Algebra und Dynamik von Quantensystemen Blatt Nr. 1

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Problem 1 (Gaussian integral for bosonic fields). Verify the functional integral

$$\int \mathcal{D}\phi \, \exp\left(-\frac{1}{2}\phi^T M \phi + J^T \phi\right) = (2\pi)^{N/2} (\det M)^{-1/2} \exp\left(\frac{1}{2} J^T M^{-1} J\right),$$

where $\phi^T = (\phi_1, \ldots, \phi_N)$ and $J^T = (J_1, \ldots, J_N)$ are real vectors, M a symmetric $N \times N$ matrix, and the integration is carried out over all the fields ϕ_i, $i = 1, \ldots, N$,

$$\int \mathcal{D}\phi \equiv \int d\phi_1 \ldots d\phi_N.$$

Proof. Since M is symmetric, we can diagonalise it orthogonally with a matrix U such that

$$\phi^T M \phi = \phi^T U^T D U \phi$$

with D diagonal. Call $\eta = U\phi$. Then □

Problem 2 (Green's function of Laplacian in two dimensions). Verify

$$\partial \bar{\partial} \ln(z\bar{z}) = \pi \delta(\tau) \delta(x),$$

where $z = x + i\tau$, $\bar{z} = x - i\tau$.

Problem 3 (Dirac Lagrangian in two dimensions). Consider the 2D Dirac Lagrangian in Minkowski space,

$$\mathcal{L}_{D,M} = \frac{1}{\pi} \bar{\Psi}_D i \partial\!\!\!/ \Psi_D,$$

where $\Psi_D = \Psi_D^\dagger \gamma^0$, $\Psi_D^\dagger = (\bar{\psi}^*, \psi^*)$, $\partial\!\!\!/ = \gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \gamma^1 \partial_1$, and

$$\Psi_D = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) Show that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, where $g^{00} = -g^{11} = 1$ is the Minkowski metric.

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(b) Obtain the equation of motion from $\mathcal{L}_{D,M}$.

(c) Verify that $\mathcal{L}_{D,M}$ is invariant under the U(1) vector symmetry $\Psi_D \to e^{i\lambda}\Psi_D$ and obtain the associated conserved current, the vector current J^μ_V.

(d) Verify that $\mathcal{L}_{D,M}$ is invariant under the U(1) axial symmetry $\Psi_D \to e^{i\lambda\gamma^5}\Psi_D$ and obtain the associated conserved current, the axial current J^μ_A.

(e) Compare the Lagrangian, as well as the results from (b)-(d), to the results we obtained in Euclidean space with complex space time coordinates $z = \tau + ix$, $\bar{z} = \tau - ix$ in class.

Problem 4 (Partial integration in the complex plane). Determine the coefficients a and b in the formula

$$\frac{1}{\pi}\int \mathrm{d}\tau\,\mathrm{d}x\,\left(\bar{\partial}f(z) + \partial\bar{f}(\bar{z})\right) = a\oint dz f(z) + b\oint d\bar{z}\bar{f}(\bar{z}),$$

where $f(z)$ and $\bar{f}(\bar{z})$ are independent functions, the $\mathrm{d}\tau\,\mathrm{d}x$ integration extends over the entire plane and the contour integrals are taken counter-clockwise around the entire z or $\bar{z}$ planes in the respective terms.