Lecture Notes

Operator-Algebraic Methods in Quantum Physics

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Preface

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Chapter 1

Observables, States, and Representations

In this introductory chapter we recall some of the basic structural aspects of quantum mechanics to clarify the notions of observables and states. To this end, we assume some mild familiarity with the basic notions of quantum mechanics. At this stage we will not yet delve into the technical analytic details, our focus is more on the *algebraic features* required by quantum theory. Thus, from a physical point of view, our definitions of the notions of observables and states are still tentative and will experience several refinements in later chapters. Even without having at least a vague interest in physical applications of operator-algebraic techniques in quantum theory, this chapter still provides the relevant algebraic definitions for the following and can be read fast-forward by those interested only in the underlying mathematical structures.

It will be the purpose of this chapter to indicate that a purely algebraic approach might lead very far but, ultimately, not far enough: for a complete picture of quantum physics analytic techniques are unavoidable in order to exclude pathological behaviour in many places and to make physically relevant predictions.

As general references on quantum mechanics and quantum field theory in a more mathematically oriented approach one may consult e.g. [1, 9, 10, 18, 19, 57].

1.1 Observables and *-Algebras

At first glance, an *observable* is a rule how a certain quantity of a physical system can be measured. Usually, this rule consists in a description of a measurement apparatus. Now, a certain quantity like e.g. the distance between two particles can be measured by very different means. Thus different measurement procedures may be used for the same abstract quantity. In this situation we call such measurements equivalent. Therefore, it appears that the honest observable should better be identified with the abstract quantity, like "distance" in our example. However, the danger with this point of view is that one might try to speak about an "observable" which actually can not be observed at all since there is no possible apparatus which can perform the corresponding measurement. This is not by the lack of technical experience but there are examples in quantum physics where one has a principle obstruction: the seemingly harmless abstract quantity like the trajectory of an electron in an atom is probably the most prominent one. There is simply no way to measure such a trajectory as one would need to observe position and velocity instantaneously. Thus we stick to the slightly more clumsy operational definition of a physical observable as an equivalence class of (at least in principle) possible measurement procedures.

The task of mathematical physics is now to obtain a mathematical model for the above physical considerations. In textbook quantum mechanics one proceeds as follows: Physical observables will be represented by certain operators A on a certain Hilbert space[1] $\mathfrak{H}$ like e.g. self-adjoint operators

[1] At the moment we use these words in a rather unspecified way without proper definitions. In the course of the

$(\operatorname{dom} A, A)$ with a suitable dense domain $\operatorname{dom} A \subseteq \mathfrak{H}$. If $\operatorname{dom} A = \mathfrak{H}$ then A will be even a continuous operator, a situation which we shall assume for the moment just for simplicity. Then, for two observables A and B also any real linear combination $\alpha A + \beta B$ is a continuous self-adjoint operator. It is one of the axioms of quantum mechanics that this should still be regarded as a valid observable, even though it is not at all clear in general how one should build the corresponding measurement apparatus. In conclusion, the observables form a real vector space. Concerning composition the situation is slightly more complicated: AB is still a continuous operator but in general it will be no longer self-adjoint since $(AB)^* = B^*A^* = BA \neq AB$. However, the anti-commutator $AB + BA$ as well as i times the commutator $\mathrm{i}[A, B]$ will be self-adjoint again. This motivates to enlarge the framework and also allow for complex linear combinations of observables as well as arbitrary compositions. This way, we obtain a complex vector space $\mathscr{A}_{\mathrm{QM}}$ of operators on $\mathfrak{H}$ which is closed under products and the involution $A \mapsto A^*$. We speak of the *observable algebra* $\mathscr{A}_{\mathrm{QM}}$ even though only those $A \in \mathscr{A}_{\mathrm{QM}}$ with $A = A^*$ are the honest observables.

It will be the observable algebra $\mathscr{A}_{\mathrm{QM}}$ and not so much the Hilbert space $\mathfrak{H}$ which carries the physically relevant structure and information about the quantum mechanical system. Hence this motivates to study the observable algebra $\mathscr{A}_{\mathrm{QM}}$ directly as the primary object of interest. It is important already at this point to stress the role played by *idealizations* in the process of building physical theories: we have never really discussed how a measurement apparatus has to be specified in order to qualify as an observable, it is not clear why all linear combinations of observables should again be observables instead of only a more selective class, and many more questions. We will come back to these matters from time to time when it will be important to motivate or even to justify certain assumptions about the mathematical nature of the models. But for the time being, we stay with the observable as the true physical information we can have about a physical system.

This shift in the point of view turned out to be extremely useful and successful in quantum physics. The following definition axiomatizes what an observable algebra should be. In particular, the reference to the Hilbert space is eliminated completely. We will have to discuss later whether this was a good idea or not.

Definition 1.1.1 (*-Algebra) *Let $\mathscr{A}$ be a vector space over $\mathbb{C}$.*

i.) The space $\mathscr{A}$ is called an algebra if it is equipped with a bilinear map

$$\mu\colon \mathscr{A} \times \mathscr{A} \ni (a, b) \mapsto \mu(a, b) \in \mathscr{A}, \tag{1.1.1}$$

called the multiplication or product of $\mathscr{A}$.

ii.) The multiplication of an algebra is called associative if

$$\mu(a, \mu(b, c)) = \mu(\mu(a, b), c) \tag{1.1.2}$$

for all $a, b, c \in \mathscr{A}$. In this case we write $ab = \mu(a, b)$.

iii.) The multiplication of an algebra is called commutative if for all $a, b \in \mathscr{A}$ one has

$$\mu(a, b) = \mu(b, a). \tag{1.1.3}$$

iv.) An element $\mathbb{1} \in \mathscr{A}$ is called unit or identity for μ if

$$\mu(a, \mathbb{1}) = a = \mu(\mathbb{1}, a) \tag{1.1.4}$$

for all $a \in \mathscr{A}$. If $\mathscr{A}$ has a unit $\mathbb{1} \neq 0$ we call it a unital algebra.

following chapters we will make precise all these notions. It is perfectly safe to think of a Hilbert space as a finite-dimensional complex vector space endowed with a positive definite sesquilinear inner product at this stage.

v.) An algebra $\mathscr{A}$ is called a *-algebra if it is an associative algebra equipped with a *-involution, i.e. a map $^*\colon \mathscr{A} \longrightarrow \mathscr{A}$ such that for all $a, b \in \mathscr{A}$ and $z, w \in \mathbb{C}$ one has

$$(za + wb)^* = \overline{z}a^* + \overline{w}b^*, \quad (ab)^* = b^*a^*, \quad \textit{and} \quad (a^*)^* = a. \tag{1.1.5}$$

There are many more interesting types of algebras than just the associative ones. In particular, the *Lie algebras* form a very important class: here the multiplication is called a *bracket*, written as $[a, b] = \mu(a, b)$, and satisfies the antisymmetry $[a, b] = -[b, a]$ as well as the *Jacobi identity*

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]] \tag{1.1.6}$$

instead of associativity. It is also clear that one can consider algebras over arbitrary (commutative) rings instead of just the complex numbers. However, for us only the associative algebras will be important and only the complex numbers will be used. Thus we refer to an associative algebra over $\mathbb{C}$ typically just as "algebra".

Clearly, in an associative algebra a unit element is necessarily unique if it exists. In a unital *-algebra we necessarily have

$$\mathbb{1}^* = \mathbb{1}. \tag{1.1.7}$$

The zero vector space is always an algebra with unit element given by the zero vector. Since this case is somehow pathological, we reserve the word unital algebra for algebras with unit different from 0. If in an algebra with unit we have $\mathbb{1} = 0$, then necessarily all elements are 0 and we are back to the above case. Finally, in a unital algebra, $a \in \mathscr{A}$ is called *invertible* if there is a (necessarily unique) $a^{-1} \in \mathscr{A}$ such that $aa^{-1} = \mathbb{1} = a^{-1}a$. In this case, a^{-1} is called the *inverse* of a.

From now on, we take a unital *-algebra $\mathscr{A}_{\mathrm{QM}}$ as the mathematical model for the observables, independently of the possibility to view them as operators on a Hilbert space. The idea is that the uncertainty relations as well as the infinitesimal time evolution via the Heisenberg equation are formulated in terms of commutation relations referring to the *associative* structure only and not to a possibly underlying Hilbert space. We will see that these two fundamental aspects of quantum physics can indeed be formulated completely algebraically. This way, we can concentrate on the algebraic relations between the observables and postpone the question of Hilbert spaces for the moment.

As known for the case of operators we have particular elements of interest in a (unital) *-algebra beside the unit element:

Definition 1.1.2 (Hermitian and unitary elements) *Let $\mathscr{A}$ be a *-algebra.*

i.) An algebra element $a \in \mathscr{A}$ is called Hermitian if $a^* = a$ and anti-Hermitian if $a^* = -a$.

ii.) An algebra element $a \in \mathscr{A}$ is called normal if $a^*a = aa^*$.

iii.) An algebra element $p \in \mathscr{A}$ is called projection if $p = p^* = p^2$.

iv.) If $\mathscr{A}$ is unital, $u \in \mathscr{A}$ is called isometric if $u^*u = \mathbb{1}$.

v.) If $\mathscr{A}$ is unital, $u \in \mathscr{A}$ is called unitary if $u^*u = \mathbb{1} = uu^*$.

In the literature there is a certain confusion about the usage of the terms Hermitian, symmetric, and self-adjoint. For us, Hermitian always will refer to a surrounding *-algebra while symmetric and self-adjoint will be used exclusively for operators defined on certain subspaces of a Hilbert space, to be discussed in detail later.

For our aspired interpretation of $\mathscr{A}_{\mathrm{QM}}$ as the observable algebra of a quantum system we again note that the true observables are the Hermitian elements of $\mathscr{A}_{\mathrm{QM}}$.

Unitary elements are those invertible elements with $u^{-1} = u^*$. Note that an isometric element needs not to be unitary. All we can say is that $p = uu^*$ is a projection for an isometry u. This is easily verified but p needs not to be $\mathbb{1}$. In fact, the unitaries and the projections will play a crucial role in the understanding of the structure of $\mathscr{A}_{\mathrm{QM}}$ both from the mathematical and physical point of view.

Whenever one introduces a new structure in mathematics it is worth to study it by means of structure preserving maps. In our case we may want to be able to compare the observable algebras $\mathscr{A}_{\mathrm{QM}}$ and $\mathscr{B}_{\mathrm{QM}}$ of two different quantum systems. Here the following notions will be helpful:

Definition 1.1.3 (*-Homomorphisms) *Let $\mathscr{A}$ and $\mathscr{B}$ be algebras.*

i.) A homomorphism from $\mathscr{A}$ to $\mathscr{B}$ is a linear map $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ such that for all $a, b \in \mathscr{A}$ one has

$$\Phi(\mu_{\mathscr{A}}(a, b)) = \mu_{\mathscr{B}}(\Phi(a), \Phi(b)). \tag{1.1.8}$$

ii.) In the case of *-algebras, a *-homomorphism from $\mathscr{A}$ to $\mathscr{B}$ is a homomorphism $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ such that in addition for all $a \in \mathscr{A}$ one has

$$\Phi(a^*) = \Phi(a)^*. \tag{1.1.9}$$

iii.) In the case of unital algebras, a homomorphism $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is called unital (or unit preserving) if $\Phi(\mathbb{1}_{\mathscr{A}}) = \mathbb{1}_{\mathscr{B}}$.

In general, we do not require that a homomorphism between unital algebras preserves the unit element. However, $\Phi(\mathbb{1}_{\mathscr{A}})$ is not far from being a unit element in the following sense: $e = \Phi(\mathbb{1}_{\mathscr{A}})$ is an *idempotent* element of $\mathscr{B}$, i.e. $e^2 = e$. In the case of a *-homomorphism, $e = e^*$ is even a projection.

The following lemma is now a standard statement and requires only a simple verification for proving:

Lemma 1.1.4 *Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be *-algebras.*

i.) For *-homomorphisms $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ and $\Psi\colon \mathscr{B} \longrightarrow \mathscr{C}$ also $\Psi \circ \Phi\colon \mathscr{A} \longrightarrow \mathscr{C}$ is a *-homomorphism. ▮

ii.) The identity map $\mathrm{id}_{\mathscr{A}}\colon \mathscr{A} \longrightarrow \mathscr{A}$ is a *-homomorphism.

iii.) The *-algebras over $\mathbb{C}$ form a category with respect to the *-homomorphisms as morphisms.

We denote the resulting category by *-alg and use *-Alg for the sub-category of unital *-algebras and *-homomorphisms as morphisms. A particular role will be played by the *-*isomorphisms* and the "self *-isomorphisms" called the *-*automorphisms* of the algebra $\mathscr{A}$. We denote the set of *-automorphisms of a *-algebra by $^*\text{-}\operatorname{Aut}(\mathscr{A})$. From general nonsense it is clear that $^*\text{-}\operatorname{Aut}(\mathscr{A})$ forms a group, see also Exercise 1.5.6.

Of similar importance is the concept of an "infinitesimal homomorphism". Heuristically, this can be motivated by considering a curve of homomorphisms $\Phi_t = \mathrm{id} + tD + \cdots$ in a parameter t for small t. Then the homomorphism property of Φ_t becomes in order t the condition

$$D(\mu(a, b)) = \mu(Da, b) + \mu(a, Db) \tag{1.1.10}$$

for all $a, b \in \mathscr{A}$. A linear endomorphism of $\mathscr{A}$ with this property is called a *derivation* of $\mathscr{A}$. If $\mathscr{A}$ is even a *-algebra a derivation D is called a *-*derivation* if in addition one has

$$D(a^*) = (Da)^* \tag{1.1.11}$$

for all $a \in \mathscr{A}$. The set of derivations of $\mathscr{A}$ will be denoted by $\operatorname{Der}(\mathscr{A})$ and for a *-algebra we denote by $^*\text{-}\operatorname{Der}(\mathscr{A})$ the *-derivations of $\mathscr{A}$. Note that the notion of a homomorphism and a derivation makes sense also for e.g. Lie algebras. Some first properties of derivations can be found in Exercise 1.5.2.

There is yet another reason why we should study structure preserving maps for the observables. It will turn out that such maps can be used to encode the *time evolution* of the physical system under consideration: at first sight this seems to be quite a strange an idea. The measurement apparatus will be the same today and tomorrow so there should be no evolution of what we mean by e.g. "measuring the energy". The time evolution should better be encoded in a changing of the current *state* of the system as we shall discuss in the next section. Nevertheless, from a mathematical point of view it turns out to be an equivalent description which sometimes has enormous advantages. Thus we will state already here the following definition:

Definition 1.1.5 (Time evolution) *Let $\mathscr{A}$ be the *-algebra of observables of a physical system. Then its time evolution is encoded in a one-parameter group of *-automorphisms of $\mathscr{A}$, i.e. a map*

$$\Phi\colon \mathbb{R} \times \mathscr{A} \ni (t, a) \mapsto \Phi(t, a) = \Phi_t(a) \in \mathscr{A}, \tag{1.1.12}$$

such that

i.) for every $t \in \mathbb{R}$ the map $\Phi_t\colon \mathscr{A} \longrightarrow \mathscr{A}$ is a unital *-automorphism,

ii.) for every $t, s \in \mathbb{R}$ one has the one-parameter group properties

$$\Phi_t \circ \Phi_s = \Phi_{t+s} \quad \textit{and} \quad \Phi_0 = \mathrm{id}_{\mathscr{A}}\,. \tag{1.1.13}$$

Alternatively, we can view Φ as a group morphism

$$\Phi\colon \mathbb{R} \ni t \mapsto \Phi_t \in {}^*\text{-}\operatorname{Aut}(\mathscr{A}) \tag{1.1.14}$$

from the additive group $(\mathbb{R}, +)$ into the group of *-automorphisms of the observable algebra $\mathscr{A}$.

The reasons to chose a one-parameter group is justified for an *autonomous system* not interacting with an environment: here it should make no difference whether we just wait two times one second or once two seconds to see the evolution of the system. This of course changes when we want to incorporate also *open systems* where the way how the system evolves also depends on the absolute time: we can switch on an interaction with the environment at some point and switch it of some time later. Then the evolution of the system after one second will depend on whether this one second was during the time of interaction or not.

We arrive at the conclusion that the *kinematics* of a physical system consists in describing its observable algebra while the *dynamics* also requires to specify a time evolution in the sense of Definition 1.1.5, i.e. a one-parameter group of automorphisms of the observable algebra.

1.2 States for *-Algebras

For a physical system the observables are those quantities which can in principle be measured. The *state* of the system describes what the current situation of the system actually is. In the state of the system, all possible answers to the measurements of observables should be encoded. From this point of view, the purpose of a state is to assign to a given observable a *number* which is the outcome of the corresponding measurement. While this idea is perfectly adequate for a classical system in a "pure" state we have to be slightly more precise for quantum systems and also for "mixed" states. In quantum mechanics the best we can hope for is that after many repetitions of the measurement in an identically prepared system, i.e. in one with the same state again and again, the measured numbers produce a reliable expectation value. Thus a state should give an *expectation value* for a given observable. The idea is therefore to axiomatize the properties of the expectation values as known from textbook quantum mechanics in a reasonable way such that we can re-use it also for general *-algebras.

We briefly recall the situation from quantum mechanics. In order to avoid analytic difficulties at the present stage, one should again think of a finite-dimensional Hilbert space in the following: in this case we are on the save terrain of linear algebra.

Thus let A be a Hermitian operator on $\mathfrak{H}$ and $\psi \in \mathfrak{H}$ a non-zero vector. Quantum mechanics tells us that the expectation value of the observable corresponding to A in the state corresponding to ψ is given by

$$\mathrm{E}_\psi(A) = \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle}. \tag{1.2.1}$$

Note that $\mathrm{E}_\psi(A)$ does not depend on ψ itself but only on the ray through ψ, i.e. on the one-dimensional subspace spanned by ψ. More generally, a mixed state is described by means of a *density matrix* ϱ on $\mathfrak{H}$: this is a positive matrix with trace normalized to $\mathrm{tr}(\varrho) = 1$. Then the expectation value of A in the mixed state corresponding to ϱ is given by

$$\mathrm{E}_\varrho(A) = \mathrm{tr}(\varrho A). \tag{1.2.2}$$

Note that (1.2.1) is a particular case of (1.2.2) if we use for ϱ the orthogonal projection P_ψ onto the one-dimensional subspace spanned by ψ, i.e.

$$P_\psi \phi = \frac{\langle \psi, \phi \rangle}{\langle \psi, \psi \rangle} \psi. \tag{1.2.3}$$

For a finite-dimensional Hilbert space one has the following properties which are easy to deduce. Note that for the more realistic and more interesting infinite-dimensional case we have not yet even defined what a trace or a density matrix should be: we have to postpone this till Section 6.2.4.

Lemma 1.2.1 *Let $\mathfrak{H}$ be a finite-dimensional Hilbert space, ϱ a density matrix, and A an operator on $\mathfrak{H}$. Then the expectation value E_ϱ as in* (1.2.2) *satisfies*

i.) $\mathrm{E}_\varrho \colon A \mapsto \mathrm{E}_\varrho(A)$ *is linear,*

ii.) $\mathrm{E}_\varrho(A^*A) \geq 0$,

iii.) $\mathrm{E}_\varrho(\mathbb{1}) = 1$.

In fact, the three properties characterize the expectation value functionals completely, see Exercise 1.5.9.

We take these three properties now as motivation to define what a state of an abstractly given *-algebra should be. First we *identify* a state with the expectation value functional. Thus a state will be a particular type of map from the observable algebra to the scalars. Then the required properties are the following:

Definition 1.2.2 (State) *Let $\mathscr{A}$ be a *-algebra.*

i.) A linear functional $\omega \colon \mathscr{A} \longrightarrow \mathbb{C}$ is called positive if for all $a \in \mathscr{A}$ one has

$$\omega(a^*a) \geq 0. \tag{1.2.4}$$

ii.) For a unital *-algebra a positive functional ω is called a state if

$$\omega(\mathbb{1}) = 1. \tag{1.2.5}$$

As we shall see, it is not a big requirement to assume that a physical observable algebra is unital: the unit element plays the role of an idealistic "measurement" which will produce the expectation value 1 in all possible states of the system. Thus, as soon as we have a positive linear functional ω with $\omega(\mathbb{1}) \neq 0$ we can rescale it to be state.

Remarkable, with this definition of states the states appear as a *derived concept* while the observables are the primary object attached to a physical system. Indeed, once $\mathscr{A}_{\mathrm{QM}}$ is specified as a *-algebra the set of states is *determined* via Definition 1.2.2. This point of view is quite different from the textbook approaches to quantum mechanics where the Hilbert space and thus the pure states as the one-dimensional subspaces are the starting point. We will have to discuss and justify our framework in the following. In fact, both approaches will turn out to be essentially equivalent.

Both points of view have their advantages and their difficulties. To focus on the observable algebra turns out to be advantageous e.g. in quantization theories where the starting point is the classical observable algebra, as well as in more axiomatic approaches to quantum field theories.

We will discuss now some first simple properties of positive linear functionals. The following statement is obvious:

Lemma 1.2.3 *Let $\omega_1, \omega_2 \colon \mathscr{A} \longrightarrow \mathbb{C}$ be positive linear functionals and let $\alpha_1, \alpha_2 > 0$. Then*

$$\omega = \alpha_1\omega_1 + \alpha_2\omega_2 \colon \mathscr{A} \longrightarrow \mathbb{C} \tag{1.2.6}$$

is again a positive linear functional. If in addition $\mathscr{A}$ is unital, ω_1, ω_2 are states, and $\alpha_1 + \alpha_2 = 1$ then ω is a state again.

Thus the positive linear functionals form a *convex cone* in the dual space of $\mathscr{A}$ which we shall denote sometimes by $\mathscr{A}^*_+$. The states of a unital *-algebra form a convex subset of the dual $\mathscr{A}^*$, though no longer a cone. For such convex subsets the *extremal* points deserve particular interest:

Definition 1.2.4 (Mixed and pure states) *Let $\mathscr{A}$ be a unital *-algebra and ω a state. Then ω is called a mixed state if there are states $\omega_1 \neq \omega_2$ and $0 < \alpha < 1$ such that*

$$\omega = \alpha\omega_1 + (1-\alpha)\omega_2. \tag{1.2.7}$$

If ω is not mixed then ω is called a pure state.

In our motivating example of a finite-dimensional Hilbert space as in Lemma 1.2.1 one immediately observes that the state E_ϱ corresponding to a density matrix ϱ is pure iff $\varrho = P_\psi$ with a non-zero vector $\psi \in \mathfrak{H}$, see Exercise 1.5.10. Thus the above definition indeed gives a good characterization of pure and mixed states as required by quantum mechanics. We will see later that the analogous result still holds for infinite-dimensional Hilbert spaces and the *-algebra of bounded operators, at least under some mild regularity assumptions on the states.

The next property is slightly less obvious but of central importance for the understanding of positive functionals:

Lemma 1.2.5 (Cauchy-Schwarz inequality) *Let $\omega \colon \mathscr{A} \longrightarrow \mathbb{C}$ be a positive linear functional. Then we have*

$$\omega(a^*b) = \overline{\omega(b^*a)} \tag{1.2.8}$$

and

$$\omega(a^*b)\overline{\omega(a^*b)} \leq \omega(a^*a)\omega(b^*b) \tag{1.2.9}$$

for all $a, b \in \mathscr{A}$. If $\mathscr{A}$ is unital we have

$$\omega(a^*) = \overline{\omega(a)}, \tag{1.2.10}$$

and $\omega(\mathbb{1}) = 0$ implies $\omega = 0$.

Proof: The proof relies on the following elementary fact on positive quadratic forms: for $\alpha, \beta, \beta', \gamma \in \mathbb{C}$ one considers the polynomial

$$p(z,w) = \alpha z\overline{z} + \beta z\overline{w} + \beta'\overline{z}w + \gamma w\overline{w}$$

for $z, w \in \mathbb{C}$. Then $p(z,w) \geq 0$ for all $z, w \in \mathbb{C}$ implies

$$\alpha \geq 0, \quad \gamma \geq 0, \quad \text{and} \quad \beta' = \overline{\beta}, \tag{$*$}$$

as well as

$$\beta\overline{\beta} \leq \alpha\gamma. \tag{$**$}$$

Indeed, this statement is well-known and can be obtained by evaluating p on suitable values for z and w, see also Exercise 1.5.11. We apply this now for $p(z, w) = \omega((za+wb)^*(za+wb)) \geq 0$. Sesquilinear evaluation of p gives immediately (1.2.8) and (1.2.9) from $(*)$ and $(**)$. Setting $b = \mathbb{1}$ in the unital case gives (1.2.10). Finally, $\omega(\mathbb{1}) = 0$ implies $|\omega(a)|^2 \leq 0$ by (1.2.9) and (1.2.10) for all $a \in \mathscr{A}$. ☐

It is this Cauchy-Schwarz inequality which makes a state $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ behave like an expectation value functional. We have the following properties:

Remark 1.2.6 (Expectation values, variance, and covariance) Let $\mathscr{A}$ be a unital *-algebra and let $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a state.

i.) For a Hermitian element $a = a^* \in \mathscr{A}$ one considers the *expectation value* of a in the state ω defined by

$$\mathrm{E}_\omega(a) = \omega(a). \tag{1.2.11}$$

It is a real quantity in accordance with our interpretation that a is an observable which should have a real expectation value in every state. Moreover, the expectation value of a "square" a^*a is non-negative.

ii.) The *variance* $\mathrm{Var}_\omega(a)$ of a in the state ω is defined as usual by

$$\mathrm{Var}_\omega(a) = \omega((a - \omega(a)\mathbb{1})^*(a - \omega(a)\mathbb{1})). \tag{1.2.12}$$

The positivity of ω gives $\mathrm{Var}_\omega(a) \geq 0$ and a bilinear evaluation yields

$$\mathrm{Var}_\omega(a) = \omega(a^*a) - \overline{\omega(a)}\omega(a). \tag{1.2.13}$$

Note that if we want to interpret $\omega(a)$ as an expectation value this only makes sense if the variance is non-negative: the positivity of ω guarantees this feature.

iii.) The covariance matrix for $a_1, \ldots, a_n \in \mathscr{A}$ is defined to be the $n \times n$ matrix with entries given by

$$\mathrm{Cov}_\omega(a_i, a_j) = \omega((a_i - \omega(a_i)\mathbb{1})^*(a_j - \omega(a_j)\mathbb{1})). \tag{1.2.14}$$

Again, as required by any reasonable interpretation, the matrix $(\mathrm{Cov}_\omega(a_i, a_j)) \in \mathrm{M}_n(\mathbb{C})$ is a positive semi-definite matrix. This follows again from the positivity of ω, see also Exercise 1.5.14.

For the variances of observables $a, b \in \mathscr{A}$ we have inequalities which are determined by the *algebraic* structure of $\mathscr{A}$ alone. The most prominent one is the uncertainty relation:

Proposition 1.2.7 (Uncertainty relation) *Let $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a state on a unital *-algebra $\mathscr{A}$. Then*

$$4\,\mathrm{Var}_\omega(a)\,\mathrm{Var}_\omega(b) \geq \left|\omega([a, b])\right|^2 \tag{1.2.15}$$

holds for all Hermitian elements $a, b \in \mathscr{A}$.

The proof of this simple proposition is contained in Exercise 1.5.13. This formulation shows that the algebraic features, namely the commutation relations in $\mathscr{A}$, determine the uncertainty relations in a *universal* manner for all states. The state ω is only responsible for the precise numerical values in (1.2.15).

In fact, this observation is the key to construct quantum mechanical observable algebras. Usually one has a reasonable, physically motivated intuition how the uncertainty relations among the relevant physical observables should look like. From this idea one tries to construct a *-algebra structure for these observables whose commutation relations guarantee the uncertainty relations one is looking for. In general, this is a highly non-trivial task lacking a general construction beyond this heuristics.

Already at this state one can use the positive linear functionals of a *-algebra to define *positive algebra elements*. Having in mind that $\omega(a)$ is the expectation value of a in the state ω we define "positivity by measurement":

Definition 1.2.8 (Positive algebra elements) *Let $\mathscr{A}$ be a *-algebra over $\mathbb{C}$. Then $a \in \mathscr{A}$ is called positive if for all positive linear functionals $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ one has*

$$\omega(a) \geq 0. \tag{1.2.16}$$

The set of positive algebra elements in $\mathscr{A}$ is denoted by $\mathscr{A}^+$.

Remark 1.2.9 Let $\mathscr{A}$ be a *-algebra over $\mathbb{C}$.

i.) It is clear that the notion of positive linear functionals is invariant under *-homomorphisms in the sense that for a *-homomorphism $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ into another *-algebra $\mathscr{B}$ and for a positive linear functional $\omega\colon \mathscr{B} \longrightarrow \mathbb{C}$ also the *pull-back* $\Phi^*\omega = \omega \circ \Phi\colon \mathscr{A} \longrightarrow \mathbb{C}$ is positive. In the unital case, states pull back to states for unital *-homomorphisms.

ii.) For $b \in \mathscr{A}$ and a positive linear functional $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ also the functional

$$\omega_b\colon \mathscr{A} \ni a \;\mapsto\; \omega_b(a) = \omega(b^*ab) \in \mathbb{C} \tag{1.2.17}$$

is positive, since $\omega_b(a^*a) = \omega(b^*a^*ab) = \omega((ab)^*(ab)) \geq 0$.

iii.) The positive elements $\mathscr{A}^+ \subseteq \mathscr{A}$ form a convex cone and are mapped to positive elements under *-homomorphisms. Moreover, for $a \in \mathscr{A}^+$ and $b \in \mathscr{A}$ we have $b^*ab \in \mathscr{A}^+$. These claims follow immediately from *i.)* and *ii.)*.

iv.) Elements of the particular form

$$a = \sum_{i=1}^{n} \alpha_i a_i^* a_i \tag{1.2.18}$$

with $\alpha_i > 0$ and $a_i \in \mathscr{A}$ are clearly positive. One calls them *algebraically positive elements* or *sums of squares (SOS)*. We denote them by $\mathscr{A}^{++}$. Again, $\mathscr{A}^{++}$ forms a convex cone which is mapped into the algebraically positive elements under *-homomorphisms and under the operation $a \mapsto b^*ab$ for arbitrary $b \in \mathscr{A}$. It is an interesting and highly non-trivial question whether $\mathscr{A}^{++} = \mathscr{A}^+$, the answer of which strongly depends on $\mathscr{A}$, see also Exercise 1.5.16.

As the positive elements of a *-algebra $\mathscr{A}$ play a crucial role in the understanding of many aspects of $\mathscr{A}$, one may be interested in those linear maps which preserve them. Clearly, *-homomorphisms map positive elements to positive elements as noted in Remark 1.2.9, *iii.)*. However, there may be more general maps with this feature motivating the following definition:

Definition 1.2.10 (Positive maps) *Let $\mathscr{A}$ and $\mathscr{B}$ be *-algebras over $\mathbb{C}$ and let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a linear map.*

i.) The map Φ is called positive if $\Phi(\mathscr{A}^+) \subseteq \mathscr{B}^+$, i.e. if Φ maps positive elements to positive elements.

ii.) The map Φ is called n-positive if the componentwise map

$$\Phi^{(n)}\colon \mathrm{M}_n(\mathscr{A}) \longrightarrow \mathrm{M}_n(\mathscr{B}) \tag{1.2.19}$$

is positive, where the $n \times n$-matrices with entries in a *-algebra are endowed with the canonical *-algebra structure given by matrix multiplication and $(a_{ij})^* = (a_{ji}^*)$ as *-involution.

iii.) The map Φ is called completely positive if Φ is n-positive for all $n \in \mathbb{N}$.

It is a remarkable fact that for non-commutative *-algebras the notions of positive, n-positive, and completely positive maps typically *differ*: already for the complex 2×2-matrices $\mathrm{M}_2(\mathbb{C})$ the transposition ${}^{\mathrm{T}}\colon \mathrm{M}_2(\mathbb{C}) \longrightarrow \mathrm{M}_2(\mathbb{C})$ is positive but not 2-positive. In general, $(n+1)$-positive maps are also n-positive but not necessarily vice versa. Since the positive elements $\mathscr{A}^+$ are a convex cone stable under the maps $a \mapsto b^*ab$, the same is true for all three sorts of positive maps, more details can be found in Exercise 1.5.17.

The physical interpretation is now that completely positive maps correspond to all sort of operations which can be performed on a quantum system including measurements. This point of view is central in quantum information theory, see e.g. [6, 25, 40] for introductory textbooks. We will come back to this interpretation when dealing with compound systems and tensor products.

Remark 1.2.11 (Strong positivity) For a *-algebra $\mathscr{A}$ there might be other, more restrictive notions of positive linear functionals which are more relevant: one might want to require e.g. continuity in some a priori given topology or compatibility with other structures present in a particular situation. This way, one can specify a *sub-cone* $\mathscr{K}$ of particular positive linear functionals on $\mathscr{A}$, which for technical reasons will be required to be stable under $a \mapsto b^*ab$. Having *less* positive functionals this results by the analogous definition in *more* positive elements, now referred to as $\mathscr{K}$-positive elements. Also, one has now $\mathscr{K}$-positive maps and completely $\mathscr{K}$-positive maps. Note however, that unlike the canonical choice for positive linear functionals these notions depend on the choice of $\mathscr{K}$. In particular, it will be more difficult to get a good functorial behaviour. Nevertheless, in the theory of unbounded operator *-algebras (O^*-algebras) this plays a crucial role and leads to the notion of strong positivity, see e.g. [51, Sect. 2.6] as well as [52].

Let us now come back to the question of time evolution: in the more conventional approaches both to classical and quantum mechanics it will be the states which evolve while the observables stay the same. From that point of view, a time evolution would be a one-parameter group of bijections between states, probably subject to additional conditions. However, from an operational point of view, the only thing which may evolve in time are the expectation values themselves, i.e. the pairing of the expectation value functionals with the observables. Now if the time evolution is encoded by a one-parameter group of *-automorphisms Φ_t of $\mathscr{A}$ and if ω is a state on $\mathscr{A}$, then also the pull-back

$$\omega_t = \omega \circ \Phi_t = \Phi_t^* \omega \tag{1.2.20}$$

of ω by Φ_t is a state, see Remark 1.2.9, *i.)*. Thus the time evolution of the observable algebra *induces* one for the states. In the expectation value

$$\mathrm{E}_{\omega_t}(a) = \omega_t(a) = \omega(\Phi_t(a)) = \mathrm{E}_\omega(\Phi_t(a)) \tag{1.2.21}$$

the difference between the two points of view disappears: both lead to the same results for the observable quantities and hence will be physically indistinguishable. Note, however, that there will be other time evolutions of states not based on automorphisms of the observable algebra since we can replace Φ_t by a curve of arbitrary *positive maps*. In fact, in the theory of open systems this is one way to encode time evolution.

1.3 *-Representations and the GNS Construction

Having replaced the operators on a Hilbert space $\mathfrak{H}$ by an abstract *-algebra as observable algebra and having also replaced the expectation value functionals with respect to state vectors in $\mathfrak{H}$ or density matrices by general positive linear functionals one should of course ask: why do we need a Hilbert space at all?

At a first glance it seems that we indeed have everything we need to get a full quantum mechanical description: the observable algebra describes the possible measurements, the states encode the expectation values of the observables, the commutation relations guarantee the uncertainty relations between the expectation values and their variances in all possible states, and the dynamics is encoded as a one-parameter group of *-automorphisms of the observable algebra.

However, one crucial feature of quantum physics is not yet implemented: it is probably the most puzzling phenomenon of quantum physics that there is a *superposition principle* for the pure states of a quantum system. In the usual approach this means that for two non-zero vectors $\phi, \psi \in \mathfrak{H}$ one can form any linear combination $z\psi + w\phi$ which results again in a physically valid state vector if non-zero. A similar structure does not seem to be possible on the level of expectation value functionals: here we can form convex combinations but this results in a mixed state while the superposition $z\psi + w\phi$ is pure. Since the cross-terms $\langle \phi, \psi \rangle$ play physically the role of transition amplitudes we indeed need the above possibility of superpositions. More mathematically speaking, the above superposition $z\psi + w\phi$ depends on the vectors ψ and ϕ themselves and not just on the physical states they represent, i.e. the complex one-dimensional subspaces they span. Since there is physical information encoded in $z\psi + w\psi$ we need the Hilbert space itself and not just its projective space.

For this reason one is interested in passing from the abstract notion of a *-algebra and positive linear functionals to the more concrete one of operators and Hilbert spaces. However, even beyond this motivation from physics the question will be interesting from a structural point of view: one would like to compare the abstractly given *-algebras with the somehow more easy *-algebras of certain linear endomorphisms. It is a general approach to understand an algebra by looking at its modules. Since at the moment it is not yet clear where and how analytic features enter the game we stay on a purely algebraic level. We recall the following definition:

Definition 1.3.1 (Pre-Hilbert space) *A complex vector space $\mathfrak{H}$ endowed with a positive definite sesquilinear inner product, i.e. a map*

$$\langle \,\cdot\,, \,\cdot\, \rangle \colon \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{C}, \tag{1.3.1}$$

such that for all $z, w \in \mathbb{C}$ and $\phi, \psi, \chi \in \mathfrak{H}$ one has

i.) $\langle \phi, z\psi + w\chi \rangle = z\langle \phi, \psi \rangle + w\langle \phi, \chi \rangle$,

ii.) $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$,

iii.) $\langle \phi, \phi \rangle > 0$ *for* $\phi \neq 0$,

is called a pre-Hilbert space.

A map with *i.)* and *ii.)* is called a *sesquilinear form.* We follow the convention that inner products on complex vector spaces are linear in the *second* argument, and thus, by *ii.)*, antilinear in the first argument. If we drop the condition *iii.)* and only require to have a non-degenerate sesquilinear form, i.e. $\langle \phi, \psi \rangle = 0$ for all ψ implies $\phi = 0$, then we call $\mathfrak{H}$ an *inner product space.* If we have a sesquilinear form with $\langle \phi, \phi \rangle \geq 0$ then we call it *positive semi-definite.*

As already for *-algebras we are looking for an appropriate notion of structure compatible maps between pre-Hilbert spaces. It turns out that the adjointable maps are the good choice:

Definition 1.3.2 (Adjointable map) *Let $\mathfrak{H}_1, \mathfrak{H}_2$ be pre-Hilbert spaces and let $A \colon \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$ be a map. Then A is called adjointable if there is a map $A^* \colon \mathfrak{H}_2 \longrightarrow \mathfrak{H}_1$ such that*

$$\langle \phi, A\psi \rangle_2 = \langle A^*\phi, \psi \rangle_1 \tag{1.3.2}$$

for all $\psi \in \mathfrak{H}_1$ and $\phi \in \mathfrak{H}_2$. The set of adjointable maps is denoted by $\mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$.

For a single pre-Hilbert space we set

$$\mathfrak{B}(\mathfrak{H}) = \mathfrak{B}(\mathfrak{H}, \mathfrak{H}) \tag{1.3.3}$$

for abbreviation. Note that in the definition we require A as well as A^* to be defined on all of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. The choice of possibly smaller domains will play a crucial role later but shall not be considered in the current discussion, see Section 5.2 instead. The following properties are now deduced easily:

Proposition 1.3.3 *Let $\mathfrak{H}_1$, $\mathfrak{H}_2$, and $\mathfrak{H}_3$ be pre-Hilbert spaces over $\mathbb{C}$ and let $A, B \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $C \in \mathfrak{B}(\mathfrak{H}_2, \mathfrak{H}_3)$ be adjointable maps.*

i.) The map A is linear and the adjoint A^* is unique.

ii.) For all $z, w \in \mathbb{C}$ the operators $zA + wB$, CA, and A^* are adjointable and

$$(zA + wB)^* = \overline{z}A^* + \overline{w}B^*, \tag{1.3.4}$$

$$(CA)^* = A^*C^*, \tag{1.3.5}$$

$$(A^*)^* = A. \tag{1.3.6}$$

iii.) For a pre-Hilbert space $\mathfrak{H}$ the adjointable operators $\mathfrak{B}(\mathfrak{H})$ form a unital *-algebra with unit element $\mathbb{1} = \mathrm{id}_{\mathfrak{H}}$ and *-involution $A \mapsto A^*$.

Proof: Let A be adjointable and $\phi \in \mathfrak{H}_2$ and $\psi, \chi \in \mathfrak{H}_1$ as well as $z, w \in \mathbb{C}$. Then we compute

$$\langle \phi, A(z\psi + w\chi)\rangle_2 = \langle A^*\phi, z\psi + w\chi\rangle_1 = z\langle A^*\phi, \psi\rangle + w\langle A^*\phi, \chi\rangle_1 = \langle \phi, zA\psi + wA\psi\rangle_2,$$

which is enough to show that A is linear since the inner product is non-degenerate. Analogously, one observes that for two adjoints A_1^* and A_2^* of A we have $\langle A_1^*\phi, \psi\rangle_1 = \langle \phi, A\psi\rangle_2 = \langle A_2^*\phi, \psi\rangle_1$. Again, by non-degeneracy of the inner product we conclude that the adjoint is unique. For the second part, one shows that $zA + wB$, CA, and A^* are adjointable by explicitly verifying the properties (1.3.4), (1.3.5), and (1.3.6), respectively. Then the last part is clear. □

We can thus transfer the notions of Hermitian, normal, isometric, and unitary elements from *-algebras to $\mathfrak{B}(\mathfrak{H})$. Note that then unitary and isometric elements really become the unitary and isometric maps, i.e. those which preserve inner products, see also Exercise 1.5.18. We also can speak now of unitary and isometric maps between different pre-Hilbert spaces.

Since for a pre-Hilbert space $\mathfrak{H}$ the adjointable operators $\mathfrak{B}(\mathfrak{H})$ are a (unital) *-algebra it is now easy to define what a *-representation of a given *-algebra $\mathscr{A}$ on $\mathfrak{H}$ should be:

Definition 1.3.4 (*-Representation) *Let $\mathscr{A}$ be a *-algebra over $\mathbb{C}$.*

i.) A *-representation of $\mathscr{A}$ on a pre-Hilbert space $\mathfrak{H}$ is a *-homomorphism

$$\pi\colon \mathscr{A} \longrightarrow \mathfrak{B}(\mathfrak{H}). \tag{1.3.7}$$

ii.) An intertwiner $T\colon (\mathfrak{H}, \pi) \longrightarrow (\mathfrak{H}', \pi')$ between two *-representations of $\mathscr{A}$ is an adjointable map $T \in \mathfrak{B}(\mathfrak{H}, \mathfrak{H}')$ such that for all $a \in \mathscr{A}$ one has

$$T\pi(a) = \pi'(a)T. \tag{1.3.8}$$

iii.) The category of *-representations of $\mathscr{A}$ on pre-Hilbert spaces with intertwiners as morphisms is denoted by *-$\mathsf{rep}(\mathscr{A})$ and called the *-representation theory of $\mathscr{A}$.

Explicitly, a map $\pi\colon \mathscr{A} \longrightarrow \mathfrak{B}(\mathfrak{H})$ is a *-representation iff for all $a, b \in \mathscr{A}$ and $z, w \in \mathbb{C}$

$$\pi(za + wb) = z\pi(a) + w\pi(b), \tag{1.3.9}$$

$$\pi(ab) = \pi(a)\pi(b), \tag{1.3.10}$$

$$\pi(a^*) = \pi(a)^*. \tag{1.3.11}$$

This way, the algebraic relations in $\mathscr{A}$ are reflected by the operators $\pi(a)\colon \mathfrak{H} \longrightarrow \mathfrak{H}$. We have realized the abstract algebra by more "concrete" operators.

We call a *-representation $\pi\colon \mathscr{A} \longrightarrow \mathfrak{B}(\mathfrak{H})$ *faithful* if the linear map π is injective.

The notion of intertwiners will allow us to relate different *-representations of $\mathscr{A}$: this is an entirely new feature. Only having the algebra $\mathscr{A}$ given *independently* of any pre-Hilbert space realization allows us to consider *different* such realizations. Physically, this will be very important in the discussion of e.g. superselection rules in algebraic quantum field theory [18, Chap. 4] and topological effects in quantum mechanics like the Aharonov-Bohm effect. The obvious task arising is to classify *-representations up to (unitary) intertwiners. If there is such a unitary intertwiner, the *-representations have to be considered "the same" from a physical point of view. Conversely, given any unitary map $U\colon \mathfrak{H} \longrightarrow \mathfrak{H}'$ and a *-representation π on $\mathfrak{H}$ it is straightforward to see that

$$\pi'(a) = U\pi(a)U^{-1} = U\pi(a)U^* \tag{1.3.12}$$

defines a *-representation on $\mathfrak{H}'$, which, by construction, is equivalent to the original one, see Exercise 1.5.24.

Finally, note that $^*\text{-}\mathsf{rep}(\mathscr{A})$ indeed forms a category: the identity map $\mathrm{id}_{\mathfrak{H}}$ is always an intertwiner and the composition of intertwiners is again an intertwiner. It is this category which contains all the information needed to understand the "representation theory" of $\mathscr{A}$, hence the name, see also Exercise 1.5.24 for some more details about this category.

Having a *-representation $(\mathfrak{H}, \pi)$ of $\mathscr{A}$ we can use the vectors $\psi \in \mathfrak{H}$ to construct positive linear functionals of $\mathscr{A}$: assume $\langle \psi, \psi \rangle = 1$ then the functional $\omega_\psi\colon \mathscr{A} \longrightarrow \mathbb{C}$ with

$$\omega_\psi(a) = \langle \psi, \pi(a)\psi \rangle \tag{1.3.13}$$

is easily shown to be a positive linear functional. Indeed, the linearity is clear and we have $\omega_\psi(a^*a) = \langle \pi(a)\psi, \pi(a)\psi \rangle \geq 0$ by the *-representation properties of π. Moreover, if the *-representation π is unital for a unital *-algebra $\mathscr{A}$, i.e. $\pi(\mathbb{1}_{\mathscr{A}}) = \mathrm{id}_{\mathfrak{H}}$, then ω_ψ is a state. This brings us back to the usual formulation of quantum physics. In particular, inside $\mathfrak{H}$ we can now form superpositions of the state vectors to gain new state vectors and, hence, new states.

The question is now how one can obtain or even construct *-representations of $\mathscr{A}$. Here we will present one of the fundamental constructions, the *Gel'fand-Naimark-Segal construction* of a *-representation out of a positive linear functional. Let $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a positive linear functional, the existence of which we take for granted at the moment. In fact, it turns out to be a quite non-trivial task to guarantee the existence of sufficiently many interesting positive linear functionals for a given *-algebra.

The GNS construction has now two aspects, an algebraic and an analytic one. To be conform with the main objective of this chapter, we focus on the algebraic part and come back to the analytic features in more specific situations later. We recall that a subspace $\mathscr{J} \subseteq \mathscr{A}$ is called a *left ideal* of $\mathscr{A}$ if for all $a \in \mathscr{A}$ and $b \in \mathscr{J}$ we have $ab \in \mathscr{J}$ again. We shortly write $\mathscr{A} \cdot \mathscr{J} \subseteq \mathscr{J}$ in this situation. Analogously, one defines a *right ideal* by the condition $\mathscr{J} \cdot \mathscr{A} \subseteq \mathscr{J}$. A left ideal which is also a right ideal is called a *two-sided ideal* or simply an *ideal* of $\mathscr{A}$. Finally, a **-ideal* in a *-algebra is an ideal $\mathscr{J}$ with $b^* \in \mathscr{J}$ for all $b \in \mathscr{J}$. With this preparation we can now formulate the following statement:

Lemma 1.3.5 (Gel'fand ideal) *Let $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a positive linear functional on a *-algebra $\mathscr{A}$. Then*

$$\mathscr{J}_\omega = \left\{ a \in \mathscr{A} \;\middle|\; \omega(a^*a) = 0 \right\} \tag{1.3.14}$$

is a left ideal of $\mathscr{A}$, the so-called Gel'fand ideal. We have $a \in \mathscr{J}_\omega$ iff $\omega(b^*a) = 0$ for all $b \in \mathscr{A}$ iff $\omega(a^*b) = 0$ for all $b \in \mathscr{A}$.

Proof: Let $a \in \mathscr{J}_\omega$ then the Cauchy-Schwarz inequality implies $|\omega(a^*b)|^2 \leq \omega(a^*a)\omega(b^*b) = 0$ for all $b \in \mathscr{A}$. Since $\omega(a^*b) = \overline{\omega(b^*a)}$ we see that this implies $\omega(a^*b) = 0 = \omega(b^*a)$. The converse is obvious, showing thereby the equivalence of the three characterizations of the Gel'fand ideal. Since the second characterization is clearly a set of linear conditions, $\mathscr{J}_\omega$ is a subspace. Moreover, for $c \in \mathscr{A}$ we get $\omega(b^*ca) = \omega((c^*b)^*a) = 0$. Hence also $ca \in \mathscr{J}_\omega$, completing the proof. □

Note that in general $\mathscr{J}_\omega$ is only a left ideal but not a right ideal. We will see examples later on and in the Exercises 1.5.29 and 1.5.30. Having the left ideal $\mathscr{J}_\omega$ we can consider the quotient vector space

$$\mathfrak{H}_\omega = \mathscr{A} \big/ \mathscr{J}_\omega, \tag{1.3.15}$$

whose elements are equivalence classes of vectors in $\mathscr{A}$ where a is equivalent to a' iff $a - a' \in \mathscr{J}_\omega$ as usual. Traditionally, the class of $a \in \mathscr{A}$ is denoted by $\psi_a \in \mathfrak{H}_\omega$ in this context. Since $\mathscr{J}_\omega \subseteq \mathscr{A}$ is a left ideal the quotient $\mathfrak{H}_\omega$ is, by general arguments, a *left module* for $\mathscr{A}$: we can multiply elements $\psi_b \in \mathfrak{H}_\omega$ by $a \in \mathscr{A}$ from the left via

$$\pi_\omega(a)\psi_b = \psi_{ab}. \tag{1.3.16}$$

Of course, one has to check that this is indeed well-defined: it is precisely the property of a left ideal which guarantees this. Then the following is obvious:

Lemma 1.3.6 *The map $\pi_\omega(a)\colon \mathfrak{H}_\omega \longrightarrow \mathfrak{H}_\omega$ is well-defined, linear, and we have for all $z, w \in \mathbb{C}$ and $a, b \in \mathscr{A}$*

$$\pi_\omega(za + wb) = z\pi_\omega(a) + w\pi_\omega(b), \tag{1.3.17}$$

$$\pi_\omega(ab) = \pi_\omega(a)\pi_\omega(b). \tag{1.3.18}$$

These properties look already very promising if we want a *-representation. The next step establishes now a positive definite inner product on $\mathfrak{H}_\omega$. One defines

$$\langle \psi_a, \psi_b \rangle_\omega = \omega(a^*b) \tag{1.3.19}$$

for $\psi_a, \psi_b \in \mathfrak{H}_\omega$. Again, we have to show first that this is well-defined. In fact, it turns out that this makes $\mathfrak{H}_\omega$ a pre-Hilbert space:

Lemma 1.3.7 *The quotient space $\mathfrak{H}_\omega$ becomes a pre-Hilbert space via $\langle \cdot\,, \cdot \rangle_\omega$.*

Again, this is a simple argument based on the Cauchy-Schwarz inequality. The last step consists in showing that π_ω is a *-representation of $\mathscr{A}$ on $\mathfrak{H}_\omega$. This is now an easy computation:

Lemma 1.3.8 *We have $\pi_\omega(a) \in \mathfrak{B}(\mathfrak{H}_\omega)$ and π_ω is a *-representation of $\mathscr{A}$ on $\mathfrak{H}_\omega$.*

Proof: For $\psi_b, \psi_c \in \mathfrak{H}_\omega$ and $a \in \mathscr{A}$ we have

$$\langle \psi_b, \pi_\omega(a)\psi_c \rangle_\omega = \omega(b^*(ac)) = \omega((a^*b)^*c) = \langle \pi_\omega(a^*)\psi_b, \psi_c \rangle_\omega.$$ □

Thus one obtains a *-representation of $\mathscr{A}$ out of every positive linear functional ω of $\mathscr{A}$. This construction is referred to as the *GNS construction*, the *-representation π_ω is called the *GNS representation* of $\mathscr{A}$ corresponding to ω.

If $\mathscr{A}$ is unital, which we can typically assume for an observable algebra, then one has a distinguished vector in $\mathfrak{H}_\omega$, namely the equivalence class

$$\Omega = \psi_{\mathbb{1}} \tag{1.3.20}$$

of the unit element $\mathbb{1} \in \mathscr{A}$. Then we have

$$\omega(a) = \langle \Omega, \pi_\omega(a)\Omega \rangle_\omega \tag{1.3.21}$$

for all $a \in \mathscr{A}$. This way, we succeeded in realizing the abstractly given expectation value functional ω as an expectation value with respect to a particular state vector in the GNS pre-Hilbert space and with respect to the GNS representation which realizes the abstract element of the observable algebra as a concrete operator. We see that the abstraction from operators and state vectors to *-algebras and positive functionals also allows for the way back: the circle closes. Some more details on the GNS construction as well as first examples can be found in the Exercises 1.5.25, 1.5.26, 1.5.27, 1.5.29, and 1.5.30.

1.4 Beyond Algebra...

According to our discussion so far, the task of building a quantum theory for a particular physical system can be summarized as follows: first, one should identify the basic and important observables and construct the observable algebra as a (unital) *-algebra on the basis of the commutation relations. These commutation relations are the core input from the physical side as they encode the uncertainty relations one expects for the system. In most situations this is the truly non-trivial part from the point of view of mathematical physics. In a next step one investigates the structure of this *-algebra, in particular its space of states. Then, to encode the superposition principle for states, a *-representation on a pre-Hilbert space has to be found and chosen. Here the GNS construction will provide a systematic way of finding *-representations.

However, at least at this point new conceptual questions arise: first it is not clear at all whether the algebra $\mathscr{A}$ has interestingly many states at all. This may well happen and then $\mathscr{A}$ may not even have any faithful *-representation. Second, even if we have many states and hence hopefully a faithful *-representation, in general the algebra $\mathscr{A}$ might have many inequivalent *-representations. So which one has to be taken in order to describe the "correct" physical situation? This is far from being obvious and will typically depend very much on the example of the *-algebra as well as on additional information about the particular physical situation one wants to describe. Thus, ideally, one would like to have a "complete list" of all *-representations of $\mathscr{A}$ out of which one can choose the adequate one matching the physical needs. This would imply that one has a good or even complete understanding of the representation theory $^*\text{-}\mathsf{rep}(\mathscr{A})$ of $\mathscr{A}$.

Clearly, here one faces serious difficulties. The complexity of $\mathscr{A}$ might simply forbid to get hands on $^*\text{-}\mathsf{rep}(\mathscr{A})$ in an efficient way. But things are even worse and here analytic questions enter the arena: in classical mechanics one can describe the observables as certain functions on the classical phase space, say on $\mathbb{R}^{2n}$. Here one has several possibilities like polynomial functions, real-analytic functions, or smooth functions to name just a few. From the physical point of view things should not depend too much on such a choice. But from their mathematical structure these three algebras of functions differ very much in many aspects. Similar effects have to be expected also in the noncommutative world of quantum systems: we are always dealing with idealizations when passing to a mathematical model and the best we can hope for is that all choices we have to make on the way will lead to "robust" predictions in the end. Note that this is a much more profound difficulty than the dependence of predictions on the values of some parameters of a single model like coupling constants.

Now back to our original task in finding the "correct" *-representation, analytic techniques might help in the following way. At first sight, two *-representations might look very different. But after some completion process with respect to certain topologies things might become equivalent. Then the idea is that this completion is physically justifiable since, hopefully, the topologies are somehow natural for the problem. To be slightly more concrete: there are much to many non-isometric pre-Hilbert spaces around but completing them with respect to the underlying norm coming from the inner product one obtains Hilbert spaces which now are easily classified. This way, analytic considerations like completions will simplify the problem as many formerly inequivalent things may become equivalent after completion. Thus the complexity of understanding $^*\text{-}\mathsf{rep}(\mathscr{A})$ and presenting the "complete list of inequivalent *-representations" will be simplified by a clever use of additional analytic techniques.

Beside this simplifying aspect of analytic tools there is yet another conceptual task to be accomplished for quantum mechanics: up to now we can formulate expectation values $\omega(a)$ for observables $a \in \mathscr{A}$ with respect to given states ω of the observable algebra. Stopping here would yield a generic sort of probabilistic theory. But quantum physics provides more that just expectation values: it is one of the crucial predictions that for a given observable a not all possible values, say in $\mathbb{R}$, can occur in a measurement. Only a particular subset of $\mathbb{R}$, characteristic for a, is allowed. This subset is called the (physical) *spectrum* of the observable a and will be denoted by $\operatorname{spec}(a)$. In particular, this quantity should *not* depend on the actual state or representation but only on the observable alone. The state enters now in the following way: for a (pure) state ω not only the expectation value of a is predicted to be $\omega(a)$ but also a probability distribution μ_ω on the set of possible outcomes of the repeated measurements corresponding to a. This probability distribution is such that the value $\omega(a)$ is indeed the expectation value of this distribution, i.e.

$$\omega(a) = \int_{\lambda \in \operatorname{spec}(a)} \lambda \, \mathrm{d}\mu_\omega. \tag{1.4.1}$$

Summarizing, quantum physics does not predict the outcome of a single, particular measurement of an observable a in a state ω except that the result λ is in the spectrum of a. However, if the system is prepared to be in that particular state ω over and over again then the repeated measurements of a in this identically prepared state results in a distribution of outcomes, all in $\operatorname{spec}(a)$, which ultimately approximate a probability distribution μ_ω with expectation value $\omega(a)$.

Analysis enters now in providing such a measure μ_ω for a given state. In a purely algebraic framework it is not possible to establish a reasonable notion of "spectrum" on the mathematical side such that the above requirements could be fulfilled. Even more difficult is the problem of defining a spectral measure μ_ω guaranteeing (1.4.1). Only with the use of (quite non-trivial) analytic techniques and only for particular classes of *-algebra with nice analytic properties such a *spectral theory* can be established.

Up to this point we only considered difficulties arising from the *kinematic* description of the physical system. However, there is yet one more challenge to be taken when it comes to the question of describing *dynamics*. According to Definition 1.1.5 the time evolution is encoded by a one-parameter group Φ_t of *-automorphisms of the observable algebra $\mathscr{A}$. Now physical theories usually do not yield this one-parameter group directly. Instead, they provide a *differential equation* which determines Φ_t. Ignoring all analytic issues for a moment, it is easy to see that the one-parameter group property

$$\Phi_t \circ \Phi_s(a) = \Phi_{t+s}(a) \quad \text{and} \quad \Phi_0(a) = a \tag{1.4.2}$$

leads, by differentiating, to the following equivalent equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi_t(a) = D\Phi_t(a) \quad \text{where} \quad Da = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_t(a), \tag{1.4.3}$$

with a *-*derivation* D of the algebra $\mathscr{A}$. Conversely, given a *-derivation D of $\mathscr{A}$ the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t) = Da(t) \tag{1.4.4}$$

with initial condition $a(0) = a$ has $a(t) = \exp(tD)a$ as its unique solution and $\Phi_t = \exp(tD)$ is a one-parameter group of *-automorphisms. It turns out that in most physical theories the time evolution is given in its *infinitesimal form* (1.4.4) with a specific *infinitesimal generator* D being the central piece of information about the dynamics. In many cases, the derivation D is even an inner derivation $D = -\mathrm{i}\operatorname{ad}(H)$ with a particular Hermitian element $H = H^* \in \mathscr{A}$ called the *Hamiltonian* of the dynamics. In this case, the infinitesimal form (1.4.4) of the time evolution is also called the *Heisenberg Equation*. The solution of the Heisenberg Equation (1.4.4) is formally given by use of the exponential

$$U_t = \exp(-\mathrm{i}tH), \tag{1.4.5}$$

and the automorphism Φ_t is then simply the *inner* *-*automorphism*

$$\Phi_t(a) = U_t a U_t^{-1}, \tag{1.4.6}$$

where $a \in \mathscr{A}$. It is clear that a justification of the above heuristics will require a solid analytic foundation since the algebras we are interested both in classical and quantum mechanics are infinite-dimensional and hence the standard theorems on differential equations in $\mathbb{C}^n$ do not apply directly: neither the differentiation needed to define D from Φ_t nor the exponentiation of D or H to re-construct Φ_t or U_t are trivial. In fact, we will need quite some analytic efforts to find mathematically rigorous and physically suitable solutions to these questions.

After these heuristic arguments we are well motivated to enter the analytic arena: the aim is to provide mathematical structures and techniques which allow us to distinguish particular *-algebras and their *-representations having nice analytic properties including a reasonable notion of spectrum with an appropriate spectral theory.

1.5 Exercises

Exercise 1.5.1 (Unitary group) Let $\mathscr{A}$ be a unital associative algebra.

i.) First show that the unit element is unique. Denote the set of invertible elements in $\mathscr{A}$ by $\mathrm{GL}(\mathscr{A})$ and show that $\mathrm{GL}(\mathscr{A})$ becomes a group with respect to the inherited multiplication from $\mathscr{A}$.

ii.) If $\mathscr{A}$ is in addition a *-algebra, show that $\mathbb{1}^* = \mathbb{1}$. Then consider the unitary elements $\mathrm{U}(\mathscr{A})$ of $\mathscr{A}$ and show that they form a subgroup of $\mathrm{GL}(\mathscr{A})$.

Exercise 1.5.2 (The commutator) Let $\mathscr{A}$ be an associative algebra over a field $\mathbb{k}$. Define the *commutator* of $a, b \in \mathscr{A}$ as usual by

$$[a, b] = ab - ba, \tag{1.5.1}$$

and set $\operatorname{ad}(a)\colon b \mapsto [a, b]$.

i.) Show that the commutator $[\,\cdot\,,\,\cdot\,]$ makes $\mathscr{A}$ a Lie algebra, i.e. it is an anti-symmetric bilinear map which satisfies the *Jacobi identity* (1.1.6).

ii.) Show $[\Phi(a), \Phi(b)] = \Phi([a, b])$ for $a, b \in \mathscr{A}$ and an algebra homomorphism $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ into another associative algebra $\mathscr{B}$. Conclude that one obtains a functor $\mathsf{alg} \longrightarrow \mathsf{LieAlg}$ into the category of complex Lie algebras.

iii.) Show that for the *left* and *right multiplications* $\mathsf{L}_a, \mathsf{R}_b\colon \mathscr{A} \longrightarrow \mathscr{A}$ one has $[\mathsf{L}_a, \mathsf{R}_b] = 0$ for all $a, b \in \mathscr{A}$, where $\mathsf{L}_a\colon c \mapsto ac$ and $\mathsf{R}_b\colon c \mapsto cb$. Show that $\operatorname{ad}(a) = \mathsf{L}_a - \mathsf{R}_a$.

iv.) Show that for any Lie algebra $\mathfrak{g}$ the adjoint map $\mathrm{ad}\colon \mathfrak{g} \ni \xi \mapsto (\eta \mapsto \mathrm{ad}(\xi)\eta = [\xi, \eta]) \in \mathrm{End}(\mathfrak{g})$ is a homomorphism of Lie algebras
$$\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}), \tag{1.5.2}$$
where $\mathrm{End}(\mathfrak{g})$ is endowed with the commutator Lie bracket.

v.) Show that for every $a \in \mathscr{A}$ the map $\mathrm{ad}(a)$ is a derivation with respect to the associative product. Show that the set of all derivations of $\mathscr{A}$ forms a Lie subalgebra $\mathrm{Der}(\mathscr{A}) \subseteq \mathrm{End}(\mathscr{A})$. Show furthermore that $\mathrm{ad}\colon \mathscr{A} \longrightarrow \mathrm{Der}(\mathscr{A})$ is a Lie algebra homomorphism.

vi.) Derivations of the form $\mathrm{ad}(a)$ are called *inner derivations* and the set of inner derivations is denoted by $\mathrm{InnDer}(\mathscr{A})$. Show that
$$[D, \mathrm{ad}(a)] = \mathrm{ad}(Da) \tag{1.5.3}$$
for every $D \in \mathrm{Der}(\mathscr{A})$ and $a \in \mathscr{A}$. Conclude that $\mathrm{OutDer}(\mathscr{A}) = \mathrm{Der}(\mathscr{A}) \big/ \mathrm{InnDer}(\mathscr{A})$ becomes a Lie algebra, the *outer derivations* of $\mathscr{A}$.

vii.) Suppose $\mathscr{A}$ is in addition a *-algebra over $\mathbb{C}$. Compute $[a, b]^*$ explicitly and show that $\mathrm{ad}(a)$ is a *-derivation for an anti-Hermitian element $a \in \mathscr{A}$.

Exercise 1.5.3 (Ideals and quotients) Let $\mathscr{A}$ be an associative algebra over a field $\mathbb{k}$. Recall that a vector space $\mathscr{M}$ is a *left module* over $\mathscr{A}$ if it is equipped with a bilinear map, the *module multiplication*,
$$\mu\colon \mathscr{A} \times \mathscr{M} \longrightarrow \mathscr{M}, \tag{1.5.4}$$
such that $\mu(a, \mu(b, m)) = \mu(ab, m)$ for all $a, b \in \mathscr{A}$ and $m \in \mathcal{M}$. We write usually $a \cdot m = \mu(a, m)$. Analogously, one defines a *right module* by requiring $\mu(a, \mu(b, m)) = \mu(ba, m)$ instead. In this case it is more suggestive to write $\mu(a, m) = m \cdot a$.

i.) Let $\mathscr{J} \subseteq \mathscr{A}$ be a left ideal. Show that the quotient $\mathscr{A} \big/ \mathscr{J}$ becomes a left $\mathscr{A}$-module via the definition $a \cdot [b] = [ab]$ where $a \in \mathscr{A}$ and $[b] \in \mathscr{A} \big/ \mathscr{J}$ denotes the equivalence class of $b \in \mathscr{A}$. Show that the quotient by a right ideal becomes a right module in a similar way.

ii.) Let $\mathscr{J} \subseteq \mathscr{A}$ be a two-sided ideal. Show that the quotient $\mathscr{A} \big/ \mathscr{J}$ becomes an associative algebra via $[a][b] = [ab]$. Show that the quotient map $\mathrm{pr}\colon \mathscr{A} \longrightarrow \mathscr{A} \big/ \mathscr{J}$ is an algebra homomorphism in this case and the multiplication is the unique one on $\mathscr{A} \big/ \mathscr{J}$ such that pr is an algebra homomorphism.

iii.) Show that for a homomorphism $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ there is a (uniquely determined) homomorphism $\phi\colon \mathscr{A} \big/ \mathscr{J} \longrightarrow \mathscr{B}$ with $\Phi = \phi \circ \mathrm{pr}$ iff $\mathscr{J} \subseteq \ker \Phi$.

iv.) If $\mathscr{A}$ is unital, show that the quotient algebra by a two-sided ideal is again unital.

v.) If $\mathscr{A}$ is a *-algebra over $\mathbb{C}$, show that the quotient algebra by a *-ideal becomes a *-algebra again in a unique way such that the quotient map is a *-homomorphism. Formulate and prove the analog of *iii.)* also in this situation.

Exercise 1.5.4 (The polynomial calculus I) Let $\mathscr{A}$ be a unital associative algebra over some field $\mathbb{k}$ and let $a \in \mathscr{A}$ be a fixed element. For a polynomial $p \in \mathbb{k}[x]$ one defines $p(a) \in \mathscr{A}$ as usual by substituting the variable x by the algebra element a. If $\mathscr{A}$ is not unital, then this is only possible for polynomials $p \in x\mathbb{k}[x]$ with vanishing constant part.

i.) Show that the map
$$\mathbb{k}[x] \ni p \;\mapsto\; p(a) \in \mathscr{A} \tag{1.5.5}$$
is a unital algebra homomorphism.

ii.) Show that if $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a unital homomorphism into some other unital associative algebra $\mathscr{B}$ over $\mathbb{k}$, then
$$\Phi(p(a)) = p(\Phi(a)) \tag{1.5.6}$$
for all $a \in \mathscr{A}$ and $p \in \mathbb{k}[x]$. In which sense does this still hold in the non-unital situation?

Exercise 1.5.5 (The polynomial calculus II) Assume that $\mathscr{A}$ is a unital *-algebra over $\mathbb{C}$ and let $a \in \mathscr{A}$ be a normal element. Consider polynomials $\mathbb{C}[z, \overline{z}]$ in two variables.

i.) Show that the algebra $\mathbb{C}[z, \overline{z}]$ becomes a *-algebra if one defines $z^* = \overline{z}$ for the generators, thereby explaining the notation.

ii.) Define for $p \in \mathbb{C}[z, \overline{z}]$ the algebra element $p(a, a^*) \in \mathscr{A}$ by substituting z by a and $\overline{z}$ by a^*. Show that this is well-defined by using the fact that a is normal.

iii.) Show that the map

$$\mathbb{C}[z, \overline{z}] \ni p \ \mapsto \ p(a, a^*) \in \mathscr{A} \tag{1.5.7}$$

is a unital *-homomorphism.

iv.) Formulate and prove an analogous statement for the case where $\mathscr{A}$ is non-unital.

Exercise 1.5.6 (Isomorphisms and automorphisms) Use the result that *-alg and *-Alg form categories to prove that the collection of all *-isomorphisms between *-algebras forms a groupoid. Conclude that for every *-algebra $\mathscr{A}$ the *-automorphisms $^*\text{-}\operatorname{Aut}(\mathscr{A})$ form a group. Prove that the set $^*\text{-}\operatorname{Iso}(\mathscr{A}, \mathscr{B})$ of *-isomorphisms from $\mathscr{A}$ to $\mathscr{B}$ is either empty or in bijection to $^*\text{-}\operatorname{Aut}(\mathscr{A})$. Show that in the latter case the two groups $^*\text{-}\operatorname{Aut}(\mathscr{A})$ and $^*\text{-}\operatorname{Aut}(\mathscr{B})$ are isomorphic. This reasoning is valid in any category and the corresponding isomorphism groupoid. We will meet many situations where we use these facts without further mentioning.

Exercise 1.5.7 (The category of pre-Hilbert spaces) Show that complex pre-Hilbert spaces form a category $\mathsf{PreHilbert}$ when one uses adjointable maps between them as morphisms.

Exercise 1.5.8 (Complex conjugate pre-Hilbert space) Let $\mathfrak{H}$, $\mathfrak{H}_1$, and $\mathfrak{H}_2$ be pre-Hilbert spaces. Consider now the complex conjugate vector space $\overline{\mathfrak{H}}$ defined as usual: as an additive group we set $\overline{\mathfrak{H}} = \mathfrak{H}$ and denote the identity map by $\mathfrak{H} \ni \phi \mapsto \overline{\phi} \in \overline{\mathfrak{H}}$. Then the vector space structure on $\overline{\mathfrak{H}}$ is defined by $z\overline{\phi} = \overline{\overline{z}\phi}$.

i.) Show that $\overline{\mathfrak{H}}$ is indeed a complex vector space. Show that $\operatorname{id}\colon \mathfrak{H} \longrightarrow \overline{\mathfrak{H}}$ is an *antilinear* isomorphism.

ii.) Consider a linear map $A\colon \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$ and define

$$\overline{A}\colon \overline{\mathfrak{H}}_1 \ni \overline{\phi} \ \mapsto \ \overline{A\phi} \in \overline{\mathfrak{H}}_2. \tag{1.5.8}$$

Show that this is again a linear map and that $\operatorname{Hom}(\mathfrak{H}_1, \mathfrak{H}_2) \ni A \mapsto \overline{A} \in \operatorname{Hom}(\overline{\mathfrak{H}}_1, \overline{\mathfrak{H}}_2)$ is an antilinear isomorphism.

iii.) Define $\langle \cdot\,, \cdot \rangle\colon \overline{\mathfrak{H}} \times \overline{\mathfrak{H}} \longrightarrow \mathbb{C}$ by

$$\langle \overline{\phi}, \overline{\psi} \rangle = \overline{\langle \phi, \psi \rangle}. \tag{1.5.9}$$

Show that this makes $\overline{\mathfrak{H}}$ a pre-Hilbert space such that $\operatorname{id}\colon \mathfrak{H} \longrightarrow \overline{\mathfrak{H}}$ is an antilinear isometric isomorphism.

iv.) Show that $A\colon \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$ is adjointable iff $\overline{A}\colon \overline{\mathfrak{H}}_1 \longrightarrow \overline{\mathfrak{H}}_2$ is adjointable. Compute the adjoint of $\overline{A}$. Use this to discuss the functoriality of complex conjugation in the category $\mathsf{PreHilbert}$.

Exercise 1.5.9 (States and density matrices) Consider the finite-dimensional pre-Hilbert space $\mathfrak{H} = \mathbb{C}^n$ with its canonical inner product.

i.) Show that the matrices $\mathrm{M}_n(\mathbb{C})$ act on $\mathbb{C}^n$ by adjointable operators and determine the induced *-involution. We will always endow $\mathrm{M}_n(\mathbb{C})$ with this *-involution.

ii.) Let $\omega\colon \mathrm{M}_n(\mathbb{C}) \longrightarrow \mathbb{C}$ be a positive linear functional in the sense of Definition 1.2.2. Prove that there exists a matrix $\varrho \in \mathrm{M}_n(\mathbb{C})$ with the property $\langle \phi, \varrho\phi \rangle \geq 0$ for all $\phi \in \mathbb{C}^n$ such that $\omega(A) = \operatorname{tr}(\varrho A)$. Show that ω is a state iff $\operatorname{tr}(\varrho) = 1$. Such a matrix ϱ is called a *density matrix.*

iii.) Conversely, show that every density matrix $\varrho \in \mathrm{M}_n(\mathbb{C})$ gives a state on $\mathrm{M}_n(\mathbb{C})$ via the definition $A \mapsto \operatorname{tr}(\varrho A)$.

iv.) Show that for a matrix $A \in \mathrm{M}_n(\mathbb{C})$ the following statements are equivalent:

(a) One has $\langle \phi, A\phi \rangle \geq 0$ for all $\phi \in \mathbb{C}^n$.

(b) One has $A = A^*$ and all eigenvalues of A are non-negative.

(c) There is a Hermitian matrix $B = B^*$ with non-negative eigenvalues and $A = B^2$.

(d) There is a Hermitian matrix $B = B^*$ with $A = B^2$.

(e) There is a matrix $B \in \mathrm{M}_n(\mathbb{C})$ with $A = B^*B$.

(f) One has $\omega(A) \geq 0$ for all states ω, i.e. A is positive in the sense of Definition 1.2.8.

Hint: Details can be found in e.g. [64, Sect. 7.8].

Exercise 1.5.10 (Pure and mixed states of $\mathrm{M}_n(\mathbb{C})$) Consider again a state ω_ϱ of the matrices $\mathrm{M}_n(\mathbb{C})$ with corresponding density matrix ϱ according to Exercise 1.5.9. Show that ω_ϱ is pure iff $\varrho^2 = \varrho$. Show that in this case, ϱ is a projection onto a one-dimensional subspace of $\mathbb{C}^n$.

Hint: Details can be found in e.g. [64, Sect. 7.8].

Exercise 1.5.11 (A positive quadratic polynomial) Consider complex numbers $a, b, b', c \in \mathbb{C}$ with

$$p(z, w) = a\overline{z}z + bz\overline{w} + b'\overline{z}w + cw\overline{w} \geq 0 \tag{1.5.10}$$

for all $z, w \in \mathbb{C}$. Show that this implies $a \geq 0$, $c \geq 0$, $\overline{b} = b'$ and $ac \geq b\overline{b}$.

Exercise 1.5.12 (Polarization identity) Let V and W be two vector spaces over $\mathbb{C}$ and $S\colon V \times V \longrightarrow W$ a *sesquilinear* map, i.e. assume that

$$S(\alpha u + \beta v, w) = \overline{\alpha} S(u, w) + \overline{\beta} S(v, w) \quad \text{and} \quad S(u, \alpha v + \beta w) = \alpha S(u, v) + \beta S(u, w) \tag{1.5.11}$$

hold for all $\alpha, \beta \in \mathbb{C}$ and $u, v, w \in V$.

i.) Show that the *polarization identity*

$$S(v, w) = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k \cdot S\big(v + \mathrm{i}^{-k} w, v + \mathrm{i}^{-k} w\big) \tag{1.5.12}$$

holds for all $v, w \in V$. Conclude that S is constant 0 iff $S(v, v) = 0$ for all $v \in V$.

ii.) Now let $W = \mathbb{C}$. A sesquilinear map $S\colon V \times V \longrightarrow \mathbb{C}$ is usually called a sesquilinear form. Such a sesquilinear form is said to be Hermitian if $\overline{S(v, w)} = S(w, v)$ holds for all $v, w \in V$. Show that a sesquilinear form S on V is Hermitian if and only if $S(v, v) \in \mathbb{R}$ holds for all $v \in V$.

iii.) Let finally $\mathscr{A}$ be a unital *-algebra over $\mathbb{C}$. Show that for every $a \in \mathscr{A}$ there exist algebraically positive elements $b_0, b_1, b_2, b_3 \in \mathscr{A}^{++}$ such that $a = \sum_{k=0}^{3} \mathrm{i}^k b_k$ holds.

Exercise 1.5.13 (Uncertainty relations) Prove Proposition 1.2.7.

Exercise 1.5.14 (Covariance and correlation) Let $\mathscr{A}$ be a unital *-algebra and let $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a state.

i.) Show that for any $a_1, \ldots, a_n \in \mathscr{A}$ the covariance matrix $\operatorname{Cov}_\omega(a_i, a_j)$ is a positive matrix in $\mathrm{M}_n(\mathbb{C})$.

Hint: Use the characterizations of positive matrices as in Exercise 1.5.9, *iv.)*.

ii.) Let $a, b \in \mathscr{A}$ with $\operatorname{Var}_\omega(a) \neq 0 \neq \operatorname{Var}_\omega(b)$ be given. Show that the *correlation*

$$\operatorname{Cor}_\omega(a, b) = \frac{\operatorname{Cov}_\omega(a, b)}{\sqrt{\operatorname{Var}_\omega(a) \operatorname{Var}_\omega(b)}} \tag{1.5.13}$$

satisfies $|\operatorname{Cor}_\omega(a, b)| \leq 1$.

Exercise 1.5.15 (Matrix algebras) Let $\mathscr{A}$ be an associative algebra and denote by $\mathrm{M}_n(\mathscr{A})$ the $n \times n$ matrices with entries in $\mathscr{A}$.

i.) Show that the usual matrix multiplication makes $\mathrm{M}_n(\mathscr{A})$ again an associative algebra containing $\mathscr{A}$ as subalgebra of those diagonal matrices with the same element on all diagonal positions.

ii.) Show that $\mathrm{M}_n(\mathscr{A})$ is unital if $\mathscr{A}$ is unital.

iii.) Show that $\mathrm{M}_n(\mathrm{M}_m(\mathscr{A})) \cong \mathrm{M}_{nm}(\mathscr{A})$ as associative algebras.

iv.) Suppose that $\mathscr{A}$ is in addition a *-algebra. Define $(a_{ij})^* = (a_{ji}^*)$ for a matrix $(a_{ij}) \in \mathrm{M}_n(\mathscr{A})$ and show that this makes $\mathrm{M}_n(\mathscr{A})$ a *-algebra again. Show that $\mathscr{A}$ is a *-subalgebra.

v.) Show that $\mathrm{M}_n(\mathrm{M}_m(\mathscr{A})) \cong \mathrm{M}_{nm}(\mathscr{A})$ also holds as *-algebras for a *-algebra $\mathscr{A}$.

Exercise 1.5.16 (Sums of squares and the Motzkin polynomial)

Exercise 1.5.17 (Complete positivity)

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Exercise 1.5.18 (Unitary and isometric maps)

Exercise 1.5.19 (Multilinear algebra and the tensor product) Let $\Bbbk$ be a field (or a commutative ring) and let $V_1, \ldots, V_n, W$ be vector spaces (or modules) over $\Bbbk$ where $n \in \mathbb{N}$.

i.) Let $\Phi\colon V_1 \times \cdots \times V_n \longrightarrow W$ be an n-linear map, i.e. linear in each argument while the remaining arguments are kept fix. Show that under the pointwise operations such multilinear maps form a subspace of the vector space of all maps $V_1 \times \cdots \times V_n \longrightarrow W$. Denote the vector space of n-linear maps by $\operatorname{Hom}(V_1, \ldots, V_n; W)$.

ii.) Suppose $\Bbbk$ is a field. Show that Φ is uniquely determined by is values on bases $\{\Phi(b_1, \ldots, b_n)\}_{b_1 \in B_1, \ldots, b_n \in B_n}$, where $B_1 \subseteq V_1, \ldots B_n \subseteq V_n$ are bases. Show conversely, that for any specified values $\{\Phi(b_1, \ldots, b_n)\}_{b_1 \in B_1, \ldots, b_n}$ on bases there exists a uniquely determined n-linear map Φ with these values.

iii.) A vector space $\tilde{V}$ together with an n-linear map $\otimes\colon V_1 \times \cdots \times V_n \longrightarrow \tilde{V}$ is called *tensor product* of the $V_1, \ldots, V_n$ if for every n-linear map $\Phi\colon V_1 \times \cdots \times V_n \longrightarrow W$ there exists a unique linear map $\phi\colon \tilde{V} \longrightarrow W$ such that $\phi \circ \otimes = \Phi$. Prove that the tensor product $(\tilde{V}, \otimes)$ is unique up to a unique isomorphism $\iota\colon (\tilde{V}, \otimes) \longrightarrow (\tilde{V}', \otimes')$ such that $\iota \circ \otimes = \otimes'$.

We denote the tensor product from now on by $V_1 \otimes \cdots \otimes V_n$ and speak of *the* tensor product by a mild abuse of language. The image of $v_1 \in V_1, \ldots, v_n \in V_n$ under $\otimes$ will be denoted by $v_1 \otimes \cdots \otimes v_n \in V_1 \otimes \cdots \otimes V_n$. Vectors in $V_1 \otimes \cdots \otimes V_n$ are also called *tensors* and elements in the image of $\otimes$ are called *elementary tensors* or *factorizing tensors*.

iii.) Show that there exists a tensor product $(V_1 \otimes \cdots \otimes V_n, \otimes)$.

Hint: There are two possible ways to show its existence: the quick 'n' dirty way is to chose bases $B_1 \subseteq V_1$, $\ldots$, $B_n \subseteq V_n$ and consider the $\Bbbk$-span of the Cartesian product $B_1 \times \cdots \times B_n$. Then *ii.)* allows to construct the map $\otimes$ explicitly in terms of the bases. This construction relies on $\Bbbk$ being a field. According to *iii.)* the construction is essentially independent on the choices of the bases. The other path is to consider the really huge vector space spanned by $V_1 \times \cdots \times V_n$. Inside this one considers the subspace spanned by the multilinearity relations. The corresponding quotient is then the tensor product. The second construction also works if $\Bbbk$ is only a commutative ring and the $V_1, \ldots, V_n$ are modules over $\Bbbk$.

iv.) Let $B_1 \subseteq V_1$, $\ldots$, $B_n \subseteq V_n$ be bases. Show that the set of all the factorizing tensors $b_1 \otimes \cdots \otimes b_n \in V_1 \otimes \cdots \otimes V_n$ forms a basis of $V_1 \otimes \cdots \otimes V_n$ where $b_1 \in B_1$, $\ldots$, $b_n \in B_n$.

v.) Show that there is are canonical isomorphism

$$\mathsf{a}_1 \colon V_1 \otimes (V_2 \otimes V_3) \longrightarrow V_1 \otimes V_2 \otimes V_3 \tag{1.5.14}$$

and

$$\mathsf{a}_2 \colon (V_1 \otimes V_2) \otimes V_3 \longrightarrow V_1 \otimes V_2 \otimes V_3 \tag{1.5.15}$$

mapping $v_1 \otimes (v_2 \otimes v_3)$ as well as $(v_1 \otimes v_2) \otimes v_3$ both to $v_1 \otimes v_2 \otimes v_3$. The resulting isomorphism $\mathsf{asso} = \mathsf{a}_2^{-1} \circ \mathsf{a}_1$ is called the associativity of the tensor product. Show that there are also canonical isomorphisms between tensor products with more factors.

vi.) Assume now that $\phi_1 \colon V_1 \longrightarrow W_1, \ldots, \phi_n \colon V_n \longrightarrow W_n$ are linear maps into some other vector spaces. Show that there is a unique linear map, denoted by

$$\phi_1 \otimes \cdots \otimes \phi_n \colon V_1 \otimes \cdots \otimes V_n \longrightarrow W_1 \otimes \cdots \otimes W_n, \tag{1.5.16}$$

such that on elementary tensors one has

$$(\phi_1 \otimes \cdots \otimes \phi_n)(v_1 \otimes \cdots \otimes v_n) = \phi_1(v_1) \otimes \cdots \otimes \phi_n(v_n). \tag{1.5.17}$$

vii.) Formulate and prove useful compatibilities between the tensor product of linear maps and the associativity of the tensor product.

In the following, the associativity isomorphism will not be mentioned in the notation anymore. Instead, we identify $V_1 \otimes (V_2 \otimes V_3)$ and $(V_1 \otimes V_2) \otimes V_3$ directly with $V_1 \otimes V_2 \otimes V_3$ and similarly for higher tensor products. A more detailed introduction to multilinear algebra and tensor products can be found e.g. in the [16, 65].

Exercise 1.5.20 (Tensor products of algebras) Let $\mathscr{A}$ and $\mathscr{B}$ be associative algebras.

i.) Show that the bilinear extension of

$$(a \otimes b)(a' \otimes b') = (aa') \otimes (bb') \tag{1.5.18}$$

for $a, a' \in \mathscr{A}$ and $b, b' \in \mathscr{B}$ gives a well-defined associative multiplication on $\mathscr{A} \otimes \mathscr{B}$. We endow $\mathscr{A} \otimes \mathscr{B}$ with this algebra structure from now on.

ii.) Show that $\mathscr{A} \otimes \mathscr{B}$ is unital if $\mathscr{A}$ and $\mathscr{B}$ are unital. Show that in this case one has algebra homomorphisms

$$\mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{B} \longleftarrow \mathscr{B}, \tag{1.5.19}$$

by mapping a to $a \otimes \mathbb{1}_{\mathscr{B}}$ and b to $\mathbb{1}_{\mathscr{A}} \otimes b$. Show that the images of $\mathscr{A}$ and $\mathscr{B}$ in $\mathscr{A} \otimes \mathscr{B}$ commute.

iii.) For a further associative algebra $\mathscr{C}$ show that canonically (how?) $\mathscr{A} \otimes (\mathscr{B} \otimes \mathscr{C}) \cong (\mathscr{A} \otimes \mathscr{B}) \otimes \mathscr{C}$ as associative algebras.

iv.) Show that $\mathrm{M}_n(\mathscr{A}) \cong \mathscr{A} \otimes \mathrm{M}_n(\mathbb{C})$ and interpret the results of Exercise 1.5.15, *iii.)*, from this point of view.

v.) For two *-algebras $\mathscr{A}$ and $\mathscr{B}$ show that $\mathscr{A} \otimes \mathscr{B}$ becomes a *-algebra again via the antilinear extension of

$$(a \otimes b)^* = a^* \otimes b^* \tag{1.5.20}$$

for $a \in \mathscr{A}$ and $b \in \mathscr{B}$. Show that $\mathrm{M}_n(\mathscr{A}) \cong \mathscr{A} \otimes \mathrm{M}_n(\mathbb{C})$ as *-algebras in this case.

xercise: More onstructions: m, Cartesian ap$(X, \mathscr{A})$, all heir universal properties

Exercise 1.5.21 (The tensor algebra)

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Exercise 1.5.22 (The free algebra) Let V be a vector space over some field $\mathbb{k}$.

i.) An algebra $\tilde{V}$ together with a linear map $\iota\colon V \longrightarrow \tilde{V}$ is called *free associative unital algebra generated by* V if for every associative unital algebra $\mathscr{A}$ over $\Bbbk$ and every linear map $\phi\colon V \longrightarrow \mathscr{A}$ there is a unital algebra homomorphism $\Phi\colon \tilde{V} \longrightarrow \mathscr{A}$ such that the diagram

$$\begin{tikzcd} \tilde{V} \arrow[r, dashed, "\Phi"] & \mathscr{A} \\ V \arrow[u, "\iota"] \arrow[ur, "\phi"'] & \end{tikzcd} \tag{1.5.21}$$

commutes. Show that the free algebra is unique up to a unique isomorphism by using the universal property alone. This will allow to speak of *the* free algebra generated by V in the following.,

ii.) Show that for every vector space V there exists the free associative unital algebra generated by V.

Hint: Use the tensor algebra $\mathrm{T}^\bullet(V)$ and its universal properties.

iii.) Let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a unit-preserving homomorphism between two unital algebras. A linear map $D\colon \mathscr{A} \longrightarrow \mathscr{B}$ is called a *derivation along* Φ if

$$D(aa') = D(a)\Phi(a') + \Phi(a)D(a') \tag{1.5.22}$$

for all $a, a' \in \mathscr{A}$. If $\mathscr{A} = \mathscr{B}$ and $\Phi = \mathrm{id}$, then a derivation along id is simply a derivation. Show that a derivation along Φ vanishes on $\mathbb{1} \in \mathscr{A}$ if $\Bbbk$ has characteristic different from 2. Show that for every two linear maps $\phi, \psi\colon V \longrightarrow \mathscr{A}$ there is a unique derivation $D_\psi\colon \mathrm{T}^\bullet(V) \longrightarrow \mathscr{A}$ along Φ given as in (1.5.21) such that the diagram

$$\begin{tikzcd} \mathrm{T}^\bullet(V) \arrow[r, dashed, "D_\psi"] & \mathscr{A} \\ V \arrow[u, "\iota"] \arrow[ur, "\psi"'] & \end{tikzcd} \tag{1.5.23}$$

commutes.

Exercise 1.5.23 (Associativity via tensor products) Let $\mathscr{A}$ be a vector space with a bilinear map $\mu_{\mathscr{A}}\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$. Denote the linear map induced by $\mu_{\mathscr{A}}$ by $\mu_{\mathscr{A}}\colon \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A}$, too.

i.) Show that $\mu_{\mathscr{A}}$ is associative iff $\mu_{\mathscr{A}} \circ (\mu_{\mathscr{A}} \otimes \mathrm{id}) = \mu_{\mathscr{A}} \circ (\mathrm{id} \otimes \mu_{\mathscr{A}})$ holds as an equation on $\mathscr{A}^{\otimes 3}$.

ii.) Assume $\mu_{\mathscr{A}}$ is associative. Moreover, let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a linear map and assume that also on $\mathscr{B}$ there is an associative multiplication $\mu_{\mathscr{B}}$. Show that Φ is a homomorphism iff $\Phi \circ \mu_{\mathscr{A}} = \mu_{\mathscr{B}} \circ (\Phi \otimes \Phi)$ holds as an equation on $\mathscr{A}^{\otimes 2}$.

iii.) Suppose in addition that $D\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a linear map. Show that D is a derivation along Φ iff $D \circ \mu_{\mathscr{A}} = \mu_{\mathscr{B}} \circ (D \otimes \Phi + \Phi \otimes D)$.

Exercise 1.5.24 (The category *-$\mathsf{rep}(\mathscr{A})$) Consider a *-algebra $\mathscr{A}$ and its representation theory *-$\mathsf{rep}(\mathscr{A})$.

i.) Let $(\mathfrak{H}, \pi)$ be a *-representation of $\mathscr{A}$ on a pre-Hilbert space $\mathfrak{H}$ and let $U\colon \mathfrak{H} \longrightarrow \mathfrak{H}'$ be a unitary map. Show that $\pi'(a) = U\pi(a)U^*$ is a *-representation on $\mathfrak{H}'$.

ii.) Show that unitary equivalence of *-representations is indeed an equivalence relation.

iii.) Show that $^*\text{-}\mathsf{rep}(\mathscr{A})$ is indeed a category.

iv.) Show that the space of intertwiners from one *-representation $(\mathfrak{H}, \pi)$ to another one $(\mathfrak{H}', \pi')$ is a subspace of $\mathfrak{B}(\mathfrak{H}, \mathfrak{H}')$. Show that $A \mapsto A^*$ induces an antilinear bijection from the intertwiners $(\mathfrak{H}, \pi) \longrightarrow (\mathfrak{H}', \pi')$ to the intertwiners $(\mathfrak{H}', \pi') \longrightarrow (\mathfrak{H}, \pi)$.

v.) Show that the self-intertwiners of a given *-representation $(\mathfrak{H}, \pi)$ form a *-subalgebra of $\mathfrak{B}(\mathfrak{H})$.

Exercise 1.5.25 (Properties of the GNS representation) Let $\mathscr{A}$ be a *-algebra and let $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a positive functional.

i.) Show that the Gel'fand ideal $\mathscr{J}_\omega$ is indeed a left ideal and show that the inner product on the GNS pre-Hilbert $\mathfrak{H}_\omega = \mathscr{A} \big/ \mathscr{J}_\omega$ is well-defined and positive definite.

ii.) Assume that $\mathscr{A}$ is unital with unit $\mathbb{1}$. Show that $\psi_{\mathbb{1}}$ is an *algebraically cyclic* vector for π_ω, i.e. every other vector can be written in the form $\pi_\omega(a)\psi_{\mathbb{1}}$.

iii.) Assume $(\mathfrak{H}, \pi)$ is another *-representation of $\mathscr{A}$ with an algebraically cyclic vector Ω such that $\omega(a) = \langle \Omega, \pi(a)\Omega \rangle_\omega$ holds for all $a \in \mathscr{A}$. Show that $(\mathfrak{H}, \pi)$ is unitarily equivalent to the GNS representation $(\mathfrak{H}_\omega, \pi_\omega)$ by constructing an explicit intertwiner mapping Ω to $\psi_{\mathbb{1}}$.

Exercise 1.5.26 (Functorial properties of the GNS construction) Consider two *-algebras $\mathscr{A}$ and $\mathscr{B}$ together with a *-homomorphism $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$. Furthermore, let $\omega\colon \mathscr{B} \longrightarrow \mathbb{C}$ be a positive linear functional.

i.) Verify that the pull-back $\Phi^*\omega = \omega \circ \Phi$ is a positive linear functional on $\mathscr{A}$.

ii.) Show that for the Gel'fand ideals one has

$$\Phi(\mathscr{J}_{\Phi^*\omega}) \subseteq \mathscr{J}_\omega. \tag{1.5.24}$$

Conclude that Φ descends to a well-defined linear map $U_\Phi\colon \mathfrak{H}_{\Phi^*\omega} \longrightarrow \mathfrak{H}_\omega$.

iii.) Show that U_Φ is isometric. Note that it may well happen that U_Φ is not adjointable.

iv.) Prove that U_Φ is a (not necessarily adjointable) *intertwiner along* Φ, i.e. for all $a \in \mathscr{A}$

$$\pi_\omega(\Phi(a))U_\Phi = U_\Phi \pi_{\Phi^*\omega}(a), \tag{1.5.25}$$

between the corresponding GNS representations of $\mathscr{A}$ and $\mathscr{B}$.

v.) Suppose now in addition that Φ is surjective. Show that in this case U_Φ is unitary and hence adjointable.

vi.) If $\Psi\colon \mathscr{C} \longrightarrow \mathscr{A}$ is yet another *-homomorphism, what can one say about the relations between U_Ψ, U_Φ, and $U_{\Phi\circ\Psi}$?

Exercise 1.5.27 (Time evolution and the GNS construction) Let $\mathscr{A}$ be a unital *-algebra and let $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a state. Moreover, suppose that $\Phi_t\colon \mathscr{A} \longrightarrow \mathscr{A}$ is a time evolution, i.e. a one-parameter group of *-automorphisms, such that the state ω is *invariant* under Φ_t meaning that $\Phi_t^*\omega = \omega$ for all $t \in \mathbb{R}$. Show that in this case there is a one-parameter group of unitary maps $U_t\colon \mathfrak{H}_\omega \longrightarrow \mathfrak{H}_\omega$ such that

$$\pi_\omega(\Phi_t(a)) = U_t \pi_\omega(a) U_t^* \tag{1.5.26}$$

for all $t \in \mathbb{R}$ and $a \in \mathscr{A}$. In this case we say that the time evolution has been *unitarily implemented* in the GNS representation.

Hint: Use Exercise 1.5.26.

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Exercise 1.5.28 (Unitary group representation)

Exercise 1.5.29 (GNS construction for a pure state of $\mathrm{M}_n(\mathbb{C})$) Consider the unital *-algebra $\mathrm{M}_n(\mathbb{C})$ and a unit vector $\psi \in \mathbb{C}^n$.

i.) Describe explicitly the Gel'fand ideal of the positive functional $\omega_\psi\colon \mathrm{M}_n(\mathbb{C}) \longrightarrow \mathbb{C}$ given by $\omega_\psi(A) = \langle \psi, A\psi \rangle$.

ii.) Determine the GNS pre-Hilbert space $\mathfrak{H}_{\omega_\psi}$ and show that there is an explicit unitary intertwiner from the GNS representation π_{ω_ψ} to the defining *-representation of $\mathrm{M}_n(\mathbb{C})$ on $\mathbb{C}^n$.

Exercise 1.5.30 (GNS construction for δ-functionals) Consider a classical phase space like e.g. $\mathbb{R}^{2n}$ with $\mathscr{A} = \mathscr{C}(\mathbb{R}^{2n})$ or $\mathscr{C}^\infty(\mathbb{R}^{2n})$ as observable algebra with the complex conjugation as *-involution.

i.) Show that the δ-functional $\delta_x\colon \mathscr{A} \ni f \mapsto f(x) \in \mathbb{C}$ is a positive linear functional.

ii.) Describe the Gel'fand ideal of δ_x explicitly.

iii.) Find an explicit description of the GNS pre-Hilbert space. Is the GNS representation injective?

Exercise 1.5.31 (GNS construction for a *-ideal) Let $\mathscr{A}$ be a *-algebra and let $\mathscr{B} \subseteq \mathscr{A}$ be a *-ideal. Suppose $\omega\colon \mathscr{B} \longrightarrow \mathbb{C}$ is a positive linear functional.

i.) Show that the Gel'fand ideal $\mathscr{J}_\omega \subseteq \mathscr{B}$ of ω is also a left ideal in $\mathscr{A}$.

ii.) Show that the GNS representation π_ω of $\mathscr{B}$ on the GNS pre-Hilbert space $\mathfrak{H}_\omega$ extends to a *-representation of $\mathscr{A}$ by setting $\pi_\omega(a)\psi_b = \psi_{ab}$ for $a \in \mathscr{A}$ and $\psi_b \in \mathfrak{H}_\omega$ being the equivalence class of $b \in \mathscr{B}$.

This version of the GNS construction is particularly useful whenever the positive linear functional ω can not be extended to the ambient *-algebra: the GNS representation nevertheless extends.

Exercise 1.5.32 (GNS construction for $\mathfrak{B}(\mathfrak{H})$) Consider a pre-Hilbert space $\mathfrak{H}$ and a non-zero vector $\phi \in \mathfrak{H}$. Moreover, consider the unital *-algebra $\mathscr{A} = \mathfrak{B}(\mathfrak{H})$ of all adjointable operators on $\mathfrak{H}$.

i.) Show that the linear functional $\omega_\phi\colon A \mapsto \langle \phi, A\phi \rangle$ is positive.

ii.) Compute the Gel'fand ideal of ω_ϕ explicitly.

iii.) Show that the GNS pre-Hilbert space $\mathfrak{H}_{\omega_\phi}$ is canonically isometrically isomorphic to the pre-Hilbert space $\mathfrak{H}$ via $U\colon \mathfrak{H}_{\omega_\psi} \ni \psi_A \mapsto A\phi \in \mathfrak{H}$.

Hint: For the well-definedness of this map you have to use the explicit characterization of the Gel'fand ideal. For the surjectivity the following operators $\Theta_{\psi,\chi}\colon \mathfrak{H} \ni \xi \mapsto \psi\langle \chi, \xi \rangle \in \mathfrak{H}$ for $\psi, \chi \in \mathfrak{H}$ might be helpful. Show that $\Theta_{\psi,\chi}$ is indeed adjointable.

iv.) Show that the GNS representation of $\mathscr{A}$ is unitarily equivalent to the defining representation of $\mathscr{A}$ on $\mathfrak{H}$ via U.

Exercise 1.5.33 stefan: GNS and Schrödin

Exercise 1.5.34 (Heuristics on the time evolution) In this exercise *no* analytic justification for the desired algebraic manipulations should be given, the focus is just on the heuristics. Consider a unital *-algebra $\mathscr{A}$ and let Φ_t be a one-parameter group of *-automorphisms of $\mathscr{A}$.

i.) Show by heuristic algebraic manipulations that D defined by $D(a) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\Phi_t(a)$ is a *-derivation of $\mathscr{A}$.

ii.) Show that Φ_t provides the solution to the initial value problem (1.4.3).

iii.) Conversely, let $D \in {}^*\text{-}\mathrm{Der}(\mathscr{A})$ be a *-derivation. Show by heuristic algebraic manipulations that the solution to the initial value problem (1.4.4) gives a one-parameter group Φ_t of *-automorphisms of $\mathscr{A}$, whose derivative at $t = 0$ reproduces D by (1.4.3).

iv.) Suppose in addition that $D = -\mathrm{i}\,\mathrm{ad}(H)$ is an inner derivation with a Hermitian element $H \in \mathscr{A}$. Show that in this case $U_t = \exp(-\mathrm{i}tH)$ is a one-parameter group of unitaries in $\mathscr{A}$ such that the corresponding inner automorphisms $\Phi_t(a) = U_t a U_t^{-1}$ provide a one-parameter group of *-automorphisms solving the initial value problem (1.4.4).

It will require substantial analytic efforts to justify these heuristics. In fact, we will find appropriate formulations step by step in the subsequent chapters.

Chapter 2

From Topological Vector Spaces to Banach Spaces

In this preparatory chapter we collect some results on topological vector spaces with particular emphasize on locally convex spaces and Banach spaces. We only give a selective presentation instead of a more systematic study which would lead us much too far. For such, we refer to the textbooks and monographs like e.g. [26, 37, 49, 66]. We will need some notions and results from point set topology as they can be found in any textbook like e.g. [31, 42]. Most of the results we need can be found in [61].

In view of later applications to spectral theory we will essentially focus on the case of complex vector spaces. However, the case of real vector spaces can be developed essentially the same way with some minor modifications at a few places. Thus it suffices to indicate the differences between the real and complex situation only at these rare instances.

The starting point of our investigations will be a complex vector space together with a topology such that the vector space operations become continuous maps. Typically, we insist on a Hausdorff topology. Requiring linear maps to be continuous, too, yields a subcategory of the category of complex vector spaces. Notions of Cauchy nets and completeness can be formulated in this context immediately leading to the first non-trivial result that a Hausdorff topological vector space always has a unique completion.

In a second step we require that the topology will be determined by means of a collection of seminorms. This will bring us into the realm of locally convex vector spaces. We will base our discussion mainly on the usage of seminorms and come to more geometric characterizations of locally convex topologies only in the last section of this chapter. In the locally convex setting, the first important theorem is the Hahn-Banach Theorem which guarantees the existence of many continuous linear functionals. While general locally convex spaces can show still a rather wild behaviour, things become much nicer if one adds not only completeness but also first countability of the topology. This will bring us to Fréchet spaces. The countability allows to use Baire theory, resulting in the important theorems of Banach-Steinhaus, the Open Mapping Theorem, and the Closed Graph Theorem. We conclude our study of locally convex spaces with some first results on initial and final topologies, in particular on locally convex quotients.

Among the Fréchet spaces, those which allow to define the topology with a single norm instead of countably many seminorms play a particular role. After choosing this norm one arrives at the notion of a Banach space. Here the linear maps between Banach spaces carry an additional structure, the operator norm. The easily defined completion of a normed space yields a Banach space. We use this construction to prove the existence of a completion of an arbitrary Hausdorff locally convex space. The study of locally convex spaces and, in particular, of Banach spaces requires a good understanding of the dual space. We introduce some first notions from duality theory and discuss the arising weak topologies. A first important application will be the Banach-Alaoglu Theorem which we formulate both for general Hausdorff locally convex spaces and for Banach spaces.

The last part of this chapter will not be needed immediately but only in Section ??. Here we discuss several more geometric aspects of the theory of locally convex spaces. First we give a geometric interpretation of the Hahn-Banach Theorem in terms of separation properties together with several applications. Second, we study bounded subsets and the Heine-Borel property and prove the Bipolar Theorem. One fundamental application will be that the bounded subsets with respect to the original and the weak topology agree. Finally, convex subsets and their extreme points are studied. The Krein-Milman Theorem as well as the Milman Theorem give far-reaching characterizations of the extreme points.

2.1 Topological Vector Spaces

This preliminary section will not be a general introduction to topological vector spaces, see e.g. [26, Chap. 2] instead. Here we will just collect the relevant definitions to proceed. As in the end we are mainly interested in complex Hilbert spaces and *-algebras over $\mathbb{C}$ we choose $\mathbb{C}$ as our field of scalars throughout. However, it will be clear that all definitions and most of the theorems will have their real counterpart.

2.1.1 Topological Vector Spaces and Continuous Linear Maps

We start with the central definition of a topological vector space:

Definition 2.1.1 (Topological vector space) *A topological vector space V is a vector space endowed with a topology such that the addition of vectors and the multiplication of vectors with scalars are continuous maps.*

More explicitly, this means that with respect to the product topology the maps

$$+ \colon V \times V \ni (v, w) \;\mapsto\; v + w \in V \tag{2.1.1}$$

and

$$\cdot \colon \mathbb{C} \times V \ni (z, v) \;\mapsto\; zv \in V \tag{2.1.2}$$

are continuous, Here $\mathbb{C}$ is of course always equipped with its natural topology. It follows that the inverse

$$V \ni v \;\mapsto\; -v \in V \tag{2.1.3}$$

and the translations by a fixed vector $v \in V$, i.e.

$$\tau_v \colon V \ni w \;\mapsto\; \tau_v(w) = w + v \in V, \tag{2.1.4}$$

are continuous as well. Thus $(V, +)$ becomes a *topological commutative group* which acts on itself by homeomorphisms τ_v. Analogously, the multiplication with $z \in \mathbb{C} \setminus \{0\}$ is a homeomorphism whose inverse is the multiplication with $\frac{1}{z}$.

Remark 2.1.2 Let V be a topological vector space.

i.) If $W \subseteq V$ is a subspace then its *closure* $W^{\mathrm{cl}} \subseteq V$ is still a subspace. This follows most easily by approximating given $v, w \in W^{\mathrm{cl}}$ with suitable nets $(v_i)_{i \in I}$ and $(w_j)_{j \in J}$ and using the continuity of (2.1.1) and (2.1.2) to show $v_i + w_j \longrightarrow v + w$ as well as $zv_i \longrightarrow zv$. Thus $v + w, zv \in W^{\mathrm{cl}}$ follows.

ii.) Analogously, one shows that the *sequential closure* $W^{\mathrm{scl}} \subseteq V$ of a subspace is a subspace again. We have $W^{\mathrm{scl}} \subseteq W^{\mathrm{cl}}$ and the sequential closure will typically be strictly smaller than the closure.

iii.) Let $U \subseteq V$ be an (open) neighbourhood of $0 \in V$ then $\tau_v(U) = v + U = \{v + u \mid u \in U\}$ is again an (open) neighbourhood of v by the homeomorphism property of τ_v. Thus we can move the whole neighbourhood system of 0 to any other point and vice versa. With other words, the topology is *translationally invariant.*

iv.) For $z \neq 0$ and an (open) neighbourhood $U \subseteq V$ of 0 also $zU = \{zu \mid u \in U\}$ is an (open) neighbourhood of 0.

v.) In general, we will be mainly interested in Hausdorff topologies. However, for many constructions one has as intermediate steps non-Hausdorff topologies and hence we do not include the Hausdorff property as part of the definition of a topological vector space. Note however, that this in not uniform in the literature, compare e.g. with [49, Def. 1.6] and [26, Sect. 2.1].

The translationally invariant topology allows to transfer two concepts from metric spaces: uniform continuity and Cauchy nets. We start with uniform continuity:

Definition 2.1.3 (Uniformly continuous maps) *Let V and W be topological vector spaces and let $\phi\colon V \longrightarrow W$ be a not necessarily linear map. Then ϕ is called uniformly continuous if for all neighbourhoods $U \subseteq W$ of 0 there is a neighbourhood $Z \subseteq V$ of 0 such that*

$$\phi(v) - \phi(v') \in U \quad \text{for} \quad v - v' \in Z. \tag{2.1.5}$$

For vector spaces we are of course mainly interested in linear maps. The following proposition discusses now the continuity properties of linear maps:

Proposition 2.1.4 *Let V and W be topological vector spaces and let $\phi\colon V \longrightarrow W$ be a linear map. Then the following statements are equivalent:*

i.) The map ϕ is continuous at 0.

ii.) The map ϕ is continuous at some $v \in V$.

iii.) The map ϕ is continuous.

iv.) The map ϕ is uniformly continuous.

PROOF: Assume *i.)* and let $v \in V$ be given. If $(v_i)_{i \in I}$ is a net in V converging to v then $(v_i - v)_{i \in I} = (\tau_{-v}(v_i))_{i \in I}$ is a net converging to 0 since the vector space operations are continuous. Hence $\phi(v_i) - \phi(v) = \phi(v_i - v)$ converges to 0 in W by the linearity and continuity at zero of ϕ. Since the vector space operations in W are continuous as well, $\phi(v_i) \longrightarrow \phi(v)$ which shows that ϕ is net-continuous at v and hence continuous at v. This shows *ii.)*. The reverse implication is done analogously and thus the equivalence with *iii.)* follows as well since $v \in V$ was arbitrary. Assume still *i.)* then the continuity at zero together with $\phi(v) - \phi(v') = \phi(v - v')$ implies *iv.)* immediately. The converse is trivial, by setting $v' = 0$. □

Proposition 2.1.5 *Let V and W be topological vector spaces. Then the continuous linear maps from V to W form a subspace of all linear maps from V to W.*

PROOF: Let $\phi, \psi\colon V \longrightarrow W$ be continuous linear maps and let $z, w \in \mathbb{C}$. Then the linear map $z\phi + w\psi$ is continuous at zero: indeed, let $(v_i)_{i \in I}$ be a net converging to $0 \in V$. Then the image net $((z\phi + w\psi)(v_i))_{i \in I}$ still converges to $0 \in W$ since $(z\phi + w\psi)(v_i) = z\phi(v_i) + w\psi(v_i)$ and the vector space operations as well as ϕ and ψ are continuous. By Proposition 2.1.4 this is all we have to show. □

Definition 2.1.6 (Continuous linear maps and the topological dual) *Let V, W be topological vector spaces.*

i.) The vector space of continuous linear maps from V to W is denoted by $\mathrm{L}(V, W)$.

ii.) *The continuous linear functionals on V, i.e.* $\mathrm{L}(V, \mathbb{C})$, *is called the topological dual V' of V.*

As usual, all linear maps from V to W are denoted by $\mathrm{Hom}(V, W)$, and the algebraic dual of V is denoted by V^*. Then by Proposition 2.1.5 we have the subspaces

$$\mathrm{L}(V, W) \subseteq \mathrm{Hom}(V, W) \tag{2.1.6}$$

and

$$V' = \mathrm{L}(V, \mathbb{C}) \subseteq V^* = \mathrm{Hom}(V, \mathbb{C}). \tag{2.1.7}$$

In general, these subspaces will be proper subspaces: not all linear maps will be continuous like this is the case in finite dimensions. Unfortunately, the notation is not uniform in the literature at all: sometimes V' denotes the algebraic dual while V^* stands for the topological dual.

Since the identity map is certainly both, linear and continuous, and since the composition of linear and continuous maps is again linear and continuous, the following statement holds:

Proposition 2.1.7 (The category $\mathsf{topVect}$) *The topological vector spaces over $\mathbb{C}$ together with the continuous linear maps as morphisms form a category, denoted by* $\mathsf{topVect}$.

The full subcategory of Hausdorff topological vector spaces is denoted by $\mathsf{TopVect}$.

2.1.2 Completeness

Analogously to metric spaces we can define Cauchy nets and Cauchy sequences in a topological vector space.

Definition 2.1.8 (Cauchy nets and Cauchy sequences) *Let V be a topological vector space and let $(v_i)_{i\in I}$ be a net in V. Then $(v_i)_{i\in I}$ is called a Cauchy net if for all neighbourhoods $U \subseteq V$ of 0 there is an index $i \in I$ such that $v_j - v_{j'} \in U$ for all $j, j' \succcurlyeq i$. A Cauchy net with index set $I = \mathbb{N}$ is called a Cauchy sequence.*

Remark 2.1.9 Let V be a topological vector space and let $(v_i)_{i\in I}$be a net in V.

i.) If $\phi\colon V \longrightarrow W$ is a uniformly continuous map and if $(v_i)_{i\in I}$ is a Cauchy net, then $(\phi(v_i))_{i\in I}$ is a Cauchy net in W. In particular, by Proposition 2.1.4, this is true for any continuous linear map ϕ.

ii.) A convergent net is a Cauchy net, see Exercise 2.5.6.

iii.) In general, the concept of a Cauchy net is more general than that of a Cauchy sequence. However, if the topology of V happens to be *first countable* (at zero and hence at every other point by translations) then we can fix a countable neighbourhood basis $\{U_n\}_{n\in\mathbb{N}}$ for $0 \in V$. This gives raise to a *sequence* of indices $i_n \in I$ with $v_j - v_{j'} \in U_n$ for all $j, j' \succcurlyeq i_n$. In this sense we end up with a Cauchy sequence again and the Cauchy net we started with is completely controlled by the behaviour of this sequence: it converges iff the sequence converges. Nevertheless, we will also meet situations where V fails to be first countable and hence the notion of Cauchy nets becomes truly more general, see also Remark 2.3.26.

tefan: Check!

As in the metric case it is interesting to ask whether every Cauchy net or Cauchy sequence converges. This motivates the following definition:

Definition 2.1.10 (Completeness) *Let V be a topological vector space.*

i.) *The space V is called complete if every Cauchy net converges.*

ii.) *The space V is called sequentially complete if every Cauchy sequence converges.*

Completeness and closure work well together in both versions:

Proposition 2.1.11 *Let V be a topological vector space and let $U \subseteq V$ be a subspace.*

i.) If V is complete then U^{cl} is complete as well.

ii.) If V is sequentially complete then U^{scl} as well as U^{cl} are sequentially complete.

PROOF: First we note by Remark 2.1.2, *i.)* and *ii.)*, that the closures U^{cl} and U^{scl} are again subspaces of V. If $(v_i)_{i \in I}$ is a Cauchy net in U^{cl} then it has a limit in V by completeness. By closeness of U^{cl} the limit is in U^{cl}. The second case follows analogously. □

As in elementary calculus the importance of completeness will be that we will often be able to show that a certain net or sequence is Cauchy. Then the corresponding completeness will give us a limit, even though in many cases we can not explicitly compute it.

Remark 2.1.12 If V is first countable then V is complete iff it is sequentially complete, see Exercise 2.5.16. In general, completeness implies sequential completeness but not vice versa. In fact, many interesting topological vector spaces will be sequentially complete but *not* complete. We will see such examples when we start discussing weak topologies in Subsection 2.3.3.

A metric spaces can always be completed by a fairly explicit construction by considering the space of all Cauchy sequences in it modulo the equivalence relation that their mutual distances become small, see Appendix **??**. For a general topological vector space one can also define and construct a completion, essentially along the same lines. However, as expected, we have to take care to consider sufficiently general Cauchy nets instead of just Cauchy sequences. If V is not Hausdorff, the concept of completion becomes somewhat pathological as we could add arbitrarily many limit points. In the Hausdorff situation, we call a Hausdorff topological vector space $\widehat{V}$ together with a continuous linear map

$$\iota \colon V \longrightarrow \widehat{V} \tag{2.1.8}$$

a *completion* of V if $\widehat{V}$ is complete, ι is a homeomorphism onto its image $\iota(V) \subseteq \widehat{V}$ and the image $\iota(V)$ is dense in $\widehat{V}$. As in the case of metric spaces we can always build a completion of V:

Theorem 2.1.13 (Completion) *Let V be a Hausdorff topological vector space. Then there exists a completion $(\widehat{V}, \iota)$ which is unique up to a uniquely determined linear homeomorphism.*

In particular, the uniqueness statement allows to speak of *the* completion of V in the following. Typically, we write $V \subseteq \widehat{V}$ for simplicity and omit the usage of the embedding map ι. We shall not prove the theorem in this generality but refer e.g. to [26, Thm. 3.3.3]. Later on, we will come back to a more specific situation where a completion can be obtained more easily. Finally note that the uniqueness statement would clearly not be possible without the Hausdorff property.

The compatibility of completion with maps is expressed in the following statement, which shows that completion has good functorial properties:

Theorem 2.1.14 (Extension to the completion) *Let V and W be Hausdorff topological vector spaces and let $\phi \colon V \longrightarrow W$ be a continuous linear map. Then there exists a unique continuous linear extension*

$$\widehat{\phi} \colon \widehat{V} \longrightarrow \widehat{W} \tag{2.1.9}$$

of ϕ to the completions of V and W, respectively. If ϕ is a homeomorphism then $\widehat{\phi}$ is a homeomorphism, too.

Again, we refer to [26, Thm. 3.4.2] for a proof of this statement. The essential idea is to approximate $v \in \widehat{V}$ by some Cauchy net $(v_i)_{i \in I}$ in V, converging to v. Then one defines $\widehat{\phi}(v) = \lim_{i \in I} \phi(v_i)$ and has to show that this is well-defined, continuous, and linear. The uniqueness is clear by Lemma A.2.1, *ii.)*. The last part follows as we can apply the completion to ϕ and to ϕ^{-1}.

Remark 2.1.15 There is also an abstract concept taking care of uniform continuity, Cauchy nets, and completion: the notion of uniform spaces, see e.g. [31, Chap. 6] or [42, Chap. 11].

2.2 Locally Convex Spaces

The category TopVect is still somewhat too large and arbitrary for interesting theorems. We will now discuss a more specific class of topological vector spaces, the locally convex spaces. Our approach is mainly based on the usage of seminorms and not so much on the characterization of the open neighbourhood systems via convexity. Both points of view are completely equivalent. We prefer the seminorm point of view as in many applications the topologies will be defined by seminorms in a rather direct way. Moreover, in many situations the vector space comes with seminorms which have a direct and independent meaning and importance. This will be typically the case for function spaces in analysis but also for various sequence spaces. Thus estimates involving these particular seminorms may carry numerical information beyond the purely topological content. Of course, relying on particular systems of seminorms may be useful in applications but spoils the good functorial behaviour.

2.2.1 Seminorms and Locally Convex Topologies

The topology of $\mathbb{C}^n$ is defined by means of a norm and any choice of a norm gives the same topology, see Exercise 2.5.9. Having this in mind, the idea is now to allow for many norms or even seminorms to specify what "open balls" should be. We recall first the following definition:

Definition 2.2.1 (Seminorm) *Let V be a complex vector space. A seminorm on V is a map $\mathrm{p}\colon V \longrightarrow \mathbb{R}$ with*

i.) $\mathrm{p}(v) \geq 0$ *for all* $v \in V$*,*

ii.) $\mathrm{p}(zv) = |z|\mathrm{p}(v)$ *for all* $v \in V$ *and* $z \in \mathbb{C}$*,*

iii.) $\mathrm{p}(v+w) \leq \mathrm{p}(v) + \mathrm{p}(w)$ *for all* $v, w \in V$*.*

If in addition $\mathrm{p}(v) > 0$ *for* $v \neq 0$ *then* p *is called a norm.*

For a seminorm p on V we define the *kernel* of it to be

$$\ker \mathrm{p} = \big\{ v \in V \bigm| \mathrm{p}(v) = 0 \big\}. \tag{2.2.1}$$

It follows easily from the defining properties of a seminorm that $\ker \mathrm{p}$ is a subspace of V. Then p is a norm iff $\ker \mathrm{p} = \{0\}$.

If p is a seminorm on V and $v \in V$ then we define the *open ϵ-ball* around v with respect to p for $\epsilon > 0$ by

$$\mathrm{B}_{\mathrm{p},\epsilon}(v) = \big\{ w \in V \bigm| \mathrm{p}(v-w) < \epsilon \big\}. \tag{2.2.2}$$

Analogously, one defines the *closed ϵ-ball* around v with respect to p for $\epsilon > 0$ by

$$\mathrm{B}_{\mathrm{p},\epsilon}(v)^{\mathrm{cl}} = \big\{ w \in V \bigm| \mathrm{p}(v-w) \leq \epsilon \big\}. \tag{2.2.3}$$

We note already here that the "balls" are perhaps better called "cylinders" since it is clear that $\ker \mathrm{p} \subseteq \mathrm{B}_{\mathrm{p},\epsilon}(0)$ for all $\epsilon > 0$. Thus if $\ker \mathrm{p}$ is nontrivial, $\mathrm{B}_{\mathrm{p},\epsilon}(0)$ looks more like a cylinder than a ball.

The idea is now to justify these notions of "open" and "closed" by the construction of a topology in such a way that the open (closed) balls are indeed open (closed), at least of certain choices of the seminorms.

To this end we do not just consider one seminorm but allow for a whole collection $\mathcal{P}$ of seminorms on V. One calls $\mathcal{P}$ *filtrating* if for finitely many $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ there is a *dominating* seminorm $\mathrm{q} \in \mathcal{P}$, i.e.

$$\mathrm{p}_1(v), \ldots, \mathrm{p}_n(v) \leq \mathrm{q}(v) \tag{2.2.4}$$

for all $v \in V$. Moreover, one calls $\mathcal{P}$ *Hausdorff* if for every $v \in V$ there is a $\mathrm{p} \in \mathcal{P}$ with

$$\mathrm{p}(v) > 0. \tag{2.2.5}$$

Note that we do not require to have a norm p: the choice of p may depend on v.

Given a collection of seminorms $\mathcal{P}$ on V we can always extend it to a filtrating collection in the following canonical way:

Lemma 2.2.2 *Let $\mathcal{P}$ be a collection of seminorms on V. Then there exists a unique collection of seminorms $\overline{\mathcal{P}}$ with the property that $\mathrm{q} \in \overline{\mathcal{P}}$ iff there exists a $c > 0$ and seminorms $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ such that*

$$\mathrm{q}(v) \leq c \max_{i=1}^{n} \mathrm{p}_i(v) \tag{2.2.6}$$

for all $v \in V$. Then $\overline{\mathcal{P}}$ contains $\mathcal{P}$, is filtrating, and it is closed under convex combinations and finite maxima. Moreover, $\overline{\overline{\mathcal{P}}} = \overline{\mathcal{P}}$. If $\mathcal{Q} \subseteq \mathcal{P}$ is another set of seminorms then $\overline{\mathcal{Q}} \subseteq \overline{\mathcal{P}}$.

PROOF: Clearly, we can take the condition (2.2.6) as definition for $\overline{\mathcal{P}}$ and obtain the unique maximal collection with this property. Obviously, it contains $\mathcal{P}$. Now let $\mathrm{q}_1, \ldots, \mathrm{q}_m \in \overline{\mathcal{P}}$ be given with corresponding $\mathrm{q}_j \leq c_j \max_i\{\mathrm{p}_{ji}\}$. Then for all $\alpha_1, \ldots, \alpha_m > 0$ we have

$$\alpha_1 \mathrm{q}_1 + \cdots + \alpha_m \mathrm{q}_m \leq C \max_{i,j} \mathrm{p}_{ji}$$

with $C = \max_j \alpha_j c_j$. This shows that $\overline{\mathcal{P}}$ is closed under convex combinations. Since $\mathrm{q}_i \leq \max_j \mathrm{q}_j \leq \mathrm{q}_1 + \cdots + \mathrm{q}_n$ it follows that $\overline{\mathcal{P}}$ is also closed under finite maximums and it is filtrating. The remaining statements are clear. □

Depending on the choice of a collection $\mathcal{P}$ of seminorms we declare now a topology be requiring the open ϵ-balls with respect to the seminorms $\mathrm{p} \in \mathcal{P}$ to be open: thus we consider

$$\mathcal{B}(\mathcal{P}) = \left\{ \mathrm{B}_{\mathrm{p},\epsilon}(v) \;\middle|\; v \in V,\ \epsilon > 0, \text{ and } \mathrm{p} \in \mathcal{P} \right\} \tag{2.2.7}$$

as the subbasis of a topology on V. We call the corresponding topology the *topology induced by* $\mathcal{P}$. This topology has now the following features:

Theorem 2.2.3 (Topology induced by seminorms) *Let V be a complex vector space and let $\mathcal{P}$ be a collection of seminorms on V.*

i.) The collection $\mathcal{B}(\mathcal{P})$ is even a basis of the topology induced by $\mathcal{P}$ if $\mathcal{P}$ is filtrating.

ii.) The topologies induced by $\mathcal{P}$ and $\overline{\mathcal{P}}$ coincide and $\mathcal{B}(\overline{\mathcal{P}})$ is a basis of it. Another basis of this topology is given by $\{\mathrm{B}_{\mathrm{p},1}(v) \mid v \in V \text{ and } \mathrm{p} \in \overline{\mathcal{P}}\}$.

iii.) A subset $U \subseteq V$ is open iff for all $v \in U$ there is a $\epsilon > 0$ and a seminorm $\mathrm{q} \in \overline{\mathcal{P}}$ with $\mathrm{B}_{\mathrm{q},\epsilon}(v) \subseteq U$. If $\mathcal{P}$ was already filtrating, then we can take $\mathrm{q} \in \mathcal{P}$.

iv.) The topology induced by $\mathcal{P}$ makes V a topological vector space.

v.) The topology induced by $\mathcal{P}$ is Hausdorff iff $\mathcal{P}$ is Hausdorff.

vi.) A seminorm p on V is continuous with respect to the topology induced by $\mathcal{P}$ iff $\mathrm{p} \in \overline{\mathcal{P}}$.

PROOF: First we show that $\mathcal{B}(\mathcal{P})$ is already a basis and not just a subbasis of a topology provided that the collection of seminorms is filtrating. Indeed, let $\mathrm{B}_{\mathrm{p}_1,\epsilon_1}(v_1), \ldots, \mathrm{B}_{\mathrm{p}_n,\epsilon_n}(v_n)$ be a finite number of open balls with $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ and suppose that their intersection

$$U = \mathrm{B}_{\mathrm{p}_1,\epsilon_1}(v_1) \cap \cdots \cap \mathrm{B}_{\mathrm{p}_n,\epsilon_n}(v_n)$$

is non-empty. For $u \in U$ we have $\mathrm{p}_i(v_i - u) < \epsilon_i$ for $i = 1, \ldots, n$ and hence $\delta_i = \epsilon_i - \mathrm{p}_i(v_i - u) > 0$. We consider a seminorm $\mathrm{q} \in \mathcal{P}$ dominating the p_i for $i = 1, \ldots, n$. Moreover, let $\delta = \min_{i=1}^n \delta_i$ be the minimum of the above constants. We claim that the open ball $\mathrm{B}_{\mathrm{q},\delta}(u)$ is still contained entirely in U. Indeed, let $u' \in \mathrm{B}_{\mathrm{q},\delta}(u)$ be given. Then

$$\mathrm{p}_i(u' - u) \leq \mathrm{q}(u' - u) < \delta \leq \delta_i.$$

By the definition of δ_i and the triangle inequality for the seminorm p_i this gives

$$\mathrm{p}_i(u' - v_i) \leq \mathrm{p}_i(u' - u) + \mathrm{p}_i(u - v_i) < \epsilon_i$$

and thus $u' \in U$ follows. This implies that for a filtrating $\mathcal{P}$ the collection $\mathcal{B}(\mathcal{P})$ is a basis. In particular, this applies to $\overline{\mathcal{P}}$. For the second part it is clear that $\mathcal{B}(\mathcal{P}) \subseteq \mathcal{B}(\overline{\mathcal{P}})$ and hence the topology induced by $\mathcal{P}$ is coarser. Thus we have to show that every $U \in \mathcal{B}(\overline{\mathcal{P}})$ is also open with respect to the topology induced by $\mathcal{P}$. Hence let $\mathrm{q} \in \overline{\mathcal{P}}$ with $\epsilon > 0$ be given. Then we find $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ and $c > 0$ with $\mathrm{q} \leq c \max_{i=1}^n \mathrm{p}_i$. But this implies that for $u \in \mathrm{B}_{\mathrm{q},\epsilon}(v)$ we have

$$\mathrm{B}_{\mathrm{p}_1,\delta}(u) \cap \cdots \cap \mathrm{B}_{\mathrm{p}_n,\delta}(u) \subseteq \mathrm{B}_{\mathrm{q},\epsilon}(v), \tag{$*$}$$

where we set $\delta = \frac{\epsilon - \mathrm{q}(v-u)}{c} > 0$. Indeed, let u' be in the intersection on the left hand side then

$$\mathrm{q}(v - u') \leq \mathrm{q}(v - u) + \mathrm{q}(u - u') \leq \mathrm{q}(v - u) + c \max_{i=1}^n \mathrm{p}_i(u - u') < \mathrm{q}(v - u) + c\frac{\epsilon - \mathrm{q}(v - u)}{c} = \epsilon.$$

From $(*)$ we conclude that $\mathrm{B}_{\mathrm{q},\epsilon}(v)$ is open in the topology induced by $\mathcal{P}$. The last statement of the second part is clear since all the rescalings of a seminorm form $\overline{\mathcal{P}}$ are already contained in $\overline{\mathcal{P}}$. For part *iii.)* it is clear that a subset $U \subseteq V$ with the described property is open as it is the union of all the open balls $\mathrm{B}_{\mathrm{q},\epsilon}(v)$. Conversely, let U be open and $v \in U$. Since the $\mathcal{B}(\overline{\mathcal{P}})$ are already a basis we conclude that $v \in \mathrm{B}_{\mathrm{q},\delta}(v') \subseteq U$ for some suitable $\delta > 0$, some $\mathrm{q} \in \overline{\mathcal{P}}$, and some $v' \in U$. Setting $\epsilon = \delta - \mathrm{q}(v - v') > 0$ yields $\mathrm{B}_{\mathrm{q},\epsilon}(v) \subseteq \mathrm{B}_{\mathrm{q},\delta}(v') \subseteq U$ again by the triangle inequality for q. This shows the third part. For the fourth part, let $v \in V$ and let $\mathrm{B}_{\mathrm{q},\epsilon}(v) \subseteq V$ be an open ϵ-ball with $\mathrm{q} \in \overline{\mathcal{P}}$ and $\epsilon > 0$. We have to show that its preimage under $+\colon V \times V \longrightarrow V$ is open in $V \times V$. This preimage consists of those $(u, u') \in V \times V$ with $\delta = \mathrm{q}(u + u' - v) < \epsilon$. For such (u, u') we claim that $\mathrm{B}_{\mathrm{q},\frac{\epsilon-\delta}{2}}(u) \times \mathrm{B}_{\mathrm{q},\frac{\epsilon-\delta}{2}}(u')$ is mapped into $\mathrm{B}_{\mathrm{q},\epsilon}(v)$. Indeed, let $w \in \mathrm{B}_{\mathrm{q},\frac{\epsilon-\delta}{2}}(u)$ and $w' \in \mathrm{B}_{\mathrm{q},\frac{\epsilon-\delta}{2}}(u')$. Then the triangle inequality gives immediately $\mathrm{q}(w + w' - v) \leq \mathrm{q}(w - u) + \mathrm{q}(w' - u') + \mathrm{q}(u + u' - v) < \frac{\epsilon-\delta}{2} + \frac{\epsilon-\delta}{2} + \delta = \epsilon$. Since $\mathcal{B}(\overline{\mathcal{P}})$ is a basis, this is enough to conclude that $+$ is continuous. For the multiplication with scalars we consider $(z, u) \in \mathbb{C} \times V$ with $\delta = \mathrm{q}(zu - v) < \epsilon$, i.e. the preimage of $\mathrm{B}_{\mathrm{q},\epsilon}(v)$ under the multiplication with scalars. We claim that

$$\mathrm{B}_{\frac{\epsilon-\delta}{2}}(z) \times \left(\mathrm{B}_{\mathrm{q},\frac{\epsilon-\delta}{2(|z|+1)}}(u) \cap \mathrm{B}_{\mathrm{q},1}(0)\right) \subseteq \mathbb{C} \times V$$

is mapped into $\mathrm{B}_{\mathrm{q},\epsilon}(v)$. Again, this is a consequence of the homogeneity and the triangle inequality for the seminorm q: let (z', w) be in this, obviously open, subset of $\mathbb{C} \times V$ then

$$\begin{aligned} \mathrm{q}(z'w - v) &\leq \mathrm{q}(z'w - zw) + \mathrm{q}(zw - zu) + \mathrm{q}(zu - v) \\ &\leq |z' - z|\mathrm{q}(w) + |z|\mathrm{q}(w - u) + \delta \end{aligned}$$

$$< \frac{\epsilon - \delta}{2} \cdot 1 + |z| \frac{\epsilon - \delta}{2(|z| + 1)} + \delta$$
$$\leq \epsilon.$$

Thus we conclude that the preimage of $\mathrm{B}_{\mathrm{q},\epsilon}(v)$ is open, proving that also the multiplication with scalars is continuous. The fifth part is easy: assume first that $\mathcal{P}$ is Hausdorff and let $u \neq v$ be given. Then let $\mathrm{p} \in \mathcal{P}$ be a seminorm with $\epsilon = \mathrm{p}(u - v) > 0$. The triangle inequality shows immediately that the open $\frac{\epsilon}{2}$-balls around u and v with respect to p are disjoint. Thus the topology is Hausdorff. Conversely, assume the induced topology is Hausdorff and let $v \neq 0$ be given. Since $\mathcal{B}(\overline{\mathcal{P}})$ is a basis we find a $\mathrm{q} \in \overline{\mathcal{P}}$ and some $\epsilon > 0$ with $0 \notin \mathrm{B}_{\mathrm{q},\epsilon}(v)$. But this just means $\mathrm{q}(v) \geq \epsilon > 0$. By (2.2.6) there must be at least one $\mathrm{p} \in \mathcal{P}$ with $\mathrm{p}(v) > 0$. For the last part, let $\mathrm{q} \in \overline{\mathcal{P}}$ be given. We have to check that for $a, b \in \mathbb{R}$ the preimage $\mathrm{q}^{-1}((a, b))$ of the open interval (a, b) is open. Let $v \in \mathrm{q}^{-1}((a, b))$, i.e. $\mathrm{q}(v) \in (a, b)$. Choose $\epsilon > 0$ such that $a < \mathrm{q}(v) - \epsilon$ and $\mathrm{q}(v) + \epsilon < b$. Then $\mathrm{B}_{\mathrm{q},\epsilon}(v) \subseteq \mathrm{q}^{-1}((a, b))$ proving that $\mathrm{q}^{-1}((a, b))$ is open. Conversely, let p be a continuous seminorm and let $\epsilon > 0$. Then $\mathrm{B}_{\mathrm{p},1}(0) = \mathrm{p}^{-1}([0, 1))$ is open. Since $\mathcal{B}(\overline{\mathcal{P}})$ is a basis, we have an open ball $\mathrm{B}_{\mathrm{q},\delta}(0) \subseteq \mathrm{B}_{\mathrm{p},\epsilon}(0)$ for some $\mathrm{q} \in \overline{\mathcal{P}}$ and some $\delta > 0$. This means that $\mathrm{q}(v) < \delta$ implies $\mathrm{p}(v) < 1$ for $v \in V$. We claim that in this situation we have for all $v \in V$ the estimate

$$\mathrm{p}(v) \leq \frac{1}{\delta} \mathrm{q}(v). \tag{$*$}$$

Indeed, first consider the case $\mathrm{q}(v) = 0$. Then also $\mathrm{q}(zv) = |z|\mathrm{q}(v) = 0$ for all z. Thus $|z|\mathrm{p}(v) = \mathrm{p}(zv) < 1$ for all z which implies $\mathrm{p}(v) = 0$. Thus $(*)$ holds in this case. Second, assume $\mathrm{q}(v) > 0$. Then $\mathrm{q}\Big(\frac{\delta}{2\mathrm{q}(v)} v\Big) = \frac{\delta}{2} < \delta$ implies $\mathrm{p}\Big(\frac{\delta}{c\mathrm{q}(v)} v\Big) = \frac{\delta}{c\mathrm{q}(v)} \mathrm{p}(v) < 1$ for every $c > 1$. Since $c > 1$ is arbitrary, this shows $(*)$. But $(*)$ clearly implies that $\mathrm{p} \in \overline{\mathcal{P}}$. □

Topological vector spaces arising from such systems of seminorms provide a class with many additional and pleasant features.

Definition 2.2.4 (Locally convex space) *A topological vector space V is called locally convex if its topology can be induced by a collection of seminorms. The category of such locally convex topological vector spaces is denoted by* $\mathsf{lcVect} \subseteq \mathsf{topVect}$ *while the full subcategory of the Hausdorff ones is denoted by* $\mathsf{LCVect} \subseteq \mathsf{TopVect}$.

For simplicity, we shall refer to a locally convex topological vector space sometimes as *locally convex space*. The name comes from the fact that the open ϵ-balls $\mathrm{B}_{\mathrm{p},\epsilon}(v)$ are clearly *convex subsets*: for $w, w' \in \mathrm{B}_{\mathrm{p},\epsilon}(v)$ and $\lambda \in [0, 1]$ one has $\lambda w + (1 - \lambda)w' \in \mathrm{B}_{\mathrm{p},\epsilon}(v)$ by the triangle inequality and the homogeneity of the seminorm p. Thus every point in a locally convex topological vector space has a basis of open *convex* neighbourhoods. In fact, one can show that this feature actually characterizes the locally convex topological vector spaces completely, see Exercise **??**.

Corollary 2.2.5 *Let $\mathcal{P}$ and $\mathcal{Q}$ be two collections of seminorms on V. Then the following statements are equivalent:*

i.) The topology induced by $\mathcal{P}$ is finer than the topology induced by $\mathcal{Q}$.

ii.) $\overline{\mathcal{Q}} \subseteq \overline{\mathcal{P}}$.

iii.) For every $\mathrm{q} \in \mathcal{Q}$ there is a $c > 0$ and $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ with

$$\mathrm{q} \leq c \max_{i=1}^{n} \mathrm{p}_i. \tag{2.2.8}$$

Proof: Suppose the topology induced by $\mathcal{P}$ is finer. Then every continuous seminorm q with respect to the topology induced by $\mathcal{Q}$, i.e. $\mathrm{q} \in \overline{\mathcal{Q}}$ by Theorem 2.2.3, *vi.)*, is also continuous with respect to the finer topology. Hence $\mathrm{q} \in \overline{\mathcal{P}}$, again by Theorem 2.2.3, *vi.)*. Now suppose $\overline{\mathcal{Q}} \subseteq \overline{\mathcal{P}}$. Then every $\mathrm{q} \in \overline{\mathcal{Q}}$ is continuous with respect to the topology induced by $\mathcal{P}$. Hence $\mathrm{B}_{\mathrm{q},\epsilon}(v)$ is open in the topology induced by $\mathcal{P}$ which implies that this topology is finer than the one induced by $\mathcal{Q}$ since the open ϵ-balls $\mathrm{B}_{\mathrm{q},\epsilon}(v)$ form a basis for the topology induced by $\mathcal{Q}$. This shows the equivalence of *i.)* and *ii.)*. The equivalence of *ii.)* and *iii.)* is clear by Lemma 2.2.2. □

If, in addition, $\mathcal{P}$ is filtrating then we can omit the maximum on the right hand side of (2.2.8) but use just a single seminorm instead.

Corollary 2.2.6 *Let V be a topological vector space. Then the following statements are equivalent:*

i.) The topology of V is locally convex.

ii.) The open ϵ-balls $\mathrm{B}_{\mathrm{p},\epsilon}(v)$ for $\epsilon > 0$, $v \in V$, and continuous seminorms p form a basis of the topology.

iii.) The open ϵ-balls $\mathrm{B}_{\mathrm{p},\epsilon}(0)$ around 0 for $\epsilon > 0$ and continuous seminorms p form a basis of open neighbourhoods of 0.

Remark 2.2.7 (Equivalent systems of seminorms) In application one tries to find systems of seminorms which on one hand are large enough to characterize the (locally convex) topology one is interested in. On the other hand, one tries to keep the system small and simple. In particular, there might be seemingly different choices $\mathcal{P}$ and $\mathcal{Q}$ yielding the same topology as described by Corollary 2.2.5, i.e. we have to have mutual estimates (2.2.8) of the seminorms from $\mathcal{P}$ and $\mathcal{Q}$. In this case we call the two systems of seminorms *equivalent*. If $\mathcal{P}$ yields the a priori given topology of V then we call $\mathcal{P}$ also a *defining system of seminorms*.

Without presenting a counterexample it should be mentioned that there are indeed topological vector spaces which are *not* locally convex. In some sense, in these examples there are not enough continuous seminorms to determine the topology as required by Corollary 2.2.6, see e.g. [26, Sect. 6.10]. In the context of the Hahn-Banach Theorem in Section 2.2.3 we will see that such topological vector spaces behave rather badly for many reasons.

In a next step, one characterizes the continuity of linear maps in the locally convex setting in terms of seminorms:

Proposition 2.2.8 *Let $\phi\colon V \longrightarrow W$ be a linear map between locally convex spaces and let $\mathcal{P}$ and $\mathcal{Q}$ be defining systems of seminorms for V and W, respectively. Then ϕ is continuous iff for all $\mathrm{q} \in \mathcal{Q}$ there is a $c > 0$ and some $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ such that*

$$\mathrm{q}(\phi(v)) \leq c \max_{i=1}^{n} \mathrm{p}_i(v) \tag{2.2.9}$$

for all $v \in V$. If in addition $\mathcal{P}$ is filtrating then it is sufficient to consider a single seminorm $\mathrm{p} \in \mathcal{P}$ instead of the maximum on the right hand side of (2.2.9).

PROOF: Suppose ϕ is continuous. Then $\mathrm{q} \circ \phi\colon V \longrightarrow \mathbb{R}$ is a continuous seminorm on V. This is clear from the linearity of ϕ. Thus $\mathrm{q} \circ \phi \in \overline{\mathcal{P}}$ by Theorem 2.2.3, *vi.)*, and this gives (2.2.9) by the characterization of $\overline{\mathcal{P}}$ according to Lemma 2.2.2. Conversely, assume (2.2.9) holds. This means that the open neighbourhood $\mathrm{B}_{\mathrm{p}_1,\frac{\epsilon}{c}}(0) \cap \cdots \cap \mathrm{B}_{\mathrm{p}_n,\frac{\epsilon}{c}}(0)$ of 0 in V is mapped into the open neighbourhood $\mathrm{B}_{\mathrm{q},\epsilon}(0)$ in W. Thus $\phi^{-1}(\mathrm{B}_{\mathrm{q},\epsilon}(0))$ contains an neighbourhood of 0 and therefore is a neighbourhood of 0 itself. Since the neighbourhoods $\mathrm{B}_{\mathrm{q},\epsilon}(0)$ form a neighbourhood subbasis of 0 we conclude that ϕ is continuous at 0. By Proposition 2.1.4 this implies continuity of ϕ everywhere. Finally, the last statement is clear by the definition of "filtrating". □

Corollary 2.2.9 *Let $\phi\colon V \longrightarrow W$ be a linear map between locally convex spaces. Then ϕ is continuous iff for every continuous seminorm q on W also $\mathrm{q} \circ \phi$ is continuous.*

Corollary 2.2.10 *Let $\phi\colon V \longrightarrow W$ be a linear map between locally convex spaces. Then ϕ is continuous iff for every continuous seminorm* q *on W there is a continuous seminorm* p *on V such that* $\mathrm{q}(\phi(v)) \le \mathrm{p}(v)$ *for all $v \in V$.*

As a first application we note that for a locally convex space V a linear functional $\varphi\colon V \longrightarrow \mathbb{C}$ is continuous iff there is a continuous seminorm p on V such that for all $v \in V$

$$|\varphi(v)| \le \mathrm{p}(v). \tag{2.2.10}$$

2.2.2 Completeness and Completion

Let V be a locally convex space with a system $\mathcal{P}$ of seminorms defining the topology. Then we want to translate the notions of convergent sequences and nets as well as completeness into the language of seminorms.

Proposition 2.2.11 *Let V be a locally convex space with defining system of seminorms $\mathcal{P}$.*

i.) A net $(v_i)_{i\in I}$ is convergent to $v \in V$ iff for all $\epsilon > 0$ and any seminorm $\mathrm{p} \in \mathcal{P}$ there exists an index $i \in I$ such that for $j \succcurlyeq i$ one has

$$\mathrm{p}(v_j - v) < \epsilon. \tag{2.2.11}$$

ii.) A net $(v_i)_{i\in I}$ is a Cauchy net iff for all $\epsilon > 0$ and any seminorm $\mathrm{p} \in \mathcal{P}$ there is an index $i \in I$ such that for $j, k \succcurlyeq i$ one has

$$\mathrm{p}(v_j - v_k) < \epsilon. \tag{2.2.12}$$

PROOF: Indeed, the condition (2.2.11) simply means $v_j \in \mathrm{B}_{\mathrm{p},\epsilon}(v)$. Since the open ϵ-balls with respect to seminorms $\mathrm{p} \in \mathcal{P}$ around v form a subbasis of neighbourhoods, this is all we need to establish convergence. The argument for Cauchy nets is analogous. ☐

In this proposition it is of course useful to choose a defining system $\mathcal{P}$ which is *small* in order to efficiently check the condition (2.2.11) or (2.2.12).

Concerning completions we would like to extend the seminorms on V to its completion $\widehat{V}$ as obtained from Theorem 2.1.13. To this end we note the following trivial observation:

Lemma 2.2.12 *Let V be a locally convex space and* p *a continuous seminorm on V. Then* p *is uniformly continuous.*

PROOF: Let $\epsilon > 0$ be given and consider $v, v' \in V$ with $v - v' \in \mathrm{B}_{\mathrm{p},\epsilon}(0)$. Then by the triangle inequality for p we have $\mathrm{p}(v) \le \mathrm{p}(v - v') + \mathrm{p}(v')$ and hence

$$|\mathrm{p}(v) - \mathrm{p}(v')| \le \mathrm{p}(v - v') < \epsilon.$$

Since the open intervals $(-\epsilon, \epsilon)$ form a basis of open neighbourhoods of $0 \in \mathbb{R}$ this is precisely the uniform continuity of p. ☐

We mention already at this stage the following result on the completion of a Hausdorff locally convex space:

Theorem 2.2.13 (Completion) *Let V be a Hausdorff locally convex space and let $(\widehat{V}, \iota)$ be its completion. Then $\widehat{V}$ is again locally convex and every continuous seminorm $\widehat{\mathrm{p}}$ on $\widehat{V}$ is obtained as canonical extension of a continuous seminorm* p *on V.*

We note that for every continuous seminorm $\widehat{\mathrm{p}}$ on $\widehat{V}$ also $\mathrm{p} = \widehat{\mathrm{p}} \circ \iota \colon V \longrightarrow \mathbb{R}$ is a continuous seminorm which is trivial iff $\widehat{\mathrm{p}}$ is trivial since $\iota(V) \subseteq \widehat{V}$ is dense, see again Lemma A.2.1, *ii.)*. Moreover, every seminorm p on V extends canonically to $\widehat{\widehat{V}}$ by Lemma 2.2.12 and yields again a continuous seminorm. The only thing to show is that the continuous seminorms arising this way already determine the topology of $\widehat{V}$. We do not prove this as well as the existence of a completion at the present stage as we want to rely on some Banach space technology. Thus we postpone the proof until Subsection 2.3.2.

The complete locally convex spaces form a subcategory of the Hausdorff locally convex space LCVect which we shall denote by $\mathsf{CLCVect}$.

As a first application of completeness we discuss the following useful criterion whether we can exchange limits: we consider two directed sets I, J and equip $I \times J$ with the canonical direction as usual, see also Appendix **??**. Then we consider a net $(v_{ij})_{(i,j)\in I\times J}$ indexed by $I \times J$ in a complete Hausdorff locally convex space V. We say that $(v_{ij})_{j\in J}$ converges *uniformly* for all $i \in I$ to v_i if for every continuous seminorm p on V and all $\epsilon > 0$ there is a $j_0 \in J$ with $\mathrm{p}(v_{ij} - v_i) < \epsilon$ for all $j \succcurlyeq j_0$ and all $i \in I$. The point is that j_0 serves for all $i \in I$ at once. Clearly, it suffices to have this feature for a defining system $\mathcal{P}$ of seminorms.

Proposition 2.2.14 *Let V be a complete Hausdorff locally convex space and let $(v_{ij})_{(i,j)\in I\times J}$ be a net in V such that $(v_{ij})_{j\in J}$ converges uniformly to some $v_i \in V$ for all $i \in I$ and such that $\lim_{i\in I} v_{ij} = v_j$ exists for all $j \in J$. Then also $(v_{ij})_{(i,j)\in I\times J}$ and $(v_j)_{j\in J}$ converge and*

$$\lim_{(i,j)\in I\times J} v_{ij} = \lim_{i\in I} \lim_{j\in J} v_{ij} = \lim_{j\in J} \lim_{i\in I} v_{ij}. \tag{2.2.13}$$

PROOF: We consider a continuous seminorm p on V and fix $j_0 \in J$ such that $\mathrm{p}(v_{ij} - v_i) < \epsilon$ for all $j \succcurlyeq j_0$ and all $i \in I$. Then we have

$$\mathrm{p}(v_{ij} - v_{ij_0}) \leq \mathrm{p}(v_{ij} - v_i) + \mathrm{p}(v_i - v_{ij_0}) < 2\epsilon$$

for $j \succcurlyeq j_0$ and all $i \in I$. For the given j_0 we fix now $i_0 \in I$ such that $\mathrm{p}(v_{ij_0} - v_{j_0}) < \epsilon$ for $i \succcurlyeq i_0$. Note that i_0 will depend on j_0 as we do not require uniform convergence for this direction. But then for all $i \succcurlyeq i_0$ we conclude

$$\mathrm{p}(v_{ij_0} - v_{i_0j_0}) \leq \mathrm{p}(v_{ij_0} - v_{j_0}) + \mathrm{p}(v_{j_0} - v_{i_0j_0}) < 2\epsilon,$$

where we used again uniform convergence in the second estimate. Let $i, i' \succcurlyeq i_0$ and $j, j' \succcurlyeq j_0$ then

$$\mathrm{p}(v_{ij} - v_{i'j'}) \leq \mathrm{p}(v_{ij} - v_{ij_0}) + \mathrm{p}(v_{ij_0} - v_{i_0j_0}) + \mathrm{p}(v_{i_0j_0} - v_{i'j_0}) + \mathrm{p}(v_{i'j_0} - v_{i'j'}) < 8\epsilon.$$

Since this holds for every continuous seminorm we conclude that $(v_{ij})_{(i,j)\in I\times J}$ is a Cauchy net and hence convergent. This establishes the existence of the first limit. Since the single limits $\lim_{i\in I} v_{ij}$ and $\lim_{j\in J} v_{ij}$ exist (the second even uniformly) for all $i \in I$ and $j \in J$, respectively, we conclude by Lemma A.4.1 that the iterated limits also exist and coincide with the limit of the whole net. Note that a Hausdorff locally convex space is clearly T_3 as this is required by Lemma A.4.1, see also Exercise 2.5.17. ☐

There are several variations of this statement. In particular, if we are only dealing with double sequences (or nets allowing for cofinal subsequences) then *sequential completeness* would be sufficient.

2.2.3 The Hahn-Banach Theorem

For a locally convex space V one can show that the topological dual V' is "large": more precisely, for every continuous linear functional on a subspace $U \subseteq V$ there exists an extension to a continuous linear functional on the whole space V. This statement is the famous Hahn-Banach Theorem which we will now discuss. We follow essentially [26, Chap.7].

Definition 2.2.15 (Sublinear functional) *Let V be a real or complex vector space. Then a map $p\colon V \longrightarrow \mathbb{R}$ is called sublinear if*

$$p(v+w) \leq p(v) + p(w) \tag{2.2.14}$$

$$p(\lambda v) = \lambda p(v) \tag{2.2.15}$$

for $v, w \in V$ and $\lambda \geq 0$. The set of sublinear functionals is denoted by $V^\sharp$.

Clearly, any $\mathbb{R}$-linear functional is sublinear, also every seminorm is sublinear. We can endow $V^\sharp$ with a partial ordering in the following way: for $p, q \in V^\sharp$ we write

$$p \preccurlyeq q \quad \text{if} \quad p(v) \leq q(v) \quad \text{for all} \quad v \in V. \tag{2.2.16}$$

The following lemma characterizes the minimal elements in $V^\sharp$ with respect to this ordering.

Lemma 2.2.16 *Let V be a real vector space. Then the algebraic dual $V^* \subseteq V^\sharp$ coincides with the set of minimal elements with respect to the partial ordering* (2.2.16)*.*

PROOF: Let $\varphi \in V^*$ be a linear functional and assume there is a $p \in V^\sharp$ with $p \preccurlyeq \varphi$. Then $0 = p(0) \leq p(v) + p(-v)$ for all $v \in V$. Since φ is linear we have $0 = \varphi(v) + \varphi(-v)$ and by $p \preccurlyeq \varphi$ we get $p(v) + p(-v) = 0$. This shows $p(v) = -p(-v)$. But then $-p(-v) = p(v) \leq \varphi(v)$ while $-p(v) = p(-v) \leq \varphi(-v) = -\varphi(v)$. Together, these inequalities imply $p(v) = \varphi(v)$ which shows that φ is minimal. Conversely, assume $p \in V^\sharp$ is minimal. We want to show that p is actually linear. We fix a vector $w \in V$ and consider the map

$$q\colon V \ni v \;\mapsto\; \inf_{\lambda \geq 0} \{p(v + \lambda w) - \lambda p(w)\}. \tag{$*$}$$

We claim that this infimum is actually finite and hence $q\colon V \longrightarrow \mathbb{R}$. Indeed, from $\lambda p(w) = p(\lambda w) \leq p(v + \lambda w) + p(-v)$ we see that $-p(-v) \leq p(v + \lambda w) - \lambda p(w)$ for all $\lambda \geq 0$ and $v, w \in V$. Thus $q(v) > -\infty$ for all $v \in V$. Moreover, we have $q(v) \leq p(v)$ for all $v \in V$ by taking $\lambda = 0$ in the expression $(*)$. In a next step, we claim that $q \in V^\sharp$. We have for $\mu > 0$

$$\begin{aligned}
q(\mu v) &= \inf_{\lambda \geq 0} \{p(\mu v + \lambda w) - \lambda p(w)\} \\
&= \inf_{\lambda \geq 0} \Big\{\mu p\Big(v + \tfrac{\lambda}{\mu} w\Big) - \mu \tfrac{\lambda}{\mu} p(w)\Big\} \\
&= \mu \inf_{\lambda' \geq 0} \big\{p(v + \lambda' w) - \lambda' p(w)\big\} \\
&= \mu q(v).
\end{aligned}$$

Clearly, this is still true for $\mu = 0$ since $q(0) = 0$. This shows (2.2.15) for q. Next, let $\epsilon > 0$ and $v_1, v_2 \in V$. By the definition of the infimum we find $\lambda_1, \lambda_2 \geq 0$ with

$$q(v_i) + \frac{\epsilon}{2} \geq p(v_i + \lambda_i w) - \lambda_i p(w) \tag{$**$}$$

for $i = 1, 2$. Combining these two estimates gives

$$\begin{aligned}
q(v_1) + q(v_2) &\geq p(v_1 + \lambda_1 w) - \lambda_1 p(w) p(v_2 + \lambda_2 w) - \lambda_2 p(w) - \epsilon \\
&\geq p(v_1 + v_2 + (\lambda_1 + \lambda_2) w) - (\lambda_1 + \lambda_2) p(w) - \epsilon \\
&\geq q(v_1 + v_2) - \epsilon
\end{aligned}$$

by the sublinearity of p. Since $\epsilon > 0$ is arbitrary, we get $q(v_1 + v_2) \leq q(v_1) + q(v_2)$ and hence $q \in V^\sharp$ as claimed. Since p is minimal and $q \preccurlyeq p$ we have $p = q$. Thus we can take $\lambda = 1$ on the right hand side of $(*)$ to get $p(v) = q(v) \leq p(v + w) - p(w) \leq p(v)$. Hence $p(v + w) = p(v) + p(w)$ follows. Together with $\lambda p(v) = p(\lambda v)$ for $\lambda \geq 0$ this gives linearity since $p(-v) + p(v) = 0$. This completes the proof.□

Lemma 2.2.17 *For $p \in V^\sharp$ there exits a $\varphi \in V^*$ with $\varphi \preccurlyeq p$.*

PROOF: This is the crucial step where we shall use Zorn's Lemma. Let $p \in V^\sharp$ be given and consider the set $P = \{q \in V^\sharp \mid q \preccurlyeq p\}$ endowed with its partial ordering inherited from $V^\sharp$. In order to apply Zorn's Lemma we have to show that every totally ordered subset of this (clearly non-empty) set has a lower bound. Thus let $\{q_i\}_{i\in I} \subseteq P$ be totally ordered. Now let $v \in V$ be given. Then we claim that $\{q_i(v)\} \subseteq \mathbb{R}$ is bounded from below. Assume the converse, then there exists a sequence q_n with $q_n(v) \leq -n$. Since $\{q_i\}$ is totally ordered we can take the minimum $p_n = \min(q_1, \ldots, q_n)$ of the first n members of the sequence and have $p_n(v) \leq -n$ and $p_{n+1} \preccurlyeq p_n$. Using the sublinearity this gives $0 \leq p_n(v) + p_n(-v) \leq -n + p_1(-v)$ and hence $p_1(-v) \geq n$ for all n which is nonsense. Thus we have a contradiction and the $\{q_i(v)\}$ are bounded from below. This allows to define

$$p_{\inf}(v) = \inf_{i\in I}\{q_i(v)\} \in \mathbb{R}.$$

It is clear that an arbitrary pointwise infimum of sublinear maps is again sublinear, if the infimum is finite at all. Thus $p_{\inf} \in V^\sharp$. Since furthermore $p_{\inf} \preccurlyeq p$ is obvious, we have $p_{\inf} \in P$. This shows that every totally ordered subset of P has an infimum. By Zorn's Lemma, we have minimal elements in P. If φ is such a minimal element, it is also minimal in $V^\sharp$ for if there would be another $\tilde{p} \preccurlyeq \varphi$, then also $\tilde{p} \preccurlyeq p$ and hence $\tilde{p} \in P$. By the previous lemma we conclude that φ is linear. □

We are now in the position to prove the Hahn-Banach Theorem. In finite or countable dimensions, there are simpler proofs avoiding Zorn's Lemma. Our presentation makes use of it in form of Lemma 2.2.17.

