3 Renormalisation group : Basic concepts.

Question: How can we understand the critical phenomena beyond the mean-field + Gaussian fluctuation level? As we have seen this approximation fails for dimensions lesser than $d=4$. The failure lies in the fact that fluctuations are taken into account poorly in the vicinity of T_c.

The renormalization group approach pioneered by Kadanoff (1966), Wilson and Kogut (1974) provides a tool tackle this problem.

Basic idea. One relevant length scale in the vicinity of the critical point. We will formulate a theory on this length scale. That means that we will concentrate

on long wave-length, low energy fluctuations.

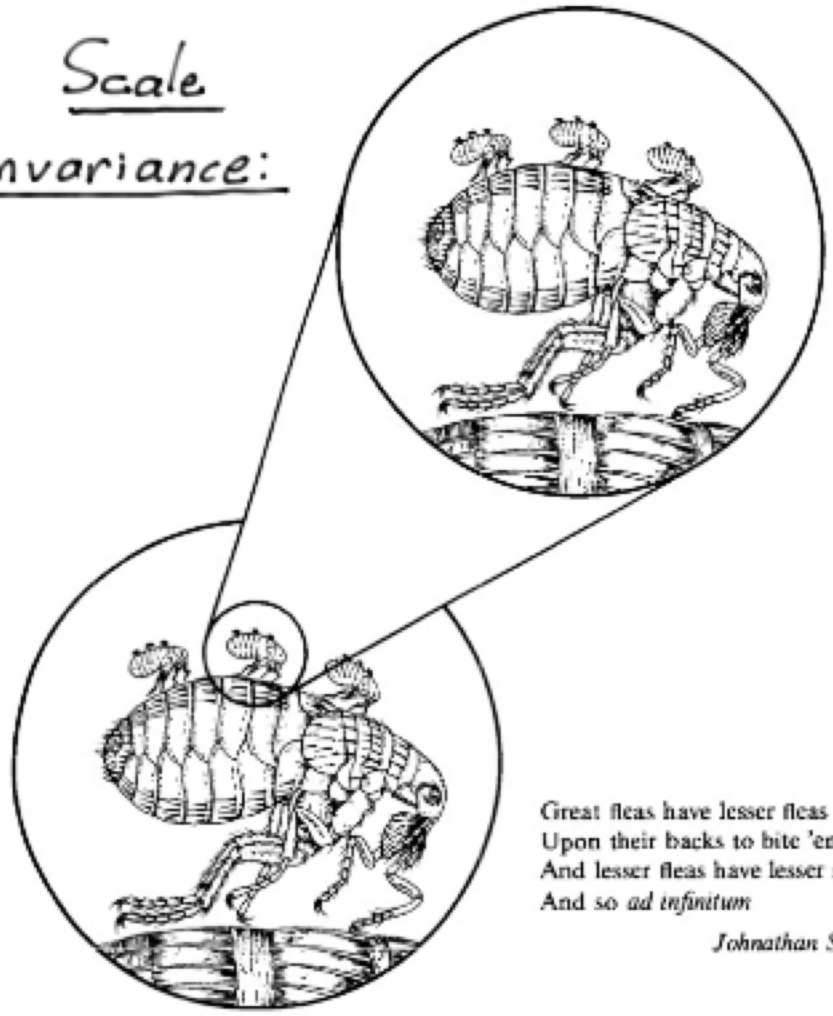
New unit length.

Under a scale transformation, $a' = ba$, the low energy effective Hamiltonian $\mathcal{H} = \beta H \to \mathcal{H}'$ thus define a flow in "model" space. At each step we enlarge the length scale at which the system is studied thus effectively moving away from the critical point.(*)

$\xi' a' = \xi a \Rightarrow \xi' ba = \xi a \Rightarrow$ Under the renormalisation group step $\boxed{\xi' = \xi / b}$

The critical points are fixpoints of this renormalisation group transformation. At a fix point $\xi^* = \infty$ or $\xi^* = 0$, and the physics is scale invariant. (We will concentrate on non-trivial fixpoints $\xi^* = \infty$)

- **Scale invariance:**



Great fleas have lesser fleas
Upon their backs to bite 'em.
And lesser fleas have lesser still.
And so *ad infinitum*

Johnathan Swift

$f(r)$ is scale invariant if under a scale transformation $r' = br$

if $f(r') = b^{\alpha} f(r)$

Examples:

a) $f(r) = r^{\alpha} \Rightarrow$

$f(r') = (br)^{\alpha} = b^{\alpha} f(r)$

$\Rightarrow$ a powerlaw is scale invariant. No matter on what length scale you probe it, it always looks the same up to a global amplitude factor.

b) $f(r) = r^{\alpha} e^{-r/\xi}$ is not scale invariant.

