Notes in

Field Theory

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CHAPTER ONE

Introduction

1.1 Mean Field Theory & Phase Transitions

We characterise phases by order parameters. An order parameter is a quantity that vanishes throughout a disordered phase, but spontaneously breaks a symmetry to take a finite value in an ordered phase.

1.2 Linear Responses

We consider a Hamiltonian H_0. Then, we introduce a perturbation $f(t)O_1$ such that

$$H(t) = H_0 + f(t)O_1.$$

We seek to compute the perturbation. We do so by expanding the time ordered exponential

$$\begin{aligned}
T(e^{-i\sum_j \Delta t_j (H_0+f(t_j)O_1)}) &= \prod_j e^{-i\Delta t_j (H_0+f(t_j)O_1)} \\
&= \prod_j e^{-i\Delta t_j H_0}(1 - i\Delta t_j f(t_j)O_1) \\
&= e^{-i\sum_j \Delta t_j H_0} - i\sum_j \Delta t_j \left(\prod_{k\geq j} e^{-i\Delta t_k H_0}\right) f(t_j)O_1 \left(\prod_{k<j} e^{-i\Delta t_k H_0}\right).
\end{aligned}$$

By converting this back into an integral, we see that

$$|\psi_n(t)\rangle = \left(e^{-i\int_{-\infty}^{t}\mathrm{d}t' H_0} - i\int_{-\infty}^{t}\mathrm{d}t'\, f(t')e^{-i\int_{t'}^{t}\mathrm{d}t'' H_0}O_1 e^{-i\int_{-\infty}^{t'}\mathrm{d}t'' H_0}\right)|\psi_n\rangle\,.$$

Hence, we have

$$\begin{aligned}
\delta\,|\psi_n(t)\rangle &= -i\int_{t_{-\infty}}^{t}\mathrm{d}t'\, f(t')e^{-iH_0(t-t')}O_1 e^{-iH_0(t'-t_{-\infty})}\,|\psi_n\rangle \\
&= -i\int_{t_{-\infty}}^{t}\mathrm{d}t'\, f(t')e^{-iH_0(t-t_{-\infty})}O_1(t')\,|\psi_n\rangle
\end{aligned}$$

where we define the operator O_1 in the interaction picture to be

$$O_1(t) = e^{iH_0(t-t_{-\infty})}O_1 e^{-iH_0(t-t_{-\infty})}.$$

Then, we consider an observable O_2 and define the change in O_2 to be

$$\begin{aligned}\delta \langle\psi_n(t)|O_2|\psi_n(t)\rangle := &\ \langle\psi_n(t)|O_2|\psi_n(t)\rangle - \langle\psi_n|e^{iH_0(t-t_{-\infty})}O_2e^{-iH_0(t-t_{-\infty})}|\psi_n\rangle \\ = &\ \langle\delta\psi_n(t)|O_2e^{-iH_0(t-t_{-\infty})}|\psi_n\rangle + \langle\psi_n|e^{iH_0(t-t_{-\infty})}O_2|\delta\psi_n(t)\rangle \\ = &-i\int_{-\infty}^{t} \mathrm{d}t'\, f(t')\,\langle\psi_n|\,[O_2(t), O_1(t')]\,|\psi_n\rangle + O(f^2) \\ = &\int_{-\infty}^{\infty} \mathrm{d}t'\, D(t,t')f(t')\end{aligned}$$

where

$$D(t,t') = -i\theta(t-t')\,\langle\psi_n|[O_2(t), O_1(t')]|\psi_n\rangle\,.$$

Because H_0 is not time dependent, we have Comment: figure out how this works

$$[O_2(t), O_1(t')] = e^{iH(t-t_{-\infty})}O_2e^{iH(t-t')}O_1e^{-iH(t'-t_{-\infty})} - e^{iH(t'-t_{-\infty})}O_1e^{iH(t-t')}O_2e^{-iH(t-t_{-\infty})}$$

1.3 Superconductivity

We consider an operator creating a pair of electrons with 0 total momentum above the Fermi sea:

$$\Lambda^\dagger = \int \mathrm{d}^3x\,\mathrm{d}^3x'\,\phi(\vec{\mathbf{x}}-\vec{\mathbf{x}}')\psi^\dagger_\downarrow(\vec{\mathbf{x}})\psi^\dagger_\uparrow(\vec{\mathbf{x}}').$$

By performing a Fourier transform, we get the momentum space representation

$$\Lambda^\dagger = \sum_{\vec{\mathbf{k}}} \phi_{\vec{\mathbf{k}}} c^\dagger_{\vec{\mathbf{k}}\downarrow} c^\dagger_{-\vec{\mathbf{k}}\uparrow}.$$

The structure of Φ represents the type of superconductor; for example, when $\phi(\vec{\mathbf{k}}) = \phi(k)$, this is known as an s-wave superconductor.

