Lineare Algebra 1 Hausaufgabenblatt Nr. 6

Jun Wei Tan*

Julius-Maximilians-Universität Würzburg

(Dated: January 15, 2024)

Problem 1. Bestimmen Sie für $b \in \left\{ \begin{pmatrix} i+2 \\ 2 \\ -i-1 \end{pmatrix}, \begin{pmatrix} -2i-6 \\ -5 \\ 3i+2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{C}^3$ jeweils die Lösungsmenge des Gleichungssystems $Ax = b$ mit $x \in \mathbb{C}^3$.

(a)

$$A = \begin{pmatrix} 1 & i & 2 \\ i & 1 & 1 \\ 0 & i & 1 \end{pmatrix} \in \mathbb{C}^{3\times 3}.$$

(b)

$$A = \begin{pmatrix} 6+2i & -i & -2 \\ 5 & -1 & -1 \\ -3-2i & i & 1 \end{pmatrix} \in \mathbb{C}^{3\times 3}.$$

Proof. (a)

$$\left(\begin{array}{ccc|c} 1 & i & 2 & 2+i \\ i & 1 & 1 & 2 \\ 0 & i & 1 & -1-i \end{array}\right) \xrightarrow{R_2 - iR_1} \left(\begin{array}{ccc|c} 1 & i & 2 & 2+i \\ 0 & 2 & 1-2i & 3-2i \\ 0 & i & 1 & -1-i \end{array}\right) \xrightarrow{R_3 - \frac{i}{2}R_2} \left(\begin{array}{ccc|c} 1 & i & 2 & 2+i \\ 0 & 2 & 1-2i & 3-2i \\ 0 & 0 & -\frac{i}{2} & -2-\frac{5i}{2} \end{array}\right)$$

$$\xrightarrow{R_1 - \frac{i}{2}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1-\frac{i}{2} & 1-\frac{i}{2} \\ 0 & 2 & 1-2i & 3-2i \\ 0 & 0 & -\frac{i}{2} & -2-\frac{5i}{2} \end{array}\right) \xrightarrow{R_2 - (4+2i)R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1-\frac{i}{2} & 1-\frac{i}{2} \\ 0 & 2 & 0 & 6+12i \\ 0 & 0 & -\frac{i}{2} & -2-\frac{5i}{2} \end{array}\right) \xrightarrow{R_1 - (1+2i)R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2+6i \\ 0 & 2 & 0 & 6+12i \\ 0 & 0 & -\frac{i}{2} & -2-\frac{5i}{2} \end{array}\right) \xrightarrow{R_3 \times 2i} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2+6i \\ 0 & 2 & 0 & 6+12i \\ 0 & 0 & 1 & 5-4i \end{array}\right) \xrightarrow{R_2 \times \frac{1}{2}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2+6i \\ 0 & 1 & 0 & 3+6i \\ 0 & 0 & 1 & 5-4i \end{array}\right)$$

* jun-wei.tan@stud-mail.uni-wuerzburg.de

Die Lösungsmenge ist $\{(-2+yi, 3+6i, 5-4i)^T\}$.

$$\left(\begin{array}{ccc|c} 1 & i & 2 & -6-2i \\ i & 1 & 1 & -5 \\ 0 & i & 1 & 2+3i \end{array}\right) \xrightarrow{R_2 \times i} \left(\begin{array}{ccc|c} 1 & i & 2 & -6-2i \\ -1 & i & i & -5i \\ 0 & i & 1 & 2+3i \end{array}\right) \xrightarrow{R_2+R_1} \left(\begin{array}{ccc|c} 1 & i & 2 & -6-2i \\ 0 & 2i & 2+i & -6-7i \\ 0 & i & 1 & 2+3i \end{array}\right)$$

