Konforme Feldtheorie Blatt Nr. 1

Jun Wei Tan*

Julius-Maximilians-Universität Würzburg

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Problem 1 (Gaussian integral for bosonic fields). Verify the functional integral

$$\int \mathcal{D}\phi \, \exp\left(-\frac{1}{2}\phi^T M\phi + J^T\phi\right) = (2\pi)^{N/2}(\det M)^{-1/2}\exp\left(\frac{1}{2}J^T M^{-1}J\right),$$

where $\phi^T = (\phi_1, \ldots, \phi_N)$ and $J^T = (J_1, \ldots, J_N)$ are real vectors, M a symmetric $N \times N$ matrix, and the integration is carried out over all the fields ϕ_i, $i = 1, \ldots, N$,

$$\int \mathcal{D}\phi \equiv \int d\phi_1 \ldots d\phi_N.$$

Proof. Since M is symmetric, we can diagonalise it orthogonally with a matrix U such that

$$\phi^T M\phi = \phi^T U^T D U\phi$$

with D diagonal. Call $\eta = U\phi$. Then we have

$$\begin{aligned}
\int \mathcal{D}\phi \exp\left(-\frac{1}{2}\phi^T M\phi + J^T\phi\right) &= \int \mathcal{D}\phi \exp\left(-\frac{1}{2}\eta D\eta + J^T U^T\eta\right) \\
&= \int \mathcal{D}\eta \exp\left(-\frac{1}{2}\eta^T D\eta + (UJ)^T\eta\right) \\
&= \int \mathcal{D}\eta \exp\left(-\frac{1}{2}\sum D_i\eta_i^2 + \sum (UJ)_i\eta_i\right) \\
&= \prod_i \int \mathrm{d}\eta_i \exp\left(-\frac{1}{2}D_i\eta_i^2 + (UJ)_i\eta_i\right) \\
&= \prod_i \sqrt{\frac{2\pi}{D_i}} e^{\frac{(UJ)_i^2}{2D_i}} \\
&= (2\pi)^{N/2}\left(\prod_i D_i\right)^{-1/2} \exp\left(\frac{1}{2}\sum \frac{(UJ)_i^2}{D_i}\right) \\
&= (2\pi)^{N/2}(\det M)^{-1/2}\exp\left(\frac{1}{2}J^T M^{-1}J\right).
\end{aligned}$$

□

* jun-wei.tan@stud-mail.uni-wuerzburg.de

Problem 2 (Green's function of Laplacian in two dimensions). Verify

$$\partial\bar{\partial}\ln(z\bar{z}) = \pi\delta(\tau)\delta(x),$$

where $z = x + i\tau$, $\bar{z} = x - i\tau$.

Proof. We begin by expanding to get

$$\partial\bar{\partial}\ln z\bar{z} = \frac{1}{4}(\partial_\tau^2 + \partial_x^2)\ln\left(\tau^2 + x^2\right).$$

Up to some constants, this is just the Laplace's equation for the potential of a line charge at $(x, y) = (0, 0)$, at which point the properties are obvious. By direct computation, it can be seen that this vanishes for $(\tau, x) \neq (0, 0)$.

The interesting bit is to show that it is truly the dirac delta, and we do so using Gauss's Theorem:

$$\begin{aligned}\int_{D_\epsilon(0)} \ln r^2 \,\mathrm{d}A &= \int_{\partial D_\epsilon(0)} \frac{2}{r} r \,\mathrm{d}\theta \\ &= 2\pi\end{aligned}$$

whereupon normalisation yields the desired result. □

Problem 3 (Dirac Lagrangian in two dimensions). Consider the 2D Dirac Lagrangian in Minkowski space,

$$\mathcal{L}_{D,M} = \frac{1}{\pi}\bar{\Psi}_D i\partial\!\!\!/\Psi_D,$$

where $\overline{\Psi}_D = \Psi_D^\dagger\gamma^0$, $\Psi_D^\dagger = (\bar{\psi}^*, \psi^*)$, $\partial\!\!\!/ = \gamma^\mu\partial_\mu = \gamma^0\partial_0 + \gamma^1\partial_1$,

$$\Psi_D = \begin{pmatrix}\overline{\psi} \\ \psi\end{pmatrix}, \quad \gamma^0 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}, \text{ and } \gamma_5 = \gamma^0\gamma^1 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

(a) Show that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, where $g^{00} = -g^{11} = 1$ is the Minkowski metric.