There are many possible implementations of the R.G.

Here we will concentrate on a real space (block spin)

Formulation.

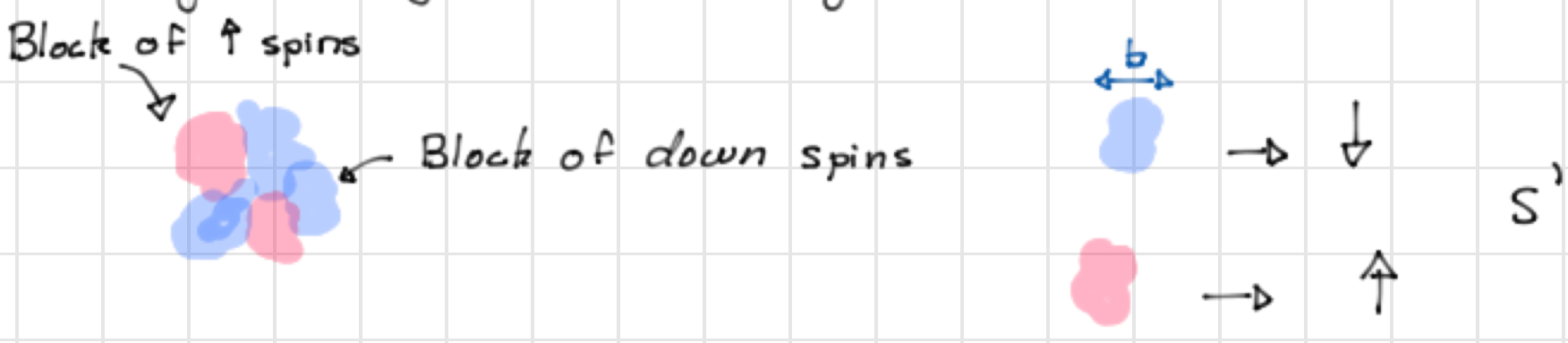
Block spin scheme

- $$Z(\vec{K}, N) = \sum_{S_1 \cdots S_N = \pm 1} \exp\left\{ \mathcal{H}(\vec{K}, \{S_i\}, N) \right\}, \quad \mathcal{H} = -\beta H$$

$$\mathcal{H}(\vec{K}, \{S_i\}, N) = N K_0 + K_1 \sum_i S_i + K_2 \sum_{\langle i,j \rangle} S_i S_j + K_3 \sum_{\langle i,j,k \rangle} S_i S_j S_k$$

$\vec{K}$: couplings.

Assume that we are in the vicinity of a critical point and that the characteristic length scale of say ferromagnetic correlations is given by ξ, then we would like to study the physics on this length scale:



Divide the system into blocks of size b^d

To each block assign a new net spin $S'_I = \pm 1$ and let $X(S_I)$ be the remaining degrees of freedom in the block.

$$\Rightarrow Z(\vec{K}, N) = \sum_{S'_1 \cdots S'_{N/b^d}} \sum_{X_1 \cdots X_{N/b^d}} \exp\left[\mathcal{H}(\vec{K}, \{S'\}, \{X\}, N) \right]$$

$= \cdots$ Trace out the high energy degrees of freedom. $\cdots =$

$$= \sum_{\{S'\}} \exp\left[\mathcal{H}(\vec{K}', \{S'\}, N b^{-d}) \right] = Z\left[\vec{K}', N b^{-d} \right]$$

This corresponds to the coarse graining step which maps one Hamiltonian onto another.

The free energy density $f \equiv \frac{1}{k_B T} \cdot \frac{1}{N} F = -\frac{1}{N} \ln Z$

$$f(\vec{K}) = -\lim_{N \to \infty} \frac{1}{N} \ln\left[Z(\vec{K}, N) \right] =$$

$$- \lim_{N\to\infty} \frac{1}{N} \ln Z(\vec{K}', N b^{-d}) = b^{-d} \left\{ - \lim_{N\to\infty} \frac{1}{N b^{-d}} \ln Z(\vec{K}', N b^{-d}) \right\}$$

$$= b^{-d} f(\vec{K}')$$

The RG transformation defines a <u>semi-group</u>. $[R_b]$

with: $\vec{K}' = \vec{R}_b(\vec{K})$

Group properties: $\vec{R}_b \circ \vec{R}_{b'} = \vec{R}_{b \cdot b'}$ However, there is no unique way to undo the R.G. step since we have thrown away high energy degrees of freedom: $\Rightarrow R_b^{-1} \nexists$

$\Rightarrow$ Semi-group.

<u>Fixpoints</u>, <u>relevant</u>, <u>irrelevant</u>, and <u>marginal operators</u>.

