7. Problemset "Quantum Algebra & Dynamics"
November 28, 2025

GNS Construction

7.1 Spins

Consider again the C^*-algebra $\mathcal{M}_2$ of 2×2-Matrices $M(a_0, \vec{a})$ parametrized by four complex numbers $(a_0, \vec{a})$, using the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

with

$$M(a_0, \vec{a}) = a_0 \mathbf{1} + \vec{a}\vec{\sigma}\,. \tag{2}$$

As you have shown in problem 5.1, the states on $\mathcal{M}_2$ can be parametrized by three real numbers $\vec{\alpha}$ with $|\vec{\alpha}| \leq 1$ and

$$\begin{aligned} \omega_{\vec{\alpha}} : \mathcal{M}_2 &\to \mathbf{C} \\ M &\mapsto \frac{1}{2} \operatorname{tr}(M\rho(1, \vec{\alpha})) \end{aligned} \tag{3}$$

using $\rho(\alpha_0, \vec{\alpha}) \in \mathcal{M}_2$. The pure states are those with $|\vec{\alpha}| = 1$.

1. Perform the GNS construction of cyclic representations $(\mathcal{H}_{\vec{\alpha}}, \pi_{\vec{\alpha}}, \Omega_{\vec{\alpha}}) = (\mathcal{H}_{\omega_{\vec{\alpha}}}, \pi_{\omega_{\vec{\alpha}}}, \Omega_{\omega_{\vec{\alpha}}})$ in the special cases

$$\vec{\alpha} = (0, 0, 0) \tag{4a}$$
$$\vec{\alpha} = (0, 0, a) \qquad \text{(with } |a| < 1) \tag{4b}$$
$$\vec{\alpha} = (0, 0, 1)\,. \tag{4c}$$

2. Give the concrete matrix realizations of

$$\pi_{\vec{\alpha}}(M(a_0, \vec{a}))\,. \tag{5}$$

3. Show that (4c) leads to an irreducible and (4a) and (4b) to reducible representations. Determine the invariant subspace(s) in the latter cases.

4. Try to repeat the above problems in the general case

$$|\vec{\alpha}| < 1 \tag{6a}$$
$$|\vec{\alpha}| = 1\,. \tag{6b}$$

7.2 Circle

Consider the algebra $C(S^1)$ of bounded complex valued continuous functions $f : S^1 \to \mathbf{C}$ on the unit circle. Perform the GNS construction of a representation for the following linear functionals $\omega : C(S^1) \to \mathbf{C}$:

$$f \mapsto \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} f(\phi) \tag{7a}$$

$$f \mapsto f(0) \tag{7b}$$