5. Problemset "Quantum Algebra & Dynamics"
November 14, 2025

States

NB: the notions of *positive linear functional* and *state* will be introduced formally in the lecture on Tuesday, November 18, 2025.

> The space of continuous linear functionals $\omega : \mathcal{A} \to \mathbf{C}$ on the C^*-algebra $\mathcal{A}$ is denoted $\mathcal{A}^*$.

We can define a natural norm on $\mathcal{A}^*$ by

$$\|\omega\| = \sup_{A\in\mathcal{A}, \|A\|=1} |\omega(A)| \,. \tag{1}$$

A linear functional $\omega : \mathcal{A} \to \mathbf{C}$ on the C^*-algebra $\mathcal{A}$ is called positive, iff

$$\forall A \in \mathcal{A} : \omega(A^*A) \geq 0 \,. \tag{2}$$

A positive $\omega : \mathcal{A} \to \mathbf{C}$ with $\|\omega\| = 1$ is called a state.

5.1 Spins

Consider again the C^*-algebra $\mathcal{M}_2$ of 2×2-Matrices parametrized by four complex numbers $(a_0, \vec{a})$, using the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3}$$

Define a family of linear functionals

$$\begin{aligned} \omega_{a_0,\vec{a}} &: \mathcal{M}_2 \to \mathbf{C} \\ &M(b_0, \vec{b}) \mapsto \mathrm{tr}(M(b_0, \vec{b})\rho(a_0, \vec{a})) \end{aligned} \tag{4}$$

for suitable $\rho(a_0, \vec{a}) \in \mathcal{M}_2$. Derive the conditions on $(a_0, \vec{a})$ for $\omega_{a_0,\vec{a}}$ to be ...

1. ... continuous?
2. ... positive?
3. ... a state?
4. ... maximal in the sense that $\omega_{a_0,\vec{a}}$ can *not* be written

   $$\omega_{a_0,\vec{a}} = p\omega_{b_0,\vec{b}} + (1-p)\omega_{c_0,\vec{c}} \tag{5}$$

   with $0 < p < 1$ and $\omega_{b_0,\vec{b}}$ and $\omega_{c_0,\vec{c}}$ states?

5.2 Circle

Consider the algebra $C(S^1)$ of bounded complex valued continuous functions $f : S^1 \to \mathbf{C}$ on the unit circle.

1. Show that $\|f\| = \sup |f(x)|$ turns $C(S^1)$ into a C^*-algebra.
2. Define linear functionals $\omega : C(S^1) \to \mathbf{C}$ via
$$\omega(f) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \, \overline{\omega(\phi)} f(\phi) \,. \tag{6}$$
What are the conditions on $\omega : S^1 \to \mathbf{C}$ for ω to be ...
   (a) ... continuous?
   (b) ... positive?
   (c) ... a state?
   (d) ... maximal in the sense that ω can *not* be written
$$\omega = p\omega_1 + (1-p)\omega_2 \tag{7}$$
with $0 < p < 1$ and $\omega_{1/2}$ states as in (6)?
3. Are there states $\omega : C(S^1) \to \mathbf{C}$ that can not be written as in (6)?
4. If yes, give examples and repeat the second subproblem!