Funktionalanalysis Notizen

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I. MATHEMATICS

Gamma Function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t$$
$$\Gamma(z+1) = z\Gamma(z)$$
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$
$$\Gamma(n) = (n-1)!,\ n \in \mathbb{N}$$

Gaussian Integral

$$\int_0^\infty x^n e^{-kx^2}\,\mathrm{d}x = \frac{1}{2}k^{-(n+1)/2}\Gamma\left(\frac{1+n}{2}\right)$$

Fermi-Dirac Integral

$$f_\alpha(z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{\xi^{\alpha-1}}{e^\xi/z+1}\,\mathrm{d}\xi$$

General Series Expansion

$$f_\alpha(z) \approx f_\alpha(z_0) + \frac{f_{\alpha-1}(z_0)}{z_0}(z-z_0) + \frac{f_{\alpha-2}(z_0) - f_{\alpha-1}(z_0)}{2z_0^2}(z-z_0)^2$$

Expansion around 0

$$f_\alpha(z) = \sum_{n=1}^\infty \frac{(-1)^{n-1}z^n}{n^\alpha}$$

Derivative

$$f_{n-1}(z) = z\frac{\partial}{\partial z}f_n(z)$$

Bose-Einstein Integral

$$g_\alpha(z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{\xi^{n-1}}{e^\xi/z-1}\,\mathrm{d}\xi$$

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General Series Expansion

$$g_\alpha(z) \approx g_\alpha(z_0) + \frac{g_{\alpha-1}(z_0)}{z_0}(z - z_0) + \frac{g_{\alpha-2}(z_0) - g_{\alpha-1}(z_0)}{2z_0^2}(z - z_0)^2$$

Around 0

$$g_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}$$

Derivatives

$$g_{\alpha-1}(z) = z\frac{\partial}{\partial z} g_\alpha(z)$$

II. THERMODYNAMIC POTENTIALS

Energy

$$U = TS - PV + \mu N$$

$$\mathrm{d}U = T\,\mathrm{d}S - P\,\mathrm{d}V + \mu\,\mathrm{d}N$$

Helmholtz Free Energy

Helmholtz free energy is the maximal work in an isothermal process.

$$F = U - TS$$

$$F = -PV + \mu N$$

$$\mathrm{d}F = -S\,\mathrm{d}T - P\,\mathrm{d}V + \mu\,\mathrm{d}N$$

Enthalpy

$$H = U + PV$$

$$H = TS + \mu N$$

$$\mathrm{d}H = T\,\mathrm{d}S + V\,\mathrm{d}P + \mu\,\mathrm{d}N$$

Gibbs Free Energy

$$G = U + PV - TS$$

$$G = \mu N$$

$$\mathrm{d}G = -T\,\mathrm{d}S + V\,\mathrm{d}P + \mu\,\mathrm{d}N$$

Grand Canonical Potential

$$
\begin{aligned}
J &= F - \mu N \\
J &= -PV \\
\mathrm{d}J &= -S\,\mathrm{d}T - P\,\mathrm{d}V - N\,\mathrm{d}\mu
\end{aligned}
$$

III. ENSEMBLES

General Definitions

Gibbs Entropy

$$S = -k_B \sum p_n \ln p_n = -k_B \operatorname{tr}(\hat{\rho} \ln \hat{\rho})$$

Microcanonical Ensemble

Energy fixed,

$$\mathbb{P} \propto \Omega$$

Partition function

$$\Omega = e^{S(E)}$$

Canonical Ensemble

Partition function

$$Z = \operatorname{tr} e^{-\beta \hat{H}}$$

Thermodynamic Potential

$$F = -k_B T \ln Z$$

Grand Canonical Ensemble

Partition Function

$$
\begin{aligned}
Z_G &= \operatorname{tr}\Big(e^{-\beta(H-\mu N)}\Big)(\text{quantum}) \\
&= \sum_{n=0}^{\infty} z^n Z_k(n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} z^n (Z_k)^n (\text{classical indistinguishable particles only})
\end{aligned}
$$

IV. QUANTUM STATISTICS

Determining the DOS:

1. Find the number of states with energy $E \leq \epsilon$

2. Differentiate

Momentum Space Sums

$$\mathrm{d}k_x = \frac{2\pi}{L_x},\ \vec{\mathbf{k}} = \frac{\vec{\mathbf{p}}}{\hbar}$$

$$\sum_{\vec{\mathbf{p}}} f(\vec{\mathbf{p}}) = \frac{V}{h^3}\int f(\vec{\mathbf{p}})\,\mathrm{d}\vec{\mathbf{p}}$$

Fermi-Dirac Distribution

$$\begin{aligned}
Z &= \sum_{n_1,n_1,\ldots,n_N=0}^{1} \langle n_1,\ldots,n_N|\, e^{-\beta(H-\mu N)}\, |n_1,\ldots,n_N\rangle \\
&= \sum_{n_1,n_1,\ldots,n_N=0}^{1} \langle n_1,\ldots,n_N|\prod_{k=1}^{N} e^{-\beta(E_k-\mu)c_k^\dagger c_k}\, |n_1,\ldots,n_N\rangle \\
&= \prod_{k=1}^{n}\sum_{n_k=0}^{1} e^{-\beta(E_k-\mu)c_k^\dagger c_k} \\
&= \prod_{k=1}^{n}[1+e^{-\beta(E_k-\mu)}]
\end{aligned}$$