3. Problemset "Quantum Algebra & Dynamics"

October 31, 2025

2×2-Matrices

3.1 Norm

Which of the maps $\mathcal{M}_2 \to [0, \infty)$

$$M \mapsto |\det M| \tag{1a}$$

$$M \mapsto |\operatorname{tr} M| \tag{1b}$$

$$M \mapsto \sup_{ij} |M_{ij}| \tag{1c}$$

$$M \mapsto \sup_{v \in \mathbf{C}^2, \|v\|=1} \|Mv\| \qquad \left(\text{with } \|v\| = \sqrt{|v_1|^2 + |v_2|^2}\right) \tag{1d}$$

define a norm on the algebra $\mathcal{M}_2$ of complex 2×2-matrices? Which turn $\mathcal{M}_2$ into a C^*-algebra?

3.2 Spectrum and Resolvent

Use the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

to parametrize a general complex 2×2-matrix $M \in \mathcal{M}_2$ by four complex numbers $(a_0, \vec{a})$:

$$M(a_0, \vec{a}) = a_0 \mathbf{1} + \vec{a}\vec{\sigma} = a_0 \mathbf{1} + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = \begin{pmatrix} a_0 + a_3 & a_1 - \mathrm{i}a_2 \\ a_1 + \mathrm{i}a_2 & a_0 - a_3 \end{pmatrix}. \tag{3}$$

1. For which $(a_0, \vec{a}) \in \mathbf{C}^4$ is $M(a_0, \vec{a})$
   (a) normal,
   (b) isometric,
   (c) unitary,
   (d) self-adjoint,
   (e) positive?

2. Determine the resolvent set $r_{\mathcal{M}_2}(M(a_0,\vec{a}))$ and spectrum $\sigma_{\mathcal{M}_2}(M(a_0,\vec{a}))$ for all $(a_0,\vec{a})$. Handle exceptional cases.

3. Test the general results for the spectrum of normal, isometric, unitary, self-adjoint and positive matrices.

4. Compute the resolvent

$$\begin{aligned} R^{a_0,\vec{a}} : r_{\mathcal{M}_2}(M(a_0,\vec{a})) &\to \mathcal{M}_2 \\ z &\mapsto (z\mathbf{1} - M(a_0,\vec{a}))^{-1} \end{aligned} \tag{4}$$

   as a 2×2-matrix (i. e. perform the matrix inversion explicitely!).

5. *NB: in the lecture on Tuesday, November 4, 2025, it will be shown that $P_{\mathcal{C}}^{M(a_0,\vec{a})}$ is indeed a projection, if $\mathcal{C}$ encircles a part of the spectrum $\sigma(M(a_0,\vec{a}))$. You can wait until then to complete the exercise, take a peek at the script or just do the integral choosing typical examples for $\mathcal{C}$ based on your earlier results for $\sigma(M(a_0,\vec{a}))$.*

   Compute the projections

$$P_{\mathcal{C}}^{M(a_0,\vec{a})} = \int_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, R^{a_0,\vec{a}}(z) \tag{5}$$

   for "interesting" $\mathcal{C}$ explicitely. Are there qualitatively different cases to consider?