$$\xrightarrow{R_2 \times \frac{1}{2}} \left(\begin{array}{ccc|c} 1 & i & 2 & -6-2i \\ 0 & i & 1+\frac{i}{2} & -3-\frac{7i}{2} \\ 0 & i & 1 & 2+3i \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & i & 2 & -6-2i \\ 0 & i & 1+\frac{i}{2} & -3-\frac{7i}{2} \\ 0 & 0 & -\frac{i}{2} & 5+\frac{13i}{2} \end{array}\right) \xrightarrow{R_1-R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1-\frac{i}{2} & -3+\frac{3i}{2} \\ 0 & i & 1+\frac{i}{2} & -3-\frac{7i}{2} \\ 0 & 0 & -\frac{i}{2} & 5+\frac{13i}{2} \end{array}\right) \xrightarrow{R_2+(1-2i)R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1-\frac{i}{2} & -3+\frac{3i}{2} \\ 0 & i & 0 & 15-7i \\ 0 & 0 & -\frac{i}{2} & 5+\frac{13i}{2} \end{array}\right) \xrightarrow{R_1-(1+2i)R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5-15i \\ 0 & i & 0 & 15-7i \\ 0 & 0 & -\frac{i}{2} & 5+\frac{13i}{2} \end{array}\right) \xrightarrow{R_2 \times -i} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5-15i \\ 0 & 1 & 0 & -7-15i \\ 0 & 0 & -\frac{i}{2} & 5+\frac{13i}{2} \end{array}\right) \xrightarrow{R_3 \times 2i} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5-15i \\ 0 & 1 & 0 & -7-15i \\ 0 & 0 & 1 & -13+10i \end{array}\right)$$

also die Lösungsmenge ist $\{(5-15i, -5-15i, -13+10i)^T\}$.

$$\left(\begin{array}{ccc|c} 1 & i & 2 & 0 \\ i & 1 & 1 & 0 \\ 0 & i & 1 & 1 \end{array}\right) \xrightarrow{R_2 \times i} \left(\begin{array}{ccc|c} 1 & i & 2 & 0 \\ -1 & i & i & 0 \\ 0 & i & 1 & 1 \end{array}\right) \xrightarrow{R_2+R_1} \left(\begin{array}{ccc|c} 1 & i & 2 & 0 \\ 0 & 2i & 2+i & 0 \\ 0 & i & 1 & 1 \end{array}\right) \xrightarrow{R_2 \times \frac{1}{2}}$$

$$\left(\begin{array}{ccc|c} 1 & i & 2 & 0 \\ 0 & i & 1+\frac{i}{2} & 0 \\ 0 & i & 1 & 1 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & i & 2 & 0 \\ 0 & i & 1+\frac{i}{2} & 0 \\ 0 & 0 & -\frac{i}{2} & 1 \end{array}\right) \xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1-\frac{i}{2} & 0 \\ 0 & i & 1+\frac{i}{2} & 0 \\ 0 & 0 & -\frac{i}{2} & 1 \end{array}\right)$$

$$\xrightarrow{R_2+(1-2i)R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1-\frac{i}{2} & 0 \\ 0 & i & 0 & 1-2i \\ 0 & 0 & -\frac{i}{2} & 1 \end{array}\right) \xrightarrow{R_1-(1+2i)R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1-2i \\ 0 & i & 0 & 1-2i \\ 0 & 0 & -\frac{i}{2} & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \times -i} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1-2i \\ 0 & 1 & 0 & -2-i \\ 0 & 0 & -\frac{i}{2} & 1 \end{array}\right) \xrightarrow{R_3 \times 2i} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1-2i \\ 0 & 1 & 0 & -2-i \\ 0 & 0 & 1 & 2i \end{array}\right)$$

also die Lösungsmenge ist $\{(-1-2i, -2-i, 2i)^T\}$.

(b)

$$\left(\begin{array}{ccc|c} 6+2i & -i & -2 & 2+i \\ 5 & -1 & -1 & 2 \\ -3-2i & i & 1 & -1-i \end{array}\right) \xrightarrow{R_3+R_1} \left(\begin{array}{ccc|c} 6+2i & -i & -2 & 2+i \\ 5 & -1 & -1 & 2 \\ 3 & 0 & -1 & 1 \end{array}\right) \xrightarrow{R_3-\frac{3}{5}R_2}$$

$$\left(\begin{array}{ccc|c} 6+2i & -i & -2 & 2+i \\ 5 & -1 & -1 & 2 \\ 0 & \frac{3}{5} & -\frac{2}{5} & -\frac{1}{5} \end{array}\right) \xrightarrow{R_1\times\frac{3}{20}-\frac{i}{20}} \left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & \frac{7}{20}+\frac{i}{20} \\ 5 & -1 & -1 & 2 \\ 0 & \frac{3}{5} & -\frac{2}{5} & -\frac{1}{5} \end{array}\right)$$

