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**Problem Sheet 4**
for the tutorial on May 30th, 2025
**Quantum Mechanics II**
Summer term 2025

Sheet handed out on May 20th, 2025; to be handed in on May 27th, 2025 until 2 pm

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**Exercise 4.1**:    **The dipole approximation and the $\vec{r} \cdot \vec{E}$ Hamiltonian**    [10 P.]

We now examine the problem of an electron bound by a central potential $V(r)$ to a nucleus located at $\vec{r}_0$. Let us start from the minimal coupling Hamiltonian (1)

$$\left\{ -\frac{\hbar^2}{2m} \left[ \nabla - i\frac{e}{\hbar}\vec{A}(\vec{r},t) \right]^2 + e\Phi(\vec{r},t) \right\} \Psi = i\hbar\frac{\partial\Psi}{\partial t}\,, \tag{1}$$

considered in the Coulomb gauge and apply the dipole approximation $\vec{A}(\vec{r},t) \simeq \vec{A}(\vec{r}_0,t)$. By performing a wave function transformation

$$\Psi(\vec{r},t) = \exp\left[ \frac{ie}{\hbar}\vec{A}(\vec{r}_0,t) \cdot \vec{r} \right] \tilde{\Psi}(\vec{r},t) \tag{2}$$

show that the total Hamiltonian can be separated in the unperturbed Hamiltonian of the electron and the interaction term given by $\mathcal{H}_{int} = -e\vec{r} \cdot \vec{E}(\vec{r}_0,t)$.

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**Exercise 4.2**:    **Phenomenological spontaneous decay in semi-classical theory**    [5 P.]

As long as the electromagnetic field is treated classically, the spontaneous decay of the system can only be implemented phenomenologically. Let us consider now once more the simple case of a two-level system with states $|1\rangle$ and $|2\rangle$ interacting with a classical electromagnetic field. The finite lifetime of the atomic levels can be described very well by adding phenomenological decay terms to the density operator equation. The decay rates can be incorporated by a relaxation matrix $\Gamma$ defined as $\langle n|\Gamma|m\rangle = \gamma_n \delta_{nm}$, where $\{n,m\} \in \{1,2\}$. With this addition, we can write the density matrix equation of motion as

$$\dot{\rho} = -\frac{i}{\hbar}[\mathcal{H},\rho] - \frac{1}{2}\{\Gamma,\rho\}\,, \tag{3}$$

where

$$\mathcal{H} = \hbar\Delta\,|2\rangle\langle2| - \hbar\Omega\,(|2\rangle\langle1| + |1\rangle\langle2|) \tag{4}$$

is the Hamiltonian for a two-level system and $\{\Gamma,\rho\} = \Gamma\rho + \rho\Gamma$. Write down the Bloch equations resulting from Eq. (3).

**Exercise 4.3**: **Hadamard lemma and interaction picture of the JC model** [5+5 P.]

Consider the Hamiltonian of the quantum mechanical harmonic oscillator

$$\mathcal{H}_O = \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) \tag{5}$$

with the ladder operators $a$, $a^\dagger$ satisfying $[a, a^\dagger] = 1$, and $[a, a] = [a^\dagger, a^\dagger] = 0$. In the lecture, we will also find this to be the Hamiltonian for a single mode of the quantized electromagnetic field.

a) Using the time evolution operator $U = e^{-\frac{i}{\hbar}\mathcal{H}_O t}$, calculate $U^\dagger a\, U$ and $U^\dagger a^\dagger\, U$.

   *Hint:* Use that according to the Hadamard lemma for two operators $A$ and $B$

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, , B]]] + \dots . \tag{6}$$

b) Use $\mathcal{H}_T = \mathcal{H}_O + \hbar x_g A_{gg} + \hbar x_e A_{ee}$ to transform the Jaynes-Cummings model $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ with

$$\mathcal{H}_0 = \mathcal{H}_O + \hbar\omega_g A_{gg} + \hbar\omega_e A_{ee}\,, \tag{7}$$

$$\mathcal{H}_I = \hbar \left( g\, A_{eg}\, a + g^*\, a^\dagger A_{ge} \right)\,, \tag{8}$$

   to an interaction picture Hamiltonian without explicit time dependence. Here, $A_{ij} = |i\rangle\langle j|$ and $g$ is the atom-field mode coupling constant.