Quantum field theory in the solid state, Exercise sheet 7

Corrections: Monday June 23$^{\text{rd}}$

Berry phase

The Berry phase is a very general concept present in the classical and quantum worlds. It refers to a phase that is acquired when a system is transported adiabatically around a closed path. For example the Aharonov-Bohm effect is a Berry phase. A Dirac cone is characterized by a Berry phase. Here we will discuss two further Berry phases for boson and spin-1/2 systems. For your convenience I have uploaded some notes for general definition of the Berry phase, in the realm of quantum mechanics.

1. Bosons

(a) In Exercise sheet 5, we considered the real time coherent state path integral for the harmonic oscillator. Carry out the Wick rotation to show that the partition reads:

$$Z = \int D\{\alpha\} e^{-\int_0^\beta d\tau \left[\left(\overline{\alpha}\frac{\partial}{\partial\tau}\alpha - \alpha\overline{\frac{\partial}{\partial\tau}\alpha}\right) + \omega\overline{\alpha}\alpha\right]} \tag{1}$$

with $\alpha(\beta) = \alpha(0)$.

(b) The first term is the Berry phase. Show that it can be written as

$$\int_0^\beta d\tau \left[\overline{\alpha}\frac{\partial}{\partial\tau}\alpha - \alpha\overline{\frac{\partial}{\partial\tau}\alpha}\right] \equiv \int_0^\beta d\tau \left(\langle\alpha(\tau)|\frac{d}{d\tau}|\alpha(\tau)\rangle - \text{H.c.}\right) = i\oint_\gamma \boldsymbol{A}\cdot d\boldsymbol{l}. \tag{2}$$

Compute the corresponding *magnetic* field $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$. Hint: express the complex variable α in terms of a vector in $\mathbb{R}^2$.

2. Coherent states for a spin 1/2.

Consider a spin 1/2 degree of freedom where the Hilbert space is spanned by the states $\{|\uparrow\rangle, |\downarrow\rangle\}$. The spin operator satisfies the algebra $\left[\hat{S}^\alpha, \hat{S}^\beta\right] = i\sum_{\gamma=1}^3 \epsilon^{\alpha\beta\gamma}\hat{S}^\gamma$ with $\hat{\boldsymbol{S}}^{\boldsymbol{2}} = s(s+1)$ with $s = 1/2$.

(a) Find normalized states, $|\boldsymbol{n}\rangle$, such that:

$$\langle\boldsymbol{n}|\hat{\boldsymbol{S}}|\boldsymbol{n}\rangle = s\boldsymbol{n}. \tag{3}$$

These states correspond to spin coherent states, and $\boldsymbol{n}$. is a vector on the unit sphere. Hint: Consider $|\boldsymbol{n}\rangle = z_1|\uparrow\rangle + z_2|\downarrow\rangle$ with $z\in\mathbb{C}$. With this Ansatz $\boldsymbol{n} = \boldsymbol{z}^\dagger\boldsymbol{\sigma}\overline{\boldsymbol{z}}$ with $\boldsymbol{z}^\dagger = (\overline{z}_1, \overline{z}_2)$.

(b) Show that a resolution of unity reads:

$$\int_{S^2}\frac{d^2\boldsymbol{n}}{2\pi}|\boldsymbol{n}\rangle\langle\boldsymbol{n}| = \hat{1}. \tag{4}$$

Here the integration runs over the unit sphere S^2.

(c) Consider the Hamiltonian

$$\hat{H} = \boldsymbol{B}(t) \cdot \hat{S} \tag{5}$$

describing the spin degree of freedom in a magnetic field. Show that the real time evolution can be expressed as:

$$\langle \boldsymbol{n}_b | \hat{U}(t,0) | \boldsymbol{n}_a \rangle = \int \prod_{n=1}^{N-1} \frac{d^2\boldsymbol{n}_n}{2\pi} e^{iS(\{\boldsymbol{n}_n\})} \tag{6}$$

with

$$S = \int_0^t dt' \left[i\langle \boldsymbol{n}(t')| \frac{d}{dt'} |\boldsymbol{n}(t')\rangle - s\boldsymbol{B}(t') \cdot \boldsymbol{n}(t') \right] \tag{7}$$

The first term corresponds to the Berry phase:

$$S_B = \int_0^t dt' i\langle \boldsymbol{n}(t')| \frac{d}{dt'} |\boldsymbol{n}(t')\rangle. \tag{8}$$

(d) Show that the Berry phase is a gauge dependent object for open paths, but has a well defined meaning for closed ones.

(e) Consider a closed path. Show that

$$S_B = \int_\Omega d\boldsymbol{\Omega} \cdot \boldsymbol{B} \quad \text{with} \quad \boldsymbol{B}(\boldsymbol{x}) = \frac{s}{r^2}\hat{n} \tag{9}$$

In the above $\boldsymbol{x} = r\boldsymbol{n}$ with $\boldsymbol{n}$ a unit vector, and Ω is the domain on the unit sphere enclosed by the path. The above is valid for a general value of s. For things to be consistent one will require that $4\pi s = 1$ thus leading to the known quantization of s.

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Compute the corresponding *magnetic* field $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$. Hint: express the complex variable α in terms of a vector in $\mathbb{R}^2$.

$$\langle\alpha_b|e^{-it\hat{H}}|\alpha_a\rangle \propto \int D\{\alpha(t)\}\, e^{i\int_{t_a}^{t_b} dt L(\alpha,\dot\alpha,t)}$$

$$L(\alpha,\dot\alpha,t) = \frac{i}{2}\left(\overline{\alpha}\dot\alpha - \alpha\dot{\overline{\alpha}}\right) - \omega\overline{\alpha}\alpha$$

$$t = -i\tau$$

$$\frac{d}{dt} = i\frac{d}{d\tau}$$

$$i\int_{t_a}^{t_b} \frac{i}{2}\left(\overline{\alpha}\dot\alpha - \alpha\dot{\overline{\alpha}}\right) - \omega\overline{\alpha}\alpha\, dt$$

$$= i\int_0^\beta -\frac{1}{2}\left(\overline{\alpha}\,\partial_\tau\alpha - \alpha\,\partial_\tau\overline{\alpha}\right) - \omega\overline{\alpha}\alpha\,(-i\,d\tau)$$

$$= -\int_0^\beta \frac{1}{2}\left(\overline{\alpha}\,\partial_\tau\alpha - \alpha\,\partial_\tau\overline{\alpha}\right) + \omega\overline{\alpha}\alpha\, d\tau$$

$$\int_0^\beta \left(\overline{\alpha}\,\partial_\tau\alpha - \alpha\,\partial_\tau\overline{\alpha}\right) d\tau =$$

$$\langle\alpha|\frac{d}{d\tau}|\alpha\rangle =$$

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$|\vec{n}\rangle = z_1|\uparrow\rangle + z_2|\downarrow\rangle$