(b) Obtain the equation of motion from $\mathcal{L}_{D,M}$.

(c) Verify that $\mathcal{L}_{D,M}$ is invariant under the U(1) vector symmetry $\Psi_D \to e^{i\lambda}\Psi_D$ and obtain the associated conserved current, the vector current J_V^μ.

(d) Verify that $\mathcal{L}_{D,M}$ is invariant under the U(1) axial symmetry $\Psi_D \to e^{i\lambda\gamma^5}\Psi_D$ and obtain the associated conserved current, the axial current J_A^μ.

(e) Compare the Lagrangian, as well as the results from (b)-(d), to the results we obtained in Euclidean space with complex space time coordinates $z = \tau + ix$, $\bar{z} = \tau - ix$ in class.

Proof. (a) Direct computation in mathematica.

(b) Direct variation of the action with respect to $\overline{\psi}_D$ yields

$$\delta S \propto i\partial\!\!\!/ \Psi_D = 0.$$

This is the Dirac equation. Similarly, we have

$$\begin{aligned}\delta S &\propto \int \mathrm{d}\tau\,\mathrm{d}x\,\overline{\Psi}_D i\partial\!\!\!/(\delta\Psi_D)\\ &= \int \mathrm{d}\tau\,\mathrm{d}x\,\overline{\Psi}_D i\gamma^\mu\partial_\mu(\delta\Psi_D)\\ &= \int \mathrm{d}\tau\,\mathrm{d}x\,(\partial_\mu\overline{\Psi}_D)\gamma^\mu\delta\psi_D = 0\end{aligned}$$

and hence

$$(\partial_\mu\overline{\Psi}_D)\gamma^\mu = 0.$$

(c) We have (in the notation of the lecture)

$$D\Psi_D = i\Psi_D, \qquad D\overline{\Psi}_D = -i\Psi_D.$$

Noting that $\mathcal{L}_{D,M}(\lambda) = \mathcal{L}_{D,M}$, i.e. $\mathcal{L}_{D,M}$ does not depend on λ at all, we have

$$J^\mu = \frac{\delta\mathcal{L}_{D,M}}{\delta(\partial_\mu\phi)}D\phi, \qquad J^\mu \propto \overline{\Psi}_D\gamma^\mu\Psi_D.$$

For purposes of comparison later, the components are

$$\begin{aligned}J^0 &= \overline{\psi}^*\overline{\psi} + \psi^*\psi,\\ J^1 &= \overline{\psi}^*\overline{\psi} - \psi^*\psi.\end{aligned}$$

(d) First we note that

$$\overline{\Psi}_D = \overline{\Psi}_D e^{i\lambda\gamma^5}.$$

Thus, we have

$$\mathcal{L}_{D,M}(\lambda) = \frac{1}{\pi}\overline{\Psi}_D e^{i\lambda\gamma^5} i\partial\!\!\!/ e^{i\lambda\gamma^5}\Psi_D.$$

Since

$$\gamma^5\gamma^0\gamma^5 = -\gamma^0, \qquad \gamma^5\gamma^1\gamma^0,$$

by direct expansion of the exponential we find that

$$e^{i\lambda\gamma^5}\gamma^0 e^{i\lambda\gamma^5} = \gamma^0, \qquad e^{i\lambda\gamma^5}\gamma^1 e^{i\lambda\gamma^5} = \gamma^1.$$

