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Problem Sheet 10
for the tutorial on July 11th, 2025
Quantum Mechanics II
Summer term 2025

Sheet handed out on July 1st, 2025; to be handed in on July 8th, 2025 until 2 pm

Exercise 10.1: Radial component of the Pauli matrix operator [2+2+2+2+2 P.]

The radial component of the Pauli matrix operator $\boldsymbol{\sigma}$ is defined as $\sigma_r := \boldsymbol{e}_r \cdot \boldsymbol{\sigma}$.

a) Show that
$$\sigma_r = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}.$$

b) In the lecture it was stated that σ_r commutes with the total angular momentum operator $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$. Show that explicitly for the component $\hat{J}_z$[1].

c) Show that the functions $\psi_{\kappa m}$ defined in the lecture are eigenfunctions of the parity operator $\hat{P}$. What are the corresponding eigenvalues[2]?

d) Explain why $\sigma_r \psi_{\kappa m}$ and $\psi_{\kappa m}$ have opposite parity.

e) According to part b) we can write that $\sigma_r \psi_{\kappa m} = a\psi_{\kappa m} + b\psi_{-\kappa m}$ with constants $a, b \in \mathbb{C}$ With the help of c) and d) show that $a = 0$.

Exercise 10.2: The K operator [7 P.]

Consider the electromagnetic 4-potential $A_\mu = \delta_{\mu 0}\phi(r)/c$. Show that the operator introduced in the lecture

$$\hat{K} = \beta\left(\frac{2}{\hbar^2}\hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{L}} + 1\right) \tag{1}$$

does commute with the Dirac Hamiltonian $\mathcal{H}_D = \gamma^\mu(\hat{p}_\mu + eA_\mu) - mc$.

[1] In spherical coordinates $\hat{L}_z = -i\hbar\,\partial/\partial\phi$.

[2] The parity operator P acting on a wave function $\psi(\boldsymbol{r})$ gives $\hat{P}\psi(\boldsymbol{r}) = \psi(-\boldsymbol{r})$. For the spherical harmonics it gives $\hat{P}Y_{lm} = (-1)^l Y_{lm}$.

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e) According to part b) we can write that $\sigma_r \psi_{\kappa m} = a\psi_{\kappa m} + b\psi_{-\kappa m}$ with constants $a, b \in \mathbb{C}$. With the help of c) and d) show that $a = 0$.

$$\hat{e}_r = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

Hence

$$\hat{e}_r \cdot \vec{\sigma} = \begin{pmatrix} \cos(\theta) & \sin(\theta)\cos(\phi) - i\sin(\theta)\sin(\phi) \\ \sin(\theta)\cos(\phi) + i\sin(\theta)\sin(\phi) & -\cos(\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta) & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos(\theta) \end{pmatrix}$$

b) $L_z = -i\hbar \frac{\partial}{\partial\phi}$

$$L_z \hat{\sigma}_r = \begin{pmatrix} 0 & J_z \sin\theta e^{-i\phi} \\ J_z \sin\theta e^{i\phi} & 0 \end{pmatrix}$$

$$\sigma_r L_z = \begin{pmatrix} 0 & \sin\theta e^{-i\phi} J_z \\ \sin\theta e^{i\phi} J_z & 0 \end{pmatrix}$$

Hence

$$[L_z, \sigma_r] = -i\hbar \begin{pmatrix} 0 & -i\sin\theta e^{-i\phi} \\ i\sin\theta e^{i\phi} & 0 \end{pmatrix}$$

$$= \hbar \begin{pmatrix} 0 & -\sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & 0 \end{pmatrix}$$

$$S_z = \frac{\hbar}{2}\sigma_z$$

$$[S_z, \sigma_r] = \frac{\hbar}{2}\begin{pmatrix} 0 & 2\sin(\theta)(\cos(\phi) - i\sin(\phi)) \\ -2\sin(\theta)(\cos(\phi) + i\sin(\phi)) & 0 \end{pmatrix}$$

$$= \hbar \begin{pmatrix} 0 & \sin\theta e^{-i\phi} \\ -\sin\theta e^{i\phi} & 0 \end{pmatrix}$$

