Homework for the Lecture

Functional Analysis

Stefan Waldmann
Christopher Rudolph

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Homework Sheet No 3

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(24 Points. Discussion 04. 11. 2024)

Homework 3-1: Normification of Seminorms

Let V be a $\mathbb{K}$-valued vector space. A seminorm on V is a homogeneous map $p : V \to [0, \infty)$ satisfying the triangle inequality, i.e.

$$p(v + w) \leq p(v) + p(w) \tag{3.1}$$

and

$$p(\lambda v) = |\lambda| p(v) \tag{3.2}$$

for any two vectors $v, w \in V$ and every scalar $\lambda \in \mathbb{K}$.

i.) **(1 Point)** Show that the kernel of a seminorm p is a subspace of V.

Given a seminorm p, we say that two vectors $v, w \in V$ are equivalent if there is a vector $u \in \ker p$ such that $w = v + u$. Make yourself clear that this yields an equivalence relation $\sim$ on V.

ii.) **(1 Point)** Show that the quotient space $V/\ker p := V/\sim$ carries a canonical linear structure.

iii.) **(1 Point)** Show that the map

$$\overline{p} : V/\ker p \ni [v] \mapsto p(v) \tag{3.3}$$

yields a well-defined norm on the quotient space.

Homework 3-2: The Space $\mathscr{C}_\mathrm{b}(M)$

(5 Points) Let M be a topological space. Show that the space $(\mathscr{C}_\mathrm{b}(M), \|\cdot\|_\infty)$ of continuous and bounded $\mathbb{K}$-valued functions endowed with the supremum norm is complete.

Homework 3-3: The Space of Essentially Bounded Functions

In this exercise we weaken the conditions of Homework 3-2 by considering functions that are only essentially bounded. The goal is to find a suitable seminorm on this function space such that the corresponding quotient becomes a Banach space. But first, we shall settle the term "essentially bounded". To this end, we need the following definitions:
Let X be a set and $\mathfrak{a} \subseteq 2^X$. We call $\mathfrak{a}$ a σ-algebra if

- $\emptyset \in \mathfrak{a}$,
- $X \backslash A \in \mathfrak{a}$ for every $A \in \mathfrak{a}$ and
- $\bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{a}$ for every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathfrak{a}$.

The pair $(X, \mathfrak{a})$ is called a measurable space. One can check that for every $A \in \mathfrak{a}$ one obtains a new σ-algebra $\mathfrak{a}\big|_{X\backslash A} \subseteq 2^{X\backslash A}$, where $B \in \mathfrak{a}\big|_{X\backslash A}$ iff there is some $C \in \mathfrak{a}$ such that $B = C\backslash A$.
A function $f : (X, \mathfrak{a}) \to \mathbb{K}$ is called measurable if $f^{-1}(\mathrm{B}_r(z)) \subseteq \mathfrak{a}$ for every $z \in \mathbb{K}$ and $r > 0$. We denote the set of measurable $\mathbb{K}$-valued functions by $\mathscr{M}(X, \mathfrak{a})$. Clearly, the restriction of a measurable function $f \in \mathscr{M}(X, \mathfrak{a})$ to $X\backslash A$ yields a measurable function $f\big|_{X\backslash A} \in \mathscr{M}\big(X\backslash A, \mathfrak{a}\big|_{X\backslash A}\big)$.
Finally, a subset $\mathfrak{n} \subseteq \mathfrak{a}$ is called a σ-ideal if

- $\emptyset \in \mathfrak{n}$,
- $\bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{a}$ for every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathfrak{n}$ and
- for all $A \in \mathfrak{n}$ and $B \in \mathfrak{a}$ one has the implication $B \subseteq A \Rightarrow B \in \mathfrak{n}$.

i.) **(3 Points)** For $f \in \mathscr{M}(X, \mathfrak{a})$ we define the essential range

$$\operatorname{ess\,range}(f) := \big\{z \in \mathbb{K} : f^{-1}(\mathrm{B}_r(z)) \notin \mathfrak{n} \text{ for all } r > 0\big\} \tag{3.4}$$

and the essential supremum

$$\operatorname{ess\,sup}(f) := \sup\{|z| : z \in \operatorname{ess\,range}(f)\}. \tag{3.5}$$

Show that $\operatorname{ess\,range}(f) \subseteq \mathbb{K}$ is closed and $f^{-1}(\mathbb{K}\backslash \operatorname{ess\,range}(f)) \in \mathfrak{n}$.

ii.) **(1 Point)** Show that two functions $f, g \in \mathscr{M}(X, \mathfrak{a})$ have the same essential range if the essential range of $f - g$ contains only zero.

iii.) **(6 Points)** The set of essentially bounded functions on X is defined as

$$\mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n}) := \{f \in \mathscr{M}(X, \mathfrak{a}) : \|f\|_{\operatorname{ess\,sup}} := \operatorname{ess\,sup}(f) < \infty\}. \tag{3.6}$$

Show that $\|\cdot\|_{\operatorname{ess\,sup}}$ defines a seminorm on $\mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$ and compute its kernel. Moreover, show that the essential supremum of $f \in \mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$ is given by

$$\operatorname{ess\,sup}(f) = C_f := \inf\big\{C > 0 : |f|^{-1}([C, \infty)) \in \mathfrak{n}\big\}. \tag{3.7}$$

Hint: You can use that $\mathscr{M}(X, \mathfrak{a})$ and $\mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$ are $\mathbb{K}$-vector spaces without proof.

iv.) **(6 Points)** Show that $\mathrm{L}^\infty(X, \mathfrak{a}, \mathfrak{n}) := \mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n}) / \ker\|\cdot\|_{\operatorname{ess\,sup}}$ is a Banachspace, i.e. a complete normed space.
Hint: Consider the sequence $(f_n)_n$ on a suitable subset of X and copy your proof of Homework 3-2. You can use that a pointwise limit of a sequence of measurable functions is again measurable without proof.