map
$$\Bbbk[x] \ni p \;\mapsto\; p(a) \in \mathscr{A} \tag{1.6}$$
is a unital algebra homomorphism.

ii.) Show that if $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a unital homomorphism into some other unital associative algebra $\mathscr{B}$ over $\Bbbk$, then
$$\Phi(p(a)) = p(\Phi(a)) \tag{1.7}$$
for all $a \in \mathscr{A}$ and $p \in \Bbbk[x]$. In which sense does this still hold in the non-unital situation?

Homework 1-4: The polynomial calculus II

Assume that $\mathscr{A}$ is a unital *-algebra over $\mathbb{C}$ and let $a \in \mathscr{A}$ be a normal element. Consider polynomials $\mathbb{C}[z, \overline{z}]$ in two variables.

i.) Show that the algebra $\mathbb{C}[z, \overline{z}]$ becomes a *-algebra if one defines $z^* = \overline{z}$ for the generators, thereby explaining the notation.

ii.) Define for $p \in \mathbb{C}[z, \overline{z}]$ the algebra element $p(a, a^*) \in \mathscr{A}$ by substituting z by a and $\overline{z}$ by a^*. Show that this is well-defined by using the fact that a is normal.

iii.) Show that the map
$$\mathbb{C}[z, \overline{z}] \ni p \;\mapsto\; p(a, a^*) \in \mathscr{A} \tag{1.8}$$
is a unital *-homomorphism.

iv.) Formulate and prove an analogous statement for the case where $\mathscr{A}$ is non-unital.

Homework 2-1: States and density matrices

Consider the finite-dimensional pre-Hilbert space $\mathfrak{H} = \mathbb{C}^n$ with its canonical inner product.

i.) Show that the matrices $\mathrm{M}_n(\mathbb{C})$ act on $\mathbb{C}^n$ by adjointable operators and determine the induced *-involution. We will always endow $\mathrm{M}_n(\mathbb{C})$ with this *-involution.

ii.) Let $\omega\colon \mathrm{M}_n(\mathbb{C}) \longrightarrow \mathbb{C}$ be a positive linear functional. Prove that there exists a matrix $\varrho \in \mathrm{M}_n(\mathbb{C})$ with the property $\langle \phi, \varrho\phi \rangle \geq 0$ for all $\phi \in \mathbb{C}^n$ such that $\omega(A) = \operatorname{tr}(\varrho A)$. Show that ω is a state iff $\operatorname{tr}(\varrho) = 1$. Such a matrix ϱ is called a *density matrix*.

iii.) Conversely, show that every density matrix $\varrho \in \mathrm{M}_n(\mathbb{C})$ gives a state on $\mathrm{M}_n(\mathbb{C})$ via the definition $A \mapsto \operatorname{tr}(\varrho A)$.

iv.) Show that for a matrix $A \in \mathrm{M}_n(\mathbb{C})$ the following statements are equivalent:

(a) One has $\langle \phi, A\phi \rangle \geq 0$ for all $\phi \in \mathbb{C}^n$.

(b) One has $A = A^*$ and all eigenvalues of A are non-negative.

(c) There is a Hermitian matrix $B = B^*$ with non-negative eigenvalues and $A = B^2$.

(d) There is a Hermitian matrix $B = B^*$ with $A = B^2$.

(e) There is a matrix $B \in \mathrm{M}_n(\mathbb{C})$ with $A = B^*B$.

(f) There are matrices $B_1, \ldots, B_N \in \mathrm{M}_n(\mathbb{C})$ with $A = B_1^*B_1 + \cdots B_N^*B_N$, i.e. A is algebraically positive.

(g) One has $\omega(A) \geq 0$ for all states ω, i.e. A is a positive algebra element.

The content of this homework should be well-known (at least in parts) from linear algebra courses. One can safely skip this homework if familiar with the results. Details can be found in e.g. [3, Sect. 7.8].

Homework 2-2: Polarization identity

Let V and W be two vector spaces over $\mathbb{C}$ and $S\colon V \times V \longrightarrow W$ a *sesquilinear* map, i.e. assume that

$$S(\alpha u + \beta v, w) = \overline{\alpha} S(u, w) + \overline{\beta} S(v, w) \quad \text{and} \quad S(u, \alpha v + \beta w) = \alpha S(u, v) + \beta S(u, w) \tag{2.1}$$

hold for all $\alpha, \beta \in \mathbb{C}$ and $u, v, w \in V$.

i.) Show that the *polarization identity*

$$S(v, w) = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k \cdot S\big(v + \mathrm{i}^{-k} w, v + \mathrm{i}^{-k} w\big) \tag{2.2}$$

holds for all $v, w \in V$. Conclude that S is constant 0 iff $S(v, v) = 0$ for all $v \in V$.

ii.) Now let $W = \mathbb{C}$. A sesquilinear map $S\colon V \times V \longrightarrow \mathbb{C}$ is usually called a sesquilinear form. Such a sesquilinear form is said to be Hermitian if $\overline{S(v, w)} = S(w, v)$ holds for all $v, w \in V$. Show that a sesquilinear form S on V is Hermitian if and only if $S(v, v) \in \mathbb{R}$ holds for all $v \in V$.

iii.) Let finally $\mathscr{A}$ be a unital *-algebra over $\mathbb{C}$. Show that for every $a \in \mathscr{A}$ there exist algebraically positive elements $b_0, b_1, b_2, b_3 \in \mathscr{A}^{++}$ such that $a = \sum_{k=0}^{3} \mathrm{i}^k b_k$ holds.

Homework 2-3: Positivity in the commutative *-algebra $\mathbb{C}[x]$

Recall that $\mathbb{C}[x]$ with *-involution $\big(\sum_{n=0}^{\infty} a_n x^n\big)^* = \sum_{n=0}^{\infty} \overline{a}_n x^n$ is a commutative *-algebra. Show that for a polynomial $a \in \mathbb{C}[x]$ the following statements are equivalent:

i.) The polynomial a is an algebraically positive element of $\mathbb{C}[x]$.

ii.) The polynomial a is a positive element of $\mathbb{C}[x]$.

iii.) The polynomial a is pointwise positive, i.e. $a(y) \geq 0$ for all $y \in \mathbb{R}$.

Hint: You might want to make use of the evaluation functionals at $y \in \mathbb{C}$, defined as

$$\delta_y \colon \mathbb{C}[x] \ni a \;\mapsto\; a(y) \in \mathbb{C}. \tag{2.3}$$

The fundamental theorem of algebra might also be useful.

Homework 3-1: Sums of squares and the Motzkin polynomial

We consider the *-algebra $\mathscr{C}^\infty(\mathbb{R}^2)$ of complex-valued smooth functions on the plane.

i.) Show that the polynomial $p \in \mathbb{C}[x, y] \subseteq \mathscr{C}^\infty(\mathbb{R}^2)$

$$p(x, y) = 1 + x^4 y^2 + x^2 y^4 - 3x^2 y^2 \tag{3.1}$$

is non-negative on $\mathbb{R}^2$ but not a sum of squares of polynomials. This an explicit example to the 17th Hilbert problem due to Motzkin [1].

Hint: Use the AM-GM inequality to check that p is actually pointwise non-negative.

ii.) Show that p is not even a sum of squares inside the bigger algebra $\mathscr{C}^\infty(\mathbb{R}^2)$.

Hint: Write $p = \sum_{i=1}^{n} \overline{f}_i f_i$ with smooth functions $f_i \in \mathscr{C}^\infty(\mathbb{R}^2)$ and use Taylor expansions of the f_i around 0.

Homework 3-2: Uncertainty, characters and eigenvectors

Let $\mathscr{A}$ be a unital *-algebra and let $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a state.

i.) Let $a, b \in \mathscr{A}$ be given. Show that

$$\big|\omega(a^* b) - \omega(a^*)\omega(b)\big|^2 \leq \operatorname{Var}_\omega(a) \operatorname{Var}_\omega(b) \tag{3.2}$$

holds.

ii.) Show the *uncertainty relation*

$$4\,\mathrm{Var}_\omega(a)\,\mathrm{Var}_\omega(b) \geq \big|\omega([a,b])\big|^2. \tag{3.3}$$

for Hermitian $a, b \in \mathscr{A}$.

iii.) Show that $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ is a unital *-homomorphism if and only if ω is a state that fulfills $\mathrm{Var}_\omega(a) = 0$ for all $a \in \mathscr{A}$.

iv.) Finally, let $\mathcal{H}$ be a pre-Hilbert space and $\mathscr{A} \subseteq \mathscr{B}(\mathcal{H})$ a unital *-subalgebra of all adjointable endomorphisms of $\mathcal{H}$. Given $A \in \mathscr{A}$ and $\phi \in \mathcal{H}$ with $\langle \phi, \phi \rangle = 1$, then show that the positive linear functional

$$\omega_\phi\colon \mathscr{A} \ni B \;\mapsto\; \omega_\phi(B) = \langle \phi, B\phi \rangle \in \mathbb{C} \tag{3.4}$$

fulfills $\mathrm{Var}_{\omega_\phi}(A) = 0$ if and only if ϕ is an eigenvector of A to the eigenvalue $\omega_\phi(A)$.

