Theory and Phenomenology of Superconductivity Homework 5

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Problem 1. Show that the Ginzburg-Landau free energy of a domain wall can be written as

$$\Delta F = A\frac{u}{4}\int dx \left[\psi_0^4 - \psi^4(x)\right],$$

where $A = L^{d-1}$ is the area of the domain wall. Using this result, show that the surface tension $\sigma = \Delta F/A$ is given by

$$\sigma = \frac{\sqrt{8}}{3}\xi u \psi_0^4,$$

with ξ the correlation length. To get the final result, you will need to recall the functional form of the soliton, i.e., the solution to the Ginzburg-Landau equation through the domain wall.

Proof. Landau theory consists of expanding the free energy as a function of the order parameter near to the phase transition, when the free energy is small:

$$F_L(\psi) = \int \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 - h\psi \,\mathrm{d}V,$$

where h describes a conjugate field such as a magnetic field. To describe the possible effects of a spatially varying order parameter, we include a gradient term

$$F_{GL}(\psi) = \int \frac{s}{2}(\boldsymbol{\nabla}\psi)^2 + \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 - h\psi \,\mathrm{d}V.$$

We seek solutions, i.e. order parameter fields that minimise the Ginzburg-Landau functional. This is done by functional differentiation to obtain

$$\begin{aligned}
\delta F_{GL}(\psi) &= \int s(\boldsymbol{\nabla}\psi)\cdot\boldsymbol{\nabla}\delta\psi + r\psi\delta\psi + u\psi^3\delta\psi - h\delta\psi \,\mathrm{d}V \\
&= \int s\boldsymbol{\nabla}\cdot(\delta\psi\boldsymbol{\nabla}\psi) - s(\nabla^2\psi)\delta\psi + r\psi\delta\psi + u\psi^3\delta\psi - h\delta\psi \,\mathrm{d}V \\
&= \int -s(\nabla^2\psi)\delta\psi + r\psi\delta\psi + u\psi^3\delta\psi - h\delta\psi \,\mathrm{d}V
\end{aligned}$$

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This leads to the equation

$$-s(\nabla^2\psi) + r\psi + u\psi^3 - h = 0.$$

The domain wall we seek is a solition like solution where the order parameter goes from one minimum to the other within a finite range of x. For $h = 0$ (no external field), these minima are symmetric, and we will call them ψ_0. Defining the derivative of the Landau free energy as

$$f_L'[\psi] = r\psi + u\psi^3,$$

we can summarise by saying that a domain wall is a solution of the equation

$$s\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = f_L'[\psi(x)]$$

with asymptotic boundary conditions

$$\psi \xrightarrow{x\to\infty} = \psi_0, \qquad \psi \xrightarrow{x\to-\infty} = -\psi_0.$$

Due to the symmetry in all directions other than x, the free energy is the Ginzburg-Landau energy multiplied by the area A of the other dimensions., we find that

$$\begin{aligned}
F[\psi] &= A\int \frac{s}{2}(\boldsymbol{\nabla}\psi)^2 + \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 \,\mathrm{d}V \\
&= A\int -\frac{s}{2}\psi\nabla^2\psi + \ \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 \,\mathrm{d}V \\
&= A\int -\frac{\psi}{2}\left(r\psi + u\psi^3\right) + \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 \,\mathrm{d}V \\
&= -A\int \frac{u}{4}\psi^4 \,\mathrm{d}V
\end{aligned}$$

We need to subtract off the energy associated with a uniform configuration ψ_0, which is clearly given by

$$F_0[\psi_0] = A\int \frac{r}{2}\psi_0^2 + \frac{u}{4}\psi_0^4 \,\mathrm{d}V\,.$$

Since ψ_0 must minimise the Landau free energy, we must have

$$r\psi_0 + u\psi_0^3 = 0$$

which leads to

$$F_0[\psi_0] = A\int -\frac{u}{2}\psi_0^4 + \frac{u}{4}\psi_0^4 \,\mathrm{d}V = -A\int \frac{u}{4}\psi_0^4.$$

This, of course, could also have been derived by substituting ψ_0 for ψ in the earlier expression for $F[\psi]$. Subtracting, we find that the energy of the domain wall is given by

$$\Delta F = A\frac{u}{4}\int \mathrm{d}x\,[\psi_0^4 - \psi^4(x)].$$

Before we proceed further, we will need to solve the equation for the order parameter field explicitly. We can solve this by analogy to Newton's Second Law with f_L' as the potential. Defining $v = \frac{\mathrm{d}\psi}{\mathrm{d}x}$, we can rewrite the equation using the chain rule as

$$sv\frac{\mathrm{d}v}{\mathrm{d}\psi} = f_L'[\psi] = r\psi + u\psi^3.$$

We note that the initial conditions are that at the start, $v = 0$ and $\psi = -\psi_0$. Thus, separating variables yields

$$\begin{aligned}
\int_0^v sv\,\mathrm{d}v &= \int_{-\psi_0}^{\psi} r\psi + u\psi^3\,\mathrm{d}\psi \\
\frac{1}{2}sv^2 &= \left[\frac{r}{2}\psi^2 + \frac{u}{4}\psi^4\right]_{-\psi_0}^{\psi} \\
&= \frac{r}{2}(\psi^2 - \psi_0^2) + \frac{u}{4}(\psi^4 - \psi_0^4)
\end{aligned}$$

