Funktionalanalysis Notizen

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CONTENTS

I. TOPOLOGY

A. Basic Concepts

Theorem 1 (List of Useful Identities)**.**

(a)
$$(A \cup B)^{\mathrm{cl}} = A^{\mathrm{cl}} \cup B^{\mathrm{cl}}.$$

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(b)
$$(A \cup B)^\circ \supseteq A^\circ \cup B^\circ.$$

(c)
$$(A \cap B)^{\mathrm{cl}} \subseteq A^{\mathrm{cl}} \cap B^{\mathrm{cl}}.$$

(d)
$$(A \cap B)^\circ = A^\circ \cap B^\circ.$$

(e)
$$(M \setminus A)^{\mathrm{cl}} = M \setminus A^\circ.$$

(f)
$$(M \setminus A)^\circ = M \setminus A^{\mathrm{cl}}.$$

Proof.   (a) If all neighbourhoods of x intersects A, then they must certainly intersect $A \cup B$. The same thing happens if all neighbourhoods intersect B. Conversely, we suppose that x is not in A^{cl} or B^{cl}. Then we have an open neighbourhood not intersecting A, and an open neighbourhood not intersecting B. Considering the intersection of these two neighbourhoods shows that x is not in the closure of $A \cup B$ either.

(b) $A^\circ \cup B^\circ$ is an open set contained in $A \cup B$.

(c) Suppose every neighbourhood of x intersects $A \cap B$. Then every neighbourhood intersects A, and also intersects B.

(d) Clearly, $A^\circ \cap B^\circ$ is an open set contained in $A \cap B$. Conversely, it is also true that $(A \cap B)^\circ$ is an open set contained in A, and thus its interior, and it is also contained in B.

(e) Clearly, $M \setminus A^\circ$ is a closed set containing $M \setminus A$. This shows one inclusion.

Now suppose $x \in M \setminus A^\circ$. Since it is not in the interior, no open neighbourhood of x is completely contained in A; in particular, every neighbourhood must intersect $M \setminus A$. This shows the reverse inclusion.

(f) Clearly, $M \setminus A^{\mathrm{cl}}$ is an open subset of $M \setminus A$.

Conversely, suppose x is an element of $(M \setminus A)^\circ$. Then there is an open neighbourhood of x contained in $M \setminus A$, in particular not intersecting A. This shows that x is in $M \setminus A^{\mathrm{cl}}$. $\qquad\square$

Definition 2 (Preorder). A relation $\preceq$ on a set A is a preorder if the following conditions hold:

  (a) $\alpha \preceq \alpha$ for all $\alpha \in A$

  (b) If $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$.

Definition 3. A *directed set* A is a set with a preorder and the following additional condition: For all $\alpha, \beta \in A$, there is some γ with $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

Remark 4. The additional condition $\alpha \preceq \beta$ and $\beta \preceq \alpha \implies \beta = \alpha$ is known as a *partial order*. This is sometimes too restrictive, and is usually not important. The important property is the defining property of a directed set.

Definition 5. A map from a directed set I into a set X is called a *net* (indexed by I). The elements of this net are denoted by $(x_i)_{i \in I}$.

Definition 6 (Convergence). Let X now be a topological space. We say that a net $(x_i)_{i \in I}$ converges to x if for every neighbourhood U of x there is $\alpha \in I$ such that $\beta \succeq \alpha \implies x_\beta \in U$.

Theorem 7. The limit of a net is unique if the space X is Hausdorff.

Proof. The proof is analogous to that of a sequence. $\qquad\square$

It is obvious that all sequences are nets. The difference is as follows: A sequence only has countably many elements. Thus, it is possible that there will be too many open sets. This leads us to our next theorem:

Theorem 8. Let X be a topological space and $(x_i)_{i \in I}$ be a net converging to x. If X is first countable, then there is a sequence converging to x.

In general, we have the following theorem

Theorem 9. Let A be a subset of the topological space X. $x \in A^{\mathrm{cl}}$ if and only if there is a net in A converging to x.

B. Baire Spaces

Definition 10 (Nowhere Dense). A subset A of a topological space is called nowhere dense if the interior of its closure is open, $(A^{\mathrm{cl}})^{\circ} = \varnothing$.