Homework 3-3: Functorial properties of the GNS construction

Let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a morphism of *-algebras $\mathscr{A}$ and $\mathscr{B}$ as well as $\omega\colon \mathscr{B} \longrightarrow \mathbb{C}$ be a positive linear functional.

i.) Verify that the pull-back $\Phi^*\omega = \omega \circ \Phi$ is a positive linear functional on $\mathscr{A}$.

ii.) Prove that for the Gel'fand ideals we have

$$\Phi(\mathscr{J}_{\Phi^*\omega}) \subseteq \mathscr{J}_\omega. \tag{3.5}$$

Conclude that Φ descends to a well-defined linear map $U_\phi\colon \mathcal{H}_{\Phi^*\omega} \longrightarrow \mathcal{H}_\omega$.

iii.) Show that the map U_Φ is isometric. Note that it may well happen that U_Φ is *not* adjointable.

iv.) Prove that the isometry U_Φ is a *intertwiner* along Φ, i.e. for all $a \in A$ we have

$$\pi_\omega(\Phi(a))U_\Phi = U_\Phi \pi_{\Phi^*\omega}(a) \tag{3.6}$$

as maps between the corresponding GNS representations of $\mathscr{A}$ and $\mathscr{B}$.

v.) Suppose now in addition that Φ is surjective. Show that in this case the intertwiner U_Φ is unitary.

vi.) If $\Psi\colon \mathscr{C} \longrightarrow \mathscr{A}$ is yet another *-homomorphism, what can we say about the relations of U_Ψ, U_Φ and $U_{\Psi\circ\Phi}$? What is $U_{\mathrm{id}_\mathscr{A}}$?

Homework 3-4: Unitary group representations

Let G be a group and let $\mathscr{A}$ be a *-algebra over $\mathbb{C}$. Moreover, assume that $\Phi\colon G \longrightarrow {}^*\text{-}\mathrm{Aut}(\mathscr{A})$ is a group morphism, i.e. G acts on $\mathscr{A}$ by *-automorphisms.

i.) Show that the properties of Φ are equivalent to a map $\Phi\colon G \ni g \mapsto \Phi_g \in {}^*\text{-}\mathrm{Aut}(\mathscr{A})$ with $\Phi_e = \mathrm{id}_\mathscr{A}$ and $\Phi_g \circ \Phi_h = \Phi_{gh}$ for all $g, h \in G$.

ii.) Suppose now that $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ is a positive functional on $\mathscr{A}$ which is G-invariant, i.e. $\Phi_g^*\omega = \omega$ for all $g \in G$. Show that the construction of $U_g = U_{\Phi_g}$ form Homework 3-3 yields a unitary representation of G on the GNS pre-Hilbert space $\mathcal{H}_\omega$.

iii.) Show that the GNS representation π_ω of $\mathscr{A}$ is G-covariant in the sense that

$$\pi_\omega(\Phi_g(a)) = U_g \pi_\omega(a) U_g^* \tag{3.7}$$

holds for all $g \in G$ and $a \in \mathscr{A}$.

Homework 4-1: Entire Functions

Consider the entire functions $\mathscr{O}(\mathbb{C}) \subseteq \mathscr{C}(\mathbb{C})$ as a subspace of all continuous functions on $\mathbb{C}$.

i.) Show that $\mathscr{O}(\mathbb{C})$ is a closed subspace of $\mathscr{C}(\mathbb{C})$ with respect to the Fréchet topology of locally uniform convergence, i.e. the locally convex topology induced by the seminorms $\mathrm{p}_{K,0} = \|\cdot\|_K$ defined by

$$\mathrm{p}_{K,0}(f) = \max_{x \in K} |f(x)| \tag{4.1}$$

for compact subsets $K \subseteq \mathbb{C}$.

Hint: Cite the needed well-known results form complex analysis without further proofs.

ii.) Write $f \in \mathscr{O}(\mathbb{C})$ as a convergent Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f}{\partial z^n}(0) z^n \tag{4.2}$$

around 0. Show that the maps $\mathrm{p}_R \colon \mathscr{O}(\mathbb{C}) \longrightarrow \mathbb{R}$ defined by

$$\mathrm{p}_R(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \left| \frac{\partial^n f}{\partial z^n}(0) \right| R^n \tag{4.3}$$

are well-defined and yield seminorms on $\mathscr{O}(\mathbb{C})$ for all $R \geq 0$.

iii.) Show that the locally convex topology defined by the seminorms (4.3) with $R \geq 0$ turns $\mathscr{O}(\mathbb{C})$ into a Fréchet space.

Hint: For the completeness you can argue as for the well-known completeness of the sequence space ℓ^1.

iv.) Prove that the polynomials $\mathbb{C}[z]$ are dense in $\mathscr{O}(\mathbb{C})$ with respect to the locally convex topology defined by the seminorms (4.3).

Hint: Show that the Taylor expansion converges in this topology.

v.) Let $K \subseteq \mathbb{C}$ be compact. Show that there is an $R \geq 0$ such that we can estimate $\mathrm{p}_{K,0}(f)$ by $\mathrm{p}_R(f)$ for all $f \in \mathscr{O}(\mathbb{C})$. Conversely, given $R > 0$ find a compact $K \subseteq \mathbb{C}$ and $c > 0$ such that

$$\mathrm{p}_R(f) \leq c\,\mathrm{p}_{K,0}(f)$$

holds for all $f \in \mathscr{O}(\mathbb{C})$. Conclude that the locally convex topology defined by the seminorms (4.3) with $R \geq 0$ coincides with the one inherited from $\mathscr{C}(\mathbb{C})$ via the restrictions of the seminorms (4.1) for compact $K \subseteq \mathbb{C}$.

vi.) Consider the derivative operator

$$\frac{\partial}{\partial z} \colon \mathscr{O}(\mathbb{C}) \ni f \mapsto \frac{\partial f}{\partial z} \in \mathscr{O}(\mathbb{C}). \tag{4.4}$$

Show that this is continuous linear map.

Hint: Choose your defining systems of seminorms wisely.

vii.) Finally, consider the pointwise multiplication

$$\mu \colon \mathscr{O}(\mathbb{C}) \times \mathscr{O}(\mathbb{C}) \ni (f, g) \mapsto \mu(f, g) = fg \in \mathscr{O}(\mathbb{C}). \tag{4.5}$$

Show that this is a continuous bilinear map.

Hint: Again, a wise choice of the defining systems of seminorms helps drastically.

Homework 4-2: Locally convex quotient

Let V be a locally convex space and $U \subseteq V$ a not necessarily closed subspace.

i.) Define [p] by

$$[\mathrm{p}]\big([v]\big) = \inf\big\{\mathrm{p}(v+u) \;\big|\; u \in U\big\}, \tag{4.6}$$

and show that this is a well-defined seminorm on $V\big/U$.

ii.) Prove that the set $[\mathcal{P}]$ of all the seminorms [p] with p a continuous seminorm on V is already saturated. The corresponding locally convex topology on $V\big/U$ is called the *locally convex quotient topology.*

iii.) Show that, if $\mathcal{Q}$ is just a defining subset of all continuous seminorms of V, then the quotient seminorms obtained from $\mathcal{Q}$ form a defining system of seminorms for the locally convex quotient topology.

Hint: Prove that $[\mathrm{p}] \leq [\mathrm{q}]$ whenever $\mathrm{p} \leq \mathrm{q}$. Why does this solve the problem?

iv.) Let W be another locally convex space and let $\Phi\colon V \longrightarrow W$ be a continuous linear function with $U \subseteq \ker\Phi$. Show that the induced map $\check{\Phi}\colon V\big/U \longrightarrow W$ is continuous.

v.) Show that $V\big/U$ is Hausdorff iff U is closed. Note that one does not need V to be Hausdorff for this conclusion.

