Übung Quantenmechanik 2 WS 2024/25

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1. Harmonic Oscillator

The hamiltonian of a (quantum mechanical) harmonic oscillator of mass m and frequency ω is given by:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2, \tag{1}$$

where $\hat{p} = \frac{\hbar}{i}\partial_x$ is the operator of momentum.

(a) - Rewrite $\hat{H}$ by using the ladder operators (is $\hat{a}$ hermitian?)

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} + x_0\partial_x\right) \tag{2}$$

$$\text{und} \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - x_0\partial_x\right) \tag{3}$$

with $x_0 = \sqrt{\frac{\hbar}{\omega m}}$.

- Calculate the commutator $\left[\hat{a}, \hat{a}^\dagger\right]_-$.

(b) Consider the states $|\varphi_n\rangle = C_n(\hat{a}^\dagger)^n|\varphi_0\rangle$ with $n \geq 0$ and $n \in \mathbb{N}$, where $|\varphi_0\rangle$ is the normalized vacuum state of $\hat{H}$. The eigenvalue of the ground state is given by $\hbar\omega/2$ and C_n is a not yet specified normalization constant.

- Show that $|\varphi_n\rangle$ is an eigenstate of $\hat{H}$. What is it's respective energy?
- Calculate the normalization constant C_n of the eigenstate $|\varphi_n\rangle$.

(c) The ground state wave function can be found by the condition

$$a|\phi_0\rangle = 0. \tag{4}$$

- Explain why.
  Hint: $a^{(\dagger)}$ are ladder operators, applying them to an eigenstate leads to another eigenstate with higher/lower energy. The bottom of the ladder is determined by exploiting that the norm is positive semidefinite by definition $\langle a\phi_n|a\phi_n\rangle \geq 0$.
- Explicitly calculate the normalized ground state wave function $\phi_0(x) = \langle x|\phi_0\rangle$ by solving the according differential equation.
- Using your results from problem (b) calculate the wave function $\phi_1(x)$ of the lowest excited state (no explicit normalization required).
  All other excited states can be calculated in the same way, the occurring polynimials are called Hermite Polynomials.

(d) Something to think (no calculations required): Let's say we forget about quantum mechanics for a second and search for solutions of the differential equation given in Eq.(1) as a purely mathematical exercise, would we find the same number of solutions as we did previously?

2. Angular momentum

Consider the angular momentum $\hat{\mathbf{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)^T$ and it's commutation relation

$$\left[\hat{J}_i, \hat{J}_j\right]_- = \left[\hat{J}_i, \hat{J}_j\right] = i\hbar \sum_k^3 \epsilon_{ijk} \hat{J}_k = i\hbar \epsilon_{ijk} \hat{J}_k, \quad i, j, k \in \{x, y, z\} \,. \tag{5}$$

Further consider the ladder operators $\hat{J}_\pm \equiv \hat{J}_x \pm i\hat{J}_y$. We will from now on use on the Einstein notation ("Einsteinsche Summenkonvention").

(a) Show the following commutation relations

- $\left[\hat{J}_+, \hat{J}_-\right] = 2\hbar \hat{J}_z$,
- $\left[\hat{J}_\pm, \hat{J}_z\right] = \mp\hbar \hat{J}_\pm$,
- $\left[\hat{\mathbf{J}}^2, \hat{J}_\pm\right] = 0$

by using the properties of the ϵ-tensor.

(b) Show, using the relations from above, that the states $\hat{J}_\pm |j, m\rangle$ are eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{J}_z$. You may further impose

$$\hat{\mathbf{J}}^2 |j, m\rangle = j(j+1)\hbar^2 |j, m\rangle \,, \tag{6}$$

$$\hat{J}_z |j, m\rangle = m\hbar |j, m\rangle \,. \tag{7}$$

What are the respective eigenvalues?

(c) Show the identity

$$\hat{\mathbf{J}}^2 = \hat{J}_\pm \hat{J}_\mp + \hat{J}_z^2 \mp \hbar \hat{J}_z \tag{8}$$

(d)
- Use (c) to calculate the normalization constant $N^\pm_{j,m}$ of the state $\hat{J}_\pm |j, m\rangle$.
- Combine all previous calculations to conclude

$$\hat{J}_\pm |j, m\rangle = \hbar\sqrt{(j \pm m + 1)(j \mp m)}\, |j, m \pm 1\rangle \,. \tag{9}$$

3. Adding angular momenta

The spin operator $\hat{S} = \hat{S}_1 + \hat{S}_2$ defines the total spin of a non interacting two-electron system. The eigenvalues of $\hat{S}^2$, $\hat{S}_1^2$, $\hat{S}_2^2$ and there respective z-components are (as usual) given by s, s_1, s_2 and m, m_1, m_2. The square of the total spin $\hat{S}^2$ commutes with $\hat{S}_1^2$ and $\hat{S}_2^2$ (you don't have to show that). Therefore these operators share their eigenfunctions $|s, m\rangle \equiv \left|s, m, s_1 = \frac{1}{2}, s_2 = \frac{1}{2}\right\rangle$.

Calculate a linear combination of the states $\left|1, 1, \frac{1}{2}, \frac{1}{2}\right\rangle = |1, 1\rangle$, $|1, 0\rangle$, $|1, -1\rangle$ and $|0, 0\rangle$ comprised of the product states $|s_1, m_1\rangle \otimes |s_2, m_2\rangle$.

Hint: Start with $|s, m\rangle$ as a linear combination of the product states and then use the ladder operators $\hat{S}_\pm$. Construct $|0, 0\rangle$ by using the orthonormality of the system.

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