$$\xrightarrow{R_2-5R_1} \left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & \frac{7}{20}+\frac{i}{20} \\ 0 & -\frac{3}{4}+\frac{3i}{4} & \frac{1}{2}-\frac{i}{2} & \frac{1}{4}-\frac{i}{4} \\ 0 & \frac{3}{5} & -\frac{2}{5} & -\frac{1}{5} \end{array}\right) \xrightarrow{R_2\times-\frac{2}{3}-\frac{2i}{3}}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & \frac{7}{20}+\frac{i}{20} \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{3}{5} & -\frac{2}{5} & -\frac{1}{5} \end{array}\right) \xrightarrow{R_3-\frac{3}{5}R_2} \left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & \frac{7}{20}+\frac{i}{20} \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1\times 20} \left(\begin{array}{ccc|c} 20 & -1-3i & -6+2i & 7+i \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1+(1+3i)R_2}$$

$$\left(\begin{array}{ccc|c} 20 & 0 & -\frac{20}{3} & \frac{20}{3} \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1\times\frac{1}{20}} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Die Lösungsmenge ist dann

$$\left\{ \begin{pmatrix} \frac{1}{3}t+\frac{1}{3} \\ \frac{2}{3}t-\frac{1}{3} \\ t \end{pmatrix}, t \in \mathbb{C} \right\}.$$

$$\left(\begin{array}{ccc|c} 6+2i & -i & -2 & -6-2i \\ 5 & -1 & -1 & -5 \\ -3-2i & i & 1 & 2+3i \end{array}\right) \xrightarrow{R_3+R_1} \left(\begin{array}{ccc|c} 6+2i & -i & -2 & -6-2i \\ 5 & -1 & -1 & -5 \\ 3 & 0 & -1 & -4+i \end{array}\right)$$

$$\xrightarrow{R_1\times\frac{3}{20}-\frac{i}{20}} \left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & -1 \\ 5 & -1 & -1 & -5 \\ 3 & 0 & -1 & -4+i \end{array}\right) \xrightarrow{R_3-3R_1}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & -1 \\ 5 & -1 & -1 & -5 \\ 0 & \frac{3}{20}+\frac{9i}{20} & -\frac{1}{10}-\frac{3i}{10} & -1+i \end{array}\right) \xrightarrow{R_2-5R_1} \left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & -1 \\ 0 & -\frac{3}{4}+\frac{3i}{4} & \frac{1}{2}-\frac{i}{2} & 0 \\ 0 & \frac{3}{20}+\frac{9i}{20} & -\frac{1}{10}-\frac{3i}{10} & -1+i \end{array}\right)$$

$$\xrightarrow{R_3\times 20} \left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & -1 \\ 0 & -\frac{3}{4}+\frac{3i}{4} & \frac{1}{2}-\frac{i}{2} & 0 \\ 0 & 3+9i & -2-6i & -20+20i \end{array}\right) \xrightarrow{R_3+4R_2}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & -1 \\ 0 & -\frac{3}{4}+\frac{3i}{4} & \frac{1}{2}-\frac{i}{2} & 0 \\ 0 & 12i & -8i & -20+20i \end{array}\right) \xrightarrow{R_2\times-\frac{2}{3}-\frac{2i}{3}}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & -1 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 12i & -8i & -20+20i \end{array}\right) \xrightarrow{R_3+-12iR_2} \left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & -1 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & -20+20i \end{array}\right)$$

Das Gleichungssystem besitzt keine Lösung, die Lösungsmenge ist $\varnothing$.

$$\left(\begin{array}{ccc|c} 6+2i & -i & -2 & 0 \\ 5 & -1 & -1 & 0 \\ -3-2i & i & 1 & 1 \end{array}\right) \xrightarrow{R_3+R_1} \left(\begin{array}{ccc|c} 6+2i & -i & -2 & 0 \\ 5 & -1 & -1 & 0 \\ 3 & 0 & -1 & 1 \end{array}\right) \xrightarrow{R_1\times\frac{3}{20}-\frac{i}{20}}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & 0 \\ 5 & -1 & -1 & 0 \\ 3 & 0 & -1 & 1 \end{array}\right) \xrightarrow{R_3-3R_1} \left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & 0 \\ 5 & -1 & -1 & 0 \\ 0 & \frac{3}{20}+\frac{9i}{20} & -\frac{1}{10}-\frac{3i}{10} & 1 \end{array}\right) \xrightarrow{R_2-5R_1}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & 0 \\ 0 & -\frac{3}{4}+\frac{3i}{4} & \frac{1}{2}-\frac{i}{2} & 0 \\ 0 & \frac{3}{20}+\frac{9i}{20} & -\frac{1}{10}-\frac{3i}{10} & 1 \end{array}\right) \xrightarrow{R_3\times\frac{2}{3}-2i} \left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & 0 \\ 0 & -\frac{3}{4}+\frac{3i}{4} & \frac{1}{2}-\frac{i}{2} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{2}{3}-2i \end{array}\right)$$

$$\xrightarrow{R_2\times-\frac{2}{3}-\frac{2i}{3}} \left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{2}{3}-2i \end{array}\right) \xrightarrow{R_3-R_2}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{20}-\frac{3i}{20} & -\frac{3}{10}+\frac{i}{10} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{2}{3}-2i \end{array}\right) \xrightarrow{R_1+\frac{1}{20}+\frac{3i}{20}R_2} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{2}{3}-2i \end{array}\right)$$