Homework 4-3: Quotients of Fréchet spaces

Consider a Fréchet space V and a closed subspace $U \subseteq V$. The aim is to prove that the locally convex quotient $V\big/U$ is again a Fréchet space. Fix a countable defining system of seminorms p_n with $\mathrm{p}_n \leq \mathrm{p}_m$ for $n \leq m$ on V.

i.) Show that in this case also $[\mathrm{p}_n] \leq [\mathrm{p}_m]$ for $n \leq m$.

ii.) Let $\big([v_n]\big)_{n\in\mathbb{N}}$ be a Cauchy sequence in the quotient $V\big/U$. Show that there is a monotonously increasing sequence n_k with

$$[\mathrm{p}_k]\big([v_{n_k}] - [v_{n_{k+1}}]\big) < \frac{1}{2^k} \tag{4.7}$$

for all $k \in \mathbb{N}$.

iii.) Find representatives $w_k \in [v_{n_k}]$ such that we have

$$\mathrm{p}_k(w_k - w_{k+1}) < \frac{1}{2^k} \tag{4.8}$$

by recursively constructing w_{k+1} out of w_k for an arbitrary starting point w_1.

iv.) Show that the sequence $(w_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in V.

v.) Conclude that $V\big/U$ is again a Fréchet space and show that in the case where V was a Banach space, also $V\big/U$ is a Banach space.

Homework 5-1: The Fréchet algebra $\mathscr{C}^\infty(X)$

Let $X \subseteq \mathbb{R}^d$ be a non-empty open subset. For $f \in \mathscr{C}^\infty(X)$ one defines

$$\mathrm{p}_{K,\ell}(f) = \sup_{x\in K, |\alpha|\leq\ell} \left|\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)\right|, \tag{5.1}$$

where $K \subseteq X$ is a compact subset and $\ell \in \mathbb{N}_0$. Here we use multi-index notation for partial derivatives, i.e. $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$.

i.) Show that the set of all such $\mathrm{p}_{K,\ell}$ defines a (filtrating) system of seminorms specifying a first countable locally convex topology on $\mathscr{C}^\infty(X)$, the *$\mathscr{C}^\infty$-topology.*

Hint: How many compact subsets K does one actually need? Here you can quote features of open subsets of $\mathbb{R}^d$ specific to this situation.

ii.) Show that standard results from analysis imply that $\mathscr{C}^\infty(X)$ is a Fréchet space.

Hint: Quote the relevant theorems from analysis on convergence of smooth functions without further proofs.

iii.) Use the Leibniz rule for partial derivative to show that the seminorms $\mathrm{p}_{K,\ell}$ are, up to a constant factor, submultiplicative. How can one redefine the seminorms to arrive at a system of submultiplicative seminorms specifying the $\mathscr{C}^\infty$-topology? Conclude that $\mathscr{C}^\infty(X)$ is a lmc Fréchet algebra.

iv.) Let

$$D = \sum_{r=0}^{k} \sum_{|\alpha|=r} D_r^\alpha \frac{\partial^\alpha}{\partial x^\alpha} \tag{5.2}$$

be a differential operator of order $k \in \mathbb{N}_0$ with smooth coefficients $D_r^\alpha \in \mathscr{C}^\infty(X)$. Show that

$$D\colon \mathscr{C}^\infty(X) \longrightarrow \mathscr{C}^\infty(X) \tag{5.3}$$

is a continuous linear map.

Homework 5-2: The Schwartz space $\mathscr{S}(\mathbb{R}^d)$

The space of *rapidly decreasing functions* or the *Schwartz space* is defined by

$$\mathscr{S}(\mathbb{R}^d) = \left\{ f \in \mathscr{C}^\infty(\mathbb{R}^d, \mathbb{C}) \;\middle|\; \sup_{x \in \mathbb{R}^d} (1+x^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| < \infty \text{ for all } m \in \mathbb{N}_0 \text{ and all } \alpha \in \mathbb{N}_0^d \right\}. \tag{5.4}$$

Here $x^2 = \langle x, x \rangle$ is the Euclidean norm square of $x \in \mathbb{R}^d$. Moreover, we use the common multi-index notation for partial derivatives. We define

$$\mathrm{r}_{m,\ell}(f) = \sup_{x \in \mathbb{R}^d, |\alpha| \le \ell} (1+x^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| \tag{5.5}$$

for $m, \ell \in \mathbb{N}_0$ and

$$\mathrm{r}_{\alpha\beta}(f) = \sup_{x \in \mathbb{R}^d} \left| x^\alpha \frac{\partial^{|\beta|} f}{\partial x^\beta}(x) \right| \tag{5.6}$$

for $\alpha, \beta \in \mathbb{N}_0^d$ as well as

$$\mathrm{r}_{p,\beta}(f) = \sup_{x \in \mathbb{R}^d} \left| p(x) \frac{\partial^{|\beta|} f}{\partial x^\beta}(x) \right| \tag{5.7}$$

for a polynomial $p \in \mathbb{C}[x^1, \ldots, x^d]$ and $\beta \in \mathbb{N}_0^d$.

i.) Show that $\mathrm{r}_{m,\ell}$, $\mathrm{r}_{\alpha,\beta}$ as well as $\mathrm{r}_{p,\beta}$ define seminorms on $\mathscr{S}(\mathbb{R}^d)$ for all the specified values of the parameters.

ii.) Find mutual estimates between the systems $\{\mathrm{r}_{m,\ell}\}_{m,\ell}$, $\{\mathrm{r}_{\alpha,\beta}\}_{\alpha,\beta}$, and $\{\mathrm{r}_{p,\beta}\}_{p,\beta}$ to show that they induce the same locally convex topology on $\mathscr{S}(\mathbb{R}^d)$. This is the *Schwartz topology* or *$\mathscr{S}$-topology.*

iii.) Show that the $\mathscr{S}$-topology is first countable.

iv.) Show that the $\mathscr{S}$-topology is finer than the $\mathscr{C}^\infty$-topology inherited from inclusion $\mathscr{S}(\mathbb{R}^d) \subseteq \mathscr{C}^\infty(\mathbb{R}^d)$.

Hint: Here you need to show that every continuous seminorm from the $\mathscr{C}^\infty$-topology (from a defining system) can be estimated by a continuous seminorm of the $\mathscr{S}$-topology.

v.) Conclude that $\mathscr{S}$-convergent sequences are $\mathscr{C}^\infty$-convergent and $\mathscr{S}$-Cauchy sequences are $\mathscr{C}^\infty$-Cauchy sequences.

vi.) Show that $\mathscr{S}(\mathbb{R}^d)$ is a Fréchet space.

Hint: Only completeness remains to be checked: consider a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$. Conclude that $f_n \to f$ with a smooth function $f \in \mathscr{C}^\infty(\mathbb{R}^d)$ in the $\mathscr{C}^\infty$-topology. This gives the candidate for the limit. Check that $f \in \mathscr{S}(\mathbb{R}^d)$ and $f_n \to f$ in the $\mathscr{S}$-topology.

Homework 5-3: The Fréchet algebra $\mathscr{S}(\mathbb{R}^d)$

Show that the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ is a non-unital locally multiplicatively convex algebra with respect to the pointwise product of functions.

Hint: One needs to show first that $fg \in \mathscr{S}(\mathbb{R}^d)$ for $f, g \in \mathscr{S}(\mathbb{R}^d)$. Finding the relevant estimates then shows that the defining systems of seminorms of the Schwartz topology are already submultiplicative, up to a possible rescaling.

Homework 6-1: Approximating the square root and the absolute value

By recursion, define the polynomials

$$p_0(x) = 0, \quad \text{and} \quad p_{n+1}(x) = p_n(x) + \frac{1}{2}\big(x - p_n^2(x)\big). \tag{6.1}$$

i.) Show $p_n(0) = 0$ and the estimates

$$p_n(x) \geq 0, \quad \text{and} \quad 0 \leq \sqrt{x} - p_n(x) \leq \frac{2\sqrt{x}}{2 + n\sqrt{x}} \tag{6.2}$$

for $x \in [0, 1]$.

Hint: First show the coarser estimates $0 \leq p_n(x) \leq 1$ for $x \in [0, 1]$ by induction. Use this in a second induction to improve the estimates.

ii.) Conclude that $(p_n)_{n\in\mathbb{N}}$ converges uniformly to the square root function on the interval $[0, 1]$.

iii.) Let $\alpha > 0$. Construct a sequence of polynomials that converges uniformly to the square root function on $[0, \alpha]$.

iv.) Let $\alpha > 0$. Construct a sequence of polynomials that converges uniformly to the absolute value function on $[-\alpha, \alpha]$.

