2. Problemset "Quantum Algebra & Dynamics"
October 24, 2025

Adjoined Units, Ideals and Factor Algebras

2.1 Adjoining a Unit

Just like in Theorem 2.1, let $\mathcal{A}$ be a C^*-algebra *without* identity and $\bar{\mathcal{A}}$ denote the set of pairs

$$\bar{\mathcal{A}} = \{(\alpha, A) : \alpha \in \mathbf{C}, A \in \mathcal{A}\} \,. \tag{1}$$

The $*$-algebra operations are again

$$\mu(\alpha, A) + \lambda(\beta, B) = (\mu\alpha + \lambda\beta, \mu A + \lambda B) \tag{2a}$$
$$(\alpha, A)(\beta, B) = (\alpha\beta, \alpha B + \beta A + AB) \tag{2b}$$
$$(\alpha, A)^* = (\bar{\alpha}, A^*) \tag{2c}$$

We can define a norm via

$$\|(\alpha, A)\|_{\bar{\mathcal{A}}} = \sup_{B\in\mathcal{A}, \|B\|=1} \|\alpha B + AB\|_{\mathcal{A}} \,. \tag{3}$$

1. Show that (3) satisfies the triangle inequality

$$\|(\alpha, A) + (\beta, B)\|_{\bar{\mathcal{A}}} \le \|(\alpha, A)\|_{\bar{\mathcal{A}}} + \|(\beta, B)\|_{\bar{\mathcal{A}}} \,. \tag{4}$$

2. Show that (3) satisfies the product inequality

$$\|(\alpha, A)(\beta, B)\|_{\bar{\mathcal{A}}} \le \|(\alpha, A)\|_{\bar{\mathcal{A}}}\|(\beta, B)\|_{\bar{\mathcal{A}}} \,. \tag{5}$$

2.2 Ideals

A subspace $\mathcal{B} \subseteq \mathcal{A}$ is called a left ideal, if $\forall A \in \mathcal{A}, B \in \mathcal{B} : AB \in \mathcal{B}$. A subspace $\mathcal{B} \subseteq \mathcal{A}$ is called a right ideal, if $\forall A \in \mathcal{A}, B \in \mathcal{B} : BA \in \mathcal{B}$. If $\mathcal{B}$ is both a left and a right ideal it is called a two sided ideal.

1. Show that every ideal is a (sub-)algebra.

2. Show that if $\mathcal{B}$ is self adjoint and a left or right ideal, it is necessarily two sided.

2.3 Factor Algebras

Let $\mathcal{I}$ be a two sided ideal of an algebra $\mathcal{A}$.

1. Show that the factor space $\mathcal{A}/\mathcal{I}$ is also an algebra, i. e. that the algebra operations are well defined for the equivalence classes

$$[A] = \{A + I : I \in \mathcal{I}\} . \tag{6}$$

2. Show that this is also true for $\mathcal{A}/\mathcal{I}$ if $\mathcal{A}$ is a
   (a) $*$-algebra and $\mathcal{I} = \mathcal{I}^*$
   (b) Banach algebras and $\mathcal{I}$ is complete.