Thus, the Lagrangian does not depend on λ. Then, the differentials are given by

$$D\Psi_D = i\gamma^5\Psi_D, \qquad D\overline{\Psi}_D = i\Psi_D\gamma^5.$$

Then the conserved current is

$$J^\mu = \frac{1}{\pi}\overline{\Psi}_D i\gamma^\mu(i\gamma^5\Psi_D) \propto \overline{\Psi}_D\gamma^\mu\gamma^5\Psi_D.$$

(e) For the Lagrangian, we had

$$\begin{aligned}\mathcal{L}_D &= \mathcal{L}_{M1} + \mathcal{L}_{M2} \\ &= \frac{1}{2\pi}(\overline{\psi}_1\partial\overline{\psi}_1 + \psi_1\bar{\partial}\psi_1) + \frac{1}{2\pi}(\overline{\psi}_2\partial\overline{\psi}_2 + \psi_2\bar{\partial}\psi_2)\end{aligned}$$

Here, we have

$$\begin{aligned}\mathcal{L}_{D,M} &= \frac{1}{\pi}\overline{\Psi}_D i\partial\!\!\!/\Psi_D \\ &= \frac{i}{\Pi}\left[\begin{pmatrix}\overline{\psi}^* \\ \psi^*\end{pmatrix}\cdot\partial_0\begin{pmatrix}\overline{\psi} \\ \psi\end{pmatrix} + \begin{pmatrix}\overline{\psi}^* \\ \psi^*\end{pmatrix}\cdot\gamma^5\partial_1\begin{pmatrix}\overline{\psi} \\ \psi\end{pmatrix}\right] \\ &= \frac{i}{\pi}\left[\overline{\psi}^*\partial_0\overline{\psi} + \psi^*\partial_0\psi + \overline{\psi}^*\partial_1\overline{\psi} - \psi^*\partial_1\psi\right].\end{aligned}$$

Noting that

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \qquad \overline{\psi} = \frac{1}{\sqrt{2}}(\overline{\psi}_1 - i\overline{\psi}_2),$$

we can substitute this to get

$$\begin{aligned}\mathcal{L}_{D,M} = \frac{i}{2\pi}[&(\overline{\psi}_1 + i\overline{\psi}_2)\partial_0(\overline{\psi}_1 - i\overline{\psi}_2) + (\psi_1 - i\psi_2)\partial_0(\psi_1 + i\psi_2) \\ &+ (\overline{\psi}_1 + i\overline{\psi}_2)\partial_1(\overline{\psi}_1 - i\overline{\psi}_2) - (\psi_1 - i\psi_2)\partial_1(\psi_1 + i\psi_2)]\end{aligned}$$

(I have no clue how to simplify this further). $\square$

Problem 4 (Partial integration in the complex plane). Determine the coefficients a and b in the formula

$$\frac{1}{\pi}\int \mathrm{d}\tau\,\mathrm{d}x\,\left(\bar{\partial}f(z)+\partial\bar{f}(\bar{z})\right)=a\oint dz f(z)+b\oint d\bar{z}\bar{f}(\bar{z}),$$

where $f(z)$ and $\bar{f}(\bar{z})$ are independent functions, the $\mathrm{d}\tau\,\mathrm{d}x$ integration extends over the entire plane and the contour integrals are taken counter-clockwise around the entire z or $\bar{z}$ planes in the respective terms.

Proof. The constants can be determined by substituting in the functions $1/z$ and $1/\bar{z}$. We know that

$$\oint \frac{1}{z}\,\mathrm{d}z = 2\pi i.$$

On the other hand, we have

$$\int \mathrm{d}\tau\,\mathrm{d}x\,\bar{\partial}\frac{1}{z}=\int \mathrm{d}\tau\,\mathrm{d}x\,\pi\delta(\tau)\delta(x)=\pi.$$

Thus, by comparison of coefficients, we have $a=\frac{1}{2\pi i}$. A similar argument extends to yield $b=\frac{1}{2\pi i}$. □