Homework for the Lecture

Functional Analysis

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Homework Sheet No 11

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(27 Points. Discussion 13. 01. 2025)

Homework 11-1: Bijective Bounded Linear Operators Into Banach Spaces

(4 Points) Let $(V, \|\cdot\|_V)$ be a normed and $(W, \|\cdot\|_W)$ be a Banach space. Prove the following:
If there exists a bijective bounded linear operator ϕ mapping from V to W, then V is a Banach space.
Hint: Use ϕ to construct a second norm $\|\cdot\|'_V$ on V. Then consider the completions of $(V, \|\cdot\|_V)$ and $(V, \|\cdot\|'_V)$.

Homework 11-2: Open Mapping, Closed Graph and Inverse Mapping: Counter Examples

(5 Points) Let $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ be normed spaces and $\phi: V \to W$ be a bounded linear operator. Give examples of V, W and ϕ such that

- the open mapping theorem
- the closed graph theorem
- the inverse mapping theorem

does not hold.

Homework 11-3: The Parallelogram Identity

(4 Points) Let $(V, \|\cdot\|)$ be a complex normed space whose norm satisfies the parallelogram identity, that is

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{11.1}$$

for any two vectors $x, y \in V$. Show that there is a positive definite inner product $\langle \cdot, \cdot \rangle$ on V such that $\|x\|^2 = \langle x, x \rangle$ for every $x \in V$.

Homework 11-4: The Bargmann-Fock Space: Part I

Let $n \in \mathbb{N}$ and $z = (z_1, \ldots, z_n) = (x_1 + \mathrm{i}y_1, \ldots, x_n + \mathrm{i}y_n) \in \mathbb{C}^n$. Consider the differential operators

$$\partial_i := \frac{\partial}{\partial z_i} := \frac{1}{2}\Big(\frac{\partial}{\partial x_i} - \mathrm{i}\frac{\partial}{\partial y_i}\Big) \quad \text{and} \quad \overline{\partial}_i := \frac{\partial}{\partial \overline{z}_i} := \frac{1}{2}\Big(\frac{\partial}{\partial x_i} + \mathrm{i}\frac{\partial}{\partial y_i}\Big), \tag{11.2}$$

$i \in \{1, \ldots, n\}$. Analogously to holomorphic functions, we call a function $f : D \to \mathbb{C}$ on an open subset D of $\mathbb{C}^n$ antiholomorphic if it is real differentiable and $\partial_i f = 0$ for all i. The set of antiholomorphic functions on D is denoted by $\overline{\mathscr{O}}(D)$. One can show that for every antiholomorphic function $f \in \overline{\mathscr{O}}(D)$ there is a unique holomorphic function $g \in \mathscr{O}(D)$ such that $f = \overline{g}$. In particular, all well-known results from complex analysis also hold true for antiholomorphic functions.
We now define the Bargmann-Fock space as

$$\mathfrak{H}_{BF} := \Big\{ f \in \overline{\mathscr{O}}(\mathbb{C}^n) : \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{C}^n} \overline{f(\overline{z})} f(\overline{z}) \mathrm{e}^{-\frac{\overline{z}z}{2\hbar}} \,\mathrm{d}z \,\mathrm{d}\overline{z} < \infty \Big\}, \tag{11.3}$$

where $\overline{z}z := \sum_{i=1}^n \overline{z}_i z_i$ and $\hbar := \frac{h}{2\pi}$ is the reduced Planck constant. Given a differentiable bijection $\mathbb{R}^2 \supset D \ni (r, s) \mapsto (z_i(r, s), \overline{z}_i(r, s))$ with Jacobian J, we set

$$\int_{\mathbb{C}} g(z_i, \overline{z}_i) \,\mathrm{d}z_i \,\mathrm{d}\overline{z}_i := \int_D g(z_i(r, s), \overline{z}_i(r, s)) \frac{|\det(J(r, s))|}{2} \,\mathrm{d}r \,\mathrm{d}s \tag{11.4}$$

for every integrable function g.

i.) **(1 Point)** Show that the map

$$\langle \cdot, \cdot \rangle_{BF} : \mathfrak{H}_{BF}^2 \ni (f, g) \mapsto \langle f, g \rangle_{BF} := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{C}^n} \overline{f(\overline{z})} g(\overline{z}) \mathrm{e}^{-\frac{\overline{z}z}{2\hbar}} \,\mathrm{d}z \,\mathrm{d}\overline{z} \tag{11.5}$$

defines a positive definite inner product that turns $\mathfrak{H}_{BF}$ into a pre-Hilbert space.

ii.) **(5 Points)** Compute the integral

$$\frac{1}{2\pi\hbar} \int_{\mathrm{B}_r(0)^{\mathrm{cl}}} a z_i^k \overline{z_i}^l \mathrm{e}^{-\frac{\overline{z}_i z_i}{2\hbar}} \,\mathrm{d}z_i \,\mathrm{d}\overline{z}_i, \tag{11.6}$$

with $a \in \mathbb{C} \backslash \{0\}$, $r > 0$ and $k, l \in \mathbb{N}_0$. Conclude that

$$\langle f, g \rangle_{BF} = \sum_{r=0}^{\infty} \frac{(2\hbar)^r}{r!} \sum_{k_1, \ldots, k_r} \overline{\frac{\partial^r f}{\partial \overline{z}_{k_1} \cdots \partial \overline{z}_{k_r}}}\Big|_{z=0} \frac{\partial^r g}{\partial \overline{z}_{k_1} \cdots \partial \overline{z}_{k_r}}\Big|_{z=0} \tag{11.7}$$

for all $f, g \in \mathfrak{H}_{BF}$.
Hint: Use polar coordinates on each copy of $\mathbb{C}$.

iii.) **(1 Point)** Show that the set

$$\Big\{ e_{k_1 \ldots k_n}(\overline{z}) := \frac{1}{\sqrt{(2\hbar)^{k_1 + \cdots + k_n} k_1! \cdots k_n!}} \overline{z}_1^{k_1} \cdots \overline{z}_n^{k_n} : k_1, \ldots, k_n \in \mathbb{N}_0 \Big\} \subset \mathfrak{H}_{BF} \tag{11.8}$$

satisfies

$$\langle e_{k_1 \ldots k_n}, e_{\ell_1 \ldots \ell_n} \rangle_{BF} = \prod_{i=1}^n \delta_{k_i \ell_i}. \tag{11.9}$$

iv.) **(3 Points)** Prove that the delta functional $\delta_{\overline{w}} : \mathfrak{H}_{BF} \to \mathbb{C}$ is continuous for every $w \in \mathbb{C}^n$.
Hint: Find a function $f_{\overline{w}} \in \mathfrak{H}_{BF}$ such that $\delta_{\overline{w}} = \langle f_{\overline{w}}, \cdot \rangle_{BF}$.

v.) **(4 Points)** Show that $\mathfrak{H}_{BF}$ is a Hilbert space.
Hint: Consider a Cauchy sequence $(f_n)_{n \in \mathbb{N}} \subset \mathfrak{H}_{BF}$ and show that it is a Cauchy sequence with respect to the supremum norm on every compact subset on $\mathbb{C}^n$.