Das Gleichungssystem besitzt keine Lösungen. Die Lösungsmenge ist $\varnothing$. □

Problem 2. (a) Bestimmen Sie für $m = 3$ die Elementarmatrizen $S_1(\lambda), Q_{12}(\lambda), P_{12}$ aus 3.6.1 als explizite 3 × 3- Matrizen und überprüfen Sie für damit die Behauptungen 1-6 aus 3.6.2 exemplarisch im Fall $i = 1, j = 2, m = 3$.

(b) Beweisen Sie für alle $i, j, m \in \mathbb{N}, \lambda \in K$ mit $i \neq j$ und $i, j \leq m$ die Behauptungen

$$S_i(\lambda)^{-1} = S_i(\lambda^{-1}),\ Q_{ij}(\lambda)^{-1} = Q_{ij}(-\lambda),\ P_{ij}^{-1} = P_{ij}$$

aus Bemerkung 3.6.2.

Proof. (a)

$$S_1(\lambda) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q_{12}(\lambda) = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_{12}(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) Es gilt

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ A_{31} & A_{32} & \dots & A_{3n} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \dots & \lambda A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ A_{31} & A_{32} & \dots & A_{3n} \end{pmatrix}.$$

(2) Es gilt

$$\begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ A_{31} & A_{32} & \dots & A_{3n} \end{pmatrix}$$
$$= \begin{pmatrix} A_{11} + \lambda A_{21} & A_{12} + \lambda A_{22} & \dots & A_{1n} + \lambda A_{2n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ A_{31} & A_{32} & \dots & A_{3n} \end{pmatrix}$$

(3) Es gilt

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ A_{31} & A_{32} & \dots & A_{3n} \end{pmatrix} = \begin{pmatrix} A_{21} & A_{22} & \dots & A_{2n} \\ A_{11} & A_{12} & \dots A_{1n} & \\ A_{31} & A_{32} & \dots & A_{3n} \end{pmatrix}.$$

(4) Es gilt

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & A_{n3} \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & A_{12} & A_{13} \\ \lambda A_{21} & A_{22} & A_{23} \\ \vdots & \vdots & \vdots \\ \lambda A_{n1} & A_{n2} & A_{n3} \end{pmatrix}.$$

(5) Es gilt

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & A_{n3} \end{pmatrix} \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} + \lambda A_{11} & A_{13} \\ A_{21} & A_{22} + \lambda A_{21} & A_{23} \\ \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} + \lambda A_{n1} & A_{n3} \end{pmatrix}.$$

(6)

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & A_{n3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_{12} & A_{11} & A_{13} \\ A_{22} & A_{21} & A_{23} \\ \vdots & \vdots & \vdots \\ A_{n2} & A_{n1} & A_{n3} \end{pmatrix}.$$

(b) Es gilt

$$\begin{aligned} (E_m + (\lambda-1)E_{i,i}) \left(E_m + \left(\frac{1}{\lambda} - 1 \right) E_{i,i} \right) =& \cancelto{E_m}{E_m^2} + (\lambda - 1)E_{i,i} \\ &+ \left(\frac{1}{\lambda} - 1 \right) E_{i,i} \\ &+ (\lambda - 1) \left(\frac{1}{\lambda} - 1 \right) \cancelto{E_{i,i}}{E_{i,i}^2} \\ =& E_m \end{aligned}$$

Es gilt

$$(E_m + \lambda E_{ij})(E_m - \lambda E_{ij}) = \cancelto{E_m}{E_m^2} + \lambda E_{ij} - \lambda E_{ij} + \lambda^2 \cancelto{0}{E_{ij}^2}$$

$$=E_m$$

Aus (3) wissen wir, dass $P_{ij}^2 A = A$ für alle $A \in K^{m\times n}$. Daraus folgt: $P_{ij}^2 = E_m$ und $P_{ij} = P_{ij}^{-1}$. $\square$