Theorem 2.2.18 (Hahn-Banach) *Let V be a real or complex vector space and let $U \subseteq V$ be a subspace. Let $p \in V^\sharp$ and $\varphi \in U^*$ be given such that*

$$\operatorname{Re}(\varphi(u)) \leq p(u) \tag{2.2.17}$$

for all $u \in U$. Then there exists an extension $\Phi \in V^*$ of φ to V such that for all $v \in V$ we have

$$\operatorname{Re}(\Phi(v)) \leq p(v). \tag{2.2.18}$$

PROOF: First we consider the real case in which we simply have $\operatorname{Re}(\varphi(u)) = \varphi(u)$. We have by assumption $\varphi(u) = -\varphi(-u) \geq -p(-u)$ for all $u \in U$ as well as $p(-u) \leq p(v-u) + p(-v)$ by sublinearity. Together, this gives

$$\begin{aligned}
-p(-v) &\leq p(-u) - p(-v) - p(-u) \\
&\leq p(v-u) + p(-v) - p(-v) + \varphi(u) \\
&= p(v-u) + \varphi(u).
\end{aligned}$$

It follows that

$$\tilde{p}(v) = \inf_{u\in U}\{p(v-u) - \varphi(u)\} > -\infty \tag{$*$}$$

is finite. With a similar argument as in the proof of Lemma 2.2.16 we see that $\tilde{p} \in V^\sharp$. Moreover, by taking $u = 0$ we have $\tilde{p} \preccurlyeq p$. From Lemma 2.2.17 we obtain a $\Phi \in V^*$ with $\Phi \preccurlyeq \tilde{p}$. We see from the very definition $(*)$ by taking $u = 0$ that $\tilde{p}(v) \leq \varphi(v)$ for all $v \in U$. Thus the restriction of Φ to U satisfies $\Phi\big|_U \preccurlyeq \varphi$. However, $\varphi \in U^*$ is already minimal by Lemma 2.2.16 and hence $\Phi\big|_U = \varphi$ follows. This shows the real case. For the complex case, let $\varphi \in U^*$ be such that $\operatorname{Re}(\varphi(u)) \leq p(u)$ for $u \in U$. Then we have already seen that there is an extension $\Psi \in \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$ of $\operatorname{Re}(\varphi)$ where we consider the complex vector space as a real one. Then one defines $\Phi(v) = \Psi(v) - \mathrm{i}\Psi(\mathrm{i}v)$ and checks that Φ is now $\mathbb{C}$-linear. We have clearly $\mathrm{Re}(\Phi) = \Psi$ and for $u \in U$ we get

$$\Phi(u) = \Psi(u) - \mathrm{i}\Psi(\mathrm{i}u) = \mathrm{Re}(\varphi(u)) - \mathrm{i}\,\mathrm{Re}(\varphi(\mathrm{i}u)) = \varphi(u),$$

since φ was already $\mathbb{C}$-linear. This shows $\Phi\big|_U = \varphi$. Finally, $\mathrm{Re}(\Phi(v)) \leq p(v)$ is clear by construction since $\mathrm{Re}(\Phi(v)) = \Psi(v) \leq p(v)$. This shows the complex case, too. □

The above formulation of the Hahn-Banach Theorem still looks rather technical. However, the following corollaries immediately show how useful the Hahn-Banach Theorem will be. In fact, we shall meet numerous applications further on.

Corollary 2.2.19 *Let V be a real or complex vector space and $U \subseteq V$ a subspace. If $\mathrm{p}\colon V \longrightarrow \mathbb{R}$ is a seminorm and $\varphi \in U^*$ is a linear functional with*

$$|\varphi(u)| \leq \mathrm{p}(u) \tag{2.2.19}$$

for all $u \in U$ then there exists an extension $\Phi \in V^*$ of φ with

$$|\Phi(v)| \leq \mathrm{p}(v) \tag{2.2.20}$$

for all $v \in V$.

PROOF: We consider the complex case, the real case is even simpler. Let $\Phi \in V^*$ be as in the Hahn-Banach Theorem, i.e. we have $\mathrm{Re}(\Phi(v)) \leq \mathrm{p}(v)$ for all $v \in V$ and $\Phi\big|_U = \varphi$. Write $\Phi(v) = r\mathrm{e}^{\mathrm{i}\alpha}$ with $r \geq 0$ and a phase $\mathrm{e}^{i\alpha} \in \mathbb{S}^1$. Then

$$|\Phi(v)| = r = \mathrm{e}^{-\mathrm{i}\alpha}\Phi(v) = \Phi(\mathrm{e}^{-\mathrm{i}\alpha}v) = \mathrm{Re}(\Phi(\mathrm{e}^{-\mathrm{i}\alpha}v)) \leq \mathrm{p}(\mathrm{e}^{-\mathrm{i}\alpha}v) = \mathrm{p}(v),$$

since p is a seminorm. This shows the claim. □

Corollary 2.2.20 *Let $U \subseteq V$ be a subspace of a locally convex space V and let $\varphi \in U'$ be a continuous linear functional on U. Then there exists a continuous extension $\Phi \in V'$ of φ to V.*

PROOF: Indeed, the continuity of φ means that there is a continuous seminorm p on V with $|\varphi(u)| \leq \mathrm{p}(u)$ for all $u \in U$ by (2.2.10). Then Corollary 2.2.19 can be applied. □

More precisely, this corollary shows that we get even the same seminorm estimate for the Hahn-Banach extension Φ of φ, i.e. if

$$|\varphi(u)| \leq \mathrm{p}(u) \quad \text{then} \quad |\Phi(v)| \leq \mathrm{p}(v) \tag{2.2.21}$$

for all $v \in V$ with the *same* seminorm p. We can use this corollary now to show that the topological dual of a Hausdorff locally convex space is large:

Corollary 2.2.21 *Let V be a locally convex space. Then V is Hausdorff iff for every $v \neq 0$ there is a continuous linear functional $\varphi \in V'$ with $\varphi(v) \neq 0$.*

PROOF: Assume V is Hausdorff and let $v \neq 0$. Then there is a seminorm p with $\mathrm{p}(v) > 0$ by Theorem 2.2.3, *v.)*. Consider now $U = \mathbb{C}v \subseteq V$ and let $\varphi\colon U \longrightarrow \mathbb{C}$ be the unique linear functional with $\varphi(v) = \mathrm{p}(v)$. Then for all $u \in U$ we have $|\varphi(u)| \leq \mathrm{p}(u)$. Hence we find a continuous linear extension of φ which will do the job. The converse is trivial: if φ is such a continuous linear functional then $\mathrm{p}(v) = |\varphi(v)|$ defines a continuous seminorm with the desired properties. □

2.2.4 Fréchet Spaces

We come now to a particularly nice class of locally convex spaces, the Fréchet spaces. Clearly, on the wishlist for a nice locally convex topology one wants a Hausdorff topology as well as completeness. However, the locally convex spaces with these two properties can still be rather complicated, so one more ingredient will simplify many things: first countability. This will be the definition of a Fréchet space:

Definition 2.2.22 (Fréchet space) *A locally convex space V is called Fréchet space if the following holds:*

i.) The space V is Hausdorff.

ii.) The space V is complete.

iii.) The space V is first countable.

The category of Fréchet spaces with continuous linear maps as morphisms is denoted by $\mathsf{Fréchet}$.

The following theorem shows that in the first-countable case we do not need too many seminorms to specify the topology:

Theorem 2.2.23 (Metrizable locally convex space) *Assume V is a Hausdorff locally convex space. Then the following statements are equivalent:*

i.) The space V is first countable.

ii.) There are countably many continuous seminorms

$$\mathrm{p}_1 \leq \mathrm{p}_2 \leq \cdots, \tag{2.2.22}$$

which already specify the topology.

iii.) The space V is metrizable by a translation invariant metric d, i.e. the topology is the metric topology of d and d satisfies for all $v, w, u \in V$

$$d(v + w, u + w) = d(v, u). \tag{2.2.23}$$

PROOF: Assume V is first countable. Then we have a countable open neighbourhood basis $U_n \subseteq V$ of 0, i.e. open subsets such that every other neighbourhood of 0 contains at least one U_n. Since V is locally convex we know that the topology is induced by the continuous seminorms. In particular, we find a continuous seminorm q_n such that $\mathrm{B}_{\mathrm{q}_n,1}(0) \subseteq U_n$, since all open balls (of radius one, if we allow for all continuous seminorms) give a basis of neighbourhoods as well. This way we arrive at countably many seminorms q_n. If now p is any other continuous seminorm then the open ball $\mathrm{B}_{\mathrm{p},1}(0)$ must contain at least one $\mathrm{B}_{\mathrm{q}_n,1}(0)$ as the latter balls form a neighbourhood basis. Thus $\mathrm{q}_n(v) < 1$ implies $\mathrm{p}(v) < 1$. As we have seen in the proof of Theorem 2.2.3 this implies $\mathrm{p} \leq \mathrm{q}_n$. By Corollary 2.2.5 we see that the system of seminorms $\{\mathrm{q}_n\}_{n\in\mathbb{N}}$ already determines the topology. Finally, we set $\mathrm{p}_n = \max_{i=1}^n \mathrm{q}_i$ which gives yet another equivalent system of seminorms, now satisfying in addition (2.2.22). Now assume that the second statement is true. Then define for $v, w \in V$

$$d(v, w) = \max_{n=1}^{\infty} \frac{1}{2^n} \frac{\mathrm{p}_n(v - w)}{1 + \mathrm{p}_n(v - w)}.$$

with the sum
l of the max?

Since $\frac{1}{2^n}$ converges to zero and $\frac{\mathrm{p}_n(v-w)}{1+\mathrm{p}_n(v-w)}$ is bounded, the maximum actually exists. Clearly, d satisfies (2.2.23). Moreover, $d(v, w) \geq 0$ and $d(v, w) = 0$ iff $v = w$ follows easily from the Hausdorff property of the seminorms $\{\mathrm{p}_n\}_{n\in\mathbb{N}}$. Next, it is a routine check, see Exercise 2.5.18, that d satisfies the triangle inequality and therefore is a metric. Finally, one checks that the induced topology is the same. The last implication from *iii.)* to *i.)* is trivial. $\square$

Remark 2.2.24 It is now clear that for a locally convex space with these equivalent properties the notion of Cauchy nets and Cauchy sequences coincide with the metric concepts: it is the translation invariance of the metric which guarantees this. In more fancy terms, not only the topological structures agree but also the uniform structures. In particular, the question about completeness is the same as for metric completeness. Note also that in this situation, completeness and sequential completeness are the same thing. Also for continuity we can rely on sequential continuity. Finally, note that the conditions *i.)* and *ii.)* in the theorem are equivalent even if V is not Hausdorff.

The fact that Fréchet spaces are, by Theorem 2.2.23, complete metric spaces allows to apply Baire's Theorem. A first application of this is the principle of uniform boundedness. In our present context, this is the content of the Banach-Steinhaus Theorem. To formulate this theorem we consider a collection $\phi_i \colon V \longrightarrow W$ of linear maps from one topological vector space to another one. The $\{\phi_i\}_{i\in I}$ are called *equicontinuous* if for every neighbourhood Z of $0 \in W$ there is a neighbourhood U of $0 \in V$ such that

$$\phi_i(U) \subseteq Z \tag{2.2.24}$$

for all $i \in I$. In particular, this implies that every ϕ_i is continuous by Proposition 2.1.4. The new feature is that the neighbourhood U does only depend on Z but is universal for all $i \in I$. If I is finite, then a collection of continuous linear maps will be automatically equicontinuous, see Exercise 2.5.19.

Lemma 2.2.25 *Let V and W be locally convex spaces. Then a collection $\{\phi_i \colon V \longrightarrow W\}_{i\in I}$ of linear maps is equicontinuous iff for every continuous seminorm q on W there exists a continuous seminorm p on V such that for all $i \in I$ and all $v \in V$ one has*

$$\mathrm{q}(\phi_i(v)) \le \mathrm{p}(v). \tag{2.2.25}$$

PROOF: Clearly, the condition of equicontinuity has only to be tested on a basis of neighbourhoods of $0 \in W$. Thus we take the balls $\mathrm{B}_{\mathrm{q},1}(0) \subseteq W$. Then we have equicontinuity iff we find a neighbourhood of $0 \in V$ which is mapped into $\mathrm{B}_{\mathrm{q},1}(0)$ by all the ϕ_i. Since every neighbourhood of $0 \in V$ contains a $\mathrm{B}_{\mathrm{p},1}(0)$ for a suitable continuous seminorm p, the equivalence is clear. □

We can now formulate the Banach-Steinhaus Theorem as follows using Baire's Theorem in the form of Corollary A.5.2:

Theorem 2.2.26 (Banach-Steinhaus) *Let V be a Fréchet space and W a locally convex space. Let $\{\phi_i \colon V \longrightarrow W\}_{i\in I}$ be a collection of continuous linear maps such that for every $v \in V$ and every continuous seminorm q on W there is a constant $C_{v,\mathrm{q}} > 0$ with*

$$\mathrm{q}(\phi_i(v)) \le C_{v,\mathrm{q}} \tag{2.2.26}$$

for all $i \in I$. Then $\{\phi_i\}_{i\in I}$ is equicontinuous.

PROOF: The maps $\mathrm{q} \circ \phi_i \colon V \longrightarrow \mathbb{R}$ are continuous and satisfy the conditions of Corollary A.5.2 since V is a complete metric space. Thus there is a constant $\tilde{C}$ and an open subset of V on which all the maps $\mathrm{q}\circ\phi_i$ are bounded by $\tilde{C}$. We find an open ball $\mathrm{B}_{\mathrm{p},1}(v_0)$ in this open subset, where p is a suitable continuous seminorm and v_0 some point in this open subset. Thus $v \in \mathrm{B}_{\mathrm{p},1}(v_0)$ implies $\mathrm{q}(\phi_i(v)) \le \tilde{C}$ for all $i \in I$. But then for $v \in \mathrm{B}_{\mathrm{p},1}(0)$ we get the estimate

$$\mathrm{q}(\phi_i(v)) = \mathrm{q}(\phi_i(v - v_0) + \phi_i(v_0)) \le \mathrm{q}(\phi_i(v - v_0)) + \mathrm{q}(\phi_i(v_0)) \le \tilde{C} + C_{v_0,\mathrm{q}} = C$$

by (2.2.26) applied for v_0. Thus there is a constant depending only on q but not on $i \in I$ such that

$$\mathrm{q}(\phi_i(v)) \le C$$

for all $v \in \mathrm{B}_{\mathrm{p},1}(0)$. We already know that this implies $\mathrm{q}(\phi_i(v)) \le C\mathrm{p}(v)$ for all $v \in V$. Rescaling the continuous seminorm by this constant we get (2.2.25) and hence Lemma 2.2.25 shows that the $\{\phi_i\}_{i\in I}$ are equicontinuous. □

As a first application of the Banach-Steinhaus Theorem we have the following result for which we will find an interpretation in terms of the weak* topology later.

Corollary 2.2.27 *Suppose V is a Fréchet space and W is an arbitrary locally convex space. Let $\phi_n \colon V \longrightarrow W$ be a sequence of continuous linear maps such that for every $v \in V$ the pointwise limit*

$$\phi(v) = \lim_{n \longrightarrow \infty} \phi_n(v) \tag{2.2.27}$$

exists. Then $\phi \colon V \longrightarrow W$ is a continuous linear map, too.

PROOF: Since the vector space operations in W are continuous it follows at once that ϕ is linear. Now let q be a continuous seminorm on W then the real numbers $\mathrm{q}(\phi_n(v))$ converge to $\mathrm{q}(\phi(v))$. Hence there is a constant $C_{v,\mathrm{q}} > 0$ with $\mathrm{q}(\phi_n(v)) \leq C_{v,q}$ as convergent sequences of real numbers are bounded. By the Banach-Steinhaus Theorem the $\{\phi_n\}_{n\in\mathbb{N}}$ are equicontinuous which means, by Lemma 2.2.25, that we find a continuous seminorm p on V with $\mathrm{q}(\phi_n(v)) \leq \mathrm{p}(v)$ for all $v \in V$. Taking the (existing) limit of these real numbers for $n \longrightarrow \infty$ gives the estimate $\mathrm{q}(\phi(v)) \leq \mathrm{p}(v)$ which implies the continuity of ϕ. ☐

Note that the above argument works only for sequences but not for general nets: a net in $\mathbb{R}$, even if converging, needs not to be bounded. Nevertheless, there are more general locally convex spaces beyond Fréchet spaces where the above statement is still true, see Exercise 2.5.21 for examples and a further discussion.

perhaps in a section? Too put this into lso needed in symbols etc.

The next consequence of Baire's Theorem is the open mapping theorem which we again formulate for Fréchet spaces. First we note the following criterion for open maps:

Lemma 2.2.28 *Let $\phi \colon V \longrightarrow W$ be a continuous linear map between topological vector spaces. Then ϕ is open iff for every neighbourhood $Z \subseteq V$ of 0 there is a neighbourhood $U \subseteq W$ of 0 with*

$$U \subseteq \phi(Z). \tag{2.2.28}$$

It suffices to consider Z from a neighbourhood basis only.

PROOF: If ϕ is open then for every open neighbourhood of $0 \in V$ also the image is open. From this, (2.2.28) is clear. Conversely, assume (2.2.28) holds for neighbourhoods Z from a neighbourhood basis of $0 \in V$. Then let $Y \subseteq V$ be an arbitrary open subset and $v \in Y$. It follows that $Y - v \subseteq V$ is an open neighbourhood of 0 and hence we find by assumption some neighbourhood $U \subseteq W$ of 0 with $U \subseteq \phi(Y - v) = \phi(Y) - \phi(v)$. Hence $\phi(v) + U \subseteq \phi(Y)$. Since $\phi(v) + U$ is a neighbourhood of $\phi(v)$ the subset $\phi(Y)$ is a neighbourhood of $\phi(v)$ as well. Since $v \in Y$ was arbitrary, $\phi(Y)$ is a neighbourhood of all its points and hence open. Thus ϕ is an open map. ☐

Theorem 2.2.29 (Open Mapping Theorem) *Let $\phi \colon V \longrightarrow W$ be a continuous linear map between Fréchet spaces. If ϕ is surjective then ϕ is open.*

PROOF: We proceed in several steps. First we consider an open ball $\mathrm{B}_{\mathrm{p},1}(0)$ around $0 \in V$ where p is a continuous seminorm. Then for every vector $v \in V$ we have either $\mathrm{p}(v) = 0$ and hence $v \in \mathrm{B}_{\mathrm{p},1}(0)$ or $\mathrm{p}(v) > 0$. Then $\frac{v}{2\mathrm{p}(v)} \in \mathrm{B}_{\mathrm{p},1}(0)$. It follows that

$$V = \bigcup_{n=1}^{\infty} n\mathrm{B}_{\mathrm{p},1}(0)$$

is the countable union of the balls $n\mathrm{B}_{\mathrm{p},1}(0) = \mathrm{B}_{\mathrm{p},n}(0)$, see also Exercise 2.5.1, *i.)*, for a more general statement. Since ϕ is linear and surjective, we have

$$W = \bigcup_{n=1}^{\infty} n\phi(\mathrm{B}_{\mathrm{p},1}(0)),$$

and hence also
$$W = \bigcup_{n=1}^{\infty} (n\phi(\mathrm{B}_{\mathrm{p},1}(0)))^{\mathrm{cl}}.$$
Thus W is a countable union of closed subsets which allows to apply Baire's Theorem, see Theorem A.5.1. Hence we have at least one $(n\phi(\mathrm{B}_{\mathrm{p},1}(0)))^{\mathrm{cl}}$ with a non-empty interior. Since rescaling with n is a homeomorphism for all $n \neq 0$, already $(\phi(\mathrm{B}_{\mathrm{p},1}(0)))^{\mathrm{cl}}$ has a non-empty open interior.

In a second step we show that $0 \in W$ is an interior point of $(\phi(\mathrm{B}_{\mathrm{p},1}(0)))^{\mathrm{cl}}$. Since the difference map $-\colon V \times V \longrightarrow V$ is continuous and $\mathrm{B}_{\mathrm{p},1}(0)$ is open, there is an open neighbourhood $Z \subseteq V$ of 0 with
$$Z - Z = \{v - w \mid v, w \in Z\} \subseteq \mathrm{B}_{\mathrm{p},1}(0).$$
Indeed, according to the definition of the product topology, any neighbourhood of $(0,0) \in V \times V$ contains a neighbourhood $Z \times Z$ with $Z \subseteq V$ being a neighbourhood of $0 \in V$, see also Exercise 2.5.1, *iii.)*. Thus,
$$(\phi(\mathrm{B}_{\mathrm{p},1}(0)))^{\mathrm{cl}} \supseteq (\phi(Z - Z))^{\mathrm{cl}} \supseteq (\phi(Z))^{\mathrm{cl}} - (\phi(Z))^{\mathrm{cl}}.$$
Applying the first part to the open subset Z instead of $\mathrm{B}_{\mathrm{p},1}(0)$ we find a non-empty open subset $U \subseteq (\phi(Z))^{\mathrm{cl}}$. Thus $(\phi(\mathrm{B}_{\mathrm{p},1}(0)))^{\mathrm{cl}} \supseteq U - U$. Now
$$U - U = \bigcup_{u \in U} U - u.$$
Since U is open each $U - u$ is open, too, since translations are homeomorphisms. Thus $U - U$ is open as well. But clearly $0 \in U - U$. Hence 0 is an interior point of $(\phi(\mathrm{B}_{\mathrm{p},1}(0)))^{\mathrm{cl}}$.

In the third step we want to show that we can omit the closure of $\phi(\mathrm{B}_{\mathrm{p},1}(0))$ in the above argument: we want to show that the image of every neighbourhood of 0 contains a neighbourhood of zero. Up to now, we first have to take the closure of the image. To prove this third claim we choose translation invariant metrics d_V and d_W on V and W, respectively, according to Theorem 2.2.23. Moreover, we fix a sequence $\epsilon_n > 0$ of radii with $\epsilon_0 = \sum_{n=1}^{\infty} \epsilon_n < \infty$. Now we apply our second claim to the *metric* open balls $\mathrm{B}_{d_V,\epsilon_n}(0)$. Thus we know that there is a $\eta_n > 0$ with
$$\mathrm{B}_{d_W,\eta_n}(0) \subseteq (\phi(\mathrm{B}_{d_V,\epsilon_n}(0)))^{\mathrm{cl}}, \tag{$*$}$$
since the metric open balls form a basis of neighbourhoods of 0 as well. Clearly, we can choose the η_n to be a zero sequence, i.e. $\eta_n \longrightarrow 0$. We want to show that $\mathrm{B}_{d_W,\eta_0}(0)$ is in the image of the *open* ball $\mathrm{B}_{d_V,2\epsilon_0}(0)$., i.e.
$$\mathrm{B}_{d_W,\eta_n}(0) \subseteq \phi(\mathrm{B}_{d_V,2\epsilon_n}(0)). \tag{$**$}$$
Establishing this result will prove our third claim. We construct now a sequence $v_n \in \mathrm{B}_{d_V,\epsilon_n}(0)$ and a sequence $w_n \in \mathrm{B}_{d_W,\eta_n}(0)$ as follows: Pick $w_0 \in \mathrm{B}_{d_W,\eta_0}(0)$. Then we know by $(*)$ that we find some $v_0 \in \mathrm{B}_{d_V,\epsilon_0}(0)$ with w_0 being close to $\phi(v_0)$. Hence we can choose v_0 such that
$$w_1 = w_0 - \phi(v_0) \in \mathrm{B}_{d_W,\eta_1}(0).$$
Inductively, we can find v_n and w_n such that
$$w_{n+1} = w_n - \phi(v_n) \in \mathrm{B}_{d_W,\eta_{n+1}}(0)$$
with $v_n \in \mathrm{B}_{d_V,\epsilon_n}(0)$. We consider the series of the v_n. By the triangle inequality and the translation invariance of d_V we have for $N, M \geq 1$
$$d_V\left(0, \sum_{n=N}^{M} v_n\right) \leq d_V(0, v_N) + d_V\left(v_N, \sum_{n=N}^{M} v_n\right) < \epsilon_N + d_V\left(0, \sum_{n=N+1}^{M} v_n\right) < \cdots < \epsilon_N + \cdots + \epsilon_M.$$

This means that $\sum_{n=N}^{M} v_n \in \mathrm{B}_{d_V,\epsilon_N+\cdots+\epsilon_M}(0)$. Since the series $\sum_{n=1}^{\infty} \epsilon_n = \epsilon_0$ converges by assumption this implies that the series $\sum_{n=0}^{N} v_n$ is a Cauchy series and hence convergent since we assume V to be a Fréchet space. Denoting the limit of the series by v then we get $v \in \mathrm{B}_{d_V,2\epsilon_0}(0)$, again by the triangle inequality. By construction, we have

$$w_0 = w_{n+1} + \phi(v_n) + \cdots + \phi(v_0)$$

for all n and thus $w_0 - \sum_{n=0}^{N} \phi(v_n) \in \mathrm{B}_{d_W,\eta_{N+1}}(0)$. By continuity of ϕ it follows that

$$w_0 = \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} \phi(v_n) = \phi\left(\sum_{n=0}^{\infty} v_n\right) = \phi(v).$$

Since $v \in \mathrm{B}_{d_V,2\epsilon_0}(0)$ and w_0 was arbitrary, we conclude that $(**)$ holds. Since $\epsilon_0 > 0$ was arbitrary, we showed our third claim. Thus, the last step consists in applying Lemma 2.2.28 to conclude that ϕ is open. ☐

The Open Mapping Theorem has several immediate corollaries which are clearly very useful:

Corollary 2.2.30 *Let $\phi\colon V \longrightarrow W$ be a continuous linear bijection between Fréchet spaces. Then ϕ is a homeomorphism, i.e. the inverse map $\phi^{-1}\colon W \longrightarrow V$ is continuous as well.*

PROOF: This is clear by Theorem 2.2.29 and Lemma A.1.2. ☐

Corollary 2.2.31 *Suppose that V carries two Fréchet topologies $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathcal{V}_1 \subseteq \mathcal{V}_2$, i.e. the second is finer. Then $\mathcal{V}_1 = \mathcal{V}_2$.*

PROOF: The identity map $\mathrm{id}_V\colon (V, \mathcal{V}_2) \longrightarrow (V, \mathcal{V}_1)$ is continuous, hence Corollary 2.2.30 applies. ☐

The third fundamental theorem about Fréchet spaces, which is based on Baire's Theorem in the form of the Open Mapping Theorem, is the Closed Graph Theorem. Recall that for a map $f\colon M \longrightarrow N$ between two sets the set

$$\mathrm{graph}(f) = \big\{(p, f(p)) \bigm| p \in M\big\} \subseteq M \times N \tag{2.2.29}$$

is called the *graph* of f. Visualizing this for a map $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ immediately explains this name. For a linear map $\phi\colon V \longrightarrow W$ between vector spaces the graph of ϕ is a subspace of $V \times W$. For topological spaces, the graph of a continuous map is closed as soon as the target space is Hausdorff, see Proposition A.1.1. The Closed Graph Theorem now gives a reverse statement to this statement:

Theorem 2.2.32 (Closed Graph Theorem) *Let $\phi\colon V \longrightarrow W$ be a linear map between Fréchet spaces. Then ϕ is continuous iff* $\mathrm{graph}(f)$ *is closed.*

PROOF: One direction is clear from Proposition A.1.1. Thus assume that $\mathrm{graph}(\phi) \subseteq V \times W$ is closed. On $V \times W$ we still have the structure of a Fréchet space: indeed, we define the seminorms

$$(\mathrm{p} \oplus \mathrm{q})(v, w) = \mathrm{p}(v) + \mathrm{q}(w)$$

for $v \in V$ and $w \in W$ where p and q are continuous seminorms on V and W, respectively. A simple check shows that the resulting locally convex topology on $V \times W$ is the Cartesian product topology. Clearly, it is Hausdorff and again complete. If $\{\mathrm{p}_n\}_{n\in\mathbb{N}}$ and $\{\mathrm{q}_m\}_{m\in\mathbb{N}}$ are countable systems of seminorms defining the topologies of V and W, respectively, then the seminorms $\{\mathrm{p}_n \oplus \mathrm{q}_m\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ form a defining set of seminorms for $V \times W$. Hence $V \times W$ is Fréchet again, see also Proposition 2.2.36. In fact, this is a particular case of the Cartesian product construction for locally convex spaces which we will investigate later on in more detail, see Subsection 2.2.5. Now the subspace $\operatorname{graph}(\phi) \subseteq V \times W$ is closed and by Proposition 2.1.11 complete. Clearly, it is still first countable as this property is inherited to subspaces in general. Thus $\operatorname{graph}(\phi)$ is a Fréchet space itself. Now we consider the following linear projection maps

$$V \xleftarrow{\mathrm{pr}_V} V \times W \xrightarrow{\mathrm{pr}_W} W,$$

mapping (v, w) to v and w, respectively. Clearly, both are continuous and surjective, thereby yielding continuous linear maps when restricted to $\operatorname{graph}(\phi)$. By the very definition of a graph,

$$\mathrm{pr}_V \big|_{\operatorname{graph}(\phi)} \colon \operatorname{graph}(\phi) \longrightarrow V$$

is a bijection. By Corollary 2.2.30 we get a *continuous* inverse $\big(\mathrm{pr}_V \big|_{\operatorname{graph}(\phi)}\big)^{-1} \colon V \longrightarrow \operatorname{graph}(\phi)$ which is explicitly given by $v \mapsto (v, \phi(v))$. Hence $\phi = \mathrm{pr}_W \circ \big(\mathrm{pr}_V \big|_{\operatorname{graph}(\phi)}\big)^{-1} \colon V \longrightarrow W$ is continuous as well. ☐

2.2.5 Initial and Quotient Topologies

In this subsection, we collect several constructions how new locally convex spaces can be obtained from old ones.

We start with the construction of the initial topology as discussed in Lemma A.3.3. In the case of linear maps and (locally convex) topological vector spaces we have the following result:

Proposition 2.2.33 (Initial topology) *Let $\{V_i\}_{i \in I}$ be a set of topological vector spaces and let $\phi_i \colon V \longrightarrow V_i$ be linear maps from a vector space V into V_i for all $i \in I$.*

i.) The initial topology on V with respect to the maps $\{\phi_i\}_{i \in I}$ makes V a topological vector space.

ii.) If $\mathcal{B}_i$ are subbases of neighbourhoods of $0 \in V_i$ for all $i \in I$ then

$$\mathcal{B} = \big\{ \phi_i^{-1}(U_i) \bigm| U_i \in \mathcal{B}_i, i \in I \big\} \tag{2.2.30}$$

forms a subbasis of neighbourhoods of 0 in V.

iii.) If all the V_i are locally convex then the initial topology of V is locally convex again. More precisely, if $\mathcal{P}_i$ are defining systems of seminorms for V_i for $i \in I$ then

$$\mathcal{P} = \big\{ \mathrm{p} \circ \phi_i \bigm| \mathrm{p} \in \mathcal{P}_i, i \in I \big\} \tag{2.2.31}$$

is a defining system of seminorms for the initial topology on V.

PROOF: We have to show that the vector space operations $+$ and $\cdot$ of V are continuous. Since the ϕ_i are linear, we have $\phi_i \circ + = +_i \circ (\phi_i \times \phi_i)$ as well as $\phi_i \circ \cdot = \cdot_i \circ (\mathrm{id}_{\mathbb{C}} \times \phi_i)$ for all $i \in I$ where $+_i$ and $\cdot_i$ denote the vector space operations on V_i. Since for the initial topology the ϕ_i are continuous by the very definition, and since the $+_i$ and $\cdot_i$ are continuous, we have continuous maps $\phi_i \circ +$ and $\phi_i \circ \cdot$. By the universal property of the initial topology according to Lemma A.3.3, *i.)*, we conclude that $+$ and $\cdot$ are continuous themselves. This shows the first part. For the second part, we first observe that $0 \in \phi_i^{-1}(U_i)$ for a neighbourhood U_i of $0 \in V_i$ since ϕ_i is linear. Then the statement is again a general consequence of Lemma A.3.3. For the third part, we consider

$$\mathcal{B}_i = \big\{ \mathrm{B}_{\mathrm{p},\epsilon}(0) \bigm| \mathrm{p} \in \mathcal{P}_i, \epsilon > 0 \big\}$$

for $i \in I$. This is a subbasis of neighbourhoods of 0 in V_i. Then the corresponding $\mathcal{B}$ as in *ii.)* consists of subsets of the form $\phi_i^{-1}(\mathrm{B}_{\mathrm{p},\epsilon}(0)) = \mathrm{B}_{\mathrm{p} \circ \phi_i, \epsilon}(0)$, i.e. the open ϵ-balls with respect to the seminorms $\mathrm{p} \circ \phi_i$. This gives the third part. ☐

As an application of the initial topology construction we can take arbitrary Cartesian products of locally convex spaces:

Example 2.2.34 (Cartesian product of locally convex spaces) Let $\{V_i\}_{i \in I}$ be locally convex spaces. Then their *Cartesian product*

$$V = \prod_{i \in I} V_i \tag{2.2.32}$$

is again a locally convex space with respect to the initial topology induced by all the projections onto the i-th components

$$\mathrm{pr}_i \colon V \longrightarrow V_i. \tag{2.2.33}$$

The Cartesian product preserves many nice features of its factors and conversely, the factors inherit those from the product:

Proposition 2.2.35 *Let $\{V_i\}_{i \in I}$ be topological vector spaces and let $V = \prod_{i \in I} V_i$ be their Cartesian product.*

i.) The Cartesian product V is Hausdorff iff all the V_i are Hausdorff.

ii.) A net $(v^\alpha)_{\alpha \in J}$ is a Cauchy net in V iff all the components $(\mathrm{pr}_i(v^\alpha))_{\alpha \in J}$ are Cauchy nets in V_i.

iii.) A net $(v^\alpha)_{\alpha \in J}$ in V is convergent to some $v \in V$ iff all the components $(\mathrm{pr}_i(v^\alpha))_{\alpha \in J}$ converge to some v_i. If V is Hausdorff, $v_i = \mathrm{pr}_i(v)$.

iv.) The Cartesian product V is complete iff all the V_i are complete.

v.) The Cartesian product V is sequentially complete iff all the V_i are sequentially complete.

PROOF: The first part is true for general topological spaces, see Lemma A.3.1, *i.)*. For the second part, suppose we have a net $(v^\alpha)_{\alpha \in J}$ in V with i-th components denoted by $v_i^\alpha = \mathrm{pr}_i(v^\alpha) \in V_i$. Then $(v^\alpha)_{\alpha \in J}$ is Cauchy iff for all neighbourhoods $\mathrm{pr}_i^{-1}(U_i)$ of 0 of the subbasis (2.2.30) we have $v^\alpha - v^\beta \in \mathrm{pr}_i^{-1}(U_i)$ for all indices $\alpha, \beta \succcurlyeq \gamma$ with an appropriate γ depending only on U_i. This is equivalent to say that $\mathrm{pr}_i(v^\alpha - v^\beta) = v_i^\alpha - v_i^\beta \in U_i$, which means that the net $(v_i^\alpha)_{\alpha \in J}$ is Cauchy in V_i for all $i \in I$. This shows the second part, the third is analogous and even true for general Cartesian products of topological spaces. But then the fourth and fifth part are clear. ☐

In other words, the Cartesian product topology is such that we can do everything "componentwise".

If we have uncountably many $\{V_i\}_{i \in I}$ and all are non-trivial then the Cartesian product topology will not be first countable. If, on the other hand, we only have a countable set $\{V_n\}_{n \in \mathbb{N}}$ and each V_n is first countable, then also V is first countable:

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Proposition 2.2.36 *Let $\{V_n\}_{n \in \mathbb{N}}$ be a sequence of first countable locally convex spaces. Then their Cartesian product $V = \prod_{n=1}^\infty V_n$ is first countable, too. In particular, a countable Cartesian product of Fréchet spaces is again a Fréchet space.*

PROOF: Let $\{\mathrm{p}_{nm} \colon V_n \longrightarrow \mathbb{R}\}_{m \in \mathbb{N}}$ be a set of seminorms on V_n which determines the topology. Then by Proposition 2.2.33, *iii.)*, the seminorms $\{\mathrm{p}_{nm} \circ \mathrm{pr}_n\}_{n,m \in \mathbb{N}}$ already determine the topology on V. Clearly, they are still countable and hence Theorem 2.2.23 shows that the topology of V is first countable, too. Note that by Remark 2.2.24 we do not need the V_n to be Hausdorff to conclude that V is first countable. The remaining statement is then clear by Proposition 2.2.35, *iv.)*. ☐

Example 2.2.37 The space $\mathbb{C}^{\mathbb{N}}$ of all complex sequences is a Fréchet space with respect to the seminorms

$$|(a_n)_{n \in \mathbb{N}}|_m = |a_m| \tag{2.2.34}$$

where $m \in \mathbb{N}$. In fact, these seminorms are precisely those which arise from the usual norm on $\mathbb{C}$ according to the construction in Proposition 2.2.33, *iii.)*. The resulting topology is the one of componet-wise convergence.

Another important application of the initial topology is the following: for a locally convex space or just a topological vector space V we have already introduced the topological dual V' of continuous linear functionals on V. Up to now, there is not yet a topology on V'. Since every $v \in V$ can also be considered as a linear (evaluation) map

$$v \colon V^* \ni \varphi \mapsto \varphi(v) \in \mathbb{C} \tag{2.2.35}$$

on the algebraic dual, one can consider the initial topology on V' (or even on V^*) with respect to all these linear functionals. This leads to the following definition:

Definition 2.2.38 (Weak* topology) *Let V be a topological vector space and V' its topological dual. The initial topology on V' with respect to all the evaluation maps given by $v \in V$ as in* (2.2.35) *is called the weak* topology on V'.*

Remark 2.2.39 Let V be a topological vector space. Then the weak* topology on V' is locally convex. Indeed, $\mathbb{C}$ is locally convex and hence we can apply Proposition 2.2.33, *iii.)*. Moreover, a defining system of seminorms is given by

$$\mathcal{P} = \left\{ \mathrm{p}_v \colon \varphi \mapsto |\varphi(v)| \;\middle|\; v \in V \right\}. \tag{2.2.36}$$

The weak* topology is necessarily Hausdorff since for $\varphi \neq 0$ we find a $v \in V$ with $\varphi(v) \neq 0$. Thus $\mathrm{p}_v(\varphi) > 0$. Convergence of a net $(\varphi_i)_{i \in I}$ in the weak* topology simply means pointwise convergence

$$\varphi(v)_i \longrightarrow \varphi(v) \tag{2.2.37}$$

for all $v \in V$. It will be of major importance to study the dual space V' with its weak* topology even if one is interested primarily in V. The reason is that V' behaves much nicer in many aspects. We will frequently come back to this point of view, in particular in Subsection 2.3.3. Finally, we note that also the algebraic dual V^* can be equipped with this weak* topology. It becomes a Hausdorff and even a complete locally convex space this way, see also Exercise 2.5.22.

The analog of the final topology as discussed in Lemma A.3.5 is more involved: the reason is that the topological final topology needs not to make the target vector space a topological vector space at all: the final topology might be too fine. There is a modification of this topological final topology, the linear final topology, and also the final locally convex topology, see e.g. [26, Chap. 4] for a detailed discussion. Here, we do not need them in this generality. Instead, we just discuss a very particular case of the final locally convex topology: the quotient topology.

Let V be a locally convex space and let $U \subseteq V$ be a subspace. Then we want to endow the quotient vector space V/U with a reasonable locally convex topology. In particular, we want the projection

$$\mathrm{pr} \colon V \longrightarrow V/U \tag{2.2.38}$$

to be continuous. If $\mathcal{P}$ is a defining system of seminorms on V then we consider

$$[\mathrm{p}]([v]) = \inf\left\{ \mathrm{p}(v + u) \;\middle|\; u \in U \right\} \tag{2.2.39}$$

for $\mathrm{p} \in \mathcal{P}$. The following is an elementary check, see Exercise 2.5.23.

Lemma 2.2.40 *Let $U \subseteq V$ be a subspace and* p *a seminorm on V. Then* [p] *is a well-defined seminorm on the quotient V/U.*

This suggest to take the seminorms $[\mathrm{p}]$ to define a locally convex topology on the quotient:

Definition 2.2.41 (Locally convex quotient) *Let $U \subseteq V$ be a subspace of a locally convex space V and let $\mathrm{pr}\colon V \longrightarrow V/U$ be the quotient. Then the locally convex quotient topology on V/U is defined by the collection of seminorms*

$$[\mathcal{P}] = \big\{[\mathrm{p}] \bigm| \mathrm{p} \textit{ is a continuous seminorm on } V\big\}. \tag{2.2.40}$$

It is clear that $[\mathcal{P}] = \overline{[\mathcal{P}]}$, see Exercise 2.5.23. By the very definition we get a locally convex topology on V/U. However, in general one can not say much about it: many nice properties like the Hausdorff property might be lost when passing to the quotient. We collect some of the relevant features:

Proposition 2.2.42 *Let $U \subseteq V$ be a subspace of a locally convex space V and endow the quotient $\mathrm{pr}\colon V \longrightarrow V/U$ with the locally convex quotient topology.*

i.) The projection map pr is continuous.

ii.) The locally convex quotient topology is the finest locally convex topology such that pr is continuous.

iii.) If $U = U^{\mathrm{cl}}$ is closed then V/U is Hausdorff.

iv.) A linear map $\phi\colon V/U \longrightarrow W$ into another locally convex space is continuous iff $\phi \circ \mathrm{pr}\colon V \longrightarrow W$ is continuous.

PROOF: Clearly, we have $[\mathrm{p}]([v]) \leq \mathrm{p}(v)$ for all $v \in V$ and all continuous seminorms. This shows the continuity of pr. Suppose now that $\mathcal{Q}$ is another system of seminorms on V/U for which pr is continuous. Then $\mathrm{q} \circ \mathrm{pr}$ is a continuous seminorm on V. We compute now $[\mathrm{q} \circ \mathrm{pr}]$ according to (2.2.39) yielding

$$[\mathrm{q} \circ \mathrm{pr}]([v]) = \inf_{u \in U}\{\mathrm{q} \circ \mathrm{pr}(v+u)\} = \inf_{u \in U}\{\mathrm{q} \circ \mathrm{pr}(v)\} = \mathrm{q}([v]),$$

since $\mathrm{pr}(v+u) = \mathrm{pr}(v)$ for all $u \in U$. Thus this reproduces q. By definition of $[\mathcal{P}]$ we have $\mathcal{Q} \subseteq [\mathcal{P}]$ proving the second part. For the third part, let $[v] \in V/U$ be such that $[\mathrm{p}]([v]) = 0$ for all $[\mathrm{p}] \in [\mathcal{P}]$. This means $\inf_{u \in U}\{\mathrm{p}(v+u)\} = 0$ and hence we find for every continuous seminorm p on V and every $\epsilon > 0$ a $u_{\mathrm{p},\epsilon} \in U$ with $\mathrm{p}(v - u_{\mathrm{p},\epsilon}) < \epsilon$. Hence in every open neighbourhood of v we have a vector from U implying that $v \in U^{\mathrm{cl}} = U$. Thus $[v] = 0$ follows, implying that V/U is Hausdorff. Finally, let $\phi\colon V/U \longrightarrow W$ be linear. Then ϕ is continuous iff for every continuous seminorm q on W we have a continuous seminorm $[\mathrm{p}] \in \hat{\mathcal{P}}$ such that $\mathrm{q}(\phi([v])) \leq [\mathrm{p}]([v])$. This is equivalent to $\mathrm{q}(\phi(\mathrm{pr}(v))) \leq ([\mathrm{p}] \circ \mathrm{pr})(v)$, which is the continuity of $\phi \circ \mathrm{pr}$ since $[\mathrm{p}] \circ \mathrm{pr}$ is a continuous seminorm on V. □

Remark 2.2.43 (Hausdorffization) For a locally convex space V we can use the third part to obtain the *Hausdorffization* of V: Clearly, V is Hausdorff iff the closure of the trivial subspace $\{0\}$ is just $\{0\}^{\mathrm{cl}} = \{0\}$, see Exercise 2.5.24. Thus, if V is not Hausdorff then the quotient $V/\{0\}^{\mathrm{cl}}$ will be a Hausdorff locally convex space, the so-called Hausdorffization of V. Note that it follows from this discussion that in general

$$\{0\}^{\mathrm{cl}} = \bigcap_{\substack{\mathrm{p}\text{ continuous} \\ \text{seminorm}}} \ker \mathrm{p}. \tag{2.2.41}$$

It is now easy to see that Hausdorffization works well together with continuous linear maps, ultimately resulting in functorial behaviour, see Exercise 2.5.24.

Corollary 2.2.44 *Let V be a locally convex space and $U \subseteq V$ a closed subspace. If $v \in V$ is not in U then there is a continuous linear functional $\varphi \in V'$ with $\varphi(v) \neq 0$ but $\varphi\big|_U = 0$.*

PROOF: This is a combination of the Hahn-Banach Theorem and Proposition 2.2.42: we know that V/U is a Hausdorff locally convex space and $[v] \in V/U$ is not the zero class. Thus by Corollary 2.2.21 we get a continuous linear functional $\Phi\colon V/U \longrightarrow \mathbb{C}$ with $\Phi([v]) \neq 0$. Denote the projection by $\mathrm{pr}\colon V \longrightarrow V/U$ then $\varphi = \Phi \circ \mathrm{pr}$ will do the job. ☐

2.3 Banach Spaces

We come now to a very particular class of locally convex spaces, the Banach spaces. For this class many aspects of the general theory will simplify drastically. Some aspects of the theory of Banach spaces will certainly be familiar from elementary calculus courses. Here we will focus on the question how Banach space theory embeds into the more general framework of locally convex spaces.

2.3.1 First Properties of Banach Spaces

We begin with the definition of a Banach space. As usual we only consider the complex version:

Definition 2.3.1 (Banach space) *Let V be a complex vector space.*

i.) *If $\|\cdot\|$ is a norm on V then the pair $(V, \|\cdot\|)$ is called a normed space.*

ii.) *A topological vector space V is called normable if its topology can be induced by a norm.*

iii.) *A complete normed space is called Banach space.*

Remark 2.3.2 (Normed space)

i.) A normable space is certainly locally convex, Hausdorff, and first countable.

ii.) A Banach space is a particular case of a Fréchet space. Hence all the results on Fréchet spaces apply to Banach spaces.

iii.) A normed space $(V, \|\cdot\|)$ has a canonical translationally invariant metric, namely

$$d(v, w) = \|v - w\| \tag{2.3.1}$$

for $v, w \in V$. Note that for a general Fréchet space we had to choose first a sequence of defining seminorms and then a way to build a metric out of them, see Theorem 2.2.23.

iv.) In a normed space $(V, \|\cdot\|)$ every continuous seminorm p satisfies an estimate of the form

$$\mathrm{p}(v) \leq c\|v\| \tag{2.3.2}$$

for all $v \in V$ with an appropriate constant $c \geq 0$ depending on p but of course not on v.

Note that for a normed space it is really part of the data to specify the norm. In particular, $(V, \|\cdot\|)$ and $(V, 2\|\cdot\|)$ will be treated as *different* normed spaces while the underlying locally convex spaces are the same. Sometimes we will indicate the norm on a normed space V by $\|\cdot\|_V$. If it is clear from the context, we shall speak of the normed space V without mentioning the norm explicitly. As for locally convex spaces we have a corresponding category of Banach spaces which we denote by Banach. Note that the forgetful functor

$$\mathsf{Banach} \longrightarrow \mathsf{Fr\acute{e}chet} \tag{2.3.3}$$

is not injective on objects as in $\mathsf{Fr\acute{e}chet}$ we are only interested in the underlying locally convex topology while in Banach we have the norm being part of the data.

Remark 2.3.3 The usage of one norm instead of a system of seminorms will also simplify the notions of convergence and completeness. A sequence $(v_n)_{n\in\mathbb{N}}$ in a normed space V is a Cauchy seuqence iff for every $\epsilon > 0$ there is a $N \in \mathbb{N}$ such that $\|v_n - v_m\| < \epsilon$ for all $n, m \geq N$. It is convergent to v iff $\|v_n - v\| < \epsilon$ for all $n \geq N$. Thus we arrive at the common formulations known from elementary calculus.

2.3.2 Continuous Linear Maps and the Operator Norm

The continuity of linear maps between normed spaces can now be described as follows:

Proposition 2.3.4 *Let V, W be normed spaces and let $A \colon V \longrightarrow W$ be a linear map. Then the following statements are equivalent:*

i.) The map A is continuous.

ii.) There exists a constant $c \geq 0$ such that for all $v \in V$ one has

$$\|A(v)\| \leq c\|v\|. \tag{2.3.4}$$

iii.) One has

$$\sup_{v \in V \setminus \{0\}} \frac{\|A(v)\|}{\|v\|} < \infty. \tag{2.3.5}$$

PROOF: Since for a normed space we can take the multiples of the norm as a filtrating, defining system of seminorms the equivalence of the first two statements is clear by Proposition 2.2.8. Assuming *ii.)*, we see that for all $v \in V \setminus \{0\}$ we have $\frac{\|A(v)\|}{\|v\|} \leq c$ since $\|v\| > 0$ for a norm. Thus the supremum is $\leq c$ and hence finite. Conversely, assume *iii.)* holds and let c be the value of the supremum. Then (2.3.4) holds for all $v \neq 0$. But for $v = 0$, the estimate (2.3.4) trivially holds, too. □

Since the supremum in (2.3.5) encodes the continuity of a linear map, the precise value of it plays a special role:

Definition 2.3.5 (Operator norm) *Let V and W be normed spaces. For a continuous linear map $A \in \mathrm{L}(V, W)$ one defines the operator norm $\|A\|$ by*

$$\|A\| = \sup_{v \in V \setminus \{0\}} \frac{\|A(v)\|}{\|v\|}. \tag{2.3.6}$$

Thus the operator norm is the optimal constant c in the estimate (2.3.4), i.e. we have

$$\|A(v)\| \leq \|A\|\|v\| \tag{2.3.7}$$

for all $v \in V$. Here we see that the operator norm really depends on the choices of the norms on V and W, the topologies themselves are not sufficient: different choices of the norms will lead to different numerical values for the operator norm. Later on, the precise numerical value turns out to be important. Hence the sole existence of a supremum like (2.3.5) will not be sufficient for several purposes.

Remark 2.3.6 Sometimes the following reformulation of the definition of the operator norm is useful. Since A is linear, we can rescale $v \in V \setminus \{0\}$ by its norm to get

$$\|A\| = \sup_{\|v\|=1} \|A(v)\|. \tag{2.3.8}$$

We show now some basic properties of the operator norm, in particular, that it is a norm:

Proposition 2.3.7 *Let V, W, U be normed spaces and let $A \in \mathrm{L}(V, W)$ and $B \in \mathrm{L}(W, U)$ be continuous linear maps.*

i.) The operator norm is a norm for $\mathrm{L}(V, W)$.

ii.) One has

$$\|B \circ A\| \leq \|B\|\|A\|. \tag{2.3.9}$$

iii.) If W is a Banach space then $\mathrm{L}(V, W)$ is a Banach space with respect to the operator norm, too.

PROOF: We clearly have $\|A\| \geq 0$ and $\|A\| > 0$ for $A \neq 0$ since in this case we find a $v \neq 0$ with $A(v) \neq 0$ and hence $\|A(v)\| > 0$. For $z \in \mathbb{C}$ we have

$$\|zA\| = \sup_{v \in V \setminus \{0\}} \frac{\|(zA)(v)\|}{\|v\|} = \sup_{v \in V \setminus \{0\}} \frac{|z| \|A(v)\|}{\|v\|} = |z| \|A\|.$$

For the triangle inequality we consider for $v \in V \setminus \{0\}$

$$\|(A + A')(v)\| = \|A(v) + A'(v)\| \leq \|A(v)\| + \|A'(v)\| \leq (\|A\| + \|A'\|)\|v\|.$$

Dividing by $\|v\|$ and taking the supremum gives immediately the triangle inequality for the operator norm. For the second part, applying twice (2.3.7) gives for all $v \in V \setminus \{0\}$

$$\|(B \circ A)(v)\| \leq \|B\| \|A(v)\| \leq \|B\| \|A\| \|v\|.$$

Again, dividing by $\|v\|$ and taking the supremum gives (2.3.9). For the last part, assume that W is complete and let $A_n \in \mathrm{L}(V, W)$ form a Cauchy sequence: note that we are in a first countable situation anyway, hence sequences will do the job. This means that for $\epsilon > 0$ we have a $N \in \mathbb{N}$ with $\|A_n - A_m\| < \epsilon$ for $n, m \geq N$. For $v \in V$ we get for such n, m the estimate

$$\|A_n(v) - A_m(v)\| \leq \|A_n - A_m\| \|v\| < \epsilon \|v\|.$$

This shows that the sequence $(A_n(v))_{n \in \mathbb{N}}$ is a Cauchy sequence in W and hence convergent. Thus we can define a map $A\colon V \longrightarrow W$ by the pointwise limit

$$A(v) = \lim_{n \longrightarrow \infty} A_n(v).$$

Since the A_n are all linear and since the vector space operations of W are continuous it is a simple check that A is linear itself. We fix $v \in V$. For $M \geq N$ large enough we have $\|A_M(v) - A(v)\| < \epsilon$. Thus for $n \geq N$ we have

$$\|A_n(v) - A(v)\| \leq \|A_n(v) - A_M(v)\| + \|A_M(v) - A(v)\| < 2\epsilon. \tag{$*$}$$

Note that the index M depends on the choice of v but the resulting estimate $(*)$ is uniform for all $v \in V$. This shows $\|A - A_n\| < 2\epsilon$ and hence $A - A_n \in \mathrm{L}(V, W)$. But then also $A - A_n + A_n = A \in \mathrm{L}(V, W)$ and $A_n \longrightarrow A$ with respect to the operator norm. Alternatively, we could have used an adapted version of Corollary 2.2.27 to conclude the continuity of A. □

Remark 2.3.8 We can define $\|A\|$ by (2.3.6) for any linear map $A\colon V \longrightarrow W$ whether A is continuous or not if we allow the value $+\infty$ for the supremum. Then the continuous ones are precisely those where the operator norm is *finite*. Note that for the triangle inequality the suprema do not need to be finite. Note also that the algebraic properties of $\|\cdot\|$ will still be valid: we have

$$\|zA\| = |z| \|A\| \quad \text{and} \quad \|A + B\| \leq \|A\| + \|B\| \tag{2.3.10}$$

for all linear maps $A, B\colon V \longrightarrow W$ and all $z \in \mathbb{C}$ if we use the usual arithmetic and order structure of $[0, +\infty]$.

Corollary 2.3.9 *The dual space $V' = \mathrm{L}(V, \mathbb{C})$ of a normed space V is a Banach space with respect to the norm*

$$\|\phi\| = \sup_{v \neq 0} \frac{|\phi(v)|}{\|v\|} = \sup_{\|v\|=1} |\phi(v)|. \tag{2.3.11}$$

We can now use the completeness of V' for a normed space V to construct a completion of V itself. Of course, we can take the set of Cauchy sequences in V modulo zero sequences and show that this will provide a completion, see also Exercise 2.5.26. However, the following alternative construction is a nice application of the Hahn-Banach Theorem. To this end, we first state the Hahn-Banach Theorem in the context of normed spaces:

Theorem 2.3.10 (Hahn-Banach) *Let V be a normed space and $U \subseteq V$ a subspace. If $\varphi\colon U \longrightarrow \mathbb{C}$ is a continuous linear functional then there is a continuous linear extension $\Phi\colon V \longrightarrow \mathbb{C}$ of φ with*

$$\|\Phi\| = \|\phi\|. \tag{2.3.12}$$

PROOF: We apply Corollary 2.2.19 to the seminorm $\mathrm{p}(v) = \|\varphi\|\|v\|$: Since $|\varphi(u)| \leq \|\varphi\|\|u\|$ we get by that corollary a Φ with $|\Phi(v)| \leq \|\varphi\|\|v\|$ for all $v \in V$. Clearly, the norm of Φ can not be smaller than the one of $\varphi = \Phi\big|_U$. ☐

As a first application of the Hahn-Banach Theorem we get the following result on the *transposed* or *dual map*

$$A'\colon W' \longrightarrow V' \tag{2.3.13}$$

of a continuous linear map $A\colon V \longrightarrow W$, which is defined as usual by

$$(A'\varphi)(v) = \varphi(Av). \tag{2.3.14}$$

Corollary 2.3.11 *Let $A\colon V \longrightarrow W$ be a continuous linear map between normed spaces. Then $A'\colon W' \longrightarrow V'$ is a well-defined continuous linear map with*

$$\|A\| = \sup_{\substack{\|v\|=1\\ \|\varphi\|=1}} |\varphi(Av)| = \sup_{\substack{\|v\|=1\\ \|\varphi\|=1}} |(A'\varphi)(v)| = \|A'\|. \tag{2.3.15}$$

PROOF: For $\varphi \in W'$ the composition $\varphi \circ A$ is again continuous and linear. Hence $A'\varphi \in V'$ as claimed. The linearity of the map A' is obvious. Thus we have to check the equality (2.3.15). For $v \in V$ there is a $\varphi \in W'$ with $\|\varphi\| = 1$ and $\varphi(Av) = \|Av\|$ by the Hahn-Banach Theorem in form of Theorem 2.3.10. Thus

$$\|A\| = \sup_{\|v\|=1} \|Av\| = \sup_{\substack{\|v\|=1\\ \|\varphi\|=1}} |\varphi(Av)| = \sup_{\substack{\|v\|=1\\ \|\varphi\|=1}} |(A'\varphi)(v)| = \sup_{\|\varphi\|=1} |(A'\varphi)(v)| = \|A'\|$$

follows at once since clearly $|\varphi(Av)| \leq \|\varphi\|\|A\|\|v\|$ as well as $\|(A'\varphi)(v)\| \leq \|A'\varphi\|\|v\|$. ☐

By Corollary 2.2.21 and the fact that a normed space is clearly Hausdorff the Hahn-Banach Theorem shows that for every vector $v \in V \setminus \{0\}$ we have a continuous linear functional $\varphi \in V'$ with $\varphi(v) \neq 0$. We will use this fact now to embed V into the dual space V'' of V'.

Proposition 2.3.12 *Let V be a normed space. Then the map $\iota\colon V \longrightarrow V''$ defined by*

$$\iota(v)\colon V' \ni \varphi \;\mapsto\; \varphi(v) \in \mathbb{C} \tag{2.3.16}$$

for every $v \in V$ is linear, injective, and norm-preserving.

Proof: First it is clear that $\iota(v)\colon V^* \ni \varphi \mapsto \varphi(v) \in \mathbb{C}$ is a linear functional on the algebraic dual of V since the pairing $\varphi(v)$ depends linearly on φ, too. Thus, the restriction to $V' \subseteq V^*$ is still a linear map. We have to show that this is continuous. Thus we compute the norm

$$\|\iota(v)\| = \sup_{\varphi \neq 0} \frac{|\iota(v)\varphi|}{\|\varphi\|} = \sup_{\varphi \neq 0} \frac{|\varphi(v)|}{\|\varphi\|} \leq \sup_{\varphi \neq 0} \frac{\|\varphi\| \|v\|}{\|\varphi\|} = \|v\| < \infty.$$

This shows $\|\iota(v)\| \leq \|v\|$. Up to this point we have not yet used the Hahn-Banach Theorem. Without it, it could well be that V' and hence V'' are very small, possibly just $\{0\}$. Then the map ι could not be injective. However, to show that this is not the case, let $v \neq 0$ be given. Then we find a $\varphi \in V'$ with $\varphi(v) = \|v\|$ and $\|\varphi\| = 1$. Indeed, on the one-dimensional subspace $\mathbb{C}v \subseteq V$ this is no problem and hence by Theorem 2.3.10 we can extend it to all of V. For this φ we have $|\iota(v)\varphi| = |\varphi(v)| = \|v\|$ and hence the operator norm of $\iota(v)$ is at least $\|v\|$ since $\|\varphi\| = 1$. Together with the above estimate this shows $\|\iota(v)\| = \|v\|$ and hence also the injectivity follows. □

As a first application of this statement we get an "explicit" construction of the completion of V:

Proposition 2.3.13 *Let V be a normed space. Then the closure $\iota(V)^{\mathrm{cl}} \subseteq V''$ of the image of V under the map $\iota\colon V \longrightarrow V''$ is a completion of V. It is a Banach space.*

Proof: We know from general arguments that the closure of a subspace in a complete topological vector space is again complete, see Proposition 2.1.11, *i.)*. Since V'' is a Banach space by Corollary 2.3.9 we see that on one hand $\iota(V)^{\mathrm{cl}} \subseteq V''$ is complete. On the other hand, it is normed again and hence a Banach space. Since by definition $\iota(V) \subseteq \iota(V)^{\mathrm{cl}}$ is a dense subspace and $\iota\colon V \longrightarrow \iota(V)$ is an isometric isomorphism by Proposition 2.3.12 we have indeed found a completion. □

Corollary 2.3.14 *Every normed space V has a completion $(\widehat{V}, \iota)$ such that $\widehat{V}$ is a Banach space and the inclusion map ι is isometric.*

This way we have found an independent construction of the completion of a normed space not relying on the general statement from Theorem 2.1.13. In addition, the completion is even a Banach space in this case, a fact which can also be deduced from the general statement of Theorem 2.1.13 and the uniform continuity of the norm. Alternatively, one can construct the completion of a normed space as the space of all Cauchy sequences modulo the space of zero sequences, as this is familiar from the construction of the real numbers out of the rational ones. The details of this alternative construction can be found in Exercise 2.5.26.

We are now in the position to complete the proof of the general completion statement for *locally convex spaces*, i.e. the proof of Theorem 2.2.13. We start with the following considerations: let V be a locally convex space which we assume to be Hausdorff. If p is a continuous seminorm on V then its kernel $\ker \mathrm{p} \subseteq V$ is not only a subspace according to (2.2.1) but even a closed subspace of V. Indeed, the closedness is clear from $\mathrm{p}^{-1}(\{0\}) = \ker \mathrm{p}$, see also Exercise 2.5.24.

Lemma 2.3.15 *Let V be a locally convex space and p a continuous seminorm on V.*

i.) Then on

$$V_{\mathrm{p}} = V \big/ \ker \mathrm{p} \tag{2.3.17}$$

the seminorm $\|\cdot\|_{\mathrm{p}} = [\mathrm{p}]$ defined by

$$\|[v]\|_{\mathrm{p}} = \inf\big\{\mathrm{p}(v+u) \bigm| u \in \ker \mathrm{p}\big\} \tag{2.3.18}$$

is a norm.

ii.) The quotient map $V \longrightarrow V_{\mathrm{p}}$ is continuous.

iii.) For all $v \in V$ one has

$$\|[v]\|_{\mathrm{p}} = \mathrm{p}(v). \tag{2.3.19}$$

Proof: Recall that $\|\cdot\|_{\mathrm{p}}$ is indeed a seminorm by Lemma 2.2.40. Suppose now that $\|[v]\|_{\mathrm{p}} = 0$. Then we find a net $(v_i)_{i\in I}$ in $\ker \mathrm{p}$ with $\mathrm{p}(v + v_i) \longrightarrow 0$ by definition of the infimum. But $\mathrm{p}(v) \leq \mathrm{p}(v + v_i) + \mathrm{p}(v_i) = \mathrm{p}(v + v_i)$ shows $\mathrm{p}(v) = 0$. Hence $[v] = 0$ and $\|\cdot\|_{\mathrm{p}}$ is a norm. We clearly have $\|[v]\|_{\mathrm{p}} \leq \mathrm{p}(v)$ which gives the second part. Finally, suppose $\mathrm{p}(v) > 0$ and $\|[v]\|_{\mathrm{p}} < \mathrm{p}(v)$. Then there exists a $w \in \ker \mathrm{p}$ with $\mathrm{p}(v + w) < \mathrm{p}(v)$. This implies $\mathrm{p}(v) \leq \mathrm{p}(v + w) + \mathrm{p}(w) = \mathrm{p}(v + w) < \mathrm{p}(v)$ which is a contradiction. This gives the remaining estimate to conclude the third part. □

Note that in general this norm topology on V_{p} is *not* the canonical locally convex quotient topology of the quotient $V/\ker \mathrm{p}$ according to Definition 2.2.41. The quotient topology is typically strictly finer than the norm topology of V_{p}. However, if we endow V with the locally convex topology determined by the single seminorm p alone, then the norm topology on V_{p} is the locally convex quotient for this (typically coarser) locally convex topology of V.

We consider now *all* continuous seminorms $\mathcal{P}$ on V. Then each V_{p} is a normed space which we can complete into a Banach space $\widehat{V_{\mathrm{p}}}$ according to Corollary 2.3.14. Then we take their Cartesian product

$$\mathcal{V} = \prod_{\mathrm{p}\in\mathcal{P}} \widehat{V}_{\mathrm{p}} \tag{2.3.20}$$

with the canonical projections $\mathrm{pr}_{\mathrm{p}} \colon \mathcal{V} \longrightarrow \widehat{V}_{\mathrm{p}}$ as usual. By Proposition 2.2.33 we can endow $\mathcal{V}$ with the Cartesian product topology which is locally convex, Hausdorff, and complete by Proposition 2.2.35. A system of defining seminorms on this Cartesian product is explicitly given by

$$\mathcal{P}_{\mathcal{V}} = \left\{ \tilde{\mathrm{p}} = \|\cdot\|_{\mathrm{p}} \circ \mathrm{pr}_{\mathrm{p}} \;\middle|\; \mathrm{p} \in \mathcal{P} \right\}. \tag{2.3.21}$$

We can now consider the map

$$\iota \colon V \ni v \;\mapsto\; ([v]_{\mathrm{p}})_{\mathrm{p}\in\mathcal{P}} \in \mathcal{V} = \prod_{\mathrm{p}\in\mathcal{P}} \widehat{V}_{\mathrm{p}}, \tag{2.3.22}$$

i.e. we map v into each of the normed spaces V_{p}, include it in its completion $\widehat{V_{\mathrm{p}}}$, and then put things together inside the big Cartesian product.

Lemma 2.3.16 *The map ι from* (2.3.22) *is linear, continuous, and injective. Moreover, the induced topology on the image coincides with the original topology of V, i.e. ι is an embedding.*

Proof: The linearity of ι is clear. We have for $v \in V$

$$\tilde{\mathrm{p}}(\iota(v)) = \|\mathrm{pr}_{\mathrm{p}}(\iota(v))\|_{\mathrm{p}} = \|[v]_{\mathrm{p}}\|_{\mathrm{p}} = \mathrm{p}(v) \tag{$*$}$$

according to Lemma 2.3.15, *iii.)*. Thus ι is continuous since we can estimate the value of the seminorm $\tilde{\mathrm{p}}(\iota(v))$ by a continuous seminorm of v, namely $\mathrm{p}(v)$. Since V is Hausdorff, $(*)$ shows that for $v \neq 0$ we have some $\mathrm{p} \in \mathcal{P}$ with $\tilde{\mathrm{p}}(\iota(v)) > 0$. Thus ι is injective. Finally, the induced topology on the image is obtained by restricting the continuous seminorms of $\mathcal{V}$ to the image. Since the seminorms in $\mathcal{P}_{\mathcal{V}}$ are a defining system already, $(*)$ shows that this reproduces the original seminorms of V. Thus we have an embedding. □

The lemma allows to identify V with its image $\iota(V)$ inside $\mathcal{V}$. Since we already know that $\mathcal{V}$ is complete, the following statement is straightforward:

Proposition 2.3.17 *Let V be a Hausdorff locally convex space.*

i.) *The locally convex space V can be identified with a subspace of a Cartesian product of Banach spaces.*

ii.) *The locally convex space V has a completion which is again locally convex.*

iii.) *If V is first countable then V can be identified as a subspace of a countable Cartesian product of Banach spaces. Its completion is then a Fréchet space.*

PROOF: The first part is clear by the Lemma 2.3.16. The second is then a consequence of Proposition 2.1.11, *i.)*, as we can take $\iota(V)^{\mathrm{cl}}$ inside the complete Cartesian product $\mathcal{V}$. For the last statement we observe that for the above construction a defining system of seminorms $\mathcal{P}$ instead of all continuous seminorms would be sufficient. Hence in the situation of a first countable space, countably many will do the job. □

Thus the proposition includes the statement of Theorem 2.2.3 and even provides a rather natural construction of the completion. An alternative route is outlined in Exercise 2.5.27.

After this excursion back to locally convex spaces we continue our study of continuous maps between normed spaces and, in particular, between Banach spaces. As a first application we give the following version of the Banach-Steinhaus Theorem: now we only have a single norm to take care of which simplifies the general statement of Theorem 2.2.26 considerably:

Theorem 2.3.18 (Banach-Steinhaus) *Let V be a Banach space and W a normed space. Moreover, let $\{\phi_i\}_{i\in I}$ be a collection of continuous linear maps $\phi_i \in \mathrm{L}(V, W)$ such that for every $v \in V$ there is a $C_v > 0$ with*

$$\|\phi_i(v)\| \leq C_v \tag{2.3.23}$$

for all $i \in I$. Then there is a $C > 0$ such that for all $i \in I$ we have

$$\|\phi_i\| \leq C. \tag{2.3.24}$$

PROOF: Since a Banach space is a particular case of a Fréchet space, this is just the reformulation of Theorem 2.2.26 for the case where we only have one norm on V and W. In fact, every continuous seminorm p on V satisfies an estimate $\mathrm{p}(v) \leq C\|v\|$ with some $C > 0$. Thus the equicontinuity from (2.2.25) in Lemma 2.2.25 boils down to $\|\phi_i(v)\| \leq C\|v\|$. But this gives (2.3.24) for the operator norms of the ϕ_i. □

Finally, we also have the Open Mapping Theorem and the Closed Graph Theorem in the context of Banach spaces. We repeat the formulations just for convenience:

Theorem 2.3.19 (Open Mapping Theorem) *A surjective continuous linear map $\phi\colon V \longrightarrow W$ between Banach spaces is open.*

Theorem 2.3.20 (Closed Graph Theorem) *A linear map $\phi\colon V \longrightarrow W$ between Banach spaces is continuous iff its graph is closed.*

2.3.3 Weak Topologies

In this last subsection we discuss some aspects of duality theory and weak topologies. Later on, we mainly focus on the case where the underlying spaces are Banach space. However, much of the theory can equally well be developed for general topological vector spaces. We refer to [26, Chap. 8] for a more detailed introduction to dualities.

We consider the following situation. Let V and W be complex vector spaces and let

$$\langle\,\cdot\,,\,\cdot\,\rangle\colon V \times W \longrightarrow \mathbb{C} \tag{2.3.25}$$

be a bilinear map. Then $\langle \cdot\,, \cdot \rangle$ is called a *pairing* if it is *non-degenerate*, i.e. if the two conditions

$$\langle v, w \rangle = 0 \quad \text{for all} \quad w \in W \quad \text{implies} \quad v = 0, \tag{2.3.26}$$

and

$$\langle v, w \rangle = 0 \quad \text{for all} \quad v \in V \quad \text{implies} \quad w = 0 \tag{2.3.27}$$

hold. In this case we call $(V, W, \langle \cdot\,, \cdot \rangle)$ a *dual pair*[1].

Remark 2.3.21 (Dual pairs) Let V and W be complex vector spaces with a pairing $\langle \cdot\,, \cdot \rangle$.

i.) The pairing $\langle \cdot\,, \cdot \rangle$ induces injective linear maps

$$V \ni v \;\mapsto\; \langle v, \cdot \rangle \in W^* \tag{2.3.28}$$

and

$$W \ni w \;\mapsto\; \langle \cdot\,, w \rangle \in V^* \tag{2.3.29}$$

into the algebraic duals. This follows from the non-degeneracy.

ii.) If V is finite-dimensional then the injectivity of (2.3.29) implies $\dim W \leq \dim V^* = \dim V$. Thus also W is finite-dimensional and hence analogously we get $\dim V \leq \dim W$. Thus $\dim V = \dim W$ and (2.3.28) and (2.3.29) are isomorphisms. Thus $V \cong W^*$ via the pairing in this case. In the infinite-dimensional case this needs not to be the case and makes a pairing an interesting object.

iii.) Dual pairs come always in pairs: if $(V, W, \langle \cdot\,, \cdot \rangle)$ is a dual pair then also $(W, V, \langle \cdot\,, \cdot \rangle^{\mathrm{opp}})$ is a dual pair where $\langle w, v \rangle^{\mathrm{opp}} = \langle v, w \rangle$. The definition is obviously symmetric in V and W.

Example 2.3.22 (Dual pairs) Let V be a topological vector space and $V' \subseteq V^*$ its topological dual. Moreover, let

$$\langle \cdot\,, \cdot \rangle \colon V \times V' \ni (v, \varphi) \;\mapsto\; \varphi(v) \in \mathbb{C} \tag{2.3.30}$$

be the usual natural pairing of linear forms and vectors.

i.) If V is just a topological vector space then (2.3.30) might be very degenerate. The reason is that there might be only very few continuous linear functionals. The non-degeneracy condition (2.3.29) is trivially fulfilled.

ii.) If V is in addition locally convex then by the Hahn-Banach Theorem in the incarnation of Corollary 2.2.21 the natural pairing is non-degenerate iff V is Hausdorff. Thus in this case we have a dual pair $(V, V', \langle \cdot\,, \cdot \rangle)$. By Remark 2.3.21, *iii.)*, we have also the dual pair $(V', V, \langle \cdot\,, \cdot \rangle^{\mathrm{opp}})$. These two dual pairs will clearly be the most important examples for the following.

The main purpose of dual pairs is to induce new locally convex topologies on the spaces V and W.

Definition 2.3.23 (Weak topology for dual pair) *Let $(V, W, \langle \cdot\,, \cdot \rangle)$ be a dual pair. Then the initial topology on V with respect to the linear functionals $\langle \cdot\,, w \rangle \in V^*$ for all $w \in W$ is denoted by $\sigma(V, W)$ and called the weak topology of V with respect to the dual pairing of V and W.*

More explicitly, from Proposition 2.2.33 we see that the initial topology is locally convex since $\mathbb{C}$ is locally convex. We have a defining system of seminorms given by the seminorms

$$\mathrm{p}_w(v) = |\langle v, w \rangle| \tag{2.3.31}$$

[1]At this point, it is hard to resist to mention that there are three types of mathematicians: those who can count to three and those who can't.

for all $w \in W$. In other words, all the linear functionals $v \mapsto \langle v, w \rangle$ are continuous with respect to the weak topology and it is the coarsest one with this property. A linear map $\phi\colon U \longrightarrow V$ is continuous with respect to the weak topology iff all the evaluations $u \mapsto \langle \phi(u), w \rangle$ on $w \in W$ are continuous linear functionals on U. In particular, W can be viewed as a subspace of the topological dual of V with respect to the weak topology. To show that W actually coincides with this topological dual we need the following lemma from linear algebra:

Lemma 2.3.24 *Let $\varphi, \varphi_1, \ldots, \varphi_n \in V^*$ be linear functionals on a complex vector space V. Then $\varphi \in \operatorname{span}_{\mathbb{C}}\{\varphi_1, \ldots, \varphi_n\}$ iff*

$$\bigcap_{i=1}^{n} \ker \varphi_i \subseteq \ker \varphi. \tag{2.3.32}$$

PROOF: The easy proof is the content of Exercise 2.5.45. ☐

Theorem 2.3.25 (Weak topological dual) *Let $(V, W, \langle \cdot\,, \cdot \rangle)$ be a dual pair and endow V with the corresponding weak topology. Then $V' \cong W$ via the pairing $\langle \cdot\,, \cdot \rangle$ according to* (2.3.29)*. Moreover, the weak topology $\sigma(V, W)$ is Hausdorff and locally convex.*

PROOF: We have already seen that $\sigma(V, W)$ is locally convex. Moreover, it is Hausdorff since we assume the non-degeneracy property (2.3.26): this immediately implies that for $v \neq 0$ there is a seminorm p_w in the defining set (2.3.31) with $\mathrm{p}_w(v) > 0$. We also note that $W \ni w \mapsto \langle \cdot\,, w \rangle$ is an injective linear map into V'. It remains to show that this map is surjective. Thus let $\varphi \in V'$ be weakly continuous. This implies that there are $w_1, \ldots, w_n \in W$ and a $c > 0$ such that

$$|\varphi(v)| \leq c \max_{i=1,\ldots,n} \mathrm{p}_{w_i}(v)$$

for all $v \in V$. Note that the system of seminorms of the form p_w might not yet be filtrating. For these w_i we see that $\mathrm{p}_{w_i}(v) = 0$ for all $i = 1, \ldots, n$ implies $\varphi(v) = 0$. Hence we are in the position of Lemma 2.3.24 with respect to the linear functionals $\varphi_i = \langle \cdot\,, w_i \rangle$ and conclude $\varphi = \sum_{i=1}^n z_i \varphi_i = \langle \cdot\,, \sum_{i=1}^n z_i w_i \rangle$ with some $z_i \in \mathbb{C}$. This completes the proof. ☐

The weak topologies tend to be non-complete. Even worse, the completion is usually non-interestingly huge:

Remark 2.3.26 Let $(V, W, \langle \cdot\,, \cdot \rangle)$ be a dual pair and equip V with the corresponding weak topology. Then V is complete iff $V \cong W^*$ under the identification (2.3.28). Thus it will typically make no sense to complete with respect to a weak topology, see Exercise 2.5.22 as well as [26, Prop. 8.1.5]. However, we will meet situations where the weak topology is at least *sequentially* complete.

The most important case is if V is already equipped with a topology and $W = V'$ is the topological dual. In this situation we have a dual pair under the assumptions of Example 2.3.22, *ii.)*.

Definition 2.3.27 (Weak and weak* topology) *Let V be a Hausdorff locally convex space.*

i.) The weak topology of V is the weak topology $\sigma(V, V')$ with respect to the usual pairing of V and V'.

ii.) The weak* topology of V' is the weak topology $\sigma(V', V)$ with respect to the usual pairing of V' and V.

This clearly reproduces the weak* topology from Definition 2.2.38. Thus, in the following, we mainly consider the topologies $\sigma(V, V')$ and $\sigma(V', V)$ and simply call them *the* weak and weak* topology, omitting the explicit reference to the dual pair. The name "weak" topology comes then from the following observation:

Proposition 2.3.28 *Let V be a Hausdorff locally convex space. Then the weak topology on V is coarser than the original topology.*

PROOF: By definition of the initial topology, the weak topology on V is the coarsest topology such that all the linear functionals $\varphi \in V'$ are continuous. Since they are also continuous with respect to the original topology, the claim follows. $\square$

In general, the weak topology is strictly coarser than the original topology of V. For practical use we mention the following characterization of the weak topology in terms of convergence:

Proposition 2.3.29 *Let V be a Hausdorff locally convex space.*

i.) A net $(v_i)_{i \in I}$ in V is a weak Cauchy net iff for all $\varphi \in V'$ the net $(\varphi(v_i))_{i \in I}$ is a Cauchy net in $\mathbb{C}$. It is weakly convergent to $v \in V$ iff for all $\varphi \in V'$ we have

$$\lim_{i \in I} \varphi(v_i) = \varphi(v). \tag{2.3.33}$$

ii.) A net $(\varphi_i)_{i \in I}$ is a weak Cauchy net iff for all $v \in V$ the net $(\varphi_i(v))_{i \in I}$ is a Cauchy net in $\mathbb{C}$. It is weak* convergent to $\varphi \in V'$ iff for all $v \in V$ we have*

$$\lim_{i \in I} \varphi_i(v) = \varphi(v). \tag{2.3.34}$$

PROOF: This is clear from the very definitions of the seminorms (2.3.31) characterizing the weak and weak* topology on V and V', respectively. $\square$

With other words, the weak* topology is the topology of *pointwise* convergence on V. From this we get immediately the following important result for Fréchet spaces:

Theorem 2.3.30 (Sequential completeness of V') *Let V be a Fréchet space. Then V' is sequentially complete with respect to the weak* topology.*

PROOF: Let $\varphi_n \in V'$ be a weak* Cauchy sequence. This means that the sequence $(\varphi_n(v))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for all $v \in V$. It follows that

$$\varphi(v) = \lim_{n \longrightarrow \infty} \varphi_n(v)$$

converges pointwise for every $v \in V$. By Corollary 2.2.27 we conclude that $\varphi \in V'$. Here the condition of having a first countable topology enters in a crucial way. Then it is easy to see that $\varphi_n \longrightarrow \varphi$ in the weak* topology, proving the theorem. $\square$

Note however that this statement is only true for sequential completeness: Cauchy nets need not to converge. In fact, here we are again in the situation of Remark 2.3.26 and get V^* as the weak* completion of V', see Exercise 2.5.22. Nevertheless, sequential completeness will already be very useful at many places. Moreover, since the conclusion of Corollary 2.2.27 holds for more general locally convex spaces than only for Fréchet ones, we get the important result of the theorem also in these situations, see Exercise **??**.

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We specialize now to the situation of a Banach space V. In this case, V' is also a Banach space and thus we have already two topologies on V': the norm topology and the weak* topology $\sigma(V', V)$. Moreover, since V'' is the dual of a Banach space V' and hence a Banach space itself, we have also the weak topology $\sigma(V', V'')$ on V', now with respect to the pairing of V' and V''. The remaining part of this subsection is devoted to the study of the relations between these topologies. We start with the following simple observation:

Proposition 2.3.31 *Let V be a Banach space.*

i.) The norm topology on V' is finer than the weak topology $\sigma(V', V'')$.

ii.) The weak topology $\sigma(V', V'')$ on V' is finer than the weak topology $\sigma(V', V)$ on V'.*

iii.) A weakly convergent sequence $v_n \in V$ is bounded in norm, i.e. there is a $c > 0$ with $\|v_n\| \leq c$.

PROOF: The first part is clear from Proposition 2.3.28 applied to V'. For the second part, we consider V as a subspace of V'' according to Proposition 2.3.12 via the embedding ι. Then the weak* topology on V' is determined by the seminorms $\mathrm{p}_v(\varphi) = |\varphi(v)| = |\iota(v)(\varphi)| = \mathrm{p}_{\iota(v)}(\varphi)$. Thus all the defining seminorms of the weak* topology appear as seminorms of the weak topology. This gives the second part. For the third part, suppose $v_n \longrightarrow v$ in the weak topology. Hence $\varphi(v_n) \longrightarrow \varphi(v)$ for all $\varphi \in V'$. Thus for every $\varphi \in V'$ the sequence $(\varphi(v_n))_{n\in\mathbb{N}}$ is bounded in $\mathbb{C}$: there is a $C_\varphi > 0$ with

$$|\iota(v)(\varphi)| = |\varphi(v_n)| \leq C_\varphi \tag{$*$}$$

for all $n \in \mathbb{N}$, where we view the v_n as elements of V'' via ι as before. Thus we can apply the Banach-Steinhaus Theorem 2.3.18 to conclude that $\|\iota(v_n)\| \leq c$ for all $n \in \mathbb{N}$ with some appropriate $c > 0$. Since ι is isometric, the third part follows. □

The next fundamental theorem emphasizes the role of the weak* topology on V'. We formulate this theorem of Banach-Alaoglu first for general locally convex spaces and pass to a more familiar Banach space version later on.

Theorem 2.3.32 (Banach-Alaoglu) *Let V be a Hausdorff locally convex space and let $U \subseteq V$ be a neighbourhood of 0. Then*

$$K = \left\{\varphi \in V' \;\middle|\; |\varphi(u)| \leq 1 \textit{ for } u \in U\right\} \subseteq V' \tag{2.3.35}$$

is a weakly compact subset of V'.*

PROOF: Let p be a continuous seminorm on V such that U contains the closed unit ball $\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$. Then we have for all $\varphi \in K$ the property that $|\varphi(v)| \leq \mathrm{p}(v)$ for all $v \in V$ since $\frac{v}{\mathrm{p}(v)} \in \mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$. The main idea of the proof is now to find an embedding of K into some very big compact space and show that K is a closed subset of it. To this end, we consider

$$\mathcal{V} = \prod_{v\in V} \mathbb{C}_v \quad \text{with} \quad \mathbb{C}_v = \mathbb{C} \quad \text{for all} \quad v \in V$$

with the Cartesian product topology. Since $\mathbb{C}$ is a Banach space, $\mathcal{V}$ is a complete locally convex space by Proposition 2.2.35, *iv.)*, though no longer a Banach space. We have a linear map

$$\jmath\colon V' \ni \varphi \;\mapsto\; (\varphi(v))_{v\in V} \in \mathcal{V},$$

which is clearly injective. The seminorms defining the Cartesian product topology are in our case simply given by

$$\mathrm{q}_v(\Phi) = |\Phi_v| \quad \text{for} \quad \Phi = (\Phi_v)_{v\in V} \in \mathcal{V}.$$

Indeed, it is sufficient to choose a defined system of seminorms on each $\mathbb{C}_v$ and hence the usual absolute value will do the job. Now this collection of seminorms gives, evaluated on some $\jmath(\varphi)$, simply $\mathrm{q}_v(\jmath(\varphi)) = |\varphi(v)| = \mathrm{p}_v(\varphi)$, i.e. the defining seminorms of the weak* topology. Thus we conclude that $\jmath$ is continuous with respect to the weak* topology on V'. Since we have even equality, we see that $\jmath$ is an embedding. Inside $\mathcal{V}$ we have a compact subset: we consider

$$\mathcal{K} = \prod_{v\in V} K_v \quad \text{with} \quad K_v = \left\{z \in \mathbb{C} \;\middle|\; |z| \leq \mathrm{p}(v)\right\},$$

i.e. the Cartesian product of closed discs of radii given by the seminorm of the indexing $v \in V$. The fact that $\mathcal{K}$ is compact is a consequence of the obvious compactness of the K_v and Tikhonov's Theorem, see Theorem A.3.2. For $\varphi \in K$ we clearly have $\jmath(\varphi) \in \mathcal{K}$. Thus $\jmath$ maps K into the compact subset $\mathcal{K}$. It remains to show that the image is actually closed. Thus let $\Phi \in \mathcal{V}$ be in the closure of $\jmath(K)$. Then there is a net $(\varphi_i)_{i\in I}$ in K with $\jmath(\varphi_i) \longrightarrow \Phi$ in the Cartesian product topology. This simply means $\Phi_v = \lim_{i\in I} \varphi_i(v)$ for all $v \in V$ according to Proposition 2.2.35, *iii.)*. Now let $z, w \in \mathbb{C}$ and $v, u \in V$. Then

$$\Phi_{zv+wu} = \lim_{i\in I} \varphi_i(zv + wu) = \lim_{i\in I}(z\varphi_i(v) + w\varphi_i(u)) = z \lim_{i\in I} \varphi_i(v) + w \lim_{i\in I} \varphi_i(u) = z\Phi_v + w\Phi_u$$

shows that Φ is actually corresponding to a linear functional $\varphi \in V^*$ on V via $\Phi = \jmath(\varphi)$, see also Exercise 2.5.22. Moreover, for $u \in U$ we have

$$|\Phi_u| = \lim_{i\in I} |\varphi_i(u)| \leq \lim_{i\in I} 1 = 1,$$

since for all φ_i we have $\varphi_i(u) \leq 1$ by $\varphi_i \in K$. This shows on one hand that $\varphi \in V'$ is continuous and on the other hand, $\varphi \in K$. Thus $\jmath(K)$ is closed and hence compact as claimed. □

Remark 2.3.33 (Polar) For a dual pair $(V, W, \langle \cdot\,, \cdot \rangle)$ and a subset $U \subseteq V$ the subset

$$U^\circ = \big\{ w \in W \;\big|\; |\langle u, w\rangle| \leq 1 \text{ for all } u \in U \big\} \subseteq W \tag{2.3.36}$$

is called the *polar* of U. The Banach-Alaoglu Theorem then states that for a Hausdorff locally convex space V and the canonical dual pair $(V, V', \langle \cdot\,, \cdot \rangle)$ the polar U° of any neighbourhood U of $0 \in V$ is weak* compact. Some elementary properties of polars are discussed in Exercise 2.5.25.

As an immediate corollary we obtain the following particular case of the Banach-Alaoglu Theorem, adapted to the Banach space situation:

Corollary 2.3.34 (Banach-Alaoglu) *Let V be a Banach space. Then the closed unit ball*

$$\mathrm{B}_1(0)^{\mathrm{cl}} = \big\{ \varphi \in V' \;\big|\; \|\varphi\| \leq 1 \big\} \subseteq V' \tag{2.3.37}$$

in the dual space V' is weak compact.*

PROOF: By Remark 2.3.6 we know that

$$\|\varphi\| = \sup_{\|v\|\leq 1} |\varphi(v)|.$$

Hence $\varphi \in \mathrm{B}_1(0)^{\mathrm{cl}}$ iff $|\varphi(v)| \leq 1$ for all $v \in V$ with $\|v\| \leq 1$. But this just means that the closed unit ball $\mathrm{B}_1(0)^{\mathrm{cl}}$ in V' is the polar of the closed unit ball in V. Since $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V$ is a neighbourhood of zero, we can apply Theorem 2.3.32. □

Remark 2.3.35 In general, $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V'$ is compact but not sequentially compact in the weak* topology. The reason is that compactness and sequential compactness are quite unrelated in general. This will sometimes limit the usability of the Banach-Alaoglu Theorem and requires extra care.

A first way to guarantee the additional and nice feature of sequential compactness is to consider separable Banach spaces. These are in some sense still very close to the finite-dimensional situation. Recall that a topological space M is called *separable* if there is a countable subset $X \subseteq M$ with $X^{\mathrm{cl}} = M$. Every finite-dimensional vector space is separable since we can take e.g. $(\mathbb{Q} + \mathrm{i}\mathbb{Q})^n \subseteq \mathbb{C}^n$ as countable dense subset after choosing an isomorphism to $\mathbb{C}^n$.

Proposition 2.3.36 *Let V be a separable Banach space. Then $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V'$ is first countable in the weak* topology and hence sequentially compact.*

PROOF: Choose a countable dense subset $\{v_1, v_2, \ldots\} \subseteq V$ and let $\varphi \in \mathrm{B}_1(0)^{\mathrm{cl}}$ be given. Then a neighbourhood basis of φ is obtained by taking the finite intersections of open 1-balls around φ with respect to the weak* seminorms p_v for $v \in V$. Note that it suffices to consider open balls of radii 1 only since $\epsilon \mathrm{p}_v = \mathrm{p}_{\epsilon v}$ for all $\epsilon > 0$. Since we are interested in the induced topology of $\mathrm{B}_1(0)^{\mathrm{cl}}$ this results in the open sets

$$U_{w_1,\ldots,w_n}(\varphi) = \mathrm{B}_{\mathrm{p}_{w_1},1}(\varphi) \cap \cdots \cap \mathrm{B}_{\mathrm{p}_{w_n},1}(\varphi) \cap \mathrm{B}_1(0)^{\mathrm{cl}}$$

as basis of neighbourhoods of φ in $\mathrm{B}_1(0)^{\mathrm{cl}}$. Since $\{v_1, v_2, \ldots\}$ is dense we find v_{m_i} in this subset with

$$\|w_i - v_{m_i}\| < \tfrac{1}{3} \tag{$*$}$$

for $i = 1, \ldots, n$. We claim that $U_{3v_{m_1},\ldots,3v_{m_n}}(\varphi)$ is contained in $U_{w_1,\ldots,w_n}(\varphi)$. Indeed, let $\psi \in U_{3v_{m_1},\ldots,3v_{m_n}}(\varphi)$ be given then $\|\psi\| \leq 1$ and $|\psi(v_{m_i}) - \varphi(v_{m_i})| < \frac{1}{3}$. Thus

$$\begin{aligned}
|\psi(w_i) - \varphi(w_i)| &\leq |\psi(w_i) - \psi(v_{m_i})| + |\psi(v_{m_i}) - \varphi(v_{m_i})| + |\varphi(v_{m_i}) - \varphi(w_i)| \\
&< \|\psi\|\|w_i - v_{m_i}\| + \tfrac{1}{3} + \|\varphi\|\|v_{m_i} - w_i\| \\
&< 1
\end{aligned}$$

for $i = 1, \ldots, n$. This shows $\psi \in U_{w_1,\ldots,w_n}(\varphi)$ and hence the claim. Since the set of all the neighbourhoods $U_{3v_{m_1},\ldots,3v_{m_n}}(\varphi)$ with $n \in \mathbb{N}$ and $m_1, \ldots, m_n \in \mathbb{N}$ is still countable, we have a countable basis of neighbourhoods of φ. This shows that $\mathrm{B}_1(0)^{\mathrm{cl}}$ is first countable in the weak* topology. Note that it is important to use the induced topology on $\mathrm{B}_1(0)^{\mathrm{cl}}$. The weak* topology of V' might fail to be first countable. From this the second statement is clear for general reasons: a compact and first countable topological space is sequentially compact, see Proposition **??**. ☐

We come now to a particular class of Banach spaces where we have additional nice properties for the weak* topology, the *reflexive* Banach spaces:

Definition 2.3.37 (Reflexive normed space) *A normed space V is called reflexive if the canonical embedding $\iota \colon V \longrightarrow V''$ is surjective.*

Proposition 2.3.38 *Let V be a normed space.*

i.) If V is reflexive then V is complete and hence a Banach space.

ii.) If V is reflexive then the weak topology $\sigma(V', V)$ and the weak topology $\sigma(V', V'')$ on V' coincide.*

iii.) If V is reflexive and $U \subseteq V$ is a closed subspace then U is reflexive, too.

iv.) If V is complete then V is reflexive iff V' is reflexive.

PROOF: The first part is clear since $\iota \colon V \longrightarrow V''$ is a norm-preserving bijection and V'' is always complete according to Corollary 2.3.9. The second part is also clear from our considerations in the proof of Proposition 2.3.31 together with the additional input that $\iota(V) = V''$. Thus let $U \subseteq V$ be a closed subspace of a reflexive (Banach) space. We denote the inclusion map by $j \colon U \longrightarrow V$ which is clearly a norm-preserving map by definition. Then the dual map gives the continuous map

$$j' \colon V' \ni \varphi \mapsto \big(u \mapsto j'(\varphi)(u) = \varphi(j(u))\big) \in U', \tag{$*$}$$

see Corollary 2.3.11. By the Hahn-Banach Theorem in the version of Theorem 2.3.10 the map j' is *surjective*. Dualizing this once more gives the *injective* map $j'' \colon U'' \longrightarrow V''$. Now let $X \in U''$ be given.

Then we want to show that there is a $u \in U$ with $\iota_U(u) = X$. Since by assumption $\iota_V \colon V \longrightarrow V''$ is bijective we have a unique $v \in V$ with $\iota_V(v) = j''(X)$. Evaluating this equation on V' gives for all $\varphi \in V'$

$$\varphi(v) = \iota_V(v)\varphi = j''(X)\varphi = X(j'(\varphi)) = X(\varphi\big|_U), \tag{$**$}$$

since $j'(\varphi)$ simply means to restrict φ to U according to $(*)$. We claim that this can only be the case if $v \in U$. Indeed, assume $v \notin U$. Then we can find a continuous linear functional $\Phi \colon V\big/U \longrightarrow \mathbb{C}$ with $\Phi([v]) \neq 0$ since $[v] \neq 0$ in the Banach space $V\big/U$, again by the Hahn-Banach Theorem. Note that at this point it is crucial that U is closed. If $\pi \colon V \longrightarrow V\big/U$ denotes the canonical projection then $\varphi = \Phi \circ \pi$ is a continuous linear functional on V with $\varphi(v) \neq 0$ but $\varphi\big|_U = 0$. Applying $(**)$ to this φ gives a contradiction. Thus $v \in U$ and $X = \iota_U(v)$ follows at once, proving that U is reflexive. The last part is easy: assume first that V is reflexive and hence $\iota_V \colon V \longrightarrow V''$ is an isomorphism. Let $\Psi \in V'''$ be given. Since every $X \in V''$ is of the form $X = \iota_V(v)$ with a unique $v \in V$ the functional $\varphi = \Psi \circ \iota_V = \iota_V'(\Psi) \in V'$ satisfies

$$(\iota_{V'}(\varphi))(X) = X(\varphi) = \iota_V(v)\varphi = \varphi(v) = \Psi(\iota_V(v)) = \Psi(X),$$

and thus $\iota_{V'}(\varphi) = \Psi$. This shows that $\iota_{V'} \colon V' \longrightarrow V'''$ is surjective and hence V' is reflexive. Conversely, let V' be reflexive and V be complete. Then we know that $\iota(V) \subseteq V''$ is a closed subspace. Applying the above result to V' we conclude that V'' is reflexive and hence the closed subspace $\iota(V)$ is reflexive as well, by the third part. Since $\iota \colon V \longrightarrow \iota(V)$ is an isometric isomorphism, V is reflexive. □

The following technical lemma is also of independent interest and will be used in the next proposition:

Lemma 2.3.39 *Let V be a normed space with separable dual space V'. Then V is separable, too.*

Proof: Let $\{\varphi_1, \varphi_2, \ldots\} \subseteq V'$ be a countable dense subset where we can assume that $\varphi_n \neq 0$ for all $n \in \mathbb{N}$. This allows to consider the normalized functionals $\psi_n = \frac{\varphi_n}{\|\varphi_n\|} \in \partial\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V'$. Since the φ_n are dense in V' it follows that the ψ_n are dense in the sphere $\partial\mathrm{B}_1(0)^{\mathrm{cl}}$, always with respect to the norm topology. Since $\|\psi_n\| = 1$ we can choose $v_n \in V$ with $|\psi_n(v_n)| \geq \frac{1}{2}$ and $\|v_n\| = 1$. We claim that $U = \operatorname{span}_{\mathbb{C}}\{v_n\}_{n\in\mathbb{N}}$ is dense in V. Suppose the contrary and choose a vector $v \in V$ with $\|v\| = 1$ but $v \notin U^{\mathrm{cl}}$. As in the proof of Proposition 2.3.38 we find a $\varphi \in V'$ with $\varphi(v) = 1$ and $\|\varphi\| = 1$ but $\varphi\big|_{U^{\mathrm{cl}}} = 0$. Since the ψ_n are dense in the sphere $\partial\mathrm{B}_1(0)^{\mathrm{cl}}$ we can approximate φ such that $\|\psi_n - \varphi\| < \frac{1}{2}$ for some suitable n. This gives a contradiction since with $\varphi(v_n) = 0$ we get

$$\frac{1}{2} \leq |\psi_n(v_n)| = |\varphi(v_n) - \psi_n(v_n)| \leq \|\varphi - \psi_n\| < \frac{1}{2}.$$

Hence $U^{\mathrm{cl}} = V$ and then $\operatorname{span}_{\mathbb{Q}+\mathrm{i}\mathbb{Q}}\{v_n\}_{n\in\mathbb{N}}$ is a countable dense subset. □

With this lemma we can now easily show the following result on reflexive Banach spaces. It will have important applications in the theory of Hilbert spaces in the next chapter.

Proposition 2.3.40 *For a reflexive Banach space V the closed unit ball $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V$ is sequentially compact with respect to the weak topology.*

Proof: Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V$ for which we have to find a convergent subsequence. Let $U = \operatorname{span}_{\mathbb{C}}\{v_n\}_{n\in\mathbb{N}}$ be the subspace spanned by the elements of the sequence. Then U^{cl} is separable. By Proposition 2.3.38, *iii.)*, the closed subspace U^{cl} of V is reflexive again. Since every $\varphi \in (U^{\mathrm{cl}})'$ has a norm-preserving extension to a $\Phi \in V'$ by the Hahn-Banach Theorem, we see that the seminorm p_φ and the seminorm $\mathrm{p}_\Phi\big|_{U^{\mathrm{cl}}}$ coincide. Conversely, every $\Phi \in V'$ restricts to $\Phi\big|_{U^{\mathrm{cl}}} \in (U^{\mathrm{cl}})'$.

Thus the weak topology of U^{cl} and the induced topology from the weak topology of V coincide. This shows that the inclusion $U^{\mathrm{cl}} \longrightarrow V$ is not only an embedding in the norm topologies but also in the weak topologies. From this we conclude that the question whether $(v_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence can entirely be discussed inside the reflexive and separable U^{cl}. Since $(U^{\mathrm{cl}})'' \cong U^{\mathrm{cl}}$ is separable, also $(U^{\mathrm{cl}})'$ is separable by Lemma 2.3.39, applied to $(U^{\mathrm{cl}})'$. Thus the closed unit ball $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq (U^{\mathrm{cl}})''$ is sequentially compact in the weak* topology by Proposition 2.3.36. Since $(U^{\mathrm{cl}})'$ is reflexive, too, by Proposition 2.3.38, *iv.)*, we conclude that the weak and the weak* topologies on its dual $(U^{\mathrm{cl}})''$ coincide by Proposition 2.3.38, *ii.)*. This shows that $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq (U^{\mathrm{cl}})''$ is sequentially compact in the weak topology of $(U^{\mathrm{cl}})''$. Since by reflexivity $(U^{\mathrm{cl}})' \cong (U^{\mathrm{cl}})'''$, the weak topology of $(U^{\mathrm{cl}})''$ becomes the weak topology of U^{cl} under the identification $U^{\mathrm{cl}} \cong (U^{\mathrm{cl}})''$. This concludes the proof. □

2.4 Geometric Aspects of Locally Convex Spaces

In this section we collect some further aspects of locally convex spaces which are mainly of geometric nature: we are interested in separation properties, bounded subsets, and convex subsets. The material will not be needed until Section **??** and can be skipped in a first reading.

2.4.1 Separation Theorems

The following separation theorems will be a geometric formulation of the Hahn-Banach Theorem. We will need some preparations and vocabulary as established in the Exercises 2.5.4 and 2.5.13:

Lemma 2.4.1 *Let $U \subseteq V$ be an absorbing and convex subset of a vector space V. Then the Minkowski functional*

$$\mathrm{p}_U(v) = \inf\big\{r > 0 \bigm| v \in rU\big\} \tag{2.4.1}$$

is a sublinear functional $\mathrm{p}_U\colon V \longrightarrow [0,\infty)$ which is a seminorm if in addition U is circled.

Proof: The proof is done in Exercise 2.5.13. □

We are now able to prove the first version of the separation theorem:

Theorem 2.4.2 (Separation theorem I) *Let $A, B \subseteq V$ be two disjoint non-empty convex subsets of a topological vector space V.*

i.) *If A is open then there exists a continuous linear functional $\Phi \in V'$ and a real number $\alpha \in \mathbb{R}$ such that*

$$\mathrm{Re}(\Phi(v)) < \alpha \le \mathrm{Re}(\Phi(u)) \tag{2.4.2}$$

for all $v \in A$ and $u \in B$.

ii.) *If A and B are both open we get even*

$$\mathrm{Re}(\Phi(v)) < \alpha < \mathrm{Re}(\Phi(u)) \tag{2.4.3}$$

for all $v \in A$ and $u \in B$.

Proof: As in the proof of the Hahn-Banach Theorem it is advantageous to treat V as a real vector space only. If V is complex and if we can find a $\mathbb{R}$-linear functional Ψ with the property (2.4.2) then $\Phi(v) = \Psi(v) - \mathrm{i}\Psi(\mathrm{i}v)$ will be the corresponding $\mathbb{C}$-linear functional still satisfying (2.4.2) as $\mathrm{Re}(\Phi) = \Psi$ by construction. Clearly, Ψ is continuous iff Φ is continuous. First we consider the case where only A is open. We fix two points $v_0 \in A$ and $w_0 \in B$ and set $u_0 = w_0 - v_0$. Then the subset

$$C = A - B + u_0 = \bigcup_{w\in B} (A - w) + u_0 \tag{$*$}$$

is still convex as a simple verification shows. Moreover, $0 \in C$ by taking $v_0 \in A$ and $w_0 \in B$ in the difference $(*)$. Finally, taking only $w_0 \in B$ shows that C contains the open subset $A - v_0$ which contains 0. Thus C is a neighbourhood of 0. Since A is open also C is open and thus absorbing. We consider now the Minkowski functional p_C of C which is sublinear according to Lemma 2.4.1. Since $A \cap B = \emptyset$ we have $u_0 \notin C$ and thus $\mathrm{p}_C(u_0) \geq 1$. On the $\mathbb{R}$-span of u_0 we define a linear functional $\psi\colon \operatorname{span}_{\mathbb{R}}\{u_0\} \longrightarrow \mathbb{R}$ by $\psi(\lambda u_0) = \lambda$. For $\lambda \geq 0$ we have

$$\psi(\lambda u_0) = \lambda \leq \lambda \mathrm{p}_C(u_0) = \mathrm{p}_C(\lambda u_0),$$

while for $\lambda < 0$ we get $\psi(\lambda u_0) = \lambda < 0 \leq \mathrm{p}_C(\lambda u_0)$. Thus for the sublinear functional p_C and the $\mathbb{R}$-linear functional ψ on $\operatorname{span}_{\mathbb{R}}\{u_0\}$ the conditions for the Hahn-Banach Theorem as in Theorem 2.2.18 are fulfilled: we get a $\mathbb{R}$-linear extension Ψ of ψ to the whole space V such that

$$\Psi(v) \leq \mathrm{p}_C(v)$$

for all $v \in V$. For $v \in C$ we know that $\mathrm{p}_C(v) \leq 1$ and thus $\Psi(v) \leq 1$. Since Ψ is $\mathbb{R}$-linear we get for $v \in -C$ the property $\Psi(v) \geq -1$, resulting in the estimate

$$|\Psi(v)| \leq 1$$

for all $v \in C \cap (-C)$. Since this subset is still a neighbourhood of 0, the functional Ψ is continuous. In fact, this is one way to express continuity of a linear functional on a general topological vector space, see also Exercise 2.5.5. Now let $v \in A$ and $u \in B$ then we get

$$\Psi(v) - \Psi(u) + 1 = \Psi(v - u + u_0) \leq \mathrm{p}_C(v - u + u_0) < 1,$$

since by the openness of C we get from $v - u + u_0 \in C$ also $v - u + u_0 \in (1 - \epsilon)C$ from some small $\epsilon > 0$. Thus the Minkowski functional is strictly smaller than 1 on all points of C. It follows that

$$\Psi(v) < \Psi(u) \tag{$**$}$$

for all $v \in A$ and $u \in B$. Since Ψ is $\mathbb{R}$-linear it maps the convex subsets A and B to convex subsets $\Psi(A), \Psi(B) \subseteq \mathbb{R}$. Thus $\Psi(A)$ and $\Psi(B)$ are necessarily open, closed, or half-open intervals. Since A is open, and Ψ is necessarily an open map by Exercise 2.5.15, $\Psi(A)$ is an *open* interval. Thus $\Psi(A) = (\omega, \alpha)$ for some $\alpha \in \mathbb{R}$ and $\omega \in [-\infty, \alpha)$. Here we use that $\Psi(A)$ is bounded to the right thanks to $(**)$. It follows from $(**)$ that this α will do the job. If B is also open then $\Psi(B) = (\beta, \omega')$ with $\alpha \leq \beta$ and $\omega' \in (\beta, \infty]$. Thus any real number in $[\alpha, \beta]$, e.g. α itself, will give (2.4.3). □

Before showing the second separation theorem we discuss the following property of disjoint closed and compact subsets valid in a general topological vector space:

Proposition 2.4.3 *Let V be a topological vector space and let $K \subseteq V$ be compact and $A \subseteq V$ be closed. If $K \cap A = \emptyset$ then one finds an open neighbourhood $U \subseteq V$ of 0 such that*

$$(K + U) \cap (A + U) = \emptyset. \tag{2.4.4}$$

PROOF: In the proof of the Open Mapping Theorem 2.2.29 we showed that for any open neighbourhood $W \subseteq V$ of 0 there is an open neighbourhood $Z \subseteq V$ of 0 such that $Z - Z \subseteq W$. This follows directly from the continuity of the vector space operations. Now we can take $X = Z \cap (-Z)$ which is still an open neighbourhood of 0 such that $X = -X$. Moreover, $X + X \subseteq W$ holds. Repeating this argument with X instead of W gives an open neighbourhood $Y = -Y$ of 0 such that $Y + Y \subseteq X$ and hence

$$Y + Y + Y + Y \subseteq W.$$

Now let K be non-empty to avoid trivialities and fix $v \in K$. Then v is in the open subset $V \setminus A$ and hence $0 \in (V \setminus A) - v = W_v$. By the above argument we get an open neighbourhood $Y_v = -Y_v$ of 0 with

$$Y_v + Y_v + Y_v + Y_v \subseteq W_v = (V \setminus A) - v.$$

This implies that $v + Y_v + Y_v + Y_v \subseteq V \setminus A$. By $Y_v = -Y_v$ we conclude that

$$(v + Y_v + Y_v) \cap (A + Y_v) = \emptyset. \tag{$*$}$$

Since K is compact, finitely many $v_1, \ldots, v_n \in K$ will be sufficient such that

$$K \subseteq (v_1 + Y_{v_1}) \cup \cdots \cup (v_n + Y_{v_n}).$$

Thus we have the open neighbourhood $U = Y_{v_1} \cap \cdots \cap Y_{v_n}$ of 0 with the property

$$K + U \subseteq \bigcup_{i=1}^{n} (v_i + Y_{v_i} + U) \subseteq \bigcup_{i=1}^{n} (v_i + Y_{v_i} + Y_{v_i}).$$

By $(*)$ we know that $v_i + Y_{v_i} + Y_{v_i}$ has trivial intersection with $A + U$. Hence also the union of the $v_i + Y_{v_i} + Y_{v_i}$ has trivial intersection which shows that U satisfies (2.4.4). $\square$

Note that we have not used any Hausdorff assumption here. Note also that from $0 \in U$ we immediately get

$$(K + U) \cap A = \emptyset. \tag{2.4.5}$$

The second version of the separation theorem involves a locally convex space instead of a general topological vector space. Here we get the following result:

Theorem 2.4.4 (Separation theorem II) *Let V be a locally convex space and let $K, C \subseteq V$ be disjoint non-empty convex subsets such that K is compact and C is closed. Then there exists a continuous linear functional $\Phi \in V'$ and real numbers $\alpha < \beta$ such that*

$$\operatorname{Re}(\Phi(v)) < \alpha < \beta < \operatorname{Re}(\Phi(u)) \tag{2.4.6}$$

for all $v \in K$ and $u \in U$.

Proof: By Proposition 2.4.3 we can choose an open neighbourhood U of 0 such that $(K + U) \cap (C + U) = \emptyset$. Since V is assumed to be locally convex, we can even find a U which is convex itself by taking an open ball with respect to a suitable continuous seminorm. Thus we can apply the previous Theorem 2.4.2 to the convex open subset $A = K + U = \bigcup_{x \in K}(x + U)$ and $B = C$. Let Ψ be the corresponding $\mathbb{R}$-linear functional and let $\beta \in \mathbb{R}$ be the corresponding point between the two subsets $\Psi(A)$ and $\Psi(B)$ of $\mathbb{R}$. Again, $\Psi(A)$ is open. Since K is compact and Ψ is continuous we have a compact subset $\Psi(K) \subseteq \Psi(A)$. Thus if $\Psi(A) \subseteq (\omega, \beta)$ for some $\omega < \beta$ we have $\Psi(K) = [\omega', \alpha] \subseteq (\omega, \beta)$ with $\alpha < \beta$. Hence (2.4.6) follows for the real case and passing from Ψ to Φ defined by $\Phi(v) = \Psi(v) - \mathrm{i}\Psi(\mathrm{i}v)$ as usual gives the complex case. $\square$

For a geometric interpretation of the separation theorems one should consult Exercise 2.5.46. We will now discuss some useful corollaries of the separation theorems.

Remark 2.4.5 Exchanging Φ with $-\Phi$ we get the inequality

$$\operatorname{Re}\Phi(u) < \alpha' < \beta' < \operatorname{Re}\Phi(v) \tag{2.4.7}$$

for all $u \in C$ and $v \in K$ with appropriate constants $\alpha' = -\beta$ and $\beta' = -\alpha$. Thus the two sets K and C enter symmetrically in the inequality.

Corollary 2.4.6 *Let V be a locally convex space and let $K, C \subseteq V$ be disjoint non-empty convex subsets such that K is compact and C is closed and absolutely convex. Then there exists a continuous linear functional $\Phi \in V'$ such that*

$$\sup_{u\in C}|\Phi(u)| < \inf_{v\in K}|\Phi(v)|. \tag{2.4.8}$$

Proof: Let Φ, α', and β' be as in (2.4.7). Then the fact that C is absolutely convex shows that $\mathrm{e}^{\mathrm{i}\varphi}u \in C$ for all phases $\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{C}$. For a given $u \in C$ let $\mathrm{e}^{\mathrm{i}\varphi}$ be the phase such that $\mathrm{e}^{\mathrm{i}\varphi}\Phi(u) = |\Phi(u)|$. Then

$$\operatorname{Re}\Phi(\mathrm{e}^{\mathrm{i}\varphi}u) = \operatorname{Re}(\mathrm{e}^{\mathrm{i}\varphi}\Phi(u)) = \operatorname{Re}|\Phi(u)| = |\Phi(u)|.$$

Since $\mathrm{e}^{\mathrm{i}\varphi}u \in C$ we get from (2.4.7) the estimate $|\Phi(u)| < \alpha'$ for all $u \in C$. Thus $\sup_{u\in C}|\Phi(u)| \leq \alpha'$ holds. Since $\operatorname{Re}\Phi(v) > \beta'$ for $v \in K$ we also have $|\Phi(v)| \geq |\operatorname{Re}\Phi(v)| \geq \operatorname{Re}\Phi(v) > \beta'$. Thus the infimum on the right hand side of (2.4.8) is still at least β' proving the claim since $\alpha' < \beta'$. ☐

The next corollary is a particular though useful case where the compact subset consists of a single point:

Corollary 2.4.7 *Let V be a locally convex space and $C \subseteq V$ an absolutely convex closed subset. If $v \in V$ is not in C then there exists a $\Phi \in V'$ with $|\Phi(u)| \leq 1$ for all $u \in C$ but $\Phi(v) > 1$.*

Proof: Set $K = \{v\}$ which is clearly convex and compact. Then we find a continuous linear functional $\tilde{\Phi} \in V'$ such that

$$\gamma = \sup_{u\in C}|\tilde{\Phi}(u)| < |\tilde{\Phi}(v)|$$

by the previous corollary. Rescaling $\tilde{\Phi}$ by γ and by the phase of $\tilde{\Phi}(v)$ gives the linear functional Φ we are looking for. ☐

The last version of the separation theorem uses the weak topology of a topological vector space: let V be a topological vector space and V' its dual. In general, V' can be very small. But if we assume that V' *separates points*, i.e. for every non-zero $v \in V$ there is a $\varphi \in V'$ with $\varphi(v) \neq 0$, then we have a dual pair $(V, V', \langle\,\cdot\,,\,\cdot\,\rangle)$ according to Example 2.3.22. In this case the weak topology on V is Hausdorff and locally convex such that the weak topological dual of V still coincides with V' according to Theorem 2.3.25. Finally, by a slight adaption of Proposition 2.3.28 we see that the weak topology is coarser than the original topology, whether V was locally convex or not: indeed, by the continuity of $\varphi \in V'$ we see that the weak open balls $\mathrm{B}_{\mathrm{p}_\varphi,\epsilon}(0) = \{v \in V \mid \mathrm{p}_\varphi(v) = |\varphi(v)| < \epsilon\}$ are open in the original topology. Using these statements, the following corollary is immediate:

Corollary 2.4.8 *Let V be a topological vector space such that $(V, V', \langle\,\cdot\,,\,\cdot\,\rangle)$ is a dual pair, i.e. V' separates points. If $K_1, K_2 \subseteq V$ are non-empty disjoint convex and compact subsets then there exists a $\Phi \in V'$ with*

$$\sup_{v_1\in K_1}\operatorname{Re}\Phi(v_1) < \inf_{v_2\in K_2}\operatorname{Re}\Phi(v_2). \tag{2.4.9}$$

Proof: Since the weak topology is coarser but still Hausdorff the subsets K_1 and K_2 are still weakly compact and weakly closed. Thus we can apply the Separation Theorem 2.4.4 to this situation with respect to the weak topology. This gives us a weakly continuous Φ with the property (2.4.9). But then Theorem 2.3.25 shows $\Phi \in V'$. ☐

Alternatively, we could have asked for a weakly closed K_2 instead of a compact one. But this is usually a tricky question whether a subset is weakly closed or not as the weak topology is typically rather coarse.

Note also that a Hausdorff locally convex space V clearly meets the conditions of this corollary since here V' separates points by the Hahn-Banach Theorem, see again Example 2.3.22.

We turn now to some applications of the separation theorems: first we note that the weak topology on a Hausdorff locally convex space V is typically strictly coarser than the original topology. Thus weak closures tend to be strictly larger than closures in the original topology. The following fact is thus remarkable:

Proposition 2.4.9 *Let V be a Hausdorff locally convex space and let $A \subseteq V$ be a convex subset. Then the weak closure coincides with the closure of A in the original topology.*

PROOF: Denote the weak closure by $A^{\text{weak-cl}}$ and the original closure by A^{cl} as usual. We know $A^{\text{cl}} \subseteq A^{\text{weak}-\text{cl}}$. Suppose we have a $v_0 \in A^{\text{weak}-\text{cl}}$ which is not in A^{cl}. Since closures in a topological vector space are clearly compatible with convexity, see Exercise 2.5.3, *iii.)*, we apply the Separation Theorem 2.4.4 to the compact subset $\{v_0\}$ and the closed convex subset A^{cl}. Thus we get a $\Phi \in V'$ with

$$\operatorname{Re}\Phi(v_0) < \alpha < \operatorname{Re}\Phi(v) \tag{$*$}$$

for all $v \in A^{\text{cl}}$. Since Φ is also weakly continuous the subset

$$U = \left\{ v \in V \;\middle|\; \operatorname{Re}\Phi(v) < \alpha \right\} \subseteq V$$

is an open subset in the weak topology and hence a weak neighbourhood of v_0. But then $(*)$ gives $U \cap A \subseteq U \cap A^{\text{cl}} = \emptyset$ and thus there is no point in A which is in this weak neighbourhood of v_0. This contradicts $v_0 \in A^{\text{weak}-\text{cl}}$, finishing the proof. ☐

Thus for *convex* subsets the notions of closure, closedness and also density with respect to the original and the weak topology coincide. Note that this applies in particular to subspaces of V:

Corollary 2.4.10 *Let V be a Hausdorff locally convex space and let $U \subseteq V$ be a subspace. Then one has $U^{\text{weak}-\text{cl}} = U^{\text{cl}}$.*

2.4.2 Bounded Subsets

In a Banach space one calls a subset $B \subseteq V$ bounded if $B \subseteq \mathrm{B}_R(0)$ for some large enough $R > 0$, i.e. for all $v \in B$ we have $\|v\| < R$. Analogously, we say that B is *bounded* in a locally convex space if for all continuous seminorms p on V one finds a $R_\mathrm{p} > 0$ with $B \subseteq \mathrm{B}_{\mathrm{p},R_\mathrm{p}}(0)$, i.e. $\mathrm{p}(v) < R_\mathrm{p}$ for all $v \in B$. Here the numerical value of the bound R_p depends of course on the seminorm p. These ideas lead to the following general definition:

Definition 2.4.11 (Bounded set) *Let V be a topological vector space and let $B \subseteq V$. Then B is called bounded if for all open neighbourhoods $U \subseteq V$ of 0 we find a $r > 0$ with $B \subseteq rU$.*

This is indeed a good generalization of the concept of bounded sets also beyond locally convex spaces as the following proposition shows:

Proposition 2.4.12 *Let V be a topological vector space and $B \subseteq V$ a subset. Then the following statements are equivalent:*

i.) *B is bounded.*

ii.) *For every sequence $(v_n)_{n\in\mathbb{N}}$ in B and every zero sequence $(z_n)_{n\in\mathbb{N}}$ of complex numbers one has*

$$\lim_{n\longrightarrow\infty} z_n v_n = 0. \tag{2.4.10}$$

iii.) *Every countable subset of B is bounded.*

Proof: Since every topological vector space has a basis of zero neighbourhoods consisting of circled open subsets, see Exercise **??**, it suffices to test everything for circled open neighbourhoods of 0 only. Thus let B be bounded and $v_n \in B$ and $z_n \in \mathbb{C}$ for $n \in \mathbb{N}$ with $z_n \longrightarrow 0$. If $U \subseteq V$ is a circled open neighbourhood of 0 we fix $r > 0$ with $B \subseteq rU$. Thus for those z_n with $|z_n| < \frac{1}{r}$ we have $z_n v_n \in U$ showing $z_n v_n \longrightarrow 0$. Conversely, if B is unbounded we find at least one open neighbourhood U of 0 such that for all n we find $v_n \in B$ with $v_n \notin nU$. Hence $\frac{1}{n} v_n \notin U$ for all n which shows that $\frac{1}{n} v_n$ can not converge to 0. Thus the equivalence of *i.)* and *ii.)* is shown. Then the equivalence of *ii.)* and *iii.)* is clear. □

In the locally convex case the notion of bounded subsets reproduces the above, more intuitive, definition:

Proposition 2.4.13 *Let V be a locally convex space and $B \subseteq V$. Then B is bounded iff for a defining systems of continuous seminorms $\mathcal{P}$ we have for every $\mathrm{p} \in \mathcal{P}$ a bound $r > 0$ with $B \subseteq \mathrm{B}_{\mathrm{p},r}(0)$.*

Proof: Let $\mathrm{p} \in \mathcal{P}$. Since $\mathrm{B}_{\mathrm{p},r}(0) = r\mathrm{B}_{\mathrm{p},1}(0)$ are open neighbourhoods of 0 the boundedness of B gives $B \subseteq r\mathrm{B}_{\mathrm{p},1}(0)$ for some $r > 0$ by the very definition. For the converse we note that it suffices to check the boundedness condition for a subbasis of neighbourhoods, e.g. for the open balls $\mathrm{B}_{\mathrm{p},r}(0)$ of a defining system of seminorms. □

Proposition 2.4.14 *Let $B \subseteq V$ be a bounded subset of a topological vector space. Then B^{cl} is bounded again.*

Proof: In a locally convex space one has a simple argument discussed in Exercise 2.5.47. In the general case we first show the following statement: if $U \subseteq V$ is an open neighbourhood of 0 then there is another open neighbourhood W of 0 with $W^{\mathrm{cl}} \subseteq U$. Indeed, consider the closed subset $A = V \setminus U$ and the compact subset $K = \{0\}$. Then by Proposition 2.4.3 we find an open neighbourhood W of 0 with $(K + W) \cap (A + W) = \emptyset$. Thus $W \subseteq V \setminus (A + W)$. Since $A + W = \bigcup_{v \in A}(v + W)$ is open, W is contained in the closed subset $V \setminus (A + W)$ and hence the closure W^{cl} is still in $V \setminus (A + W)$. Since $A \subseteq A + W$ by $0 \in W$ we conclude $W \subseteq W^{\mathrm{cl}} \subseteq V \setminus (A + W) \subseteq V \setminus A = U$, proving the claim. Now let B be bounded and U be given. Choose W as above then we find $r > 0$ with $B \subseteq rW$. Hence $B^{\mathrm{cl}} \subseteq (rW)^{\mathrm{cl}} = rW^{\mathrm{cl}} \subseteq rU$ shows that B^{cl} is again bounded. □

Example 2.4.15 (Bounded subsets) Let V be a topological vector space.

i.) The finite union and an arbitrary intersection of bounded subsets is again bounded. This is clear from the definition.

ii.) Any finite subset is bounded. More generally, any compact subset is bounded. Indeed, let U be an open neighbourhood of 0 and let $K \subseteq V$ be compact. Since $\{nU\}_{n \in \mathbb{N}}$ cover all of V they provide an open cover of K, too. Hence finitely many will cover K and thus the one with the largest n will do the job already since $nU \subseteq mU$ for $n \leq m$. This shows that K is bounded, see also Exercise 2.5.47 for the locally convex case.

iii.) If B is bounded then B is bounded in any coarser topology as well.

iv.) Any subset of a bounded subset is again bounded.

Since in a Hausdorff topological vector space a compact subset is also closed we see that any compact subset is bounded and closed in this case. For subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ this is precisely the characterization of compact subsets according to the classical Theorem of Heine and Borel. This motivates the following definition:

Definition 2.4.16 (Heine-Borel property) *Let V be a Hausdorff topological vector space. If every bounded and closed subset of V is compact then V is said to have the Heine-Borel property.*

In infinite dimensions the Heine-Borel property is usually a non-trivial feature. To see this in examples we need some preparation. Recall that a Hausdorff topological space is called *locally compact* if every point has a compact neighbourhood. In case of a topological vector space we only have to check this at one point, say at $0 \in V$, by the translation invariance.

Theorem 2.4.17 (Heine-Borel property) *Let V be a Hausdorff topological vector space.*

i.) If V is locally compact then V is finite-dimensional.

ii.) If V has a bounded neighbourhood of 0 and the Heine-Borel property then V is finite-dimensional.

iii.) If V is normable and has the Heine-Borel property then V is finite-dimensional.

Proof: Let $U \subseteq V$ be an open neighbourhood of 0 with compact closure U^{cl}. Then U as well as U^{cl} are bounded by Example 2.4.15, *ii.)* and *iv.)*. Since $\frac{1}{2}U$ is still an open neighbourhood of 0, the sets of the form $v + \frac{1}{2}U$ with $v \in V$ provide an open cover of V and hence of U^{cl}. Thus the compactness of U^{cl} implies that

$$U^{\mathrm{cl}} \subseteq (v_1 + \tfrac{1}{2}U) \cup \cdots \cup (v_n + \tfrac{1}{2}U)$$

for some $v_1, \ldots, v_n \in V$. We consider the finite-dimensional subspace $W = \operatorname{span}_{\mathbb{C}}\{v_1, \ldots, v_n\}$ which is closed, see Exercise **??**, since we assume that V is Hausdorff. Now we have $U \subseteq W + \frac{1}{2}U$ and thus $\frac{1}{2}U \subseteq \frac{1}{2}W + \frac{1}{4}U = W + \frac{1}{4}U$, since W is a subspace. This gives $U \subseteq W + \frac{1}{2}U \subseteq W + W + \frac{1}{4}U = W + \frac{1}{4}U$, again using that W is a subspace. By induction we see that

$$U \subseteq W + \frac{1}{2^n}U$$

for all $n \in \mathbb{N}$. Since U is bounded we know that for every open neighbourhood O of $0 \in V$ we have $U \subseteq 2^n O$ for large enough $n \in \mathbb{N}$. Hence $\frac{1}{2^n}U \subseteq O$ shows that the open sets $\{\frac{1}{2^n}U\}_{n \in \mathbb{N}}$ form a basis of open neighbourhoods of 0. Thus

$$\bigcap_{n=1}^{\infty}\left(W + \frac{1}{2^n}U\right) = W^{\mathrm{cl}} = W$$

follows, see also Exercise 2.5.10, and we get $U \subseteq W^{\mathrm{cl}} = W$. Hence $V \subseteq W$ since the sets nU cover V. But this means that V is finite-dimensional, showing the first part. For the second, let $U \subseteq V$ be a bounded neighbourhood of 0. By Proposition 2.4.14 also U^{cl} is bounded and hence compact by the Heine-Borel property. By the first part, V is finite-dimensional. The third part is a particular case as here the unit ball $\mathrm{B}_1(0)$ is a bounded neighbourhood of 0. □

Corollary 2.4.18 *Let V be a Banach space.*

i.) If V is infinite-dimensional then $\mathrm{B}_1(0)^{\mathrm{cl}}$ is non-compact.

ii.) If V is infinite-dimensional then V does not have the Heine-Borel property.

Proof: If $\mathrm{B}_1(0)^{\mathrm{cl}}$ would be compact then by Theorem 2.4.17, *i.)*, we have $\dim V < \infty$. Then the second part is clear as well. □

In Proposition 6.1.5 we will give an alternative proof of this statement in a different context. In the exercises we discuss examples of infinite-dimensional topological vector spaces which have the Heine-Borel property, see Exercise 2.5.48.

The last statement on bounded subsets we are interested in compares the original topology with the weak topology: it will typically give a good criterion to handle boundedness. As preparation we need the following statements also known as the Bipolar Theorem. We formulate it only for a particular case.

Theorem 2.4.19 (Bipolar Theorem) *Let V be a Hausdorff locally convex space and let p be a continuous seminorm on V. Then for all $r > 0$ the bipolar*

$$\mathrm{B}_{\mathrm{p},r}(0)^{\circ\circ} = \left\{ v \in V \;\middle|\; |\varphi(v)| \leq 1 \textit{ for all } \varphi \in \mathrm{B}_{\mathrm{p},r}(0)^{\circ} \right\} \tag{2.4.11}$$

of the open r-ball with respect to p is the closed r-ball

$$\mathrm{B}_{\mathrm{p},r}(0)^{\circ\circ} = \mathrm{B}_{\mathrm{p},r}(0)^{\mathrm{cl}}. \tag{2.4.12}$$

PROOF: First recall that the polar of $\mathrm{B}_{\mathrm{p},r}(0)$ consists of all those $\varphi \in V'$ with $|\varphi(v)| \leq 1$ for $v \in \mathrm{B}_{\mathrm{p},r}(0)$, see Remark 2.3.33. Moreover, the condition $|\varphi(v)| \leq 1$ for a given $\varphi \in V'$ gives a weakly closed subset of v's and hence $\mathrm{B}_{\mathrm{p},r}(0)^{\circ\circ}$ is weakly closed. Since the weak topology is coarser than the original one we see that $\mathrm{B}_{\mathrm{p},r}(0)^{\circ\circ}$ is closed with respect to the original topology and thus we have the inclusion $\mathrm{B}_{\mathrm{p},r}(0)^{\mathrm{cl}} \subseteq \mathrm{B}_{\mathrm{p},r}(0)^{\circ\circ}$. Thus assume that we have a $v_0 \in \mathrm{B}_{\mathrm{p},r}(0)^{\circ\circ}$ not in $\mathrm{B}_{\mathrm{p},r}(0)^{\mathrm{cl}}$. By Corollary 2.4.7 applied to the absolutely convex closed subset $\mathrm{B}_{\mathrm{p},r}(0)^{\mathrm{cl}}$ we find a $\varphi \in V'$ with $|\varphi(v)| \leq 1$ for all $v \in \mathrm{B}_{\mathrm{p},r}(0)^{\mathrm{cl}}$ but $\varphi(v_0) > 1$. Hence $\varphi \in \mathrm{B}_{\mathrm{p},r}(0)^{\circ}$ and we have a contradiction to $v_0 \in \mathrm{B}_{\mathrm{p},r}(0)^{\circ\circ}$. Therefore such a v_0 can not exist and we have (2.4.12). □

A sightly more general formulation of the Bipolar Theorem can be found in [26, Thm. 8.2.2]. We use the Bipolar Theorem in combination with the Banach-Alaoglu Theorem to prove the following statement on weakly bounded subsets:

Theorem 2.4.20 (Boundedness and weak boundedness) *Let V be a Hausdorff locally convex space and let $B \subseteq V$ be a subset. Then B is bounded iff B is weakly bounded.*

PROOF: Clearly, a bounded subset is weakly bounded by Example 2.4.15, *iii.)*. Thus assume that B is weakly bounded and let p be a continuous seminorm on V. Then we consider the polar $K = \mathrm{B}_{\mathrm{p},1}(0)^{\circ} \subseteq V'$ of the open unit p-ball in V. By the Banach-Alaoglu Theorem 2.3.32 this is a weakly* compact subset of V'. Being a polar, K is convex, too. Since B is weakly bounded we find a $c_\varphi > 0$ for every $\varphi \in V'$ such that

$$|\varphi(v)| \leq c_\varphi \tag{$*$}$$

for all $v \in B$. Via the natural pairing we can view $v \in B$ as weakly* continuous functional $\iota(v)\colon V' \longrightarrow \mathbb{C}$ as usual. In a first step we want to show that

$$\sup_{\substack{\varphi \in K \\ v \in B}} |\varphi(v)| < \infty, \tag{$**$}$$

i.e. the optimal constants c_φ in $(*)$ are bounded as φ ranges over the weakly* compact set K. For $v \in B$ we take the intersection of all the weakly* closed balls

$$A = \bigcap_{v \in B} \mathrm{B}_{\mathrm{p}_v,1}(0)^{\mathrm{cl}} = \bigcap_{v \in B} \left\{ \varphi \in V' \;\middle|\; \mathrm{p}_v(\varphi) = |\varphi(v)| \leq 1 \right\},$$

which gives a weakly* closed subset of V'. Since for all $v \in B$ we have $(*)$ we get for $\varphi \in K$

$$\varphi \in \bigcap_{v \in B} \mathrm{B}_{\mathrm{p}_v,c_\varphi}(0)^{\mathrm{cl}} = c_\varphi \bigcap_{v \in B} \mathrm{B}_{\mathrm{p}_v,1}(0)^{\mathrm{cl}} = c_\varphi A.$$

Thus for every $\varphi \in K$ we have a $n \in \mathbb{N}$ with $\varphi \in nA$ by taking $n \geq c_\varphi$. Hence K can be written as a countable union

$$K = \bigcup_{n=1}^{\infty} (K \cap nA)$$

of closed subsets. Since K is compact and hence locally compact in the weak* topology we can apply Baire's Theorem in the version of Theorem **??** to this situation: we find at least one n_0 with $K \cap n_0 A$ having a non-empty open interior (in the relative topology of K). Thus let $\varphi_0 \in K \cap n_0 A$ be an interior point. Then we get a weakly* open neighbourhood of φ_0 whose intersection with K is contained in $K \cap n_0 A$. Without restriction we can choose this neighbourhood to be of the form $\varphi_0 + U$ with

$$U = \mathrm{B}_{\mathrm{p}_{w_1},1}(0) \cap \cdots \cap \mathrm{B}_{\mathrm{p}_{w_N},1}(0)$$

for some suitably chosen $w_1, \ldots, w_N \in V$ since the open balls $\mathrm{B}_{\mathrm{p}_w,1}(0)$ form a subbasis of open neighbourhoods of 0 for the weak* topology. Since K is compact, it is also bounded by Example 2.4.15, *ii.)*, with respect to the weak* topology. Thus also $K - \varphi_0$ is bounded and we find a $r > 0$ with $K - \varphi_0 \subseteq rU$. Without restriction we can take $r > 1$. This allows to form the convex combination $\psi = (1 - \frac{1}{r})\varphi_0 + \frac{1}{r}\varphi$ for any other $\varphi \in K$ which yields again a point $\psi \in K$ by convexity. Thus we get $\psi - \varphi_0 = \frac{1}{r}(\varphi - \varphi_0) \in \frac{1}{r}(K - \varphi_0) \subseteq U$ by our choice of r. By the choice of U this means

$$\psi = \varphi_0 + \frac{1}{r}(\varphi - \varphi_0) \in \varphi_0 + U \subseteq K \cap n_0 A,$$

i.e. $|\psi(v)| \leq n_0$ for all $v \in B$. Together with $|\varphi_0(v)| \leq n_0$ we get for all $v \in B$ the estimate

$$|\varphi(v)| = \big|r\psi(v) - r\varphi_0(v) + \varphi_0(v)\big| \leq rn_0 + (r-1)n_0 = c,$$

with c independent of φ. Since $\varphi \in K$ was arbitrary this shows $(**)$ with the supremum being $\leq c$. Now the second step is very simple: for all $v \in B$ we have $|\varphi(\frac{1}{c}v)| \leq 1$ for all $\varphi \in \mathrm{B}_{\mathrm{p},1}(0)^\circ$ and thus $\frac{1}{c}v \in \mathrm{B}_{\mathrm{p},1}(0)^{\circ\circ} = \mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ by the Bipolar Theorem 2.4.19. Thus $B \subseteq c\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}} \subseteq (c+1)\mathrm{B}_{\mathrm{p},1}(0)$ shows that B is bounded in the original topology. □

Since it is usually much easier to test the boundedness in the weak topology this theorem gives an efficient tool to get bounded subsets in Hausdorff locally convex spaces. One of the most important application of this is the characterization of weakly holomorphic functions, see Section B.6.

2.4.3 Extreme Points of Convex Subsets

In this subsection we continue our study of geometric properties of locally convex spaces with the main focus on the geometry of convex subsets. We first recall some geometric definitions:

Definition 2.4.21 (Convex hull and extreme points) *Let $K \subseteq V$ be a subset of a vector space.*

i.) The convex hull of K is the subset

$$\operatorname{conv}(K) = \big\{v = \lambda_1 v_1 + \cdots + \lambda_n v_n \;\big|\; n \in \mathbb{N}, v_i \in K, \lambda_i \geq 0, \lambda_1 + \cdots + \lambda_n = 1\big\}. \tag{2.4.13}$$

ii.) A point $v \in K$ is called extreme if $v = \lambda v_1 + (1-\lambda)v_2$ with $\lambda \in (0,1)$ and $v_1, v_2 \in K$ implies $v_1 = v_2$. The set of extreme points of K is denoted by extreme(K).

With other words, $\operatorname{conv}(K)$ is the smallest convex subset of V which contains K. It can also be obtained as the intersection of all convex subsets containing K, see Exercise 2.5.3. Extreme points in a convex subset can not be decomposed as convex combinations in a non-trivial way. We have already encountered extreme points of a convex subset:

Remark 2.4.22 Let $\mathscr{A}$ be a unital *-algebra. Then the states form a convex subset of the algebraic dual $\mathscr{A}^*$ and the pure states are precisely the extreme points of it, see Definition 1.2.4.

In general, a convex subset of a vector space does not have any extreme points, examples can be found already in $\mathbb{R}^n$ very easily, see Exercise 2.5.49. In a topological vector space, extreme points are necessarily boundary points:

Proposition 2.4.23 *Let V be a topological vector space and let $K \subseteq V$ be a convex subset. If $v \in K$ is extreme then $v \in \partial K$, i.e.*

$$\operatorname{extreme}(K) \subseteq \partial K. \tag{2.4.14}$$

PROOF: Let $v \in K^\circ$ be an interior point of K. By the continuity of the vector space operations one finds a $\epsilon > 0$ such that for all $|\lambda| < \epsilon$ we have $(1+\lambda)v \in K^\circ$, too. But then for such a $\lambda \neq 0$ we have

$$v = \tfrac{1}{2}(1+\lambda)v + \tfrac{1}{2}(1-\lambda)v,$$

which is a non-trivial convex combination since $(1-\lambda)v \neq (1+\lambda)v$. Thus v can not be extreme. □

So if we want to find extreme points we should look for *closed* convex subsets. However, even if K is closed and hence the whole boundary belongs to K, there might be no extreme points: V itself is closed but has no extreme points, see also Exercise 2.5.49 for more examples. The famous Krein-Milman Theorem shows that for a *compact* convex subset we have many extreme points: in fact they span the whole convex subset up to closure:

Theorem 2.4.24 (Krein-Milman, I) *Let V be a topological vector space such that V' separates points. Then for a non-empty compact convex subset $K \subseteq V$ one has*

$$K = (\operatorname{conv}(\operatorname{extreme}(K)))^{\mathrm{cl}}. \tag{2.4.15}$$

PROOF: We consider the following set of all compact extreme subsets of K

$$\mathcal{P} = \big\{A \subseteq K \;\big|\; A \text{ non-empty, compact and extreme}\big\},$$

where A is called *extreme* if $\lambda v + (1-\lambda)w \in A$ for $v, w \in K$ and $\lambda \in (0,1)$ implies $v, w \in A$. Clearly, $K \in \mathcal{P}$ and thus $\mathcal{P}$ is non-empty. We establish now two properties of the extreme subsets of K:

First, we claim that for any collection $\{A_i\}_{i \in I}$ of sets $A_i \in \mathcal{P}$ their intersection $A = \bigcap_{i \in I} A_i$ is either empty or again in $\mathcal{P}$. Indeed, assume that A is non-empty. Then the compactness of A is clear as it is a closed subset of a compact subset in a Hausdorff situation (since V' separates points). Now let $v, w \in K$ and $\lambda \in (0,1)$ with $\lambda v + (1-\lambda)w \in A$. Then also $\lambda v + (1-\lambda)w \in A_i$ for all $i \in I$ and hence $v, w \in A_i$ by $A_i \in \mathcal{P}$.

The second property of $\mathcal{P}$ is more subtle: we claim that for $A \in \mathcal{P}$ and $\varphi \in V'$ the subset

$$A_\varphi = \big\{v \in A \;\big|\; \operatorname{Re}\varphi(v) = m_A = \max\nolimits_{w \in A} \operatorname{Re}\varphi(w)\big\} \tag{$*$}$$

belongs to $\mathcal{P}$ again, where we note that the maximum m_A is well-defined by the compactness of A and hence $A_\varphi \neq \emptyset$. To prove this second claim, let $v, w \in K$ and $\lambda \in (0,1)$ be given such that $\lambda v + (1-\lambda)w \in A_\varphi$. Since $A_\varphi \subseteq A$ we get $v, w \in A$ by $A \in \mathcal{P}$. Thus $\operatorname{Re}\varphi(v), \operatorname{Re}\varphi(w) \leq m_A$. By assumption we have

$$m_A = \operatorname{Re}\varphi(\lambda v + (1-\lambda)w) = \lambda \operatorname{Re}\varphi(v) + (1-\lambda)\operatorname{Re}\varphi(w).$$

Together, this gives $\operatorname{Re}\varphi(v) = \operatorname{Re}\varphi(w) = m_A$. But then $v, w \in A_\varphi$ by definition and the second claim follows.

Let now $A \in \mathcal{P}$ be given. Then we denote by $\mathcal{P}_A \subseteq \mathcal{P}$ those elements of $\mathcal{P}$ which are subsets of A. Clearly, $A \in \mathcal{P}_A$ and hence $\mathcal{P}_A$ is non-empty. We can partially order $\mathcal{P}_A$ by inclusion. Now let $\{A_i\}_{i \in I} \subseteq \mathcal{P}_A$ be a (decreasing) totally ordered subset. Then we consider the intersection

$$B = \bigcap_{i \in I} A_i.$$

Since $A_i \subseteq A_j$ for $i \succcurlyeq j$ we see that for finitely many indices the intersection $A_{i_1} \cap \cdots \cap A_{i_r}$ is just A_{i_r} if $i_r \succcurlyeq i_{r-1} \succcurlyeq \cdots \succcurlyeq i_1$. Hence the finite intersection is non-empty. Next, $A \setminus A_i = B_i$ is an open subset of A. If $B = \emptyset$ then the B_i would cover A and thus, by compactness, already finitely many, say $B_{i_1}, \ldots, B_{i_r}$ with $i_r \succcurlyeq \cdots \succcurlyeq i_1$ would cover A. But

$$B_{i_1} \cup \cdots \cup B_{i_r} = (A \setminus A_{i_1}) \cup \cdots \cup (A \setminus A_{i_r}) = A \setminus (A_{i_1} \cap \cdots \cap A_{i_r}) = A \setminus A_{i_r}$$

can not be A since we have $A_{i_r} \neq \emptyset$. Thus $B \neq \emptyset$ follows. By our first claim about $\mathcal{P}$ we conclude that $B \in \mathcal{P}$ again. Since by construction $B \subseteq A$ we even have $B \in \mathcal{P}_A$. It is clear that B is the infimum for the decreasing totally ordered subset $\{A_i\}_{i \in I}$. Thus we can apply Zorn's Lemma to $\mathcal{P}_A$ and conclude that $\mathcal{P}_A$ has minimal elements. Let B be such a minimal element. Then B can not have any proper subset also belonging to $\mathcal{P}_A$ by minimality. Thus for all $\varphi \in V'$ the subsets B_φ coincide with B, i.e. for all $v \in B$ we already have $\operatorname{Re} \varphi(v) = \max_{w \in B} \operatorname{Re} \varphi(w)$. In particular, $\operatorname{Re} \varphi$ is constant on B. Replacing φ by $\mathrm{i}\varphi$ shows that φ itself is constant on B. Since we assume that V' separates points this means that B consists of a single point. Thus this point is an extreme point and we have shown that $\operatorname{extreme}(K) \neq \emptyset$.

Actually, we have shown more: every extreme subset $A \in \mathcal{P}$ of K contains extreme points since $B \subseteq A$ for the above B, i.e.

$$\operatorname{extreme}(K) \cap A \neq \emptyset \tag{☺}$$

for all $A \in \mathcal{P}$. Now K is convex and thus $\operatorname{conv}(\operatorname{extreme}(K)) \subseteq \operatorname{conv}(K) = K$. Furthermore, K is closed as V is Hausdorff. Thus

$$\operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}} \subseteq K^{\mathrm{cl}} = K.$$

This shows that $\operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}}$ is a compact subset itself. Since the closure of a convex subset is again convex we can now use the separation theorem to show that we have equality in (2.4.15): Assume the converse and let $v_0 \in K$ be a point not in $\operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}}$. Then apply Corollary 2.4.8 to the two compact subsets $\{v_0\}$ and $\operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}}$. Thus we get a $\varphi \in V'$ with

$$\sup_{v \in \operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}}} \operatorname{Re} \varphi(v) < \operatorname{Re} \varphi(v_0). \tag{$**$}$$

We take this φ and consider K_φ as defined in $(*)$. We know that $K_\varphi \in \mathcal{P}$. From $(**)$ we see that the maximum of $\operatorname{Re} \varphi$ on K is strictly larger than the values of $\operatorname{Re} \varphi$ on $\operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}}$. Hence we conclude that

$$K_\varphi \cap \operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}} = \emptyset, \tag{$\star$}$$

as K_φ consists of those points realizing the maximum. But $(\star)$ contradicts (☺). Hence we have (2.4.15). ☐

Remark 2.4.25 (Krein-Milman Theorem) The importance of the Krein-Milman Theorem lies, first of all, in the fact that compact convex subsets do have extreme points at all, i.e. $\operatorname{extreme}(K) \neq \emptyset$ for $K \neq \emptyset$. Second, it shows that we even have *many* extreme points, enough to recover K from convex combinations of the extreme points. Note however, that in general $\operatorname{conv}(\operatorname{extreme}(K))$ is not yet all of K, the topological closure will be necessary. It will also be necessary to take the closure and not just the sequential closure unless we are in a first countable situation, see also Exercise C.6.10.

Most of the effort in proving the Krein-Milman Theorem was due to showing that the convex and closed subset $\operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}}$ is compact again, in order to apply Corollary 2.4.8. If the ambient space V is even Hausdorff and locally convex we can alternatively argue with Theorem 2.4.4, leading to the following version of the Krein-Milman Theorem:

Theorem 2.4.26 (Krein-Milman, II) *Let V be a Hausdorff locally convex space and $K \subseteq V$ a non-empty compact subset. Then*

$$K \subseteq \operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}}. \tag{2.4.16}$$

Proof: Note that we do not assume that K is convex here, the notion of extreme points still makes sense. Since we are in a locally convex situation and $\operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}}$ is closed as well as convex we can apply Theorem 2.4.4 to the compact convex subset $\{v_0\}$ for $v_0 \in K \setminus \operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}}$ and the closed convex subset $\operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}}$, assuming that (2.4.16) is not true. This gives us a continuous linear functional $\varphi \in V'$ separating via its real part the two subsets: this yields the same statement $(**)$ as in the proof of Theorem 2.4.24. Note that for the definition and the properties of $\mathcal{P}$ of compact extreme subsets of K we did not need the convexity of K yet. Thus we can proceed from here as in the previous proof from $(**)$ onward. □

Again, this shows that a compact subset has already a lot of extreme points: they generate the same closed convex subsets as K itself, i.e. (2.4.16) gives immediately

$$\operatorname{conv}(K)^{\mathrm{cl}} = \operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}}, \tag{2.4.17}$$

since the right hand side is both: convex and closed. Before investigating the compactness of of $\operatorname{conv}(K)^{\mathrm{cl}}$ we need a better understanding of the convex hull of a union of compact convex subsets, which is also of independent interest:

Proposition 2.4.27 *Let V be a topological vector space with compact convex subsets $K_1, \ldots, K_n \subseteq V$. Then the convex hull $\operatorname{conv}(K_1 \cup \cdots \cup K_n)$ is again compact.*

Proof: Since each of the subsets $K_1, \ldots, K_n$ is assumed to be convex, the convex hull of their union is obtained from convex combinations as

$$\operatorname{conv}(K_1 \cup \cdots \cup K_n) = \left\{ \sum_{i=1}^{n} \lambda_i v_i \;\middle|\; v_i \in K_i \text{ and } \lambda_i \geq 0 \text{ for all } i = 1, \ldots, n \text{ with } \lambda_1 + \cdots + \lambda_n = 1 \right\}.$$

Consider now the map

$$\psi \colon V^n \times \Delta_n \longrightarrow V,$$

defined by

$$\psi(v_1, \ldots, v_n, \lambda) = \sum_{i=1}^{n} \lambda_i v_i,$$

where

$$\Delta_n = \left\{ \lambda \in \mathbb{R}^n \;\middle|\; \lambda_i \geq 0 \text{ for all } i = 1, \ldots, n \text{ with } \lambda_1 + \cdots + \lambda_n = 1 \right\}$$

is the standard simplex in $\mathbb{R}^n$. Clearly, $\Delta_n \subseteq \mathbb{R}^n$ is compact and

$$\operatorname{conv}(K_1 \cup \cdots \cup K_n) = \psi\big(K_1 \times \cdots \times K_n \times \Delta_n\big).$$

Hence the convex hull is the image of a compact subset under the obviously continuous map ψ, thus compact itself. □

In general, it might happen that $\operatorname{conv}(K)^{\mathrm{cl}}$ is no longer compact for a compact subset $K \subseteq V$. If one can guarantee that $\operatorname{conv}(K)^{\mathrm{cl}}$ is compact then one gets not more extreme points for $\operatorname{conv}(K)^{\mathrm{cl}}$ than for K itself. This is the statement of Milman's Theorem:

Theorem 2.4.28 (Milman) *Let V be a Hausdorff locally convex space and let $K \subseteq V$ be a compact subset such that $\operatorname{conv}(K)^{\mathrm{cl}}$ is still compact. Then*

$$\operatorname{extreme}(\operatorname{conv}(K)^{\mathrm{cl}}) \subseteq K. \tag{2.4.18}$$

PROOF: Assume $v_0 \in \operatorname{extreme}(\operatorname{conv}(K)^{\mathrm{cl}})$ is an extreme point not contained in K, i.e. $v_0 \notin K$. Since V is Hausdorff we can separate v_0 from K by an absolutely convex zero neighbourhood $U \subseteq V$ via

$$(v_0 + U^{\mathrm{cl}}) \cap K = \emptyset,$$

according to Proposition 2.4.3 and the fact that in a locally convex space we have a basis of closed absolutely convex zero neighbourhoods. The compactness of K gives finitely many points $v_1, \ldots, v_n \in K$ such that $K \subseteq (v_1 + U) \cup \cdots \cup (v_n + U)$. Consider now the subsets

$$A_i = \operatorname{conv}(K \cap (v_i + U))^{\mathrm{cl}} \subseteq \operatorname{conv}(K)^{\mathrm{cl}}.$$

By construction, they are closed and stay convex. Since $A_i \subseteq \operatorname{conv}(K)^{\mathrm{cl}}$, these subsets are compact. Finally, we have

$$K = (K \cap (v_1 + U)) \cup \cdots \cup (K \cap (v_n + U)) \subseteq A_1 \cup \cdots \cup A_n.$$

Thus taking the convex hull and then the closure gives

$$\operatorname{conv}(K)^{\mathrm{cl}} \subseteq \operatorname{conv}(A_1 \cup \cdots \cup A_n)^{\mathrm{cl}} = \operatorname{conv}(A_1 \cup \cdots \cup A_n),$$

where the last equality holds by the compactness of the subsets $A_1, \ldots, A_n$ and Proposition 2.4.27: in the Hausdorff case, the compact subset $\operatorname{conv}(A_1 \cup \cdots \cup A_n)$ is necessarily closed. Conversely, $A_i \subseteq \operatorname{conv}(K)^{\mathrm{cl}}$ shows that we have the equality

$$\operatorname{conv}(K)^{\mathrm{cl}} = \operatorname{conv}(A_1 \cup \cdots \cup A_n).$$

Thus for v_0 we find a convex combination

$$v_0 = \lambda_1 w_1 + \cdots + \lambda_n w_n$$

for some $w_i \in A_i$, $\lambda_i \geq 0$ and $\lambda_1 + \cdots + \lambda_n = 1$. Without restriction, we can assume $\lambda_2 + \cdots + \lambda_n > 0$ and hence

$$v_0 = \lambda_1 w_1 + (1 - \lambda_1) \frac{\lambda_2 w_2 + \cdots + \lambda_n w_n}{\lambda_2 + \cdots + \lambda_n} = \lambda_1 w_1 + (1 - \lambda_1) w,$$

where w is a convex combination of $w_2, \ldots, w_n$ and hence $w \in \operatorname{conv}(K)^{\mathrm{cl}}$. By assumption, v_0 is an extreme point of $\operatorname{conv}(K)^{\mathrm{cl}}$. Thus either $v_0 = w_1$ or $v_0 = w$. If $v_0 = w$, we can repeat the argument with $w_2, \ldots, w_n$ instead of $w_1, \ldots, w_n$ and ultimately find an index $i = 1, \ldots, n$ with $v_0 = w_i \in A_i$. Since U^{cl} is (absolutely) convex, we note that $A_i \subseteq v_i + U^{\mathrm{cl}}$ and thus $v_0 \in v_i + U^{\mathrm{cl}} \subseteq K + U^{\mathrm{cl}}$. Since $U^{\mathrm{cl}} = -U^{\mathrm{cl}}$ by assumption, we see that $v_0 + U^{\mathrm{cl}} \cap K \neq \emptyset$, a contradiction. □

2.5 Exercises

Exercise 2.5.1 (Open neighbourhoods of 0**)** Let V be a topological vector space and let $U \subseteq V$ be a neighbourhood of $0 \in V$.

i.) Show that for every sequence $r_n > 0$ converging to $+\infty$, one has

$$V = \bigcup_{n=0}^{\infty} r_n U. \tag{2.5.1}$$

Hint: Use the continuity of $\mathbb{C} \ni z \mapsto zv \in V$ for a fixed $v \in V$ to conclude that the set of complex numbers z with $zv \in U$ is open and contains 0.

ii.) Show that $+: V \times V \longrightarrow V$ is an open map.

Hint: Write $U + W = \bigcup_{x \in W}(U + x)$.

iii.) Show that there is a neighbourhood $O \subseteq U$ of 0 such that the neighbourhood $O - O$ is contained in U. Why is $O - O$ still a neighbourhood of 0?

Hint: Consider the continuity of the map $V \times V \ni (v, w) \mapsto v - w \in V$.

Exercise 2.5.2 (Operations with seminorms) Let V be a vector space and let $\mathrm{p}, \mathrm{p}_1, \ldots, \mathrm{p}_N$ be seminorms on V. Moreover, let $\alpha \geq 0$ and $p, p' \geq 1$.

i.) Show that $\alpha\mathrm{p}$ is again a seminorm.

ii.) Show that $\mathrm{q}_{\max} = \max(\mathrm{p}_1, \ldots, \mathrm{p}_N)$ is again a seminorm.

iii.) Show that $\mathrm{p}_1 + \cdots + \mathrm{p}_N$ is again a seminorm.

iv.) Show that $\mathrm{q}_p = \sqrt[p]{p_1^p + \cdots \mathrm{p}_N^p}$ is again a seminorm.

v.) Find explicit estimates between $\mathrm{q}_{\max}$ and q_p as well as $\mathrm{q}_{p'}$.

vi.) Describe the kernels of $\mathrm{q}_{\max}$ and q_p using the kernels of $\mathrm{p}_1, \ldots, \mathrm{p}_N$.

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Exercise 2.5.3 (Convex subsets: basics) Let V be a vector space.

i.) Show that an arbitrary intersection of convex subsets is again convex.

ii.) Show that the convex hull of an arbitrary subset $K \subseteq V$ is the intersection of all convex subsets containing K.

iii.) Suppose in addition that V is a topological vector space. Show that the closure as well as the sequential closure of a convex subset is again convex.

iv.) Let $K, K' \subseteq V$ be convex and $z, w \in \mathbb{C}$. Show that $zK + wK'$ is again convex.

Exercise 2.5.4 (Circled and convex subsets) Let V be a topological vector space. Recall that U is called *absorbing* if for every $v \in V$ there is a $\lambda \in \mathbb{R}^+$ with $\lambda v \in U$. Moreover, U is called *circled* (or *balanced*) if for $v \in U$ also $zv \in U$ for all $z \in \mathbb{C}$ with $|z| \leq 1$.

i.) Show that every neighbourhood of 0 contains a circled neighbourhood. Conclude that V has a neighbourhood basis of 0 consisting of circled neighbourhoods.

ii.) Show that the closure as well as the open interior (if non-empty) of a convex or a circled subset is again convex or circled, respectively.

iii.) Show that the convex closure $\operatorname{conv}(U)$ of an open subset U is open.

iv.) Show that the circled closure U, i.e. the smallest circled subset containing U, of an open subset is again open.

v.) Show that every neighbourhood of 0 is absorbing.

vi.) Show that a circled subset of $\mathbb{C}$ different from $\mathbb{C}$ is necessarily bounded.

Exercise 2.5.5 (Continuous linear functionals) Consider a topological vector space V and a linear functional $\varphi: V \longrightarrow \mathbb{C}$. Show that φ is continuous iff φ is bounded on a neighbourhood of 0 iff $\ker \varphi \subseteq V$ is a closed subspace. Interpret this result for a locally convex space and compare with (2.2.10).

Hint: Assume φ is bounded on a neighbourhood $U \subseteq V$ of 0 by a bound $C > 0$. Show that $v \in \epsilon U$ implies $|\varphi(v)| \leq \epsilon C$ in this case. Deduce that φ is continuous at 0 by e.g. using convergent nets. If $\ker \varphi$ is closed and different from V, take a point $u \in V \setminus \ker \varphi$. Find a circled open neighbourhood $U \subseteq V$ of 0 such that $u + U \subseteq V \setminus \ker \varphi$. Show that φ is bounded on U by using Exercise 2.5.4, *vi.)*.

Exercise 2.5.6 (Convergent nets are Cauchy nets) Let V be a topological vector space. Show that a convergent net $(v_i)_{i \in I}$ in V is a Cauchy net.

Exercise 2.5.7 (Completion is functorial) Formulate and prove the statement that completion yields a functor $\mathsf{LCVect} \longrightarrow \mathsf{CLCVect}$.

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Exercise 2.5.8 (Closed subspaces) Let V be a Hausdorff topological vector space, not necessarily complete. Show that a complete subspace $U \subseteq V$ is necessarily closed.

Exercise 2.5.9 (Finite-dimensional topological vector spaces) Let V be a Hausdorff topological vector space of finite dimension $\dim V = n$.

i.) Let a topology on $\mathbb{C}$ be given which makes it a topological vector space. Show that either the topology is the indiscrete one or the canonical Hausdorff one.

ii.) Endow $\mathbb{C}^n$ with its canonical topology. Show that every linear isomorphism $\Phi\colon \mathbb{C}^n \longrightarrow V$ is a homeomorphism.

Hint: First use the basis vectors $v_i = \Phi(\mathrm{e}_i)$ of V and the very definition of a topological vector space to show that Φ is continuous. The more tricky part is to show that Φ^{-1} is continuous as well: the sport is to avoid the usage of the Open Mapping Theorem or alike as here we need some additional assumptions on the topology of V. Proceed by induction on the dimension n. The case $n = 0$ is clear. Then consider an arbitrary $n-1$-dimensional subspace U of V. By induction, U and $\mathbb{C}^{n-1}$ are homeomorphic via Φ. Use Exercise 2.5.8 to conclude that U is closed. Use Exercise 2.5.5 to conclude that the corresponding linear functional $\varphi\colon V \longrightarrow \mathbb{C}$ with $\ker \varphi = U$ is continuous. Deduce from this that the inverse of Φ is continuous, too.

iii.) Simplify the above proof drastically for the case where V is in addition a Hausdorff locally convex space.

iv.) Conclude that a finite-dimensional subspace of a Hausdorff topological vector space is closed by using Exercise 2.5.8.

Exercise 2.5.10 (Closure of a subset) Let V be a topological vector space and $W \subseteq V$ an arbitrary subset. Consider the open neighbourhood $W + U$ of W for an open neighbourhood U of $0 \in V$. Prove that $\bigcap_U (W + U) = W^{\mathrm{cl}}$ where the intersection is taken over a basis of open neighbourhoods of 0.

Exercise 2.5.11 (Existence of norms) Let V be a vector space. Show that there exists a norm on V.

Exercise 2.5.12 (Open and closed balls) Let V be a locally convex space and let p be a continuous seminorm on V.

i.) Show that the open interior of the closed ball $\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ as in (2.2.3) is given by the open ball $\mathrm{B}_{\mathrm{p},1}(0)$.

