## Quantum field theory in the solid state, Exercise sheet 8 $$\rm Corrections{:}\ June\ 30^{th}$}$

## Grassmann numbers and path integral for non-interacting fermions

1. Normal ordering. Consider a product of fermion creation and annihilation operators,

$$\hat{A} = \hat{c}_{\alpha_1}^{\#_1} \cdots \hat{c}_{\alpha_N}^{\#_N}. \tag{1}$$

Here  $\#=\dagger$ , such that  $\hat{c}_{\alpha_1}^{\#}=\hat{c}_{\alpha_1}^{\dagger}$  if  $\#=\dagger$  and  $\hat{c}_{\alpha_1}^{\#}=\hat{c}_{\alpha_1}$  if #=. The normal ordering of the operator  $\hat{A}$  is denoted by :  $\hat{A}$ : and defined as:

$$: \hat{A} := (-1)^{\pi} \hat{c}_{\alpha_{\pi(1)}}^{\#_{\pi(1)}} \cdots \hat{c}_{\alpha_{\pi(N)}}^{\#_{\pi(N)}}. \tag{2}$$

where  $\pi$  is an permutation of N numbers chosen such that all the destruction operators are on the right.

a) For fermion coherent states  $|\xi\rangle$  and  $|\xi'\rangle$ , show that:

$$\langle \xi' | : \hat{A} : | \xi \rangle = \xi_{\alpha_1}^{\#_1} \cdots \xi_{\alpha_N}^{\#_N} \langle \xi' | \xi \rangle$$
with  $\xi_{\alpha}^{\#} = \begin{cases} \xi_{\alpha}^{\prime \dagger} & \text{if } \# = \dagger \\ \xi_{\alpha} & \text{if } \# = . \end{cases}$ 
(3)

- **b)** Show that normal ordering is not a linear operation.
- c) Show that

$$\langle \xi | e^{\hat{c}^{\dagger} A \hat{c}} | \xi' \rangle = e^{\xi^{\dagger} e^A \xi'}. \tag{4}$$

b. Seemed ordering. Conclude a quantitative operation $A = 2^{(t-t)} - 2^{(t)}$ . (1)  Then $\phi = 1$ , and that $Z = I_0$ , $Z = I_0$ and $Z = I_0$ and $Z = I_0$ . The natural architects of the experiment of $Y_0$ . At an ellipse $I_0$ and the observation of the experiment of $Y_0$ and a distinct two speciments are into the experiment of $Y_0$ . At an ellipse $I_0$ and $I_0$ are into a colorest consequence are in $I_0$ and $I_0$ are into a colorest consequence are in $I_0$ and $I_0$ are increased ordered in one of the experiment of $I_0$ and $I_0$ are increased ordered in one of the experiment of $I_0$ and $I_0$ are increased ordered in one of the experiment of $I_0$ and $I_0$ are increased ordered in one of the experiment of $I_0$ and $I_0$ are increased ordered in one of the experiment of $I_0$ and $I_0$ are increased ordered in one of the experiment of $I_0$ and $I_0$ are increased ordered in one of the experiment of $I_0$ and $I_0$ are increased ordered in one of the experiment of $I_0$ and $I_0$ are increased ordered in one of the experiment of $I_0$ and $I_0$ are increased ordered in one of the experiment of $I_0$ and $I_0$ are increased ordered in one of the experiment of $I_0$ and $I_0$ are increased ordered in one of the experiment of $I_0$ and $I_0$ are increased ordered in $I_0$ and $I_0$ are increased in	
where $\pi$ is an permutation of N numbers chosen such that all the destruction operators are on the right.  a) For fermion coherent states [O] and $\{C\}$ , show that: $(C_1 : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ with $C_2^{C_1} = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_1^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C   C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C  C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C  C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C  C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C  C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C  C)$ $(C_2^{C_1} : \hat{A} :  C  = C_2^{C_1} \cdots C_n^{C_n}  C  C)$ $(C_2^{C_1} : $	
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## 2. Coherent state path integrals for Gaussian systems

Consider the quadratic Hamiltonian:

$$\hat{H} = \sum_{i,j} \hat{c}_i^{\dagger} T(i - j) \hat{c}_j. \tag{5}$$

(a) Show that the partition function reads:

$$Z = \operatorname{Tr} e^{-\beta \hat{H}} = \int \left\{ \prod_{i,\tau=1}^{L} d\xi_{i,\tau}^{\dagger} d\xi_{i,\tau} \right\} \exp \left[ -\int_{0}^{\beta} d\tau \sum_{i,j} \xi_{i}^{\dagger}(\tau) \left( \delta_{i,j} \frac{\partial}{\partial \tau} + T(i-j) \right) \xi_{j}(\tau) \right]$$
(6)

**(b)** Show that

$$\langle \mathcal{T} \hat{c}_{i}(\tau) \hat{c}_{j}^{\dagger}(\tau') \rangle =$$

$$\frac{1}{Z} \int \left\{ \prod_{i,\tau=1}^{L} d\xi_{i,\tau}^{\dagger} d\xi_{i,\tau} \right\} \exp \left[ -\int_{0}^{\beta} d\tau \sum_{i,j} \xi_{i}^{\dagger}(\tau) \left( \delta_{i,j} \frac{\partial}{\partial \tau} + T(i-j) \right) \xi_{j}(\tau) \right] \xi_{i}(\tau) \xi_{j}^{\dagger}(\tau')$$
(7)

In the above equation the time ordering is defined as:

$$\mathcal{T}\hat{c}_{\boldsymbol{i}}(\tau)\hat{c}_{\boldsymbol{j}}^{\dagger}(\tau') = \begin{cases} \hat{c}_{\boldsymbol{i}}(\tau)\hat{c}_{\boldsymbol{j}}^{\dagger}(\tau') & \text{if } \tau \geq \tau' \\ -\hat{c}_{\boldsymbol{j}}^{\dagger}(\tau')\hat{c}_{\boldsymbol{i}}(\tau) & \text{if } \tau < \tau' \end{cases}$$
(8)

(c) Show that:

$$\langle \mathcal{T} \hat{c}_{i}(\tau) \hat{c}_{j}^{\dagger}(\tau') \rangle = M_{(i,\tau),(i,\tau')}^{-1} \tag{9}$$

In the above we have discretized the imaginary time so as to obtain

$$\int_{0}^{\beta} d\tau \sum_{i,j} \xi_{i}^{\dagger}(\tau) \left( \delta_{i,j} \frac{\partial}{\partial \tau} + T(i-j) \right) \xi_{j}(\tau) = \sum_{(i,\tau),(j,\tau')} \xi_{i}^{\dagger}(\tau) M_{(i,\tau),(j,\tau')} \xi_{j}(\tau')$$
(10)

(d) Consider now a hypercubic lattice of linear length L with periodic boundary conditions and consider the transformed Grassmann variables:

$$\eta_{\mathbf{k},i\omega_m} = \frac{1}{\sqrt{\beta N}} \int_0^\beta d\tau \sum_{\mathbf{j}} e^{i\omega_m \tau - i\mathbf{k} \cdot \mathbf{j}} \xi_{\mathbf{j}}(\tau)$$
 (11)

with  $N = L^d$  and d the dimension of the hyper-cubic lattice. Determine the quantization of the crystal momenta, k, and of the so called fermionic Matsubara frequencies  $\omega_m$ . Show that the above transformation diagonalizes M.