Of course, the uniform convergence of the above p_n can be inferred from Dini's Theorem as their convergence is easily seen to be monotonous and pointwise. The explicit estimate (6.2) is a bit more complicated.

Homework 6-2: The Stone-Weierstraß Theorem

Let X be a compact Hausdorff space. We once again consider the continuous functions $\mathscr{C}(X) = \mathscr{C}(X, \mathbb{C})$ with the usual supremum norm.

i.) For two functions $f, g \in \mathscr{C}(X, \mathbb{R})$ write $\max(f, g)$ and $\min(f, g)$ as a linear combination of $f \pm g$ and $|f \pm g|$ to show $\max(f, g), \min(f, g) \in \mathscr{C}(X, \mathbb{R})$ again.

Let now $\mathscr{A} \subseteq \mathscr{C}(X)$ be a point-separating *-subalgebra, i.e. for different $x, y \in X$ there is a function $g \in \mathscr{A}$ with $g(x) \neq g(y)$. We consider a fixed $f \in \mathscr{C}(X)$ in the sequel.

iii.) Use Homework 6-1 to conclude that for $f = \overline{f}$ and $g = \overline{g}$ both in $\mathscr{A}$ one has $\max(f, g), \min(f, g) \in \mathscr{A}^{\mathrm{cl}}$.

iv.) Let $y, z \in X$. Show that there is a function $g \in \mathscr{A}$ with $g(y) = f(y)$ as well as $g(z) = f(z)$.

Hint: Consider the function $\tilde{g}(x) = f(y)h(x) - f(z)h(x) - f(y)h(z) + f(z)h(y)$ for a suitable $h \in \mathscr{A}$.

v.) Assume now that $f = \overline{f} \in \mathscr{C}(X)$ is real-valued and $\epsilon > 0$ as well as $z \in X$ are given. Show that there is a real-valued function $h_z \in \mathscr{A}$ with $h_z(z) = f(z)$ as well as $h(x) \leq f(x) + \epsilon$ for all $x \in X$.

Hint: Part *iv.)* gives us functions $g_y \in \mathscr{C}(X)$ with $g_y(z) = f(z)$ as well as $g_y(y) = f(y)$. By continuity, they are not too different from f in a small neighbourhood of y. Use then the compactness of X and approximate the resulting functions by means of *iii.)*.

vi.) Let again $f = \overline{f} \in \mathscr{C}(X, \mathbb{R})$. Prove that for every $\epsilon > 0$ there is a real-valued $g \in \mathscr{A}$ with $\|f - g\|_\infty < \epsilon$.

Hint: Let $h_z \in \mathscr{A}$ be chosen as in *v.)* for every $z \in X$. Use continuity to show $h_z(x) > f(x) - \epsilon$ in a small neighbourhood of z. Use then again the compactness of X and *iii.)* to find a candidate for g.

vii.) Conclude the *Stone-Weierstraß Theorem*: every point-separating unital *-subalgebra is dense in $\mathscr{C}(X, C)$.

viii.) Conclude the classical *approximation Theorem of Weierstraß*: every continuous real-valued function on $[0, 1]$ is the uniform limit of polynomials $p_n \in \mathbb{R}[x]$.

ix.) Show that the Fourier modes $\{f_n(x) = \mathrm{e}^{inx}\}_{n \in \mathbb{Z}}$ span a dense subspace of $\mathscr{C}(\mathbb{S}^1)$, where we interpret f_n for $x \in [0, 2\pi]$ as continuous functions on the circle.

Homework 6-3: Stone-Weierstraß counterexamples

We show that the assumptions in the Stone-Weierstraß Theorem can not be weakened in the most naive ways.

i.) Find an example of a unital point-separating subalgebra $\mathscr{A} \subseteq \mathscr{C}(X, \mathbb{C})$, whose closure is strictly less than $\mathscr{C}(X, \mathbb{C})$.

ii.) Find an example of a unital *-subalgebra $\mathscr{A} \subseteq \mathscr{C}(X, \mathbb{C})$, whose closure is strictly less than $\mathscr{C}(X, \mathbb{C})$.

iii.) Let $K \subseteq \mathbb{R}$ be compact. Consider the polynomials without a constant term $x\mathbb{C}[x]$, restricted to K. Show that they are dense in $\mathscr{C}(K, \mathbb{C})$ iff $0 \notin K$. Which condition in the Stone-Weierstraß Theorem fails in $x\mathbb{C}[x]$?

Homework 7-1: The C^*-algebra $\mathscr{C}(X)$

Let X be a compact Hausdorff space. We consider the Banach *-algebra of complex-valued continuous functions $\mathscr{C}(X)$ on X with the supremum norm $\|\cdot\|_\infty$.

i.) Show that the supremum norm satisfies the C^*-property

$$\left\|f^* f\right\|_\infty = \|f\|_\infty^2 \tag{7.1}$$

for all $f \in \mathscr{C}(X)$.

ii.) Let $f \in \mathscr{C}(X)$. Compute $\operatorname{spec}_{\mathscr{C}(X)}(f)$.

iii.) Let $f \in \mathscr{C}(X)$ be fixed. Show that the map

$$\mathscr{C}(\operatorname{spec}_{\mathscr{C}(X)}(f)) \ni g \ \mapsto \ g \circ f \in \mathscr{C}(X) \tag{7.2}$$

is a unital *-algebra morphism. Compute $\|g \circ f\|_\infty$ to show that (7.2) is also continuous. Show that it extends the polynomial, the entire, and the general holomorphic calculus.

iv.) Compute $\operatorname{spec}_{\mathscr{C}(X)}(g \circ f)$ for $g \in \mathscr{C}(\operatorname{spec}_{\mathscr{C}(X)}(f))$.

v.) Suppose now that X is only a locally compact Hausdorff space and consider the bounded continuous functions $\mathscr{C}_b(X)$. Indicate how to modify the above proofs to show that also $\mathscr{C}_b(X)$ becomes a unital commutative C^*-algebra with respect to the pointwise operations and the supremum norm.

Homework 7-2: Vector-valued Riemann integrals

Let V be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ (or, slightly more general, a sequentially complete locally convex space). The aim of this exercise is to understand vector-valued Riemann integrals of nice functions.

Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ be a closed n-dimensional interval with non-empty open interior, i.e. $a_1 < b_1, \ldots, a_n < b_n$. Recall that a *partition* of I consists of finitely many closed intervals $\mathcal{I} = \{I_1, \ldots, I_N\}$ such that $I_i^\circ \cap I_j^\circ = \emptyset$ for $i \neq j$ and $I = I_1 \cup \cdots \cup I_N$. We write $I_j = [a_1^j, b_1^j] \times \cdots \times [a_n^j, b_n^j]$. For a given partition $\mathcal{I}$ one chooses points $\Xi = \{\xi_1, \ldots, \xi_N\}$ with $\xi_j \in I_j$. A pair $(\mathcal{I}, \Xi)$ is then called a *marked partition* and the points in Ξ are the *marked points*. The set of all marked partitions $(\mathcal{I}, \Xi)$ will be denoted by $\mathfrak{Z}(I)$. One says that a marked partition $(\mathcal{I}, \Xi)$ is *finer* than $(\mathcal{I}', \Xi')$ if every interval from $\mathcal{I}$ is contained in some interval from $\mathcal{I}'$. We write $(\mathcal{I}, \Xi) \succcurlyeq (\mathcal{I}', \Xi')$ in this case. Note that we do not require any relation between the marked points in Ξ and Ξ'.

i.) Show that the set $\mathfrak{Z}(I)$ is directed with respect to the relation $\succcurlyeq$.