Hint: Suppose there is an interior point $v \in \mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ which is not in $\mathrm{B}_{\mathrm{p},1}(0)$, i.e. $\mathrm{p}(v) = 1$. Use the continuity of the multiplication with scalars to find a contradiction.

ii.) Show that $\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ is indeed the topological closure of $\mathrm{B}_{\mathrm{p},1}(0)$ and prove that it coincides with the sequential closure.

iii.) Show that the topological boundary of the open ball $\mathrm{B}_{\mathrm{p},1}(0)$ is the sphere

$$\mathbb{S}_{\mathrm{p},1} = \big\{ v \in V \bigm| \mathrm{p}(v) = 1 \big\}. \tag{2.5.2}$$

Exercise 2.5.13 (The Minkowski functional) Let V be a vector space and $U \subseteq V$ a subset.

i.) Prove that the *Minkowski functional* p_U of a convex and absorbing $U \subseteq V$, defined by

$$\mathrm{p}_U(v) = \inf\big\{\lambda > 0 \bigm| v \in \lambda U\big\}, \tag{2.5.3}$$

is sublinear.

ii.) Show that p_U is a seminorm if U is in addition circled.

iii.) Now let p be a seminorm on V and consider its closed unit ball $U = \mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$. Show that U is convex, absorbing and circled. Compute the Minkowski functional p_U.

e: Minkowski and relation $\subseteq \mathrm{B}_{\mathrm{p}_U,1}(0)^c l$ $U \subseteq \mathrm{B}_{\mathrm{p}_U,1}(0)$ $,1(0)^{\mathrm{cl}} \subseteq U^{\mathrm{cl}}$. y when p_U is continuous.

Exercise 2.5.14 (Characterization of locally convex spaces) Let V be a topological vector space. Show that V is locally convex iff V has a neighbourhood basis of 0 consisting of convex and circled subsets iff V has a neighbourhood basis of 0 consisting of closed, convex, and circled subsets.

Hint: Exercise 2.5.4 together with Exercise 2.5.13 may be useful.

Exercise 2.5.15 (Linear functionals are open) Let V be a topological vector space and let $\varphi\colon V \longrightarrow \mathbb{C}$ be a linear functional (not necessarily continuous). Show that φ is open iff $\varphi \neq 0$.

Hint: Consider $\varphi \neq 0$ and let $U \subseteq V$ be an open neighbourhood of 0. Show that there is a $v \in U$ with $\varphi(v) \neq 0$. Next, show that for all $z \in \mathbb{C}$ with $|z - 1|$ small enough also $z\varphi(v)$ is in the image of U. Why is this already sufficient?

Exercise 2.5.16 (Completeness and sequential completeness) Let V be a first countable topological vector space. Show that V is complete iff V is sequentially complete.

Hint: Assume V is sequentially complete and fix a countable basis $U_n \subseteq V$ of neighbourhoods of 0. Let $(v_i)_{i\in I}$ be a Cauchy net. Define now a sequence $(v_{i_n})_{n\in\mathbb{N}}$ inductively by choosing i_1 arbitrary and i_n such that $v_{i_{n-1}} - v_i \in U_n$ for all $i \succcurlyeq i_n$. Show that this gives a Cauchy sequence. By assumption it converges to some $v \in V$. Show that also $v_i \longrightarrow v$.

Exercise 2.5.17 (Separation properties of locally convex spaces) Let V be a locally convex space. Show that the following statements are equivalent:

i.) V is Hausdorff.

ii.) V is T_1.

iii.) V is T_3.

iv.) V is regular.

In fact, the above statements are still correct for a general topological vector space, see also Exercise 2.5.24.

Exercise 2.5.18 (A metric for a Fréchet space) Let V be a locally convex space with a defining sequence of seminorms $\mathrm{p}_1 \leq \mathrm{p}_2 \leq \cdots$. Define for $v, w \in V$

$$d(v, w) = \max_{n=1}^{\infty} \lambda_n \frac{\mathrm{p}_n(v - w)}{1 + \mathrm{p}_n(v - w)}, \tag{2.5.4}$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is a fixed choice of a zero sequence with $\lambda_n > 0$.

i.) Show that d is well-defined and yields a translation invariant metric on V.

Hint: To validate the triangle inequality for d we first note that the function $f\colon \xi \mapsto \frac{\xi}{1+\xi}$ is monotonously increasing on $\mathbb{R}_0^+$ and we have $f(\alpha + \beta) \leq f(\alpha) + f(\beta)$ for all $\alpha, \beta \geq 0$.

ii.) Show that the metric balls $\mathrm{B}_{d,\epsilon}(0) = \{v \mid d(0, v) < \epsilon\} \subseteq V$ for $\epsilon > 0$ are open.

Hint: It is important that d is defined by a maximum and not just a supremum.

iii.) Show that in every open neighbourhood $U \subseteq V$ of 0 one has an open metric ball $\mathrm{B}_{d,\epsilon}(0) \subseteq U$. Conclude that d induced the same topology for V.

iv.) Show that Cauchy sequences in V coincide with metric Cauchy sequences with respect to d.

Exercise 2.5.19 (Finitely many continuous linear maps are equicontinuous) Let V and W be topological vector spaces and let $\phi_1, \ldots, \phi_n \colon V \longrightarrow W$ be finitely many continuous linear maps. Show that the collection $\{\phi_1, \ldots, \phi_n\}$ is equicontinuous.

Exercise 2.5.20 (Inductive limits)

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Exercise 2.5.21 (Strict inductive limits and LF spaces)

Exercise 2.5.22 (The weak* topology of V^*) Let V be a vector space with algebraic dual V^*.

i.) Show that V^* is a complete Hausdorff locally convex space with respect to the weak* topology.

ii.) Suppose now that V is a locally convex Hausdorff space and consider the topological dual $V' \subseteq V^*$. Show that for every linear functional $\Phi \in V^*$, for every $n \in \mathbb{N}$, and for all vectors $v_1, \ldots, v_n \in V$ there is a continuous linear functional $\varphi \in V'$ such that

$$\varphi(v_i) = \Phi(v_i) \tag{2.5.5}$$

for $i = 1, \ldots, n$.

Hint: Use an appropriate version of the Hahn-Banach Theorem.

iii.) Conclude that $V' \subseteq V^*$ is dense with respect to the weak* topology.

Since in many situations V' is sequentially complete and hence sequentially closed in V^* this illustrates drastically the difference between completeness and sequential completeness.

Exercise 2.5.23 (Locally convex quotient) Let V be a locally convex space and $U \subseteq V$ a not necessarily closed subspace.

i.) Define $[\mathrm{p}]$ by (2.2.39) and show that this is a well-defined seminorm on V/U.

ii.) Prove that the set $[\mathcal{P}]$ of all the seminorms $[\mathrm{p}]$ with p a continuous seminorm on V satisfies $\overline{[\mathcal{P}]} = [\mathcal{P}]$.

iii.) Suppose now that U is closed with respect to the seminorm p. Show that if p is a norm then $[\mathrm{p}]$ is again a norm.

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Exercise 2.5.24 (Hausdorffization) Let V be a topological vector space.

i.) Show that V is Hausdorff iff $\{0\}^{\mathrm{cl}} = \{0\}$ iff every point forms a closed subset.

ii.) Let p be a seminorm on V. Show that the kernel of p is a subspace of V.

iii.) Show that in the case where V is locally convex one has (2.2.41).

iv.) Let now $\phi \colon V \longrightarrow W$ be a continuous linear map between locally convex spaces. Show that ϕ descends to a well-defined continuous linear map $\phi \colon V/\{0\}^{\mathrm{cl}} \longrightarrow W/\{0\}^{\mathrm{cl}}$.

v.) Formulate and prove that Hausdorffization is a functor $\mathsf{lcVect} \longrightarrow \mathsf{LCVect}$.

Exercise 2.5.25 (Polars) Let $(V, W, \langle \cdot\,, \cdot \rangle)$ be a dual pair.

i.) Let $U_1 \subseteq U_2 \subseteq V$. Show that $U_2^\circ \subseteq U_1^\circ$.

ii.) Show that $U \subseteq U^{\circ\circ}$ for every subset $U \subseteq V$. Here $U^{\circ\circ} = (U^\circ)^\circ$ is the *bipolar* of U.

iii.) Show that for all subsets $U \subseteq V$ one has $U^\circ = U^{\circ\circ\circ}$. Hence taking the polar stabilizes after two steps.

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Exercise 2.5.26 (Completion of a normed space) Let V be a normed space. Denote by $\mathcal{V}$ the set of all Cauchy sequences in V.

i.) Show that $\mathcal{V}$ becomes a vector space itself and V can be included into $\mathcal{V}$ as the subspace of constant sequences. Moreover, show that the set $\mathcal{N} \subseteq \mathcal{V}$ of zero sequences is a subspace, too.

ii.) Consider the quotient space $\widehat{V} = \mathcal{V}/\mathcal{N}$. Show that

$$\|[(v_n)_{n\in\mathbb{N}}]\| = \lim_{n\longrightarrow\infty} \|v_n\| \tag{2.5.6}$$

is a well-defined norm on $\widehat{V}$.

iii.) Show that $\iota\colon V \longrightarrow \widehat{V}$ defined by $v \mapsto [(v_n)_{n\in\mathbb{N}}]$ with $v_n = v$ for all $n \in \mathbb{N}$ is an isometric linear map.

iv.) Show that $\widehat{V}$ is complete with respect to the norm (2.5.6).

v.) Show that the image of V under ι is dense in $\widehat{V}$.

This way, one has constructed a completion of the normed space V to the Banach space $\widehat{V}$, independently of the construction in Corollary 2.3.14.

Exercise 2.5.27 (Completion of a Hausdorff locally convex space) Let V be a Hausdorff locally convex space. Alternatively to the construction of the completion $\widehat{V}$ as in Proposition 2.3.17 one can construct the completion in the same spirit as the completion of a normed space outlined in Exercise 2.5.26. The only difficulty is that we need Cauchy nets instead of Cauchy sequences: of course we can not just take the "set" of all Cauchy nets in V as this is not even a set. One first has to limit the index set of the Cauchy nets in a reasonable way.

i.) Show that the system $\mathcal{V}(0)$ of open neighbourhoods of $0 \in V$ forms a directed set which one can use to index nets in V.

ii.) Argue that it is sufficient to consider Cauchy nets in V indexed by (directed) subsets I of $\mathcal{V}(0)$. Construct the completion of V along the lines of Exercise 2.5.26 for such particular Cauchy nets and show that this gives indeed a completion.

Hint: Construct for an arbitrary Cauchy net $(v_i)_{i\in I}$ a suitable subnet with index set by a suitable directed subset of $\mathcal{V}(0)$.

Exercise 2.5.28 (Bounded functions) Let X be a non-empty set. Then consider the space of bounded functions

$$\mathscr{B}(X) = \big\{f\colon X \longrightarrow \mathbb{C} \bigm| f \text{ is bounded}\big\}. \tag{2.5.7}$$

i.) Show that $\mathscr{B}(X)$ is a vector space for the pointwise operations.

ii.) Show that the supremum norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \tag{2.5.8}$$

is a norm for $\mathscr{B}(X)$.

iii.) Show that $\mathscr{B}(X)$ becomes a Banach space with respect to the supremum norm $\|\cdot\|_\infty$.

iv.) Show that the bounded functions on X coincide with the bounded continuous functions $\mathscr{C}_b(X)$ when X is equipped with the discrete topology.

Exercise 2.5.29 (The space $\mathscr{C}^k(X)$) Let $X \subseteq \mathbb{R}^d$ be a nonempty open subset and consider the set

$$\mathscr{C}^k(X) = \big\{f\colon X \longrightarrow \mathbb{C} \bigm| f \text{ is } k\text{-times continuously differentiable }\big\} \tag{2.5.9}$$

of $\mathscr{C}^k$-functions on X. Here one allows also $k = +\infty$ in which case the functions are called *smooth*. Define for every compact subset $K \subseteq X$ and every $\ell \in \mathbb{N}_0$ with $\ell \leq k$ for $f \in \mathscr{C}^k(X)$

$$\mathrm{p}_{K,\ell}(f) = \sup_{\substack{x \in K \\ \alpha\alpha \leq \ell}} |\partial^\alpha f(x)|, \tag{2.5.10}$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ is a multiindex and $\partial^\alpha f$ denotes the α-th partial derivative as usual.

i.) Show that $\mathrm{p}_{K,\ell}$ defines a system of seminorms on $\mathscr{C}^k(X)$ inducing a Hausdorff topology, where K runs through all compact subsets of X and $\ell \in \mathbb{N}_0$ satisfies $\ell \leq k$. The locally convex topology induced by this system of seminorms will be called the $\mathscr{C}^k$-topology. In the case $k = \infty$ we also call it the smooth topology.

ii.) Show that a sequence $f_n \in \mathscr{C}^k(X)$ converges to $f \in \mathscr{C}^k(X)$ iff all partial derivatives of order $\ell \in \mathbb{N}_0$ with $\ell \leq k$ converge locally uniformly to the corresponding derivative of f.

iii.) Show that there exists a sequence K_n of compact subsets of X with $K_n \subseteq K_{n+1}^\circ$ such that $X = \bigcup_{n \in \mathbb{N}} K_n$.

Hint: Consider the closed balls $\mathrm{B}_r(x)^{\mathrm{cl}} \subseteq X$ inside X with rational radius and rational $x \in \mathbb{Q}^n$ and take appropriate unions and closures.

iv.) Show that already countably many of the seminorms $\mathrm{p}_{K,\ell}$ suffice to define the topology.

v.) Show that $\mathscr{C}^k(X)$ is complete with respect to the $\mathscr{C}^k$-topology and hence a Fréchet space.

Hint: Use *ii.)* and known statements on the convergence of $\mathscr{C}^k$-functions from elementary calculus.

vi.) Let $X \subseteq Y$ be open subsets of $\mathbb{R}^n$. Show that the restriction map

$$\mathscr{C}^k(Y) \longrightarrow \mathscr{C}^k(X) \tag{2.5.11}$$

is a continuous linear map. What can one say about the injectivity or surjectivity?

vii.) Let $k \leq k'$ and show that the inclusion

$$\mathscr{C}^{k'}(X) \longrightarrow \mathscr{C}^k(X) \tag{2.5.12}$$

is a continuous injective linear map.

Exercise 2.5.30 (The test functions $\mathscr{C}_0^k(X)$) Consider again $k \in \mathbb{N}_0 \cup \{+\infty\}$ and a nonempty open subset $X \subseteq \mathbb{R}^d$. For a compact subset $K \subseteq X$ one defines

$$\mathscr{C}_K^k(X) = \Big\{ \phi \in \mathscr{C}^k(X) \;\Big|\; \operatorname{supp} \phi \subseteq K \Big\}, \tag{2.5.13}$$

where as usual the *support* $\operatorname{supp} \phi$ is the closure of the points where ϕ is different from zero. Endow $\mathscr{C}^k(X)$ with the $\mathscr{C}^k$-topology of Exercise 2.5.29 in the following.

i.) Show that $\mathscr{C}_K^k(X)$ is a closed subspace of $\mathscr{C}^k(X)$ and hence a Fréchet space itself. Moreover, show that for $k \in \mathbb{N}_0$ the induced topology is even a Banach topology. Is this also true for $\mathscr{C}_K^\infty(X)$?

ii.) Let $K' \subseteq K$ be another compact subset. Show that the inclusion map

$$\mathscr{C}_{K'}^k(X) \longrightarrow \mathscr{C}_K^k(X) \tag{2.5.14}$$

is an injective continuous linear map. Furthermore, show that this is an embedding with closed image.

iii.)

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Exercise 2.5.31 (Existence of test functions) Let $X \subseteq \mathbb{R}^n$ be a nonempty open subset. Since the space $\mathscr{C}_0^\infty(X) = \bigcap_{k \in \mathbb{N}} \mathscr{C}_0^k(X)$ is a infinite intersection, it is not completely obvious, how large and rich this space actually is.

i.) Consider the function

$$h(t) = \begin{cases} \mathrm{e}^{-\frac{1}{t}} & t > 0 \\ 0 & t \leq 0, \end{cases} \tag{2.5.15}$$

and show that $h \in \mathscr{C}^\infty(\mathbb{R})$. Conclude that for all $\epsilon > 0$ the functions

$$g_\epsilon(t) = \frac{h(t)}{h(t) + h(\epsilon - t)} \tag{2.5.16}$$

are smooth, satisfy $g(t) = 0$ for $t \leq 0$ and $g_\epsilon(t) = 1$ for $t \geq \epsilon$, and fulfill $0 \leq g_\epsilon(t) \leq 1$ everywhere.

ii.) Sketch the graphs of the functions h and g_ϵ.

iii.) Show that the following *bump functions*

$$\varphi_{x_0,r,\epsilon}(x) = 1 - g_\epsilon(\|x - x_0\| - r) \tag{2.5.17}$$

for $x \in \mathbb{R}^n$, $r > 0$, $\epsilon > 0$ are smooth, have (compact) support in $\mathrm{B}_{r+\epsilon}(x_0)$, and satisfy $\varphi_{x_0,r,\epsilon}\big|_{\mathrm{B}_r(x_0)} = 1$. Conclude that $\mathscr{C}_0^\infty(\mathbb{R}^n)$ is infinite-dimensional and separates points in $\mathbb{R}^n$.

iv.) Let $x_0 \in \mathbb{R}^n$ and $r > 0$ be given. Construct a function $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subseteq \mathrm{B}_r(x_0)$ and $\|\varphi\|_{\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)} = 1$, where

$$\|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)} = \int_{\mathbb{R}^n} |\varphi(x)|^2 \, \mathrm{d}^n x. \tag{2.5.18}$$

chwartz space **Exercise 2.5.32 (The Schwartz functions $\mathscr{S}(\mathbb{R}^n)$)**

Exercise 2.5.33 (The entire functions $\mathscr{O}(\mathbb{C})$) Consider the entire functions $\mathscr{O}(\mathbb{C}) \subseteq \mathscr{C}(\mathbb{C})$ as a subspace of all continuous functions on $\mathbb{C}$.

i.) Show that $\mathscr{O}(\mathbb{C})$ is a closed subspace of $\mathscr{C}(\mathbb{C})$ with respect to the Fréchet topology of locally uniform convergence, see Exercise 2.5.29, and hence a Fréchet space itself.

ii.) Write $f \in \mathscr{O}(\mathbb{C})$ as convergent Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f}{\partial z^n}(0) z^n \tag{2.5.19}$$

around 0. Show that

$$\mathrm{p}_R(f) = \sum_{n=0}^{\infty} \frac{R^n}{n!} \left| \frac{\partial^n f}{\partial z^n}(0) \right| \tag{2.5.20}$$

defines a seminorm on $\mathscr{O}(\mathbb{C})$ for all $R \geq 0$.

iii.) Let $K \subseteq \mathbb{C}$ be compact. Show that there is an $R \geq 0$ such that we can estimate $\mathrm{p}_{K,0}(f)$ by $\mathrm{p}_R(f)$ for all $f \in \mathscr{O}(\mathbb{C})$. Conclude that the locally convex topology defined by the seminorms $\{\mathrm{p}_R\}_{R \geq 0}$ is finer than the one inherited from $\mathscr{C}(\mathbb{C})$ according to Exercise 2.5.29.

iv.) Show that the locally convex topology induced by the seminorms $\{\mathrm{p}_R\}_{R \geq 0}$ turns $\mathscr{O}(\mathbb{C})$ into a Fréchet space. Moreover, show that the polynomials $\mathbb{C}[z]$ are dense in $\mathscr{O}(\mathbb{C})$.

Hint: Here the argument is very similar to the completeness argument for ℓ^1 from Exercise 2.5.40.

v.) Conclude that the two Fréchet topologies on $\mathscr{O}(\mathbb{C})$ coincide.

Hint: Either find an explicit estimate for the seminorms in the other direction, or use the Open Mapping Theorem.

vi.) Consider now the derivative operator

$$\frac{\partial}{\partial z}\colon \mathscr{O}(\mathbb{C}) \ni f \;\mapsto\; \frac{\partial f}{\partial z} \in \mathscr{O}(\mathbb{C}). \tag{2.5.21}$$

Show that this is a continuous linear endomorphism of the Fréchet space $\mathscr{O}(\mathbb{C})$.

Hint: Use a suitable defining system of continuous seminorms to show this.

Exercise 2.5.34 (The Fréchet space $\mathscr{O}(X)$) We want to extend now the results of Exercise 2.5.33 to holomorphic functions $\mathscr{O}(X)$ on an arbitrary non-empty open subset $X \subseteq \mathbb{C}$.

i.) Define for $w \in X$ and $r > 0$ small enough such that $\mathrm{B}_r(w) \subseteq X$ for $f \in \mathscr{O}(X)$

$$\mathrm{p}_{w,r}(f) = \sum_{n=0}^{\infty} \frac{r^n}{n!}\Big|\frac{\partial^n f}{\partial z^n}(w)\Big|, \tag{2.5.22}$$

and show that this gives a well-defined seminorm on $\mathscr{O}(X)$.

ii.) Give examples for X where the system of all the seminorms $\mathrm{p}_{w,r}$ is filtrating and where it is not filtrating.

iii.) Show that the system of all the seminorms $\mathrm{p}_{w,r}$ with $w \in X$ and $r > 0$ small enough is equivalent to the system of seminorms $\mathrm{p}_K(f) = \max_{z\in K}|f(z)|$ for all $K \subseteq X$ compact.

iv.) Show that $\mathscr{O}(X) \subseteq \mathscr{C}(X)$ is a closed embedded subspace and hence a Fréchet space with respect to the above (equivalent) systems of seminorms where $\mathscr{C}(X)$ carries the topology from Exercise 2.5.29. Alternatively, one can show the completeness with respect to one of the above systems of seminorms directly.

v.) Show that the derivative

$$\frac{\partial}{\partial z}\colon \mathscr{O}(X) \ni f \;\mapsto\; \frac{\partial f}{\partial z} \in \mathscr{O}(X) \tag{2.5.23}$$

is continuous.

vi.) Prove the following theorem of Weierstraß: if a sequence of holomorphic functions $f_k \in \mathscr{O}(X)$ converges locally uniformly in X then the limit f is holomorphic and for all derivatives one has the locally uniform convergence

$$\frac{\partial^n f_k}{\partial z^n} \longrightarrow \frac{\partial^n f}{\partial z^n}. \tag{2.5.24}$$

Exercise 2.5.35 (Dense subspaces) The following considerations should avoid some pitfalls. Suppose that V is a locally convex space with a defining system of seminorms $\mathcal{P}$, i.e. the open balls for seminorms of $\mathcal{P}$ generate the topology. We do *not* assume that $\mathcal{P}$ is filtrating. Construct an example of V with suitable non-filtrating $\mathcal{P}$ and a subspace $U \subseteq V$ such that for all $\epsilon > 0$, all $\mathrm{p} \in \mathcal{P}$ and all $v \in V$ we find a $u \in U$ with $\mathrm{p}(v - u) < \epsilon$ but U is *not* dense in V.

Hint: Consider holomorphic functions on $\mathbb{C} \setminus \{0\}$ and the holomorphic polynomials $\mathbb{C}[z] \subseteq \mathscr{O}(X)$, where $\mathscr{O}(X)$ is endowed with its usual Fréchet topology of locally uniform convergence from Exercise 2.5.34. One calls $X \subseteq \mathbb{C}$ a *Runge domain* if $\mathbb{C}[z] \subseteq \mathscr{O}(X)$ is dense.

Exercise 2.5.36 (Pointwise convergence is rather pointless) One may wonder which function space is best suited to the locally convex topology responsible for pointwise convergence. Consider e.g. a non-empty open subset $X \subseteq \mathbb{R}^n$ and the space of all functions $\mathrm{Map}(X, \mathbb{C})$ on it.

i.) Show that for all $x \in X$ the δ-functional δ_x yields a seminorm

$$\mathrm{p}_x(f) = |\delta_x(f)|. \tag{2.5.25}$$

ii.) Show that the induced locally convex topology on $\mathrm{Map}(X, \mathbb{C})$ is Hausdorff and complete. Show that it is the coarsest locally convex topology which makes all the δ-functionals continuous.

iii.) Show that essentially all interesting function spaces on X like $\mathscr{C}_0^\infty(X)$, $\mathscr{C}^k(X)$, etc., are dense in $\mathrm{Map}(X, \mathbb{C})$ with respect to this topology.

Hint: It is important to note that the topology is *not* first countable and hence we need general nets to approach points in the boundary. To this end, formulate what it means for a net $(f_i)_{i \in I}$ to converge to another function f.

Thus the natural locally convex topology related to pointwise convergence to by far too coarse to be of any interest, though it is still Hausdorff. Note however, that the above nice function spaces are typically *not* sequentially dense: their sequential closures inside $\mathrm{Map}(X, \mathbb{C})$ is typically much smaller and rather non-trivial to characterize.

Exercise 2.5.37 (The sequence spaces c, $c_\circ$ and $c_{\circ\circ}$) Consider the following sets of sequences in $\mathbb{C}$

$$c = \left\{ a = (a_n)_{n \in \mathbb{N}} \;\middle|\; \lim_{n \longrightarrow \infty} a_n \text{ exists} \right\} \tag{2.5.26}$$

and

$$c_\circ = \left\{ a = (a_n)_{n \in \mathbb{N}} \;\middle|\; \lim_{n \longrightarrow \infty} a_n = 0 \right\} \subseteq c. \tag{2.5.27}$$

Moreover, denote by $c_{\circ\circ}$ the set of those sequences with only finitely many entries different from 0. Clearly, c is a vector space, $c_\circ$ is a subspace of c, and $c_{\circ\circ}$ is a subspace of $c_\circ$.

i.) Show that the supremum

$$\|a\|_\infty = \sup\{ |a_n| \mid n \in \mathbb{N} \} \tag{2.5.28}$$

defines a norm on c. In the following c will always be equipped with this norm.

ii.) Show that c is complete with respect to $\|\cdot\|_\infty$.

iii.) Show that taking the limit defines a continuous linear functional

$$\lim \colon c \ni (a_n)_{n \in \mathbb{N}} \mapsto \lim_{n \longrightarrow \infty} a_n \in \mathbb{C}. \tag{2.5.29}$$

iv.) Show that $c_\circ$ is a closed subspace of c and hence a Banach space, too.

v.) Show that the linear functional $\epsilon_n \colon a \mapsto a_n$ is continuous for all $n \in \mathbb{N}$.

vi.) Let e_n be the sequence $(\delta_{nk})_{k \in \mathbb{N}}$, i.e. a 1 at the n-th position and zeros elsewhere. Show that

$$a = \sum_{n \in \mathbb{N}} \epsilon_n(a) \mathrm{e}_n \tag{2.5.30}$$

converges unconditionally for every $a \in c_\circ$. Such a collection of vectors $\{\mathrm{e}_n\}_{n \in \mathbb{N}}$ together with continuous evaluation functionals ϵ_n is also called an *unconditional Schauder basis*, see e.g. [26, Chap. 14] for more details on topological bases.

vii.) Show that

$$c_{\circ\circ} = \mathrm{span}_{\mathbb{C}}\{\mathrm{e}_n\}_{n \in \mathbb{N}}. \tag{2.5.31}$$

Conclude that $c_{\circ\circ}$ is a dense subspace of $c_\circ$.

viii.) Consider the sequence $\mathrm{e}_\infty \in c$ consisting of 1's everywhere. Show that together with the previous e_n this gives a Schauder basis for c. What is the corresponding evaluation functional ϵ_∞?

ix.) Show that $c_\circ$ and c are separable.

Exercise 2.5.38 (The dual space of c and $c_\circ$) Consider again the Banach spaces c and $c_\circ$ from Exercise 2.5.37.

i.) Let $\varphi\colon c_\circ \longrightarrow \mathbb{C}$ be a continuous linear functional. Show that it is uniquely determined by its values on the Schauder basis $\varphi_n = \varphi(\mathrm{e}_n)$ from Exercise 2.5.37, *vi.)*.

ii.) Show that a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of complex numbers arises as $\varphi_n = \varphi(\mathrm{e}_n)$ for a continuous linear functional $\varphi \in c_\circ'$ iff

$$\big\|(\varphi_n)_{n\in\mathbb{N}}\big\|_{\ell^1} = \sum_{n=0}^{\infty} |\varphi_n| < \infty. \tag{2.5.32}$$

We will identify the linear functional φ with the corresponding sequence $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ in the following.

iii.) Show that $\|\varphi\|_{\ell^1}$ coincides with the functional norm $\|\varphi\|$ of $\varphi \in c_\circ'$.

iv.) Conclude that the set of sequences $(\varphi_n)_{n\in\mathbb{N}}$ satisfying (2.5.32) is a Banach space, denoted by ℓ^1.

v.) Show that every continuous linear functional $\varphi \in c_\circ'$ has a canonical extension to a continuous linear functional $\varphi \in c'$ with the same functional norm by setting

$$\varphi(a) = \sum_{n=0}^{\infty} \varphi_n a_n. \tag{2.5.33}$$

vi.) Show that c has continuous linear functionals not of the form (2.5.33). Find an explicit description of them and of the dual space c'.

Exercise 2.5.39 (Hölder and Minkowski Inequality) Let $1 < p < \infty$ be given and let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, let $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ be sequences of complex numbers.

i.) Show that for all $\alpha, \beta \geq 0$ one has

$$\alpha^{1/p}\beta^{1/q} \leq \frac{\alpha}{p} + \frac{\beta}{q}. \tag{2.5.34}$$

ii.) Show that one has *Hölder's inequality*

$$\sum_{n\in\mathbb{N}} |a_n b_n| \leq \left(\sum_{n\in\mathbb{N}} |a_n|^p\right)^{\frac{1}{p}} \left(\sum_{n\in\mathbb{N}} |b_n|^q\right)^{\frac{1}{q}}, \tag{2.5.35}$$

viewed as in inequality in $[0, +\infty]$.

iii.) Consider the sequence $c = (c_n)_{n\in\mathbb{N}}$ with $c_n = |a_n + b_n|^{p-1}$. Use Hölder's inequality for $|a_n|c_n$ and $|b_n|c_n$ to arrive at the inequality

$$\sum_{n\in\mathbb{N}} |a_n + b_n|^p \leq \left(\sum_{n\in\mathbb{N}} |a_n + b_n|^p\right)^{\frac{1}{q}} \left(\left(\sum_{n\in\mathbb{N}} |a_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n\in\mathbb{N}} |b_n|^p\right)^{\frac{1}{p}}\right), \tag{2.5.36}$$

again viewed as inequality in $[0, +\infty]$.

iv.) Use (2.5.36) and distinguish the cases where some of the series might diverge to $+\infty$ and where all are finite in order to show Minkowski's inequality

$$\left(\sum_{n\in\mathbb{N}} |a_n + b_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n\in\mathbb{N}} |a_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n\in\mathbb{N}} |b_n|^p\right)^{\frac{1}{p}}. \tag{2.5.37}$$

Does this inequality also hold for $p = 1$?

Exercise 2.5.40 (The sequence spaces ℓ^p) Let $1 \le p < \infty$ and consider the following set

$$\ell^p = \left\{ a = (a_n)_{n \in \mathbb{N}} \mid \|a\|_{\ell^p} < \infty \right\}, \quad \text{where} \quad \|a\|_{\ell^p} = \sqrt[p]{\sum\nolimits_{n \in \mathbb{N}} |a_n|^p}, \tag{2.5.38}$$

called the *p-summable complex sequences.*

i.) Show that ℓ^p is a vector space and $\|\cdot\|_{\ell^p}$ is a norm on it.

Hint: Use Exercise 2.5.39, *iv.)*.

ii.) Show that ℓ^p is a Banach space with respect to the norm $\|\cdot\|_{\ell^p}$.

iii.) Let $p \le p'$. Show that one has the continuous inclusion maps

$$\ell^1 \longrightarrow \ell^p \longrightarrow \ell^{p'} \longrightarrow c_\circ. \tag{2.5.39}$$

Show that the inclusions are all proper for $p \neq p'$.

iv.) Consider again the evaluation functionals $a \mapsto a_n$ as in Exercise 2.5.37, *v.)* and show that they are continuous linear functionals on ℓ^p.

v.) Show that the sequences e_n from Exercise 2.5.37, *vi.)*, are also elements of ℓ^p and form an unconditional Schauder basis of each ℓ^p. Conclude that $c_{\circ\circ} \subseteq \ell^p$ is a dense subspace.

vi.) Show that the Schauder basis is even *absolute* in the case of ℓ^1, i.e. the map $a \mapsto \sum_{n \in \mathbb{N}} |a_n| \|\mathrm{e}_n\|_{\ell^1}$ defines a continuous seminorm on the Banach space ℓ^1. Is the same true for the other ℓ^p?

Exercise 2.5.41 (The dual space of ℓ^p) Consider again the Banach spaces ℓ^p for $1 \le p < \infty$ as in Exercise 2.5.40. Moreover, we consider the space of all *bounded* sequences

$$\ell^\infty = \left\{ a = (a_n)_{n \in \mathbb{N}} \mid \|a\|_\infty < \infty \right\} \tag{2.5.40}$$

with $\|\cdot\|_\infty$ as in (2.5.28).

i.) Show that ℓ^∞ is a vector space and $\|\cdot\|_\infty$ is a norm on it turning it into a Banach space.

ii.) Let $\varphi\colon \ell^p \longrightarrow \mathbb{C}$ be a continuous linear functional. Show that φ is uniquely determined by the sequence $(\varphi(\mathrm{e}_n))_{n \in \mathbb{N}}$.

iii.) Consider first $p = 1$ and show that the sequence $(\varphi_n)_{n \in \mathbb{N}}$ defines a continuous linear functional on ℓ^1 iff it is bounded, i.e. $(\varphi_n)_{n \in \mathbb{N}} \in \ell^\infty$. Show that the functional norm of φ coincides with the supremum norm $\|(\varphi_n)_{n \in \mathbb{N}}\|_\infty$. Conclude that $(\ell^1)' = \ell^\infty$ as Banach spaces.

iv.) Consider now $p > 1$ and show that the sequence $(\varphi_n)_{n \in \mathbb{N}}$ defines a continuous linear functional on ℓ^p iff $(\varphi_n)_{n \in \mathbb{N}} \in \ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Show that the functional norm of φ coincides with $\|(\varphi_n)_{n \in \mathbb{N}}\|_{\ell^q}$.

Hint: Hölder's inequality.

v.) Show that ℓ^p is reflexive for $1 < p < \infty$. What happens for $p = 1$?

vi.) Show that ℓ^p is separable for $1 \le p < \infty$.

ill to be done

Exercise 2.5.42 (The Schwartz sequence space s)

Exercise 2.5.43 (Completeness of quotients) Consider a Fréchet space V and a closed subspace $U \subseteq V$. We endow the quotient V/U with the locally convex quotient topology as usual. The aim of this exercise is to show that V/U is a Fréchet space again. As a warning one should note that there are counterexamples of this statement for general complete locally convex space: locally convex quotients of complete locally convex spaces need not to be complete beyond this Fréchet case. We fix a countable defining system of seminorms p_n with $\mathrm{p}_n \le \mathrm{p}_m$ for $n \le m$ on V.

i.) Show that in this case also $[\mathrm{p}_n] \le [\mathrm{p}]_m$ for $n \le m$.

ii.) Let $([v_n])_{n\in\mathbb{N}}$ be a Cauchy sequence in the quotient $V\big/U$. Show that there is a monotonously increasing sequence n_k with

$$[\mathrm{p}_k]([v_{n_k}] - [v_{n_{k+1}}]) < \frac{1}{2^k} \tag{2.5.41}$$

for all $k \in \mathbb{N}$.

iii.) Find representatives $w_k \in [v_{n_k}]$ such that

$$\mathrm{p}_k(w_k - w_{k+1}) < \frac{1}{2^k}, \tag{2.5.42}$$

by recursively constructing w_{k+1} out of $w_1, \ldots, w_k$ for an arbitrary starting point w_1.

iv.) Show that the sequence $(w_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in V.

v.) Conclude that $V\big/U$ is again a Fréchet space and show that in the case where V was a Banach space, also $V\big/U$ is a Banach space.

Exercise 2.5.44 (Functoriality of dualizing) Let V, W, and U be Banach space. Formulate and prove the statement that dualizing is a contravariant functor $\mathsf{Banach} \longrightarrow \mathsf{Banach}$. Consider also the cases of general linear maps, i.e. the functoriality of $\mathrm{L}(V, \cdot\,)$ as well as $\mathrm{L}(\,\cdot\,, V)$ with a fixed Banach space V.

Exercise 2.5.45 (Proof of Lemma 2.3.24) Prove Lemma 2.3.24.

Hint: Consider the map $\Phi\colon V \longrightarrow \mathbb{C}^n$ whose i-th component is φ_i. Then Φ descends to a bijective map $[\Phi]$ from $V\big/\ker\Phi$ to $\operatorname{im}\Phi \subseteq \mathbb{C}^n$. Also φ descends to a linear functional $[\varphi]$ on $V\big/\ker\Phi$ by assumption. Consider $[\varphi]\circ[\Phi]$.

Exercise 2.5.46 (The geometry of the separation theorems) Consider for visualization $V = \mathbb{R}^2$.

i.) Consider non-empty convex subsets A and B with A being open and give a geometric interpretation of Theorem 2.4.2 by using the linear functional Ψ to define (open or closed) affine half spaces. Explain the name "Separation Theorem". Show by geometric considerations that the estimates in Theorem 2.4.2 can not be improved in general.

ii.) Consider now non-empty convex subsets K and C with K being compact and C being closed. Proceed analogously to *i.)* to explain Theorem 2.4.4 geometrically. Find also here geometric situations which show that the estimates can not be improved. What happens if K is only assumed to be closed but both subsets are non-compact?

Exercise 2.5.47 (Bounded subsets) Let V be a locally convex space and let $B \subseteq V$ be a subset.

i.) Show that $B^{\mathrm{cl}} \subseteq V$ is bounded iff B is bounded by using a seminorm based argument.

ii.) Find an argument based on seminorms to show that a compact subset of V is bounded.

Exercise 2.5.48 (The Heine-Borel property) Exercise: Ne examples?

Exercise 2.5.49 (Extreme points of convex subsets I)

i.) Find a non-empty bounded convex subset with no extreme points in $\mathbb{R}^2$.

ii.) Find a non-empty closed convex subset $K \neq \mathbb{R}^2$ with no extreme points.

iii.) Find a non-empty bounded convex subset $K \neq \mathbb{R}^2$ with boundary points which are not extreme points.

iv.) Give an example of a compact subset K for which we have $K \neq \operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}}$ in $\mathbb{R}^2$.

v.) Now let V be a topological vector space such that V' separates points. Let $K \subseteq V$ be a compact convex subset with precisely one extreme point $v \in K$. Show that $K = \operatorname{extreme}(K) = \{v\}$ in this case. Show by a simple counterexample that the compactness assumption is essential.

Chapter 3

Hilbert Spaces

Even though for the formulation of quantum mechanical expectation values by vector states pre-Hilbert spaces were sufficient, we will need more analytic techniques for the representation space $\mathfrak{H}$ than just the algebraic features of a pre-Hilbert space. Many of the interesting and desirable results only become available when we pass to a Hilbert space: here we have to add completeness. In this chapter, we will discuss some elementary features and properties of Hilbert spaces as well as central constructions.

However, one should keep in mind that the passage from a pre-Hilbert space to its completion might be a physically hard to motivate idealization: if we are interested in certain unbounded observables like energy, momenta or positions, then they will be defined as operators only on the original pre-Hilbert space but do not allow for an extension to the completion. Thus there will be vectors in the Hilbert space completion which give undefined expectation values for such observables and hence do not describe physically realizable states. Alternatively, one can abandon unbounded observables as being not physically relevant as measurements always yield finite numbers as answers and finitely many measurements can not detect a possible failure in boundedness. Thus unbounded observables do not have an operational justification and only provide an idealization themselves. Either way, it is always a good advice to keep in mind the role and importance of idealizations in a physical model. When difficulties with the model start to get out of hands, it is thus a good option to re-think about the justification of certain idealizations which in the beginning led to the model. In our present discussion we shall no longer follows this but present both, a Hilbert space way of describing things and a way to incorporate unbounded observables afterwards. In this chapter we focus on the basic properties of Hilbert spaces.

We begin with a more detailed discussion of the geometry of pre-Hilbert spaces. This will lead to a clear picture what is still missing, thereby motivating the notion of completeness and completion of a pre-Hilbert space to a Hilbert space. This additional feature ultimately guarantees many nice properties. Essentially, a Hilbert space behaves very much like a finite-dimensional vector space. We will then pass to the geometry of Hilbert spaces discussing the lattice of closed subspaces and the dual. We prove the existence of a Hilbert basis and obtain a classification by the cardinality of the Hilbert basis. This extremely simple classification will allow us to speak of *the* Hilbert space in quantum mechanics. There is, up to isometries, only one relevant Hilbert space around. Many seemingly different examples will lead to this distinguished Hilbert space in the end. Constructions like the direct sum and tensor products of Hilbert spaces will be important for many-particle systems. The adjointable operators on a Hilbert space will now provide a *-algebra with many additional and nice properties: it is the prototype of a C^*-algebra. We conclude this chapter with some features of projections and bounded operators on direct sums and tensor products.

3.1 From Pre-Hilbert Spaces to Hilbert Spaces

The geometry of pre-Hilbert spaces includes several deficits which we cure when passing to Hilbert spaces. In this section we will deal with this transition.

3.1.1 The Geometry of Pre-Hilbert Spaces

In the following, we consider a pre-Hilbert space $\mathfrak{H}$ over $\mathbb{C}$ with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$. The reason for the choice of the complex numbers as number field is that spectral theory will become much simpler in this case: this we can anticipate already from the finite-dimensional case known from linear algebra. Nevertheless, pre-Hilbert spaces as well as Hilbert spaces can also be defined over $\mathbb{R}$ and play an important role also in this case. We leave it as an exercise to formulate the definitions as well as the basic results also for this case, see also Exercise 3.6.6 for further relations between the real and complex cases.

The first important statement about $\langle\,\cdot\,,\,\cdot\,\rangle$ is the Cauchy-Schwarz inequality which reads a follows:

Proposition 3.1.1 (Cauchy-Schwarz inequality) *Let $\mathfrak{H}$ be a complex vector space with a sesquilinear, positive semi-definite $\langle\,\cdot\,,\,\cdot\,\rangle$.*

i.) For all $\phi, \psi \in \mathfrak{H}$ we have the Cauchy-Schwarz inequality

$$|\langle\phi,\psi\rangle|^2 \le \langle\phi,\phi\rangle\langle\psi,\psi\rangle. \tag{3.1.1}$$

ii.) The set $\{\phi \in \mathfrak{H} \mid \langle\phi,\phi\rangle = 0\}$ is a subspace of $\mathfrak{H}$ which coincides with

$$\mathfrak{H}^\perp = \big\{\phi \in \mathfrak{H} \;\big|\; \langle\psi,\phi\rangle = 0 \textit{ for all } \psi \in \mathfrak{H}\big\}. \tag{3.1.2}$$

iii.) The quotient space $\mathfrak{H}\big/\mathfrak{H}^\perp$ becomes a pre-Hilbert space via

$$\langle[\phi],[\psi]\rangle_{\mathfrak{H}/\mathfrak{H}^\perp} = \langle\phi,\psi\rangle. \tag{3.1.3}$$

iv.) In a pre-Hilbert space we have equality in (3.1.1) *iff ϕ and ψ are linearly dependent.*

Proof: For the first part we consider $z, w \in \mathbb{C}$ and have

$$0 \le p(z,w) = \langle z\phi, +w\psi, z\phi + w\psi\rangle = z\overline{z}\langle\phi,\phi\rangle + \overline{z}w\langle\phi,\psi\rangle + z\overline{w}\langle\psi,\phi\rangle + w\overline{w}\langle\psi,\psi\rangle$$

for all $\phi, \psi \in \mathfrak{H}$. From here we can argue as in the proof of Lemma 1.2.5 to show the Cauchy-Schwarz inequality. For the second part, let $\phi \in \mathfrak{H}$ satisfy $\langle\phi,\phi\rangle = 0$. Then by (3.1.1) we have $\langle\psi,\phi\rangle = \overline{\langle\phi,\psi\rangle} = 0$ for all $\psi \in \mathfrak{H}$, proving $\phi \in \mathfrak{H}^\perp$. The converse is trivial. For the third part one first shows that (3.1.3) is well-defined at all. But from the characterization of $\mathfrak{H}^\perp$ as in (3.1.2) this is clear. Then the properties of a positive semi-definite sesquilinear form are easily checked on representatives. We have $\langle[\phi],[\phi]\rangle = \langle\phi,\phi\rangle = 0$ iff $\phi \in \mathfrak{H}^\perp$ iff $[\phi] = 0$. Thus we have a pre-Hilbert space as claimed. Finally, let $\mathfrak{H}$ be already a pre-Hilbert space and let $\phi, \psi \in \mathfrak{H}$ with $|\langle\phi,\psi\rangle|^2 = \langle\phi,\phi\rangle\langle\psi,\psi\rangle$ be given. Without restriction we can assume $\phi \ne 0$, otherwise ϕ and ψ are linearly dependent in a trivial way. Then a simple computation gives

$$\left\langle \psi - \tfrac{\langle\phi,\psi\rangle}{\langle\phi,\phi\rangle}\phi, \psi - \tfrac{\langle\phi,\psi\rangle}{\langle\phi,\phi\rangle}\phi \right\rangle = 0.$$

By the non-degeneracy of the $\langle\,\cdot\,,\,\cdot\,\rangle$ we get $\psi = \frac{\langle\phi,\psi\rangle}{\langle\phi,\phi\rangle}\phi$. This completes the proof. ∎

For a general sesquilinear form $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{H}$ one calls $\mathfrak{H}^\perp$ the *degeneracy space* of $\langle \cdot\,, \cdot \rangle$. Dividing by $\mathfrak{H}^\perp$ yields a non-degenerate sesquilinear form on the quotient $\mathfrak{H}/\mathfrak{H}^\perp$, which is a pre-Hilbert space whenever $\langle \cdot\,, \cdot \rangle$ was positive semi-definite. It is important to emphasize that in this case the Cauchy-Schwarz inequality already holds for the positive semi-definite sesquilinear form. It allows to turn the single quadratic equation $\langle \phi, \phi \rangle = 0$ into a (typically infinite) system of linear equations $\langle \psi, \phi \rangle = 0$ where $\psi \in \mathfrak{H}$.

Remark 3.1.2 The Cauchy-Schwarz inequality is a particular case of a more general statement: for $\phi_1, \ldots, \phi_n \in \mathfrak{H}$ one considers the matrix

$$\Phi = \big(\langle \phi_i, \phi_j \rangle\big)_{i,j=1,\ldots,n} \in \mathrm{M}_n(\mathbb{C}). \tag{3.1.4}$$

Then Φ turns our to be a *positive* matrix, see Exercise 3.6.1. Evaluating this positivity for only two vectors gives us the Cauchy-Schwarz inequality.

The Cauchy-Schwarz inequality allows us to define a seminorm on the vector space $\mathfrak{H}$:

Lemma 3.1.3 *Let $\mathfrak{H}$ be equipped with a positive semi-definite sesquilinear form $\langle \cdot\,, \cdot \rangle$. Then*

$$\|\phi\| = \sqrt{\langle \phi, \phi \rangle} \tag{3.1.5}$$

defines a seminorm on $\mathfrak{H}$ which is a norm iff $\mathfrak{H}$ is a pre-Hilbert space.

PROOF: First it is clear that $\|\phi\| \geq 0$ as well as $\|z\phi\| = |z|\|\phi\|$ for all $\phi \in \mathfrak{H}$ and $z \in \mathbb{C}$. To prove the triangle inequality we observe that

$$\begin{aligned}
\|\phi + \psi\|^2 &= \langle \phi, \phi \rangle + \langle \phi, \psi \rangle + \langle \psi, \phi \rangle + \langle \psi, \psi \rangle \\
&= \|\phi\|^2 + 2\,\mathrm{Re}\langle \phi, \psi \rangle + \|\psi\|^2 \\
&\leq \|\phi\|^2 + 2\|\phi\|\|\psi\| + \|\psi\|^2 \\
&= (\|\phi\| + \|\psi\|)^2,
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the form

$$|\mathrm{Re}\langle \phi, \psi \rangle| \leq |\langle \phi, \psi \rangle| \leq \|\phi\|\|\psi\|.$$

Taking the square root gives the triangle inequality, proving that $\|\cdot\|$ is a seminorm. Clearly, $\|\phi\| > 0$ for all $\phi \neq 0$ iff $\mathfrak{H}$ is a pre-Hilbert space. □

In the following, we will always endow a pre-Hilbert space with this norm. It turns out that the inner product does not only determine the norm by (3.1.5) but also vice versa:

Lemma 3.1.4 *Let $\mathfrak{H}$ be a pre-Hilbert space. Then for $\phi, \psi \in \mathfrak{H}$ one has*

$$\langle \phi, \psi \rangle = \frac{1}{4} \sum_{r=0}^{3} \mathrm{i}^r \|\mathrm{i}^r \phi + \psi\|^2. \tag{3.1.6}$$

The proof consists in a simple evaluation of the right hand side. The following important property of the norm characterizes those norms which actually come from a positive definite inner product:

Proposition 3.1.5 (Parallelogram identity) *Let $\mathfrak{H}$ be a normed vector space. Then there exists a positive definite sesquilinear form $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{H}$ with* (3.1.5) *iff the norm satisfies the parallelogram identity*

$$\|\phi + \psi\|^2 + \|\phi - \psi\|^2 = 2\|\phi\|^2 + 2\|\psi\|^2 \tag{3.1.7}$$

for all $\phi, \psi \in \mathfrak{H}$.

The proof as well as an elementary geometric interpretation and justification of the name of the parallelogram identity is the content of Exercise 3.6.2.

A pre-Hilbert space does not only have a norm which will allow to implement analytic concepts. We also have geometric notions very similar to Euclidean spaces: in particular, the Cauchy-Schwarz inequality allows to define an *angle* between two non-zero vectors. In fact, this can only be done for a *real* pre-Hilbert space, a concept which we will not need in the sequel but whose definition should be clear as analog of Definition 1.3.1. In the real case, we also have a Cauchy-Schwarz inequality which now implies that

$$-1 \leq \frac{\langle \phi, \psi \rangle}{\|\phi\| \|\psi\|} \leq 1 \tag{3.1.8}$$

for non-zero vectors. As known from the finite-dimensional case this allows to define the *angle* $\sphericalangle(\phi, \psi)$ between ϕ and ψ by

$$\sphericalangle(\phi, \psi) = \arccos\left(\frac{\langle \phi, \psi \rangle}{\|\phi\| \|\psi\|} \right) \in [0, \pi]. \tag{3.1.9}$$

In the complex case, the inequality (3.1.8) only holds for absolute values as $\langle \phi, \psi \rangle$ takes now values in $\mathbb{C}$. Therefore we will not speak of the angle $\sphericalangle(\phi, \psi)$ itself. Nevertheless, the notion of orthogonal vectors still makes perfect sense. We call ϕ and ψ *orthogonal* if

$$\langle \phi, \psi \rangle = 0, \tag{3.1.10}$$

and write this as $\phi \perp \psi$. If $\phi_1, \ldots, \phi_n \in \mathfrak{H}$ are pairwise orthogonal we find immediately the *Theorem of Pythagoras*, i.e. we have

$$\|\phi_1\|^2 + \cdots + \|\phi_n\|^2 = \|\phi_1 + \cdots + \phi_n\|^2. \tag{3.1.11}$$

The notion of orthogonal vectors is crucial throughout the theory of (pre-) Hilbert spaces. In particular, it will be important to know which vectors are actually orthogonal to a given vector or a given subset of vectors. This motivates the following definition:

Definition 3.1.6 (Orthogonal complement) *Let $\mathfrak{H}$ be a pre-Hilbert space and $U \subseteq \mathfrak{H}$ a subset. Then*

$$U^\perp = \left\{ \phi \in \mathfrak{H} \;\middle|\; \langle \psi, \phi \rangle = 0 \textit{ for all } \psi \in U \right\} \tag{3.1.12}$$

is called the orthogonal complement of U.

In principle, we do not need that $\langle \cdot\,, \cdot \rangle$ is non-degenerate. With this interpretation the previous notion of $\mathfrak{H}^\perp$ as in Proposition 3.1.1, *ii.)*, is still consistent with Definition 3.1.6.

Proposition 3.1.7 *Let $\mathfrak{H}$ be a pre-Hilbert space and let $U, V \subseteq \mathfrak{H}$ be subsets.*

i.) One has $U \cap U^\perp = \{0\}$ if $0 \in U$ and $U \cap U^\perp = \emptyset$ otherwise.

ii.) The subset $U^\perp$ is a subspace of $\mathfrak{H}$.

iii.) If $U \subseteq V$ then $V^\perp \subseteq U^\perp$.

iv.) One has $U \subseteq (U^\perp)^\perp$.

PROOF: Let $\phi \in U \cap U^\perp$ then $\langle \phi, \phi \rangle = 0$ by definition of $U^\perp$. This shows the first part. The second part is clear since the condition $\psi \in U^\perp$, i.e. $\langle \phi, \psi \rangle = 0$ for all $\phi \in U$, is a system of linear equations. For the third part, let $\phi \in V^\perp$ and $\psi \in U$. Then $\psi \in V$, too, and hence $\langle \psi, \phi \rangle = 0$. Thus $\phi \in U^\perp$ follows since ψ was arbitrary. Finally, let $\phi \in U^\perp$ and $\psi \in U$, i.e. $\langle \phi, \psi \rangle = 0$. Thus $\psi \in (U^\perp)^\perp$ since ϕ was arbitrary. ☐

Remark 3.1.8 (Orthogonal complement) Let $\mathfrak{H}$ be a pre-Hilbert space and $U \subseteq \mathfrak{H}$ a subset. In most applications, U is already a subspace. We will write $U^{\perp\perp}$ for $(U^\perp)^\perp$ etc.

i.) Form part *iii.)* and *iv.)* of Proposition 3.1.7 we conclude that

$$U^{\perp\perp\perp} = U^{\perp}, \tag{3.1.13}$$

since on one hand $U \subseteq U^{\perp\perp}$ implies $U^{\perp\perp\perp} \subseteq U^{\perp}$ by the third part and on the other hand $U^{\perp} \subseteq U^{\perp\perp\perp}$ by the fourth part. Thus taking orthogonal complements stabilizes after three steps.

ii.) We have

$$U^{\perp\perp} \cap U^{\perp} = \{0\}. \tag{3.1.14}$$

Indeed, from the first part we get $U^{\perp\perp} \cap U^{\perp\perp\perp} = \{0\}$. Hence (3.1.13) implies (3.1.14). It follows that the sum of the two subspaces $U^{\perp}$ and $U^{\perp\perp}$ inside $\mathfrak{H}$ is necessarily *direct*, i.e.

$$U^{\perp\perp} + U^{\perp} = U^{\perp\perp} \oplus U^{\perp}. \tag{3.1.15}$$

However, this direct sum does not necessarily exhaust the whole pre-Hilbert space, i.e.

$$U^{\perp\perp} \oplus U^{\perp} \subseteq \mathfrak{H} \tag{3.1.16}$$

is typically a *proper* inclusion. This is an effect of infinite dimensions: for finite-dimensional pre-Hilbert spaces one has equality in (3.1.16) as a simple counting of dimensions shows. In this situations $U^{\perp}$ and $U^{\perp\perp}$ are indeed *complementary* subspaces. The possible failure of this in infinite dimensions will be yet another reason to move on to Hilbert spaces.

Before we proceed we mention the following two fundamental types of examples:

Example 3.1.9 (Function spaces as pre-Hilbert spaces)

i.) Consider the continuous complex-valued functions $\mathscr{C}^0([a, b])$ on some compact interval $[a, b] \subseteq \mathbb{R}$ with $a < b$. Then one defines

$$\langle \phi, \psi \rangle = \int_a^b \overline{\phi(x)} \psi(x) \, \mathrm{d}x, \tag{3.1.17}$$

and checks immediately that $\mathscr{C}^0([a, b])$ becomes a pre-Hilbert space. Note that in (3.1.17) we can take the most naive definition for the integral as the integrand is continuous and the domain of integration is compact. Hence a Riemann integral will suffice.

ii.) Let $k \in \mathbb{N}_0 \cup \{+\infty\}$ and consider the k-times continuously differentiable complex-valued functions $\mathscr{C}^k_0(\mathbb{R}^n)$ on $\mathbb{R}^n$ which have *compact support*, i.e. the *support*

$$\operatorname{supp} \phi = \left\{ x \in \mathbb{R}^n \;\middle|\; \phi(x) \neq 0 \right\}^{\mathrm{cl}} \subseteq \mathbb{R}^n \tag{3.1.18}$$

is a compact subset of $\mathbb{R}^n$, see also Exercise 2.5.30 for more details on these spaces. Analogously to (3.1.17) one defines

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}^n} \overline{\phi(x)} \psi(x) \, \mathrm{d}^n x, \tag{3.1.19}$$

and obtains a pre-Hilbert space structure. Again, the most naive Riemann integral will do the job. We have for $0 < k < k'$ the inclusions

$$\mathscr{C}^\infty_0(\mathbb{R}^n) \subseteq \mathscr{C}^{k'}_0(\mathbb{R}^n) \subseteq \mathscr{C}^k_0(\mathbb{R}^n) \subseteq \mathscr{C}^0_0(\mathbb{R}^n) \tag{3.1.20}$$

which are all proper. Note that these inclusions are compatible with the inner products in the sense that the inner product of the larger space restricts to the inner product of the included one.

iii.) Yet another variation of the above theme is the following: we consider the *Schwartz space*

$$\mathscr{S}(\mathbb{R}^n) = \left\{ f \in \mathscr{C}^\infty(\mathbb{R}^n) \;\middle|\; \sup_{x\in\mathbb{R}^n} \left| x^\alpha \partial^\beta f(x) \right| < \infty \text{ for all } \alpha, \beta \in \mathbb{N}_0^n \right\} \tag{3.1.21}$$

of rapidly decreasing smooth functions, see also Exercise 2.5.32. Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex of length n and we set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ as well as $\frac{\partial^\beta}{\partial x^\beta} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}$ as usual. For two such functions we can define the inner product by

$$\langle f, g \rangle = \int_{\mathbb{R}^n} \overline{f(x)} g(x) \, \mathrm{d}^n x, \tag{3.1.22}$$

where now the integral is slightly more tricky: we need an improper Riemann integral instead of a Riemann integral as in (3.1.17) or (3.1.19). However, the functions in the Schwartz space decrease so fast that the non-compactness of the domain of integration does not cause any further problems. This way, we obtain yet another pre-Hilbert space and

$$\mathscr{C}_0^\infty(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n) \tag{3.1.23}$$

is a proper subspace. The inner products are again compatible.

In view of quantum mechanical needs and wishes, all the versions of Example 3.1.9, *ii.)* and *iii.)* seem to yield plausible candidates for nice pre-Hilbert spaces of *wave functions* on $\mathbb{R}^n$. Note that in particular the Gaussian functions $x \mapsto \mathrm{e}^{-x^{\mathrm{T}} A x}$ with a positive definite matrix $A > 0$ are all in $\mathscr{S}(\mathbb{R}^n)$, though not in $\mathscr{C}_0^\infty(\mathbb{R}^n)$. The compactly supported function spaces seem to be appropriate since experiments and laboratories are taking place in compact regions of space[1]. However, the next example shows that all the above pre-Hilbert spaces have a certain bad behaviour with respect to orthogonal complements:

Example 3.1.10 Consider again the pre-Hilbert space $\mathscr{C}^0([a, b])$ from Example 3.1.9, *i.)*. We claim that there are subspaces $U \subseteq \mathscr{C}^0([a, b])$ for which $U^{\perp\perp} \oplus U^\perp$ is not yet the whole pre-Hilbert space. Indeed, let $c \in (a, b)$ be an interior point of the interval. Then consider the subspace

$$U = \left\{ \phi \in \mathscr{C}^0([a, b]) \;\middle|\; \operatorname{supp} \phi \subseteq [a, c] \right\}. \tag{3.1.24}$$

Obviously, we have $\psi \in U^\perp$ iff $\operatorname{supp} \psi \subseteq [c, b]$ and hence

$$U^\perp = \left\{ \psi \in \mathscr{C}^0([a, b]) \;\middle|\; \operatorname{supp} \psi \subseteq [c, b] \right\}. \tag{3.1.25}$$

Moreover, we clearly have $U = U^{\perp\perp}$ in this case. However, the constant function $f(x) = 1$ is clearly continuous on $[a, b]$ but we can not find functions $\phi \in U$ and $\psi \in U^\perp$ with $\phi + \psi = 1$. By continuity of ϕ and ψ we necessarily have $\phi(c) = 0 = \psi(c)$. It is clear that also for all the other above function space we can construct subspaces U with this unpleasant feature, see Exercise 3.6.4.

3.1.2 Completeness and Completion

The main flaws of the pre-Hilbert spaces as in Example 3.1.9 is that they are not complete. This the the ultimate reason for all other sorts of weird behaviour which we want to exclude. We consider the following example:

[1]Limited budgets in science yield an immediate proof of this statement.

Example 3.1.11 Consider again $\mathscr{C}_0^\infty(\mathbb{R})$ and let $a < b$ be real numbers. Then we denote by

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{for } x \in [a,b] \\ 0 & \text{else} \end{cases} \tag{3.1.26}$$

the *characteristic function* of the interval $[a,b] \subseteq \mathbb{R}$. It is clear that such a characteristic function is Riemann integrable with integral given just by $b-a$. We can extend our inner product (3.1.17) also to this more general type of functions, say the piecewise continuous ones with compact support. We claim that there is a sequence $\phi_n \in \mathscr{C}_0^\infty(\mathbb{R})$ such that ϕ_n is a Cauchy sequence with respect to the pre-Hilbert space norm and we have

$$\lim_{n\longrightarrow\infty} \big\|\phi_n - \chi_{[a,b]}\big\| = 0, \tag{3.1.27}$$

i.e. in the enlarged pre-Hilbert space ϕ_n converges to $\chi_{[a,b]}$. The construction of such a sequence is very easy. Any choice of smooth functions $0 \le \phi_n \le 1$ with $\operatorname{supp}\phi_n \subseteq [a-\frac{1}{n}, b+\frac{1}{n}]$ and $\phi_n\big|_{[a,b]} = 1$ will do the job. The actual construction of smooth functions with these features is well-known from elementary calculus. The verification of the convergence to $\chi_{[a,b]}$ with respect to the pre-Hilbert space norm $\|\cdot\|$ is simple. We conclude that $\mathscr{C}_0^\infty(\mathbb{R})$ is not complete and the completion should at least contain also functions like $\chi_{[a,b]}$, see Exercise 3.6.5.

We take this example as one of the main motivations for the definition of a Hilbert space:

Definition 3.1.12 (Hilbert space) *A complete pre-Hilbert space is called a Hilbert space.*

Here the completeness refers of course to the norm build out of the inner product according to Lemma 3.1.3. Since we have just a single norm which characterizes the topology of a Hilbert space, we see that Hilbert spaces are particular cases of Banach spaces. As an immediate consequence of Proposition 3.1.5 we obtain the following:

Corollary 3.1.13 *A Banach space is a Hilbert space iff its norm satisfies the parallelogram identity.*

Starting from a possibly non-complete pre-Hilbert space we can always complete it as a normed space and obtain a Banach space according to Proposition 2.3.13. It turns our that the resulting Banach space is then a Hilbert space:

Proposition 3.1.14 (Completion of a pre-Hilbert space) *Let $\mathfrak{H}$ be a pre-Hilbert space.*

i.) For $R > 0$, the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ is uniformly continuous on $\mathrm{B}_R(0) \times \mathrm{B}_R(0) \subseteq \mathfrak{H} \times \mathfrak{H}$.

ii.) The inner product extends canonically to the Banach space completion $\widehat{\mathfrak{H}}$ of $\mathfrak{H}$ and yields a Hilbert space $\widehat{\mathfrak{H}}$.

iii.) Every pre-Hilbert space can be completed uniquely to a Hilbert space (up to isomorphism).

Proof: Let $\phi, \psi, \phi', \psi' \in \mathrm{B}_R(0) \subseteq \mathfrak{H}$ and $\epsilon > 0$ be given. If $\|\phi - \phi'\| < \delta$ and $\|\psi - \psi'\| < \delta$ with $\delta = \frac{\epsilon}{2R}$ then we have

$$|\langle\phi,\psi\rangle - \langle\phi',\psi'\rangle| = \big|\langle\phi,\psi-\psi'\rangle + \langle\phi-\phi',\psi'\rangle\big| \le \|\phi\|\|\psi-\psi'\| + \|\psi'\|\|\phi-\phi'\| < 2R\delta = \epsilon.$$

This is uniform continuity as claimed. Note that on all of $\mathfrak{H} \times \mathfrak{H}$ the inner product is not uniformly continuous. Now let $(\phi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ be Cauchy sequences in $\mathfrak{H}$ converging to limits $\phi, \psi \in \widehat{\mathfrak{H}}$ in the completion of $\mathfrak{H}$, respectively. We claim that

$$\langle\phi,\psi\rangle = \lim_{n\longrightarrow\infty} \langle\phi_n,\psi_n\rangle \tag{$*$}$$

gives a well-defined positive definite inner product on $\widehat{\mathfrak{H}}$. We first check that this is well-defined. Thus let $(\tilde{\phi}_n)_{n\in\mathbb{N}}$ and $(\tilde{\psi}_{n\in\mathbb{N}})$ be other choices of Cauchy sequences in $\mathfrak{H}$ converging to the same limits ϕ and ψ, respectively. In this case, we know that $\phi_n - \tilde{\phi}_n$ as well as $\psi_n - \tilde{\psi}_n$ converge to zero for $n \longrightarrow \infty$. Since all the involved vectors are contained in some large enough ball $\mathrm{B}_R(0)$ we can apply the first part to this situation. The uniform continuity gives immediately

$$\lim_{n\longrightarrow\infty} \langle \phi_n, \psi_n - \tilde{\psi}_n \rangle = 0 = \lim_{n\longrightarrow\infty} \langle \phi_n - \tilde{\phi}_n, \tilde{\psi}_n \rangle$$

from which we conclude that the limits of $\langle \phi_n, \psi_n \rangle$ and $\langle \tilde{\phi}_n, \tilde{\psi}_n \rangle$ coincide. This shows that $(*)$ is well-defined. Then the properties of a positive semi-definite sesquilinear form are easily obtained from a standard continuity argument. Moreover, for the norm on $\widehat{\mathfrak{H}}$ we get

$$\|\phi\| = \lim_{n\longrightarrow\infty} \|\phi_n\| = \lim_{n\longrightarrow\infty} \sqrt{\langle \phi_n, \phi_n \rangle} = \sqrt{\langle \phi, \phi \rangle},$$

whenever $\phi_n \in \mathfrak{H}$ is a sequence converging to $\phi \in \widehat{\mathfrak{H}}$. Thus the norm on the completion is determined by the extension of the inner product. In particular, we know that $\widehat{\mathfrak{H}}$ is a Banach space and hence Hausdorff, implying that $\langle \phi, \phi \rangle > 0$ for $\phi \neq 0$. Thus $\widehat{\mathfrak{H}}$ is indeed a Hilbert space. Finally, the continuity according to the first part shows that the above extension is the unique possibility for a fixed choice of the Banach space completion. □

There is also an alternative way to prove this proposition, namely by using Proposition 3.1.5: by continuity it is easy to see that the Banach norm on the completion $\widehat{\mathfrak{H}}$ of $\mathfrak{H}$ still satisfies the parallelogram identity. Thus the completion is a pre-Hilbert space with inner product given by (3.1.6). It is now easy to see that the inner product obtained this way on $\widehat{\mathfrak{H}}$ restricts to the original inner product on $\mathfrak{H}$. In the above proof we took advantage to illustrate the usage of the uniform continuity of the inner product on bounded subsets of $\mathfrak{H}$.

3.1.3 Examples of Hilbert Spaces

In this subsection we collect some basic examples of Hilbert spaces relevant for quantum physics.

The first is actually more a construction than an example. We consider a non-empty set I and the vector space

$$\mathfrak{H} = \bigoplus_{i\in I} \mathfrak{H}_i \quad \text{with} \quad \mathfrak{H}_i = \mathbb{C}, \tag{3.1.28}$$

i.e. the direct sum of I copies of $\mathbb{C}$. We denote this direct sum sometimes by $\mathbb{C}^{(I)}$ in order to distinguish it from the Cartesian product $\mathbb{C}^I$ of I copies of $\mathbb{C}$. Note that $\mathbb{C}^{(I)} \subseteq \mathbb{C}^I$ is (by the very definition) the subspace of those $\phi = (\phi_i)_{i\in I}$ for which $\phi_i = 0$ except for finitely many $i \in I$. We will view $\mathfrak{H}$ now as this subspace. On $\mathfrak{H}$ we have the inner product

$$\langle \phi, \psi \rangle = \sum_{i\in I} \overline{\phi}_i \psi_i. \tag{3.1.29}$$

Indeed, this sum is well-defined since only finitely many terms may be different from zero. The properties of a pre-Hilbert space are then easily verified.

Claim 3.1.15 $\langle \cdot\,, \cdot \rangle$ *as in* (3.1.29) *makes* $\mathfrak{H}$ *a pre-Hilbert space.*

We consider now the vectors $\mathrm{e}_i \in \mathfrak{H}$ with 1 at the i-th position and zeros elsewhere. Then every $\phi \in \mathfrak{H}$ can be written as

$$\phi = \sum_{i\in I} \phi_i \mathrm{e}_i, \tag{3.1.30}$$

with the $\phi_i \in \mathbb{C}$ being the i-th component of $\phi = (\phi_i)_{i\in I}$.

Claim 3.1.16 *The vectors* $\{\mathrm{e}_i\}_{i\in I}$ *form a vector space basis of* $\mathfrak{H}$ *with*

$$\langle \mathrm{e}_i, \mathrm{e}_j \rangle = \delta_{ij}. \tag{3.1.31}$$

Indeed, this is obvious from the definitions.

In a next step we want to determine the completion of $\mathfrak{H}$ explicitly. To this end, we define the following subset

$$\widehat{\mathfrak{H}} = \left\{ \phi \in \mathbb{C}^I \;\middle|\; \phi_i = 0 \text{ for all except countably many } i \in I \text{ and } \sum\nolimits_{i\in I} |\phi_i|^2 < \infty \right\} \tag{3.1.32}$$

of the Cartesian product $\mathbb{C}^I$. Since we require $\phi_i = 0$ for all but countable many indices the condition

$$\sum_{i\in I} |\phi_i|^2 < \infty \tag{3.1.33}$$

boils down to the convergence of an ordinary *series*. Since $|\phi_i|^2 \geq 0$ the convergence of (3.1.33) is either absolute and hence unconditional or we have "absolute divergence to $+\infty$". In any case, the order of the summation in (3.1.33) will not play any role. We clearly have

$$\mathfrak{H} \subseteq \widehat{\mathfrak{H}}. \tag{3.1.34}$$

Claim 3.1.17 *For* $\phi, \psi \in \widehat{\mathfrak{H}}$ *the series* $\sum_{i\in I} \overline{\phi_i}\psi_i$ *has at most countably many non-zero terms and converges absolutely. Moreover, one has*

$$\left(\sum_{i\in I} |\overline{\phi_i}\psi_i| \right)^2 \leq \left(\sum_{i\in I} |\phi_i|^2 \right) \left(\sum_{i\in I} |\psi_i|^2 \right). \tag{3.1.35}$$

The first part of this claim is clear and the second is the usual Hölder inequality for series, see Exercise 2.5.39.

Claim 3.1.18 $\widehat{\mathfrak{H}}$ *is a subspace of* $\mathbb{C}^I$ *and for* $\phi, \psi \in \widehat{\mathfrak{H}}$ *one has*

$$\sqrt{\sum_{i\in I} |\phi_i + \psi_i|^2} \leq \sqrt{\sum_{i\in I} |\phi_i|^2} + \sqrt{\sum_{i\in I} |\psi_i|^2}. \tag{3.1.36}$$

Again, in $\phi + \psi \in \mathbb{C}^I$ we still have at most countably many non-trivial coefficients. Then the inequality (3.1.36) is the usual Minkowski inequality for series which follows as usual from (3.1.35). But this shows that $\phi + \psi \in \widehat{\mathfrak{H}}$ whenever $\phi, \psi \in \widehat{\mathfrak{H}}$. Since in this situation clearly $z\phi \in \widehat{\mathfrak{H}}$ for all $z \in \mathbb{C}$, the claim is established.

Claim 3.1.19 $\widehat{\mathfrak{H}}$ *becomes a pre-Hilbert space with the inner product*

$$\langle \phi, \psi \rangle = \sum_{i\in I} \overline{\phi_i}\psi_i. \tag{3.1.37}$$

Since by Claim 3.1.18 we know that $\widehat{\mathfrak{H}}$ is indeed a subspace and since by Claim 3.1.17 the series converges absolutely, the remaining properties of a positive definite inner product follow easily. We note that this definition extends the inner product of $\mathfrak{H}$ to $\widehat{\mathfrak{H}}$. The corresponding norm is just

$$\|\phi\| = \sqrt{\sum_{i\in I} |\phi_i|^2}. \tag{3.1.38}$$

Comparing this with the defining property of $\widehat{\mathfrak{H}}$ we see that a general vector $\phi \in \mathbb{C}^I$ is actually in $\widehat{\mathfrak{H}}$ iff the "norm" of ϕ as in (3.1.38) is finite. We are now in the position to establish the crucial property of $\widehat{\mathfrak{H}}$:

Claim 3.1.20 *$\widehat{\mathfrak{H}}$ is complete, i.e. a Hilbert space.*

To prove this we consider a Cauchy sequence $\phi_n \in \widehat{\mathfrak{H}}$. Since in any $\phi_n = (\phi_{n,i})_{i\in I}$ at most countable many entries are non-zero, and since we have countably many $(\phi_n)_{n\in\mathbb{N}}$ in a *sequence*, there are at most countable many $\phi_{n,i}$ different from zero. We will use this in the following without further mentioning, in particular, all occurring sums $\sum_{i\in I}$ will be ordinary series thanks to this fact. Now let $\epsilon > 0$ be given. Then for large enough $n, m \geq N$ we have $\|\phi_n - \phi_m\| < \epsilon$ by assumption and hence

$$\sum_{i\in I} |\phi_{n,i} - \phi_{m,i}|^2 < \epsilon^2, \tag{3.1.39}$$

which means in particular that for all $i \in I$ we have $|\phi_{n,i} - \phi_{m,i}| < \epsilon$. Thus for fixed $i \in I$ the sequence $(\phi_{n,i})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$, having a limit

$$\phi_i = \lim_{n\longrightarrow\infty} \phi_{n,i} \tag{3.1.40}$$

by the completeness of $\mathbb{C}$. We claim that $\phi = (\phi_i)_{i\in I}$ is in $\widehat{\mathfrak{H}}$ and $\phi_n \longrightarrow \phi$. To see this, let $J \subseteq I$ be an arbitrary *finite* subset. Then we have

$$\sum_{j\in J} |\phi_{n,j} - \phi_{m,j}|^2 \leq \sum_{i\in I} |\phi_{n,i} - \phi_{m,i}|^2 < \epsilon^2. \tag{3.1.41}$$

Hence the limit of the left hand side for $n \longrightarrow \infty$ exists and obeys the estimate

$$\sum_{j\in J} |\phi_j - \phi_{m,j}|^2 \leq \epsilon^2. \tag{3.1.42}$$

Since this estimate holds for all finite J and since we only have countably many non-trivial contributions for $i \in I$ we can conclude that

$$\sum_{i\in I} |\phi_i - \phi_{m,i}|^2 \leq \epsilon^2. \tag{3.1.43}$$

But this simply means that $\|\phi - \phi_m\|^2 \leq \epsilon^2$ from which we conclude $\phi - \phi_m \in \widehat{\mathfrak{H}}$. Since $\widehat{\mathfrak{H}}$ is a vector space and $\phi_m \in \widehat{\mathfrak{H}}$ we get $\phi \in \widehat{\mathfrak{H}}$ as well. Then the estimate (3.1.43) shows that indeed $\phi_m \longrightarrow \phi$ as claimed.

Thus we have established that $\widehat{\mathfrak{H}}$ is a Hilbert space. It remains to show that the pre-Hilbert space $\mathfrak{H}$ we started with is actually dense in $\widehat{\mathfrak{H}}$. This is the last claim:

Claim 3.1.21 *$\mathfrak{H}$ is dense in $\widehat{\mathfrak{H}}$.*

Indeed, let $\phi \in \widehat{\mathfrak{H}}$ be given. Then we have already seen that the set $I_\phi \subseteq I$ of those indices i with $\phi_i \neq 0$ is countable. If $\# I_\phi < \infty$ then $\phi \in \mathfrak{H}$ and nothing is to be shown. Thus assume I_ϕ is infinite and choose a bijection $\mathbb{N} \ni n \mapsto i_n \in I_\phi$. This allows to define

$$\phi_n = (\phi_{n,i})_{i\in I} \quad \text{with} \quad \phi_{n,i} = \begin{cases} \phi_{i_m} & \text{if } i = i_m \text{ with } m \leq n \\ 0 & \text{else.} \end{cases} \tag{3.1.44}$$

Clearly, $\phi_n \in \mathfrak{H}$ and a simple argument shows that $\phi_n \longrightarrow \phi$. This establishes that last claim.

In the following, the Hilbert space $\widehat{\mathfrak{H}}$ will be denoted by

$$\ell^2(I) = \widehat{\mathfrak{H}}. \tag{3.1.45}$$

In the particular case of $I = \mathbb{N}$ we simply write $\ell^2 = \ell^2(\mathbb{N})$. One calls ℓ^2 the Hilbert space of *square summable sequences*, see also our general discussion of sequence spaces in Exercise 2.5.40. Our discussion can be summarized as follows:

Proposition 3.1.22 (The Hilbert space $\ell^2(I)$) *Let I be a set. Then $\ell^2(I) \subseteq \mathbb{C}^I$ with the inner product* (3.1.37) *is a Hilbert space. The vectors $\mathrm{e}_i \in \ell^2(I)$ form an orthonormal set and their $\mathbb{C}$-span*

$$\mathbb{C}^{(I)} = \operatorname{span}_{\mathbb{C}}\{\mathrm{e}_i\}_{i \in I} \subseteq \ell^2(I) \tag{3.1.46}$$

is a dense subspace of $\ell^2(I)$.

Here, orthonormal refers to the property (3.1.31). The importance of such an orthonormal set of vectors will become clear in Subsection 3.3.2 where we will see that any Hilbert space is isometrically isomorphic to $\ell^2(I)$ for an appropriate index set I.

Remark 3.1.23 The above proof of completeness can be cast into a bigger picture when interpreting I as a measure space with the counting measure: then it is known that for any measure space the square integrable functions on it form a Hilbert space. The above argument is somehow the shadow of the general measure-theoretic approach to square integrability for this very particular counting measure. For more details we refer to Appendix C.3.2.

The next and probably most important example for quantum mechanics are the square integrable functions. We do not give the corresponding proofs here, which can be found in full generality in Appendix C.3.2.

The Hilbert space completion of the pre-Hilbert spaces from Example 3.1.9 are given by the Hilbert space of square integrable functions with respect to the Lebesgue measure on $[a, b] \subseteq \mathbb{R}$ or on $\mathbb{R}^n$, respectively. In some more detail, one considers those functions $\psi \in \mathscr{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ on $\mathbb{R}^n$ which are measurable for the Borel σ-algebra and have the property that $\overline{\psi}\psi$ is integrable (the case of functions on a compact or open subset of $\mathbb{R}^n$ is analogous). For such functions,

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}^n} \overline{\phi(x)} \psi(x) \, \mathrm{d}^n x \tag{3.1.47}$$

turns out to be a well-defined positive semi-definite inner product while

$$\|\phi\|_2 = \sqrt{\int_{\mathbb{R}^n} |\phi(x)|^2 \, \mathrm{d}^n} \tag{3.1.48}$$

is the corresponding seminorm. Note that there are functions $\psi \in \mathscr{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ which are *not* zero but have $\|\psi\|_2 = 0$, so $\|\cdot\|_2$ is indeed only a *seminorm*. From Remark 2.2.43 we know how to handle this situation: we pass to the quotient

$$\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x) = \mathscr{L}^2(\mathbb{R}^n, \mathrm{d}^n x) \big/ \{0\}^{\mathrm{cl}}, \tag{3.1.49}$$

where $\{0\}^{\mathrm{cl}}$ is the topological closure of 0. It turns out that this consists precisely of those functions $\psi \in \mathscr{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ which vanish outside a set of measure zero. Equivalently, they satisfy $\|\psi\|_2 = 0$. Then $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ is Hausdorff and $\|\cdot\|_2$ becomes a well-defined norm on it. It is now easy to see that this norm still satisfies the parallelogram identity and hence $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ is a pre-Hilbert space. Equivalently, one can see directly that the inner product (3.1.47) passes to the quotient. By standard theorems of measure theory one confirms that $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ is complete and hence a Hilbert space.

In a last step, one notices that the spaces $\mathscr{C}^k_0(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0 \cup \{+\infty\}$ and $\mathscr{S}(\mathbb{R}^n)$ are part of $\mathscr{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$. Hence they are mapped to the quotient $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$. The crucial point is that the seminorm $\|\cdot\|_2$ of $\mathscr{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ is already a *norm* on the subspace $\mathscr{C}^k_0(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n)$. Thus the induced map

$$\mathscr{C}^k_0(\mathbb{R}^n), \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x) \tag{3.1.50}$$

is *injective*. In fact, one can show that any square integrable *continuous* function, i.e. the intersection $\mathscr{C}(\mathbb{R}^n) \cap \mathscr{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ is injectively mapped into $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$. This allows to identify continuous square integrable functions with their images in the quotient $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$. With some slight abuse of notation we therefore write for all $k \in \mathbb{N}_0 \cup \{+\infty\}$

$$\mathscr{C}_0^k(\mathbb{R}^n), \mathscr{S}(\mathbb{R}^n) \subseteq \mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x). \tag{3.1.51}$$

Moreover, since we can approximate the characteristic functions of e.g. compact intervals $[a_1, b_1] \times \cdots \times [a_n, b_n]$ by smooth and compactly supported functions with respect to the norm $\|\cdot\|_2$, see Example 3.1.11, one can show that the image of $\mathscr{C}_0^\infty(\mathbb{R}^n)$ in $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ is already *dense*, see Exercise **??**. Thus $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ turns out to be the *completion* of $\mathscr{C}_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_2$. Clearly, also all the other function spaces are dense as they contain $\mathscr{C}_0^\infty(\mathbb{R}^n)$. Hence all these function spaces have the same completion, namely $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$. Note that we could have defined $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ directly, without any measure theoretic background, as the Hilbert space completion of, say, $\mathscr{C}_0^\infty(\mathbb{R}^n)$. We summarize this discussion as follows:

pletion of $\mathscr{C}_0^\infty$

Proposition 3.1.24 (The Hilbert space $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$) *The square integrable functions with respect to the Lebesgue measure modulo the zero-functions form a Hilbert space $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ with respect to the inner product*

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}^n} \overline{\phi(x)} \psi(x) \, \mathrm{d}^n x. \tag{3.1.52}$$

All the function spaces $\mathscr{C}_0^k(\mathbb{R}^n)$ for $k \in \mathbb{N}_0 \cup \{+\infty\}$ and $\mathscr{S}(\mathbb{R}^n)$ are mapped injectively into the Hilbert space $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ and have dense image therein.

The last example we want to discuss is again a function space but of some different nature. While square integrable functions can have a very wild behaviour the following class is particularly nice: recall that a function $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ is called *anti-holomorphic* if $\overline{f}$ is holomorphic. This means that $\overline{f}$ is continuously differentiable and satisfies the Cauchy-Riemann differential equations, see e.g. the textbooks [44] or [48] for a more detailed introduction to holomorphic functions as well as Appendix B.6. It is a non-trivial consequence that such a (anti-) holomorphic function has a convergent Taylor expansion on $\mathbb{C}^n$ in z or $\overline{z}$, respectively. Thus, if f is anti-holomorphic we have

$$f(z) = \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{i_1, \ldots, i_r = 1}^{n} \frac{\partial^r f}{\partial \overline{z}^{i_1} \cdots \partial \overline{z}^{i_r}}(0) \, \overline{z}^{i_1} \cdots \overline{z}^{i_r} \tag{3.1.53}$$

for all $z \in \mathbb{C}^n$. The anti-holomorphic functions on $\mathbb{C}^n$ will then be denoted by $\overline{\mathscr{O}}(\mathbb{C}^n)$. For a fixed parameter $\hbar > 0$ one can now define the *Bargmann-Fock space* $\mathfrak{H}_{\mathrm{BF}}$ by

$$\mathfrak{H}_{\mathrm{BF}} = \left\{ f \in \overline{\mathscr{O}}(\mathbb{C}^n) \;\middle|\; \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{C}^n} |f(z)|^2 \mathrm{e}^{-\frac{\overline{z}z}{2\hbar}} \, \mathrm{d}^n z \, \mathrm{d}^n \overline{z} < \infty \right\}, \tag{3.1.54}$$

where $\mathrm{d}^n z \, \mathrm{d}^n \overline{z}$ denotes the usual Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\overline{z}z = \overline{z}^1 z^1 + \cdots + \overline{z}^n z^n$. Since the integrand is clearly continuous and non-negative an improper Riemann integral will actually do the job. Thanks to the rapidly decaying Gaussian function we see that the anti-holomorphic polynomials $\mathbb{C}[\overline{z}^1, \ldots, \overline{z}^n]$ are contained in $\mathfrak{H}_{\mathrm{BF}}$.

With the usual estimates one can show that $\mathfrak{H}_{\mathrm{BF}}$ is actually a subspace of $\overline{\mathscr{O}}(\mathbb{C}^n)$ and

$$\langle f, g \rangle_{\mathrm{BF}} = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{C}^n} \overline{f(z)} g(z) \mathrm{e}^{-\frac{\overline{z}z}{2\hbar}} \, \mathrm{d}^n z \, \mathrm{d}^n \overline{z} \tag{3.1.55}$$

is a well-defined positive definite inner product on $\mathfrak{H}_{\mathrm{BF}}$. This way, $\mathfrak{H}_{\mathrm{BF}}$ becomes a pre-Hilbert space. Less evident is the fact that $\mathfrak{H}_{\mathrm{BF}}$ is already complete:

Proposition 3.1.25 (Bargmann-Fock space) *The Bargmann-Fock space $\mathfrak{H}_{\mathrm{BF}}$ is a closed subspace*

$$\mathfrak{H}_{\mathrm{BF}} \subseteq \mathrm{L}^2\left(\mathbb{C}^n, \frac{1}{(2\pi\hbar)^n}\mathrm{e}^{-\frac{\overline{z}z}{2\hbar}}\,\mathrm{d}^n z\,\mathrm{d}^n\overline{z}\right) \tag{3.1.56}$$

of the Hilbert space of square integrable functions on $\mathbb{C}^n$ with respect to the Gaussian measure.

The proof of this fact as well as a discussion of other interesting properties of this Hilbert space will be the topic of the Exercises 3.6.7 and 3.6.8.

3.2 The Geometry of Hilbert Spaces

In this section we will discuss how the completeness of a Hilbert space changes its geometry compared to a general pre-Hilbert space. The first major difference will be the fact that orthogonal complements now behave much better: every close subspace induces an orthogonal decomposition of the Hilbert space. The set of all closed subspaces forms an orthomodular lattice in a natural way, explaining the various operations we have for closed subspaces in a conceptual way. The second important difference is that a Hilbert space will canonically be isomorphic to its topological dual. This Theorem of Riesz is of fundamental importance in quantum theory. We will conclude this section with some remarks on the weak topology of a Hilbert space.

3.2.1 The Lattice of Closed Subspaces

The first instance where completeness improves the features of a pre-Hilbert space is the following theorem which gives a strengthening of the results of Proposition 3.1.7:

Theorem 3.2.1 (Orthogonal complement) *Let $\mathfrak{H}$ be a Hilbert space and let $U \subseteq \mathfrak{H}$ be a subspace.*

i.) *With respect to the restriction of the inner product to U the space U is a Hilbert space iff U is closed.*

ii.) *The orthogonal complement $U^\perp$ is a closed subspace and hence a Hilbert subspace, too.*

iii.) *One has $(U^{\mathrm{cl}})^\perp = U^\perp$.*

iv.) *Assume $U = U^{\mathrm{cl}}$. Then for every $\phi \in \mathfrak{H}$ there exist unique vectors $\phi_\perp \in U^\perp$ and $\phi_\parallel \in U$ with*

$$\phi = \phi_\perp + \phi_\parallel. \tag{3.2.1}$$

For their norms one has

$$\|\phi_\perp\| = \inf_{\psi \in U} \|\phi - \psi\| \tag{3.2.2}$$

$$\|\phi_\parallel\| = \inf_{\psi \in U^\perp} \|\phi - \psi\|, \tag{3.2.3}$$

where the infimum is a minimum obtained by $\psi = \phi_\parallel$ and $\psi = \phi_\perp$, respectively.

v.) *One has $U \oplus U^\perp = \mathfrak{H}$ iff $U = U^{\mathrm{cl}}$.*

vi.) *One has $U^{\mathrm{cl}} = U^{\perp\perp}$ and thus U is dense in $U^{\perp\perp}$.*

PROOF: For the first part we know by Proposition 2.1.11, *i.)*, that a closed subspace of a complete vector space is complete itself. Thus a closed U is a Hilbert space by its own. The converse is true as well: if U is complete and $\psi \in U^{\mathrm{cl}}$ then choose $\psi_n \in U$ with $\psi_n \longrightarrow \psi$. Since ψ_n is clearly a Cauchy sequence we conclude that its limit ψ belongs to U by completeness. This shows the first part. For the second part, let $\psi \in U$. Then the map

$$\phi \mapsto \langle \psi, \phi \rangle$$

is a continuous linear functional on $\mathfrak{H}$ by Proposition 3.1.14, *i.)*. Thus its kernel $\ker\langle\psi, \cdot\,\rangle$ is a *closed* subspace of $\mathfrak{H}$. It follows that the intersection

$$U^\perp = \bigcap_{\psi \in U} \ker\langle\psi, \cdot\,\rangle \subseteq \mathfrak{H}$$

is still closed. For the third part we know on the one hand $U \subseteq U^{\mathrm{cl}}$ and hence $(U^{\mathrm{cl}})^\perp \subseteq U^\perp$ by Proposition 3.1.7, *iii.)*. On the other hand, let $\phi \in U^\perp$ and $\psi \in U^{\mathrm{cl}}$ be arbitrary. Choose a sequence $\psi_n \in U$ with $\psi_n \longrightarrow \psi$. Then $\langle\phi, \psi\rangle = \lim_{n\longrightarrow\infty}\langle\phi, \psi_n\rangle = 0$ by the continuity of the inner product. Thus $\phi \in (U^{\mathrm{cl}})^\perp$ follows, proving the third part. The fourth part is really the crucial point. We assume $U = U^{\mathrm{cl}}$ and let $\phi \in \mathfrak{H}$ be given. Consider now the *distance* of ϕ to U, i.e.

$$d = \inf_{\psi \in U} \|\phi - \psi\| \leq \|\phi - \psi\|, \tag{$*$}$$

where the last inequality holds for all $\psi \in U$. Choose a sequence $\psi_n \in U$ with $\|\phi - \psi_n\| \longrightarrow d$ according to the definition of d as an infimum. We claim that this sequence ψ_n is a Cauchy sequence. Indeed, by the parallelogram identity (3.1.7) we have

$$\|\psi_n - \psi_m\|^2 = \|\psi_n - \phi + \phi - \psi_m\|^2 = 2\|\psi_n - \phi\|^2 + 2\|\psi_m - \phi\|^2 - \|\psi_n - \phi - (\phi - \psi_m)\|^2.$$

For the last term we have $\|2\phi - \psi_n - \psi_m\|^2 \geq 2d^2$ by $(*)$. This gives

$$\|\psi_n - \psi_m\|^2 \leq 2\|\psi_n - \phi\|^2 + 2\|\psi_m - \phi\|^2 - 4d^2 \longrightarrow 0$$

for $n, m \longrightarrow \infty$ since $\|\psi_n - \phi\| \longrightarrow d$ by assumption. By the closedness of $U = U^{\mathrm{cl}}$ and hence by completeness according to *i.)* we conclude that $\psi_n \longrightarrow \psi \in U$ for some ψ. We set $\phi_\parallel = \psi$ and $\phi_\perp = \phi - \psi$. Observe that $d = \lim_{n\longrightarrow\infty}\|\phi - \psi_n\| = \|\phi - \phi_\parallel\| = \|\phi_\perp\|$, proving (3.2.2). For $z \in \mathbb{C}$ and $\chi \in U$ we have

$$\begin{aligned}
d^2 &\leq \|\phi - (\phi_\parallel - z\chi)\|^2 \\
&= \|\phi_\perp - z\chi\|^2 \\
&= \langle\phi_\perp, \phi_\perp\rangle - z\langle\phi_\perp, \chi\rangle - \overline{z}\langle\chi, \phi_\perp\rangle + \overline{z}z\langle\chi, \chi\rangle \\
&= d^2 + \overline{z}z\langle\chi, \chi\rangle - z\langle\phi_\perp, \chi\rangle - \overline{z}\langle\chi, \phi_\perp\rangle.
\end{aligned}$$

But this is only possible for all $z \in \mathbb{C}$ if $\langle\phi_\perp, \chi\rangle = 0$. This shows $\phi_\perp \in U^\perp$ proving the orthogonal decomposition (3.2.1). Since in the whole discussion U and $U^\perp$ enter in an entirely symmetric way, we can exchange their roles. This exchanges the notion of parallel and orthogonal components $\phi_\parallel \leftrightarrow \phi_\perp$. From this and (3.2.2) we get (3.2.3). This completes the proof of the fourth part. For the fifth part, we note that $U = U^{\mathrm{cl}}$ gives $U \oplus U^\perp = \mathfrak{H}$ by the fourth part. Conversely, assume that $U \neq U^{\mathrm{cl}}$ and pick $\phi \in U^{\mathrm{cl}}$ not being in U. For such a ϕ we have $\phi_\perp = 0$ since $(U^{\mathrm{cl}})^\perp = U^\perp$ and hence ϕ can not be in $U \oplus U^\perp$. For the last part we note that $\mathfrak{H} = U^{\mathrm{cl}} \oplus U^\perp$ by the third and fourth part. But then it is clear that U^{cl} is the orthogonal complement of $U^\perp$, i.e. $U^{\mathrm{cl}} = U^{\perp\perp}$. □

Corollary 3.2.2 *Let $U \subseteq \mathfrak{H}$. Then U is dense iff $U^\perp = \{0\}$.*

This corollary will provide an efficient tool to determine dense subspaces of a Hilbert space, see e.g. Exercise 3.6.10 for an application.

While being extremely useful at many places, Theorem 3.2.1 is also at the core of the *quantum physical interpretation* of subspaces in a Hilbert space. To a closed subspace $U \subseteq \mathfrak{H}$ one typically assigns a property like "U is certainly true". Then the orthogonal complement $U^\perp$ means "U is certainly false". For this to make sense $U \oplus U^\perp = \mathfrak{H}$ is crucial. Now the new "quantum" feature is that a state vector $\phi \in \mathfrak{H}$ can describe a pure physical state in which both properties "U" and "not U" are both a little bit true: $\phi_\parallel \in U$ tells us ultimately the probability for finding the property "U" when measuring in the state ϕ while $\phi_\perp$ gives the probability for measuring "not U". In particular, in the generic situation both probabilities will be non-zero which is the new feature of quantum physics compared to classical physics: here a certain property is either true or false in a pure state. The mathematical structure behind this "quantum logic" is that of a lattice[2]:

Definition 3.2.3 (Lattice) *A set V with two maps $\wedge, \vee \colon V \times V \longrightarrow V$ is called a lattice if for all $a, b, c \in V$ one has*

i.) $a \vee b = b \vee a$ *and* $a \vee (b \vee c) = (a \vee b) \vee c$,

ii.) $a \wedge b = b \wedge a$ *and* $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,

iii.) $a \vee a = a = a \wedge a$,

iv.) $(a \vee b) \wedge a = a = (a \wedge b) \vee a$.

In particular, the compositions $\vee$ and $\wedge$ are both commutative and associative. The interpretation is that $\vee$ corresponds to a logical OR while $\wedge$ corresponds to a logical AND, which explains the third and fourth axiom.

In a lattice, we can define a partial order as follows:

Lemma 3.2.4 *Let V be a lattice.*

i.) One has $a \vee b = b$ iff $a \wedge b = a$ for all $a, b \in V$.

ii.) Define $a \leq b$ if $a \vee b = b$. Then $(V, \leq)$ becomes a partially ordered set.

iii.) The partial order $\leq$ admits inf *and* sup *for each two elements $a, b \in V$, explicitly given by*

$$\inf(a, b) = a \wedge b \quad \text{and} \quad \sup(a, b) = a \vee b. \tag{3.2.4}$$

Conversely, if $(V, \leq)$ is a partially ordered set with inf *and* sup *for any two elements then* (3.2.4) *define a lattice structure on V whose partial order according to ii.) reproduces the given order of $(V, \leq)$.*

PROOF: The proof is the content of Exercise 3.6.13. $\square$

Thus we have an equivalent definition for a lattice. In the following we will use both characterizations interchangeably. The following additional features of lattices will play an important role:

Definition 3.2.5 (Maximal and minimal elements) *Let $(V, \vee, \wedge)$ be a lattice.*

i.) $1 \in V$ is called maximal element (or unit) if $a \leq 1$ for all $a \in V$.

ii.) $0 \in V$ is called minimal element (or zero) if $0 \leq a$ for all $a \in V$.

An element 1 is maximal iff $a \wedge 1 = a$ for all a and analogously, 0 is minimal iff $a \vee 0 = a$ for all a. If 1 or 0 exist then they are necessarily unique. Hence we can speak of *the* unit and *the* zero element.

Definition 3.2.6 (Distributive lattice) *Let $(V, \vee, \wedge)$ be a lattice. The lattice V is called distributive if for all $a, b, c \in V$ one has*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{and} \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \tag{3.2.5}$$

[2]One should be aware that there are at least two notions of "lattice" in the mathematical language, both relevant for quantum physics. The other option is used for the $\mathbb{Z}$-span of a basis in a finite-dimensional real vector space, as e.g. $\mathbb{Z}^n \subseteq \mathbb{R}^n$, needed e.g. to describe crystals in solid state physics.

Definition 3.2.7 (Orthomodular lattice) *Let $(V, \wedge, \vee, 1, 0)$ be a lattice with 0 and 1 and a map $'\colon V \longrightarrow V$ such that*

i.) $a \leq b$ *implies* $b' \leq a'$,

ii.) $(a')' = a$,

iii.) $a \wedge a' = 0$,

iv.) $a \vee a' = 1$

for all $a, b \in V$. Then $(V, \wedge, \vee, 1, 0,')$ is called an orthocomplemented lattice. If in addition

v.) $a \leq b$ *implies* $(a \vee b') \wedge b = a$ *for all* $a, b \in V$

then the lattice is called orthomodular.

The interpretation of the map $'$ is that of a logical NOT. The standard example of an orthomodular lattice is the one underlying classical logic:

Example 3.2.8 (The lattice 2^M) Let M be a non-empty set and 2^M its power set. Then 2^M is an orthomodular, distributive lattice with respect to the operations

$$U \wedge V = U \cap V \quad \text{and} \quad U \vee V = U \cup V, \tag{3.2.6}$$

$$0 = \emptyset \quad \text{and} \quad 1 = M, \tag{3.2.7}$$

$$U' = M \setminus U, \tag{3.2.8}$$

where $U, V \subseteq M$ are subsets. The canonical partial order is then just the set-theoretic inclusion, i.e. $U \leq V$ iff $U \subseteq V$. The verification of the claimed properties is elementary. The interpretation of this in classical physics is that M represents a phase space, the points in it correspond to classical states, and U is a "property" which is satisfied if the state corresponds to a point belonging to U.

Contrary to a distributive lattice the "quantum logic" requires more freedom. It is this distributive law which may fail in quantum physics. We have the following result for the lattice of the closed subspaces of a Hilbert space:

Theorem 3.2.9 (The lattice of closed subspaces) *Let $\mathfrak{H}$ be a Hilbert space. Then the set of all closed subspaces of $\mathfrak{H}$ forms an orthomodular lattice with respect to the operations*

$$U \wedge V = U \cap V \quad \textit{and} \quad U \vee V = (U + V)^{\mathrm{cl}} = (U + V)^{\perp\perp}, \tag{3.2.9}$$

$$U' = U^{\perp}, \tag{3.2.10}$$

$$0 = \{0\} \quad \textit{and} \quad 1 = \mathfrak{H}, \tag{3.2.11}$$

where $U, V \subseteq \mathfrak{H}$ are closed subspaces. Then the ordering is given by $U \leq V$ iff $U \subseteq V$. If $\dim \mathfrak{H} \geq 2$ then this lattice is not distributive.

PROOF: First we note that $U \wedge V$, $U \vee V$ and $U^{\perp}$ are indeed again closed subspaces of $\mathfrak{H}$. The verification is now rather straightforward by using Theorem 3.2.1. We leave the details as exercise. For the last statement consider the orthonormal vectors $e_1, e_2 \in \mathbb{C}^2$ and the three one-dimensional subspaces $U = \mathbb{C}e_1$, $V = \mathbb{C}e_2$, and $W = \mathbb{C}(e_1 + e_2)$. Then it is immediate that the distributive law fails for them. □

The existence of infimum and supremum for any two elements in the lattice of closed subspaces of $\mathfrak{H}$ can be strengthened as follows:

Proposition 3.2.10 *In the lattice of closed subspaces of a Hilbert space $\mathfrak{H}$ the* inf *of arbitrary decreasing and the* sup *of arbitrary increasing nets exist. More precisely, let I be a directed set and $\{U_i\}_{i \in I}$ a family of closed subspaces.*

i.) *If* $U_i \le U_j$ *for* $j \preccurlyeq i$ *then*

$$U_{\inf} = \bigcap_{i \in I} U_i \tag{3.2.12}$$

is the infimum of $\{U_i\}_{i \in I}$.

ii.) *If* $U_i \le U_j$ *for* $i \preccurlyeq j$ *then*

$$U_{\sup} = \left(\bigcup_{i \in I} U_i\right)^{\mathrm{cl}} \tag{3.2.13}$$

is the supremum of $\{U_i\}_{i \in I}$.

PROOF: The case of a decreasing net is simple. For an increasing net we first note that the union of all the U_i is indeed a subspace thanks to the filtration property. Then $U_{\sup}$ is a closed subspace containing all the U_i, by definition. Hence $U_i \le U_{\sup}$ since "$\le$" coincides with "$\subseteq$". Clearly, any other closed subspace V with this property has to contain $\bigcup_{i \in I} U_i$ as well. By closedness, also $U_{\sup} \subseteq V$, which shows that $U_{\sup}$ is indeed the smallest such subspace and hence the supremum. $\square$

3.2.2 The Dual of a Hilbert Space

In general, the explicit description of the topological dual of a given locally convex space or even a Banach space can be quite involved, see e.g. Appendix C.4 for several examples as well as the Exercises 2.5.38 and 2.5.41. In the case of a Hilbert space things are much nicer: in this short subsection we shall determine the topological dual of a Hilbert space. To this end, we note that for $\phi \in \mathfrak{H}$ the linear map

$$\jmath(\phi) \colon \mathfrak{H} \ni \psi \ \mapsto \ \langle \phi, \psi \rangle \in \mathbb{C} \tag{3.2.14}$$

is continuous. This is clear by the continuity of the inner product according to the Cauchy-Schwarz inequality $|\langle \phi, \psi \rangle| \le \|\phi\| \|\psi\|$. More specifically, this estimate shows that the norm of the linear functional $\jmath(\phi)$ satisfies $\|\jmath(\phi)\| \le \|\phi\|$. Taking $\psi = \phi$ shows

$$\|\jmath(\phi)\| = \|\phi\| \tag{3.2.15}$$

for all $\phi \in \mathfrak{H}$. This shows that the map

$$\jmath \colon \mathfrak{H} \longrightarrow \mathfrak{H}' \tag{3.2.16}$$

is an *antilinear* norm-preserving injection. In terms of the complex-conjugate Hilbert space we can also interpret this map as a *linear* and norm-preserving injection

$$\jmath \colon \overline{\mathfrak{H}} \longrightarrow \mathfrak{H}'. \tag{3.2.17}$$

The following theorem of Riesz shows that for a Hilbert space this is even a *surjective* map, giving thereby the explicit description of the dual we are looking for:

Theorem 3.2.11 (Riesz) *Let* $\mathfrak{H}$ *be a Hilbert space. Then the map* $\jmath$ *from* (3.2.16) *is an antilinear norm-preserving bijection.*

PROOF: It remains to show the surjectivity. Thus let $\varphi \in \mathfrak{H}'$ be given. Then $\ker \varphi \subseteq \mathfrak{H}$ is a closed subspace of $\mathfrak{H}$ by linearity and continuity of φ. To avoid trivialities we can assume $\varphi \neq 0$ and hence $\ker \varphi \neq \mathfrak{H}$. By Theorem 3.2.1, *v.)*, we have

$$\mathfrak{H} = \ker \varphi \oplus (\ker \varphi)^{\perp}. \tag{$*$}$$

Now we fix $\chi \in \mathfrak{H}$ with $\varphi(\chi) \neq 0$. By $(*)$ we can assume $\chi = \chi_{\perp} \in (\ker \varphi)^{\perp}$ without restriction. Let $\psi \in \mathfrak{H}$ be arbitrary. Then

$$\varphi\Big(\psi - \tfrac{\varphi(\psi)}{\varphi(\chi)} \chi\Big) = 0,$$

and hence $\psi - \frac{\varphi(\psi)}{\varphi(\chi)}\chi \in \ker\varphi$. One gets $0 = \left\langle \chi, \psi - \frac{\varphi(\psi)}{\varphi(\chi)}\chi \right\rangle = \langle \chi, \psi \rangle - \frac{\varphi(\psi)}{\varphi(\chi)}\langle \chi, \chi \rangle$ and hence

$$\varphi(\psi) = \left\langle \overline{\frac{\varphi(\chi)}{\langle \chi,\chi \rangle}}\chi, \psi \right\rangle.$$

This shows that we have found $\phi = \overline{\frac{\varphi(\chi)}{\langle \chi,\chi \rangle}}\chi$ with $\jmath(\phi) = \varphi$ as wanted. Note that the crucial point is the decomposition $(*)$ with a non-trivial orthogonal complement of the kernel. □

Remark 3.2.12 It follows from the proof that for $\varphi \in \mathfrak{H}'$ we have $(\ker\varphi)^\perp = \mathbb{C}\jmath^{-1}(\varphi)$.

Remark 3.2.13 (Dirac's bra and ket notation) This non-trivial result is the justification for the commonly used bra and ket notation in quantum physics. Vectors in $\mathfrak{H}$ are denoted as *kets* $|\psi\rangle$ and their counterparts in $\mathfrak{H}'$ by *bras* $\jmath(\psi) = \langle \psi|$. Then the notation for the inner product becomes the usual *bra-ket* $\jmath(\psi)(\phi) = \langle \psi|(|\phi\rangle) = \langle \psi \,|\, \phi \rangle$. Note that it is crucial that every element in $\mathfrak{H}'$ is actually a bra in order to give a meaning to this identification.

3.2.3 The Weak Topology

We shall now discuss the weak topology of a Hilbert space: here things simplify drastically compared to a general Banach space since the dual $\mathfrak{H}'$ is isomorphic to $\mathfrak{H}$ via Riesz' Theorem. As an immediate consequence we obtain the following corollary:

Corollary 3.2.14 (Weak topology) *Let $\mathfrak{H}$ be a Hilbert space.*

i.) The weak topology of $\mathfrak{H}$ is determined by the seminorms

$$\mathrm{p}_\psi(\phi) = |\langle \psi, \phi \rangle| \quad \textit{for all} \quad \psi \in \mathfrak{H}. \tag{3.2.18}$$

ii.) A net $(\varphi_i)_{i\in I}$ converges to $\varphi \in \mathfrak{H}$ in the weak topology iff for all $\psi \in \mathfrak{H}$ one has

$$\lim_{i\in I}\langle \psi, \varphi_i \rangle = \langle \psi, \varphi \rangle. \tag{3.2.19}$$

PROOF: Since every continuous linear functional in $\mathfrak{H}'$ is of the form $\varphi \mapsto \langle \psi, \varphi \rangle$ for a uniquely determined $\psi \in \mathfrak{H}$ by Theorem 3.2.11, the first part follows at once from the general characterization of the weak topology according to Subsection 2.3.3. By the same argument, the second part follows, see also Proposition 2.3.29, *i.)*. □

As we see from Exercise 3.6.12 the weak topology is in fact strictly coarser than the norm topology once $\mathfrak{H}$ is infinite-dimensional. In the context of Hilbert spaces, the norm topology is also called the *strong topology* in contrast to the weak topology. To distinguish the two types of convergence one frequently uses the symbols

$$\text{s-lim} \quad \text{and} \quad \text{w-lim} \tag{3.2.20}$$

to denote a limit with respect to the norm, i.e. the strong topology, and the weak topology, respectively. Note, however, that also other symbols are used in the literature. We shall not make much use of this notation but explicitly state the topologies involved.

Since the norm topology is finer, a norm convergent sequence is always weakly convergent. For the reverse implication to be true one needs an additional requirement:

Proposition 3.2.15 *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in a Hilbert space. Then the following statements are equivalent:*

i.) In the weak topology one has $\lim_{n\longrightarrow\infty}\varphi_n = \varphi$ and $\lim_{n\longrightarrow\infty}\|\varphi_n\| = \|\varphi\|$.

ii.) One has the norm convergence $\lim\varphi_{n\longrightarrow\infty} = \varphi$.

PROOF: The implication *ii.)* $\implies$ *i.)* is trivial since the norm is continuous and the norm topology is finer than the weak one. For the reverse, assume *i.)*. Then

$$\|\varphi_n - \varphi\|^2 = \|\varphi_n\|^2 + \|\varphi\|^2 - \langle\varphi_n, \varphi\rangle - \langle\varphi, \varphi_n\rangle \longrightarrow 0,$$

since $\langle\varphi_n, \varphi\rangle \longrightarrow \langle\varphi, \varphi\rangle$ for a weakly convergent sequence and the additional assumption $\|\varphi_n\| \longrightarrow \|\varphi\|$. □

Since a Hilbert space is canonically (antilinearly) isometric to its topological dual we compare the weak topology of $\mathfrak{H}$ and the weak* topology of $\mathfrak{H}'$ under this isomorphism:

Theorem 3.2.16 (Weak and weak* topology of a Hilbert space) *The canonical antilinear norm-preserving isomorphism $j\colon \mathfrak{H} \longrightarrow \mathfrak{H}'$ is a homeomorphism with respect to the weak and weak* topologies of $\mathfrak{H}$ and $\mathfrak{H}'$, respectively.*

PROOF: The weak* topology of $\mathfrak{H}'$ is determined by the seminorms $\mathrm{p}_\psi(\varphi) = |\varphi(\psi)|$ where $\psi \in \mathfrak{H}$ and $\varphi \in \mathfrak{H}'$. Thus for $\phi \in \mathfrak{H}$ we get

$$\mathrm{p}_\psi(\jmath(\phi)) = |\jmath(\phi)(\psi)| = |\langle\phi, \psi\rangle| = \mathrm{p}_\psi(\phi)$$

with the weak seminorm p_ψ as in (3.2.18). Under the identification via $\jmath$, the two systems of defining seminorms simply coincide. This is clearly enough to guarantee that $\jmath$ and $\jmath^{-1}$ are continuous. □

Corollary 3.2.17 *A Hilbert space is sequentially complete in the weak topology.*

PROOF: First we note that a Hilbert space is a particular case of a Fréchet space and hence Theorem 2.3.30 can be applied: it follows that $\mathfrak{H}'$ is sequentially complete with respect to the weak* topology. By Theorem 3.2.16 we can transfer this to $\mathfrak{H}$. □

Remark 3.2.18 This corollary is of immediate applicability to quantum physics: in many situations one wants to define a vector $\phi \in \mathfrak{H}$ by specifying its "components" $\langle\psi, \phi\rangle$, i.e. the inner products with all other vectors. Moreover, in many situations the definition of $\langle\psi, \phi\rangle$ comes from a limiting process. Hence we are automatically in the realm of the weak topology in this case. Nevertheless, Corollary 3.2.17 should be taken with some care since not all weak Cauchy (and hence weakly convergent) sequences converge also in the norm topology. Here the additional requirement from Proposition 3.2.15 is essential, see also Exercise 3.6.12 for further illustrating examples.

Riesz' Theorem is also the key to prove that Hilbert spaces are reflexive:

Proposition 3.2.19 *A Hilbert space is reflexive.*

PROOF: By Riesz' Theorem we know that $\jmath_{\mathfrak{H}}\colon \mathfrak{H} \longrightarrow \mathfrak{H}'$ is a norm-preserving antilinear isomorphism. It follows that the norm of $\mathfrak{H}'$ satisfies the parallelogram identity as well. Hence, $\mathfrak{H}'$ is a Hilbert space by its own and $\jmath_{\mathfrak{H}}$ becomes an isometric (though still antilinear) map, i.e. we have

$$\langle\jmath_{\mathfrak{H}}(\phi), \jmath_{\mathfrak{H}}(\psi)\rangle_{\mathfrak{H}'} = \langle\psi, \phi\rangle_{\mathfrak{H}}$$

for all $\phi, \psi \in \mathfrak{H}$. This follows from Proposition 3.1.5 and the obvious completeness of $\mathfrak{H}'$ as the dual of $\mathfrak{H}$. Thus we can apply Riesz' Theorem also to $\mathfrak{H}'$ and get an antilinear isomorphism $\jmath_{\mathfrak{H}'}\colon \mathfrak{H}' \longrightarrow \mathfrak{H}''$. Now let $\psi \in \mathfrak{H}$ and $\varphi \in \mathfrak{H}'$ be given. Then we compute

$$\jmath_{\mathfrak{H}'}(\jmath_{\mathfrak{H}}(\psi))\varphi = \langle j_{\mathfrak{H}}(\psi), \varphi\rangle_{\mathfrak{H}'} = \varphi(\psi) = \iota(\psi)\varphi,$$

with the canonical map $\iota\colon \mathfrak{H} \longrightarrow \mathfrak{H}''$ as usual. Thus $\iota = \jmath_{\mathfrak{H}'} \circ \jmath_{\mathfrak{H}}$ follows. Since both maps are bijections, ι is bijective as well, proving the claim. □

This fact allows to apply our general results on reflexive Banach spaces to Hilbert spaces. We list the relevant conclusions:

Corollary 3.2.20 *On the dual $\mathfrak{H}'$ of a Hilbert space $\mathfrak{H}$ the weak and the weak* topology coincide.*

PROOF: This is Proposition 2.3.38, *ii.)*. $\square$

Corollary 3.2.21 *The closed unit ball in a Hilbert space is weakly compact.*

PROOF: By the Banach-Alaoglu Theorem in form of Corollary 2.3.34 the closed unit ball in $\mathfrak{H}'$ is weak* compact. Since $\jmath\colon \mathfrak{H} \longrightarrow \mathfrak{H}'$ is a norm-preserving isomorphism by Riesz' Theorem, the closed unit ball in $\mathfrak{H}$ corresponds to the one in $\mathfrak{H}'$. Since $\jmath$ is also a homeomorphism in the weak and weak* topologies, see Theorem 3.2.16, the compactness is preserved under $\jmath^{-1}$. $\square$

Corollary 3.2.22 *The closed unit ball in a Hilbert space is sequentially compact in the weak topology.*

PROOF: This is Proposition 2.3.40. $\square$

3.3 Hilbert Bases and Classification

The existence of a Hilbert basis will allow us to give a complete and even rather easy classification of Hilbert spaces.

3.3.1 The Notion of a Hilbert Basis

The aim of this subsection is to establish one of the most important tools in Hilbert space analysis, namely the existence of a Hilbert basis. We have already met a first example, namely the vectors $\{\mathrm{e}_i\}_{i\in I}$ in $\ell^2(I)$: they had the remarkable feature of being orthonormal and having a dense $\mathbb{C}$-span. From Corollary 3.2.2 we see that the orthogonal complement of their span is necessarily trivial, i.e. there are no further non-zero vectors in $\ell^2(I)$ which are orthogonal to all the e_i. This maximality gives the motivation for the following definition which we can even state for a pre-Hilbert space:

Definition 3.3.1 (Hilbert basis) *Let $\mathfrak{H}$ be a pre-Hilbert space. Then a non-empty set $\{\mathrm{e}_i\}_{i\in I}$ of vectors in $\mathfrak{H}$ is called orthonormal if*

$$\langle \mathrm{e}_i, \mathrm{e}_j \rangle = \delta_{ij} \tag{3.3.1}$$

for all $i, j \in I$. If in addition $\{\mathrm{e}_i\}_{i\in I}$ is maximal with respect to this property then $\{\mathrm{e}_i\}_{i\in I}$ is called a Hilbert basis.

Note that we do not make any restrictions on the size of I, every index set is, in principle, allowed. Maximality of $\{\mathrm{e}_i\}_{i\in I}$ means that if $\{\mathrm{f}_j\}_{j\in J}$ is another orthonormal system with $\{\mathrm{e}_i\}_{i\in I} \subseteq \{\mathrm{f}_j\}_{j\in J}$ then already $\{\mathrm{e}_i\}_{i\in I} = \{\mathrm{f}_j\}_{j\in J}$.

Note also that an orthonormal system is necessarily linearly independent. Indeed, if we have a relation of the form

$$\sum_{i\in I} z_i \mathrm{e}_i = 0 \tag{3.3.2}$$

with only finitely many z_i different from zero, then taking the inner product with a fixed e_j gives $z_j = 0$ for all $j \in I$. This is the usual argument known from linear algebra. Thus an orthonormal system is always a vector space basis of its span $\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_i\}_{i\in I}$. However, as we shall see, a Hilbert basis will, in general, not be a vector space basis at all. If $\mathfrak{H}$ is complete, i.e. a Hilbert space, this is only the case in finite dimensions. Nevertheless, a Hilbert basis behaves very much the same as a vector space basis.

The following simple recursion scheme known from elementary linear algebra guarantees the existence of an orthonormal system in an at most countable situation.

Proposition 3.3.2 (Gram-Schmidt) *Let $\mathfrak{H}$ be a pre-Hilbert space and let $\{\varphi_1, \varphi_2, \ldots,\}$ be an at most countable set of linearly independent vectors in $\mathfrak{H}$. Then the recursively defined vectors*

$$\mathrm{e}_n = \frac{1}{\|f_n\|} f_n \tag{3.3.3}$$

with $f_1 = \varphi_1$ and

$$f_{n+1} = \varphi_{n+1} - \sum_{i=1}^{n} \langle \varphi_{n+1}, \mathrm{e}_i \rangle \mathrm{e}_i \tag{3.3.4}$$

form an orthonormal set $\{\mathrm{e}_1, \mathrm{e}_2, \ldots\}$ such that for all N one has

$$\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_1, \ldots, \mathrm{e}_N\} = \operatorname{span}_{\mathbb{C}}\{\varphi_1, \ldots, \varphi_N\}. \tag{3.3.5}$$

PROOF: This is a straightforward computation, see also Exercise 3.6.9. □

Let us consider the case of countably many vectors to avoid trivialities. Then it is easy to see that if the $\{\varphi_n\}_{n\in\mathbb{N}}$ are a vector space basis of the pre-Hilbert space $\mathfrak{H}$ then the Gram-Schmidt procedure yields a maximal orthonormal set of vectors $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ which in addition spans $\mathfrak{H}$. It is then easy to see that they also form a Hilbert basis for the Hilbert space completion $\widehat{\mathfrak{H}}$ of $\mathfrak{H}$: if there would be yet another vector $\mathrm{f} \in \widehat{\mathfrak{H}}$ orthonormal to the $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ we get $\mathrm{f} \in (\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_n\}_{n\in\mathbb{N}})^{\perp}$. This gives immediately $\mathrm{f} = 0$ by Theorem 3.2.1, contradicting $\|\mathrm{f}\| = 1$. Thus for Hilbert spaces with a dense subspace of countable vector space dimension we have shown the existence of a Hilbert basis constructively:

Corollary 3.3.3 *Let $\mathfrak{H}$ be a Hilbert space with a dense subspace $U \subseteq \mathfrak{H}$ spanned by at most countably many vectors, i.e. $U = \operatorname{span}_{\mathbb{C}}\{\varphi_n\}_{n\in I\subseteq\mathbb{N}}$. Then $\mathfrak{H}$ has a Hilbert basis.*

In fact, many Hilbert spaces are of this form as we shall discuss in the Exercises 3.6.10. We emphasize this point because the general situation of "larger" Hilbert spaces requires an application of Zorn's Lemma to prove the existence of a Hilbert basis:

Theorem 3.3.4 (Existence of a Hilbert basis) *Every pre-Hilbert space has a Hilbert basis. More specifically, if $\{\mathrm{e}_i\}_{i\in I}$ is an orthonormal set then there exists a Hilbert basis which contains $\{\mathrm{e}_i\}_{i\in I}$.*

PROOF: Clearly, it suffices to show the second statement. Thus let $\{\mathrm{e}_i\}_{i\in I}$ be an orthonormal set. We consider now the set $\mathcal{O}$ of all orthonormal sets containing $\{\mathrm{e}_i\}_{i\in I}$. Clearly, $\mathcal{O}$ is non-empty as it contains at least $\{\mathrm{e}_i\}_{i\in I}$. By set-theoretic inclusion we obtain a partial ordering $\subseteq$ for $\mathcal{O}$. If $\{O_\alpha\}_{\alpha\in A}$ is now a totally ordered subset of $\mathcal{O}$ then also $\bigcup_{\alpha\in A} O_\alpha$ is in $\mathcal{O}$ and it contains every O_α. This shows that every totally ordered subset in $\mathcal{O}$ has a maximal element. Thus, by Zorn's Lemma, $\mathcal{O}$ itself has a maximal element. It is now easy to see that this is the Hilbert basis we are looking for. □

Not yet using the maximality of a Hilbert basis, an orthonormal system of vectors in $\mathfrak{H}$ behaves very much like in finite dimensions. We collect some important properties, most of which hold even for a pre-Hilbert space:

Theorem 3.3.5 (Orthonormal system) *Let $\mathfrak{H}$ be a pre-Hilbert space and $\{\mathrm{e}_i\}_{i\in I}$ an orthonormal system of vectors in $\mathfrak{H}$.*

i.) For every finite subset $J \subseteq I$ one has for all $\varphi \in \mathfrak{H}$

$$\sum_{j\in J} |\langle \mathrm{e}_j, \varphi \rangle|^2 \leq \|\varphi\|^2. \tag{3.3.6}$$

ii.) *For every $\varphi \in \mathfrak{H}$ the set*

$$I_\varphi = \left\{ i \in I \;\middle|\; \langle \mathrm{e}_i, \varphi \rangle \neq 0 \right\} \subseteq I \tag{3.3.7}$$

is a countable subset of I.

iii.) *For all $\varphi \in \mathfrak{H}$ one has Bessel's inequality*

$$\sum_{i \in I} |\langle \mathrm{e}_i, \varphi \rangle|^2 \leq \|\varphi\|^2, \tag{3.3.8}$$

in the sense that all except for countably many terms are zero and the remaining series converges. The convergence is necessarily absolute.

iv.) *For all $\varphi, \psi \in \mathfrak{H}$ one has*

$$\sum_{i \in I} |\langle \varphi, \mathrm{e}_i \rangle \langle \mathrm{e}_i, \psi \rangle| \leq \|\varphi\| \|\psi\|, \tag{3.3.9}$$

with the same interpretation of the summation as in part iii.).

v.) *Assume that $\mathfrak{H}$ is complete, i.e. a Hilbert space, and let $U = (\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_i\}_{i \in I})^{\mathrm{cl}}$ be the closed subspace spanned by the orthonormal system. Then for every $\varphi \in \mathfrak{H}$ the series*

$$\sum_{i \in I} \langle \mathrm{e}_i, \varphi \rangle \mathrm{e}_i = \varphi_\parallel \tag{3.3.10}$$

converges unconditionally to the parallel component of φ with respect to the orthogonal splitting $\mathfrak{H} = U \oplus U^\perp$. Again, the series has at most countably many non-trivial terms.

PROOF: Let $\varphi \in \mathfrak{H}$ be given. For a finite subset $J \subseteq I$ we consider the vector

$$\varphi_J = \varphi - \sum\nolimits_{j \in J} \langle \mathrm{e}_j, \varphi \rangle \mathrm{e}_j.$$

Since $\langle \mathrm{e}_{j_0}, \mathrm{e}_j \rangle = \delta_{j_0 j}$ we have for every $j_0 \in J$

$$\langle \mathrm{e}_{j_0}, \varphi_J \rangle = \langle \mathrm{e}_{j_0} \varphi \rangle - \sum\nolimits_{j \in J} \langle \mathrm{e}_j, \varphi \rangle \langle \mathrm{e}_{j_0}, \mathrm{e}_j \rangle = 0.$$

It follows that φ_J is orthogonal to all e_j for $j \in J$. This gives by Pythagoras' Theorem

$$\|\varphi\|^2 = \|\varphi_J\|^2 + \left\| \sum\nolimits_{j \in J} \langle \mathrm{e}_j, \varphi \rangle \mathrm{e}_j \right\|^2 = \|\varphi_J\|^2 + \sum\nolimits_{j \in J} |\langle \mathrm{e}_j, \varphi \rangle|^2,$$

since also the terms in the sum are pairwise orthogonal. This gives immediately the estimate (3.3.6). For the second part we consider

$$I_\varphi^{(n)} = \left\{ i \in I \;\middle|\; |\langle \mathrm{e}_j, \varphi \rangle| > \tfrac{1}{n} \right\} \subseteq I. \tag{$*$}$$

In view of the estimate (3.3.6), each $I_\varphi^{(n)}$ can contain at most finitely many indices. Thus their union $I_\varphi = \bigcup_{n \geq 1} I_\varphi^{(n)}$ is at most countable which proves the second part. For the third part we notice that the sum contains at most countably many non-trivial terms according to the second part. Moreover, if we show convergence it will be automatically absolute as the series contains only non-negative terms. If the set I_φ is finite then (3.3.8) is just (3.3.6). Thus assume that $I_\varphi \cong \mathbb{N}$ and choose an enumeration $\{i_n\}_{n \in \mathbb{N}}$ of I_φ. Then by (3.3.6) we have

$$\sum_{n=1}^{k} |\langle \mathrm{e}_{i_n}, \varphi \rangle|^2 \leq \|\varphi\|^2$$

for all k. Hence the convergence and the estimate follow at once. We move to the fourth part: first we note that only indices in the at most countable set $I_\varphi \cap I_\psi$ play a role on the left hand side of (3.3.9). This explains the remark about the way the sum is to be understood. Then we get by Hölder's inequality

$$\begin{aligned}\sum\nolimits_{i\in I} |\langle\varphi, \mathrm{e}_i\rangle\langle\mathrm{e}_i, \psi\rangle| &= \sum\nolimits_{i\in I_\varphi\cap I_\psi} |\langle\varphi, \mathrm{e}_i\rangle\langle\mathrm{e}_i, \psi\rangle| \\ &\le \sqrt{\sum\nolimits_{i\in I_\varphi\cap I_\psi} |\langle\varphi, \mathrm{e}_i\rangle|^2}\sqrt{\sum\nolimits_{i\in I_\varphi\cap I_\psi} |\langle\mathrm{e}_i, \psi\rangle|^2} \\ &\overset{(3.3.8)}{\le} \|\varphi\|\|\psi\|.\end{aligned}$$

For the last part we first consider the convergence of the left hand side of (3.3.10). Clearly, by (3.3.7) we have at most countably many indices $i \in I_\varphi$ contributing to (3.3.10) non-trivially. If I_φ is finite nothing is to show. Thus we may assume that $I_\varphi \cong \mathbb{N}$ and we choose again an enumeration $\{i_n\}_{n\in\mathbb{N}}$ of I_φ. Then the series $\sum_{n=1}^\infty \langle\mathrm{e}_{i_n}, \varphi\rangle\mathrm{e}_{i_n}$ is a Cauchy series in $\mathfrak{H}$: indeed, for $N \le M$ we have

$$\left\|\sum_{n=N}^M \langle\mathrm{e}_{i_n}, \varphi\rangle\mathrm{e}_{i_n}\right\|^2 = \sum_{n=N}^M |\langle\mathrm{e}_{i_n}, \varphi\rangle|^2 \tag{$**$}$$

by Pythagoras. Since (3.3.8) converges absolutely with bound given by $\|\varphi\|^2$ we see that for N, M sufficiently large the sum ($**$) becomes small. This gives a Cauchy series. By completeness, $\psi = \sum_{n=1}^\infty \langle\mathrm{e}_{i_n}, \varphi\rangle\mathrm{e}_{i_n}$ exists. We have to show that ψ does not depend on the enumeration of I_φ in order to establish unconditional convergence. Thus let $\pi\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a bijection and let $\tilde{\psi} = \sum_{n=1}^\infty \langle\mathrm{e}_{i_{\pi(n)}}, \varphi\rangle\mathrm{e}_{i_{\pi(n)}}$ be the resulting limit for the permuted summation. For $\chi \in \mathfrak{H}$ we have

$$\begin{aligned}\langle\psi, \chi\rangle &= \left\langle \sum_{n=1}^\infty \langle\mathrm{e}_{i_n}, \varphi\rangle\mathrm{e}_{i_n}, \chi \right\rangle \\ &\overset{(a)}{=} \sum_{n=1}^\infty \langle\varphi, \mathrm{e}_{i_n}\rangle\langle\mathrm{e}_{i_n}, \chi\rangle \\ &\overset{(b)}{=} \sum_{n=1}^\infty \langle\varphi, \mathrm{e}_{i_{\pi(n)}}\rangle\langle\mathrm{e}_{i_{\pi(n)}}, \chi\rangle \\ &\overset{(a)}{=} \left\langle \sum_{n=1}^\infty \langle\mathrm{e}_{i_{\pi(n)}}, \varphi\rangle\mathrm{e}_{i_{\pi(n)}}, \chi \right\rangle \\ &= \langle\tilde{\psi}, \chi\rangle,\end{aligned}$$

where in (a) we use the convergence of the series and the continuity of the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ while (b) uses the fact that the series converges absolutely by (3.3.9). Now χ was arbitrary and hence $\psi = \tilde{\psi}$ follows. Thus the left hand side of (3.3.10) converges unconditionally to some vector $\psi \in \mathfrak{H}$. It remains to show that $\psi = \varphi_\parallel$. By construction, it is clear that $\psi \in U$. We have to show that $\varphi - \psi \in U^\perp$. For all $k \in I$ we compute

$$\langle\mathrm{e}_k, \varphi - \psi\rangle = \langle\mathrm{e}_k, \varphi\rangle - \sum_{i\in I} \langle\mathrm{e}_k, \langle\mathrm{e}_i, \varphi\rangle\mathrm{e}_i\rangle = \langle\mathrm{e}_k, \varphi\rangle - \langle\mathrm{e}_k, \varphi\rangle = 0,$$

by the convergence of the series, the continuity of the inner product, and $\langle\mathrm{e}_k, \mathrm{e}_i\rangle = \delta_{ki}$. Thus $\varphi - \psi \in (\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_i\}_{i\in I})^\perp = \left(\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_i\}_{i\in I}^{\mathrm{cl}}\right)^\perp = U^\perp$ follows from Theorem 3.2.1, *iii.)*. This shows $\psi = \varphi_\parallel$. ☐

Remark 3.3.6 (Unconditional versus absolute convergence) Recall that in a finite-dimensional vector space, in particular in $\mathbb{R}$ or $\mathbb{C}$, unconditional and absolute convergence coincide: if a series $z = \sum_{n=1}^{\infty} z_n$ of complex numbers converges unconditionally, i.e. every resummation converges again to the same limit z, then it converges absolutely, i.e. $\sum_{n=1}^{\infty} |z_n|$ converges, and vice versa. In infinite-dimensional Banach spaces we only have the implication that absolute convergence, i.e. convergence of $\sum_{n=1}^{\infty} \|\varphi_n\|$ implies unconditional convergence of $\sum_{n=1}^{\infty} \varphi_n$, but *not* the other way round. In fact, Exercise 3.6.11 provides a simple example in Hilbert space theory.

If the orthonormal system $\{\mathrm{e}_i\}_{i\in I}$ is even maximal, i.e. a Hilbert basis, the situation can be simplified even further. The next theorem clarifies this and justifies also the name "basis":

Theorem 3.3.7 (Hilbert basis) *Let $\{\mathrm{e}_i\}_{i\in I}$ be an orthonormal system in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

i.) The set $\{\mathrm{e}_i\}_{i\in I}$ is a Hilbert basis.

ii.) One has $(\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_i\}_{i\in I})^{\perp} = \{0\}$.

iii.) The subspace $\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_i\}_{i\in I}$ is dense in $\mathfrak{H}$.

iv.) For all $\varphi \in \mathfrak{H}$ we have

$$\varphi = \sum_{i\in I} \langle \mathrm{e}_i, \varphi\rangle \mathrm{e}_i. \tag{3.3.11}$$

v.) For all $\varphi, \psi \in \mathfrak{H}$ we have

$$\langle \varphi, \psi\rangle = \sum_{i\in I} \langle \varphi, \mathrm{e}_i\rangle\langle \mathrm{e}_i, \psi\rangle. \tag{3.3.12}$$

vi.) For all $\varphi \in \mathfrak{H}$ one has Parseval's identity

$$\|\varphi\|^2 = \sum_{i\in I} |\langle \mathrm{e}_i, \varphi\rangle|^2. \tag{3.3.13}$$

For the convergence of the series the same properties hold as in Theorem 3.3.5.

PROOF: We prove *i.)* $\implies$ *ii.)* $\implies$ *iii.)* $\implies$ *iv.)* $\implies$ *v.)* $\implies$ *vi.)* $\implies$ *i.)*. The main parts have already been achieved. Suppose *i.)* and let ψ be orthogonal to all $\{\mathrm{e}_i\}_{i\in I}$. If $\psi \neq 0$ then we could take $e_0 = \frac{\psi}{\|\psi\|}$ as an additional orthonormal vector contradicting the maximality of $\{\mathrm{e}_i\}_{i\in I}$. Thus $\psi = 0$ which is *ii.)*. The implication *ii.)* $\implies$ *iii.)* is Corollary 3.2.2. By Theorem 3.3.5 we know that the series converges to $\varphi_{\parallel}$. But with *iii.)* the closure is all of $\mathfrak{H}$ and hence $\varphi_{\parallel} = \varphi$. This shows *iii.)* $\implies$ *iv.)*. The next implication is just the continuity of the inner product. Then *v.)* $\implies$ *vi.)* follows trivially by setting $\psi = \varphi$. Finally, assume *vi.)* and assume $e_0 \in \mathfrak{H}$ is yet another orthonormal vector such that $\{e_0\} \cup \{\mathrm{e}_i\}_{i\in I}$ is still an orthonormal system. Then setting $\varphi = e_0$ with $\|e_0\| = 1$ gives an immediate contradiction in (3.3.13). Thus $\{\mathrm{e}_i\}_{i\in I}$ was already maximal. □

Example 3.3.8 (Hilbert basis) Let I be a set and consider the Hilbert space $\ell^2(I)$ from Proposition 3.1.22. Then this proposition shows that the canonical vectors $\{\mathrm{e}_i\}_{i\in I}$ form a Hilbert basis of $\ell^2(I)$.

3.3.2 The Classification of Hilbert Spaces

Unlike the situation of Banach, Fréchet, or even general locally convex spaces the variety of Hilbert spaces is rather easy to understand: they can be classified by the cardinality of their Hilbert bases:

Theorem 3.3.9 (Classification of Hilbert spaces) *Let $\mathfrak{H}$ be a Hilbert space.*

i.) If $\{\mathrm{e}_i\}_{i\in I}$ and $\{\mathrm{f}_j\}_{j\in J}$ are Hilbert bases of $\mathfrak{H}$ then $\#I = \#J$.

ii.) *If* $\{\mathrm{e}_i\}_{i\in I}$ *is a Hilbert basis of* $\mathfrak{H}$ *then the map*

$$U\colon \mathfrak{H} \ni \phi \;\mapsto\; U\phi = (\langle \mathrm{e}_i, \phi\rangle)_{i\in I} \in \ell^2(I) \tag{3.3.14}$$

defines an isometric linear bijection onto $\ell^2(I)$.

iii.) *Two Hilbert spaces are isometrically isomorphic iff their Hilbert bases have the same cardinality.*

PROOF: For the first part, we note that the statement is trivial if one (and hence necessarily the other) index set is finite. Thus we may assume that both index sets are infinite. For $i \in I$ the subset

$$J_i = \left\{ j \in J \;\middle|\; \langle \mathrm{e}_i, \mathrm{f}_j\rangle \neq 0 \right\} \subseteq J$$

is at most countable by Theorem 3.3.5, *ii.)*. Since the $\{\mathrm{e}_i\}_{i\in I}$ form a Hilbert basis, for every $j \in J$ there is at least one inner product $\langle \mathrm{e}_i, \mathrm{f}_j\rangle \neq 0$. Hence there is at least one $i \in I$ with $j \in J_i$. It follows that we have an injective map

$$J \longrightarrow \coprod_{i\in I} J_i$$

into the disjoint union of all the J_i. Since they are at most countable we see that $\#J \leq \#I\#\mathbb{N} = \#I$. Exchanging the role of I and J gives $\#I \leq \#J$ and thus $\#I = \#J$ by the *Schröder-Bernstein Theorem*, see [, ???]. For the second part we first note that $U\phi \in \mathbb{C}^I$ is actually in $\ell^2(I)$: indeed, by Theorem 3.3.7, *vi.)*, we have the equality

$$\|U\phi\|^2_{\ell^2(I)} = \sum_{i\in I} |\langle \mathrm{e}_i, \phi\rangle|^2 = \|\phi\|^2 < \infty,$$

which shows $U\phi \in \ell^2(I)$. Clearly, U is linear and isometric since

$$\langle U\phi, U\psi\rangle_{\ell^2(I)} = \sum_{i\in I} \overline{\langle \mathrm{e}_i, \phi\rangle}_{\mathfrak{H}} \langle \mathrm{e}_i, \psi\rangle_{\mathfrak{H}} = \langle \phi, \psi\rangle_{\mathfrak{H}},$$

again by Theorem 3.3.7, *v.)*. Conversely, if $(a_i)_{i\in I} \in \ell^2(I)$ is given then $\sum_{i\in I} |a_i|^2 < \infty$ with at most countably many a_i different from zero. We claim that $\phi = \sum_{i\in I} a_i \mathrm{e}_i$ is a well-defined vector in $\mathfrak{H}$. Indeed, the series is a Cauchy series since for every finite subset Pythagoras' Theorem gives

$$\left\| \sum_{\text{finite}} a_i \mathrm{e}_i \right\|^2 = \sum_{\text{finite}} |a_i|^2.$$

By completeness of $\mathfrak{H}$ the series converges to some vector $\phi \in \mathfrak{H}$. Then $a_i = \langle \mathrm{e}_i, \phi\rangle$ is clear by (3.3.12) and hence $U\phi = (a_i)_{i\in I}$ follows. This shows that U is also surjective, proving the second part. The third part is now clear since $\ell^2(I)$ is isometrically isomorphic to $\ell^2(J)$ iff $\#I = \#J$. Indeed, suppose $U\colon \ell^2(I) \longrightarrow \ell^2(J)$ is an isometric isomorphism. Then $\{U\mathrm{e}_i\}_{i\in I}$ is a Hilbert basis of $\ell^2(J)$, resulting in $\#I = \#J$ by the first part. The converse is clear by the second part. By the second part this particular case of $\ell^2(I)$ is all we have to consider to conclude the general case. □

Remark 3.3.10 The classification results as well as Theorem 3.3.7 show that a Hilbert basis behaves indeed similar to a vector space basis. Note however, that in infinite dimensions a Hilbert basis is *not* a vector space basis since the series in (3.3.11) is *not* a finite sum in general. In fact, one can show that as a vector space $\ell^2(\mathbb{N})$ has uncountable dimension as a vector space in the sense of linear algebra while the Hilbert basis is countable.

Exercise: Pro

Definition 3.3.11 (Hilbert space dimension) *For a Hilbert space* $\mathfrak{H}$ *we define its Hilbert space dimension* $\dim \mathfrak{H}$ *to be* $\#I$ *for a Hilbert basis* $\{\mathrm{e}_i\}_{i\in I}$.

By Theorem 3.3.9 this is indeed well-defined and we see that two Hilbert spaces are isometrically isomorphic iff their Hilbert space dimensions coincide. By some abuse of notation we shall refer to the Hilbert space dimension simply as "dimension" as long as we are dealing with Hilbert spaces since the dimension in the sense of linear algebra is usually entirely irrelevant in the context of Hilbert spaces.

For applications in quantum physics it is sometimes required that the Hilbert space should be separable as a topological space. The reason is that one would like to be able to approach every state vector at least approximately by a *finite* number of measurements. Thus one requires a separable Hilbert space where we have a countable dense subset. The next simple proposition shows that the only separable Hilbert spaces are those with at most countable dimension:

Proposition 3.3.12 (Separable Hilbert space) *A Hilbert space* $\mathfrak{H}$ *is separable iff* $\dim \mathfrak{H} \leq \#\mathbb{N}$.

PROOF: Suppose that $\dim \mathfrak{H} \leq \#\mathbb{N}$ and fix a Hilbert basis $\{\mathrm{e}_i\}_{i\in I}$. By Theorem 3.3.7, *iii.)*, the subspace $\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_i\}_{i\in I}$ is dense. But then $\operatorname{span}_{\mathbb{Q}+\mathrm{i}\mathbb{Q}}\{\mathrm{e}_i\}_{i\in I}$ is still dense and countable. Conversely, assume $\#I > \#\mathbb{N}$ is uncountable. Suppose there is a countable dense subset $\{\varphi_n\} \subseteq \mathfrak{H}$. Then for every $i \in I$ there is a subsequence $\varphi_{n_k} \longrightarrow \mathrm{e}_i$ and hence a φ_{n_i} with $\|\varphi_{n_i} - \mathrm{e}_i\| < \frac{1}{2}$. But for $i \neq j$ we have $\|\mathrm{e}_i - \mathrm{e}_j\| = \sqrt{2}$ as they are orthonormal. This shows for $i \neq j$

$$\|\varphi_{n_i} - \varphi_{n_j}\| \geq \sqrt{2} - \frac{1}{2} - \frac{1}{2} > 0$$

by the triangle inequality. Hence the φ_{n_i} are different for different $i \in I$. But this is a contradiction since there are only countably many different φ_n at hand. □

In the Exercises 3.6.10 we will discuss explicit examples of Hilbert bases for various Hilbert spaces used in quantum mechanics.

3.4 Constructions of Hilbert Spaces

In this section we present some elementary constructions of how one can obtain new Hilbert spaces out of given ones. The general idea is to perform a construction for pre-Hilbert spaces in the realm of linear algebra and complete afterwards to a Hilbert space. One should note that many of the constructions and isomorphism are "trivial" in the sense that one only has to count vectors of Hilbert bases to conclude that the resulting Hilbert spaces are isomorphic. However, the emphasize here is on the observation that there are *canonical* isomorphisms, independent of the choices of Hilbert bases.

3.4.1 The Direct Sum of Hilbert Spaces

Suppose we are given a family $\{\mathfrak{H}_i\}_{i\in I}$ of pre-Hilbert spaces. Then one considers their (algebraic) direct sum

$$\mathfrak{H} = \bigoplus_{i\in I} \mathfrak{H}_i. \tag{3.4.1}$$

Recall that the direct sum is defined to be the subset of the Cartesian product $\prod_{i\in I} \mathfrak{H}_i$ where at most finitely many components are non-zero. Of course, we could now use the Cartesian product topology which is locally convex. However, it will not be Banach and, even worse, on $\prod_{i\in I} \mathfrak{H}_i$ there is no reasonable definition of an inner product unless I is finite. On the direct sum, however, we can easily define an inner product $\langle \,\cdot\,, \,\cdot\, \rangle \colon \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{C}$ by

$$\langle (\phi_i)_{i\in I}, (\psi_j)_{j\in I} \rangle_{\mathfrak{H}} = \sum_{i\in I} \langle \phi_i, \psi_i \rangle_{\mathfrak{H}_i}. \tag{3.4.2}$$

Since for $(\phi_i)_{i\in I}, (\psi_j)_{j\in J} \in \mathfrak{H}$ only finitely many entries ϕ_i, ψ_j can be different from zero the sum is finite and hence well-defined.

Lemma 3.4.1 *Let $\{\mathfrak{H}_i\}_{i\in I}$ be a family of pre-Hilbert spaces and endow their direct sum $\mathfrak{H} = \bigoplus_{i\in I} \mathfrak{H}_i$ with $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{H}}$ as in* (3.4.2).

i.) The map $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{H}}$ is a positive definite inner product and $\mathfrak{H}$ becomes a pre-Hilbert space.

ii.) The canonical inclusion $\mathfrak{H}_i \longrightarrow \mathfrak{H}$, placing a vector from $\mathfrak{H}_i$ at the i-th position and zeros elsewhere, is isometric for all $i \in I$.

iii.) Viewing the $\mathfrak{H}_i$ as subspaces of $\mathfrak{H}$ via the canonical inclusion we have for $i \neq j$

$$\mathfrak{H}_i \perp \mathfrak{H}_j. \tag{3.4.3}$$

Proof: All statements of the lemma only require very simple verifications which we leave as an exercise. □

Thus we have a pre-Hilbert space $\bigoplus_{i\in I} \mathfrak{H}_i$ which we can complete to a Hilbert space by our general results from Proposition 3.1.14. However, in this particular situation we have a very explicit description of the completion:

Theorem 3.4.2 (Direct orthogonal sum of Hilbert spaces) *Let $\{\mathfrak{H}_i\}_{i\in I}$ be a family of Hilbert spaces and let $\mathfrak{H} = \bigoplus_{i\in I} \mathfrak{H}_i$ be their direct sum with inner product* (3.4.2).

i.) The subset

$$\widehat{\mathfrak{H}} = \left\{ (\phi_i)_{i\in I} \;\middle|\; \textit{at most countably many } \phi_i \neq 0 \textit{ and } \sum\nolimits_{i\in I} \|\phi_i\|^2_{\mathfrak{H}_i} < \infty \right\} \subseteq \prod_{i\in I} \mathfrak{H}_i \tag{3.4.4}$$

is a subspace of the Cartesian product.

ii.) On $\widehat{\mathfrak{H}}$ the inner product

$$\langle (\phi_i)_{i\in I}, (\psi_j)_{j\in I} \rangle = \sum_{i\in I} \langle \phi_i, \psi_i \rangle_{\mathfrak{H}_i} \tag{3.4.5}$$

is well-defined and positive definite.

iii.) We have $\mathfrak{H} \subseteq \widehat{\mathfrak{H}}$ and $\mathfrak{H}^{\mathrm{cl}} = \widehat{\mathfrak{H}}$.

iv.) The pre-Hilbert space $\widehat{\mathfrak{H}}$ is complete and hence a Hilbert space completion of $\mathfrak{H}$.

v.) If $\{\mathrm{e}_{ij}\}_{j\in J_i}$ are Hilbert bases of $\mathfrak{H}_i$ for all $i \in I$ then $\{\mathrm{e}_{ij}\}_{i\in I, j\in J_i}$ is a Hilbert basis of $\mathfrak{H}$.

Proof: For the parts *i.) – iv.)* we can proceed completely analogously to the case of the construction of $\ell^2(I)$ in Subsection 3.1.3. In fact, we can literally copy all arguments by replacing the absolute value $|\phi_i|$ of the complex numbers there with the norms $\|\phi_i\|_{\mathfrak{H}_i}$ here. We will not repeat the details. For the last part it is clear by (3.4.3) that the vectors $\{\mathrm{e}_{ij}\}_{i\in I, j\in J_i}$ for an orthonormal system. Moreover, for fixed i the span $\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_{ij}\}_{j\in J_i}$ is dense in $\mathfrak{H}_i$. From this we see that the span of all of them is dense in the direct sum $\bigoplus_{i\in I} \mathfrak{H}_i$ since here we only have to approximate finitely many components. But with *iii.)* we see that their span is also dense in $\widehat{\mathfrak{H}}$ which means that we have a Hilbert basis by Theorem 3.3.7, *iii.)*. □

Definition 3.4.3 (Direct orthogonal Hilbert space sum) *Let $\{\mathfrak{H}_i\}_{i\in I}$ be a family of Hilbert spaces. Then the Hilbert space $\mathfrak{H}$ as in* (3.4.4) *is called the direct orthogonal Hilbert space sum of the $\mathfrak{H}_i$. We also write*

$$\mathfrak{H} = \widehat{\bigoplus_{i\in I}} \mathfrak{H}_i. \tag{3.4.6}$$

Since in the context of Hilbert space we only encounter this direct sum construction one refers to (3.4.6) also as the *direct sum* of Hilbert spaces or as the *orthogonal sum* of Hilbert spaces. If the $\mathfrak{H}_i$ are only pre-Hilbert spaces then it is easy to see that canonically

$$\widehat{\bigoplus_{i\in I} \mathfrak{H}_i} \cong \widehat{\bigoplus_{i\in I}} \widehat{\mathfrak{H}}_i, \tag{3.4.7}$$

see also the discussion in Exercise 3.6.14 for further details on the direct sum of Hilbert spaces. Note also that for a set I the Hilbert space $\ell^2(I)$ can be viewed as the orthogonal Hilbert space sum of I copies of the one-dimensional Hilbert space $\mathbb{C}$.

3.4.2 The Tensor Product of Hilbert Spaces

More important for applications in physics is the tensor product of Hilbert spaces. First, recall that the *tensor product* of V and W is a vector space $V \otimes W$ together with a bilinear map $\otimes\colon V \times W \longrightarrow V \otimes W$ with the universal property that any bilinear map $\Phi\colon V \times W \longrightarrow U$ into another vector space U factors through $V \otimes W$, i.e. there is a unique linear map ϕ with

$$\phi \circ \otimes = \Phi. \tag{3.4.8}$$

This universal property of $\otimes$ can also be encoded in the fact that the diagram

$$\begin{tikzcd}
V \otimes W \arrow[r, dashed, "\phi"] & U \\
V \times W \arrow[u, "\otimes"] \arrow[ur, "\Phi"'] &
\end{tikzcd} \tag{3.4.9}$$

commutes. The pair $(V \otimes W, \otimes)$ is then uniquely determined up to a unique isomorphism, see e.g. Exercise 1.5.19 for more details on the (algebraic) tensor product or the monographs [16, 63] for a thorough discussion on various aspects of multilinear algebra.

In the case of pre-Hilbert spaces the tensor product is again a pre-Hilbert space:

Lemma 3.4.4 *Let $\mathfrak{H}$ and $\mathfrak{K}$ be pre-Hilbert spaces. Then on $\mathfrak{H} \otimes \mathfrak{K}$ there is a unique positive definite inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{H}\otimes\mathfrak{K}}$ such that on factorizing tensors one has*

$$\langle \phi \otimes \phi', \psi \otimes \psi' \rangle_{\mathfrak{H}\otimes\mathfrak{K}} = \langle \phi, \psi \rangle_{\mathfrak{H}} \langle \phi', \psi' \rangle_{\mathfrak{K}} \tag{3.4.10}$$

for $\phi, \psi \in \mathfrak{H}$ and $\phi', \psi' \in \mathfrak{K}$.

PROOF: First we note that the left hand side is indeed bilinear in ψ and ψ' as well as anti-bilinear in ϕ and ϕ'. Thus (3.4.10) extends uniquely by general nonsense to a sesquilinear form on $\mathfrak{H} \otimes \mathfrak{K}$. We have to show positive definiteness: let $\Phi = \sum_{i=1}^n \phi_i \otimes \phi_i'$ be a general tensor. In the finite-dimensional span of the ϕ_i and ϕ_i' we find orthonormal bases by the Gram-Schmidt algorithm. Thus let $\mathrm{e}_1, \ldots \mathrm{e}_k \in \mathfrak{H}$ be an orthonormal basis for $\operatorname{span}_{\mathbb{C}}\{\phi_i\}_{i=1,\ldots,n}$ as well as $\mathrm{e}_1', \ldots, \mathrm{e}_\ell' \in \mathfrak{K}$ an orthonormal basis for $\operatorname{span}_{\mathbb{C}}\{\phi_i'\}_{i=1,\ldots,n}$. Then there are unique coefficients $\Phi_{ij} \in \mathbb{C}$ such that

$$\Phi = \sum_{i=1}^k \sum_{j=1}^\ell \Phi_{ij} \mathrm{e}_i \otimes \mathrm{e}_j',$$

which are obtained by expressing $\phi_1, \ldots, \phi_n$ as well as $\phi_1', \ldots, \phi_n'$ in terms of the bases. From the definition of the inner product on $\mathfrak{H} \otimes \mathfrak{K}$ we see

$$\langle \mathrm{e}_i \otimes \mathrm{e}_j', \mathrm{e}_r \otimes \mathrm{e}_s' \rangle_{\mathfrak{H}\otimes\mathfrak{K}} = \delta_{ir}\delta_{jr},$$

which gives immediately

$$\langle \Phi, \Phi \rangle_{\mathfrak{H}\otimes\mathfrak{K}} = \sum_{i=1}^k \sum_{j=1}^\ell |\Phi_{ij}|^2.$$

From this we see the positive definiteness immediately. $\square$

Since we have a pre-Hilbert space structure on the tensor product $\mathfrak{H} \otimes \mathfrak{K}$ of two (pre-) Hilbert spaces we can complete it to a Hilbert space. This gives the following notion of a topological tensor product:

Definition 3.4.5 (Hilbert space tensor product) *The Hilbert space tensor product of two Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ is the Hilbert space completion of the algebraic tensor product $\mathfrak{H} \otimes \mathfrak{K}$, denoted by $\mathfrak{H} \hat{\otimes} \mathfrak{K}$.*

We collect now a few properties of the Hilbert space tensor product.

Theorem 3.4.6 (Hilbert space tensor product) *Let $\mathfrak{H}$, $\mathfrak{K}$, and $\mathfrak{L}$ be Hilbert spaces.*

i.) The tensor product $\otimes \colon \mathfrak{H} \times \mathfrak{K} \longrightarrow \mathfrak{H} \hat{\otimes} \mathfrak{K}$ is continuous.

ii.) If $U \subseteq \mathfrak{H}$ and $U' \subseteq \mathfrak{K}$ are dense subspaces then $U \otimes U'$ is dense in $\mathfrak{H} \hat{\otimes} \mathfrak{K}$.

iii.) If $\{\mathrm{e}_i\}_{i \in I}$ and $\{\mathrm{e}'_j\}_{j \in J}$ are orthonormal systems (Hilbert bases) of $\mathfrak{H}$ and $\mathfrak{K}$, respectively, then $\{\mathrm{e}_i \otimes \mathrm{e}'_j\}_{i \in I, j \in J}$ is an orthonormal system (Hilbert basis) of $\mathfrak{H} \hat{\otimes} \mathfrak{K}$.

iv.) For the Hilbert space dimension one has

$$\dim(\mathfrak{H} \hat{\otimes} \mathfrak{K}) = \dim \mathfrak{H} \cdot \dim \mathfrak{K}. \tag{3.4.11}$$

v.) The canonical associativity of the algebraic tensor product induces an isometric isomorphism

$$\mathfrak{H} \hat{\otimes} (\mathfrak{K} \hat{\otimes} \mathfrak{L}) \cong (\mathfrak{H} \hat{\otimes} \mathfrak{K}) \hat{\otimes} \mathfrak{L}. \tag{3.4.12}$$

Proof: We first will show the continuity: here we have a bilinear map and thus our previous considerations do not apply directly. For $\phi \in \mathfrak{H}$ and $\phi' \in \mathfrak{K}$ we have

$$\|\phi \otimes \phi'\|^2_{\mathfrak{H} \hat{\otimes} \mathfrak{K}} = \langle \phi \otimes \phi', \phi \otimes \phi' \rangle_{\mathfrak{H} \hat{\otimes} \mathfrak{K}} = \langle \phi, \phi \rangle_{\mathfrak{H}} \langle \phi', \phi' \rangle_{\mathfrak{K}} = \|\phi\|^2_{\mathfrak{H}} \|\phi'\|^2_{\mathfrak{K}}$$

and hence $\|\phi \otimes \phi'\|_{\mathfrak{H} \hat{\otimes} \mathfrak{K}} = \|\phi\|_{\mathfrak{H}} \|\phi'\|_{\mathfrak{H}}$. Anticipating the results from Theorem 4.1.3 we conclude that this equality shows the continuity of the bilinear tensor product. Alternatively, one can use this to show the continuity at every point $(\phi, \phi') \in \mathfrak{H} \times \mathfrak{K}$ directly. Then the second part is clear: every vector in $\mathfrak{H} \otimes \mathfrak{K}$ is a finite sum of factorizing tensors $\phi \otimes \phi'$. Now we can approximate $\phi_n \longrightarrow \phi$ and $\phi'_n \longrightarrow \phi'$ by $\phi_n \in U$ and $\phi'_n \in U'$, respectively. The continuity of $\otimes$ then ensures that $\phi_n \otimes \phi'_n \longrightarrow \phi \otimes \phi'$. This shows that $U \otimes U'$ is dense in $\mathfrak{H} \otimes \mathfrak{K}$ and hence also in $\mathfrak{H} \hat{\otimes} \mathfrak{K}$. For the third part we already computed in the proof of Lemma 3.4.4 that the set $\{\mathrm{e}_i \otimes \mathrm{e}'_j\}_{i \in I, j \in J}$ is still orthonormal. If in addition we have Hilbert bases we can apply the second part to $U = \operatorname{span}_{\mathbb{C}}\{\mathrm{e}_i\}_{i \in I}$ and $U' = \operatorname{span}_{\mathbb{C}}\{\mathrm{e}'_j\}_{j \in J}$ to conclude that $\{\mathrm{e}_i \otimes \mathrm{e}'_j\}_{i \in I, j \in J}$ is a Hilbert basis, too. This also implies the fourth part as $\#(I \times J) = \#I \cdot \#J$ (for infinite sets this is essentially the definition of the product). The fifth part is now a simple computation: recall that on the algebraic tensor product the canonical linear isomorphism asso is defined by the linear extension of

$$\mathfrak{H} \otimes (\mathfrak{K} \otimes \mathfrak{L}) \ni \phi \otimes (\phi' \otimes \phi'') \ \mapsto \ (\phi \otimes \phi') \otimes \phi'' \in (\mathfrak{H} \otimes \mathfrak{K}) \otimes \mathfrak{L}.$$

It is easily shown to be isometric: it is sufficient to check this on elementary tensors where it is a trivial computation. Analogously, the inverse is isometric as well and both extend to the completions. Note that $\mathfrak{H} \otimes (\mathfrak{K} \otimes \mathfrak{L})$ is dense in $\mathfrak{H} \hat{\otimes} (\mathfrak{K} \hat{\otimes} \mathfrak{L})$ and also for the other way of putting brackets by the second part. □

Remark 3.4.7 (Entanglement) The tensor product $\mathfrak{H} \hat{\otimes} \mathfrak{K}$ is used in quantum physics to describe a system which consists of two subsystems each of which is described by $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Thus the study of tensor products is of crucial interest in all contemporary discussions of the measurement process, quantum information theory, and quantum computing. In fact, one of the most puzzling features of quantum physics is that there are pure states of the combined system $\mathfrak{H} \hat{\otimes} \mathfrak{K}$ which are not just factorizing into states of the two subsystems. They give "non-classical" correlations called *entanglement*. Even worse, in a certain sense most states are of this entangled form, see also Exercise **??** as well as [].

stefan: Refer quantum info needed

3.4.3 The Fock Space Construction

We consider now the following quantum mechanical situation: a certain system is described by a Hilbert space $\mathfrak{H}$, so two copies of it will require $\mathfrak{H} \mathbin{\hat{\otimes}} \mathfrak{H}$, three copies $\mathfrak{H} \mathbin{\hat{\otimes}} \mathfrak{H} \mathbin{\hat{\otimes}} \mathfrak{H}$, and so on. Here we will leave out the brackets according to the associativity result from Theorem 3.4.6, *v.)*. In quantum field theory, one has the situation that one wants to consider states of all possible "particle numbers" at once on equal footing. Thus on considers $\mathfrak{H}$ for one "particle", $\mathfrak{H} \mathbin{\hat{\otimes}} \mathfrak{H}$ for two, and

$$\mathfrak{H}^{\hat{\otimes} k} = \underbrace{\mathfrak{H} \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \mathfrak{H}}_{k} \tag{3.4.13}$$

for k particles. States with different particle numbers are then combined into the direct sum

$$\mathsf{F}(\mathfrak{H}) = \widehat{\bigoplus_{n \in \mathbb{N}_0}} \mathfrak{H}^{\hat{\otimes} n}, \tag{3.4.14}$$

where by convention $\mathfrak{H}^{\hat{\otimes} 0} = \mathbb{C}$ corresponds to "no" particle, i.e. the "vacuum". Here we have combined the two constructions of tensor products and direct orthogonal sum, so $\mathsf{F}(\mathfrak{H})$ is again a Hilbert space.

