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2. Open quantum systems

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Throughout this exercise sheet, we adopt the convention $\boxed{\hbar = 1}$.

Exercise 1 *Bit-phase flip map* 2 P.

If $\{|0\rangle, |1\rangle\}$ is an orthonormal basis for the qubit, the three types of errors can be characterized as:

- **Bit flip error:** $\begin{array}{l} |0\rangle \mapsto |1\rangle \\ |1\rangle \mapsto |0\rangle \end{array}$ or $|\psi\rangle \mapsto \hat{\sigma}_x|\psi\rangle, \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- **Phase flip error:** $\begin{array}{l} |0\rangle \mapsto |0\rangle \\ |1\rangle \mapsto -|1\rangle \end{array}$ or $|\psi\rangle \mapsto \hat{\sigma}_z|\psi\rangle, \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- **Bit-phase flip error:** $\begin{array}{l} |0\rangle \mapsto +i|1\rangle \\ |1\rangle \mapsto -i|0\rangle \end{array}$ or $|\psi\rangle \mapsto \hat{\sigma}_y|\psi\rangle, \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\hat{\sigma}_x\hat{\sigma}_z,$
  as the name indicates, this is a combination of phase flip and bit flip errors.

The bit-phase flip channel is characterized by the Kraus operators

$$\hat{K}_0 = \sqrt{p}\,\mathbb{1} = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \hat{K}_1 = \sqrt{1-p}\hat{\sigma}_y = \sqrt{1-p}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Calculate M and $\vec{c}$ via

$$\rho' = \sum_{\alpha=0} \hat{K}_\alpha \left(\frac{1}{2}\left(\mathbb{1} + \vec{v}\cdot\vec{\sigma}\right)\right)\hat{K}_\alpha^\dagger = \frac{1}{2}\left(\mathbb{1} + \vec{v'}\cdot\vec{\sigma}\right), \text{ where } \vec{v'} = M\vec{v} + \vec{c},$$

as was done in the lecture. Interpret your results.

Exercise 2 *$Z \otimes Z$ coupling* 4 P.

Consider the system-bath interaction Hamiltonian $H = \lambda\,\hat{\sigma}_S^z \otimes \hat{\sigma}_B^z$. For this choice, since $\hat{\sigma}^z$ is diagonal, only the Kraus operators $\hat{K}_{00}$ and $\hat{K}_{11}$ are non-zero, and they have the form:

$$\begin{aligned} \hat{K}_{00} &= \sqrt{\lambda_0}\left(\cos\theta\cdot\mathbb{1} - i\sin\theta\cdot\sigma^z\right) = \sqrt{\lambda_0}\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \\ \hat{K}_{11} &= \sqrt{\lambda_1}\left(\cos\theta\cdot\mathbb{1} + i\sin\theta\cdot\sigma^z\right) = \sqrt{\lambda_1}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}. \end{aligned} \tag{1}$$

How does the density matrix evolve under the action of these two Kraus operators? Also compute the purity $P = \mathrm{Tr}(\rho^2)$. Interpret your results.

Hint: Use the equation from the lecture:

$$\rho_S(t) = \sum_{\mu\nu} K_{\mu\nu}(t)\,\rho_S(0)\,K^\dagger_{\mu\nu}(t). \tag{2}$$

Note: Choose $\rho_S(0)$ as a pure state, i.e., $\rho_S(0) = |\psi\rangle\langle\psi|$.

Exercise 3 *Decoherence* **5 P.**

A spin-1/2 particle is exposed to a magnetic field directed along the z-axis, characterized by the Hamiltonian

$$\hat{H} = -B \cdot \hat{\sigma}_z$$

As a result of interaction with the environment, the magnetic field experiences random fluctuations

$$B \equiv B_0 + \delta B(t),$$

which we assume to be Gaussian with zero mean value and finite correlator

$$\langle \delta B(t)\delta B(t')\rangle = \Gamma\delta(t-t').$$

Let the density matrix at the initial time be in the most general form

$$\hat{\rho}(0) = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}.$$

Determine the time dependence of the averaged density matrix $\langle\hat{\rho}(t)\rangle$ due to these fluctuations. Take the limit, where $t \to \infty$. Interpret your results.

Note: The averaging $\langle\rangle$ over the fluctuations refers to the expectation value of the random fluctuations $\delta B(t)$. During the solution process, you will need to compute averages of the type

$$\langle \exp\left(i\int_{t_1}^{t_2} \delta B(t')\,dt'\right)\rangle.$$

Hint: Use Wick's probability theorem, which states that for any Gaussian variable with zero mean value, the following holds $\langle e^A\rangle = e^{\frac{1}{2}\langle A^2\rangle}$.