Theory and Phenomenology of Superconductivity Homework 3

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Problem 1 (Problem 1)**.** Consider two reservoirs for electrons at thermal equilibrium. Refer to them as left L and right R leads which extend in the x direction. Imagine we apply small voltages to the two leads which tunes their chemical potential by eV_L and eV_R.

- Starting from the scattering matrix approach, relate the states entering and leaving each lead at the interface.
- Within the second quantization formalism, write the formula for the electrical current. **Hint:** it will be useful later to write it as an energy integration.
- Taking advantage of the elements of the scattering matrix, identify in the previous formula the transmission function.
- Identify the equilibrium conductance

  $$G = \frac{e^2}{h} \operatorname{Tr}\left[\mathbf{t}^\dagger(E_F)\,\mathbf{t}(E_F)\right], \tag{1}$$

  where $t(E_F)$ is the transmission matrix of our system evaluated at the Fermi energy E_F. What is the meaning of the eigenvalues of the hermitian matrix $\mathbf{t}^\dagger(E_F)\mathbf{t}(E_F)$?

Proof. (a) We take each lead to have n channels, each with a quantum number $\pm$ denoting whether a certain wave is moving right $(+)$ or left $(-)$. Thus, the outgoing waves for the L (R) lead are those going in the $-$ $(+)$ direction, and the incoming waves are those going in the $+$ $(-)$ direction.

Then, denoting the amplitudes for each as $\vec{\mathbf{a}}^{\pm}_{L/R}$, where $\vec{\mathbf{a}}$ is a vector denoting the n channels, the S-matrix relates the amplitudes of the ingoing and outgoing waves

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as

$$\begin{pmatrix} \vec{\mathbf{a}}_L^- \\ \vec{\mathbf{a}}_R^+ \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{r} & \mathbf{t}' \\ \mathbf{t} & \mathbf{r}' \end{pmatrix}}_{S} \begin{pmatrix} \vec{\mathbf{a}}_L^+ \\ \vec{\mathbf{a}}_R^- \end{pmatrix}.$$

Due to this constraint, the Hilbert space dimension is $2n$ instead of the $4n$ that we would get from $4n$ independent modes. We can write creation operators for the $2n$ eigenstates as $b^\dagger_{\lambda,R/L}$, where λ denotes the channel and

$$\begin{aligned} b^\dagger_{\lambda,L} &= (c^+_{\lambda,L})^\dagger + \sum_{\lambda'=1}^{n} \left[r_{\lambda'\lambda}(c^-_{\lambda',L})^\dagger + t_{\lambda'\lambda}(c^+_{\lambda',R})^\dagger \right] \\ b^\dagger_{\lambda,R} &= (c^-_{\lambda,R})^\dagger + \sum_{\lambda'=1}^{n} \left[(t')_{\lambda'\lambda}(c^-_{\lambda',L})^\dagger + (r')_{\lambda'\lambda}(c^+_{\lambda',R})^\dagger \right]. \end{aligned}$$

(b) In general, the electrical current can be determined through the derivative $j^\mu = \frac{\delta S[A]}{\delta A^\mu}$, and leads to the expression for free particles coupled to an electric field

$$\begin{aligned} \vec{\mathbf{J}} &= \vec{\mathbf{J}}^{\boldsymbol{\nabla}} + \vec{\mathbf{J}}^{\vec{\mathbf{A}}} \\ \vec{\mathbf{J}}^{\boldsymbol{\nabla}} &= \frac{\hbar}{2mi}\left[\psi^\dagger \boldsymbol{\nabla}\psi - (\boldsymbol{\nabla}\psi^\dagger)\psi\right] \\ \vec{\mathbf{J}}^{\vec{\mathbf{A}}} &= -\frac{q}{m}\vec{\mathbf{A}}\psi^\dagger\psi \end{aligned}$$

In the case of a DC current, we usually have an electric field but no magnetic field. Hence, we can set $\vec{\mathbf{A}} = 0$. We can write this as the current operator

$$\mathbf{J} = \frac{\hbar}{2mi}\left[\vec{\partial} + \overleftarrow{\partial}\right]$$

where the very dumb arrow denotes that the derivative acts to the left or to the right, because we have to write it as a matrix before second quantising it :((

We second quantise it in the typical fashion by taking the matrix elements

$$\hat{\mathbf{J}} = \sum_{\lambda\lambda'} \langle\lambda|\mathbf{J}|\lambda'\rangle\, b^\dagger_\lambda b_{\lambda'}$$

where we have cheated by letting λ include the quantum number L/R because this is a formal expression anyway. In the case of a continuous spectrum, we replace the sum by an integral

$$\hat{\mathbf{J}} = \iint_{\lambda\lambda'} \langle\lambda|\mathbf{J}|\lambda'\rangle\, b^\dagger_\lambda b_{\lambda'}\, \mathrm{d}\lambda\, \mathrm{d}\lambda'.$$

Now, we assume that the leads are coupled to thermal baths, i.e that the correlation functions are given by, with $\eta \in \{L, R\}$

$$\langle b^\dagger_{\lambda,\eta} b_{\lambda',\eta'} \rangle = \delta_{\eta\eta'} \delta_{\lambda\lambda'} f_{\mu_\eta}(E_\lambda).$$

The function f_μ is the fermi function

$$f_\mu(x) = \frac{1}{e^{\beta(x-\mu)} + 1}$$

and the chemical potentials are

$$\mu_{L/R} = eV_{L/R}.$$

□

Problem 2 (Problem 2)**.** Consider a wide conductor along x, e.g. width in y direction is large W while the height in z direction is small. The length in the x direction is large L.

Given information: The density of states of a 2D spin degenerate system is

$$\mathcal{D}_0 = \frac{m}{\pi\hbar^2}.$$

The transmission through a wire with length L is

$$T = \frac{L_0}{L + L_0},$$

where L_0 is the mean free path. The conductivity is

$$\sigma = e^2 \mathcal{D}_0 D,$$

where

$$D = \frac{v_F L_0}{\pi}$$

is the diffusion coefficient and v_F is the Fermi velocity.

Using this information, relate the previous results for conductance G with Ohm's law. I.e., does the relationship between G and conductivity σ correspond to Ohm's law?