Let $\vec{K}_c$ satisfy: $\vec{R}(\vec{K}_c) = \vec{K}_c$

At the fixpoint the correlation length diverges since

From $\xi(K') = \xi(K)/b$ follows $\xi(K_c) = \xi(K_c)/b$

$\Rightarrow \xi(K_c) = 0$ (trivial fixpoint), $\xi(K_c) = \infty$.

What can we learn from the behaviour of the flow in the vicinity of a fixpoint?

Let $\vec{K} = \vec{K}_c + \delta\vec{K}$ $\Rightarrow$ With Taylor:

$$R_{b,\alpha}(\vec{K}) = R_{b,\alpha}(\vec{K}_c + \delta\vec{K}) \cong K_{c,\alpha} + \sum_\beta \frac{\partial R_{b,\alpha}}{\partial K_\beta}\bigg|_{K=K_c} \delta K_\beta$$

$$\Rightarrow \vec{R}_b(\vec{K}) = \vec{K}_c + M_b \delta\vec{K} \quad \text{with} \quad M_{b\alpha,\beta} = \frac{\partial R_{b,\alpha}}{\partial K_\beta}\bigg|_{K=K_c}$$

The Matrix M_b is real (since coupling constants are real) but it is generally not symmetric. We will nevertheless assume that M is symmetric for simplicity.

Since $\bar{R}_b \circ \bar{R}_{b'}(\bar{K}) = \bar{R}_{bb'}(K)$ we have:

$$\bar{R}_b \circ \bar{R}_{b'}(\bar{K}_c + \delta\bar{K}) = \bar{R}_b(\bar{K}_c + M_{b'}\delta\bar{K}) = \bar{K}_c + M_b M_{b'}\delta\vec{K}$$

$$\overset{!}{=} \vec{K}_c + M_{bb'}\delta\vec{K} \Rightarrow M_b M_{b'} = M_{bb'} \Rightarrow M_b M_{b'} = M_{bb'} = M_{b'b} = M_{b'} M_b$$

Since $[M_b, M_{b'}] = 0$ we can find a b-independent basis in which M_b is diagonal.

$\Rightarrow$ Let $M_b \vec{e}^{\,\nu} = \Lambda_b^{\nu} \vec{e}^{\,\nu}$, $\Lambda_b^{\nu} \in \mathbb{R} \Rightarrow$

$$M_b \cdot M_{b'} = M_{b \cdot b'} \quad \text{so that:} \quad \Lambda_b^{\nu} \Lambda_{b'}^{\nu} = \Lambda_{bb'}^{\nu}$$

$$\Rightarrow \boxed{\Lambda_b^{\nu} = b^{y_\nu} \quad \text{since} \quad \Lambda_{b=1}^{\nu} = 1}$$

<u>Proof</u> $\ulcorner$ $\frac{\partial}{\partial b'} \Lambda_b^{\nu} \Lambda_{b'}^{\nu}\Big|_{b'=1} = \Lambda_b^{\nu} \frac{\partial}{\partial b'} \Lambda_{b'}^{\nu}\Big|_{b'=1} = \frac{\partial}{\partial b'} \Lambda_{bb'}^{\nu}\Big|_{b'=1} \underset{(*)}{=} b \frac{\partial \Lambda_b^{\nu}}{\partial b}$

Let:

$$y_\nu = \frac{\partial \Lambda_b^{\nu}}{\partial b}\Big|_{b=1} \Rightarrow \Lambda_b^{\nu} y_\nu = b \frac{\partial \Lambda^{\nu}}{\partial b} \Rightarrow \Lambda_b^{\nu} = b^{y_\nu} \lrcorner$$

$$(*) \quad b \frac{\partial}{\partial (bb')} \Lambda_{bb'}^{\nu}\Big|_{b'=1} = b \frac{\partial}{\partial x} \Lambda_x^{\nu}\Big|_{x=b} = b \frac{\partial}{\partial b} \Lambda_b^{\nu}$$

$\ulcorner$ $b \frac{\partial \Lambda^{\nu}}{\partial b} = b \frac{\partial}{\partial b} \exp(y_\nu \ln(b)) = b\, y_\nu \frac{1}{b} \Lambda_b^{\nu} \lrcorner$ ✓

Under the linearized RG transformation:

$$\delta \vec{K}' = M \delta \vec{K} = M \sum_{\nu} a^{\nu} \vec{e}^{\nu} = \sum_{\nu} \underbrace{a^{\nu} b^{y_{\nu}}}_{\tilde{a}^{\nu}} \vec{e}^{\nu}$$

$$= \sum_{\nu} \tilde{a}^{\nu} \vec{e}^{\nu}.$$

$\Rightarrow$ • If $y_{\nu} > 0 \Rightarrow \tilde{a}^{\nu}$ grows and diverges as a function of repeated linearized RG steps. $\Rightarrow$ <u>relevant operator</u>

• If $y_{\nu} < 0 \Rightarrow \tilde{a}^{\nu}$ decreases and ultimately vanishes as a function of repeated linearized R.G. steps $\Rightarrow$ <u>irrelevant operator</u> (Universality !)