Hence $[J_z, \sigma_r] = 0$

c) Show that the functions $\psi_{\kappa m}$ defined in the lecture are eigenfunctions of the parity operator $\hat{P}$. What are the corresponding eigenvalues[2]?

$$\Psi_{jm} = \sum_{m_1=-l}^{l} \sum_{m_2=-1/2}^{1/2} (l\ m_1\ \tfrac{1}{2}\ m_2|j\ m) Y_{lm_1}(\theta,\varphi)\chi_{m_2}\,,$$

Under the parity operator, $\varphi \to \varphi + \pi$

$\theta \to \pi - \theta$

Thus $\Psi_{jm} \to (-1)^j \Psi_{jm}$

Eigenfunction with eigenvalue 1^j

d) Explain why $\sigma_r \psi_{\kappa m}$ and $\psi_{\kappa m}$ have opposite parity.

Under the parity transformation,

$\cos\theta \to \cos(\pi-\theta) = -\cos\theta$

$\sin\theta \to \sin(\pi-\theta) = \sin\theta$

$e^{i\phi} \to e^{i\pi}e^{i\phi} = -e^{i\phi}$

$$\begin{pmatrix} \cos(\theta) & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos(\theta) \end{pmatrix} \to \begin{pmatrix} -\cos\theta & -\sin\theta e^{-i\phi} \\ -\sin\theta e^{i\phi} & \cos\theta \end{pmatrix} = -\sigma_r$$

Hence when applying the parity operator to both, we pick up a negative sign when pulling the parity operator through σ_r, flipping the parity

e) According to part b) we can write that $\sigma_r\psi_{\kappa m} = a\psi_{\kappa m} + b\psi_{-\kappa m}$ with constants $a, b \in \mathbb{C}$. With the help of c) and d) show that $a = 0$.

$$\hat{P}(\sigma_r \Psi_{\kappa m}) = \hat{P}(a\Psi_{\kappa m}) + b\hat{P}(\Psi_{-\kappa m})$$

By cancelling out all the powers of -1

$$(-1)\sigma_r \Psi_{\kappa m} = a\Psi_{\kappa m} + (-1) b \Psi_{-\kappa m}$$

adding the two equations, we get

$$a\Psi_{\kappa m} = 0 \Rightarrow a = 0$$

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does commute with the Dirac Hamiltonian $\mathcal{H}_D = \gamma^\mu(\hat{p}_\mu + eA_\mu) - mc$.

$$\mathcal{H}_D = \gamma^0\left(p_0 + e\phi(r)/c\right) + \gamma^i p_i - mc$$

$$\hat{K} = \beta + \frac{\cancel{2}\beta}{\hbar^{\cancel{2}}}\,\frac{\cancel{\hbar}}{\cancel{2}}\,\sigma_i L_i \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

Clearly mc commutes with all parts of K

Exercise 10.3: Helicities [4+2+2 P.]

In this task the operator $\hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{p}}$ is to be examined where the spin operator reads

$$\hat{\boldsymbol{S}} = \frac{\hbar}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} . \tag{2}$$

a) The wave function of a free relativistic particle with momentum $\boldsymbol{p} = \hbar \boldsymbol{k}$ is given by

$$\psi_{\pm 1/2} = \begin{pmatrix} \chi_{\pm 1/2} \\ \frac{c\hbar\vec{k}\vec{\sigma}}{E+mc^2} \chi_{\pm 1/2} \end{pmatrix} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \tag{3}$$

where $\chi_{\pm 1/2}$ is the eigenspinor of σ_z with eigenvalue ± 1. Show that the wave functions $\psi_{\pm 1/2}$ for $\boldsymbol{k} = k\boldsymbol{e}_z$ are eigenfunctions of the operator $\hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{p}}$. What are the corresponding eigenvalues?

b) Dividing the eigenvalues from a) by $\hbar^2 k$ gives us the so called helicities of the wave functions. What is the physical meaning of the helicity in this context?

c) Does $\hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{p}}$ commute with the free Dirac Hamiltonian?