CHAPTER TWO

The Path Integral

2.1 Phase Space Path Integrals

2.2 The Grassman Algebra

2.2.1 Introduction

Because every operator can be written in the formalism of second quantisation as a product of creation and annihilation operators, coherent states turn these operators into scalars, which are then easier to deal with. We define a fermionic coherent state by the usual equation

$$a_k |\phi\rangle = \phi_k |\phi\rangle .$$

Because annihilation operators for different k anticommute rather than commute, we must have

$$\phi_i \phi_j = -\phi_j \phi_i.$$

Thus, the ϕ_is cannot be part of a field, because they must anticommute rather than commute! We define the Grassman algebra to be generated by n generators ξ_i, with the basis coming from all products $\xi_i \xi_j$ etc. We will assume that there is an even number of generators, and to each generator ξ_i we assign an inversion $(\xi_i)^* = \xi_j$ such that the inversion satisfies $(\xi^*)^* = \xi$ and $(\xi_i \xi_j)^* = \xi_j^* \xi_i^*$.

Because of the anticommutativity, we have $\xi^2 = -\xi^2 = 0$ for all Grassman numbers ξ. Explicitly, we can construct the Grassman algebra as the exterior algebra on some differential forms. Thus, all analytic functions can be expressed in terms of their Taylor series

$$f(\xi) = f_0 + f_1 \xi.$$

All operators are then bilinear:

$$A(\xi^*, \xi) = a_0 + a_1 \xi + a_2 \xi^* + a_{12} \xi^* \xi.$$

We define the derivatives to be equal to the integral

$$\frac{\partial}{\partial \xi} f(\xi) = f_1 = \int \mathrm{d}\xi \, f(\xi).$$

Notably, we work in spirit analogously to the Wirtinger derivatives, and let ξ and ξ^* be independent. For reasons of anticommutativity, we require that the derivative be next to ξ in order to act on it, for example

$$\frac{\partial}{\partial \xi}(\xi^* \xi) = \frac{\partial}{\partial \xi}(-\xi \xi^*) = -\xi^*.$$

Next, we seek to deal with Gaussian integrals. We will see how they pop up later; for now, it suffices to say that the partition function is the integral of an exponential. After substituting in the fermionic coherent states, we will get something that looks like a Gaussian integral. The desired result is

Gaussian Integrals

$$\int \pi_\alpha \, d\xi^*_\alpha \, d\xi_\alpha \exp\left[-\sum_{\alpha,\beta} \xi^*_\alpha M_{\alpha,\beta} \xi_\alpha + \sum_\alpha (J^*_\alpha \xi_\alpha + \xi^*_\alpha J_\alpha)\right] = \det(M) \exp\left(\sum_{\alpha,\beta} J^*_\alpha (M^{-1})_{\alpha,\beta} J_\beta\right)$$

where the Js are Grassman variables and M is Hermitian.

We show this by diagonalising $\lambda = (\lambda_i)_{ii} = UMU^\dagger$. Then,

$$\begin{aligned} -\xi^\dagger M\xi + J^\dagger \xi + \xi^\dagger J &= -\xi^\dagger U^\dagger \lambda U \xi + J^\dagger U^\dagger U \xi + \xi^\dagger U^\dagger U J \\ &= \sum_\alpha (-\lambda_\alpha \eta^*_\alpha \eta_\alpha + \tilde{J}^\dagger_\alpha + \eta_\alpha + \eta^\dagger_\alpha \tilde{J}_\alpha \end{aligned}$$

and hence the integral simplifies to

$$\begin{aligned} &\int \prod_\alpha d\eta^\dagger_\alpha \, d\eta_\alpha \exp\left[\sum_\alpha -\lambda_\alpha \eta^\dagger_\alpha \eta_\alpha + \tilde{J}^*_\alpha \eta_\alpha + \eta^*_\alpha \tilde{J}_\alpha\right] \\ =& \prod_\alpha \int d\eta^\dagger_\alpha \, d\eta_\alpha \exp\left[-\lambda \eta^\dagger_\alpha \eta_\alpha\right] \exp\left[J^*_\alpha \eta_\alpha + \eta^*_\alpha J_\alpha\right] \\ =& \det(M) \exp\left(J^\dagger M^{-1} J\right) \end{aligned}$$

