Funktionalanalysis Hausaufgaben Blatt 1

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Problem 1. Let (M, d) be a metric space. Consider a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathrm{Map}(N, M)$ of Cauchy sequences in M i.e. $a_n = (a_{mn})_{m\in\mathbb{N}} \subset M$ for every $n \in \mathbb{N}$.

1. Show that the sequence $(d_k^{(mn)})_{k\in N} \subset \mathbb{R}$ defined by.

$$d_k^{(mn)} := d(a_{nk}, a_{mk})$$

   is convergent. In the following, we assume that for every $\epsilon > 0$ there is a natural number $N \in \mathbb{N}$ such that $\lim_{k\to\infty} d_k^{(nm)} < \epsilon$ for every $n, m \geq N$.

2. For a strictly monotously increasing sequence $(m_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, we define the diagonal sequence $(D_k)_{k\in\mathbb{N}}$ as follows

$$D_k := a_{km_k}.$$

   Show that there exists a diagonal Cauchy sequence $(D_k)_k$ such that $\lim_{k\to\infty} d(a_{nk}, D_k)$ converges to zero in the limit $n \to \infty$. Moreover, show that every other diagonal Cauchy sequence $(D'_k)_k$ with the same property satisfies $\lim_{k\to\infty} d(D_k, D'_k) = 0$.

3. Assume now that M is complete. Show that $(D_k)_k$ converges and compute its limit.

Proof. 1. We show that the sequence is Cauchy. Choose $N \in \mathbb{N}$ such that for all $k_1, k_2 \geq N$, we have $d(a_{nk_1}, a_{nk_2}) < \epsilon$ and $d(a_{mk_1}, a_{mk_2}) < \epsilon$. Then

$$d(a_{nk_1}, a_{nk_2}) - d(a_{mk_1}, a_{mk_2})$$

□

Problem 2. Let (M, d) be a metric space. We write $\tilde{M}$ for the set of Cauchy sequences in M.

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1. We say that two Cauchy sequences $(a_n)_n, (b_n)_n \in \tilde{M}$ are equivalent if

$$\lim_{n\to\infty} d(a_n, b_n) = 0$$

and write $(a_n)_n \sim (b_n)_n$. Show that this defines an equivalence relation on $\tilde{M}$

2. Show that there exists a metric $\hat{d}$ on the quotient space $\hat{M} := \tilde{M}/\sim$ such that $(\hat{M}, \hat{d})$ is a completion of (M, d).

3. Let (M', D') be another completion of (M, d). Show that M' is isometrically isomorphic to $\hat{M}$, i.e. there exists a bijective isometry $\phi : \hat{M} \to M'$.

4. Now, assume (M', d') to be another complete metric space and let $\Phi : M \to M'$ be a uniformly continuous map. Show that there is a unique continuous map $\phi : \hat{M} \to M'$ such that

$$\Phi = \phi \circ \iota.$$

Conclude that ϕ is even uniformly continuous.