The *Riemann integral* of a function $f\colon I \longrightarrow V$ is then defined as the net-limit

$$\int_I f(x)\, \mathrm{d}^n x = \lim_{(\mathcal{I},\Xi)\in\mathfrak{Z}(I)} \Sigma_{\mathcal{I},\Xi}(f), \tag{7.3}$$

whenever this limit exists where

$$\Sigma_{(\mathcal{I},\Xi)}(f) = \sum_{k=1}^{N} f(\xi_k) \operatorname{vol}(I_k) \tag{7.4}$$

is the *Riemann sum* of f for the marked partition $(\mathcal{I}, \Xi)$. Here $\operatorname{vol}(I_k) = (b_1^k - a_1^k) \cdots (b_n^k - a_n^k)$ is the usual n-dimensional volume of the k-th interval I_k of the partition $\mathcal{I}$. A function is called *Riemann integrable* if the above limit exists. The set of all Riemann integrable functions is denoted by

$$\mathscr{R}(I, V) = \big\{ f\colon I \longrightarrow V \bigm| f \text{ is Riemann integrable} \big\}. \tag{7.5}$$

ii.) Show that constant functions are Riemann integrable and compute their Riemann integral.

iii.) Let $f\colon I \longrightarrow V$ be of the form $f(x) = \chi(x)v$ for a scalar Riemann integrable function $\chi \in \mathscr{R}(I, \mathbb{K})$ and a fixed vector $v \in V$. Show that f is Riemann integrable and compute its Riemann integral.

iv.) Show that the Riemann integrable functions form a subspace of all maps $\operatorname{Map}(I, V)$ and show that the Riemann integral

$$\int_I \cdot\, \mathrm{d}^n x\colon \mathscr{R}(I, V) \longrightarrow \mathbb{K} \tag{7.6}$$

is a linear functional.

v.) Let $A\colon V \longrightarrow W$ be a continuous linear map into another Banach space. Show that $A \circ f \in \mathscr{R}(I, W)$ for every $f \in \mathscr{R}(I, V)$ and prove

$$A\bigg(\int_I f(x)\, \mathrm{d}^n x\bigg) = \int_I (A \circ f)(x)\, \mathrm{d}^n x. \tag{7.7}$$

vi.) Show that continuous functions are Riemann integrable, i.e.

$$\mathscr{C}(I, V) \subseteq \mathscr{R}(I, V). \tag{7.8}$$

Hint: The strategy is to show that the net of Riemann sums is a Cauchy net and thus convergent by the completeness of a Banach space. Fix $\epsilon > 0$ and find a corresponding $\delta > 0$ such that $\|f(x) - f(y)\| < \epsilon$ whenever

$|x - y| < \delta$. Here you need the uniform continuity of f on the *compact* interval I. Fix a partition $(\mathcal{I}_0, \Xi_0)$ with $\Delta\mathcal{I}_0 < \delta$. Then show that for every other partition $(\mathcal{I}, \Xi)$ finer than $(\mathcal{I}_0, \Xi_0)$ one has

$$\left\|\Sigma_{(\mathcal{I}_0,\Xi_0)}(f) - \Sigma_{(\mathcal{I},\Xi)}(f)\right\| < \epsilon \operatorname{vol}(I).$$

Conclude that for two refinements $(\mathcal{I}, \Xi)$ and $(\mathcal{I}', \Xi')$ of $(\mathcal{I}_0, \Xi_0)$ one then has $\|\Sigma_{(\mathcal{I},\Xi)}(f) - \Sigma_{(\mathcal{I}',\Xi')}(f)\| < 2\epsilon \operatorname{vol}(I)$. Conclude that this implies to have a Cauchy net.

vii.) Show that for every continuous functions $f \in \mathscr{C}(I, V)$ one has

$$\left\| \int_I f(x) \, \mathrm{d}^n x \right\| \leq \operatorname{vol}(I) \|f\|_\infty, \tag{7.9}$$

where $\|f\|_\infty = \sup_{x \in I} \|f(x)\|$ is the supremum norm of f. Conclude that the Riemann integral is a continuous linear map

$$\int_I \cdot \, \mathrm{d}^n x \colon \mathscr{C}(I, V) \longrightarrow V. \tag{7.10}$$

The moral of the story is that Riemann integrals with values in Banach spaces work exactly the same way as scalar Riemann integrals. A slightly closer look even shows that all the above arguments are still valid if V is a sequentially complete locally convex space. In particular, as long as one integrates nice enough functions like continuous functions, no Lebesgue theory is needed here. In fact, a measure-theoretic approach to vector-valued integrals is substantially more complicated than the scalar case.

viii.) As a special challenge one can try to enhance the above construction for sequentially complete locally complex vector spaces V instead of Banach spaces: here a locally convex vector space V is called sequentially complete if every Cauchy sequence converges. This is strictly weaker than completeness as soon as the topology is no longer first countable. Nevertheless, many interesting examples are sequentially complete without being complete.

Hint: Replace the norm $\|\cdot\|$ by a continuous seminorm of the target everywhere. Conclude that for a continuous function the Riemann sums becomes a Cauchy net, thus converging whenever V is complete. The sport is to show that sequential completeness is actually sufficient. The idea is to construct a subsequence of the above Cauchy net which is *cofinal* enough so that its convergence implies the convergence of the Cauchy net itself.

Homework 8-1: Deradicalization

Consider the category of commutative unital Banach algebras. Formulate and prove that the *deradicalization* $\mathscr{A} \rightsquigarrow \mathscr{A} \big/ \operatorname{Rad}(\mathscr{A})$ gives a functor into the subcategory of semisimple commutative unital Banach algebras.

Homework 8-2: The logarithm in a Banach algebra

Let $\mathscr{A}$ be a unital Banach algebra.

i.) Use the holomorphic calculus and show that for $\|a\| < 1$ the logarithm series

$$\log(\mathbb{1} + a) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} a^k \tag{8.1}$$

converges absolutely.

ii.) Show that exp and log are inverse to each other on certain open subsets of $\mathscr{A}$ and characterize the open subsets by explicit norm estimates.

iii.) Suppose $u \in \mathscr{A}$ is invertible such that $\operatorname{spec}_{\mathscr{A}}(u)$ does not contain a closed curve around 0, i.e. 0 can be joined by a continuous curve with infinity inside the complement of $\operatorname{spec}_{\mathscr{A}}(u)$. Show that there exists an algebra element $a \in \mathscr{A}$ with $u = \exp(a)$.

Hint: Use the holomorphic calculus.

iv.) Show that every invertible matrix $U \in \mathrm{GL}_n(\mathbb{C})$ is of the form $U = \exp(A)$ for some $A \in \mathrm{M}_n(\mathbb{C})$.

Homework 8-3: Spec is a functor

Consider a continuous unital homomorphism $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ between two commutative unital Banach algebras.

i.) Show that the pull-back $\Phi^*\varphi = \varphi \circ \Phi$ of a character $\varphi \in \mathrm{Spec}(\mathscr{B})$ is a character of $\mathscr{A}$.

ii.) Show that the pull-back induces a continuous map $\Phi^*\colon \mathrm{Spec}(\mathscr{B}) \longrightarrow \mathrm{Spec}(\mathscr{A})$.

iii.) Conclude that passing to the spectrum yields a contravariant functor from the category of commutative unital Banach algebras to the category of compact Hausdorff spaces.

Homework 8-4: The Fuglede-Putnam-Rosenblum Theorem

Let $\mathscr{A}$ be a C^*-algebra and let $a, b \in \mathscr{A}$ with a normal. Prove that $[a, b] = 0$ iff $[a^*, b] = 0$. Give simple examples in $\mathrm{M}_2(\mathbb{C})$ that the statement can fail if a is not normal.

Hint: Pass to the unitization if $\mathscr{A}$ is non-unital. Assume $[a, b] = 0$ and show that $\exp(\mathrm{i}\bar{z}a)$ commutes with b for all $z \in \mathbb{C}$. Use the holomorphic calculus to show that

$$\exp(-\mathrm{i}za^*)b\exp(\mathrm{i}za^*) = \exp(-\mathrm{i}(za^* + \bar{z}a))b\exp(\mathrm{i}(za^* + \bar{z}a)) \tag{$*$}$$

by $a^*a = aa^*$. Next show that this exponential is unitary for all $z \in \mathbb{C}$. Use this to show that the right hand side of $(*)$ is bounded by $\|b\|$. Show that the left hand side is entire and use a Liouville argument to conclude that the left hand side is just b for all $z \in \mathbb{C}$. Deduce from this $[a^*, b] = 0$.

Homework 8-5: $\mathscr{C}(\mathbb{S}^2)$ with exotic *-involution

Consider again the continuous functions $\mathscr{C}(\mathbb{S}^2)$ on the 2-sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ with respect to their usual Banach algebra structure with respect to the pointwise operations and the supremum norm $\|\cdot\|_\infty$. Define for $f \in \mathscr{C}(\mathbb{S}^2)$

$$f^*(x) = \overline{f(I(x))} \quad \text{where} \quad I(x) = -x \tag{8.2}$$

is the antipodal map $I\colon \mathbb{S}^2 \longrightarrow \mathbb{S}^2$.

i.) Show that * is a well-defined *-involution for $\mathscr{C}(\mathbb{S}^2)$.

ii.) Show that $\mathscr{C}(\mathbb{S}^2)$ becomes a Banach *-algebra $\mathscr{A}$ with respect to this *-involution.

iii.) Show that this *-involution will *not* satisfy the C^*-condition with respect to $\|\cdot\|_\infty$.

iv.) Characterize the Hermitian elements in $\mathscr{A}$ and show that there is a non-vanishing function $f = f^* \in \mathscr{A}$ with spectrum in $\mathrm{i}\mathbb{R}$.

v.) Characterize the unitary elements in $\mathscr{A}$ and show that for every $\lambda \in \mathbb{R} \setminus \{0\}$ there exists a unitary element $u \in \mathscr{A}$ with $\lambda \in \mathrm{spec}_{\mathscr{A}}(u)$.

vi.) Show that for every character φ of $\mathscr{A}$ there exists a function $f \in \mathscr{A}$ with $\varphi(f^*) \neq \overline{\varphi(f)}$. In particular, the characters are not states.

