Homework for the Lecture

Functional Analysis

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Homework Sheet No 7

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(23 Points. Discussion 02. 12. 2024)

Homework 7-1: Stone-Weierstraß Counterexamples

We show that the assumptions in the Stone-Weierstraß Theorem can not be weakened in the most naive ways.

i.) **(1 Point)** Find an example of a unital point-separating subalgebra $\mathscr{A} \subseteq \mathscr{C}(X, \mathbb{C})$, whose closure is strictly less than $\mathscr{C}(X, \mathbb{C})$.

ii.) **(1 Point)** Find an example of a unital *-subalgebra $\mathscr{A} \subseteq \mathscr{C}(X, \mathbb{C})$, whose closure is strictly less than $\mathscr{C}(X, \mathbb{C})$.

iii.) **(2 Points)** Let $K \subseteq \mathbb{R}$ be compact. Consider the polynomials without a constant term $x\mathbb{C}[x]$, restricted to K. Show that they are dense in $\mathscr{C}(K, \mathbb{C})$ iff $0 \notin K$. Which condition in the Stone-Weierstraß Theorem fails in $x\mathbb{C}[x]$?

Homework 7-2: Continuous Functions and Separability

i.) **(1 Point)** Let M be a topological space and $N \subseteq M$ be a dense subset. Show that every dense subset of N (with respect to the subspace topology) is dense in M.

ii.) **(2 Points)** Let $n \in \mathbb{N}$ and K be a compact subset of $\mathbb{K}^n$. Without using part *iii.)*, prove that $\mathscr{C}(K, \mathbb{C}) = \mathscr{C}_{\mathrm{b}}(K, \mathbb{C})$ (endowed with the usual supremum norm topology) is separable, that is it contains a countable dense subset.

iii.) **(5 Points)** Let (M, d) be a compact metric space. Assume that M is separable. In this case, show that also $\mathscr{C}(M, \mathbb{C})$ is separable.

Hint: Urysohn's Lemma

iv.) **(6 Points)** Let now M be a separable compact Hausdorff space. Cand you find a sufficient condition for M under which $\mathscr{C}(M, \mathbb{C})$ becomes separable?

Hint: Every compact space is locally compact.

Homework 7-3: The Limit Functional

In Homework 5-3, we showed that the multiplication of sequences induces a linear homeomorphism $\phi : \ell^q \to (\ell^p)'$ for any two conjugated numbers $p, q \in [1, \infty]$ with $p \in [1, \infty)$. In this exercise, we show that this map fails to be an isomorphism in the case of $p = \infty$.

i.) **(1 Point)** Use Homework 5-3 to show that there is an isometry $\iota : \ell^1 \to (\ell^\infty)'$.

ii.) **(4 Points)** Consider the space $c \subset \ell^\infty$ of convergent $\mathbb{K}$-valued sequences. Show that the limit functional

$$L : c \ni (x_n)_{n\in\mathbb{N}} \mapsto x := \lim_{n\to\infty} x_n \tag{7.1}$$

defines a bounded linear operator and compute its operator norm. Conclude that ι is not surjective.