Theorem 11. A subset A of a topological space $(M, \mathcal{M})$ is nowhere dense iff its complement contains a dense open set.

Proof. We perform the following computation:

$$
\begin{aligned}
A \text{ nowhere dense} &\iff (A^{\mathrm{cl}})^{\circ} = \varnothing \\
&\iff M \setminus (A^{\mathrm{cl}})^{\circ} = M \\
&\iff (M \setminus A^{\mathrm{cl}})^{\mathrm{cl}} = M \\
&\iff ((M \setminus A)^{\circ})^{\mathrm{cl}} = M \\
&\iff (M \setminus A)^{\circ} \text{ is dense.} \qquad \square
\end{aligned}
$$

Definition 12 (Meager). A subset is called meager if it is a countable union of nowhere dense sets.

Theorem 13. Let $(M, \mathcal{M})$ be a topological space. Then the following are equivalent:

(a) Any countable union of closed subsets of M without inner points has no inner points.

(b) Any countable intersection of open dense subsets of M is dense.

(c) Every non-empty open subset of M is not meager

(d) The complement of every meager subset of M is dense.

Proof. The proof follows (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) $\implies$ (a).

1. Let $(U_i)_{i\in\mathbb{N}}$ be a collection of dense open subsets of M. We consider their complements, which all have empty interior. Since

$$M \setminus \left(\bigcap_{i=1}^{\infty} U_i \right) = \bigcup_{i=1}^{\infty} (M \setminus U_i)$$

and the sets in the union on the right are all closed subsets without interior points, their union has no interior points, hence the countable intersection is dense.

2. Suppose we had a meager nonempty open subset U of M, that is, we have $U = \bigcup_{i=1}^{\infty} A_i$ with A_i nowhere dense sets. Then $M \setminus A_i$ is a dense subset for all i, by (b), their intersection is still dense. Then

$$\varnothing = M \setminus \left(\bigcap_{i=1}^{\infty} (M \setminus A_i^{\mathrm{cl}}) \right)^{\mathrm{cl}}$$

$$= M \setminus \left(M \setminus \bigcup_{i=1}^{\infty} A_i^{\mathrm{cl}} \right)^{\mathrm{cl}}$$

$$= M \setminus \left(M \setminus \left(\bigcup_{i=1}^{\infty} A_i^{\mathrm{cl}} \right)^{\circ} \right)$$

$$= \left(\bigcup_{i=1}^{\infty} A_i^{\mathrm{cl}} \right)^{\circ},$$

which is a contradiction, since we had U as a nonempty open subset of the final union.

3. Suppose we have a meager subset A such that $M \setminus A$ is not dense. Then $(M \setminus A)^{\mathrm{cl}} = M \setminus A^{\circ} \neq M$. Then A has nonempty interior, contradicting (c).

4. Finally, let us consider a sequence of closed sets $(A_n)_{n\in\mathbb{N}}$ without interior points, and we suppose that their union has a nonempty interior. Then

$$\left(M \setminus \bigcup_{n=1}^{\infty} A_n \right)^{\mathrm{cl}} = M \setminus \left(\bigcup_{n=1}^{\infty} A_n \right)^{\circ} \neq M,$$

contradicting (d), since $\bigcup_{n=1}^{\infty} A_n$ is meager and hence has a dense complement. $\qquad \square$

Definition 14 (Baire Spaces). A topological space X is a Baire space if one (and hence all) of the conditions of Theorem 13 hold.

Theorem 13 is clearly unwieldy, and one must first show that a space is a Baire space before even beginning to apply this theorem. Thus, we seek easier conditions with which we can verify that a space is a Baire space. These theorems are known as Baire's Theorems.

Theorem 15 (Baire I). A complete metric space (M, d) is a Baire space.