Homework 9-1: Weyl operators on Schwartz space

We consider again the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ as in Homework 5-2.

i.) Show that the position operators $Q^i\colon \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$ defined by

$$(Q^i\psi)(x) = x^i\psi(x) \tag{9.1}$$

are continuous linear maps with respect to the $\mathscr{S}$-topology for all $i = 1, \ldots, d$.

ii.) Show that the momentum operators $P_i \colon \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$ defined by

$$(P_i\psi)(x) = \frac{\hbar}{\mathrm{i}} \frac{\partial \psi}{\partial x^i}(x) \tag{9.2}$$

are continuous linear maps with respect to the $\mathscr{S}$-topology for all $i = 1, \ldots, d$.

iii.) Define the Weyl operators $W(t,s) \colon \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$ by

$$(W(t,s)\psi)(x) = \mathrm{e}^{\mathrm{i}t\cdot x}\psi(x+s) \tag{9.3}$$

for parameters $t, s \in \mathbb{R}^d$. Show that $W(t,s)$ is a continuous endomorphism of the Schwartz space for all values of the parameters $t, s \in \mathbb{R}^d$.

Hint: To keep the notation efforts manageable you can consider $d = 1$. Moreover, it suffices to consider $W(t,0)$ and $W(0,s)$ separately. Show that there is a quadratic function q with $1 + (x-s)^2 \le (1+x^2)q(\|s\|)$ for all $x, s \in \mathbb{R}^d$ and $q(0) = 1$.

iv.) Show that $W(0,0) = \mathrm{id}$ and

$$W(t,s)W(t',s') = \mathrm{e}^{\mathrm{i}t\cdot s'}W(t+t', s+s') \tag{9.4}$$

for all $t, t', s, s' \in \mathbb{R}^d$. Compare this to $W(t',s')W(t,s)$ and show that $W(t,s)$ is invertible with continuous inverse. These are the *Weyl relations*.

v.) Show that for every $\psi \in \mathscr{S}(\mathbb{R}^d)$ the map

$$\mathbb{R}^{2d} \ni (t,s) \ \mapsto \ W(t,s)\psi \in \mathscr{S}(\mathbb{R}^d) \tag{9.5}$$

is continuous.

Hint: To keep the notation efforts manageable you can consider $d = 1$ for simplicity. Why is it then sufficient to show continuity at $(t,s) = (0,0)$?

vi.) Show that

$$\lim_{\sigma \to 0} \frac{1}{\sigma}\big(W(0, \sigma e_i)\psi - \psi\big) = \frac{\partial \psi}{\partial x^i} \tag{9.6}$$

in the $\mathscr{S}$-topology for all $i = 1, \ldots, d$. Use the Weyl relations and the continuity of $W(0,s)$ to show that

$$\frac{\partial}{\partial s^i}W(0,s)\psi = \frac{\partial}{\partial x^i}W(0,s)\psi \tag{9.7}$$

holds in the $\mathscr{S}$-topology for all $i = 1, \ldots, d$ and for all $s \in \mathbb{R}^d$.

vii.) Show that

$$\lim_{\tau \to 0} \frac{1}{\tau}\big(W(\tau e_i)\psi - \psi\big) = \mathrm{i}x^i\psi \tag{9.8}$$

in the $\mathscr{S}$-topology for all $i = 1, \ldots, d$. Again, use the Weyl relations and the continuity of $W(t,0)$ to show that

$$\frac{\partial}{\partial t^i}W(t,0)\psi = \mathrm{i}x^i W(t,0)\psi \tag{9.9}$$

holds in the $\mathscr{S}$-topology for all $i = 1, \ldots, d$ and for all $t \in \mathbb{R}^d$.

viii.) Consider the standard L^2-space $\mathrm{L}^2(\mathbb{R}^d)$ with respect to the canonical Lebesgue measure. Show that

$$\mathscr{S}(\mathbb{R}^d) \subseteq \mathrm{L}^2(\mathbb{R}^d) \tag{9.10}$$

is a continuous inclusion. Show that the Weyl operators are unitary with respect to the inner product inherited from $\mathrm{L}^2(\mathbb{R}^d)$ by computing $W(t,s)^*$ explicitly.

ix.) Conclude that the derivatives (9.6) and (9.8) also hold in the sense of the L^2-topology.

The last part then shows that the Weyl operators can be extended to all of $L^2(\mathbb{R}^d)$ by continuity and the density of $\mathscr{S}(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$, which we have not yet shown. They stay unitary and still satisfy the Weyl relations.

Homework 10-1: Four unitaries

Let $\mathscr{A}$ be a unital C^*-algebra. Show that every element $a \in \mathscr{A}$ can be written as a linear combination of four unitary elements in $\mathscr{A}$.

Hint: First decompose a into real and imaginary part. By rescaling one can assume $\|\mathrm{Re}(a)\|, \|\mathrm{Im}(a)\| \leq 1$. Then consider the functions

$$f_{\pm}\colon [-1, 1] \ni x \mapsto f_{\pm}(x) = x \pm \mathrm{i}\sqrt{1 - x^2} \in \mathbb{C}. \tag{10.1}$$

Homework 10-2: Squaring breaks inequalities

Consider the complex matrix

$$A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{C}). \tag{10.2}$$

i.) Compute the spectrum of A to show that $A \geq 0$.

ii.) Find another positive matrix $B \geq 0$ with $B \leq A$ such that $B^2 \leq A^2$ does not hold.

Homework 10-3: Matrix algebra of a C^*-algebra

Let $\mathscr{A}$ be a C^*-algebra, no necessarily unital. Moreover, let $n \in \mathbb{N}$ and consider $\mathscr{A}^n$ as right $\mathscr{A}$-module via

$$\mathscr{A}^n \times \mathscr{A} \ni (x, a) \mapsto x \cdot a \in \mathscr{A}^n \tag{10.3}$$

with componentwise multiplication $(x \cdot a)_i = x_i a$ for $i = 1, \ldots, n$. Moreover, define the map

$$\langle \cdot\,, \cdot \rangle_{\mathscr{A}} \colon \mathscr{A}^n \times \mathscr{A}^n \longrightarrow \mathscr{A} \tag{10.4}$$

by

$$\langle x, y \rangle_{\mathscr{A}} = \sum_{i=1}^{n} x_i^* y_i. \tag{10.5}$$

i.) Show that $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is linear in the second argument, right $\mathscr{A}$-linear in the second argument, Hermitian in the sense that $(\langle x, y \rangle_{\mathscr{A}})^* = \langle y, x \rangle_{\mathscr{A}}$ and positive in the sense that

$$\langle x, x \rangle_{\mathscr{A}} \in \mathscr{A}^{++}. \tag{10.6}$$

Conclude that $\langle x, \cdot a, y \rangle_{\mathscr{A}} = a^* \langle x, y \rangle_{\mathscr{A}}$ for all $a \in \mathscr{A}$ and $x, y \in \mathscr{A}^n$.

ii.) Show that $\langle x, x \rangle_{\mathscr{A}} = 0$ iff $x = 0$.

Hint: This is not completely obvious but requires some usage of the order relation in $\mathscr{A}$.

iii.) Define $\|x\|_{\mathscr{A}^n} = \sqrt{\|\langle x, x \rangle_{\mathscr{A}}\|_{\mathscr{A}}}$ for $x \in \mathscr{A}^n$ and show that this is a norm turning $\mathscr{A}^n$ into a Banach space. Show that in each component it induces the original C^*-norm of $\mathscr{A}$.

iv.) Show the Cauchy-Schwarz inequality

$$\langle x, y \rangle_{\mathscr{A}} \langle y, x \rangle_{\mathscr{A}} \leq \|\langle x, x \rangle_{\mathscr{A}}\| \langle y, y \rangle_{\mathscr{A}} \tag{10.7}$$

as inequality between Hermitian (positive) elements of $\mathscr{A}$.