Definition 3.4.8 (Fock space) *For a Hilbert space $\mathfrak{H}$ the Hilbert space $\mathsf{F}(\mathfrak{H})$ is called the Fock space over $\mathfrak{H}$.*

It is a good point to recall some basic facts on the tensor algebra at this stage: for a general vector space V (over some field $\mathbb{k}$ or even for a module over some commutative ring) the vector space

$$\mathrm{T}(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n} \tag{3.4.15}$$

is called the *tensor algebra* over V. The tensor product itself induces a multiplication

$$\otimes \colon \mathrm{T}(V) \times \mathrm{T}(V) \longrightarrow \mathrm{T}(V), \tag{3.4.16}$$

which makes $\mathrm{T}(V)$ an associative unital algebra with unit $\mathbb{1} = 1 \in V^{\otimes 0} = \mathbb{k}$. This follows from the associativity properties of the tensor product. More specifically, on elementary tensors $v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$ and $w_1 \otimes \cdots \otimes w_\ell \in V^{\otimes \ell}$ one has

$$(v_1 \otimes \cdots \otimes v_k) \otimes (w_1 \otimes \cdots \otimes w_\ell) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell, \tag{3.4.17}$$

which is then extended to $\mathrm{T}(V) \times \mathrm{T}(V)$ by bilinearity. Note that $\mathrm{T}(V)$ is graded in the sense that for $\mathrm{T}^k(V) = V^{\otimes k}$ we have for the tensor product

$$\otimes \colon \mathrm{T}^k(V) \times \mathrm{T}^\ell(V) \longrightarrow \mathrm{T}^{k+\ell}(V). \tag{3.4.18}$$

The fact that $\mathrm{T}(V)$ is graded is sometimes expressed by the notation $\mathrm{T}^\bullet(V)$. Elements in $\mathrm{T}^k(V)$ are also called *homogeneous of degree k*.

The tensor algebra $\mathrm{T}^\bullet(V)$ is the most non-trivial algebra generated by V in the following sense: if $\mathscr{A}$ is another associative unital algebra and $\phi \colon V \longrightarrow \mathscr{A}$ a linear map then there is a unique unital algebra homomorphism

$$\Phi \colon \mathrm{T}^\bullet(V) \longrightarrow \mathscr{A}, \tag{3.4.19}$$

such that $\Phi\big|_{\mathrm{T}^1(V)=V} = \phi$. Explicitly, on factorizing elements it is given by

$$\Phi(v_1 \otimes \cdots \otimes v_k) = \phi(v_1) \cdots \phi(v_k), \tag{3.4.20}$$

which extends to $\mathrm{T}^k(V)$ and then to $\mathrm{T}^\bullet(V)$ by multilinearity. Because of this property the tensor algebra is also called the *free algebra* generated by V, see also Exercise 1.5.22.

Similarly to the construction of a homomorphism as in (3.4.19) we can extend linear maps also to *derivations*. We do not need the most general statement but only the following situation: given a linear map $\phi\colon V \longrightarrow V$ there is a unique derivation

$$D_\phi\colon \mathrm{T}(V) \longrightarrow \mathrm{T}(V) \tag{3.4.21}$$

with $D_\phi\big|_V = \phi$. Explicitly, one sets $D_\phi(\mathbb{1}) = 0$ and

$$D_\phi(v_1 \otimes \cdots \otimes v_k) = \sum_{\ell=1}^{k} v_1 \otimes \cdots \otimes \phi(v_\ell) \otimes \cdots \otimes v_k, \tag{3.4.22}$$

and checks immediately that the linear extension of this gives the derivation as wanted, see also Exercise 1.5.22. Clearly, the map $\phi \mapsto D_\phi$ is linear. A particular case of this is obtained for the linear map $\mathrm{id}\colon V \longrightarrow V$. The resulting derivation is denoted by

$$\deg = D_{\mathrm{id}}\colon \mathrm{T}(V) \longrightarrow \mathrm{T}(V), \tag{3.4.23}$$

explicitly given on homogeneous tensors by

$$\deg\big|_{\mathrm{T}^k(V)} = k\,\mathrm{id}_{\mathrm{T}^k(V)}. \tag{3.4.24}$$

The derivation property of this *degree derivation* just means that the tensor algebra is graded, i.e. (3.4.18).

Back to the case of Hilbert spaces we see that the tensor algebra $\mathrm{T}(\mathfrak{H})$ is a dense subspace of the Fock space $\mathsf{F}(\mathfrak{H})$. The degree derivation deg clearly extends to

$$\deg\colon \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\hat{\otimes} n} \longrightarrow \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\hat{\otimes} n} \tag{3.4.25}$$

with the same property of being n times the identity on $\mathfrak{H}^{\hat{\otimes} n}$. However, deg does not extend further: to see this we fix a sequence $\phi_n \in \mathfrak{H}^{\hat{\otimes} n}$ with $\|\phi_n\| = 1$. Then $\deg \phi_n = n\phi_n$ implies $\|\deg \phi_n\| = n\|\phi_n\|$. Thus deg is *not continuous* and does not extend to the whole Fock space $\mathsf{F}(\mathfrak{H})$.

In quantum field theory, where $\mathsf{F}(\mathfrak{H})$ corresponds to the Hilbert space hosting all states corresponding to an arbitrary number of particles of the type described by the single particle Hilbert space $\mathfrak{H}$, the operator deg corresponds to the *number operator*. It is an *unbounded* operator whose eigenspaces correspond to the states with a fixed number of particles. The number itself is given by the corresponding eigenvalue. This is clearly the very definition of deg as in (3.4.24). We will come back to this in Exercise **??**.

Exercise: Exe eigenvalues a theory of deg

Exercise: Fu space constru

3.4.4 The Bosonic and Fermionic Fock Spaces

The physical interpretation of $\mathfrak{H} \hat{\otimes} \mathfrak{K}$ as Hilbert space corresponding to the total system consisting of two subsystems described by $\mathfrak{H}$ and $\mathfrak{K}$, respectively, is only correct for subsystems which are *distinguishable* like the macroscopic measurement apparatus and the measured atom or the proton and the electron in the hydrogen atom. Things become more subtle when we are dealing with *identical particles* like only electrons. Then not every vector in $\mathfrak{H} \hat{\otimes} \mathfrak{H}$ corresponds to a physically realizable state. Indeed, we have to take into account the symmetry properties according to the spin of the particle. We do not enter the discussion of the famous *Spin-Statistic Theorem* which asserts that bosonic particles require symmetrization while fermionic particles correspond to antisymmetrization.

Thus the Fock space $\mathsf{F}(\mathfrak{H})$ as discussed in the previous subsection was not yet the final answer for a many-particle system of identical particles. For further information we refer to [18, Sect. II.5] or [53].

Here, we only focus on the two types of symmetrization and explain their features. We start again with the algebraic situation first.

Let V be a vector space and $n \in \mathbb{N}$. On the n-fold tensor product $V^{\otimes n}$ one defines a linear map

$$\sigma \triangleright : V^{\otimes n} \longrightarrow V^{\otimes n} \tag{3.4.26}$$

for every permutation $\sigma \in S_n$ by the linear extension of

$$\sigma \triangleright (v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}, \tag{3.4.27}$$

where $v_1, \ldots, v_n \in V$. It is easily verified that this defines a representation of the symmetric group S_n, see also Exercise 3.6.15. Using this representation, one considers the two maps

$$\mathrm{Sym}_n = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \triangleright \tag{3.4.28}$$

and

$$\mathrm{Alt}_n = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \sigma \triangleright\,, \tag{3.4.29}$$

where $\mathrm{sign}(\sigma)$ is the signum of the permutation σ as usual.

Lemma 3.4.9 *Let V be a vector space. Then for all $n \in \mathbb{N}$ the maps*

$$\mathrm{Sym}_n, \mathrm{Alt}_n \colon V^{\otimes n} \longrightarrow V^{\otimes n} \tag{3.4.30}$$

are idempotents, i.e. $\mathrm{Sym}_n^2 = \mathrm{Sym}_n$ *and* $\mathrm{Alt}_n^2 = \mathrm{Alt}_n$, *satisfying* $\mathrm{Sym}_1 = \mathrm{Alt}_1$ *and*

$$\mathrm{Sym}_n \mathrm{Alt}_n = 0 = \mathrm{Alt}_n \mathrm{Sym}_n \tag{3.4.31}$$

for all $n \geq 2$. If in addition $V = \mathfrak{H}$ is a Hilbert space then Sym_n *and* Alt_n *are continuous with operator norms*

$$\|\mathrm{Sym}_n\| = 1 \quad \textit{and} \quad \|\mathrm{Alt}_n\| = 1 \quad \textit{provided} \quad \mathrm{Alt}_n \neq 0. \tag{3.4.32}$$

PROOF: The algebraic features of the maps Sym_n and Alt_n are well-known and easily verified, see Exercise 3.6.15. To obtain (3.4.32) we note that for $\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_n \in \mathfrak{H}$ we have

$$\begin{aligned}
\langle \phi_1 \otimes \cdots \otimes \phi_n, \mathrm{Sym}_n(\psi_1 \otimes \cdots \otimes \psi_n)\rangle &= \frac{1}{n!} \sum_{\sigma \in S_n} \langle \phi_1 \otimes \cdots \otimes \phi_n, \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(n)}\rangle \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} \langle \phi_1, \psi_{\sigma(1)}\rangle \cdots \langle \phi_n, \psi_{\sigma(n)}\rangle \\
&= \frac{1}{n!} \sum_{\pi \in S_n} \langle \phi_{\pi(1)}, \psi_1\rangle \cdots \langle \phi_{\pi(n)}, \psi_n\rangle \\
&= \langle \mathrm{Sym}_n(\phi_1 \otimes \cdots \otimes \phi_n), \psi_1 \otimes \cdots \otimes \psi_n\rangle.
\end{aligned}$$

By sesquilinearity we conclude that for all $\Phi, \Psi \in \mathfrak{H}^{\otimes n}$ we have

$$\langle \Phi, \mathrm{Sym}_n \Psi\rangle = \langle \mathrm{Sym}_n \Phi, \Psi\rangle, \tag{$*$}$$

and analogously for Alt_n since the signum is real and coincides for σ and σ^{-1}. Then $(*)$ and $\mathrm{Sym}_n^2 = \mathrm{Sym}_n$ implies

$$\|\mathrm{Sym}_n \Phi\|^2 = |\langle \mathrm{Sym}_n \Phi, \mathrm{Sym}_n \Phi\rangle|^2 = |\langle \Phi, \mathrm{Sym}_n \Phi\rangle|^2 \leq \|\Phi\| \|\mathrm{Sym}_n \Phi\|$$

by the Cauchy-Schwarz inequality. Hence $\|\mathrm{Sym}_n \Phi\| \leq \|\Phi\|$ follows and therefore Sym_n is continuous with $\|\mathrm{Sym}_n\| \leq 1$. However, for a continuous idempotent operator we have in general

$$\|\mathrm{Sym}_n\| = \|\mathrm{Sym}_n^2\| \leq \|\mathrm{Sym}_n\|^2.$$

Hence $\|\mathrm{Sym}_n\| \geq 1$ as soon as $\mathrm{Sym}_n \neq 0$, see also Exercise 4.5.16. Thus the claim (3.4.32) for the symmetrization map follows since Sym_n is never the zero map. The argument for Alt_n is analogous with the only difference that $\mathrm{Alt}_n = 0$ if $n > \dim \mathfrak{H}$ in *finite* dimensions and hence $\|\mathrm{Alt}_n\| = 0$ in this case. ☐

Thus, in the Hilbert space setting the operators Sym_n and Alt_n extend from $\mathfrak{H}^{\otimes n}$ to the completed tensor powers $\mathfrak{H}^{\hat{\otimes} n}$ yielding operators

$$\mathrm{Sym}_n, \mathrm{Alt}_n \colon \mathfrak{H}^{\hat{\otimes} n} \longrightarrow \mathfrak{H}^{\hat{\otimes} n}, \tag{3.4.33}$$

still obeying the algebraic properties of the lemma by continuity. From the proof of the lemma we also see that Sym_n and Alt_n are Hermitian and hence *projections*.

In a next step, we extend Sym_n and Alt_n to the whole tensor algebra by setting

$$\mathrm{Sym} = \bigoplus_{n=0}^{\infty} \mathrm{Sym}_n \colon \mathrm{T}^\bullet(V) \longrightarrow \mathrm{T}^\bullet(V) \tag{3.4.34}$$

and

$$\mathrm{Alt} = \bigoplus_{n=0}^{\infty} \mathrm{Alt}_n \colon \mathrm{T}^\bullet(V) \longrightarrow \mathrm{T}^\bullet(V), \tag{3.4.35}$$

i.e. using the maps Sym_n and Alt_n as diagonal blocks of Sym and Alt, respectively. By convention, we set $\mathrm{Sym}_0 = \mathrm{id}_{\mathbb{C}} = \mathrm{Alt}_0$. Since the different blocks do not mix we still have

$$\mathrm{Sym}^2 = \mathrm{Sym} \quad \text{and} \quad \mathrm{Alt}^2 = \mathrm{Alt}\,. \tag{3.4.36}$$

In the Hilbert space setting we first get Sym and Alt on the direct orthogonal sum $\bigoplus_{n=0}^{\infty} \mathfrak{H}^{\hat{\otimes} n}$. Then, due to the block structure and the fact that the different blocks are now orthogonal, we obtain

$$\|\mathrm{Sym}\| = 1 = \|\mathrm{Alt}\|. \tag{3.4.37}$$

Hence both idempotents extend to the whole Fock space $\mathsf{F}(\mathfrak{H})$ as idempotents. Again, the orthogonality of the blocks ensures that Sym and Alt are still Hermitian and thus projections.

In a next step, we recall again some facts from multilinear algebra, see e.g. [16, Chap. V and VII] or [63, Chap. 3] for further details. The *symmetric algebra* over a vector space V is defined by

$$\mathrm{S}^\bullet(V) = \bigoplus_{n=0}^{\infty} \mathrm{S}^n(V) \quad \text{with} \quad \mathrm{S}^n(V) = \operatorname{im} \mathrm{Sym}_n \subseteq V^{\otimes n}. \tag{3.4.38}$$

Equivalently, we have $\mathrm{S}^\bullet(V) = \operatorname{im}\mathrm{Sym} \subseteq \mathrm{T}^\bullet(V)$. Analogously, the *Graßmann algebra* over V is defined by

$$\Lambda^\bullet(V) = \bigoplus_{n=0}^{\infty} \Lambda^n(V) \quad \text{with} \quad \Lambda^n(V) = \operatorname{im} \mathrm{Alt}_n \subseteq V^{\otimes n}, \tag{3.4.39}$$

or $\Lambda^\bullet(V) = \operatorname{im}\mathrm{Alt} \subseteq \mathrm{T}^\bullet(V)$. Tensors in $\mathrm{S}(V)$ are called *totally symmetric* while the tensors in $\Lambda(V)$ are called *totally antisymmetric*, see also Exercise 3.6.15.

For $\Phi \in \mathrm{S}^k(V)$ and $\Psi \in \mathrm{S}^\ell(V)$ one defines their *symmetric (tensor) product* by

$$\Phi \vee \Psi = \mathrm{Sym}_{k+\ell}(\Phi \otimes \Psi) \in \mathrm{S}^{k+\ell}(V). \tag{3.4.40}$$

Note that there are different conventions in the literature concerning the pre-factor. Our version results in

$$v \vee w = \frac{1}{2}(v \otimes w + w \otimes v) \tag{3.4.41}$$

for $v, w \in V$. Analogously, one defines the *Graßmann product* (also called *antisymmetric tensor product* or *wedge product*) of $\Phi \in \Lambda^k(V)$ and $\Psi \in \Lambda^\ell(V)$ by

$$\Phi \wedge \Psi = \mathrm{Alt}_{k+\ell}(\Phi \otimes \Psi). \tag{3.4.42}$$

Then these products are extended to $\mathrm{S}^\bullet(V)$ and $\Lambda^\bullet(V)$, respectively, by bilinearity. One obtains associative multiplications such that $(\mathrm{S}^\bullet(V), \vee)$ is a commutative associative unital graded algebra while $(\Lambda^\bullet(V), \wedge)$ is super-commutative in the sense that

$$\Phi \wedge \Psi = (-1)^{k\ell} \Psi \wedge \Phi \quad \text{for} \quad \Phi \in \Lambda^k(V), \Psi \in \Lambda^\ell(V). \tag{3.4.43}$$

The symmetric and the Graßmann algebra over V have now similar universal properties as the tensor algebra, now only within the categories of commutative and super-commutative algebras, respectively, see also Exercise 4.5.13.

After this excursion into the realm of linear algebra we discuss now again the case of the symmetric and the Graßmann algebra over a Hilbert space. Here the topological properties of the maps Sym and Alt enter:

Lemma 3.4.10 *Let $\mathfrak{H}$ be a Hilbert space. Then*

$$\widehat{\mathrm{S}}(\mathfrak{H}) = \operatorname{im} \mathrm{Sym} \subseteq \mathsf{F}(\mathfrak{H}) \tag{3.4.44}$$

and

$$\widehat{\Lambda}(\mathfrak{H}) = \operatorname{im} \mathrm{Alt} \subseteq \mathsf{F}(\mathfrak{H}) \tag{3.4.45}$$

are closed subspaces and hence Hilbert spaces. For homogeneous tensors $\Phi \in \operatorname{im} \mathrm{Sym}_k$ and $\Psi \in \mathrm{Sym}_\ell$ one has

$$\|\Phi \vee \Psi\| \leq \|\Phi\| \|\Psi\| \tag{3.4.46}$$

and analogously for $\Phi \in \mathrm{Alt}_k$ and $\Psi \in \mathrm{Alt}_\ell$

$$\|\Phi \wedge \Psi\| \leq \|\Phi\| \|\Psi\|. \tag{3.4.47}$$

PROOF: Since $\mathrm{Sym}^2 = \mathrm{Sym}$ we know by algebraic manipulations that $\operatorname{im} \mathrm{Sym} = \ker(\mathrm{id} - \mathrm{Sym})$. Now Sym is continuous and so is $\mathrm{id} - \mathrm{Sym}$, But then $\operatorname{im} \mathrm{Sym}$ being the kernel of a continuous linear map is a closed subspace. Finally, it is easy to see that $\mathrm{S}(\mathfrak{H})$ is dense in $\operatorname{im} \mathrm{Sym}$. This shows (3.4.44) and (3.4.45) follows analogously. In the proof of Theorem 3.4.6 we have seen that for $\Phi \in \mathfrak{H}^{\hat{\otimes} k}$ and $\Psi \in \mathfrak{H}^{\hat{\otimes} \ell}$ we have

$$\|\Phi \otimes \Psi\|_{\mathfrak{H}^{\hat{\otimes} k} \otimes \mathfrak{H}^{\hat{\otimes} \ell}} = \|\Phi\|_{\mathfrak{H}^{\hat{\otimes} k}} \|\Psi\|_{\mathfrak{H}^{\hat{\otimes} \ell}}.$$

Now $\mathrm{Sym}_{k+\ell}$ has operator norm one and hence

$$\|\Phi \vee \Psi\| = \|\mathrm{Sym}_{k+\ell}(\Phi \otimes \Psi)\| \leq \|\Phi \otimes \Psi\| \leq \|\Phi\| \|\Psi\|$$

and analogously for the $\wedge$-product. $\square$

The totally symmetric and totally antisymmetric tensors constitute now the physically relevant parts of the Fock space $\mathsf{F}(\mathfrak{H})$ for either bosonic or fermionic particles, respectively.

Definition 3.4.11 (Bosonic and fermionic Fock space) *Let $\mathfrak{H}$ be a Hilbert space. Then*

$$\mathsf{F}_+(\mathfrak{H}) = \widehat{\mathrm{S}}(\mathfrak{H}) \subseteq \mathsf{F}(\mathfrak{H}) \tag{3.4.48}$$

is called the bosonic Fock space over $\mathfrak{H}$ while

$$\mathsf{F}_-(\mathfrak{H}) = \widehat{\Lambda}(\mathfrak{H}) \subseteq \mathsf{F}(\mathfrak{H}) \tag{3.4.49}$$

is called the fermionic Fock space over $\mathfrak{H}$.

Remark 3.4.12 Let $\mathfrak{H}$ be a Hilbert space.

i.) The bosonic and fermionic Fock spaces $\mathsf{F}_\pm(\mathfrak{H})$ over $\mathfrak{H}$ are the basic ingredients to build what is called *second quantization*. Here one passes from the quantum mechanical description of a single quantum particle to a quantum field theoretical description of all possible multi-particle states. This passage $\mathfrak{H} \rightsquigarrow \mathsf{F}_\pm(\mathfrak{H})$ has indeed good properties essentially originating from the universal properties of the symmetric algebra and Graßmann algebra, respectively. In fact, one obtains a functor in a rather specific meaning, see Exercise **??**. For further reading one may consult e.g. [43, Sect. X.7].

Exercise: Lat aspects of lift and self-adjoi

ii.) The bosonic and fermionic Fock spaces are also the arena on which the canonical (anti-) commutation relation algebras act. For further details on these CCR and CAR algebras we refer to [10, Sect. 5.3].

Exercise: Exe

iii.) Even for a fixed particle number, the Hilbert spaces $\widehat{\mathrm{S}}^k(\mathfrak{H}) = \operatorname{im}\mathrm{Sym}_k \subseteq \mathfrak{H}^{\hat{\otimes} n}$ and $\widehat{\Lambda}^k(\mathfrak{H}) = \operatorname{im}\mathrm{Alt}_k \subseteq \mathfrak{H}^{\hat{\otimes} n}$ play an important role in quantum mechanics. In atomic and molecular physics the factorizing tensors $\phi_1 \wedge \cdots \wedge \phi_n \in \Lambda^n(\mathfrak{H})$ with $\phi_1, \ldots, \phi_n \in \mathfrak{H}$ are sometimes referred to as *Slater determinants*.

Exercise: Als here? (Dis-)c and $\wedge$ for the topology.

3.5 Bounded Operators on Hilbert Spaces

In this section we start to investigate first properties of continuous linear operators on Hilbert spaces. Since the continuity of $A\colon \mathfrak{H} \longrightarrow \mathfrak{K}$ can be expressed as

$$\sup_{\phi \neq 0} \frac{\|A\phi\|_{\mathfrak{K}}}{\|\phi\|_{\mathfrak{H}}} = \|A\| < \infty, \tag{3.5.1}$$

the continuous operators are also referred to as *bounded* operators. Note however, that the map $\phi \mapsto A\phi$, being linear, is of course not bounded at all in norm. Instead, it maps bounded subsets of $\mathfrak{H}$ to bounded subsets of $\mathfrak{K}$, see also Section 2.4.2.

3.5.1 The Hellinger-Toeplitz Theorem

The first fundamental property of continuous linear maps between Hilbert spaces which goes beyond the general results on Banach spaces is the following statement:

Theorem 3.5.1 (Hellinger-Toeplitz) *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces and let $A\colon \mathfrak{H} \longrightarrow \mathfrak{K}$ be a map. Then the following statements are equivalent:*

i.) The map A is linear and continuous.

ii.) The map A is adjointable.

Proof: Assume first that A is linear and continuous. For a fixed $\phi \in \mathfrak{K}$ the map

$$\mathfrak{H} \ni \psi \ \mapsto \ \langle \phi, A\psi \rangle \in \mathbb{C}$$

is then linear and continuous. Thus by Riesz' Theorem 3.2.11 there is a unique vector in $\mathfrak{H}$, denoted by $A^*\phi$, such that $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$. This defines the map $A^* \colon \mathfrak{K} \longrightarrow \mathfrak{H}$ and A is adjointable. Conversely, assume A is adjointable. Then we know already from Proposition 1.3.3, *i.)*, that A is necessarily linear. We consider the graph

$$\operatorname{graph} A = \big\{ (\phi, A\phi) \;\big|\; \phi \in \mathfrak{H} \big\} \subseteq \mathfrak{H} \times \mathfrak{K}$$

of A and show that it is closed: indeed, let $\Phi_n \in \operatorname{graph} A$ be a sequence converging to some $\Phi = (\phi, \psi) \in \mathfrak{H} \times \mathfrak{K}$. This means $\Phi_n = (\phi_n, A\phi_n)$ with some sequence $\phi_n \in \mathfrak{H}$ with $\phi_n \longrightarrow \phi$ and $A\phi_n \longrightarrow \psi$. We only have to show $A\phi = \psi$. Thus let $\chi \in \mathfrak{K}$ then

$$\begin{aligned}
\langle \chi, \psi \rangle_{\mathfrak{K}} &= \Big\langle \chi, \lim_{n \longrightarrow \infty} A\phi_n \Big\rangle_{\mathfrak{K}} \\
&= \lim_{n \longrightarrow \infty} \langle \chi, A\phi_n \rangle_{\mathfrak{K}} \\
&= \lim_{n \longrightarrow \infty} \langle A^*\chi, \phi_n \rangle_{\mathfrak{H}} \\
&= \Big\langle A^*\chi, \lim_{n \longrightarrow \infty} \phi_n \Big\rangle_{\mathfrak{H}} \\
&= \langle A^*\chi, \phi \rangle_{\mathfrak{H}} \\
&= \langle \chi, A\phi \rangle_{\mathfrak{K}},
\end{aligned}$$

using twice the continuity of the inner products. Since χ is arbitrary $\psi = A\phi$ follows. Hence the graph of A is closed and by the Closed Graph Theorem 2.3.20 we conclude that A is continuous. ☐

Remark 3.5.2 The theorem relies for both directions heavily on the completeness of a Hilbert space. It is easy to find counter-examples for the case of pre-Hilbert spaces which are not Hilbert spaces, see also Exercise 3.6.16.

Remark 3.5.3 This equivalence will be of crucial importance in operator algebra theory as it builds a bridge between algebraic features (existence of an adjoint) and analytic features (continuity). We can restate the theorem therefore also as

$$\mathfrak{B}(\mathfrak{H}, \mathfrak{K}) = \mathrm{L}(\mathfrak{H}, \mathfrak{K}) \tag{3.5.2}$$

to emphasize this equivalence. Since $\mathrm{L}(\mathfrak{H}, \mathfrak{K})$ is a Banach space with respect to the operator norm, see Proposition 2.3.7, the adjointable operators become a Banach space. We will see many more statements that algebraic and analytic features are intimately linked in the theory of operator algebras.

Beside the general properties of the operator norm discussed in Proposition 2.3.7 the case of continuous linear maps between Hilbert spaces leads to the following additional feature:

Theorem 3.5.4 (C^*-property of the operator norm) *Let $\mathfrak{H}, \mathfrak{K}$ be Hilbert spaces and let $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$ be a continuous linear map. Then*

$$\|A^*A\| = \|A\|^2, \tag{3.5.3}$$

and hence

$$\|A^*\| = \|A\|. \tag{3.5.4}$$

PROOF: Let $\phi \in \mathfrak{H}$. Then

$$\|A\phi\|^2 = |\langle A\phi, A\phi \rangle| = |\langle \phi, A^*A\phi \rangle| \leq \|\phi\| \|A^*A\phi\| \leq \|A^*A\| \|\phi\|$$

This implies $\|A\| \leq \sqrt{\|A^*A\|}$. Conversely, we use the fact that the operator norm can be characterized by

$$\|A\| = \sup_{\|\psi\|=1, \|\phi\|=1} |\langle \psi, A\phi \rangle|,$$

according to Exercise 3.6.18. With this result it immediately follows that $\|A^*\| = \|A\|$. This gives $\|A^*A\| \leq \|A^*\|\|A\| = \|A\|^2$ by the general continuity property of the operator product according to Proposition 2.3.7, *ii.)*. Together, this gives (3.5.3). ☐

Remark 3.5.5 (C^*-property) The property (3.5.3) of the operator norm is called the C^**-property*. It is the motivation for the definition of a C^*-algebra. We come back to this feature in great detail in Chapter 4. Having (3.5.3) and the continuity of the product $\|AB\| \leq \|A\|\|B\|$ the feature (3.5.4) is of course an easy consequence. It expresses the fact that also the antilinear map $A \mapsto A^*$ is continuous.

We conclude this subsection with the computation of the operator norms of some types of operators. Note that we do only use the defining algebraic relations from Definition 1.1.2 as well as the C^*-property of the operator norm.

Proposition 3.5.6 *Let $\mathfrak{H}$ and $\mathfrak{K}$ be non-zero Hilbert spaces.*

i.) *If $V \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$ is isometric then $\|V\| = 1$.*

ii.) *If $U \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$ is unitary then $\|U\| = 1$.*

iii.) *If $P \in \mathfrak{B}(\mathfrak{H})$ is a non-zero projection then $\|P\| = 1$.*

PROOF: We know $\|\mathbb{1}\| = 1$ from the very definition of the operator norm. If V is isometric then $1 = \|\mathbb{1}\| = \|V^*V\| = \|V\|^2$ gives the first part. The second is a particular case of the first. If P is a non-zero projection then $\|P\| = \|P^*P\| = \|P\|^2$ gives immediately the third part. ☐

Note that we have seen these properties already in the examples of the symmetrizer Sym and the anti-symmetrizer Alt in Lemma 3.4.9.

3.5.2 The Lattice of Projections

Let $\mathfrak{H}$ be a Hilbert space. In this subsection we focus on the projections in $\mathfrak{B}(\mathfrak{H})$, i.e. those linear maps $P \colon \mathfrak{H} \longrightarrow \mathfrak{H}$ with $P = P^* = P^2$.

Proposition 3.5.7 *The map which assigns to a projection $P \in \mathfrak{B}(\mathfrak{H})$ its image $\operatorname{im} P \subseteq \mathfrak{H}$ is a bijection between the set of projections and the lattice of closed subspaces of $\mathfrak{H}$. The inverse map assigns to a closed subspace $U = U^{\mathrm{cl}} \subseteq \mathfrak{H}$ the projection P_U along the direct orthogonal sum $U \oplus U^\perp = \mathfrak{H}$.*

PROOF: Since a projection P is continuous, also $\mathbb{1} - P$ is continuous. Hence $\operatorname{im} P = \ker(\mathbb{1} - P)$ is a closed subspace of $\mathfrak{H}$. Thus the map is well-defined. Suppose $U = U^{\mathrm{cl}}$ and consider the decomposition $\mathfrak{H} = U \oplus U^\perp$ according to Theorem 3.2.1, *v.)*. For $\phi \in \mathfrak{H}$ this gives $\phi = \phi_\parallel + \phi_\perp$ with $\phi_\parallel \in U$ and $\phi_\perp \in U^\perp$. This defines a linear map $P_U \colon \phi \mapsto \phi_\parallel$. Clearly, P_U is idempotent since $(\phi_\parallel)_\parallel = \phi_\parallel$. Let $\phi, \psi \in \mathfrak{H}$ then

$$\langle \psi, P_U \phi \rangle = \langle \psi, \phi_\parallel \rangle = \langle \psi_\parallel, \phi_\parallel \rangle = \langle \psi_\parallel, \phi \rangle = \langle P_U \psi, \phi \rangle$$

by the orthogonality of the decomposition $\mathfrak{H} = U \oplus U^\perp$. Thus $P_U = P_U^*$ and P_U is a projection. Being Hermitian it is adjointable and hence continuous by Theorem 3.5.1. By the very definition the image of P_U is U. This shows that the above map is surjective. The injectivity is clear since a projection is uniquely determined by its image: indeed, let P be given. Then $\operatorname{im} P \oplus \operatorname{im}(\mathbb{1} - P) = \mathfrak{H}$ which is true for every idempotent. Now, let $\psi \in \operatorname{im}(\mathbb{1} - P)$ then $\psi = \phi - P\phi$ for some ϕ and hence

$$\langle \psi, P\chi \rangle = \langle \phi, P\chi \rangle - \langle P\phi, P\chi \rangle = 0$$

shows that $\psi \in (\operatorname{im} P)^\perp$. Thus $\operatorname{im}(\mathbb{1} - P) \subseteq (\operatorname{im} P)^\perp$. Since by Theorem 3.2.1, *v.)*, we know $\mathfrak{H} = \operatorname{im} P \oplus (\operatorname{im} P)^\perp$, we conclude that $\operatorname{im}(\mathbb{1} - P) = (\operatorname{im} P)^\perp$. But this shows that the projection onto $\operatorname{im} P$ along $\operatorname{im} P \oplus (\operatorname{im} P)^\perp$ is indeed just P itself, i.e. we have $P_{\operatorname{im} P} = P$. Thus the claim is shown.□

Since the closed subspaces form a lattice according to Theorem 3.2.9 we can transfer the lattice operations via Proposition 3.5.7 to the set of projection operators in $\mathfrak{B}(\mathfrak{H})$ such that the map

$$U \mapsto P_U \tag{3.5.5}$$

becomes a lattice isomorphism. Explicitly, the lattice operations read as follows:

Theorem 3.5.8 (The lattice of projections) *Let $\mathfrak{H}$ be a Hilbert space. Then the bijection* (3.5.5) *between closed subspaces and projections establishes the structure of an orthomodular lattice with arbitrary* sup *and* inf *on the set of projections. Explicitly, one has for projections $P, Q \in \mathfrak{B}(\mathfrak{H})$*

$$P \wedge Q = P_{\operatorname{im} P \cap \operatorname{im} Q}, \tag{3.5.6}$$

$$P \vee Q = P_{(\operatorname{im} P + \operatorname{im} Q)^{\mathrm{cl}}}, \tag{3.5.7}$$

$$P \leq Q \quad \Leftrightarrow \quad PQ = P \quad \Leftrightarrow \quad QP = P, \tag{3.5.8}$$

$$P' = \mathbb{1} - P. \tag{3.5.9}$$

PROOF: The verification of the above explicit statements is straightforward using Theorem 3.2.9 and the isomorphism (3.5.5), see also Exercise 3.6.21. □

rary Sup and inf

3.5.3 Bounded Operators on Direct Sums and Tensor Products

In this short subsection we consider the relation of bounded operators on Hilbert space direct sums and tensor products.

Proposition 3.5.9 *Let $\{\mathfrak{H}_i\}_{i \in I}$ and $\{\mathfrak{K}'_j\}_{j \in J}$ be two families of Hilbert spaces and let $\mathfrak{H} = \widehat{\bigoplus}_{i \in I} \mathfrak{H}_i$ and $\mathfrak{K} = \widehat{\bigoplus}_{j \in J} \mathfrak{K}_j$ be their direct sums.*

i.) Let $P_i \in \mathfrak{B}(\mathfrak{H})$ and $Q_j \in \mathfrak{B}(\mathfrak{K})$ be the projections onto $\mathfrak{H}_i$ and $\mathfrak{K}_j$ which we view as subspaces of $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Then for $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$ we have bounded operators

$$A_{ji} = Q_j A P_i \colon \mathfrak{H}_i \longrightarrow \mathfrak{K}_j \tag{3.5.10}$$

with adjoints given by

$$A^*_{ji} = P_i A^* Q_j \colon \mathfrak{K}_j \longrightarrow \mathfrak{H}_i, \tag{3.5.11}$$

satisfying $\|A_{ji}\| \leq \|A\|$ for all $i \in I$ and $j \in J$.

ii.) If conversely $A_{ji} \in \mathfrak{B}(\mathfrak{H}_i, \mathfrak{K}_j)$ is a given bounded operator then we can view A_{ji} also as bounded operator $A_{ji} \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$ by setting $A_{ji}\big|_{\mathfrak{H}_{i'}} = 0$ for $i' \neq i$. In this case $Q_{j'} A_{ji} P_{i'} = A_{ji} \delta_{jj'} \delta_{ii'}$.

iii.) Suppose $A_{ji} \in \mathfrak{B}(\mathfrak{H}_i, \mathfrak{K}_j)$ is given for all $i \in I$ and $j \in J$ such that

$$c_j = \sup_{i \in I} \|A_{ji}\| < \infty \tag{3.5.12}$$

yields a square summable sequence $(c_j)_{j \in J} \in \ell^2(J)$. Then one has for the block-wise defined operator

$$A = \bigoplus_{i \in I, j \in J} A_{ji} \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K}). \tag{3.5.13}$$

iv.) Let $A_i \in \mathfrak{B}(\mathfrak{H}_i)$ be given for all $i \in I$ such that

$$\sup_{i \in I} \|A_i\| < \infty. \tag{3.5.14}$$

Then the block-diagonal operator

$$A = \bigoplus_{i \in I} A_i \tag{3.5.15}$$

yields a bounded operator $A \in \mathfrak{B}(\mathfrak{H})$ with operator norm

$$\|A\| = \sup_{i \in I} \|A_i\|. \tag{3.5.16}$$

PROOF: Since P_i and Q_j are continuous, A_{ji} is continuous as well and hence identifiable with an operator in $\mathfrak{B}(\mathfrak{H}_i, \mathfrak{K}_j)$. Clearly $A_{ji}^* = P_i A^* Q_j$ since $P_i = P_i^*$ and $Q_j = Q_j^*$ are projections. The norm estimate is clear since $\|P_i\|, \|Q_j\| \leq 1$. The second part is clear as well. For the third part, we first note that at most countably many c_j are different from zero. First, let $\phi \in \bigoplus_{i \in I} \mathfrak{H}_i \subseteq \mathfrak{H}$ be given then

$$\begin{aligned}
\|A\phi\|_{\mathfrak{K}}^2 &= \sum_{j \in J} \left\| \sum_{i \in I} A_{ji}\phi_i \right\|_{\mathfrak{K}_j}^2 \\
&\leq \sum_{j \in J} \sum_{i \in I} \|A_{ji}\phi_i\|_{\mathfrak{K}_j}^2 \\
&\leq \sum_{j \in J} \sum_{i \in I} c_j^2 \|\phi_i\|_{\mathfrak{H}_i}^2 \\
&= \|(c_j)_{j \in J}\|_{\ell^2(J)}^2 \|\phi\|_{\mathfrak{H}}^2,
\end{aligned}$$

where the sum over the index $i \in I$ is always finite as ϕ is assumed to be in the (algebraic) direct sum. But this shows that A, defined on the dense domain $\bigoplus_{i \in I} \mathfrak{H}_i$, is a bounded operator with operator norm $\|A\| \leq \|(c_j)_{j \in J}\|_{\ell^2(J)}$. Thus it extends to a bounded operator on $\mathfrak{H}$. For the last part, we estimate for $\phi \in \bigoplus_{i \in I} \mathfrak{H}_i$

$$\|A\phi\|_{\mathfrak{H}}^2 = \sum_{i \in I} \|A_i\phi_i\|_{\mathfrak{H}_i}^2 \leq \sup_{i \in I} \|A_i\|^2 \sum_{i \in I} \|\phi_i\|_{\mathfrak{H}_i}^2 = \sup_{i \in I} \|A_i\|^2 \|\phi\|_{\mathfrak{H}}^2,$$

which shows that A extends to a bounded operator on $\mathfrak{H}$ with operator norm $\|A\| \leq \sup_{i \in I} \|A_i\|$. Now let $\epsilon > 0$ be given and fix $i_0 \in I$ with $\|A_{i_0}\| + \epsilon > \sup_{i \in I} \|A_i\|$. Then pick a unit vector $\phi_{i_0} \in \mathfrak{H}_{i_0}$ with $\|A_{i_0}\phi_{i_0}\| + \epsilon > \|A_{i_0}\|$. For such a vector we get

$$\|A\phi_{i_0}\| = \|A_{i_0}\phi_{i_0}\| > \|A_{i_0}\| + \epsilon > \sup_{i \in I} \|A_i\| + 2\epsilon,$$

which implies the equality in (3.5.16). ☐

Remark 3.5.10 The condition in part *iii.)* is typically not necessary but only sufficient. In fact, it is very hard to characterize the continuity of $A\colon \mathfrak{H} \longrightarrow \mathfrak{K}$ in terms of its components A_{ji} in general. Clearly, if $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$ then for all $i \in I$ and $j \in J$ we have $\|A_{ji}\| \leq \|A\|$ by part *i.)* but the boundedness of the norms $\{\|A_{ji}\|\}_{i \in I, j \in J}$ will in general not yet be sufficient to guarantee the continuity of A.

Next we pass to tensor products. Thus consider two Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ as well as their tensor product $\mathfrak{H} \hat{\otimes} \mathfrak{K}$. Here an operator on $\mathfrak{H} \hat{\otimes} \mathfrak{K}$ can no longer be split into parts acting on $\mathfrak{H}$ or $\mathfrak{K}$ separately. However, we have the following statements:

Proposition 3.5.11 *Let $\mathfrak{H}_i$ and $\mathfrak{K}_i$ for $i = 1, 2, 3$ be Hilbert spaces and let $A \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$, $B \in \mathfrak{B}(\mathfrak{H}_2, \mathfrak{H}_3)$ as well as $\tilde{A} \in \mathfrak{B}(\mathfrak{K}_1, \mathfrak{K}_2)$, $\tilde{B} \in \mathfrak{B}(\mathfrak{K}_2, \mathfrak{K}_3)$ be given.*

i.) The map $A \otimes \tilde{A} \colon \mathfrak{H}_1 \otimes \mathfrak{K}_1 \longrightarrow \mathfrak{H}_2 \otimes \mathfrak{K}_2$ is continuous and extends to a continuous map

$$A \mathbin{\hat{\otimes}} \tilde{A} \colon \mathfrak{H}_1 \mathbin{\hat{\otimes}} \mathfrak{K}_1 \longrightarrow \mathfrak{H}_2 \mathbin{\hat{\otimes}} \mathfrak{K}_2. \tag{3.5.17}$$

ii.) One has $\|A \mathbin{\hat{\otimes}} \tilde{A}\| = \|A\|\|\tilde{A}\|$ for the operator norm of $A \mathbin{\hat{\otimes}} \tilde{A}$ and hence

$$\hat{\otimes} \colon \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2) \times \mathfrak{B}(\mathfrak{K}_1, \mathfrak{K}_2) \longrightarrow \mathfrak{B}(\mathfrak{H}_1 \mathbin{\hat{\otimes}} \mathfrak{K}_1, \mathfrak{H}_2 \mathbin{\hat{\otimes}} \mathfrak{K}_2) \tag{3.5.18}$$

is a continuous bilinear map.

iii.) One has

$$(B \mathbin{\hat{\otimes}} \tilde{B})(A \mathbin{\hat{\otimes}} \tilde{A}) = (BA) \mathbin{\hat{\otimes}} (\tilde{B}\tilde{A}), \tag{3.5.19}$$

and

$$(A \mathbin{\hat{\otimes}} \tilde{A})^* = A^* \mathbin{\hat{\otimes}} (\tilde{A})^*. \tag{3.5.20}$$

PROOF: To show the first part we choose Hilbert bases $\{\mathrm{e}_i\}_{i \in I}$ of $\mathfrak{H}_1$ and $\{\mathrm{e}'_{i'}\}_{i' \in I'}$ of $\mathfrak{K}_1$ as well as Hilbert bases $\{\mathrm{f}_j\}_{j \in J}$ of $\mathfrak{H}_2$ and $\{\mathrm{f}'_{j'}\}_{j' \in J'}$ of $\mathfrak{K}_2$, respectively. Then we know that every vector $\Phi \in \mathfrak{H}_1 \mathbin{\hat{\otimes}} \mathfrak{K}_1$ can be written as

$$\Phi = \sum_{i,i'} \Phi_{ii'} \mathrm{e}_i \otimes e_{i'} \quad \text{with} \quad \Phi_{ii'} = \langle \mathrm{e}_i \otimes e_{i'}, \Phi \rangle_{\mathfrak{H}_1 \hat{\otimes} \mathfrak{K}_1}$$

by Theorem 3.4.6, *iii.)*, and analogously for vectors in $\mathfrak{H}_2 \mathbin{\hat{\otimes}} \mathfrak{K}_2$. We fix now a vector $\Psi \in \mathfrak{H}_2 \mathbin{\hat{\otimes}} \mathfrak{K}_2$. Then for a fixed vector $\psi' \in \mathfrak{K}_2$ the linear functional

$$\mathfrak{H}_2 \ni \psi \; \mapsto \; \langle \Psi, \psi \otimes \psi' \rangle_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2} \in \mathbb{C}$$

is continuous since the tensor product is continuous by Theorem 3.4.6, *i.)*, and the inner product is continuous anyway. Thus, by Riesz' Theorem, we find a unique vector $u \in \mathfrak{H}_2$ depending on ψ' and Ψ such that

$$\langle \Psi, \psi \otimes \psi' \rangle_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2} = \langle u, \psi \rangle_{\mathfrak{H}_2}.$$

Analogously, we find for fixed ψ a vector $u' \in \mathfrak{K}_2$ such that for all $\psi' \in \mathfrak{K}_2$ we get

$$\langle \Psi, \psi \otimes \psi' \rangle_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2} = \langle u', \psi' \rangle_{\mathfrak{K}_2}.$$

In the following, the vectors $u_{i'} \in \mathfrak{H}_2$ and $u'_j \in \mathfrak{K}_2$ will be chosen according to these properties. We have

$$\begin{aligned}
\sum_{\substack{i \in I \\ i' \in I'}} \left| \langle \Psi, (A \otimes \tilde{A})(\mathrm{e}_i \otimes \mathrm{e}'_{i'}) \rangle_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2} \right|^2 &= \sum_{\substack{i \in I \\ i' \in I'}} \left| \langle \Psi, A\mathrm{e}_i \otimes A\mathrm{e}'_{i'} \rangle \right|^2 \\
&= \sum_{\substack{i \in I \\ i' \in I'}} \left| \langle u_{i'}, A\mathrm{e}_i \rangle_{\mathfrak{H}_2} \right|^2 \\
&= \sum_{\substack{i \in I \\ i' \in I'}} \left| \langle A^* u_{i'}, \mathrm{e}_i \rangle_{\mathfrak{H}_1} \right|^2 \\
&\overset{(a)}{=} \sum_{i' \in I'} \|A^* u_{i'}\|^2_{\mathfrak{H}_1}
\end{aligned}$$

$$
\begin{aligned}
&\leq \|A\|^2 \sum_{i' \in I'} \|u_{i'}\|_{\mathfrak{H}_2} \\
&\overset{(a)}{=} \|A\|^2 \sum_{\substack{i' \in I' \\ j \in J}} \left|\langle u_{i'}, \mathrm{f}_j \rangle_{\mathfrak{H}_2}\right|^2 \\
&= \|A\|^2 \sum_{\substack{i' \in I' \\ j \in J}} \left|\langle \Psi, \mathrm{f}_j \otimes \tilde{A}\mathrm{e}'_{i'} \rangle_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2}\right|^2 \\
&= \|A\|^2 \sum_{\substack{i' \in I' \\ j \in J}} \left|\langle u'_j, \tilde{A}\mathrm{e}'_{i'} \rangle_{\mathfrak{K}_2}\right|^2 \\
&= \|A\|^2 \sum_{\substack{i' \in I' \\ j \in J}} \left|\langle (\tilde{A})^* u'_j, \mathrm{e}'_{i'} \rangle_{\mathfrak{K}_1}\right|^2 \\
&\overset{(a)}{=} \|A\|^2 \sum_{j \in J} \left\|(\tilde{A})^* u'_j\right\|^2_{\mathfrak{K}_1} \\
&\leq \|A\|^2 \|\tilde{A}\|^2 \sum_{j \in J} \|u'_j\|^2_{\mathfrak{K}_2} \\
&\overset{(a)}{=} \|A\|^2 \|\tilde{A}\|^2 \sum_{\substack{j \in J \\ j' \in J'}} \left|\langle u'_j, \mathrm{f}'_{j'} \rangle_{\mathfrak{K}_2}\right|^2 \\
&= \|A\|^2 \|\tilde{A}\|^2 \sum_{\substack{j \in J \\ j' \in J'}} \left|\langle \Psi, \mathrm{f}_j \otimes \mathrm{f}'_{j'} \rangle_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2}\right|^2 \\
&\overset{(a)}{=} \|A\|^2 \|\tilde{A}\|^2 \|\Psi\|^2_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2},
\end{aligned}
$$

where in (a) we use Parseval's identity (3.3.13). Thus we arrive at the estimate

$$
\sum_{\substack{i \in I \\ i' \in I'}} \left|\left\langle \Psi, (A \otimes \tilde{A})(\mathrm{e}_i \otimes \mathrm{e}'_{i'}) \right\rangle_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2}\right|^2 \leq \|A\|^2 \|\tilde{A}\|^2 \|\Psi\|^2_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2} \tag{$*$}
$$

for all Hilbert bases of $\mathfrak{H}_1$ and $\mathfrak{K}_1$ and all $\Psi \in \mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2$. This allows now to estimate for $\Phi \in \operatorname{span}_{\mathbb{C}}\{\mathrm{e}_i \otimes \mathrm{e}'_{i'}\}_{i \in I, i' \in I'} \subseteq \mathfrak{H}_1 \otimes \mathfrak{K}_1$ by the Cauchy-Schwarz inequality

$$
\begin{aligned}
\left|\left\langle \Psi, (A \otimes \tilde{A})\Phi \right\rangle_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2}\right|^2 &= \left|\sum_{i \in I, i' \in I'} \Phi_{ii'} \left\langle \Psi, (A \otimes \tilde{A})(\mathrm{e}_i \otimes \mathrm{e}'_{i'}) \right\rangle_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2}\right|^2 \\
&\leq \left(\sum_{i \in I, i' \in I'} |\Phi_{ii'}|^2\right) \left(\sum_{i \in I, i' \in I'} \left|\left\langle \Psi, (A \otimes \tilde{A})(\mathrm{e}_i \otimes \mathrm{e}'_{i'}) \right\rangle_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2}\right|^2\right) \\
&\overset{(*)}{\leq} \|\Phi\|^2_{\mathfrak{H}_1 \otimes \mathfrak{K}_1} \|A\|^2 \|\tilde{A}\|^2 \|\Psi\|^2_{\mathfrak{H}_2 \hat{\otimes} \mathfrak{K}_2}.
\end{aligned}
$$

By the characterization of the operator norm as in Exercise 3.6.18, see also the proof of Theorem 3.5.4, and the fact that the span of a Hilbert basis is dense, we see that $A \otimes \tilde{A}$ is continuous and

$$
\|A \otimes \tilde{A}\| \leq \|A\| \|\tilde{A}\|. \tag{$**$}
$$

Thus it extends to a continuous operator $A \hat{\otimes} \tilde{A}$ as required. For the second part we notice that for factorizing tensors $\phi \otimes \phi'$ we have

$$
\|(A \otimes \tilde{A})(\phi \otimes \phi')\| = \|A\phi \otimes \tilde{A}\phi'\| = \|A\phi\| \|\tilde{A}\phi'\|,
$$

as well as $\|\phi \otimes \phi'\| = \|\phi\|\|\phi'\|$. From this we see that in (∗∗) we actually get equality. But then the map $\hat{\otimes}$ as in (3.5.18) is continuous, again anticipating the characterization of continuous bilinear maps as discussed in Theorem 4.1.3 in general. The last part is now clear since we can check these identities on the algebraic tensor products and extend to the completion by the usual continuity argument. □

Corollary 3.5.12 *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces. Then the maps*

$$\mathfrak{B}(\mathfrak{H}) \ni A \;\mapsto\; A \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}} \in \mathfrak{B}(\mathfrak{H} \mathbin{\hat{\otimes}} \mathfrak{K}) \tag{3.5.21}$$

and

$$\mathfrak{B}(\mathfrak{K}) \ni \tilde{A} \;\mapsto\; \mathrm{id}_{\mathfrak{H}} \mathbin{\hat{\otimes}} \tilde{A} \in \mathfrak{B}(\mathfrak{H} \mathbin{\hat{\otimes}} \mathfrak{K}) \tag{3.5.22}$$

are unital norm-preserving *-homomorphisms such that for all A, $\tilde{A}$ one has

$$(\mathrm{id}_{\mathfrak{H}} \mathbin{\hat{\otimes}} \tilde{A})(A \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}}) = (A \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}})(\mathrm{id}_{\mathfrak{H}} \mathbin{\hat{\otimes}} \tilde{A}). \tag{3.5.23}$$

In fact, this follows from Exercise 1.5.20 by continuity and completion.

3.6 Exercises

Exercise 3.6.1 (Positivity of the inner product) Let $\mathfrak{H}$ be a vector space with a positive semi-definite sesquilinear form $\langle \cdot\,, \cdot \rangle$ on it. Show that for $n \in \mathbb{N}$ and all $\phi_1, \ldots, \phi_n \in \mathfrak{H}$ the matrix $(\langle \phi_i, \phi_j \rangle)_{i,j=1,\ldots,n} \in \mathrm{M}_n(\mathbb{C})$ is positive.

Exercise 3.6.2 (The parallelogram identity) Give a proof for Proposition 3.1.5 and give an elementary geometric interpretation of this identity.

Hint: First it is easy to check that (3.1.7) holds for a pre-Hilbert space. For the converse define $\langle \cdot\,, \cdot \rangle$ by (3.1.6). Show that this is a continuous map. Next show that the parallelogram identity gives additivity in the second argument. Deduce from this $\mathbb{Q}$-linearity in the second argument and use the continuity to complete the proof.

Exercise 3.6.3 (Polarization formula) Consider a real pre-Hilbert space $\mathfrak{H}$. Formulate and prove the analog of Lemma 3.1.4 in the real case. Show also Proposition 3.1.5 in this case.

Exercise 3.6.4 (Quantum mechanical need for good complements) Consider the pre-Hilbert spaces from Example 3.1.9 and show that in all of them one finds subspaces U with the property that $U^\perp \oplus U^{\perp\perp}$ is not yet the whole pre-Hilbert space. Argue why this would be a very desirable and in fact crucial feature for a consistent quantum mechanical interpretation.

Exercise 3.6.5 (Approximation of $\chi_{[a,b]}$) Let $a < b$ and consider the characteristic function $\chi_{[a,b]}$ of the interval $[a, b] \subseteq \mathbb{R}$. Construct an explicit sequence $\varphi_n \in \mathscr{C}^\infty_{[a,b]}(\mathbb{R})$ of smooth functions with support in $[a, b]$ such that $\varphi_n \longrightarrow \chi_{[a,b]}$ with respect to the pre-Hilbert norm induced by the inner product given by the Riemann integral over $[a, b]$.

Exercise 3.6.6 (Complexification of a real Hilbert space) Let $\mathfrak{H}$ be a *real* pre-Hilbert space, also known under the name *Euclidean space.*

i.) Define the complexified vector space $\mathfrak{H}_{\mathbb{C}} = \mathfrak{H} \otimes_{\mathbb{R}} \mathbb{C}$, i.e. its elements can be written as $\mathbb{R}$-linear combinations of $\phi \otimes z$ with $\phi \in \mathfrak{H}$ and $z \in \mathbb{C}$. Show that this becomes a complex vector space by setting $w \cdot (\phi \otimes z) = \phi \otimes (wz)$ for $w \in \mathbb{C}$. One calls $\mathfrak{H}_{\mathbb{C}}$ the complexification of the real vector space $\mathfrak{H}$.

ii.) Show that there is a unique $\mathbb{R}$-linear map
$$\overline{}: \mathfrak{H}_{\mathbb{C}} \longrightarrow \mathfrak{H}_{\mathbb{C}} \tag{3.6.1}$$
with $\overline{\phi \otimes z} = \phi \otimes \overline{z}$ for all $\phi \in \mathfrak{H}$ and $z \in \mathbb{C}$. One calls this map the complex conjugation of $\mathfrak{H}_{\mathbb{C}}$. Show that it is involutive and complex anti-linear. Prove that every vector $\Psi \in \mathfrak{H}_{\mathbb{C}}$ can be written as $\Psi = \operatorname{Re}(\Psi) + \mathrm{i}\operatorname{Im}(\Psi)$ with $\operatorname{Re}(\Psi), \operatorname{Im}(\Psi)$ begin real, i.e. stable under complex conjugation. Show that this decomposition yields a $\mathbb{R}$-linear isomorphism
$$\mathfrak{H}_{\mathbb{C}} \cong \mathfrak{H} \oplus \mathfrak{H}. \tag{3.6.2}$$

iii.) Show that the $\mathbb{R}$-bilinear inner product $\langle \cdot\,, \cdot \rangle$ of $\mathfrak{H}$ extends uniquely to a sesquilinear inner product
$$\langle \cdot\,, \cdot \rangle_{\mathbb{C}}: \mathfrak{H}_{\mathbb{C}} \times \mathfrak{H}_{\mathbb{C}} \longrightarrow \mathbb{C} \tag{3.6.3}$$
such that $\langle \phi \otimes z, \psi \otimes w \rangle_{\mathbb{C}} = \langle \phi, \psi \rangle \overline{z} w$ for $\phi, \psi \in \mathfrak{H}$ and $z, w \in \mathbb{C}$. Show that $\langle \cdot\,, \cdot \rangle_{\mathbb{C}}$ is positive definite.

iv.) Show that the complex conjugation of $\mathfrak{H}_{\mathbb{C}}$ is an anti-unitary map.

v.) Let $\mathfrak{K}$ be another real pre-Hilbert space and let $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$ be an adjointable map. Show that there is unique linear map
$$A_{\mathbb{C}}: \mathfrak{H}_{\mathbb{C}} \longrightarrow \mathfrak{K}_{\mathbb{C}} \tag{3.6.4}$$
such that $A_{\mathbb{C}}(\phi \otimes z) = (A\phi) \otimes z$ for all $\psi \in \mathfrak{H}$ and $z \in \mathbb{C}$. Show that $A_{\mathbb{C}}$ is adjointable and compute its adjoint on elementary tensors $\psi \otimes z$ with $\psi \in \mathfrak{K}$. Conclude that *complexification* of real pre-Hilbert spaces to complex ones is functorial.

vi.) Denote the adjoint of $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$ by A^{T}. What is the relation of $(A^{\mathrm{T}})_{\mathbb{C}}$, $A_{\mathbb{C}}^*$, and the complex conjugation on $\mathfrak{H}_{\mathbb{C}}$ and $\mathfrak{K}_{\mathbb{C}}$, respectively?

vii.) Suppose now in addition that $\mathfrak{H}$ was already complete, i.e. a real Hilbert space. Show that $\mathfrak{H}_{\mathbb{C}}$ is complete, too.

Exercise 3.6.7 (The Bargmann-Fock space I)

Exercise 3.6.8 (The Bargmann-Fock space II)

Exercise 3.6.9 (Gram-Schmidt algorithm) Give a proof of Proposition 3.3.2.

Exercise 3.6.10 (Hilbert bases of L^2-spaces) The aim of this exercise is to provide a powerful tool to construct Hilbert bases of various L^2-spaces occurring in quantum mechanical applications. The construction does not use the Lebesgue integral at all; instead the L^2-spaces can be viewed as the completion of square-integrable functions with respect to the inner product coming from a (possibly improper) Riemann integral. Let $I = (a, b) \subseteq \mathbb{R}$ be an open interval with $-\infty \leq a < b \leq +\infty$ and let $\varrho \in \mathscr{C}^\infty(I)$ be a smooth function with $\varrho(x) > 0$ for all $x \in I$. We require that ϱ has exponential decay at infinity, i.e. there are constants $\alpha, c > 0$ with
$$\varrho(x) \leq c\mathrm{e}^{-\alpha|x|}. \tag{3.6.5}$$
If $I \neq \mathbb{R}$ we extend ϱ by 0 to a (possibly discontinuous) function on $\mathbb{R}$.

i.) Consider now $\varphi \in \mathscr{C}_0^\infty(I)$ and the corresponding Fourier transform F of $\overline{\varphi}\varrho$ defined by
$$F(k) = \int_{\mathbb{R}} \overline{\varphi(x)} \varrho(x) \mathrm{e}^{\mathrm{i}kx} \,\mathrm{d}x, \tag{3.6.6}$$
where $k \in \mathbb{R}$. Show that the compact support of φ allows to exchange the integration with the Taylor expansion of the exponential series. Conclude that F can be extended to a entire function for $k \in \mathbb{C}$ and compute the derivatives of F at $k = 0$. This is the most simple version of the *Palais-Wiener Theorem*, see e.g. [49, Chap. 7].

ii.) Let $\mathfrak{H}$ be the Hilbert space completion of $\mathscr{C}_0^\infty(I)$. Show that the functions $\mathrm{e}_n(x) = x^n \varrho(x)$ are square-integrable over I. Moreover, show that the functions e_n can be approximated by functions from $\mathscr{C}_0^\infty(I)$. Conclude that they can be viewed as elements of $\mathfrak{H}$.

iii.) Consider now a $\varphi \in \mathscr{C}_0^\infty(I)$ which is perpendicular to all the e_n. Use the injectivity of the Fourier transform and the fact that F is holomorphic to show that $\varphi = 0$.

iv.) Conclude that the span of the e_n in $\mathfrak{H}$ is dense. Use the Gram-Schmidt algorithm to produce a countable Hilbert basis of $\mathfrak{H}$.

Remarkably, the above argument does not require the Lebesgue integral at all for the price of an "abstract" completion $\mathfrak{H}$ of $\mathscr{C}_0^\infty(I)$. With little effort this construction can be used for various other situations in quantum mechanics.

Reference to ...ate Section in the Lebesgue ...sure context.

v.) Apply the above construction to $I_1 = \mathbb{R}$ and $\varrho_1(x) = \mathrm{e}^{-\frac{1}{2}x^2}$, to $I_2 = (0, \infty)$ and $\varrho_2(x) = \mathrm{e}^{-\frac{1}{2}x}$, as well as to $I_3 = (-1, 1)$ and $\varrho_3(x) = 1$. Which known types of orthogonal functions do arise?

Exercise 3.6.11 (Unconditional and absolute convergence) Consider the Hilbert space $\ell^2(\mathbb{N})$ with its canonical Hilbert basis $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$.

i.) Show that in a Banach space an absolutely convergent series is always unconditionally convergent, i.e. every resummation converges to the same limit.

ii.) Show that the harmonic sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ is in $\ell^2(\mathbb{N})$.

iii.) Show that the expansion of the harmonic sequence in terms of the Hilbert basis $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ is not absolutely convergent.

Exercise 3.6.12 (The weak topology of a Hilbert space) Let $\mathfrak{H}$ be a Hilbert space.

i.) Let $\{\mathrm{e}_i\}_{i\in I}$ be a Hilbert basis of $\mathfrak{H}$ and put some direction $\preccurlyeq$ on I. Show that

$$\lim_{i\in I} \mathrm{e}_i = 0 \tag{3.6.7}$$

in the weak topology. Show that also $\|\mathrm{e}_i\|$ converges.

Hint: No matter which direction you put on I, you have to show the following statement: for every $\epsilon > 0$ and all $\varphi \in \mathfrak{H}$ there exists an index $i \in I$ such that for all $j \succcurlyeq i$ one has $|\langle \varphi, \mathrm{e}_j\rangle| < \epsilon$. Use now that there are only finitely many $k \in I$ with $|\langle \varphi, \mathrm{e}_k\rangle| \geq \epsilon$.

ii.) Show that for an infinite-dimensional Hilbert space the weak topology is strictly coarser than the norm topology.

iii.) Show that $\langle \cdot\,, \cdot \rangle\colon \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{C}$ is separately continuous in the weak topology.

iv.) Show that for an infinite-dimensional Hilbert space the scalar product is discontinuous in the weak topology.

Exercise 3.6.13 (Lattices) Let $(V, \vee, \wedge)$ be a lattice.

i.) Prove that $a \vee b = b$ iff $a \wedge b = a$. Use this to show that the definition $a \leq b$ if $a \wedge b = a$ gives a partial ordering on V.

ii.) Show that $\leq$ allows for inf and sup.

iii.) Prove now the converse for a partially ordered set with inf and sup: the definition $a \wedge b = \inf(a, b)$ and $a \vee b = \sup(a, b)$ induce the structure of a lattice, whose partial order according to *i.)* reproduces the given partial order.

iv.) Define a lattice homomorphism $\phi\colon V \longrightarrow W$ as a map preserving $\vee$ and $\wedge$. Show that this is equivalent to an order preserving map. Conclude that the lattices form a category $\mathsf{Lattice}$ which can be viewed as a full subcategory of the category of all partially ordered sets Poset.

Exercise 3.6.14 (Direct sum of Hilbert spaces) Let I be a nonempty set and let $\{\mathfrak{H}_i\}_{i\in I}$ be a collection of pre-Hilbert spaces indexed by I. Denote by $\widehat{\mathfrak{H}} \subseteq \prod_{i\in I} \mathfrak{H}_i$ the completion of the direct sum to a Hilbert space according to Theorem 3.4.2.

i.) Show that the norm topology of the Hilbert space $\widehat{\mathfrak{H}}$ is strictly finer than the Cartesian product topology iff the index set I is infinite.

ii.) Show that there is a canonical isometric isomorphism (3.4.7).

iii.) Assume that the $\mathfrak{H}_i$ are already complete and let $\mathfrak{K}$ be another Hilbert space. Moreover, let $A_i\colon \mathfrak{H}_i \longrightarrow \mathfrak{K}$ be bounded maps. Show that there is a unique bounded map $A\colon \mathfrak{H} \longrightarrow \mathfrak{K}$ such that

$$A \circ \iota_i = A_i \tag{3.6.8}$$

for all $i \in I$ where $\iota_i\colon \mathfrak{H}_i \longrightarrow \mathfrak{H}$ is the canonical injection, *provided* the index set I is finite. What happens for infinite index sets? Conclude that the category $\mathsf{Hilbert}$ has finite *coproducts*, see also [34, Sect. III.3].

Exercise 3.6.15 (Tensors and symmetry) Let V be a vector space over a field $\Bbbk$ of characteristic 0 and let $n \in \mathbb{N}$.

i.) Let $\sigma \in S_n$ be a permutation. Show that $\sigma \triangleright$ as in (3.4.26) yields a well-defined linear endomorphism of $V^{\otimes n}$.

ii.) Show that $\sigma \mapsto \sigma \triangleright$ is a representation of the group S_n, i.e. a group homomorphism into the general linear group of $V^{\otimes n}$.

iii.) Define the symmetrizer and the antisymmetrizer as in (3.4.28) and (3.4.29), respectively, and show that they are idempotents satisfying $\mathrm{Sym}_n \circ \mathrm{Alt}_n = 0 = \mathrm{Alt}_n \circ \mathrm{Sym}_n$ whenever $n \geq 2$.

Hint: It will be enough to use that fact that $\sigma \mapsto \sigma \triangleright$ is a group representation and $\sigma \mapsto \mathrm{sign}(\sigma)$ is a group homomorphism.

iv.) Show that a tensor $v \in V^{\otimes n}$ is in the image of Sym_n iff $\sigma \triangleright v = v$ for all $\sigma \in S_n$. Analogously, show that the image of Alt_n consists of those tensors v satisfying $\sigma \triangleright v = \mathrm{sign}(\sigma) v$ for all $\sigma \in S_n$. Tensors in $\operatorname{im} \mathrm{Sym}_n$ are therefore called *symmetric tensors* while those in $\operatorname{im} \mathrm{Alt}_n$ are called *antisymmetric tensors.*

v.) Assume in addition that V is finite dimensional. Compute the dimensions of the images of Sym_n and Alt_n.

Exercise 3.6.16 (Adjointable and bounded linear maps) Show that the condition of completeness in Theorem 3.5.1 is necessary for both directions:

i.) Consider a pre-Hilbert space $\mathfrak{H}$ which is *not* complete and let $\iota\colon \mathfrak{H} \longrightarrow \widehat{\mathfrak{H}}$ denote its completion. Prove that ι is a continuous linear map which is not adjointable.

ii.) Consider the pre-Hilbert space $\mathfrak{H} = \mathrm{span}_{\mathbb{C}}\{\mathrm{e}_n\}_{n\in\mathbb{N}} \subseteq \ell^2(\mathbb{N})$. Show that the operator $A\colon \mathfrak{H} \longrightarrow \mathfrak{H}$ defined by $A\mathrm{e}_n = n\mathrm{e}_n$ is adjointable (and in fact Hermitian) but not continuous.

Exercise 3.6.17 (Image and kernel of a bounded operator) Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces and let $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$. Show that

$$(\operatorname{im} A^*)^\perp = \ker A \quad \text{and} \quad (\operatorname{im} A^*)^{\mathrm{cl}} = (\ker A)^\perp. \tag{3.6.9}$$

Exercise 3.6.18 (The operator norm: alternatives) Let $A\colon V \longrightarrow W$ be a continuous linear map between Banach spaces.

i.) Show that the operator norm of A is given by

$$\|A\| = \sup_{\varphi\in W'\setminus\{0\}} \sup_{v\in V\setminus\{0\}} \frac{|\varphi(Av)|}{\|\varphi\|\|v\|} = \sup_{\substack{\varphi\in W' \\ \|\varphi\|=1}} \sup_{\substack{v\in V \\ \|v\|=1}} |\varphi(Av)|. \tag{3.6.10}$$

ii.) Assume now that V and W are Hilbert spaces and show that the operator norm of A is given by

$$\|A\| = \sup_{w \in W \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{|\langle w, Av \rangle|}{\|w\| \|v\|} = \sup_{\substack{w \in W \\ \|w\|=1}} \sup_{\substack{v \in V \\ \|v\|=1}} |\langle w, Av \rangle|. \tag{3.6.11}$$

Exercise 3.6.19 (The norm of the inverse) Show that for an invertible bounded operator $A \in \mathfrak{B}(\mathfrak{H})$ one always has $\|A\| \|A^{-1}\| \geq 1$. Find an example for A such that $\|A\| = \|A^{-1}\| = 2$.

Lax-Milgram

Exercise 3.6.20 (The Lax-Milgram Theorem)

Exercise 3.6.21 (Lattice of projections) Prove Theorem 3.5.8.

Exercise 3.6.22 (The bicategory of Hilbert spaces) This exercise requires some knowledge about bicategories as it can be found e.g. in [5]. Formulate and prove that Hilbert spaces with adjointable operators form a bicategory with one object, the Hilbert spaces as 1-morphisms, and the adjointable operators as 2-morphisms, where the tensor product of Hilbert spaces serves as the composition of 1-morphisms.

Hint: This is essentially the content of Proposition 3.5.11.

Chapter 4

From Topological Algebras to C^*-Algebras

In this chapter we shall investigate now observable algebras more closely with the aim to find appropriate topological contexts which guarantee "good" behaviour. Again, we chose to work over the complex numbers $\mathbb{C}$ even though many definitions and results work well over $\mathbb{R}$, too. However, when it comes to the spectral calculus then the complex numbers are needed. The final aim is to come as close to $\mathfrak{B}(\mathfrak{H})$ as possible, but we will meet interesting and more general algebras on the way. One important question will be how a good notion of a spectrum can be defined. Closely related is the question whether we can define a functional calculus for algebra elements which goes beyond the purely algebraic polynomial calculus. We will discuss several calculi in increasing generality concerning the type of functions which will be allowed. This will also shine some light on the question how many positive elements one has concerning the purely algebraic definition from Chapter 1. Finally, we will discuss first ideas of *-representation theory, now on Hilbert spaces.

For all mentioned issues the class of C^*-algebras will turn out to be of particular convenience. This will lead to the point of view that in quantum mechanical systems the observables should be a C^*-algebra. Even though this is very much desirable, things are typically not that easy. Already in simple examples it is not at all obvious how to achieve this nice C^*-situation. From a quantization point of view things are indeed very difficult. Here already other and less convenient types of observable algebras are not so simple to get, not to speak of C^*-algebras at all. This motivates and justifies to consider also more general types of topological algebras on the way.

Thus in a first step we just implement a topology on an algebra such that the product becomes continuous. This yields a topological algebra being the direct cousin of a topological vector space. As already for topological vector spaces, not much interesting can be said about topological algebras per se. Hence we move on to the more particular case where the underlying topology is locally convex. In order to study locally convex algebras we investigate the continuity properties of bilinear and also multilinear maps in general leading to the projective tensor product. Specializing even further brings us to the class of locally multiplicatively convex algebras. Here one has a first important structural result: they allow for an entire calculus extending the polynomial calculus present in any algebra. We discuss several examples and show that, unfortunately, the canonical commutation relations will not allow for a locally multiplicatively convex algebra structure. While this is a certain draw-back, we nevertheless move on to an even nicer class of topological algebras, the Banach algebras and Banach *-algebras. Here we meet a good notion of spectrum and a holomorphic calculus extending the entire calculus. For commutative Banach algebras we formulate and study the properties of the Gel'fand transform in quite some detail. Finally, the C^*-algebras have the closest similarity to bounded operators on a Hilbert space: the C^*-property of the norm is responsible for better spectral properties than in general Banach algebras. As a consequence, commutative unital C^*-algebras turn out to be just the continuous functions on a compact Hausdorff space, this is the famous Gel'fand-Naimark Theorem. Based on this observation we can then introduce a continuous calculus for normal elements in a C^*-algebra. We conclude this chapter with a discussion of positive elements, states, and *-representations of C^*-algebras. Automatic continuity of *-homomorphisms leads to continuity of *-representations. The existence of sufficiently many states then shows that a C^*-algebra is always isomorphic to a closed *-subalgebra of the bounded operators on a Hilbert space.

4.1 Locally Convex Algebras and Entire Calculus

In the following, $\mathscr{A}$ will denote an associative algebra over $\mathbb{C}$. Later on, we shall equip $\mathscr{A}$ also with a *-involution. We are now interested in finding an appropriate compatibility of algebraic structures with concepts of convergence. Here the following definition is an obvious idea:

Definition 4.1.1 (Topological algebra) *A topological algebra $\mathscr{A}$ over $\mathbb{C}$ is a topological vector space with an (associative) algebra structure*

$$\mu\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}, \tag{4.1.1}$$

which is continuous.

In this definition we do not necessarily rely on μ being associative. In fact, one can define topological Lie algebras etc. the same way. However, we shall focus on associative algebras in the following whence in this context we simply speak of "topological algebras".

The requirement of the continuity of μ seems quite reasonable at first glance. Nevertheless, we will meet situations where we have an associative algebra structure on a topological vector space such that the *left* and *right multiplications* with a fixed algebra element $a \in \mathscr{A}$

$$\mathsf{L}_a\colon \mathscr{A} \ni b \ \mapsto\ ab \in \mathscr{A}, \tag{4.1.2}$$

$$\mathsf{R}_a\colon \mathscr{A} \ni b \ \mapsto\ ba \in \mathscr{A}, \tag{4.1.3}$$

are continuous linear maps but μ itself is *not* continuous. In this case we call μ *separately continuous*. We have already seen examples of bilinear maps with this feature: the inner product on an infinite-dimensional Hilbert space is weakly continuous if we fix one entry but not weakly continuous as bilinear map, see Exercise 3.6.12. With this caution in mind we will now study the continuity features of bilinear maps in general.

Remark 4.1.2 (Categories of topological algebras) Using continuous homomorphisms for topological algebras we arrive at some obvious categories like the topological algebras topalg, the unital topological algebras topAlg, the Hausdorff topological algebras Topalg, and the unital Hausdorff topological algebras TopAlg. We leave it as an exercise to formulate the mutual relations between these categories.

4.1.1 Bilinear Maps and the Projective Tensor Product

In this more technical subsection we develop some ideas about continuity features of bilinear maps. This will directly lead to the notion of the projective tensor product. In the following, all occurring topological vector spaces will be locally convex and mostly also Hausdorff.

For three locally convex spaces V, W, and U we consider a bilinear map

$$\phi\colon V \times W \longrightarrow U, \tag{4.1.4}$$

like e.g. the multiplication μ of a topological algebra or a dual pairing etc. If ϕ is continuous we call ϕ a continuous bilinear map. The vector space of continuous bilinear maps is denoted by

$$\mathrm{Bil}(V, W; U) = \{\phi\colon V \times W \longrightarrow U \mid \phi \text{ bilinear and continuous}\}. \tag{4.1.5}$$

If ϕ has the feature that for every $v \in V$ and for every $w \in W$ the linear maps

$$\phi(v, \cdot\,)\colon W \longrightarrow U \quad \text{and} \quad \phi(\,\cdot\,, w)\colon V \longrightarrow U \tag{4.1.6}$$

are continuous then ϕ is called *separately continuous*. Again, it is easy to see that the separately continuous bilinear maps form a subspace inside all bilinear maps which we denote by

$$\mathcal{B}il(V, W; U) = \{\phi\colon V \times W \longrightarrow U \mid \phi \text{ bilinear and separately continuous}\}. \tag{4.1.7}$$

Clearly, one has

$$\mathrm{Bil}(V, W; U) \subseteq \mathcal{B}il(V, W; U), \tag{4.1.8}$$

but the inclusion will be proper in general. Thus a considerable part of the theory of bilinear maps on topological vector spaces deals with the question under which conditions on V, W, and U one has actually equality in (4.1.8), see e.g. [26, Part. III]. We shall not enter this discussion very deeply here but focus essentially on $\mathrm{Bil}(V, W; U)$. It is also clear that k-linear maps with k larger than 2 are of interest. Here the theory can be developed essentially parallel to the case of $k = 2$, see also the Exercises 4.5.1 and 4.5.2.

Theorem 4.1.3 (Continuity of bilinear maps) *Let V, W, and U be locally convex spaces and $\phi\colon V \times W \longrightarrow U$ bilinear. Then the following statements are equivalent:*

i.) The map ϕ is continuous.

ii.) The map ϕ is continuous at $(0, 0) \in V \times W$.

iii.) For every continuous seminorm r on U there are continuous seminorms p on V and q on W such that for all $v \in V$ and $w \in W$ we have

$$\mathrm{r}(\phi(v, w)) \le \mathrm{p}(v)\mathrm{q}(w). \tag{4.1.9}$$

As usual, it suffices to check this for a defining system of seminorms on U.

PROOF: Clearly, *i.)* implies *ii.)*. So assume *ii.)*. This means that for every neighbourhood O of $0 \in U$ the inverse image $\phi^{-1}(O) \subseteq V \times W$ is a neighbourhood of $(0, 0)$. In particular, for every open unit ball $\mathrm{B}_{\mathrm{r},1}(0)$ with respect to a continuous seminorm r on U the inverse image $\phi^{-1}(\mathrm{B}_{\mathrm{r},1}(0))$ is an open neighbourhood of $(0, 0)$. Thus, by definition of the product topology, there are open balls $\mathrm{B}_{\mathrm{p},1}(0) \subseteq V$ and $\mathrm{B}_{\mathrm{q},1}(0) \subseteq W$ with

$$\mathrm{B}_{\mathrm{p},1}(0) \times \mathrm{B}_{\mathrm{q},1}(0) \subseteq \phi^{-1}(\mathrm{B}_{\mathrm{r},1}(0)), \tag{$*$}$$

where p and q are suitably chosen continuous seminorms on V and W, respectively. Then $(*)$ means for $v \in V$ and $w \in W$

$$\mathrm{p}(v) < 1 \quad \text{and} \quad \mathrm{q}(w) < 1 \quad \Longrightarrow \quad \mathrm{r}(\phi(v, w)) < 1. \tag{$**$}$$

We claim that this implies the estimate (4.1.9). We proceed analogously to the proof of Theorem 2.2.3. Assume first $v \in V$ satisfies $\mathrm{p}(v) = 0$. Then also $\mathrm{p}(zv) = 0$ for all $z \in \mathbb{C}$. For arbitrary $w \in W$ we find a $\lambda > 0$ with $\lambda w \in \mathrm{B}_{\mathrm{q},1}(0)$, taking e.g. $\lambda = \frac{1}{2\mathrm{q}(w)}$ if $\mathrm{q}(w) \ne 0$ and $\lambda = 1$ else. Then $(zv, \lambda w) \in \mathrm{B}_{\mathrm{p},1}(0) \times \mathrm{B}_{\mathrm{q},1}(0)$ and thus $\mathrm{r}(\phi(zv, \lambda w)) < 1$ by $(**)$. Since ϕ is linear in the first argument and r is a seminorm this gives $\mathrm{r}(\phi(zv, \lambda w)) = |z|\mathrm{r}(\phi(v, \lambda w)) < 1$ for all z. Hence $\mathrm{r}(\phi(v, \lambda w)) = 0$. By the linearity of ϕ in the second argument we conclude that also $\mathrm{r}(\phi(v, w)) = 0$. Thus (4.1.9) holds. Similarly, one shows that (4.1.9) holds for the case where $\mathrm{q}(w) = 0$. Thus we are left with the case $\mathrm{p}(v) \neq 0 \neq \mathrm{q}(w)$. Let $c > 1$ then $\frac{v}{c\mathrm{p}(v)} \in \mathrm{B}_{\mathrm{p},1}(0)$ and $\frac{w}{c\mathrm{q}(w)} \in \mathrm{B}_{\mathrm{q},1}(0)$, respectively. Hence $(*)$ applies to these rescaled vectors which gives

$$\mathrm{r}(\phi(v, w)) < c^2 \mathrm{p}(v)\mathrm{q}(w)$$

by the bilinearity of ϕ and the fact that r is a seminorm. Since $c > 1$ was arbitrary we conclude (4.1.9) also in this case. Finally, we assume *iii.)*. To show continuity of ϕ at an arbitrary point we use the net continuity criterion: let $(v_i, w_i)_{i\in I} \longrightarrow (v, w)$ be a net in the Cartesian product converging to (v, w). We know that for the product topology this means $(v_i)_{i\in I} \longrightarrow v$ and $(w_i)_{i\in I} \longrightarrow w$, see Proposition 2.2.35, *iii.)*. Now we consider a continuous seminorm r on U. Then by bilinearity and (4.1.9) we get

$$\begin{aligned} \mathrm{r}(\phi(v_i, w_i) - \phi(v, w)) &= \mathrm{r}(\phi(v_i - v, w_i) + \phi(v, w_i - w)) \\ &\leq \mathrm{r}(\phi(v_i - v, w_i)) + \mathrm{r}(\phi(v, w_i - w)) \\ &\leq \mathrm{p}(v_i - v)\mathrm{q}(w_i) + \mathrm{p}(v)\mathrm{q}(w_i - w). \end{aligned}$$

Now $\mathrm{p}(v_i - v) \longrightarrow 0$ and $\mathrm{q}(w_i) \longrightarrow \mathrm{q}(w)$. Hence the first term converges to zero. The second clearly converges to zero as well. Thus $\mathrm{r}(\phi(v_i, w_i) - \phi(v, w))$ converges to zero which ultimately proves the convergence $\phi(v_i, w_i) \longrightarrow \phi(v, w)$. This shows *i.)*. $\square$

Typically, the last part gives a good and manageable criterion for continuity of bilinear maps. With a little additional effort one can generalize this now to multilinear maps as well, see also Exercise 4.5.1.

The study of bilinear maps can be traced back to the case of linear maps by using the tensor product. Every bilinear map $\phi\colon V \times W \longrightarrow U$ induces a unique linear map $\Phi\colon V \otimes W \longrightarrow U$ with

$$\phi(v, w) = \Phi(v \otimes w) \tag{4.1.10}$$

for all $v \in V$ and $w \in W$. Thus we only have to understand the continuity properties of *one* bilinear map, namely $\otimes$, in the following sense: we want $\otimes$ to be continuous and hence we want on $V \otimes W$ a topology in such a way that the correspondence (4.1.10) gives a continuous Φ whenever ϕ is continuous. This will simplify things drastically as now we only have to deal with one *universal* bilinear map $\otimes$ and have linear maps instead of bilinear ones afterwards.

Thus we first need to construct a reasonable topology on the algebraic tensor product $V \otimes W$. To this end we need the following lemma:

Lemma 4.1.4 *Let* p *be a seminorm on* V *and* q *a seminorm on* W. *Then*

$$(\mathrm{p} \otimes \mathrm{q})(u) = \inf\Big\{\sum\nolimits_i \mathrm{p}(v_i)\mathrm{q}(w_i) \;\Big|\; u = \sum\nolimits_i v_i \otimes w_i\Big\} \tag{4.1.11}$$

defines a seminorm on $V \otimes W$ *with the property that on elementary tensors we have*

$$(\mathrm{p} \otimes \mathrm{q})(v \otimes w) = \mathrm{p}(v)\mathrm{q}(w). \tag{4.1.12}$$

PROOF: Here the infimum is taken over all finite linear combinations $\sum_i v_i \otimes w_i = u$ with $v_i \in V$ and $w_i \in W$. Clearly, $(\mathrm{p} \otimes \mathrm{q})(u) \geq 0$. For $z \in \mathbb{C}$ we have

$$zu = \sum\nolimits_i (zv_i) \otimes w_i \quad \text{iff} \quad u = \sum\nolimits_i v_i \otimes w_i.$$

Thus the infimum needed for zu is taken over the same linear combinations as the infimum needed for u including the additional pre-factor z. Thus

$$(\mathrm{p} \otimes \mathrm{q})(zu) = \inf \sum\nolimits_i \mathrm{p}(zv_i)\mathrm{q}(w_i) = \inf \sum\nolimits_i |z|\mathrm{p}(v_i)\mathrm{q}(w_i) = |z|(\mathrm{p} \otimes \mathrm{q})(u).$$

For the triangle inequality we observe that for

$$u = \sum\nolimits_i v_i \otimes w_i \quad \text{and} \quad \tilde{u} = \sum\nolimits_j \tilde{v}_j \otimes \tilde{w}_j$$

the linear combination $\sum_i v_i \otimes w_i + \sum_j \tilde{v}_j \otimes \tilde{w}_j$ gives one possibility to get $u + \tilde{u}$. Thus the possible linear combinations for $u + \tilde{u}$ contain those coming from linear combinations needed for u and $\tilde{u}$ separately. This gives for the infimum

$$\begin{aligned}
(\mathrm{p} \otimes \mathrm{q})(u + \tilde{u}) &= \inf \sum\nolimits_\ell \mathrm{p}(\hat{v}_\ell)\mathrm{q}(\hat{w}_\ell) \\
&\le \inf \sum\nolimits_i \mathrm{p}(v_i)\mathrm{q}(w_i) + \inf \sum\nolimits_j \mathrm{p}(\tilde{v}_j)\mathrm{q}(\tilde{w}_j) \\
&= (\mathrm{p} \otimes \mathrm{q})(u) + (\mathrm{p} \otimes \mathrm{q})(\tilde{u}),
\end{aligned}$$

and hence the triangle inequality holds. This shows that $\mathrm{p} \otimes \mathrm{q}$ is indeed a seminorm. Now let $v \otimes w$ be an elementary tensor. Then we find linear functionals $\varphi \in V^*$ and $\psi \in W^*$ in the algebraic duals with

$$\varphi(v) = \mathrm{p}(v) \quad \text{and} \quad |\varphi(v')| \le \mathrm{p}(v')$$

for all $v' \in V$ as well as

$$\psi(w) = \mathrm{q}(w) \quad \text{and} \quad |\psi(w')| \le \mathrm{q}(w')$$

for all $w' \in W$ by the Hahn-Banach Theorem, see Corollary 2.2.19. Suppose now that $v \otimes w = \sum_i v_i \otimes w_i$ is another way to write the tensor $v \otimes w$. Then

$$\begin{aligned}
\mathrm{p}(v)\mathrm{q}(w) &= |\varphi(v)\psi(w)| \\
&= |\varphi \otimes \psi)(v \otimes w)| \\
&= \left|(\varphi \otimes \psi)\Big(\sum\nolimits_i v_i \otimes w_i\Big)\right| \\
&= \left|\sum\nolimits_i \varphi(v_i)\psi(w_i)\right| \\
&\le \sum\nolimits_i |\varphi(v_i)||\psi(w_i)| \\
&\le \sum\nolimits_i \mathrm{p}(v_i)\mathrm{q}(w_i).
\end{aligned}$$

Taking now the infimum over all possible ways to decompose $v \otimes w$ we conclude that

$$\mathrm{p}(v)\mathrm{q}(w) \le \inf \sum\nolimits_i \mathrm{p}(v_i)\mathrm{q}(w_i) = (\mathrm{p} \otimes \mathrm{q})(v \otimes w).$$

Since trivially the opposite estimate holds as well, we get the claimed equality in (4.1.12). ☐

We can now use the seminorms $\mathrm{p} \otimes \mathrm{q}$ to construct a locally convex topology on the algebraic tensor product $V \otimes W$ of two locally convex spaces. First, we note that if $\mathrm{p} \le \mathrm{p}'$ and $\mathrm{q} \le \mathrm{q}'$ then

$$\mathrm{p} \otimes \mathrm{q} \le \mathrm{p}' \otimes \mathrm{q}' \tag{4.1.13}$$

clearly holds. This shows that if $\mathrm{p} \in \mathcal{P}$ and also $\mathrm{q} \in \mathcal{Q}$ are from filtrating system of seminorms then the system

$$\mathcal{P} \otimes \mathcal{Q} = \big\{\mathrm{p} \otimes \mathrm{q} \bigm| \mathrm{p} \in \mathcal{P} \text{ and } \mathrm{q} \in \mathcal{Q}\big\} \tag{4.1.14}$$

is again filtrating. The following theorem characterizes now the locally convex topology determined by the system $\mathcal{P} \otimes \mathcal{Q}$ of seminorms:

Theorem 4.1.5 (Continuity of $\otimes$) *Let V and W be locally convex spaces and let $\mathcal{P}$ and $\mathcal{Q}$ be defining systems of continuous seminorms for V and W, respectively.*

i.) *If $\mathcal{P}$ and $\mathcal{Q}$ are filtrating then $\mathcal{P} \otimes \mathcal{Q}$ is filtrating, too.*

ii.) *The locally convex topology determined by $\mathcal{P} \otimes \mathcal{Q}$ is the finest locally convex topology such that*

$$\otimes \colon V \times W \longrightarrow V \otimes W \tag{4.1.15}$$

is a continuous bilinear map.

iii.) *A bilinear map $\phi \colon V \times W \longrightarrow U$ into some other locally convex space U is continuous iff the corresponding linear map $\Phi \colon V \otimes W \longrightarrow U$ is continuous with respect to the topology determined by $\mathcal{P} \otimes \mathcal{Q}$.*

iv.) *The topology determined by $\mathcal{P} \otimes \mathcal{Q}$ is the unique locally convex topology with respect to the property iii.).*

PROOF: This first part was already discussed. It is clear from the property (4.1.13) that the resulting locally convex topology on $V \otimes W$ does not depend on the particular choices of the defining systems of seminorms $\mathcal{P}$ and $\mathcal{Q}$. This is mainly a technical point important for applications as it allows to use "small" systems $\mathcal{P}$ and $\mathcal{Q}$ to describe the topology on $V \otimes W$. For the second statement we note that (4.1.12) shows that the bilinear map $\otimes$ satisfies the criterion (4.1.9) from Theorem 4.1.3 for continuity. Thus $\otimes$ is continuous as claimed. If we have now another locally convex topology for which $\otimes$ is continuous then for any continuous seminorm r of this topology we have by Theorem 4.1.3, *iii.)*, continuous seminorms p and q on V and W, respectively, with

$$\mathrm{r}(v \otimes w) \le \mathrm{p}(v)\mathrm{q}(w) = (\mathrm{p} \otimes \mathrm{q})(v \otimes w) \tag{$*$}$$

for all $v \in V$ and $w \in W$. If now $u = \sum_i v_i \otimes w_i$ this gives

$$\mathrm{r}(u) = \mathrm{r}\Big(\sum\nolimits_i v_i \otimes w_i\Big) \le \sum\nolimits_i \mathrm{r}(v_i \otimes w_i) \overset{(*)}{\le} \sum\nolimits_i \mathrm{p}(v_i)\mathrm{q}(w_i).$$

Since this holds for any decomposition of u we can take the infimum and get

$$\mathrm{r}(u) \le \inf \sum\nolimits_i \mathrm{p}(v_i)\mathrm{q}(w_i) = (\mathrm{p} \otimes \mathrm{q})(u).$$

This shows that the seminorm r is continuous with respect to the topology determined by $\mathcal{P} \otimes \mathcal{Q}$, proving that the latter gives the finer topology. For the third part, we note that if Φ is continuous then also $\phi = \Phi \circ \otimes$ is continuous as composition of continuous maps. It is the reverse implication which is the non-trivial point. So suppose ϕ is continuous and let r be a continuous seminorm on U. Then we find continuous seminorms p and q on V and W, respectively, with $\mathrm{r}(\phi(v, w)) \le \mathrm{p}(v)\mathrm{q}(w)$ according to Theorem 4.1.3, *iii.)*. Thus for the induced map Φ and some $u = \sum_i v_i \otimes w_i$ we get

$$\mathrm{r}(\Phi(u)) = \mathrm{r}\Big(\sum\nolimits_i \Phi(v_i \otimes w_i)\Big) = \mathrm{r}\Big(\sum\nolimits_i \phi(v_i, w_i)\Big) \le \sum\nolimits_i \mathrm{r}(\phi(v_i, w_i)) \le \sum\nolimits_i \mathrm{p}(v_i)\mathrm{q}(w_i).$$

Again, since this holds for all decompositions of u we can take the infimum to conclude

$$\mathrm{r}(\Phi(u)) \le (\mathrm{p} \otimes \mathrm{q})(u)$$

and thereby the continuity of Φ. For the last part, suppose there is another locally convex topology with this property, denoted by $\mathcal{T}$. Then the identity map $\mathrm{id} \colon (V \otimes W, \mathcal{T}) \longrightarrow V \otimes W$ corresponds to the map $\otimes \colon V \times W \longrightarrow V \otimes W$, which we have shown to be continuous. Thus id is continuous by the assumed property of $\mathcal{T}$ and hence $\mathcal{T}$ is finer than the above constructed topology. Conversely, $\mathrm{id} \colon (V \otimes W, \mathcal{T}) \longrightarrow (V \otimes W, \mathcal{T})$ is clearly continuous if both source and target carry the same topology $\mathcal{T}$. By part *iii.)*, we know that $\otimes \colon V \times W \longrightarrow (V \otimes W, \mathcal{T})$ is continuous. By part *ii.)*, we know that in this case $\mathcal{T}$ is necessarily coarser that the topology obtained from $\mathcal{P} \otimes \mathcal{Q}$. Thus they actually coincide. ☐

Definition 4.1.6 (π-Topology) *For two locally convex spaces V and W the topology described in Theorem 4.1.5 is called the π-topology (or projective topology) on $V \otimes W$. We denote the tensor product with this topology also by $V \otimes_\pi W$.*

We collect now a few properties of the π-topology. First we recall that for linear maps $\phi\colon V \longrightarrow \tilde{V}$ and $\psi\colon W \longrightarrow \tilde{W}$ there is also a unique linear map denoted by $\phi \otimes \psi\colon V \otimes W \longrightarrow \tilde{V} \otimes \tilde{W}$ with

$$(\phi \otimes \psi)(v \otimes w) = \phi(v) \otimes \psi(w), \tag{4.1.16}$$

as we have used this already implicitly in the proof of Lemma 4.1.4. Concerning the continuity of $\phi \otimes \psi$ we have the following statement:

Corollary 4.1.7 *Suppose $\phi\colon V \longrightarrow \tilde{V}$ and $\psi : W \longrightarrow \tilde{W}$ are continuous linear maps. Then also*

$$\phi \otimes \psi\colon V \otimes_\pi W \longrightarrow \tilde{V} \otimes_\pi \tilde{W} \tag{4.1.17}$$

is continuous.

PROOF: First it is clear that $\phi \times \psi\colon V \times W \longrightarrow \tilde{V} \times \tilde{W}$ is continuous, this is true for general topological spaces and continuous maps, see Exercise A.7.1. Thus also $\otimes \circ (\phi \times \psi)\colon V \times W \longrightarrow \tilde{V} \otimes_\pi \tilde{W}$, viewed as bilinear map, is continuous by the continuity of $\otimes$. Then Theorem 4.1.5, *iii.)*, shows that the induced linear map $\phi \otimes \psi$ corresponding to this bilinear map is continuous as well. $\square$

Occasionally, we write $\phi \otimes_\pi \psi$ instead of $\phi \otimes \psi$ to emphasize the continuity properties of $\phi \otimes_\pi \psi$.

Corollary 4.1.8 *For locally convex spaces V and W different from $\{0\}$ their tensor product $V \otimes_\pi W$ is Hausdorff iff V and W are Hausdorff.*

PROOF: Suppose $V \otimes_\pi W$ is Hausdorff. Then for $v \otimes w$ there is a seminorm $\mathrm{p} \otimes \mathrm{q}$ with continuous seminorms p and q on V and W with $(\mathrm{p} \otimes \mathrm{q})(v \otimes w) > 0$. But this gives $\mathrm{p}(v)\mathrm{q}(w) > 0$ and hence V and W are Hausdorff, too. Conversely, let $u = \sum_i v_i \otimes w_i$ be a non-zero vector in $V \otimes W$. Without restriction we assume that the vectors $v_1, \ldots, v_n$ are linearly independent and $w_1 \neq 0$. Then we find continuous linear functionals $\varphi\colon V \longrightarrow \mathbb{C}$ and $\psi\colon W \longrightarrow \mathbb{C}$ such that $\varphi(v_1) = 1$ and $\varphi(v_i) = 0$ for $i \neq 1$ as well as $\psi(w_1) = 1$, respectively, by the usual Hahn-Banach argument. Then by Corollary 4.1.7 the linear map

$$\varphi \otimes \psi\colon V \otimes_\pi W \longrightarrow \mathbb{C} \otimes \mathbb{C} = \mathbb{C}$$

is continuous and we have $(\varphi \otimes \psi)(u) = 1$ by construction. By Corollary 2.2.21 we know that $V \otimes_\pi W$ is Hausdorff. $\square$

Corollary 4.1.9 *Two seminorms p on V and q on W are norms iff $\mathrm{p} \otimes \mathrm{q}$ is a norm on $V \otimes W$.*

PROOF: Let e.g. p be not a norm and $0 \neq v \in V$ with $\mathrm{p}(v) = 0$ be given. Take $w \neq 0$ then $v \otimes w \neq 0$ but $(\mathrm{p} \otimes \mathrm{q})(v \otimes w) = \mathrm{p}(v)\mathrm{q}(w) = 0$. Thus $\mathrm{p} \otimes \mathrm{q}$ can not be a norm, too. Conversely, assume p and q are norms then we can equip V and W with the corresponding norm topologies which are Hausdorff. Since the single seminorm $\mathrm{p} \otimes \mathrm{q}$ determines the π-topology on $V \otimes W$ in this case, it has to be a norm by Corollary 4.1.8. $\square$

Corollary 4.1.10 *The topological dual of $V \otimes_\pi W$ is given by the space $\mathrm{Bil}(V, W; \mathbb{C})$ of continuous bilinear forms on $V \times W$.*

PROOF: This is Theorem 4.1.5, *iii.)*, applied to $U = \mathbb{C}$. $\square$

For the algebraic tensor product one knows that taking successive tensor products gives canonically isomorphic results. For three (and analogously for more) vector spaces the linear extension of the map

$$\mathsf{asso}\colon (V \otimes W) \otimes U \ni (v \otimes w) \otimes u \ \mapsto\ v \otimes (w \otimes u) \in V \otimes (W \otimes U) \tag{4.1.18}$$

is the canonical isomorphism. The next proposition shows that the π-topology is compatible with this isomorphism.

Proposition 4.1.11 (Associativity of $\otimes_\pi$) *Let V, W, and U be locally convex spaces.*

i.) For any seminorms p on V, q on W, and r on U we have for all $z \in (V \otimes W) \otimes U$

$$((\mathrm{p} \otimes \mathrm{q}) \otimes \mathrm{r})(z) = (\mathrm{p} \otimes (\mathrm{q} \otimes \mathrm{r}))(\mathsf{asso}(z)). \tag{4.1.19}$$

ii.) The linear isomorphism $\mathsf{asso}\colon (V \otimes_\pi W) \otimes_\pi U \longrightarrow V \otimes_\pi (W \otimes_\pi U)$ is continuous with continuous inverse.

Proof: First it is clear that for factorizing elements $(v \otimes w) \otimes u$ the identity (4.1.19) holds thanks to the factorization property (4.1.12). Now let $z \in (V \otimes W) \otimes U$ be arbitrary, written as

$$z = \sum\nolimits_i x_i \otimes u_i = \sum\nolimits_i \Big(\sum\nolimits_j v_{ij} \otimes w_{ij} \Big) \otimes u_i. \tag{$*$}$$

Relabeling, this sum is termwise equal to a sum of the form

$$z = \sum\nolimits_k (v_k \otimes w_k) \otimes u_k. \tag{$**$}$$

Thus for all representations of z as a sum of factorizing tensors $(*)$ we can assume to have it written as $(**)$. Analogously, all representations of $\mathsf{asso}(z)$ are of the form

$$\mathsf{asso}(z) = \sum\nolimits_k v_k \otimes (w_k \otimes u_k).$$

This shows that the relevant infima agree, i.e.

$$((\mathrm{p} \otimes \mathrm{q}) \otimes \mathrm{r})(z) = \inf \sum\nolimits_k \mathrm{p}(v_k)\mathrm{q}(w_k)\mathrm{r}(u_k) = (\mathrm{p} \otimes (\mathrm{q} \otimes \mathrm{r}))(\mathsf{asso}(z)).$$

This shows the first part. Applying this to continuous seminorms p, q, and r gives immediately the second part. □

Remark 4.1.12 The statement of the proposition also holds for finitely many tensor factors $V_1, \ldots, V_N$: the π-topology is compatible with the associativity of the algebraic tensor product. Therefore, we shall simply write $V_1 \otimes_\pi \cdots \otimes_\pi V_N$ in the following and identify all possible ways to put brackets, see also Exercise 4.5.2 for a more conceptual discussion of this. Moreover, $V_1 \otimes_\pi \cdots \otimes_\pi V_N$ has the universal property for continuous N-linear maps analogously to the linear algebraic case as discussed in Exercise 1.5.19. Denoting the N-linear continuous maps $V_1 \times \cdots \times V_N \longrightarrow W$ with values in some locally convex space W by $\mathrm{L}(V_1, \ldots, V_N; W) \subseteq \mathrm{Hom}(V_1, \ldots, V_N; W)$ we thereby obtain

$$\mathrm{L}(V_1, \ldots, V_N; W) \cong \mathrm{L}(V_1 \otimes_\pi \cdots \otimes_\pi V_N, W) \tag{4.1.20}$$

by the universal property.

Since $V \otimes_\pi W$ is again Hausdorff for Hausdorff locally convex spaces we can complete the tensor product to a complete Hausdorff locally convex space. In general, this will be necessary even if V and W were already complete.

Definition 4.1.13 (Completed π-tensor product) *Let V and W be Hausdorff locally convex spaces. Then the completion of $V \otimes_\pi W$ will be called the completed π-tensor product. It is denoted by $V \hat{\otimes}_\pi W$.*

For continuous linear maps $\phi\colon V_1 \longrightarrow V_2$ and $\psi\colon W_1 \longrightarrow W_2$ between Hausdorff locally convex spaces the canonical extension of the continuous linear map $\phi \otimes_\pi \psi\colon V_1 \otimes_\pi V_2 \longrightarrow W_1 \otimes_\pi W_2$ to the completions is denoted by

$$\phi \hat{\otimes}_\pi \psi\colon V_1 \hat{\otimes}_\pi W_1 \longrightarrow W_1 \hat{\otimes}_\pi W_2. \tag{4.1.21}$$

We will now discuss the compatibility of the π-tensor product and completion: it turns out that everything matches very nicely:

Theorem 4.1.14 (Completed π-tensor product) *Let V, W, and U be Hausdorff locally convex spaces.*

i.) Canonically, we have the isomorphisms

$$\widehat{V} \hat{\otimes}_\pi \widehat{W} \cong V \hat{\otimes}_\pi \widehat{W} \cong \widehat{V} \hat{\otimes}_\pi W \cong V \hat{\otimes}_\pi W, \tag{4.1.22}$$

where $\widehat{V}$ and $\widehat{W}$ denote the completions of V and W, respectively.

ii.) The canonical associativity isomorphism extends to an isomorphism

$$\widehat{\mathsf{asso}}\colon \big(V \hat{\otimes}_\pi W\big) \hat{\otimes}_\pi U \longrightarrow V \hat{\otimes}_\pi \big(W \hat{\otimes}_\pi U\big). \tag{4.1.23}$$

iii.) If $\phi\colon V_1 \longrightarrow V_2$ and $\psi\colon W_1 \longrightarrow W_2$ are continuous linear maps between Hausdorff locally convex spaces then we have

$$\widehat{\phi} \hat{\otimes}_\pi \widehat{\psi} = \phi \hat{\otimes}_\pi \widehat{\psi} = \widehat{\phi} \hat{\otimes}_\pi \psi = \phi \hat{\otimes}_\pi \psi \tag{4.1.24}$$

under the identifications as in (4.1.22).

PROOF: We only have to check $V \hat{\otimes}_\pi W \cong \widehat{V} \hat{\otimes}_\pi \widehat{W}$ since the remaining two isomorphism are then obtained by replacing V or W by their completions $\widehat{V}$ or $\widehat{W}$, respectively. We have to show that $\widehat{V} \hat{\otimes}_\pi \widehat{W}$ is a completion of $V \otimes_\pi W$ as well. A priori, it could be too large since clearly $V \otimes_\pi W \subseteq \widehat{V} \otimes_\pi \widehat{W} \subseteq \widehat{V} \hat{\otimes}_\pi \widehat{W}$. First we recall that $V \longrightarrow \widehat{V}$ and $W \longrightarrow \widehat{W}$ are embeddings with the seminorms on $\widehat{V}$ and $\widehat{W}$ being just the canonical extensions of the seminorms on V and W, respectively. Thus the defining system of seminorms on $\widehat{V} \otimes_\pi \widehat{W}$ is obtained from $\widehat{\mathrm{p}} \otimes \widehat{\mathrm{q}}$ with p and q being continuous seminorms on V and W, respectively. Then $\widehat{\mathrm{p}} \otimes \widehat{\mathrm{q}}$, extended further to $\widehat{V} \hat{\otimes}_\pi \widehat{W}$, is again determined by p and q and provides extensions of $\mathrm{p} \otimes \mathrm{q}$ on $V \otimes_\pi W$. This shows that $V \otimes_\pi W \longrightarrow \widehat{V} \hat{\otimes}_\pi \widehat{W}$ is continuous and the induced topology on the image is the original π-topology. Moreover, it is clear that the image of $V \otimes_\pi W$ is dense as it is clearly dense in $\widehat{V} \otimes_\pi \widehat{W}$: indeed, for a finite linear combinations of $\widehat{v} \otimes \widehat{w}$ we can approximate each $\widehat{v} \in \widehat{V}$ and $\widehat{w} \in \widehat{W}$ by $v \in V$ and $w \in W$, respectively. Since the tensor product is continuous for the π-topologies, the corresponding $v \otimes w$ approximates $\widehat{v} \otimes \widehat{w}$, showing that $V \otimes_\pi W$ is dense in $\widehat{V} \otimes_\pi \widehat{W}$. Since the latter is dense in its completion by the very definition, we conclude that $V \otimes W$ is dense in $\widehat{V} \hat{\otimes}_\pi \widehat{W}$. This shows that $\widehat{V} \hat{\otimes}_\pi \widehat{W}$ is a completion of $V \hat{\otimes}_\pi W$ and hence the first part follows. The second part is now easy: the associativity $\mathsf{asso}\colon (V \otimes_\pi W) \otimes_\pi U \longrightarrow V \otimes_\pi (W \otimes_\pi U)$ extends to the completions giving continuous maps

$$\widehat{\mathsf{asso}}\colon (V \otimes_\pi W) \hat{\otimes}_\pi U \longrightarrow V \hat{\otimes}_\pi (W \otimes_\pi U)$$

as well as $\widehat{\mathsf{asso}^{-1}}$ which are still mutually inverse continuous isomorphisms since this is true on a dense subspace. By the first part, the left hand side is $(V \hat{\otimes}_\pi W) \hat{\otimes}_\pi U$ and the right hand side is $V \hat{\otimes}_\pi (W \hat{\otimes}_\pi U)$. The last part is clear since all four maps are continuous and agree on the dense domain $V \otimes_\pi W$. ☐

In conclusion, also for the completed π-tensor product $\hat{\otimes}_\pi$ we can automatically implement the above isomorphisms and simplify our notation that way. Needless to say, there are analogous statements for tensor products with more factors, see again Exercise 4.5.2.

Exercise: Bic properties

As a first application we obtain the following extension of continuous bilinear maps to the completions:

Corollary 4.1.15 *Let V, W, and U be Hausdorff locally convex spaces and let $\phi\colon V \times W \longrightarrow U$ be a continuous bilinear map. Then there exists a unique continuous bilinear extension $\widehat{\phi}\colon \widehat{V} \times \widehat{W} \longrightarrow \widehat{U}$.*

PROOF: Since $V \times W$ is dense in $\widehat{V} \times \widehat{W}$ any such extension is necessarily unique by the Hausdorff property of $\widehat{U}$, see Lemma **??**. Now let $\Phi\colon V \otimes_\pi W \longrightarrow U$ be the corresponding continuous linear map with $\phi = \Phi \circ \otimes$ according to Theorem 4.1.5, *iii.)*. As a linear map, it has a continuous extension $\widehat{\Phi}\colon V \hat{\otimes}_\pi W = \widehat{V} \hat{\otimes}_\pi \widehat{W} \longrightarrow \widehat{U}$ by our general statements from Theorem 2.1.14. Then $\widehat{\phi} = \widehat{\Phi} \circ \hat{\otimes}_\pi\colon \widehat{V} \times \widehat{W} \longrightarrow \widehat{U}$ is the extension we are looking for. ☐

Of course, this last corollary could have been obtained without the usage of the π-tensor product by directly investigating the continuity properties of ϕ according to Theorem 4.1.3 carefully. Note again that by taking successive tensor products one can easily extend all the results from bilinear maps to arbitrary multilinear maps as well, see also the Exercises 4.5.1 and 4.5.2.

4.1.2 Locally Convex Algebras

With the knowledge on continuous bilinear maps it is now fairly easy to characterize topological algebras where the underlying topological vector space is locally convex. First we establish some vocabulary:

Definition 4.1.16 (Locally convex algebra) *Let $\mathscr{A}$ be a topological algebra.*

i.) The algebra $\mathscr{A}$ is called locally convex algebra if the underlying topological vector space is locally convex. The category of locally convex algebras with continuous algebra homomorphisms as morphisms is denoted by lcalg.

ii.) The algebra $\mathscr{A}$ is called Fréchet algebra if the underlying topological vector space is a Fréchet space.

As usual, we shall mainly be interested in the Hausdorff case which gives the full subcategory LCalg. Nevertheless, as intermediate steps also non-Hausdorff locally convex algebras may occur. The corresponding subcategories of unital (Hausdorff) locally convex algebras are denoted by lcAlg and LCAlg, respectively. Moreover, we have the subcategories of complete (unital) locally convex algebras CLCalg and CLCAlg, respectively. Finally, a special role will be played by the subcategories of (unital) Fréchet algebras Fréchetalg and FréchetAlg. It is a good exercise to visualize the relations between these various types of topological algebras, see also Exercise 4.5.6.

We give now an equivalent formulation of a locally convex algebra using seminorms explicitly to control the continuity of the product:

Proposition 4.1.17 *Let $\mathscr{A}$ be a locally convex space and let $\mu\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$ be an (associative) multiplication. Then $(\mathscr{A}, \mu)$ is a locally convex algebra iff for all continuous seminorms p on $\mathscr{A}$ one finds a continuous seminorm q on $\mathscr{A}$ such that*

$$\mathrm{p}(\mu(a, b)) \leq \mathrm{q}(a)\mathrm{q}(b) \tag{4.1.25}$$

for all $a, b \in \mathscr{A}$. Again, it suffices to check this for p from a defining system of seminorms.

PROOF: Let $\mathcal{P}$ be a defining system of seminorms. Then the continuity of μ is equivalent to the statement that for $\mathrm{p} \in \mathcal{P}$ we find continuous seminorms q' and q'' on $\mathscr{A}$ with $\mathrm{p}(\mu(a, b)) \leq \mathrm{q}'(a)\mathrm{q}''(b)$ for all $a, b \in \mathscr{A}$. Taking the continuous seminorm $\mathrm{q} = \max\{\mathrm{q}', \mathrm{q}''\}$ then gives (4.1.25). □

Proposition 4.1.18 *Let $\mathscr{A}$ be a locally convex algebra and let $\mathscr{J} \subseteq \mathscr{A}$ be a two-sided ideal. Then the locally convex quotient $\mathscr{A}/\mathscr{J}$ with its canonical algebra structure is a locally convex algebra and the quotient map*

$$\mathscr{A} \longrightarrow \mathscr{A}/\mathscr{J} \tag{4.1.26}$$

is a continuous algebra homomorphism.

PROOF: We know that $\mathscr{A}/\mathscr{J}$ is an algebra again such that (4.1.26) is an algebra homomorphism. Moreover, the locally convex quotient topology is the finest locally convex topology on $\mathscr{A}/\mathscr{J}$ making (4.1.26) continuous. It remains to check whether the algebra multiplication on $\mathscr{A}/\mathscr{J}$ is continuous. Thus let p be a continuous seminorm on $\mathscr{A}$ and chose a continuous seminorm q with $\mathrm{p}(ab) \leq \mathrm{q}(a)\mathrm{q}(b)$ according to (4.1.25). Then in the quotient we have with $[a][b] = [ab]$

$$\begin{aligned}
[\mathrm{p}]([a][b]) &= \inf\{\mathrm{p}(ab + c) \mid c \in \mathscr{J}\} \\
&\leq \inf\{\mathrm{p}((a + c)(b + c')) \mid c, c' \in \mathscr{J}\} \\
&\leq \inf\{\mathrm{q}(a + c)\mathrm{q}(b + c') \mid c, c' \in \mathscr{J}\} \\
&= \inf\{\mathrm{q}(a + c) \mid c \in \mathscr{J}\} \inf\{\mathrm{q}(b + c') \mid c' \in \mathscr{J}\} \\
&= [\mathrm{q}]([a])[\mathrm{q}]([b]),
\end{aligned}$$

from which we get $[\mathrm{p}]([a][b]) \leq [\mathrm{q}]([a])[\mathrm{q}]([b])$. Since the seminorms of the form $[\mathrm{p}]$ define the locally convex quotient topology, this establishes the continuity of the product. □

Corollary 4.1.19 *Let $\mathscr{A}$ be a locally convex algebra. Then $\{0\}^{\mathrm{cl}}$ is a closed ideal in $\mathscr{A}$ and hence the Hausdorffization $\mathscr{A}/\{0\}^{\mathrm{cl}}$ becomes a Hausdorff locally convex algebra with the canonical projection*

$$\mathscr{A} \longrightarrow \mathscr{A}/\{0\}^{\mathrm{cl}} \tag{4.1.27}$$

being a continuous algebra homomorphism.

PROOF: We know that the closure of $\{0\}$ are those algebra elements a with $\mathrm{p}(a) = 0$ for all continuous seminorms p, see Remark 2.2.43. Suppose that $a \in \{0\}^{\mathrm{cl}}$ and $b \in \mathscr{A}$. Then for an arbitrary continuous seminorm p we find a continuous seminorm q with $\mathrm{p}(ab) \leq \mathrm{q}(a)\mathrm{q}(b) = 0$ by (4.1.25) and hence $ab \in \{0\}^{\mathrm{cl}}$. Analogously, one shows $ba \in \{0\}^{\mathrm{cl}}$. Thus $\{0\}^{\mathrm{cl}}$ is a two-sided ideal and hence the locally convex quotient becomes again a locally convex algebra, now being Hausdorff. The continuity of (4.1.27) is true by the general Remark 2.2.43. □

The next proposition deals with the relation of topological closures and completions on one hand and algebraic features on the other hand.

Proposition 4.1.20 *Let $\mathscr{A}$ be a Hausdorff locally convex algebra. Then the product extends canonically to the completion which becomes then a complete Hausdorff locally convex algebra.*

PROOF: By Corollary 4.1.15 it is clear that the multiplication μ extends to a continuous bilinear map $\widehat{\mu}\colon \widehat{\mathscr{A}} \times \widehat{\mathscr{A}} \longrightarrow \widehat{\mathscr{A}}$. It remains to check the associativity. Note that the associativity can be expressed as

$$\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu), \tag{$*$}$$

viewed as an identity on $\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A}$, see Exercise 1.5.23. Since the identity map is clearly continuous, both sides of $(*)$ consist of continuous linear maps. Thus by Theorem 4.1.14 and the functoriality of completion we conclude that

$$\widehat{\mu} \circ (\widehat{\mu} \mathbin{\hat{\otimes}_\pi} \mathrm{id}) = \mu \circ \widehat{(\mu \otimes_\pi} \mathrm{id}) = \mu \circ \widehat{(\mathrm{id} \otimes_\pi \mu)} = \widehat{\mu} \circ (\mathrm{id} \mathbin{\hat{\otimes}_\pi} \widehat{\mu})$$

as an equation on $\widehat{\mathscr{A}} \mathbin{\hat{\otimes}_\pi} \widehat{\mathscr{A}} \mathbin{\hat{\otimes}_\pi} \widehat{\mathscr{A}}$. But this includes the associativity of $\widehat{\mu}$. □

The general idea behind the proof of associativity is that an algebraic property should be expressible as identity of *linear* maps on certain tensor powers. As such, it survives the completion provided all structure maps are continuous. It is clear that commutativity is preserved as well as the Jacobi identity in locally convex Lie algebras etc., see also Exercise **??**.

ly convex Lie algebras

From the last corollary and from Proposition 4.1.20 we see that we can always achieve to end up with a Hausdorff and complete locally convex algebra. The next proposition extends this idea also to homomorphisms and derivations of algebras:

Proposition 4.1.21 *Let $\mathscr{A}$ and $\mathscr{B}$ be locally convex algebras.*

i.) If $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a continuous homomorphism then $\Phi(\{0\}^{\mathrm{cl}}) \subseteq \{0\}^{\mathrm{cl}}$ and Φ induces a continuous algebra homomorphism

$$\Phi\colon \mathscr{A}/\{0\}^{\mathrm{cl}} \longrightarrow \mathscr{B}/\{0\}^{\mathrm{cl}}. \tag{4.1.28}$$

ii.) If $\mathscr{A}$ and $\mathscr{B}$ are in addition Hausdorff and $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a continuous algebra homomorphism then

$$\widehat{\Phi}\colon \widehat{\mathscr{A}} \longrightarrow \widehat{\mathscr{B}} \tag{4.1.29}$$

is still a continuous algebra homomorphism.

iii.) If $\mathscr{A}$ is Hausdorff and $D\colon \mathscr{A} \longrightarrow \mathscr{A}$ a continuous derivation then also

$$\widehat{D}\colon \widehat{\mathscr{A}} \longrightarrow \widehat{\mathscr{A}} \tag{4.1.30}$$

is a continuous derivation.

Proof: Any continuous linear map Φ satisfies $\Phi(\{0\}^{\mathrm{cl}}) \subseteq \{0\}^{\mathrm{cl}}$ since $\Phi^{-1}(\{0\}^{\mathrm{cl}}) \subseteq \mathscr{A}$ is a closed subspace and hence contains 0. By the closedness it necessarily contains $\{0\}^{\mathrm{cl}}$ as well. From this it follows that we have a well-defined continuous linear map (4.1.28) on the quotients. If in addition Φ was a homomorphism of algebras, so is (4.1.28) since we can check this on representatives. For the second part we note that Φ being a homomorphism means

$$\Phi \circ \mu_{\mathscr{A}} = \mu_{\mathscr{B}} \circ (\Phi \otimes \Phi) \tag{$*$}$$

with $\mu_{\mathscr{A}}\colon \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A}$ and $\mu_{\mathscr{B}}\colon \mathscr{B} \otimes \mathscr{B} \longrightarrow \mathscr{B}$ being the multiplications, see also Exercise 1.5.23. If Φ is continuous, then $(*)$ is an equality between continuous linear maps $\mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{B}$. Thus it survives completion as we already argued for the associativity in the proof of Proposition 4.1.20. The third part is done in the same way by noting that D is a derivation iff $D \circ \mu_{\mathscr{A}} = \mu_{\mathscr{A}} \circ (D \otimes \mathrm{id} + \mathrm{id} \otimes D)$, see again Exercise 1.5.23. □

To conclude this subsection we also introduce the notion of a locally convex *-algebra:

Definition 4.1.22 (Locally convex *-algebra) *A locally convex algebra with a continuous *-involution is called a locally convex *-algebra.*

As for locally convex algebras we get the categories of locally convex *-algebras lcalg^*, of Hausdorff locally convex *-algebras LCalg^*, of complete locally convex *-algebras CLCalg^*, of locally convex unital *-algebras lcAlg^*, Hausdorff locally convex unital *-algebras LCAlg^*, and complete locally convex unital *-algebras CLCAlg^*, always with continuous *-homomorphisms as morphisms.

Remark 4.1.23 Let $\mathscr{A}$ be a locally convex *-algebra.

i.) The continuity of an antilinear map gives the same condition as for linear maps. Thus the *-involution is continuous iff for all continuous seminorms p there is a continuous seminorm q with

$$\mathrm{p}(a^*) \le \mathrm{q}(a) \tag{4.1.31}$$

for all $a \in \mathscr{A}$. Since in this case, $a \mapsto \mathrm{q}(a^*)$ is a continuous seminorm for all continuous seminorms on $\mathscr{A}$, also $\mathrm{r}(a) = \max(\mathrm{q}(a), \mathrm{q}(a^*))$ is a continuous seminorm, now satisfying $\mathrm{r}(a) = \mathrm{r}(a^*)$. Since $\mathrm{q} \le \mathrm{r}$, the collection of all seminorms r satisfying $\mathrm{r}(a) = \mathrm{r}(a^*)$ still yields the same locally convex topology. Thus we can find a defining system of continuous seminorms with the additional property $\mathrm{p}(a^*) = \mathrm{p}(a)$ from the beginning.

ii.) The *-involution also preserves $\{0\}^{\mathrm{cl}}$ and extends to the completion. We do not formulate the rather obvious details but it is clear that our present technology ensures that the *-involution and also *-homomorphisms and *-derivations behave well under Hausdorffization and completion, see Exercise 4.5.5 for further details.

iii.) There are still more compatibilities between the algebraic world and the locally convex one, which we have not spelled out in detail. Nevertheless, it is a good exercise to check that the closure of a left ideal is a left ideal etc., see Exercise 4.5.7.

4.1.3 Locally Multiplicatively Convex Algebras

While for generic locally convex algebras many compatibilities between algebra and analysis work very well, there is yet another, more specific class of locally convex algebras with many nicer features not present in the general case:

Definition 4.1.24 (Locally multiplicatively convex algebra) *Let $\mathscr{A}$ be an algebra.*

i.) A seminorm p is called submultiplicative if for all $a, b \in \mathscr{A}$ one has

$$\mathrm{p}(ab) \le \mathrm{p}(a)\mathrm{p}(b). \tag{4.1.32}$$

ii.) A locally convex algebra is called locally multiplicatively convex (for short: lmc) if it has a defining system of continuous submultiplicative seminorms.

The corresponding categories of lmc algebras, Hausdorff lmc algebras, complete lmc algebras, unital lmc algebras, unital Hausdorff lmc algebras, and complete unital lmc algebras are denoted by lmcalg, LMCalg, $\mathsf{CLMCalg}$, lmcAlg, LMCAlg, and $\mathsf{CLMCAlg}$, respectively.

The main point of the definition is that we have the *same* seminorm on the right hand side in (4.1.32). Note that if we have

$$\mathrm{p}(ab) \le c\mathrm{p}(a)\mathrm{p}(b) \tag{4.1.33}$$

for some constant $c > 0$ instead then we can rescale the seminorm and $\mathrm{q} = c\mathrm{p}$ will be submultiplicative. Thus we can absorb a positive constant as in (4.1.33) into the definition from the beginning. Note also that for a unital lmc algebra we have for any submultiplicative seminorm p either $\mathrm{p} = 0$ or

$$\mathrm{p}(\mathbb{1}) \ge 1, \tag{4.1.34}$$

which follows immediately from $\mathrm{p}(a) \le \mathrm{p}(\mathbb{1})\mathrm{p}(a)$ for all $a \in \mathscr{A}$.

Before we discuss further properties of lmc algebras we present some important examples:

Example 4.1.25 ($\mathscr{C}^k$-Functions) Let $X \subseteq \mathbb{R}^n$ be an open subset and consider the algebra of k-times continuously differentiable functions $\mathscr{C}^k(X)$ on X where $k \in \mathbb{N}_0 \cup \{+\infty\}$. Moreover, for any compact subset $K \subseteq X$ and $\ell \in \mathbb{N}_0$ with $\ell \leq k$ one defines the seminorm

$$\mathrm{p}_{K,\ell}(f) = \max_{\substack{x \in K \\ |\alpha| \leq \ell}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right|. \tag{4.1.35}$$

As we have always an exhausting sequence of countably many compact subsets of X, countably many of the seminorms $\mathrm{p}_{K,\ell}$ will already determine the same topology as all of them, see also Exercise 2.5.29 as well as Appendix B.4.2 for more information on the resulting topology. It is now a standard result in analysis that $\mathscr{C}^k(X)$ becomes a complete (Hausdorff) locally convex space by these seminorms, and hence a Fréchet space. For the product a straightforward computation via the Leibniz rule gives the estimate

$$\mathrm{p}_{K,\ell}(fg) \leq 2^\ell \mathrm{p}_{K,\ell}(f)\mathrm{p}_{K,\ell}(g). \tag{4.1.36}$$

Rescaling the seminorms by 2^ℓ gives submultiplicative seminorms. Hence $\mathscr{C}^k(X)$ is a lmc algebra. A more general construction is given in Corollary B.4.12. Finally, we note that on a differentiable manifold M the space $\mathscr{C}^k(M)$ carries a natural locally convex topology as well, where we use seminorms of the form (4.1.35) with respect to all coordinate charts of an atlas. This makes $\mathscr{C}^k(M)$ also a complete locally multiplicatively convex algebra which, since manifolds are required to be second countable, will be a Fréchet space. These algebras can be seen as the observable algebras of a *classical* theory where the underlying manifold M (or the open subset X) plays the role of the classical phase space. Of course, for this interpretation, M has to carry additional structures like a Poisson bracket for the functions.

Example 4.1.26 (Entire functions) We consider the entire holomorphic functions $\mathscr{O}(\mathbb{C})$. With the $\mathscr{C}^0$-topology coming from the seminorms $\mathrm{p}_{K,0}$ of continuous functions we know from Exercise 2.5.33 that $\mathscr{O}(\mathbb{C})$ is a Fréchet space. From Example 4.1.25 we conclude that $\mathscr{O}(\mathbb{C})$ is a closed lmc subalgebra of $\mathscr{C}^0(\mathbb{C})$. In fact, this holds for holomorphic functions on any non-empty open subset $U \subseteq \mathbb{C}$. Equivalently, we can check that also the seminorms

$$\mathrm{p}_R(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \left| \frac{\partial^n f}{\partial z^n}(0) \right| R^n \tag{4.1.37}$$

are submultiplicative: here we simply compute for $f, g \in \mathscr{O}(\mathbb{C})$

$$\begin{aligned}
\mathrm{p}_R(fg) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left| \frac{\partial^n (fg)}{\partial z^n}(\,\cdot\,) \right| R^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left| \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k f}{\partial z^k}(\,\cdot\,) \frac{\partial^{n-k} g}{\partial z^{n-k}}(\,\cdot\,) \right| R^n \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!} \left| \frac{\partial^k f}{\partial z^k}(\,\cdot\,) \right| R^k \frac{1}{(n-k)!} \left| \frac{\partial^{n-k} g}{\partial z^{n-k}}(\,\cdot\,) \right| R^{n-k} \\
&= \mathrm{p}_R(f)\mathrm{p}_R(g).
\end{aligned}$$

Hence the system of seminorms $\{\mathrm{p}_R\}_{R>0}$ is submultiplicative directly. From Exercise 2.5.33 we know that the two topologies are the same. The seminorms p_R will play a crucial role later one.

Example 4.1.27 (Holomorphic functions) As a slight variation we consider the holomorphic functions $\mathscr{O}(\mathrm{B}_\rho(0))$ on an open ball of radius $\rho > 0$. Again, the $\mathscr{C}^0$-topology makes this a closed subalgebra of $\mathscr{C}^0(\mathrm{B}_\rho(0))$ which is therefore again lmc. Also here we have the alternative description with the seminorms p_R using the Taylor coefficients: we just have to use those p_R with $R < \rho$, yielding the same Fréchet topology. We have the restriction map

$$\mathscr{O}(\mathbb{C}) \longrightarrow \mathscr{O}(\mathrm{B}_\rho(0)), \tag{4.1.38}$$

which is an injective continuous unital algebra homomorphism for every $\rho > 0$. Note however that (4.1.38) is not surjective. Note also that for other domains $X \subseteq \mathbb{C}$ we have to use the analogous seminorms to (4.1.37) involving the Taylor expansion of f around all points in X, see also Exercise 2.5.34.

The standard constructions behave well with respect to locally multiplicatively convex algebras. We have already used the first part in the above examples:

Proposition 4.1.28 *Let $\mathscr{A}$ be a locally multiplicatively convex algebra.*

i.) If $\mathscr{B} \subseteq \mathscr{A}$ is a subalgebra then $\mathscr{B}$ is again lmc with respect to the induced topology.

ii.) If $\mathscr{J} \subseteq \mathscr{A}$ is a two-sided ideal then the locally convex quotient $\mathscr{A}/\mathscr{J}$ is again lmc.

iii.) The Hausdorffization of $\mathscr{A}$ is again lmc.

iv.) If $\mathscr{A}$ is Hausdorff then the completion of $\mathscr{A}$ is again lmc.

PROOF: The first part is clear as we just have to restrict the seminorms. For the second part, let p be a submultiplicative seminorm on $\mathscr{A}$. Analogously to the proof of Proposition 4.1.18, the corresponding seminorm $[\mathrm{p}]$ on $\mathscr{A}/\mathscr{J}$ according to (2.2.39) is easily shown to be submultiplicative again, from which the second part follows. The third part is a particular case of the second by taking $\mathscr{J} = \{0\}^{\mathrm{cl}}$. The last part is also easy as the estimate $\mathrm{p}(ab) \le \mathrm{p}(a)\mathrm{p}(b)$ holds on the dense subset $\mathscr{A}$ of the completion $\widehat{\mathscr{A}}$ of $\mathscr{A}$ and hence everywhere. □

We come now to a slightly more sophisticated example of a lmc algebra. We start with a given locally convex space V and consider the tensor powers $V^{\otimes_\pi n}$ with the π-topology. Then on the whole tensor algebra

$$\mathrm{T}(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n} \tag{4.1.39}$$

we consider the following seminorm

$$\mathrm{T}(\mathrm{p})(v) = \sum_{n=0}^{\infty} \mathrm{p}^n(v_n), \tag{4.1.40}$$

where $v = \sum_{n=0}^{\infty} v_n$ is decomposed into its homogeneous parts as usual, i.e. $v_n \in V^{\otimes n}$. Moreover, $\mathrm{p}^n = \mathrm{p} \otimes \cdots \otimes \mathrm{p}$ is the seminorm on $V^{\otimes_\pi n}$ obtained from a continuous seminorm p on V by n-fold tensoring as in Lemma 4.1.4. Finally, by convention, p^0 is just the usual absolute value on $V^{\otimes 0} = \mathbb{C}$. Since we are dealing with a direct sum the series (4.1.40) contains at most finitely many non-zero contributions. Thus it follows that

$$\mathrm{T}(\mathrm{p})\colon \mathrm{T}(V) \longrightarrow \mathbb{R} \tag{4.1.41}$$

is a well-defined seminorm. Clearly, for all $n \in \mathbb{N}_0$ we have

$$\mathrm{T}(\mathrm{p})\Big|_{V^{\otimes n}} = \mathrm{p}^n. \tag{4.1.42}$$

Finally, it is easy to see that for two seminorms p and q on V we have

$$\mathrm{p} \le \mathrm{q} \quad \text{iff} \quad \mathrm{T}(\mathrm{p}) \le \mathrm{T}(\mathrm{q}). \tag{4.1.43}$$

The construction of the seminorm $\mathrm{T}(\mathrm{p})$ out of a given seminorm p on V motivates now the following definition:

Definition 4.1.29 (Tensor algebra topology) *Let V be a locally convex space. Then the tensor algebra topology on $\mathrm{T}^\bullet(V)$ is determined by the system of seminorms of all $\mathrm{T}(\mathrm{p})$ where the corresponding seminorm p is an arbitrary continuous seminorm on V.*

There is some caution necessary when dealing with the tensor algebra topology: if p and q are continuous seminorms with $\mathrm{p} \leq c\mathrm{q}$ for some constant $c > 0$ then there might be *no* constant $C > 0$ with $\mathrm{T}(\mathrm{p}) \leq C\mathrm{T}(\mathrm{q})$. Indeed, we have for the continuous seminorm $\mathrm{r} = c\mathrm{q}$ the corresponding seminorm

$$\mathrm{T}(\mathrm{r})(v) = \sum_{n=0}^{\infty} (c\mathrm{q})^{\otimes n}(v_n) = \sum_{n=0}^{\infty} c^n \mathrm{q}^{\otimes n}(v_n), \tag{4.1.44}$$

which does not allow for a simple estimate by $\mathrm{T}(\mathrm{q})(v)$ for $c > 1$. Thus we conclude that even if $V \neq \{0\}$ is normed then the tensor algebra topology is *not* a normable topology. Nevertheless, the tensor algebra topology has the following remarkable properties:

Proposition 4.1.30 (Tensor algebra topology) *Let V be a locally convex space and let $\mathrm{T}^\bullet(V)$ be its tensor algebra with the tensor algebra topology.*

i.) *For all $n \in \mathbb{N}_0$ the canonical inclusion $V^{\otimes_\pi n} \longrightarrow \mathrm{T}^\bullet(V)$ is an embedding with closed image.*

ii.) *For all $n \in \mathbb{N}_0$ the canonical projection $\mathrm{T}^\bullet(V) \longrightarrow V^{\otimes_\pi n}$ is continuous.*

iii.) *The tensor algebra $\mathrm{T}^\bullet(V)$ is Hausdorff iff V is Hausdorff.*

iv.) *The tensor algebra topology is finer than the Cartesian product topology inherited from $\mathrm{T}^\bullet(V) \subseteq \prod_{n=0}^\infty V^{\otimes_\pi n}$.*

v.) *For $v, w \in \mathrm{T}^\bullet(V)$ one has*
$$\mathrm{T}(\mathrm{p})(v \otimes w) \leq \mathrm{T}(\mathrm{p})(v)\mathrm{T}(\mathrm{p})(v) \tag{4.1.45}$$
for all continuous seminorms p on V. Hence $\mathrm{T}^\bullet(V)$ becomes a lmc algebra with respect to the tensor product.

vi.) *Let $\phi\colon V \longrightarrow \mathscr{A}$ be a continuous linear map into a unital lmc algebra. Then the canonical extension*
$$\Phi\colon \mathrm{T}(V) \longrightarrow \mathscr{A} \tag{4.1.46}$$
as unital algebra homomorphism is continuous.

vii.) *For every continuous linear map $\phi\colon V \longrightarrow V$ the canonical extension*
$$D_\phi\colon \mathrm{T}^\bullet(V) \longrightarrow \mathrm{T}^\bullet(V) \tag{4.1.47}$$
as derivation is continuous.

viii.) *The degree derivation $\deg\colon \mathrm{T}^\bullet(V) \longrightarrow \mathrm{T}^\bullet(V)$ is continuous.*

PROOF: The first and second part is clear from (4.1.42). Suppose now that V is Hausdorff. Then we know that $V^{\otimes_\pi n}$ is Hausdorff for all $n \in \mathbb{N}_0$ by Corollary 4.1.8. Let $v = \sum_{n=0}^\infty v_n \neq 0$ be some nonzero vector with some nonzero component $v_m \neq 0$. Then we have a continuous seminorm p on V with $\mathrm{p}^m(v_m) > 0$ which gives immediately $\mathrm{T}(\mathrm{p})(v) > 0$, too. Thus $\mathrm{T}^\bullet(V)$ is Hausdorff. The converse is clear from the first part. Since the Cartesian product topology is the coarsest one for which all the projections are continuous, the fourth part follows from the second at once. For the fifth part, consider $v = \sum_{n=0}^\infty v_n$ and $w = \sum_{n=0}^\infty w_n$ with $v_n, w_n \in V^{\otimes_\pi n}$ and only finitely many of them nonzero. Then $\sum_{k=0}^n v_k \otimes w_{n-k}$ is the homogeneous component of $v \otimes w$ in degree n. Hence

$$\mathrm{T}(\mathrm{p})(v \otimes w) = \mathrm{T}(\mathrm{p})\left(\sum_{n=0}^\infty \left(\sum_{k=0}^n v_k \otimes w_{n-k}\right)\right)$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \mathrm{p}^n \left(\sum_{k=0}^{n} v_k \otimes w_{n-k} \right) \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathrm{p}^k(v_k) \mathrm{p}^{n-k}(w_{n-k}) \\
&= \sum_{n=0}^{\infty} \mathrm{p}^n(v_n) \sum_{m=0}^{\infty} \mathrm{p}^m(w_m) \\
&= \mathrm{T}(\mathrm{p})(v) \mathrm{T}(\mathrm{p})(w)
\end{aligned}
$$

shows (4.1.45) and thus the fifth part. Now let $\phi\colon V \longrightarrow \mathscr{A}$ be a continuous linear map. Thus, for a given submultiplicative seminorm q on $\mathscr{A}$ we find a seminorm p on V with $\mathrm{q}(\phi(v)) \leq \mathrm{p}(v)$ for all $v \in V$. Without restriction we can assume $\mathrm{q} \neq 0$ and hence $\mathrm{q}(\mathbb{1}_{\mathscr{A}}) = c \geq 1$ by (4.1.34). From the algebraic properties of the tensor algebra we already know that there is a unique algebra homomorphism Φ extending ϕ. We only have to show its continuity. For $v = z\mathbb{1} + \sum_{n=1}^{\infty} v_n$ with $v_n = \sum_{i_1,\ldots,i_n} v_{i_1}^{(n)} \otimes \cdots \otimes v_{i_n}^{(n)}$ where $v_{i_k}^{(n)} \in V$ and $z \in \mathbb{C}$ we get

$$
\begin{aligned}
\mathrm{q}(\Phi(v)) &= \mathrm{q}\left(z\mathbb{1}_{\mathscr{A}} + \sum_{n=1}^{\infty} \phi\left(v_{i_1}^{(n)}\right) \cdots \phi\left(v_{i_n}^{(n)}\right) \right) \\
&\leq |z|c + \sum_{n=1}^{\infty} \mathrm{q}\left(\phi\left(v_{i_1}^{(n)}\right)\right) \cdots \mathrm{q}\left(\phi\left(v_{i_n}^{(n)}\right)\right) \\
&\leq c|z| + c \sum_{n=1}^{\infty} \mathrm{p}\left(v_{i_1}^{(n)}\right) \cdots \mathrm{p}\left(v_{i_1}^{(n)}\right).
\end{aligned}
$$

Since this is true for all possible decompositions of the v_n into factorizing tensors we can take the infimum over all those decompositions and still have the same estimate. This results in

$$
\mathrm{q}(\Phi(v)) \leq c|z| + c \sum_{n=1}^{\infty} \mathrm{p}^n(v_n) = c\mathrm{T}(\mathrm{p})(v).
$$

Since the submultiplicative seminorms on $\mathscr{A}$ determine the topology by assumption, this shows the continuity of Φ. Part *vii.)* is shown analogously and the last part is a special case of this for $\phi = \mathrm{id}$.□

Remark 4.1.31 In particular, part *vi.)* shows that the tensor algebra topology gives nice categorical properties of $\mathrm{T}^\bullet(V)$ inside the category of unital lmc algebras. More details can be found in Exercise 4.5.10.

Remark 4.1.32 For a Hilbert space $\mathfrak{H}$ we see from part *viii.)* on the one hand and from the discussion following (3.4.25) on the other hand, that the Fock space topology on $\mathrm{T}^\bullet(\mathfrak{H})$ differs essentially from the tensor algebra topology, see also Exercise 4.5.11.

While the importance of the tensor product topology is already clear from the universal property inside the category lmcAlg, the next example gives a more concrete interpretation of this topology: Exercise: Ter Hilbert space topologies

Example 4.1.33 We consider $V = \mathbb{C}$ with its usual norm topology. Already in this case the tensor algebra topology on $\mathrm{T}^\bullet(\mathbb{C})$ is interesting. We have $\mathbb{C}^{\otimes n} = \mathbb{C}$ for all $n \in \mathbb{N}_0$ and hence $\mathrm{T}^\bullet(\mathbb{C})$ can canonically be identified with the polynomial algebra $\mathbb{C}[t]$ where t^n corresponds to $1 \otimes \cdots \otimes 1 \in \mathbb{C}^{\otimes n}$. The crucial point for the tensor algebra topology is that we have to take *all* continuous seminorms on $\mathbb{C}$, given explicitly by $|z|_R = R|z|$ with $R > 0$. The corresponding seminorms on $\mathrm{T}^\bullet(\mathbb{C}) = \mathbb{C}[t]$ according to the definition evaluated on a vector of the form $a = a_n t^n + \cdots + a_1 t + a_0$ gives

$$\mathrm{T}(|\cdot|_R)(a) = \sum_{n=0}^{\infty} |\cdot|_R^n(a_n t^n) = \sum_{n=0}^{\infty} |a_n||t|_R \cdots |t|_R = \sum_{n=0}^{\infty} |a_n| R^n, \tag{4.1.48}$$

since t corresponds to $1 \in \mathbb{C}$ and the tensor products of seminorms on factorizing tensors factorize. By viewing the polynomials $\mathbb{C}[t]$ as particular holomorphic functions $\mathbb{C}[t] \subseteq \mathscr{O}(\mathbb{C})$ we see that under this inclusion the seminorm $\mathrm{T}(|\cdot|_R)$ corresponds literally to the seminorm p_R from Example 4.1.26. Thus the tensor algebra topology of $\mathrm{T}(\mathbb{C})$ is the canonical topology of the entire holomorphic functions $\mathscr{O}(\mathbb{C})$ restricted to polynomials.

Remark 4.1.34 Since with the tensor algebra topology $\mathrm{T}^\bullet(V)$ is a Hausdorff locally multiplicatively convex algebra for a Hausdorff locally convex space V we can pass to the completion which we denote by $\hat{\mathrm{T}}^\bullet(V)$. We know that it is again lmc by Proposition 4.1.28, *iv.)*. Without going into the details one can show that $\hat{\mathrm{T}}^\bullet(V) \cong \hat{\mathrm{T}}^\bullet(\widehat{V})$ and it also contains $\bigoplus_{n=0}^{\infty} V^{\hat{\otimes}_\pi n}$ as an even *sequentially* dense subspace. In fact, extending the definition of $\mathrm{T}(\mathrm{p})$ to the Cartesian product $\prod_{n=0}^{\infty} V^{\hat{\otimes}_\pi n}$ by allowing $+\infty$ as a value, the completion $\hat{\mathrm{T}}^\bullet(V)$ can be characterized explicitly as the following subspace of the Cartesian product

$$\hat{\mathrm{T}}^\bullet(V) = \left\{ v \in \prod\nolimits_{n=0}^{\infty} V^{\hat{\otimes}_\pi n} \;\middle|\; \mathrm{T}(\mathrm{p})(v) < +\infty \text{ for all continuous seminorms } \mathrm{p} \right\}. \tag{4.1.49}$$

Details of this completion and also further features and universal properties of $\hat{\mathrm{T}}^\bullet(V)$ are discussed in Exercise 4.5.10.

Remark 4.1.35 The tensor algebra topology also induces locally multiplicatively convex topologies on the symmetric algebra $\mathrm{S}^\bullet(V)$ and the Graßmann algebra $\Lambda^\bullet(V)$. In fact, it is easy to see that

$$\mathrm{T}(\mathrm{p})(v \vee w) \le \mathrm{T}(\mathrm{p})(v)\mathrm{T}(\mathrm{p})(w) \quad \text{and} \quad \mathrm{T}(\mathrm{p})(v \wedge w) \le \mathrm{T}(\mathrm{p})(v)\mathrm{T}(\mathrm{p})(w) \tag{4.1.50}$$

for v, w either in $\mathrm{S}^\bullet(V)$ or $\Lambda^\bullet(V)$, respectively. This way, one can transfer the above results on $\mathrm{T}^\bullet(V)$ easily to $\mathrm{S}^\bullet(V)$ and $\Lambda^\bullet(V)$, see also Exercise 4.5.13.

We come now to the main reason why locally multiplicatively convex algebras are nicer than just locally convex ones:

Theorem 4.1.36 (Entire holomorphic calculus) *Let $\mathscr{A}$ be a complete unital lmc algebra and let $a \in \mathscr{A}$.*

i.) For every $f \in \mathscr{O}(\mathbb{C})$ the series

$$f(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f}{\partial z^n}(0)\, a^n \tag{4.1.51}$$

converges absolutely in $\mathscr{A}$.

ii.) The map

$$\mathscr{O}(\mathbb{C}) \ni f \;\mapsto\; f(a) \in \mathscr{A} \tag{4.1.52}$$

is a unital algebra homomorphism, extending the polynomial calculus.

iii.) The algebra homomorphism (4.1.52) is continuous. More precisely, for every continuous submultiplicative seminorm q on $\mathscr{A}$ we have

$$\mathrm{q}(f(a)) \le \mathrm{q}(\mathbb{1})\mathrm{p}_{\mathrm{q}(a)}(f), \tag{4.1.53}$$

with the seminorms p_R of $\mathscr{O}(\mathbb{C})$ as in Example 4.1.26.

iv.) If $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a continuous unital algebra homomorphism into another complete unital lmc algebra $\mathscr{B}$ then for every $f \in \mathscr{O}(\mathbb{C})$ and $a \in \mathscr{A}$ we have

$$f(\Phi(a)) = \Phi(f(a)). \tag{4.1.54}$$

Proof: Let $a \in \mathscr{A}$ be given and let q be a continuous submultiplicative seminorm. To prove absolute convergence we estimate

$$\sum_{n=0}^{\infty} \mathrm{q}\left(\frac{1}{n!}\frac{\partial^n f}{\partial z^n}(0)a^n\right) \le \sum_{n=0}^{\infty} \frac{1}{n!}\left|\frac{\partial^n f}{\partial z^n}(0)\right| c\mathrm{q}(a)^n \le c\mathrm{p}_{\mathrm{q}(a)}(f),$$

by $\mathrm{q}(a^n) \le \mathrm{q}(a)^n$ and where we need the constant $c = \mathrm{q}(\mathbb{1})$ only because of the first term. Thus we have on one hand the absolute convergence of (4.1.51) and on the other hand the continuity of the linear map (4.1.52) including the estimate (4.1.53). It remains to check the homomorphism property. Clearly, if $f \in \mathbb{C}[z]$ is just a polynomial then $f(a)$ according to (4.1.51) is the usual polynomial calculus which we know to be a homomorphism $\mathbb{C}[z] \longrightarrow \mathscr{A}$, see Exercise 1.5.4. Since $\mathbb{C}[z] \subseteq \mathscr{O}(\mathbb{C})$ is a dense subspace and since we have already shown the continuity the extension (4.1.52) to all entire functions is still a homomorphism. For the last part we note that the claim is true for polynomials f by the homomorphism property, see again Exercise 1.5.4. Then the general situation follows by continuity and the density of $\mathbb{C}[z] \subseteq \mathscr{O}(\mathbb{C})$. □

The last part shows that the entire calculus behaves natural with respect to continuous unital algebra homomorphisms. This functional calculus allows us to speak of elements like $\exp(a)$ in a complete locally multiplicatively convex algebra. In the non-unital case one still has an entire calculus, now only for those entire functions which vanish at zero, see Exercise 4.5.14.

While this situation is clearly very nice, the following example shows that the reality of quantum physics requires some more sophisticated tools:

Proposition 4.1.37 (Canonical commutation relations) *Assume $\mathscr{A}$ is a unital algebra with two elements $P, Q \in \mathscr{A}$ such that*

$$[Q, P] = \mathrm{i}\hbar\mathbb{1}, \tag{4.1.55}$$

where $\hbar \in \mathbb{C} \setminus \{0\}$ is a non-zero complex number. Then the only submultiplicative seminorm on $\mathscr{A}$ is $\mathrm{q} = 0$.

Proof: Let $n \in \mathbb{N}$ be given. Then we compute by successive use of the commutation relations the n-fold commutator

$$\underbrace{[Q, \cdots, [Q}_{n \text{ times}}, P^n] \cdots] = \underbrace{[Q, \cdots, [Q}_{n-1 \text{ times}}, n\mathrm{i}\hbar P^{n-1}] \cdots] = \cdots = n!(\mathrm{i}\hbar)^n \mathbb{1}, \tag{$*$}$$

since clearly $[Q, P^n] = n\mathrm{i}\hbar P^{n-1}$. Now for any submultiplicative seminorm we have

$$\begin{aligned}
\mathrm{q}\Big(\underbrace{[Q, \cdots [Q}_{n \text{ times}}, P^n] \cdots]\Big) &= \mathrm{q}\Big(Q\underbrace{[Q, \cdots [Q}_{n-1 \text{ times}}, P^n] \cdots] - \underbrace{[Q, \cdots [Q}_{n-1 \text{ times}}, P^n] \cdots]Q\Big) \\
&\le 2\mathrm{q}(Q)\mathrm{q}\Big(\underbrace{[Q, \cdots [Q}_{n-1 \text{ times}}, P^n] \cdots]\Big) \\
&\le 2^n \mathrm{q}(Q)^n \mathrm{q}(P^n) \\
&\le 2^n \mathrm{q}(Q)^n \mathrm{q}(P)^n.
\end{aligned}$$

Thus combining this with the computation ($*$) gives immediately $\mathrm{q}(\mathbb{1}) = 0$, since the factorial grows faster than any exponential. But a submultiplicative seminorm vanishing on the unit element is identically zero, see (4.1.34). □

Corollary 4.1.38 *There is no Hausdorff locally multiplicatively convex algebra with elements satisfying the canonical commutation relations.*

Remark 4.1.39 Even though this seems to be really bad news there are two ways out: On one hand, one considers the canonical commutation relations in their formally "exponentiated form". Here one can show that a lmc topology is easily available and even better: one can have a C^*-topology. The details of this construction will be postponed until Section **??**. On the other hand, there are Hausdorff locally convex topologies on the algebra generated by P and Q with the canonical commutation relations (4.1.55). However, the corresponding Fréchet completions do not allow for an entire holomorphic calculus. For more details on this construction see [4, 60].

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rious versions

4.2 Banach Algebras

In this section we will pass from general locally convex algebras to more specific ones, the Banach algebras. Here we will see a good notion for the spectrum of an algebra element allowing also for a refined version of the holomorphic calculus.

4.2.1 Definition and First Examples

Recall that for a normed space we do not just require the locally convex topology to be normable. Instead the choice of a particular norm is part of the data.

Definition 4.2.1 (Banach and Banach *-algebra) *Let $\mathscr{A}$ be an associative algebra over $\mathbb{C}$ and $\|\cdot\|$ a norm on $\mathscr{A}$.*

i.) The pair $(\mathscr{A}, \|\cdot\|)$ is called normed algebra if for all $a, b \in \mathscr{A}$ one has

$$\|ab\| \leq \|a\|\|b\|. \tag{4.2.1}$$

ii.) The pair $(\mathscr{A}, \|\cdot\|)$ is called a Banach algebra if $(\mathscr{A}, \|\cdot\|)$ is a complete normed algebra.

iii.) If $\mathscr{A}$ has in addition a *-involution then $(\mathscr{A}, \|\cdot\|, ^*)$ is called a normed *-algebra if in addition to (4.2.1) one has for all $a \in \mathscr{A}$

$$\|a^*\| = \|a\|. \tag{4.2.2}$$

iv.) A Banach *-algebra is a complete normed *-algebra.

v.) For a unital normed (or Banach) algebra we require for the unit element

$$\|\mathbb{1}\| = 1. \tag{4.2.3}$$

As usual we get corresponding categories $\mathsf{Banachalg}$, $\mathsf{BanachAlg}$, $\mathsf{Banachalg}^*$, and $\mathsf{BanachAlg}^*$ of Banach algebras and Banach *-algebras with or without units where for the morphisms we take the usual algebra homomorphisms which are continuous.

Remark 4.2.2 (Banach algebra) Let $\mathscr{A}$ be an associative algebra.

i.) If $\|\cdot\|$ is a norm on $\mathscr{A}$ making it a normed algebra then for all $c > 1$ also $c\|\cdot\|$ is such a norm. However, for $0 < c < 1$ the property (4.2.1) might be lost even though we have the same underlying locally convex algebra.

ii.) In the unital case we have by (4.2.1) the property $\|\mathbb{1}\| \geq 1$, where we use $\mathbb{1} \neq 0$ according to our convention. Thus we may replace the norm by the equivalent norm $\frac{1}{\|\mathbb{1}\|}\|\cdot\|$ to achieve (4.2.3). However, then (4.2.1) might be spoiled: thus the unital normed algebra might not be a normed unital algebra.

iii.) Clearly, a normed algebra is Hausdorff and locally multiplicatively convex. A normed *-algebra is a Hausdorff lmc *-algebra. Thus the product and the *-involution are both continuous in the norm topology. However, there are normed spaces with a continuous algebra multiplication which are not normed algebras as the continuity itself only means $\|ab\| \leq c\|a\|\|b\|$. Thus a rescaling of the norm will yield a normed algebra which is, as normed space, different from the original one.

iv.) Being a complete lmc algebra every Banach algebra allows for an entire calculus according to Theorem 4.1.36.

Normed algebras behave well under quotients and completions. Note that we are in a Hausdorff situation anyway.

Proposition 4.2.3 *Let $\mathscr{A}$ be a normed algebra.*

i.) If $\mathscr{J} \subseteq \mathscr{A}$ is a closed ideal then $\mathscr{A}/\mathscr{J}$ is a normed algebra again.

ii.) The completion $\widehat{\mathscr{A}}$ is a Banach algebra.

iii.) If $\mathscr{A}$ is a normed *-algebra and $\mathscr{J} \subseteq \mathscr{A}$ a closed *-ideal then $\mathscr{A}/\mathscr{J}$ is a normed *-algebra again.

iv.) If $\mathscr{A}$ is a normed *-algebra then its completion $\widehat{\mathscr{A}}$ is a Banach *-algebra.

PROOF: We know that the quotient locally convex topology of $\mathscr{A}/\mathscr{J}$ is determined by the single seminorm

$$\|[a]\| = \inf\{\|a + c\| \mid c \in \mathscr{J}\}, \tag{$*$}$$

since all other continuous seminorms p on $\mathscr{A}$ are dominated by multiples of $\|\cdot\|$ and hence their quotient seminorms [p] are dominated by multiples of $(*)$. Since $\mathscr{J}$ is assumed to be closed the quotient is Hausdorff by Proposition 2.2.42, *iii.)*. Hence $(*)$ is actually a norm. The submultiplicativity of $(*)$ was already shown in the general situation of lmc algebras in Proposition 4.1.28, *ii.)*. Now let $\mathscr{A}$ be unital then $[\mathbb{1}]$ is a unit for the quotient algebra $\mathscr{A}/\mathscr{J}$. Unless $\mathscr{J} = \mathscr{A}$ it will be different from $[0]$. Thus necessarily $\|[\mathbb{1}]\| \geq 1$ by Remark 4.2.2, *ii.)*. However, by $(*)$ the infimum in $\|[\mathbb{1}]\|$ is clearly ≤ 1 since this value is already obtained for $c = 0$. Thus $\|[\mathbb{1}]\| = 1$ as wanted and we get a normed algebra also in the unital case. The statement about the completion is clear by the same arguments as for Proposition 4.1.28, *iv.)*. For the third part we have to check $\|[a^*]\| = \|[a]\|$. But since $\mathscr{J}$ is stable under the *-involution this is clear since

$$\inf\{\|a^* + c\| \mid c \in \mathscr{J}\} = \inf\{\|a + c^*\| \mid c \in \mathscr{J}\},$$

and we have $\|a^*\| = \|a\|$ in a normed algebra. The last part is clear. □

Remark 4.2.4 From Exercise 2.5.43 we know that for a Banach space the quotient by a closed subspace is again a Banach space. Thus a Banach algebra modulo a closed ideal gives again a Banach algebra and not just a normed algebra, i.e. completeness is preserved. The same holds for a Banach *-algebra modulo a closed *-ideal.

We discuss now several examples. The first one is in some sense the model for an observable algebra in *classical* physics while the fourth is the prototype for quantum physics.

Example 4.2.5 (The Banach *-algebra $\mathscr{C}(X)$) Let X be a compact Hausdorff space and denote by $\mathscr{C}(X)$ the continuous complex-valued functions on X as usual. Then $\mathscr{C}(X)$ is a unital commutative associative *-algebra with respect to the pointwise multiplication, the pointwise complex conjugation as *-involution, and the constant function 1 as unit. With the maximum norm

$$\|f\|_\infty = \max_{x \in X} |f(x)| \tag{4.2.4}$$

we get a Banach *-algebra with unit. The relevant compatibilities between the norm $\|\cdot\|_\infty$ and the algebraic structures are easily verified, see Exercise 4.5.17. Moreover, we have the additional property

$$\|\overline{f}f\|_\infty = \|f\|_\infty^2, \tag{4.2.5}$$

which is not required for a Banach *-algebra. The completeness of $\mathscr{C}(X)$ is the well-known fact that the (locally) uniform limits of continuous functions are again continuous, see also Appendix B.1.

Example 4.2.6 (Algebra-valued functions) Let again X be a compact Hausdorff space and let $\mathscr{A}$ be a Banach algebra (or a Banach *-algebra). Then we consider the continuous maps from X to $\mathscr{A}$, denoted by

$$\mathscr{C}(X, \mathscr{A}) = \{f \colon X \longrightarrow \mathscr{A} \mid f \text{ is continuous}\}. \tag{4.2.6}$$

We can endow $\mathscr{C}(X, \mathscr{A})$ with the structure of an associative algebra (unital or commutative if $\mathscr{A}$ was unital or commutative, respectively) by the pointwise operations. For the norm we take the combination of the norm of $\mathscr{A}$ and the maximum norm, i.e. we set

$$\|f\|_\infty = \max_{x\in X} \|f(x)\|_{\mathscr{A}}. \tag{4.2.7}$$

It is then easy to check that we get a Banach algebra or a Banach *-algebra by this construction, see also Exercise 4.5.21.

Example 4.2.7 (The Banach algebra $\mathrm{L}(V)$) Let V be a Banach space. Then by Proposition 2.3.7 we obtain a unital Banach algebra structure for the continuous linear endomorphisms $\mathrm{L}(V)$ of V with the operator norm as Banach norm.