Proof. We seek to show the condition in Theorem 13(b). Let us consider a collection of dense open sets O_n and a point $x \in M$, as well as an open neighbourhood U of x. Since U intersects O_1, we can find an open ball $B_{\epsilon_1}(p_1)$ whose closure is contained in $U \cap O_1$. Inductively, we construct open balls $B_{\epsilon_n}(p_n)$ such that $B_{\epsilon_n}(p_n)^{\text{cl}} \subseteq B_{\epsilon_{n-1}}(p_{n-1})$. Then,

$$\bigcap_{i=1}^{\infty} B_{\epsilon_n}(p_n)^{\text{cl}} \subseteq U \cap \bigcap_{i=1}^{\infty} O_n$$

But since the metric space is complete, the intersection on the left side is nonempty. Thus, the intersection on the right is nonempty. Since U was an arbitrary open set, the intersection is dense. $\square$

Theorem 16 (Baire II). A locally compact Hausdorff space $(M, \mathcal{M})$ is a Baire space.

II. TOPOLOGICAL VECTOR SPACES

Definition 17. A topological vector space is a vector space with a topology such that addition and scalar multiplication are continuous.

Theorem 18. Translation $T_v : x \mapsto x + v$ and multiplication $\lambda : x \mapsto \lambda x$ with $\lambda \neq 0$ are homeomorphisms.

Proof. They are invertible with continuous inverse T_{-v} and $\frac{1}{\lambda}$ respectively $\square$

It is defined in some books that a topological vector space must be T_1 (or T_0). The usefulness of this definition comes from the fact that T_0 topological vector spaces are automatically T_2 and T_3. This is, in fact, not a statement about topological vector spaces, but about topological groups. Before we prove the result, we will need the following lemmas

Lemma 19. Let G be a group that is also a T_1 topological space. G is a topological group iff the map $G \times G \to G, (x, y) \mapsto xy^{-1}$ is continuous.

Proof. Clearly, if G is a topological group, then the map is continuous.

Conversely, we consider $y \mapsto (e, y) \mapsto y^{-1}$, which is continuous as a composition of continuous functions. Then $(x, y) \mapsto (x, y^{-1}) \mapsto xy$ is continuous, again as a composition of continuous functions. $\qquad \square$

Lemma 20. A topological space X is Hausdorff iff the diagonal $X \times X$ is closed.

Theorem 21. Let G be a T_0 topological group. Then it is T_1, T_2 and T_3.

Proof. We will prove $T_0 \implies T_1 \implies T_2$.

The goal is: We show that $\{e\}$ is closed. Pick $g \neq e$. Because G is T_0, either $G \setminus \{e\}$ is a neighbourhood of g, or $G \setminus \{g\}$ is a neighbourhood of e. Suppose the latter. Then applying the translation $T_{g^{-1}}$ shows that $G \setminus \{e\}$ is a neighbourhood of g^{-1}, and applying the inversion map shows that $G \setminus \{e\}$ is a neighbourhood of g. Thus $\{e\}$ is closed.

Then the translations show that every singleton set is closed.TODO

Note: If the group is abelian, then the proof is significantly easier: We suppose U is a neighbourhood of x not containing y. Then $x + y - U$ is a neighbourhood of y not containing x.

Now let G be T_1 $\qquad \square$

A.   Continuity

Definition 22 (Uniform Continuity). Let V, W be topological vector spaces and $\phi : V \to W$ be a map. Then ϕ is called uniformly continuous if for all neighbourhoods $U \subseteq W$ of 0 there exists a neighbourhood $Z \subseteq V$ such that $v - v' \in Z \implies \phi(v) - \phi(v') \in U$.

Theorem 23 (Equivalence of Continuity Conditions). Let V and W be topological vector spaces and let $\phi : V \to W$ be a linear map. Then the following are equivalent

  (a) The map ϕ is continuous at 0.

  (b) The map ϕ is continuous at some $v \in V$.

  (c) The map ϕ is continuous.

  (d) The map ϕ is uniformly continuous.

Proof.   1. Assume ϕ is continuous at 0.

    Recall: This means that for all neighbourhoods U of 0_W, there is a neighbourhood Z of 0_V such that $\phi(Z) \subseteq V$. Now choose $v \in V$ and a neighbourhood $U \ni \phi(v)$. Then $T_{-v}(U)$ is a neighbourhood of 0_W, and we choose a neighbourhood Z of 0 such that $\phi(Z) \subseteq T_{-v}(U)$ by continuity at 0. Then $T_v(Z) \subseteq U$ and contains v, as desired.

