Prof. Dr. Adriana Pálffy-Buß
Fabian Richter
Julius-Maximilians-Universität Würzburg
Institute for Theoretical Physics and Astrophysics

Problem Sheet 10
for the tutorial on July 11th, 2025
Quantum Mechanics II
Summer term 2025

Sheet handed out on July 1st, 2025; to be handed in on July 8th, 2025 until 2 pm

Exercise 10.1: Radial component of the Pauli matrix operator [2+2+2+2+2 P.]

The radial component of the Pauli matrix operator $\boldsymbol{\sigma}$ is defined as $\sigma_r := \boldsymbol{e}_r \cdot \boldsymbol{\sigma}$.

a) Show that

$$\sigma_r = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix} .$$

b) In the lecture it was stated that σ_r commutes with the total angular momentum operator $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$. Show that explicitly for the component $\hat{J}_z$[1].

c) Show that the functions $\psi_{\kappa m}$ defined in the lecture are eigenfunctions of the parity operator $\hat{P}$. What are the corresponding eigenvalues[2]?

d) Explain why $\sigma_r \psi_{\kappa m}$ and $\psi_{\kappa m}$ have opposite parity.

e) According to part b) we can write that $\sigma_r \psi_{\kappa m} = a\psi_{\kappa m} + b\psi_{-\kappa m}$ with constants $a, b \in \mathbb{C}$. With the help of c) and d) show that $a = 0$.

Exercise 10.2: The K operator [7 P.]

Consider the electromagnetic 4-potential $A_\mu = \delta_{\mu 0}\phi(r)/c$. Show that the operator introduced in the lecture

$$\hat{K} = \beta \left(\frac{2}{\hbar^2} \hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{L}} + 1 \right) \tag{1}$$

does commute with the Dirac Hamiltonian $\mathcal{H}_D = \gamma^\mu(\hat{p}_\mu + eA_\mu) - mc$.

[1] In spherical coordinates $\hat{L}_z = -i\hbar\, \partial/\partial\phi$.

[2] The parity operator P acting on a wave function $\psi(\boldsymbol{r})$ gives $\hat{P}\psi(\boldsymbol{r}) = \psi(-\boldsymbol{r})$. For the spherical harmonics it gives $\hat{P}Y_{lm} = (-1)^l Y_{lm}$.

Exercise 10.3: Helicities [4+2+2 P.]

In this task the operator $\hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{p}}$ is to be examined where the spin operator reads

$$\hat{\boldsymbol{S}} = \frac{\hbar}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} . \tag{2}$$

a) The wave function of a free relativistic particle with momentum $\boldsymbol{p} = \hbar \boldsymbol{k}$ is given by

$$\psi_{\pm 1/2} = \begin{pmatrix} \chi_{\pm 1/2} \\ \frac{c\hbar \vec{k}\vec{\sigma}}{E+mc^2} \chi_{\pm 1/2} \end{pmatrix} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \tag{3}$$

where $\chi_{\pm 1/2}$ is the eigenspinor of σ_z with eigenvalue ± 1. Show that the wave functions $\psi_{\pm 1/2}$ for $\boldsymbol{k} = k\boldsymbol{e}_z$ are eigenfunctions of the operator $\hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{p}}$. What are the corresponding eigenvalues?

b) Dividing the eigenvalues from a) by $\hbar^2 k$ gives us the so called helicities of the wave functions. What is the physical meaning of the helicity in this context?

c) Does $\hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{p}}$ commute with the free Dirac Hamiltonian?