Example 4.2.8 (The Banach *-algebra $\mathfrak{B}(\mathfrak{H})$) Let $\mathfrak{H}$ be a Hilbert space. Then the Banach algebra $\mathrm{L}(\mathfrak{H})$ from the previous example coincides with the *-algebra $\mathfrak{B}(\mathfrak{H})$ by Theorem 3.5.1. Moreover, Theorem 3.5.4 tells us that in addition we have

$$\|A^*A\| = \|A\|^2, \tag{4.2.8}$$

and hence $\|A^*\| = \|A\|$. Thus $\mathfrak{B}(\mathfrak{H})$ is a unital Banach *-algebra with the additional property (4.2.8), analogously to (4.2.5).

Example 4.2.9 (The Banach *-algebra $\Lambda(\mathbb{C}^n)$) Let $\Lambda(\mathbb{C}^n)$ be the Graßmann algebra over $\mathbb{C}^n$. For the canonical basis $\mathrm{e}_1, \ldots, \mathrm{e}_n \in \mathbb{C}^n \subseteq \Lambda(\mathbb{C}^n)$ one defines

$$\mathrm{e}_i^* = \mathrm{e}_i \tag{4.2.9}$$

and also sets $\mathbb{1}^* = \mathbb{1}$. Then it is easy to see that this extends uniquely to a *-involution on $\Lambda(\mathbb{C}^n)$. Thus we obtain a unital *-algebra structure on the Graßmann algebra which, in addition, is super-commutative. Writing $\alpha = \sum_r \sum_{i_1,\ldots,i_r} \alpha^{i_1\cdots i_r} \mathrm{e}_{i_1} \wedge \cdots \wedge \mathrm{e}_{i_r}$ one sets

$$\|\alpha\| = \sum_{i_1,\ldots,i_r} |\alpha^{i_1\cdots i_r}|, \tag{4.2.10}$$

and verifies that this gives a submultiplicative norm compatible with the *-involution. Since $\Lambda(\mathbb{C}^n)$ is finite-dimensional, it is complete and we end up with a Banach *-algebra. However, one can show that this Banach *-algebra contains elements α with

$$\|\alpha^*\alpha\| < \|\alpha\|^2, \tag{4.2.11}$$

unlike for $\mathscr{C}(X)$ or $\mathfrak{B}(\mathfrak{H})$. In particular the basis vectors e_i will satisfy $\mathrm{e}_i^*\mathrm{e}_i = 0$. We will see a more conceptual reason for this when discussing C^*-algebras in Section 4.3.

After these examples we can now discuss the following construction of *adjoining a unit* or *unitization* of a non-unital algebra. Thus let $\mathscr{A}$ be a non-unital algebra and set

$$\widetilde{\mathscr{A}} = \mathscr{A} \oplus \mathbb{C} \tag{4.2.12}$$

as vector space. Then one defines the product

$$(a, z)(b, w) = (ab + zb + wa, zw) \tag{4.2.13}$$

for $(a, z), (b, w) \in \widetilde{\mathscr{A}}$. The following lemma is then a straightforward verification, see Exercise 4.5.22:

Lemma 4.2.10 (Unitization) *The vector space $\widetilde{\mathscr{A}}$ becomes a unital associative algebra over $\mathbb{C}$ with unit given by $\mathbb{1} = (0, 1)$. Moreover, $\widetilde{\mathscr{A}}$ contains $\mathscr{A}$ as a two-sided ideal via $\mathscr{A} \ni a \mapsto (a, 0) \in \widetilde{\mathscr{A}}$.*

Thanks to the embedding of $\mathscr{A}$ into $\widetilde{\mathscr{A}}$ we simply write $a + z\mathbb{1}$ instead of (a, z) in the following. If $\mathscr{A}$ is even a *-algebra then it is easy to see that

$$(a + z\mathbb{1})^* = a^* + \overline{z}\mathbb{1} \tag{4.2.14}$$

endows $\widetilde{\mathscr{A}}$ with the structure of a unital *-algebra such that $\mathscr{A}$ is a *-ideal in $\widetilde{\mathscr{A}}$.

We shall now investigate how this algebraic construction matches with topological properties. Though the following is not the only possibility, one considers the norm on $\widetilde{\mathscr{A}}$ defined by

$$\|a + z\mathbb{1}\| = \|a\| + |z|. \tag{4.2.15}$$

Proposition 4.2.11 (Unitization of Banach algebras) *Let $\mathscr{A}$ be a normed algebra and let $\widetilde{\mathscr{A}}$ be its unitization.*

i.) *With* (4.2.15) *the unital algebra $\widetilde{\mathscr{A}}$ becomes a normed algebra containing $\mathscr{A}$ as a closed ideal.*

ii.) *The algebra $\mathscr{A}$ is complete iff $\widetilde{\mathscr{A}}$ is complete.*

iii.) *If in addition $\mathscr{A}$ is a normed *-algebra then the *-involution* (4.2.14) *and the norm* (4.2.15) *make $\widetilde{\mathscr{A}}$ a unital normed *-algebra.*

PROOF: First we note that $\|\cdot\|$ as in (4.2.15) is indeed a norm. We compute

$$\begin{aligned}
\|(a + z\mathbb{1})(b + w\mathbb{1})\| &= \|ab + zb + wa + zw\mathbb{1}\| \\
&= \|ab + zb + wa\| + |z||w| \\
&\le \|a\|\|b\| + |z|\|b\| + |w|\|a\| + |z||w| \\
&= (\|a\| + |z|)(\|b\| + |w|) \\
&= \|a + z\mathbb{1}\|\|b + w\mathbb{1}\|,
\end{aligned}$$

and hence (4.2.15) is submultiplicative. Finally, $\|\mathbb{1}\| = 1$ is clear by construction. Thus $\widetilde{\mathscr{A}}$ is a normed algebra. Since the evaluation functional $a + z\mathbb{1} \mapsto z$ is clearly continuous, by $|z| \le \|a + z\mathbb{1}\| = \|a\| + |z|$, we see that its kernel, being $\mathscr{A}$, is closed. This proves the first part. The second is clear as $\mathscr{A}$ and $\mathbb{C}$ are both complete and we have a finite direct sum. For the last part we observe that

$$\|(a + z\mathbb{1})^*\| = \|a^* + \overline{z}\mathbb{1}\| = \|a^*\| + |\overline{z}| = \|a\| + |z| = \|a + z\mathbb{1}\|. \qquad \square$$

In Exercise 4.5.24, we will see other possibilities to endow $\widetilde{\mathscr{A}}$ with a compatible norm. But for the time being this will do the job. Moreover, in Exercise 4.5.23 the unitization for locally convex and lmc algebras is discussed.

4.2.2 The Spectrum and the Holomorphic Calculus

In this subsection we introduce the central definition of the spectrum of an element in a Banach algebra. As a motivation, we recall several equivalent statements for spectral values of linear endomorphisms of $\mathbb{C}^n$: for a given linear map $A\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ and a complex number $z \in \mathbb{C}$ the following statements are well-known to be equivalent:

1. The complex number z is an eigenvalue of A, i.e. there is a non-zero vector $v \in \mathbb{C}^n$ with $Av = zv$.
2. The linear map $z\mathbb{1} - A$ is not injective.
3. The linear map $z\mathbb{1} - A$ is not surjective.
4. The linear map $z\mathbb{1} - A$ is not bijective.
5. One has $\det(z\mathbb{1} - A) = 0$.
6. The linear map $z\mathbb{1} - A$ is not invertible in $\operatorname{End}_{\mathbb{C}}(\mathbb{C}^n)$.

For a general Banach or Hilbert space $\mathfrak{H}$ instead of $\mathbb{C}^n$ and a general linear, say at least continuous, map $A\colon \mathfrak{H} \longrightarrow \mathfrak{H}$ the first two statements are still equivalent. But the second and the third become completely independent and hence the fourth is typically strictly stronger than both the second and the third. Moreover, the fifth is simply not defined at all. The relation between the sixth and the fourth is at least delicate as we have to specify where we want the inverse to be. Here the open mapping theorem simplifies the situation as we will see later: the fourth and sixth are still equivalent when interpreting the inverse as an element in the continuous endomorphisms, too.

If we are now interested in the notion of spectrum for an abstractly given algebra, not represented by linear maps on some vector space, then all options except for the last are simply not available. These considerations motivate the following definition:

Definition 4.2.12 (Spectrum and resolvent) *Let $\mathscr{A}$ be a unital associative algebra over $\mathbb{C}$ and let $a \in \mathscr{A}$.*

i.) The resolvent set of a is defined by

$$\operatorname{res}_{\mathscr{A}}(a) = \left\{ z \in \mathbb{C} \;\middle|\; z\mathbb{1} - a \textit{ is invertible in } \mathscr{A} \right\}. \tag{4.2.16}$$

ii.) The spectrum of a is defined as the complement of the resolvent set

$$\operatorname{spec}_{\mathscr{A}}(a) = \mathbb{C} \setminus \operatorname{res}_{\mathscr{A}}(a). \tag{4.2.17}$$

iii.) The resolvent of a is the function

$$\operatorname{Res}\colon \operatorname{res}_{\mathscr{A}}(a) \ni z \;\mapsto\; \operatorname{Res}_z(a) = (z\mathbb{1} - a)^{-1} \in \mathscr{A}. \tag{4.2.18}$$

Remark 4.2.13 (Resolvent and spectrum) Let $\mathscr{A}$ be a unital associative algebra.

i.) The definition of the spectrum and the resolvent is entirely *algebraic* and would work equally well for a unital associative algebra over some other field than $\mathbb{C}$. However, the definition is rather meaningless unless one adds some requirements on $\mathscr{A}$: one can easily find examples where the spectrum of a non-trivial element is either empty or behaves strangely in other ways.

ii.) The resolvent as well as the spectrum depend very much on the whole algebra in the following sense: suppose $\mathscr{A} \subseteq \mathscr{B}$ is a unital subalgebra of some larger algebra $\mathscr{B}$. Then it might happen that in $\mathscr{B}$ the element $z\mathbb{1} - a \in \mathscr{A}$ is invertible but the inverse is not in $\mathscr{A}$. If $z\mathbb{1} - a$ is invertible in $\mathscr{A}$ then clearly also in $\mathscr{B}$. So in general we expect

$$\operatorname{spec}_{\mathscr{A}}(a) \supseteq \operatorname{spec}_{\mathscr{B}}(a) \tag{4.2.19}$$

for $a \in \mathscr{A}$, with a typically *proper* inclusion.

iii.) One instance where (4.2.19) may be a proper inclusion is for an incomplete topological algebra $\mathscr{A}$ inside its completion $\mathscr{B} = \widehat{\mathscr{A}}$. Here a might not be invertible in $\mathscr{A}$ since the construction of the inverse would require some completion procedure first. Thus it will be reasonable to consider only complete algebras from the beginning.

In the general case, even for complete topological or complete locally convex algebras, not much can be said about the spectrum and the resolvent: under the additional assumption of a locally multiplicatively convex topology, there are results generalizing the ones for Banach algebras. However, the theory is far from being easy and will go beyond the scope of our investigations. One should consult instead e.g. [36, 38]. The situation becomes tractable if one considers Banach algebras. Here we have the following fundamental result:

Theorem 4.2.14 (Spectrum in a Banach algebra) *Let $\mathscr{A}$ be a unital Banach algebra and let $a, b \in \mathscr{A}$.*

i.) *If b is invertible and a is close to b in the sense that*

$$\|a - b\| < \frac{1}{\|b^{-1}\|}, \tag{4.2.20}$$

then a is invertible, too. In particular, if $\|a\| < 1$ then $\mathbb{1} - a$ is invertible.

ii.) *The resolvent set $\operatorname{res}_{\mathscr{A}}(a) \subseteq \mathbb{C}$ is open.*

iii.) *The resolvent $\operatorname{Res}\colon z \mapsto \operatorname{Res}_z(a)$ is holomorphic on $\operatorname{res}_{\mathscr{A}}(a)$.*

iv.) *For $z, w \in \operatorname{res}_{\mathscr{A}}(a)$ one has the resolvent identity*

$$\operatorname{Res}_z(a) - \operatorname{Res}_w(a) = (w - z) \operatorname{Res}_w(a) \operatorname{Res}_z(a). \tag{4.2.21}$$

v.) *For $z, w \in \operatorname{res}_{\mathscr{A}}(a)$ one has*

$$\operatorname{Res}_z(a) \operatorname{Res}_w(a) = \operatorname{Res}_w(a) \operatorname{Res}_z(a), \tag{4.2.22}$$

and

$$a \operatorname{Res}_z(a) = \operatorname{Res}_z(a) a. \tag{4.2.23}$$

vi.) *For $\lambda \in \operatorname{spec}_{\mathscr{A}}(a)$ one has*

$$|\lambda| \leq \|a\|. \tag{4.2.24}$$

It follows that $\operatorname{spec}_{\mathscr{A}}(a)$ is compact.

vii.) *The spectrum $\operatorname{spec}_{\mathscr{A}}(a)$ is non-empty.*

Proof: Apparently, we have the identity $a = b(\mathbb{1} - b^{-1}(b - a))$. The idea is now to invert the second factor by means of a geometric series. We have by assumption

$$\|b^{-1}(b - a)\| \leq \|b^{-1}\| \|b - a\| < 1.$$

Thus it will be sufficient to consider the invertibility of the second factor which amounts to consider the case $b = \mathbb{1}$ and a with $\|a\| < 1$. For such a the series $\sum_{n=0}^{\infty} a^n$ clearly converges absolutely since $\|a^n\| \leq \|a\|^n$ and $\|a\| < 1$. This way, we obtain $\sum_{n=0}^{\infty} a^n \in \mathscr{A}$ by completeness of $\mathscr{A}$ and thus we have an inverse of $\mathbb{1} - a$. For the second part let $z \in \operatorname{res}_{\mathscr{A}}(a)$ be in the resolvent set and let $w \in \mathbb{C}$ with $|z - w| < \frac{1}{\|\operatorname{Res}_z(a)\|}$ be a complex number close to z. Then $z\mathbb{1} - a$ is invertible and we have

$$\|z\mathbb{1} - a - (w\mathbb{1} - a)\| = \|(z - w)\mathbb{1}\| = |z - w| < \frac{1}{\|\operatorname{Res}_z(a)\|} = \frac{1}{\|(z\mathbb{1} - a)^{-1}\|}.$$

By the first part we conclude that also $w\mathbb{1} - a$ is invertible and hence $\operatorname{res}_{\mathscr{A}}(a)$ is open. For the third part we explicitly expand $\operatorname{Res}_z(a)$ into its Taylor series around some $z_0 \in \operatorname{res}_{\mathscr{A}}(a)$ for z being close enough to z_0. Consider $z \in \mathbb{C}$ with

$$|z - z_0| < \frac{1}{\|\operatorname{Res}_{z_0}(a)\|},$$

then by the proof of the second part we know that $z \in \operatorname{res}_{\mathscr{A}}(a)$ as well. We have

$$z - a = (z_0 - a)(\mathbb{1} - \operatorname{Res}_{z_0}(a)(z_0 - z)),$$

and hence by the geometric series argument the second factor is invertible, too, leading to

$$\begin{aligned} \operatorname{Res}_z(a) &= (\mathbb{1} - \operatorname{Res}_{z_0}(a)(z_0 - z))^{-1} \operatorname{Res}_{z_0}(a) \\ &= \left(\sum_{n=0}^{\infty} \operatorname{Res}_{z_0}(a)^n (z_0 - z)^n \right) \operatorname{Res}_{z_0}(a) \\ &= \sum_{n=0}^{\infty} \operatorname{Res}_{z_0}(a)^{n+1} (z_0 - z)^n \end{aligned} \tag{$*$}$$

as an absolutely convergent series. Note that we have used the continuity of the product in the last step. This yields a power series expansion around z_0 showing that $\operatorname{Res}_z(a)$ is holomorphic in the open ball $\mathrm{B}_{\frac{1}{\|\operatorname{Res}_{z_0}(a)\|}}(z_0)$ around $z_0 \in \operatorname{res}_{\mathscr{A}}(a)$. Since z_0 was arbitrary the third part follows. The fourth part is a straightforward algebraic computation, see Exercise 4.5.31. From this the fifth part is immediate. Now let $z \in \mathbb{C}$ with $|z| > \|a\|$ be given. Then $\mathbb{1} - \frac{1}{z}a$ is invertible by the first part. Thereby, also $z - a$ is invertible and $z \in \operatorname{res}_{\mathscr{A}}(a)$ follows. Thus $\operatorname{spec}_{\mathscr{A}}(a)$ has to be in $\mathrm{B}_{\|a\|}(0)^{\mathrm{cl}} \subseteq \mathbb{C}$. Since the resolvent set is open, its complement is closed and by the boundedness, $\operatorname{spec}_{\mathscr{A}}(a)$ is compact. The last part is now the really interesting one. We assume that the spectrum is empty. Then $\operatorname{Res}_z(a)$ is holomorphic on $\operatorname{res}_{\mathscr{A}}(a) = \mathbb{C}$. By the argument in the proof of the third part we know that for $z_0 \in \mathbb{C}$ we have for all $z \in \mathrm{B}_{\frac{1}{\|\operatorname{Res}_{z_0}(a)\|}}(z_0)$ a convergent Taylor expansion ($*$). We apply now a continuous linear functional $\varphi \in \mathscr{A}'$ to ($*$) which gives a function $f_\varphi(z) = \varphi(\operatorname{Res}_z(a))$ being entire holomorphic since the resolvent is entire. For z close to z_0 we have the Taylor expansion

$$f_\varphi(z) = \sum_{n=0}^{\infty} \varphi\big(\operatorname{Res}_{z_0}(a)^{n+1}\big)(z_0 - z)^n,$$

by linearity and continuity of φ. Moreover, consider now $|z| > 2\|a\|$. Then by the geometric series argument we compute

$$f_\varphi(z) = \varphi\left(\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \right) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{z^n}, \tag{$**$}$$

using the linearity and continuity of φ again. From this we get the estimate

$$|f_\varphi(z)| \le \frac{1}{|z|} \sum_{n=0}^{\infty} \frac{|\varphi(a^n)|}{|z|^n} \le \frac{1}{|z|} \sum_{n=0}^{\infty} \frac{\|\varphi\| \|a^n\|}{|z|^n} \le \frac{1}{|z|} \sum_{n=0}^{\infty} \frac{\|\varphi\| \|a\|^n}{|z|^n} \le \frac{\|\varphi\|}{\|a\|}$$

for $|z| \ge 2\|a\|$. Since for $|z| \le 2\|a\|$ the holomorphic function f_φ is bounded anyway, we have a bounded holomorphic function on the whole complex plane. By Liouville's Theorem the function has to be constant. Thus the Taylor coefficients in ($**$) have to vanish for all $n \ge 1$ by the uniqueness of the Taylor expansion. In particular, for every z_0 we have $\varphi(\operatorname{Res}_{z_0}(a)^2) = 0$. Since $\varphi \in \mathscr{A}'$ was arbitrary we conclude by the Hahn-Banach Theorem in form of Corollary 2.2.21 that $\operatorname{Res}_{z_0}(a)^2 = 0$. But $\operatorname{Res}_{z_0}(a)^2$ is invertible with inverse $(z_0 - a)^2$, a contradiction. Thus the spectrum is non-empty. $\square$

From the proof we also note the following corollary which is sometimes useful to handle the resolvent:

Corollary 4.2.15 *Let $\mathscr{A}$ be a unital Banach algebra and let $a \in \mathscr{A}$. If $z_0 \in \operatorname{res}_{\mathscr{A}}(a)$ and $|z - z_0| < \frac{1}{\|\operatorname{Res}_{z_0}(a)\|}$ then $z \in \operatorname{res}_{\mathscr{A}}(a)$ and we have the absolutely convergent Taylor expansion of the resolvent*

$$\operatorname{Res}_z(a) = \sum_{n=0}^{\infty} \operatorname{Res}_{z_0}(a)^{n+1}(z_0 - z)^n. \tag{4.2.25}$$

The notion of spectrum can also be applied to non-unital algebras. Here the definition uses the unitization according to (4.2.12).

Definition 4.2.16 (Spectrum and resolvent, non-unital case) *Let $\mathscr{A}$ be a non-unital associative algebra and $\widetilde{\mathscr{A}}$ its (algebraic) unitization. Then the resolvent set of $a \in \mathscr{A}$ is defined by*

$$\operatorname{res}_{\mathscr{A}}(a) = \operatorname{res}_{\widetilde{\mathscr{A}}}(a) \tag{4.2.26}$$

and the spectrum is defined by

$$\operatorname{spec}_{\mathscr{A}}(a) = \operatorname{spec}_{\widetilde{\mathscr{A}}}(a). \tag{4.2.27}$$

Corollary 4.2.17 *Let $\mathscr{A}$ be a non-unital algebra and $a \in \mathscr{A}$. Then $0 \in \operatorname{spec}_{\mathscr{A}}(a)$.*

PROOF: Since $\mathscr{A} \subseteq \widetilde{\mathscr{A}}$ is a proper two-sided ideal, no element $a \in \mathscr{A}$ has an inverse in $\widetilde{\mathscr{A}}$. ☐

In the following we will always extend definitions and results for unital algebras to the non-unital case by passing to the unitization if necessary. In particular, for the next definition it is possible to treat the non-unital case as well:

Definition 4.2.18 (Spectral radius) *Let $\mathscr{A}$ be a Banach algebra and let $a \in \mathscr{A}$. Then the spectral radius $\varrho_{\mathscr{A}}(a)$ of a is defined by*

$$\varrho_{\mathscr{A}}(a) = \sup\bigl\{|\lambda| \bigm| \lambda \in \operatorname{spec}_{\mathscr{A}}(a)\bigr\}. \tag{4.2.28}$$

With other words, $\varrho_{\mathscr{A}}(a)$ is the radius of the smallest closed ball around $0 \in \mathbb{C}$ containing the spectrum of a. Note that (4.2.28) is well-defined since we have shown that the spectrum is non-empty. The next theorem shows that the spectral radius as well as the spectrum behave well with respect to algebraic manipulations, see also the Exercises 4.5.25 and 4.5.27:

Theorem 4.2.19 (Spectrum and spectral radius) *Let $\mathscr{A}$ be a unital Banach algebra and let $a, b \in \mathscr{A}$. Then one has:*

i.) For the spectrum of the product ab one has

$$\operatorname{spec}_{\mathscr{A}}(ab) \cup \{0\} = \operatorname{spec}_{\mathscr{A}}(ba) \cup \{0\}, \tag{4.2.29}$$

and hence

$$\varrho_{\mathscr{A}}(ab) = \varrho_{\mathscr{A}}(ba). \tag{4.2.30}$$

ii.) For any polynomial $p \in \mathbb{C}[x]$ we have the polynomial spectral mapping theorem

$$\operatorname{spec}_{\mathscr{A}}(p(a)) = \bigl\{p(\lambda) \bigm| \lambda \in \operatorname{spec}_{\mathscr{A}}(a)\bigr\} = p(\operatorname{spec}_{\mathscr{A}}(a)). \tag{4.2.31}$$

iii.) If a is invertible then

$$\operatorname{spec}_{\mathscr{A}}\left(a^{-1}\right) = \bigl\{\lambda^{-1} \bigm| \lambda \in \operatorname{spec}_{\mathscr{A}}(a)\bigr\} = \operatorname{spec}_{\mathscr{A}}(a)^{-1}. \tag{4.2.32}$$

iv.) For the spectral radius one has

$$\varrho_{\mathscr{A}}(a) = \lim_{n \longrightarrow \infty} \|a^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} \leq \|a\|. \tag{4.2.33}$$

v.) There exists a $\lambda \in \operatorname{spec}_{\mathscr{A}}(a)$ with

$$\varrho_{\mathscr{A}}(a) = |\lambda|. \tag{4.2.34}$$

vi.) If $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a unital homomorphism into another unital algebra then for all $a \in \mathscr{A}$ one has

$$\operatorname{spec}_{\mathscr{B}}(\Phi(a)) \subseteq \operatorname{spec}_{\mathscr{A}}(a). \tag{4.2.35}$$

PROOF: We consider $z \in \operatorname{res}_{\mathscr{A}}(ab)$ with $z \neq 0$. Hence $z - ab$ is invertible. Then a simple algebraic computation shows that $\frac{1}{z}(\mathbb{1} + b\frac{1}{z-ab}a)$ is a left and a right inverse of $z - ba$. Hence $z \in \operatorname{res}_{\mathscr{A}}(ba)$, too. By symmetry we get also the reverse inclusion and hence $\operatorname{res}_{\mathscr{A}}(ab) \setminus \{0\} = \operatorname{res}_{\mathscr{A}}(ba) \setminus \{0\}$ which is (4.2.29), see also Exercise 4.5.27. But then (4.2.30) is clear. For the second part let $p(x) = \alpha_n x^n + \cdots + \alpha_1 x + \alpha_0$ with $\alpha_n \neq 0$ be a polynomial of degree n and fix $z \in \mathbb{C}$. Then we have

$$z - p(x) = \alpha_n(x - \lambda_1) \cdots (x - \lambda_n)$$

for some appropriate $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ by the Fundamental Theorem of Algebra. Of course, the $\lambda_1, \ldots, \lambda_n$ depend on z and need not to be pairwise different. For $a \in \mathscr{A}$ we get

$$z - p(a) = \alpha_n(a - \lambda_1) \cdots (a - \lambda_n).$$

We claim that $z - p(a)$ is invertible iff all the factors $a - \lambda_i$ are invertible: clearly, if $a - \lambda_i$ is invertible for all $i = 1, \ldots, n$ then their product is invertible as well. Conversely, if $z - p(a)$ is invertible then it is easy to see that $(z - p(a))^{-1}\alpha_n(a - \lambda_1) \cdots (a - \lambda_{n-1})$ is an inverse of $a - \lambda_n$ since all the factors *commute*. Since by commutativity we can arrange any order, the same argument holds for every $a - \lambda_i$. Thus we have $z \in \operatorname{res}_{\mathscr{A}}(p(a))$ iff $\lambda_i \in \operatorname{res}_{\mathscr{A}}(a)$ for all $i = 1, \ldots, n$. Since $p(\lambda_i) = z$ for all $i = 1, \ldots, n$ we obtain $\operatorname{res}_{\mathscr{A}}(p(a)) = p(\operatorname{res}_{\mathscr{A}}(a))$ and hence (4.2.31). For the third part, let a be invertible which means $0 \notin \operatorname{spec}_{\mathscr{A}}(a)$. For $z \neq 0$ we have $z^{-1} - a^{-1} = (za)^{-1}(a - z)$. This shows that $z^{-1} - a^{-1}$ is invertible iff $a - z$ is invertible which gives (4.2.32). For the fourth part consider the sequence $\|a^n\|^{\frac{1}{n}} \geq 0$ and denote by

$$\alpha = \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}}$$

its infimum. For a given $\epsilon > 0$ we choose $N \in \mathbb{N}$ such that $\left\|a^N\right\|^{\frac{1}{N}} \leq \alpha + \epsilon$. Now let $n = Nk + r$ with $0 \leq r \leq N - 1$ then

$$\|a^n\|^{\frac{1}{n}} = \left\|a^{Nk+r}\right\|^{\frac{1}{n}} \leq \left(\left\|a^N\right\|^k \|a^r\|\right)^{\frac{1}{n}} \leq \left((\alpha + \epsilon)^{Nk} \|a^r\|\right)^{\frac{1}{n}} \leq (\alpha + \epsilon)^{1 - \frac{r}{n}} \left(\max_{r=0}^{N-1} \|a^r\|\right)^{\frac{1}{n}}.$$

Since for $r = 0$ we get $\|a^0\| = 1$ the maximum is at least 1. Let now $n \longrightarrow \infty$ then the range of the possible r is still finite and given by $0, \ldots, N - 1$. Thus the value of the maximum does not change but its n-th root converges to 1. Also the first factor converges to $\alpha + \epsilon$. Hence can conclude the estimate

$$\|a^n\|^{\frac{1}{n}} \leq \alpha + \epsilon'$$

for every given $\epsilon' > 0$ and large enough n. Since on the other hand $\alpha \leq \|a^n\|^{\frac{1}{n}}$ for all n by definition of the infimum, we get the convergence

$$\lim_{n \longrightarrow \infty} \|a^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}}.$$

We have to show now that this limit coincides with the spectral radius. Let $z \in \mathbb{C}$ with $|z| > \lim_{n \longrightarrow \infty} \|a^n\|^{\frac{1}{n}}$ be given. We claim that the series $\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$ converges absolutely. Indeed, we have for the norms

$$\limsup_{n \in \mathbb{N}} \left\| \left(\frac{a}{z}\right)^n \right\|^{\frac{1}{n}} = \lim_{n \longrightarrow \infty} \left\| \left(\frac{a}{z}\right)^n \right\|^{\frac{1}{n}} = \frac{1}{|z|} \lim_{n \longrightarrow \infty} \|a^n\|^{\frac{1}{n}} < 1, \tag{$*$}$$

proving that the series is a Cauchy series and hence convergent (even absolutely) by the usual root test. But the series is the geometric series yielding the inverse of $z - a$. Hence $z \in \operatorname{res}_{\mathscr{A}}(a)$ and $\varrho_{\mathscr{A}}(a) \leq \lim_{n \longrightarrow \infty} \|a^n\|^{\frac{1}{n}}$ follows. To show the reverse estimate we consider again $z \in \mathbb{C}$ with $|z| > \lim_{n \longrightarrow \infty} \|a^n\|^{\frac{1}{n}}$. In particular, $z \in \operatorname{res}_{\mathscr{A}}(a)$ as we have just seen. Hence for a continuous linear functional $\varphi \in \mathscr{A}'$ we get the convergent series

$$f_\varphi(z) = \varphi(\operatorname{Res}_z(a)) = \varphi\left(\frac{1}{z} \frac{1}{1 - \frac{a}{z}}\right) = \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{z^{n+1}} \tag{$**$}$$

by the convergence result from $(*)$ and the continuity of φ. On the other hand, we know that $z \mapsto \operatorname{Res}_z(a)$ is holomorphic. Hence, by the continuity of φ the function f_φ is a holomorphic scalar function on $\operatorname{res}_{\mathscr{A}}(a)$. Thus we have found a convergent Laurent expansion of the holomorphic function f_φ in $(**)$ valid for $|z| > \lim_{n \longrightarrow \infty} \|a^n\|^{\frac{1}{n}}$. From complex analysis one knows that the Laurent expansion of a holomorphic function is *unique* and convergent on the *largest* open ring around 0 which is still contained in the domain where the function is holomorphic. Since f_φ is holomorphic on $\operatorname{res}_{\mathscr{A}}(a)$ we see that $(**)$ has to hold also for all z with $|z| > \varrho_{\mathscr{A}}(a)$ and not just for those z with $|z| > \lim_{n \longrightarrow \infty} \|a^n\|^{\frac{1}{n}}$. In particular, the convergence of $(**)$ implies that for $|z| > \varrho_{\mathscr{A}}(a)$ we have

$$\lim_{n \longrightarrow \infty} \frac{\varphi(a^n)}{z^{n+1}} = 0.$$

Since φ was arbitrary this means that $\frac{a^n}{z^{n+1}}$ is a weak zero sequence. By the Banach-Steinhaus argument from Proposition 2.3.31, *iii.)*, we conclude that the sequence $\frac{a^n}{z^{n+1}}$ is bounded in norm, i.e. we have $\|\frac{a^n}{z^{n+1}}\| \leq K$ for some $K > 0$. But then

$$\|a^n\|^{\frac{1}{n}} \leq K^{\frac{1}{n}} |z|^{\frac{n+1}{n}} \longrightarrow |z|$$

for $n \longrightarrow \infty$. This shows that $\lim_{n \longrightarrow \infty} \|a^n\|^{\frac{1}{n}} \leq |z|$ for all z with $|z| > \varrho_{\mathscr{A}}(a)$. But this implies $\lim_{n \longrightarrow \infty} \|a^n\|^{\frac{1}{n}} \leq \varrho_{\mathscr{A}}(a)$ and hence (4.2.33) is shown at last. The next part is clear since the continuous function $z \mapsto |z|$ assumes maximum and minimum on the compact subset $\operatorname{spec}_{\mathscr{A}}(a)$. The last part is in fact again purely algebraic and does not require Banach algebras and continuity at all: it is discussed in Exercise 4.5.26. □

In the case of a Banach *-algebra we have also a good compatibility of the spectrum with the *-involution.

Proposition 4.2.20 *Let $\mathscr{A}$ be a Banach *-algebra and $a \in \mathscr{A}$. Then*

$$\operatorname{spec}_{\mathscr{A}}(a^*) = \left\{ \overline{\lambda} \;\middle|\; \lambda \in \operatorname{spec}_{\mathscr{A}}(a) \right\} = \overline{\operatorname{spec}_{\mathscr{A}}(a)}, \tag{4.2.36}$$

and hence

$$\varrho_{\mathscr{A}}(a^*) = \varrho_{\mathscr{A}}(a). \tag{4.2.37}$$

PROOF: First we may adjoin a unit if necessary. Then $z - a$ is invertible with inverse given by $\operatorname{Res}_z(a)$ iff $\overline{z} - a^*$ is invertible with inverse $(\operatorname{Res}_z(a))^*$. Clearly, (4.2.37) is a direct consequence of (4.2.36). □

Note that for a Hermitian element $a = a^*$ in a Banach *-algebra this only means that the spectrum as a set is invariant under reflection with respect to the real axis. In may well happen that the spectral values themselves are e.g. purely imaginary, see Exercise 4.5.28.

We come now to the main result of this subsection, the holomorphic calculus. Since a Banach algebra $\mathscr{A}$ is a particular case of a complete lmc algebra we have an entire calculus for $\mathscr{A}$ according to Theorem 4.1.36. Since we have now only a single norm instead of a system of seminorms we can use the value $\|a\|$ to sharpen the result of Theorem 4.1.36 in the sense that we can allow more general holomorphic functions for the calculus than just the entire ones. In fact, not $\|a\|$ will be relevant but the spectral radius $\varrho_{\mathscr{A}}(a)$. Then we will get a holomorphic calculus for functions from $\mathscr{O}(\mathrm{B}_\varrho(0))$ equipped with its Fréchet structure as in Example 4.1.27, provided $\varrho > \varrho_{\mathscr{A}}(a)$. More precisely, we have the following statement:

Theorem 4.2.21 (Holomorphic calculus, I) *Let $\mathscr{A}$ be a unital Banach algebra and $a \in \mathscr{A}$. Let $\varrho > \varrho_{\mathscr{A}}(a)$ be fixed.*

i.) For every $f \in \mathscr{O}(\mathrm{B}_\varrho(0))$ the series

$$f(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f}{\partial z^n}(0)\, a^n \tag{4.2.38}$$

converges absolutely in $\mathscr{A}$.

ii.) The map

$$\mathscr{O}(\mathrm{B}_\varrho(0)) \ni f \;\mapsto\; f(a) \in \mathscr{A} \tag{4.2.39}$$

is a unital algebra homomorphism extending the polynomial and the entire holomorphic calculus.

iii.) The algebra homomorphism (4.2.39) is continuous. More precisely, for every $\epsilon > 0$ with $\varrho_{\mathscr{A}}(a)+\epsilon < \varrho$ one finds a $c > 0$ such that for all $f \in \mathscr{O}(\mathrm{B}_\varrho(0))$ one has

$$\|f(a)\| \le c\mathrm{p}_{\varrho_{\mathscr{A}}(a)+\epsilon}(f). \tag{4.2.40}$$

iv.) If $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a continuous unital algebra homomorphism into another unital Banach algebra $\mathscr{B}$ then for all $f \in \mathscr{O}(\mathrm{B}_\varrho(0))$ one has

$$\Phi(f(a)) = f(\Phi(a)). \tag{4.2.41}$$

Proof: First we estimate the convergence of the series $\sum_{n=0}^{\infty} \frac{1}{n!}\left|\frac{\partial^n f}{\partial z^n}(0)\right|\|a^n\|$. By assumption, we know that

$$\lim_{n\longrightarrow\infty} \|a^n\|^{\frac{1}{n}} = \varrho_{\mathscr{A}}(a) < \varrho \tag{$*$}$$

by Theorem 4.2.19, *iv.)*. We fix $\epsilon > 0$ such that we still have $\varrho_{\mathscr{A}}(a) + \epsilon < \varrho$. Then we have some $N \in \mathbb{N}$ with the property

$$\|a^n\|^{\frac{1}{n}} \le \varrho_{\mathscr{A}}(a) + \epsilon$$

for all $n \ge N$ by the convergence $(*)$. Thus, we get the estimate

$$\sum_{n=N}^{\infty} \frac{1}{n!}\left|\frac{\partial^n f}{\partial z^n}(0)\right|\|a^n\| \le \sum_{n=N}^{\infty} \frac{1}{n!}\left|\frac{\partial^n f}{\partial z^n}(0)\right|(\varrho_{\mathscr{A}}(a)+\epsilon)^n \le \mathrm{p}_{\varrho_{\mathscr{A}}(a)+\epsilon}(f)$$

by the very definition of the seminorms $\mathrm{p}_{\varrho_{\mathscr{A}}(a)+\epsilon}$ as in (4.1.37). Since $f \in \mathscr{O}(\mathrm{B}_\varrho(0))$ with $\varrho_{\mathscr{A}}(a)+\epsilon < \varrho$ it follows that $\mathrm{p}_{\varrho_{\mathscr{A}}(a)+\epsilon}(f)$ is finite and hence the series converges absolutely. To get (4.2.40) we have to estimate the remaining finitely many terms. Let

$$\alpha = \max_{n=0}^{N-1} \|a^n\|^{\frac{1}{n}},$$

which gives $\|a^n\| \leq \alpha^n$ for all $n = 0, \ldots, N-1$. Define now

$$\beta = \max\left\{\frac{\alpha}{\varrho_{\mathscr{A}}(a) + \epsilon}, 1\right\}.$$

Then we get for all $n = 0, \ldots, N-1$ the estimate $\alpha^n \leq \beta^{N-1}(\varrho_{\mathscr{A}}(a) + \epsilon)^n$ and thus

$$\sum_{n=0}^{N-1} \frac{1}{n!}\left|\frac{\partial^n f}{\partial z^n}(0)\right| \|a^n\| \leq \beta^{N-1} \sum_{n=0}^{N-1} \frac{1}{n!}\left|\frac{\partial^n f}{\partial z^n}(0)\right| (\varrho_{\mathscr{A}}(a) + \epsilon)^n.$$

Combining the two estimates we can take $c = \beta^{N-1}$ to get (4.2.40). Then the continuity is clear. The homomorphism property can now safely be checked on polynomials (where it is trivial) and then extended by continuity as usual using the fact that $\mathbb{C}[z] \subseteq \mathscr{O}(\mathrm{B}_\varrho(0))$ is dense. Clearly, (4.2.39) extends the entire calculus. For the last part, recall that $\mathrm{spec}_{\mathscr{B}}(\Phi(a)) \subseteq \mathrm{spec}_{\mathscr{A}}(a)$. Hence the right hand side is well-defined since the open ball $\mathrm{B}_\varrho(0)$ still contains the spectrum of $\Phi(a)$. Then the proof of the identity (4.2.41) is literally the same as for (4.1.54). □

Remark 4.2.22 Let $\mathscr{A}$ be a unital Banach algebra and let $a \in \mathscr{A}$.

i.) The holomorphic calculus allows now to speak of functions like e.g. $\frac{1}{1-a}$ or $\sqrt{1+a}$ whenever $\varrho_{\mathscr{A}}(a) < 1$. Moreover, we do not just get a definition of these algebra elements but a reasonable way to approximate them: the Taylor expansion of the corresponding holomorphic function converges as good as possible, namely absolutely.

ii.) There is also a holomorphic calculus for non-unital Banach algebras. For a function $f \in \mathscr{O}(\mathrm{B}_\varrho(0))$ with $\varrho > \varrho_{\mathscr{A}}(a)$ and the *additional* requirement that $f(0) = 0$ one can show that the holomorphic calculus of the unitization $\widetilde{\mathscr{A}}$ actually yields an element $f(a)$ of $\mathscr{A}$ for $a \in \mathscr{A}$. Indeed, we have

$$f(a) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n f}{\partial z^n}(0)\, a^n \tag{4.2.42}$$

as an absolutely convergent series in $\widetilde{\mathscr{A}}$ which consists of terms in $\mathscr{A}$ thanks to $f(0) = 0$. Since $\mathscr{A} \subseteq \widetilde{\mathscr{A}}$ is a *closed* ideal we get $f(a) \in \mathscr{A}$. Thus for the vanishing ideal at 0 in $\mathscr{O}(\mathrm{B}_\varrho(0))$ we get a holomorphic calculus which takes place in $\mathscr{A}$ only. Also here the calculus is natural for homomorphisms.

In a second step we want to extend the holomorphic calculus even further. Up to now we can only use holomorphic functions which are holomorphic on an open disk containing the spectrum of the algebra element. The idea is now that *any* open subset containing the spectrum should be sufficient, not only the disks. To see this, we have to use some tools from complex analysis for vector-valued functions as outlined in Appendix B.6. In particular, the Cauchy formula for curve integrals will be used to establish the holomorphic calculus we are interested in. The key is now the invariance of the curve integrals under suitable deformations of the curve into a homologous curve.

Lemma 4.2.23 *Let $\mathscr{A}$ be a unital Banach algebra and let $a \in \mathscr{A}$. Suppose $\gamma \subseteq \mathbb{C}$ is a simply closed curve such that $\mathrm{spec}_{\mathscr{A}}(a)$ is in the interior of γ. Then for all $n \in \mathbb{N}_0$ we have*

$$a^n = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{z^n}{z-a}\, \mathrm{d}z. \tag{4.2.43}$$

PROOF: First we can deform the curve γ into a homologous curve given by the boundary of a large enough disk $\mathrm{B}_\varrho(0)$ around $0 \in \mathbb{C}$, where $\varrho > 0$ is large such that $\mathrm{spec}_{\mathscr{A}}(a) \subseteq \mathrm{B}_\varrho(0)$. To be sure, we can chose $\varrho > \|a\|$. By Corollary **??**, **??**, the integral does not change its value. Note that under the assumption on the curve the function $z \mapsto \frac{z^n}{z-a}$ is indeed holomorphic on $\mathbb{C} \setminus \operatorname{spec}(a)$ by Theorem 4.2.14, *iii.)*, and hence holomorphic on an open subset containing the curve. If $z \in \partial \mathrm{B}_\varrho(0)$ then $|z| = \varrho$ is large enough to make the geometric series

$$\frac{1}{z-a} = \frac{1}{z}\frac{1}{1-\frac{a}{z}} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k \tag{$*$}$$

converge absolutely in $\mathscr{A}$ since $\|\frac{a}{z}\| = \frac{1}{\varrho}\|a\|$ is less than one by our choice of ϱ. Moreover, the needed estimate holds for all $z \in \partial \mathrm{B}_\varrho(0)$ and hence the convergence of $(*)$ is uniform on the curve $\partial \mathrm{B}_\varrho(0)$. In fact, it is even (locally) uniform on $\mathbb{C} \setminus \mathrm{B}_{\varrho-\epsilon}(0)$ where $\epsilon > 0$ is chosen such that $\|a\| < \varrho - \epsilon$. By the continuity of the curve integral as in Corollary **??** we see that we can exchange the limit $(*)$ in $\mathscr{C}(\mathbb{C} \setminus \mathrm{B}_{\varrho-\epsilon}(0), \mathscr{A})$ with the integral. This gives

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{z^n}{z-a}\,\mathrm{d}z &= \frac{1}{2\pi \mathrm{i}}\int_{\partial \mathrm{B}_\varrho(0)} \frac{z^n}{z-a}\,\mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}}\int_{\partial \mathrm{B}_\varrho(0)} z^{n-1}\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k \mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}}\sum_{k=0}^{\infty} a^k \int_{\partial \mathrm{B}_\varrho(0)} z^{n-k-1}\,\mathrm{d}z \\
&= a^n,
\end{aligned}$$

where in the last step we have used the well-known integral over powers of z: only z^{-1} gives a non-trivial contribution equal to $2\pi\mathrm{i}$. Hence the lemma is shown. ☐

Lemma 4.2.24 *Let $\mathscr{A}$ be a unital Banach algebra and let $a \in \mathscr{A}$. If $\gamma \subseteq \mathbb{C}$ is a simply closed curve with $\operatorname{spec}_{\mathscr{A}}(a)$ in its interior then the polynomial calculus with respect to a can be written as*

$$p(a) = \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{p(z)}{z-a}\,\mathrm{d}z \tag{4.2.44}$$

for all $p \in \mathbb{C}[z]$.

Proof: With the linearity of the integral this follows immediately from the previous lemma. ☐

The idea is now to use this Cauchy formula to actually *define* an algebra element $f(a)$ for holomorphic functions beyond polynomials.

Theorem 4.2.25 (Holomorphic calculus II) *Let $\mathscr{A}$ be a unital Banach algebra and let $a \in \mathscr{A}$. Let $X \subseteq \mathbb{C}$ be an open subset containing $\operatorname{spec}_{\mathscr{A}}(a)$. If γ is a simply closed curve in X with $\operatorname{spec}_{\mathscr{A}}(a)$ in the interior of γ then*

$$\mathscr{O}(X) \ni f \;\mapsto\; f(a) = \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}z \in \mathscr{A} \tag{4.2.45}$$

yields a continuous unital algebra homomorphism extending the polynomial calculus and the previous holomorphic calculus of a. The integral does not depend on the choice of γ.

Proof: First of all, the integral is well-defined since $z \mapsto f(z)(z-a)^{-1}$ is at least continuous along the curve γ. Moreover, if we deform the curve into another curve with the same properties then the value of the integral does not change according to Corollary **??**,

iv. Lemma 4.2.24 shows that the integral extends the polynomial calculus. We prove now that (4.2.45) is a unital algebra homomorphism. Clearly, it is linear and unital. So let $f, g \in \mathscr{O}(X)$ be given. Then we choose two simply closed curves γ and $\tilde{\gamma}$ such that γ is entirely inside the interior of $\tilde{\gamma}$. Thanks to the openness of X this is possible. We compute

$$
\begin{aligned}
f(a)g(a) &= \frac{1}{(2\pi\mathrm{i})^2}\left(\int_\gamma \frac{f(z)}{(z-a)}\,\mathrm{d}z\right)\left(\int_{\tilde{\gamma}} \frac{g(w)}{w-a}\,\mathrm{d}w\right) \\
&\overset{(a)}{=} \frac{1}{(2\pi\mathrm{i})^2}\int_\gamma\int_{\tilde{\gamma}} \frac{f(z)g(w)}{(z-a)(w-a)}\,\mathrm{d}z\,\mathrm{d}w \\
&\overset{(b)}{=} \frac{1}{(2\pi\mathrm{i})^2}\int_\gamma\int_{\tilde{\gamma}} f(z)g(w)\frac{1}{w-z}\left(\frac{1}{z-a}-\frac{1}{w-a}\right)\mathrm{d}z\,\mathrm{d}w \\
&= \frac{1}{(2\pi\mathrm{i})^2}\int_\gamma f(z)\frac{1}{z-a}\int_{\tilde{\gamma}}\frac{g(w)}{w-z}\,\mathrm{d}w\,\mathrm{d}z - \frac{1}{(2\pi\mathrm{i})^2}\int_{\tilde{\gamma}}\frac{g(w)}{w-a}\left(\int_\gamma \frac{f(z)}{w-z}\,\mathrm{d}z\right)\mathrm{d}w \\
&\overset{(c)}{=} \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f(z)g(z)}{z-a}\,\mathrm{d}z \\
&= (fg)(a),
\end{aligned}
$$

where in (a) we used the Fubini's Theorem to put everything under a double integration, in (b) we used essentially the resolvent identity, and in (c) we used the fact that in the first contribution the integral over $\tilde{\gamma}$ gives $g(z)$ according to the Cauchy formula since z is inside of $\tilde{\gamma}$ and in the second term the integration along γ results in 0 since w is outside of γ by assumption. This proves that the Cauchy integration in (4.2.45) is a homomorphism as claimed. The continuity of this holomorphic calculus is now a standard estimate for the integrals according to Corollary **??**: we have

$$
\|f(a)\| \le \frac{1}{2\pi}\,\mathrm{L}(\gamma)\max_{z\in\gamma}\left\|f(z)(z-a)^{-1}\right\| \le \frac{1}{2\pi}\,\mathrm{L}(\gamma)\max_{z\in\gamma}\left\|(z-a)^{-1}\right\|\|f\|_\gamma,
$$

where $\mathrm{L}(\gamma)$ denotes the length of γ and $\|(z-a)^{-1}\|$ is bounded on the compact subset $\gamma \subseteq \operatorname{res}_{\mathscr{A}}(a)$ since $z \mapsto (z-a)^{-1}$ is holomorphic on the resolvent set and thus continuous. This gives an estimate of $\|f(a)\|$ by a seminorm of the $\mathscr{O}$-topology of $\mathscr{O}(X)$ as wanted. Finally, if f is holomorphic on an open ball $\mathrm{B}_\varrho(0)$ with $\varrho > \varrho_{\mathscr{A}}(a)$, i.e. containing the spectrum, then we can exchange the Taylor expansion of f around $z = 0$ with the above integral thanks to the proven continuity properties. This reduces the computation of $f(a)$ according to (4.2.45) to the polynomial case of Lemma 4.2.23. Hence for such holomorphic functions, the two calculi coincide. □

Remark 4.2.26 If $X \subseteq \mathbb{C}$ happens to be a Runge domain, i.e. the polynomials $\mathbb{C}[z] \subseteq \mathscr{O}(X)$ are dense, see e.g. the discussion in [48, Thm. 13.11], then the above proof follows more directly from Lemma 4.2.23 and the continuity of the integral: we simply conclude all algebraic properties of $f \mapsto f(a)$ for general f from those for polynomials by approximation and continuity.

We conclude this section with the following version of the spectral mapping theorem adapted to the holomorphic calculus:

Theorem 4.2.27 (Spectral Mapping Theorem) *Let $\mathscr{A}$ be a unital Banach algebra and let $a \in \mathscr{A}$. Let f be holomorphic on an open neighbourhood of $\operatorname{spec}_{\mathscr{A}}(a)$. Then*

$$
\operatorname{spec}_{\mathscr{A}}(f(a)) = f(\operatorname{spec}_{\mathscr{A}}(a)). \tag{4.2.46}
$$

Proof: Let $f \in \mathscr{O}(X)$ with $\operatorname{spec}_{\mathscr{A}}(a) \subseteq X$ and $X \subseteq \mathbb{C}$ open. Then $f(a) \in \mathscr{A}$ is defined by means of the holomorphic calculus as in Theorem 4.2.25. First, let $\lambda \in \operatorname{spec}_{\mathscr{A}}(a)$. Then the left or right ideal generated by $\lambda - a$ has to be a proper ideal: if both ideals would be equal to $\mathscr{A}$ then there would be elements $b, b' \in \mathscr{A}$ with $b(\lambda - a) = \mathbb{1} = (\lambda - a)b'$ and hence $b = b' = (\lambda - a)^{-1}$ would be the inverse of $\lambda - a$, contradicting $\lambda \in \operatorname{spec}_{\mathscr{A}}(a)$. Thus, without restriction, we assume that $(\lambda - a)\mathscr{A} \neq \mathscr{A}$. Now we fix a simply closed curve $\gamma \subseteq X$ with $\operatorname{spec}_{\mathscr{A}}$ in its interior. By the holomorphic calculus (4.2.45) and the scalar Cauchy formula we get

$$\begin{aligned} f(\lambda)\mathbb{1} - f(a) &= \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{z - \lambda} \, \mathrm{d}z \mathbb{1} - \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{z - a} \, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_\gamma f(z) \left(\frac{1}{z - \lambda} - \frac{1}{z - a} \right) \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_\gamma (\lambda - a) \frac{f(z)}{(z - \lambda)(z - a)} \, \mathrm{d}z \\ &= (\lambda - a) \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{(z - \lambda)(z - a)} \, \mathrm{d}z, \end{aligned}$$

since the integral in the last line exists by the continuity of $z \mapsto \frac{f(z)}{(z-\lambda)(z-a)}$ for $z \in \gamma$ and since the left multiplication with $\lambda - a$ is a continuous endomorphism of $\mathscr{A}$ which can therefore be taken outside the integral. But this shows that $f(\lambda)\mathbb{1} - f(a)$ is contained in the proper right ideal $(\lambda - a)\mathscr{A}$ implying that it can not be invertible. Thus $f(\lambda) \in \operatorname{spec}_{\mathscr{A}}(f(a))$. The converse is simpler: if $\lambda \notin f(\operatorname{spec}_{\mathscr{A}}(a))$ then the function $z \mapsto \lambda - f(z)$ is invertible on $\operatorname{spec}_{\mathscr{A}}(a)$ and by continuity also on an open neighbourhood of $\operatorname{spec}_{\mathscr{A}}(a)$. Hence on this neighbourhood it has a holomorphic inverse $z \mapsto (\lambda - f(z))^{-1}$. On the intersection of the original X with this neighbourhood the holomorphic calculus yields an inverse $(\lambda - f(a))^{-1}$ of $\lambda - f(a)$ as the calculus is a unital homomorphism according to Theorem 4.2.25. Thus $\lambda \in \operatorname{res}_{\mathscr{A}}(f(a))$, showing the opposite inclusion and thus proving the equality in (4.2.46). □

A first application is now that we can iterate the holomorphic calculus in such a way that it becomes compatible with the composition of holomorphic functions. As usual, we have to be a bit careful concerning the domains of definition where the compositions are actually defined:

Corollary 4.2.28 *Let $\mathscr{A}$ be a unital Banach algebra and let $a \in \mathscr{A}$. Let g be a holomorphic function on an open neighbourhood of $\operatorname{spec}_{\mathscr{A}}(a)$ and let f be a holomorphic function on an open neighbourhood of $g(\operatorname{spec}_{\mathscr{A}}(a))$. Then there is an open neighbourhood of $\operatorname{spec}_{\mathscr{A}}(a)$ on which $f \circ g$ is defined and holomorphic and we have*

$$(f \circ g)(a) = f(g(a)). \tag{4.2.47}$$

PROOF: Let $\operatorname{spec}_{\mathscr{A}}(a) \subseteq X \subseteq \mathbb{C}$ with X open and $g \in \mathscr{O}(X)$. Moreover, let $g(\operatorname{spec}_{\mathscr{A}}(a))$, which is still compact, be contained in an open subset $Y \subseteq \mathbb{C}$ such that $f \in \mathscr{O}(Y)$. Then the pre-image $g^{-1}(Y)$ is open by continuity of g and $\operatorname{spec}_{\mathscr{A}}(a) \subseteq g^{-1}(Y)$. Hence $Z = X \cap g^{-1}(Y)$ is again an open neighbourhood of $\operatorname{spec}_{\mathscr{A}}(a)$. The composition $f \circ g$ is defined and holomorphic on Z, settling the issue of domains. From the Spectral Mapping Theorem 5.4.11 we known that the spectrum of $g(a)$ is indeed contained in the domain Y where f is holomorphic. So the right hand side of (4.2.47) is defined by means of the holomorphic calculus. We choose now two simply closed curves: first let γ be in Z with $\operatorname{spec}(a)$ in the interior of γ. Next, choose an open subset $U \subset Y$ such that $g^{-1}(U)$ is an open neighbourhood of γ and all the interior points of γ, in particular of $\operatorname{spec}(a)$. Let now $\tilde{\gamma}$ be a simply closed curve in Y around U. In particular, $g(\operatorname{spec}(a))$ and $g \circ \gamma$ are in the interior of $\tilde{\gamma}$. Hence for every fixed $w \in \tilde{\gamma}$, the function $z \mapsto (w - g(z))^{-1}$ is holomorphic on $g^{-1}(U)$, thus defining $(w - g(a))^{-1}$. We compute

$$\begin{aligned} f(g(a)) &= \frac{1}{2\pi \mathrm{i}} \int_{\tilde{\gamma}} \frac{f(w)}{w - g(a)} \, \mathrm{d}w \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\tilde{\gamma}} f(w) \left(\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{w - g(z)} \frac{1}{z - a} \, \mathrm{d}z \right) \mathrm{d}w \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{z-a} \left(\frac{1}{2\pi \mathrm{i}} \int_{\tilde{\gamma}} \frac{f(w)}{w - g(z)} \, \mathrm{d}w \right) \mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(g(z))}{z-a} \, \mathrm{d}z \\
&= (f \circ g)(a),
\end{aligned}
$$

where we again exchanged the orders of integration and used the Cauchy integral formula. □

Remark 4.2.29 Again, if the function f is defined on a Runge domain, it suffices to check the relation (4.2.47) for polynomials f by continuity and density of the polynomials. This simplifies the proof in this case.

Another application is obtained from elements $a \in \mathscr{A}$ having a *disconnected* spectrum: in this case locally constant and hence holomorphic functions can take different values on each connected component leading to the existence of idempotents in $\mathscr{A}$. To illustrate this we only consider a simple case: assume that $\operatorname{spec}(a) = C_1 \cup C_2$ is the union of two non-trivial *disjoint* parts $C_1 \cap C_2 = \emptyset$. Then C_1 and C_2 are both compact themselves and we find by the T_4-property of $\mathbb{C}$ open subsets $X_1, X_2 \subseteq \mathbb{C}$ with $C_i \subseteq X_i$ for $i = 1, 2$ and $X_1 \cap X_2 = \emptyset$. Thus the characteristic functions χ_{X_i} are holomorphic on the union $X = X_1 \cup X_2$ since this is a disjoint union. Both characteristic functions take both values 0 and 1 as we assume that the disjoint parts are non-trivial. Hence by the holomorphic calculus the algebra elements $P_i = \chi_{X_i}(a) \in \mathscr{A}$ are idempotents and we have $\operatorname{spec}_{\mathscr{A}}(P_i) = \{0, 1\}$ by the Spectral Mapping Theorem. It follows that P_1 and P_2 are non-trivial idempotents and we have $\mathbb{1} = P_1 + P_2$ together with $P_1 P_2 = 0 = P_2 P_1$.

4.2.3 Commutative Banach Algebras

In this concluding subsection on Banach algebras we focus on the commutative case: in some sense this corresponds to classical observable algebras like the continuous functions on a classical phase space. Here we have as principal example the complex-valued continuous functions $\mathscr{C}(X)$ on a compact Hausdorff space. The results of Exercise 4.5.17 show that $\mathscr{C}(X)$ is a commutative unital Banach *-algebra. The aim of this subsection is now to show that *any* commutative unital Banach algebra is closely related to this example. Moreover, the commutative case of a C^*-algebra will be of crucial importance for the continuous calculus later on.

We start with the following theorem of Gel'fand and Mazur:

Theorem 4.2.30 (Gel'fand-Mazur) *Let $\mathscr{A}$ be a unital Banach algebra. If every element $a \neq 0$ of $\mathscr{A}$ is invertible then $\mathscr{A} \cong \mathbb{C}$.*

PROOF: Let $a \neq 0$. Since $\operatorname{spec}_{\mathscr{A}}(a) \neq \emptyset$ by Theorem 4.2.14, *vii.)*, we have a spectral value $\lambda \in \mathbb{C}$ different from 0 since a is invertible. Thus $\lambda \mathbb{1} - a$ is not invertible and hence, by assumption, equal to zero. Thus $a = \lambda \mathbb{1}$, establishing the isomorphism $\mathscr{A} \cong \mathbb{C}$. □

For the next lemma we first recall that a *maximal ideal* in an associative algebra $\mathscr{A}$ is a two-sided ideal $\mathscr{J} \subseteq \mathscr{A}$ different from $\mathscr{A}$ such that every other proper ideal containing $\mathscr{J}$ has to coincide with $\mathscr{J}$. Note that $\mathbb{C}$ has one maximal ideal, namely $\{0\}$. We are now interested in the relation between maximal ideals and unital homomorphisms into the complex numbers:

Lemma 4.2.31 *Let $\mathscr{A}$ be a unital Banach algebra and let $\varphi \colon \mathscr{A} \longrightarrow \mathbb{C}$ be a not necessarily unital algebra homomorphism.*

i.) *Either $\varphi = 0$ or $\varphi(\mathbb{1}) = 1$.*

ii.) *The homomorphism φ is necessarily continuous and $\|\varphi\| = 1$ for $\varphi \neq 0$.*

iii.) *The kernel* $\ker\varphi \subseteq \mathscr{A}$ *is a closed ideal of* $\mathscr{A}$*.*

iv.) *If* $\varphi \neq 0$ *then* $\ker\varphi$ *is a maximal ideal.*

PROOF: The first part is clear since for $\varphi \neq 0$ we find by rescaling an element $a \in \mathscr{A}$ with $\varphi(a) = 1$. Then $1 = \varphi(a) = \varphi(a\mathbb{1}) = \varphi(a)\varphi(\mathbb{1}) = \varphi(\mathbb{1})$ gives the first part. For the second, let $\varphi \neq 0$ to avoid trivialities. Since $\varphi(\mathbb{1}) = 1$ we have $\|\varphi\| \in [1, +\infty]$ where $+\infty$ corresponds to a discontinuous φ as usual. Thus assume $1 < \|\varphi\| \leq \infty$. Then there is an element $a \in \mathscr{A}$ with $\|a\| < 1$ but $|\varphi(a)| = 1$. Rescaling by a phase gives $\|a\| < 1$ and $\varphi(a) = 1$. Consider now $b = \sum_{n=0}^{\infty} a^n$ which converges absolutely to the inverse of $\mathbb{1} - a$ since $\|a\| < 1$. Thus $a + ab = b$. This relation gives

$$\varphi(b) = \varphi(a) + \varphi(ab) = 1 + \varphi(a)\varphi(b) = 1 + \varphi(b),$$

which is a contradiction. Thus $\|\varphi\| = 1$ follows and φ is continuous. Then the third part is clear as well since the kernel of a continuous linear functional is closed and the kernel of a algebra homomorphism is a two-sided ideal. For the last part, let $\varphi \neq 0$ and hence $\varphi(\mathbb{1}) = 1$. Thus $\mathbb{1} \notin \ker\varphi$ and hence $\ker\varphi$ is a proper ideal. Assume that $\mathscr{J} \subseteq \mathscr{A}$ is another proper ideal containing $\ker\varphi$. Then assume there is an $a \in \mathscr{J}$ with $\varphi(a) \neq 0$. Since clearly $a - \varphi(a)\mathbb{1} \in \ker\varphi \subseteq \mathscr{J}$ we have $\mathbb{1} = \frac{1}{\varphi(a)}(a - \varphi(a)\mathbb{1} - a) \in \mathscr{J}$, contradicting that $\mathscr{J}$ is a proper ideal. Thus $\ker\varphi$ was already maximal. $\square$

For a general (Banach or not) algebra there might be only very few or even no unital homomorphisms into the scalars $\mathbb{C}$: the non-commutativity will typically prevent this. Thus we can hope for interestingly many homomorphisms into $\mathbb{C}$ only for commutative algebras. Even then, there might be very few such homomorphisms since nilpotency typically spoils this, see Exercise **??**.

For a unital associative algebra $\mathscr{A}$ the unital homomorphisms

$$\varphi\colon \mathscr{A} \longrightarrow \mathbb{C} \tag{4.2.48}$$

are also called *characters*. In the case of a topological algebra one typically requires a character to be continuous in addition. Remarkably, it is a complicated and yet unsolved question of Michael whether on a commutative lmc Fréchet algebra a character is necessarily continuous, see [38]. From this point of view Lemma 4.2.31 tells us that for a *Banach* algebra a character is automatically continuous.

Remark 4.2.32 A closer look at the proof shows that part *i.)* and *iv.)* are true for a character of an arbitrary unital associative algebra, even over an arbitrary field of scalars.

The next lemma provides a sort of reverse statement relating maximal ideals to characters in case of a *commutative* Banach algebra:

Lemma 4.2.33 *Let* $\mathscr{A}$ *be a unital commutative Banach algebra.*

i.) *Every ideal is contained in a maximal ideal.*

ii.) *The closure of an ideal is again an ideal.*

iii.) *The closure of a proper ideal is again a proper ideal.*

iv.) *A maximal ideal is closed.*

v.) *Let* $\mathscr{J} \subseteq \mathscr{A}$ *be a maximal ideal. Then* $\mathscr{A}\big/\mathscr{J}$ *is isomorphic to* $\mathbb{C}$ *as unital Banach algebra.*

vi.) *If* $\mathscr{J} \subseteq \mathscr{A}$ *is a maximal ideal then there is a unique character* $\varphi\colon \mathscr{A} \longrightarrow \mathbb{C}$ *with* $\mathscr{J} = \ker\varphi$*.*

PROOF: The first part is true for every unital associative algebra and originates from Zorn's Lemma: let $\mathscr{J} \subseteq \mathscr{A}$ be a proper ideal and consider the set

$$\mathfrak{J} = \big\{\mathscr{I} \subseteq \mathscr{A} \;\big|\; \mathscr{I} \text{ is a proper ideal containing } \mathscr{J}\big\}.$$

Clearly, $\mathfrak{J}$ is not empty as $\mathscr{J} \in \mathfrak{J}$. Moreover, it is partially ordered by inclusion. If $\{\mathscr{I}_i\}_{i\in I}$ is a linearly ordered subset of $\mathfrak{J}$ then it has a supremum, namely the union $\mathscr{I} = \bigcup_{i\in I} \mathscr{I}_i$. Indeed, this is a well-defined ideal since we have a linearly ordered set of ideals. Then it is clear that $\mathscr{I}$ is the smallest ideal containing all the $\mathscr{I}_i$. Finally, since all ideals $\mathscr{I}_i$ are proper, none contains $\mathbb{1}$. But then also their union $\mathscr{I}$ does not contain $\mathbb{1}$. Hence $\mathscr{I}$ is proper again and we have shown the existence of a supremum. By Zorn's Lemma we conclude that $\mathfrak{J}$ has maximal elements which are clearly the maximal ideals we are looking for. The second part holds for every locally convex algebra, see e.g. our discussion in Remark 4.1.23, *iii.)*, and Exercise 4.5.7. We have just repeated this statement here for convenience. For the third, let $\mathscr{J}$ be a proper ideal and assume that its closure $\mathscr{J}^{\mathrm{cl}}$ is all of $\mathscr{A}$. Then we can find a sequence $a_n \in \mathscr{J}$ with $\lim_{n\longrightarrow\infty} a_n = \mathbb{1}$. In particular, for some large enough n_0 we have $\|a_{n_0} - \mathbb{1}\| < \frac{1}{2}$. By Theorem 4.2.14, *i.)*, we conclude that a_{n_0} is invertible. But an ideal containing an invertible element is already the whole algebra, giving a contradiction. Then also the fourth part is clear. For the fifth part we first note that $\mathscr{A}\big/\mathscr{J}$ is again a unital Banach algebra, see Remark 4.2.4, since $\mathscr{J}$ is closed. We claim that every non-zero element in $\mathscr{A}\big/\mathscr{J}$ invertible. Indeed, assume that $[a] \neq 0$ is not invertible. Then $(\mathscr{A}\big/\mathscr{J})[a]$ is a proper ideal in $\mathscr{A}\big/\mathscr{J}$ which contains $[a]$ as the quotient is still unital. But then the pre-image of this proper ideal in $\mathscr{A}$ is easily shown to be a proper ideal in $\mathscr{A}$ containing $\mathscr{J}$. By the maximality of $\mathscr{J}$ this implies that the pre-image has to coincide with $\mathscr{J}$. But this means that every representative a of $[a]$ is in $\mathscr{J}$ and thus $[a] = 0$, a contradiction. Thus $[a] \neq 0$ is invertible and by the Theorem 4.2.30 of Gel'fand-Mazur this shows $\mathscr{A}\big/\mathscr{J} \cong \mathbb{C}$. Then the last part is clear: we consider the canonical projection map $\varphi\colon \mathscr{A} \longrightarrow \mathscr{A}\big/\mathscr{J} \cong \mathbb{C}$ onto the quotient. In general, this is a unital algebra homomorphism with $\ker\varphi = \mathscr{J}$, and hence a character after identifying $\mathscr{A}\big/\mathscr{J}$ with $\mathbb{C}$. Now assume that φ' is another character with $\ker\varphi' = \mathscr{J}$. Then for all $a \in \mathscr{A}$ one has $a - \varphi(a)\mathbb{1} \in \ker\varphi = \ker\varphi'$ and hence $0 = \varphi'(a) - \varphi(a)\varphi'(\mathbb{1}) = \varphi'(a) - \varphi(a)$. Thus $\varphi = \varphi'$. ☐

Remark 4.2.34 Again, the first part is entirely algebraic and holds for general unital associative algebras. The second part holds for locally convex algebras. The third and fourth part do not yet require commutativity.

In conclusion, we see that for a unital commutative Banach algebra there is a one-to-one correspondence between characters and maximal ideals. Since characters $\varphi\colon \mathscr{A} \longrightarrow \mathbb{C}$ satisfy in addition $\|\varphi\| = 1$ we can view them as particular elements of the unit sphere in the dual $\mathscr{A}'$ of $\mathscr{A}$ and hence in the closed unit ball $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq \mathscr{A}'$. This will now open the door for geometric techniques as $\mathscr{A}'$ and hence $\mathrm{B}_1(0)^{\mathrm{cl}}$ carry the weak* topology in addition to the norm topology: this will turn out to be very useful. Before investigating these features in detail, we first state the following definition:

Definition 4.2.35 (Spectrum of $\mathscr{A}$) *Let $\mathscr{A}$ be a unital commutative Banach algebra. Then the set of characters is called the spectrum of $\mathscr{A}$, denoted by*

$$\mathrm{Spec}(\mathscr{A}) = \big\{\varphi\colon \mathscr{A} \longrightarrow \mathbb{C} \;\big|\; \varphi \text{ is a character}\big\} \subseteq \mathscr{A}'. \tag{4.2.49}$$

It will be endowed with the weak topology inherited from $\mathscr{A}'$.*

Alternatively, and perhaps more traditionally, we can define the spectrum as the *set of maximal ideals* of $\mathscr{A}$. Note that in algebraic geometry yet another definition is used: one considers the set of *prime ideals* instead of the maximal ones as the spectrum of a ring, see e.g. [21] or any other classical textbooks on algebraic geometry. Since we are dealing with a unital situation, a maximal ideal is always prime but not necessarily the other way round: the prime spectrum in algebraic geometry is typically strictly larger than the maximal spectrum.

Note that for the description of the spectrum of $\mathscr{A}$ by maximal ideals the easy access to the weak* topology is missing hence we prefer (4.2.49). In any case, the relation between the two points of view is via

$$\mathscr{J} = \ker\varphi \leftrightarrow \varphi, \tag{4.2.50}$$

according to Lemma 4.2.33, *vi.)*. Note that $\{0\} \subseteq \mathscr{A}$ is always a proper ideal and hence contained in a maximal ideal by Lemma 4.2.33, *i.)*. This shows that the spectrum $\operatorname{Spec}(\mathscr{A})$ is *non-empty*. The structure of $\operatorname{Spec}(\mathscr{A})$ is now clarified by the following theorem:

Theorem 4.2.36 (Gel'fand Representation Theorem) *Let $\mathscr{A}$ be a unital commutative Banach algebra.*

i.) *The spectrum $\operatorname{Spec}(\mathscr{A})$ is weakly* closed in $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq \mathscr{A}'$ and hence a compact Hausdorff space with respect to the weak* topology.*

ii.) *The Gel'fand transform defined by*

$$\mathscr{A} \ni a \;\mapsto\; \hat{a} = \big(\varphi \mapsto \hat{a}(\varphi) = \varphi(a)\big) \in \mathscr{C}(\operatorname{Spec}(\mathscr{A})) \tag{4.2.51}$$

is a continuous unital algebra homomorphism with

$$\|\hat{a}\|_\infty \leq \|a\|. \tag{4.2.52}$$

iii.) *The kernel of the Gel'fand transform is the radical of $\mathscr{A}$, i.e. the intersection of all maximal ideals*

$$\operatorname{Rad}(\mathscr{A}) = \bigcap_{\substack{\mathscr{J} \subseteq \mathscr{A} \\ \textit{maximal ideal}}} \mathscr{J} \quad = \bigcap_{\varphi \in \operatorname{Spec}(\mathscr{A})} \ker \varphi. \tag{4.2.53}$$

iv.) *For all $a \in \mathscr{A}$ we have*

$$\operatorname{spec}_{\mathscr{C}(\operatorname{Spec}(\mathscr{A}))}(\hat{a}) = \hat{a}(\operatorname{Spec}(\mathscr{A})) = \operatorname{spec}_{\mathscr{A}}(a). \tag{4.2.54}$$

v.) *For the spectral radius of $a \in \mathscr{A}$ one has*

$$\varrho_{\mathscr{A}}(a) = \|\hat{a}\|_\infty = \varrho_{\mathscr{C}(\operatorname{Spec}(\mathscr{A}))}(\hat{a}). \tag{4.2.55}$$

Since $\operatorname{Spec}(\mathscr{A})$ is a compact Hausdorff space the continuous functions $\mathscr{C}(\operatorname{Spec}(\mathscr{A}))$ are a unital commutative Banach algebra with respect to the pointwise algebraic operations and the supremum norm $\|\cdot\|_\infty$, according to Example 4.2.5. In fact, it is even a Banach *-algebra satisfying (4.2.5), which we will not need at the moment. After establishing the compactness of $\operatorname{Spec}(\mathscr{A})$ as claimed in the first part, we will always endow $\mathscr{C}(\operatorname{Spec}(\mathscr{A}))$ with this Banach algebra structure.

PROOF (OF THEOREM 4.2.36): For the first part we already argued that $\operatorname{Spec}(\mathscr{A}) \subseteq \mathrm{B}_1(0)^{\mathrm{cl}} \subseteq \mathscr{A}'$. By the Banach-Alaoglu Theorem in form of Corollary 2.3.34 we know that $\mathrm{B}_1(0)^{\mathrm{cl}}$ is weakly* compact. It remains to show that $\operatorname{Spec}(\mathscr{A})$ is weakly* closed in $\mathrm{B}_1(0)^{\mathrm{cl}}$. But this is easy: by the very definition of the weak* topology, the map $\varphi \mapsto \varphi(a)$ is a weakly* continuous map $\mathscr{A}' \longrightarrow \mathbb{C}$ for every $a \in \mathscr{A}$. Hence also the (non-linear) maps

$$\varphi \mapsto \varphi(ab) - \varphi(a)\varphi(b) \quad \text{and} \quad \varphi \mapsto \varphi(\mathbb{1}) - 1$$

are weakly* continuous as they are compositions of continuous maps. Clearly, $\operatorname{Spec}(\mathscr{A})$ is the intersection of the zero loci of all these maps for all $a, b \in \mathscr{A}$. Each zero locus is weakly* closed by continuity and hence the intersection is closed in the weak* topology as well. This proves the first part. For the second, note that the function $\hat{a}\colon \operatorname{Spec}(\mathscr{A}) \longrightarrow \mathbb{C}$ is just the restriction of the linear function $\iota(a)\colon \mathscr{A}' \longrightarrow \mathbb{C}$ which we know to be continuous in the weak* topology by the very definition of the weak* topology, see Definition 2.2.38. It follows that $\hat{a} \in \mathscr{C}(\operatorname{Spec}(\mathscr{A}))$. Clearly, (4.2.51) is linear and for $\varphi \in \operatorname{Spec}(\mathscr{A})$ and $a, b \in \mathscr{A}$ we have

$$\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \hat{a}(\varphi)\hat{b}(\varphi)$$

as well as $\hat{\mathbb{1}}(\varphi) = \varphi(\mathbb{1}) = 1$. Hence (4.2.51) is a unital algebra homomorphism. To prove continuity we consider $\varphi \in \operatorname{Spec}(\mathscr{A})$. Then we have for $a \in \mathscr{A}$

$$|\hat{a}(\varphi)| = |\varphi(a)| \leq \|a\|,$$

since $\|\varphi\| = 1$. Hence taking the supremum over $\operatorname{Spec}(\mathscr{A})$ gives immediately the estimate (4.2.52) and thus the continuity of the Gel'fand transform. For the third part, let $a \in \mathscr{A}$ be in the kernel, i.e. $\hat{a} = 0$. This means $0 = \hat{a}(\varphi) = \varphi(a)$ and hence $a \in \ker \varphi$ for all characters φ. This gives $a \in \operatorname{Rad}(\mathscr{A})$. Conversely, let $a \in \operatorname{Rad}(\mathscr{A})$ then $\hat{a}(\varphi) = \varphi(a) = 0$ for all $\varphi \in \operatorname{Spec}(\mathscr{A})$ and thus $\hat{a} = 0$. Since the kernels of the characters are just the maximal ideals, the third part follows. To show the fourth part, let $a \in \mathscr{A}$ be given. Then $a - \varphi(a)\mathbb{1} \in \ker \varphi$ for $\varphi \in \operatorname{Spec}(\mathscr{A})$. Thus $a - \varphi(a)\mathbb{1}$ is in a (proper since maximal) ideal and can therefore not be invertible. It follows that $\varphi(a) \in \operatorname{spec}_{\mathscr{A}}(a)$. But $\varphi(a) = \hat{a}(\varphi)$ which shows $\hat{a}(\operatorname{Spec}(\mathscr{A})) \subseteq \operatorname{spec}_{\mathscr{A}}(a)$. Conversely, let $\lambda \in \operatorname{spec}_{\mathscr{A}}(a)$ be given. Then $\mathscr{J} = \{b(\lambda\mathbb{1} - a) \mid b \in \mathscr{A}\}$ is a proper *two-sided* ideal in $\mathscr{A}$ which is, by Lemma 4.2.33, *i.)*, contained in a maximal ideal. Thus there exists a not necessarily unique $\varphi \in \operatorname{Spec}(\mathscr{A})$ with $\mathscr{J} \subseteq \ker \varphi$ by Lemma 4.2.31, *iv.)*. Since φ is a character we have $0 = \varphi(\lambda\mathbb{1} - a) = \lambda - \varphi(a)$ by taking $b = \mathbb{1}$. This shows that $\hat{a}(\varphi) = \varphi(a) = \lambda$, proving the reverse inclusion $\operatorname{spec}_{\mathscr{A}}(a) \subseteq \hat{a}(\operatorname{Spec}(\mathscr{A}))$. We finally remark that for a compact Hausdorff space X and $f \in \mathscr{C}(X)$ we have in general $\operatorname{spec}_{\mathscr{C}(X)}(f) = f(X)$. This is not hard to see and explains the first equation in (4.2.54). Equally simple we have $\varrho_{\mathscr{C}(X)}(f) = \|f\|_\infty$ in this case, see also Exercise 4.5.17, *vi.)*, which gives the second equation in (4.2.55). From (4.2.54) we also get the first equality in (4.2.55) at once. □

In general, the Gel'fand transform is neither injective nor surjective. Nevertheless, it provides the most important tool in the study of commutative Banach algebras. The name "spectrum" of the algebra is justified by the fourth part of the theorem: the spectrum of every element $a \in \mathscr{A}$ can be obtained from $\operatorname{Spec}(\mathscr{A})$ by applying $\hat{a}$.

The case of an injective Gel'fand transform deserves special attention: in this case the Banach algebra is a subalgebra of $\mathscr{C}(\operatorname{Spec}(\mathscr{A}))$:

Definition 4.2.37 (Semisimple Banach algebra) *A unital commutative Banach algebra $\mathscr{A}$ is called semisimple if* $\operatorname{Rad}(\mathscr{A}) = \{0\}$. *In general, the elements $a \in \operatorname{Rad}(\mathscr{A})$ are called the generalized zero elements.*

Thus a unital commutative Banach algebra is semisimple iff its Gel'fand transform is injective in which case we can identify $\mathscr{A}$ with a subalgebra of $\mathscr{C}(\operatorname{Spec}(\mathscr{A}))$.

Corollary 4.2.38 *Let $\mathscr{A}$ be a unital commutative Banach algebra. Then $\mathscr{A}$ is semisimple iff for every nonzero element $a \in \mathscr{A}$ there is a character φ with $\varphi(a) \neq 0$.*

Up to now we have not used a *-involution but this corollary already suggests that characters should not be too different from positive functionals, i.e. states. Then $\mathscr{A}$ is semisimple iff a nonzero element can also be "measured" to be nonzero in the sense of a non-trivial expectation value. In the general case this needs not to be the case at all:

Example 4.2.39 Consider the Graßmann algebra in one variable, i.e. $\mathscr{A} = \mathbb{C}\mathbb{1} \oplus \mathbb{C}e$ where $e \wedge e = 0$ is the only non-trivial product rule. This is clearly a commutative unital Banach algebra of dimension 2 if we use e.g. the norm

$$\|\alpha\mathbb{1} + \beta e\| = |\alpha| + |\beta|. \tag{4.2.56}$$

Since $e \wedge e = 0$ we have for every character $0 = \varphi(e \wedge e) = \varphi(e)\varphi(e)$ and thus $\varphi(e) = 0$. Hence the map $\varphi(\alpha\mathbb{1} + \beta e) = \alpha$ is the only character of $\mathscr{A}$. We conclude that

$$\operatorname{Rad}(\mathscr{A}) = \mathbb{C}e \quad \text{and} \quad \operatorname{Spec}(\mathscr{A}) = \{\varphi\}, \tag{4.2.57}$$

with $\mathbb{C}e$ being the generalized zero elements.

Another consequence of the Gel'fand Representation Theorem is the following:

Corollary 4.2.40 *Let $\mathscr{A}$ be a unital commutative Banach algebra. Then the spectral radius*

$$\varrho\colon \mathscr{A} \ni a \ \mapsto\ \varrho_{\mathscr{A}}(a) \in \mathbb{R} \tag{4.2.58}$$

is a continuous submultiplicative seminorm on $\mathscr{A}$ which is a norm iff $\mathscr{A}$ is semisimple.

PROOF: Since $a \mapsto \hat{a}$ is a continuous algebra homomorphism and the supremum norm $\|\cdot\|_\infty$ on $\mathscr{C}(\mathscr{A})$ is a submultiplicative norm, the spectral radius $\varrho_{\mathscr{A}}$ is a continuous submultiplicative seminorm by (4.2.55). We have $\varrho_{\mathscr{A}}(a) > 0$ for all $a \neq 0$ iff the Gel'fand transform $a \mapsto \hat{a}$ is injective. □

Thus we have yet another characterization of the generalized zero elements of $\mathscr{A}$, namely

$$\operatorname{Rad}(\mathscr{A}) = \ker \varrho_{\mathscr{A}}\,. \tag{4.2.59}$$

This also explains the name "generalized zero elements". It also suggests that we can get rid of these elements by passing to the quotient:

Proposition 4.2.41 (Deradicalization) *Let $\mathscr{A}$ be a unital commutative Banach algebra. Then* $\operatorname{Rad}(\mathscr{A})$ *is a closed ideal and* $\mathscr{A}\big/\operatorname{Rad}(\mathscr{A})$ *is semisimple with*

$$\operatorname{Spec}(\mathscr{A}) \cong \operatorname{Spec}(\mathscr{A}\big/\operatorname{Rad}(\mathscr{A})). \tag{4.2.60}$$

PROOF: It is clear that $\operatorname{Rad}(\mathscr{A})$ is a closed ideal, e.g. by the definition (4.2.53) or by the characterization as kernel of the continuous seminorm $\varrho_{\mathscr{A}}$. Thus $\mathscr{A}\big/\operatorname{Rad}(\mathscr{A})$ is again a unital commutative Banach algebra. Let $[a] \in \mathscr{A}\big/\operatorname{Rad}(\mathscr{A})$ and $\varphi \in \operatorname{Spec}(\mathscr{A})$ be given. Then $\varphi(a)$ only depends on $[a]$ since $\operatorname{Rad}(\mathscr{A}) \subseteq \ker\varphi$ by (4.2.53). Thus $\varphi\colon [a] \mapsto \varphi(a)$ is well-defined and clearly a character on $\mathscr{A}\big/\operatorname{Rad}(\mathscr{A})$. This defines a map

$$\operatorname{Spec}(\mathscr{A}) \longrightarrow \operatorname{Spec}(\mathscr{A}\big/\operatorname{Rad}(\mathscr{A})). \tag{$*$}$$

We claim that this map is a homeomorphism. It is clear that for any character $\psi \in \operatorname{Spec}(\mathscr{A}\big/\operatorname{Rad}(\mathscr{A}))$ the map $a \mapsto \psi([a])$ is again a character on $\mathscr{A}$ which maps to ψ under $(*)$. Thus $(*)$ is surjective. If $\varphi, \tilde{\varphi} \in \operatorname{Spec}(\mathscr{A})$ a two different characters then we have an element $a \in \mathscr{A}$ with $\varphi(a) \neq \tilde{\varphi}(a)$. But then also their images under $(*)$ take different values on the class $[a]$, showing the injectivity. Now let $(\varphi_i)_{i\in I}$ be a net in $\operatorname{Spec}(\mathscr{A})$ converging in the weak* topology to some other character φ. This means $\varphi_i(a) \longrightarrow \varphi(a)$ for all $a \in \mathscr{A}$ by the very definition of the weak* topology. But then also $\varphi_i([a]) \longrightarrow \varphi([a])$ for all $[a] \in \mathscr{A}\big/\operatorname{Rad}(\mathscr{A})$ showing that $(*)$ is continuous. Since a continuous bijection between compact Hausdorff spaces is necessarily a homeomorphism, see Lemma **??**, the claim is established. □

Remark 4.2.42 The passage $\mathscr{A} \rightsquigarrow \mathscr{A}\big/\operatorname{Rad}(\mathscr{A})$ is physically very desirable as the generalized zero elements behave in many aspects as rather "bad" observables, see also Exercise 4.5.38 for further details.

4.3 C^*-Algebras and the Continuous Calculus

While Banach algebras and in particular Banach *-algebras already have nice features concerning the notions of spectrum and holomorphic calculus, there are still certain "pathological" effects possible as we have seen this in the examples and in particular in the Exercises 4.5.28, 4.5.30, and 4.5.36. In this section we will demonstrate that the C^*-condition for a Banach *-algebra will help to cure most of these difficulties. We will end up with a class of *-algebras, the C^*-algebras, which behave indeed very nicely and allow for a continuous functional calculus as well as a rich *-representation theory. Nevertheless, concerning quantum theory one should always keep in mind that the way from heuristic commutation relations to an honest construction of a C^*-algebra as observables is usually a long and arduous path rather than a comfortable highway.

4.3.1 C^*-Algebras

We state the definition of a C^*-algebra now in several steps. The main motivation is the property (4.2.8) of the operator norm for bounded operators on a Hilbert space.

Definition 4.3.1 (Pro C^*-algebra) *Let $\mathscr{A}$ be a *-algebra.*

i.) A seminorm p on $\mathscr{A}$ is called a C^*-seminorm if p is submultiplicative and satisfies for all $a \in \mathscr{A}$

$$\mathrm{p}(a^*a) = \mathrm{p}(a)^2. \tag{4.3.1}$$

ii.) $\mathscr{A}$ is called a pro C^*-algebra if $\mathscr{A}$ is equipped with a complete Hausdorff locally convex topology which is determined by a system of C^*-seminorms.

Alternatively, pro C^*-algebras are called *projective C^*-algebras* or *locally C^*-algebras*.

Lemma 4.3.2 *Let p be a C^*-seminorm on a *-algebra $\mathscr{A}$. Then for all $a \in \mathscr{A}$ one has*

$$\mathrm{p}(a^*) = \mathrm{p}(a). \tag{4.3.2}$$

If $\mathscr{A}$ is unital then either $\mathrm{p}(\mathbb{1}) = 0$ or $\mathrm{p}(\mathbb{1}) = 1$.

PROOF: We have $\mathrm{p}(a)^2 = \mathrm{p}(a^*a) \leq \mathrm{p}(a^*)\mathrm{p}(a)$ from which we deduce $\mathrm{p}(a) \leq \mathrm{p}(a^*)$. Exchanging the roles of a and a^* gives (4.3.2). In the unital case we have $\mathrm{p}(\mathbb{1}) = \mathrm{p}(\mathbb{1}^*\mathbb{1}) = \mathrm{p}(\mathbb{1})^2$ and hence $\mathrm{p}(\mathbb{1})$ is either 0 or 1. ☐

Corollary 4.3.3 *A pro C^*-algebra is a complete lmc *-algebra.*

Indeed, both the product and the *-involution are continuous as the defining system of seminorms is submultiplicative and satisfies (4.3.2). However, the converse to Corollary 4.3.3 is not true at all: there are complete lmc *-algebras which are *not* pro C^*-algebras, see e.g. Exercise 4.5.28 for such an example which is even a Banach *-algebra.

We are again most interested in the case where a single (semi-) norm already suffices:

Definition 4.3.4 (C^*-Algebra) *A Banach *-algebra with the norm being a C^*-norm is called a C^*-algebra.*

Note that by Lemma 4.3.2 the requirement $\|a^*\| = \|a\|$ for a Banach *-algebra is already implied by the C^*-property. Moreover, in the unital case, $\|\mathbb{1}\| = 1$ is also implied by the C^*-condition.

Remark 4.3.5 Let $\mathscr{A}$ be a locally convex Hausdorff *-algebra with a defining system of C^*-seminorms. Then its completion $\widehat{\mathscr{A}}$ is a pro C^*-algebra. If $\mathscr{A}$ is a normed *-algebra with a C^*-norm then its completion is a C^*-algebra. Such a not necessarily complete normed *-algebra with a C^*-norm is sometimes called a *pre C^*-algebra*.

Remark 4.3.6 A pro C^*-algebra is always *non-degenerate* in the sense that $ba = 0$ for all b implies $a = 0$. Indeed, taking $b = a^*$ we get $0 = \mathrm{p}(a^*a) = \mathrm{p}(a)^2$ for all C^*-seminorms. Since the topology is required to be Hausdorff this gives $a = 0$, see also Exercise 4.5.36. We also conclude that $a^*a = 0$ implies $a = 0$ in any pro C^*-algebra. In particular, the Graßmann algebra can not be a pro C^*-algebra and hence not a C^*-algebra while it is a Banach *-algebra, see Example 4.2.9.

We collect now some examples of C^*-algebras which we have already seen and which lead to new examples by standard constructions:

Example 4.3.7 (Pro C^*-algebras and C^*-algebras)

i.) If X is a compact Hausdorff space then $\mathscr{C}(X)$ is a unital commutative C^*-algebra. This was already established in Example 4.2.5 and Exercise 4.5.17.

ii.) If X is only a locally compact Hausdorff space then we can still consider the complex-valued continuous functions $\mathscr{C}(X)$, now possibly containing unbounded functions. One considers the seminorms

$$\|f\|_K = \mathrm{p}_{K,0}(f) = \max_{x \in K} |f(x)| \tag{4.3.3}$$

for every compact subset $K \subseteq X$ and $f \in \mathscr{C}(X)$. Clearly, they form a system of C^*-seminorms. Since every point has a compact neighbourhood one can repeat the proof of the completeness as in the compact case and obtains a unital commutative pro C^*-algebra which is in general *not* a C^*-algebra, see also Theorem B.1.6.

iii.) According to Example 4.2.8 we conclude that the bounded operators $\mathfrak{B}(\mathfrak{H})$ on a Hilbert space form a unital C^*-algebra with respect to the operator norm.

iv.) A slight modification of Example 4.2.6 for a compact Hausdorff space X and a C^*-algebra $\mathscr{A}$ shows that also $\mathscr{C}(X, \mathscr{A})$ is again a C^*-algebra. Indeed, the C^*-property of the norm (4.2.7) can first be checked pointwise in $\mathscr{A}$ and then the maximum of the norms of $f(x)$ and $f(x)^* f(x)$ are obtained at the same points.

v.) If $\mathscr{A}$ is a C^*-algebra and $\mathscr{B} \subseteq \mathscr{A}$ a closed *-subalgebra then also $\mathscr{B}$ is a C^*-algebra. In particular, every norm-closed *-subalgebra of $\mathfrak{B}(\mathfrak{H})$ is a C^*-algebra. Such a C^*-algebra is sometimes called a *concrete C^*-algebra*. Later we will see that every C^*-algebra is isomorphic to a concrete C^*-algebra.

vi.) Let $\{\mathscr{A}_i\}_{i \in I}$ be a family of pro C^*-algebras. Then the Cartesian product $\mathscr{A} = \prod_{i \in I} \mathscr{A}_i$ becomes a *-algebra by the obvious componentwise *-operation and multiplication, i.e. we set

$$(a_i)_{i \in I}^* = (a_i^*)_{i \in I} \tag{4.3.4}$$

and

$$(a_i)_{i \in I} (b_j)_{j \in I} = (a_i b_i)_{i \in I}. \tag{4.3.5}$$

Since the Cartesian product topology is determined by the seminorms coming from each $\mathscr{A}_i$, we see that

$$\mathrm{p}_i((a_j)_{j \in I}) = \mathrm{p}_i(a_i) \tag{4.3.6}$$

for each C^*-seminorm p_i on $\mathscr{A}_i$ gives a defining system of C^*-seminorms for the Cartesian product topology consisting of C^*-seminorms on $\mathscr{A}$. Since the Cartesian product is complete iff each component is complete, $\mathscr{A}$ becomes a pro C^*-algebra this way. Clearly, if all the $\mathscr{A}_i$ are unital $\mathscr{A}$ is unital again. Moreover, note that the projections

$$\mathrm{pr}_j \colon \prod_{i \in I} \mathscr{A}_i \longrightarrow \mathscr{A}_j \tag{4.3.7}$$

as well as the inclusion maps

$$\iota_j \colon \mathscr{A}_j \longrightarrow \prod_{i \in I} \mathscr{A}_i \tag{4.3.8}$$

are continuous *-homomorphisms. In the unital case the pr_j are unital while the ι_j are not unital unless I has only one entry.

vii.) Let $\{\mathscr{A}_i\}_{i \in I}$ be a family of C^*-algebras. Then the Cartesian product will only be a pro C^*-algebra in general, unless the index set I is *finite*. On the *-subalgebra

$$\bigoplus_{i \in I} \mathscr{A}_i \subseteq \prod_{i \in I} \mathscr{A}_i \tag{4.3.9}$$

of the Cartesian product we can define now a C^*-norm by taking the *supremum*

$$\|(a_i)_{i\in I}\|_\infty = \sup_{i\in I}\|a_i\|_{\mathscr{A}_i} \tag{4.3.10}$$

over the C^*-norms of each component. Since in the direct sum we have only finitely many nonzero components, this supremum is actually a maximum. It is now easily verified that $\|\cdot\|_\infty$ is a submultiplicative norm satisfying the C^*-property. Hence the direct sum becomes a pre C^*-algebra which we can complete to a C^*-algebra: we denote this completion by $\widehat{\bigoplus}_{i\in I}\mathscr{A}_i$. More explicitly, we can realize the completion as a subalgebra of the Cartesian product by

$$\widehat{\bigoplus_{i\in I}}\mathscr{A}_i = \Big\{(a_i)_{i\in I} \;\Big|\; \forall\epsilon>0\exists K\subseteq I \text{ finite } \sup_{i\in I\setminus}\|a_i\|_{\mathscr{A}_i}<\epsilon\Big\} \subseteq \prod_{i\in I}\mathscr{A}_i. \tag{4.3.11}$$

One calls $\widehat{\bigoplus}_{i\in I}\mathscr{A}_i$ the *C^*-algebraic direct sum* of the C^*-algebras $\mathscr{A}_i$. Again, $\widehat{\bigoplus}_{i\in I}\mathscr{A}_i$ is unital if all the $\mathscr{A}_i$ are unital. If the index set is finite we have

$$\bigoplus_{i\in I}\mathscr{A}_i = \widehat{\bigoplus_{i\in I}}\mathscr{A}_i = \prod_{i\in I}\mathscr{A}_i. \tag{4.3.12}$$

For a non-unital Banach algebra $\mathscr{A}$ we have seen that the unitization $\widetilde{\mathscr{A}} = \mathscr{A}\oplus\mathbb{C}\mathbb{1}$ becomes again a Banach algebra. For a non-unital C^*-algebra $\mathscr{A}$ however, we have to use a different norm instead of the most simple choice (4.2.15): for this choice the C^*-property is typically no longer valid. We will proceed differently.

Proposition 4.3.8 (Unitization of C^*-algebras) *Let $\mathscr{A}$ be a non-unital C^*-algebra. On the unitization $\widetilde{\mathscr{A}} = \mathscr{A}\oplus\mathbb{C}\mathbb{1}$ one defines the usual *-algebra structure and*

$$\|a+z\mathbb{1}\| = \sup\big\{\|ab+zb\| \;\big|\; b\in\mathscr{A}, \|b\|=1\big\}. \tag{4.3.13}$$

Then $\widetilde{\mathscr{A}}$ becomes a unital C^*-algebra and $\mathscr{A}\subseteq\widetilde{\mathscr{A}}$ is a closed *-ideal in $\widetilde{\mathscr{A}}$.

PROOF: We have to show that (4.3.13) defines a C^*-norm. To this end we observe that the left multiplication $\mathsf{L}_a\colon \mathscr{A}\longrightarrow\mathscr{A}$ with $a\in\mathscr{A}$ is a continuous linear map with operator norm

$$\|\mathsf{L}_a\| = \sup_{\|b\|=1}\|\mathsf{L}_a b\| = \sup_{\|b\|=1}\|ab\|.$$

With this interpretation we see that (4.3.13) is the operator norm of $\mathsf{L}_a + z\,\mathrm{id}_{\mathscr{A}}$, i.e.

$$\|a+z\mathbb{1}\| = \|\mathsf{L}_a + z\,\mathrm{id}_{\mathscr{A}}\|.$$

Since clearly $a+z\mathbb{1}\mapsto\mathsf{L}_a+z\,\mathrm{id}_{\mathscr{A}}$ defines an algebra homomorphism $\widetilde{\mathscr{A}}\longrightarrow\mathrm{L}(\mathscr{A})$ into all continuous endomorphisms of $\mathscr{A}$, the definition (4.3.13) gives a submultiplicative seminorm on $\widetilde{\mathscr{A}}$. Moreover, since $\|a^*a\| = \|a\|^2$ we see that $\|\mathsf{L}_a\|\geq\|a\|$ and hence $a\mapsto\mathsf{L}_a$ is injective. Since by assumption, $\mathscr{A}$ has no unit element, it follows that also $a+z\mathbb{1}\mapsto\mathsf{L}_a+z\,\mathrm{id}_{\mathscr{A}}$ is injective. Thus (4.3.13) defines indeed a norm. Moreover, $\|\mathsf{L}_a b\| = \|ab\|\leq\|a\|\|b\|$ shows that $\|\mathsf{L}_a\| = \|a\|$ and thus (4.3.13) restricts to the original C^*-norm on $\mathscr{A}$. The property $\|\mathrm{id}_{\mathscr{A}}\| = 1$ gives $\|\mathbb{1}\| = 1$. To show the C^*-property we check

$$\begin{aligned}\|a+z\mathbb{1}\|^2 &= \sup_{\|b\|=1}\|ab+zb\|^2\\ &= \sup_{\|b\|=1}\|(ab+zb)^*(ab+zb)\|\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\|b\|=1} \|b^*(a^*ab + za^*b + \overline{z}ab + \overline{z}zb)\| \\
&\leq \sup_{\|b\|=1} \|a^*ab + za^*b + \overline{z}ab + \overline{z}zb\| \\
&= \|a^*a + za^* + \overline{z}a + \overline{z}z\mathbb{1}\| \\
&= \|(a + z\mathbb{1})^*(a + z\mathbb{1})\|. \qquad (*)
\end{aligned}
$$

This shows $\|A\|^2 \leq \|A^*A\|$ for $A \in \widetilde{A}$. Since (4.3.13) is submultiplicative this implies $\|A\|^2 \leq \|A\|\|A^*\|$ and hence $\|A\| \leq \|A^*\|$. Exchanging $A \leftrightarrow A^*$ gives $\|A\| = \|A^*\|$. Then the estimate $(*)$ implies $\|A^*A\| = \|A\|^2$ at once. The fact that $\mathscr{A} \subseteq \widetilde{\mathscr{A}}$ is a closed *-ideal is now easy. □

plier algebra, definition and realization as domorphisms $\mathfrak{B}(\mathscr{A})$

It follows that for non-unital C^*-algebras we can pass to the unitization without leaving the C^*-algebraic framework. In this sense the following statements about unital C^*-algebras can be transferred to the non-unital case easily. In particular, the considerations about spectra will remain valid in the non-unital case, too.

The first improvement in comparison to the unital Banach *-algebra case is the following: the spectra of unitary and Hermitian elements behave exactly as we want. Note that the following theorem is, in general, wrong for a unital Banach *-algebra, see Exercise 4.5.28.

Theorem 4.3.9 (Spectra in a C^*-algebra) *Let $\mathscr{A}$ be a C^*-algebra.*

i.) If $a \in \mathscr{A}$ is normal then $\varrho_{\mathscr{A}}(a) = \|a\|$.

ii.) If $\mathscr{A}$ is unital and $u \in \mathscr{A}$ is isometric then $\varrho_{\mathscr{A}}(u) = 1$.

iii.) If $\mathscr{A}$ is unital and $u \in \mathscr{A}$ is unitary then

$$
\operatorname{spec}_{\mathscr{A}}(u) \subseteq \big\{ z \in \mathbb{C} \bigm| |z| = 1 \big\} = \mathrm{U}(1). \tag{4.3.14}
$$

iv.) If $a \in \mathscr{A}$ is Hermitian then

$$
\operatorname{spec}_{\mathscr{A}}(a) \subseteq \big[-\|a\|, \|a\| \big] \tag{4.3.15}
$$

and

$$
\operatorname{spec}_{\mathscr{A}}(a^2) \subseteq \big[0, \|a\|^2\big]. \tag{4.3.16}
$$

PROOF: In the non-unital case we first pass to the unitization $\widetilde{\mathscr{A}}$ as usual. Then Theorem 4.2.19, *iv.)*, gives us $\varrho_{\mathscr{A}}(a) = \lim_{n\longrightarrow\infty}\|a^n\|^{\frac{1}{n}}$ for all $a \in \mathscr{A}$. We compute the (existing) limit by evaluating it only on a suitable subsequence. Since a and a^* commute for a normal element we have

$$
\begin{aligned}
\|a^{2^n}\|^2 &= \big\|\big(a^{2^n}\big)^* a^{2^n}\big\| = \big\|(a^*a)^{2^n}\big\| = \big\|(a^*a)^{2^{n-1}}(a^*a)^{2^{n-1}}\big\| = \big\|(a^*a)^{2^{n-1}}\big\|^2 \\
&= \cdots = \|a^*a\|^{2^n} = \|a\|^{2^{n+1}},
\end{aligned}
$$

by using the C^*-property of the norm several times. But then the first claim follows since for this subsequence we clearly have $\|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|$ already identically. For the second part let $u^*u = \mathbb{1}$. Then $\|u^n\|^2 = \|(u^n)^*u^n\| = \|(u^*)^n u^n\| == \|u^* \cdots u^*u \cdot u\| = \|\mathbb{1}\| = 1$. This gives $\|u^n\| = 1$ for all $n \in \mathbb{N}$ and hence the result by Theorem 4.2.19, *iv.)*. Note however, that isometric elements need *not* to be normal. For a unitary element u we have on one hand $\operatorname{spec}_{\mathscr{A}}(u^*) = \overline{\operatorname{spec}_{\mathscr{A}}(u)}$ by Proposition 4.2.20 and $\operatorname{spec}_{\mathscr{A}}(u^{-1}) = \frac{1}{\operatorname{spec}_{\mathscr{A}}(u)}$ by Theorem 4.2.19, *iii.)*. Since $u^* = u^{-1}$ we conclude that $\overline{\operatorname{spec}_{\mathscr{A}}(u)} = \frac{1}{\operatorname{spec}_{\mathscr{A}}(u)}$ or more explicitly

$$
\lambda \in \operatorname{spec}_{\mathscr{A}}(u) \Leftrightarrow \frac{1}{\overline{\lambda}} \in \operatorname{spec}_{\mathscr{A}}(u). \tag{$*$}
$$

So far, this is true for every unitary element in a Banach *-algebra. In a C^*-algebra we know in addition that $\varrho_{\mathscr{A}}(u) = 1$ by the second part. Thus $|\lambda| \leq 1$ for all $\lambda \in \operatorname{spec}_{\mathscr{A}}(u)$. Together with

(∗) this gives $\frac{1}{|\lambda|} \leq 1$ as well and hence $|\lambda| = 1$. This shows the third part. Now let $a = a^*$ be Hermitian. By the first part we know $\varrho_{\mathscr{A}}(a) = \|a\|$. It remains to show that the spectrum is real. Let $\lambda = \lambda_1 + \mathrm{i}\lambda_2 \in \operatorname{spec}_{\mathscr{A}}(a)$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ be given. Then consider the normal elements $b_n = a - \lambda_1 + \mathrm{i}n\lambda_2$. We see that

$$\mathrm{i}(n+1)\lambda_2 - b_n = \mathrm{i}(n+1)\lambda_2 - a + \lambda_1 - \mathrm{i}n\lambda_2 = (\lambda_1 + \mathrm{i}\lambda_2) - a = \lambda - a$$

is not invertible. Hence $\mathrm{i}(n+1)\lambda_2 \in \operatorname{spec}_{\mathscr{A}}(b_n)$. Now we have the estimate

$$\begin{aligned} |\mathrm{i}(n+1)\lambda_2|^2 &= (n^2 + 2n + 1)\lambda_2^2 \\ &\leq \varrho_{\mathscr{A}}(b_n)^2 \\ &= \|b_n\|^2 \\ &= \|b_n^* b_n\| \\ &= \|(a - \lambda_1 - \mathrm{i}n\lambda_2)(a - \lambda_1 + \mathrm{i}n\lambda_2)\| \\ &= \|(a - \lambda_1)^2 + n^2\lambda_2^2\| \\ &\leq \|a - \lambda_1\|^2 + n^2\lambda_2^2. \end{aligned}$$

It follows that for all n we have $(2n+1)\lambda_2^2 \leq \|a - \lambda_1\|^2$ which is only possible for $\lambda_2 = 0$. This shows (4.3.15) and (4.3.16) is a simple consequence of Theorem 4.2.19, *ii.)*. □

4.3.2 Commutative C^*-Algebras

In this subsection we shall see how the results of Subsection 4.2.3 can be specialized from general commutative Banach algebras to commutative C^*-algebras. Again, we will focus on the unital case and get the results for the non-unital case by unitization.

As a first step we consider once more the continuous functions $\mathscr{C}(X)$ on a compact Hausdorff space. Though this seems to be a rather particular commutative C^*-algebra we shall investigate its spectrum and its Gel'fand transform in some detail. To this end we first show the correspondence between the closed ideals in $\mathscr{C}(X)$ and the closed subsets of X:

Theorem 4.3.10 (Closed ideals in $\mathscr{C}(X)$) *Let X be a compact Hausdorff space.*

i.) The vanishing ideal $\mathscr{J}_A$ of a subset $A \subseteq X$ is a closed *-ideal of $\mathscr{C}(X)$ and one has $\mathscr{J}_A = \mathscr{J}_{A^{\mathrm{cl}}}$.

ii.) If $\mathscr{J} \subseteq \mathscr{C}(X)$ is a closed ideal then there is a unique closed subset $A \subseteq X$ with $\mathscr{J} = \mathscr{J}_A$. In particular, $\mathscr{J}$ is necessarily a *-ideal.

PROOF: If $A \subseteq X$ is an arbitrary subset and $f_n \in \mathscr{C}(X)$ is a sequence satisfying $f_n\big|_A = 0$ then a uniform limit $f_n \longrightarrow f$ is in particular a pointwise limit and thus $f\big|_A = 0$ follows. This shows that the vanishing ideal $\mathscr{J}_A$ is a closed ideal. Since $f(x) = 0$ iff $\overline{f}(x) = \overline{f(x)} = 0$ we see that $\mathscr{J}_A$ is necessarily a *-ideal. Moreover, if $f\big|_A = 0$ then also $f\big|_{A^{\mathrm{cl}}} = 0$ by the continuity of f. This shows $\mathscr{J}_A \subseteq \mathscr{J}_{A^{\mathrm{cl}}}$. The converse inclusion is trivial and thus the first part follows. Conversely, let $\mathscr{J} \subseteq \mathscr{C}(X)$ be a closed ideal. Then we consider the common zero locus of all functions in $\mathscr{J}$, i.e. we define

$$A = \bigcap_{f \in \mathscr{J}} f^{-1}(\{0\}) \subseteq X.$$

Since every f is continuous, $f^{-1}(\{0\})$ is a closed subset and thus A is closed. It follows that $\mathscr{J} \subseteq \mathscr{J}_A$. We want to prove that for $f \in \mathscr{J}_A$ we also have $f \in \mathscr{J}$. To this end, fix $\epsilon > 0$ and consider

$$B_\epsilon = \big\{x \in X \bigm| |f(x)| \geq \epsilon\big\} \subseteq X,$$

which is a closed subset of X and hence compact. Moreover, $A \cap B_\epsilon = \emptyset$ since f does not vanish on B_ϵ. Thus for every point $x \in B_\epsilon$ we have a function $g_x \in \mathscr{J}$ with $g_x(x) \neq 0$ since otherwise $x \in A$. Thus $\overline{g_x} g_x \geq 0$ is strictly positive at x and by continuity also strictly positive in a small open neighbourhood of x. Finitely many of these neighbourhoods cover the compact subset B_ϵ and thus

$$g_\epsilon = \overline{g_{x_1}} g_{x_1} + \cdots + \overline{g_{x_n}} g_{x_n} \in \mathscr{J}$$

is strictly positive on B_ϵ and still in $\mathscr{J}$ as $\mathscr{J}$ is an ideal. Note that we do *not* need a *-ideal at this point. Let now $y = \min_{x \in B_\epsilon} g_\epsilon(x) > 0$ be the minimum of g_ϵ on the compact subset B_ϵ and define $h_\epsilon = \max\{g_\epsilon, y\}$. Then h_ϵ coincides with g_ϵ on B_ϵ and we have $h_\epsilon \geq y > 0$ outside of B_ϵ. In particular, $h_\epsilon > 0$ everywhere and the continuous function $\frac{g_\epsilon}{h_\epsilon} \in \mathscr{J}$ coincides with 1 on B_ϵ and satisfies $0 \leq \frac{g_\epsilon}{h_\epsilon} \leq 1$ everywhere. Now $f \frac{g_\epsilon}{h_\epsilon} \in \mathscr{J}$ and we have $\left| f - f \frac{g_\epsilon}{h_\epsilon} \right| = 0$ on B_ϵ and $< \epsilon$ on the complement of B_ϵ. Thus $\| f - f \frac{g_\epsilon}{h_\epsilon} \|_\infty < \epsilon$ shows that a function in $\mathscr{J}$ approximates f up to ϵ. By the closedness of $\mathscr{J}$ we conclude that $f \in \mathscr{J}$ and hence $\mathscr{J} = \mathscr{J}_A$ follows. It remains to show that A is the unique closed subset with this property. Thus let $x \in X \setminus A$ then we can find a continuous function $f \in \mathscr{C}(X)$ with $f(x) = 1$ but $f\big|_A = 0$ by Urysohn's Lemma, see Theorem A.2.2, as a compact Hausdorff space is T_4, see **??**. But then $f \in \mathscr{J} = \mathscr{J}_A$ but f is not in $\mathscr{J}_{A \cup \{x\}}$. Thus the uniqueness follows. ☐

Corollary 4.3.11 (Characters of $\mathscr{C}(X)$) *Let X be a compact Hausdorff space. Any character φ of $\mathscr{C}(X)$ is of the form $f \mapsto \varphi(f) = \delta_x(f) = f(x)$ for some unique point $x \in X$. In particular, φ is a state, i.e.*

$$\varphi(\overline{f} f) \geq 0 \quad \textit{and} \quad \varphi(\mathbb{1}) = 1. \tag{4.3.17}$$

PROOF: Clearly, the maximal ideals correspond to the smallest proper closed subsets, the points of X. By Lemma 4.2.33 this gives the characters being the δ-functionals. Obviously, they are states. ☐

Thus we have established two things: on one hand the closed ideals of $\mathscr{C}(X)$ are in one-to-one correspondence with the closed subsets of X. On the other hand, the characters of $\mathscr{C}(X)$ correspond precisely to the points of X. In particular, we have established a set-theoretic bijection of X and the spectrum $\operatorname{Spec}(\mathscr{C}(X))$ of $\mathscr{C}(X)$. Since $\operatorname{Spec}(\mathscr{C}(X)) \subseteq \mathscr{C}(X)'$ is a compact Hausdorff space itself with respect to the weak* topology of the dual $\mathscr{C}(X)'$ it is tempting to claim that

$$X \ni x \ \mapsto \ \delta_x \in \operatorname{Spec}(\mathscr{C}(X)) \tag{4.3.18}$$

is in fact a homeomorphism and not just a bijection. This is indeed the case as the next theorem shows:

Theorem 4.3.12 (Gel'fand transform for $\mathscr{C}(X)$) *Let X be a compact Hausdorff space.*

i.) The Gel'fand transform of $\mathscr{C}(X)$ is given by

$$\mathscr{C}(X) \ni f \ \mapsto \ \hat{f} = (\delta_x \mapsto \delta_x(f) = f(x)) \in \mathscr{C}(\operatorname{Spec}(\mathscr{C}(X))), \tag{4.3.19}$$

i.e. the identity up to the identification of X with $\operatorname{Spec}(\mathscr{C}(X))$ according to (4.3.18).

ii.) The weak topology of $\operatorname{Spec}(\mathscr{C}(X))$ and the topology of X coincide under the identification* (4.3.18).

PROOF: The first part is clear from the general construction of the Gel'fand transform according to Theorem 4.2.36, *ii.)*, and the bijection (4.3.18). Thus we have to check the second part. First, let $x_i \longrightarrow x$ be a convergent net in X and let $f \in \mathscr{C}(X)$. Then we have

$$\mathrm{p}_f(\delta_{x_i} - \delta_x) = |\delta_{x_i}(f) - \delta_x(f)| = |f(x_i) - f(x)| \longrightarrow 0$$

by the continuity of f. But this shows $\delta_{x_i} \longrightarrow \delta_x$ in the weak* topology as the seminorms p_f define the weak* topology. Thus (4.3.18) is net-continuous and hence continuous. By Lemma **??** this is already enough to show that the bijection (4.3.18) is a homeomorphism. ☐

Thus in this particular case we can determine the spectrum and the Gel'fand transform explicitly: the spectrum reproduces X and the Gel'fand transform is essentially the identity (up to the above identification). The next theorem now shows that our example is far more than an example: it already describes the generic commutative unital C^*-algebra.

Theorem 4.3.13 (Gel'fand-Naimark) *Let $\mathscr{A}$ be a commutative unital C^*-algebra. Then the Gel'fand transform*

$$\mathscr{A} \ni a \;\mapsto\; \hat{a} \in \mathscr{C}(\operatorname{Spec}(\mathscr{A})) \tag{4.3.20}$$

is a norm-preserving *-isomorphism. Hence $\mathscr{A} \cong \mathscr{C}(\operatorname{Spec}(\mathscr{A}))$ via (4.3.20).

Proof: Since $\mathscr{A}$ is commutative, all elements are normal. Hence by Theorem 4.3.9, *i.)*, we have $\varrho_{\mathscr{A}}(a) = \|a\|$ for all $a \in \mathscr{A}$. By Theorem 4.2.36, *v.)*, this implies that the Gel'fand transform is injective and even norm-preserving since $\varrho_{\mathscr{A}}(a) = \|\hat{a}\|_\infty$. For $a = a^* \in \mathscr{A}$ the spectrum of a is real by Theorem 4.3.9, *iv.)*, and thus the spectrum of $\hat{a}$ is real by Theorem 4.2.36, *iv.)*, as well since it coincides with $\operatorname{spec}_{\mathscr{A}}(a)$. This means that the function $\hat{a} \in \mathscr{C}(\operatorname{Spec}(\mathscr{A}))$ is real-valued, i.e. $\overline{\hat{a}} = \hat{a}$. For a general $a \in \mathscr{A}$ we decompose a into the real and imaginary part by

$$a = \operatorname{Re}(a) + \mathrm{i}\operatorname{Im}(a) \quad \text{where} \quad \operatorname{Re}(a) = \frac{1}{2}(a + a^*) \quad \text{and} \quad \operatorname{Im}(a) = \frac{1}{2\mathrm{i}}(a - a^*)$$

as usual. We get

$$\hat{a}(\varphi) = \widehat{\operatorname{Re}(a)}(\varphi) + \mathrm{i}\widehat{\operatorname{Im}(a)}(\varphi) \quad \text{and} \quad \widehat{a^*}(\varphi) = \widehat{\operatorname{Re}(a)}(\varphi) - \mathrm{i}\widehat{\operatorname{Im}(a)}(\varphi),$$

by the linearity of the Gel'fand transform. Since the real and imaginary part of a are Hermitian and the Gel'fand transform maps Hermitian elements to real-valued functions we see that the Gel'fand transform is actually a unital *-homomorphism, i.e. $\widehat{a^*} = \overline{\hat{a}}$ for all $a \in \mathscr{A}$. It remains to show the surjectivity. Thus let $\varphi_1 \neq \varphi_2$ be two different characters. Then there is an element $a \in \mathscr{A}$ with $\hat{a}(\varphi_1) = \varphi_1(a) \neq \varphi_2(a) = \hat{a}(\varphi_2)$. We conclude that the image of the Gel'fand transform is a unital *-subalgebra of the continuous functions on $\operatorname{Spec}(\mathscr{A})$ which separates points. By the Stone-Weierstraß Theorem, see Theorem A.2.3, we conclude that $\widehat{\mathscr{A}} \subseteq \mathscr{C}(\operatorname{Spec}(\mathscr{A}))$ is *dense*. However, $\mathscr{A}$ is complete and the Gel'fand transform $\widehat{\ }$ is norm-preserving showing that $\widehat{\mathscr{A}}$ is also closed. Thus we have $\widehat{\mathscr{A}} = \mathscr{C}(\operatorname{Spec}(\mathscr{A}))$, finishing the proof. □

The Gel'fand-Naimark Theorem and Theorem 4.3.12 together establish a one-to-one correspondence between compact Hausdorff spaces on one hand and unital commutative C^*-algebras on the other hand. This correspondence can be easily shown to be functorial in the following sense:

Theorem 4.3.14 (Commutative topology)

i.) *For two compact Hausdorff spaces X and Y every continuous map $\phi\colon X \longrightarrow Y$ gives a continuous unital *-homomorphism $\phi^*\colon \mathscr{C}(Y) \longrightarrow \mathscr{C}(X)$ via pull-back and with $\psi\colon Y \longrightarrow Z$ being another continuous map into a compact Hausdorff space Z we have*

$$(\psi \circ \phi)^* = \phi^* \circ \psi^* \quad \textit{and} \quad \mathrm{id}_X^* = \mathrm{id}_{\mathscr{C}(X)}. \tag{4.3.21}$$

ii.) *For two commutative unital C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ every unital homomorphism $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ induces a unique continuous map $\phi\colon \operatorname{Spec}(\mathscr{B}) \longrightarrow \operatorname{Spec}(\mathscr{A})$ such that Φ corresponds to ϕ^* under the Gel'fand transform. In particular, Φ is necessarily a continuous *-homomorphism.*

iii.) *The categories of compact Hausdorff spaces and commutative unital C^*-algebras are equivalent.*

Proof: The first part is discussed in Exercise 4.5.18 and yields a contravariant functor. For the second part, let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a unital homomorphism. If $\varphi \in \operatorname{Spec}(\mathscr{B})$ is a character the clearly $\varphi \circ \Phi\colon \mathscr{A} \longrightarrow \mathbb{C}$ is a character as well. Thus there is an induced map $\phi\colon \operatorname{Spec}(\mathscr{B}) \longrightarrow \operatorname{Spec}(\mathscr{A})$ with $\phi\colon \varphi \mapsto \varphi \circ \Phi$. Moreover, for $a \in \mathscr{A}$ we have

$$(\phi^* \hat{a})(\varphi) = \hat{a}(\phi(\varphi)) = \hat{a}(\varphi \circ \Phi) = \varphi(\Phi(a)) = \widehat{\Phi(a)}(\varphi)$$

for all $\varphi \in \operatorname{Spec}(\mathscr{B})$. Hence with the Gel'fand transforms of the two algebras we have

$$\phi^* \circ \widehat{\ }^{\mathscr{A}} = \widehat{\ }^{\mathscr{B}} \circ \Phi. \tag{$*$}$$

We claim that ϕ is continuous: we clearly can extend ϕ to a map $\mathscr{B}^* \longrightarrow \mathscr{A}^*$ between the algebraic duals. By definition, for an arbitrary linear functional $\chi \in \mathscr{B}^*$ we have

$$\mathrm{p}_a(\phi(\chi)) = \mathrm{p}_a(\chi \circ \Phi) = |\chi(\Phi(a))| = \mathrm{p}_{\Phi(a)}(\chi),$$

showing that ϕ is continuous in the weak* topology. In fact, this is true for the transpose of *any* linear map $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$. But we already know that ϕ maps $\operatorname{Spec}(\mathscr{B})$ into $\operatorname{Spec}(\mathscr{A})$ and since the topologies of the spectra are the weak* topologies we conclude that ϕ is continuous. Since the Gel'fand transforms are both continuous *-isomorphisms (with continuous inverses), we conclude from $(*)$ that Φ is continuous, too. Now assume there is another map $\psi\colon \operatorname{Spec}(\mathscr{B}) \longrightarrow \operatorname{Spec}(\mathscr{A})$ satisfying $(*)$ with $\psi \neq \phi$. Let $\varphi \in \operatorname{Spec}(\mathscr{B})$ be a point with $\psi(\varphi) \neq \phi(\varphi)$. The means that there is an algebra element $a \in \mathscr{A}$ with $(\psi(\varphi))(a) \neq (\phi(\varphi))(a)$. But $(\psi(\varphi))(a) = \hat{a}(\psi(\varphi)) = \widehat{\Phi(a)}(\varphi)$ by $(*)$ gives a contradiction. Hence ϕ is uniquely determined, showing the second part. For the third part we first observe that also the assignment of the second part, i.e. $\mathscr{A} \rightsquigarrow \operatorname{Spec}(\mathscr{A})$ and $(\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}) \rightsquigarrow (\phi\colon \operatorname{Spec}(\mathscr{B}) \longrightarrow \operatorname{Spec}(\mathscr{A}))$ is a contravariant functor. This is clear form the properties of the pull-back and the uniqueness statement. Moreover, we have seen that the Gel'fand transform

$$\widehat{\ }\colon \mathscr{A} \longrightarrow \mathscr{C}(\operatorname{Spec}(\mathscr{A}))$$

as well as the "evaluation"

$$\delta\colon X \longrightarrow \operatorname{Spec}(\mathscr{C}(X))$$

are isomorphisms in the corresponding categories. The relation $(*)$ shows that the Gel'fand transform is natural. Analogously, the obvious relation

$$\delta_x \circ \phi^* = \delta_{\phi(x)}$$

for a continuous map $\phi\colon X \longrightarrow Y$ and $x \in X$ shows that the evaluation is natural, too. All together this shows that we have a (contravariant) equivalence of categories. ☐

Remark 4.3.15 It is a general paradigm, both in mathematics and physics, that *spaces* and *functions on spaces* are the incarnation of one and the same entity. The above theorem turns this into a precise mathematical statement: the functorial equivalence of the two categories says indeed that everything we can do in one of them can be done in the other as well. So we are free to choose the more convenient setting: sometimes the topological structure of a space X can be understood easily, sometimes a unital commutative C^*-algebra $\mathscr{A}$ is rather nice but has a very inaccessible spectrum $\operatorname{Spec}(\mathscr{A})$.

Remark 4.3.16 (Noncommutative topology) Since commutative unital C^*-algebras correspond to *classical* topological spaces one extends this correspondence and "defines" a *noncommutative* topological space as corresponding to a *noncommutative* C^*-algebra which then describes the "functions" on the noncommutative space. Of course, there is nothing like the noncommutative space itself but only its algebra of functions. However, this is far from being just a name, as a first glance suggests.

In fact, *noncommutative topology* provides fruitful analogies from commutative spaces transferred to the realm of general C^*-algebras. Here one can get a good intuition for interesting structures by this analogy, helping to understand noncommutative C^*-algebras. Conversely, and perhaps even more surprising, the extension to noncommutative C^*-algebras also allows to understand the topology of classical spaces which behave badly from a classical point of view: the passage to noncommutative C^*-algebras as functions on noncommutative spaces opens new possibilities here, too. Finally, one can also extend this from purely topological concepts to more geometric notions including metric aspects leading to *noncommutative geometry*. Beside being a mathematically very appealing and rich theory, it is considered to be an interesting candidate to understand fundamental physics at very small length scales. More on this topic can be found in Connes' seminal book [12] as well as in several textbooks like [11, 15, 32, 35, 59].

Corollary 4.3.17 *Unital algebra homomorphisms between unital commutative C^*-algebras are continuous *-homomorphisms.*

Proof: The crucial point in the proof of Theorem 4.3.14, *ii.)*, was that for a character φ also the composition of φ with a unital algebra homomorphism is a character. □

Corollary 4.3.18 *In a unital commutative C^*-algebra the *-involution and the C^*-norm are uniquely determined by the unital algebra structure.*

Proof: This means that if $\mathscr{A}$ is a unital commutative algebra and if $\|\cdot\|_i$ and ^{*_i} are norms and *-involutions on $\mathscr{A}$ such that $(\mathscr{A}, \|\cdot\|_i, ^{*_i})$ are both C^*-algebras for $i = 1, 2$ then $\|\cdot\|_1 = \|\cdot\|_2$ and $^{*_1} = ^{*_2}$. Indeed, the identity $\mathrm{id}_{\mathscr{A}}$ is a unital algebra homomorphism between these two C^*-algebras. Hence Corollary 4.3.17 applies. □

We will extend these automatic continuity statements later also to the noncommutative case. In fact, C^*-algebras are very rigid in the sense that the algebraic structure determines essentially everything. It is a good exercise to interpret the above two Corollaries in the light of Exercise 4.5.28.

We get an even sharper statement by allowing a different norm on $\mathscr{C}(X)$ or, more generally, on a commutative C^*-algebra which might be non-unital:

Theorem 4.3.19 (Uniqueness of the C^*-norm) *Let $\mathscr{A}$ be a commutative C^*-algebra and let $\|\cdot\|_1$ be another norm on $\mathscr{A}$ making it a normed algebra. Then for all $a \in \mathscr{A}$ we have $\|a\| \le \|a\|_1$.*

Proof: First we consider the case where $\mathscr{A}$ is non-unital. Since the unitization $\widetilde{\mathscr{A}}$ is on one hand a unital C^*-algebra by Proposition 4.3.8 and on the other hand, by extending $\|\cdot\|_1$, a unital normed algebra by Proposition 4.2.11 we arrive in the unital situation. Therefore it suffices to consider the unital case from the beginning. Thus let $\mathscr{A} = \mathscr{C}(X)$ with a compact Hausdorff space X. We consider the completion $\widehat{\mathscr{A}}$ of $\mathscr{A}$ with respect to the (not necessarily complete) norm $\|\cdot\|_1$ yielding now a unital Banach algebra. The points in X corresponding to the characters of $\mathscr{A}$ but not all of them might extend to $\widehat{\mathscr{A}}$. Thus let $X_1 \subseteq X$ be the subset of those points such that $\delta_x\colon \mathscr{C}(X) \longrightarrow \mathbb{C}$ is continuous with respect to $\|\cdot\|_1$, too. Then δ_x will extend to a character of $\widehat{\mathscr{A}}$ by continuity. Now assume $X_1^{\mathrm{cl}} \subseteq X$ is not all of X. Then $X \setminus X_1^{\mathrm{cl}}$ is a non-empty open subset. Hence we find a non-empty open subset $O \subseteq X$ with $O^{\mathrm{cl}} \subseteq X \setminus X_1^{\mathrm{cl}}$ by the usual separation properties of compact Hausdorff spaces. Moreover, by Urysohn's Lemma we find two functions $\chi, \psi \in \mathscr{C}(X)$ with $\chi\big|_{X_1^{\mathrm{cl}}} = 1$ but $\chi\big|_{O^{\mathrm{cl}}} = 0$ and $\psi \ne 0$ but $\operatorname{supp}\psi \subseteq O$. We claim that $\chi \in \widehat{\mathscr{A}}$ is invertible. Indeed, suppose χ is not invertible then $\widehat{\mathscr{A}}\chi \subseteq \widehat{\mathscr{A}}$ is a proper ideal therefore contained in a maximal ideal. Hence there is a character δ_{x_1} of $\widehat{\mathscr{A}}$ with $\delta_{x_1}(\chi) = 0$ by Lemma 4.2.33. Since δ_{x_1} is still a character for the subalgebra $\mathscr{A} \subseteq \widehat{\mathscr{A}}$ we have a $x_1 \in X$ and $\delta_{x_1}(\chi) = \chi(x_1)$, thereby justifying our abuse of notation. Clearly, $x_1 \in X_1$.

However, $\chi = 1$ on X_1^{cl} which is a contradiction. Thus χ is invertible in $\widehat{\mathscr{A}}$. Now we have $\chi\psi = 0$ which implies $\psi = 0$ as χ is invertible, again a contradiction. Thus we conclude $X_1^{\mathrm{cl}} = X$. But then the continuity of characters according to Lemma 4.2.31, *ii.)*, gives for all $f \in \mathscr{A}$ and $x \in X_1$ the estimate $|f(x)| \leq \|f\|_1$ and hence also $\sup_{x\in X_1}|f(x)| \leq \|f\|_1$. Since f is continuous and $X_1^{\mathrm{cl}} = X$ this implies $\|f\|_\infty \leq \|f\|_1$, proving the claim. □

Corollary 4.3.20 *For commutative C^*-algebras the *-involution and the C^*-norm are uniquely determined.*

PROOF: Suppose that $(\mathscr{A}, \|\cdot\|, ^*)$ is a C^*-algebra with another norm $\|\cdot\|_1$ and another *-involution ^{*_1} making it a C^*-algebra as well. If $\mathscr{A}$ is non-unital then $\widetilde{\mathscr{A}}$ becomes a C^*-algebra as usual, for both choices. Then the identity $\mathrm{id}\colon \widetilde{\mathscr{A}} \longrightarrow \widetilde{\mathscr{A}}$ is a unital homomorphism which is, by Corollary 4.3.18, a *-isomorphism. It also maps $\mathscr{A}$ to $\mathscr{A}$. □

Corollary 4.3.21 *If $\|\cdot\|_1$ is another C^*-norm on a commutative C^*-algebra then $\|\cdot\|_1 = \|\cdot\|$.*

PROOF: Note that we do not assume that $\|\cdot\|_1$ is a *complete* norm. Nevertheless, completing $\mathscr{A}$ with respect to $\|\cdot\|_1$ gives a C^*-algebra $\widehat{\mathscr{A}}$. Since in general spectra only shrink when passing to a bigger algebra, see Theorem 4.2.19, *vi.)*, we get $\varrho_{\widehat{\mathscr{A}}}(a) \leq \varrho_{\mathscr{A}}(a)$ for all $a \in \mathscr{A}$. Here we might first pass to the unitization in the non-unital case. Since for a commutative C^*-algebra the spectral radius is just the C^*-norm we have $\|a\|_1 \leq \|a\|$ which gives equality by Theorem 4.3.19. □

4.3.3 The Continuous Calculus

Let $\mathscr{A}$ be a unital C^*-algebra, not necessarily commutative. For an element $a \in \mathscr{A}$ we denote by $\mathbb{C}[a] \subseteq \mathscr{A}$ the unital subalgebra of all polynomials in a. In general, this will not be a *-subalgebra unless e.g. $a^* = a$. Thus by $\mathbb{C}\langle a, a^*\rangle \subseteq \mathscr{A}$ we denote the unital *-subalgebra of all (noncommutative) polynomials in a and a^*. Note that we have to take care of the order as a and a^* do not commute in general, unless a is normal. To emphasize the commutativity in the normal case we write $\mathbb{C}[a, a^*]$ in the normal case. Clearly, $\mathbb{C}\langle a, a^*\rangle$ is the smallest unital *-subalgebra containing a (and hence also a^*). In general, we define $\mathbb{C}\langle a_i\rangle_{i\in I}$ to be the unital (noncommutative) subalgebra of $\mathscr{A}$ which consists of all (noncommutative) polynomials in the elements $\{a_i\}_{i\in I}$. It will be a *-subalgebra if with a_i also a_i^* is in the set of *generators* $\{a_i\}_{i\in I}$. Again, $\mathbb{C}\langle a_i\rangle_{i\in I}$ is the smallest unital subalgebra containing all the elements $\{a_i\}_{i\in I}$ and $\mathbb{C}\langle a_i, a_i^*\rangle_{i\in I}$ is the smallest unital *-subalgebra containing all the $\{a_i\}_{i\in I}$.

These subalgebras will be commutative iff all the generators commute among each other. In particular, for $\mathbb{C}\langle a_i, a_i^*\rangle_{i\in I}$ this can only happen if the a_i are all normal elements. Finally, we denote by

$$\mathsf{C}^*\langle a_i\rangle_{i\in I} = \left(\mathbb{C}\langle a_i, a_i^*\rangle_{i\in I}\right)^{\mathrm{cl}} \subseteq \mathscr{A} \tag{4.3.22}$$

the topological closure of $\mathbb{C}\langle a_i, a_i^*\rangle_{i\in I}$ inside $\mathscr{A}$. Clearly, this is the smallest unital C^*-subalgebra of $\mathscr{A}$ containing all the generators $\{a_i\}_{i\in I}$. Moreover, $\mathsf{C}^*\langle a_i\rangle_{i\in I}$ is commutative iff all the generators and their adjoints commute, i.e. if

$$a_i a_j = a_j a_i \quad \text{and} \quad a_i a_j^* = a_j^* a_i \tag{4.3.23}$$

for all $i, j \in I$. This is clear by a simple continuity argument. In fact, the second requirement follows actually from the weaker property $a_i a_j = a_j a_i$ and a_i being normal, see Exercise 4.5.43. The importance of the C^*-subalgebra $\mathsf{C}^*\langle a_i\rangle_{i\in I}$ comes in particular from the following proposition:

Proposition 4.3.22 *Let $\mathscr{A}$ be a unital C^*-algebra and let $a \in \mathscr{A}$ be invertible. Then $a^{-1} \in \mathsf{C}^*\langle a\rangle$.*

PROOF: We consider the case $a = a^*$ first. Then we know that $\operatorname{spec}_{\mathscr{A}}(a) \subseteq \big[-\|a\|, \|a\|\big]$ by Theorem 4.3.9, *iv.)*, and $0 \notin \operatorname{spec}_{\mathscr{A}}(a)$ by assumption. Let $z_0 = 2\mathrm{i}\|a\|$ then

$$\operatorname{Res}_{z_0}(a) = (z_0 - a)^{-1} = \frac{1}{z_0} \sum_{n=0}^{\infty} \left(\frac{a}{z_0}\right)^n,$$

by the usual geometric series argument. Since each term of the series is in $\mathsf{C}^*\langle a \rangle$ we conclude that $\operatorname{Res}_{z_0}(a) \in \mathsf{C}^*\langle a \rangle$, too, since $\mathsf{C}^*\langle a \rangle$ is a *closed* subalgebra. Now $\mathsf{C}^*\langle a \rangle$ is commutative as a was assumed to be Hermitian and thus $\operatorname{Res}_{z_0}(a)$ is normal. Hence $\|\operatorname{Res}_{z_0}(a)\| = \varrho_{\mathscr{A}}(\operatorname{Res}_{z_0}(a))$. By the calculus for spectral values according to Theorem 4.2.19 we see that $\lambda \in \operatorname{spec}_{\mathscr{A}}(\operatorname{Res}_{z_0}(a))$ iff there is a $\mu \in \operatorname{spec}_{\mathscr{A}}(a)$ with $\lambda = \frac{1}{z_0 - \mu}$. This allows to determine the spectral radius of $\operatorname{Res}_{z_0}(a)$ to be

$$\begin{aligned}
\varrho_{\mathscr{A}}(\operatorname{Res}_{z_0}(a)) &= \sup\big\{|\lambda| \;\big|\; \lambda \in \operatorname{spec}_{\mathscr{A}}(\operatorname{Res}_{z_0}(a))\big\} \\
&= \sup\left\{\tfrac{1}{z_0-\mu} \;\middle|\; \mu \in \operatorname{spec}_{\mathscr{A}}(a)\right\} \\
&= \frac{1}{\operatorname{dist}(z_0, \operatorname{spec}_{\mathscr{A}}(a))},
\end{aligned}$$

where $\operatorname{dist}(z_0, \operatorname{spec}_{\mathscr{A}}(a))$ is the distance from z_0 to the compact subset $\operatorname{spec}_{\mathscr{A}}(a) \subseteq \big[-\|a\|, \|a\|\big] \subseteq \mathbb{C}$. Since 0 is not in the spectrum and since $\operatorname{spec}_{\mathscr{A}}(a)$ is closed there is a $\epsilon > 0$ with $(-\epsilon, \epsilon) \cap \operatorname{spec}_{\mathscr{A}}(a) = \emptyset$. Then elementary triangle geometry in the complex plane shows that the distance from z_0 on the imaginary axis to $\operatorname{spec}_{\mathscr{A}}(a)$ on the real axis is *strictly* larger than $|z_0| = 2\|a\|$. Thus $|z_0| < \frac{1}{\|\operatorname{Res}_{z_0}(a)\|}$ follows and we can apply Corollary 4.2.15 for the value $z = 0$ yielding

$$a^{-1} = \operatorname{Res}_0(a) = \sum_{n=0}^{\infty} (\operatorname{Res}_{z_0}(a))^{n+1} z_0^n.$$

As we have already shown $\operatorname{Res}_{z_0}(a) \in \mathsf{C}^*\langle a \rangle$ this implies $a^{-1} \in \mathsf{C}^*\langle a \rangle$, proving the Hermitian case. Now assume $a \in \mathscr{A}$ is an arbitrary invertible element. Then a^*a is Hermitian and contained in $\mathsf{C}^*\langle a \rangle$. Moreover, since $(a^*)^{-1} = (a^{-1})^*$ whenever one of the inverses exists, we conclude that a^*a is invertible as well. Hence $(a^*a)^{-1} \in \mathsf{C}^*\langle a \rangle$ and hence also $a^{-1} = (a^*a)^{-1}a^* \in \mathsf{C}^*\langle a \rangle$. □

Corollary 4.3.23 *Let $a \in \mathscr{B} \subseteq \mathscr{A}$ be an element in a C^*-subalgebra $\mathscr{B}$ of a unital C^*-algebra $\mathscr{A}$ with $\mathbb{1} \in \mathscr{B}$. Then*

$$\operatorname{spec}_{\mathscr{B}}(a) = \operatorname{spec}_{\mathscr{A}}(a). \tag{4.3.24}$$

PROOF: Since the relevant inverses are, if existing, already contained in the smallest unital C^*-subalgebra $\mathsf{C}^*\langle a \rangle$ generated by a, the statement is clear. □

Remark 4.3.24 This corollary allows us to speak of *the* spectrum of an algebra element in a C^*-algebra without reference to the precise ambient C^*-algebra. Thus for $a \in \mathscr{A}$ in a C^*-algebra we will write

$$\operatorname{spec}(a) = \operatorname{spec}_{\mathscr{A}}(a), \tag{4.3.25}$$

$$\operatorname{res}(a) = \operatorname{res}_{\mathscr{A}}(a), \tag{4.3.26}$$

and

$$\varrho(a) = \varrho_{\mathscr{A}}(a) \tag{4.3.27}$$

in the following. Note however, that this corollary will, in general, be false for Banach algebras.

Since the whole spectral information of $a \in \mathscr{A}$ is contained in $\mathsf{C}^*\langle a\rangle$ the situation simplifies drastically for *normal* elements: in this case $\mathsf{C}^*\langle a\rangle$ is *commutative* and hence given by $\mathscr{C}(\operatorname{Spec}(\mathsf{C}^*\langle a\rangle))$. This is now the crucial idea of the continuous functional calculus: for continuous functions f on a compact Hausdorff space X we get again a continuous function $g \circ f$ whenever g is a continuous function on the set of values of f, see also Exercise 4.5.20. Since on the other hand $\mathscr{C}(\operatorname{Spec}(\mathsf{C}^*\langle a\rangle)) \cong \mathsf{C}^*\langle a\rangle$ we can perform such operations on all elements of $\mathsf{C}^*\langle a\rangle$ and in particular on a itself. Making this idea precise leads to the following theorem:

Theorem 4.3.25 (Continuous calculus) *Let $\mathscr{A}$ be a unital C^*-algebra and let $a \in \mathscr{A}$ be normal.*

i.) The map

$$\hat{a}\colon \operatorname{Spec}(\mathsf{C}^*\langle a\rangle) \longrightarrow \operatorname{spec}(a) \tag{4.3.28}$$

is a homeomorphism.

ii.) This homeomorphism induces via pull-back a C^*-algebra isomorphism

$$\mathscr{C}(\operatorname{spec}(a)) \ni f \;\mapsto\; f(a) \in \mathsf{C}^*\langle a\rangle, \tag{4.3.29}$$

such that polynomials $p \in \mathbb{C}[z, \overline{z}]$ on $\operatorname{spec}(a)$ are mapped to the corresponding polynomials $p(a, a^*)$ in a and a^*.

Proof: We know from Theorem 4.2.36, *iv.)*, that $\operatorname{spec}(\hat{a}) = \operatorname{spec} a$. Since $\hat{a} \in \mathscr{C}(\operatorname{Spec}(\mathsf{C}^*\langle a\rangle))$ is a continuous function on a compact Hausdorff space its spectrum is simply the set of values $\hat{a}(\operatorname{Spec}(\mathsf{C}^*\langle a\rangle))$, see also Exercise 4.5.17, *vi.)*. Thus $\hat{a}$ maps $\operatorname{Spec}(\mathsf{C}^*\langle a\rangle)$ continuously into $\operatorname{spec}(\hat{a}) = \operatorname{spec}(a)$. Clearly, $\hat{a}$ is surjective onto its values so it remains to show injectivity of (4.3.28). Let $\varphi, \psi \in \operatorname{Spec}(\mathsf{C}^*\langle a\rangle)$ be two characters with $\hat{a}(\varphi) = \hat{a}(\psi)$, i.e. $\varphi(a) = \psi(a)$. Since characters of a commutative C^*-algebra are automatically *-homomorphisms by Corollary 4.3.11 we also have $\varphi(a^*) = \psi(a^*)$. But then $\varphi\big|_{\mathbb{C}[a,a^*]} = \psi\big|_{\mathbb{C}[a,a^*]}$ follows at once. By continuity of the characters we get $\varphi = \psi$. This shows the injectivity. Since both spaces are compact Hausdorff spaces, the inverse of (4.3.28) is necessarily continuous, too, and hence we get a homeomorphism, see Lemma **??**. This proves the first part. For the second part we know from the equivalence of categories according to Theorem 4.3.14 that the pull-back with $\hat{a}$ gives an isomorphism of unital C^*-algebras $\mathscr{C}(\operatorname{spec}(a)) \longrightarrow \mathscr{C}(\operatorname{Spec}(\mathsf{C}^*\langle a\rangle))$. Moreover, the Gel'fand-Naimark Theorem tells us that the Gel'fand transform $\hat{\ }\colon \mathsf{C}^*\langle a\rangle \longrightarrow \mathscr{C}(\operatorname{Spec}(\mathsf{C}^*\langle a\rangle))$ is an isomorphism of C^*-algebras as well. Thus we get an isomorphism (4.3.29) as claimed. Now let $p(z, \overline{z}) = z$ be the identity viewed as a polynomial function on $\operatorname{spec}(a)$. Then its image in $\mathscr{C}(\operatorname{Spec}(\mathsf{C}^*\langle a\rangle))$ under the pull-back $\hat{a}^*$ is given by

$$(\hat{a}^* p)(\varphi) = p\Big(\hat{a}(\varphi), \overline{\hat{a}(\varphi)}\Big) = \hat{a}(\varphi)$$

for every character φ. Thus $\hat{a}^* p = \hat{a}$, which shows that the image of this polynomial p under (4.3.29) is just a. Since (4.3.29) is a *-homomorphism, this is enough to obtain the claim of the second part. □

Remark 4.3.26 (Continuous calculus) Let $\mathscr{A}$ be a unital C^*-algebra and let $a \in \mathscr{A}$ be normal.

i.) Since the continuous calculus is a norm preserving *-homomorphism we can approximate $f \in \mathscr{C}(\operatorname{spec}(a))$ uniformly by some $f_n \in \mathscr{C}(\operatorname{spec}(a))$ and get $f_n(a) \longrightarrow f(a)$ in the norm topology of $\mathscr{A}$.

ii.) Since every $f \in \mathscr{C}(\operatorname{spec}(a))$ can be approximated on the compact spectrum $\operatorname{spec}(a) \subseteq \mathbb{C}$ by polynomials $p_n \in \mathbb{C}[z, \overline{z}]$ by the Stone-Weierstraß Theorem, the corresponding polynomials in a and a^* approximate $f(a)$. This gives an efficient way to compute $f(a)$ in applications.

iii.) Clearly, for a holomorphic function $f \in \mathscr{O}(\mathrm{B}_\varrho(0))$ with $\varrho > \varrho(a) = \|a\|$ we reproduce the holomorphic calculus by (4.3.29). Indeed, the approximation of f by its Taylor series is uniform on the compact subset $\operatorname{spec}(a) \subseteq \mathrm{B}_\varrho(0)$. Thus the two calculi give the same result by continuity.

iv.) Let $\mathscr{B}$ be another unital C^*-algebra and let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a (continuous) unital *-homomorphism. In fact, we will see later that the requirement of continuity is superfluous as *-homomorphisms are automatically continuous. Then we have $\Phi(f(a)) = f(\Phi(a))$ for polynomials f on $\operatorname{spec}(a)$. By continuity of Φ, we can therefore pass to the continuous functions by uniform approximations. Thus we have for general $f \in \mathscr{C}(\operatorname{spec}(a))$

$$\Phi(f(a)) = f(\Phi(a)). \tag{4.3.30}$$

Note that f is indeed continuous on $\operatorname{spec}(\Phi(a))$ by (4.2.35) as usual.

The non-unital case requires some minor modifications. Of course, for a non-unital C^*-algebra $\mathscr{A}$ we can use the continuous functional calculus of its unitization $\widetilde{\mathscr{A}}$ in order to define $f(a)$ for a normal $a \in \mathscr{A}$. But then $f(a) \in \widetilde{\mathscr{A}}$ and the question is when we achieve $f(a) \in \mathscr{A}$. To this end we note the following proposition which is also of independent interest:

Proposition 4.3.27 *Let $K \subseteq \mathbb{R}^n$ be compact and $0 \in K$. Then the closure of the polynomials which vanish at 0 inside $\mathscr{C}(K)$ is given by the continuous functions which vanish at 0.*

PROOF: Since the uniform limit of polynomials which vanish at 0 is also a pointwise limit, the limit will still vanish at 0. Thus the closure of those polynomials consists of functions $\mathscr{J} \subseteq \mathscr{C}(K)$ which vanish at 0. In particular, $\mathscr{J} \subseteq \ker \delta_0$ is in the kernel of δ_0. Moreover, $\mathscr{J}$ is an ideal: indeed, if $f = \lim_{n\longrightarrow\infty} p_n$ can be written as such a uniform limit and $g \in \mathscr{C}(K)$ is arbitrary, then we can obtain $g = \lim_{n\longrightarrow\infty} g_n$ as a uniform limit of polynomials g_n, again by the Stone-Weierstraß Theorem. Thus $fg = \lim_{n\longrightarrow\infty} f_n g_n$ but $f_n g_n$ vanishes at 0. Hence $\mathscr{J}$ is an ideal and even a closed ideal in $\mathscr{C}(K)$. By Theorem 4.3.10 we know that $\mathscr{J}$ is the vanishing ideal $\mathscr{J}_A$ of some closed subset $A \subseteq K$ which clearly contains 0. Since for every point $x \in K \setminus \{0\}$ we clearly have a polynomial with $p(x) = 1$ but $p(0) = 0$ we see that A can not contain any other point than 0. Thus the claim follows. ☐

Corollary 4.3.28 *Let $\mathscr{A}$ be a non-unital C^*-algebra and let $a \in \mathscr{A}$ be a normal element. Then the continuous calculus inside $\widetilde{\mathscr{A}}$ gives $f(a) \in \mathscr{A}$ for those $f \in \mathscr{C}(\operatorname{spec}(a))$ with $f(0) = 0$.*

PROOF: We know that $f(a)$ can be approximated by polynomials p_n in a and a^* for suitable $p_n \in \mathbb{C}[z, \overline{z}]$ if $p_n \longrightarrow f$ uniformly on $\operatorname{spec}(a)$. If $f(0) = 0$ then we can choose p_n with $p_n(0) = 0$ by the proposition. But this means that p_n has no constant term and hence, evaluated on a and a^* we get an element in $\mathscr{A}$. Since $\mathscr{A}$ is closed in $\widetilde{\mathscr{A}}$, we conclude that also the limit $f(a)$ is in $\mathscr{A}$. ☐

When applying a continuous function to a normal element its spectrum behaves well. This is the content of the Spectral Mapping Theorem:

Theorem 4.3.29 (Spectral Mapping Theorem) *Let $\mathscr{A}$ be a unital C^*-algebra and let $a \in \mathscr{A}$ be a normal element. Then for all $f \in \mathscr{C}(\operatorname{spec}(a))$ we have*

$$\operatorname{spec}(f(a)) = f(\operatorname{spec}(a)). \tag{4.3.31}$$

PROOF: Since the continuous calculus is a *-isomorphism from $\mathscr{C}(\operatorname{spec}(a))$ to $\mathsf{C}^*\langle a\rangle$ the spectrum of $f(a) \in \mathsf{C}^*\langle a\rangle$ coincides with the spectrum of $f \in \mathscr{C}(\operatorname{spec}(a))$, given by $f(\operatorname{spec}(a))$. But the spectrum of an element $b \in \mathscr{A}$ is already determined by the smallest unital C^*-subalgebra containing b. Hence the spectrum of $f(a)$ in $\mathsf{C}^*\langle a\rangle$ coincides with the spectrum of $f(a)$ with respect to $\mathscr{A}$. ☐

Corollary 4.3.30 *Let $\mathscr{A}$ be a unital C^*-algebra and let $a \in \mathscr{A}$ be a normal element. For $g \in \mathscr{C}(\operatorname{spec}(a))$ and $f \in \mathscr{C}(g(\operatorname{spec}(a)))$ one has $f \circ g \in \mathscr{C}(\operatorname{spec}(a))$ and*

$$(f \circ g)(a) = f(g(a)). \tag{4.3.32}$$

PROOF: Clearly, $f \circ g \in \mathscr{C}(\operatorname{spec}(a))$ and hence $(f \circ g)(a)$ is defined by the continuous calculus. Moreover, $g(\operatorname{spec}(a)) = \operatorname{spec}(g(a))$ by the Spectral Mapping Theorem 4.3.29 and thus $f(g(a))$ is defined by the continuous calculus, too. Since the continuous calculus takes place in the commutative C^*-algebra $\mathsf{C}^*\langle a \rangle \cong \mathscr{C}(\operatorname{spec}(a))$, we only have to prove the claim (4.3.32) for the continuous function $\hat{a} \in \mathscr{C}(\operatorname{Spec}(\mathsf{C}^*\langle a \rangle)) = \mathscr{C}(\operatorname{spec}(a))$, which is trivial, see also Exercise 4.5.20. $\square$

Summarizing, the continuous calculus provides a vast generalization of our previous entire holomorphic or holomorphic calculus. However, the price one has to pay is the restriction to *normal* elements of a C^*-algebra.

4.4 States and *-Representations of C^*-Algebras

We will continue our discussion of C^*-algebras and investigate their states and representations according to our general ideas from Chapter 1. The first crucial step will be to understand the positive elements in a C^*-algebra: here we have at least three different competing definitions: the positive and the algebraic positive elements according to Definition 1.2.8 and now also the Hermitian elements with positive spectrum. We will show some additional properties and prove that all three characterizations actually coincide. In a next step we consider approximate identities. They provide a good replacement for a unit element in C^*-algebras without units. Of course, we know that we can always adjoint units but sometimes this is not a suitable choice. Instead the approximate identities prove to be a valuable tool. We will then show that C^*-algebras always have sufficiently many positive linear functionals to separate elements. One consequence will be that there is always a faithful *-representation on a pre-Hilbert space. Since we have now analytic tools around, the last step in this section consists in showing that *-representations (and also *-homomorphisms) of C^*-algebra enjoy automatic continuity properties. This leads to a refined representation theory for C^*-algebras.

4.4.1 Positivity in a C^*-Algebra

Before discussing the states of a C^*-algebra we focus on the positive elements. More precisely, in the terminology of Chapter 1, we consider the algebraically positive elements, i.e. those which are convex combinations of squares a^*a. Since we have a well-behaved notion of spectrum we can also consider the "spectrally positive" elements, i.e. those with spectrum in $[0, \infty)$. In a general Banach *-algebra this will not lead to anything interesting. For a C^*-algebra, however, we have the following characterization:

Proposition 4.4.1 *Let $\mathscr{A}$ be a C^*-algebra and let $a = a^*$ be a Hermitian element in $\mathscr{A}$. Then the following statements are equivalent:*

i.) *One has* $\operatorname{spec}(a) \subseteq [0, \|a\|]$.

ii.) *There is a Hermitian $b = b^*$ with $a = b^2$.*

iii.) *For all $t \geq \|a\|$ one has*
$$\|t\mathbb{1} - a\| \leq t. \tag{4.4.1}$$

iv.) *There is a $t \geq \|a\|$ with*
$$\|t\mathbb{1} - a\| \leq t. \tag{4.4.2}$$

PROOF: In the non-unital case, part *iii.)* and *iv.)* refer to the unitization $\widetilde{\mathscr{A}}$ while in the second statement one requires $b \in \mathscr{A}$ and not only $b \in \widetilde{\mathscr{A}}$. For the first implication *i.)* $\implies$ *ii.)* we note that $x \mapsto \sqrt{x}$ is a continuous function on $\operatorname{spec}(a)$, vanishing at 0. By the continuous calculus there is a Hermitian square root $\sqrt{a} \in \mathsf{C}^*\langle a \rangle$ with $\sqrt{a}\sqrt{a} = a$ and $\sqrt{a} \in \mathscr{A}$ in the non-unital case. We can take this as b. Note that without the continuous calculus this would hardly be achievable. The reverse implication *ii.)* $\Longrightarrow$ *i.)* is clear from Theorem 4.3.9, *iv.)*. For *i.)* $\Longrightarrow$ *iii.)* we first adjoin a unit if necessary. Thus let $t \geq \|a\|$ be given and observe that

$$\|t\mathbb{1} - a\| = \varrho(t\mathbb{1} - a) = \sup\bigl\{|t - \lambda| \bigm| \lambda \in \operatorname{spec}(a)\bigr\} \leq t,$$

since $0 \leq \lambda \leq \|a\| \leq t$. The implication *iii.)* $\Longrightarrow$ *iv.)* is trivial. Now assume $t \geq \|a\|$ satisfies (4.4.2). Then for $\lambda \in \operatorname{spec}(a)$ we have $t - \lambda \in \operatorname{spec}(t\mathbb{1} - a)$ by Theorem 4.2.19, *ii.)*. Hence $|t - \lambda| \leq \|t\mathbb{1} - a\|$ by the general fact (4.2.24) and thus $|t - \lambda| \leq t$ by assumption. But this is only possible for $\lambda \geq 0$ showing the remaining implication. □

Corollary 4.4.2 *Let $a, b \in \mathscr{A}$ be Hermitian elements of a C^*-algebra $\mathscr{A}$. If* $\operatorname{spec}(a), \operatorname{spec}(b) \subseteq [0, \infty)$ *and $\alpha, \beta \geq 0$ then* $\operatorname{spec}(\alpha a + \beta b) \subseteq [0, \infty)$*, too.*

PROOF: Let $t = \|a\| + \|b\| \geq \|a + b\|$. Then we have

$$\|t\mathbb{1} - (a + b)\| \leq \bigl\|\|a\|\mathbb{1} - a\bigr\| + \bigl\|\|b\|\mathbb{1} - b\bigr\| \leq \|a\| + \|b\| = t$$

by Proposition 4.4.1, *iii.)*. Thus by the fourth part of the same proposition we get the conclusion for $a + b$. The rescaling of a by $\alpha \geq 0$ clearly preserves the positivity of the spectrum by the general statement (4.2.31). □

Corollary 4.4.3 *Let $\mathscr{A}$ be a C^*-algebra. The set of Hermitian elements with spectrum in* $[0, \infty)$ *is closed.*

PROOF: For a Hermitian element $a = a^* \in \mathscr{A}$ we know that $\operatorname{spec}(a) \subseteq [0, \infty)$ iff $\bigl\|\|a\|\mathbb{1} - a\bigr\| \leq \|a\|$ viewed as relation in the unitization if $\mathscr{A}$ is non-unital: this is obtained by setting $t = \|a\|$ in Proposition 4.4.1, *iii.)*. Now this inequality is clearly build out of continuous terms and hence the algebra elements fulfilling it yield a closed subset. □

In other words, for a C^*-algebra we have found yet another, now even closed, convex cone of "positive elements". As for a general *-algebra, we have $\mathscr{A}^+$ defined by means of the positive linear functionals, $\mathscr{A}^{++}$ defined as sums of squares, and now for a C^*-algebra we have also the Hermitian elements with spectrum in the non-negative real numbers. It will turn out that all these three cones actually coincide.

Using our continuous calculus we consider the following continuous functions on the real line and on $\mathbb{R}_0^+$: for $x \geq 0$ we have the square root $\sqrt{x}$ as usual. For $x \in \mathbb{R}$ we have the absolute value $|x|$ of x as well as the positive and negative part $x_\pm$ defined by $x_\pm = \frac{1}{2}(|x| \pm x)$. All these continuous functions vanish at $x = 0$ which will allow to use them for the continuous calculus also in the non-unital situation.