• $y_{\nu} = 0$ <u>Marginal.</u>

The origin of scaling and critical exponents.

a) One relevant variable. which we will assume to be the temperature.

$\Rightarrow$ At the fixpoint we have $T^* = R_b(T^*)$

$$\delta K' = M_b \delta K \Rightarrow (T' - T^*) = \Lambda_b (T - T^*), \quad \Lambda_b = b^{y_t}$$

($y_t > 0$. Since relevant operator).

$$\Rightarrow \quad t' \equiv \frac{T' - T^*}{T^*} = b^{y_t} t \quad \rightarrow \quad n \text{ iterations:}$$

$$\boxed{t^{(n)} = (b^{y_t})^n t}$$

To make contact with the critical exponents we consider the correlation length which under a scale transformation transforms as:

$$\xi' \equiv \xi(t') = \xi(t)/b \quad \Rightarrow \quad \text{After } n\text{-}$$

transformations: $\xi(t^{(n)}) = \xi(t^{(n-1)})/b = \xi(t)/b^n$

$$\Rightarrow \quad b^n \xi(t^{(n)}) = \xi(t) \Rightarrow \boxed{\xi(t) = b^n \xi(b^{n y_t} t)}.$$

$$\boxed{\xi(t) = b^n \xi(b^{n y_t} t)}.$$

Let $b^{n y_t} t = 1 \Rightarrow b^n = \left(\frac{1}{t}\right)^{\frac{1}{y_t}} \Rightarrow$

$\xi(t) = t^{-\frac{1}{y_t}} \xi(1) \Rightarrow \boxed{\nu = \frac{1}{y_t}} \Rightarrow$ We have

determined the correlation length exponent with the R.G.

approach !

Free energy density at criticality. $[f] = \frac{1}{\text{Volume}} \overset{?}{\sim} \frac{1}{\xi^d}$

<u>Hyperscaling.</u> We had seen that $f(t) = b^{-d} f(t')$

$\Rightarrow f(t) = b^{-nd} f(t^{(n)})$ after n-iterations. $\Rightarrow$

since $t^{(n)} = (b^{y_t})^n t$ we have: $f(t) = b^{-nd} f(b^{n y_t} t)$

Let $b^n = \left(\frac{1}{t}\right)^{\frac{1}{y_t}} \Rightarrow f(t) = t^{\frac{d}{y_t}} f(1)$

<u>Josephson:</u> $f(t) \sim \left(t^{\frac{1}{y_t}}\right)^d \sim \xi^{-d} \sim |t|^{\nu d}$

Two relevant variables

Consider the more representative case of two relevant variables.

$$\vec{K} = \begin{pmatrix} T \\ H \end{pmatrix} \begin{matrix} \leftarrow \text{Temperature} \\ \leftarrow \text{Mag.} \end{matrix} \quad \text{with} \quad f(\vec{K}) = b^{-d} f(\vec{K}')$$

Linearization around a fixpoint, $\vec{R}_b(\vec{K}^*) = \vec{K}^*$,

gives:
$$R_{b,\alpha}(\vec{K}^* + \Delta\vec{K}) = R_{b,\alpha}(\vec{K}^*) + \sum_\beta \left.\frac{\partial R_{b,\alpha}}{\partial K_\beta}\right|_{\vec{K}=\vec{K}^*} \Delta K_\beta$$

such that:

$$\Delta\vec{K}' \equiv \vec{R}_b(\vec{K}^* + \Delta\vec{K}) - \vec{R}_b(\vec{K}^*) = M\,\Delta\vec{K} \quad \text{with} \quad M_{\alpha\beta} = \left.\frac{\partial R_{b,\alpha}}{\partial K_\beta}\right|_{\vec{K}=\vec{K}^*}$$

For simplicity, let us assume that M is diagonal $\Rightarrow$

$$M = \begin{bmatrix} \Lambda_b^t & \\ & \Lambda_b^h \end{bmatrix}$$ Since we are considering the case of two relevant variables, we can write as before:

$\Lambda_b^t = b^{y_t}$, $\Lambda_b^h = b^{y_h}$ with $y_h > 0$, $y_t > 0$

<u>Correlation length</u> Consider $\delta\vec{K} = \left((T-T_c)/T_c , \left(\frac{H-H_c}{H_c}\right)\right)^T = (t, h)^T$

