Quantum field theory in the solid state, Exercise sheet 3

Corrections: Monday 19^{th} of May

Mean-field theories

1. Transverse field Ising model

In class we carried out the mean-field theory of the transverse field Ising model

$$\hat{H} = -J \sum_{\langle i,j \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z - h \sum_i \hat{\sigma}_i^x, \text{ with } J > 0 \text{ and } h \geq 0. \tag{1}$$

Go through the notes and check the calculations that lead to the Ginzburg-Landau free energy at $T = 0$:

$$f(\varphi) = \frac{1}{2} \frac{\Delta h}{h_c} \varphi^2 + \frac{1}{8} \varphi^4 + \cdots \tag{2}$$

with $\varphi = \langle \hat{\sigma}_i^z \rangle$ and $\Delta h = h - h_c$.

(a) Show that the Ising symmetry of the original model implies that the free energy satisfies:

$$f(-\varphi) = f(\varphi). \tag{3}$$

(b) Show that for the explicit form of the free energy given in the above equation that the transition is continuous. That is: the value of the order parameter φ that minimizes the free energy, φ^*, is a continuous function of the external field h.

(c) Consider a case where you add a cubic term to the free energy:

$$f(\varphi) = \frac{1}{2} \frac{\Delta h}{h_c} \varphi^2 + a\varphi^3 + \frac{1}{8} \varphi^4 + \cdots \tag{4}$$

Show that in this case the transition can become first order. That is φ^* is a discontinuous function of h.

(d) Cubic terms are not symmetry allowed! Can one obtain a first order transition without including a cubic term? Consider the following free energy:

$$f(\varphi) = \frac{1}{2} \frac{\Delta h}{h_c} \varphi^2 + u_4 \varphi^4 + u_6 \varphi^6 \cdots \tag{5}$$

with $u_6 > 0$. Show that the transition is continuous for $u_4 > 0$ and that $u_4 < 0$ enables first order transitions.

2. Mean-field theory of the Heisenberg model

We have seen in class that the Heisenberg model on a square lattice of linear length L and lattice constant a is given by:

$$\hat{H} = -\frac{J}{4} \sum_{b=\langle i,j \rangle} \left(\hat{D}_b^\dagger \hat{D}_b + \hat{D}_b \hat{D}_b^\dagger \right) \tag{6}$$

with $\hat{D}_b = \sum_{\sigma=1}^{N} \hat{c}_{i,\sigma}^\dagger \hat{c}_{j,\sigma}$ and $\sum_{\sigma=1}^{N} \hat{c}_{i,\sigma}^\dagger \hat{c}_{i,\sigma} \equiv \hat{n}_i = \frac{N}{2}$.

(a) Carry out the mean-field Ansatz

$$\left\langle \hat{D}_{b=\langle i,j \rangle} \right\rangle = \chi_b \equiv |\chi_b| e^{\frac{2\pi i}{\Phi_0} \int_i^j \boldsymbol{A} \cdot d\boldsymbol{l}} \tag{7}$$

where $\Phi_0 = \frac{h}{e}$ is the flux quanta, and $\boldsymbol{A}$ a vector potential.

Show that the mean-field Hamiltonian takes the form:

$$\hat{H}_{MF} = -\frac{J}{4} \sum_{b=\langle i,j \rangle} \left(\chi_b \hat{D}_b^\dagger + \chi_b^\dagger \hat{D}_b \right) + \lambda \sum_i \left(\hat{n}_i^\dagger - \frac{N}{2} \right) + C(\chi_b) \tag{8}$$

(b) Show that the the saddle point equations $\partial F / \partial \chi_b = \partial F / \partial \lambda = 0$, where F is the free energy, amount to the self-consistent equations:

$$\chi_b = \langle \hat{D}_b \rangle \text{ and } \frac{1}{L^d} \sum_i \langle \hat{n}_i \rangle = \frac{N}{2} \tag{9}$$

(c) Can you show that $\lambda = 0$ is required to impose the particle number constraint. Here is a hint. Consider particle-hole symmetry $\hat{T} \hat{c}_{i,\sigma}^\dagger \hat{T}^{-1} = e^{i\boldsymbol{Q} \cdot \boldsymbol{i}} \hat{c}_{i,\sigma}$ with $\boldsymbol{Q} = \frac{\pi}{a}(1,1)$

(d) Show invariance under gauge transformations: $\boldsymbol{A}(\boldsymbol{x}) \to \boldsymbol{A}(\boldsymbol{x}) + \nabla \chi(\boldsymbol{x})$. Beware of the boundary conditions!

(e) We will now consider a special solution in which $|\chi_b| = |\chi|$ (a bound independent constant) and $\boldsymbol{A} = B(y,0,0)$ with $Ba^2 = \Phi_0/2$. Show that for this Ansatz the Hamiltonian is time reversal symmetric. Can you diagonalize the Hamiltonian and determine the band structure? The result of your calculation should produce a Dirac dispersion at low energies.

This mean field approach to the Heisenberg model was pioneered by J. B. Marston and I. Affleck, Phys. Rev. B 39 (1989), 11538. The elementary excitations are particle-hole excitations of the f-fermions. This is very different from the spin-wave result, but a very good starting point to understand so called spin liquids. Finally if you wonder why we have chosen the π-flux, ($Ba^2 = \Phi_0/2$) here is the reason: Elliott H. Lieb, Flux phase of the half-filled band, Phys. Rev. Lett. 73 (1994), 2158.