Proposition 4.4.4 *Let $\mathscr{A}$ be a C^*-algebra and let $a = a^*$ be a Hermitian element in $\mathscr{A}$.*

i.) If $\operatorname{spec}(a) \subseteq [0, \|a\|]$ *then there exists a unique Hermitian element $\sqrt{a} \in \mathscr{A}$ with the properties*

$$\sqrt{a}^2 = a \quad \textit{and} \quad \operatorname{spec}(\sqrt{a}) \subseteq \bigl[0, \sqrt{\|a\|}\bigr]. \tag{4.4.3}$$

Moreover, one has $\sqrt{a} \in \mathsf{C}^*\langle a\rangle$.

ii.) There are Hermitian elements $|a|$, a_+, and a_- with the properties

$$a = a_+ - a_-, \tag{4.4.4}$$

$$a_+ a_- = 0 = a_- a_+, \tag{4.4.5}$$

$$\operatorname{spec}(a_\pm) \subseteq [0, \|a\|], \tag{4.4.6}$$

$$|a| = a_+ + a_- = \sqrt{a^2}, \tag{4.4.7}$$

$$\operatorname{spec}(|a|) \subseteq [0, \|a\|], \tag{4.4.8}$$

$$|a|^2 = a^2, \tag{4.4.9}$$

$$a_\pm = \tfrac{1}{2}(|a| \pm a). \tag{4.4.10}$$

Moreover, one has $|a|, a_+, a_- \in \mathsf{C}^*\langle a\rangle$. *The elements* $|a|, a_+, a_-$ *are uniquely determined by* (4.4.8) *and* (4.4.9) *as well as by* (4.4.4), (4.4.5), *and* (4.4.6), *respectively.*

PROOF: The existence of the elements $\sqrt{a}$ as well as $|a|$, a_+, and a_- with the described properties follows at once form the continuous calculus and the Spectral Mapping Theorem. It remains to check the uniqueness of the square root as well as the uniqueness of $|a|$, a_+, and a_-. Thus assume first that $b \in \mathscr{A}$ is another Hermitian square root of a with $\operatorname{spec}(b) \subseteq \big[0, \sqrt{\|a\|}\big]$. We choose a sequence p_n of polynomials which approximate the square root function $x \mapsto \sqrt{x}$ uniformly on $[0, \|a\|]$ by evoking the Stone-Weierstraß Theorem or the explicitly given sequence from Exercise 4.5.19. Next we define $q_n(x) = p_n(x^2)$. On $\big[0, \sqrt{\|a\|}\big]$ it follows that q_n converges uniformly to the identity function $x \mapsto x$. Thus we have by the continuity properties of the continuous calculus for b according to Remark 4.3.26, *ii.)*,

$$b = \lim_{n \longrightarrow \infty} q_n(b) = \lim_{n \longrightarrow \infty} p_n(b^2) = \lim_{n \longrightarrow \infty} p_n(a) = \sqrt{a},$$

showing the uniqueness of the square root with non-negative spectrum. Now let $a = a^*$ be a general Hermitian element. By $|a|^2 = a^2$ and $\operatorname{spec}(|a|) \subseteq [0, \|a\|]$ we see from the uniqueness statement of the first part that $|a|$ is the unique square root of a^2 with non-negative spectrum. Next assume that $b_\pm^* = b_\pm$ satisfy (4.4.4), (4.4.5), and (4.4.6). In fact, $b_+ b_- = 0$ implies $b_- b_+ = 0$ and vice versa by applying the *-involution. We have $a^2 = (b_+ - b_-)^2 = b_+^2 + b_-^2 = (b_+ + b_-)^2$ by (4.4.5) for $b_\pm$. Since $\operatorname{spec}(b_+ + b_-) \subseteq \mathbb{R}_0^+$ by Corollary 4.4.2, we see that $b_+ + b_-$ is the unique square root of a^2 and hence $b_+ + b_- = |a|$. Thus $b_\pm = \frac{1}{2}(|a| \pm a) = a_\pm$ follows, proving also the last uniqueness statement. □

This decomposition of a Hermitian element a into its *positive* and *negative* part a_+ and a_-, respectively, allows to determine the algebraically positive elements $\mathscr{A}^{++}$ as follows:

Theorem 4.4.5 (Positive elements) *Let* $\mathscr{A}$ *be a* C^**-algebra. Then the following statements for* $a \in \mathscr{A}$ *are equivalent:*

i.) $a \in \mathscr{A}^{++}$, *i.e. there are* $b_i \in \mathscr{A}$ *and* $\alpha_i > 0$ *with* $a = \sum_{i=1}^n \alpha_i b_i^* b_i$.

ii.) $a = a^*$ *and* $\operatorname{spec}(a) \subseteq \mathbb{R}_0^+$ *(and hence even* $\operatorname{spec}(a) \subseteq [0, \|a\|]$*).*

iii.) $a = b^2$ *for some* $b = b^* \in \mathscr{A}$.

iv.) $a = a^*$ *and* $a = a_+$.

v.) $a = a^*$ *and* $a = |a|$.

vi.) $a = c^2$ *with* $c^* = c$ *and* $\operatorname{spec}(c) \subseteq \mathbb{R}_0^+$ *(which means* $c = \sqrt{a}$*).*

PROOF: Assume first $a = b^* b$. Then $a = a^*$ is clearly fulfilled and we have $a = a_+ - a_-$ according to Proposition 4.4.4, *ii.)*. We have to show $a_- = 0$. We have

$$(b\sqrt{a_-})^*(b\sqrt{a_-}) = \sqrt{a_-} b^* b \sqrt{a_-} = \sqrt{a_-}(a_+ - a_-)\sqrt{a_-} = -a_-^2,$$

since $\sqrt{a_-} a_+ = 0$ by the continuous calculus of a. Since $\operatorname{spec}(a_-^2) \subseteq \mathbb{R}_0^+$ we have

$$\operatorname{spec}((b\sqrt{a_-})^*(b\sqrt{a_-})) = \operatorname{spec}(-a_-^2) \subseteq \mathbb{R}_0^-. \tag{$*$}$$

Now let us decompose $b\sqrt{a_-}$ into its real and imaginary part, i.e. we write $b\sqrt{a_-} = c + \mathrm{i}d$ with $c = \operatorname{Re}(b\sqrt{a_-})$ and $d = \operatorname{Im}(b\sqrt{a_-})$, both Hermitian. Then

$$(b\sqrt{a_-})(b\sqrt{a_-})^* = cc^* + dd^* + \mathrm{i}dc^* - \mathrm{i}cd^* = c^2 + d^2 + \mathrm{i}(dc - cd),$$

while on the other hand $(b\sqrt{a_-})^*(b\sqrt{a_-}) = c^2 + d^2 - \mathrm{i}(dc - cd)$. This shows

$$(b\sqrt{a_-})(b\sqrt{a_-})^* = 2(c^2 + d^2) - (b\sqrt{a_-})^*(b\sqrt{a_-}).$$

Since $\operatorname{spec}(c^2 + d^2) \subseteq \mathbb{R}_0^+$ by Corollary 4.4.2 and Theorem 4.3.9, *iv.)*, and since for $-(b\sqrt{a_-})^*(b\sqrt{a_-})$ we have $\operatorname{spec}(-(b\sqrt{a_-})^*(b\sqrt{a_-})) \subseteq \mathbb{R}_0^+$ by $(*)$ we obtain by another application of Corollary 4.4.2 that $\operatorname{spec}((b\sqrt{a_-})(b\sqrt{a_-})^*) \subseteq \mathbb{R}_0^+$. But from the general statement of Theorem 4.2.19, *i.)*, we know that the spectra of $(b\sqrt{a_-})^*(b\sqrt{a_-})$ and $(b\sqrt{a_-})(b\sqrt{a_-})^*$ coincide except for the possible spectral value 0. We conclude that the Hermitian element $(b\sqrt{a_-})^*(b\sqrt{a_-}) = -a_-^2$ has spectrum 0. But for a Hermitian element the norm equals the spectral radius according to Theorem 4.3.9, *i.)*, and hence $-a_-^2 = 0$ follows. Again, since a_- is Hermitian, this shows $a_- = 0$. Hence we have showed $\operatorname{spec}(a) \subseteq \mathbb{R}_0^+$ as wanted. The general case for $a \in \mathscr{A}^{++}$ follows from this by another application of Corollary 4.4.2. Hence the implication *i.)* $\implies$ *ii.)* is shown. The implication *ii.)* $\implies$ *iii.)* was contained in Proposition 4.4.4 by taking $b = \sqrt{a}$. The reverse implication was already proved in Theorem 4.3.9, *iv.)*. Clearly, *iv.)* and *v.)* are equivalent by $|a| = a_+ + a_-$ and $a = a_+ - a_-$. Since $\operatorname{spec}(|a|) \subseteq \mathbb{R}_0^+$ we have the implication *v.)* $\implies$ *ii.)*. Conversely, assuming *ii.)* we see that a itself has the properties (4.4.8) and (4.4.9) which characterize $|a|$ uniquely. Thus $a = |a|$ and hence *v.)*. Clearly, *ii.)* also implies *vi.)* by taking $c = \sqrt{a}$. Finally, the implication *vi.)* $\implies$ *i.)* is trivial. ☐

We conclude that two of the convex cones, namely $\mathscr{A}^{++}$ and the spectrally positive elements, coincide. Note that the above statement uses the C^*-properties in an essential way. None of the conclusions need to be true in a general Banach *-algebra. Here the example of Exercise 4.5.28 is very illustrative.

Corollary 4.4.6 *Let $\mathscr{A}$ be a C^*-algebra. Then every element is a linear combination of four algebraically positive elements. More precisely, $a = \operatorname{Re}(a)_+ - \operatorname{Re}(a)_- + \mathrm{i}\operatorname{Im}(a)_+ - \mathrm{i}\operatorname{Im}(a)_-$ with $\operatorname{Re}(a)_\pm$ being the positive and negative parts of the real part $\operatorname{Re}(a)$ while $\operatorname{Im}(a)_\pm$ are the positive and negative parts of the imaginary part $\operatorname{Im}(a)$. One has*

$$\|\operatorname{Re}(a)_\pm\|, \|\operatorname{Im}(a)_\pm\| \le \|a\|. \tag{4.4.11}$$

Proof: Let $a \in \mathscr{A}$ then we know $a = \operatorname{Re}(a) + \mathrm{i}\operatorname{Im}(a)$ as this holds in a general *-algebra with $\operatorname{Re}(a)$ and $\operatorname{Im}(a)$ Hermitian. According to Proposition 4.4.4, *ii.)*, we can write $\operatorname{Re}(a) = \operatorname{Re}(a)_+ - \operatorname{Re}(a)_-$ and $\operatorname{Im}(a) = \operatorname{Im}(a)_+ - \operatorname{Im}(a)_-$. The norm estimate follows form $\|\operatorname{Re}(a)\| \le \|a\|$ and $\|\operatorname{Im}(a)\| \le \|a\|$ as this holds in every normed *-algebra together with the norm estimate from Proposition 4.4.4, *ii.)*, which we get from $\operatorname{spec}(a_\pm) \subseteq [0, \|a\|]$, i.e. $\|a_\pm\| \le \|a\|$ for every Hermitian element a. ☐

Corollary 4.4.7 *Let $\mathscr{A}$ be a C^*-algebra. Then $\mathscr{A}$ is idempotent, i.e. every element is a linear combination of products.*

Proof: Since every algebraically positive element is a linear combination of products, the proof follows at once from Corollary 4.4.6. Note that the statement becomes trivial if $\mathscr{A}$ is unital. ☐

We use the convex cone $\mathscr{A}^{++}$ to define an order relation on the Hermitian elements of $\mathscr{A}$. In principle, this can be done for every *-algebra by either using $\mathscr{A}^{++}$ or $\mathscr{A}^{+}$. In the C^*-algebra case both coincide anyway, as we shall see later. Hence we do not need to worry about this choice.

Definition 4.4.8 (Order for Hermitian elements) *Let $\mathscr{A}$ be a C^*-algebra and let $a, b \in \mathscr{A}$ be Hermitian. Then we write $a \le b$ if $b - a \in \mathscr{A}^{++}$.*

While for a general *-algebra, this order will not lead to anything particularly interesting, for a C^*-algebra, the following proposition shows that this defines a partial order with nice properties:

Proposition 4.4.9 *Let $\mathscr{A}$ be a C^*-algebra and let $a, b, c, d \in \mathscr{A}$ be Hermitian elements.*

i.) One has $a \geq 0$ iff $a \in \mathscr{A}^{++}$.

ii.) If $a \geq 0$ and $a \leq 0$ then $a = 0$.

iii.) One has $a \leq b$ and $b \leq c$ implies $a \leq c$. If $a \leq b$ and $c \leq d$ then $a + c \leq b + d$.

iv.) If $\mathscr{A}$ is unital then for $a \geq 0$ one has $a \leq \|a\|\mathbb{1}$.

v.) If $0 \leq a \leq b$ then $\|a\| \leq \|b\|$.

vi.) For $a \geq 0$ one has $a^2 \leq a\|a\|$.

vii.) If $0 \leq a \leq b$ then for every $x \in \mathscr{A}$ one has

$$0 \leq x^* a x \leq x^* b x. \tag{4.4.12}$$

viii.) Let $\mathscr{A}$ be unital. If $0 \leq a \leq b$ and $\lambda > 0$ then $a + \lambda\mathbb{1}$ and $b + \lambda\mathbb{1}$ are invertible and we have

$$\frac{1}{a + \lambda\mathbb{1}} \geq \frac{1}{b + \lambda\mathbb{1}}. \tag{4.4.13}$$

ix.) For arbitrary elements $a, b \in \mathscr{A}$ one has $0 \leq (ab)^*(ab) \leq \|a\|^2 b^* b$.

Proof: The first part is clear by definition. By spectral properties of a we know that $a, -a \in \mathscr{A}^{++}$ implies $\operatorname{spec}(a) \subseteq \mathbb{R}_0^+ \cap \mathbb{R}_0^- = \{0\}$ and hence $\varrho(a) = 0$. Thus $\|a\| = 0$ since a is normal, proving the second part. Geometrically, this means that the two cones $\mathscr{A}^{++}$ and $-\mathscr{A}^{++}$ intersect only in $\{0\}$, a feature which is far from being true in general *-algebras, see e.g. the example in Exercise 4.5.29. The third part is just a consequence of $\mathscr{A}^{++}$ being a convex cone as $b - a, c - d \in \mathscr{A}^{++}$ implies $c - b + b - a = c - a \in \mathscr{A}^{++}$. Similarly, $b - a, d - c \in \mathscr{A}^{++}$ give $b + d - a - c \in \mathscr{A}^{++}$. For the fourth part, $a = a^*$ gives $\varrho(a) = \|a\|$ and $\operatorname{spec}(a) \subseteq [0, \|a\|]$ by spectral positivity. Thus by the Spectral Mapping Theorem we have $\operatorname{spec}(\|a\|\mathbb{1} - a) \subseteq [0, \|a\|]$ again. It follows that $\|a\|\mathbb{1} - a$ is still spectrally positive and hence in $\mathscr{A}^{++}$. Now let $0 \leq a \leq b$. Then on the one hand $0 \leq a \leq \|b\|\mathbb{1}$ follows from *iii.)* and *iv.)*. Thus $\operatorname{spec}(\|b\|\mathbb{1} - a) \subseteq \mathbb{R}_0^+$. On the other hand $\operatorname{spec}(\|b\|\mathbb{1} - a) = \|b\| - \operatorname{spec}(a) \subseteq [\|b\| - \|a\|, \|b\|]$ by the Spectral Mapping Theorem and since $\|a\| \in \operatorname{spec}(a)$ for a spectrally positive element a. This implies that $\|b\| - \|a\| \geq 0$ and hence *v.)*. By the Spectral Mapping Theorem we conclude that $\operatorname{spec}(\|a\|a - a^2) \subseteq \mathbb{R}_0^+$ since the function $f(x) = \|a\|x - x^2$ is non-negative on $[0, \|a\|]$. This gives the sixth part. Part *vii.)* is again true in every *-algebra since the cone $\mathscr{A}^{++}$ is mapped into $\mathscr{A}^{++}$ under $a \mapsto c^* a c$, see Remark 1.2.9, *iii.)*. From this, (4.4.12) is a simple consequence. Now, by spectral properties we know that for $\operatorname{spec}(a) \subseteq [0, \|a\|]$ we have $\operatorname{spec}(a + \lambda\mathbb{1}) \subseteq [\lambda, \lambda + \|a\|]$ and hence 0 is not in the spectrum of $a + \lambda\mathbb{1}$ for $\lambda > 0$. This means that $a + \lambda\mathbb{1}$ is invertible for $\lambda > 0$. Let $0 \leq a \leq b$ then $\lambda\mathbb{1} \leq \lambda\mathbb{1} + a \leq \lambda\mathbb{1} + b$ by *iii.)*. Using (4.4.12) we obtain

$$\mathbb{1} \leq \frac{1}{\sqrt{\lambda\mathbb{1} + a}}(\lambda\mathbb{1} + a)\frac{1}{\sqrt{\lambda\mathbb{1} + a}} \leq \frac{1}{\sqrt{\lambda\mathbb{1} + a}}(\lambda\mathbb{1} + b)\frac{1}{\sqrt{\lambda\mathbb{1} + a}} = c,$$

and hence $\operatorname{spec}(c) \subseteq [1, \infty)$ since $\operatorname{spec}(c - \mathbb{1}) \subseteq \mathbb{R}_0^+$. But this implies by Theorem 4.2.19, *iii.)*, that c is invertible with $\operatorname{spec}(c^{-1}) \subseteq (0, 1]$. It follows that $c^{-1} \leq \mathbb{1}$ by *iv.)* and thus

$$\sqrt{\lambda\mathbb{1} + a}\frac{1}{\lambda\mathbb{1} + b}\sqrt{\lambda\mathbb{1} + a} \leq \mathbb{1},$$

which becomes (4.4.13) after a further conjugation with $\frac{1}{\sqrt{\lambda\mathbb{1} + a}}$ and (4.4.12). For the last part we may first adjoin a unit if necessary. Then in a unital $\mathscr{A}$ we have $a^* a \leq \|a^* a\|\mathbb{1} = \|a\|^2\mathbb{1}$ by *iv.)*. Applying (4.4.12) gives the last part right away. If $\mathscr{A}$ was non-unital, then the inequality holds in $\widetilde{\mathscr{A}}$ and hence also in $\mathscr{A}$. □

Corollary 4.4.10 *Let $\mathscr{A}$ be a unital C^*-algebra and let $a, b \in \mathscr{A}$ be Hermitian and invertible. Then $0 \leq a \leq b$ implies*

$$0 \leq \frac{1}{b} \leq \frac{1}{a}. \tag{4.4.14}$$

PROOF: Being algebraically positive and invertible means that $\operatorname{spec}(a) \subseteq [\epsilon, \|a\|]$ for some suitable $\epsilon > 0$ by Theorem 4.4.5. The same holds for b. Now on the interval $[\epsilon, \|a\|]$ the functions

$$f_\lambda(x) = \frac{1}{\lambda + x}$$

for $\lambda > 0$ are continuous and converge uniformly to $f_0(x) = x^{-1}$ as $\lambda \longrightarrow 0$. Hence by the continuity of the Continuous Calculus as in Theorem 4.3.25, we get $f_\lambda(a) = \frac{1}{\lambda + a} \longrightarrow f_0(a) = a^{-1}$ in the norm topology of $\mathscr{A}$. By Proposition 4.4.9, *viii.)*, we have for all $\lambda > 0$

$$0 \leq \frac{1}{b + \lambda \mathbb{1}} \leq \frac{1}{a + \lambda \mathbb{1}}.$$

Finally, we know that $\mathscr{A}^{++}$ is closed by Corollary 4.4.3 and hence the inequality still holds in the limit $\lambda \longrightarrow 0$. □

The inequality (4.4.13) can be used to obtain many other interesting inequalities between functions of elements in $\mathscr{A}$. We illustrate this for the following case:

Proposition 4.4.11 *Let $\mathscr{A}$ be a unital C^*-algebra and let $a, b \in \mathscr{A}$ be Hermitian with $0 \leq a \leq b$. Then for all $\alpha \in (0, 1)$ we have*

$$0 \leq a^\alpha \leq b^\alpha. \tag{4.4.15}$$

PROOF: We consider the following function of two real variables

$$f(\lambda, x) = \lambda^{\alpha - 1} \frac{x}{x + \lambda}$$

where $x \geq 0$ and $\lambda > 0$. Clearly, f is continuous. Now let $I = [\epsilon, T]$ and $J = [0, \beta]$ be compact intervals where $0 < \epsilon < T$ and $0 < \beta$. Thus we can consider the restriction of f to a continuous function $f_{\epsilon,T} \in \mathscr{C}(I \times J)$. By Lemma B.2.7 we can view $f_{\epsilon,T}$ also as a continuous vector-valued function $f_{\epsilon,T} \in \mathscr{C}(I, \mathscr{C}(J))$ since if $\lambda_n \longrightarrow \lambda$ is a convergent sequence in I then $f_{\epsilon,T}(\lambda_n, \cdot\,) \longrightarrow f_{\epsilon,T}(\lambda, \cdot\,)$ in the uniform topology on $\mathscr{C}(J)$. Now we know that the Riemann integral over the variable λ exists by Proposition B.2.5 and is given by convergent Riemann sums in the topology of $\mathscr{C}(J)$. Hence the function

$$g_{\epsilon,T}(x) = \int_{I=[\epsilon,T]} \lambda^{\alpha-1} \frac{x}{x + \lambda} \, \mathrm{d}\lambda \tag{$*$}$$

defines an element in $\mathscr{C}(J)$ and the Riemann sums of the integral converge *uniformly* to $g_{\epsilon,T}$. This can of course also be obtained by elementary estimates but we take the advantage to illustrate the power of the techniques developed in Appendix B.2. There are now several consequences of this: since the Riemann sums converge uniformly on J we can compare the continuous calculus for $a \in \mathscr{A}$ with the vector-valued integration of the function

$$I \ni \lambda \;\mapsto\; \lambda^{\alpha-1} \frac{a}{a + \lambda} \in \mathscr{A}.$$

Clearly, the property $a \geq 0$ shows that this is a well-defined and continuous function in $\mathscr{C}(I, \mathscr{A})$. Thus we can integrate it over I yielding an element in $\mathscr{A}$. The uniform convergence of the Riemann sums in $(*)$ together with the continuity of the continuous calculus as in Remark 4.3.26, *i.)*, now shows that $g_{\epsilon,T}(a)$ defined by means of the continuous calculus from Theorem 4.3.25 coincides with the Riemann integral

$$g_{\epsilon,T}(a) = \int_\epsilon^T \lambda^{\alpha-1} \frac{a}{a+\lambda} \, \mathrm{d}\lambda.$$

In a next step we consider the limits $\epsilon \longrightarrow 0$ and $T \longrightarrow \infty$ of the function $g_{\epsilon,T}$. We claim that uniformly on J we have

$$\lim_{\substack{\epsilon \longrightarrow 0 \\ T \longrightarrow \infty}} g_{\epsilon,T} = g$$

with

$$g(x) = \int_0^\infty \lambda^{\alpha-1} \frac{x}{x+\lambda} \, \mathrm{d}\lambda \quad \text{for} \quad x \in J. \tag{$**$}$$

Indeed, these are standard estimates: we consider for $0 < \epsilon' \leq \epsilon$

$$0 \leq \int_{\epsilon'}^{\epsilon} \lambda^{\alpha-1} \frac{x}{x+\lambda} \, \mathrm{d}\lambda \leq \frac{x}{x+\epsilon'} \frac{1}{\alpha} \big(\epsilon^\alpha - (\epsilon')^\alpha \big)$$

and for $1 \leq T' \leq T$

$$0 \leq \int_{T'}^{T} \lambda^{\alpha-1} \frac{x}{x+\lambda} \, \mathrm{d}\lambda \leq \frac{x}{1-\alpha} \big((T')^{\alpha-1} - T^{\alpha-1} \big).$$

From these estimates and the uniform bound $0 \leq x \leq \beta$ it is easy to see that $g_{\epsilon,T}$ is a Cauchy net with respect to the uniform topology of $\mathscr{C}(J)$. Moreover, the limit is obviously given by $g \in \mathscr{C}(J)$ as in $(**)$. In particular, the improper Riemann integral in $(**)$ exists. By the continuity of the continuous calculus we get also for the algebra element a

$$g_{\epsilon,T}(a) \longrightarrow g(a)$$

in the norm topology of $\mathscr{A}$, since we can adjust β such that $\|a\| \leq \beta$. We compute $g(x)$ now explicitly. By a standard change of variables we get

$$g(x) = \frac{1}{\alpha} \int_0^\infty \frac{x}{x + u^{1/\alpha}} \, \mathrm{d}u = \frac{x^\alpha}{\alpha} \int_0^\infty \frac{1}{1 + v^{1/\alpha}} \, \mathrm{d}v.$$

Setting

$$c(\alpha) = \frac{1}{\alpha} \int_0^\infty \frac{1}{1 + v^{1/\alpha}} \, \mathrm{d}v > 0$$

gives then the formula

$$x^\alpha = \frac{1}{c(\alpha)} g(x) = \frac{1}{c(\alpha)} \int_0^\infty \lambda^{\alpha-1} \frac{x}{x+\lambda} \, \mathrm{d}\lambda.$$

The main point is now that this (easy) formula also applies to a^α defined by the continuous calculus. Hence we have for every $a \geq 0$

$$a^\alpha = \frac{1}{c(\alpha)} \int_0^\infty \lambda^{\alpha-1} \frac{a}{a+\lambda} \, \mathrm{d}\lambda. \tag{☺}$$

The whole effort about the uniform convergence above was to establish this coincidence of the continuous calculus and the vector-valued Riemann integral. The final step is now to observe that Riemann integrals of $\mathscr{A}$-valued functions preserve inequalities with respect to the order of $\mathscr{A}$ as in Definition 4.4.8. Indeed, if for all $\lambda \in I$ one has $a(\lambda) \leq b(\lambda)$ for two continuous functions of $\lambda \in I$ taking values in the Hermitian elements of $\mathscr{A}$ then also

$$\int_I a(\lambda) \leq \int_I b(\lambda).$$

This follows from the fact that the order relation is compatible with summations according to Proposition 4.4.9, *iii.)*, and the fact that the order relation is preserved under limits, see Corollary 4.4.3 and Exercise 4.5.44. Now we apply this to $0 \leq a \leq b$ and hence for $\lambda > 0$ we get

$$\frac{a}{a+\lambda} = \mathbb{1} - \frac{1}{a+\lambda} \leq \mathbb{1} - \frac{1}{b+\lambda} = \frac{b}{b+\lambda}$$

according to the inequality (4.4.13) from Proposition 4.4.9, *viii.)*. Hence we get $g_{\epsilon,T}(a) \leq g_{\epsilon,T}(b)$ for all $0 < \epsilon < T$. Taking once more limits gives finally $g(a) \leq g(b)$ from which the claim (4.4.15) follows by (☺). □

The case $\alpha = \frac{1}{2}$ is of particular interest for us where we get

$$0 \leq \sqrt{a} \leq \sqrt{b}, \tag{4.4.16}$$

whenever $0 \leq a \leq b$. This particular case has a slightly more elementary proof, see Exercise **??**. Note also that for $\alpha = \frac{1}{2}$ the constant $c(\alpha)$ can be computed explicitly yielding the explicit formula

$$\sqrt{a} = \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{\lambda}} \frac{a}{a+\lambda} \, \mathrm{d}\lambda \tag{4.4.17}$$

for the square root of a as a vector-valued Riemann integral.

Interesting enough, not all monotonous functions of a positive real variable give raise to monotonous functions of positive elements in a C^*-algebra: already for the function $f(x) = x^2$ one finds immediate counterexamples, see Exercise 4.5.45. For a further discussion of such *operator-monotonous functions* we refer to [20].

4.4.2 Approximate Identities

For a non-unital C^*-algebra we have seen how to adjoin a unit and pass to the unitization. While this is sometimes a convenient way to obtain a unital situation it turns out to be not yet sufficient for other purposes. Here the concept of an approximate identity gives another option where one stays somehow inside the non-unital framework.

Definition 4.4.12 (Approximate identity) *Let $\mathscr{A}$ be a C^*-algebra and let $\mathscr{J} \subseteq \mathscr{A}$ be a right ideal. Then a net $\{\mathrm{e}_i\}_{i \in I}$ of elements in $\mathscr{J}$ is called a left approximate identity for $\mathscr{J}$ if*

i.) *for all $i \in I$ one has $\mathrm{e}_i \geq 0$,*

ii.) *for all $i \preccurlyeq j$ one has $\mathrm{e}_i \leq \mathrm{e}_j$,*

iii.) *for all $i \in I$ one has $\|\mathrm{e}_i\| \leq 1$,*

iv.) *for all $a \in \mathscr{J}$ one has*

$$\lim_{i \in I} \mathrm{e}_i a = a. \tag{4.4.18}$$

Remark 4.4.13 (Approximate identity) Let $\mathscr{A}$ be a C^*-algebra.

i.) For a left ideal one defines a right approximate identity in the analogous way by replacing the last requirement (4.4.18) by

$$\lim_{i \in I} a \mathrm{e}_i = a. \tag{4.4.19}$$

ii.) If $\mathscr{A}$ is unital and $\mathscr{J} = \mathscr{A}$ then $\{\mathbb{1}\}$ is a left and right approximate identity for $\mathscr{J}$.

iii.) If $\mathscr{J}$ is a *-ideal then any left approximate identity for $\mathscr{J}$ is also a right approximate identity and vice versa. Indeed, if $\lim_{i \in I} \mathrm{e}_i a = a$ for all $a \in \mathscr{J}$ then also $\lim_{i \in I} a^* \mathrm{e}_i^* = a^*$ by the continuity of the *-involution. But $\mathrm{e}_i^* = \mathrm{e}_i$ and $a^* \in \mathscr{J}$ iff $a \in \mathscr{J}$. Thus in this case we call $\{\mathrm{e}_i\}_{i \in I}$ simply an approximate identity.

iv.) If $\mathscr{A}$ is non-unital then the case $\mathscr{J} = \mathscr{A}$ is already non-trivial and interesting. In fact, this case is one of the main motivations for the concept of an approximate identity.

We will now show that there is always a left approximate identity for any right ideal (and, by an analogous argument: a right approximate identity for any left ideal):

Proposition 4.4.14 *Let $\mathscr{A}$ be a C^*-algebra and let $\mathscr{J} \subseteq \mathscr{A}$ be a right ideal. Then there exists a left approximate identity for $\mathscr{J}$. Analogously, every left ideal admits a right approximate identity.*

Proof: We adjoin a unit if $\mathscr{A}$ is non-unital. Clearly, $\mathscr{J}$ stays a right ideal also in the unitization. The construction uses the positivity results from the last subsection in an essential way. First we take as index set I the set of all finite subsets of $\mathscr{J}$, directed by inclusion. Thus if $i = \{a_1, \ldots, a_n\}$ is a finite subset then

$$f_i = \sum_{k=1}^{n} a_k a_k^* \in \mathscr{A}^{++}$$

is an element of $\mathscr{J}$ by the right ideal property. One defines

$$\mathrm{e}_i = n f_i (\mathbb{1} + n f_i)^{-1} = \mathbb{1} - \frac{1}{\mathbb{1} + n f_i}. \tag{$*$}$$

Indeed, this is well-defined by Proposition 4.4.9, *viii.)*, since the algebra element $\mathbb{1} + n f_i$ is invertible and the inverse is still in $\mathscr{A}^{++}$. Moreover, by definition, e_i is an element of $\mathscr{J}$ since f_i clearly is an element in this right ideal. By the Spectral Mapping Theorem we still have $\mathrm{e}_i \geq 0$ since the continuous function $x \mapsto nx(1 + nx)^{-1}$ is non-negative for $x \geq 0$ and thus $\operatorname{spec}(\mathrm{e}_i) \subseteq \mathbb{R}_0^+$. Moreover, since $nx(1 + nx)^{-1} \leq 1$ for all $x \geq 0$ we see that $\operatorname{spec}(\mathrm{e}_i) \subseteq [0, 1]$ and hence $\|\mathrm{e}_i\| \leq 1$. Next, let $i \preccurlyeq j$ and thus $i = \{a_1, \ldots, a_n\} \subseteq \{b_1, \ldots, b_m\} = j$. It follows that $f_i \leq f_j$ since we simply have more squares in the sum for f_j than for f_i. But then also $n f_i \leq m f_j$. From Proposition 4.4.9, *viii.)*, we know that $(\mathbb{1} + n f_i)^{-1} \geq (\mathbb{1} + m f_j)^{-1}$. This implies, analogously to the arguments in the proof of Proposition 4.4.11, that

$$\mathrm{e}_i = \frac{n f_i}{\mathbb{1} + n f_i} = \mathbb{1} - \frac{1}{\mathbb{1} + n f_i} \leq \mathbb{1} - \frac{1}{\mathbb{1} + m f_j} = \frac{m f_j}{\mathbb{1} + m f_j} = \mathrm{e}_j$$

as wanted. It remains to show the approximation property (4.4.18). Thus let $a \in \mathscr{J}$ be given and assume that $i = \{a_1, \ldots, a_n\}$ is a subset containing a. In this case we clearly have

$$a a^* \leq \sum_{k=1}^{n} a_k a_k^*,$$

since a is among the a_k. Moreover, for every $j \in I$ there is such a i with $i \succcurlyeq j$, simply by adding a to the subset j. We compute in the sense of the order relation

$$\begin{aligned}
(\mathrm{e}_i a - a)(\mathrm{e}_i a - a)^* &= (\mathrm{e}_i - \mathbb{1}) a a^* (\mathrm{e}_i - \mathbb{1})^* \\
&\leq (\mathrm{e}_i - \mathbb{1}) \sum_{k=1}^{n} a_k a_k^* (\mathrm{e}_i - \mathbb{1})^* \\
&= (\mathrm{e}_i - \mathbb{1}) f_i (\mathrm{e}_i - \mathbb{1})^* \\
&= \frac{1}{\mathbb{1} + n f_i} f_i \frac{1}{\mathbb{1} + n f_i} \\
&= \frac{f_i}{(\mathbb{1} + n f_i)^2},
\end{aligned}$$

where we used $(*)$. Since $f_i \geq 0$ we get, again by the Spectral Mapping Theorem,

$$\operatorname{spec}\left(\frac{f_i}{(\mathbb{1} + nf_i)^2}\right) \subseteq [0, \tfrac{1}{n}],$$

since for all $x \geq 0$ the continuous function $x \mapsto \frac{x}{(1+nx)^2}$ is bounded by $\frac{1}{n}$. Thus we get by Proposition 4.4.9, *v.)*, the estimate

$$\|\mathrm{e}_i a - a\|^2 = \|(\mathrm{e}_i a - a)(\mathrm{e}_i a - a)^*\| \leq \left\|\frac{f_i}{(\mathbb{1} + nf_i)^2}\right\| \leq \frac{1}{n}.$$

But as we can take finite subsets of arbitrary length containing a we conclude that $\lim_{i\in I} \mathrm{e}_i a = a$. The case of a left ideal $\mathscr{J} \subseteq \mathscr{A}$ follows at once since in this case $\mathscr{J}^*$ is a right ideal. ☐

Corollary 4.4.15 *Let $\mathscr{A}$ be a C^*-algebra and let $a \in \mathscr{A}$. Then there exists a sequence $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ with $0 \leq \mathrm{e}_n$, $\|\mathrm{e}\|_n \leq 1$, $\mathrm{e}_n \leq \mathrm{e}_{n+1}$, and*

$$\lim_{n\longrightarrow\infty} \mathrm{e}_n a = a. \tag{4.4.20}$$

PROOF: Indeed, the above proof as to be modified only slightly: we simply define $\mathrm{e}_n = naa^*(\mathbb{1} + naa^*)^{-1}$. Then we can follow the proof literally. Alternatively, we choose an approximate identity $\{\mathrm{e}_i\}_{i\in I}$ and choose for every $n \in \mathbb{N}$ an $i_n \in I$ with $i_n \preccurlyeq i_{n+1}$ and $\|\mathrm{e}_{i_n} a - a\| < \frac{1}{n}$. Then $\mathrm{e}_n = \mathrm{e}_{i_n}$ will do the job. ☐

Corollary 4.4.16 (Closed ideals of a C^*-algebra) *Let $\mathscr{J} \subseteq \mathscr{A}$ be a closed two-sided ideal in a C^*-algebra $\mathscr{A}$. Then $\mathscr{J}$ is a closed *-ideal.*

PROOF: Choose a left approximate identity $\{\mathrm{e}_i\}_{i\in I}$ for $\mathscr{J}$ and let $a \in \mathscr{J}$. Then one has $\lim_{i\in I} a^*\mathrm{e}_i = (\lim_{i\in I} \mathrm{e}_i a)^* = a^*$ since $\mathrm{e}_i^* = \mathrm{e}_i$ and the *-involution is continuous. Since $\mathscr{J}$ is a two-sided ideal, $a^*\mathrm{e}_i \in \mathscr{J}$ and since $\mathscr{J}$ is closed, also the limit a^* of the above convergent net is contained in $\mathscr{J}$. ☐

Note that for a commutative unital C^*-algebra we had a completely different proof of this fact in Theorem 4.3.10, *ii.)*, since every unital commutative C^*-algebra is isomorphic to the continuous functions on its spectrum by Theorem 4.3.13.

As a last corollary we consider quotients of C^*-algebras by closed *-ideals. We know already that the quotient is a Banach *-algebra according to Proposition 4.2.3, *iii.)*, and according to Exercise 2.5.43. We use the approximate identity to show that it is again a C^*-algebra:

Corollary 4.4.17 (Quotient C^*-algebra) *Let $\mathscr{J} \subseteq \mathscr{A}$ be a closed *-ideal in a C^*-algebra. Then the Banach *-algebra $\mathscr{A}/\mathscr{J}$ is a C^*-algebra and the quotient norm is given by*

$$\|[a]\| = \lim_{i\in I} \|a - \mathrm{e}_i a\|, \tag{4.4.21}$$

where $\{\mathrm{e}_i\}_{i\in I}$ is a approximate identity for $\mathscr{J}$.

PROOF: We first prove (4.4.21). Thus let $a \in \mathscr{A}$ and let an approximate identity $\{\mathrm{e}_i\}_{i\in I}$ for $\mathscr{J}$ be given. We first consider the unital case. By the Spectral Mapping Theorem we know $\mathbb{1} - \mathrm{e}_i \geq 0$ and hence $\|\mathbb{1} - \mathrm{e}_i\| \leq 1$ for all $i \in I$. Since $\lim_{i\in I} \mathrm{e}_i b = b$ for all $b \in \mathscr{J}$ we get

$$\limsup_{i\in I} \|a - \mathrm{e}_i a\| = \limsup_{i\in I} \|a - \mathrm{e}_i a + b - \mathrm{e}_i b\| = \limsup_{i\in I} \|(\mathbb{1} - \mathrm{e}_i)(a + b)\| \leq \|a + b\|.$$

Thus we get

$$\|[a]\| = \inf\{\|a + b\| \mid b \in \mathscr{J}\} \geq \limsup_{i\in I} \|a - \mathrm{e}_i a\| \geq \liminf_{i\in I} \|a - \mathrm{e}_i a\| \geq \inf\{\|a + b\| \mid b \in \mathscr{J}\} = \|[a]\|,$$

since $\mathrm{e}_i a \in \mathscr{J}$. But this implies (4.4.21). In particular, the limes superior and the limes inferior coincide and hence are a limit. Clearly, the arguments remain valid in the non-unital case after adjoining a unit. Next, for an arbitrary $b \in \mathscr{J}$ and $a \in \mathscr{A}$ we have

$$\|[a]\|^2 = \lim_{i \in I} \|a - \mathrm{e}_i a\|^2 = \lim_{i \in I} \|(a - \mathrm{e}_i a)^*(a - \mathrm{e}_i a)\| = \lim_{i \in I} \|(\mathbb{1} - \mathrm{e}_i)(a^*a - b)(\mathbb{1} - \mathrm{e}_i)\| \leq \|a^*a + b\|, \quad (*)$$

since again $\lim_{i \in I}(\mathbb{1} - \mathrm{e}_i)b = 0$ does not contribute and $\|\mathbb{1} - \mathrm{e}_i\| \leq 1$. But this shows $\|[a]\|^2 \leq \|[a]^*[a]\|$ by taking the infimum over all $b \in \mathscr{J}$. Since we already know that the quotient $\mathscr{A}/\mathscr{J}$ is a Banach *-algebra, this is all we need to show. Again, the inequality shown in $(*)$ also holds in the non-unital case by first adjoining a unit. □

Proposition 4.4.18 *Let $\mathscr{J} \subseteq \mathscr{A}$ be a closed *-ideal in a C^*-algebra $\mathscr{A}$ and $[b] \in \mathscr{A}/\mathscr{J}$. Then there is a representative $a \in \mathscr{A}$ of $[b]$ with $\|a\| = \|[b]\|$. If $[b]$ is Hermitian we find also a Hermitian $a \in \mathscr{A}$ with this property.*

PROOF: We may assume $\|[b]\| = 1$ without restriction. First it is clear that if $[b]^* = [b]$ then we find also a Hermitian representative: since the quotient map $\mathscr{A} \longrightarrow \mathscr{A}/\mathscr{J}$ is a *-homomorphism $\frac{1}{2}(b + b^*)$ also represents $[b]$. Thus let $b \in \mathscr{A}$ be any (Hermitian) representative. Then we consider the continuous function $x \mapsto \min(1 + x, 2)$ on the real axis and use the continuous calculus to define

$$a = b \frac{\min(1 + b^*b, 2)}{1 + b^*b}.$$

Indeed, $1 + b^*b$ is always invertible and hence this is well-defined. Clearly, if $b = b^*$ then also $a = a^*$. Note that if $\mathscr{A}$ is non-unital then $\frac{\min(1+b^*b,2)}{1+b^*b}$ is in the unitization but a itself is again in $\mathscr{A}$ since $b \in \mathscr{A}$ and $\mathscr{A} \subseteq \tilde{\mathscr{A}}$ is a *-ideal. In the quotient we have $\|[b]\| = 1$ by assumption and thus $\|[b]^*[b]\| = 1$ as well. Thus

$$[\min(1 + b^*b, 2)] = \min(1 + [b]^*[b], 2)] = 1 + [b]^*[b]$$

since the continuous calculus is compatible with (unital) *-homomorphisms: here we have to pass to the unitization if $\mathscr{A}$ is non-unital. Thus $[a] = [b]$ follows and a is also a representation of $[b]$. Finally, we claim that the norm of a can be estimated by

$$\|a\|^2 = \|a^*a\| = \left\| \frac{\min(1 + b^*b, 2)}{1 + b^*b} b^*b \frac{\min(1 + b^*b, 2)}{1 + b^*b} \right\|$$

and the maximum of the function

$$x \mapsto \left(\frac{\min(1 + x, 2)}{1 + x} \right)^2 x$$

for $x \geq 0$ is given by 1 obtained for $x = 1$. Thus $\|a\| \leq 1$ follows from the continuous calculus. But in any case $1 = \|[b]\| = \|[a]\| \leq \|a\|$ giving $\|a\| = 1$ as wanted. □

4.4.3 States for C^*-Algebras

In a next step we want to establish $\mathscr{A}^+ = \mathscr{A}^{++}$ for a C^*-algebra $\mathscr{A}$. To this end we need to discuss the positive linear functionals $\omega \colon \mathscr{A} \longrightarrow \mathbb{C}$.

Proposition 4.4.19 (Positive functionals on C^*-algebras) *Let $\mathscr{A}$ be a C^*-algebra and let $\{\mathrm{e}_i\}_{i \in I}$ be an approximate identity for $\mathscr{A}$. Then for a linear functional $\omega \colon \mathscr{A} \longrightarrow \mathbb{C}$ the following two statements are equivalent:*

i.) The functional ω is positive, i.e. $\omega(a^*a) \geq 0$ for all $a \in \mathscr{A}$.

ii.) The functional ω is continuous and

$$\|\omega\| = \lim_{i \in I} \omega(\mathrm{e}_i). \tag{4.4.22}$$

In this case we have for all $a, b \in \mathscr{A}$

$$\overline{\omega(a)} = \omega(a^*), \tag{4.4.23}$$

$$|\omega(a)|^2 \le \omega(a^*a)\|\omega\|, \tag{4.4.24}$$

$$|\omega(a^*ba)| \le \omega(a^*a)\|b\|, \tag{4.4.25}$$

$$\|\omega\| = \sup\big\{\omega(a^*a) \bigm| \|a\| \le 1\big\}, \tag{4.4.26}$$

$$\|\omega\| = \lim_{i \in I} \omega(\mathrm{e}_i^2). \tag{4.4.27}$$

PROOF: Assume that ω is positive. We first check that ω is continuous. To this end, assume that ω is discontinuous on the cone $\mathscr{A}^{++}$. Thus let $a_n \in \mathscr{A}^{++}$ be a sequence of algebraically positive elements with $\|a_n\| \le 1$ but $\omega(a_n) \ge 2^n$ for all $n \in \mathbb{N}$. Then on the one hand, the series $a = \sum_{n=1}^{\infty} \frac{1}{2^n} a_n$ converges absolutely and hence defines an element $a \in \mathscr{A}^{++}$ by the closedness of $\mathscr{A}^{++}$ according to Corollary 4.4.3, see also Theorem 4.4.5. On the other hand, $\sum_{n=N+1}^{\infty} \frac{1}{2^n} a_n$ is still in $\mathscr{A}^{++}$ for all $N \in \mathbb{N}$ by the same argument. Hence

$$0 \le \omega\left(\sum_{n=N+1}^{\infty} \frac{1}{2^n} a_n\right) = \omega\left(a - \sum_{n=1}^{N} \frac{1}{2^n} a_n\right) = \omega(a) - \sum_{n=1}^{N} \frac{1}{2^n} \omega(a_n) \le \omega(a) - N,$$

by $\omega(a_n) \ge 2^n$. This is clearly absurd and hence there is a constant $C > 0$ with

$$C = \sup\big\{\omega(a) \bigm| \|a\| \le 1 \text{ and } a \in \mathscr{A}^{++}\big\} < \infty.$$

For a general element $a \in \mathscr{A}$ we have $a = a_1 - a_2 + \mathrm{i}a_3 - \mathrm{i}a_4$ with $a_k \in \mathscr{A}^{++}$ and $\|a_k\| \le \|a\|$ by Corollary 4.4.6. Thus for a with $\|a\| \le 1$ we get $|\omega(a)| \le 4C$. It follows that ω is bounded on $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq \mathscr{A}$ and hence continuous. Now we consider the approximate identity $\{\mathrm{e}_i\}_{i \in I}$. Then $0 \le \mathrm{e}_i \le \mathrm{e}_j$ for all $i, j \in I$ with $i \preccurlyeq j$. Since moreover $\|\mathrm{e}_i\| \le 1$ we conclude

$$0 \le \omega(\mathrm{e}_i) \le \omega(\mathrm{e}_j) \le \|\omega\|$$

whenever $i \preccurlyeq j$. Note that a positive linear functional respects the order of Hermitian elements in $\mathscr{A}$. Thus the real numbers $\{\omega(\mathrm{e}_i)\}_{i \in I}$ form an increasing net in the interval $[0, \|\omega\|]$ which therefore converges, i.e.

$$c = \lim_{i \in I} \omega(\mathrm{e}_i)$$

exists. Clearly, $c \le \|\omega\|$. Conversely, let $\epsilon > 0$ be given. Then we find an element $a \in \mathscr{A}$ with $\|a\| \le 1$ and $|\omega(a)| \ge \|\omega\| - \epsilon$ by the definition of the functional norm $\|\omega\|$ as a supremum. By continuity of ω we have

$$|\omega(a)|^2 = \lim_{i \in I} |\omega(\mathrm{e}_i a)|^2 \le \limsup_{i \in I} \omega(\mathrm{e}_i^2)\omega(a^*a) \le \limsup_{i \in I} \omega(\mathrm{e}_i)\omega(a^*a) = c\omega(a^*a) \le c\|\omega\|, \tag{$*$}$$

since $\mathrm{e}_i^2 \le \mathrm{e}_i$ by the Spectral Mapping Theorem and since $\|a^*a\| \le 1$. Thus for all $\epsilon > 0$ we have $(\|\omega\| - \epsilon)^2 \le c\|\omega\|$ and hence $c \ge \|\omega\|$. This shows that actually $c = \|\omega\|$ as claimed which proves the implication *i.)* $\implies$ *ii.)*. Denote now by $\tilde{c}$ the limes superior of the net $\{\omega(\mathrm{e}_i^2)\}_{i \in I}$. Then we know $\tilde{c} \le c$. But the estimate in $(*)$ actually shows that $|\omega(a)|^2 \le \tilde{c}\omega(a^*a)$ and hence $\|\omega\| \le \tilde{c}$ follows analogously to $\|\omega\| \le c$. Together, this shows $\tilde{c} = c = \|\omega\|$ and hence we showed (4.4.27) with limes replaced by limes superior, i.e.

$$\|\omega\| = \limsup_{i \in I} \omega(\mathrm{e}_i^2). \tag{$**$}$$

Note that $\{\mathrm{e}_i^2\}_{i\in I}$ needs not to be an approximate identity since we do not know whether it is increasing, see the discussion after Proposition 4.4.11. In the second step, we assume *ii.)*. Then we can extend ω to the unitization $\widetilde{\mathscr{A}}$ of $\mathscr{A}$, if we are in the non-unital situation, by setting

$$\widetilde{\omega}(a + z\mathbb{1}) = \omega(a) + z\|\omega\|$$

for $a + z\mathbb{1} \in \widetilde{\mathscr{A}}$. We claim that $\widetilde{\omega}$ is still continuous. Indeed, we have

$$\begin{aligned} |\widetilde{\omega}(a + z\mathbb{1})| &= \big|\omega(a) + z\|\omega\|\big| \\ &= \lim_{i\in I}\big|\omega(a\mathrm{e}_i) + z\omega(\mathrm{e}_i)\big| \\ &\leq \|\omega\| \sup_{i\in I}\|a\mathrm{e}_i + z\mathrm{e}_i\| \\ &\leq \|\omega\| \sup\big\{\|ab + zb\| \bigm| \|b\| \leq 1\big\} \\ &= \|\omega\|\|a + z\mathbb{1}\|, \end{aligned}$$

and hence $\|\widetilde{\omega}\| \leq \|\omega\|$. But since $\tilde{\omega}(\mathbb{1}) = \|\omega\|$ we see that the two norms coincide. Thus $\widetilde{\omega}$ is a continuous linear functional on $\widetilde{\mathscr{A}}$ with $\widetilde{\omega}(\mathbb{1}) = \|\omega\| = \|\widetilde{\omega}\|$. If we already started with the unital case then for an arbitrary approximate identity one has

$$\lim_{i\in I} \mathrm{e}_i = \lim_{i\in I} \mathrm{e}_i\mathbb{1} = \mathbb{1},$$

and hence (4.4.22) implies that also here $\|\omega\| = \omega(\mathbb{1})$ by continuity of ω. Thus we may assume the unital case from the beginning with a continuous linear functional such that $\omega(\mathbb{1}) = \|\omega\|$. By rescaling, we can assume even $\|\omega\| = 1$, since clearly the only remaining case $\omega = 0$ is trivially a positive linear functional. For such a linear functional we first show that on Hermitian elements $a = a^*$ we have $\omega(a) \in \mathbb{R}$. Write $\omega(a) = \alpha + \mathrm{i}\beta$ with $\alpha, \beta \in \mathbb{R}$. Then we want to show $\beta = 0$. For all $\gamma \in \mathbb{R}$ we have $\omega(a + \mathrm{i}\gamma\mathbb{1}) = \alpha + \mathrm{i}(\beta + \gamma)$ since $\|\omega\| = 1 = \omega(\mathbb{1})$ by assumption. Now $a + \mathrm{i}\gamma\mathbb{1}$ is normal and $\operatorname{spec}(a + \mathrm{i}\gamma\mathbb{1}) \subseteq [-\|a\|, \|a\|] + \mathrm{i}\gamma \subseteq \mathbb{C}$. The spectral radius is therefore given by

$$\|a + \mathrm{i}\gamma\mathbb{1}\| = \varrho(a + \mathrm{i}\gamma) = \sqrt{\|a\|^2 + \gamma^2},$$

since at least one of the two end points $\|a\|$ or $-\|a\|$ belongs to the spectrum of a. Now $|\omega(a+\mathrm{i}\gamma\mathbb{1})| = |\alpha + \mathrm{i}(\beta + \gamma)| \geq |\beta + \gamma|$ and on the other hand $|\omega(a + \mathrm{i}\gamma\mathbb{1})| \leq \|a + \mathrm{i}\gamma\mathbb{1}\|$ by $\|\omega\| = 1$. Thus together we get

$$|\beta + \gamma| \leq \sqrt{\|a\|^2 + \gamma^2}.$$

But this implies $\beta^2 + 2\beta\gamma + \gamma^2 \leq \|a\|^2 + \gamma^2$ which is only possible for arbitrary γ if $\beta = 0$. Thus $\omega(a)$ is real as claimed. By Proposition 4.4.9, *iv.)*, we have $0 \leq a^*a \leq \|a^*a\|\mathbb{1} = \|a\|^2\mathbb{1}$. Hence, for $a \neq 0$, we get $0 \leq \frac{a^*a}{\|a\|^2} \leq \mathbb{1}$ and thus

$$\left\|\mathbb{1} - \frac{a^*a}{\|a\|^2}\right\| \leq 1$$

as well. Since $\|\omega\| = 1$ it follows that

$$1 \geq \left|\omega\left(\mathbb{1} - \frac{a^*a}{\|a\|^2}\right)\right| = \left|1 - \frac{\omega(a^*a)}{\|a\|^2}\right|.$$

Since we already know that $\omega(a^*a) \in \mathbb{R}$ this is only possible for $\omega(a^*a) \geq 0$. Thus ω is a positive linear functional as wanted and *ii.)* $\implies$ *i.)* is shown, too. For the remaining properties and claims one proceeds as follows: first we can assume to have a unital C^*-algebra since otherwise we can pass from ω to the positive $\widetilde{\omega}$ as above. Then $\widetilde{\omega}$ and hence also ω satisfy (4.4.23) as this is true in any unital *-algebra by Lemma 1.2.5. Moreover, in the unital case, (4.4.24) is again just the Cauchy-Schwarz inequality applied to $\omega(a\mathbb{1})$ using $\omega(\mathbb{1}) = \|\omega\|$. Again we can restrict from the unitization to the original non-unital $\mathscr{A}$ if we started in the non-unital situation. For (4.4.25) we first use Proposition 4.4.9, *ix.)*, to get $0 \leq a^*b^*ba \leq \|b\|^2 a^*a$ and thus $\omega(a^*b^*ba) \leq \|b\|^2\omega(a^*a)$. By the Cauchy-Schwarz inequality we get

$$|\omega(a^*ba)|^2 \leq \omega(a^*a)\omega(a^*b^*ba) \leq \omega(a^*a)\|b\|^2\omega(a^*a),$$

and hence (4.4.25). Finally, the supremum on the right hand side of (4.4.26) is clearly bounded by $\|\omega\|$. Then the already proven estimate $(**)$ shows that it is equal to $\|\omega\|$. We still have to show that in (4.4.27) we actually have a limit instead of a limes superior: but this is now a consequence of the Cauchy-Schwarz inequality. Without restriction we can assume to be in the unital situation. Then $|\omega(\mathrm{e}_i)|^2 \leq \omega(\mathrm{e}_j^2)\omega(\mathbb{1}) = \omega(\mathrm{e}_i^2)\|\omega\|$. It follows that $\|\omega\|\omega(\mathrm{e}_i^2) \geq \omega(\mathrm{e}_i)^2$ and hence $\|\omega\| \liminf_{i\in I}\omega(\mathrm{e}_i^2) \geq \lim_{i\in I}\omega(\mathrm{e}_i)^2 = \|\omega\|^2$ according to (4.4.22). Hence the limes inferior and the limes superior coincide, proving (4.4.27). ☐

Remark 4.4.20 Note that the formulation of the proposition as well as its proof simplifies significantly if one has a unital C^*-algebra and considers the trivial approximate identity $\{\mathbb{1}\}$ only. In some sense, the true difficulty of the proposition consist in showing that for a positive linear functional ω on a non-unital C^*-algebra the canonical extension

$$\widetilde{\omega}(a + z\mathbb{1}) = \omega(a) + z\|\omega\| \tag{4.4.28}$$

to the unitization $\widetilde{\mathscr{A}}$ is still a positive linear functional. Conversely, any positive linear functional $\widetilde{\omega}$ on $\widetilde{\mathscr{A}}$ restricts to a positive linear functional $\omega = \widetilde{\omega}\big|_{\mathscr{A}}$ on $\mathscr{A}$. Clearly, $\widetilde{(\widetilde{\omega}\big|_{\mathscr{A}})} = \widetilde{\omega}$ again. Thus we have a bijective correspondence for the positive linear functionals on the non-unital C^*-algebra $\mathscr{A}$ and the ones on its unitization $\widetilde{\mathscr{A}}$. Moreover, the above characterization suggests that we define a *state* on a non-unital C^*-algebra to be a positive linear functional ω with $\|\omega\| = 1$. Thus ω is a state on $\mathscr{A}$ iff $\widetilde{\omega}$ is a state on $\mathscr{A}$.

Contrary to the general algebraic situation where one might have only very few positive linear functionals, on a C^*-algebra the situation is much nicer. In fact, the second characterization of positive linear functionals in Proposition 4.4.19 is the key for the following statement:

Theorem 4.4.21 (Existence of states) *Let $\mathscr{A}$ be a C^*-algebra and let $a \in \mathscr{A}$ be a normal element. For any $\lambda \in \operatorname{spec}(a)$ there exists a state $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ with*

$$\omega(a) = \lambda. \tag{4.4.29}$$

PROOF: Without restriction we can assume that $\mathscr{A}$ is unital, otherwise we adjoin a unit and pass to the unitization. Then we consider the commutative unital C^*-subalgebra $\mathsf{C}^*\langle a\rangle$ generated by a inside the unital C^*-algebra $\mathscr{A}$. Since $\mathsf{C}^*\langle a\rangle \cong \mathscr{C}(\operatorname{spec}(a))$ by the Gel'fand-Naimark Theorem we have for every $\lambda \in \operatorname{spec}(a)$ a δ-functional $\delta_\lambda\colon \mathsf{C}^*\langle a\rangle \cong \mathscr{C}(\operatorname{spec}(a)) \longrightarrow \mathbb{C}$ which is clearly a state with $\delta_\lambda(a) = \lambda$ since $a \in \mathsf{C}^*\langle a\rangle$ corresponds to the identity map $\hat{a}\colon z \mapsto z$ on $\operatorname{spec}(a)$ under the Gel'fand transform, see also Theorem 4.3.25, *ii.)*. Thus we have found a continuous linear functional $\delta_\lambda\colon \mathsf{C}^*\langle a\rangle \longrightarrow \mathbb{C}$ with $\delta_\lambda(\mathbb{1}) = 1 = \|\delta_\lambda\|$. By the Hahn-Banach Theorem 2.3.10 we have a continuous extension denoted by $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ such that $\omega\big|_{\mathsf{C}^*\langle a\rangle} = \delta_\lambda$ and $\|\omega\| = \|\delta_\lambda\| = 1$. But since $\omega(\mathbb{1}) = \delta_\lambda(\mathbb{1}) = 1$ we see by Proposition 4.4.19 that ω is positive and even a state. Being an extension of δ_λ we still have $\omega(a) = \lambda$ as wanted. ☐

Corollary 4.4.22 *Let $\mathscr{A}$ be a C^*-algebra and let $a \in \mathscr{A}$. Then we have $a = 0$ iff for all states $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ we have $\omega(a) = 0$.*

PROOF: One direction is trivial. Thus assume $\omega(a) = 0$ for all states. We see that in this case also $\omega(a^*) = 0$ and hence $\omega(a + a^*) = 0 = \omega(a - a^*)$. Thus for the Hermitian elements $\mathrm{Re}(a) = \frac{1}{2}(a + a^*)$ and $\mathrm{Im}(a) = \frac{1}{2\mathrm{i}}(a - a^*)$ we also have this property $\omega(\mathrm{Re}(a)) = 0 = \omega(\mathrm{Im}(a))$. By Theorem 4.4.21 we conclude that their spectrum is just $\{0\}$ and hence $\mathrm{Re}(a) = 0 = \mathrm{Im}(a)$. But then $a = 0$ follows. $\square$

Thus we have enough states on a C^*-algebra in order to separate points. In more physical terms, we have enough possible states to decide by measurements of expectation values whether an observable is zero or not. This is of course very much desirable for every physically meaningful observable algebra.

Another corollary to the above theorem is that we only have one notion of positivity in a C^*-algebra:

Corollary 4.4.23 *Let $\mathscr{A}$ be a C^*-algebra. Then $\mathscr{A}^+ = \mathscr{A}^{++}$.*

PROOF: We know that $\mathscr{A}^{++} \subseteq \mathscr{A}^+$ as this is the case for every *-algebra. For $\mathscr{A}^{++}$ we have the characterization by Theorem 4.4.5 that $a \in \mathscr{A}^{++}$ iff $a = a^*$ and $\mathrm{spec}(a) \subseteq \mathbb{R}_0^+$. If now $a \in \mathscr{A}^+$ then $\omega(a) \geq 0$ for all states. But then also $\omega(a - a^*) = \omega(a) - \overline{\omega(a)} = 0$ since $\omega(a)$ is real. By the last corollary this means $a - a^* = 0$, i.e. a is Hermitian and hence normal. By Theorem 4.4.21 we conclude that $\mathrm{spec}(a) \subseteq \mathbb{R}_0^+$ and thus $a \in \mathscr{A}^{++}$. $\square$

At last we arrive at the point where we can simply speak of *the* positive elements in a C^*-algebra since all notions of positivity coincide.

4.4.4 *-Representations and *-Homomorphisms

We have seen that for a general *-algebra and a positive linear functional one can construct a pre-Hilbert space and a *-representation of this *-algebra on the pre-Hilbert space. This was the GNS construction as presented in Section 1.3. Of course, the obtained operators on the pre-Hilbert space are adjointable on this *pre*-Hilbert space but typically they are *not* continuous and thus they do not extend to the Hilbert space completion. Note that this is not in contradiction to the Hellinger-Toeplitz Theorem as for this the condition of a Hilbert space is crucial. In the case of a C^*-algebra the situation is much nicer and we obtain continuous operators automatically:

Theorem 4.4.24 (Continuity of *-representations) *Let $\mathscr{A}$ be a C^*-algebra and let $\pi\colon \mathscr{A} \longrightarrow \mathfrak{B}(\mathfrak{H})$ be a *-representation of $\mathscr{A}$ on a pre-Hilbert space $\mathfrak{H}$. Then for all $a \in \mathscr{A}$ one has*

$$\|\pi(a)\| \leq \|a\|, \tag{4.4.30}$$

and π extends to a *-representation $\widehat{\pi}$ of $\mathscr{A}$ on the completion $\widehat{\mathfrak{H}}$ of $\mathfrak{H}$.

PROOF: First we note that if $\mathscr{A}$ is non-unital then $\widetilde{\pi}(a + z\mathbb{1}) = \pi(a) + z\,\mathrm{id}_{\mathfrak{H}}$ gives a *-representation of the unitization $\widetilde{\mathscr{A}}$ on $\mathfrak{H}$ which is unital, i.e. $\widetilde{\pi}(\mathbb{1}) = \mathrm{id}_{\mathfrak{H}}$. Note that, in general, for a unital C^*-algebra, $\pi(\mathbb{1}) = P$ is a projection on $\mathfrak{H}$ but needs not to be the identity. In any case, we may assume that $\mathscr{A}$ is unital. By Proposition 4.4.9, *iv.)*, we have $0 \leq a^*a \leq \|a\|^2\mathbb{1}$ for all $a \in \mathscr{A}$. Since for a given vector $\phi \in \mathfrak{H}$ the map

$$a \mapsto \langle \phi, \pi(a)\phi \rangle$$

is a positive linear functional on $\mathscr{A}$, we get the following estimate

$$\|\pi(a)\phi\|^2 = \langle \pi(a)\phi, \pi(a)\phi \rangle = \langle \phi, \pi(a^*a)\phi \rangle \leq \langle \phi, \pi(\mathbb{1})\phi \rangle \|a\|^2 = \|P\phi\|^2\|a\|^2,$$

where $P = \pi(\mathbb{1})$. Since any non-zero projection has operator norm 1 (and norm zero for $P = 0$) we conclude

$$\|\pi(a)\| = \sup_{\phi \neq 0} \frac{\|\pi(a)\phi\|}{\|\phi\|} \leq \|a\|,$$

establishing the continuity of $\pi(a)$ together with the estimate (4.4.30) on $\mathfrak{H}$. Thus $\pi(a)$ extends to the completion $\widehat{\mathfrak{H}}$, still obeying (4.4.30). By the usual continuity arguments we see that $\widehat{\pi}$ is still a *-representation. $\square$

Thus we can automatically extend any *-representation of a C^*-algebra on a pre-Hilbert space to its Hilbert space completion. In the following we will frequently do that. In the end, this theorem allows to consider *only* *-representations on Hilbert spaces from the beginning. This point of view will be taken in Chapter 7 where we will refine our considerations on the representation theory of C^*-algebras considerably.

Remark 4.4.25 (GNS construction for C^*-algebras) Let $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a positive linear functional on a C^*-algebra $\mathscr{A}$. In Section 1.3 we have constructed out of ω a pre-Hilbert space $\mathfrak{H}_\omega = \mathscr{A}\big/\mathscr{J}_\omega$ together with a *-representation π_ω of $\mathscr{A}$ on $\mathfrak{H}_\omega$, the GNS representation corresponding to ω. Now we can complete this to a *-representation $\widehat{\pi}_\omega$ on $\widehat{\mathfrak{H}}_\omega$ by Theorem 4.4.24, still called the GNS representation of ω.

Corollary 4.4.26 *Let $\pi\colon \mathscr{A} \longrightarrow \mathfrak{B}(\mathfrak{H})$ be a *-representation of a unital C^*-algebra on a Hilbert space with $\pi \neq 0$. Then*

$$\|\pi\| = 1. \tag{4.4.31}$$

PROOF: We know already $\|\pi(a)\| \leq \|a\|$ from (4.4.30), which implies $\|\pi\| \leq 1$. Since for $\pi \neq 0$ we have $P = \pi(\mathbb{1}) \neq 0$ and since the norm of a non-zero projection necessarily is 1, we get (4.4.31). $\square$

In a next step we deal with general *-homomorphisms between C^*-algebras. In fact, for the following continuity statement it will be sufficient to have a C^*-algebra as target and an arbitrary Banach *-algebra as source:

Proposition 4.4.27 *Let $\mathscr{A}$ be a unital Banach *-algebra and let $\mathscr{B}$ be a unital C^*-algebra. If $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a unital *-homomorphism then Φ is continuous and for all $a \in \mathscr{A}$ one has*

$$\|\Phi(a)\| \leq \|a\| \tag{4.4.32}$$

and $\|\Phi\| = 1$.

PROOF: We use $\operatorname{spec}_{\mathscr{B}}(\Phi(a)) \subseteq \operatorname{spec}_{\mathscr{A}}(a)$ according to Theorem 4.2.19, *vi.)*, and $\varrho_{\mathscr{A}}(a) \leq \|a\|$ for Banach algebras according to *iv.)* of the same theorem. For a normal element $a \in \mathscr{A}$ also $\Phi(a)$ is normal and we get

$$\|\Phi(a)\| = \varrho_{\mathscr{B}}(\Phi(a)) = \sup\big\{|\lambda| \bigm| \lambda \in \operatorname{spec}_{\mathscr{B}}(\Phi(a))\big\} \leq \sup\big\{|\lambda| \bigm| \lambda \in \operatorname{spec}_{\mathscr{A}}(a)\big\} = \varrho_{\mathscr{A}}(a) \leq \|a\|,$$

and hence (4.4.32). For an arbitrary element a we know that a^*a is normal and hence by the C^*-property in $\mathscr{B}$ we have

$$\|\Phi(a)\|^2 = \|\Phi(a)^*\Phi(a)\| = \|\Phi(a^*a)\| \leq \|a^*a\| \leq \|a\|^2,$$

proving (4.4.32) in general, Since $\Phi(\mathbb{1}) = \mathbb{1}$ by assumption, we have $\|\Phi\| = 1$ and not just $\|\Phi\| \leq 1$.$\square$

Thus *-homomorphisms *into* C^*-algebras are necessarily continuous. Moreover, we note that if $\mathscr{A}$ is non-unital then we can extend Φ to the unitization $\widetilde{\mathscr{A}}$ by setting $\widetilde{\Phi}(a + z\mathbb{1}) = \Phi(a) + z\mathbb{1}_{\mathscr{B}}$. Hence we get the same conclusion also if $\mathscr{A}$ is non-unital. If $\mathscr{B}$ is non-unital the situation is slightly more complicated. We can pass from $\mathscr{B}$ to $\widetilde{\mathscr{B}}$ but if $\mathscr{A}$ was already unital then $\Phi(\mathbb{1}_{\mathscr{A}}) = P \in \mathscr{B}$ is only a projection different from $\mathbb{1}_{\widetilde{\mathscr{B}}} \in \widetilde{\mathscr{B}}$. In this situation not much can be said about the spectrum of $\Phi(a)$. We summarize this discussion in a separate corollary:

Corollary 4.4.28 *Let $\mathscr{A}$ be a non-unital Banach *-algebra and let $\mathscr{B}$ be a unital C^*-algebra. If $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a *-homomorphism then Φ is continuous and $\|\Phi\| \le 1$.*

In the case of two C^*-algebras we can prove the continuity of *-homomorphism also in the non-unital case in a more simple way, relying on the uniqueness of the C^*-norm according to Theorem 4.3.19.

Proposition 4.4.29 *Let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a *-homomorphism between C^*-algebras. Then Φ is continuous with $\|\Phi\| \le 1$.*

PROOF: We define a new norm on $\mathscr{A}$ by

$$\|a\|_1 = \max\left\{\|a\|_{\mathscr{A}}, \|\Phi(a)\|_{\mathscr{B}}\right\}.$$

Indeed, this is a norm and we have $\|ab\|_1 \le \|a\|_1\|b\|_1$ by the homomorphism properties of Φ. Moreover, we have

$$\begin{aligned}
\|a^*a\|_1 &= \max\left\{\|a^*a\|_{\mathscr{A}}, \|\Phi(a^*a)\|_{\mathscr{B}}\right\} \\
&= \max\left\{\|a\|_{\mathscr{A}}^2, \|\Phi(a)\Phi(a)^*\|_{\mathscr{B}}\right\} \\
&= \max\left\{\|a\|_{\mathscr{A}}^2, \|\Phi(a)\|_{\mathscr{B}}^2\right\} \\
&= \|a\|_1^2,
\end{aligned}$$

since both algebras are C^*-algebras and Φ is a *-homomorphism. Hence $\|\cdot\|_1$ is a C^*-norm on $\mathscr{A}$, too. Thus on the (probably non-unital) commutative C^*-subalgebra of $\mathscr{A}$ generated by the Hermitian element a^*a we get $\|a^*a\|_1 = \|a^*a\|_{\mathscr{A}}$ by Corollary 4.3.21. Thus, by the C^*-property we get $\|a\|_1 = \|a\|_{\mathscr{A}}$ for all $a \in \mathscr{A}$ and hence the norms coincide. But this implies $\|\Phi(a)\|_{\mathscr{B}} \le \|a\|_{\mathscr{A}}$. ☐

In a next step we consider two C^*-algebras. In this situation we can characterize the injectivity of a (necessarily continuous) *-homomorphism very easily:

Proposition 4.4.30 *Let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a *-homomorphism between C^*-algebras. Then the following statements are equivalent:*

i.) The *-homomorphism Φ is injective.

ii.) For all $a \in \mathscr{A}$ one has $\|\Phi(a)\| = \|a\|$.

iii.) For all $a \in \mathscr{A}$ with $a \ge 0$ and $\Phi(a) = 0$ one has $a = 0$.

PROOF: Assume first that Φ is injective but there is a $b \in \mathscr{A}$ with $\|\Phi(b)\| < \|b\|$. By the C^*-property we also know $\|\Phi(b^*b)\| = \|\Phi(b)\|^2 < \|b\|^2 = \|b^*b\|$. Hence we also find a positive element $a \ge 0$ with $\|\Phi(a)\| < \|a\|$. Since $\operatorname{spec}(a) \subseteq [0, \|a\|]$ and $\operatorname{spec}(\Phi(a)) \subseteq [0, \|\Phi(a)\|]$ we conclude that $\|a\| \notin \operatorname{spec}(\Phi(a))$. However, for a positive element a, we necessarily have $\|a\| \in \operatorname{spec}(a)$. Thus there is a continuous function $f \in \mathscr{C}(\operatorname{spec}(a))$ with $f(\|a\|) = 1$ but $f\big|_{\operatorname{spec}(\Phi(a))} = 0$ by Urysohn's Lemma. Note that we can arrange things such that $f(0) = 0$ so the continuous calculus stays inside $\mathscr{A}$ even in the non-unital case. We consider now the element $f(a) \in \mathscr{A}$. Clearly, $f(a) \ne 0$ since f is not the zero-function on $\operatorname{spec}(a)$. On the other hand, $f(\Phi(a)) = 0$ by the vanishing of f on $\operatorname{spec}(\Phi(a))$. Now Remark 4.3.26, *iv.)*, tells us $0 = f(\Phi(a)) = \Phi(f(a))$ which contradicts the injectivity of Φ. This shows *i.)* $\implies$ *ii.)*. The implication *ii.)* $\implies$ *iii.)* is trivial. Thus assume *iii.)* and let $a \in \ker\Phi$. Then $a^*a \ge 0$ and $\Phi(a^*a) = \Phi(a)^*\Phi(a) = 0$. Thus $a^*a = 0$ by assumption and hence $a = 0$ follows by the C^*-property. ☐

An immediate consequence of this proposition and the quotient construction from Corollary 4.4.17 is the following:

Corollary 4.4.31 *Let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a (unital) *-homomorphism between (unital) C^*-algebras. Then $\Phi(\mathscr{A}) \subseteq \mathscr{B}$ is a (unital) C^*-subalgebra of $\mathscr{B}$.*

PROOF: If Φ is not injective we can pass to the quotient $\mathscr{A} \big/ \ker \Phi$ which is again a (unital) C^*-algebra since $\ker \Phi$ is clearly a closed *-ideal as Φ is automatically continuous. But then the induced map $\Phi\colon \mathscr{A} \big/ \ker \Phi \longrightarrow \mathscr{B}$ is still a (unital) *-homomorphism with the same image $\Phi(\mathscr{A})$. However, now this map is injective. Thus we can assume without restriction that we have an injective *-homomorphism from the beginning. Then Φ is norm-preserving by Proposition 4.4.30 and hence the image is *closed*. Since the image is a (unital) *-algebra anyway, the result follows. □

Corollary 4.4.32 *Between C^*-algebras *-isomorphisms are norm-preserving. In particular, the *-automorphisms of a C^*-algebra are norm-preserving.*

We can use the results on *-representations and *-homomorphism together with the GNS construction in order to obtain a faithful *-representation of a C^*-algebra. This finally shows that C^*-algebras are nothing else than closed *-subalgebras of $\mathfrak{B}(\mathfrak{H})$ for some suitably chosen Hilbert space $\mathfrak{H}$.

Theorem 4.4.33 (Faithful *-representation) *Let $\mathscr{A}$ be a C^*-algebra. Then there exists a faithful, i.e. injective, *-representation of $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$ turning $\mathscr{A}$ into a C^*-subalgebra of $\mathfrak{B}(\mathfrak{H})$. More specifically, the *-representation $(\mathfrak{H}, \pi)$ with*

$$\mathfrak{H} = \widehat{\bigoplus_{\omega \text{ is a state on } \mathscr{A}}} \widehat{\mathfrak{H}}_\omega \quad \text{and} \quad \pi = \widehat{\bigoplus_{\omega \text{ is a state on } \mathscr{A}}} \widehat{\pi}_\omega \tag{4.4.33}$$

is faithful, where $(\widehat{\mathfrak{H}}_\omega, \widehat{\pi}_\omega)$ is the completed GNS representation for the state ω and the direct sum runs over all states of $\mathscr{A}$.

PROOF: Clearly, it suffices to show the second, more specific claim. If $\mathscr{A}$ is non-unital, we pass to the unitization first. We already know that the GNS construction yields a *-representation of $\mathscr{A}$ on the completed GNS pre-Hilbert space $\widehat{\mathfrak{H}}_\omega$ according to Remark 4.4.25. Moreover, it is not difficult to see that $\bigoplus_\omega \widehat{\pi}_\omega$ is still a *-representation of $\mathscr{A}$ on the pre-Hilbert space $\bigoplus_\omega \widehat{\mathfrak{H}}$ thanks to the block-diagonal structure. Thus by Theorem 4.4.24 it extends to the completed direct orthogonal sum $\widehat{\bigoplus}_\omega \widehat{\mathfrak{H}}_\omega$ yielding a *-representation $\pi = \widehat{\bigoplus}_\omega \widehat{\pi}_\omega$. It remains to check that this (quite enormous) *-representation is faithful. Thus let $a \in \mathscr{A}$ be non-zero. Then $a^*a \neq 0$, too, and there is a state ω_0 of $\mathscr{A}$ with $\omega_0(a^*a) = \|a\|^2$ by Theorem 4.4.21. But this means that for the GNS representation π_{ω_0} we have

$$0 < \|a\|^2 = \omega_0(a^*a) = \langle \Omega_{\omega_0}, \pi_{\omega_0}(a^*a)\Omega_{\omega_0} \rangle = \|\pi_{\omega_0}(a)\Omega_{\omega_0}\|^2,$$

where $\Omega_{\omega_0} = \psi_{\mathbb{1}}$ is the equivalence class of $\mathbb{1} \in \mathscr{A}$ in $\mathscr{A} \big/ \mathscr{J}_{\omega_0} = \mathfrak{H}_{\omega_0}$ as usual. But this shows that the operator $\pi_{\omega_0}(a)$ is different from zero. Since ω_0 occurs in the collection of all states, also $\pi(a)$ is different from zero as at least the block entry $\pi_{\omega_0}(a)$ is. Thus π is injective as claimed. In the unital case we know that $\mathscr{A} \cong \pi(\mathscr{A}) \subseteq \mathfrak{B}(\mathfrak{H})$ is *-isomorphic to the closed *-subalgebra $\pi(\mathscr{A})$ by Corollary 4.4.31. In the non-unital case we have $\widetilde{\mathscr{A}} \cong \pi(\widetilde{\mathscr{A}})$ and $\pi(\widetilde{\mathscr{A}})$ is closed. But since π is norm-preserving and $\mathscr{A}$ is closed in the unitization $\widetilde{\mathscr{A}}$, the image $\pi(\mathscr{A})$ is closed in $\pi(\widetilde{\mathscr{A}})$ and hence closed in $\mathfrak{B}(\mathfrak{H})$ also in this case. □

Remark 4.4.34 (*-Representation theory of C^*-algebras)

i.) This theorem is the starting point for the *-representation theory of C^*-algebras. It shows that C^*-algebras are in some sense very easy compared to general Banach *-algebras: every C^*-algebra is (isomorphic to) a closed *-subalgebra of $\mathfrak{B}(\mathfrak{H})$ for a suitable, large Hilbert space $\mathfrak{H}$. Thus the Example 4.3.7, *v.)*, was already generic.

ii.) The construction in Theorem 4.4.33 is, however, of rather limited use: the direct sum over *all* states is typically a Hilbert space of enormous size. Already for $\mathscr{A} = \mathrm{M}_2(\mathbb{C})$ every ray in $\mathbb{C}^2$ gives a state. Hence we have at least $\mathbb{CP}^1$-many states. Thus the cardinality of the set of states is at least the continuum (in fact, equal to it) and hence $\mathfrak{H}$ is clearly non-separable. However, already one state would have yielded a faithful GNS representation in this case. Thus we expect that for a given C^*-algebra a suitably small collection of states yields already a faithful *-representation.

iii.) For quantum physics the class of C^*-algebras seems very attractive now: even though we can (and want to) treat the observables as an abstract concept based solely on the commutation relations in a first step, in the end we get back our operators on Hilbert spaces without losses on the way.

iv.) Even though we have reached a very satisfactory stage of the theory there are at least two problems unsolved: for a physically reasonable theory we still have to find criteria which is the correct and physically meaningful representation of the given observable algebra. This indeed stays an open question and is discussed e.g. in algebraic quantum field theory within the framework of super-selection rules. The second question is probably even more severe: even though C^*-algebras are very much desirable to have as observable algebras, they are typically not easy to obtain as we have seen in the discussion of the canonical commutation relations in Proposition 4.1.37. In more sophisticated examples of quantum systems it is an ongoing research program to find the reasonable C^*-algebraic description for the observables. However, it would go much beyond our present, and still quite elementary, discussion to enter such questions more thoroughly.

4.5 Exercises

Exercise 4.5.1 (Continuity of multilinear maps) Consider locally convex spaces $V_1, \ldots, V_k$ and W together with a k-linear map

$$\phi\colon V_1 \times \cdots \times V_k \longrightarrow W. \tag{4.5.1}$$

Show that the following statements are equivalent by proceeding analogously to the proof of Theorem 4.1.3:

i.) The map ϕ is continuous.

ii.) The map ϕ is continuous at $(0, \ldots, 0) \in V_1 \times \cdots \times V_k$.

iii.) For every continuous seminorm q from a defining system on W there exist continuous seminorms $\mathrm{p}_1, \ldots, \mathrm{p}_k$ on $V_1, \ldots, V_k$, respectively, such that

$$\mathrm{q}(\phi(v_1, \ldots, v_k)) \le \mathrm{p}_1(v_1) \cdots \mathrm{p}_k(v_k) \tag{4.5.2}$$

for all $v_1 \in V_1, \ldots, v_k \in V_k$.

Exercise 4.5.2 (Associativity of $\otimes_\pi$ and $\hat{\otimes}_\pi$) Formulate and prove the associativity properties for the π-tensor product $\otimes_\pi$ and the completed π-tensor product $\hat{\otimes}_\pi$ of locally convex spaces.

Exercise 4.5.3 (Functorial properties of $\otimes_\pi$ and $\hat{\otimes}_\pi$)

Exercise 4.5.4 (Tensor product of Banach and Fréchet spaces) Let V and W be locally convex spaces.

i.) Suppose that V and W are in addition Fréchet spaces. Show that then $V \hat{\otimes}_\pi W$ is again a Fréchet space.

ii.) Suppose in addition that V and W are normed spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. Show that the π-topology on $V \otimes_\pi W$ is obtained by the norm $\|\cdot\|_V \otimes \|\cdot\|_W$.

iii.) Show that the completed π-tensor product $V \mathbin{\hat{\otimes}_\pi} W$ of Banach spaces is again a Banach space in a natural way.

Exercise 4.5.5 (Hausdorffization and completion functor) Show that locally convex algebras form a category with respect to continuous algebra homomorphisms. Formulate and show that Hausdorffization of locally convex algebras as well as completion of Hausdorff locally convex algebras are functorial. Proceed analogously for locally convex *-algebras.

Exercise 4.5.6 (Diagram of categories of topological algebras) Draw a big diagram with the categories topalg, topAlg, Topalg, TopAlg, lcalg, LCalg, lcAlg, LCAlg, CLCalg, CLCAlg, Fréchetalg, and FréchetAlg and indicate which category is a (full) subcategory of which other. Include also the Hausdorffization and completion functors into the picture.

Exercise 4.5.7 (Closure of ideals) Let $\mathscr{A}$ be a locally convex algebra and $\mathscr{J} \subseteq \mathscr{A}$. Show that if $\mathscr{J}$ is a subalgebra, a left, a right, or a two-sided ideal, respectively, then the closure $\mathscr{J}^{\mathrm{cl}}$ is again a subalgebra, a left, a right, or a two-sided ideal, respectively. Show also that in the case of a locally convex *-algebra the closure of a *-ideal is again a *-ideal.

Exercise: Mo lmc algebras: space, $\mathscr{C}_0^\infty$ is complicated, maps are (dis for test funct distributions

Exercise 4.5.8 (The trace of finite-rank operators I) Let V, W, and U be locally convex spaces with topological duals V', W' and U', respectively. Recall that a linear map $\Phi\colon V \longrightarrow W$ has *finite rank* if $\dim \Phi < \infty$.

i.) Show that a continuous linear map $\Phi\colon V \longrightarrow W$ has finite-rank iff there exist vectors $w_1, \ldots, w_n \in W$ and continuous linear functionals $\varphi_1, \ldots, \varphi_n \in V'$ such that

$$\Phi(v) = \sum_{i=1}^{n} w_i \varphi_i(v) \tag{4.5.3}$$

for all $v \in V$. Denote the continuous finite-rank maps by $\mathfrak{F}(V, W) \subseteq \mathrm{L}(V, W)$. Show that $\mathfrak{F}(V, W) \subseteq \mathrm{L}(V, W)$ is a subspace.

ii.) Show that the composition of finite-rank operators is again a finite-rank operator. More general, let $A\colon V \longrightarrow W$ and $B\colon W \longrightarrow U$ be continuous linear maps and let $\Phi \in \mathfrak{F}(V, W)$ as well as $\Psi \in \mathfrak{F}(W, U)$. Show that $\Phi \circ A$ and $B \circ \Psi$ are finite-rank operators. Conclude that $\mathfrak{F}(V) = \mathfrak{F}(V, V)$ is a two-sided ideal of $\mathrm{L}(V)$.

iii.) Show that the linear map

$$\Theta\colon W \otimes V' \longrightarrow \mathfrak{F}(V, W), \tag{4.5.4}$$

defined on elementary tensors by $\Theta(w \otimes \varphi)(v) = w\varphi(v)$ for $v \in V$, $\varphi \in V'$ and $w \in W$, is a well-defined linear map. We write $\Theta_{w,\varphi} = \Theta(w \otimes \varphi)$ and call $\Theta_{w,\varphi}$ a *rank-one* map.

iv.) Show that Θ is in fact bijective.

Hint: For the injectivity you have to use a basis $B \subseteq W$ and use the fact that every tensor $\alpha \in W \otimes V'$ can be written as

$$\alpha = \sum_{b \in B} b \otimes \varphi_b$$

with uniquely determined $\varphi_b \in V'$ such that all but finitely many φ_b are zero, see also [63, Prop. 3.40].

v.) Compute explicitly the composition $\Theta_{u,\psi} \circ \Theta_{w,\varphi}$ for $u \in U$, $\psi \in W'$, $w \in W$ and $\varphi \in V'$. Moreover, compute explicitly $\Theta_{u,\psi} \circ A$ and $B \circ \Theta_{w,\varphi}$ for continuous linear maps $B\colon W \longrightarrow U$ and $A\colon V \longrightarrow W$.

vi.) Show that the evaluation map

$$\mathrm{ev}\colon V \otimes V' \longrightarrow \mathbb{C} \tag{4.5.5}$$

defined on elementary tensors by $\mathrm{ev}(v \otimes \varphi) = \varphi(v)$ is a well-defined linear map.

vii.) Consider the linear functional

$$\mathrm{tr} = \mathrm{ev} \circ \Theta^{-1} \colon \mathfrak{F}(V) = \mathfrak{F}(V, V) \longrightarrow \mathbb{C}. \tag{4.5.6}$$

Prove that tr is a trace, i.e. vanishes on commutators with general continuous linear maps.

Hint: Why is it enough to show $\mathrm{tr}(\Phi\Psi) = \mathrm{tr}(\Psi\Phi)$ for a rank-one operators Φ and an arbitrary $\Psi \in \mathrm{L}(V)$? Use now *v.)*.

Exercise 4.5.9 (Locally convex Lie algebra)

Exercise 4.5.10 (Completed tensor algebra) Let V be a Hausdorff locally convex space.

i.) Consider the subset $\hat{\mathrm{T}}(V)$ of $\prod_{n=0}^{\infty} V^{\hat{\otimes}_\pi n}$ as in (4.1.49) and show first that it is a subspace of the Cartesian product. Show also that $\mathrm{T}(\mathrm{p})$ is a well-defined seminorm on it for all continuous seminorms p on V. Endow it with the resulting locally convex topology and show that this topology is finer than the Cartesian product topology.

ii.) Show that $V^{\hat{\otimes}_\pi n}$ is canonically embedded into $\hat{\mathrm{T}}(V)$ and the image is a closed subspace.

iii.) Show that $\hat{\mathrm{T}}(V)$ is complete. To this end, investigate the relation between the component nets $(v_{n,i})_{i \in I}$ for $n \in \mathbb{N}_0$ of a net $(v_i)_{i \in I}$ in $\hat{\mathrm{T}}(V)$.

iv.) Show that $\mathrm{T}(V)$ and $\mathrm{T}(\widehat{V})$ are dense subspaces of $\hat{\mathrm{T}}(V)$. Moreover, show that $\bigoplus_{n=0}^{\infty} V^{\hat{\otimes}_\pi n}$ is even a sequentially dense subspace.

v.) Conclude that $\hat{\mathrm{T}}(V)$ is the completion of $\mathrm{T}(V)$ and hence a complete unital locally multiplicatively convex algebra.

vi.) Use Proposition 4.1.30, *vi.)*, to formulate and show that the construction $V \rightsquigarrow \mathrm{T}(V)$ gives a functor $\mathsf{lcVect} \longrightarrow \mathsf{lmcAlg}$. Show analogously that one gets a functor $\mathsf{LCVect} \longrightarrow \mathsf{LMCAlg}$.

vii.) Formulate in detail and prove the statement that $\hat{\mathrm{T}}(V)$ is the free unital complete locally multiplicatively convex algebra generated by V.

viii.) Formulate and show that also $V \rightsquigarrow \hat{\mathrm{T}}(V)$ has nice functorial properties resulting in a functor $\mathsf{LCVect} \longrightarrow \mathsf{CLMCAlg}$.

ix.) Discuss in which sense the completed tensor algebra $\hat{\mathrm{T}}^\bullet(V)$ is still "graded", thereby justifying our notation.

ensor algebra roperty is the nent as entire alculus, since $\mathscr{O}(\mathbb{C})$. Prove 6 along these lines. Also $f(g(a))$ using c calculus for $\mathscr{O}(\mathbb{C})$ itself. space differs ensor algebra completion

Exercise 4.5.11 (The tensor algebra of a Hilbert space)

Exercise 4.5.12 (Seminorms for a Fréchet algebra) Let $\mathscr{A}$ be a Fréchet algebra (in fact, the completeness is not essential here: first countable and Hausdorff locally convex would be sufficient). Show that there is a sequence of seminorms $\mathrm{p}_1 \leq \mathrm{p}_2 \leq$ defining the topology such that for all $n \in \mathbb{N}$ one has

$$\mathrm{p}_n(ab) \leq \mathrm{p}_{n+1}(a)\mathrm{p}_{n+1}(b) \tag{4.5.7}$$

for $a, b \in \mathscr{A}$.

Exercise 4.5.13 (LMC symmetric and Graßmann algebra)

Exercise 4.5.14 (Non-unital entire calculus)

Exercise 4.5.15 (Finite dimensional algebras) Show that a finite-dimensional algebra $\mathscr{A}$ can always be endowed with the structure of a Banach algebra, though it might not be a unital Banach algebra.

Hint: First use that all norms on $\mathbb{C}^n$ yield the same topology. Prove that any bilinear map is continuous. Use the continuity estimate for the multiplication and rescale the norm.

Exercise 4.5.16 (Idempotents in a normed algebra) Let $\mathscr{A}$ be a normed algebra and let $e \in \mathscr{A}$ be an idempotent. Show that either $e = 0$ or $\|e\| \geq 1$. Give an example where $\|e\| > 1$.

Exercise 4.5.17 (The Banach *-algebra $\mathscr{C}(X)$) Let X be a compact Hausdorff space and $\mathscr{C}(X)$ the continuous complex-valued functions on X.

i.) Verify that under the pointwise multiplication and with respect to the pointwise complex conjugation $\mathscr{C}(X)$ becomes a unital commutative *-algebra.

ii.) Use the compactness of X to show that $\|f\|_\infty = \sup_{x\in X}|f(x)|$ is a well-defined norm for $\mathscr{C}(X)$ which makes $\mathscr{C}(X)$ a unital normed *-algebra.

iii.) Show that the supremum in the definition of $\|f\|_\infty$ can be replaced by the maximum. Conclude (4.2.5) from this.

iv.) Show that $\mathscr{C}(X)$ is complete with respect to $\|\cdot\|_\infty$. Show that it becomes a closed subspace of $\mathscr{B}(X)$ this way.

Hint: Show first that a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ with respect to $\|\cdot\|_\infty$ converges pointwise to a bounded function f. Then use a standard $\frac{\epsilon}{3}$-argument to show that f is continuous at a given point $x \in X$.

v.) Conclude that $\mathscr{C}(X)$ is a unital commutative Banach *-algebra.

vi.) Now let $f \in \mathscr{C}(X)$ and show that the spectrum of f is given by the set of its values, i.e.

$$\operatorname{spec}_{\mathscr{C}(X)}(f) = f(X). \tag{4.5.8}$$

Conclude that this implies $\varrho_{\mathscr{C}(X)}(f) = \|f\|_\infty$.

vii.) Suppose now that X is only a locally compact Hausdorff space and consider the *bounded* continuous functions $\mathscr{C}_b(X)$. Modify the above proofs to show that also $\mathscr{C}_b(X)$ becomes a unital commutative Banach *-algebra with respect to the pointwise operations and the supremum norm. Is (4.2.5) also valid in this case?

Exercise 4.5.18 (The functor $\mathscr{C}$) Consider a continuous map $\phi\colon X \longrightarrow Y$ between compact Hausdorff spaces.

i.) Show that the pull-back $\phi^* f = f \circ \phi$ for a continuous function $f \in \mathscr{C}(Y)$ gives a continuous function on X and conclude that $\phi^*\colon \mathscr{C}(Y) \longrightarrow \mathscr{C}(Y)$ is a continuous unital *-homomorphism.

Hint: The continuity follows from the quite obvious estimate $\|\phi^* f\|_\infty \leq \|f\|_\infty$.

ii.) Show that the assignment $X \rightsquigarrow \mathscr{C}(X)$ and $(\phi\colon X \longrightarrow Y) \rightsquigarrow (\phi^*\colon \mathscr{C}(Y) \longrightarrow \mathscr{C}(X))$ gives a contravariant functor from the category of compact Hausdorff spaces into the category of commutative unital C^*-algebras.

Exercise 4.5.19 (Approximating the square root) Show that there is a sequence of real polynomials $p_n \in \mathbb{R}[x]$ with $p_n(0) = 0$ which approximate the function $x \mapsto \sqrt{x}$ uniformly on $[0, 1]$. Extend this result for an arbitrary interval $[0, a]$ with $a > 0$.

Hint: By recursion, define the polynomials

$$p_0(x) = 0, \quad \text{and} \quad p_{n+1}(x) = p_n(x) + \frac{1}{2}\left(x - p_n^2(x)\right). \tag{4.5.9}$$

Show $p_n(0) = 0$ and the estimates

$$p_n(x) \geq 0, \quad \text{and} \quad 0 \leq \sqrt{x} - p_n(x) \leq \frac{2\sqrt{x}}{2 + n\sqrt{x}} \tag{4.5.10}$$

for $x \in [0, 1]$.

ise: including composition

Exercise 4.5.20 (Continuous calculus for $\mathscr{C}(X)$)

Exercise 4.5.21 (The Banach algebra $\mathscr{C}(X, \mathscr{A})$) Let $\mathscr{A}$ be a normed algebra and X a compact Hausdorff space. Denote the continuous $\mathscr{A}$-valued functions on X by $\mathscr{C}(X, \mathscr{A})$.

i.) Show that under the pointwise operations $\mathscr{C}(X, \mathscr{A})$ becomes an associative algebra which is unital or commutative or a *-algebra if $\mathscr{A}$ was unital or commutative or a *-algebra, respectively.

ii.) Show that the supremum $\|f\|_\infty = \sup_{x \in X} \|f(x)\|_{\mathscr{A}}$ is actually a finite maximum and defines a norm on $\mathscr{C}(X, \mathscr{A})$ making $\mathscr{C}(X, \mathscr{A})$ a normed algebra again.

iii.) Show that $\mathscr{C}(X, \mathscr{A})$ is complete whenever $\mathscr{A}$ was complete by transferring the ideas from Exercise 4.5.17, **??**, to this more general situation.

iv.) Conclude that $\mathscr{C}(X, \mathscr{A})$ is a Banach algebra or a Banach *-algebra whenever $\mathscr{A}$ was a Banach or a Banach *-algebra, respectively.

v.) Suppose that $\mathscr{A}$ satisfies $\|a^*a\|_{\mathscr{A}} = \|a\|^2_{\mathscr{A}}$ for all $a \in \mathscr{A}$. Show that in this situation also $\mathscr{C}(X, \mathscr{A})$ has this property with respect to $\|\cdot\|_\infty$.

vi.) Suppose $\mathscr{A}$ is unital. Show that there is a canonical algebra homomorphism from $\mathscr{C}(X)$ into the *center* of $\mathscr{C}(X, \mathscr{A})$. Show that in the non-unital case $\mathscr{C}(X, \mathscr{A})$ becomes a $\mathscr{C}(X)$-module such that the module structure and the algebra structure of $\mathscr{C}(X, \mathscr{A})$ are compatible (in which sense?). Show that this module multiplication enjoys nice continuity properties by estimating $\|fg\|_\infty$ for $f \in \mathscr{C}(X)$ and $g \in \mathscr{C}(X, \mathscr{A})$ appropriately.

vii.) Transfer the above results to a locally compact Hausdorff space X and the bounded continuous functions $\mathscr{C}_b(X, \mathscr{A})$ analogously to Exercise 4.5.17, *vii.)*.

Exercise 4.5.22 (Unitization) Let $\mathscr{A}$ be an associative algebra over a field $\mathbb{k}$ without unit. Define $\widetilde{\mathscr{A}} = \mathscr{A} \oplus \mathbb{k}$ and denote the vector $(0, 1)$ by $\mathbb{1}$.

i.) Show that the product defined by (4.2.13) turns $\widetilde{\mathscr{A}}$ into a unital associative algebra over $\mathbb{k}$ which contains $\mathscr{A}$ as a two-sided ideal.

ii.) Show that the unitization has the following universal property: for every unital algebra $\mathscr{B}$ over $\mathbb{k}$ and every algebra homomorphism $\phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ there exists a unique extension $\Phi\colon \widetilde{\mathscr{A}} \longrightarrow \mathscr{B}$ as an unital algebra homomorphism. If ϕ was injective, then Φ is injective, too. In this sense, $\widetilde{\mathscr{A}}$ is the *smallest unital algebra* containing $\mathscr{A}$.

iii.) Consider now the case of a non-unital *-algebra $\mathscr{A}$ over $\mathbb{C}$ and show that the unitization becomes a unital *-algebra obeying again a universal property analogous to *ii.)*.

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Exercise 4.5.23 (Unitization of locally convex algebras)

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Exercise 4.5.24 (Norms on the unitization) Let $\mathscr{A}$ be a normed non-unital algebra and let $\widetilde{\mathscr{A}}$ be its unitization. Let $p \geq 1$ and consider

$$\|a + z\mathbb{1}\|_p = \sqrt[p]{\|a\|^p + |z|^p} \quad \text{as well as} \quad \|a + z\mathbb{1}\|_\infty = \max(\|a\|, |z|). \tag{4.5.11}$$

i.) Show that $\|\cdot\|_p$ and $\|\cdot\|_\infty$ define norms on the unitization $\widetilde{\mathscr{A}}$.

ii.) Show that $\mathscr{A}$ is a closed ideal in $\widetilde{\mathscr{A}}$ and the induced norm on $\mathscr{A}$ coincides with the original norm of $\mathscr{A}$.

iii.) Show that as a locally convex space, all the topologies coming from the norms $\|\cdot\|_p$ and $\|\cdot\|_\infty$ agree and coincide with the Cartesian product topology. Deduce statements about completeness etc. from this observation. However, give examples that the normed spaces are all different in general.

iv.) Investigate whether the norms $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are submultiplicative.

Exercise 4.5.25 (Spectrum in a non-unital algebra) Let $\mathscr{A}$ be a non-unital associative algebra.

i.) In which sense does part *i.)* of Theorem 4.2.19 still hold in the non-unital case?

ii.) For which polynomials $p(x) \in \mathbb{C}[x]$ can one formulate an analogous statement as Theorem 4.2.19, *ii.)*?

Exercise 4.5.26 (Spectrum and homomorphisms) Let $\mathscr{A}$ and $\mathscr{B}$ be unital associative algebras over a field $\mathbb{k}$ and let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a unital homomorphism. Show that for all $a \in \mathscr{A}$ one has (4.2.35).

Exercise 4.5.27 (The spectrum of a product) This exercise illustrates the statement of Theorem 4.2.19, *i.)*.

i.) Let $A, B \in \mathrm{M}_n(\mathbb{C})$. Show that $0 \in \operatorname{spec}(AB)$ iff $0 \in \operatorname{spec}(BA)$. This strengthens the statement of Theorem 4.2.19, *i.)*, for the case $\mathscr{A} = \mathrm{M}_n(\mathbb{C})$.

ii.) Consider again the shift operator $V \in \mathfrak{B}(\ell^2)$ acting on the canonical Hilbert basis as $V\mathrm{e}_n = \mathrm{e}_{n+1}$. Prove that for $A = V$ and $B = V^*$ the spectra of AB and BA are different. This gives an example where in Theorem 4.2.19, *i.)*, the full generality is needed. stefan: Wo v

Exercise 4.5.28 ($\mathscr{C}(\mathbb{S}^2)$ as Banach *-algebra) Consider the continuous functions $\mathscr{C}(\mathbb{S}^2)$ on the sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ with its usual Banach algebra structure. Define for $f \in \mathscr{C}(\mathbb{S}^2)$ stefan: Upda *-rep script?

$$f^*(x) = \overline{f(I(x))} \quad \text{where} \quad I(x) = -x \tag{4.5.12}$$

is the antipodal map $I\colon \mathbb{S}^2 \longrightarrow \mathbb{S}^2$.

i.) Show that * gives a well-defined *-involution for $\mathscr{C}(\mathbb{S}^2)$.

ii.) Show that $\mathscr{C}(\mathbb{S}^2)$ becomes a Banach *-algebra with respect to this *-involution, denoted by $\mathscr{A}$.

iii.) Show that the supremum norm does *not* satisfy the C^*-condition with respect to this new involution.

iv.) Characterize the Hermitian elements in $\mathscr{A}$ and show that there exists a non-zero $f = f^* \in \mathscr{A}$ with purely imaginary spectrum.

v.) Characterize the unitary elements in $\mathscr{A}$ and show that for every $\lambda \in \mathbb{R} \setminus \{0\}$ there exists a unitary $u = u^* \in \mathscr{A}$ with $\lambda \in \operatorname{spec}_{\mathscr{A}}(u)$.

vi.) Show that there are characters φ of $\mathscr{A}$ with $\varphi(f^*) \neq \overline{\varphi(f)}$. In particular, they are not states.

This example shows that all the nice properties of commutative unital C^*-algebras we discovered in the Subsections 4.3.1 and 4.3.2 may fail for commutative Banach *-algebras.

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Exercise 4.5.29 ($-\mathbb{1}$ as a sum of squares) Let $\mathscr{A} = \mathscr{C}(\mathbb{S}^1)$ be the unital Banach algebra of continuous functions on the circle with the *-involution given by

$$f^\dagger(z) = \overline{f(-z)} \tag{4.5.13}$$

for $f \in \mathscr{C}(\mathbb{S}^1$ and $z \in \mathbb{S}^1$. Show that $-\mathbb{1} \in \mathscr{A}^{++}$.

Exercise 4.5.30 (Positive functionals of $\Lambda^\bullet(\mathbb{C}^n)$) We consider the Graßmann algebra $\Lambda^\bullet(\mathbb{C}^n)$ in n variables. Denote the standard basis of $\mathbb{C}^n$ by $\mathrm{e}_1, \ldots, \mathrm{e}_n$, then the Graßmann algebra is generated by the $\mathrm{e}_1, \ldots, \mathrm{e}_n$ having the relations $\mathrm{e}_i \wedge \mathrm{e}_j = -\mathrm{e}_j \wedge \mathrm{e}_i$.

Exercise 4.5.31 (The resolvent identity) Show the resolvent identity (4.2.21).

Exercise 4.5.32 (The exponential map) Let $\mathscr{A}$ be a unital complete locally multiplicatively convex algebra.

i.) Show that for all $a \in \mathscr{A}$ the exponential series $\exp(a)$ converges absolutely and defines an element in $\mathscr{A}$. Prove that for every continuous submultiplicative seminorm p and all $a \in \mathscr{A}$ one has

$$\mathrm{p}(\exp(a)) \leq \exp(\mathrm{p}(a)). \tag{4.5.14}$$

ii.) Show that the exponential map

$$\exp\colon \mathscr{A} \longrightarrow \mathscr{A} \tag{4.5.15}$$

satisfies the usual properties of the exponential function, i.e. we have for all $a \in \mathscr{A}$ and $s, t \in \mathbb{C}$

$$\exp(0) = \mathbb{1} \tag{4.5.16}$$

$$\exp(a) \text{ is invertible with } \exp(a)^{-1} = \exp(-a) \tag{4.5.17}$$

$$\exp((s+t)a) = \exp(sa)\exp(ta). \tag{4.5.18}$$

iii.) Let $u \in \mathscr{A}$ be invertible. Show that for all $a \in \mathscr{A}$ one has

$$u\exp(a)u^{-1} = \exp(uau^{-1}), \tag{4.5.19}$$

both by a direct computation and by an application of Theorem 4.1.36, *iv.)*.

iv.) Show that for every $a \in \mathscr{A}$ the map $z \mapsto \exp(za)$ is entire with respect to the topology of $\mathscr{A}$ and satisfies

$$\frac{\partial}{\partial z}\exp(za) = a\exp(za) = \exp(za)a. \tag{4.5.20}$$

v.) Let $a, b \in \mathscr{A}$. Show that $[a, b] = 0$ iff $[a, \exp(tb)] = 0$ for all $t \in \mathbb{R}$ iff $[\exp(ta), \exp(sb)] = 0$ for all $t, s \in \mathbb{R}$. Show that in this case we also can use $t, s \in \mathbb{C}$.

Exercise 4.5.33 (The logarithm in a Banach algebra) Let $\mathscr{A}$ be a unital Banach algebra.

i.) Use the holomorphic calculus and show that for $\|a\| < 1$ the logarithm series

$$\log(\mathbb{1} + a) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} a^k \tag{4.5.21}$$

converges absolutely.

ii.) Use Corollary 4.2.28 to show that exp and log are inverse to each other on certain open subsets of $\mathscr{A}$ and characterize the open subsets by explicit norm estimates.

iii.) Suppose $u \in \mathscr{A}$ is invertible such that $\mathrm{spec}_{\mathscr{A}}(u)$ does not contain a closed curve around 0, i.e. 0 can be joined by a continuous curve with infinity inside the complement of $\mathrm{spec}_{\mathscr{A}}(u)$. Show that there exists an algebra element $a \in \mathscr{A}$ with $u = \exp(a)$.

Hint: Use the holomorphic calculus from Theorem 4.2.25.

iv.) Show that every invertible matrix $U \in \mathrm{GL}_n(\mathbb{C})$ is of the form $U = \exp(A)$ for some $A \in \mathrm{M}_n(\mathbb{C})$.

Exercise 4.5.34 (The holomorphic calculus for Runge domains) Recall that an open subset $X \subseteq \mathbb{C}$ is called a *Runge domain* if the polynomials $\mathbb{C}[z] \subseteq \mathscr{O}(X)$ are dense with respect to the usual $\mathscr{O}$-topology on the holomorphic functions, see [48, Thm. 13.11] for details.

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Exercise 4.5.35 (The Graßmann algebra and nilpotency) Let $n \in \mathbb{N}$ and consider the Graßmann algebra $\Lambda^\bullet(\mathbb{C}^n)$ with n generators.

i.) Show that the even part of $\Lambda^\bullet(\mathbb{C}^n)$, i.e.

$$\Lambda^{\mathrm{even}}(\mathbb{C}^n) = \bigoplus_{k=0}^{\infty} \Lambda^{2k}(\mathbb{C}^n) \tag{4.5.22}$$

is a commutative unital subalgebra of $\Lambda^\bullet(\mathbb{C}^n)$.

ii.) Show that

$$\Lambda^{+}(\mathbb{C}^n) = \bigoplus_{k=1}^{\infty} \Lambda^{2k}(\mathbb{C}^n) \tag{4.5.23}$$

is an ideal in $\Lambda^{\mathrm{even}}(\mathbb{C}^n)$ such that every element in $\Lambda^{+}(\mathbb{C}^n)$ is nilpotent.

iii.) Show that the only character of $\Lambda^{\mathrm{even}}(\mathbb{C}^n)$ is given by the projection onto the $k = 0$ part in the direct sum (4.5.22).

Exercise 4.5.36 (A degenerate Banach algebra)

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Exercise 4.5.37 (The quaternions) Consider the quaternions $\mathbb{H}$ and show that there is a submultiplicative norm on $\mathbb{H}$ making the multiplication continuous and satisfying $\|\mathbb{1}\| = 1$. Why do the quaternions $\mathbb{H}$ not provide a counter-example to the Gel'fand-Mazur Theorem?

Exercise 4.5.38 (Deradicalization) Consider the category of commutative unital Banach algebras. Formulate and prove that the *deradicalization* $\mathscr{A} \rightsquigarrow \mathscr{A}\big/\operatorname{Rad}(\mathscr{A})$ gives a functor into the subcategory of semisimple commutative unital Banach algebras.

Exercise 4.5.39 (The C^*-algebra $\mathscr{B}(X)$) Let X be a non-empty set. Prove that the bounded functions $\mathscr{B}(X)$ are a unital C^*-algebra with respect to the pointwise operations and the supremum norm $\|\cdot\|_\infty$.

Hint: This can be done either directly or by using Exercise 2.5.28, *iv.)*, and Exercise 4.5.17, *vii.)*.

Show furthermore that the set X can be seen as (generally very tiny) part of the spectrum $\operatorname{Spec}(\mathscr{B}(X))$. It is fairly complicated to imagine the spectrum $\operatorname{Spec}(\mathscr{B}(X))$ of this unital C^*-algebra, even for the case $X = \mathbb{N}$.

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Exercise 4.5.40 (Nilpotent elements) Let $\mathscr{A}$ be a C^*-algebra and let $a \in \mathscr{A}$ be a normal element.

i.) Show that if a is nilpotent, i.e. $a^n = 0$ for some $n \in \mathbb{N}$, then $a = 0$.

Hint: One can either use the spectral theorem for this or the following argument: first consider a^*a instead and show that this is a nilpotent Hermitian element. Then take a faithful *-representation of $\mathscr{A}$ on a Hilbert space (actually, a pre-Hilbert space suffices) and conclude that $a^*a = 0$ follows from the definiteness of the inner product. This argument generalizes to all *-algebras which have a faithful *-representation on a pre-Hilbert space whether a spectral theorem is available or not.

ii.) Show that the Graßmann algebra $\Lambda^\bullet(\mathbb{C}^n)$ has no faithful *-representation on a pre-Hilbert space. In particular, it can not be a C^*-algebra.

Exercise 4.5.41 (Spec is a functor) Consider a continuous unital homomorphism $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ between two commutative unital Banach algebras.

i.) Show that the pull-back $\Phi^*\varphi = \varphi \circ \Phi$ of a character $\varphi \in \operatorname{Spec}(\mathscr{B})$ is a character of $\mathscr{A}$.

ii.) Show that the pull-back induces a continuous map $\Phi^* \colon \operatorname{Spec}(\mathscr{B}) \longrightarrow \operatorname{Spec}(\mathscr{A})$.

iii.) Conclude that passing to the spectrum yields a contravariant functor from the category of commutative unital Banach algebras to the category of compact Hausdorff spaces.

Exercise 4.5.42 (Four unitaries) Let $\mathscr{A}$ be a unital C^*-algebra. Show that every element $a \in \mathscr{A}$ can be written as a linear combination of four unitary elements in $\mathscr{A}$.

Hint: First decompose a into real and imaginary part. By rescaling one can assume $\|\operatorname{Re}(a)\|, \|\operatorname{Im}(a)\| \leq 1$. Then consider the functions $f_\pm : [-1, 1] \ni x \mapsto x \pm \mathrm{i}\sqrt{1 - x^2}$.

Exercise 4.5.43 (The Fuglede-Putnam-Rosenblum Theorem) Let $\mathscr{A}$ be a C^*-algebra and let $a, b \in \mathscr{A}$ with a normal. Prove that $[a, b] = 0$ iff $[a^*, b] = 0$. Give simple examples in $\mathrm{M}_2(\mathbb{C})$ that the statement can fail if a is not normal.

Hint: Pass to the unitization if $\mathscr{A}$ is non-unital. Assume $[a, b] = 0$ and show that $\exp(\mathrm{i}\overline{z}a)$ commutes with b for all $z \in \mathbb{C}$. Use the holomorphic calculus to show that

$$\exp(-\mathrm{i}za^*) b \exp(\mathrm{i}za^*) = \exp(-\mathrm{i}(za^* + \overline{z}a)) b \exp(\mathrm{i}(za^* + \overline{z}a)) \tag{$*$}$$

by $a^*a = aa^*$. Next show that this exponential is unitary for all $z \in \mathbb{C}$. Use this to show that the right hand side of $(*)$ is bounded by $\|b\|$. Show that the left hand side is entire and use a Liouville argument to conclude that the left hand side is just b for all $z \in \mathbb{C}$. Deduce from this $[a^*, b] = 0$.

Exercise 4.5.44 (The ordering is a closed relation) Let $\mathscr{A}$ be a C^*-algebra. Prove that the ordering $\leq$ on the Hermitian elements of $\mathscr{A}$ is a norm-closed relation. This means that the subset

$$\Big\{(a, b) \in \mathscr{A} \times \mathscr{A} \;\Big|\; a \leq b\Big\} \subseteq \mathscr{A} \times \mathscr{A} \tag{4.5.24}$$

is closed. This refines the statement of Corollary 4.4.3.

Exercise 4.5.45 (Squaring violates inequalities) Consider the complex matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \tag{4.5.25}$$

and show $0 \leq A \leq B$. Compute A^2 and B^2 explicitly to show that $A^2 \leq B^2$ does *not* hold.

Chapter 5

Spectral Theory for Hilbert Space Operators

For C^*-algebras we have seen a reasonably well-behaved notion of spectrum together with a good and useful continuous calculus. Since every C^*-algebra can be realized as a closed *-subalgebra of the bounded operators $\mathfrak{B}(\mathfrak{H})$ for some suitably chosen Hilbert space $\mathfrak{H}$, one may wonder how this additional structure can be used to get a refined notion of spectrum: for operators it makes sense to ask for eigenvalues and eigenvectors. Thus we will be interested in transferring the familiar notions from finite-dimensional complex vector spaces to the general case of Hilbert spaces. The new feature, already indicated in our discussion at the beginning of Subsection 4.2.2, will be that operators may have spectral values for which no eigenvectors exist: $\lambda \mathbb{1} - A$ may be injective without being surjective.

The aim of this chapter is now to provide the appropriate notions and tools to handle these phenomena. It will turn out that the "diagonalization" of matrices can be formulated by means of spectral measures. This is the famous spectral theorem for normal operators which we will prove in Subsection 5.1.4. Closely related is the possibility to build not only continuous functions of normal operators but also (bounded) measurable functions. This is not possible in a general C^*-algebra but a new feature of $\mathfrak{B}(\mathfrak{H})$ and provides a drastic generalization of the continuous calculus.

In view of the formulas for the spectral theorem expressing the operator in terms of its spectral measure it is tempting to extend this formula from bounded measurable functions to general ones *provided* some additional integrability conditions are posed. This generalization is indeed possible and leads to unbounded but still self-adjoint operators, a notion which we will discuss in detail in Section 5.3. As a preparation, we will have to deal with unbounded operators in general in Section 5.2. Here completely new phenomena have to be taken into account. We discuss the notions of closable and closed operators together with their spectra before passing to symmetric operators. Here the notion of deficiency indices will become important to distinguish closed symmetric from self-adjoint operators. The Cayley transform will provide a universal tool to relate closed symmetric operators with isometric operators. Using the Cayley transform we will be able to formulate to spectral theorem and the measurable calculus for self-adjoint operators as well in Section 5.4. Important applications will come from dynamics: we discuss the solutions to Schrödinger's equation in terms of unitary one-parameter groups obeying a suitable continuity property. The notions of analytic and smooth vectors will play a crucial role in the understanding of self-adjointness in various examples.

Throughout this chapter, $\mathfrak{H}$ will denote a complex Hilbert space and $\mathfrak{B}(\mathfrak{H})$ is the unital C^*-algebra of all bounded, i.e. continuous, linear endomorphisms of $\mathfrak{H}$. In particular, all the results on C^*-algebras developed so far can be applied to $\mathfrak{B}(\mathfrak{H})$.

5.1 Spectral Theorem for Bounded Normal Operators

In this section we will establish the bounded measurable calculus for normal operators $A \in \mathfrak{B}(\mathfrak{H})$, thereby extending the previous continuous calculus even further. It will be the crucial ingredient for the spectral theorem in so far as the spectral projections of A turn out to be measurable but in general not continuous functions of A.

5.1.1 Topologies for $\mathfrak{B}(\mathfrak{H})$

The norm topology of $\mathfrak{B}(\mathfrak{H})$ makes it a unital C^*-algebra establishing all the nice features of C^*-algebras this way. However, for several reasons this topology may be too fine. Thus we will discuss now various other, coarser topologies on $\mathfrak{B}(\mathfrak{H})$. All of them will be locally convex: it suffices to specify some defining systems of seminorms. We begin with the strong and strong* topologies:

Definition 5.1.1 (Strong and strong* topology) *Let $A \in \mathfrak{B}(\mathfrak{H})$ and $\phi \in \mathfrak{H}$. Then one defines the seminorms*

$$\|A\|_\phi = \|A\phi\| \tag{5.1.1}$$

and

$$\|A\|^*_\phi = \|A\phi\| + \|A^*\phi\|. \tag{5.1.2}$$

The locally convex topology determined by all the seminorms $\{\|\cdot\|_\phi\}_{\phi\in\mathfrak{H}}$ is called the strong topology of $\mathfrak{B}(\mathfrak{H})$ while the locally convex topology determined by $\{\|\cdot\|^*_\phi\}_{\phi\in\mathfrak{H}}$ is called the strong* topology of $\mathfrak{B}(\mathfrak{H})$.

Thus the strong topology is the topology of *pointwise convergence* on $\mathfrak{H}$, i.e. a net $(A_i)_{i\in I}$ converges strongly to A iff $A_i\phi \longrightarrow A\phi$ for all $\phi \in \mathfrak{H}$. In the strong* topology we have pointwise convergence of A_i *and* A_i^* simultaneously. For the strong* topology it will sometimes be advantageous to use the system of seminorms

$$\|A\|^{*'}_\phi = \sqrt{\|A\phi\|^2 + \|A^*\phi\|^2}, \tag{5.1.3}$$

for all $\phi \in \mathfrak{H}$. Clearly, the two systems defined by all the seminorms $\|\cdot\|^{*'}_\phi$ and $\|\cdot\|^*_\phi$ are equivalent.

The next topologies take into account countably many vectors at once instead of finitely many:

Definition 5.1.2 (σ-strong and σ-strong* topology) *Let $A \in \mathfrak{B}(\mathfrak{H})$ and $(\phi_n)_{n\in\mathbb{N}_0}$ be a sequence in $\mathfrak{H}$ such that $\sum_{n=0}^\infty \|\phi_n\|^2 < \infty$. Then one defines the seminorms*

$$\|A\|_{(\phi_n)_{n\in\mathbb{N}_0}} = \sqrt{\sum_{n=0}^\infty \|A\phi_n\|^2} \tag{5.1.4}$$

and

$$\|A\|^*_{(\phi_n)_{n\in\mathbb{N}_0}} = \sqrt{\sum_{n=0}^\infty \|A\phi_n\|^2 + \|A^*\phi_n\|^2}. \tag{5.1.5}$$

The locally convex topology determined by all the seminorms $\{\|\cdot\|_{(\phi_n)_{n\in\mathbb{N}_0}}\}$ and $\{\|\cdot\|^*_{(\phi_n)_{n\in\mathbb{N}_0}}\}$, respectively, are called the σ-strong and the σ-strong* topology, respectively.

Here we should check that the series in (5.1.4) and (5.1.5) actually converge in order to give well-defined seminorms. This is accomplished in the next lemma:

Lemma 5.1.3 *Let $A \in \mathfrak{B}(\mathfrak{H})$ and $(\phi_n)_{n\in\mathbb{N}_0}$ be a sequence in $\mathfrak{H}$ such that $\sum_{n=0}^\infty \|\phi_n\|^2 < \infty$.*

i.) The map $\|\cdot\|_{(\phi_n)_{n\in\mathbb{N}_0}}$ *is a well-defined seminorm with*

$$\|A\|_{(\phi_n)_{n\in\mathbb{N}_0}} \leq \|A\| \sqrt{\sum_{n=0}^{\infty} \|\phi_n\|^2}. \tag{5.1.6}$$

ii.) The map $\|\cdot\|^*_{(\phi_n)_{n\in\mathbb{N}_0}}$ *is a well-defined seminorm with*

$$\|A\|^*_{(\phi_n)_{n\in\mathbb{N}_0}} \leq \|A\|_{(\phi_n)_{n\in\mathbb{N}_0}} + \|A^*\|_{(\phi_n)_{n\in\mathbb{N}_0}} \leq 2\|A\| \sqrt{\sum_{n=0}^{\infty} \|\phi_n\|^2}. \tag{5.1.7}$$

PROOF: The first estimate is clear from $\|A\phi_n\| \leq \|A\|\|\phi_n\|$, applied to every term in the series. Thus $\|A\|_{(\phi_n)_{n\in\mathbb{N}_0}}$ satisfies the estimate (5.1.6). The properties of a seminorm are immediate. Since we clearly have $(\|A\|^*_{(\phi_n)_{n\in\mathbb{N}_0}})^2 = \|A\|^2_{(\phi_n)_{n\in\mathbb{N}_0}} + \|A^*\|^2_{(\phi_n)_{n\in\mathbb{N}_0}}$ the first estimate in (5.1.7) follows. The second estimate is clear since $\|A^*\| = \|A\|$ holds in every C^*-algebra. Again, the properties of a seminorm follow from Minkowski's inequality at once. □

Remark 5.1.4 Unlike the system of seminorms used to define the strong and strong* topology, the σ-strong and σ-strong* versions are already filtrating: indeed, if $(\phi_n^k)_{n\in\mathbb{N}_0}$ for $k = 1, \ldots, N$ are sequences with $\sum_{n=0}^{\infty} \|\phi_n^k\|^2 < \infty$ then we consider the new sequence $(\psi_m)_{m\in\mathbb{N}_0}$ with

$$\psi_{Nn+k} = \phi_n^{k+1}, \tag{5.1.8}$$

i.e. we merge the N sequences into one. Then clearly

$$\sum_{m=0}^{\infty} \|A\psi_m\|^2 = \sum_{n=0}^{\infty} \|A\phi_n^1\|^2 + \cdots + \sum_{n=0}^{\infty} \|A\phi_n^k\|^2, \tag{5.1.9}$$

and hence $\|\cdot\|_{(\psi_m)_{m\in\mathbb{N}_0}}$ dominates each of the seminorms $\|\cdot\|_{(\phi_n^k)_{n\in\mathbb{N}_0}}$. The analogous result holds for the σ-strong* seminorms. Note also that instead of (5.1.5) one could equivalently take $\|A\|_{(\phi_n)_{\mathbb{N}_0}} + \|A^*\|_{(\phi_n)_{n\in\mathbb{N}_0}}$ to specify the σ-strong* topology. Finally, note that the series (5.1.4) and (5.1.5) converge absolutely and hence unconditionally. See also Exercise 5.5.2 for another interpretation of the strong and σ-strong topologies.

Definition 5.1.5 (Weak topology) *For* $A \in \mathfrak{B}(\mathfrak{H})$ *and* $\phi, \psi \in \mathfrak{H}$ *one defines the seminorm*

$$\|A\|_{\phi,\psi} = |\langle \phi, A\psi \rangle|. \tag{5.1.10}$$

The locally convex topology determined by all the seminorms $\{\|\cdot\|_{\phi,\psi}\}_{\phi,\psi\in\mathfrak{H}}$ *is called the weak topology of* $\mathfrak{B}(\mathfrak{H})$*.*

Thus the weak topology is the topology of *convergence of matrix elements*, i.e. a net $(A_i)_{i\in I}$ converges weakly to A iff for all $\phi, \psi \in \mathfrak{H}$

$$\langle \phi, A_i\psi \rangle \longrightarrow \langle \phi, A\psi \rangle. \tag{5.1.11}$$

This makes the weak topology interesting for applications in quantum mechanics: it refers immediately to physically observable quantities, the *transition amplitudes*.

Remark 5.1.6 (Weak topology) Here we have a certain clash of notations: for a general locally convex space V we already defined the weak topology to be the locally convex topology induced by all the seminorms $\mathrm{p}_\varphi(v) = |\varphi(v)|$ with $\varphi \in V'$. Now (5.1.10) is clearly coming from the (operator norm) continuous linear functional $A \mapsto \langle \phi, A\psi \rangle$ on $\mathfrak{B}(\mathfrak{H})$ and thereby defines indeed a seminorm of the weak topology in the sense of Definition 2.3.27. However, in case of an infinite-dimensional Hilbert space the topological dual $\mathfrak{B}(\mathfrak{H})'$ of the Banach space $\mathfrak{B}(\mathfrak{H})$ is much bigger than just the functionals of the form $\langle \phi, \cdot\, \psi \rangle$. Thus the weak topology in the sense of Definition 2.3.27 of $\mathfrak{B}(\mathfrak{H})$ is *strictly* finer than the weak topology in the sense of Definition 5.1.5. For historical reasons we stick to the notion "weak topology" for the one of Definition 5.1.5 in the context of $\mathfrak{B}(\mathfrak{H})$, being conform with most of the literature.

The last topology we need is the σ-version of the weak one:

Definition 5.1.7 (σ-Weak topology) *Let $A \in \mathfrak{B}(\mathfrak{H})$ and let $(\phi_n)_{n\in\mathbb{N}_0}$ and $(\psi_n)_{n\in\mathbb{N}}$ be sequences in $\mathfrak{H}$ such that $\sum_{n=0}^\infty \|\phi_n\|^2$ and $\sum_{n=0}^\infty \|\psi_n\|^2$ are finite. Then one defines the seminorm*

$$\|A\|_{(\phi_n)_{n\in\mathbb{N}_0},(\psi_n)_{n\in\mathbb{N}_0}} = \left| \sum_{n=0}^\infty \langle \phi_n, A\psi_n \rangle \right|. \tag{5.1.12}$$

The locally convex topology determined by all the seminorms $\|\cdot\|_{(\phi_n)_{n\in\mathbb{N}_0},(\psi_n)_{n\in\mathbb{N}_0}}$ for all such sequences is called the σ-weak topology of $\mathfrak{B}(\mathfrak{H})$.

Again, like the σ-strong and σ-strong* case this gives a well-defined seminorm and the system of all these is already filtrating.

Lemma 5.1.8 *Let $(\phi_n)_{n\in\mathbb{N}_0}$ and $(\psi_n)_{n\in\mathbb{N}_0}$ be sequences in $\mathfrak{H}$ with $\sum_{n=0}^\infty \|\phi_n\|^2$ and $\sum_{n=0}^\infty \|\psi_n\|^2$ finite.*

i.) For all $A \in \mathfrak{B}(\mathfrak{H})$ one has

$$\|A\|_{(\phi_n)_{n\in\mathbb{N}_0},(\psi_n)n\in\mathbb{N}_0} \le \|A\| \sqrt{\sum_{n=0}^\infty \|\phi_n\|^2} \sqrt{\sum_{n=0}^\infty \|\psi_n\|^2}, \tag{5.1.13}$$

and $\|\cdot\|_{(\phi_n)_{n\in\mathbb{N}_0},(\psi_n)n\in\mathbb{N}_0}$ is a well-defined seminorm.

ii.) The system of all seminorms $\|\cdot\|_{(\phi_n)_{n\in\mathbb{N}_0},(\psi_n)n\in\mathbb{N}_0}$ is filtrating.

Proof: By the Cauchy-Schwarz inequality we have $|\langle \phi, A\psi \rangle| \le \|A\|\|\phi\|\|\psi\|$ and hence

$$\left| \sum_{n=0}^\infty \langle \phi_n, A\psi_n \rangle \right| \le \sum_{n=0}^\infty |\langle \phi_n, A\psi_n \rangle| \le \|A\| \sum_{n=0}^\infty \|\phi_n\|\|\psi_n\|.$$

The last sum can now be estimated again by the Hölder inequality giving (5.1.13). The properties of a seminorm are immediate. With a merging trick analogously to the one for the σ-strong case one shows also the second claim. ☐

Remark 5.1.9 By the polarization identity we see that

$$\langle \psi, A\phi \rangle = \frac{1}{4} \sum_{k=0}^3 \mathrm{i}^{-k} \big\langle \phi + \mathrm{i}^k \psi, A(\phi + \mathrm{i}^k \psi) \big\rangle, \tag{5.1.14}$$

which implies that

$$|\langle \psi, A\phi \rangle| \le \frac{1}{4} \sum_{k=0}^3 \big| \big\langle \phi + \mathrm{i}^k \psi, A(\phi + \mathrm{i}^k \psi) \big\rangle \big| \tag{5.1.15}$$

as well as

$$\left|\sum_{n=0}^{\infty}\langle\psi_n, A\phi_n\rangle\right| \leq \frac{1}{4}\sum_{k=0}^{3}\left|\sum_{n=0}^{\infty}\langle\phi_n + \mathrm{i}^k\psi_n, A(\phi_n + \mathrm{i}^k\psi_n)\rangle\right|. \tag{5.1.16}$$

Thus the seminorms $\|\cdot\|_{\phi,\phi}$ and $\|\cdot\|_{(\phi_n)_{n\in\mathbb{N}_0},(\phi_n)_{n\in\mathbb{N}_0}}$ are already sufficient to determine the weak and the σ-weak topology. In more physical terms, the weak topology is the topology of *convergence of expectation values* in the usual quantum mechanical sense. Since the expectation values $\langle\phi, A\phi\rangle$ are directly linked to the (repeated) measurement of A in the state described by ϕ, the weak topology is in some sense the only directly accessible one from a quantum mechanical point of view. The σ-weak topology can be seen as arising from the idealization of countably many measurements.

We relate now the six new topologies and clarify some first properties of them in the following theorem. To make things interesting we assume that $\mathfrak{H}$ is infinite-dimensional. In finite dimensions it is rather easy to see that all the above topologies coincides with the norm topology, see also Exercise 5.5.1.

Theorem 5.1.10 (Operator topologies) *Let $\mathfrak{H}$ be a infinite-dimensional Hilbert space.*

i.) The σ-strong*, the σ-strong, the σ-weak, the strong*, the strong, and the weak topology turn $\mathfrak{B}(\mathfrak{H})$ into a Hausdorff locally convex space.

ii.) One has the following relations between the various operator topologies

$$\begin{array}{ccccccc} & & \sigma\text{-strong}^* & \supset & \sigma\text{-strong} & \supset & \sigma\text{-weak} \\ \text{norm} & \supset & \cup & & \cup & & \cup \\ & & \text{strong}^* & \supset & \text{strong} & \supset & \text{weak}, \end{array} \tag{5.1.17}$$

where $\supset$ means “strictly finer than”.

iii.) Cauchy sequences in all of the above topologies are bounded with respect to the operator norm.

iv.) The multiplication of operators is discontinuous in all of the above topologies (except in the norm topology).

v.) The restriction of the operator multiplication to

$$\mathrm{B}_1(0) \times \mathfrak{B}(\mathfrak{H}) \longrightarrow \mathfrak{B}(\mathfrak{H}) \tag{5.1.18}$$

is continuous in all of the above topologies except in the σ-weak and the weak topology.

vi.) The operator multiplication is separately continuous in all the operator algebra topologies.

vii.) The adjoint $A \mapsto A^*$ is continuous in the σ-strong*, in the strong*, in the σ-weak, and in the weak topology but it is discontinuous in the σ-strong and in the strong topology.

viii.) $\mathfrak{B}(\mathfrak{H})$ is strongly and weakly sequentially complete.

PROOF: We only sketch the proof and refer to the exercises for more details and the relevant counterexamples, see Exercise **??**, **??**, and **??**. The first part is clear by the construction of the topologies via seminorms, also the Hausdorff property is obvious. For the second part, the necessary inequalities between the defining seminorms have either been obtained already in the preceding lemmas or can be seen easily directly. To show that we have strictly finer topologies is more subtle and requires the construction of certain sequences of operators which have different convergence properties with respect to the various topologies. These examples are discussed in the above mentioned exercises. For the third part we first note that Cauchy sequences in the finer topologies are also Cauchy sequences in the coarser ones. Hence by the second part, it suffices to show that a weak Cauchy sequence is norm-bounded. Thus let $A_n \in \mathfrak{B}(\mathfrak{H})$ form a weak Cauchy sequence, i.e. $(\langle\phi, A_n\psi\rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for all $\phi, \psi \in \mathfrak{H}$. This means that for every vector $\psi \in \mathfrak{H}$ the sequence $(A_n\psi)_{n\in\mathbb{N}}$ is a weak Cauchy sequence in the Banach space $\mathfrak{H}$. By Proposition 2.3.31, *iii.)*, it is bounded in norm, i.e. we have a constant $C_\psi > 0$ with $\|A_n\psi\| \leq C_\psi$ for all $n \in \mathbb{N}$. Then a Banach-Steinhaus argument implies $\sup_{n\in\mathbb{N}}\|A_n\| < \infty$, using e.g. the formulation in Theorem 2.3.18, thereby proving the third part. Finding the relevant counter-examples for the fourth part is again content of the Exercises. For the fifth part, let $A, A_0 \in \mathrm{B}_1(0)$ be in the unit ball of $\mathfrak{B}(\mathfrak{H})$ and let $B, B_0 \in \mathfrak{B}(\mathfrak{H})$ be arbitrary. Then for $\phi \in \mathfrak{H}$ we have

$$
\begin{aligned}
\|AB - A_0B_0\|_\phi &= \|A(B - B_0) + (A - A_0)B_0\|_\phi \\
&\leq \|A(B - B_0)\phi\| + \|(A - A_0)B_0\phi\| \\
&\leq \|A\|\|B - B_0\|_\phi + \|A - A_0\|_{B_0\phi} \\
&\leq \|B - B_0\|_\phi + \|A - A_0\|_{B_0\phi},
\end{aligned}
$$

from which the continuity of (5.1.18) in the strong operator topology follows at once. Using this estimate also the other continuity statements can be shown. The discontinuity of (5.1.18) in the σ-weak and the weak topology follows again from the construction of suitable sequences which are discussed in the exercises. For the sixth part we consider the equalities

$$
\|AB\|_{\phi,\psi} = |\langle\phi, AB\psi\rangle| = \|A\|_{\phi,B\psi} = \|B\|_{A^*\phi,\psi}, \tag{$*$}
$$

which show the continuity of the left and the right multiplications in the weak topology, i.e. the separate continuity of the operator product. By summing over $(*)$ one immediately gets the separate continuity also in the σ-weak topology, see Exercise **??**. The separate continuity of the operator product for the remaining topologies is clear from the stronger statement (5.1.18) of the fifth part. For the seventh part the discontinuity of $A \mapsto A^*$ in the σ-strong and the strong topology is again exemplified by evaluation on a suitable sequence. The continuity in the other topologies is clear as the defining systems of seminorms are invariant under $A \mapsto A^*$. It remains to show the sequential completeness of $\mathfrak{B}(\mathfrak{H})$ in the strong and the weak topology. Let $(A_n)_{n\in\mathbb{N}}$ with $A_n \in \mathfrak{B}(\mathfrak{H})$ be a strong Cauchy sequence. Since $\|A_n - A_m\|_\phi = \|A_n\phi - A_m\phi\|$ this means that for very $\phi \in \mathfrak{H}$ the sequence $(A_n\phi)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $\mathfrak{H}$ is complete we have a limit of this Cauchy sequence which we denote by $A\phi$. Clearly, this defines a linear map $A\colon \phi \mapsto A\phi$. We have to show that $A \in \mathfrak{B}(\mathfrak{H})$ and $A_n \longrightarrow A$ strongly. With the Banach-Steinhaus argument already used in the third part we know that $\sup_{n\in\mathbb{N}}\|A_n\| < \infty$. Hence $\|A_n\phi\| \leq \|A_n\|\|\phi\| \leq C\|\phi\|$ with some $C > 0$ and for all $n \in \mathbb{N}$. But then also $\|A\phi\| = \lim_{n\longrightarrow\infty}\|A_n\phi\| \leq C\|\phi\|$ follows, implying $\|A\| \leq C$. Hence $A \in \mathfrak{B}(\mathfrak{H})$. But then we have

$$
\|A - A_n\|_\phi = \|A\phi - A_n\phi\| = \lim_{m\longrightarrow\infty}\|A_m\phi - A_n\phi\|,
$$

which becomes small for n large enough since $(A_n)_{n\in\mathbb{N}}$ is a strong Cauchy sequence. This establishes $A_n \longrightarrow A$ in the strong sense. For a weak Cauchy sequence $(A_n)_{n\in\mathbb{N}}$ we first conclude that for every $\phi \in \mathfrak{H}$ the sequence $(A_n\phi)_{n\in\mathbb{N}}$ is a weak Cauchy sequence in $\mathfrak{H}$. By Corollary 3.2.17 we have a limit also in this case which we denote by $A\phi \in \mathfrak{H}$. Again it is easy to see that $A\colon \phi \mapsto A\phi$ defines a linear map. From here we conclude as before by the Banach-Steinhaus argument from the third part that $A \in \mathfrak{B}(\mathfrak{H})$. Then the weak convergence $A_n \longrightarrow A$ is again a simple consequence of the weak continuity of $\langle\psi, \cdot\rangle$. □

Remark 5.1.11 There are many more features and relations between these topologies some of which can be found e.g. in [8, Sect. I.3.2]. We only mention two of them: first, on the unitary elements $\mathfrak{U}(\mathfrak{H}) \subseteq \mathfrak{B}(\mathfrak{A})$ all the operator topologies (except for the norm topology) coincide and make $\mathfrak{U}(\mathfrak{H})$ a topological group, see Exercise **??**. Second, $\mathfrak{B}(\mathfrak{H})$ is complete (and not just sequentially complete) with respect to the σ-strong* and the σ-strong topology. We will see some further properties of all these topologies in Section 6.3.2.

5.1.2 Projection-Valued Measures

We come now to the crucial notion of a projection-valued measure. The basic idea behind this definition is that we want to "diagonalize" a Hermitian or normal operator $A \in \mathfrak{B}(\mathfrak{H})$ analogously to the finite-dimensional situation: recall that for $\dim \mathfrak{H} = n < \infty$ any Hermitian or normal operator determines two quantities: its eigenspaces and the corresponding eigenvalues. If $\{\lambda_1, \ldots, \lambda_k\}$ are the distinct eigenvalues we denote by P_{λ_i} the orthogonal projection onto the corresponding eigenspace. Note that $k \leq n$ and $\dim P_{\lambda_i}$ is the multiplicity of λ_i. Then we have the following properties. First,

$$P_{\lambda_i} P_{\lambda_j} = P_{\lambda_i} \delta_{ij} = P_{\lambda_i}^* \delta_{ij}, \tag{5.1.19}$$

i.e. the projections are pairwise orthogonal. Second, we have

$$\mathbb{1} = \sum_{i=1}^{k} P_{\lambda_i}, \tag{5.1.20}$$

i.e. the collection of all eigenspaces spans the whole space. Finally, A is diagonal with respect to the decomposition (5.1.20), i.e.

$$A = \sum_{i=1}^{k} \lambda_i P_{\lambda_i}. \tag{5.1.21}$$

When we pass to infinite-dimensional Hilbert spaces we have difficulties of various kinds to extend this form of the spectral theorem: first, a spectral value does not need to be an eigenvalue at all. It may well happen that $\lambda - A$ is not invertible but nevertheless injective. Thus it is not clear what the eigenspace corresponding to such a λ should be (in fact, there is none). Thus there is no spectral projection P_λ. Moreover, there will be situations where we have not only a countably infinite number of eigenvalues but even a continuum of spectral values. This we already know from the C^*-algebraic examples: if $f \in \mathscr{C}(X)$ with some compact Hausdorff space X then $\operatorname{spec}(f) = f(X) \subseteq \mathbb{C}$ which needs not to be discrete at all. Since $\mathscr{C}(X)$, as every other C^*-algebra, has a faithful *-representation on a Hilbert space, we get the corresponding (normal) operator with the same spectrum. In conclusion, the sum in (5.1.20) and (5.1.21) has to be abandoned and re-interpreted in a suitable way. We expect that even a generalization from a finite sum to a series will not be sufficient. Finally, we can also understand why in general C^*-algebras we can not expect to have a diagonalization like (5.1.21): there may simply be *no* interesting projections at all.

Example 5.1.12 (Projections in $\mathscr{C}(X)$) Suppose that the compact Hausdorff space X is connected. Then $f(X) \subseteq \mathbb{C}$ is connected for every $f \in \mathscr{C}(X)$. If $f^2 = f$ is idempotent then at every point $x \in X$ we have $f(x)^2 = f(x)$ and hence either $f(x) = 0$ or $f(x) = 1$. Thus in the connected case, $f = 1$ and $f = 0$ are the only (obvious) projections. To have interesting projections in $\mathscr{C}(X)$, the space X has to be *disconnected*.

Luckily, $\mathfrak{B}(\mathfrak{H})$ does not suffer from having only few projections since every closed subspace $U \subseteq \mathfrak{H}$ determines a projection P_U and vice versa.

We shall now collect some preparatory material on normal operators which make use of the Hilbert space structure in the background and therefore they have no (obvious) analogs in the general C^*-algebraic framework.

Proposition 5.1.13 *Let $A \in \mathfrak{B}(\mathfrak{H})$.*

i.) The operator A is uniquely determined by the values $\langle \phi, A\phi \rangle$ for $\phi \in \mathfrak{H}$.

ii.) The operator A is normal iff $\|A^*\phi\| = \|A\phi\|$ for all $\phi \in \mathfrak{H}$.

iii.) The operator A is invertible iff $\operatorname{im} A \subseteq \mathfrak{A}$ *is dense and $\|A\phi\| \geq \delta \|\phi\|$ for some $\delta > 0$ and all $\phi \in \mathfrak{H}$.*

iv.) *Suppose A is normal. Then A is not invertible iff there exists a sequence $\phi_n \in \mathfrak{H}$ with $\|\phi_n\| = 1$ and $\|A\phi_n\| \longrightarrow 0$.*

PROOF: The first part is clear as the weak topology is Hausdorff and we have the polarization identity (5.1.14). For the second part, let $\phi \in \mathfrak{H}$ be given. Then

$$\|A\phi\|^2 = \langle \phi, A^*A\phi \rangle \quad \text{while} \quad \|A^*\phi\|^2 = \langle \phi, AA^*\phi \rangle$$

Thus the first part implies the second statement. For the third part, assume first that A is invertible. Then clearly $\operatorname{im} A = \mathfrak{H}$ is dense in $\mathfrak{H}$ and $\|\phi\| = \|A^{-1}A\phi\| \leq \|A^{-1}\|\|A\phi\|$ gives the desired estimate with $\delta = \frac{1}{\|A^{-1}\|}$. Conversely, assume A has dense image and let $\delta > 0$ satisfy $\|A\phi\| \geq \delta\|\phi\|$ for all $\phi \in \mathfrak{H}$. Then $\ker A = \{0\}$ and hence A defines a linear bijection $A\colon \mathfrak{H} \longrightarrow \operatorname{im} A$. Define now $B\colon \operatorname{im} A \longrightarrow \mathfrak{H}$ to be the inverse of this map. Then for $\psi = A\phi \in \operatorname{im} A$ we have

$$\|B\psi\| = \|BA\phi\| = \|\phi\| \leq \frac{1}{\delta}\|A\phi\| = \frac{1}{\delta}\|\psi\|.$$

This shows that the densely defined operator B is actually continuous with operator norm $\|B\| \leq \frac{1}{\delta}$. Since $\mathfrak{H}$ is complete, B extends to $\mathfrak{H} = \widehat{\operatorname{im} A}$. The extension $\widehat{B}$ then still satisfies the relations $\widehat{B}A = \mathrm{id} = A\widehat{B}$ as they hold on a dense domain. Hence $\widehat{B} = A^{-1}$ provides the inverse of A. In fact, a posteriori the proof shows that $\operatorname{im} A = \mathfrak{H}$ was already the whole space. For the last part, let A be normal. Assume first that we have such a sequence of vectors $\phi_n \in \mathfrak{H}$. If A would be invertible then $\phi_n = A^{-1}A\phi_n \longrightarrow 0$ since A^{-1} is again continuous. This contradicts $\|\phi_n\| = 1$ immediately. Thus assume that A is not invertible. If A is not injective then we can take a sequence of normalized vectors $\phi_n \in \ker A$ which will do the job. Thus assume that A is injective but not surjective. We claim that $\operatorname{im} A$ is dense. Indeed, let $\phi \in \mathfrak{H}$ be arbitrary then $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$ for all $\psi \in \mathfrak{H}$. Hence $\phi \perp \operatorname{im} A$ iff $\phi \in \ker A^*$. But $\phi \in \ker A^*$ iff $\phi \in \ker A$ by the second part. Since we assume that A is injective, we obtain $\phi = 0$ and conclude that $\operatorname{im} A$ is dense as its orthogonal complement is $\{0\}$ only. Now by the third part we conclude that for every $\epsilon > 0$ there is a ϕ_ϵ with $\|A\phi_\epsilon\| \leq \epsilon\|\phi_\epsilon\|$. Normalizing gives the desired sequence. $\square$

Corollary 5.1.14 (Approximate eigenvalues) *Let $A \in \mathfrak{B}(\mathfrak{H})$ be a normal operator. Then $\lambda \in \operatorname{spec}(A)$ iff λ is an approximate eigenvalue in the sense that there exists a sequence $\phi_n \in \mathfrak{H}$ with $\|\phi_n\| = 1$ and*

$$\lim_{n \longrightarrow \infty} (A\phi_n - \lambda\phi_n) = 0. \tag{5.1.22}$$

In general, the sequence ϕ_n does not converge at all, see also Exercise 5.5.8. We call such a sequence also a sequence of *approximate eigenvectors* for the approximate eigenvalue λ.

After this motivation we can now state the main definition of this section: we want to interpret the sum (5.1.21) as an *integration* with respect to a projection-valued measure.

Definition 5.1.15 (Projection-valued measure) *Let $(X, \mathfrak{a})$ be a measurable space. A map*

$$E\colon \mathfrak{a} \longrightarrow \mathfrak{B}(\mathfrak{H}) \tag{5.1.23}$$

is called projection-valued measure if

i.) $E_U = E_U^2 = E_U^*$ *for all* $U \in \mathfrak{a}$,

ii.) $E_\emptyset = 0$ *and* $E_X = \mathbb{1}$,

iii.) $E_{U \cup V} = E_U + E_V$ *if* $U, V \in \mathfrak{a}$ *are disjoint,*

iv.) *the map* $\langle \phi, E\psi \rangle\colon U \mapsto \langle \phi, E_U\psi \rangle$ *is a complex measure on* $(X, \mathfrak{a})$ *for every* $\phi, \psi \in \mathfrak{H}$.

If in addition X is a locally compact Hausdorff space and $\mathfrak{a}$ is the σ-algebra of Borel subsets of X then we require the complex measure $\langle \phi, E\psi \rangle$ to be regular.

definition for this?

Details on (scalar-valued) complex measures can be found in Appendix C.2.3.

Finally, we note that a projection-valued measure is sometimes also called a *resolution of the identity*.

Proposition 5.1.16 *Let $(X, \mathfrak{a})$ be a measurable space and E a projection-valued measure on it.*

i.) For all $U, V \in \mathfrak{a}$ we have

$$E_U E_V = E_{U \cap V} = E_V E_U. \tag{5.1.24}$$

ii.) The projection-valued measure E is σ-additive in the strong operator topology, i.e. for pairwise disjoint $U_n \in \mathfrak{a}$ and $\phi \in \mathfrak{H}$ we have

$$\sum_{n=0}^{\infty} E_{U_n} \phi = E_{\bigcup_{n=0}^{\infty} U_n} \phi \tag{5.1.25}$$

as a convergent series in $\mathfrak{H}$ in the norm topology. Conversely, if E is a map as in Definition 5.1.15 with the properties i.), ii.), and iii.) but with (5.1.25) instead of iv.) then E is a projection-valued measure.

iii.) For $\phi \in \mathfrak{H}$ the map $E_\phi \colon \mathfrak{a} \longrightarrow \mathbb{R}_0^+$ with

$$E_\phi(U) = \langle \phi, E_U \phi \rangle \tag{5.1.26}$$

is a finite positive measure with variational norm

$$\|E_\phi\| = \|\phi\|^2. \tag{5.1.27}$$

PROOF: We first consider the case $U \cap V = \emptyset$. In this case $E_{U \cup V} = E_U + E_V$ holds and gives a projection. Hence $E_U E_V = -E_V E_U$ and $(E_U E_V)^* = E_V E_U = -E_U E_V$. Hence $E_U E_V$ is anti-Hermitian and thus normal. Now $E_U E_V + E_U E_V E_U = 0$ as well as $E_U E_V E_U + E_V E_U = 0$ by $E_U^2 = E_U$. It follows that $E_U E_V E_U = 0$ and hence also $(E_U E_V)^2 = E_U E_V E_U E_V = 0$. Thus $E_U E_V$ is nilpotent and normal. But the only nilpotent and normal element in the C^*-algebra $\mathfrak{B}(\mathfrak{H})$ is 0, showing $E_U E_V = 0$, see also Exercise 4.5.40. Thus $E_U E_V = 0$ and hence also $E_V E_U = 0$ by symmetry. Now let $U \cap V$ be arbitrary then $E_U = E_{U \setminus (U \cap V)} + E_{U \cap V}$ and $E_V = E_{V \setminus (U \cap V)} + E_{U \cap V}$ by additivity. Then

$$\begin{aligned}
E_U E_V &= \big(E_{U \setminus (U \cap V)} + E_{U \cap V}\big)\big(E_{V \setminus (U \cap V)} + E_{U \cap V}\big) \\
&= E_{U \setminus (U \cap V)} E_{V \setminus (U \cap V)} + E_{U \setminus (U \cap V)} E_{U \cap V} + E_{U \cap V} E_{V \setminus (U \cap V)} + E_{U \cap V} E_{U \cap V} \\
&= E_{U \cap V},
\end{aligned}$$

since $E_{U \cap V}$ is a projection and all other combinations vanish as the corresponding subsets are disjoint. This proves the first part. For the second part we note that the σ-additivity of the complex measures $U \mapsto \langle \phi, E_U \psi \rangle$ is just the σ-additivity of E in the *weak* operator topology, i.e.

$$\sum_{n=0}^{\infty} \langle \phi, E_{U_n} \psi \rangle = \Big\langle \phi, E_{\bigcup_{n=0}^{\infty} U_n} \psi \Big\rangle \tag{$*$}$$

for all $\phi, \psi \in \mathfrak{H}$. By the first part we know that the projections E_{U_n} are pairwise orthogonal thanks to $U_n \cap U_m = \emptyset$ for $n \neq m$. Thus by Pythagoras' Theorem we get

$$\Big\|\sum_{n=0}^{N} E_{U_n} \psi\Big\|^2 = \sum_{n=0}^{N} \big\|E_{U_n} \psi\big\|^2 = \sum_{n=0}^{N} \langle \psi, E_{U_n} \psi \rangle \longrightarrow \sum_{n=0}^{\infty} \langle \psi, E_{U_n} \psi \rangle = \Big\langle \psi, E_{\bigcup_{n=0}^{\infty} U_n} \psi \Big\rangle,$$

using $E_{U_n}^* = E_{U_n} = E_{U_n}^2$ and the weak convergence $(*)$. But this shows that

$$\sum_{n=0}^{\infty} \left\| E_{U_n} \psi \right\|^2 = \left\| E_{\bigcup_{n=0}^{\infty} U_n} \psi \right\|^2.$$

Together with $(*)$, which we can also interpret as the weak convergence of $\sum_{n=0}^{\infty} E_{U_n} \psi$ in $\mathfrak{H}$, we get from Proposition 3.2.15 that this sum converges in the norm sense in $\mathfrak{H}$. But this is the strong convergence of the projections we want to show. The converse statement is clear since strong convergence implies weak convergence. This shows the second part. For the last part, it is clear that $E_\phi(U) = \langle \phi, E_U \phi \rangle = \langle E_U \phi, E_U \phi \rangle = \|E_U \phi\|^2 \geq 0$ gives a positive measure. Since $E_\phi(X) = \|E_X \phi\|^2 = \|\phi\|^2$ by $E_X = \mathbb{1}$, it is a finite measure. For any finite positive measure, the variational norm equals the volume of the whole measure space, see Proposition C.2.25 and (C.2.46).□

Remark 5.1.17 Except for some trivial cases the σ-additivity of a projection-valued measure E can not be in the operator-norm sense. The reason is that $\|E_{U_n}\| = 1$ unless $E_{U_n} = 0$ since E_{U_n} is a projection. Thus either the series $\sum_{n=0}^{\infty} E_{U_n}$ contains only finitely many terms with $E_{U_n} \neq 0$ or the norm sequence $\|E_{U_n}\|$ is not even a zero sequence.

We can now define integration with respect to a projection-valued measure in precisely the same way as one does so for ordinary Lebesgue integrals. Let $f \colon X \longrightarrow \mathbb{C}$ be a simple function on the measurable space $(X, \mathfrak{a})$ with distinct values $z_1, \ldots, z_n \in \mathbb{C}$, i.e.

$$f = \sum_{i=1}^{n} z_i \chi_{U_i}, \tag{5.1.28}$$

where $U_i = f^{-1}(\{z_i\})$ for $i = 1, \ldots, n$. Since by definition simple functions are measurable we have $U_i \in \mathfrak{a}$ and the decomposition (5.1.28) is the unique normal form writing f as a linear combination of characteristic functions of measurable subsets if we add the requirement that the U_i are pairwise disjoint and cover all of X.

As in the usual scalar integration theory one defines now the operator-valued integral of f with respect to the projection-valued measure by

$$\int_X f \, \mathrm{d}E = \sum_{i=1}^{n} z_i E_{U_i}. \tag{5.1.29}$$

The left hand side is nothing more than the symbol for the right hand side. In particular, as in the scalar case, there is no intrinsic meaning to the symbol "dE" alone. Note that we have

$$\int_X 1 \, \mathrm{d}E = \mathbb{1}. \tag{5.1.30}$$

Lemma 5.1.18 *Let E be a projection-valued measure on $(X, \mathfrak{a})$.*

i.) The map $\int_X \cdot \, \mathrm{d}E$ is a linear map from the vector space of simple functions to $\mathfrak{B}(\mathfrak{H})$.

ii.) For a simple function f one has

$$\int_X \overline{f} \, \mathrm{d}E = \left(\int_X f \, \mathrm{d}E \right)^*. \tag{5.1.31}$$

iii.) For two simple functions f and g one has

$$\int_X f \, \mathrm{d}E \int_X g \, \mathrm{d}E = \int_X fg \, \mathrm{d}E. \tag{5.1.32}$$

iv.) For a simple function f one has

$$\left\| \int_X f \, \mathrm{d}E \right\| \leq \|f\|_\infty. \tag{5.1.33}$$

PROOF: Let $f = \sum_{i=1}^n z_i \chi_{U_i}$ be in its normal form. If now $W_{ij} \in \mathfrak{a}$ for $j = 1, \ldots, k_i$ are measurable subsets such that $U_i = \bigcup_{j=1}^{k_i} W_{ij}$ with W_{ij} being pairwise disjoint then we compute

$$\sum_{i=1}^n \sum_{j=1}^{k_i} z_i E_{W_{ij}} = \sum_{i=1}^n z_i E_{\bigcup_{j=1}^{k_i} W_{ij}} = \sum_{i=1}^n z_i E_{U_i} = \int_X f \, \mathrm{d}E, \tag{$*$}$$

by the finite additivity of E. Now let $g = \sum_{j=1}^m w_j \chi_{V_j}$ be another simple function in its normal form and let $z, w \in \mathbb{C}$. Then we take $W_{ij} = U_i \cap V_j$. These subsets are pairwise disjoint, measurable and satisfy

$$U_i = \bigcup_{j=1}^m W_{ij} \quad \text{as well as} \quad V_j = \bigcup_{i=1}^n W_{ij}.$$

The simple function $zf + wg$ can now be written as

$$zf + wg = \sum_{i,j} (zz_i + ww_j) \chi_{W_{ij}},$$

where the W_{ij} are pairwise disjoint and contained in the pre-images of values of $zf + wg$. Hence they satisfy the condition to apply $(*)$ for $zf + wg$, f, and g simultaneously. We get

$$\begin{aligned}
\int_X (zf + wg) \, \mathrm{d}E &\overset{(*)}{=} \sum_{i,j} (zz_i + ww_j) \chi_{W_{ij}} \\
&= z \sum_i z_i \sum_j E_{W_{ij}} + w \sum_j w_j \sum_i E_{W_{ij}} \\
&\overset{(*)}{=} z \int_X f \, \mathrm{d}E + w \int_X g \, \mathrm{d}E.
\end{aligned}$$

Thus the integral is indeed linear and (5.1.29) holds for *all* ways to write a simple function as linear combination of characteristic ones and not only for the normal form. The second part is clear as $E_U = E_U^*$. The third part is again a simple computation since

$$\int_X \chi_U \, \mathrm{d}E \int_X \chi_V \, \mathrm{d}E = E_U E_V = E_{U \cap V} = \int_X \chi_{U \cap V} \, \mathrm{d}E = \int_X \chi_U \chi_V \, \mathrm{d}E \tag{$**$}$$

by Proposition 5.1.16, *i.)*, and $\chi_{U \cap V} = \chi_U \chi_V$. Now both sides of (5.1.32) are bilinear in f in g and hence $(**)$ is all we have to show. For the last part we estimate (very roughly) for $\phi \in \mathfrak{H}$

$$\begin{aligned}
\left\| \left(\int_X f \, \mathrm{d}E \right) \phi \right\|^2 &= \left\| \sum_{i=1}^n z_i E_{U_i} \phi \right\|^2 \\
&\overset{(a)}{=} \sum_{i=1}^n |z_i|^2 \left\| E_{U_i} \phi \right\|^2 \\
&\leq \max_{i=1}^n |z_i|^2 \sum_{i=1}^n \left\| E_{U_i} \phi \right\|^2 \\
&\overset{(b)}{=} \|f\|_\infty^2 \left\| \sum_{i=1}^n E_{U_i} \phi \right\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|f\|_\infty^2 \|E_X \phi\|^2 \\
&\stackrel{(c)}{=} \|f\|_\infty^2 \|\phi\|^2,
\end{aligned}
$$

where we have used twice Pythagoras' Theorem in (a) and (b) as well as $E_X = \mathbb{1}$ in (c). This shows (5.1.33). ☐

This simple lemma has now a drastic and far-reaching consequence. Recall that inside the bounded measurable functions $\mathscr{BM}(X, \mathfrak{a})$ the simple functions are *dense* with respect to the supremum norm. Moreover, $\mathscr{BM}(X, \mathfrak{a})$ is a commutative unital C^*-algebra with respect to the usual pointwise operations and the supremum norm, see Proposition C.1.25 for a detailed discussion.

