Stochastic Differential Equations

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Definition 1. A stochastic differential equation is a (formal) equation of the form

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = b(t, X_t) + \sigma(t, X_t)W_t, \tag{1}$$

where W_t is white noise.

This equation is to be interpreted as follows:

Definition 2. We say that the stochastic process X_t is a *solution* of the SDE (1) if

$$X_t = X_0 + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s$$

The standard method to solve a stochastic differential equation is the Itô formula

Theorem 3 (The 1-Dimensional Itô Formula)**.** Suppose X_t is an Itô process defined by the formula

$$\mathrm{d}X_t = u\,\mathrm{d}t + v\,\mathrm{d}B_t\,.$$

Let $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$. Then

$$Y_t = g(t, X_t)$$

is also an Itô process, and

$$\mathrm{d}Y_t = \frac{\partial g}{\partial t}\,\mathrm{d}t + \frac{\partial g}{\partial x}\,\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2} \cdot (\mathrm{d}X_t)^2$$

where $(\mathrm{d}X_t)^2$ is computed according to the rules $\mathrm{d}t \cdot \mathrm{d}t = \mathrm{d}t \cdot \mathrm{d}B_t = 0$ and $\mathrm{d}B_t \cdot \mathrm{d}B_t = \mathrm{d}t$.

where an Itô process is defined as follows

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Definition 4 (Itô Process)**.** An Itô process is a stochastic process of the form

$$X_t = X_a + \int_a^t b(s)\,\mathrm{d}s + \int_a^t \sigma(s)\,\mathrm{d}B(s)$$

Let us look at an example:

Example 5. A model of a population is given by the stochastic differential equation

$$\frac{\mathrm{d}N_t}{\mathrm{d}t} = rN_t + \alpha W_t N_t .$$

Here, r, α are constants and W_t is white noise.

This is the well known model for a population, except that we have allowed r to vary by a white noise term. The solution in the nonstochastic limit is given by a simple exponential. To solve this, we first rewrite the SDE in standard form:

$$\mathrm{d}N_t = rN_t\,\mathrm{d}t + \alpha N_t\,\mathrm{d}B_t\,,$$

or

$$\frac{\mathrm{d}N_t}{N_t} = r\,\mathrm{d}t + \alpha\,\mathrm{d}B_t\,.$$

Inspired by the solution in the deterministic case, we guess $g(t,x) = \ln x$ in Itô's formula, and let $Y_t = g(t, N_t)$. Then, we have

$$\mathrm{d}Y_t = \frac{\mathrm{d}N_t}{N_t} + \frac{1}{2}\alpha^2\,\mathrm{d}t\,.$$

Substituting, we have

$$\mathrm{d}Y_t = \left(r - \frac{1}{2}\alpha^2\right)\mathrm{d}t + \alpha B_t$$

which integrates easily to yield

$$N_t = N_0 \exp\left(\left(r - \frac{1}{2}\alpha^2\right)t + \alpha B_t\right).$$

Clearly, for $\alpha = 0$, this reduces to the well known exponential solution. This process is known as geometric Brownian motion, which, for example, is a solution to the Black-Scholes equation [1].

Now, we turn to the questions of existence and uniqueness. Recall the basic theorem for existence and uniqueness of a deterministic differential equation

Theorem 6 (Picard-Lindelöf). Let $\dot{x} = f(t, x)$ be a differential equation with f defined on a rectangle $[a, b] \times \mathbb{R}^n$. If f is Lipschitz continuous in x, with Lipschitz constant independent of time, and continuous in time, then the differential equation has a unique global solution on $[a, b]$.

Note that for uniqueness we do not need the continuity in time; the Lipschitz condition alone suffices. This extends to the stochastic case:

Theorem 7. Let $\sigma(t, x)$ and $f(t, x)$ be measurable functions on $[a, b] \times \mathbb{R}$ satisfying the Lipschitz condition in x, and $\mathcal{F}$ a filtration such that the Brownian motion is adapted to it. Suppose ξ is an $\mathcal{F}_a$-measurable random variable satisfying $\mathbb{E}[\xi^2] < \infty$. Then the stochastic differential equation

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t) + \sigma(t, X_t) W_t$$

has at most one continuous solution on $[a, b]$.

Proof. We will not go through the proof in detail. Instead, we will talk about the main steps. Assume we have two solutions X_t and Y_t. We seek to estimate $Z_t = X_t - Y_t$

1. We estimate the expectation value $\mathbb{E}(Z_t^2)$, using the Lipschitz condition.

2. We obtain an integral inequality

$$\mathbb{E}(Z_t^2) \leq 2K^2(1 + b - a) \int_a^t \mathbb{E}(Z_s^2)\,\mathrm{d}s\,,$$

   which, by the theory of classical differential equations, implies that Z_t is 0 almost surely for all t.