  2. Repeating the same proof, we can get continuity at every point in V, which is equivalent to continuity.

  3. We actually only need continuity at 0. Given an open neighbourhood Z of 0_W, we choose an open neighbourhood U of 0_V that maps into Z. Then we have, for $v - u \in Z$, that $\phi(v) - \phi(u) = \phi(v - u) \in W$.

  4. Uniform continuity directly implies continuity at 0 if we take v' to be 0 in the definition.

      $\square$

Definition 24 (Algebraic and Topological Dual). The algebraic dual of a vector space V is the well known space of linear maps from V to the field over which it is defined. We denote this by $V' = \mathrm{Hom}(V, \mathbb{K})$. The topological dual $V^* = L(V, \mathbb{K})$ is the space of all

continuous linear maps. Clearly, $L(V, \mathbb{K}) \subseteq \mathrm{Hom}(V, \mathbb{K})$.

B. Subspaces

Theorem 25. Let V be a topological vector space. If $W \subseteq V$ is a subspace, then its closure W^{cl} is also a subspace of V.

Proof. Suppose $v, w \in W^{\mathrm{cl}}$. Then there are nets $(v_i)_{i \in I}$ converging to v and $(w_i)_{i \in I}$ converging to w. Then $(v_i + w_i)_{i \in I}$ is a net converging to $v + w$, hence $v + w$ is still in W^{cl}. $\qquad \square$

III. BANACH SPACES

A. Norms & Seminorms

Definition 26 (Norm). A norm $\| \cdot \|$ on a vector space V is a function $V \to \mathbb{R}$ that satisfies

(a) (Homogenity) $\|sv\| = |s| \cdot \|v\|$ for all vectors v and scalars s

(b) (Triangle Inequality) $\|u + v\| \leq \|u\| + \|v\|$

(c) (Positivity) $\|v\| > 0$ for all $v \neq 0$.

Remark 27. A norm induces a topology through the metric defined by $d(v, w) = \|v - w\|$. If a vector space has this topology, it is known as a *normed space*.

Definition 28 (Banach Space). A Banach space is a complete normed space.

Definition 29 (Seminorm). A seminorm p is a norm without the positivity condition. Instead, we have positive semidefiniteness, i.e.

$$p(v) \geq 0 \ \forall v \in V.$$

Note that a seminorm does not form a metric space. We define the *kernel* of the seminorm as the set $\{v \in V \mid p(v) = 0\}$. Note that this is a subspace. To get an actual norm, we must "divide" by the kernel.

We can define a norm on the quotient vector space $V/\ker p$ by letting the seminorm p act on any representative of this space.

Theorem 30 (Quotient of Norms). The norm $\|\cdot\| : V/\ker p \to \mathbb{R}$, $[v] \mapsto p(v)$ is a norm.

Proof. First, we show that it is well defined. Consider $v \in V$ and let $u \in \ker p$ be arbitrary. Then we have

$$p(v + u) \leq p(v) + p(u) = p(v)$$

by the triangle inequality, and conversely

$$p(v) = p(v - u + u) \leq p(v + u) + p(u) = p(v + u)$$

which shows that the norm is independent of the choice of representative.

The triangle inequality and homogenity follow from the same properties of the seminorm. It is also true by definition that the norm is positive. $\qquad\square$

Thus, given a seminorm on a vector space, we can construct a new vector space that has a norm, and thus an induced topology.

Theorem 31 (Operator Norm). Let V, W be normed spaces and let $A : V \to W$ be a linear map. Then the following statements are equivalent:

(a) A is continuous.

(b) There exists a constant $c \geq 0$ such that

$$\|A(v)\| \leq c\|v\|$$

for all $v \in V$.

Proof. We use the fact that continuity is equivalent to continuity at 0. Then, we simply unravel the definitions. A is continuous at 0 if for every neighbourhood $Z \subseteq W$ of 0_W we have a neighbourhood $U \subseteq V$ of 0_V such that $A(U) \subseteq V$.