Theorem 5.1.19 (Bounded measurable calculus) *Let E be a projection-valued measure on a measurable space $(X, \mathfrak{a})$. Then there is a unique unital *-homomorphism*

$$
\int_X \cdot \, \mathrm{d}E \colon \mathscr{BM}(X, \mathfrak{a}) \longrightarrow \mathfrak{B}(\mathfrak{H}), \tag{5.1.34}
$$

which maps χ_U to E_U for all $U \in \mathfrak{a}$. The image of (5.1.34) *is a commutative unital C^*-subalgebra of $\mathfrak{B}(\mathfrak{H})$.*

Proof: Clearly, by Lemma 5.1.18 we have the *-homomorphism defined (in a unique way) on the subspace of simple functions. Since $\mathfrak{B}(\mathfrak{H})$ is complete in the operator norm, since the simple functions are dense in $\mathscr{BM}(X, \mathfrak{a})$, and since we have the continuity property (5.1.33), there is a unique extension of $\int_X \cdot \, \mathrm{d}E$ to a continuous linear map (5.1.34). Since the algebra products as well as the *-involutions are continuous and since (5.1.34) satisfies the *-homomorphism property on the dense subspace of simple functions, we conclude that the extension to all bounded measurable functions satisfies these properties as well. By Corollary 4.4.31 the image is closed and hence we have a C^*-subalgebra of $\mathfrak{B}(\mathfrak{H})$. ☐

Remark 5.1.20 Let E be a projection-valued measure on a measurable space $(X, \mathfrak{a})$.

i.) Being a *-homomorphism between C^*-algebras means explicitly that

$$
\int_X (zf + wg) \, \mathrm{d}E = z \int_X f \, \mathrm{d}E + w \int_X g \, \mathrm{d}E, \tag{5.1.35}
$$

$$
\int_X \overline{f} \, \mathrm{d}E = \left(\int_X f \, \mathrm{d}E \right)^*, \tag{5.1.36}
$$

$$
\int_X fg \, \mathrm{d}E = \int_X f \, \mathrm{d}E \int_X g \, \mathrm{d}E, \tag{5.1.37}
$$

for all $f, g \in \mathscr{BM}(X, \mathfrak{a})$ and by Proposition 4.4.29 we also have

$$
\left\| \int_X f \, \mathrm{d}E \right\| \leq \|f\|_\infty. \tag{5.1.38}
$$

We will call this *-homomorphism also the *spectral integral* with respect to the projection-valued measure E. We will find other examples and similar constructions where we can integral also other types of functions with respect to a projection-valued measure later on.

ii.) The images of simple functions, i.e. linear combinations of the projections E_U for $U \in \mathfrak{a}$ are norm-dense in the image of (5.1.34). This is clear from the continuity of (5.1.34).

iii.) If $A \in \mathfrak{B}(\mathfrak{H})$ is an operator then A commutes with every $\int_X f \, \mathrm{d}E$ for all $f \in \mathscr{BM}(X, \mathfrak{a})$ iff A commutes with every E_U for all $U \in \mathfrak{a}$. Indeed, one direction is trivial. For the other, if A commutes with every E_U then A commutes with a dense subset of the image and hence, by continuity, with the whole image.

iv.) Let $\phi, \psi \in \mathfrak{H}$ and $f \in \mathscr{BM}(X, \mathfrak{a})$. Then we have

$$\left\langle \phi, \int_X f \, \mathrm{d}E\psi \right\rangle = \int_X f \, \mathrm{d}\langle \phi, E\psi \rangle, \tag{5.1.39}$$

where the right hand side is the integral of f over X with respect to the complex measure $U \mapsto \langle \phi, E_U \psi \rangle$. Indeed, by continuity this is clear as (5.1.39) obviously holds for simple functions. Combining (5.1.39) with the *-homomorphism properties gives the equality

$$\left\| \int_X f \, \mathrm{d}E\phi \right\|^2 = \int_X |f|^2 \, \mathrm{d}\langle \phi, E\phi \rangle, \tag{5.1.40}$$

where now on the right hand side we have a finite positive measure $\langle \phi, E\phi \rangle$.

In a next step we want to discuss the kernel of the integration (5.1.34). As in the scalar case, there might be subsets $U \in \mathfrak{a}$ with "measure zero", i.e. $E_U = 0$. Note that now measure zero means that E_U is the zero projection. Nevertheless, the subsets of measure zero behave like the zero sets in the scalar theory: they form a *σ-ideal*. This allows to state the following definition completely analogously to Definition C.2.16:

Definition 5.1.21 (Essentially bounded functions) *Let E be a projection-valued measure on a measurable space $(X, \mathfrak{a})$.*

i.) *For $f \in \mathscr{M}(X, \mathfrak{a})$ the essential range with respect to E is defined by*

$$\operatorname{ess\,range}(f) = \left\{ z \in \mathbb{C} \;\middle|\; E_{f^{-1}(\mathrm{B}_\epsilon(z))} \neq 0 \textit{ for all } \epsilon > 0 \right\}, \tag{5.1.41}$$

and the essential supremum of f with respect to E is

$$\operatorname*{ess\,sup}_{x \in X} |f(x)| = \sup\left\{ |z| \;\middle|\; z \in \operatorname{ess\,range}(f) \right\}. \tag{5.1.42}$$

ii.) *The function f is called E-essentially bounded if*

$$\|f\|_{E,\infty} = \operatorname*{ess\,sup}_{x \in X} |f(x)| < \infty, \tag{5.1.43}$$

and the space of all E-essentially bounded functions is denoted by $\mathscr{L}^\infty(X, \mathfrak{a}, E)$.

As in the scalar case we see that

$$\|f\|_{E,\infty} \leq \|f\|_\infty \tag{5.1.44}$$

for every $f \in \mathscr{BM}(X, \mathfrak{a})$. Moreover, $\mathscr{L}^\infty(X, \mathfrak{a}, E)$ is a commutative unital *-algebra with respect to the pointwise operations and $\|\cdot\|_{E,\infty}$ is a C^*-seminorm on it. Indeed, these facts only require to know what the zero sets are and that they form a σ-ideal. Hence a projection-valued measure can be handled as a positive measure as done in Section C.2.2, see in particular Remark C.2.21. Hence we can state the following definition:

Definition 5.1.22 (The C^*-algebra $\mathrm{L}^\infty(X, \mathfrak{a}, E)$) *For a projection-valued measure E on measurable space $(X, \mathfrak{a})$ one defines*

$$\mathrm{L}^\infty(X, \mathfrak{a}, E) = \mathscr{L}^\infty(X, \mathfrak{a}, E) \big/ \ker \|\cdot\|_{E,\infty}. \tag{5.1.45}$$

Theorem 5.1.23 (Essentially bounded calculus) *For a projection-valued measure E on a measurable space $(X, \mathfrak{a})$ one has:*

i.) The quotient $\mathrm{L}^\infty(X, \mathfrak{a}, E)$ *is a unital* C^**-algebra and the canonical map*

$$\mathscr{BM}(X, \mathfrak{a}) \longrightarrow \mathrm{L}^\infty(X, \mathfrak{a}, E) \tag{5.1.46}$$

is a surjective **-homomorphism between* C^**-algebras.*

ii.) The integral (5.1.34) *factors to an injective unital* **-homomorphism*

$$\int_X \cdot \, \mathrm{d}E \colon \mathrm{L}^\infty(X, \mathfrak{a}, E) \longrightarrow \mathfrak{B}(\mathfrak{H}). \tag{5.1.47}$$

iii.) For $f \in \mathrm{L}^\infty(X, \mathfrak{a}, E)$ *we have the spectral mapping theorem*

$$\operatorname{spec}\left(\int_X f \, \mathrm{d}E\right) = \operatorname{spec}(f) = \operatorname{ess\,range}(f). \tag{5.1.48}$$

iv.) For every measurable $U \in \mathfrak{a}$ *the direct orthogonal sum*

$$\mathfrak{H} = \operatorname{im} E_U \oplus \ker E_U \tag{5.1.49}$$

is invariant under all operators $\int_X f \, \mathrm{d}E$ *for* $f \in \mathrm{L}^\infty(X, \mathfrak{a}, E)$.

PROOF: The first part is done as in the scalar case in Proposition C.2.18 since for the proof we only needed to know what the zero sets are. For the second part, let $f = \sum_{i=1}^n z_i \chi_{U_i}$ be a simple function. Then it is easy to see that

$$\|f\|_{E,\infty} = \max_{\substack{i=1 \\ E_{U_i} \neq 0}}^{n} |z_i|.$$

Redoing the estimate (5.1.33) from Lemma 5.1.18, *iv.)*, shows that

$$\left\| \int_X f \, \mathrm{d}E \phi \right\|^2 \leq \|f\|_{E,\infty}^2 \|\phi\|^2, \tag{$*$}$$

since we only have to take care of those i with $E_{U_i} \neq 0$. If i_0 is an index with $E_{U_{i_0}} \neq 0$ then there is a vector $\phi \neq 0$ with $E_{U_{i_0}} \phi = \phi$. For this ϕ we have $E_{U_i} \phi = 0$ for all other $i \neq i_0$ since the U_i are pairwise disjoint and hence the projections are pairwise orthogonal by Proposition 5.1.16, *i.)*. Thus we simply have $\int_X f \, \mathrm{d}E \phi = z_{i_0} \phi$ in this case. Taking i_0 to be the index with the largest $|z_{i_0}|$ gives

$$\left\| \int_X f \, \mathrm{d}E \phi \right\|^2 = \|f\|_{E,\infty}^2 \|\phi\|^2.$$

This shows that in $(*)$ we actually find vectors yielding equality. Together with the estimate $(*)$ this implies

$$\left\| \int_X f \, \mathrm{d}E \right\| = \|f\|_{E,\infty} \tag{☺}$$

for simple functions. Since the left hand side is continuous in f with respect to $\|\cdot\|_\infty$, and also the right hand side is continuous with respect to the supremum norm, the density of simple functions in $\mathscr{BM}(X, \mathfrak{a})$ shows that (☺) holds for all $f \in \mathscr{BM}(X, \mathfrak{a})$. We conclude that the kernel of the integral coincides with the kernel of the seminorm $\|\cdot\|_{E,\infty}$, i.e.

$$\ker \int_X \cdot \, \mathrm{d}E = \ker \|\cdot\|_{E,\infty}.$$

Thus the integral factors as claimed and remains to be a unital *-homomorphism. Since we divided precisely by the kernel of the integral, it becomes injective. This is also clear as for the essential supremum norm it is norm-preserving by (☺), see also Proposition 4.4.30. For the third part we first note that $\operatorname{spec}(f)$ coincides with the essential range of f. Indeed, the proof is entirely parallel to the scalar case as discussed in Remark C.2.19, *iii.)*. Then, under an injective unital *-homomorphism like (5.1.47) the spectrum is preserved, see e.g. Corollary 4.3.23. Finally, recall that (5.1.49) is indeed an orthogonal direct sum since E_U is a projection. Now $E_U = \int_X \chi_U \, \mathrm{d}E$ is in the image of $\mathrm{L}^\infty(X, \mathfrak{a}, E)$ which is a commutative C^*-algebra. Thus every $\int_X f \, \mathrm{d}E$ commutes with E_U and hence the decomposition (5.1.49) is invariant. □

Up to now the continuity statements of the integral $\int_X \cdot \, \mathrm{d}E$ refer always to the operator norm and hence to a C^*-algebraic concept. In a next step we extend this to a weaker concept of convergence:

Proposition 5.1.24 *Let E be a projection-valued measure on a measurable space $(X, \mathfrak{a})$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of bounded measurable functions $f_n \in \mathscr{BM}(X, \mathfrak{a})$ with $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. If $f(x) = \lim_{n\longrightarrow\infty} f_n(x)$ exists pointwise for every $x \in X$ then*

$$\int_X f_n \, \mathrm{d}E \longrightarrow \int_X f \, \mathrm{d}E \tag{5.1.50}$$

in the strong operator topology.

PROOF: Let $\phi, \psi \in \mathfrak{H}$ be given. Then the complex measure $\langle \phi, E\psi \rangle$ can be written as a complex linear combination

$$\begin{aligned} \langle \phi, E_U\psi \rangle = &\frac{1}{2}\big(\langle \phi + \psi, E_U(\phi + \psi)\rangle - \langle \phi, E_U\phi\rangle - \langle \psi, E_U\psi\rangle\big) \\ &+ \frac{\mathrm{i}}{2}\big(\langle \phi + \mathrm{i}\psi, E_U(\phi + \mathrm{i}\psi)\rangle - \langle \phi, E_U\phi\rangle - \langle \psi, E_U\psi\rangle\big), \end{aligned} \tag{$*$}$$

where now each measure on the right hand side is of the form $\langle \chi, E\chi \rangle$ with some $\chi \in \mathfrak{H}$. By Proposition 5.1.16, *iii.)*, these measures are now finite positive measures. Remark 5.1.20, *iv.)*, gives

$$\left\langle \chi, \int_X f_n \, \mathrm{d}E\chi \right\rangle = \int_X f_n \, \mathrm{d}\langle \chi, E\chi \rangle.$$

Since we integrate a sequence of bounded measurable complex-valued functions f_n with respect to a *finite* positive measure the condition $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ shows that we are in the situation of dominated convergence. Hence we have

$$\lim_{n\longrightarrow\infty} \int_X f_n \, \mathrm{d}\langle \chi, E\chi \rangle = \int_X f \, \mathrm{d}\langle \chi, E\chi \rangle,$$

which also shows

$$\lim_{n\longrightarrow\infty} \int_X f_n \, \mathrm{d}\langle \phi, E\psi \rangle = \int_X f \, \mathrm{d}\langle \phi, E\psi \rangle$$

by ($*$). In fact, Lebesgue's dominated convergence can also directly by applied to complex measures, see Exercise C.6.9. We conclude that (5.1.50) holds with respect to the weak operator topology. To finish the proof we note that by Remark 5.1.20, *iv.)*, we have

$$\left\| \int_X f_n \, \mathrm{d}E\phi \right\|^2 = \int_X |f_n|^2 \, \mathrm{d}\langle \phi, E\phi \rangle \longrightarrow \int_X |f|^2 \, \mathrm{d}\langle \phi, E\phi \rangle = \left\| \int_X f \, \mathrm{d}E\phi \right\|^2,$$

again using dominated convergence. Thus Proposition 3.2.15 can be applied to finally yield the strong convergence as claimed. □

5.1.3 Commutative C^*-Subalgebras of $\mathfrak{B}(\mathfrak{H})$

In the previous subsection we have seen that a projection-valued measure E gives a commutative unital C^*-subalgebra of $\mathfrak{B}(\mathfrak{H})$ isomorphic to $\mathrm{L}^\infty(X, \mathfrak{a}, E)$. The aim of this subsection is now to show a sort of converse statement: we consider a unital commutative C^*-subalgebra $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$. In general, it will *not* be isomorphic to some $\mathrm{L}^\infty(X, \mathfrak{a}, E)$ but it will not be very far from it, in a sense we shall establish now. Since $\mathscr{A}$ is commutative, one knows from the Gel'fand transform as in Theorem 4.3.13 that

$$\mathscr{A} \ni A \mapsto \hat{A} \in \mathscr{C}(\mathrm{Spec}(\mathscr{A})) \tag{5.1.51}$$

is a *-isomorphism of C^*-algebras where $\mathrm{Spec}(\mathscr{A})$ is the spectrum of $\mathscr{A}$ viewed as a compact Hausdorff space as usual. Recall that as a set $\mathrm{Spec}(\mathscr{A})$ consists of the characters of $\mathscr{A}$ and $\hat{A}(\varphi) = \varphi(A)$ is just the evaluation of A in a character. In general, $\mathrm{Spec}(\mathscr{A})$ does not carry any further structure than the weak*-topology inherited from the dual $\mathscr{A}'$. However, if $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ we can induce complex measures on $\mathrm{Spec}(\mathscr{A})$:

Lemma 5.1.25 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a commutative unital C^*-subalgebra. Then for every $\phi, \psi \in \mathfrak{H}$ there exists a unique regular complex Borel measure $\mu_{\phi,\psi}$ on $\mathrm{Spec}(\mathscr{A})$ such that for all $A \in \mathscr{A}$*

$$\langle \phi, A\psi \rangle = \int_{\mathrm{Spec}(\mathscr{A})} \hat{A} \, \mathrm{d}\mu_{\phi,\psi} \tag{5.1.52}$$

with variational norm satisfying $\|\mu_{\phi,\psi}\| \leq \|\phi\|\|\psi\|$.

Proof: Let $A \in \mathscr{A}$ and $\phi, \psi \in \mathfrak{H}$ be given. Then we have the trivial estimate

$$|\langle \phi, A\psi \rangle| \leq \|A\|\|\phi\|\|\psi\|. \tag{$*$}$$

Since $\mathscr{A}$ is isomorphic to $\mathscr{C}(\mathrm{Spec}(\mathscr{A}))$ we can view the map $\hat{A} \mapsto \langle \phi, A\psi \rangle$ as a linear functional on $\mathscr{C}(\mathrm{Spec}(\mathscr{A}))$. Since $\|\hat{A}\|_\infty = \|A\|$ we see that this linear functional is continuous with respect to the supremum norm of $\mathscr{C}(\mathrm{Spec}(\mathscr{A}))$. But then the Riesz Representation Theorem **??** ensures that the functional is actually the integration with respect to a uniquely determined regular complex measure $\mu_{\phi,\psi}$ on $\mathrm{Spec}(\mathscr{A})$. For the variational norm of $\mu_{\phi,\psi}$ we know that it coincides with the norm of the linear functional $\hat{A} \mapsto \langle \phi, A\psi \rangle$. However, $(*)$ gives an immediate estimate that this functional has norm at most $\|\phi\|\|\psi\|$. □

In some sense this was the only non-trivial part of the spectral theorem as we needed the non-trivial Riesz Representation Theorem in quite some generality: every compact Hausdorff space X can occur as spectrum of $\mathscr{C}(X)$. We collect now some easy consequences of the construction of the measures $\mu_{\phi,\psi}$:

Lemma 5.1.26 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a commutative unital C^*-subalgebra. Then the inverse Gel'fand transform extends to a *-homomorphism*

$$\mathscr{B}\mathscr{M}(\mathrm{Spec}(\mathscr{A})) \longrightarrow \mathfrak{B}(\mathfrak{H}). \tag{5.1.53}$$

Proof: Let $\phi, \psi \in \mathfrak{H}$ be given and let $\mu_{\phi,\psi}$ be the complex regular measure from Lemma 5.1.25. Then for every bounded measurable function $f \in \mathscr{B}\mathscr{M}(\mathrm{Spec}(\mathscr{A}))$ the integration with respect to $\mu_{\phi,\psi}$ is well-defined and satisfies

$$\left| \int_{\mathrm{Spec}(\mathscr{A})} f \, \mathrm{d}\mu_{\phi,\psi} \right| \leq \|f\|_\infty \|\mu_{\phi,\psi}\|,$$

where $\|\mu_{\phi,\psi}\|$ is the variational norm of $\mu_{\phi,\psi}$ as usual. Since $\|\mu_{\phi,\psi}\| \leq \|\phi\|\|\psi\|$ by Lemma 5.1.25 we get the estimate

$$\left| \int_{\mathrm{Spec}(\mathscr{A})} f \, \mathrm{d}\mu_{\phi,\psi} \right| \leq \|f\|_\infty \|\phi\|\|\psi\|. \tag{$*$}$$

Now it is clear that the map

$$\mathfrak{H} \times \mathfrak{H} \ni (\phi, \psi) \;\mapsto\; \mu_{\phi,\psi} \in \mathrm{Meas}(\mathrm{Spec}(\mathscr{A})) \tag{$**$}$$

is sesquilinear: indeed, it suffices to check this sesquilinearity evaluated on *continuous* functions $f \in \mathscr{C}(\mathrm{Spec}(\mathscr{A}))$ since the measures are completely determined by their values in continuous functions. This is precisely the uniqueness statement in the Riesz Representation Theorem. But a continuous function f is just an inverse Gel'fand transform of an element $A \in \mathscr{A}$ for which the sesquilinearity is trivial as it corresponds to the sesquilinearity of the expression $\langle \phi, A\psi \rangle$. The estimate $(*)$ together with $(**)$ shows that the map

$$\mathfrak{H} \times \mathfrak{H} \ni (\phi, \psi) \;\mapsto\; \int_{\mathrm{Spec}(\mathscr{A})} f \,\mathrm{d}\mu_{\phi,\psi} \tag{☺}$$

is a continuous sesquilinear map. Hence we can use the Lax-Milgram Theorem, see Exercise 3.6.20, to see that the map (☺) is necessarily of the form $(\phi, \psi) \mapsto \langle \phi, A_f \psi \rangle$ with some uniquely determined operator $A_f \in \mathfrak{B}(\mathfrak{H})$. This is the extension of the Gel'fand transform we are looking for since for $A \in \mathscr{A}$ we clearly have $A_{\hat{A}} = A$. It remains to show that (5.1.53) is a *-homomorphism: the linearity is clear since integrals are linear. Moreover, we have $\overline{\mu_{\phi,\psi}} = \mu_{\psi,\phi}$ as this holds when evaluated on continuous functions on $\mathrm{Spec}(\mathscr{A})$, i.e. for elements $A \in \mathscr{A}$, again by the uniqueness of the measures $\mu_{\phi,\psi}$. But then we have

$$\overline{\int_{\mathrm{Spec}(\mathscr{A})} f \,\mathrm{d}\mu_{\phi,\psi}} = \int_{\mathrm{Spec}(\mathscr{A})} \overline{f} \,\mathrm{d}\mu_{\psi,\phi}.$$

Therefore we can compute

$$\langle \phi, A_{\overline{f}} \psi \rangle = \int_{\mathrm{Spec}(\mathscr{A})} \overline{f} \,\mathrm{d}\mu_{\phi,\psi} = \overline{\int_{\mathrm{Spec}(\mathscr{A})} f \,\mathrm{d}\mu_{\psi,\phi}} = \overline{\langle \psi, A_f \phi \rangle} = \langle \phi, A_f^* \psi \rangle,$$

showing the compatibility with the *-involutions. Finally, let $f, g \in \mathscr{BM}(\mathrm{Spec}(\mathscr{A}))$ be given. If both are continuous, i.e. correspond to elements of $\mathscr{A}$, then (5.1.53) is multiplicative as it is the inverse Gel'fand transform. Thus we have for $f = \hat{A}$ and $g = \hat{B}$ with $A, B \in \mathscr{A}$ the relation

$$\int_{\mathrm{Spec}(\mathscr{A})} fg \,\mathrm{d}\mu_{\phi,\psi} = \langle \phi, AB\psi \rangle = \int_{\mathrm{Spec}(\mathscr{A})} f \,\mathrm{d}\mu_{\phi,B\psi}. \tag{$\star$}$$

For $g = \hat{B}$ fixed the map $f \mapsto \int_{\mathrm{Spec}(\mathscr{A})} fg \,\mathrm{d}\mu_{\phi,\psi}$ is a continuous linear functional as well. Hence it corresponds to an integration of f with respect to a *uniquely* determined complex regular Borel measure on $\mathrm{Spec}(\mathscr{A})$. By this uniqueness, $(\star)$ shows that the measure is just given by $\mu_{\phi,B\psi}$. But then, $(\star)$ holds for all $f \in \mathscr{BM}(\mathrm{Spec}(\mathscr{A}))$ since the measures agree. Hence we have

$$\int_{\mathrm{Spec}(\mathscr{A})} fg \,\mathrm{d}\mu_{\phi,\psi} = \langle \phi, A_f B\phi \rangle = \langle \phi, BA_f \phi \rangle.$$

This proves the homomorphism property $A_{fg} = A_f A_g$ as soon as one function, say g, is continuous. Now let $f \in \mathscr{BM}(\mathrm{Spec}(\mathscr{A}))$ be fixed and consider the map

$$\mathscr{C}(\mathrm{Spec}(\mathscr{A})) \ni g \;\mapsto\; \int_{\mathrm{Spec}(\mathscr{A})} fg \,\mathrm{d}\mu_{\phi,\psi} = \int_{\mathrm{Spec}(\mathscr{A})} g \,\mathrm{d}\mu_{\phi,A_f\psi}.$$

This is again a continuous linear functional, hence given by an integration, and by uniqueness the measure is given by $\mu_{\phi,A_f\psi}$. But with the measure being determined, the equation still holds for all $g \in \mathscr{BM}(\mathrm{Spec}(\mathscr{A}))$ from which we get

$$\langle \phi, A_{fg} \psi \rangle = \int_{\mathrm{Spec}(\mathscr{A})} fg \,\mathrm{d}\mu_{\phi,\psi} = \int_{\mathrm{Spec}(\mathscr{A})} g \,\mathrm{d}\mu_{\phi,A_f\psi} = \langle \phi, A_g A_f \psi \rangle,$$

now valid for arbitrary $f, g \in \mathscr{BM}(\mathrm{Spec}(\mathscr{A}))$. This shows the homomorphism property as claimed.□

The homomorphism (5.1.53) is apparently uniquely determined (by construction) by the property that for every $\phi, \psi \in \mathfrak{H}$ we have

$$\big\langle \phi, A_f \psi \big\rangle = \int_{\operatorname{Spec}(\mathscr{A})} f \, \mathrm{d}\mu_{\phi,\psi}. \tag{5.1.54}$$

We are now in the position to define a projection-valued measure E such that (5.1.53) is just the integral with respect to E.

Lemma 5.1.27 *Let $\mathscr{A}$ be a commutative unital C^*-subalgebra of $\mathfrak{B}(\mathfrak{H})$. Then there exists a unique projection-valued measure E on* $\operatorname{Spec}(\mathscr{A})$ *such that the *-homomorphism* (5.1.53) *is given by the integral with respect to E.*

PROOF: Suppose E is such a projection-valued measure. Then for $\phi, \psi \in \mathfrak{H}$ we have

$$\left\langle \phi, \int_{\operatorname{Spec}(\mathscr{A})} f \, \mathrm{d}E \psi \right\rangle = \int_{\operatorname{Spec}(\mathscr{A})} f \, \mathrm{d}\langle \phi, E\psi \rangle \stackrel{(*)}{=} \int_{\operatorname{Spec}(\mathscr{A})} f \, \mathrm{d}\mu_{\phi,\psi}$$

for all $f \in \mathscr{BM}(\operatorname{Spec}(\mathscr{A}))$. In particular, this holds for all continuous functions $f = \hat{A}$ with $A \in \mathscr{A}$ where the right hand side is simply $\langle \phi, A\psi \rangle$. Since for a projection-valued measure on a locally compact space like $\operatorname{Spec}(\mathscr{A})$ we insist on the measures $\langle \phi, E\psi \rangle$ to be *regular*, the equality $(*)$ for continuous functions implies already that the measures $\langle \phi, E\psi \rangle$ and $\mu_{\phi,\psi}$ coincide. This shows the uniqueness of E. For the existence we define for a measurable subset $U \subseteq \operatorname{Spec}(\mathscr{A})$ the operator

$$E_U = A_{\chi_U},$$

i.e. the image of the corresponding characteristic function under (5.1.53). Since (5.1.53) is a *-homomorphism, $E_U = E_U^2 = E_U^*$ is a projection. Clearly,

$$\big\langle \phi, E_U \psi \big\rangle = \big\langle \phi, A_{\chi_U} \psi \big\rangle = \int_{\operatorname{Spec}(\mathscr{A})} \chi_U \, \mathrm{d}\mu_{\phi,\psi} = \mu_{\phi,\psi}(U)$$

holds. If $U = \emptyset$ then $\chi_U = 0$ and hence $E_U = 0$ follows. If $U = \operatorname{Spec}(\mathscr{A})$ then $\chi_U = 1$ and hence $E_{\operatorname{Spec}(\mathscr{A})} = \mathbb{1}$ since (5.1.53) is a unital *-homomorphism. Finally, if $U \cap V = \emptyset$ then $\chi_U + \chi_V = \chi_{U \cup V}$ and thus $E_{U \cup V} = E_U + E_V$ follows from the linearity of (5.1.53). This shows that $E \colon U \mapsto E_U$ is a projection-valued measure on $\operatorname{Spec}(\mathscr{A})$ such that $\langle \phi, E\psi \rangle = \mu_{\phi,\psi}$. By the very construction of (5.1.53), i.e. (5.1.54), this means $A_f = \int_{\operatorname{Spec}(\mathscr{A})} f \, \mathrm{d}E$. ☐

We collect now the results of this construction which can be seen as a quite general version of the spectral theorem:

Theorem 5.1.28 (Spectral Theorem I) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a commutative unital C^*-subalgebra.*

i.) There exists a unique projection-valued measure E on $\operatorname{Spec}(\mathscr{A})$ *such that for $A \in \mathscr{A}$ with Gel'fand transform $\hat{A} \in \mathscr{C}(\operatorname{Spec}(\mathscr{A}))$ one has*

$$A = \int_{\operatorname{Spec}(\mathscr{A})} \hat{A} \, \mathrm{d}E. \tag{5.1.55}$$

ii.) The integral with respect to E induces an isometric *-homomorphism

$$\mathrm{L}^\infty(\operatorname{Spec}(\mathscr{A}), E) \ni f \mapsto \int_{\operatorname{Spec}(\mathscr{A})} f \, \mathrm{d}E \in \mathfrak{B}(\mathfrak{H}), \tag{5.1.56}$$

extending the inverse Gel'fand transform.

iii.) *An operator* $B \in \mathfrak{B}(\mathfrak{H})$ *commutes with all operators in* $\mathscr{A}$ *iff* B *commutes with every* E_U *for all measurable* $U \subseteq \operatorname{Spec}(\mathscr{A})$*. In this case,* B *commutes with the whole image of* $\mathrm{L}^\infty(\operatorname{Spec}(\mathscr{A}), E)$*, too.*

iv.) *If a sequence* $(f_n)_{n\in\mathbb{N}}$ *of functions* $f_n \in \mathscr{BM}(\operatorname{Spec}(\mathscr{A}))$ *converges pointwise to* f *such that* $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ *then*

$$\int_{\operatorname{Spec}(\mathscr{A})} f_n \,\mathrm{d}E \longrightarrow \int_{\operatorname{Spec}(\mathscr{A})} f \,\mathrm{d}E \tag{5.1.57}$$

in the strong operator topology.

PROOF: According to Lemma 5.1.27 there is a projection-valued measure E such that (5.1.55) holds. We have already seen that this implies $\langle \phi, E\psi \rangle = \mu_{\phi,\psi}$ with $\mu_{\phi,\psi}$ being the unique measures from Lemma 5.1.25, showing the uniqueness of E. It extends the continuous calculus of $\mathscr{A}$ to a bounded measurable one which factors to $\mathrm{L}^\infty(\operatorname{Spec}(\mathscr{A}), E)$ according to Theorem 5.1.23. In fact, we know in general that "isometric" and "injective" is the same thing for *-homomorphisms between (unital) C^*-algebras according to Proposition 4.4.30. Thus also the second part is clear. For the third part we use Remark 5.1.20, *iii.)*, to conclude that if B commutes with all E_U that then B commutes with the whole image of $\mathrm{L}^\infty(\operatorname{Spec}(\mathscr{A}), E)$ and hence, in particular, with $\mathscr{A}$. Conversely, suppose B commutes with $\mathscr{A}$. Then from (5.1.52) we see that for all $A \in \mathscr{A}$ we have

$$\int_{\operatorname{Spec}(\mathscr{A})} \hat{A} \,\mathrm{d}\mu_{B^*\phi,\psi} = \langle B^*\phi, A\psi \rangle = \langle \phi, BA\psi \rangle = \langle \phi, AB\psi \rangle = \int_{\operatorname{Spec}(\mathscr{A})} \hat{A} \,\mathrm{d}\mu_{\phi,B\psi}.$$

By the usual uniqueness argument this implies $\mu_{B^*\phi,\psi} = \mu_{\phi,B\psi}$ since $\hat{A}$ runs through all continuous functions on $\operatorname{Spec}(\mathscr{A})$. But then also for arbitrary $f \in \mathscr{BM}(\operatorname{Spec}(\mathscr{A}))$ we have

$$\begin{aligned}
\left\langle \phi, B \int_{\operatorname{Spec}(\mathscr{A})} f \,\mathrm{d}E\psi \right\rangle &= \left\langle B^*\phi, \int_{\operatorname{Spec}(\mathscr{A})} f \,\mathrm{d}E\psi \right\rangle \\
&= \int_{\operatorname{Spec}(\mathscr{A})} f \,\mathrm{d}\mu_{B^*\phi,\psi} \\
&= \int_{\operatorname{Spec}(\mathscr{A})} f \,\mathrm{d}\mu_{\phi,B\psi} \\
&= \left\langle \phi, \int_{\operatorname{Spec}(\mathscr{A})} f \,\mathrm{d}EB\psi \right\rangle.
\end{aligned}$$

Taking $f = \chi_U$ gives the claim. The last part is the content of Proposition 5.1.24 applied to our situation. □

Remark 5.1.29 (Bounded measurable calculus) One way to use this theorem is that for a commutative unital C^*-subalgebra $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ we have now a bounded measurable calculus (5.1.56). Note also that the image of (5.1.56) is isomorphic to $\mathrm{L}^\infty(\operatorname{Spec}(\mathscr{A}), E)$ and a commutative unital C^*-subalgebra of $\mathfrak{B}(\mathfrak{H})$ itself by Corollary 4.4.31.

Remark 5.1.30 (Projections in $\mathrm{L}^\infty(\operatorname{Spec}(\mathscr{A}), E)$) Since the simple functions are norm-dense in the bounded measurable ones and hence also yield a norm-dense subset of $\mathrm{L}^\infty(\operatorname{Spec}(\mathscr{A}), E)$ after the quotient procedure, we conclude that the finite linear combinations of the projections $E_U = \int_{\operatorname{Spec}(\mathscr{A})} \chi_U \,\mathrm{d}E$ are dense in the image of $\mathrm{L}^\infty(\operatorname{Spec}(\mathscr{A}), E)$ with respect to the operator norm. In particular, every element $A \in \mathscr{A}$ can be approximated by linear combinations of the E_U in the *operator norm*. The converse is more tricky: to get the characteristic functions χ_U out of the continuous ones is a much more involved process as $\mathscr{C}(\operatorname{Spec}(\mathscr{A})) \subseteq \mathscr{BM}(\operatorname{Spec}(\mathscr{A}))$ is known to be a closed subspace with respect to the supremum norm. Thus the approximation has to be done in some weaker sense. Already for $\operatorname{Spec}(\mathscr{A}) = [0, 1]$ this is known to be difficult.

Remark 5.1.31 If $O \subseteq \operatorname{Spec}(\mathscr{A})$ is open and non-empty and $\varphi \in O$ then there is a continuous function $f \in \mathscr{C}(\operatorname{Spec}(\mathscr{A}))$ with $f(\varphi) \neq 0$ but $f\big|_{\operatorname{Spec}(\mathscr{A})\setminus O} = 0$. This follows from the Urysohn Lemma A.2.2 for the compact Hausdorff space $\operatorname{Spec}(\mathscr{A})$. By the continuous calculus we have $f = \hat{A}$ for some non-zero $A \in \mathscr{A}$. This shows that $f \in \mathscr{C}(\operatorname{Spec}(\mathscr{A}))$ is mapped to some non-zero element $f \in \mathrm{L}^\infty(\operatorname{Spec}(\mathscr{A}), E)$ since $A = \int_{\operatorname{Spec}(\mathscr{A})} f \, \mathrm{d}E$. Thus the essential range $\operatorname{ess\,range}(f)$ is not just $\{0\}$. Since $f \neq 0$ only inside of O this means that

$$E_O \neq 0. \tag{5.1.58}$$

Thus the open and non-empty subsets of $\operatorname{Spec}(\mathscr{A})$ have a non-zero spectral measure.

5.1.4 The Spectral Theorem for Bounded Normal Operators

We shall now specialize the spectral theorem for commutative unital C^*-subalgebras of $\mathfrak{B}(\mathfrak{H})$ to the special case of one normal operator $A \in \mathfrak{B}(\mathfrak{H})$. Recall that the unital C^*-algebra $\mathsf{C}^*\langle A\rangle \subseteq \mathfrak{B}(\mathfrak{H})$ generated by A is the closure of $\mathbb{C}[A, A^*]$ with respect to the operator norm. Moreover, $\mathsf{C}^*\langle A\rangle$ is commutative iff A is normal. From the general C^*-algebra framework developed in Subsection 4.3.3 we know that $\operatorname{Spec}(\mathsf{C}^*\langle A\rangle) \cong \operatorname{spec}(A)$ via the Gel'fand transform

$$\hat{A}\colon \operatorname{Spec}(\mathsf{C}^*\langle A\rangle) \longrightarrow \operatorname{spec}(A) \subseteq \mathbb{C}, \tag{5.1.59}$$

and $\mathscr{C}(\operatorname{spec}(A)) \cong \mathsf{C}^*\langle A\rangle$ via the pull-back with (5.1.59) such that the polynomials $\mathbb{C}[z, \overline{z}]$ restricted to $\operatorname{spec}(A) \subseteq \mathbb{C}$ correspond to the polynomials in A and A^*, see Theorem 4.3.25. This continuous calculus extends now to a bounded measurable calculus and the spectral theorem as follows:

Theorem 5.1.32 (Spectral theorem for bounded normal operators) *Let $A \in \mathfrak{B}(\mathfrak{H})$ be a normal operator.*

i.) There exists a unique projection-valued measure E on $\operatorname{spec}(A)$ such that

$$A = \int_{\operatorname{spec}(A)} \lambda \, \mathrm{d}E, \tag{5.1.60}$$

where λ stands for the identity map $\mathbb{C} \longrightarrow \mathbb{C}$ restricted to $\operatorname{spec}(A)$.

ii.) The continuous calculus for A is given by

$$f(A) = \int_{\operatorname{spec}(A)} f \, \mathrm{d}E, \tag{5.1.61}$$

with $f \in \mathscr{C}(\operatorname{spec}(A))$, and it extends to the bounded measurable calculus for $f \in \mathscr{BM}(\operatorname{spec}(A))$ via the same formula.

iii.) An operator $B \in \mathfrak{B}(\mathfrak{H})$ commutes with A iff B commutes with all E_U for $U \subseteq \operatorname{spec}(A)$ measurable iff B commutes with all bounded measurable functions $f(A)$ of A.

tral Mapping his situation? later for the d measurable calculus?

PROOF: Since $\mathscr{A} = \mathsf{C}^*\langle A\rangle$ is a unital commutative C^*-subalgebra of $\mathfrak{B}(\mathfrak{H})$ for a normal $A \in \mathfrak{B}(\mathfrak{H})$ and since $\operatorname{Spec}(\mathsf{C}^*\langle A\rangle) \cong \operatorname{spec}(A)$ we get by Theorem 5.1.28 the existence of a projection-valued measure E with (5.1.60) since the function λ corresponds to A under the Gel'fand isomorphism $\mathscr{C}(\operatorname{Spec}(\mathsf{C}^*\langle A\rangle)) \cong \mathsf{C}^*\langle A\rangle$. Now (5.1.60) together with the usual calculus for projection-valued measures shows that for every polynomial $p(z, \overline{z})$ we have $p(A, A^*) = \int_{\operatorname{spec}(A)} p \, \mathrm{d}E$, see Remark 5.1.20, *i.)*. Since the polynomials are dense in $\mathscr{C}(\operatorname{spec}(A))$ by the Stone-Weierstraß Theorem A.2.3 and since the integration with respect to E is continuous with respect to the supremum norm we conclude that (5.1.61) holds for all $f \in \mathscr{C}(\operatorname{spec}(A))$. But this means that E is the unique projection-valued measure with (5.1.55) according to Theorem 5.1.28. This shows the uniqueness as well as the second part. For the last part, we first recall that the Fuglede-Putnam-Rosenblum Theorem guarantees that B also commutes with A^* and hence with $\mathbb{C}[A, A^*]$, see Exercise 4.5.43. By continuity, B also commutes with all elements of $\mathsf{C}^*\langle A\rangle$ and hence Theorem 5.1.28, *iii.)*, can be applied. □

Definition 5.1.33 (Spectral measure) *Let $A \in \mathfrak{B}(\mathfrak{H})$ be a normal operator. Then the projection-valued measure E from Theorem 5.1.32 is called the spectral measure of A.*

We still have to argue that Theorem 5.1.32 is a good replacement and in fact a generalization of the spectral theorem known from finite-dimensional linear algebra. First we note that for every countable decomposition

$$\operatorname{spec}(A) = \bigcup_{n=1}^{\infty} U_n \tag{5.1.62}$$

into *disjoint* measurable subsets we have a direct orthogonal sum

$$\mathfrak{H} = \widehat{\bigoplus}_{n=1}^{\infty} \operatorname{im} E_{U_n} \tag{5.1.63}$$

by Proposition 5.1.16, *ii.)*. Of course, some of the images can be trivial, i.e. $E_{U_n} = 0$. In any case, the operator A and all its bounded measurable functions $f(A)$ preserves this direct sum decomposition by Theorem 5.1.28, *iii.)*. Thus A is "block-diagonal" with respect to any such decomposition (5.1.63). The question is now whether we can make the images so small such that A is just a multiple of the identity on $\operatorname{im} E_{U_n}$. In this case we would have diagonalized A in the usual sense. Of course, we know that this can not be possible if we have spectral values which are not eigenvalues. However, we get very close to diagonalization in the naive sense. Concerning eigenvalues we can state the following:

Theorem 5.1.34 (Eigenvalues and eigenspaces) *Let $A \in \mathfrak{B}(\mathfrak{H})$ be a normal operator with spectral measure E and let $f \in \mathscr{BM}(\operatorname{spec}(A))$.*

i.) *One has $\ker f(A) = \operatorname{im} E_{f^{-1}(0)}$.*

ii.) *For $\lambda \in \mathbb{C}$ one has*

$$\ker(\lambda \mathbb{1} - A) = \operatorname{im} E_{\{\lambda\}}. \tag{5.1.64}$$

iii.) *A complex number $\lambda \in \mathbb{C}$ is an eigenvalue of A iff $E_{\{\lambda\}} \neq 0$. In this case $\operatorname{im} E_{\{\lambda\}}$ is the eigenspace corresponding to λ.*

iv.) *If $\lambda \in \mathbb{C}$ is an eigenvalue of A then for every $\phi \in \operatorname{im} E_{\{\lambda\}}$ one has*

$$f(A)\phi = f(\lambda)\phi. \tag{5.1.65}$$

v.) *Every isolated point $\lambda \in \operatorname{spec}(A)$ is an eigenvalue.*

vi.) *If the spectrum $\operatorname{spec}(A) = \{\lambda_1, \lambda_2, \ldots\}$ is countable then*

$$\mathfrak{H} = \widehat{\bigoplus_{\lambda_n \in \operatorname{spec}(A)}} E_{\{\lambda_n\}}\mathfrak{H} \tag{5.1.66}$$

and

$$f(A)E_{\{\lambda_n\}}\phi = f(\lambda_n)E_{\{\lambda_n\}}\phi \tag{5.1.67}$$

for all $\phi \in \mathfrak{H}$. Note that $E_{\{\lambda_n\}} = 0$ is still possible.

vii.) *If the spectrum $\operatorname{spec}(A) = \{\lambda_1, \lambda_2, \ldots\}$ is countable then*

$$f(A) = \sum_{\lambda_n \in \operatorname{spec}(A)} f(\lambda_n)E_{\{\lambda_n\}} \tag{5.1.68}$$

as unconditionally convergent series in the strong operator topology.

PROOF: Let $U = f^{-1}(\{0\})$ be the measurable subset of $\operatorname{spec}(A)$ on which f vanishes. Then $f\chi_U = 0$ and thus
$$0 = f(A)\chi_U(A) = f(A)E_U = E_U f(A)$$
by the measurable calculus. This shows that $\operatorname{im} E_U \subseteq \ker f(A)$. To show the converse inclusion consider
$$U_n = \left\{ \lambda \in \operatorname{spec}(A) \;\middle|\; \tfrac{1}{n} \le |f(\lambda)| < \tfrac{1}{n-1} \right\}$$
for $n \ge 1$. In particular we have $U_1 = |f|^{-1}([1, \infty))$. Then these subsets are measurable, disjoint, and we have the disjoint union
$$\operatorname{spec}(A) = U \cup \bigcup_{n=1}^{\infty} U_n. \tag{$*$}$$
We define now the functions $f_n \colon \operatorname{spec}(A) \longrightarrow \mathbb{C}$ by
$$f_n(\lambda) = \begin{cases} \frac{1}{f(\lambda)} & \text{on } U_n \\ 0 & \text{elsewhere,} \end{cases}$$
which, by the general gluing arguments from Proposition C.1.22, *ii.)*, are still measurable and satisfy $\|f_n\|_\infty \le n$. Hence $f_n \in \mathscr{BM}(\operatorname{spec}(A))$. By the very construction we have the important feature that $f_n f = \chi_{U_n}$. Thus the measurable calculus gives the relation $f_n(A)f(A) = E_{U_n}$. Now let $\phi \in \ker f(A)$ then we have $E_{U_n}\phi = f_n(A)f(A)\phi = 0$ and hence $\ker f(A) \subseteq \ker E_{U_n}$ for all $n \ge 1$ follows. On the other hand, the strong σ-additivity of E according to Proposition 5.1.16, *ii.)*, together with ($*$) implies
$$\phi = E_U\phi + \sum_{n=1}^{\infty} E_{U_n}\phi = E_U\phi$$
for all $\phi \in \mathfrak{H}$. Hence $\ker f(A) \subseteq \operatorname{im} E_U$ is shown, giving the first part. The second is clear from this by taking $f(z) = \lambda - z$. From $\ker(\lambda\mathbb{1} - A) = \operatorname{im} E_{\{\lambda\}}$ we see that we have eigenvectors iff $\operatorname{im} E_{\{\lambda\}} \ne \{0\}$ which is the case iff $E_{\{\lambda\}} \ne 0$. Then the third part is clear as well. For every $\lambda \in \mathbb{C}$ we have
$$f\chi_{\{\lambda\}} = f(\lambda)\chi_{\{\lambda\}}. \tag{☺}$$
The measurable calculus translates this into $f(A)E_{\{\lambda\}} = f(\lambda)E_{\{\lambda\}}$. If $\phi \in \operatorname{im} E_{\{\lambda\}}$ then (5.1.65) follows. Now assume that $\lambda \in \operatorname{spec}(A)$ is isolated, meaning that $\{\lambda\} \subseteq \operatorname{spec}(A)$ is an *open* subset (in the subspace topology inherited from $\mathbb{C}$). By Remark 5.1.31 we conclude that $E_{\{\lambda\}} \ne 0$. Then part *iii.)* shows that λ is an eigenvalue. Now assume that $\operatorname{spec}(A)$ is countable with distinct points $\lambda_1, \lambda_2, \ldots$. Then the strong σ-additivity of E shows that
$$\phi = \sum_{\lambda_n \in \operatorname{spec}(A)} E_{\{\lambda_n\}}\phi \tag{$\star$}$$
converges in norm (or is a finite sum) for every $\phi \in \mathfrak{H}$. Since the projections $E_{\{\lambda_n\}}$ are pairwise orthogonal by Proposition 5.1.16, *i.)*, we conclude (5.1.66) from ($\star$). While proving (5.1.65) we showed (5.1.67) in (☺) whether λ_n is an eigenvalue or not. Since $f(A) \in \mathfrak{B}(\mathfrak{H})$ is continuous we have from the convergent series ($\star$)
$$f(A)\phi \overset{(\star)}{=} f(A) \sum_{\lambda_n \in \operatorname{spec}(A)} E_{\{\lambda_n\}}\phi = \sum_{\lambda_n \in \operatorname{spec}(A)} f(A)E_{\{\lambda_n\}}\phi \overset{(5.1.65)}{=} \sum_{\lambda_n \in \operatorname{spec}(A)} f(\lambda_n)E_{\{\lambda_n\}}\phi$$
as an either finite sum or a convergent series in norm for every $\phi \in \mathfrak{H}$. Note that the convergence is clearly unconditional as we have this result for every enumeration of $\operatorname{spec}(A)$. Hence the unconditional strong convergence (5.1.68) is shown. ☐

Remark 5.1.35 If $\mathfrak{H}$ is finite-dimensional we know that every spectral value is an eigenvalue simply by counting dimensions: $\lambda\mathbb{1} - A$ is invertible iff $\lambda\mathbb{1} - A$ is surjective iff $\lambda\mathbb{1} - A$ is injective. Thus we conclude that $\operatorname{spec}(A)$ consists of eigenvalues only. Since there can be at most $\dim\mathfrak{H}$ many distinct eigenvalues we are in the situation of part *vi.)* and *vii.)* of the theorem, even with a finite number of eigenvalues. Thus we conclude

$$f(A) = \sum_{n=1}^{N} f(\lambda_n) E_{\{\lambda_n\}} \tag{5.1.69}$$

for a normal operator A in this case. Moreover, the projections $E_{\{\lambda_n\}}$ are pairwise orthogonal and

$$\mathbb{1} = E_{\{\lambda_1\}} + \cdots + E_{\{\lambda_N\}}. \tag{5.1.70}$$

Thus we have indeed recovered the finite-dimensional version of the spectral theorem on diagonalization of normal operators from our more general statements. However, in infinite dimensions this nice situation will even fail, if we have an (orthonormal) basis of eigenvectors: there are easy examples, see Exercise 5.5.6, of bounded operators on $\ell^2(\mathbb{N})$ with an orthonormal basis of eigenvectors, but a continuum as spectrum. Thus even in this seemingly harmless generalization the spectral theorem from finite dimensions has to be modified substantially.

5.1.5 The Polar Decomposition

As a first application of the measurable calculus we want to establish several versions of the polar decomposition of operators on a Hilbert space. The idea is that we seek for a multiplicative decomposition analogous to the polar decomposition $z = \mathrm{e}^{\mathrm{i}\varphi}|z|$ of a non-zero complex number into its phase and its absolute value. The absolute value can easily be defined by our notions of positivity, the phase will be generalized to a unitary and, later on, to a partial isometry.

The first version does not yet use any specific properties of the C^*-algebra $\mathfrak{B}(\mathfrak{H})$ but works for all unital C^*-algebras:

Theorem 5.1.36 (Polar decomposition in C^*-algebra) *Let $\mathscr{A}$ be a unital C^*-algebra and let $a \in \mathscr{A}$ be invertible. Then there exists a unique positive element, namely $|a|$, and a unique unitary element u such that*

$$a = u|a|. \tag{5.1.71}$$

Proof: First we note that if b is positive and satisfies $a = ub$ for some unitary u then $a^*a = b^*u^*ub = b^*b = b^2$ and hence $b = |a|$. Thus $|a|$ is the only possibility to achieve (5.1.71). Now if we want (5.1.71) then necessarily $u = a|a|^{-1}$ since with a also a^*a is invertible and hence $|a|$ is invertible, too. Thus we define u by this relation and compute $u^*u = |a|^{-1}a^*a|a|^{-1} = \mathbb{1}$ since $|a|^{-1}$ is still a Hermitian element and $a^*a = |a|^2$ by the very definition of $|a|$. This show that u is isometric. Since u being a product of two invertible elements is invertible itself, we conclude $u^* = u^{-1}$. □

While this version of the polar decomposition works for a very general algebra, the assumptions of having an invertible element a are too strong for many applications. Moreover, the three elements a, u, and $|a|$ will not commute in general. These deficits can be overcome by the next version of the polar decomposition which uses the measurable spectral calculus of bounded operators on a Hilbert space in an essential way.

Theorem 5.1.37 (Polar decomposition for normal operators) *Let $\mathfrak{H}$ be a Hilbert space and let $A \in \mathfrak{B}(\mathfrak{H})$ be a normal operator. Then there exists a unique positive operator, namely $|A|$, and a not necessarily unique unitary operator U such that*

$$A = U|A|, \tag{5.1.72}$$

and all three operators A, U, and $|A|$ commute.

Proof: Again, the uniqueness of the positive operator follows as before. Now we consider the continuous functions $f\colon z \mapsto |z|$ for $z \in \mathbb{C}$ and the continuous function $g\colon z \mapsto \frac{z}{|z|}$ for $z \in \mathbb{C} \setminus \{0\}$. We extend g in a measurable way to a function on the whole complex plane by setting e.g. $g(0) = 1$. Then $g(z) \in \mathrm{U}(1)$ and we have

$$z = g(z)f(z)$$

for *all* $z \in \mathbb{C}$. Note that g is not continuous but only measurable. Now the bounded measurable calculus for A gives us two operators $U = g(A)$ and $|A| = f(A)$ for which we have $A = U|A|$. Moreover, since $g(z)\overline{g(z)} = 1 = \overline{g(z)}g(z)$ we get $U^*U = \mathbb{1} = UU^*$, i.e. U is unitary. Being functions of the normal operator A, all three operators commute. □

Here we have the nice feature that all the three operators commute. However, the unitary map U is not uniquely defined: we had a choice of how to handle the value $z = 0$ in the definition of g. Any other complex number in $\mathrm{U}(1)$ would yield the same statement. Now the point $z = 0$ in the spectrum of A corresponds to the *kernel* of A which coincides with the kernel of $|A|$: thus on this subspace we can essentially define U as we want without disturbing the polar decomposition. An extreme example is provided by $A = 0$ where $|A| = 0$ and hence *any* unitary will satisfy the requirements of Theorem 5.1.37.

Since there is no canonical way of defining U on $\ker A$ as a *unitary*, the next version of the polar decomposition will simply set U to be zero on this subspace. This will lead to uniqueness again but requires to go beyond unitaries for the phase of A.

As a generalization of unitaries we will need partial isometries. The idea is that we have an orthogonal decomposition of the Hilbert space into two subspaces on one of which the partial isometry is an honest isometry and on the other it will simply be zero. We state the definition in a slightly different way to make it available also in a general *-algebra:

Definition 5.1.38 (Partial isometry) *Let $\mathscr{A}$ be a *-algebra. Then $u \in \mathscr{A}$ is called a partial isometry if $u^*u = p$ is a projection.*

Clearly, any projection p will be a partial isometry since $p^*p = p$ is again a projection.

In a general *-algebra this will not lead to anything particularly interesting. However, with a mild assumption we get the following result:

Lemma 5.1.39 *Let $\mathscr{A}$ be a *-algebra and assume that $a^*a = 0$ implies $a = 0$ for $a \in \mathscr{A}$. Then for any partial isometry $u \in \mathscr{A}$ also u^* is a partial isometry. For the projections $p = u^*u$ and $q = uu^*$ one has*

$$up = qu \quad \text{and} \quad pu^* = u^*q. \tag{5.1.73}$$

Proof: Let $p = u^*u$ and $q = uu^*$. Then q is clearly Hermitian. We note that $q^2 = uu^*uu^* = upu^*$ and hence

$$(q^2 - q)^*(q^2 - q) = upu^*upu^* - upu^*uu^* - uu^*upu^* - uu^*uu^* = 0,$$

since $p = u^*u$ is a projection by assumption. If $\mathscr{A}$ has the specified property then $q = q^2$ follows, showing that u^* is a partial isometry, too. Finally, (5.1.73) is obvious. □

Clearly, any C^*-algebra has the required property for Lemma 5.1.39 by the very definition of the C^*-property of the norm. Thus in a C^*-algebra partial isometries are strongly linked to projections and vice versa, a relation which we shall investigate in some more detail later in Section 7.3.1. While in this general framework the role of the projections p and q for a given partial isometry u remains unclear, for the C^*-algebra $\mathfrak{B}(\mathfrak{H})$ we get a simple geometric interpretation, thereby justifying also the name *partial isometry.* We shall even extend the framework slightly to bounded operators between possibly different Hilbert spaces. We need the following preparatory lemma:

Lemma 5.1.40 *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces and let $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$.*

i.) The operators $A^*A \in \mathfrak{B}(\mathfrak{H})$ and $AA^* \in \mathfrak{B}(\mathfrak{K})$ are positive.

ii.) Let $|A| = \sqrt{A^*A} \in \mathfrak{B}(\mathfrak{H})$ then for all $\phi \in \mathfrak{H}$ one has

$$\big\||A|\phi\big\|_{\mathfrak{H}} = \|A\phi\|_{\mathfrak{K}}. \tag{5.1.74}$$

iii.) The kernel of $|A|$ is given by

$$\ker|A| = \ker A. \tag{5.1.75}$$

Proof: Note that we can not apply our considerations to positive operators immediately, since A is a linear map between two possibly different Hilbert spaces and thus A^*A is not an algebraically positive element of $\mathfrak{B}(\mathfrak{H})$ directly. Nevertheless, for $\phi \in \mathfrak{H}$ we have

$$\langle \phi, A^*A\phi\rangle_{\mathfrak{H}} = \langle A\phi, A\phi\rangle_{\mathfrak{K}} \geq 0. \tag{$*$}$$

Using e.g. Corollary 5.1.14 we see that the spectrum of the normal operator A^*A is concentrated on $[0, \infty)$ and hence A^*A is indeed a positive element of $\mathfrak{B}(\mathfrak{H})$. The analogous argument works for AA^* by exchanging the roles of A and A^*. For the second part, we note that $|A|$ is well-defined as the (positive) square root of the positive operator A^*A. Then $(*)$ shows $\|A\phi\|^2 = \||A|\phi\|^2$ and hence (5.1.74). Then the third part is clear. □

As before we call $|A|$ the absolute value of the operator A. Note that the only new aspect is that we have operators between possibly different Hilbert spaces, see also Exercise 5.5.9 for an alternative proof of this lemma. We use this lemma now for the following characterization of partial isometries:

Proposition 5.1.41 *Let $U \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$ be a bounded operator between two Hilbert spaces. Then the following statements are equivalent:*

i.) The operator U is a partial isometry, i.e. $P = U^*U$ is a projection.

ii.) The operator U^* is a partial isometry, i.e. $Q = UU^*$ is a projection.

iii.) The restriction $U\big|_{(\ker U)^\perp}$ is an isometry.

In this case, we have the following decompositions

$$\mathfrak{H} = \ker U \oplus (\ker U)^\perp \quad \textit{and} \quad \mathfrak{K} = \operatorname{im} U \oplus (\operatorname{im} U)^\perp \tag{5.1.76}$$

with

$$(\ker U^*)^\perp = \operatorname{im} U \quad \textit{and} \quad \operatorname{im} U^* = (\ker U)^\perp. \tag{5.1.77}$$

Moreover, the projections induced by (5.1.76) *are explicitly given by*

$$U^*U = P_{(\ker U)^\perp} = P_{\operatorname{im} U^*} \quad \textit{and} \quad UU^* = P_{(\ker U^*)^\perp} = P_{\operatorname{im} U}. \tag{5.1.78}$$

Proof: Since also for bounded operators between different Hilbert spaces we have the property that $A^*A = 0$ implies $A = 0$, we can proceed as in Lemma 5.1.39 to show that the first and the second statement are equivalent. Thus we have to show the equivalence with the third statement. In general, we know that

$$(\operatorname{im} U^*)^{\mathrm{cl}} = (\ker U)^\perp \quad \text{and} \quad (\operatorname{im} U^*)^\perp = \ker U \tag{$*$}$$

for any operator $U \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$, see Exercise 3.6.17. First, let U be a partial isometry and set $P = U^*U$ as well as $Q = UU^*$. From Lemma 5.1.40, *iii.)*, we conclude that $\ker P = \ker U$ as well as $\ker Q = \ker U^*$. Since a projection is uniquely determined by its kernel, we get $U^*U = P_{(\ker U)^\perp}$ and $UU^* = P_{(\ker U^*)^\perp}$. Moreover, since $\operatorname{im} P \subseteq \operatorname{im} U^*$ by $P = U^*U$ and since on the other hand $\operatorname{im} U^* \subseteq (\ker U)^\perp = \operatorname{im} P$ by $(*)$ and the fact that P is the projection onto $(\ker U)^\perp$, we conclude that $\operatorname{im} U^* = (\ker U)^\perp$. In particular, $\operatorname{im} U^*$ is already closed. Exchanging the roles of U and U^*, this completes the proof of (5.1.76), (5.1.77), and (5.1.78). Finally, let $\phi, \phi' \in (\ker U)^\perp = \operatorname{im} U^*$ be given and write these vectors as $\phi = U^*\chi$, $\phi' = U^*\chi'$. Then

$$\langle U\phi, U\phi' \rangle = \langle UU^*\chi, UU^*\chi' \rangle = \langle \chi, UU^*UU^*\chi' \rangle = \langle \chi, UU^*\chi' \rangle = \langle U^*\chi, U^*\chi' \rangle = \langle \phi, \phi' \rangle$$

shows that U is an isometry on $(\ker U)^\perp$. This shows the implication *i.)* $\implies$ *iii.)*. Conversely, assume *iii.)* and decompose $\phi = \phi_\shortparallel + \phi_\perp$ according to the decomposition $\mathfrak{H} = \ker U \oplus (\ker U)^\perp$. Then

$$\langle \phi_\perp, \psi_\perp \rangle = \langle U\phi_\perp, U\psi_\perp \rangle = \langle \phi_\perp, U^*U\psi_\perp \rangle$$

shows that $U^*U\colon (\ker U)^\perp \longrightarrow \operatorname{im} U^* \subseteq (\ker U)^\perp$ is actually the identity. Indeed, this follows since the inner product restricted to $(\ker U)^\perp$ is still non-degenerate. Moreover, $U^*U\big|_{\ker U} = 0$ is trivial. Thus U^*U is block-diagonal with respect to the decomposition $\mathfrak{H} = \ker U \oplus (\ker U)^\perp$ and coincides with $P_{(\ker U)^\perp}$. In particular, it is a projection and hence *iii.)* $\implies$ *i.)* follows. □

The third part of Proposition 5.1.41 is the motivation for the notion of a partial isometry. Using this we can now formulate the third version of the polar decomposition: we consider an arbitrary bounded operator A between two possibly different Hilbert spaces. We want to have a polar decomposition into the absolute value $|A|$ and a partial isometry U which we would like to be determined uniquely. This is accomplished as follows:

Theorem 5.1.42 (Polar decomposition of bounded operators) *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces and let $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$. Then there exists a unique partial isometry $U \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$ such that*

$$A = U|A| \quad \textit{and} \quad \ker A = \ker U. \tag{5.1.79}$$

PROOF: To construct U we use the decomposition $\mathfrak{H} = \ker A \oplus (\ker A)^\perp$ and set $U\big|_{\ker A} = 0$ as required. On $(\ker A)^\perp$ we proceed as follows: since $|A| = |A|^*$ is Hermitian we have $(\operatorname{im}|A|)^\perp = \ker|A|$ by Exercise 3.6.17 and hence $\operatorname{im}|A| \subseteq (\ker A)^\perp$ is dense. On the image of $|A|$ we define U by

$$U|A|\phi = A\phi$$

and claim that this is well-defined. Indeed, suppose $|A|\phi = |A|\psi$ then $\phi - \psi \in \ker|A| = \ker A$ by Lemma 5.1.40, *iii.)*. Thus this makes U well-defined on $\operatorname{im}|A|$. Moreover, for all vectors in the image we have

$$\langle U|A|\phi, U|A|\psi \rangle = \langle A\phi, A\psi \rangle = \langle \phi, A^*A\psi \rangle = \langle \phi, |A|^2\psi \rangle = \langle |A|\phi, |A|\psi \rangle.$$

Thus U is isometric on $\operatorname{im}|A|$. This way, it extends uniquely to an isometry on $(\operatorname{im}|A|)^{\mathrm{cl}} = (\ker A)^\perp$. Putting these two domains together, we get the definition of U with $\ker U = \ker A$ and U being isometric on $(\ker A)^\perp = (\ker U)^\perp$. Hence U is a partial isometry. The relation $A = U|A|$ holds by construction. Finally, if V is another partial isometry with these properties, then the values of V are fixed on $\operatorname{im}|A|$ and hence on $(\operatorname{im}|A|)^{\mathrm{cl}}$ by continuity as well as on $\ker A = (\operatorname{im}|A|)^\perp$. Hence the uniqueness follows at once. □

Remark 5.1.43 In general, the three operators A, $|A|$, and U from the last version of the polar decomposition do no longer commute. In fact, if $\mathfrak{H} \neq \mathfrak{K}$, such a statement simply makes no sense anymore. From this point of view, the uniqueness of the phase U has a certain price. If $\mathfrak{H} = \mathfrak{K}$ and if A is invertible, then the phases from Theorem 5.1.36 and Theorem 5.1.42 coincide. If in addition A is normal, then they also coincide with the necessarily unique phase from Theorem 5.1.37 and hence all three operators commute.

While in general, the three operators A, $|A|$, and U from the polar decomposition as in Theorem 5.1.42 do not commute, one still has some statement on commutativity. The difficulty is now that the operators can not be composed in an unlimited way for $\mathfrak{H} \neq \mathfrak{K}$.

Lemma 5.1.44 *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces and let $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$, $B = B^* \in \mathfrak{B}(\mathfrak{H})$, and $C = C^* \in \mathfrak{B}(\mathfrak{K})$ be operators with $CA = AB$. Then one has*

$$BA^*A = A^*AB \quad \text{and} \quad CAA^* = AA^*C. \tag{5.1.80}$$

PROOF: This is just a trivial computation. We have

$$BA^*A = B^*A^*A = (AB)^*A = (CA)^*A = A^*C^*A = A^*CA = A^*AB$$

and analogously for the second claim. □

Proposition 5.1.45 *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces and let $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$, $B = B^* \in \mathfrak{B}(\mathfrak{H})$, and $C = C^* \in \mathfrak{B}(\mathfrak{K})$ be operators with $CA = AB$. Then for the polar decomposition $A = U|A|$ as in* (5.1.79) *we have*

$$UB = CU \quad \text{and} \quad B|A| = |A|B. \tag{5.1.81}$$

PROOF: The second statement is clear from (5.1.80) and the Spectral Theorem 5.1.32, *iii.)*. For $\phi \in \operatorname{im}|A|$ we get with $\phi = |A|\psi$

$$CU\phi = CU|A|\psi = CA\psi = AB\psi = U|A|B\psi = UB|A|\psi = UB\phi,$$

and hence $CU = UB$ on $\operatorname{im}|A|$. By continuity this equation still holds on $(\operatorname{im}|A|)^{\mathrm{cl}} = (\ker|A|)^{\perp}$. Next, we consider $\phi \in \ker|A|$. Then $|A|B\phi = B|A|\phi = 0$ shows $B\ker|A| \subseteq \ker|A|$. Since $\ker|A| = \ker A = \ker U$, we see that $CU\phi = 0$ for $\phi \in \ker U$ as well as $UB\phi = 0$. Thus the relation $UB = CU$ also holds on $\ker|A|$, which proves the remaining claim. □

Corollary 5.1.46 *Let $\mathfrak{H}$ be a Hilbert space and let $A \in \mathfrak{B}(\mathfrak{H})$ with polar decomposition $A = U|A|$ as in* (5.1.79).

i.) If $B = B^* \in \mathfrak{B}(\mathfrak{H})$ commutes with A then B commutes with U and $|A|$, too.

ii.) If $B \in \mathfrak{B}(\mathfrak{H})$ commutes with A and with A^*, then B commutes with U and $|A|$, too.

PROOF: For the first part we use Proposition 5.1.45 for $B = C$. For the second part we observe that $[B, A^*] = 0$ is equivalent to $[B^*, A] = 0$. Hence the real and imaginary part of B both commute with A. To those we apply the first part and conclude that the real and the imaginary part of B commute with U and $|A|$. But then also B commutes with U and $|A|$. □

We will see a more conceptual interpretation of this last corollary when we discuss von Neumann algebras in Section **??**.

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5.1.6 Applications in Quantum Mechanics

We shall now outline the interpretation of the spectral theorem in the context of quantum theory. First we consider the more abstract approach of observable algebras: assume that a quantum mechanical system has observables which can be described as (certain) normal elements in a unital C^*-algebra $\mathscr{A}$. We already know that it might take some nontrivial effort to achieve this situation but let us assume this has been done somehow. Then a Hermitian element $a = a^*$ has a compact subset $\operatorname{spec}(a) \subseteq \mathbb{R}$ as spectrum. As usual, we interpret the spectral values as those numbers which, in principle, can be obtained when measuring the observable a. Next, one fixes a state ω of $\mathscr{A}$, i.e. a normalized positive linear functional. As such we get the GNS Hilbert space $\mathfrak{H}_\omega$ on which $\mathscr{A}$ acts via the GNS representation π_ω. Since π_ω is a unital *-homomorphism we know

$$\operatorname{spec}(\pi_\omega(a)) \subseteq \operatorname{spec}(a). \tag{5.1.82}$$

Moreover, since $\pi_\omega(a) = \pi_\omega(a)^*$ we get a spectral measure E_ω defined on $\operatorname{spec}(\pi_\omega(a))$ such that

$$\pi_\omega(a) = \int_{\operatorname{spec}(\pi_\omega(a))} \lambda \, \mathrm{d}E_\omega. \tag{5.1.83}$$

For aesthetic reasons we can view E_ω also as a projection-valued measure on the possibly larger compact subset $\operatorname{spec}(a) \subseteq \mathbb{R}$ by extending it trivially, i.e. by setting $E_{\operatorname{spec}(a)\setminus\operatorname{spec}(\pi_\omega(a))} = 0$. It is an easy check that this requirement fixes a unique projection-valued measure on $\operatorname{spec}(a)$. Then we can write for $b \in \mathscr{A}$

$$\omega_b(a) = \langle \psi_b, \pi_\omega(a)\psi_b \rangle = \int_{\operatorname{spec}(a)} \lambda \, \mathrm{d}\langle \psi_b, E_\omega \psi_b \rangle \tag{5.1.84}$$

with a positive measure $\langle \psi_b, E_\omega \psi_b \rangle$ on $\operatorname{spec}(a)$ normalized to

$$\int_{\operatorname{spec}(a)} 1 \, \mathrm{d}\langle \psi_b, E_\omega \psi_b \rangle = \langle \psi_b, \psi_b \rangle = \omega(b^* b). \tag{5.1.85}$$

Thus taking $b \in \mathscr{A}$ with $\omega(b^*b) = 1$ gives a probability measure on the spectrum such that $\omega_b(a) = \langle \psi_b, \pi_\omega(a)\psi_b \rangle$ is indeed the expectation value of the identity function in the sense of probability measures. In particular, $b = \mathbb{1}$ has this property and gives

$$\omega(a) = \int_{\operatorname{spec}(a)} \lambda \, \mathrm{d}\langle \psi_{\mathbb{1}}, E_\omega \psi_{\mathbb{1}} \rangle. \tag{5.1.86}$$

Thus we arrive at the situation that every state ω induces a probability measure on the spectrum $\operatorname{spec}(a)$ such that the expectation value $\omega(a)$ becomes indeed an expectation value (first moment) in the sense of this probability measure. In quantum theory the probability measure $\langle \psi_{\mathbb{1}}, E_\omega \psi_{\mathbb{1}} \rangle$, which we can denote by μ_ω as in Section 1.4, describes the distribution of spectral values as they are found after successively repeated measurements of a in the state ω. Moreover, the expectation values of powers of a correspond to the higher moments of the probability measure μ_ω. From this point of view we have reached the goals set up in Section 1.4 to complete satisfaction. Note that the measure E_ω can only be formulated using the GNS Hilbert space as we know that for a general C^*-algebra $\mathscr{A}$ there need not to be any nontrivial projections at all.

The second, more familiar approach is to work directly with bounded operators on a fixed Hilbert space $\mathfrak{H}$. In this case, a Hermitian operator $A = A^* \in \mathfrak{B}(\mathfrak{H})$ has a spectral measure E such that

$$A = \int_{\operatorname{spec}(A)} \lambda \, \mathrm{d}E, \tag{5.1.87}$$

and for every $\psi \in \mathfrak{H}$ we have

$$\langle \psi, A\psi \rangle = \int_{\operatorname{spec}(A)} \lambda \, \mathrm{d}\langle \psi, E\psi \rangle, \tag{5.1.88}$$

with a positive measure $\langle \psi, E\psi \rangle$ on $\operatorname{spec}(A)$. Again, for $\|\psi\| = 1$ this measure is normalized to 1 and hence a probability measure. Note that $\langle \psi, E\psi \rangle$ depends only on the class $[\psi] \in \mathbb{P}\mathfrak{H}$ for $\|\psi\| = 1$, i.e. a complex phase $\mathrm{e}^{\mathrm{i}\varphi}$ yields the same measure

$$\langle \mathrm{e}^{\mathrm{i}\varphi}\psi, E\mathrm{e}^{\mathrm{i}\varphi}\psi \rangle = \langle \psi, E\psi \rangle, \tag{5.1.89}$$

see also Exercise 5.5.7. We have the same quantum theoretical interpretation of $\langle \psi, E\psi \rangle$ being the probability distribution of the possible values of the observable A when measured repeatedly in the state ψ. This distribution gives then the correct expectation value by (5.1.88).

5.2 Unbounded Operators

The typical Hamiltonian operators but also the infinitesimal generators of symmetries in quantum mechanics are not continuous: they contain operations like differentiation of wave functions which can clearly not be defined in a straightforward way for all square-integrable functions. Also the potentials contain multiplication operators with unbounded functions, hence not leading to any bounded operator on L^2-spaces. While these may still be just particular features of the examples and of the way they are presented, our investigations in Proposition 4.1.37 showed that there is a more profound difficulty: we have to deal with unbounded operators in all realistic quantum mechanical systems. In this section we will collect some first examples of unbounded operators and discuss their adjoints and closures. Throughout this section $\mathfrak{H}$ will be a Hilbert space, typically infinite-dimensional.

5.2.1 First Examples from Quantum Mechanics

We have seen in Proposition 4.1.37 that there is no unital Banach algebra with two elements P and Q such that $[Q, P] = \mathrm{i}\hbar\mathbb{1}$ unless $\hbar = 0$. Thus, in particular, there are no operators $P, Q \in \mathfrak{B}(\mathfrak{H})$ with this commutation relation. On first sight, this is a very disappointing result as it seems that all the nice C^*-algebraic technology and the results from the Sections 4.2, 4.3, and 5.1 will not find any application to operators relevant in quantum mechanics. Thus we are faced to deal with *unbounded operators* and it will take a certain detour to see that our efforts in Section 5.1 *do* apply also in this case, though however, not in a direct and naive way.

We first have to establish some vocabulary. The following definition is the central notion of this section:

Definition 5.2.1 (Unbounded operator) *An (unbounded) operator in a Hilbert space $\mathfrak{H}$ is a pair $(\operatorname{dom} A, A)$ with a subspace $\operatorname{dom} A \subseteq \mathfrak{H}$ and a linear map $A\colon \operatorname{dom} A \longrightarrow \mathfrak{H}$. The subspace $\operatorname{dom} A$ is called the domain of definition of the operator. An (unbounded) operator in $\mathfrak{H}$ is called densely defined if $\operatorname{dom} A$ is a dense subspace.*

It is a slight abuse of notation that we will speak of an *unbounded operator* even in the case where the linear map $A\colon \operatorname{dom} A \longrightarrow \mathfrak{H}$ happens to be bounded. The important point is that the domain will typically not be the whole Hilbert space. The specification of the domain of definition (or domain for short) of an unbounded operator is a crucial part of the data. By some (quite common) abuse of notation we will refer to A as the unbounded operator if it is clear from the context which domain $\operatorname{dom} A$ belongs to it. However, in this sense a definition like "the quantum mechanical momentum operator is $P = -\mathrm{i}\hbar\frac{\partial}{\partial x}$" is rather meaningless until we specify the domain in $\mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$ on which P should act. Note that the naive and obvious choice to take the whole Hilbert space $\mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$ as domain is not possible for the momentum operator P.

Definition 5.2.2 (Extension) *Let $(\operatorname{dom} A, A)$ and $(\operatorname{dom} B, B)$ be operators in a Hilbert space $\mathfrak{H}$. Then $(\operatorname{dom} B, B)$ is called an extension of $(\operatorname{dom} A, A)$ if*

$$\operatorname{dom} A \subseteq \operatorname{dom} B \tag{5.2.1}$$

and

$$B\big|_{\operatorname{dom} A} = A. \tag{5.2.2}$$

In this case we write $A \subseteq B$ for short.

In the following it will be important to make the domain $\operatorname{dom} A$ of an operator "as large as possible" without destroying the features of A one is interested in. Of course, choosing a vector space basis of $\operatorname{dom} A \subseteq \mathfrak{H}$ and extending it to a vector space basis of $\mathfrak{H}$ by the usual application of Zorn's Lemma always allows us to extend A to a linear map $A\colon \mathfrak{H} \longrightarrow \mathfrak{H}$ in *some* way, e.g. by fixing its values on the remaining basis vectors not in $\operatorname{dom} A$ somehow. But then usually all interesting properties of A will be lost and we end up with just a linear map. Thus it will be the interplay with the inner product which makes an extension interesting.

Remark 5.2.3 (Algebraic features of unbounded operators) While the everywhere defined operators on $\mathfrak{H}$ form an algebra under the usual composition laws this is no longer true for unbounded operators in $\mathfrak{H}$ as we have to take care of the domains. Thus let $(\operatorname{dom} A, A)$, $(\operatorname{dom} B, B)$, and $(\operatorname{dom} C, C)$ be unbounded operators in $\mathfrak{H}$.

i.) For every $z \in \mathbb{C}$ one defines $(\operatorname{dom} zA, zA)$ by $\operatorname{dom} zA = \operatorname{dom} A$ and $zA\colon \phi \mapsto zA\phi$ for $z \neq 0$ but $\operatorname{dom} 0A = \mathfrak{H}$ and hence $0A = 0$ for $z = 0$. We could have also defined $0A$ only on $\operatorname{dom} A$ but the extension of the zero map to all of $\mathfrak{H}$ is only too reasonable to ignore it.

ii.) The addition is more complicated. The best we can do is to take $\operatorname{dom} A + B = \operatorname{dom} A \cap \operatorname{dom} B$ and define $A + B$ on this intersection as usual.

iii.) The product can only be defined on

$$\operatorname{dom} AB = \left\{\phi \in \operatorname{dom} B \;\middle|\; B\phi \in \operatorname{dom} A\right\} \tag{5.2.3}$$

by $(AB)\phi = A(B\phi)$.

It is now an easy check that these new domains are indeed subspaces and hence one gets new unbounded operators. Moreover, we note that

$$A + (B + C) = (A + B) + C, \tag{5.2.4}$$

$$A(BC) = (AB)C, \tag{5.2.5}$$

$$(A + B)C = AC + BC, \tag{5.2.6}$$

but only

$$A(B + C) \supseteq AB + AC. \tag{5.2.7}$$

To see that in (5.2.7) we can have the strict inclusion, consider operators A and B with $B(\operatorname{dom} B) \cap \operatorname{dom} A = \{0\}$ and B injective. Then $\operatorname{dom} AB = \{0\}$. Finally, set $C = -B$. Then $B\phi + C\phi = 0$ for all $\phi \in \operatorname{dom} B = \operatorname{dom} C = \operatorname{dom} B + C$ and hence the left hand side is 0 on the domain $\operatorname{dom} B = \operatorname{dom} A(B + C)$. On the other hand, the right hand side is only defined on $\operatorname{dom} AB + AC = \operatorname{dom} AB \cap \operatorname{dom} AC = \operatorname{dom} AB = \{0\}$.

Up to this point we were just doing linear algebra and consequently, the above definitions all make sense not only for operators in a Hilbert space $\mathfrak{H}$ but in a general vector space. This changes in the following remark:

Remark 5.2.4 If $(\operatorname{dom} A, A)$ is an unbounded operator in $\mathfrak{H}$ such that on $\operatorname{dom} A$ it is actually bounded, i.e.

$$\sup_{\phi \in \operatorname{dom} A \setminus \{0\}} \frac{\|A\phi\|}{\|\phi\|} < \infty, \tag{5.2.8}$$

then A extends uniquely to a continuous operator on the closure $\operatorname{dom} A^{\mathrm{cl}} \subseteq \mathfrak{H}$ of $\operatorname{dom} A$ inside $\mathfrak{H}$. In particular, if A was densely defined, then we have a bounded extension of A to $\mathfrak{H}$. Conversely, if such a bounded extension of A exists then it is necessarily unique by $\operatorname{dom} A^{\mathrm{cl}} = \mathfrak{H}$ and hence A was already bounded on $\operatorname{dom} A$. Thus in this situation we are back to the case of bounded operators. From now on our main interest is in operators $(\operatorname{dom} A, A)$ in $\mathfrak{H}$ which are truly unbounded on $\operatorname{dom} A$, i.e. the supremum in (5.2.8) is *not* finite. Again, we can make the same observations for a general topological vector space and in particular for a Banach space instead of a Hilbert space.

The next example shows that essentially all interesting operators occurring in the Schrödinger formulation of quantum mechanics are unbounded:

Theorem 5.2.5 (Differential operators) *Let $U \subseteq \mathbb{R}^n$ be a non-empty open subset and consider $\mathfrak{H} = \mathrm{L}^2(U, \mathrm{d}^n x)$. Moreover, let*

$$A = \sum_{|\alpha| \le r} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} \tag{5.2.9}$$

be a differential operator of order $\le r$ and with continuous coefficients $A^\alpha \in \mathscr{C}(U)$, defined on the dense domain $\operatorname{dom} A = \mathscr{C}_0^\infty(U) \subseteq \mathrm{L}^2(U, \mathrm{d}^n x)$. *Then A is bounded on* $\operatorname{dom} A$ *iff*

$$\|A^0\|_\infty < \infty \quad \textit{and} \quad A^\alpha = 0 \quad \textit{for} \quad \alpha \ne 0. \tag{5.2.10}$$

PROOF: First we recall that $\mathscr{C}_0^\infty(U)$ is indeed a dense subspace of $\mathrm{L}^2(U, \mathrm{d}^n x)$, see also Appendix **??**. Moreover, A is clearly a well-defined linear map $A\colon \mathscr{C}_0^\infty(U) \longrightarrow \mathscr{C}_0(U) \subseteq \mathrm{L}^2(U, \mathrm{d}^n x)$ and hence a densely defined operator in $\mathrm{L}^2(U, \mathrm{d}^n x)$. Now assume that A satisfies the condition (5.2.10). Then for $\varphi \in \mathscr{C}_0^\infty(U)$ we have

$$\|A\varphi\|^2_{\mathrm{L}^2(U,\mathrm{d}^n x)} = \int_U \overline{A^0(x)\varphi(x)} A^0(x)\varphi(x) \,\mathrm{d}^n x \le \sup_{x \in U} \left|A^0(x)\right|^2 \int_U |\varphi(x)|^2 \,\mathrm{d}^n x = \left\|A^0\right\|^2_\infty \|\varphi\|^2_{\mathrm{L}^2(U,\mathrm{d}^n x)},$$

from which we see that A is bounded with operator norm $\|A\| \le \|A^0\|_\infty$. In fact, a little more effort shows here even equality, see also Exercise 5.5.13. For the converse, we first assume that $r > 0$ and there is one $A^{\alpha_0}(x_0) \ne 0$ with $|\alpha_0| = r$. By continuity, $A^{\alpha_0}(x) \ne 0$ on some open ball $\mathrm{B}_R(x_0) \subseteq U$. We choose a nonzero $\varphi \in \mathscr{C}_0^\infty(U)$ with support $\operatorname{supp}\varphi \subseteq \mathrm{B}_R(x_0)$. For the function $\varphi_k\colon x \mapsto \mathrm{e}^{\mathrm{i}kx}\varphi(x)$ with $k \in \mathbb{R}^n$ we get

$$\begin{aligned}
\|A\varphi_k\|^2 &= \int_U \overline{\sum_{|\alpha| \le r} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}(\mathrm{e}^{\mathrm{i}kx}\varphi(x))} \sum_{|\beta| \le r} A^\beta \frac{\partial^{|\beta|}}{\partial x^\beta}\left(\mathrm{e}^{\mathrm{i}kx}\varphi(x)\right) \mathrm{d}^n x \\
&= \sum_{|\alpha|,|\beta| \le r} \sum_{\gamma \le \alpha} \sum_{\tilde{\gamma} \le \beta} \binom{\alpha}{\gamma}\binom{\beta}{\tilde{\gamma}} \int_U \overline{A^\alpha (\mathrm{i}k)^\gamma \frac{\partial^{|\alpha-\gamma|}\varphi}{\partial x^{\alpha-\gamma}}} A^\beta (\mathrm{i}k)^{\tilde{\gamma}} \frac{\partial^{|\beta-\tilde{\gamma}|}\varphi}{\partial x^{\beta-\tilde{\gamma}}} \,\mathrm{d}^n x,
\end{aligned}$$

by the usual Leibniz rule for higher derivatives. As a function of k this is a polynomial with top degree $2r$. More precisely, the coefficient of $(k^{\alpha_0})^2$ is given by $\int_U |A^{\alpha_0}\varphi|^2 \,\mathrm{d}^n x$ which is nonzero by assumption. Thus we see that letting k grow into the direction determined by the multiindex α_0 makes $\|A\varphi_k\|^2$ diverge to $+\infty$. However, as $\mathrm{e}^{\mathrm{i}kx}$ has absolute value 1 we have $\|\varphi_k\|_{\mathrm{L}^2(U,\mathrm{d}^n x)} = \|\varphi\|_{\mathrm{L}^2(U,\mathrm{d}^n x)}$ for all k. Hence A can not be bounded. Finally, assume that $r = 0$ but A^0 is unbounded on U. Let $x_n \in U$ be a sequence with $|A^0(x_n)| \ge 2n$. Then we have even an open neighbourhood $\mathrm{B}_{r_n}(x_n)$ with $|A^0(x)| \ge n$ for $x \in \mathrm{B}_{r_n}(x_n)$. We choose now a test function $\varphi_n \in \mathscr{C}_0^\infty(U)$ with support $\operatorname{supp}\varphi_n \subseteq \mathrm{B}_{r_n}(x_n)$ and $\|\varphi\|_{\mathrm{L}^2(U,\mathrm{d}^n x)} = 1$. Note that such functions actually exist, see e.g Exercise 2.5.31, *iv.)*, for an explicit construction. Then we get

$$\begin{aligned}
\|A\varphi_n\|^2_{\mathrm{L}^2(U,\mathrm{d}^n x)} &= \int_U \left|A^0(x)\varphi_n(x)\right|^2 \mathrm{d}^n x \\
&= \int_{\mathrm{B}_{r_n}(x_n)} \left|A^0(x)\varphi_n(x)\right|^2 \mathrm{d}^n x \\
&\ge n^2 \int_{\mathrm{B}_{r_n}(x_n)} |\varphi_n(x)|^2 \,\mathrm{d}^n x \\
&= n^2,
\end{aligned}$$

by our condition on the support of φ_n. But this shows that A is unbounded also in this case. $\square$

The typical Hamiltonian operator in quantum mechanics is usually something like

$$H = -\frac{\hbar^2}{2m}\Delta + V, \tag{5.2.11}$$

with the *Laplacian* $\Delta = \sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial x^i x^j}$ with respect to a positive definite metric g and a potential $V \in \mathscr{C}(U)$, acting on wave functions in $\mathrm{L}^2(U, \mathrm{d}^n x)$ where typically $U \subseteq \mathbb{R}^n$ is either all of $\mathbb{R}^n$ or $\mathbb{R}^n$ minus some points where V might be singular. By the theorem we are thus forced to consider unbounded operators in quantum mechanics.

Nevertheless, this statement refers only to continuity properties with respect to the pre-Hilbert space structure of $\mathscr{C}_0^\infty(U)$: there are other topologies on the test functions $\mathscr{C}_0^\infty(U)$ for which differential operators with smooth coefficients are indeed continuous operators. These questions are addressed in distribution theory and will not be our concern here, see e.g. [23, 49, 58] for more information on distribution theory.

5.2.2 The Adjoint

For spectral calculus we have seen that the normal or even Hermitian operators play a crucial role. To formulate a condition like $A = A^*$ we need a notion for the adjoint of an unbounded operator $(\operatorname{dom} A, A)$. The idea is that we want to make sense out of the relation

$$\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle \tag{5.2.12}$$

for as many vectors $\phi, \psi \in \mathfrak{H}$ as possible. First it is clear that we only can use $\psi \in \operatorname{dom} A$. If we will find a reasonable definition of $A^*\phi$ then the value will be uniquely determined by the right hand side if $\operatorname{dom} A \subseteq \mathfrak{H}$ is a *dense* domain. Otherwise we could add vectors from $\operatorname{dom} A^\perp$ to $A^*\phi$ without changing the numerical value of the right hand side. Thus, from now on, we shall always assume that $(\operatorname{dom} A, A)$ is a *densely* defined operator. Then the following definition is now well motivated:

Definition 5.2.6 (Adjoint operator) *Let* $(\operatorname{dom} A, A)$ *be a densely defined operator in* $\mathfrak{H}$. *Then the adjoint operator* $(\operatorname{dom} A^*, A^*)$ *of* $(\operatorname{dom} A, A)$ *is defined by*

$$\operatorname{dom} A^* = \left\{ \phi \in \mathfrak{H} \;\middle|\; \textit{there exists a } \chi \in \mathfrak{H} \textit{ with } \langle \phi, A\psi \rangle = \langle \chi, \psi \rangle \textit{ for all } \psi \in \operatorname{dom} A \right\} \tag{5.2.13}$$

and $A^*\phi = \chi$ *with* χ *such that* $\langle \phi A\psi \rangle = \langle \chi, \psi \rangle$ *for all* $\psi \in \operatorname{dom} A$.

Proposition 5.2.7 *Let* $(\operatorname{dom} A, A)$ *be a densely defined operator in* $\mathfrak{H}$.

i.) The adjoint $(\operatorname{dom} A^*, A^*)$ *is a well-defined operator in* $\mathfrak{H}$.

ii.) One has

$$\operatorname{dom} A^* = \left\{ \phi \in \mathfrak{H} \;\middle|\; \operatorname{dom} A \ni \psi \;\mapsto\; \langle \phi, A\psi \rangle \textit{ is continuous} \right\}. \tag{5.2.14}$$

iii.) If $(\operatorname{dom} B, B)$ *is another operator in* $\mathfrak{H}$ *such that*

$$\langle \phi, A\psi \rangle = \langle B\phi, \psi \rangle \tag{5.2.15}$$

for all $\phi \in \operatorname{dom} B$ *and* $\psi \in \operatorname{dom} A$ *then* $B \subseteq A^*$.

iv.) One has

$$\ker A^* = (\operatorname{im} A)^\perp. \tag{5.2.16}$$

In particular, $\ker A^*$ *is closed.*

PROOF: It is clear that $\operatorname{dom} A^*$ is a subspace as the conditions in (5.2.13) are linear. Since we assume $\operatorname{dom} A$ to be dense there can be at most one χ with $\langle \phi, A\psi \rangle = \langle \chi, \psi \rangle$ for all $\psi \in \operatorname{dom} A$. This makes A^* as map $\operatorname{dom} A^* \longrightarrow \mathfrak{H}$ well-defined. Then it is easy to see that A^* is actually linear. For the second part, assume $\psi \mapsto \langle \phi, A\psi \rangle$ is continuous. Then this linear functional has a unique continuous linear extension to $\operatorname{dom} A^{\mathrm{cl}} = \mathfrak{H}$. By Riesz' Theorem 3.2.11 it is of the form $\psi \mapsto \langle \chi, \psi \rangle$ with a unique $\chi \in \mathfrak{H}$. Hence $\phi \in \operatorname{dom} A^*$ follows. The converse is obvious. For the third part, let $(\operatorname{dom} B, B)$ be such an operator and let $\phi \in \operatorname{dom} B$. Then $\phi \in \operatorname{dom} A^*$ is clear by either of the characterizations (5.2.13) and (5.2.14) of $\operatorname{dom} A^*$. By the density of $\operatorname{dom} A$ we conclude that $B\phi = A^*\phi$. Finally, let $\phi \in \operatorname{dom} A^*$ with $A^*\phi = 0$ be given. Then for all $\psi \in \operatorname{dom} A$ we have $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle = 0$ showing $\phi \in (\operatorname{im} A)^\perp$. Conversely, assume $\phi \in (\operatorname{im} A)^\perp$. Then $\psi \mapsto \langle \phi, A\psi \rangle = 0$ is clearly continuous giving $\phi \in \operatorname{dom} A^*$. Moreover, $A^*\phi = 0$ is clear by the defining relation (5.2.12) as $\operatorname{dom} A$ is dense. □

The second statement can be interpreted as follows. Even though $A\colon \operatorname{dom} A \ni \psi \mapsto A\psi \in \mathfrak{H}$ is not continuous, taking the scalar product with $\phi \in \operatorname{dom} A^*$ makes the composed map $\psi \mapsto \langle \phi, A\psi \rangle$ continuous. Thus ϕ "cures" the unboundedness of A. The third part simply means that the definition of the adjoint operator is optimal with respect to the condition (5.2.12): any (possibly existing) proper extension of A^* would destroy this property. Concerning the last part we note that the image of A will not be closed in general, even if the operator is bounded: take a dense proper subspace and consider the unit operator on it. However, it generalizes the properties of bounded operators according to Exercise 3.6.17 to this particular case of unbounded operators.

In the next proposition we investigate the behaviour of the adjoint with respect to the algebraic manipulations as in Remark 5.2.3.

Proposition 5.2.8 *Let* $(\operatorname{dom} A, A)$ *and* $(\operatorname{dom} B, B)$ *be densely defined operators in* $\mathfrak{H}$ *and let* $z \in \mathbb{C}$.

i.) One has $(zA)^* = \overline{z}A^*$.

ii.) Suppose $A + B$ *is still densely defined. Then one has*

$$(A + B)^* \supseteq A^* + B^*. \tag{5.2.17}$$

iii.) Suppose AB *is still densely defined. Then one has*

$$(AB)^* \supseteq B^*A^*. \tag{5.2.18}$$

If in addition $A \in \mathfrak{B}(\mathfrak{H})$ *is a bounded operator then we have*

$$(AB)^* = B^*A^*. \tag{5.2.19}$$

iv.) If $A \subseteq B$ *then* $B^* \subseteq A^*$.

PROOF: The first part is clear. Thus assume $\operatorname{dom} A + B = \operatorname{dom} A \cap \operatorname{dom} B$ is still dense so we have an adjoint operator $(A + B)^*$. Let $\phi \in \operatorname{dom} A^* + B^* = \operatorname{dom} A^* \cap \operatorname{dom} B^*$ and $\psi \in \operatorname{dom} A + B$. Then we have $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$ as well as $\langle \phi, B\psi \rangle = \langle B^*\phi, \psi \rangle$ and hence $\langle \phi, (A+B)\psi \rangle = \langle A^*\phi + B^*\phi, \psi \rangle$. This shows $\phi \in \operatorname{dom}(A + B)^*$ and also $A^*\phi + B^*\phi = (A + B)^*\phi$, by uniqueness. This proves (5.2.17). Now assume that AB is densely defined and let $\psi \in \operatorname{dom} AB$, i.e. $\psi \in \operatorname{dom} B$ with $B\psi \in \operatorname{dom} A$. Moreover, let $\phi \in \operatorname{dom} B^*A^*$. Then we have $\langle \phi, AB\psi \rangle = \langle A^*\phi, B\psi \rangle = \langle B^*A^*\phi, \psi \rangle$ showing that $\phi \in \operatorname{dom}(AB)^*$ and $(AB)^*\phi = B^*A^*\phi$. If in addition A is bounded, then AB is densely defined and $\operatorname{dom} A^* = \mathfrak{H}$ as usual. Then $\operatorname{dom} B^*A^*$ consists of all those $\phi \in \mathfrak{H}$ with $A^*\phi \in \operatorname{dom} B^*$. Conversely, $\operatorname{dom} AB$ is just $\operatorname{dom} B$. Thus $\operatorname{dom}(AB)^*$ consists of those vectors ϕ with $\psi \mapsto \langle \phi, AB\psi \rangle = \langle A^*\phi, B\psi \rangle$ being continuous for all $\psi \in \operatorname{dom} B$. But this is precisely the condition $A^*\phi \in \operatorname{dom} B^*$. Hence (5.2.19) follows. For the last part, let $(\operatorname{dom} B, B)$ be an extension of $(\operatorname{dom} A, A)$. Then the continuity of $\psi \mapsto \langle \phi, B\psi \rangle$ for all $\psi \in \operatorname{dom} B \supseteq \operatorname{dom} A$ implies the continuity of $\psi \mapsto \langle \phi, A\psi \rangle$ for all $\psi \in \operatorname{dom} A$ since here $B\big|_{\operatorname{dom} A} = A$ by assumption. Hence $\phi \in \operatorname{dom} B^*$ implies $\phi \in \operatorname{dom} A^*$ which gives $\operatorname{dom} B^* \subseteq \operatorname{dom} A^*$. Then $A^*\big|_{\operatorname{dom} B^*} = B^*$ is clear. □

In general, the domain $\operatorname{dom} A^*$ can be very small, in fact, too small to be of any interest as the following example shows:

Example 5.2.9 Let $\mathfrak{H}$ be a separable infinite-dimensional Hilbert space and fix a Hilbert basis $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$. Let $\{\mathrm{f}_{nm}\}_{n,m\in\mathbb{N}}$ be another Hilbert basis which we shall label by $\mathbb{N}\times\mathbb{N}$. Then define an unbounded operator $(\operatorname{dom} A, A)$ by

$$\operatorname{dom} A = \operatorname{span}_{\mathbb{C}}\{\mathrm{f}_{nm}\}_{n,m\in\mathbb{N}\times\mathbb{N}}, \tag{5.2.20}$$

and let the mapping be the linear extension of

$$A\mathrm{f}_{nm} = m\mathrm{e}_n. \tag{5.2.21}$$

Since the $\{\mathrm{f}_{nm}\}_{n,m\mathbb{N}\times\mathbb{N}}$ form a vector space basis of their span this determines a unique linear map $A\colon \operatorname{dom} A \longrightarrow \mathfrak{H}$. Moreover, $\operatorname{dom} A$ is dense in $\mathfrak{H}$ as we have fixed a Hilbert basis. It is now a simple argument, see Exercise 5.5.12, that the domain of A^* is trivial, i.e.

$$\operatorname{dom} A^* = \{0\}. \tag{5.2.22}$$

From this example we see that a densely defined unbounded operator may have no interesting adjoint at all. However, the adjointable operators on a pre-Hilbert space behave much nicer: we denote by dom a pre-Hilbert space and let $\mathfrak{H} = \widehat{\operatorname{dom}}$ be its completion to a Hilbert space. Recall that, by Definition 1.3.2, a (necessarily linear) map $A\colon \operatorname{dom} \longrightarrow \operatorname{dom}$ is called adjointable if there is a (necessarily linear) map $A^{*_{\operatorname{dom}}}\colon \operatorname{dom} \longrightarrow \operatorname{dom}$ with $\langle \phi, A\psi\rangle = \langle A^{*_{\operatorname{dom}}}\phi, \psi\rangle$ for all $\phi, \psi \in \operatorname{dom}$. Note that we indicate the adjoint in the sense of Definition 1.3.2 with $^{*_{\operatorname{dom}}}$ to distinguish it from the adjoint in the sense of Definition 5.2.6. The set of adjointable operators on dom is then a unital *-algebra $\mathfrak{B}(\operatorname{dom})$ with *-involution $^{*_{\operatorname{dom}}}$. Viewing A as an operator $(\operatorname{dom} A = \operatorname{dom}, A)$ in $\mathfrak{H}$ gives then the following result:

Proposition 5.2.10 *Let* $\operatorname{dom} \subseteq \mathfrak{H}$ *be a dense subspace of a Hilbert space and let* $A \in \mathfrak{B}(\operatorname{dom})$.

i.) The map A can be viewed as a densely defined operator in $\mathfrak{H}$ by $\operatorname{dom} A = \operatorname{dom}$ *and* $A\colon \operatorname{dom} A \longrightarrow \mathfrak{H}$ *unchanged.*

ii.) The adjoint $(\operatorname{dom} A^*, A^*)$ *of* $(\operatorname{dom} A, A)$ *is an extension of the operator* $(\operatorname{dom}, A^{*_{\operatorname{dom}}})$ *with* $A^{*_{\operatorname{dom}}} \in \mathfrak{B}(\operatorname{dom})$ *being the adjoint map of A in the sense of Definition 1.3.2.*

iii.) The adjoint $(\operatorname{dom} A^*, A^*)$ *is densely defined.*

PROOF: The first part we already discussed. For the second we observe that for all $\phi, \psi \in \operatorname{dom}$ we have the equality

$$\langle \phi, A\psi\rangle = \langle A^{*_{\operatorname{dom}}}\phi, \psi\rangle.$$

Since $\operatorname{dom} A = \operatorname{dom}$ by definition, this implies that $\phi \in \operatorname{dom} A^*$ with $A^*\phi = A^{*_{\operatorname{dom}}}\phi$. Hence $(\operatorname{dom}, A^{*_{\operatorname{dom}}}) \subseteq (\operatorname{dom} A^*, A^*)$ follows. Since dom is dense, also $\operatorname{dom} A^* \supseteq \operatorname{dom}$ is dense. ☐

We have here a certain clash of notations concerning what we want to call the adjoint: for $A \in \mathfrak{B}(\operatorname{dom})$ we have the adjoint $A^{*_{\operatorname{dom}}} \in \mathfrak{B}(\operatorname{dom})$ in the sense of Definition 1.3.2 and the adjoint A^* in the sense of Definition 5.2.6. According to the proposition we see that

$$A^{*_{\operatorname{dom}}} \subseteq A^*. \tag{5.2.23}$$

In general, and we will see more explicit examples later, this is a proper extension. One should also note that for two operators $A, B \in \mathfrak{B}(\operatorname{dom})$ the adjoints $A^{*_{\operatorname{dom}}}$ and $B^{*_{\operatorname{dom}}}$ are defined on the common domain dom and enjoy simple algebraic properties: $\mathfrak{B}(\operatorname{dom})$ is a honest *-algebra. However, the true adjoints A^* and B^* may have domains $\operatorname{dom} A^*$ and $\operatorname{dom} B^*$, respectively, which do *not* coincide. They only have a dense intersection $\operatorname{dom} \subseteq \operatorname{dom} A^* \cap \operatorname{dom} B^*$. Also the algebraic relations between A^* and B^* are now subject to the general constructions in Remark 5.2.3.

Corollary 5.2.11 *Let $\mathscr{A}$ be a *-algebra and let $\pi\colon \mathscr{A} \longrightarrow \mathfrak{B}(\operatorname{dom})$ be a *-representation on a pre-Hilbert space* dom *. Viewing* $(\operatorname{dom}, \pi(a))$ *as densely defined operator in the Hilbert space completion* $\mathfrak{H}$ *of* dom *these operators have densely defined adjoints* $(\operatorname{dom}\pi(a)^*, \pi(a)^*)$ *for all* $a \in \mathscr{A}$ *with*

$$\pi(a^*) \subseteq \pi(a)^*. \tag{5.2.24}$$

Thus the *-representations of *-algebras will, on one hand, yield unbounded operators in general, but, on the other hand, they do not behave as bad as the one in Example 5.2.9: they automatically have densely defined adjoints. This gives some hope that the *-representation theory of *-algebras as defined in Chapter 1 is accessible even though we have to deal with unbounded operators in general. Nevertheless, the theory is much more involved than the nice framework of *-representations of C^*-algebras where we can rely on Theorem 4.4.24. However, there exists a well-developed representation theory also in this much more technical setting. We refer to the monograph of Schmüdgen [51] for further reading.

5.2.3 Closed Operators

The pathologies concerning adjoints as they can occur for general unbounded operators do not arise for closed operators. We recall that for a linear map $A\colon \mathfrak{H} \longrightarrow \mathfrak{H}$ defined on the whole Hilbert space the graph $\operatorname{graph}(A)$ is a closed subspace iff A is continuous, i.e. $A \in \mathfrak{B}(\mathfrak{H})$. This is precisely the Closed Graph Theorem 2.3.20 for Banach spaces. For an operator $(\operatorname{dom} A, A)$ in $\mathfrak{H}$ we will view its graph always as a subspace of $\mathfrak{H} \oplus \mathfrak{H}$ in the following. Thus it is reasonable to believe that the following definition provides a good replacement for continuity:

Definition 5.2.12 (Closed operator) *Let* $(\operatorname{dom} A, A)$ *be a not necessarily densely defined operator in* $\mathfrak{H}$*.*

i.) *The operator A is called closed if* $\operatorname{graph}(A)$ *is a closed subspace of* $\mathfrak{H} \oplus \mathfrak{H}$*.*

ii.) *The operator A is called closable if it has a closed extension.*

iii.) *If A is closable then the closure A^{cl} (also denoted by $\overline{A}$) is the smallest closed extension of A.*

Remark 5.2.13 (Closed Operator) Let $(\operatorname{dom} A, A)$ be an operator in $\mathfrak{H}$.

i.) The graph $\operatorname{graph}(A)$ of A allows to reconstruct A completely: if we denote by

$$\operatorname{pr}_{1/2}\colon \mathfrak{H} \oplus \mathfrak{H} \longrightarrow \mathfrak{H} \tag{5.2.25}$$

the projections onto the first and second summand then $\operatorname{dom} A = \operatorname{pr}_1(\operatorname{graph}(A))$ and for $\phi \in \operatorname{dom} A$ we have $A\phi = \operatorname{pr}_2(\Psi)$ where $\Psi \in \operatorname{graph}(A)$ is the unique vector in the graph of A with $\operatorname{pr}_1(\Psi) = \phi$. Note that a subspace $U \subseteq \mathfrak{H} \oplus \mathfrak{H}$ is a graph iff for every $\Psi \in U$ the projection $\operatorname{pr}_1(\Psi)$ determines Ψ, i.e. $\operatorname{pr}_1\big|_U$ is injective.

ii.) The operator A is closed iff for all sequences $\phi_n \in \operatorname{dom} A$ with $\phi_n \longrightarrow \phi$ and $A\phi_n \longrightarrow \psi$ we have $\phi \in \operatorname{dom} A$ and $A\phi = \psi$.

iii.) If A is closable we consider all its closed extensions $\{\operatorname{dom} A_i, A_i\}_{i\in I}$. With the criterion from *i.)* it is clear that the intersection

$$\bigcap_{i\in I} \operatorname{graph}(A_i) \subseteq \operatorname{graph}(A) \tag{5.2.26}$$

is again a graph of some operator. Since it contains $\operatorname{graph}(A)$ this operator is an extension of A. Since the intersection of closed subspaces is again closed, this extension is closed itself. Thus we have found the obviously smallest closed extension of A, i.e. the closure A^{cl}. This shows that the closure of a closable operator is well-defined. Moreover, for a closable operator A we clearly have $\operatorname{graph}(A)^{\mathrm{cl}} \subseteq \operatorname{graph}(A^{\mathrm{cl}})$. Conversely, if $(\phi_n, A\phi_n) \in \operatorname{graph}(A)$ is a sequence converging to some $(\phi, \psi) \in \operatorname{graph}(A)^{\mathrm{cl}}$, we know that $\psi = A^{\mathrm{cl}}(\phi)$ since A^{cl} is a closed extension of A. Hence $(\phi, \psi) \in \operatorname{graph}(A^{\mathrm{cl}})$ shows the reverse inclusion and we get

$$\operatorname{graph}(A^{\mathrm{cl}}) = \operatorname{graph}(A)^{\mathrm{cl}}. \tag{5.2.27}$$

iv.) There are (even densely defined) operators $(\operatorname{dom} A, A)$ in $\mathfrak{H}$ which are not closable as soon as $\mathfrak{H}$ is infinite-dimensional. We shall see later that the operator in Example 5.2.9 is not closable. Note that the closure of the subspace $\operatorname{graph}(A)$ is always a closed subspace. Hence an operator is non-closable iff the right hand side of (5.2.27) is not the graph of an operator.

v.) Using the definition

$$\langle \psi, \phi \rangle_A = \langle \psi, \phi \rangle + \langle A\psi, A\phi \rangle \tag{5.2.28}$$

for $\psi, \phi \in \operatorname{dom} A$, we can turn $\operatorname{dom} A$ into a pre-Hilbert space with inner product $\langle \,\cdot\,, \,\cdot\, \rangle_A$. Indeed, (5.2.28) is clearly a positive-definite sesquilinear inner product on $\operatorname{dom} A$. Then the map

$$(\operatorname{dom} A, \langle \,\cdot\,, \,\cdot\, \rangle_A) \ni \phi \ \mapsto \ (\phi, A\phi) \in \mathfrak{H} \oplus \mathfrak{H} \tag{5.2.29}$$

is isometric with respect to the canonical orthogonal sum structure of $\mathfrak{H} \oplus \mathfrak{H}$. The image (5.2.29) is precisely the graph $\operatorname{graph}(A)$. Hence we conclude that A is closed iff $(\operatorname{dom} A, \langle \,\cdot\,, \,\cdot\, \rangle_A)$ is already complete, i.e. a Hilbert space. The corresponding norm

$$\|\phi\|_A = \sqrt{\langle \phi, \phi \rangle + \langle A\phi, A\phi \rangle} \tag{5.2.30}$$

is also called the *graph norm* on $\operatorname{dom} A$. Note that $A\colon \operatorname{dom} A \longrightarrow \mathfrak{H}$ is continuous when we endow $\operatorname{dom} A$ with the graph norm instead of the original Hilbert space norm.

vi.) The kernel of a closed operator is a closed subspace. Indeed, this is immediate from the definition: if $\phi_n \in \ker A \subseteq \operatorname{dom} A$ converge to $\phi \in \mathfrak{H}$ then $(\phi_n, 0) \in \operatorname{graph}(A)$ converge to $(\phi, 0)$ which is therefore in the graph of A and $A\phi = 0$. This is a very pleasant feature of closed operators when it comes to spectral analysis. Note that one can define unbounded operators with non-closed kernels very easily by choosing a non-closed subspace and a (non-orthogonal) complementary subspace to it. The (non-orthogonal) projection onto the complementary subspace would provide an example.

We shall now investigate the relation between closability and closure on the one hand and the adjoint on the other hand. To this end we consider the canonical *complex structure*

$$J\colon \mathfrak{H} \oplus \mathfrak{H} \ni (\phi, \psi) \ \mapsto \ (-\psi, \phi) \in \mathfrak{H} \oplus \mathfrak{H}. \tag{5.2.31}$$

A simple check shows that this linear map satisfies

$$J^* = J^{-1} \quad \text{and} \quad J^2 = -\operatorname{id}_{\mathfrak{H}\oplus\mathfrak{H}}, \tag{5.2.32}$$

i.e. J is unitary and behaves like multiplication with i if we interpret $\mathfrak{H} \oplus \mathfrak{H}$ as a decomposition into real and imaginary part, see also Exercise 5.5.14.

We can now formulate the following principal result on the relation between closure and adjoints:

Theorem 5.2.14 (Adjoint and closure) *Let $(\operatorname{dom} A, A)$ be a densely defined operator in $\mathfrak{H}$.*

i.) The adjoint operator $(\operatorname{dom} A^*, A^*)$ is a closed operator and one has

$$\operatorname{graph}(A^*) = (J(\operatorname{graph}(A)))^{\perp}. \tag{5.2.33}$$

ii.) The operator $(\operatorname{dom} A, A)$ is closable iff $\operatorname{dom} A^*$ is dense. In this case we have

$$A^{\mathrm{cl}} = A^{**}. \tag{5.2.34}$$

iii.) If $(\operatorname{dom} A, A)$ *is closable, then*

$$(A^{\mathrm{cl}})^* = A^*. \tag{5.2.35}$$

PROOF: As the orthogonal complement of an arbitrary subspace is always closed, we only have to show (5.2.33) for the first part. Thus let $(\phi, \psi) \in \mathfrak{H} \oplus \mathfrak{H}$. Then we have $(\phi, \psi) \in (J(\operatorname{graph}(A)))^\perp$ iff for all $\chi \in \operatorname{dom} A$ we have

$$0 = \langle (\phi, \psi), J(\chi, A\chi) \rangle = \langle (\phi, \psi), (-A\chi, \chi) \rangle = -\langle \phi, A\chi \rangle + \langle \psi, \chi \rangle,$$

i.e. $\langle \psi, \chi \rangle = \langle \phi, A\chi \rangle$. But this precisely means $\phi \in \operatorname{dom} A^*$ and $\psi = A^*\phi$. For the second part, we first note that A^{**} can only be define if $\operatorname{dom} A^*$ is dense. For the closure of the graph of A we have

$$\operatorname{graph}(A)^{\mathrm{cl}} \stackrel{(a)}{=} \operatorname{graph}(A)^{\perp\perp} \stackrel{(b)}{=} (J^2 \operatorname{graph}(A))^{\perp\perp} \stackrel{(c)}{=} \Big(J\Big((J \operatorname{graph}(A))^\perp\Big)\Big)^\perp \stackrel{(d)}{=} (J(\operatorname{graph}(A^*)))^\perp, \quad (*)$$

where we used the general fact that the closure coincides with the double orthogonal complement according to Theorem 3.2.1, *vi.)*, in (a), the fact that $J^2 = -\operatorname{id}$ in (b), the unitarity of J in (c), and the first part in (d). Thus we can iterate this if $\operatorname{dom} A^*$ is dense and get by the first part

$$\operatorname{graph}(A)^{\mathrm{cl}} = \operatorname{graph}(A^{**}).$$

Thus the closure of $\operatorname{graph}(A)$ is again a graph, namely the graph of A^{**}. Hence A is closable with closure given by $A^{\mathrm{cl}} = A^{**}$ thanks to (5.2.27). Conversely, assume that $\operatorname{dom} A^*$ is not dense and choose a vector $\psi \in \operatorname{dom} A^{*\perp}$ with $\psi \neq 0$. This means that $\langle (\psi, 0), (\chi, A^*\chi) \rangle = \langle \psi, \chi \rangle = 0$ for all $\chi \in \operatorname{dom} A^*$ and thus $(\psi, 0) \in \operatorname{graph}(A^*)^\perp$. This implies $J(\psi, 0) \in J(\operatorname{graph}(A^*)^\perp) = (J(\operatorname{graph}(A^*)))^\perp$ by the unitarity of J. Since $J(\psi, 0) = (0, \psi)$ with $\psi \neq 0$ we see that $(J(\operatorname{graph}(A^*)))^\perp$ can not be a graph. By $(*)$ this means that $\operatorname{graph}(A)^{\mathrm{cl}}$ is not the graph of an operator and hence A is not closable. The last part is now simple, we have

$$A^* \stackrel{i.)}{=} (A^*)^{\mathrm{cl}} \stackrel{(5.2.34)}{=} (A^*)^{**} = (A^{**})^* = (A^{\mathrm{cl}})^*. \qquad \square$$

Since we are interested in operators with a reasonable adjoint it is clear that at least we want to require $\operatorname{dom} A^*$ to be dense again. Thus we are led to closable operators naturally. In particular, the theorem shows that the operator from Example 5.2.9 is very far from being closable. The following corollary is obvious:

Corollary 5.2.15 *Let* $(\operatorname{dom} A, A)$ *be a closed and densely defined operator in* $\mathfrak{H}$. *Then*

$$\mathfrak{H} \oplus \mathfrak{H} = J(\operatorname{graph}(A)) \oplus \operatorname{graph}(A^*) \tag{5.2.36}$$

is an orthogonal decomposition.

While densely defined operators need not to be closable at all, the operators arising from *-representations of *-algebras are always closable. This is a very important feature which shows that the algebraic structure of a *-algebra and hence also of its image under a *-representation already contain certain analytic properties:

Corollary 5.2.16 *Let* $\mathscr{A}$ *be a* **-algebra and let* $\pi\colon \mathscr{A} \longrightarrow \mathfrak{B}(\mathfrak{D})$ *be a* **-representation of* $\mathscr{A}$ *on a pre-Hilbert space* $\mathfrak{D}$ *with Hilbert space completion* $\widehat{\mathfrak{D}} = \mathfrak{H}$. *Then all the operators* $(\operatorname{dom}, \pi(a))$ *for* $a \in \mathscr{A}$ *are closable.*

PROOF: Indeed, by Corollary 5.2.11 we have densely defined adjoints $(\operatorname{dom} \pi(a)^*, \pi(a)^*) \supseteq (\operatorname{dom}, \pi(a^*))$.$\square$ ▮

As an application of this idea, the next example shows that all the typical operators from quantum mechanics are closable:

Example 5.2.17 (Differential operators are closable) Let $A^\alpha \in \mathscr{C}^\infty(\mathbb{R}^n)$ be smooth functions for multiindices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq r$. Then the differential operator

$$A = \sum_{|\alpha| \leq r} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} \tag{5.2.37}$$

with domain $\operatorname{dom} A = \mathscr{C}_0^\infty(\mathbb{R}^n) \subseteq \mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ is closable. Its adjoint A^* is an extension of the differential operator

$$\sum_{|\alpha| \leq r} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \overline{A^\alpha} \subseteq A^*. \tag{5.2.38}$$

This follows from an elementary integration by parts needed to show

$$\langle \phi, A\psi \rangle = \left\langle \sum\nolimits_{|\alpha| \leq r} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} (\overline{A^\alpha} \phi), \psi \right\rangle. \tag{5.2.39}$$

Note however, that the precise domain $\operatorname{dom} A^*$ of the adjoint is much more complicated to describe. In general, it will be strictly larger than $\mathscr{C}_0^\infty(\mathbb{R}^n)$. Also the closure of A will be a proper extension of A and $\operatorname{dom} A^{\mathrm{cl}}$ will be difficult to compute in general. It depends on the detailed properties of the coefficient functions A^α of A.

e: Find other fops on other , less regular functions etc

In general, it will be difficult to find the precise domain of the closure of a closable operator. Thus one is interested in smaller subspaces of $\operatorname{dom} A^{\mathrm{cl}}$ which allow to reconstruct the closure. This leads to the following definition:

Definition 5.2.18 (Core) *Let $(\operatorname{dom} A, A)$ be a closed operator in $\mathfrak{H}$. Then a subspace $\mathfrak{D} \subseteq \operatorname{dom} A$ is called a core of A if the closure of the restriction of A to $\mathfrak{D}$ is again A, i.e.*

$$\left(A\big|_{\mathfrak{D}}\right)^{\mathrm{cl}} = A. \tag{5.2.40}$$

Note that the restriction $A\big|_{\mathfrak{D}}$ has at least one closed extension, namely A, Thus for every subspace $\mathfrak{D} \subseteq \operatorname{dom} A$ the operator $(\mathfrak{D}, A\big|_{\mathfrak{D}})$ is closable with

$$\left(A\big|_{\mathfrak{D}}\right)^{\mathrm{cl}} \subseteq A. \tag{5.2.41}$$

Hence a core is large enough to recover all of A. We shall see examples of cores later in Section 5.4.4 in particular in Theorem 5.4.38

5.2.4 The Spectrum of Closed Operators

We shall now discuss the notion of spectral values for a closed operator. For an operator $(\operatorname{dom} A, A)$ in $\mathfrak{H}$ we can clearly ask whether $z - A\colon \operatorname{dom} A \longrightarrow \mathfrak{H}$ is injective. Hence the notion of eigenvalues is not problematic. Moreover, for a closed operator, the kernel is a closed subspace by Remark 5.2.13, *vi.)*. Hence we get orthogonal projections onto the eigenspaces of a closed operator. Concerning the surjectivity, and hence the notion of spectral values which are not eigenvalues, things are slightly more subtle. In particular, why should we require surjectivity of $z - A$ if A is not even defined on all of $\mathfrak{H}$? To approach this question, we first define the inverse of an operator in $\mathfrak{H}$ in the spirit of not everywhere defined operators as follows: suppose that A is injective. Then

$$A\colon \operatorname{dom} A \longrightarrow \operatorname{im}(A) \subseteq \mathfrak{H} \tag{5.2.42}$$

is a linear bijection. This way, we obtain an inverse map $A^{-1}\colon \operatorname{im}(A) \longrightarrow \operatorname{dom} A \subseteq \mathfrak{H}$ with domain of definition given by $\operatorname{dom} A^{-1} = \operatorname{im}(A)$. In general, $\operatorname{im}(A)$ needs not to be dense even if A was injective and densely defined. Hence A^{-1} needs not to be densely defined. We have now the following simple properties of the inverse of an injective operator:

mple for this,
r bounded A

Proposition 5.2.19 *Let* $(\operatorname{dom} A, A)$ *be an injective operator in* $\mathfrak{H}$ *and let* $(\operatorname{dom} A^{-1}, A^{-1})$ *be its inverse.*

i.) Denote by $\tau\colon \mathfrak{H} \oplus \mathfrak{H} \ni (\phi, \psi) \mapsto (\psi, \phi) \in \mathfrak{H} \oplus \mathfrak{H}$ *the canonical (unitary) flip. Then one has*

$$\operatorname{graph}(A^{-1}) = \tau(\operatorname{graph}(A)). \tag{5.2.43}$$

ii.) The inverse $(\operatorname{dom} A^{-1}), A^{-1})$ *is densely defined iff* A *has dense image.*

iii.) The inverse $(\operatorname{dom} A^{-1}), A^{-1})$ *is closed iff* $(\operatorname{dom} A, A)$ *is closed.*

iv.) If $(\operatorname{dom} A, A)$ *is densely defined and closed such that* $\operatorname{im}(A) = \mathfrak{H}$ *then* A^{-1} *is continuous.*

v.) If $(\operatorname{dom} A, A)$ *is densely defined with dense image then also* A^* *is injective and*

$$(A^*)^{-1} = (A^{-1})^*. \tag{5.2.44}$$

Proof: The first part is clear since τ is unitary , see also Exercise 5.5.14. Then we compute

$$\operatorname{graph}(A^{-1}) = \big\{(\phi, A^{-1}\phi) \bigm| \phi \in \operatorname{im}(A)\big\} = \big\{(A\psi, \psi) \bigm| \psi \in \operatorname{dom} A\big\} = \tau(\operatorname{graph}(A)),$$

proving the first part. The second is clear by definition,. The third part follows from (5.2.43) and the fact that the unitary τ maps closed subspaces to closed subspaces. For the fourth part we have a closed operator A^{-1} defined on $\mathfrak{H}$ by the third part. Hence A^{-1} is continuous by the Closed Graph Theorem 2.3.20. For the fifth part we note that A^{-1} is densely defined and hence it is meaningful to speak of $(A^{-1})^*$ at all. By Theorem 5.2.14 and $\tau J = -J\tau$ we have

$$\begin{aligned}
\operatorname{graph}((A^{-1})^*) &= (J(\operatorname{graph}(A^{-1})))^{\perp} \\
&= (J(\tau(\operatorname{graph}(A))))^{\perp} \\
&= (\tau(J(\operatorname{graph}(A))))^{\perp} \\
&= \tau((J(\operatorname{graph}(A)))^{\perp}) \\
&= \tau(\operatorname{graph}(A^*)) \\
&= \operatorname{graph}((A^*)^{-1}),
\end{aligned}$$

where we used again that τ is unitary. Since the graph determines its operator uniquely, we get (5.2.44) once we have showed that A^* is injective at all. But this is easy, we have $\ker A^* = (\operatorname{im}(A))^{\perp}$ by Proposition 5.2.7, *iv.)*. Since $\operatorname{im}(A)$ is dense, A^* is injective. □

For operators which are not closed or not even closable the spectral analysis becomes quickly very pathological. In quantum mechanics we have see that all relevant operators are at least closable. Hence we restrict the definition and the further investigation of spectra to the class of closed operators.

Definition 5.2.20 (Spectrum and resolvent) *Let* $(\operatorname{dom} A, A)$ *be a closed operator in* $\mathfrak{H}$. *Then*

$$\operatorname{res}(A) = \big\{z \in \mathbb{C} \bigm| z - A\colon \operatorname{dom} A \longrightarrow \mathfrak{H} \text{ is bijective}\big\} \tag{5.2.45}$$

is called the resolvent set of $(\operatorname{dom} A, A)$. *Its complement*

$$\operatorname{spec}(A) = \mathbb{C} \setminus \operatorname{res}(A) \tag{5.2.46}$$

is called the spectrum of $(\operatorname{dom} A, A)$ *and the map*

$$\operatorname{Res}\colon \operatorname{res}(A) \ni z \mapsto (z - A)^{-1} = \operatorname{Res}_z(A) \in \mathfrak{B}(\mathfrak{H}) \tag{5.2.47}$$

is called the resolvent of $(\operatorname{dom} A, A)$.

Remark 5.2.21 By Proposition 5.2.19, *iv.)*, we know that $(z - A)^{-1}$ is continuous indeed. Hence $\operatorname{Res}_z(A)$ is a continuous operator defined on $\mathfrak{H}$. For a closable but not necessarily closed operator, one defines the resolvent set, the spectrum, and the resolvent with respect to its closure, though this is not done systematically in the literature. Note also, that A is closed iff $z - A$ is closed for one $z \in \mathbb{C}$ iff $z - A$ is closed for all $z \in \mathbb{C}$, see Exercise 5.5.15, *ii.)*.

In the Banach-algebraic or C^*-algebraic approach we have seen that the notion of resolvent set, spectrum, and resolvent has very nice features, see Theorem 4.2.14. In that theorem we never needed estimates for A but only for $\operatorname{Res}_z(A)$ for some of the statements. This allows to transfer all those statements to the unbounded situation as follows:

Theorem 5.2.22 (Resolvent of closed operators) *Let* $(\operatorname{dom} A, A)$ *be a closed operator in* $\mathfrak{H}$*.*

i.) *The resolvent set* $\operatorname{res}(A) \subseteq \mathbb{C}$ *is open.*

ii.) *The resolvent* $\operatorname{Res}\colon \operatorname{res}(A) \longrightarrow \mathfrak{B}(\mathfrak{H})$ *is holomorphic.*

iii.) *For all* $z, w \in \operatorname{res}(A)$ *one has the resolvent identity*

$$\operatorname{Res}_z(A) - \operatorname{Res}_w(A) = (w - z) \operatorname{Res}_z(A) \operatorname{Res}_w(A). \tag{5.2.48}$$

PROOF: In fact, the proof of the corresponding statements in Theorem 4.2.14 can be copied literally as we only needed (operator) norm estimates for $\operatorname{Res}_z(A)$. By Remark 5.2.21 we still have $\operatorname{Res}_z(A) \in \mathfrak{B}(\mathfrak{H})$ and hence we can proceed. □

In the bounded case, we were able to say more: in particular, the spectrum was a non-empty compact subset of $\mathbb{C}$. We shall see that this may fail in both ways: the spectrum can be non-compact (as expected for unbounded operators) but it can also be empty, which is of course even more unpleasant.

Spectrum of operators on [compact] interval

5.3 Self-Adjoint Operators

While for general closed operators in a Hilbert space the spectrum can still be very wild, we shall now pass to the self-adjoint operators. They will provide the good class of operators relevant for quantum theory with good spectral properties.

5.3.1 Symmetric Operators and Deficiency Indices

In quantum mechanics one has as fundamental example of an unbounded operator the momentum operator $P = -\mathrm{i}\hbar\frac{\partial}{\partial x}$ acting on e.g. $\mathscr{C}_0^\infty(\mathbb{R}) \subseteq \mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$. By a simple integration by parts we see that

$$\langle \phi, P\psi \rangle = \langle P\phi, \psi \rangle \tag{5.3.1}$$

for all $\phi, \psi \in \mathscr{C}_0^\infty(\mathbb{R})$. Hence, whatever the precise domain of P^* might be, we have the feature

$$P \subseteq P^*. \tag{5.3.2}$$

Operators with this property are now called *symmetric*:

Definition 5.3.1 (Symmetric operator) *Let* $(\operatorname{dom} A, A)$ *be a densely defined operator in* $\mathfrak{H}$*. Then* A *is called symmetric if* $A \subseteq A^*$*.*

Clearly, $A \subseteq A^*$ is equivalent to

$$\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle \tag{5.3.3}$$

for all $\phi, \psi \in \operatorname{dom} A$. We collect now some first properties of symmetric operators:

Remark 5.3.2 (Symmetric operators) Let $(\operatorname{dom} A, A)$ be a symmetric operator.

i.) The operator $(\operatorname{dom} A, A)$ is closable since $\operatorname{dom} A^* \supseteq \operatorname{dom} A$ is dense.

ii.) By Theorem 5.2.14, *ii.)*, its closure is given by $A^{\mathrm{cl}} = A^{**}$. Since $A \subseteq A^*$ implies $A^{**} \subseteq A^*$ by Proposition 5.2.8, *iv.)*, and $(A^{\mathrm{cl}})^* = A^*$ by Theorem 5.2.14, *iii.)*, we see that

$$A \subseteq \overline{A} = A^{**} \subseteq A^* = (\overline{A})^*. \tag{5.3.4}$$

Hence also $\overline{A}$ is symmetric. In general, both extensions in (5.3.4) will be proper.

In general, $A^{**} \subseteq A^*$ is a proper extension of the closed operator A^{**} for a symmetric operator A. Thus even for a closed symmetric operator A its adjoint provides a proper extension in general. If this is not the case then A is called (essentially) self-adjoint:

Definition 5.3.3 (Self-adjoint operator) *A densely defined operator* $(\operatorname{dom} A, A)$ *is called self-adjoint if*

$$(\operatorname{dom} A, A) = (\operatorname{dom} A^*, A^*), \tag{5.3.5}$$

and essentially self-adjoint if A is symmetric and its closure $A^{\mathrm{cl}} = A^{**}$ is self-adjoint.

Remark 5.3.4 Let $(\operatorname{dom} A, A)$ be an operator in $\mathfrak{H}$.

i.) The operator A is self-adjoint iff it is densely defined and both, A and A^* are closed and symmetric. Indeed, for a self-adjoint operator A^* is closed and symmetric. Conversely, if A^* is symmetric then $A^* \subseteq A^{**} = A$ since A is closed and symmetric. Moreover, $A \subseteq A^*$ holds if A is symmetric. Thus $A = A^*$ follows.

ii.) If A is self-adjoint and $\mathfrak{D} \subseteq \operatorname{dom} A$ is a core for A then $A\big|_{\mathfrak{D}}$ is essentially self-adjoint. Indeed, first we note that a core of a densely defined closed operator is dense again. This is clear as the graph of $(\mathfrak{D}, A\big|_{\mathfrak{D}})$ is dense in the graph of $(\operatorname{dom} A, A)$ which implies that $\mathfrak{D}$ is dense in $\operatorname{dom} A$. Since $\operatorname{dom} A \subseteq \mathfrak{H}$ is dense, also $\mathfrak{D} \subseteq \mathfrak{H}$ is dense. Thus $A\big|_{\mathfrak{D}}$ is again densely defined and from $A\big|_{\mathfrak{D}} \subseteq A$ we get $A^* \subseteq (A\big|_{\mathfrak{D}})^*$. With $A \subseteq A^*$ this gives $A\big|_{\mathfrak{D}} \subseteq (A\big|_{\mathfrak{D}})^*$. Hence $A\big|_{\mathfrak{D}}$ is symmetric. Its closure is A by definition of a core.

iii.) Conversely, if A is essentially self-adjoint then A has a unique self-adjoint extension, namely the closure $A^{\mathrm{cl}} = A^{**}$. If $(\operatorname{dom} B, B)$ would be another (closed) symmetric, without restriction already closed, extension then we have $A \subseteq B \subseteq B^*$ and hence $A^{\mathrm{cl}} \subseteq B^{\mathrm{cl}} = B \subseteq B^*$ since a closed symmetric operator is closed. On the other hand $A \subseteq B$ implies $B \subseteq B^* \subseteq A^* = A^{***} = A^{**} = A^{\mathrm{cl}}$ since A^{**} is self-adjoint. This shows $A^{\mathrm{cl}} = B$. It also follows that a self-adjoint operator A has no non-trivial closed symmetric extensions. Thus a self-adjoint operator is *maximally symmetric*.

The next theorem clarifies the relation between closed symmetric and self-adjoint operators and gives a characterization of their spectra. For its proof, we need the following lemma which is also of independent interest:

Lemma 5.3.5 *Let $\mathfrak{H}$ be a Hilbert space and let $V, U \subseteq \mathfrak{H}$ be closed subspaces. If $V \cap U^\perp = \{0\}$ then*

$$\dim V \leq \dim U. \tag{5.3.6}$$

PROOF: Denote by $\phi = \phi_{\shortparallel} + \phi_\perp$ the orthogonal decomposition with respect to $\mathfrak{H} = U \oplus U^\perp$. Then for $\phi \in V$ we have that

$$P_U\big|_V \colon V \longrightarrow U$$

is injective. Indeed, assume $P_U \phi = \phi_{\shortparallel} = 0$ then $\phi = \phi_\perp \in U^\perp \cap V = \{0\}$ shows $\phi = 0$. This provides us an injective continuous linear map $V \longrightarrow U$. We consider the Hilbert space $V \oplus U$. Then we obtain a continuous linear map $A\colon V \oplus U \longrightarrow V \oplus U$ by setting $A\big|_V = P_U\big|_V$ and $A\big|_U = 0$. By Theorem 5.1.42 we find a (unique) partial isometry $W\colon V \oplus U \longrightarrow V \oplus U$ with $A = W\sqrt{A^*A}$ and $\ker W = \ker A = U$. Thus on V this gives an isometry $W\big|_V\colon V \longrightarrow U$, mapping any Hilbert basis of V to an orthogonal system in U. Hence (5.3.6) follows. ☐

Theorem 5.3.6 (Spectrum of a closed symmetric operator) *Let* $(\operatorname{dom} A, A)$ *be a closed symmetric operator in* $\mathfrak{H}$.

i.) *For all* $z \in \mathbb{C} \setminus \mathbb{R}$ *one has*

$$\ker(z - A^*) = \operatorname{im}(\overline{z} - A)^\perp \tag{5.3.7}$$

and $\operatorname{im}(z - A)$ *is closed.*

ii.) *For* $z \in \mathbb{C}$ *with* $\operatorname{Im}(z) > 0$ *the Hilbert space dimension* $\dim\ker(z - A^*)$ *is constant.*

iii.) *For* $z \in \mathbb{C}$ *with* $\operatorname{Im}(z) < 0$ *the Hilbert space dimension* $\dim\ker(z - A^*)$ *is constant, too.*

iv.) *For the spectrum of* A *one has the following alternatives:*

  i.) $\operatorname{spec}(A) = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$.

  ii.) $\operatorname{spec}(A) = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \leq 0\}$.

  iii.) $\operatorname{spec}(A) = \mathbb{C}$.

  iv.) $\operatorname{spec}(A)$ *is a closed subset of* $\mathbb{R}$.

v.) *The operator* A *is self-adjoint iff* $\operatorname{spec}(A) \subseteq \mathbb{R}$.

vi.) *The operator* A *is self-adjoint iff for all* $z \in \mathbb{C}$ *with* $\operatorname{Im}(z) > 0$ *one has*

$$\dim\ker(z - A^*) = 0 = \dim\ker(\overline{z} - A^*), \tag{5.3.8}$$

iff for one $z \in \mathbb{C}$ *with* $\operatorname{Im}(z) > 0$ *one has* (5.3.8).

PROOF: For the first part let $z = \alpha + \mathrm{i}\beta$ with $\beta \neq 0$. Since A is symmetric we get for $\varphi \in \operatorname{dom} A$

$$\begin{aligned}
\|(z - A)\varphi\|^2 &= \langle (z - A)\varphi, (z - A)\varphi \rangle \\
&= \langle (\alpha - A)\varphi, (\alpha - A)\varphi \rangle + \beta^2\langle \varphi, \varphi \rangle + \mathrm{i}\beta\langle (\alpha - A)\varphi, \varphi \rangle - \mathrm{i}\beta\langle \varphi, (\alpha - A)\varphi \rangle \\
&= \|(\alpha - A)\varphi\|^2 + \beta^2\|\varphi\|^2,
\end{aligned}$$

and hence in particular

$$\|(z - A)\varphi\|^2 \geq \beta^2\|\varphi\|^2. \tag{$*$}$$

We show that $\operatorname{im}(z - A)$ is closed. Let $\varphi_n \in \operatorname{dom} A$ with $(z - A)\varphi_n \longrightarrow \psi$ be given. Then $(*)$ gives

$$\|(z - A)\varphi_n - (z - A)\varphi_m\|^2 \geq \beta^2\|\varphi_n - \varphi_m\|^2,$$

and hence φ_n is a Cauchy sequence, thus convergent $\varphi_n \longrightarrow \varphi \in \mathfrak{H}$. Since clearly $z\varphi_n \longrightarrow z\varphi$ we conclude that $A\varphi_n \to z\varphi - \psi$ is convergent as well. Thus the sequence $(\varphi_n, A\varphi_n) \in \operatorname{graph}(A)$ converges to $(\varphi, z\varphi - \psi) \in \mathfrak{H} \oplus \mathfrak{H}$. Since A is closed, i.e. $\operatorname{graph}(A)$ is closed, by assumption, we conclude $\varphi \in \operatorname{dom} A$ and $A\varphi = z\varphi - \psi$. This means $\psi = (z - A)\varphi \in \operatorname{im}(z - A)$, showing that $\operatorname{im}(z - A)$ is closed. The remaining statement (5.3.7) is clear by Proposition 5.2.7, *iv.)*, even for general z. For the second part, let $z \in \mathbb{C}$ with $\operatorname{Im}(z) > 0$ be given and $w \in \mathbb{C}$ be a small complex number. We want to show that $\ker(z - A^*)$ and $\ker(z + w - A^*)$ have the same dimension as long as w is small enough. Fix a non-zero vector $\varphi \in \operatorname{dom} A^*$ with $\varphi \in \ker(z + w - A^*)$. Assume further that $\varphi \perp \ker(z - A^*)$. By the first part $\varphi \in (\ker(z - A^*))^\perp = \operatorname{im}(\overline{z} - A)^{\perp\perp} = \operatorname{im}(\overline{z} - A)$ follows. Thus let $\chi \in \operatorname{dom} A$ be such that $\varphi = (\overline{z} - A)\chi$. This gives

$$0 = \langle ((z + w) - A^*)\varphi, \chi \rangle = \langle \varphi, (\overline{z} - A)\chi \rangle + \overline{w}\langle \varphi, \chi \rangle = \langle \varphi, \varphi \rangle + \overline{w}\langle \varphi, \chi \rangle. \tag{$**$}$$

Now let $z = \alpha + \mathrm{i}\beta$ with $|w| < \beta$, then by $(*)$

$$\|\varphi\|^2 = \|(\overline{z} - A)\chi\|^2 \geq \beta^2 \|\chi\|^2 > |w|^2 \|\chi\|^2.$$

Using this, the Cauchy-Schwarz inequality gives

$$|\overline{w}\langle \varphi, \chi \rangle| \leq |w| \|\varphi\| \|\chi\| < \|\varphi\|^2.$$

But then $(**)$ can only be true for $\varphi = 0$, a contradiction. This shows that the only vector φ in $\ker((z + w) - A^*)$ orthogonal to $\ker(z - A^*)$ is $\varphi = 0$ for those w. Hence we have

$$\ker((z + w) - A^*) \cap \ker(z - A^*)^\perp = \{0\} \tag{☺}$$

for $|w| < \beta$. Replacing now $z \leftrightarrow z + w$ we get the analogous statement as long as $|w|$ is smaller then the imaginary part of $z + w$. This is certainly the case for $|w| < \frac{1}{2}\beta$. Then we get

$$\ker(z - A^*) \cap \ker((z + w) - A^*)^\perp = \{0\}. \tag{☹}$$

Using now Lemma 5.3.5, (☺) and (☹) imply that the dimensions of $\ker(z - A^*)$ and $\ker(z + w - A^*)$ coincide for $|w| < \frac{1}{2}\beta$. Since the function $z \mapsto \dim \ker(z - A^*)$ is thus locally constant it is constant on the connected open upper half plane. This shows the second part, the third goes analogously by replacing $\beta > 0$ with $-\beta > 0$. For the fourth part we use $(*)$ to see that $z - A$ is injective for $z \in \mathbb{C} \setminus \mathbb{R}$. By the first part, $\operatorname{im}(z - A) = \ker(\overline{z} - A^*)^\perp$ shows that $z - A$ is surjective onto $\mathfrak{H}$ iff $\overline{z} - A^*$ is injective, i.e. $\ker(\overline{z} - A^*) = \{0\}$. Now $\dim \ker(\overline{z} - A^*)$ is constant on both open half planes. Hence $z - A$ is surjective for all $z \in \mathbb{C}$ with $\operatorname{Im}(z) < 0$. Since the spectrum is closed, the four possibilities in part *iv.)* are the only remaining ones, showing the fourth part. Now we first assume that A is self-adjoint. Then if $\varphi \in \operatorname{dom} A = \operatorname{dom} A^*$ is an eigenvector to the eigenvalue $\lambda \in \mathbb{C}$ we have

$$\lambda \langle \varphi, \varphi \rangle = \langle \varphi, A\varphi \rangle = \langle A^*\varphi, \varphi \rangle = \langle A\varphi, \varphi \rangle = \overline{\lambda} \langle \varphi, \varphi \rangle,$$

and thus $\lambda = \overline{\lambda}$. As expected, a self-adjoint operator can have only real eigenvectors. But then $A = A^*$ shows $\ker(z - A^*) = \{0\}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$ and hence (5.3.8) follows. By the first part this implies that $z - A$ is surjective for all $z \in \mathbb{C} \setminus \mathbb{R}$ and hence $z \in \operatorname{res}(A)$. Thus $\operatorname{spec}(A) \subseteq \mathbb{R}$ is the only remaining possibility. Conversely, assume $\operatorname{spec}(A) \subseteq \mathbb{R}$ then $z - A$ is bijective for all $z \in \mathbb{C} \setminus \mathbb{R}$. Hence $\operatorname{im}(z - A) = \mathfrak{H}$ gives $\ker(\overline{z} - A^*) = \{0\}$ by the first part and thus (5.3.8) holds. It remains to show that A is actually self-adjoint. By $\operatorname{im}(z - A) = \mathfrak{H}$ for $z \in \mathbb{C} \setminus \mathbb{R}$ we find for $\varphi \in \operatorname{dom} A^*$ a $\psi \in \operatorname{dom} A$ with $(z - A)\psi = (z - A^*)\varphi$. Since $\operatorname{dom} A \subseteq \operatorname{dom} A^*$ we see that $\varphi - \psi \in \operatorname{dom} A^*$ and thus $(z - A^*)(\varphi - \psi) = 0$ follows. But $\ker(z - A^*) = \{0\}$ gives $\varphi = \psi \in \operatorname{dom} A$. Hence $\operatorname{dom} A^* \subseteq \operatorname{dom} A$ which gives $A = A^*$ as claimed. Thus the parts *v.)* and *vi.)* are shown at once. □

This theorem is of central importance for the study of self-adjoint operators. We collect now some consequences of it allowing to characterize self-adjointness:

Corollary 5.3.7 *Let* $(\operatorname{dom} A, A)$ *be a closed symmetric operator in* $\mathfrak{H}$. *Then the following statements are equivalent:*

i.) *The operator* A *is self-adjoint.*

ii.) *One has* $\ker(z \pm A^*) = \{0\}$ *for one* $z \in \mathbb{C} \setminus \mathbb{R}$.

iii.) *One has* $\ker(z \pm A^*) = \{0\}$ *for all* $z \in \mathbb{C} \setminus \mathbb{R}$.

iv.) *One has* $\operatorname{im}(z \pm A) = \mathfrak{H}$ *for one* $z \in \mathbb{C} \setminus \mathbb{R}$.

v.) *One has* $\operatorname{im}(z \pm A) = \mathfrak{H}$ *for all* $z \in \mathbb{C} \setminus \mathbb{R}$.

vi.) *One has* $\operatorname{spec}(A) \subseteq \mathbb{R}$.

Corollary 5.3.8 *Let* $(\operatorname{dom} A, A)$ *be a closed symmetric operator and* $\lambda \in \mathbb{R}$ *with* $\lambda \in \operatorname{res}(A)$*. Then* A *is self-adjoint.*

Since the size of $\ker(z \pm A^*)$ with $z \in \mathbb{C} \setminus \mathbb{R}$ determines whether A is self-adjoint or not, the Hilbert space dimensions of these subspaces are characteristic for a closed symmetric operator.

Definition 5.3.9 (Deficiency indices) *Let* $(\operatorname{dom} A, A)$ *be a closed symmetric operator in* $\mathfrak{H}$*. Then the deficiency indices* $n_\pm(A)$ *of* A *are defined by*

$$n_\pm(A) = \dim \ker(\mathrm{i} \mp A^*). \tag{5.3.9}$$

The subspaces

$$\mathfrak{K}_\pm(A) = \ker(\mathrm{i} \mp A^*) \tag{5.3.10}$$

are called the deficiency spaces of A*.*

Remark 5.3.10 (Deficiency indices) Let $(\operatorname{dom} A, A)$ be a closed symmetric operator in $\mathfrak{H}$.

i.) We have

$$\mathfrak{K}_\pm(A) = \operatorname{im}(\mathrm{i} \pm A)^\perp \tag{5.3.11}$$

and

$$\operatorname{im}(\mathrm{i} \pm A) = \mathfrak{K}_\pm(A)^\perp. \tag{5.3.12}$$

Moreover, $n_\pm(A) = \dim \mathfrak{K}_\pm(A)$ measures how much $\mathrm{i} \pm A$ is not surjective. Instead of i we can take any other number in the open upper half-plane. Another interpretation is that $\mathfrak{K}_\pm(A)$ is the eigenspace of A^* to the eigenvalue $\pm\mathrm{i}$.

ii.) The operator $(\operatorname{dom} A, A)$ is self-adjoint iff $n_\pm(A) = 0$. This is the statement of Theorem 5.3.6, *vi.)*.

iii.) As we shall see later, all combinations of values for the deficiency indices $0 \le n_\pm(A) \le \dim \mathfrak{H}$ can actually occur if $\dim \mathfrak{H}$ is not finite.

iv.) If $\mathfrak{H}$ is finite-dimensional then $n_\pm(A) = 0$ for all symmetric operators (which are necessarily defined on all of $\mathfrak{H}$) by a simple counting of dimensions.

We conclude this section with two further constructions of self-adjoint operators: in the first one obtains a self-adjoint operator out of the densely defined inverse of an injective Hermitian operator. This gives many important examples.

Proposition 5.3.11 *Let* $(\operatorname{dom} A, A)$ *be a symmetric operator in* $\mathfrak{H}$*.*

i.) If A *is self-adjoint and injective, then* $\operatorname{im} A$ *is dense and* $(\operatorname{dom} A^{-1}, A^{-1})$ *is self-adjoint, too.*

ii.) If $\operatorname{im} A$ *is dense, then* A *is injective.*

iii.) If $\operatorname{im} A = \mathfrak{H}$*, then* A *is self-adjoint and* A^{-1} *is continuous.*

PROOF: First, let $\phi \in (\operatorname{im} A)^\perp$ then for $\psi \in \operatorname{dom} A$ we have $\langle \phi, A\psi \rangle = 0$. Hence $\phi \in \operatorname{dom} A^* = \operatorname{dom} A$ follows. Moreover, $0 = \langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$ shows that $A\phi$ is orthogonal to the dense domain $\operatorname{dom} A$ and thus $A\phi = 0$. Since A is assumed to be injective, $\phi = 0$ and hence $\operatorname{im} A$ is dense. By Proposition 5.2.19, *v.)*, we get $A^{-1} = (A^*)^{-1} = (A^{-1})^*$, proving the first statement. For the second part, let $\phi \in \operatorname{dom} A$ with $A\phi = 0$ be given. Then for all $\psi \in \operatorname{dom} A$ we have $0 = \langle \psi, A\phi \rangle = \langle A\psi, \phi \rangle$. Since $\operatorname{im} A$ is dense, $\phi = 0$ follows. For the third part, we know that A is injective by the second part. Hence $A\colon \operatorname{dom} A \longrightarrow \mathfrak{H}$ is bijective and thus invertible with an inverse $A^{-1}\colon \mathfrak{H} \longrightarrow \operatorname{dom} A$. Now A^{-1} is again symmetric and thus adjointable and everywhere defined. Hence the Hellinger-Toeplitz Theorem 3.5.1 gives us a bounded and Hermitian $A^{-1} \in \mathfrak{B}(\mathfrak{H})$. In particular, A^{-1} is self-adjoint in the sense of operators in $\mathfrak{H}$, now of course with $\operatorname{dom} A^{-1} = \mathfrak{H}$. Applying the first part to A^{-1} gives a self-adjoint inverse $(A^{-1})^{-1} = A$ on the domain $\operatorname{dom} A$. Alternatively, we can argue with Proposition 5.2.19, *iii.)*, that A is closed and $0 \in \operatorname{res}(A)$. Hence A is self-adjoint by Corollary 5.3.8. □

The second construction starts with an arbitrary operator $(\operatorname{dom} A, A)$. Then one considers the operator A^*A, with its domain

$$\operatorname{dom} A^*A = \left\{\phi \in \operatorname{dom} A \;\middle|\; A\phi \in \operatorname{dom} A^*\right\}, \tag{5.3.13}$$

according to our general considerations on the composition of unbounded operators. If A would be bounded, then A^*A would be a bounded *positive* operator. In particular, $\mathbb{1} + A^*A$ is invertible and still positive by the usual spectral calculus. It turns out that also in the unbounded case this is still true. In particular, the domain of A^*A is still dense, which, on a first sight, is not obvious at all.

Theorem 5.3.12 (Self-adjointness of A^*A) *Let* $(\operatorname{dom} A, A)$ *be a densely defined, closed operator.*

i.) The operator

$$\mathbb{1} + A^*A\colon \operatorname{dom} A^*A \longrightarrow \mathfrak{H} \tag{5.3.14}$$

is bijective and there are operators $B, C \in \mathfrak{B}(\mathfrak{H})$ with $\|B\|, \|C\| \leq 1$,

$$C = AB, \tag{5.3.15}$$

and

$$B(\mathbb{1} + A^*A) \subseteq (\mathbb{1} + A^*A)B = \mathbb{1}. \tag{5.3.16}$$

ii.) The operator B is positive and $\mathbb{1} + A^*A$ as well as A^*A are self-adjoint on $\operatorname{dom} A^*A$.

iii.) The domain $\operatorname{dom} A^*A$ *is a core for* $(\operatorname{dom} A, A)$.

Proof: First let $\phi \in \operatorname{dom} A^*A$. Then $\langle \phi, \phi\rangle + \langle A\phi, A\phi\rangle = \langle \phi, \phi\rangle + \langle \phi, A^*A\phi\rangle = \langle \phi, (\mathbb{1} + A^*A)\phi\rangle$ shows

$$\|\phi\|^2 \leq \|\phi\| \|(\mathbb{1} + A^*A)\phi\|.$$

Hence $\mathbb{1} + A^*A$ is indeed injective. We use again the canonical complex structure J, i.e. the unitary $J\colon \mathfrak{H} \oplus \mathfrak{H} \longrightarrow \mathfrak{H} \oplus \mathfrak{H}$ from (5.2.31). For $\psi \in \mathfrak{H}$ we can decompose $(0, \psi) \in \mathfrak{H} \oplus \mathfrak{H}$ into its components with respect to $J(\operatorname{graph}(A))$ and $\operatorname{graph}(A^*)$ according to Corollary 5.2.15, since A is assumed to be closed. Thus there are unique vectors denoted by $B\psi \in \operatorname{dom} A$ and $C\psi \in \operatorname{dom} A^*$ such that

$$(0, \psi) = (-AB\psi, B\psi) + (C\psi, A^*C\psi). \tag{$*$}$$

From the uniqueness of the decomposition we see that the maps $B\colon \psi \mapsto B\psi$ and $C\colon \psi \mapsto C\psi$ are linear. From $(*)$ we have

$$\|\psi\|^2 = \|(0, \psi)\|^2 = \|-AB\psi\|^2 + \|B\psi\|^2 + \|C\psi\|^2 + \|A^*C\psi\|^2 \geq \|B\psi\|^2 + \|C\psi\|^2.$$

In particular, B and C are continuous and have norms at most 1. Moreover, from the first component of $(*)$ we get $0 = -AB\psi + C\psi$ and hence $C = AB$. The second component of $(*)$ gives the relation

$$\psi = B\psi + A^*C\psi = B\psi + A^*AB\psi = (\mathbb{1} + A^*A)B\psi.$$

First we note that $\operatorname{im} B \subseteq \operatorname{dom} A^*A$ holds. Moreover, B is injective and $(\mathbb{1} + A^*A)\big|_{\operatorname{im} B}$ is already surjective onto $\mathfrak{H}$. Since $\mathbb{1} + A^*A$ is injective and $(\mathbb{1} + A^*A)\big|_{\operatorname{im} B}$ is bijective, we need to have $\operatorname{im} B = \operatorname{dom} A^*A$. Indeed, if $\phi \in \operatorname{dom} A^*A$ would not be in the image of B, then there would be a $\phi' \in \operatorname{im} B$ with

$$(\mathbb{1} + A^*A)\phi' = (\mathbb{1} + A^*A)\phi$$

by the surjectivity, and hence $\phi' = \phi$ by the injectivity. Hence $\mathbb{1} + A^*A\colon \operatorname{dom} A^*A \longrightarrow \mathfrak{H}$ as well as $B\colon \mathfrak{H} \longrightarrow \operatorname{dom} A^*A$ are linear bijections with $(\mathbb{1} + A^*A)B = \operatorname{id}_{\mathfrak{H}}$. Since inverses of bijections are unique, we also get $B(\mathbb{1}+A^*A) = \mathrm{id}_{\operatorname{dom} A^*A}$, concluding the first part. For the second part, let $\psi \in \mathfrak{H}$. Then there is a unique $\phi \in \operatorname{dom} A^*A$ with $\psi = (\mathbb{1} + A^*A)\phi$. We have

$$\langle \psi, B\psi \rangle = \langle (\mathbb{1} + A^*A)\phi, B(\mathbb{1} + A^*A)\phi \rangle = \langle (\mathbb{1} + A^*A)\phi, \phi \rangle \geq 0$$

by $(*)$. Thus B is positive. Now we apply Proposition 5.3.11 to the self-adjoint (even Hermitian) operator $(\operatorname{dom} B = \mathfrak{H}, B)$ and obtain a self-adjoint inverse $(\operatorname{dom} B^{-1}, B^{-1}) = (\operatorname{dom} A^*A, \mathbb{1} + A^*A)$. But then also $(\operatorname{dom} A^*A, A^*A)$ is self-adjoint, proving the second part. Finally, since A is closed, its graph is closed in $\mathfrak{H} \oplus \mathfrak{H}$. Now let $(\phi, A\phi) \in \operatorname{graph}(A)$ be orthogonal to $\operatorname{graph}(A\big|_{\operatorname{dom} A^*A})$. Then for all $\psi \in \operatorname{dom} A^*A$ we have

$$0 = \langle (\phi, A\phi), (\psi, A\psi) \rangle = \langle \phi, \psi \rangle + \langle A\phi, A\psi \rangle = \langle \phi, (\mathbb{1} + A^*A)\psi \rangle.$$

Since $\mathbb{1} + A^*A\colon \operatorname{dom} A^*A \longrightarrow \mathfrak{H}$ is surjective, this implies $\phi = 0$ and hence $(\phi, A\phi) = 0$ as well. But then the graph of the restriction is dense in the graph of A, i.e. we have $(\operatorname{graph}(A\big|_{\operatorname{dom} A^*A}))^{\mathrm{cl}} = \operatorname{graph}(A)$ which means $(A\big|_{\operatorname{dom} A^*A})^{\mathrm{cl}} = A$, showing the last part. □

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5.3.2 Self-Adjoint Extensions

Suppose we have a closed symmetric operator $(\operatorname{dom} A, A)$. Then we want to see whether we can find (proper) symmetric extensions $(\operatorname{dom} B, B)$ of it. We already know from Remark 5.3.4, *iii.)*, that this will not be possible for a self-adjoint operator $(\operatorname{dom} A, A)$: such an operator is already maximally symmetric and has no proper symmetric extensions.

If $(\operatorname{dom} B, B)$ is a symmetric extension of $(\operatorname{dom} A, A^*)$ we can assume without restriction that it is closed as well by Remark 5.3.2, *ii.)*. We have from $A \subseteq A^*$ and $A \subseteq B$ the extension $B^* \subseteq A^*$ by Proposition 5.2.8, *iv.)*. If B is symmetric as well, $B \subseteq B^*$, this results in the chain of extensions

$$A \subseteq B \subseteq B^* \subseteq A^*, \tag{5.3.17}$$

which will, in general, all be proper. Thus the art of finding *self-adjoint extensions* consists in finding an extension B such that the domain of A is enlarged while the domain of the adjoint shrinks in such a way that they meet in the middle.

As a tool for the following discussions we consider not only the inner product $\langle \cdot\,, \cdot \rangle_A$ on $\operatorname{dom} A$ but also the antisymmetric version

$$\{\phi, \psi\}_A = \langle A\phi, \psi \rangle - \langle \phi, A\psi \rangle, \tag{5.3.18}$$

defined for $\phi, \psi \in \operatorname{dom} A$. Clearly this is sesquilinear and antisymmetric in the sense that

$$\overline{\{\phi, \psi\}_A} = \overline{\langle A\phi, \psi \rangle} - \overline{\langle \phi, A\psi \rangle} = \langle \psi, A\phi \rangle - \langle A\psi, \phi \rangle = -\{\psi, \phi\}_A. \tag{5.3.19}$$

We see that for a symmetric operator $(\operatorname{dom} A, A)$ we have

$$\{\cdot\,, \cdot\}_A = 0. \tag{5.3.20}$$

From (5.3.17) we see that every symmetric extension of A has a domain which is contained inside $\operatorname{dom} A^*$. Since also $\operatorname{dom} B^* \subseteq \operatorname{dom} A^*$ we have to characterize those subspaces of $\operatorname{dom} A^*$ which can be used as domains of symmetric extensions. Here the antisymmetric bracket (5.3.18) will turn out to be crucial.

