

8. Problemset “Quantum Algebra & Dynamics”

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8.1 Inclusions

Consider once more the C^* -algebra \mathcal{M}_2 of 2×2 -Matrices $M(a_0, \vec{a})$ parametrized by four complex numbers (a_0, \vec{a}) , using the Pauli matrices with

$$M(a_0, \vec{a}) = a_0 \mathbf{1} + \vec{a} \vec{\sigma}. \quad (1)$$

The states on \mathcal{M}_2 can be parametrized by three real numbers $\vec{\alpha}$ with $|\vec{\alpha}| \leq 1$ and

$$\omega_{\vec{\alpha}} : M \mapsto \frac{1}{2} \text{tr}(M \rho(1, \vec{\alpha})) \quad (2)$$

using $\rho(\alpha_0, \vec{\alpha}) \in \mathcal{M}_2$. The pure states are those with $|\vec{\alpha}| = 1$.

Consider also the C^* -algebra $\mathcal{M}_1 \cong \mathbf{C}$ of 1×1 -Matrices $m(a) = a$ and the direct sum

$$\mathcal{M}_1 \oplus \mathcal{M}_1 \subset \mathcal{M}_2 \quad (3)$$

with the inclusion maps

$$\begin{aligned} \iota_{\vec{e}} : \mathcal{M}_1 \oplus \mathcal{M}_1 &\hookrightarrow \mathcal{M}_2 \\ m(a) \oplus m(a') &\mapsto \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix} = \frac{1}{2} M(a + a', (a - a') \vec{e}) \end{aligned} \quad (4)$$

for $\vec{e} \in \mathbf{R}^3$ with $|\vec{e}| = 1$.

NB: iff the general formulae become too messy for your taste, you can restrict yourself in the following to the special case $\vec{e} = (0, 0, 1)$.

1. Show that $\iota_{\vec{e}}$ is a C^* -algebra homomorphism.
2. Show that $\omega_{\vec{e}, \vec{\alpha}} = \omega_{\vec{\alpha}} \circ \iota_{\vec{e}}$ is a state on $\mathcal{M}_1 \oplus \mathcal{M}_1$. Give a concrete representation in the chosen basis.
3. Give necessary and sufficient conditions on \vec{e} and $\vec{\alpha}$ for $\omega_{\vec{e}, \vec{\alpha}}$ to be pure.
4. Construct the cyclic representations $(\mathcal{H}_{\vec{e}, \vec{\alpha}}, \pi_{\vec{e}, \vec{\alpha}}, \Omega_{\vec{e}, \vec{\alpha}})$ of $\mathcal{M}_1 \oplus \mathcal{M}_1$ from $\omega_{\vec{e}, \vec{\alpha}}$ in the special cases

$$\vec{\alpha} = (0, 0, 0) \quad (5a)$$

$$\vec{\alpha} = (0, 0, 1). \quad (5b)$$

5. Give the concrete matrix realizations of

$$\pi_{\vec{e}, \vec{\alpha}}(m(a) \oplus m(a')). \quad (6)$$

6. Which representations are irreducible? Determine the invariant subspace(s) for the others.

7. Is there a map $\tilde{\iota}_{\vec{e}}$ such that

$$\tilde{\iota}_{\vec{e}} \circ \pi_{\vec{e}, \vec{\alpha}} = \pi_{\vec{\alpha}} \circ \iota_{\vec{e}} \quad (7)$$

with $\pi_{\vec{\alpha}}$ from problem 7.1?

8.2 Gaussian Integrals

Starting from

$$P_{\pi} = \int \frac{d\xi d\eta}{2\pi} e^{-\frac{\xi^2 + \eta^2}{4}} \pi(W(\xi, \eta)) \quad (8)$$

and

$$W(\xi, \eta)W(\xi', \eta') = e^{\frac{i}{2}(\xi\eta' - \eta\xi')} W(\xi + \xi', \eta + \eta') \quad (9)$$

show that

$$P_{\pi} \pi(W(\xi, \eta)) P_{\pi} = e^{-\frac{\xi^2 + \eta^2}{4}} P_{\pi} \quad (10)$$

by a suitable variable transformation and evaluating two (identical) Gaussian integrals.