Hint: Show that 10.7 trivially holds for $x = 0$ and assume $x \neq 0$ from now on. Consider the positive element $\langle x \cdot a - y, x \cdot a - y \rangle_{\mathscr{A}}$, evaluate sesquilinearly and then fix $a \in \mathscr{A}$ in a clever way. Note that $a^* b a \leq \|b\| a^* a$ for any two $a, b \in \mathscr{A}$.

v.) Show that $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ is continuous with respect to the norm $\|\cdot\|_{\mathscr{A}^n}$.

Hint: Estimate $\|\langle x, y\rangle_{\mathscr{A}}\|$ by means of the Cauchy-Schwarz inequality.

A map $A\colon \mathscr{A}^n \longrightarrow \mathscr{A}^n$ is called *adjointable* iff there is a map $A^*\colon \mathscr{A}^n \longrightarrow \mathscr{A}^n$ such that

$$\langle A(x), y\rangle_{\mathscr{A}} = \langle x, A^*(y)\rangle_{\mathscr{A}} \tag{10.8}$$

holds for all $x, y \in \mathscr{A}$. The set of such maps will be denoted by $\mathfrak{B}_{\mathscr{A}}(\mathscr{A}^n)$.

vi.) Show that adjointable maps are necessarily right $\mathscr{A}$-linear, have unique adjoints and form a unital *-algebra $\mathfrak{B}_{\mathscr{A}}(\mathscr{A}^n)$.

vii.) Show that adjointable endomorphisms of $\mathscr{A}^n$ are necessarily continuous and the operator norm gives a C^*-norm for them.

Hint: The continuity follows from the closed graph theorem and the continuity of $\langle \cdot\,, \cdot \rangle_{\mathscr{A}}$ as in the proof of the Hellinger-Toeplitz theorem for Hilbert spaces. The C^*-property can be shown as for bounded operators on a Hilbert space.

viii.) Show that the adjointable endomorphisms are complete, i.e. a unital C^*-algebra.

ix.) Suppose that $\mathscr{A}$ is unital. Show that in this case $\mathfrak{B}_{\mathscr{A}}(\mathscr{A}^n)$ is isomorphic to $\mathrm{M}_n(\mathscr{A})$ as unital *-algebra, thus turning the matrix algebra $\mathrm{M}_n(\mathscr{A})$ into a C^*-algebra.

x.) Suppose that $\mathscr{A}$ is not unital. Show that nevertheless the matrix algebra $\mathrm{M}_n(\mathscr{A})$ can be viewed as a norm-closed *-subalgebra of $\mathfrak{B}_{\mathscr{A}}(\mathscr{A}^n)$, thus inheriting the structure of a C^*-algebra as well.

Homework 11-1: Multiplier algebra I

Let $\mathscr{A}$ be a C^*-algebra. The *multiplier algebra* $\mathrm{M}(\mathscr{A}) = \mathfrak{B}_{\mathscr{A}}(\mathscr{A})$ of $\mathscr{A}$ is the C^*-algebra of adjointable endomorphisms of the canonical inner product right $\mathscr{A}$-module $\mathscr{A}$ as in Homework 10-3.

i.) Show that the left multiplications $\mathrm{L}\colon \mathscr{A} \ni a \mapsto \mathrm{L}_a \in \mathrm{M}(\mathscr{A})$ provide an injective *-homomorphism. Conclude that $\mathrm{M}(\mathscr{A}) \cong \mathscr{A}$ whenever $\mathscr{A}$ has a unit.

ii.) Show that $\mathscr{A} \subseteq \mathrm{M}(\mathscr{A})$ is a closed *-ideal.

iii.) Show that the *-ideal $\mathscr{A} \subseteq \mathrm{M}(\mathscr{A})$ is *essential*, i.e. for any non-zero closed *-ideal $\mathscr{J} \subseteq \mathrm{M}(\mathscr{A})$ one has a non-trivial intersection $\mathscr{A} \cap \mathscr{J} \neq \{0\}$.

iv.) Let $\widetilde{\mathscr{A}}$ be the unitization of a non-unital C^*-algebra $\mathscr{A}$. Show that $\mathscr{A} \subseteq \widetilde{\mathscr{A}}$ is an essential *-ideal.

Homework 11-2: The weak topology of a Hilbert space

Let $\mathfrak{H}$ be a infinite-dimensional Hilbert space and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal system. We consider the weak topology of $\mathfrak{H}$, induced by the seminorms p_ϕ for $\phi \in \mathfrak{H}$ with $\mathrm{p}_\phi(\psi) = |\langle \phi, \psi\rangle|$.

i.) Show that a basis of zero neighbourhoods in the weak topology is given by

$$U_{\phi_1,\ldots,\phi_k,\epsilon} = \big\{\psi \in \mathfrak{H} \;\big|\; |\langle \phi, \psi\rangle| < \epsilon \text{ for } i = 1, \ldots, k\big\}. \tag{11.1}$$

ii.) Show that

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{k} |\langle e_n, \phi_i\rangle|\right)^2 < \infty, \tag{11.2}$$

and conclude that for every $\epsilon > 0$ there is a $n \in \mathbb{N}$ with $|\langle e_n, \phi_i\rangle| < \frac{\epsilon}{\sqrt{n}}$ for $i = 1, \ldots, k$.

iii.) Consider the set $M = \{\sqrt{n} e_n \mid n \in \mathbb{N}\}$. Show that $0 \in \mathfrak{H}$ is in the weak closure of M.

iv.) Show that $(\sqrt{n} e_n)_{n\in\mathbb{N}}$ has no weakly convergent subsequence. How does that fit to *iii.)*? Conclude that the weak topology is not first countable and hence not metrizable.

Homework 11-3: Discontinuity of the operator multiplication

Let $\mathfrak{H}$ be a infinite-dimensional Hilbert space and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal system. Denote by $\Theta_{e_n,e_n} \in \mathfrak{B}(\mathfrak{H})$ the orthogonal projection onto e_n and set $A_n = \sqrt{n}\Theta_{e_n,e_n}$.

i.) Show that $\|A_n\| = \sqrt{n}$.

ii.) Show that for every $\phi_1, \ldots, \phi_k \in \mathfrak{H}$ and every $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that $\|A\phi_i\| < \epsilon$ for all $i = 1, \ldots, k$. Conclude that $0 \in \mathfrak{B}(\mathfrak{H})$ is in the strong closure of the set $\{A_n\}_{n\in\mathbb{N}}$.

Hint: Homework 11-2, *ii.)*.

iii.) Conclude that there is a subnet $\{A_{n_i}\}_{i\in I}$ for some suitable direct set I with $\lim_{i\in I} A_{n_i} = 0$ in the strong topology.

Hint: Do not even try to construct I, this is not necessary. Note that the *sequence* $(A_n)_{n\in\mathbb{N}}$ is norm-unbounded and hence not even a strong Cauchy sequence.

iv.) Construct a vector $\phi \in \mathfrak{H}$ with $\|A_n^2\phi\| = 1$ for all $n \in \mathbb{N}$. Conclude that $\{A_{n_i}^2\}_{i\in I}$ for the above subnet is not convergent to zero in the strong topology. In particular, the operator product is not strongly continuous.

Homework 11-4: Sequential continuity of the operator multiplication

Let $\mathfrak{H}$ be a Hilbert space. Show that the operator product of $\mathfrak{B}(\mathfrak{H})$ is sequentially continuous in the σ-strong*, the σ-strong, the strong* and the strong topology, i.e. for convergent sequences $A_n \to A$ and $B_n \to B$ in any of these four topologies, also the product sequence A_nB_n converges to AB in this topology.

Homework 12-1: Multiplier algebra II

Let $\mathscr{A}$ be a C^*-algebra with multiplier algebra $\mathrm{M}(\mathscr{A}) = \mathfrak{B}_{\mathscr{A}}(\mathscr{A})$ as in Homework 11-1. Show that the multiplier algebra has the following universal property: if $(\mathscr{B}, \iota)$ is a pair of a unital C^*-algebra $\mathscr{B}$ and an injective *-homomorphism $\iota\colon \mathscr{A} \longrightarrow \mathscr{B}$ such that $\iota(\mathscr{A}) \subseteq \mathscr{B}$ is an essential *-ideal, then there exists a unique injective unital *-homomorphism $\Phi\colon \mathscr{B} \longrightarrow \mathrm{M}(\mathscr{A})$ such that $\Phi(\iota(a)) = \mathrm{L}_a$ for all $a \in \mathscr{A}$. With other words, the multiplier algebra is the largest unital C^*-algebra containing $\mathscr{A}$ as an essential *-ideal.