$\xi(\vec{R}(\delta\vec{K})) = \xi(\delta\vec{K})/b$ $\Rightarrow$ After n-iterations we have:

$$\xi(\underbrace{\vec{R} \circ \vec{R} \circ \cdots \vec{R}(\delta\vec{K})}_{n\text{-times}}) = \xi(\delta\vec{K})/b^n = \xi(b^{ny_t}t, b^{ny_h}h)^T$$

$$\Rightarrow \boxed{\xi = b^n \xi(b^{ny_t}t, b^{ny_h}h)}$$

For $h=0$ and with $\nu = \frac{1}{y_t}$ we recover the same result as before. $\xi \sim t^{-\nu}$

Note that we can also set, $t=0$ to obtain:

$\xi = b^n \xi(b^{ny_h}h)$. Let $b^{ny_h}h = 1$ $\Rightarrow$ $b^n = \left(\frac{1}{h}\right)^{\frac{1}{y_h}}$

$\Rightarrow$ $\xi \sim h^{-1/y_h}$ at $t=0$ and $h \to 0$

<u>Free energy</u> $f(\delta\vec{K}) = b^{-d} f(\vec{R}(\delta\vec{K}))$

$$f(\underbrace{\bar{R}\circ\bar{R}\circ\cdots\circ\bar{R}(\delta\vec{k})}_{n\text{-times}}) = b^{nd} f(\delta\vec{k}) = f(b^{ny_t}t,\ b^{ny_h}h)$$

$$\Rightarrow \quad f(\delta\vec{k}) = b^{-nd} f(b^{ny_t}t,\ b^{ny_h}h)$$

- If we set $h=0$ then we obtain the previous result.

- Let $b^{ny_t}t = 1 \Rightarrow b^n = \left(\frac{1}{t}\right)^{+1/y_t} = t^{-\nu}$ and

$$\boxed{f(t,h) = t^{+\nu d} f(t^{-\nu y_h} h)}$$

This is just another form of the scaling hypothesis. We can also extract the exponents since.

- $m = \left.\frac{\partial f}{\partial h}\right|_{h=0} \simeq t^{+\nu(d-y_h)} \equiv t^{\beta} \Rightarrow \underline{\underline{\beta}} = \nu d - \nu y_h = \frac{d}{y_t} - \frac{y_h}{y_t}$

- $\chi = \left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} \simeq t^{+\nu d - 2\nu y_h} \equiv t^{-\gamma} \Rightarrow$

$+\underline{\underline{\gamma}} = 2\nu y_h - \nu d = 2\frac{y_h}{y_t} - \frac{d}{y_t}$

- $\underline{\underline{C_V}} \simeq \left.\frac{\partial^2 f}{\partial t^2}\right|_{h=0} = t^{+\nu d - 2} \overset{!}{=} t^{-\alpha} \Rightarrow \alpha = 2 - \nu d = 2 - \frac{d}{y_t}$

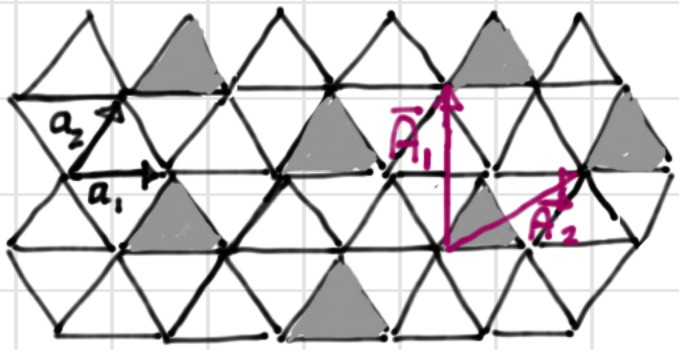
<u>Rushbrook</u>. $\alpha + 2\beta + \gamma = 2 - \cancel{\frac{d}{y_t}} + \cancel{\frac{2d}{y_t}} - \cancel{\frac{2y_h}{y_t}} + \cancel{\frac{2y_h}{y_t}} - \cancel{\frac{d}{y_t}} = 2.$

<u>etc</u>.

4 Block RG for the two dimensional Ising model.

Consider the triangular lattice

- $\vec{a}_1 = (1,0)a, \quad \vec{a}_2 = \frac{1}{2}(1, \sqrt{3})\, a$ Original spins.



$$\mathcal{H} = K \sum_{\langle i,j \rangle} s_i s_j + h \sum_i s_i$$

▲ Block spins. $\vec{A}_2 = \vec{a}_1 + \vec{a}_2 = \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) a \Rightarrow$

$$|\vec{A}_2| = |\vec{A}_1| = \sqrt{\frac{9}{4} + \frac{3}{4}}\, a = \sqrt{3}\, a \Rightarrow b = \sqrt{3}$$

Majority rule for the block spins.

