Homework for the Lecture

Functional Analysis

Stefan Waldmann
Christopher Rudolph

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Homework Sheet No 12

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(25 Points. Discussion 20. 01. 2025)

Homework 12-1: Polynomial Functions and Hilbert Bases

Let $I = [a, b]$, $-\infty \leq a < b \leq \infty$, be a real interval and $\rho \in \mathscr{C}(I, \mathbb{R}^+)$ a positive continuous function. On the Borel σ-algebra of I we define a measure μ via integration over ρ, i.e.

$$\mu(U) := \int_U \rho(x)\,\mathrm{d}x \tag{12.1}$$

for every open subset U of I. In the following, we assume that there are constants $\alpha, C > 0$ such that $\rho(x)\mathrm{e}^{\alpha|x|} \leq C$ for every $x \in I$.

i.) **(1 Point)** Does such a function ρ always exist?

ii.) **(1 Point)** Show that the monomial x^n lies in $L^2(I, \mu, \mathbb{C})$ for every $n \in \mathbb{N}_0$.

iii.) **(4 Points)** For $f \in L^2(I, \mu, \mathbb{C})$, define

$$F(p) := \int_I f(x)\mathrm{e}^{\mathrm{i}px}\,\mathrm{d}\mu(x), \tag{12.2}$$

where p lies within the strip $S_\alpha := \{z \in \mathbb{C} : |\mathrm{Im} z| < \frac{\alpha}{4}\}$. Show that the map $S_\alpha \ni p \mapsto F(p)$ is well-defined and continuous.

Hint: Find a function $h \in L(I, \mu, \mathbb{R}_0^+)$ such that $|f(x)\mathrm{e}^{\mathrm{i}px}| < h(x)$ for every $(p, x) \in S_\alpha \times I$.

iv.) **(5 Points)** Show that F is holomorphic.

Hint: First, show that F is real differentiable. To this end, use a modification of the hint from the previous part. Then conclude that F satisfies the Cauchy Riemann equations.

v.) **(4 Points)** Conclude that $F \equiv 0$ if $\langle x^n, f \rangle_2 = 0$ for every $n \in \mathbb{N}_0$.

vi.) **(4 Points)** Conclude that the closure of $\operatorname{span}_{\mathbb{C}}\{x^n \in L^2(I, \mu, \mathbb{C}) : n \in \mathbb{N}\}$ coincides with $L^2(I, \mu, \mathbb{C})$.

Hint: Here, some Fourier analysis is needed. For a Schwartz function $\phi \in \mathscr{S}(\mathbb{R})$, one defines its Fourier transform as $\mathcal{F}\phi(p) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(x) \mathrm{e}^{-\mathrm{i}px} \, \mathrm{d}x$, $p \in \mathbb{R}$. One can show that the map $\mathcal{F}$ is bijective with inverse map given by $\mathcal{L}\phi(p) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(x) \mathrm{e}^{\mathrm{i}px} \, \mathrm{d}x$. It turns out that $\mathcal{F}$ (and thus also $\mathcal{L}$) becomes an isometry if one endows the Schwartz space with the L^2-norm. Since the Schwartz functions are dense in $L^2(I, \mathrm{d}x, \mathbb{C})$, the Fourier transform extends to a bijective isometry $\mathcal{F} : L^2(I, \mathrm{d}x, \mathbb{C}) \to L^2(I, \mathrm{d}x, \mathbb{C})$. You can use all these facts without proof.

vii.) **(1 Point)** Prove the following: $L^2(I, \mu, \mathbb{C})$ has a countable Hilbert basis consisting of polynomial functions.

viii.) **(3 Points)** Prove the following: There is a continuous function $f \in \mathscr{C}(I, \mathbb{R})$ and a sequence $(p_n)_{n \in \mathbb{N}_0} \subset \mathscr{C}(I, \mathbb{R})$ of polynomial functions such that the sequence $(f p_n)_{n \in \mathbb{N}_0}$ forms a Hilbert basis of $L^2(I, \mathrm{d}x, \mathbb{C})$.

Homework 12-2: Integrability and Essential Boundedness

(2 Points) Show that for every $p \in [1, \infty)$ there is a function $f \in L^p(\mathbb{R}, \mathrm{d}x, \mathbb{R})$ which is not essentially bounded.