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Problem Sheet 9
for the tutorial on July 4th, 2025
Quantum Mechanics II
Summer term 2025

Sheet handed out on June 24th, 2025; to be handed in on July 1st, 2025 until 2 pm

Exercise 9.1: Getting familiar with the Pauli spin vector [6 + 5 P.]

a) Prove the relation

$$(\boldsymbol{\sigma} \cdot \boldsymbol{A})(\boldsymbol{\sigma} \cdot \boldsymbol{B}) = \boldsymbol{A} \cdot \boldsymbol{B} + i\boldsymbol{\sigma} \cdot (\boldsymbol{A} \times \boldsymbol{B}) \tag{1}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denotes the vector of the 2×2 Pauli- spin matrices. $\boldsymbol{A} = (A_x, A_y, A_z)^\top$ and $\boldsymbol{B} = (B_x, B_y, B_z)^\top$ are arbitrary vectors.

b) Show that

$$\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}} = \frac{1}{r^2}(\boldsymbol{\sigma} \cdot \boldsymbol{r}) \left(-i\hbar r \partial_r + i\boldsymbol{\sigma} \cdot \hat{\boldsymbol{L}}\right) \tag{2}$$

where $\hat{\boldsymbol{p}} = -i\hbar\boldsymbol{\nabla}$ and $\hat{\boldsymbol{L}} = \boldsymbol{r} \times \hat{\boldsymbol{p}}$.

Exercise 9.2: Majorana representation of the Dirac equation [8 P.]

Multiplying the Dirac equation known from the lecture by $-\frac{i}{\hbar}$ we get

$$H_D\Psi = \left(\frac{\partial}{\partial t} + \vec{\alpha} \cdot \vec{\nabla} + im_0\beta\right)\Psi = 0 \tag{3}$$

with

$$\beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \vec{\sigma} = \left(\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right).$$

Thus, some of the matrices in H_D are imaginary. Show that the transformation

$$\Psi' = U\Psi \tag{4}$$

with

$$U = \frac{1}{\sqrt{2}}(\alpha_y + \beta) \tag{5}$$

results in a representation of the Dirac-equation where $H'_D = UH_DU^{-1}$ is purely real.

Exercise 9.3: **Some properties of the γ matrices** [3 + 3 P.]

a) By considering $\mu = \nu = 0$, $\mu = \nu \neq 0$ and $\mu \neq \nu$ show that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{6}$$

where $\{,\}$ denotes the anti-commutator.

b) Show that

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0. \tag{7}$$