Let I denote a grey triangle, each of which contains 3 spins s_1^I, s_2^I, s_3^I. The majority rule for the block spin reads:

$$S_I = \text{sign}\left(S_1^I + S_2^I + S_3^I \right).$$

The internal degrees of freedom $\{X(S_I)\}$ read

$$S_I = 1 \Rightarrow X(S_I=1) = \uparrow\uparrow\uparrow,\ \downarrow\uparrow\uparrow,\ \uparrow\downarrow\uparrow,\ \uparrow\uparrow\downarrow$$

$$S_I = -1 \Rightarrow X(S_I=-1) = \downarrow\downarrow\downarrow,\ \uparrow\downarrow\downarrow,\ \downarrow\uparrow\downarrow,\ \downarrow\downarrow\uparrow$$

This <u>coarse graining</u> in terms of block spins and internal block degrees of freedom clearly conserves the total number of states. Originally we have $2^3 = 8$ states. After coarse graining we have $\underset{\text{Block spin } S_I}{2} \times \underset{X(S_I)}{4} = 8$ states.

<u>The effective Hamiltonian @ $h=0$</u>

$$\exp\left[\mathcal{H}'(\{S_I\}) \right] = \sum_{X(S_1)\cdots X(S_{N/3})} \exp\left[\mathcal{H}(\{S_I\}, \{X(S_I)\}) \right]$$

(I)

To carry out the calculation, we split the Hamiltonian in terms of interactions within the block and interactions between blocks:

$$\mathcal{H} = \mathcal{H}_0 + V, \qquad \mathcal{H}_0 = K \sum_I \sum_{i,j \in I} S_i \cdot S_j$$

$$V = K \sum_{I \neq J} \sum_{i \in I,\, j \in J} S_i S_j$$

<u>Def</u> $$\langle A(S_i) \rangle_0 = \frac{\sum_{\{x_I\}} e^{\mathcal{H}_0[S_I, x(S_I)]} A(S_I, x(S_I))}{\sum_{\{x_I\}} e^{\mathcal{H}_0(S_I, x(S_I))}}$$

With this definition, Equation (I) becomes:

$$e^{\mathcal{H}'[S_I]} = \langle e^V \rangle_0 \sum_{\{x(S_I)\}} e^{\mathcal{H}_0([S_I], [x(S_I)])}$$

For one block, $\sum_{x(S)} e^{\mathcal{H}_0(x(S))} = {}^{\Gamma}x(S) = \{S_1, S_2, S_3\}$

with $S = \mathrm{sign}(S_1 + S_2 + S_3)\rfloor = \sum_{\substack{S_1 S_2 S_3 \\ S = \mathrm{sign}(S_1+S_2+S_3)}} \exp K\{S_1 S_2 + S_2 S_3 + S_3 S_1\}$

$$= \cdots = \boxed{3e^{-K} + e^{3K} = Z_0(K)}$$

3 states with a domain wall

One fully polarized state.

The result is independent on the sign of the block spin.

With the above result, and since we have $M = N/3$ blocks.

$$\boxed{\sum_{\{X(S_I)\}} \exp\{\mathcal{H}_0(\{S_I\}, \{X(S_I)\}) = Z_0(K)^M}$$

Thus: $e^{\mathcal{H}'(\{S_I\})} = \langle e^V \rangle_0 \, Z_0(K)^M$

<u>Perturbation theory for the RG recursion relation.</u>

$$\exp \mathcal{H}'(\{S_I\}) = \exp\left[\ln \langle e^V \rangle_0 + M \ln Z_0(K)\right]$$

$\ln \langle e^V \rangle_0$? Consider $I = \ln \langle e^{\eta V} \rangle_0$ and Taylor expand around $\eta = 0$. At the end of the calculation set $\eta = 1$. That is, we are implicitly assuming that V is small.

$$\ln \langle e^{\eta V} \rangle_0 = \ln \Big[\sum_{X(S_I)} \exp \mathcal{H}_0(S_I, X(S_I)) \; e^{\eta V} \Big]$$

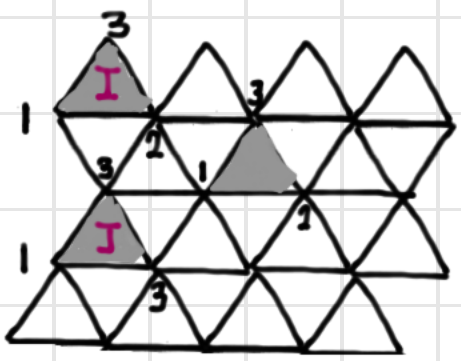
$$- \ln \Big[\sum_{X(S_I)} \exp \mathcal{H}_0(S_I, X(S_I)) \Big]$$

$$\Rightarrow \ln \langle e^{\eta V} \rangle_0 = \eta \langle V \rangle_0 + \frac{\eta^2}{2} \left[\langle V^2 \rangle_0 - \langle V \rangle_0 \langle V_0 \rangle \right] + O(\eta^3)$$