Again, we simplify this by noting that ψ_0 was the minimum of the Landau energy and hence satisfied the equation

$$r + u\psi_0^2 = 0.$$

Substituting, we find that

$$\begin{aligned}
\frac{1}{2}sv^2 &= -\frac{u\psi_0^2}{2}(\psi^2 - \psi_0^2) + \frac{u}{4}(\psi^4 - \psi_0^4) \\
&= u\left[-\frac{1}{2}\psi^2\psi_0^2 + \frac{1}{2}\psi_0^4 + \frac{1}{4}\psi^4 - \frac{1}{4}\psi_0^4\right] \\
&= \frac{u}{4}\left[\psi_0^4 + \psi^4 - 2\psi^2\psi_0^2\right] \\
&= \frac{u}{4}(\psi_0^2 - \psi^2)^2
\end{aligned}$$

Since $v = \psi'$ must always be of the same sign, we choose the positive sign to allow us to travel from $-\psi_0$ to ψ_0. Thus, this simplifies to

$$\begin{aligned}
v &= \sqrt{\frac{u}{2s}}(\psi_0^2 - \psi^2) \\
&= \frac{\psi_0}{\sqrt{2}\xi}\left(1 - \frac{\psi^2}{\psi_0^2}\right)
\end{aligned}$$

where ξ is defined as the correlation length $\xi = \sqrt{s/|r|}$. We can then integrate this differential equation to get

$$\frac{\psi_0}{\sqrt{2}\xi}\,\mathrm{d}x = \frac{1}{1-\frac{\psi^2}{\psi_0^2}}\,\mathrm{d}\xi\,.$$

This yields the desired solution

$$\psi(x) = \psi_0 \tanh\left(\frac{x-x_0}{\sqrt{2}\xi}\right).$$

Now all we need to do is substitute this back in to get

$$\begin{aligned}
\sigma = \frac{\Delta F}{A} &= \frac{u}{4}\psi_0^4 \int_{-\infty}^{\infty}\left[1-\tanh^4\left(\frac{x-x_0}{\sqrt{2}\xi}\right)\right]\mathrm{d}x \\
&= \frac{\xi u}{\sqrt{8}}\psi_0^4 \int_{-\infty}^{\infty}\left[1-\tanh^4(u)\right]\mathrm{d}u \\
&= \frac{\sqrt{8}}{3}\xi u \psi_0^4.
\end{aligned}$$

□

Problem 2. Consider a two-component Dirac electron moving in one dimension through a domain wall, described by the wave equation

$$\left(-i\sigma_1\nabla_x - m(x)\sigma_3\right)\psi = E\psi,$$

where the mass field forms a domain wall, changing sign at the origin according to

$$m(x) = m_0 \tanh\left(\frac{x}{\sqrt{2}\xi}\right).$$

Asymptotically, the energy of the excitation is gapped, with an excitation spectrum $E(k) = \sqrt{k^2+m_0^2}$. Show that the domain wall gives rise to a zero energy bound state and derive the form of its wave function.

Proof. Since it is given that the bound state has 0 energy, we seek such states directly:

$$(-i\sigma_1\nabla_x - m(x)\sigma_3)\psi = 0.$$

This corresponds to, in terms of the components, if $\psi = (\psi_1, \psi_2)^T$

$$\begin{pmatrix} i\partial_x\psi_2 + m(x)\psi_1 \\ i\partial_x\psi_1 + m(x)\psi_2 \end{pmatrix} = 0.$$

Alternatively, we could also write

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = -im(x)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = m(x)\sigma_2\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

This is a linear differential equation and can be solved using the matrix exponential

$$\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \exp\left(\sigma_2\int_0^x m(x)\,\mathrm{d}x\right)\begin{pmatrix}\psi_1(0)\\ \psi_2(0)\end{pmatrix}$$

While this matrix exponential can be explicitly evaluated, we can proceed in a more clever manner. The zero energy state must be normalisable, and hence must decay at $\pm\infty$. Since the function $m(x)$ is even, we can write

$$\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \exp\left(\sigma_2\int_0^{|x|} m(x)\,\mathrm{d}x\right)\begin{pmatrix}\psi_1(0)\\ \psi_2(0)\end{pmatrix}$$

from which it is clear that the integral is always positive. Now, since σ_2 has eigenvalues ±1, we must choose the negative eigenvector $(1, -i)$. Thus, we find that the state is given by, up to a constant prefactor,

$$\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \exp\left(\sigma_2\int_0^{|x|} m(x)\,\mathrm{d}x\right)\begin{pmatrix}1\\ -i\end{pmatrix} = \exp\left(-\int_0^{|x|} m(x)\,\mathrm{d}x\right)\begin{pmatrix}1\\ -i\end{pmatrix}. \qquad \square$$