3. Then, we extend the solution to show that Z_t is 0 almost surely, using sample path continuity. $\square$

The existence theorem is as follows:

Theorem 8. The stochastic differential equation (1) has a unique solution with initial condition ξ, where ξ^2 has finite expectation, σ and b are Lipschitz in x, with Lipschitz constant independent of t, and continuous in t.

Proof. The proof follows by Picard Iteration much as it does in the deterministic case. We define $X^{(0)} = X_0 = \xi$ and

$$X_t^{(n+1)} = X_0 + \int_0^t b(s, X_s^{(n)})\,\mathrm{d}s + \int_0^t \sigma(s, X_s^{(n)})\,\mathrm{d}B_s\,.$$

Then, the proof proceeds in two steps. First, we show that this sequence converges. If it does, it satisfies the integral equation by the dominated convergence theorem. $\square$

It is a known property of initial value problems that the future solution is not dependent on the past. In particular, we can imagine that we have some initial state $x(0)$ and let it evolve a time t to $x(t)$. Then we can let it evolve further. Alternatively, we can consider an initial value problem that has the value $x(t)$ at time t. We expect that these two solutions are identical. In a stochastic differential equation, this "memory" property is known as the Markov property.

Definition 9. A stochastic process X_t, with $a \le t \le b$, is said to have the *Markov property* if for all sequences $a < t_1 < \cdots < t_n < t < b$ and corresponding $x_1, \ldots, x_n$, we have

$$\mathbb{P}(X_t \le x | X_{t_1} = x_1, \ldots, X_{t_n} = x_n) = \mathbb{P}(X_t \le x | X_{t_n} = x_n).$$

As an example, all processes with independent increments have the Markov property. The theorem we seek is thus

Theorem 10. The solution to (1) is a Markov process.

The final property that is of interest to us is time translation invariance. For a deterministic differential equation $\dot{x} = f(x)$, we know that the solution exhibits time translation invariance. In the deterministic case, this is quite easy to see, and follows from the fact that $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t - t_0) = \varphi'(t - t_0)$.

In this case, the relevant property is called the *stationary Markov property*

Definition 11. A stochastic process X is called stationary if the moments are time translation invariant:

$$\langle X_{t_1+\tau} X_{t_2+\tau} \dots X_{t_n+\tau} \rangle = \langle X_{t_1} X_{t_2} \dots X_{t_n} \rangle$$

for all n, τ and $t_1, \dots, t_n$.

Thus, we have our final theorem

Theorem 12. Suppose that $b(x)$ and $\sigma(x)$ are functions satisfying the Lipschitz condition. Then the solution to

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = b(X_t) + \sigma(X_t) W_t, \tag{2}$$

is a stationary Markov process.

As an example of this, we solve the Langevin equation

Example 13. The Langevin equation is the SDE given by

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma \,\mathrm{d}B_t \,.$$

It has solutions

$$X_t = e^{\mu t} X_0 + \int_0^t \sigma e^{\mu(t-s)} \,\mathrm{d}B_s$$

Proof. We multiply by the "integrating factor" $e^{-\mu t}$ and consider

$$Y_t = e^{-\mu t} X_t.$$

By Itô's formula, we have

$$\begin{aligned} \mathrm{d}Y_t &= -\mu e^{-\mu t} X_t \,\mathrm{d}t + e^{-\mu t} \,\mathrm{d}X_t \\ &= -\mu e^{-\mu t} X_t \,\mathrm{d}t + e^{-\mu t} (\mu X_t \,\mathrm{d}t + \sigma \,\mathrm{d}B_t) \\ &= e^{-\mu t} \sigma \,\mathrm{d}B_t \end{aligned}$$

implying that

$$X_t = e^{\mu t} X_0 + \int_0^t \sigma e^{\mu(t-s)} \,\mathrm{d}B_s \,. \qquad \square$$

Finally, we note that some stochastic processes can be described through density functions. Where such a density function is available, it satisfies the *Fokker-Planck Equation*

Theorem 14. The probability density of the solution to Eq. (1) $p(x,t)$ satisfies the equation

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[b(x,t)p(x,t)] + \frac{\partial^2}{\partial x^2}[D(x,t)p(x,t)]$$

where $D(x,t) = \frac{\sigma^2(X_t,t)}{2}$ is the diffusion coefficient.

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[1] T. Sauer, *Numerical Solution of Stochastic Differential Equations in Finance*, pp. 529–550. 07 2012.

[2] H. Kuo, *Introduction to Stochastic Integration.* Universitext, Springer New York, 2005.

[3] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications.* Universitext, Springer Berlin Heidelberg, 2010.