Topological Field Theory WS 2025

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PROBLEM SET 4
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1. Topological action for a skyrmion configuration

Consider the skyrmion configuration $\hat{\boldsymbol{n}}_W\colon \mathbb{R} \to S^2$,

$$(x_1, x_2) \to \phi = W \tan^{-1}\left(\frac{x_2}{x_1}\right), \ \theta = 2\tan^{-1}\sqrt{\frac{a^2}{x_1^2 + x_2^2}}, \tag{1}$$

where

$$\hat{\boldsymbol{n}}(\theta, \phi) = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}. \tag{2}$$

Calculate

$$S_{\text{top}}[\hat{\boldsymbol{n}}] = \mathrm{i}\theta W, \quad W = -\frac{1}{4\pi}\int dx_1 dx_2 \hat{\boldsymbol{n}}(\partial_1 \hat{\boldsymbol{n}} \times \partial_2 \hat{\boldsymbol{n}}) \tag{3}$$

explicitly for (1).

Hint: Parametrize the plane (x_1, x_2) in terms of polar coordinates (r, φ) and avail yourself of the fact that topological actions are invariant of the metric.

2. Classical equation of motion for the O(3) model

Consider the O(3) or non-linear sigma model

$$S_0[\hat{\boldsymbol{n}}] = \frac{1}{8\pi}\int d^2x\, (\partial_\mu \hat{\boldsymbol{n}})\, (\partial^\mu \hat{\boldsymbol{n}}), \quad |\hat{\boldsymbol{n}}| = 1, \tag{4}$$

with the Euclidean metric $g^{\mu\nu} = g_{\mu\nu} = \delta_{\mu\nu}$. In the lectures, we derived the inequality

$$S_0[\hat{\boldsymbol{n}}] \geq W \tag{5}$$

where W is the “winding number” as given in (3) above. The equal sign in (5) holds if and only if

$$\partial_\mu \hat{\boldsymbol{n}} - \varepsilon_{\mu\nu}\, (\hat{\boldsymbol{n}} \times \partial^\nu \hat{\boldsymbol{n}}) = 0, \tag{6}$$

where $\varepsilon_{12} = -\varepsilon_{21} = 1$. We found that if we parametrize the target space (*i.e.*, the field vector $\hat{\boldsymbol{n}}$) via a stereographic projection by a complex number w such that

$$\hat{\boldsymbol{n}}(w) = \frac{1}{w\bar{w} + 1}\begin{pmatrix} w + \bar{w} \\ -\mathrm{i}(w - \bar{w}) \\ w\bar{w} - 1 \end{pmatrix}, \tag{7}$$

($\bar{w}$ is just the complex conjugate of w) and our base space (x_1, x_2) by another complex number $z = x_1 + \mathrm{i}x_2$, the most general solution of (6) for $W > 0$ is given by

$$w = \prod_{i=1}^{W} \frac{a_i z + b_i}{c_i z + d_i}, \quad \text{with } a_i d_i - b_i c_i = 1\ \forall\ i \in \{1, 2, \ldots, W\}. \tag{8}$$

(a) Find the solution of the classical equation of motion (6) of (5) for $W = 0$.

(b) For which values of W (and a) are the skyrmion configurations in (1) solutions of (6)?

(c) What is the general solution of (6) for $W < 0$?