Notes for the course in

FUNCTIONAL ANALYSIS

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CHAPTER ONE

C^* Algebras

Theorem 1.1. If we have an algebra $\mathcal{A}$, there is an algebra $\bar{\mathcal{A}}$ containing $\mathcal{A}$ as a sub algebra that has an identity. If $\mathcal{A}$ is C^*, then so is $\overline{\mathcal{A}}$.

Definition 1.2. The set of λ such that $\lambda 1 - A$ is invertible is called the resolvent set of A. Its complement in the complex plane is called the spectrum.

Definition 1.3 (Von-Neumann Series)**.** We can expand the resolvent using the geometric series

$$\frac{1}{\lambda - A} = \frac{1}{\lambda}\frac{1}{1 - A/\lambda} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{A}{\lambda}\right)^n.$$

Remark 1.4. This shows that for $\lambda > \|A\|$, the series is absolutely convergent. This implies that the spectrum is bounded.

Theorem 1.5. The spectral radius is given, for general A, by

$$\rho(A) = \inf_n \|A^n\|^{1/n} = \lim_{n\to\infty} \|A^n\|^{1/n} \leq \|A\|$$

Proof. The former follows from the root test. □

Theorem 1.6. Let $\mathcal{A}$ be a unital C^* algebra.

1. If A is normal then
$$\rho(A) = \|A\|$$
2. If A is an isometry, then
$$\rho(A) = 1$$

3. If A is unitary, then
$$\sigma_A \subseteq S^1$$
4. If A is self adjoint, its spectrum is real
5. For all polynomials P,
$$\sigma(P(A)) = P(\sigma(A))$$

Proof. 1. We use the C^* property to move the power out of the norm:
$$\begin{aligned}\|A^{2^n}\|^2 &= \|(A^*A)^{2^n}\| \\ &= \|(A^*A)^{2^{n-1}}\|^2 \\ &= \|A^*A\|^{2^n} \\ &= \|A\|^{2^{n+1}}\end{aligned}$$

2. We have
$$\begin{aligned}\|A^{2n}\|^2 &= \|(A^*)^{2n}A^{2n}\| \\ &= \|1\| = 1\end{aligned}$$

3. We have
$$\sigma(A) = \overline{\sigma(A^*)} = \overline{\sigma(A^{-1})} = \left(\overline{\sigma(A)}\right)^{-1}.$$
□

Theorem 1.7 (Uniqueness)**.** The norm of a C^* algebra is unique.

Proof. The norm is given by, for normal elements,
$$\|A\| = \rho(A).$$
For not normal elements, we have
$$\|A\| = \sqrt{\|A^*A\|} = \rho(A^*A).$$
□

CHAPTER TWO

Homomorphisms

Theorem 2.1. Homomorphisms preserve positivity

Theorem 2.2. The kernel of a homomorphism is a 2 sided ideal.

Definition 2.3 (Approximative Identity)**.** If $\mathcal{I}$ is a right sided ideal, an approximate identity is defined such that

$$\|E_\alpha I - I\| \xrightarrow{\alpha\to\infty} 0.$$

Theorem 2.4. Every right ideal possesses an approximate identity.

Proof. The idea is: We partially order the set of all finite families $\alpha = \{A_1, \ldots, A_{|\alpha|}\}$, then we construct

$$F_\alpha = \sum_{i=1}^{|\alpha|} A_\alpha A_\alpha^*.$$

Then, by adjoining a unit to $\mathcal{A}$ if necessary, we can define

$$E_\alpha = |\alpha| F_\alpha \frac{1}{1+|\alpha|F_\alpha} = 1 - \frac{1}{1+|\alpha|F_\alpha}.$$

The proof idea is showing that $\alpha \to \infty$, hence this looks like a 1. □

Theorem 2.5. Every closed two sided ideal is self adjoint and the factor algebra is a C^* algebra.

CHAPTER THREE

Quantum Mechanics

Definition 3.1 (Symplectic Vector Space)**.** A symplectic vector space is a real vector space with a nondegenerate antisymmetric bilinear map. Nondegenerate means that if

$$\theta(u, v) = 0$$

for all $u \in V$, then $v = 0$.

Definition 3.2 (Weyl System)**.** A Weyl system of a symplectic vector space is a map $V \to \mathcal{A}$ such that

$$W(0) = 1$$
$$(W(\phi))^* = W(-\phi)$$
$$W(\phi)W(\psi) = e^{-\frac{i}{2}\theta(\phi,\psi)}W(\phi + \psi)$$

Definition 3.3 (Weyl Algebra)**.** We define a particular vector space by letting $V = \mathbb{R}^2$, and defining the symplectic product as

$$\theta((\xi_1, \eta_1), (\xi_2, \eta_2)) = \eta_1\xi_2 - \xi_1\eta_2.$$

The algebra generated by $W(\xi, 0),\ W(0, \eta)$ is called the Weyl algebra.

Theorem 3.4. Let $(\mathcal{A}, W)$ be a Weyl system of a symplectic vector space (V, θ). Then

1. $W(\phi)$ is unitary for all $\phi \in V$
2. $\|W(\phi) - W(\psi)\| = 2$ for all $\phi \neq \psi \in V$.
3. $\mathcal{A}$ is not separable, unless $V = \{0\}$.
4. The family $\{W(\phi)\}_{\phi \in V}$ is linearly independent.

Proof. 1. We have

$$\begin{aligned}[W(\phi)]^* W(\phi) &= W(-\phi)W(\phi) \\ &= e^{-\frac{i}{2}\theta(-\phi,\phi)} W(0)\end{aligned}$$

Because θ is antisymmetric, it follows that $\theta(-\phi, \phi) = -\theta(\phi, \phi) = 0$.

2.

$\square$

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