Hint: Show first that for $b \in \mathscr{B}$ and $a \in \mathscr{A}$ there is a unique $\Phi(b)a \in \mathscr{A}$ with $\iota(\Phi(b)a) = b\iota(a)$. Show that the resulting map $a \mapsto \Phi(b)a$ is a multiplier. This defines the candidate for Φ. To show that Φ is injective you need that $\iota(\mathscr{A}) \subseteq \mathscr{B}$ is essential: consider for $b \in \mathscr{B}$ with $\Phi(b) = 0$ the ideal generated by b and show that it intersects trivially with $\iota(\mathscr{A})$.

Homework 12-2: Zero sequences

Denote by $c \subseteq \mathrm{Map}(\mathbb{N}, \mathbb{C})$ the set of convergent complex sequences and let $c_\circ \subseteq c$ be the zero sequences.

i.) Show that c is a unital commutative C^*-algebra with respect to the usual operations on sequences and the supremum norm

$$\|a\|_\infty = \sup\{|a_n| \mid n \in \mathbb{N}\}. \tag{12.1}$$

ii.) Show that $c_\circ \subseteq c$ is a maximal *-ideal. What is the corresponding character?

iii.) Show that c is the unitization of $c_\circ$.

iv.) Determine all characters of c.

Hint: Consider the one-point compactification $K = \mathbb{N} \cup \{\infty\}$ and show that $c = \mathscr{C}(K, \mathbb{C})$.

v.) Show that the multiplier algebra $\mathrm{M}(c_\circ)$ of $c_\circ$ can be identified with ℓ^∞, the bounded sequences. Show first that ℓ^∞ is indeed a unital commutative C^*-algebra.

Hint: The inclusion $\ell^\infty \subseteq \mathrm{M}(c_\circ)$ is easy to see. Suppose $A\colon c_\circ \longrightarrow c_\circ$ is a multiplier. It is now helpful to consider the particular sequences $e_n \in c_\circ$ with $e_n(m) = \delta_{nm}$. Show that their span $c_{\circ\circ} \subseteq c_\circ$ is dense. Next, show that $A(e_n) = a_n e_n$ for all $n \in \mathbb{N}$ thus defining a sequence $(a_n)_{n\in\mathbb{N}}$. This gives the candidate of which you have to show that it is a bounded sequence. Why is it sufficient to determine $A\big|_{c_{\circ\circ}}$?

Homework 12-3: Closedness of the ordering relation

Let $\mathscr{A}$ be a C^*-algebra and let $\mathfrak{H}$ be a Hilbert space.

i.) Show that the ordering relation $\le$ of the Hermitian elements of $\mathscr{A}$ is a closed subset of $\mathscr{A} \times \mathscr{A}$ with respect to the product topology induced by the norm on $\mathscr{A}$.

ii.) Consider now the C^*-algebra $\mathfrak{B}(\mathfrak{H})$. Show that in this case the ordering relation of Hermitian operators is also closed for the σ-strong*, the σ-strong, the σ-weak, the strong*, the strong, and the weak topology.

Homework 13-1: The unitary group

Let $\mathfrak{H}$ be a Hilbert space and denote the corresponding unitary group by $\mathrm{U}(\mathfrak{H})$.

i.) Show that the unitary group $\mathrm{U}(\mathfrak{H})$ is a topological group with respect to the norm topology, i.e. the product and the inversion are continuous operations.

ii.) Show that the strong and weak operator topologies coincide on $\mathrm{U}(\mathfrak{H})$.

Hint: The strong topology is finer than the weak one in general. To show the other inclusion, first show that it suffices to show that for a seminorm $\|\cdot\|_\chi$ of the strong topology with $\chi \in \mathfrak{H}$ every ϵ-ball of $U_0 \in \mathrm{U}(\mathfrak{H})$ in $\mathrm{U}(\mathfrak{H})$ contains a suitable δ-ball of U_0 with respect to a weak seminorm $\|\cdot\|_{\phi,\psi}$ and $\phi, \psi \in \mathfrak{H}$ suitably chosen.

iii.) Show that the unitary group $\mathrm{U}(\mathfrak{H})$ is a topological group with respect to the strong topology as well, see also [2].

Homework 13-2: The dual spaces of c and $c_\circ$

Consider again the sequence spaces c and $c_\circ$ from Homework 12-2.

i.) Let $\varphi\colon c_\circ \longrightarrow \mathbb{C}$ be a continuous linear functional. Show that it is uniquely determined by its values $\varphi_n = \varphi(e_n)$ for the canonical sequences e_n from Homework 12-2.

ii.) Show that a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of complex numbers arises as $\varphi_n = \varphi(e_n)$ for a continuous linear functional $\varphi \in c_\circ'$ iff

$$\big\|(\varphi_n)_{n\in\mathbb{N}}\big\|_{\ell^1} = \sum_{n=1}^{\infty} |\varphi_n| < \infty. \tag{13.1}$$

We will identify the linear functional φ with the corresponding sequence $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ in the following.

iii.) Show that $\|\varphi\|_{\ell^1}$ coincides with the functional norm $\|\varphi\|$ of $\varphi \in c_\circ'$. Conclude that the set ℓ^1 of sequences satisfying (13.1) is a Banach space.

iv.) Show that every continuous linear functional $\varphi \in c_\circ'$ has a canonical extension to a continuous linear functional $\varphi \in c'$ with the same functional norm by setting

$$\varphi(a) = \sum_{n=1}^{\infty} \varphi_n a_n. \tag{13.2}$$

v.) Show that c has continuous linear functionals not of the form (13.2). Find an explicit description of them and of the dual space c'.

Homework 13-3: The sequence spaces ℓ^p

Let $1 \leq p < \infty$ and consider the following set

$$\ell^p = \left\{a = (a_n)_{n\in\mathbb{N}} \;\middle|\; \|a\|_{\ell^p} < \infty\right\}, \quad \text{where} \quad \|a\|_{\ell^p} = \sqrt[p]{\sum\nolimits_{n\in\mathbb{N}} |a_n|^p}, \tag{13.3}$$

called the *p-summable complex sequences.*

i.) Show that ℓ^p is a vector space and $\|\cdot\|_{\ell^p}$ is a norm on it.

ii.) Show that ℓ^p is a Banach space with respect to the norm $\|\cdot\|_{\ell^p}$.

iii.) Let $p \leq p'$. Show that one has the continuous inclusion maps

$$\ell^1 \longrightarrow \ell^p \longrightarrow \ell^{p'} \longrightarrow c_\circ. \tag{13.4}$$

Show that the inclusions are all proper for $p \neq p'$.

iv.) Consider again the evaluation functionals $\epsilon_n \colon a \mapsto a_n$ and show that they are continuous linear functionals on ℓ^p.

v.) Show that the sequences e_n from Homework 12-2, *v.)*, are also elements of ℓ^p and form an unconditional Schauder basis of each ℓ^p, i.e. the sequence

$$a = \sum_{n\in\mathbb{N}} \epsilon_n(a) e_n \tag{13.5}$$

converges unconditionally with respect to the topology of ℓ^p for all $a \in \ell^p$. Conclude that $c_{\circ\circ} \subseteq \ell^p$ is a dense subspace.

vi.) Let $\varphi \colon \ell^p \longrightarrow \mathbb{C}$ be a continuous linear functional. Show that φ is uniquely determined by the sequence $(\varphi(e_n))_{n\in\mathbb{N}}$.

vii.) Consider first $p = 1$ and show that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ defines a continuous linear functional on ℓ^1 iff it is bounded, i.e. $(\varphi_n)_{n\in\mathbb{N}} \in \ell^\infty$. Show that the functional norm of φ coincides with the supremum norm $\|(\varphi_n)_{n\in\mathbb{N}}\|_\infty$. Conclude that $(\ell^1)' = \ell^\infty$ as Banach spaces.

viii.) Consider now $p > 1$ and show that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ defines a continuous linear functional on ℓ^p iff $(\varphi_n)_{n\in\mathbb{N}} \in \ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Show that the functional norm of φ coincides with $\|(\varphi_n)_{n\in\mathbb{N}}\|_{\ell^q}$.

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[2] Schottenloher, M.: *The Unitary Group In Its Strong Topology.* Adv. Pure Math. **8**.5 (2018), 508–515. 19

[3] Waldmann, S.: *Lineare Algebra I. Die Grundlagen für Studierende der Mathematik und Physik.* Springer-Verlag, Berlin, 2. edition, 2021. 4