2.2.2 Wick's Theorem

Now, we are in a good position to prove Wick's theorem, the statement of which is

Wick's Theorem

$$\frac{\int \prod_\alpha d\psi^*_\alpha \, d\psi_\alpha \, e^{-\sum_{i,j} \psi^*_i M_{ij} \psi_j + \sum_i (J^*_i \psi_i + \psi^*_i J_i)}}{\int \prod_\alpha d\psi^*_\alpha \, d\psi_\alpha \, e^{-\sum_{ij} \psi^*_i M_{ij} \psi_j}} = \sum_P \zeta^P M^{-1}_{i_{P_n}, j_n} \dots M^{-1}_{i_{P_1}, j_1}.$$

To do so, we consider the generating function

$$G(J^*, J) = \frac{\int \prod_\alpha d\psi^*_\alpha \, d\psi_\alpha \, e^{-\sum_{i,j} \psi^*_i M_{ij} \psi_j + \sum_i (J^*_i \psi_i + \psi^*_i J_i)}}{\int \prod_\alpha d\psi^*_\alpha \, d\psi_\alpha \, e^{-\sum_{ij} \psi^*_i M_{ij} \psi_j}} = e^{\sum_{ij} J^*_i (M^{-1})_{ij} J_j}$$

(note that the action of dividing is to take away the det M). We differentiate the first line Comment: TODO

2.2.3 Fermionic Coherent States

Now, we move on to construct fermionic coherent states. We define that the Grassman numbers anticommute with the annihilation operators, and define

$$|\xi\rangle = e^{-\sum_\alpha \xi_\alpha a^\dagger_\alpha} |0\rangle = \prod_\alpha (1 - \xi_\alpha a^\dagger_\alpha) |0\rangle \,.$$

To show that this is a coherent state, we simply act on this with a_β:

$$a_\beta |\xi\rangle = a_\beta \prod_\alpha (1 - \xi_\alpha a^\dagger_\alpha) |0\rangle$$

$$
\begin{aligned}
&= \prod_{\alpha \neq \beta} (1 - \xi_\alpha a_\alpha^\dagger) a_\beta (1 - \xi_\beta a_\beta^\dagger) \left|0\right\rangle \\
&= \prod_{\alpha \neq \beta} (1 - \xi_\alpha a_\alpha^\dagger) \xi_\beta (1 - \xi_\beta a_\beta^\dagger) \left|0\right\rangle \\
&= \xi_\beta \prod_{\alpha} (1 - \xi_\alpha a_\alpha^\dagger) \left|0\right\rangle
\end{aligned}
$$

Note that here we used the commutativity where $\alpha \neq \beta$ as well as the relation

$$
\begin{aligned}
\alpha_\beta (1 - \xi_\beta a_\beta^\dagger) \left|0\right\rangle &= -\alpha_\beta \xi_\beta a_\beta^\dagger \left|0\right\rangle \\
&= \xi_\beta a_\beta a_\beta^\dagger \left|0\right\rangle \\
&= \xi_\beta \left|0\right\rangle \\
&= \xi_\beta (1 - \xi_\beta a_\beta^\dagger) \left|0\right\rangle
\end{aligned}
$$

Then, we can easily show the expressions for the scalar product

$$
\begin{aligned}
\langle \xi | \xi' \rangle = &\ \langle 0 | \prod_{\alpha,\beta} (1 + \xi_\alpha a_\alpha)(1 - \xi'_\beta a_\beta^\dagger) | 0 \rangle \\
&= \prod_{\alpha} (1 + \xi_\alpha^* \xi'_\alpha) \\
&= e^{\sum_\alpha \xi_\alpha^* \xi'_\alpha}
\end{aligned}
$$

Similarly, we can show that we can produce a partition of unity using

$$
1 = \int \prod_\alpha \mathrm{d}\xi_\alpha^* \, \mathrm{d}\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \left|\xi\right\rangle \left\langle \xi\right| .
$$

CHAPTER THREE

The Functional Renormalisation Group