Equivalently, we can consider open balls in place of Z and U. This is then equivalent to the second condition. $\qquad\square$

Definition 32 (Operator Norm)**.** The operator norm $\|A\|$ is defined as

$$\|A\| = \inf\{c \mid \|Av\| \le c\|v\| \ \forall v \in V\}$$

Corollary 33 (Equivalent Statements)**.**

(a)
$$\|A\| = \sup_{V \ni v \ne 0} \frac{\|Av\|}{\|v\|}$$

(b)
$$\|A\| = \sup_{\|v\|=1} \|Av\|$$

Theorem 34 (Operator Composition)**.** Let V, W, Z be normed spaces, $A \in L(V, W)$, $B \in L(W, Z)$. Then
$$\|B \circ A\| \le \|A\|\|B\|.$$

Proof. For $v \in V$, we have

$$\|BAv\| \le \|B\|\|Av\| \le \|B\|\|A\|\|v\|. \qquad\square$$

Note that the above definition shows once again the continuity of composition.

B. Bases

Definition 35 (Hamel Basis)**.** A Hamel basis is a set $B \subseteq V$ such that for all $v \in V$, we have

$$v = \sum_{k=1}^{n} a_k e_k$$

with $a_i \in \mathbb{K}$ and $e_i \in B$ for all i, and

$$\sum_{k=1}^{n} a_k e_k = 0 \implies a_i = 0 \forall i.$$

The existence of Hamel bases is equivalent to the axiom of choice, and can be proven by Zorn's Lemma. The proof is as follows: We construct minimal spanning sets and maximally linearly independent sets by Zorn's Lemma (partial order by inclusion), and show that they are the same. These sets are Hamel bases.

A Hamel basis has all the beloved properties of a basis from finite dimensional linear algebra. For example, a linear map is uniquely defined through its action on the Hamel basis:

$$Av = \sum_{k=1}^{n} a_k A e_k.$$

However, a Hamel bases are usually difficult to come by, as we see with the following theorem:

Theorem 36. A Banach space with a countable Hamel basis is finite.

Proof. Suppose we have a countable Hamel basis $(e_n)_{n \in \mathbb{N}}$ of a Banach space V. Denote $M_n = \text{span}(\{e_1, \ldots, e_n\})$. This is a closed proper subspace of V. Thus it has empty interior. However, because by definition the e_ns form a Hamel basis, we have $V = \bigcup_{k=1}^{\infty} M_k$, contradicting Baire's category theorem. $\square$

Definition 37 (Schauder Basis)**.** A Schauder basis is a countable set $\{e_n\} \subseteq V$ such that all vectors $v \in V$ can be uniquely expressed as a sum

$$v = \sum_{k=1}^{\infty} a_k e_k,$$

where the convergence is understood to be in the topology of the vector space.

The importance of a Schauder basis is that it is countable, and we are still able to define *some* linear maps by their action on the basis. In particular, for any continuous linear map A, we have

$$Av = \sum_{k=1}^{\infty} a_k A e_k.$$

Also important to note is that this basis must be *ordered*, since the sum does not necessarily converge unconditionally.

C. Quotients of Banach Spaces

Theorem 38 (Quotient (Semi-)Norm)**.** Let V be a vector space and $U \subseteq V$ a subspace.

(a) For every seminorm p on V,

$$[p]([v]) = \inf\{p(v + u)|u \in U\}$$

defines a seminorm on V/U.

(b) If p is a norm then $[p]$ is a norm on V/U iff U is closed.

Definition 39 (Quotient (Semi-)Norm)**.**

Theorem 40 (Quotient Norm)**.**

D. Various Examples

IV. THE MAIN THEOREMS

Definition 41. Let V be a vector space over $\mathbb{K}$. A functional p is called *sublinear*, if

(a) (Homogenity) $p(rx) = r(px)$ for all real $r \geq 0$ and $v \in V$.

(b) (Subadditivity) $p(u + v) \leq p(u) + p(v)$ for all $u, v \in V$

We can define a partial order on the sublinear functions by defining p to be less than q if p is less than q pointwise. Zorn's Lemma then yields minimal elements.

Theorem 42. A sublinear function is minimal if and only if it is a real linear functional.