This corresponds to a cumulant expansion.

$$\Rightarrow \mathcal{H}'[S_I] = M \ln Z_0(K) + \langle V \rangle_0 + \frac{1}{2} \left[\langle V^2 \rangle_0 - \langle V \rangle_0^2 \right] + O(V^3)$$

Computing $\langle V \rangle_0$ $\quad V = \sum_{I \neq J} V_{IJ}$ with according to



the figure: $V_{IJ} = K \, S_3^J (S_1^I + S_2^I)$

Sine $\mathcal{H}_0$ does not couple blocks,

we have: $\langle V_{IJ} \rangle_0 = K \langle S_3^J \rangle_0 \left(\langle S_1^I \rangle_0 + \langle S_2^I \rangle_0 \right)$

$$\langle S_\alpha^J \rangle_0 = \frac{\displaystyle\sum_{\substack{S_1^J, S_2^J S_3^J \\ S^J = \mathrm{sign}(S_1^J + S_2^J + S_3^J)}} \exp \left\{ K \left(S_1^J S_2^J + S_2^J S_3^J + S_3^J S_1^J \right) \right] S_\alpha^J}{Z_0(K)}$$

The calculation: ⌜ Forget index J ⌟

$S = 1 \Rightarrow$ States: $X(S=1) = \uparrow\uparrow\uparrow,\ \downarrow\uparrow\uparrow,\ \uparrow\downarrow\uparrow,\ \uparrow\uparrow\downarrow$

$$\Rightarrow \langle S_1 \rangle_0 = \frac{e^{3K} - e^{-K} + 2e^{-K}}{e^{3K} + 3e^{-K}} = \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}} = \langle S_2 \rangle_0 = \langle S_3 \rangle_0$$

$S = -1 \Rightarrow X(S=-1) = \downarrow\downarrow\downarrow,\ \uparrow\downarrow\downarrow,\ \downarrow\uparrow\downarrow,\ \downarrow\downarrow\uparrow$

$$\langle S_1 \rangle_0 = \frac{-e^{3K} - 2e^{-K} + e^{-K}}{e^{3K} + 3e^{-K}} = -\frac{e^{3K} + e^{-K}}{e^{3K} + e^{-K}} = \langle S_2 \rangle_0 = \langle S_3 \rangle_0$$

$\Rightarrow$ Reintroduce index J

$$\langle S_\alpha^J \rangle = s^J \phi(K) \quad \text{with} \quad \phi(K) = \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}$$

$$\Rightarrow \langle V \rangle_0 = 2K\, \phi^2(K) \sum_{\langle IJ \rangle} S_I S_J \quad \text{so that:}$$

$$\mathcal{H}'(\{S_I\}) = M \ln Z_0(K) + K' \sum_{\langle I,J \rangle} S_I S_J + O(V^2)$$

$$K' = 2K\, \phi^2(K)$$

We have thus achieved our goal at least up to first order in V !

<u>Fixpoints and critical exponents.</u>

The fixpoints are determined by the Eq.

$$K^* = 2K^* \Phi^2(K^*).$$

Since $\Phi(0) = \frac{1}{2}$ and $\Phi(\infty) = 1$ we have 3 fixpoints.

$K^* = 0$ paramagnetic phase
$K^* = \infty$ ferromagnetic, phase

] „Trivial" phases at $T = 0$ or $T = \infty$

$\Phi(K_c) = 1/\sqrt{2}$ Scale invariant state at finite temperature !

$\Phi(K_c) = \frac{1}{\sqrt{2}} \Rightarrow$ Let $x = e^{K_c} \Rightarrow$ $\dfrac{x^3 + \frac{1}{x}}{x^3 + 3/x} = \frac{1}{\sqrt{2}}$

$$\frac{x^4+1}{x^4+3} = \frac{1}{\sqrt{2}} \Rightarrow x^4+1 = \frac{1}{\sqrt{2}}(x^4+3) \Rightarrow \left(1-\frac{1}{\sqrt{2}}\right)x^4 = \frac{3}{\sqrt{2}}-1$$

$$x^4 = \frac{\frac{3}{\sqrt{2}}-1}{1-\frac{1}{\sqrt{2}}} = \frac{(3-\sqrt{2})}{(\sqrt{2}-1)}\frac{(\sqrt{2}+1)}{(\sqrt{2}+1)} = \frac{3\sqrt{2}+3-2-\sqrt{2}}{2-1}$$

$$= (1+2\sqrt{2}) \Rightarrow K_c = \frac{1}{4}\ln(1+2\sqrt{2}) \cong 0.34$$

<u>Exact value.</u> Onsager $K_c = \frac{1}{4}\ln(3) = 0.27.$!

<u>Eigenvalues</u>

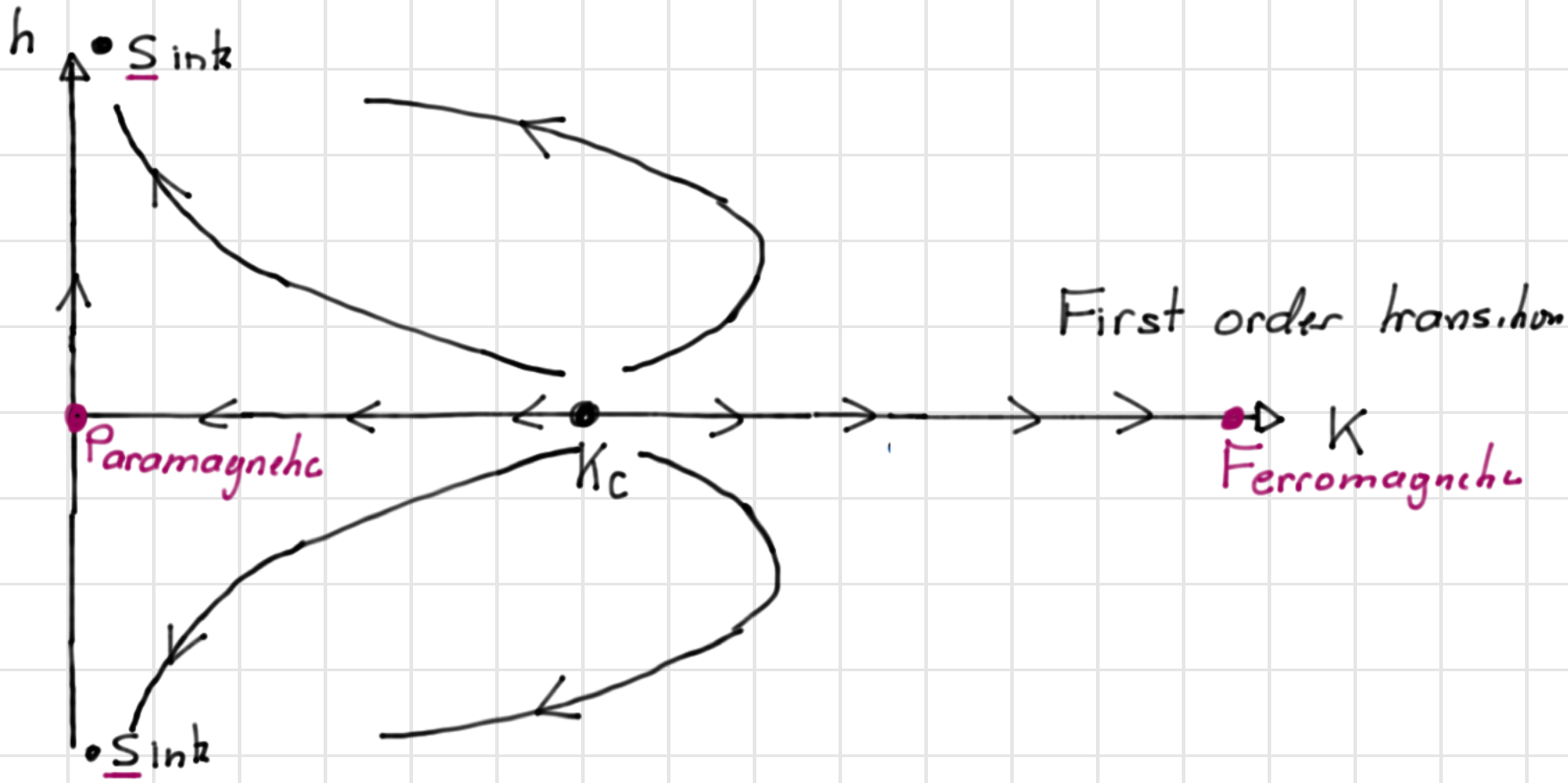
$$\Lambda_t = \left.\frac{\partial K'}{\partial K}\right|_{K=K_c} = \dots = 1.62 = b^{y_t}, \quad b = \sqrt{3}$$

$$\Rightarrow y_t = \frac{\ln(1.62)}{\ln(b)} \Rightarrow \nu = \frac{1}{y_t} = \frac{\ln(\sqrt{3})}{\ln(1.62)} \cong 1.14$$

<u>Onsager</u> $\nu = 1$!

Second order in V. $\Lambda_t = 1.77 \Rightarrow \nu \cong 0.96$

The RG Flow @ $h \neq 0$



Further reading.

a) N. Goldenfield Lectures on phase transitions and the renormalization group.

b) K. Huang Statistical mechanics, (second edition.)