Definition 5.3.13 (A-symmetric subspace) *Let $(\operatorname{dom} A, A)$ be a closed symmetric operator in $\mathfrak{H}$. Then a subspace $U \subseteq \operatorname{dom} A^*$ is called A-symmetric, if it is isotropic for $\{\cdot\,, \cdot\}_{A^*}$ i.e. if for $\phi, \psi \in U$ one has*

$$\{\phi, \psi\}_{A^*} = 0. \tag{5.3.21}$$

Though "A^*-symmetric" would probably be the more consistent notion we stick to the tradition following [43, Sect. X.1]. Analogously, one uses the terms *A-closed* and *A-orthogonal* for subspaces of $\operatorname{dom} A^*$ with respect to the scalar product $\langle \cdot\,, \cdot \rangle_{A^*}$ and the resulting graph norm $\|\cdot\|_{A^*}$ on $\operatorname{dom} A^*$, see Remark 5.2.13, *v.)*. At this point we recall again that $\operatorname{dom} A^*$ with $\langle \cdot\,, \cdot \rangle_{A^*}$ is a Hilbert space since the adjoint is always a closed operator. Since A is symmetric iff $\{\cdot\,, \cdot\}_{A^*} = 0$ on $\operatorname{dom} A$, the following theorem is not very surprising: it will be the A-symmetric subspaces of $\operatorname{dom} A^*$ which are responsible for symmetric extensions of A:

Theorem 5.3.14 (A-closed and A-symmetric subspaces) *Let $(\operatorname{dom} A, A)$ be a closed symmetric operator in $\mathfrak{H}$.*

i.) *The closed symmetric extensions $(\operatorname{dom} B, B)$ of $(\operatorname{dom} A, A)$ are in bijection to the restrictions of A^* to A-closed and A-symmetric subspaces $U \subseteq \operatorname{dom} A^*$, with $\operatorname{dom} A \subseteq U$, i.e. $\operatorname{dom} B = U$ and $B = A^*\big|_U$.*

ii.) *The domain $\operatorname{dom} A$ as well as the deficiency spaces $\mathfrak{K}_\pm(A)$ of A are A-closed subspaces of $\operatorname{dom} A^*$ and $\operatorname{dom} A^*$ is decomposed A-orthogonally into*

$$\operatorname{dom} A^* = \operatorname{dom} A \oplus_A \mathfrak{K}_+(A) \oplus_A \mathfrak{K}_-(A). \tag{5.3.22}$$

iii.) *The A-closed and A-symmetric subspaces $V \subseteq \mathfrak{K}_+(A) \oplus \mathfrak{K}_-(A)$ are in bijection to the A-closed and A-symmetric subspaces $U \subseteq \operatorname{dom} A^*$ which contain $\operatorname{dom} A$ via the map*

$$V \;\mapsto\; \operatorname{dom} A \oplus_A V \subseteq \operatorname{dom} A^*. \tag{5.3.23}$$

iv.) *One has $A^*\big|_{\mathfrak{K}_\pm(A)} = \pm\mathrm{i}\,\mathrm{id}_{\mathfrak{K}_\pm(A)}$ and a subspace $V \subseteq \mathfrak{K}_+(A) \oplus \mathfrak{K}_-(A)$ is A-closed iff V is closed.*

PROOF: If $(\operatorname{dom} B, B)$ is a symmetric extension then it is still contained in A^* by (5.3.17). Hence the graph of B is contained in the graph of A^* which is a closed subspace $\operatorname{graph} A^* \subseteq \mathfrak{H} \oplus \mathfrak{H}$ by Theorem 5.2.14, *i.)*. Thus $\operatorname{graph} B$ is closed in $\mathfrak{H} \oplus \mathfrak{H}$ iff it is a closed subspace of $\operatorname{graph} A^*$. Now $\operatorname{graph} A^*$ is isometrically isomorphic to $(\operatorname{dom} A^*, \langle \cdot\,, \cdot \rangle_{A^*})$ by Remark 5.2.13, *v.)*, via $\operatorname{dom} A^* \ni \phi \mapsto (\phi, A^*\phi) \in \operatorname{graph} A^*$ and its inverse $\mathrm{pr}_1\big|_{\operatorname{graph} A^*}\colon \operatorname{graph} A^* \subseteq \mathfrak{H} \oplus \mathfrak{H} \longrightarrow \operatorname{dom} A^* \subseteq \mathfrak{H}$. The graph of B is now mapped to $\operatorname{dom} B^*$ under the unitary $\mathrm{pr}_1\big|_{\operatorname{graph} A^*}$ which shows that B is a closed operator iff $\operatorname{dom} B^* \subseteq \operatorname{dom} A^*$ is a closed subspace with respect to the Hilbert space structure coming from $\langle \cdot\,, \cdot \rangle_{A^*}$. Moreover, $B \subseteq A^*$ implies that for $\phi, \psi \in \operatorname{dom} B$ we have

$$0 = \langle \phi, B\psi \rangle - \langle B\phi, \psi \rangle = \langle \phi, A^*\psi \rangle - \langle A^*\phi, \psi \rangle = -\{\phi, \psi\}_{A^*},$$

and hence $\operatorname{dom} B$ is A-symmetric. Conversely, let $\operatorname{dom} B$ be a A-symmetric subspace of $\operatorname{dom} A^*$ and $B = A^*\big|_{\operatorname{dom} B}$. Then for $\phi, \psi \in \operatorname{dom} B \subseteq \operatorname{dom} A^*$ we have

$$0 = \langle \phi, A^*\psi \rangle - \langle A^*\phi, \psi \rangle = \langle \phi, B\psi \rangle - \langle B\phi, \psi \rangle,$$

showing that B is symmetric. This shows the first part. Clearly, $(\operatorname{dom} A, A)$ is a closed symmetric extension of A and hence $\operatorname{dom} A$ is A-symmetric and A-closed by the first part. The deficiency spaces $\mathfrak{K}_\pm(A) = \ker(\mathrm{i} \mp A^*)$ are closed by Proposition 5.2.7, *iv.)*, in the original topology of $\mathfrak{H}$. Since $\|\phi\| \le \|\phi\|_{A^*}$ for $\phi \in \operatorname{dom} A^*$ it follows that the graph topology of A^* on $\operatorname{dom} A^*$ is finer and hence $\mathfrak{K}_\pm(A)$ are still closed in this finer topology. Thus they are A-closed subspaces of $\operatorname{dom} A^*$. We have to show the decomposition (5.3.22). First, let $\phi \in \operatorname{dom} A$ and $\psi_\pm \in \mathfrak{K}_\pm(A)$ be given. Then $A^*\psi_\pm = \pm\mathrm{i}\psi_\pm$ and we know that $\psi_\pm \perp \operatorname{im}(\mathrm{i} \pm A)$ by (5.3.11). This gives

$$\begin{aligned} \langle \phi, \psi_\pm \rangle_{A^*} &= \langle \phi, \psi_\pm \rangle + \langle A^*\phi, A^*\psi_\pm \rangle \\ &= \langle \phi, \psi_\pm \rangle \pm \mathrm{i} \langle A\phi, \psi_\pm \rangle \end{aligned}$$

$$
\begin{aligned}
&= \langle \phi, \psi_\pm \rangle \pm \mathrm{i} \langle \phi, A^* \psi_\pm \rangle \\
&= \langle \phi, \psi_\pm \rangle \pm \mathrm{i} \langle \phi, \pm \mathrm{i} \psi_\pm \rangle \\
&= 0.
\end{aligned}
$$

Moreover,

$$
\langle \psi_+, \psi_- \rangle_{A^*} = \langle \psi_+, \psi_- \rangle + \langle A^* \psi_+, A^* \psi_- \rangle = \langle \psi_+, \psi_- \rangle + \langle \mathrm{i} \psi_+, -\mathrm{i} \psi_- \rangle = 0,
$$

and hence all the three vectors are mutually orthogonal with respect to $\langle \cdot, \cdot \rangle_{A^*}$. This shows the A-orthogonal direct sum in (5.3.22). It remains to check that the sum gives all of $\operatorname{dom} A^*$. Thus let $\psi \in \operatorname{dom} A^*$ be A-orthogonal to $\operatorname{dom} A$ and to $\mathfrak{K}_\pm(A)$. For $\varphi \in \operatorname{dom} A$ we have

$$
0 = \langle \varphi, \psi \rangle_{A^*} = \langle \varphi, \psi \rangle + \langle A^* \varphi, A^* \psi \rangle = \langle \varphi, \psi \rangle + \langle A \varphi, A^* \psi \rangle,
$$

from which we conclude $A^* \psi \in \operatorname{dom} A^*$ and $A^* A^* \psi = -\psi$. Thus we have $A^* \psi - \mathrm{i} \psi \in \operatorname{dom} A^*$ and hence we can apply $A^* + \mathrm{i}$ yielding

$$
(A^* + \mathrm{i})(A^* - \mathrm{i}) \psi = (A^* A^* + \mathrm{id}) \psi = 0.
$$

This shows that $(A^* - \mathrm{i}) \psi \in \ker(A^* + \mathrm{i}) = \mathfrak{K}_-(A)$. For $\varphi \in \mathfrak{K}_-(A)$, i.e. $A^* \varphi = -\mathrm{i} \varphi$, this gives

$$
\mathrm{i} \langle \varphi, (A^* - \mathrm{i}) \psi \rangle = \langle \varphi, \psi \rangle + \langle A^* \varphi, A^* \psi \rangle = \langle \varphi, \psi \rangle_A = 0,
$$

since $\psi \in \mathfrak{K}_-(A)^\perp$ by assumption. Thus also $(A^* - \mathrm{i}) \psi \in \mathfrak{K}_-(A)^\perp$. But $(A^* - \mathrm{i}) \psi \in \mathfrak{K}_-(A)$ as just shown results in $(A^* - \mathrm{i}) \psi = 0$. Thus $\psi \in \ker(A^* - \mathrm{i}) = \mathfrak{K}_+(A)$. Since $\mathfrak{K}_-(A) \cap \mathfrak{K}_+(A) = \{0\}$ we conclude $\psi = 0$. This shows that the A-orthogonal space of the right hand side of (5.3.22) is $\{0\}$. Since we already know that all three spaces are A-closed and since $\operatorname{dom} A^*$ is a Hilbert space with respect to $\langle \cdot\,, \cdot \rangle_{A^*}$ this implies (5.3.22). For the third part, let first $V \subseteq \mathfrak{K}_+(A) \oplus \mathfrak{K}_-(A)$ be A-closed and A-symmetric. Since the A-orthogonal sum of two A-closed subspaces is again A-closed we see by (5.3.22) that $\operatorname{dom} A \oplus V$ is A-closed indeed. Moreover, for $\varphi, \psi \in \operatorname{dom} A \oplus V$ we have the A-orthogonal decomposition

$$
\varphi = \varphi_A + \varphi_V \quad \text{and} \quad \psi = \psi_A + \psi_V
$$

by (5.3.22). This gives

$$
\{\varphi_A, \psi_V\}_{A^*} = \langle A^* \varphi_A, \psi_V \rangle - \langle \varphi_A, A^* \psi_V \rangle = \langle A \varphi_A, \psi_V \rangle - \langle \varphi_A, A^* \psi_V \rangle = 0,
$$

since $\psi_V \in \operatorname{dom} A^*$ is still in the domain of A^*. By the A-symmetry of $\operatorname{dom} A$ and V we get

$$
\{\varphi, \psi\}_{A^*} = \{\varphi_A, \psi_A\}_{A^*} + \{\varphi_A, \psi_V\}_{A^*} + \{\varphi_V, \psi_A\}_{A^*} + \{\varphi_V, \psi_V\}_{A^*} = 0 + 0 + 0 + 0 = 0,
$$

proving that $\operatorname{dom} A \oplus V$ is again A-symmetric. This shows that $\operatorname{dom} A \oplus V$ is A-closed and A-symmetric for any A-closed and A-symmetric $V \subseteq \mathfrak{K}_+(A) \oplus \mathfrak{K}_-(A)$. Conversely, let $U \subseteq \operatorname{dom} A^*$ be an A-closed and A-symmetric subspace containing $\operatorname{dom} A \subseteq U$. Since we have the A-orthogonal direct sum decomposition (5.3.22) we have A-orthogonal projections onto the three subspaces which we denote by

$$
\mathrm{pr}_A \colon \operatorname{dom} A^* \to \operatorname{dom} A \quad \text{and} \quad \mathrm{pr}_\pm \colon \operatorname{dom} A^* \to \mathfrak{K}_\pm(A)
$$

in the sequel. We define now

$$
V = (\mathrm{pr}_+ \oplus \mathrm{pr}_-)(U) \subseteq \mathfrak{K}_+(A) \oplus \mathfrak{K}_-(A)
$$

to be the component of U which is A-orthogonal to $\operatorname{dom} A$. Then by (5.3.22) and $\operatorname{dom} A \subseteq U$ we have $U = \operatorname{dom} A \oplus V$. Moreover, the image of a projection is closed and given by the kernel of the complementary projection. In our case, this shows that V is A-closed as wanted. Finally, since U is A-symmetric and $V \subseteq U$ we have for $\varphi, \psi \in V$ still $\{\varphi, \psi\}_{A^*} = 0$ as this holds for all vectors in U anyway. This shows the third part. Finally, $A^*\big|_{\mathfrak{K}_\pm(A)} = \pm\mathrm{i}\,\mathrm{id}\big|_{\mathfrak{K}_\pm(A)}$ is clear by the very definition of $\mathfrak{K}_\pm(A)$ as in (5.3.10). Thus A^* is a bounded operator on $\mathfrak{K}_+(A) \oplus \mathfrak{K}_-(A)$ with operator norm 1 (or even 0 if $\mathfrak{K}_+(A) \oplus \mathfrak{K}_-(A) = \{0\}$). But this shows that for $\phi \in \mathfrak{K}_+(A) \oplus \mathfrak{K}_-(A)$ we have

$$\|\phi\|_A^2 = \langle \phi, \phi \rangle + \langle A^*\phi, A^*\phi \rangle \leq \|\phi\|^2 + \|\phi\|^2 = 2\|\phi\|^2,$$

implying that $\|\cdot\|_A$ is dominated by $\|\cdot\|$. Since trivially the converse is true on all of $\operatorname{dom} A^*$ we see that on $\mathfrak{K}_+(A) \oplus \mathfrak{K}_-(A)$ the two norms are equivalent. Hence the fourth part follows. $\square$

Thus we have to understand the A-closed and A-symmetric subspaces of $\mathfrak{K}_+(A) \oplus \mathfrak{K}_-(A)$ in order to understand the closed symmetric extensions of A. In particular, the last part even shows that we can consider closed subspaces with respect to the original Hilbert space structure. The following theorem gives now a parametrization of those A-symmetric and closed (or A-closed) subspaces in terms of partials isometries with respect to the original inner product:

Theorem 5.3.15 (Closed symmetric extensions) *Let* $(\operatorname{dom} A, A)$ *be a closed symmetric operator in* $\mathfrak{H}$.

i.) For a partial isometry $U\colon \mathfrak{K}_+(A) \longrightarrow \mathfrak{K}_-(A)$ *with respect to* $\langle \cdot\,, \cdot \rangle$

$$\operatorname{dom} A_U = \left\{ \phi + \phi_+ + U\phi_+ \;\middle|\; \phi \in \operatorname{dom} A \text{ and } \phi_+ \in (\ker U)^\perp \subseteq \mathfrak{K}_+(A) \right\} \tag{5.3.24}$$

and

$$A_U(\phi + \phi_+ + U\phi_+) = A\phi + \mathrm{i}\phi_+ - \mathrm{i}U\phi_+ \tag{5.3.25}$$

defines a closed symmetric extension of A.

ii.) One has

$$\mathfrak{K}_+(A_U) = \ker U \subseteq \mathfrak{K}_+(A) \tag{5.3.26}$$

and

$$\mathfrak{K}_-(A_U) = (\operatorname{im} U)^\perp \subseteq \mathfrak{K}_-(A). \tag{5.3.27}$$

iii.) The map $U \mapsto (\operatorname{dom} A_U, A_U)$ *establishes a bijection between the partial isometries and the closed symmetric extensions of* A.

Proof: First we note that $\langle \cdot\,, \cdot \rangle_{A^*}\big|_{\mathfrak{K}_\pm(A)} = 2\langle \cdot\,, \cdot \rangle$ by $A^*\big|_{\mathfrak{K}_\pm(A)} = \pm\mathrm{i}\,\mathrm{id}_{\mathfrak{K}_\pm(A)}$. Hence it does not matter whether we consider U as an partial isometry with respect to $\langle \cdot\,, \cdot \rangle$ or $\langle \cdot\,, \cdot \rangle_{A^*}$, both restricted to $\mathfrak{K}_\pm(A)$, respectively. Furthermore, recall that for a partial isometry $U\colon \mathfrak{K}_+(A) \longrightarrow \mathfrak{K}_-(A)$ we have the orthogonal decompositions

$$\mathfrak{K}_+(A) = \ker(U) \oplus (\ker U)^\perp \quad \text{and} \quad \mathfrak{K}_-(A) = \operatorname{im} U \oplus (\operatorname{im} U)^\perp,$$

as well as $(\ker U)^\perp = \operatorname{im} U^*$ according to Proposition 5.1.41. Since the decomposition (5.3.22) is a direct sum, the linear map A_U is well-defined by (5.3.25) on $\operatorname{dom} A_U$. Moreover, we have $A^*\phi_+ = \mathrm{i}\phi_+$ for $\phi_+ \in (\ker U)^\perp \subseteq \mathfrak{K}_+(A)$ as well as $A^*U\phi_+ = -\mathrm{i}U\phi_+$ since $U\phi_+ \in \mathfrak{K}_-(A)$. Hence we see that

$$A_U(\phi + \phi_+ + U\phi_+) = A\phi + A^*(\phi_+ + U\phi_+) = A^*\big|_{\operatorname{dom} A_U}(\phi + \phi + U\phi_+).$$

Thus A_U is the restriction of A^* to $\operatorname{dom} A_U$. By Theorem 5.3.14, *i.)*, we have to show that $\operatorname{dom} A_U$ is A-closed and A-symmetric to conclude that A_U is a closed symmetric extension of A. To do so it suffices by part *iii.)* of that theorem to show that

$$V = (\ker U)^{\perp} \oplus \operatorname{im} U \tag{$*$}$$

is A-closed and A-symmetric. First we note that $(\ker U)^{\perp}$ and $\operatorname{im} U$ are closed subspaces of $\mathfrak{K}_+(A)$ and $\mathfrak{K}_-(A)$, respectively, since U is a partial isometry. By Theorem 5.3.14, *iv.)*, this implies that they are A-closed, too. Since the decomposition $(*)$ is A-orthogonal also V is A-closed. Now let $\phi_+ + U\phi_+, \psi_+ + U\psi_+ \in V$ be given. Then

$$\begin{aligned}
\big\{\phi_+ + U\phi_+, \psi_+ + U\psi_+\big\}_{A^*} &= \big\langle A^*(\phi_+ + U\phi_+), \psi_+ + U\psi_+\big\rangle - \big\langle \phi_+ + U\phi_+, A^*(\psi_+ + U\psi_+)\big\rangle \\
&= \big\langle \mathrm{i}\phi_+ - \mathrm{i}U\phi_+, \psi_+ + U\psi_+\big\rangle - \big\langle \phi_+ + U\phi_+, \mathrm{i}\psi_+ - \mathrm{i}U\psi_+\big\rangle \\
&= -\mathrm{i}\big\langle \phi_+, \psi_+\big\rangle - \mathrm{i}\big\langle \phi_+, U\psi_+\big\rangle + \mathrm{i}\big\langle U\phi_+, \psi_+\big\rangle + \mathrm{i}\big\langle U\phi_+, U\psi_+\big\rangle \\
&\quad - \mathrm{i}\big\langle \phi_+, \psi_+\big\rangle + \mathrm{i}\big\langle \phi_+, U\psi_+\big\rangle - \mathrm{i}\big\langle U\phi_+, \psi_+\big\rangle + \mathrm{i}\big\langle U\phi_+, U\psi_+\big\rangle \\
&= 0,
\end{aligned}$$

since U is isometric on ϕ_+ and ψ_+. Hence V is also A-symmetric and we can apply Theorem 5.3.14 as wanted. This shows that $(\operatorname{dom} A_U, A_U)$ is indeed a closed symmetric extension of A, proving the first part. To obtain the second part we have to determine the adjoint A_U^* of A_U. We already know by the symmetry of A_U that $A_U^* = A^*\big|_{\operatorname{dom} A_U^*}$ and $\operatorname{dom} A_U^* \subseteq \operatorname{dom} D_{A^*}$, see (5.3.17). Thus we have to determine $\operatorname{dom} A_U^*$. We know that

$$\operatorname{dom} A_U^* = \operatorname{dom} A_U \oplus \mathfrak{K}_+(A_U) \oplus \mathfrak{K}_-(A_U)$$

by Theorem 5.3.14, *ii.)*, applied to A_U with an A_U-orthogonal direct sum decomposition. Moreover, $\mathfrak{K}_\pm(A_U) = \operatorname{im}(\mathrm{i} \pm A_U)^{\perp} = \ker(\mathrm{i} \mp A_U^*)$ and hence we have to determine the image of $\mathrm{i} \pm A_U$. Then

$$(\mathrm{i} \pm A_U)(\phi + \phi_+ + U\phi_+) = \mathrm{i}\phi + \mathrm{i}\phi_+ + \mathrm{i}U\phi_+ \pm A\phi \pm \mathrm{i}\phi_+ \mp \mathrm{i}U\phi_+ = (\mathrm{i} \pm A)\phi + \begin{cases} 2\mathrm{i}\phi_+ & \text{for } + \\ -2\mathrm{i}U\phi_+ & \text{for } - \,. \end{cases} \tag{$**$}$$

Now let $\psi \in \mathfrak{K}_+(A_U) = \operatorname{im}(\mathrm{i} + A_U)^{\perp}$. Then taking $\phi_+ = 0$ in $(**)$ gives that ψ is orthogonal to $(\mathrm{i} + A)\phi$ for all $\phi \in \operatorname{dom} A$. Hence $\psi \in \operatorname{im}(\mathrm{i} + A)^{\perp} = \mathfrak{K}_+(A)$. Second, taking now $\phi = 0$ gives ψ orthogonal to ϕ_+ for all $\phi_+ \in (\ker U)^{\perp} \subseteq \mathfrak{K}_+(A)$. Note that here we take the orthogonal space of $\ker U \subseteq \mathfrak{K}_+(A)$ inside $\mathfrak{K}_+(A)$. Since we already know $\psi \in \mathfrak{K}_+(A)$ we can conclude $\psi \in \ker U$ since $\ker U \subseteq \mathfrak{K}_+(A)$ is closed. This shows $\mathfrak{K}_+(A_U) \subseteq \ker U \subseteq \mathfrak{K}_+(A)$. Conversely, let $\psi \in \ker U$. Since $\mathfrak{K}_+(A) = \operatorname{im}(\mathrm{i} + A)^{\perp}$ we have $\psi \in \operatorname{im}(\mathrm{i} + A)^{\perp}$. This shows that ψ is orthogonal to $\operatorname{im}(\mathrm{i} + A)$ and to $(\ker U)^{\perp}$. Hence we conclude that

$$\mathfrak{K}_+(A_U) = \ker U.$$

The other case is similar for $\psi \in \mathfrak{K}_-(A_U) = \operatorname{im}(\mathrm{i} - A_U)^{\perp}$. We first have again $\psi \in \operatorname{im}(\mathrm{i} - A)^{\perp} = \mathfrak{K}_-(A)$ by taking $\phi_+ = 0$. Now, taking $\phi = 0$ gives that ψ is orthogonal to all $U\phi_+$ for $\phi_+ \in (\ker U)^{\perp} \subseteq \mathfrak{K}_+(A)$. Since U is a partial isometry, the image of U is given by $\operatorname{im} U = \operatorname{im}\big(U\big|_{(\ker U)^{\perp}}\big)$ and we conclude $\psi \in (\operatorname{im} U)^{\perp} \subseteq \mathfrak{K}_-(A)$ since we already know $\psi \in \mathfrak{K}_-(A)$. Conversely, let $\psi \in (\operatorname{im} U)^{\perp} \subseteq \mathfrak{K}_-(A)$ be given then ψ is orthogonal to $\operatorname{im}(\mathrm{i} - A)$ by definition of $\mathfrak{K}_-(A)$ and to those vectors in $\mathfrak{K}_-(A)$ which are orthogonal to $\operatorname{im} U$. Hence

$$\mathfrak{K}_-(A_U) = (\operatorname{im} U)^{\perp},$$

proving the second part. For the third part, we first note that the extension $(\operatorname{dom} A_U, A_U)$ allows to reconstruct the partial isometry $U\colon \mathfrak{K}_+(A) \longrightarrow \mathfrak{K}_-(A)$. By the second part and by the general decomposition we have

$$\operatorname{dom} A^* = \overbrace{(\underbrace{\ker U}_{\mathfrak{K}_+(A_U)} \oplus (\ker U)^\perp)}^{\mathfrak{K}_+(A)} \oplus_A \operatorname{dom} A \oplus_A \overbrace{(\operatorname{im} U \oplus \underbrace{(\operatorname{im} U)^\perp}_{\mathfrak{K}_-(A_U)})}^{\mathfrak{K}_-(A),}, \tag{$\star$}$$

where $\oplus_A$ denotes A-orthogonal direct sums while $\oplus$ are the orthogonal sums in the original sense. The orthogonal complements refer to the surrounding Hilbert spaces $\mathfrak{K}_\pm(A)$, respectively, but not to $\mathfrak{H}$. Moreover, $\operatorname{dom} A_U$ is not all of $\operatorname{dom} A \oplus (\ker U)^\perp \oplus \operatorname{im} U$ but only $\operatorname{dom} A \oplus \operatorname{graph} U$. For $\Phi \in \operatorname{dom} A_U$ we have the A-orthogonal decomposition $\Phi = \operatorname{pr}_A \Phi + \operatorname{pr}_+ \Phi + \operatorname{pr}_- \Phi$ according to ($\star$) with $\operatorname{pr}_A \Phi \in \operatorname{dom} A, \operatorname{pr}_\pm \Phi \in \mathfrak{K}_\pm(A)$. But then $\Phi_+ = \operatorname{pr}_+ \Phi \in (\ker U)^\perp$ and $U\Phi_+ = \operatorname{pr}_- \Phi$. This shows that A_U determines U completely as a partial isometry U is determined by its behaviour on $(\ker U)^\perp$ alone. For the converse, let $(\operatorname{dom} B, B)$ be a closed symmetric extension of A. We have to construct a partial isometry $U\colon \mathfrak{K}_+(A) \longrightarrow \mathfrak{K}_-(A)$ with $(\operatorname{dom} B, B) = (\operatorname{dom} A_U, A_U)$. We know that $\operatorname{dom} B = \operatorname{dom} A \oplus_A V$ with a A-closed and A-symmetric subspace $V \subseteq \mathfrak{K}_+(A) \oplus \mathfrak{K}_-(A)$ and $B = A^*\big|_{\operatorname{dom} B}$ by Theorem 5.3.14. Hence we can decompose $\Phi \in \operatorname{dom} B$ into $\operatorname{pr}_A \Phi + \operatorname{pr}_+ \Phi + \operatorname{pr}_- \Phi$ according to (5.3.22). Since V is A-symmetric we know that for $\Phi = \operatorname{pr}_+ \Phi + \operatorname{pr}_- \Phi \in V$

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$$\begin{aligned}
0 &= \{\Phi, \Phi\}_{A^*} \\
&= \langle A^*\Phi, \Phi\rangle - \langle \Phi, A^*\Phi\rangle \\
&= \big\langle A^*(\operatorname{pr}_+ \Phi + \operatorname{pr}_- \Phi), \operatorname{pr}_+ \Phi + \operatorname{pr}_- \Phi\big\rangle - \big\langle \operatorname{pr}_+ \Phi + \operatorname{pr}_- \Phi, A^*(\operatorname{pr}_+ \Phi + \operatorname{pr}_- \Phi)\big\rangle \\
&= \big\langle \mathrm{i} \operatorname{pr}_+ \Phi - \mathrm{i} \operatorname{pr}_- \Phi, \operatorname{pr}_+ \Phi + \operatorname{pr}_- \Phi\big\rangle - \big\langle \operatorname{pr}_+ \Phi + \operatorname{pr}_- \Phi, \mathrm{i} \operatorname{pr}_+ \Phi - \mathrm{i} \operatorname{pr}_+ \Phi\big\rangle \\
&= 2\mathrm{i}\|\operatorname{pr}_- \Phi\|^2 - 2\mathrm{i}\|\operatorname{pr}_+ \Phi\|^2.
\end{aligned}$$

Thus for $\Phi \in V$ we have $\|\operatorname{pr}_+ \Phi\| = \|\operatorname{pr}_- \Phi\|$. This shows two things: first, Φ is completely determined by one of its components, say $\operatorname{pr}_+ \Phi$ since $\|\Phi\| = 0$ iff $\|\operatorname{pr}_+ \Phi\| = 0$. Second, we have a isometric bijection $\operatorname{pr}_+ \Phi \mapsto \operatorname{pr}_- \Phi$, defined on $\operatorname{pr}_+(V)$ with image $\operatorname{pr}_-(V)$. Now we can extend this to a partial isometry $U\colon \mathfrak{K}_+(A) \longrightarrow \mathfrak{K}_-(A)$ by setting $U = 0$ on $(\operatorname{pr}_+(V))^\perp \subseteq \mathfrak{K}_+(A)$. In particular, $\ker U = \operatorname{pr}_+(V)^\perp$ and $\Phi \in V$ is of the form $\Phi = \operatorname{pr}_+ \Phi + \operatorname{pr}_- \Phi = \phi_+ + U\phi_+$ with $\phi_+ \in (\ker U)^\perp = \operatorname{pr}_+ V$. Finally, for $\Phi = \phi + \phi_+ + U\phi_+$ with $\phi = \operatorname{pr}_A \Phi$ and $\phi_+ = \operatorname{pr}_+ \Phi$ we have

$$B\Phi = A^*\Phi = A^*\phi + A^*\phi_+ + A^*U\phi_+ = A\phi + \mathrm{i}\phi_+ - \mathrm{i}U\phi_+ = A_U\Phi,$$

since $A^*\big|_{\mathfrak{K}_\pm(A)} = \pm\mathrm{i}$ and $B = A^*\big|_{\operatorname{dom} B}$. Thus $(\operatorname{dom} B, B) = (\operatorname{dom} A_U, A_U)$ as wanted, completing the proof. □

Using this theorem we have an efficient criterion to determine the self-adjoint extensions of a (closed) symmetric operator.

Corollary 5.3.16 *Let* $(\operatorname{dom} A, A)$ *be a closed symmetric operator in* $\mathfrak{H}$.

i.) *The operator* A *has self-adjoint extensions iff* $n_+(A) = n_-(A)$.

ii.) *If* $n_+(A) = n_-(A)$ *then the self-adjoint extensions of* A *are in bijection to the unitary maps* $U\colon \mathfrak{K}_+(A) \longrightarrow \mathfrak{K}_-(A)$ *via* $U \leftrightarrow (\operatorname{dom} A_U, A_U)$.

iii.) *If* $n_+(A) = 0 \neq n_-(A)$ *or* $n_+(A) \neq 0 = n_-(A)$ *then* A *is maximally symmetric without self-adjoint extensions.*

PROOF: A closed symmetric extension $(\operatorname{dom} A_U, A_U)$ of A via a partial isometry $U\colon \mathfrak{K}_+(A) \longrightarrow \mathfrak{K}_-(A)$ is self-adjoint iff $\mathfrak{K}_\pm(A_U) = \{0\}$. By Theorem 5.3.15, *ii.)*, this is the case iff $\ker U = \{0\} = (\operatorname{im} U)^\perp$. But this means that $U\colon \mathfrak{K}_+(A) \longrightarrow \mathfrak{K}_-(A)$ is actually a unitary. It follows that $n_+(A) =$

$n_-(A)$. Conversely, if this is the case then any unitary $U\colon \mathfrak{K}_+(A) \longrightarrow \mathfrak{K}_-(A)$ will provide a self-adjoint extension by $(\operatorname{dom} A_U, A_U)$. Thus the first and second part follows. If one of the deficiency spaces is $\{0\}$ then there is no other partial isometry than the zero map, which corresponds to the trivial extension. Hence the third part is clear. $\square$

Remark 5.3.17 (Self-adjoint extensions) For a symmetric operator we can first pass to its closure as guaranteed by Remark 5.3.2, *ii.)*, and hence we can assume that $(\operatorname{dom} A, A)$ is already closed and symmetric. Then the question about self-adjoint extensions boils down to compute $n_\pm(A)$ or, more specifically, the deficiency spaces $\mathfrak{K}_\pm(A) = \operatorname{im}(\mathrm{i} \pm A)^\perp = \ker(\mathrm{i} \mp A^*)$. Thus the lack of A being self-adjoint lies in the possible $\pm\mathrm{i}$ eigenvectors of A^*: here we have to solve an eigenvector equation

$$A^*\phi = \pm\mathrm{i}\phi \tag{5.3.28}$$

to determine $\mathfrak{K}_\pm(A)$. Then the above procedure allows to extend A to a self-adjoint operator and even parametrize the possible extensions.

The following particular situation allows to prove that the dimensions $n_\pm(A)$ are equal without actually computing them. We call a $\mathbb{C}$-antilinear, bijective map

$$T\colon \mathfrak{H} \longrightarrow \mathfrak{H} \tag{5.3.29}$$

anti-unitary if for all $\phi, \psi \in \mathfrak{H}$

$$\langle T\phi, T\psi\rangle = \overline{\langle \phi, \psi\rangle}. \tag{5.3.30}$$

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Proposition 5.3.18 *Let* $(\operatorname{dom} A, A)$ *be a symmetric operator and* $T\colon \mathfrak{H} \longrightarrow \mathfrak{H}$ *an anti-unitary operator such that* $T \operatorname{dom} A = \operatorname{dom} A$ *and* $TA = AT$. *Then*

$$n_+(A) = n_-(A). \tag{5.3.31}$$

Proof: First we note that an anti-unitary map is a $\mathbb{R}$-linear bijection $T\colon \mathfrak{H} \longrightarrow \mathfrak{H}$ with $\|T\phi\| = \|\phi\|$. In particular, T^{-1} is anti-unitary as well. In our case, T^{-1} leaves $\operatorname{dom} A$ invariant and commutes with A, too. Let $\psi \in \operatorname{dom} A$ and $\phi \in \operatorname{dom} A^*$. Then $\langle A^*\phi, \psi\rangle = \langle \phi, A\psi\rangle$ and $\langle T\phi, \psi\rangle = \overline{\langle \phi, T^{-1}\psi\rangle}$. We compute

$$\langle T\phi, A\psi\rangle = \overline{\langle \phi, T^{-1}A\psi\rangle} = \overline{\langle \phi, AT^{-1}\psi\rangle} = \overline{\langle A^*\phi, T^{-1}\psi\rangle} = \langle TA^*\phi, \psi\rangle,$$

which shows that $T\phi \in \operatorname{dom} A^*$ and $A^*T\phi = TA^*\phi$. The same argument holds for T^{-1}. Now let $\phi \in \mathfrak{K}_\pm(A) = \ker(\mathrm{i} \mp A^*)$ then $A^*\phi = \pm\mathrm{i}\phi$ and hence

$$A^*T\phi = TA^*\phi = T(\pm\mathrm{i}\phi) = \mp\mathrm{i}T\phi.$$

This gives $T\phi \in \mathfrak{K}_\mp(A)$ and we have established a $\mathbb{R}$-linear bijection $T\colon \mathfrak{K}_+(A) \longrightarrow \mathfrak{K}_-(A)$. From (5.3.30) we see that a Hilbert basis $\{\phi_j\}_{j\in J}$ of $\mathfrak{K}_+(A)$ is mapped to an orthonormal system of $\mathfrak{K}_-(A)$. If $\psi \in \mathfrak{K}_-(A)$ would be a vector with $\psi \perp T\phi_j$ for all $j \in J$ then $T^{-1}\psi \perp \phi_j$ in $\mathfrak{K}_+(A)$ which implies $T^{-1}\psi = 0$ and by $\mathbb{R}$-linearity $\psi = 0$. Thus $\{T\phi_k\}_{j\in J}$ is a Hilbert basis of $\mathfrak{K}_-(A)$ showing (5.3.31). $\square$

Example 5.3.19 (Time reversal) Consider $\mathfrak{H} = \mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ and the Hamiltonian operator

$$H = -\frac{\hbar^2}{2m}\Delta + V \tag{5.3.32}$$

with a real potential $V = \overline{V} \in \mathscr{C}(\mathbb{R}^n)$. On $\operatorname{dom} H = \mathscr{C}_0^\infty(\mathbb{R}^n)$ the usual integration by parts shows that $(\operatorname{dom} H, H)$ is symmetric (though not yet closed in general). Then we consider the *time reversal* map

$$T \colon \mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x) \ni \psi \;\mapsto\; \overline{\psi} \in \mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x), \tag{5.3.33}$$

which is easily shown to be anti-unitary with $T^2 = \mathrm{id}$. Clearly T maps $\mathscr{C}_0^\infty(\mathbb{R}^n)$ into itself and

$$TH = HT. \tag{5.3.34}$$

Thus, H has self-adjoint extensions. Note however, that the proposition does not tell how large the deficiency indices actually will be. More details will be discussed in the exercises.

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5.3.3 The Cayley Transformation

We shall give now yet another interpretation of the extension problem of closed symmetric operators. Recall that the holomorphic map, called the Cayley transformation,

$$\mathbb{C} \setminus \{-\mathrm{i}\} \in z \;\mapsto\; \frac{z - \mathrm{i}}{z + \mathrm{i}} \in \mathbb{C} \setminus \{1\} \tag{5.3.35}$$

is bijective with holomorphic inverse

$$\mathbb{C} \setminus \{1\} \ni w \;\mapsto \mathrm{i}\frac{1 + w}{1 - w} \in \mathbb{C} \setminus \{\mathrm{i}\}. \tag{5.3.36}$$

It maps the real axis $\mathbb{R} \subseteq \mathbb{C}$ onto the unit circle $S^1 \setminus \{1\}$ excluding the point $\{1\}$, see Exercise **??**. From the continuous calculus we get immediately the following result:

Proposition 5.3.20 *Let $A \in \mathfrak{B}(\mathfrak{H})$ be a Hermitian operator. Then its Cayley transform*

$$U_A = \frac{A - \mathrm{i}}{A + \mathrm{i}} \tag{5.3.37}$$

is a unitary operator whose spectrum does not contain 1*. Conversely, every unitary operator $U \in \mathfrak{B}(\mathfrak{H})$ with $1 \notin \operatorname{spec}(U)$ is of the form* (5.3.37) *with*

$$A = \mathrm{i}\frac{1 + U}{1 - U}, \tag{5.3.38}$$

establishing a bijection between such unitaries and bounded Hermitian operators.

The idea is now to extend this bijective correspondence to symmetric operators on one hand and (partial) isometries on the other hand. Thus let $(\operatorname{dom} A, A)$ be a symmetric operator. As we have seen by an elementary computation in the proof of Theorem 5.3.6, *i.)*, we have

$$\|(A + a + \mathrm{i}b)\phi\|^2 = \|(A + a)\phi\|^2 + b^2\|\phi\|^2 \tag{5.3.39}$$

for $a, b \in \mathbb{R}$. In particular,

$$\|(A \pm \mathrm{i})\phi\|^2 = \|A\phi\|^2 + \|\phi\|^2, \tag{5.3.40}$$

and hence

$$A + \mathrm{i} \colon \operatorname{dom} A \longrightarrow \operatorname{im}(A + \mathrm{i}) \tag{5.3.41}$$

is a linear bijection: symmetric operators (closed or not) do not have eigenvectors for eigenvalues $\pm\mathrm{i}$. We consider $\operatorname{dom} U = \operatorname{im}(A + \mathrm{i})$ and define the linear map $U \colon \operatorname{dom} U \longrightarrow \operatorname{im}(A - \mathrm{i})$ by

$$U(A + \mathrm{i})\phi = (A - \mathrm{i})\phi. \tag{5.3.42}$$

Since (5.3.40) is bijective, this is indeed well-defined and yields a linear map

$$U = (A - \mathrm{i})(A + \mathrm{i})^{-1} \tag{5.3.43}$$

on $\operatorname{dom} U$. From (5.3.39) we see that

$$\|U(A + \mathrm{i})\phi\|^2 = \|(A - \mathrm{i})\phi\|^2 = \|A\phi\|^2 + \|\phi\|^2 = \|(A + \mathrm{i})\phi\|^2, \tag{5.3.44}$$

which means that U is isometric on $\operatorname{dom} U$. Finally, for $\phi \in \operatorname{dom} A$ we have

$$(1 - U)(A + \mathrm{i})\phi = (A + \mathrm{i})\phi - (A - \mathrm{i})\phi = 2\mathrm{i}\phi, \tag{5.3.45}$$

showing that

$$\operatorname{im}(1 - U) = \operatorname{dom} A. \tag{5.3.46}$$

This allows now to state the following definition of the Cayley transform for symmetric operators:

Definition 5.3.21 (Cayley Transformation) *Let* $(\operatorname{dom} A, A)$ *be a symmetric operator in* $\mathfrak{H}$*. Then the operator* $(\operatorname{dom} U, U)$ *in* $\mathfrak{H}$ *with*

$$\operatorname{dom} U = \operatorname{im}(A + \mathrm{i}) \tag{5.3.47}$$

and

$$U = (A - \mathrm{i})(A + \mathrm{i})^{-1} \tag{5.3.48}$$

is called the Cayley transform of A*.*

We denote the Cayley transform of A sometimes by $(\operatorname{dom} U_A, U_A)$ to stress the dependence on A.

The following lemma investigates properties of an isometry $(\operatorname{dom} U, U)$ defined only on some arbitrary subspace $\operatorname{dom} U \subseteq \mathfrak{H}$.

Lemma 5.3.22 *Let* $(\operatorname{dom} U, U)$ *be an isometry in* $\mathfrak{H}$*.*

i.) *The isometry* $U\colon \operatorname{dom} U \longrightarrow \mathfrak{H}$ *is continuous and extends canonically to an isometry* $U\colon \operatorname{dom} U^{\mathrm{cl}} \longrightarrow \mathfrak{H}$*.*

ii.) *If one of the subspaces* $\operatorname{dom} U$*,* $\operatorname{im} U$*, or* $\operatorname{graph} U$ *is closed (viewed as subspaces of* $\mathfrak{H}$ *or* $\mathfrak{H} \oplus \mathfrak{H}$*, respectively) then so are the others.*

iii.) *The canonically extension* $U^{\mathrm{cl}}\colon \operatorname{dom} U^{\mathrm{cl}} \longrightarrow \mathfrak{H}$ *is the closure of* $(\operatorname{dom} U, U)$*.*

iv.) *If* $\operatorname{im}(1 - U)$ *is dense then* $1 - U$ *is injective.*

v.) *If* $\operatorname{im}(1 - U)$ *is dense then* $(\operatorname{dom} A, A)$ *with* $\operatorname{dom} A = \operatorname{im}(1 - U)$ *and*

$$A = \mathrm{i}(1 + U)(1 - U)^{-1} \tag{5.3.49}$$

defines a symmetric operator whose Cayley transform is given by $(\operatorname{dom} U, U)$*.*

vi.) *If* $\operatorname{im}(1 - U)$ *is dense then* $(\operatorname{dom} U, U)$ *is closed iff the corresponding* $(\operatorname{dom} A, A)$ *is closed.*

vii.) *If* $\operatorname{im}(1 - U)$ *is dense then the closure of* $(\operatorname{dom} U, U)$ *corresponds to the closure of* $(\operatorname{dom} A, A)$*.*

Proof: The first part is clear. For the second, let $\phi \in \operatorname{dom} U$. Then by the isometry of U we have

$$\|\phi\|^2_{\mathfrak{H}} = \|U\phi\|^2_{\mathfrak{H}} = \frac{1}{2}\big\|(\phi, U\phi)\big\|^2_{\mathfrak{H}\oplus\mathfrak{H}},$$

which means that the canonical norms on $\operatorname{dom} U \subseteq \mathfrak{H}$, $\operatorname{im} U \subseteq \mathfrak{H}$, and $\operatorname{graph} U \subseteq \mathfrak{H} \oplus \mathfrak{H}$ lead to the same locally convex topology on the linear space $\operatorname{dom} U \cong \operatorname{im} U \cong \operatorname{graph} U$. Thus the second part follows. The third part is then clear as well since the closure of the graph is simply given by the graph of the canonical extension $U^{\mathrm{cl}}\colon \operatorname{dom} U^{\mathrm{cl}} \longrightarrow \mathfrak{H}$. Now let $\phi \in \operatorname{dom} U$ with $U\phi = \phi$, i.e. $\phi \in \ker(1-U)$. Then for all $\psi \in \operatorname{dom} U$ we have

$$\langle \phi, (1-U)\psi \rangle = \langle \phi, \psi \rangle - \langle \phi, U\psi \rangle = \langle \phi, \psi \rangle - \langle U\phi, U\psi \rangle = 0,$$

since U is isometric. Hence $\phi = 0$ follows by the assumption that $\operatorname{im}(1-U)$ is dense. For the fifth part we note that $(\operatorname{dom} A, A)$ is a well-defined and densely defined operator in $\mathfrak{H}$ since $1-U$ is injective on $\operatorname{dom} U$ and hence $(1-U)^{-1}\colon \operatorname{im}(1-U) \longrightarrow \operatorname{dom} U$ is a well-defined linear bijection. Let $\phi, \psi \in \operatorname{dom} U$ and hence $(1-U)\phi, (1-U)\psi \in \operatorname{dom} A$. Then

$$\begin{aligned}
\big\langle (1-U)\phi, A(1-U)\psi \big\rangle &= \big\langle (1-U)\phi, \mathrm{i}(1+U)\psi \big\rangle \\
&= \mathrm{i}\langle \phi, \psi \rangle - \mathrm{i}\langle U\phi, \psi \rangle + \mathrm{i}\langle \phi, U\psi \rangle - \mathrm{i}\langle U\phi, U\psi \rangle \\
&= \mathrm{i}\langle \phi, U\psi \rangle - \mathrm{i}\langle U\phi, \psi \rangle \\
&= \overline{-\mathrm{i}\langle U\psi, \phi \rangle + \mathrm{i}\langle \psi, U\phi \rangle} \\
&= \overline{\big\langle (1-U)\psi, A(1-U)\phi \big\rangle} \\
&= \big\langle A(1-U)\psi, (1-U)\phi \big\rangle.
\end{aligned}$$

Since all vectors in $\operatorname{dom} A$ are obtained this way we conclude $A \subseteq A^*$. We have to compute the Cayley transform $(\operatorname{dom} U_A, U_A)$ of this operator $(\operatorname{dom} A, A)$. By definition $\operatorname{dom} U_A = \operatorname{im}(A + \mathrm{i})$. Now every vector in $\operatorname{dom} A$ is of the form $(1-U)\phi$ with $\phi \in \operatorname{dom} U$. Hence $(A+\mathrm{i})(1-U)\phi = \mathrm{i}(1+U)(1-U)^{-1}(1-U)\phi + \mathrm{i}(1-U)\phi = 2\mathrm{i}\phi$. This shows $\operatorname{dom} U_A = \operatorname{dom} U$ and

$$(A+\mathrm{i})(1-U)\phi = 2\mathrm{i}\phi.$$

For the Cayley transform U_A we have by definition

$$\begin{aligned}
U_A(A+\mathrm{i})(1-U)\phi &= (A-\mathrm{i})(1-U)\phi \\
&= \mathrm{i}(1+U)(1-U)^{-1}(1-U)\phi - \mathrm{i}(1-U)\phi \\
&= 2\mathrm{i}U\phi \\
&= U(A+\mathrm{i})(1-U)\phi.
\end{aligned}$$

Since every vector in $\operatorname{dom} U = \operatorname{dom} U_A$ is of the form $(A+\mathrm{i})(1-U)\phi$ with $\phi \in \operatorname{dom} U$ we conclude $U_A = U$ as wanted. For the sixth part we use once more

$$\|(A+\mathrm{i})\phi\|^2 = \|\phi\|^2 + \|A\phi\|^2 = \|\phi\|_A^2$$

for $\phi \in \operatorname{dom} A$. Thus the graph norm on $\operatorname{dom} A$ coincides with the usual norm on $\operatorname{im}(A+\mathrm{i})$ under the linear bijection $\operatorname{dom} A \cong \operatorname{im}(A+\mathrm{i})$. Since $\operatorname{im}(A+\mathrm{i}) = \operatorname{dom} U_A = \operatorname{dom} U$ the statement follows from part *ii.)*. For the last part, we have seen that $\operatorname{dom} A$ endowed with the graph norm $\|\cdot\|_A$ is isometrically isomorphic to $\operatorname{im}(A+\mathrm{i}) = \operatorname{dom} U$ with the usual norm. Moreover, on $\operatorname{dom} U$ the graph norm $\|\cdot\|_U$ is equivalent to the usual norm. Thus the closures correspond to each other as well. ☐

The Cayley transform of Hermitian operators $A \in \mathfrak{B}(\mathfrak{H})$ extends to a bijective correspondence between symmetric operators $(\operatorname{dom} A, A)$ on the one hand and isometric operators $(\operatorname{dom} U, U)$ with $\operatorname{im}(1-U) \subseteq \mathfrak{H}$ dense on the other hand. Moreover, this bijection preserves closedness and closures of the operators in question. The next lemma relates the extensions of $(\operatorname{dom} A, A)$ to those of its Cayley transform:

Lemma 5.3.23 *Let* $(\operatorname{dom} A, A)$ *be a symmetric operator in* $\mathfrak{H}$ *with Cayley transform* $(\operatorname{dom} U_A, U_A)$. *Then the symmetric extensions of* $(\operatorname{dom} A, A)$ *correspond via the Cayley transform bijectively to the isometric extensions of* $(\operatorname{dom} U_A, U_A)$.

Proof: Indeed, let $(\operatorname{dom} B, B)$ be a symmetric extension of $(\operatorname{dom} A, A)$ then $\operatorname{dom} A \subseteq \operatorname{dom} B$ and $B\big|_{\operatorname{dom} A} = A$ gives immediately $\operatorname{im}(A + \mathrm{i}) \subseteq \operatorname{im}(B + \mathrm{i})$ as well as

$$\begin{aligned} U_B\big|_{\operatorname{im}(A+\mathrm{i})} &= (B - \mathrm{i})(B + \mathrm{i})^{-1}\big|_{\operatorname{im}(A+\mathrm{i})} \\ &= (B - \mathrm{i})(A + \mathrm{i})^{-1}\big|_{\operatorname{im}(A+\mathrm{i})} \\ &= (A - \mathrm{i})(A + \mathrm{i})^{-1}\big|_{\operatorname{im}(A+\mathrm{i})} \\ &= U_A. \end{aligned}$$

Conversely, if $(\operatorname{dom} V, V)$ is an isometric extension of $(\operatorname{dom} U_A, U_A)$ then $\operatorname{im}(1 - V)$ is still dense as it contains $\operatorname{im}(1 - U_A)$. Thus $(\operatorname{dom} V, V)$ is the Cayley transform of some symmetric operator $(\operatorname{dom} B, B)$ with $\operatorname{dom} B = \operatorname{im}(1 - V) \supseteq \operatorname{im}(1 - U_A) = \operatorname{dom} A$. On $\operatorname{dom} A$ we have

$$B\big|_{\operatorname{dom} A} = \mathrm{i}(1 + V)(1 - V)^{-1}\big|_{\operatorname{im}(1-U_A)} = \mathrm{i}(1 + U_A)(1 - U_A)^{-1}\big|_{\operatorname{im}(1-U_A)} = A,$$

since $U_A = V\big|_{\operatorname{dom} U_A}$. Thus $(\operatorname{dom} A, A) \subseteq (\operatorname{dom} B, B)$ holds. □

After these preliminary considerations we can now compare the two ways of extending a symmetric operator symmetrically: the last lemma tells us that symmetric extensions correspond to isometric extensions of its Cayley transform. On the other hand, symmetric extensions correspond to partial isometries between the deficiency spaces. This suggests to relate the involved partial isometries:

Theorem 5.3.24 (Cayley transformation and extensions) *Let $(\operatorname{dom} A, A)$ be a closed symmetric operator in $\mathfrak{H}$ and let $(\operatorname{dom} U_A, U_A)$ be its Cayley transform.*

i.) One has the orthogonal decomposition

$$\mathfrak{H} = \operatorname{dom} U_A \oplus \mathfrak{K}_+(A) = \operatorname{im}(U_A) \oplus \mathfrak{K}_-(A). \tag{5.3.50}$$

ii.) Let $V\colon \mathfrak{K}_+(A) \longrightarrow \mathfrak{K}_-(A)$ be a partial isometry and $(\operatorname{dom} A_V, A_V)$ the closed symmetric extension of $(\operatorname{dom} A, A)$ as in Theorem 5.3.15, i.). Then the Cayley transform of $(\operatorname{dom} A_V, A_V)$ is given by

$$\operatorname{dom} U_{A_V} = \operatorname{dom} U_A \oplus (\ker V)^\perp \tag{5.3.51}$$

and

$$U_{A_V} = U_A \oplus (-V). \tag{5.3.52}$$

iii.) Conversely, if $(\operatorname{dom} \tilde{U}, \tilde{U})$ is a closed isometric extension of $(\operatorname{dom} U_A, U_A)$ corresponding to a closed symmetric extension $(\operatorname{dom} \tilde{A}, \tilde{A})$ of $(\operatorname{dom} A, A)$ then for the partial isometry $V\colon \mathfrak{K}_+(A) \longrightarrow \mathfrak{K}_-(A)$ with

$$V = \begin{cases} -\tilde{U} & \text{on } \mathfrak{K}_+(A) \cap \operatorname{dom} \tilde{U} \\ 0 & \text{on } (\mathfrak{K}_+(A) \cap \operatorname{dom} \tilde{U})^\perp \end{cases} \tag{5.3.53}$$

one has $\operatorname{dom} \tilde{A} = \operatorname{dom} A_V$ and $\tilde{A} = A_V$.

Proof: For the first part we note that $\operatorname{dom} U_A \subseteq \mathfrak{H}$ is closed by Lemma 5.3.22, *vi.)* and *ii.)*. Since $\operatorname{dom} U_A = \operatorname{im}(A + \mathrm{i}) = \ker(\mathrm{i} - A^*)^\perp = \mathfrak{K}_+(A)^\perp$ and $\operatorname{im} U_A = \operatorname{im}(A - \mathrm{i}) = \ker(\mathrm{i} + A^*)^\perp = \mathfrak{K}_-(A)^\perp$ by Remark 5.3.10 and Theorem 5.3.6, *i.)*, we get (5.3.50). Here, of course, we need $(\operatorname{dom} A, A)$ to be a closed symmetric operator. Now let V be such a partial isometry and let

$$\operatorname{dom} A_V = \left\{ \phi + \phi_+ + V\phi_+ \;\middle|\; \phi \in \operatorname{dom} A \text{ and } \phi_+ \in (\ker V)^\perp \subseteq \mathfrak{K}_+(A) \right\}$$

and

$$A_V(\phi + \phi_+ + V\phi_+) = A\phi + \mathrm{i}\phi_+ - \mathrm{i}V\phi_+$$

be the corresponding extension as in Theorem 5.3.15, *i.)*. Then the domain of U_{A_V} is by definition $\operatorname{im}(A_V + \mathrm{i})$ which is given by

$$
\begin{aligned}
\operatorname{im}(A_V + \mathrm{i}) &= \Big\{ A\phi + \mathrm{i}\phi_+ - \mathrm{i}V\phi_+ + \mathrm{i}\phi + \mathrm{i}\phi_+ + \mathrm{i}V\phi_+ \;\Big|\; \phi \in \operatorname{dom} A \text{ and } \phi_+ \in (\ker V)^\perp \Big\} \\
&= \Big\{ (A + \mathrm{i})\phi + 2\mathrm{i}\phi_+ \;\Big|\; \phi \in \operatorname{dom} A \text{ and } \phi_+ \in (\ker V)^\perp \Big\} \\
&= \operatorname{im}(A + \mathrm{i}) \oplus (\ker V)^\perp \\
&= \operatorname{dom} U_A \oplus (\ker V)^\perp,
\end{aligned}
$$

since we know $\operatorname{im}(A + \mathrm{i}) \perp \mathfrak{K}_+(A)$ and hence the sum is direct orthogonal. Moreover, by definition, the Cayley transform is now given by

$$
\begin{aligned}
U_{A_V}(A_V + \mathrm{i})(\phi + \phi_+ + V\phi_+) &= (A_V - \mathrm{i})(\phi + \phi_+ + V\phi_+) \\
&= A\phi + \mathrm{i}\phi_+ - \mathrm{i}V\phi_+ - \mathrm{i}\phi - \mathrm{i}\phi_+ - \mathrm{i}V\phi_+ \\
&= (A - \mathrm{i})\phi - 2\mathrm{i}V\phi_+.
\end{aligned}
$$

Conversely, we compute

$$
\begin{aligned}
(U_A \oplus (-V))(A_V + \mathrm{i})(\phi + \phi_+ + V\phi_+) &= (U_A \oplus (-V))(A\phi + \mathrm{i}\phi_+ - \mathrm{i}V\phi_+ + \mathrm{i}\phi + \mathrm{i}\phi_+ + \mathrm{i}V\phi_+) \\
&= (U_A \oplus (-V))((A + \mathrm{i})\phi + 2\mathrm{i}\phi_+) \\
&= (A - \mathrm{i})\phi - 2\mathrm{i}V\phi_+,
\end{aligned}
$$

since $(A + \mathrm{i})\phi \in \operatorname{im}(A + \mathrm{i}) = \operatorname{dom} U_A$ and $2\mathrm{i}\phi_+ \in (\ker V)^\perp$ is precisely the decomposition needed to apply the block-diagonal operator $U_A \oplus (-V)$. This shows the second part. Finally, given $(\operatorname{dom} \tilde{U}, \tilde{U})$ we have

$$
\begin{aligned}
\operatorname{dom} \tilde{A} &= \operatorname{im}(1 - \tilde{U}) \\
&= \big\{ (1 - \tilde{U})\Phi \;\big|\; \Phi \in \operatorname{dom} \tilde{U} \big\} \\
&= \big\{ (1 - \tilde{U})(\phi + \phi_+) \;\big|\; \Phi = \phi + \phi_+ \text{ with } \phi \in \operatorname{dom} U_A \text{ and } \phi_+ \in \mathfrak{K}_+(A) \cap \operatorname{dom} \tilde{U} \big\} \\
&= \big\{ (1 - U_A)\phi + (1 - \tilde{U})\phi_+ \;\big|\; \phi \in \operatorname{dom} U_A \text{ and } \phi_+ \in (\ker V)^\perp \big\} \\
&= \big\{ (1 - U_A)\phi + (1 + V)\phi_+ \;\big|\; \phi \in \operatorname{dom} U_A \text{ and } \phi_+ \in (\ker V)^\perp \big\} \\
&= \big\{ \psi + \phi_+ + V\phi_+ \;\big|\; \psi \in \operatorname{dom} A \text{ and } \phi_+ \in (\ker V)^\perp \big\} \\
&= \operatorname{dom} A_V,
\end{aligned}
$$

where V is given as in (5.3.53). On this domain we compute for $\psi = (1 - U_A)\phi \in \operatorname{dom} A$ and $\phi_+ \in (\ker V)^\perp$ and hence for $\Phi = (1 - \tilde{U})(\psi + \phi_+) = \psi + \phi_+ + V\phi_+$

$$
\begin{aligned}
\tilde{A}\Phi &= \mathrm{i}(1 + \tilde{U})(1 - \tilde{U})^{-1}\Phi \\
&= \mathrm{i}(1 + \tilde{U})(\psi + \phi_+) \\
&= \mathrm{i}(1 + U_A)\phi + \mathrm{i}(1 + (-V))\phi_+ \\
&= A\mathrm{i}(1 - U_A)\phi + \mathrm{i}\phi_+ - \mathrm{i}V\phi_+ \\
&= A_V(\psi + \phi_+ + V\phi_+) \\
&= A_V\Phi.
\end{aligned}
$$

□

Corollary 5.3.25 *A closed symmetric operator* $(\operatorname{dom} A, A)$ *is self-adjoint iff its Cayley transformation* $(\operatorname{dom} U_A, U_A)$ *is defined on* $\operatorname{dom} U_A = \mathfrak{H}$ *and* U_A *is unitary.*

Corollary 5.3.26 *The self-adjoint extensions of a closed symmetric operator* $(\operatorname{dom} A, A)$ *are in bijection to the unitary extensions of its Cayley transform* $(\operatorname{dom} U_A, U_A)$.

If one is only interested in the Cayley transform, one can give a more direct proof of this theorem avoiding the usage of Theorem 5.3.15, see e.g. [49, Thm. 13.19] for this approach.

While up to now we have not yet seen many examples of closed symmetric operators the theorem can be used to construct such, even for all combinations of deficiency indices. Here the following example provides the starting point:

Example 5.3.27 (Shift operators) Let $\mathfrak{H} = \ell^2(\mathbb{N})$ and define the shift operator

$$U_k \colon \operatorname{span}_{\mathbb{C}}\{\mathrm{e}_n\}_{n\in\mathbb{N}_0} \longrightarrow \operatorname{span}_{\mathbb{C}}\{\mathrm{e}_n\}_{n\in\mathbb{N}_0} \tag{5.3.54}$$

by

$$U_k \colon \mathrm{e}_n \;\mapsto\; \mathrm{e}_{n+k}, \tag{5.3.55}$$

and the corresponding linear extension, where $k > 0$. It is easy to see that U_k is isometric and hence extends to an isometry U_k on $\ell^2(\mathbb{N}_0)$. Consider now the operator $V_k = 1 - U_k$, then

$$V_k \mathrm{e}_n = \mathrm{e}_n - \mathrm{e}_{n+k}. \tag{5.3.56}$$

Assume that $\psi \in \mathfrak{H}$ is orthogonal to $\operatorname{im} V_k = \operatorname{im}(1 - U_k)$. Then $0 = \langle \psi, \mathrm{e}_n - \mathrm{e}_{n+k}\rangle$ implies

$$\langle \mathrm{e}_{n+k}, \psi\rangle = \langle \mathrm{e}_n, \psi\rangle \tag{5.3.57}$$

for all n and hence

$$\langle \mathrm{e}_{Nk+r}, \psi\rangle = \langle \mathrm{e}_r, \psi\rangle \tag{5.3.58}$$

for all $N \in \mathbb{N}$ and $r = 0, \ldots, k-1$. Since $|\langle \mathrm{e}_n, \psi\rangle|^2$ is a zero sequence, (5.3.58) can only be true if all coefficients $\langle \mathrm{e}_n, \psi\rangle = 0$ vanish. Thus $\psi = 0$ and $\operatorname{im}(1 - U_k)$ is dense. By Lemma 5.3.22, *v.)*, the isometry U_k is the Cayley transform of some symmetric operator $(\operatorname{dom} A_k, A_k)$. Since U_k is defined on $\mathfrak{H}$, it is already closed and thus $(\operatorname{dom} A_k, A_k)$ is closed, too. Now

$$\operatorname{im} U_k = \operatorname{span}_{\mathbb{C}}\{\mathrm{e}_0, \ldots, \mathrm{e}_{k-1}\}^{\perp}. \tag{5.3.59}$$

as well as $\operatorname{dom} U_k = \mathfrak{H}$ shows that

$$\mathfrak{K}_+(A_k) = \operatorname{dom} {U_k}^{\perp} = \{0\}, \tag{5.3.60}$$

and

$$\mathfrak{K}_-(A_k) = (\operatorname{im} U_k)^{\perp} = \operatorname{span}_{\mathbb{C}}\{\mathrm{e}_0, \ldots, \mathrm{e}_{k-1}\}, \tag{5.3.61}$$

by Theorem 5.3.24, *i.)*. Thus we have the deficiency indices

$$n_+(A_k) = 0 \text{ and } n_-(A_k) = k, \tag{5.3.62}$$

which shows that A_k is maximally symmetric but it does not have any self-adjoint extensions.

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5.4 Spectral Theorem for Self-Adjoint Operators

The importance of self-adjoint operators lies in the fact that they allow for a spectral theorem analogous to the one for bounded normal operators. Beside the spectral theorem we will see that we also have a calculus for self-adjoint operators which extends the bounded measurable calculus to a measurable calculus without any restrictions. The aspects of having a *calculus*, i.e. a *-homomorphism, have to be relaxed only slightly in order to take care of domains. Once we have established the calculus and the spectral theorem we consider as a first class of applications the Schrödinger equation and its solutions. This way we will meet unitary one-parameter groups whose continuity properties will result in a bounded or unbounded self-adjoint generator, the Hamiltonian operator. The theorem of Stone and von Neumann will not only give us the solutions to Schrödinger type equations but, in the converse, a possibility to construct self-adjoint operators very easily. Finally, we discuss the question whether the measurable calculus can also be obtained as a calculus in the spirit of the holomorphic calculus: while we have defined the exponential of a self-adjoint operator in general by means of a spectral integral, it would be nice to see whether a definition using the exponential series is possible as well. In full generality this is not the case, but for a certain subset of the domain this is indeed possible. This will lead us to the study of analytic and smooth vectors of a self-adjoint operator.

5.4.1 The Measurable Calculus

The first aim is to extend our bounded measurable calculus for projection valued measures, see Theorem 5.1.19 and Theorem 5.1.23, from the bounded to the unbounded case.

Thus let $(X, \mathfrak{a})$ be a measurable space with a projection-valued measure E on it. The aim is now to define an operator $\int_X f \, \mathrm{d}E$ for *any* measurable function, bounded or not. The price we have to pay is that this will not be a bounded operator on $\mathfrak{H}$ but only an unbounded operator with some domain in $\mathfrak{H}$, if f is essentially unbounded. The following lemma provides the tools for defining the unbounded version of $\int_X f \, \mathrm{d}E$:

Lemma 5.4.1 *Let $(X, \mathfrak{a})$ be a measurable space and E a projection-valued measure on $(X, \mathfrak{a})$.*

i.) For every measurable function $f \in \mathscr{M}(X, \mathfrak{a})$ the subset

$$\operatorname{dom} f = \left\{ \phi \in \mathfrak{H} \;\middle|\; \int_X |f|^2 \, \mathrm{d}\langle \phi, E\phi \rangle < \infty \right\} \tag{5.4.1}$$

of $\mathfrak{H}$ is a dense subspace.

ii.) For all $\phi, \psi \in \mathfrak{H}$ and all $f \in \mathscr{M}(X, \mathfrak{a})$ one has

$$\int_X |f| \, \mathrm{d}|\langle \phi, E\psi \rangle| \leq \|\phi\| \sqrt{\int_X |f|^2 \, \mathrm{d}\langle \psi, E\psi \rangle} \tag{5.4.2}$$

as inequality in $[0, +\infty]$.

PROOF: For a measurable subset $U \in \mathfrak{a}$ of X and $\phi, \psi \in \mathfrak{H}$ we know

$$\begin{aligned} \langle \phi + \psi, E_U(\phi + \psi) \rangle &= \|E_U(\phi + \psi)\|^2 \\ &\leq (\|E_U\phi\| + \|E_U\psi\|)^2 \\ &\leq 2(\|E_U\phi\|^2 + \|E_U\psi\|^2) \\ &= 2\langle \phi, E_U\phi \rangle + 2\langle \psi, E_U\psi \rangle. \end{aligned}$$

Thus the positive measure $\langle \phi+\psi, E(\phi+\psi) \rangle$ is dominated by the positive measure $2\langle \phi, E\phi \rangle + 2\langle \psi, E\psi \rangle$. But then it is clear that for all $f \in \mathscr{M}(X, \mathfrak{a})$ we have

$$\int_X |f|^2 \, \mathrm{d}\langle \phi + \psi, E(\phi + \psi) \rangle \leq 2 \int_X |f|^2 \, \mathrm{d}\langle \phi, E\phi \rangle + 2 \int_X |f|^2 \, \mathrm{d}\langle \psi, E\psi \rangle$$

as an inequality in $[0, +\infty]$. Thus $\phi, \psi \in \operatorname{dom} f$ implies that also $\phi + \psi \in \operatorname{dom} f$. Now let $z \in \mathbb{C}$ then $\langle z\phi, E_U z\phi\rangle = |z|^2 \langle \phi, E_U \phi\rangle$ for all $U \in \mathfrak{a}$ and $\phi \in \mathfrak{H}$. Thus for all $f \in \mathscr{M}(X, \mathfrak{a})$

$$\int_X |f|^2 \,\mathrm{d}\langle z\phi, Ez\phi\rangle = |z|^2 \int_X |f|^2 \,\mathrm{d}\langle \phi, E\phi\rangle$$

follows, again as an equality in $[0, +\infty]$. Hence $\phi \in \operatorname{dom} f$ implies $z\phi \in \operatorname{dom} f$. Together we see that $\operatorname{dom} f \subseteq \mathfrak{H}$ is a subspace. Next we consider the following measurable subsets

$$U_n = |f|^{-1}\big([n, n+1)\big) = \big\{x \in X \;\big|\; n \le |f(x)| < n+1\big\}$$

for $n \in \mathbb{N}_0$. We clearly have $U_n \cap U_m = \emptyset$ for $n \neq m$ and $\bigcup_{n=0}^\infty U_n = X$. For $\phi \in \operatorname{im} E_{U_n}$ we have for all $U \in \mathfrak{a}$

$$E_U \phi = E_U E_{U_n} \phi = E_{U \cap U_n} \phi.$$

Hence the measure $\langle \phi, E\phi\rangle$ is concentrated on U_n, i.e. all measurable subsets in $X \setminus U_n$ have measure zero. Thus we get

$$\begin{aligned}
\int_X |f|^2 \,\mathrm{d}\langle \phi, E\phi\rangle &= \int_{U_n} |f|^2 \,\mathrm{d}\langle \phi, E\phi\rangle \\
&\le \int_{U_n} (n+1)^2 \,\mathrm{d}\langle \phi, E\phi\rangle \\
&= (n+1)^2 \langle \phi, E_{U_n} \phi\rangle \\
&= (n+1)^2 \|\phi\|^2,
\end{aligned}$$

which is finite. Hence $\phi \in \operatorname{dom} f$ and thus $\operatorname{im} E_{U_n} \subseteq \operatorname{dom} f$. Since $\operatorname{dom} f$ is a subspace we also have

$$\bigoplus_{n=0}^{\infty} \operatorname{im} E_{U_n} \subseteq \operatorname{dom} f. \tag{$*$}$$

But from $\bigcup_{n=0}^\infty U_n = X$ and Proposition 5.1.16, *ii.)*, we get that the left hand side in $(*)$ is already dense in $\mathfrak{H}$. This completes the proof of the first part. For the second part it is convenient to use the polar decomposition of the complex measure $\langle \phi, E\psi\rangle$ and the function f. By the polar decomposition for complex measures, see Theorem C.3.47, we have a measurable function $u \in \mathscr{M}(X, \mathfrak{a})$ with $u\overline{u} = 1$ and the feature that

$$\langle \phi, E_U \psi\rangle = \int_U u \,\mathrm{d}|\langle \phi, E\psi\rangle|$$

for all $U \in \mathfrak{a}$ which implies

$$\int_X g \,\mathrm{d}\langle \phi, E\psi\rangle = \int_x gu \,\mathrm{d}|\langle \phi, E\psi\rangle|$$

for all $g \in \mathrm{L}^1(X, \mathfrak{a}, |\langle \phi, E\psi\rangle|)$. For f we get a measurable function $v \in \mathscr{M}(X, \mathfrak{a})$ such that $v\overline{v} = 1$ and $f = v|f|$ by the polar decomposition of measurable functions according to Lemma C.3.23. This allows now to compute for $f \in \mathscr{B}\mathscr{M}(X, \mathfrak{a}) \subseteq \mathrm{L}^1(X, \mathfrak{a}, |\langle \phi, E\psi\rangle|)$

$$\begin{aligned}
\int_X |f| \,\mathrm{d}|\langle \phi, E\psi\rangle| &= \int_X f\overline{v}\,\overline{u}u \,\mathrm{d}|\langle \phi, E\psi\rangle| \\
&= \int_X f\overline{v}\,\overline{u} \,\mathrm{d}\langle \phi, E\psi\rangle \\
&= \left\langle \phi, \left(\int_X f\overline{v}\,\overline{u} \,\mathrm{d}E\right)\psi \right\rangle \\
&\le \|\phi\| \left\| \int_X f\overline{v}\,\overline{u} \,\mathrm{d}E\psi \right\|
\end{aligned}$$

by Remark 5.1.20, *iv.)*. Moreover, by the same remark we get

$$\left\| \int_X f \overline{v}\, \overline{u} \, \mathrm{d}E\psi \right\|^2 = \int_X |f \overline{v}\, \overline{u}|^2 \, \mathrm{d}\langle \psi, E\psi \rangle,$$

which implies (5.4.2) since $|\overline{v}\, \overline{u}| = 1$. Now assume $f \in \mathscr{M}(X, \mathfrak{a})$ is not necessarily bounded. If g is now a non-negative simple function with $g \leq |f|$ we have

$$\int_X |g|^2 \, \mathrm{d}\langle \psi, E\psi \rangle \leq \int_X |f|^2 \, \mathrm{d}\langle \psi, E\psi \rangle.$$

For the supremum over all such g we get

$$\sup_g \int_X |g|^2 \, \mathrm{d}\langle \psi, E\psi \rangle = \int_X |f|^2 \, \mathrm{d}\langle \psi, E\psi \rangle$$

by the very definition of the Lebesgue integral of functions $|f| \colon X \longrightarrow [0, \infty]$. Conversely, since for each such g we can apply (5.4.2) we get

$$\int_x |f| \, \mathrm{d}|\langle \phi, E\psi \rangle| = \sup_g \int_X |g| \, \mathrm{d}|\langle \phi, E\psi \rangle| \leq \|\phi\|^2 \sup_g \sqrt{\int_X |g|^2 \, \mathrm{d}\langle \psi, E\psi \rangle} = \|\phi\|^2 \sqrt{\int_X |f|^2 \, \mathrm{d}\langle \psi, E\psi \rangle},$$

which is (5.4.2) for general $f \in \mathscr{M}(X, \mathfrak{a})$. $\square$

Note that we have the equivalence

$$f \in \mathrm{L}^2(X, \mathfrak{a}, \langle \psi, E\psi \rangle) \quad \text{iff} \quad \psi \in \operatorname{dom} f. \tag{5.4.3}$$

The subspace $\operatorname{dom} f$ will now serve as the maximal domain on which we can define $\int_X f \, \mathrm{d}E$ in a meaningful way. The following theorem shows that this is indeed possible. The map which assigns an operator to each measurable function becomes a drastic generalization of the bounded measurable calculus of Theorem 5.1.32.

Theorem 5.4.2 (Measurable calculus) *Let E be a projection-valued measure on $(X, \mathfrak{a})$.*

i.) For $f \in \mathscr{M}(X, \mathfrak{a})$ there exists a unique operator $\big(\operatorname{dom} f, \int_X f \, \mathrm{d}E\big)$ in $\mathfrak{H}$ such that

$$\left\langle \phi, \int_X f \, \mathrm{d}E\psi \right\rangle = \int_X f \, \mathrm{d}\langle \phi, E\psi \rangle \tag{5.4.4}$$

for all $\phi \in \mathfrak{H}$ and $\psi \in \operatorname{dom} f$.

ii.) For each $\psi \in \mathfrak{H}$ and $f_n \in \mathrm{L}^2(X, \mathfrak{a}, \langle \psi, E\psi \rangle)$ with $f_n \longrightarrow f$ in the L^2-sense we have in the norm sense of $\mathfrak{H}$

$$\lim_{n \longrightarrow \infty} \int_X f_n \, \mathrm{d}E\psi = \int_X f \, \mathrm{d}E\psi. \tag{5.4.5}$$

iii.) For $f \in \mathscr{M}(X, \mathfrak{a})$ and $\psi \in \operatorname{dom} f$ on has

$$\left\| \int_X f \, \mathrm{d}E\psi \right\|^2 = \int_X |f|^2 \, \mathrm{d}\langle \psi, E\psi \rangle. \tag{5.4.6}$$

iv.) The map $f \mapsto (\operatorname{dom} f, \int_X f \, \mathrm{d}E)$ is a *-homomorphism in the following weaker sense: for $f, g \in \mathscr{M}(X, \mathfrak{a})$ and $z \in \mathbb{C}$ one has

$$\int_X z f \, \mathrm{d}E = z \int_X f \, \mathrm{d}E, \tag{5.4.7}$$

$$\int_X (f+g)\,\mathrm{d}E \supseteq \int_X f\,\mathrm{d}E + \int_X g\,\mathrm{d}E, \tag{5.4.8}$$

$$\int_X fg\,\mathrm{d}E \supseteq \int_X f\,\mathrm{d}E \int_X g\,\mathrm{d}E, \tag{5.4.9}$$

$$\int_X \overline{f}\,\mathrm{d}E = \left(\int_X f\,\mathrm{d}E\right)^*, \tag{5.4.10}$$

and

$$\int_X f\,\mathrm{d}E \int_X \overline{f}\,\mathrm{d}E = \int_X |f|^2\,\mathrm{d}E = \int_X \overline{f}\,\mathrm{d}E \int_X f\,\mathrm{d}E. \tag{5.4.11}$$

One has

$$\operatorname{dom} \int_X f\,\mathrm{d}E \int_X g\,\mathrm{d}E = \operatorname{dom} g \cap \operatorname{dom} fg, \tag{5.4.12}$$

and in (5.4.9) *equality holds iff* $\operatorname{dom} fg \subseteq \operatorname{dom} g$. *In* (5.4.8) *we have equality iff* $\operatorname{dom} f+g = \operatorname{dom} f \cap \operatorname{dom} g$.

v.) For all $f \in \mathscr{M}(X, \mathfrak{a})$ *the operator* $\int_X f\,\mathrm{d}E$ *is closed.*

PROOF: The uniqueness in the first part is clear. For $\phi \in \mathfrak{H}$ and $\psi \in \operatorname{dom} f$ we have by Lemma 5.4.1, *ii.)*, that f is integrable with respect to $|\langle\phi, E\psi\rangle|$. By definition, f is also integrable with respect to $\langle\phi, E\psi\rangle$ and from Proposition C.3.51, *i.)*, we have

$$\left|\int_X f\,\mathrm{d}\langle\phi, E\psi\rangle\right| \le \int_X |f|\,\mathrm{d}|\langle\phi, E\psi\rangle|.$$

Combining this with the estimate (5.4.2) from Lemma 5.4.1, *ii.)*, we see that the map

$$\mathfrak{H} \ni \phi \;\mapsto\; \int_X f\,\mathrm{d}\langle\phi, E\psi\rangle \in \mathbb{C} \tag{$*$}$$

is a continuous anti-linear map: recall that the measure $\langle\phi, E\psi\rangle$ depends anti-linearly on ϕ. Moreover, the functional norm of $(*)$ is bounded by $\sqrt{\int_X |f|^2\,\mathrm{d}\langle\psi, E\psi\rangle}$ by (5.4.2). Thus the representation theorem of Riesz, see Theorem 3.2.11, shows that there is a unique vector in $\mathfrak{H}$, which we denote by $\int_X f\,\mathrm{d}E\psi$, such that (5.4.4) holds. Moreover, the norm of this vector is equal to the functional norm of $(*)$ and hence estimated by

$$\left\|\int_X f\,\mathrm{d}E\psi\right\|^2 \le \int_X |f|^2\,\mathrm{d}\langle\psi, E\psi\rangle. \tag{$**$}$$

This shows the first part. Now let $f_n \in \mathrm{L}^2(X, \mathfrak{a}, \langle\psi, E\psi\rangle)$ with $f_n \longrightarrow f$ in the L^2-sense be given. This means $\psi \in \operatorname{dom} f_n$ for all $n \in \mathbb{N}$. Moreover, $f, f_n - f \in \mathrm{L}^2(X, \mathfrak{a}, \langle\psi, E\psi\rangle)$ shows that also $\psi \in \operatorname{dom} f_n - f$ and $\psi \in \operatorname{dom} f$. Hence we first see that

$$\begin{aligned}
\left\langle \phi, \int_X f\,\mathrm{d}E\psi - \int_X f_n\,\mathrm{d}E\psi \right\rangle &= \int_X f\,\mathrm{d}\langle\phi, E\psi\rangle - \int_X f_n\,\mathrm{d}\langle\phi, E\psi\rangle \\
&= \int_X (f - f_n)\,\mathrm{d}\langle\phi, E\psi\rangle \\
&= \left\langle \phi, \int_X (f - f_n)\,\mathrm{d}\langle\phi, E\psi\rangle \right\rangle,
\end{aligned} \tag{☺}$$

since, as argued before, $f - f_n$ is integrable with respect to $\langle\phi, E\psi\rangle$ as soon as $\psi \in \operatorname{dom} f - f_n$. This shows that

$$\int_X (f - f_n)\,\mathrm{d}E\psi = \int_X f\,\mathrm{d}E\psi - \int_X f_n\,\mathrm{d}E\psi.$$

Then $(**)$ gives immediately the convergence (5.4.5), proving the second part. Note however, ψ has to be in the domains of all the operators in order to have the convergence. For the third part we have already the estimate $(**)$. Since every $f \in \mathrm{L}^2(X, \mathfrak{a}, \langle \psi, E\psi \rangle)$ can be approximated in the L^2-sense by bounded $f_n \in \mathrm{L}^2(X, \mathfrak{a}, \langle \psi, E\psi \rangle)$ we choose such a sequence f_n. Then for each f_n the equality (5.4.6) holds by the bounded measurable calculus according to Remark 5.1.20, *iv.)*. Then (5.4.5) implies (5.4.6) also for f itself. For the fourth part, the claim (5.4.7) is clear. For $f, g \in \mathscr{M}(X, \mathfrak{a})$ and $\psi \in \mathfrak{H}$ we have $\psi \in \operatorname{dom} f$ and $\psi \in \operatorname{dom} g$ iff $f, g \in \mathrm{L}^2(X, \mathfrak{a}, \langle \psi, E\psi \rangle)$. But then also $f+g \in \mathrm{L}^2(X, \mathfrak{a}, \langle \psi, E\psi \rangle)$ and hence $\psi \in \operatorname{dom} f + g$. This shows $\operatorname{dom} f \cap \operatorname{dom} g \subseteq \operatorname{dom} f + g$. With the same linearity argument as in (☺) we conclude (5.4.8). Clearly, we have equality in (5.4.8) iff the domain $\operatorname{dom} f + g$ of $\int_X (f + g) \,\mathrm{d}E$ is equal to the domain of $\int_X f \,\mathrm{d}E + \int_X g \,\mathrm{d}E$ which is $\operatorname{dom} f \cap \operatorname{dom} g$. For (5.4.9) we first assume that f is in addition bounded. Then we have $\operatorname{dom} f \subseteq \operatorname{dom} fg$ and since $\operatorname{dom} f = \mathfrak{H}$ the composition on the right hand side of (5.4.9) is defined on all of $\operatorname{dom} g$. Let now $\phi \in \mathfrak{H}$ and $\psi \in \operatorname{dom} g$. Set $\chi = \int_X \overline{f} \,\mathrm{d}E\phi$ then we want to relate the two complex measure $\langle \chi, E\psi \rangle$ and $\langle \phi, E\psi \rangle$. For $U \in \mathfrak{a}$ we have

Exercise: Exe $f_n = \chi_{U_n} f$ w $U_n = |f|^{-1}(0$ job. (for any measure!)

$$
\begin{aligned}
\langle \chi, E_U \psi \rangle &= \left\langle \phi, \int_X f \,\mathrm{d}E E_U \psi \right\rangle \\
&= \left\langle \phi, \int_X f \chi_U \,\mathrm{d}E \psi \right\rangle \\
&= \int_X f \chi_U \,\mathrm{d}\langle \phi, E\psi \rangle \\
&= \int_U f \,\mathrm{d}\langle \phi, E\psi \rangle
\end{aligned}
$$

by the bounded measurable calculus and the fact that $E_U = \int_X \chi_U \,\mathrm{d}E$. This means that

$$
\mathrm{d}\langle \chi, E_U \psi \rangle = f \,\mathrm{d}\langle \phi, E\psi \rangle
$$

in the sense of the Radon-Nikodym derivative, see Remark C.3.46, *i.)*, and hence

$$
\begin{aligned}
\left\langle \phi, \int_X f \,\mathrm{d}E \int g \,\mathrm{d}E\psi \right\rangle &= \left\langle \int_X \overline{f} \,\mathrm{d}E\phi, \int g \,\mathrm{d}E\psi \right\rangle \\
&= \left\langle \chi, \int g \,\mathrm{d}E\psi \right\rangle \\
&= \int_X g \,\mathrm{d}\langle \chi, E\psi \rangle \\
&= \int_X g f \,\mathrm{d}\langle \phi, E\psi \rangle \\
&= \left\langle \phi, \int fg \,\mathrm{d}E\psi \right\rangle.
\end{aligned}
$$

Since $\phi \in \mathfrak{H}$ was arbitrary, this shows (5.4.9) for the case of $f \in \mathscr{B}\mathscr{M}(X, \mathfrak{a})$. Now set $\eta = \int_X g \,\mathrm{d}E\psi$ then we have for a bounded $f \in \mathscr{B}\mathscr{M}(X, \mathfrak{a})$

$$
\int_X |f|^2 \,\mathrm{d}\langle \eta, E\eta \rangle = \left\| \int f \,\mathrm{d}E\eta \right\|^2 = \left\| \int fg \,\mathrm{d}E\psi \right\|^2 = \int_X |f|^2 |g|^2 \,\mathrm{d}\langle \psi, E\psi \rangle, \tag{$\star$}
$$

and hence

$$
\mathrm{d}\langle \eta, E\eta \rangle = |g|^2 \,\mathrm{d}\langle \psi, E\psi \rangle
$$

follows right away. But with this equality of positive measures we see that $(\star)$ holds for arbitrary $f \in \mathscr{M}(X, \mathfrak{a})$ as an equality in $[0, +\infty]$. In particular, the left hand side of $(\star)$ is finite iff $\eta \in \operatorname{dom} f$

while the right hand side is finite iff $\psi \in \operatorname{dom} fg$ since $\eta = \int g \,\mathrm{d}E\psi$ with $\psi \in \operatorname{dom} g$ and $\int_X g \,\mathrm{d}E\psi = \eta \in \operatorname{dom} f$ by the definition of the product of unbounded operators as in Remark 5.2.3. Thus we conclude that for all $f, g \in \mathscr{M}(X, \mathfrak{a})$ we have

$$\operatorname{dom} \int_X f \,\mathrm{d}E \int_X g \,\mathrm{d}E = \operatorname{dom} g \cap \operatorname{dom} fg.$$

Now let $\psi \in \operatorname{dom} g \cap \operatorname{dom} fg$ be given such that the composition on the right hand side is defined and set $\eta = \int_X g \,\mathrm{d}E\psi$ as before. Again we can choose $f_n \in \mathscr{BM}(X, \mathfrak{a})$ such that $|f_n| \leq |f|$ and $f_n \longrightarrow f$ pointwise. Then $\eta \in \operatorname{dom} f$ and hence $f \in \mathrm{L}^2(X, \mathfrak{a}, \langle \eta, E\eta \rangle)$ shows $f_n \longrightarrow f$ in the L^2-sense. Analogously, we have $\psi \in \operatorname{dom} fg$ and hence $fg \in \mathrm{L}^2(X, \mathfrak{a}, \langle \psi, E\psi \rangle)$ with $f_n g \longrightarrow fg$ in the L^2-sense. Then the second part together with (5.4.9) for the bounded f_n give (5.4.9) for f and g since

$$\int_X f \,\mathrm{d}E \int_X g \,\mathrm{d}E\psi \overset{\textit{ii.)}}{=} \lim_{n \longrightarrow \infty} \int_X f_n \,\mathrm{d}E \int_X g \,\mathrm{d}E\psi \overset{(5.4.9)}{=} \lim_{n \longrightarrow \infty} \int_X f_n g \,\mathrm{d}E\psi \overset{\textit{ii.)}}{=} \int_X fg \,\mathrm{d}E\psi.$$

For (5.4.10) we first note $\operatorname{dom} f = \operatorname{dom} \overline{f}$ and by $\langle \phi, E\psi \rangle = \overline{\langle \psi, E\phi \rangle}$ we get immediately from the defining property

$$\left\langle \phi, \int_X f \,\mathrm{d}E\psi \right\rangle = \int_X f \,\mathrm{d}\langle \phi, E\psi \rangle = \overline{\int_X \overline{f} \,\mathrm{d}\langle \phi, E\phi \rangle} = \overline{\left\langle \psi, \int \overline{f} \,\mathrm{d}E\phi \right\rangle} = \left\langle \int \overline{f} \,\mathrm{d}E\phi, \psi \right\rangle,$$

showing $\int_X \overline{f} \,\mathrm{d}E \subseteq \left(\int f \,\mathrm{d}E\right)^*$. To show the reverse, we again use the $U_n = \left\{x \in X \;\middle|\; |f(x)| \leq n\right\}$ and set $f_n = \chi_{U_n} f$ as before. Moreover, let $\psi \in \operatorname{dom} \left(\int_X f \,\mathrm{d}E\right)^*$ and set $\eta = \left(\int_X f \,\mathrm{d}E\right)^* \psi$: Since the χ_{U_n} as well as the f_n are bounded, we can apply (5.4.9) to this product, noting that $\operatorname{dom} \chi_{U_n} = \mathfrak{H} = \operatorname{dom} f_n$. Thus by (5.4.12) we see that we have equality in (5.4.9) in this case, i.e. for all $\phi \in \mathfrak{H}$ we get

$$\int_X f_n \,\mathrm{d}E\phi = \int_X f \,\mathrm{d}E \int \chi_{U_n} \,\mathrm{d}E\phi = \int_X f \,\mathrm{d}E E_{U_n} \phi.$$

Thus we get by Proposition 5.2.8, *iii.)*,

$$E_{U_n} \left(\int_X f \,\mathrm{d}E\right)^* \subseteq \left(\int_X f \,\mathrm{d}E E_{U_n}\right)^* = \left(\int_X f_n \,\mathrm{d}E\right)^* = \int_X \overline{f_n} \,\mathrm{d}E,$$

since for the bounded f_n we can apply the bounded measurable calculus. Hence

$$\int_X \overline{f_n} \,\mathrm{d}E\psi = E_{U_n} \left(\int_X f \,\mathrm{d}E\right)^* \psi = E_{U_n} \eta$$

follows. By monotonous convergence we conclude

$$\int |f|^2 \,\mathrm{d}\langle \psi, E\psi \rangle = \lim_{n \longrightarrow \infty} \int |f_n|^2 \,\mathrm{d}\langle \psi, E\psi \rangle = \lim_{n \longrightarrow \infty} \langle E_{U_n} \eta, E_{U_n} \eta \rangle = \langle \eta, \eta \rangle < \infty,$$

where in the second step we used again (5.4.5) for the bounded measurable function f_n. This shows $\psi \in \operatorname{dom} f = \operatorname{dom} \overline{f}$ and hence we have the equality (5.4.10). Equation (5.4.11) is now a simple consequence of (5.4.9). First we note that for a finite measure space we have that $|f|^2 \in \mathrm{L}^2(X, \mathfrak{a}, \langle \psi, E\psi \rangle)$ implies $f \in \mathrm{L}^2(X, \mathfrak{a}, \langle \psi, E\psi \rangle)$. This shows $\operatorname{dom} f \cap \operatorname{dom} |f|^2 = \operatorname{dom} |f|^2$ and thus $\operatorname{dom} |f|^2 \subseteq \operatorname{dom} f$.
ise: Exercise!
Now we apply (5.4.9) for f and $g = \overline{f}$ to get the equality (5.4.11) directly. Finally, the last part is clear since $\int_X f \,\mathrm{d}E = \left(\int_X \overline{f} \,\mathrm{d}E\right)^*$ by (5.4.10) and adjoint operators are necessarily closed, see Theorem 5.2.14, *i.)*. □

This indeed provides a tremendous generalization of Theorem 5.1.32. We list now a couple of consequences which we met during the proof:

Corollary 5.4.3 *Let E be a projection-valued measure and let $g \in \mathscr{BM}(X, \mathfrak{a})$ and $f \in \mathscr{M}(X, \mathfrak{a})$. Then*
$$\int_X g \, \mathrm{d}E \int_X f \, \mathrm{d}E \subseteq \int_X f \, \mathrm{d}E \int_X g \, \mathrm{d}E = \int_X fg \, \mathrm{d}E, \tag{5.4.13}$$
and in particular for $U \in \mathfrak{a}$
$$E_U \int_X f \, \mathrm{d}E \subseteq \int_X f \, \mathrm{d}E E_U = \int_X \chi_U f \, \mathrm{d}E. \tag{5.4.14}$$

Corollary 5.4.4 *Let E be a projection-valued measure on $(X, \mathfrak{a})$ and $f \in \mathscr{M}(X, \mathfrak{a})$. Then the subspaces $\operatorname{im} E_U \cap \operatorname{dom} f \subseteq \operatorname{dom} f$ are invariant under $\int_X f \, \mathrm{d}E$.*

Proof: This is precisely the first inclusion in (5.4.13). □

Corollary 5.4.5 *Let E be a projection-valued measure on $(X, \mathfrak{a})$ and $f = \overline{f} \in \mathscr{M}(X, \mathfrak{a})$. Then the operator $(\operatorname{dom} f, \int_X f \, \mathrm{d}E)$ is self-adjoint.*

Corollary 5.4.6 *Let E be a projection-valued measure on $(X, \mathfrak{a})$ and $f \in \mathscr{M}(X, \mathfrak{a})$. Then for every $n \in \mathbb{N}$ we have*
$$\left(\int_X f \, \mathrm{d}E \right)^n = \int_X f^n \, \mathrm{d}E. \tag{5.4.15}$$

Proof: We know from part *iv.)* of Theorem 5.4.2 the inclusion
$$\int_X f^n \, \mathrm{d}E \supseteq \int_X f \, \mathrm{d}E \int_X f^{n-1} \, \mathrm{d}E \tag{$*$}$$
and hence by induction
$$\left(\int_X f \, \mathrm{d}E \right)^n \subseteq \int_X f^n \, \mathrm{d}E.$$
Now the domains of the operators in $(*)$ coincide iff $\operatorname{dom} f f^{n-1} \subseteq \operatorname{dom} f^{n-1}$. Since we have a finite measure space $(X, \mathfrak{a}, \mathrm{d}\langle \psi, E\psi \rangle)$ for every $\psi \in \mathfrak{H}$, we know that $\mathrm{L}^p(X, \mathfrak{a}, \mathrm{d}\langle \psi, E\psi \rangle) \subseteq \mathrm{L}^{p'}(X, \mathfrak{a}, \mathrm{d}\langle \psi, E\psi \rangle)$ for all $p \geq p'$. In particular, $\psi \in \operatorname{dom} f^n$ is equivalent to $|f^n|^2$ being integrable, or $f \in \mathrm{L}^{2n}(X, \mathfrak{a}, \mathrm{d}\langle \psi, E\psi \rangle)$. But then also $f \in \mathrm{L}^{2(n-1)}(X, \mathfrak{a}, \mathrm{d}\langle \psi, E\psi \rangle)$ and hence $\psi \in \operatorname{dom} f^{n-1}$. This shows equality in (5.4.15).□

Remark 5.4.7 In general, the map $f \mapsto \int_X f \, \mathrm{d}E$ is not injective. From (5.4.6) we see that the operator vanishes on all $\psi \in \operatorname{dom} f$ iff $|f|^2$ and hence f is a zero function for all the positive measures $\langle \psi, E\psi \rangle$ with $\psi \in \operatorname{dom} f$. Since $\operatorname{dom} f$ is dense we see that for $U \in \mathfrak{a}$ we have
$$E_U = 0 \quad \text{iff} \quad \langle \psi, E_U \psi \rangle = 0 \tag{5.4.16}$$
for all $\psi \in \operatorname{dom} f$. Thus f is a zero function with respect to E. The converse is clear and was already discussed in Theorem 5.1.34, *ii.)*. Thus also in this unbounded setting we can factorize to measurable functions modulo zero functions.

Corollary 5.4.8 *Let E be a projection-valued measure on $(X, \mathfrak{a})$ and $f \in \mathscr{M}(X, \mathfrak{a})$. Then $\int_X f \, \mathrm{d}E$ is bounded iff $f \in \mathrm{L}^\infty(X, \mathfrak{a}, E)$ iff $\operatorname{dom} f = \mathfrak{H}$.*

Proof: Since we can ignore zero functions anyway and since every essentially bounded function can be made a bounded function by modifying it on a subset of measure zero, the spectral integral results in a bounded operator by the bounded measurable calculus. Now let $\int_X f \, \mathrm{d}E$ be bounded. This is the case iff $\operatorname{dom} f = \mathfrak{H}$ since $\int_X f \, \mathrm{d}E$ is a closed operator. We use again the truncations $f_n = \chi_{U_n} f$ as before in the proof of Theorem 5.4.2. Then f_n and χ_{U_n} are bounded with $\|f_n\|_\infty \leq n$ and $\|\chi_{U_n}\|_\infty \leq 1$.

Suppose that f is essentially unbounded, i.e. none of the subsets $|f|^{-1}([n,\infty)) = X \setminus U$ are zero sets. Then $E_{X\setminus U_n} \neq 0$ and we find $\psi_n \in \operatorname{im} E_{X\setminus U_n}$ with $\|\psi_n\| = 1$. Since $\operatorname{dom} f = \mathfrak{H}$ we can apply Theorem 5.4.2, *iii.)*, and get

$$
\begin{aligned}
\left\| \int_X f \,\mathrm{d}E\psi_n \right\|^2 &= \int |f|^2 \,\mathrm{d}\langle \psi_n, E\psi_n\rangle \\
&\geq \int_{X\setminus U_n} |f|^2 \,\mathrm{d}\langle \psi_n, E\psi_n\rangle \\
&\geq n^2 \int_{X\setminus U_n} \mathrm{d}\langle \psi_n, E\psi_n\rangle \\
&= n^2 \int_X \chi_{X\setminus U_n} \,\mathrm{d}\langle \psi_n, E\psi_n\rangle \\
&= n^2 \|E_{X\setminus U_n}\psi_n\|^2 \\
&= n^2.
\end{aligned}
$$

Thus $\int f \,\mathrm{d}E$ can not be bounded which is a contradiction. □

The following example is now of fundamental importance for quantum mechanics:

Example 5.4.9 (Multiplication operators) Let $(X, \mathfrak{a}, \mu)$ be a measurable space and let $\mathfrak{H} = \mathrm{L}^2(X, \mathfrak{a}, \mu)$. Then for $U \in \mathfrak{a}$ one defines the operator

$$
E_U \colon \mathfrak{H} \ni \psi \ \mapsto \ \chi_U \psi \in \mathfrak{H}, \tag{5.4.17}
$$

which is clearly a projection. It is now a simple verification that this defines a projection-valued measure E on $(X, \mathfrak{a})$, see Exercise 5.5.18. For $\phi, \psi \in \mathrm{L}^2(X, \mathfrak{a}, \mu)$ we have

$$
\langle \phi, E_U \psi\rangle = \int_X \overline{\phi} \chi_U \psi \,\mathrm{d}\mu = \int_U \overline{\phi}\psi \,\mathrm{d}\mu, \tag{5.4.18}
$$

and thus $\mathrm{d}\langle \phi, E\psi\rangle = \overline{\phi}\psi \,\mathrm{d}\mu$ follows. This shows that for $f \in \mathscr{M}(X, \mathfrak{a})$ we have

$$
\operatorname{dom} f = \left\{ \psi \in \mathrm{L}^2(X, \mathfrak{a}, \mu) \;\middle|\; \int_X |f|^2 |\psi|^2 \,\mathrm{d}\mu < \infty \right\}. \tag{5.4.19}
$$

Moreover, for $\phi \in \mathrm{L}^2(X, \mathfrak{a}, \mu)$ and $\psi \in \operatorname{dom} f$ we have

$$
\left\langle \phi, \int_X f \,\mathrm{d}E\psi \right\rangle = \int_X f \,\mathrm{d}\langle \phi, E\psi\rangle = \int_X f \overline{\phi}\psi \,\mathrm{d}\mu = \langle \phi, f\psi\rangle. \tag{5.4.20}
$$

Hence the operator $\int_X f \,\mathrm{d}E$ is nothing else than the multiplication operator

$$
\int_X f \,\mathrm{d}E \colon \operatorname{dom} f \ni \psi \ \mapsto \ f\psi \in \mathrm{L}^2(X, \mathfrak{a}, \mu). \tag{5.4.21}
$$

By Corollary 5.4.5 the multiplication operator is self-adjoint if $f = \overline{f}$ and by Corollary 5.4.8 it is bounded iff f is essentially bounded.

Example 5.4.10 (Position operators) Let $\mathfrak{H} = \mathrm{L}^2(U, \mathrm{d}^n x)$ with some non-empty open subset $U \subseteq \mathbb{R}^n$. Then the position operators

$$
Q_i \colon \operatorname{dom} Q_i \ni \psi \mapsto (x \mapsto x_i \psi(x)) \in \mathfrak{H} \tag{5.4.22}
$$

are self-adjoint, where

$$
\operatorname{dom} Q_i = \left\{ \psi \in \mathrm{L}^2(U, \mathrm{d}^n x) \;\middle|\; \int_U x_i^2 \psi(x) \,\mathrm{d}^n x < \infty \right\}. \tag{5.4.23}
$$

As already for the bounded measurable calculus we want to compute the spectrum of $\int_X f \, \mathrm{d}E$ for general $f \in \mathscr{M}(X, \mathfrak{a})$. Here we get the following analogue of Theorem 5.1.23, *iii.)*:

Theorem 5.4.11 (Spectral mapping theorem) *Let E be a projection-valued measure on $(X, \mathfrak{a})$ and $f \in \mathscr{M}(X, \mathfrak{a})$.*

i.) If $\lambda \in \operatorname{ess\,range}(f)$ and $E_{f^{-1}(\{\lambda\})} \neq 0$ then λ is an eigenvalue of $\int_X f \, \mathrm{d}E$ and

$$\operatorname{im} E_{f^{-1}(\{\lambda\})} = \ker\left(\lambda - \int_X f \, \mathrm{d}E\right). \tag{5.4.24}$$

ii.) If $\lambda \in \operatorname{ess\,range}(f)$ and $E_{f^{-1}(\{\lambda\})} = 0$ then the operator $\lambda - \int_X f \, \mathrm{d}E$ is injective. Moreover, the image $\operatorname{im}\bigl(\lambda - \int_X f \, \mathrm{d}E\bigr) \subseteq \mathfrak{H}$ is a proper dense subspace and there exists a sequence $\phi_n \in \operatorname{dom} f$ with $\|\phi_n\| = 1$ and

$$\lim_{n \longrightarrow \infty} \left(\int f \, \mathrm{d}E \phi_n - \lambda \phi_n \right) = 0. \tag{5.4.25}$$

iii.) One has

$$\operatorname{ess\,range}(f) = \operatorname{spec}\left(\int_X f \, \mathrm{d}E \right). \tag{5.4.26}$$

PROOF: Replacing f by $f - \lambda$ we may assume $\lambda = 0$ from the beginning. Assume that $E_{f^{-1}(\{0\})} \neq 0$ and let $\psi \in \operatorname{im} E_{f^{-1}(\{0\})}$ with $\|\psi\| = 1$ be given. Note that necessarily $0 \in \operatorname{ess\,range}(f)$ to have $E_{f^{-1}(\{0\})} \neq 0$. Since $\chi_{f^{-1}(\{0\})} f = 0$ we see from Corollary 5.4.3 that $\operatorname{dom} \chi_{f^{-1}(\{0\})} f = \mathfrak{H}$ and hence $\int_X f \, \mathrm{d}E E_{f^{-1}(\{0\})}$ is defined on all of $\mathfrak{H}$. Thus for $\psi = E_{f^{-1}(\{0\})} \psi \in \operatorname{dom} f$ and we have

$$\int_X f \, \mathrm{d}E E_{f^{-1}(\{0\})} \psi = \int_X f \chi_{f^{-1}(\{0\})} \, \mathrm{d}E \psi = 0,$$

and thus $\int_X f \, \mathrm{d}E$ is not injective, $\operatorname{im} E_{f^{-1}(\{0\})} \subseteq \ker \int_X f \, \mathrm{d}E$. To prove the converse statement, we can essentially repeat the argument from the bounded version of the Spectral Mapping Theorem in form of Theorem 5.1.34, *i.)*. We consider again the subsets

$$U_1 = |f|^{-1}([1, \infty)) \quad \text{and} \quad U_n = |f|^{-1}\left(\left[\tfrac{1}{n}, \tfrac{1}{n-1}\right)\right)$$

for $n \geq 2$. Hence $X = f^{-1}(\{0\}) \cup \bigcup_{n=1}^{\infty} U_n$ is a disjoint countable union. The functions

$$f_n(x) = \begin{cases} \frac{1}{f(x)} & \text{on } U_n \\ 0 & \text{else} \end{cases}$$

are again bounded by n and satisfy for all n

$$f_n f = \chi_{U_n}.$$

Thus by Theorem 5.4.2, *iv.)*, we have $\operatorname{dom} f_n = \mathfrak{H} = \operatorname{dom} \chi_{U_n}$ and hence $\int_X f_n \, \mathrm{d}E \int_X f \, \mathrm{d}E$ is defined on $\operatorname{dom} f$ while $\int_X \chi_{U_n} \, \mathrm{d}E = E_{U_n}$ is of course defined on $\mathfrak{H}$. Now $\phi \in \ker \int_X f \, \mathrm{d}E$ satisfies $E_{U_n} \phi = 0$ by Theorem 5.4.2, *iv.)*. From here we can continue as in the proof of Theorem 5.1.34, *i.)*, to conclude $\phi = E_{f^{-1}(\{0\})} \phi$. For the second part, let $0 \in \operatorname{ess\,range}(f)$ but $E_{f^{-1}(\{0\})} = 0$. Hence for every $n \in \mathbb{N}$ we have $E_{f^{-1}\left(\mathrm{B}_{\frac{1}{n}}(0)\right)} \neq 0$ by the definition of the essential range. We choose $\phi_n \in \operatorname{im} E_{f^{-1}\left(\mathrm{B}_{\frac{1}{n}}(0)\right)}$. Since $|f \chi_{f^{-1}\left(\mathrm{B}_{\frac{1}{n}}(0)\right)}| \leq \frac{1}{n}$ we have a bounded function implying $\operatorname{dom} f \chi_{f^{-1}(\mathrm{B}_{\frac{1}{n}}(0))} = \mathfrak{H}$. Again, by Corollary 5.4.3, this gives $\operatorname{im} E_{f^{-1}\left(\mathrm{B}_{\frac{1}{n}}(0)\right)} \subseteq \operatorname{dom} f$ and

$$\left\| \int_X f \, \mathrm{d}E \phi_n \right\| = \left\| \int_X f \, \mathrm{d}E E_{f^{-1}(\mathrm{B}_{\frac{1}{n}}(0))} \phi_n \right\| = \left\| \int_X f \chi_{f^{-1}(\mathrm{B}_{\frac{1}{n}}(0))} \, \mathrm{d}E \phi_n \right\| \leq \frac{1}{n} \|\phi_n\|.$$

Hence (5.4.25) follows. We have to show that $\int_X f \, \mathrm{d}E$ is injective with dense image. Suppose $\phi \in \ker \int_X f \, \mathrm{d}E$. Then by Theorem 5.4.2, *iii.)*, we have

$$\int_X |f|^2 \, \mathrm{d}\langle \phi, E\phi \rangle = 0. \tag{$*$}$$

Since $E_{f^{-1}(\{0\})} = 0$ the function $|f|^2$ is strictly positive up to a zero set with respect to E, namely $f^{-1}(\{0\})$. Since every zero set with respect to E is also a zero set with respect to any $\langle \phi, E\phi \rangle$ we see that $f^{-1}(\{0\})$ is a zero set with respect to the measure $\langle \phi, E\phi \rangle$. On the other hand, $(*)$ means that the points where f is non-zero have to be a zero-set with respect to $\langle \phi, E\phi \rangle$. Hence X is a $\langle \phi, E\phi \rangle$ zero set which means $\phi = 0$. Thus $\int_X f \, \mathrm{d}E$ is injective. By Proposition 5.2.7, *iv.)*, we see that the image of $\int_X f \, \mathrm{d}E$ is orthogonal to the kernel of $\left(\int_X f \, \mathrm{d}E\right)^* = \int_X \overline{f} \, \mathrm{d}E$. Since $\operatorname{ess\,range}(\overline{f}) = \overline{\operatorname{ess\,range}(f)}$ contains 0 as well and since the estimate $(*)$ holds for $\overline{f}$ as well, we see that $\ker \int_X \overline{f} \, \mathrm{d}E = \{0\}$, too, since the zero sets of f and $\overline{f}$ coincide: $E_{f^{-1}(\{0\})} = E_{\overline{f}^{-1}(\{0\})}$. Thus it follows that $\operatorname{im} \int_X f \, \mathrm{d}E$ is dense. If the image would be all of $\mathfrak{H}$, we would have an inverse operator $\left(\int_X f \, \mathrm{d}E\right)^{-1} \colon \mathfrak{H} \longrightarrow \operatorname{dom} f$ of a closed operator $\int_X f \, \mathrm{d}E$ which would be closed itself by Proposition 5.2.19, *iii.)*, and hence continuous by Proposition 5.2.19, *iv.)*. But this is in contradiction to the existence of the sequence (5.4.25). For the third part we know that $\operatorname{ess\,range}(f) \subseteq \operatorname{spec}(f)$. Thus let $\lambda \in \operatorname{spec}(f)$ be a point not in the essential range. Shifting things we can assume $\lambda = 0$. Then there exists $\varepsilon > 0$ with $E_{f^{-1}(\mathrm{B}_\varepsilon(0))} = 0$. On $X \setminus f^{-1}(\mathrm{B}_\varepsilon(0))$ the function $g = \frac{1}{f}$ is well-defined and bounded by $\frac{1}{\varepsilon}$. We extend g to a bounded measurable function on $(X, \mathfrak{a})$ and get $fg = 1$ on $X \setminus f^{-1}(\mathrm{B}_\varepsilon(0))$. Then on one hand $\int_X g \, \mathrm{d}E$ is bounded. On the other hand we have

$$\int_X f \, \mathrm{d}E \int_X g \, \mathrm{d}E = \int_X fg \, \mathrm{d}E = \int_{X \setminus f^{-1}(\mathrm{B}_\varepsilon(0))} \mathrm{d}E = \mathrm{id}_{\mathfrak{H}}$$

on $\operatorname{dom} g \cap \operatorname{dom} gf = \mathfrak{H} \cap \operatorname{dom} 1 = \mathfrak{H}$. Hence $\int_X f \, \mathrm{d}E$ is surjective. Moreover $|f|^2 > 0$ except on the zero set $f^{-1}(\mathrm{B}_\varepsilon(0))$ which allows to argue as in $(*)$ to conclude that $\int_X f \, \mathrm{d}E$ is also injective. Hence we have a contradiction. ☐

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5.4.2 The Spectral Theorem for Self-Adjoint Operators

We are now in the position to formulate the spectral theorem for self-adjoint operators, generalizing our previous formulation from Theorem 5.1.32 to the unbounded situation. The main idea is to use the bounded version for the unitary and hence normal Cayley transform of a self-adjoint operator: To this end we first introduce the push-forward of a projection-valued measure:

Proposition 5.4.12 *Let E be a projection-valued measure on $(X, \mathfrak{a})$ and let $\Phi \colon (X, \mathfrak{a}) \longrightarrow (Y, \mathfrak{b})$ be a measurable map.*

i.) *For $U \in \mathfrak{b}$ one defines $(\Phi_* E)(U) = E_{\Phi^{-1}(U)}$. Then $\Phi_* E$ is a projection-valued measure on $(Y, \mathfrak{b})$.*

ii.) *For every $f \in \mathscr{M}(Y, \mathfrak{b})$ one has*

$$\int_Y f \, \mathrm{d}(\Phi_* E) = \int_X (f \circ \Phi) \, \mathrm{d}E. \tag{5.4.27}$$

Proof: We have to verify the properties of a projection-valued measure. First it is clear that $(\Phi_* E)(U) = E_{\Phi^{-1}(U)}$ is a projection. Since $\Phi^{-1}(\emptyset) = \emptyset$, we have $(\Phi_* E)(\emptyset) = 0$. Moreover, $\Phi^{-1}(Y) = X$ gives $(\Phi_* E)(Y) = E_X = \mathbb{1}$. Now let $U \cap V = \emptyset$ be measurable disjoint subsets in Y. Then also $\Phi^{-1}(U) \cap \Phi^{-1}(V) = \emptyset$ and hence $(\Phi_* E)(U \cup V) = E_{\Phi^{-1}(U \cup V)} = E_{\Phi^{-1}(U) \cup \Phi^{-1}(V)} = E_{\Phi^{-1}(U)} +$

$E_{\phi^{-1}(V)} = (\Phi_* E)(U) + (\Phi_* E)(V)$. Finally, for $\varphi, \psi \in \mathfrak{H}$ and pairwise disjoint $U_1, U_2, \ldots \in \mathfrak{b}$, also $\Phi^{-1}(U_1), \Phi^{-1}(U_2), \ldots \in \mathfrak{a}$ are disjoint and hence

$$\left\langle \varphi, (\Phi_* E)\Big(\bigcup\nolimits_{n=1}^{\infty} U_n\Big)\psi \right\rangle = \left\langle \varphi, E_{\bigcup_{n=1}^{\infty} \Phi^{-1}(U_n)}\psi \right\rangle = \sum_{n=1}^{\infty} \big\langle \varphi, E_{\Phi^{-1}(U_n)}\psi \big\rangle = \sum_{n=1}^{\infty} \big\langle \varphi, (\Phi_* E)(U_n)\psi \big\rangle.$$

This shows that $\Phi_* E$ is indeed a projection-valued measure, completely parallel to the case of scalar measures as discussed in Proposition C.2.23. Moreover, this computation shows that the complex measure $\langle \varphi, E\psi \rangle$ is pushed-forward to $\langle \varphi, \Phi_* E\psi \rangle$ by Φ, i.e. we have

$$\Phi_* \langle \varphi, E\psi \rangle = \langle \varphi, \Phi_* E\psi \rangle.$$

Thus for $f \in \mathscr{M}(Y, \mathfrak{b})$ we have by ordinary scalar integration theory

$$\int_Y |f|^2 \, \mathrm{d}\langle \varphi, \Phi_* E\varphi \rangle = \int_Y |f|^2 \, \mathrm{d}\Phi_* \langle \varphi, E\varphi \rangle = \int_X |f|^2 \circ \Phi \, \mathrm{d}\langle \varphi, E\varphi \rangle = \int_X |f \circ \Phi|^2 \, \mathrm{d}\langle \varphi, E\varphi \rangle.$$

Hence $\varphi \in \operatorname{dom} f$ with respect to $\Phi_* E$ iff $\varphi \in \operatorname{dom} f \circ \Phi$ with respect to E, showing that the domains of the two operators in (5.4.27) coincide. Finally, again by scalar integration theory we get for all $\varphi \in \mathfrak{H}$ and $\psi \in \operatorname{dom} f = \operatorname{dom} f \circ \Phi$

$$\begin{aligned} \left\langle \varphi, \int_Y f \, \mathrm{d}\Phi_* E\psi \right\rangle &= \int_Y f \, \mathrm{d}\langle \varphi, \Phi_* E\psi \rangle \\ &= \int_Y f \, \mathrm{d}\Phi_* \langle \varphi, E\psi \rangle \\ &= \int_X f \circ \Phi \, \mathrm{d}\langle \varphi, E\psi \rangle \\ &= \left\langle \varphi, \int_X f \circ \Phi \, \mathrm{d}E\psi \right\rangle, \end{aligned}$$

showing that the two operators coincide. □

In combination with the Cayley transform this is the key to get the spectral theorem for self-adjoint operators: from $(\operatorname{dom} A, A)$ we pass to the Cayley transform U_A which we know to be unitary. Thus by the bounded normal spectral theorem we get a spectral measure for U_A. Pushing this forward with the inverse of the Cayley transform gives the desired spectral measure for $(\operatorname{dom} A, A)$. In detail, we have the following result:

Theorem 5.4.13 (Spectral theorem for self-adjoint operators) *Let* $(\operatorname{dom} A, A)$ *be a self-adjoint operator in* $\mathfrak{H}$. *Then there exists a unique projection-valued measure* E *on* $\operatorname{spec}(A) \subseteq \mathbb{R}$ *such that*

$$A = \int_{\operatorname{spec}(A)} \lambda \, \mathrm{d}E, \tag{5.4.28}$$

where λ *is viewed as the identity function on* $\operatorname{spec}(A)$ *as usual.*

Proof: Denote by U_A the Cayley transform of A which we know to be unitary by Corollary 5.3.25. We write

$$\Phi\colon \mathbb{C} \setminus \{-\mathrm{i}\} \ni z \;\mapsto\; \frac{z - \mathrm{i}}{z + \mathrm{i}} \in \mathbb{C} \setminus \{1\}$$

for the Cayley transform. By the spectral theorem for bounded normal operators we get a spectral measure $\tilde{E}$ for U_A defined on $\operatorname{spec}(U_A) \subseteq \mathbb{S}^1$, since U_A is unitary, see Theorem 5.1.32. From the fact $\operatorname{dom} A = \operatorname{im}(1 - U_A)$ and Lemma 5.3.22 we know that $1 - U_A$ is injective, i.e. 1 is not an eigenvalue of U_A. But then either $1 \notin \operatorname{spec}(U_A)$ at all, a case which is of marginal interest here as it corresponds to a bounded A, as we shall see later, or $1 \in \operatorname{spec}(U_A)$ but $\tilde{E}_{\{1\}} = 0$. Thus the point $\{1\} \subseteq \operatorname{spec}(U_A)$ is of measure zero with respect to the spectral measure $\tilde{E}$ of U_A. We claim now that the Cayley back-transformation of U_A to A can be expressed by the spectral integral of the real-valued function $\Phi^{-1} \in \mathscr{M}(\mathbb{S}^1)$ with respect to $\tilde{E}$. Indeed, let $\operatorname{dom} \Phi^{-1}$ and $\int_{\operatorname{spec}(U_A)} \Phi^{-1} \, \mathrm{d}\tilde{E}$ be defined according to Theorem 5.4.2. Note that the point $1 \in \mathbb{S}^1$ does not play any role by $\tilde{E}_{\{1\}} = 0$. By Corollary 5.4.5 this is a self-adjoint operator. For the bounded functions $\lambda \mapsto (1-\lambda)$ and $\lambda \mapsto \mathrm{i}(1+\lambda)$ the measurable calculus gives the equality

$$\int_{\operatorname{spec}(U_A)} \Phi^{-1} \, \mathrm{d}\tilde{E}(1 - U_A) = \mathrm{i}(1 + U_A),$$

and hence $\operatorname{dom} A = \operatorname{im}(1 - U_A) \subseteq \operatorname{dom} \Phi^{-1}$. But for $\psi \in \operatorname{dom} A$ we have $A(1 - U_A)\psi = \mathrm{i}(1 + U_A)\psi$ and hence $\int_{\operatorname{spec}(A)} \Phi^{-1} \, \mathrm{d}\tilde{E}$ coincides with A on $\operatorname{dom} A$. Since A is already self-adjoint it can not have any proper self-adjoint extensions. Thus

$$A = \int_{\operatorname{spec}(U_A)} \Phi^{-1} \, \mathrm{d}\tilde{E}$$

follows. Here we also see that if $1 \notin \operatorname{spec}(U_A)$ the function Φ^{-1} is bounded on $\operatorname{spec}(U_A)$ since this is a compact subset of $\mathbb{S}^1$. Thus A is bounded by Corollary 5.4.8 and hence we are in the non-interesting situation. But now things are very simple. We take the measure $\tilde{E}$ restricted to $\operatorname{spec}(U_A) \setminus \{1\}$. The crucial point is that this is again a projection-valued measure. Here every axiom is clear except the completeness which follows from $\tilde{E}_{\operatorname{spec}(U_A)\setminus\{1\}} = \tilde{E}_{\operatorname{spec}(U_A)} - \tilde{E}_{\{1\}} = \mathbb{1} - 0$ since $\{1\}$ is a set of measure zero. Thus by Proposition 5.4.12

$$A = \int_{\operatorname{spec}(U_A)\setminus\{1\}} \Phi^{-1} \, \mathrm{d}\tilde{E} = \int_{\Phi^{-1}(\operatorname{spec}(U_A)\setminus\{1\})} \Phi^{-1} \circ \Phi \, \mathrm{d}\Phi^{-1}_* \tilde{E} = \int_{\operatorname{spec}(A)} \lambda \, \mathrm{d}\Phi^{-1}_* \tilde{E},$$

since $\Phi^{-1} \circ \Phi = \lambda$ and $\Phi^{-1}(\operatorname{spec}(U_A)) = \operatorname{spec}(A)$ by the spectral mapping theorem, see Theorem 5.4.11, and the fact that open subsets of $\operatorname{spec}(U_A)$ have non-zero measure with respect to $\tilde{E}$ by Remark 5.1.31: this guarantees $\operatorname{ess\,range}(\Phi^{-1}) = \operatorname{range} \Phi^{-1}$ for the continuous map $\Phi^{-1} \colon \operatorname{spec}(U_A) \setminus \{1\} \longrightarrow \operatorname{spec}(A)$. Thus $E = \Phi^{-1}_* \tilde{E}$ will do the job. For the uniqueness, suppose $\tilde{E}'$ is another spectral measure for A, i.e.

$$\int_{\operatorname{spec}(A)} \lambda \, \mathrm{d}E' = A = \int_{\operatorname{spec}(A)} \lambda \, \mathrm{d}E.$$

Since Φ and Φ^{-1} implement measurable (even continuous) bijections between $\operatorname{spec}(U_A) \setminus \{1\}$ and $\operatorname{spec}(A)$ we can push-forward E and E' via Φ to a projection-valued measure on $\operatorname{spec}(U_A) \setminus \{1\}$. On one hand, we get back $\tilde{E}$, on the other hand, the measurable calculus gives us by an analogous argument as before

$$U_A = \int_{\operatorname{spec}(A)} \Phi \, \mathrm{d}E' = \int_{\Phi(\operatorname{spec}(A)) = \operatorname{spec}(U_A)\setminus\{1\}} \lambda \, \mathrm{d}\Phi_* E'.$$

Extending $\Phi_* E'$ to $\operatorname{spec}(U_A)$ by setting it equal to 0 on $\{1\}$ gives then a spectral measure on $\operatorname{spec}(U_A)$ which represents U_A. But for a normal bounded operator we know already that the spectral measure is uniquely determined, i.e. $\Phi_* E' = \Phi_* E = \tilde{E}$ holds, see Theorem 5.1.32, *i.)*, once again. But this implies $E' = E$ as well, since Φ is bijective. □

Remark 5.4.14 (Spectral theorem) Let $(\operatorname{dom} A, A)$ be a self-adjoint operator.

i.) As in the bounded case we call E the *spectral measure* of $(\operatorname{dom} A, A)$.

ii.) For a measurable function $f \in \mathscr{M}(\operatorname{spec}(A))$ one defines

$$f(A) = \int_{\operatorname{spec}(A)} f \, \mathrm{d}E. \tag{5.4.29}$$

Here we have to check a few consistencies, but it turns out to be compatible with our previous definitions of, say A^n, see Corollary 5.4.6. In particular, we can now build bounded functions of the unbounded self-adjoint operator.

iii.) If $f \in \mathscr{C}(\operatorname{spec}(A))$ is even continuous then $\operatorname{ess\,range}(f) = (\operatorname{range}(f))^{\mathrm{cl}}$, see Exercise 5.5.19. Thus the spectral mapping theorem becomes in this case

$$\operatorname{spec}\left(\int_{\operatorname{spec}(A)} f \, \mathrm{d}E\right) = \big(f(\operatorname{spec}(A))\big)^{\mathrm{cl}}. \tag{5.4.30}$$

Remark 5.4.15 (Normal operators) A densely defined operator $(\operatorname{dom} A, A)$ is called normal if $A^*A = AA^*$ which, as usual, includes the equality $\operatorname{dom} A^*A = \operatorname{dom} AA^*$. Also for such operators one has a spectral theorem with a spectral measure supported on $\operatorname{spec}(A) \subseteq \mathbb{C}$. However, we shall not enter the discussion of this generalization of the bounded normal case, see e.g. [49, Thm. 3.33] for a precise formulation.

Remark 5.4.16 The spectral theorem together with the (unbounded) measurable calculus has uncountable applications, we shall only indicate some in the sequel. Moreover, many of the results on bounded normal operators can now be transferred to the unbounded case as well, in particular our discussion of eigenvalues according to Theorem 5.1.34 can be translated immediately.

Example 5.4.17 Let $(\operatorname{dom} A, A)$ be self-adjoint with countable spectrum $\{\lambda_1, \lambda_2, \ldots\} = \operatorname{spec}(A)$. Since in this case the σ-algebra is just the power set of $\operatorname{spec}(A)$ every function f on $\operatorname{spec}(A)$ is measurable. Now let $I_N \subseteq \operatorname{spec}(A)$ be a subset with $\# I_N = N < \infty$ and define

$$f_{I_N} = \chi_{I_N} f = \begin{cases} f(\lambda_i) & \text{if } i \in I_N \\ 0 & \text{else.} \end{cases} \tag{5.4.31}$$

Clearly $|f_{I_N}|^2 \leq |f|^2$ and hence f_{I_N} will be square integrable with respect to $\langle \phi, E\phi \rangle$ for $\phi \in \operatorname{dom} f$. Moreover, $f_{I_N} \longrightarrow f$ for $N \longrightarrow \infty$ in the L^2-sense with respect to $\langle \phi, E\phi \rangle$ by Lebesgue's dominated convergence. Thus we get

$$\int_{\operatorname{spec}(A)} f_{I_N}(\lambda) \, \mathrm{d}E\phi = \sum_{i \in I_N} f(\lambda_i) E_{\{\lambda_i\}} \phi = \sum_{i \in I_N} f(\lambda_i) P_{\ker(\lambda_i - A)} \phi, \tag{5.4.32}$$

and the left hand sides converges to $\int_{\operatorname{spec}(A)} f \, \mathrm{d}E\phi = f(A)\phi$ in norm by Theorem 5.4.2, *ii.)*. This gives the (unconditional) convergence in norm of the series

$$\sum_{n=0}^{\infty} f(\lambda_n) P_{\ker(\lambda_n - A)} \phi = f(A)\phi \tag{5.4.33}$$

for all $\phi \in \operatorname{dom} f(A)$.

5.4.3 Schrödinger Equation and One-Parameter Groups

Recall that the quantum mechanical time evolution is governed by the *Schrödinger equation*

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} \psi(t) = H\psi(t), \tag{5.4.34}$$

where $\psi(t) \in \mathfrak{H}$ is a curve of vectors in Hilbert space and H denotes the *Hamiltonian* of the system under consideration. In realistic quantum mechanical models, H is an unbounded but hopefully self-adjoint operator with domain $\operatorname{dom} H \subseteq \mathfrak{H}$. The question of time evolution consists now in studying the above differential equation and in finding its solution $\psi(t)$ for a given initial condition $\psi(0) \in \mathfrak{H}$. Here we meet immediately several difficulties: to make sense, (5.4.34) can only be considered on $\operatorname{dom} H$ and not on $\mathfrak{H}$ itself. Thus we have to guarantee $\psi(t) \in \operatorname{dom} H$ for a given $\psi(0) \in \operatorname{dom} H$. Moreover, since H is unbounded, no general Banach space argument can be used to guarantee the existence of solutions, etc.

Before going into the details we proceed heuristically. As candidate for the solution one would expect

$$\psi(t) = U_t \psi(0) \tag{5.4.35}$$

with

$$U_t = \exp\left(-\frac{\mathrm{i}t}{\hbar} H\right), \tag{5.4.36}$$

since a formal differentiation of (5.4.36) gives (5.4.34) immediately. However, unless H is bounded, the exponential is not defined by the holomorphic calculus as a convergent power series but by the measurable calculus as a spectral integral. Thus we have to understand how to differentiate spectral integrals with respect to parameters before discussing (5.4.36). In any case, we expect U_t to be unitary and a *one-parameter group*, i.e. for all $t, s \in \mathbb{R}$ we have

$$U_{t+s} = U_t U_s \quad \text{and} \quad U_0 = \operatorname{id}. \tag{5.4.37}$$

Beside the fact that an exponential series clearly has these properties, (5.4.37) is of course crucial to understand (5.4.34): it expresses the deterministic character of the differential equation and its unique solvability. Rephrasing (5.4.37) we see that

$$U \colon (\mathbb{R}, +) \ni t \;\mapsto\; U_t \in \mathfrak{U}(\mathfrak{H}) \tag{5.4.38}$$

is a group homomorphism from the abelian group $(\mathbb{R}, +)$ into the group $\mathfrak{U}(\mathfrak{H})$ of unitary operators on $\mathfrak{H}$. We will take this now as a starting point to make our heuristic considerations more precise: while (5.4.38) is purely algebraic and can be considered for an arbitrary vector space $\mathfrak{H}$ instead of a Hilbert space and for arbitrary invertible endomorphisms instead of unitary ones, the following additional continuity properties refer to the topological structure we have on $\mathfrak{H}$:

Definition 5.4.18 (Continuous unitary one-parameter group) *Let $U \colon (\mathbb{R}, +) \longrightarrow \mathfrak{U}(\mathfrak{H})$ be a unitary one-parameter group for a Hilbert space $\mathfrak{H}$.*

i.) The one-parameter group U is called norm-continuous if $t \mapsto U_t$ is continuous with respect to the operator norm topology on $\mathfrak{U}(\mathfrak{H})$.

ii.) The one-parameter group U is called strongly continuous (or continuous) if $t \mapsto U_t$ is continuous in the strong operator topology on $\mathfrak{U}(\mathfrak{H})$.

iii.) The one-parameter group U is called weakly continuous if $t \mapsto U_t$ is continuous in the weak operator topology on $\mathfrak{U}(\mathfrak{H})$.

Continuous as $G \times \mathfrak{H} \longrightarrow \mathfrak{H}$

For a unitary one-parameter group we clearly have the implications

$$\text{norm-continuous} \implies \text{strongly continuous} \implies \text{weakly continuous} \tag{5.4.39}$$

according to the results of Theorem 5.1.10. In principle, one could also consider the various other topologies as well. Since we insist on unitary U_t for $t \in \mathbb{R}$ the following general fact becomes relevant:

Proposition 5.4.19 *On the unitary group $\mathfrak{U}(\mathfrak{H}) \subseteq \mathfrak{B}(\mathfrak{H})$ of a Hilbert space the strong and the weak operator topologies coincide.*

Proof: In general, the strong topology is finer than the weak one. Hence we have to show the converse. Let $\psi \in \mathfrak{H}$ and $U \in \mathfrak{U}(\mathfrak{H})$ be unitary. Then we show that the strong ε-ball $\mathrm{B}_{\varepsilon, \|\cdot\|_\psi}(U)$ around U with respect to the seminorm $\|A\|_\psi = \|A\psi\|$ from the strong topology is also open in the weak topology, when restricted to $\mathfrak{U}(\mathfrak{H})$. Thus let $V \in \mathfrak{U}(\mathfrak{H})$ then

$$
\begin{aligned}
\|U\psi - V\psi\|^2 &= \langle U\psi - V\psi, U\psi - V\psi\rangle \\
&= \langle U\psi, (U - V)\psi\rangle + \langle V\psi, V\psi\rangle - \langle V\psi, U\psi\rangle \\
&= \langle U\psi, (U - V)\psi\rangle + \langle U\psi, U\psi\rangle - \langle V\psi, U\psi\rangle \\
&= \langle U\psi, (U - V)\psi\rangle + \langle (U - V)\psi, U\psi\rangle \\
&\leq 2|\langle U\psi, (U - V)\psi\rangle|.
\end{aligned}
$$

Thus if $V \in \mathrm{B}_{\frac{\varepsilon^2}{2}, \|\cdot\|_{\psi, U\psi}}(U)$ is close to U in the weak topology, i.e. $|\langle U\psi, (U - V)\psi\rangle| \leq \frac{\varepsilon^2}{2}$ then also $V \in \mathrm{B}_{\varepsilon, \|\cdot\|_\psi}(U)$. Thus $\mathrm{B}_{\varepsilon, \|\cdot\|_\psi}(U)$ contains $\mathrm{B}_{\frac{\varepsilon^2}{2}, \|\cdot\|_{\psi, U\psi}}(U)$ and hence the strong topology is coarser than the weak one, as this is true for every $U \in \mathfrak{U}(\mathfrak{H})$. □

Corollary 5.4.20 *A unitary one-parameter group is strongly continuous iff it is weakly continuous.*

The next little result shows that thanks to the one-parameter group property we only have to check continuity at $t = 0$. Since this is a general feature even beyond the Hilbert space setting, we formulate it for locally convex spaces as follows:

Proposition 5.4.21 *Let V be a locally convex space and let $U\colon (\mathbb{R}, +) \longrightarrow L(V)$ a one-parameter group of continuous endomorphisms on V.*

i.) For $v \in V$ the map $t \mapsto U_t v$ is continuous iff it is continuous at $t = 0$.

ii.) If in addition V is a Banach space then U is norm continuous iff it is norm continuous at $t = 0$.

Proof: Clearly continuity of whatever sort implies the same continuity at $t = 0$. Now let $v \in V$ be fixed and $t \mapsto U_t v$ continuous at $t = 0$. Then for $s \in \mathbb{R}$ we have

$$
\lim_{t \longrightarrow 0} U_{s+t} v = \lim_{t \longrightarrow 0} U_s U_t v = U_s \lim_{t \longrightarrow 0} U_t v = U_s v,
$$

which is continuity at s. Note that we have used the continuity of $U_s \in L(V)$ in a crucial way. For the second case, we have the estimate

$$
\|U_{s+t} - U_s\| = \|U_s U_t - U_s\| \leq \|U_s\| \|U_t - \mathbb{1}\|,
$$

from which the continuity at s follows at once. □

Clearly the first situation gives the result that a one-parameter group is strongly continuous iff it is strongly continuous at $t = 0$. By a slight modification of the above argument, it is clear that there is nothing special about $t = 0$: continuity at one point $t \in \mathbb{R}$ implies continuity everywhere.

The norm continuous one-parameter groups, even for general Banach spaces without the unitarity requirement, are now described explicitly as exponential series as follows:

Theorem 5.4.22 (Norm continuous one-parameter groups) *Let V be a Banach space and let $U\colon \mathbb{R} \longrightarrow L(V)$ be a one-parameter group. Then the following statements are equivalent:*

i.) The one-parameter group U is norm continuous.

ii.) The one-parameter group U is norm differentiable at $t = 0$.

iii.) *The one-parameter group U is norm differentiable at $t = 0$ and*

$$A\psi = \frac{1}{\mathrm{i}} \lim_{t \longrightarrow 0} \frac{U_t\psi - \psi}{t} \tag{5.4.40}$$

defines a continuous operator $A \in L(V)$ on V.

iv.) *The one-parameter group U is norm-analytic in t and of the form*

$$U_t = \sum_{n=0}^{\infty} \frac{(\mathrm{i}tA)^n}{n!} = \mathrm{e}^{\mathrm{i}tA} \tag{5.4.41}$$

with a unique $A \in L(V)$ in the sense of the entire holomorphic calculus of the Banach algebra $L(V)$.

v.) *The one-parameter group U has a holomorphic extension to a complex one-parameter group*

$$U \colon \mathbb{C} \ni z \ \mapsto \ U_z \in L(V) \tag{5.4.42}$$

of the form

$$U_z = \sum_{n=0}^{\infty} \frac{(\mathrm{i}zA)^n}{n!} = \mathrm{e}^{\mathrm{i}zA} \tag{5.4.43}$$

with a unique $A \in L(V)$ in the sense of the entire holomorphic calculus.

If in addition $V = \mathfrak{H}$ is a Hilbert space and U_t is unitary for all t then the operator A from (5.4.40) *is Hermitian*

$$A = A^*. \tag{5.4.44}$$

PROOF: We clearly have *iv.)* $\iff$ *v.)* as the exponential series is norm convergent for $z \in \mathbb{C}$. In this case, A is uniquely determined as it can be recovered by differentiation at $z = 0$. Also *v.)* $\implies$ *ii.)* and *ii.)* $\implies$ *i.)* are clear since continuity at $t = 0$ is continuity everywhere and differentiability implies continuity. So we have to take care of *i.)* $\implies$ *iii.)* which is the non-trivial step as well as *iii.)* $\implies$ *iv.)*. In this proof we make extensive use of Riemann integrals of Banach space-valued functions on the real line, see Section B.2 for details on such integrals. The essence is that we can handle them essentially as in the usual scalar situation. Thus assume *i.)* then the norm-continuity of $t \mapsto U_t$ gives the estimate:

$$\left\| \int_a^b U_t \,\mathrm{d}t \right\| \le \int_a^b \|U_t\| \,\mathrm{d}t \le (b - a) \max_{t \in [a,b]} \|U_t\|.$$

Fix $t_0 > 0$ such that for all $|t| < t_0$ we have

$$\|U_t - \mathbb{1}\| < 1, \tag{$*$}$$

which is possible by norm-continuity. Now for $|t| < t_0$ and $t \ne 0$ we define

$$S_t = \frac{1}{t} \int_0^t U_s \,\mathrm{d}s \in L(V), \tag{☺☺}$$

which gives a norm-continuous map $t \mapsto S_t$ for $0 < |t| < t_0$. Moreover,

$$\|S_t - \mathrm{id}\| = \left\| \frac{1}{t} \int_0^t U_s \,\mathrm{d}s - \mathrm{id} \right\| = \left\| \frac{1}{t} \int_0^t (U_s - \mathrm{id}) \,\mathrm{d}s \right\| \le \max_{s \in [0,t]} \|U_s - \mathrm{id}\| \tag{$**$}$$

shows that $S_0 = \mathrm{id}$ gives a continuous extension of S_t for $t = 0$. Since $|t| < t_0$ we also conclude from $(*)$ and $(**)$ that

$$\|S_t - \mathrm{id}\| < 1,$$

and thus S_t is invertible for all $|t| < t_0$ according to Theorem 4.2.14, *i.)*. Finally, for $t, s \neq 0$ and $|t|, |s| < t_0$ we have

$$\begin{aligned}
\frac{U_s - \mathrm{id}}{s} S_t &= \frac{U_s - \mathrm{id}}{st} \int_0^t U_\sigma \, \mathrm{d}\sigma \\
&= \frac{1}{st} \int_0^t (U_{s+\sigma} - U_\sigma) \, \mathrm{d}\sigma \\
&= \frac{1}{st} \left(\int_s^{s+t} U_\sigma \, \mathrm{d}\sigma - \int_0^t U_\sigma \, \mathrm{d}\sigma \right) \\
&= \frac{1}{st} \left(\int_t^{t+s} U_\sigma \, \mathrm{d}\sigma + \int_s^t U_\sigma \, \mathrm{d}\sigma - \int_0^t U_\sigma \, \mathrm{d}\sigma \right) \\
&= \frac{1}{st} \left(\int_0^s U_{t+\sigma} \, \mathrm{d}\sigma - \int_0^s U_\sigma \, \mathrm{d}\sigma \right) \\
&= \frac{U_t - \mathrm{id}}{t} S_s.
\end{aligned}$$

Since S_t is continuous at $t = 0$ and since the operator product is continuous anyway, we get

$$\lim_{s \longrightarrow 0} \frac{U_s - \mathrm{id}}{s} S_t = \lim_{s \longrightarrow 0} \frac{U_t - \mathrm{id}}{t} S_s = \frac{U_t - \mathrm{id}}{t} S_0 = \frac{U_t - \mathrm{id}}{t}.$$

Since S_t was invertible for small t we conclude that

$$\mathrm{i}A = \lim_{s \longrightarrow 0} \frac{U_s - \mathrm{id}}{s} = \frac{U_t - \mathrm{id}}{t} S_t^{-1} \tag{☺}$$

exists as norm limit in $L(V)$ and is, obviously, independent of t. This shows *iii.)*. In a next step, we assume *iii.)*. Then U_t is differentiable at $t = 0$ and by Proposition 5.4.21, *ii.)*, it is norm-continuous everywhere. It follows that we can apply our previous results to determine A of U_t at $t = 0$. The operator A is necessarily given by (☺) with S_t given as in (☺☺). But this shows that for small t we have

$$U_t = \mathrm{id} + \mathrm{i}A \int_0^t U_s \, \mathrm{d}s.$$

Since $s \mapsto U_s$ is norm-continuous the fundamental theorem of calculus shows that U_t is, for the above small t, differentiable and its derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} U_t = \mathrm{i}A U_t.$$

We claim that the unique solution of this differential equation is given by $U_t = \exp(\mathrm{i}tA)$ if we take into account the required initial condition $U_0 = \mathrm{id}$. Clearly, $\exp(\mathrm{i}tA)$ solves it. Thus let $\tilde{U}_t$ be another solution then

$$\frac{\mathrm{d}}{\mathrm{d}t} (\exp(-\mathrm{i}tA)\tilde{U}_t) = -\mathrm{i}A \exp(-\mathrm{i}tA)\tilde{U}_t + \exp(-\mathrm{i}tA)\mathrm{i}A\tilde{U}_t = 0,$$

by the Leibniz rule and the fact that A commutes with $\exp(-\mathrm{i}tA)$. But then the fundamental theorem of calculus implies $\exp(-\mathrm{i}tA)\tilde{U}_t = \mathit{const}$ from which we get $\tilde{U}_t = \exp(\mathrm{i}tA)$ as soon as we take into account the initial condition once again. Thus for small t the one-parameter group U_t is given by (5.4.41). Since $\exp(\mathrm{i}tA)$ is clearly a one-parameter group itself, they have to coincide for all $t \in \mathbb{R}$. Let us finally assume that $V = \mathfrak{H}$ is a Hilbert space and U_t is unitary. Then applying the *-involution to (5.4.40) immediately gives $A = A^*$ since $U_t^* = U_{-t}$. □

Definition 5.4.23 (Infinitesimal generator) *Let V be a Banach space and let $U\colon \mathbb{R} \longrightarrow L(V)$ be a norm-continuous one-parameter group. Then the operator*

$$A = \frac{1}{\mathrm{i}} \lim_{t \longrightarrow 0} \frac{U_t - \mathrm{id}}{t} \tag{5.4.45}$$

is called the infinitesimal generator of U.

Coming back to the time evolution of a quantum system we see that for a Hermitian Hamiltonian $H = H^* \in \mathfrak{B}(\mathfrak{H})$ the Schrödinger equation

$$\mathrm{i}\hbar \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = H\psi(t) \tag{5.4.46}$$

has a unique solution

$$\psi(t) = \exp\left(-\frac{\mathrm{i}t}{\hbar} H\right)\psi(0) \tag{5.4.47}$$

for every initial condition $\psi(0) \in \mathfrak{H}$ and the map $U_t\colon \psi(0) \mapsto \psi(t)$ is a norm-continuous unitary one-parameter group. Conversely, every norm-continuous unitary one-parameter group arises this way.

Unfortunately, $H \in \mathfrak{B}(\mathfrak{H})$ describes a rather untypical situation for quantum mechanical applications. Luckily, we also have such a correspondence for self-adjoint operators on one hand and *strongly continuous* unitary one-parameter groups on the other hand. We start with the construction of the one-parameter group.

Theorem 5.4.24 (Strongly continuous one-parameter groups) *Let* $(\operatorname{dom} A, A)$ *be a self-adjoint operator in $\mathfrak{H}$ with spectral measure E.*

i.) For all $t \in \mathbb{R}$ the operator

$$U_t = \exp(\mathrm{i}tA) = \int_{\operatorname{spec}(A)} \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}E \tag{5.4.48}$$

is defined on $\mathfrak{H}$ and unitary. One has

$$U_t\colon \operatorname{dom} A \longrightarrow \operatorname{dom} A. \tag{5.4.49}$$

ii.) The map $U\colon t \mapsto U_t$ is a strongly continuous unitary one-parameter group.

iii.) One has

$$\operatorname{dom} A = \left\{ \phi \in \mathfrak{H} \;\middle|\; \lim_{t \longrightarrow 0} \frac{U_t - \mathrm{id}}{t}\phi \text{ exists} \right\}, \tag{5.4.50}$$

and for $\phi \in \operatorname{dom} A$

$$A\phi = \frac{1}{\mathrm{i}} \lim_{t \longrightarrow 0} \frac{U_t - \mathrm{id}}{t}\phi. \tag{5.4.51}$$

iv.) For $\phi \in \operatorname{dom} A$ and all $t \in \mathbb{R}$ one has

$$AU_t\phi = U_t A\phi. \tag{5.4.52}$$

Proof: The claim that U_t is a unitary operator with $U_t U_s = U_{t+s}$ and $U_0 = \mathrm{id}$ is immediate from the (bounded) measurable calculus. Note that (5.4.48) can not be defined as exponential series but only as spectral integral. We know that U_t commutes with every E_V for measurable $V \subseteq \operatorname{spec}(A)$ by Theorem 5.1.19. Hence for all $\phi \in \mathfrak{H}$ we have $\langle U_t\phi, E_V U_t\phi\rangle = \langle U_t\phi, U_t E_V \phi\rangle = \langle \phi, E_V\phi\rangle$, implying that the measures $\langle U_t\phi, EU_t\phi\rangle$ and $\langle \phi, E\phi\rangle$ coincide. Since $\phi \in \operatorname{dom} A$ iff the function $\lambda \mapsto \lambda^2$ is integrable with respect to $\langle \phi, E\phi\rangle$ by the spectral theorem, see Theorem 5.4.13, and the definition of the domain according to Lemma 5.4.1, *i.)*, we see that $U_t\phi \in \operatorname{dom} A$ iff $\phi \in \operatorname{dom} A$. This shows the first part. Since $\|\mathrm{e}^{\mathrm{i}\lambda t}\|_\infty = 1$ for all t and since $\mathrm{e}^{\mathrm{i}\lambda t} \longrightarrow 1$ for $t \longrightarrow 0$ pointwise for all $\lambda \in \mathbb{R}$, we get from Proposition 5.1.24 the strong convergence $U_t \longrightarrow \mathrm{id}$ for $t \longrightarrow 0$. This completes the remaining proof of part *ii.)*, since we only have to check strong continuity at $t = 0$ thanks to Proposition 5.4.21, *i.)*. Now let $\phi \in \operatorname{dom} A$ be given. Then the function $\frac{\mathrm{e}^{\mathrm{i}\lambda t}-1}{t}$ is bounded by $|\lambda|$, which follows elementary, see Exercise 5.5.20. But λ is square integrable with respect to $\langle \phi, E\phi\rangle$ and hence $\frac{\mathrm{e}^{\mathrm{i}\lambda t}-1}{t}$ is square integrable with respect to $\langle \phi, E\phi\rangle$ for all $t \in \mathbb{R}$, too. Moreover, pointwise in λ we get

$$\frac{\mathrm{e}^{\mathrm{i}\lambda t}-1}{t} \longrightarrow \mathrm{i}\lambda \tag{$*$}$$

for $t \longrightarrow 0$. By dominated convergence this shows that $(*)$ is in fact an L^2-convergence with respect to $\langle \phi, E\phi\rangle$. Then Theorem 5.4.2, *ii.)*, applies which gives

$$\lim_{t\longrightarrow 0} \frac{U_t - \mathrm{id}}{t}\phi = \lim_{t\longrightarrow 0} \int_{\operatorname{spec}(A)} \frac{\mathrm{e}^{\mathrm{i}\lambda t}-1}{t} \,\mathrm{d}E\phi = \int_{\operatorname{spec}(A)} \mathrm{i}\lambda \,\mathrm{d}E\phi = \mathrm{i}A\phi.$$

This shows that $\phi \in \operatorname{dom} A$ fulfills the differentiability requirement (5.4.51). Conversely, assume $\phi \in \mathfrak{H}$ is such that the limit on the right hand side of (5.4.51) exists. We denote the set of vectors with this property by $\operatorname{dom} B \subseteq \mathfrak{H}$ and define on $\operatorname{dom} B$ the map

$$B\phi = \frac{1}{\mathrm{i}} \lim_{t\longrightarrow 0} \frac{U_t - \mathrm{id}}{t}\phi.$$

Since U_t is linear and the vector space operations are continuous, $\operatorname{dom} B$ is a subspace and B is a linear map. Thus we arrive at an operator $(\operatorname{dom} B, B)$ which extends $(\operatorname{dom} A, A)$, as we have already shown. For $\phi, \psi \in \operatorname{dom} B$ we get

$$\begin{aligned}
\langle \phi, B\psi\rangle &= \left\langle \phi, \frac{1}{\mathrm{i}} \lim_{t\longrightarrow 0} \frac{U_t - \mathrm{id}}{t}\psi \right\rangle \\
&= \lim_{t\longrightarrow 0} \left\langle \phi, \frac{1}{\mathrm{i}} \frac{U_t - \mathrm{id}}{t}\psi \right\rangle \\
&= \lim_{t\longrightarrow 0} \left\langle -\frac{1}{\mathrm{i}} \frac{U_t^* - \mathrm{id}}{t}\phi, \psi \right\rangle \\
&= \lim_{t\longrightarrow 0} \left\langle \frac{1}{\mathrm{i}} \frac{U_t - \mathrm{id}}{t}\phi, \psi \right\rangle \\
&= \langle B\phi, \psi\rangle, ,
\end{aligned} \tag{☺}$$

since $U_t^* = U_{-t}$ by unitarity and since $\langle \,\cdot\,, \,\cdot\,\rangle$ is continuous. Thus B is a symmetric operator and hence a symmetric extension of the self-adjoint operator $(\operatorname{dom} A, A)$. Since self-adjoint operators are already maximally symmetric, see Remark 5.3.4, *iii.)*, we conclude $\operatorname{dom} B = \operatorname{dom} A$. The last claim is clear by the functional calculus according to Corollary 5.4.3 applied to the bounded function $\lambda \mapsto \mathrm{e}^{\mathrm{i}t\lambda}$ on $\operatorname{spec}(A)$. This completes the proof. □

Remark 5.4.25 (The Schrödinger equation) Let $(\operatorname{dom} H, H)$ be now a self-adjoint Hamiltonian. Then the strongly continuous unitary one-parameter group

$$U_t = \exp\left(-\frac{\mathrm{i}tH}{\hbar}\right) = \int_{\operatorname{spec}(H)} \mathrm{e}^{\frac{-\mathrm{i}t\lambda}{\hbar}} \,\mathrm{d}E \tag{5.4.53}$$

satisfies the Schrödinger equation

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} U_t\psi = HU_t\psi \tag{5.4.54}$$

for all $\psi \in \operatorname{dom} H$ and all times $t \in \mathbb{R}$. Indeed, we have by the one-parameter group property and Theorem 5.4.24, *iii.)* and *i.)*,

$$\frac{\mathrm{d}}{\mathrm{d}t} U_t \psi = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} U_s U_t \psi = -\frac{\mathrm{i}}{\hbar} H U_t \psi$$

since $U_t\psi \in \operatorname{dom} H$. Moreover, for a given $\psi(0) \in \operatorname{dom} H$ this is the unique solution as one immediately confirms by differentiating the norm square of $\phi(t) = U_t\psi(0) - \tilde{\psi}(t)$ for any other solution $\tilde{\psi}(t) \in \operatorname{dom} H$ with $\tilde{\psi}(0) = \psi(0)$. Indeed, we have $\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = H\phi(t)$ and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\phi(t)\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \phi(t), \phi(t)\rangle = \left\langle \frac{1}{\mathrm{i}\hbar} H\phi(t), \phi(t) \right\rangle + \left\langle \phi(t), \frac{1}{\mathrm{i}\hbar} H\phi(t) \right\rangle = 0,$$

since H is symmetric. Hence $\|\phi(t)\|$ is constant and by $\phi(0) = 0$ we get $\phi(t) = 0$ for all $t \in \mathbb{R}$. Note that the Schrödinger equation only makes sense on $\operatorname{dom} H$ while the time evolution operator U_t is defined on the whole Hilbert space $\mathfrak{H}$.

We see that the requirement of self-adjointness of the Hamiltonian was sufficient to guarantee a good time evolution via the Schrödinger equation. Remarkably, it is also *necessary* if we insist on a strongly continuous time evolution. This converse statement is the content of the following theorem of Stone:

Theorem 5.4.26 (Stone) *Let U_t be a strongly continuous unitary one-parameter group. Then there exists a uniquely determined self-adjoint operator* $(\operatorname{dom} A, A)$ *with* $U_t = \exp(\mathrm{i}tA)$*.*

s in the proof y, to be fixed!

PROOF: If U_t is of the form $\exp(\mathrm{i}tA)$ according to Theorem 5.4.24, we can reconstruct A and its domain by the third part of that theorem, i.e.

$$\operatorname{dom} A = \left\{ \phi \in \mathfrak{H} \;\middle|\; \lim_{t \longrightarrow 0} \frac{U_t - \mathrm{id}}{t}\phi \text{ exists} \right\} \quad \text{and} \quad A\phi = \frac{1}{\mathrm{i}} \lim_{t\longrightarrow 0} \frac{U_t - \mathrm{id}}{t}\phi \qquad (☺)$$

for $\phi \in \operatorname{dom} A$. Thus the uniqueness is clear. The idea of the proof is to define $(\operatorname{dom} A, A)$ by (☺) and show that this gives a self-adjoint operator with $U_t = \exp(\mathrm{i}tA)$. To this end, we first show that $\operatorname{dom} A$ is non-empty at all: let $f \in \mathscr{C}_0^\infty(\mathbb{R})$ be given and consider for $\phi \in \mathfrak{H}$ the vector

$$\phi_f = \int_{-\infty}^{\infty} f(t) U_t \phi \,\mathrm{d}t$$

in the sense of a Riemann integral. Since $t \mapsto U_t\phi$ is continuous and since f has compact support, the integral is well-defined. Denote by $\mathfrak{D} \subseteq \mathfrak{H}$ the $\mathbb{C}$-span of all the vectors ϕ_f obtained for all $f \in \mathscr{C}_0^\infty(\mathbb{R})$ and $\phi \in \mathfrak{H}$. We claim that $\mathfrak{D}$ is dense. Indeed, let $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subseteq [-1, 1]$ and $\chi \geq 0$ be given such that

$$\int_{-\infty}^{\infty} \chi(t)\,\mathrm{d}t = 1.$$

Then we consider the rescaled function $\chi_\varepsilon(t) = \frac{1}{\varepsilon}\chi(\frac{t}{\varepsilon})$ for $\varepsilon > 0$ and have

$$\operatorname{supp}\chi_\varepsilon \subseteq [-\varepsilon, \varepsilon], \quad \chi_\varepsilon \geq 0, \quad \text{and} \int_{-\infty}^{\infty} \chi_\varepsilon(t)\,\mathrm{d}t = 1.$$

We compute for $\phi \in \mathfrak{H}$

$$\|\phi_{\chi_\varepsilon} - \phi\| = \left\| \int_{-\infty}^{\infty} \chi_\varepsilon(t) U_t\phi \,\mathrm{d}t - \phi \right\| = \left\| \int_{-\infty}^{\infty} \chi_\varepsilon(t)(U_t\phi - \phi)\,\mathrm{d}t \right\| \leq \max_{t\in[-\varepsilon,\varepsilon]} \|U_t\phi - \phi\|,$$

since the integral over χ_ε is 1 and the support of χ_ε is in $[-\varepsilon, \varepsilon]$. Thus we see from the strong continuity of U_t at $t = 0$ that $\phi_{\chi_\varepsilon} \longrightarrow \phi$. Hence $\mathfrak{D}$ is dense in $\mathfrak{H}$. Moreover, for $\phi_f \in \mathfrak{D}$ we have

$$U_t \phi_f = U_t \int_{-\infty}^{\infty} f(s) U_s \phi \, \mathrm{d}s = \int_{-\infty}^{\infty} f(s) U_{t+s} \phi \, \mathrm{d}s = \int_{-\infty}^{\infty} f(s-t) U_s \phi \, \mathrm{d}s = \phi_{\tau_{-t} f},$$

where $(\tau_{-t} f)(s) = f(s-t)$ is the translation of f by $-t$ as usual. Since $\tau_{-t} f \in \mathscr{C}_0^\infty(\mathbb{R})$ we see that U_t maps $\mathfrak{D}$ into itself. Finally, we have

$$\frac{U_t - \mathrm{id}}{t} \phi_f = \int_{-\infty}^{\infty} f(s) \frac{U_t - \mathrm{id}}{t} U_s \phi \, \mathrm{d}s = \int_{-\infty}^{\infty} f(s) \frac{U_{t+s} - U_s}{t} \phi \, \mathrm{d}s = \int_{-\infty}^{\infty} \frac{f(\sigma - t) - f(\sigma)}{t} U_\sigma \phi \, \mathrm{d}\sigma. \quad (\star)$$

Now $f \in \mathscr{C}_0^\infty(\mathbb{R})$ implies that the difference quotient approximates the derivative uniformly, i.e. we have

$$\lim_{t \longrightarrow 0} \frac{\tau_{-t} f - f}{t} = -f'$$

in the $\mathscr{C}$-topology. The integral $(\star)$ is continuous in this topology and hence we can take the limit $t \longrightarrow 0$ of $(\star)$ to obtain

$$\lim_{t \longrightarrow 0} \frac{U_t - \mathrm{id}}{t} \phi_f = \phi_{-f'}.$$

It follows that $\mathfrak{D} \subseteq \operatorname{dom} A$ and hence $\operatorname{dom} A$ is dense, too. It is clear that $\operatorname{dom} A$ is a subspace and A is a linear map $A \colon \operatorname{dom} A \longrightarrow \mathfrak{H}$. By literally the same argument as in the proof of Theorem 5.4.24, see (☺) there, we conclude that A is symmetric. For general $\phi \in \mathfrak{H}$ we have by the substitution rule and the one-parameter group property

$$\begin{aligned} U_t \int_0^s U_\sigma \phi \, \mathrm{d}\sigma &= \int_0^s U_{t+\sigma} \phi \, \mathrm{d}\sigma \\ &= \int_t^{t+s} U_\tau \phi \, \mathrm{d}\tau \\ &= \int_0^{t+s} U_\tau \phi \, \mathrm{d}\tau - \int_0^t U_\tau \phi \, \mathrm{d}\tau \\ &= \int_{-s}^t U_{\tau+s} \phi \, \mathrm{d}\tau - \int_0^t U_\tau \phi \, \mathrm{d}\tau. \end{aligned}$$

Both terms on the right have a continuous integrand and are therefore differentiable in t by the fundamental theorem of calculus. Thus

$$\lim_{t \longrightarrow 0} \frac{U_t - \mathrm{id}}{t} \int_0^s U_\sigma \phi \, \mathrm{d}\sigma = U_s \phi - \phi \tag{$*$}$$

exists for all $\phi \in \mathfrak{H}$. Hence the vector $\int_0^s U_\sigma \phi \, \mathrm{d}\sigma$ belongs to the domain $\operatorname{dom} A$ of A. Now let $\phi \in \operatorname{dom} A$ then

$$\lim_{s \longrightarrow 0} \sup_t \left\| U_t \frac{1}{\mathrm{i}} \frac{U_s - \mathrm{id}}{s} \phi - U_t A \phi \right\| = \lim_{s \longrightarrow 0} \sup_t \left\| \frac{1}{\mathrm{i}} \frac{U_s - \mathrm{id}}{s} \phi - A \phi \right\| = 0, \tag{$**$}$$

which shows that also $U_t \frac{1}{\mathrm{i}} \frac{U_s - \mathrm{id}}{s} \phi \longrightarrow U_t A \phi$ for all $t \in \mathbb{R}$ even uniformly. This allows to compute for $\phi \in \operatorname{dom} A$

$$U_t \phi - \phi \overset{(*)}{=} \lim_{s \longrightarrow 0} \frac{U_s - \mathrm{id}}{s} \int_0^t U_\sigma \phi \, \mathrm{d}\sigma = \lim_{s \longrightarrow 0} \int_0^t U_\sigma \frac{U_s - \mathrm{id}}{s} \phi \, \mathrm{d}\sigma \overset{(**)}{=} \int_0^t U_\sigma \mathrm{i} A \phi \, \mathrm{d}\sigma, \tag{☹}$$

thanks to the uniform convergence ($**$) for which the integral is a continuous operation. Using this, we can now show that A is closed on $\operatorname{dom} A$. Thus let $\phi_n \in \operatorname{dom} A$ be given with $\phi_n \longrightarrow \phi$ and $A\phi_n \longrightarrow \psi$. Then

$$
\begin{aligned}
\frac{1}{\mathrm{i}} \frac{U_t - \mathrm{id}}{t} \phi &= \frac{1}{\mathrm{i}} \frac{U_t - \mathrm{id}}{t} \lim_{n \longrightarrow \infty} \phi - n \\
&= \frac{1}{\mathrm{i}} \lim_{n \longrightarrow \infty} \frac{U_t - \mathrm{id}}{t} \phi_n \\
&\overset{(\frownie)}{=} \frac{1}{\mathrm{i}} \lim_{n \longrightarrow \infty} \frac{1}{t} \int_0^t U_\sigma \mathrm{i} A \phi_n \, \mathrm{d}\sigma \\
&\overset{(a)}{=} \frac{1}{t} \int_0^t U_\sigma \lim_{n \longrightarrow \infty} A\phi_n \, \mathrm{d}\sigma \\
&= \frac{1}{t} \int_0^t U_\sigma \psi \, \mathrm{d}\sigma,
\end{aligned}
$$

where in (a) we have used the fact that $U_\sigma \mathrm{i} A\phi_n \longrightarrow U_\sigma \mathrm{i}\psi$ uniformly in σ thanks to the unitarity of U_σ. Since the limit of the right hand side for $t \longrightarrow 0$ exists and is given by ψ we see that $\phi \in \operatorname{dom} A$ and $A\phi = \psi$. Thus A is closed. Now let $\phi \in \operatorname{dom} A$ then the continuity of U_t and $U_t U_s = U_s U_t$ gives

$$
\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} U_s U_t \phi = U_t \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} U_s \phi = U_t \mathrm{i} A \phi. \tag{$\star\star$}
$$

This implies that $U_t\phi \in \operatorname{dom} A$ as well and

$$
U_t A = A U_t
$$

on $\operatorname{dom} A$. We use this to compute the deficiency spaces of A. Let $\phi_\pm \in \mathfrak{K}_\pm(A)$, i.e. $A^*\phi_\pm = \pm\mathrm{i}\phi_\pm$. Then for $\psi \in \operatorname{dom} A$ we have

$$
\frac{\mathrm{d}}{\mathrm{d}t} \langle U_t\psi, \phi_\pm \rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} U_s U_t \psi, \phi_\pm \right\rangle = \langle \mathrm{i} A U_t \psi, \phi_\pm \rangle = -\mathrm{i} \langle U_t \psi, A^* \phi_\pm \rangle = \pm \langle U_t\psi, \phi_\pm \rangle.
$$

Thus $\langle U_t\psi, \phi_\pm \rangle = c_\pm \mathrm{e}^{\pm t}$ with $c_\pm = \langle \psi, \phi_\pm \rangle$ follows at once. However $\|\langle U_t\psi, \phi_\pm\rangle\| \le \|\psi\| \|\phi_\pm\|$ stays bounded for $\pm t \longrightarrow \infty$ by the unitarity of U_t. Hence $c_\pm = 0$ follows. Since $\psi \in \operatorname{dom} A$ was arbitrary and $\operatorname{dom} A$ dense, $\phi_\pm = 0$ is the only possibility. This shows $n_\pm(A) = 0$. Since we already know that A is closed, we conclude that $(\operatorname{dom} A, A)$ is self-adjoint. It remains to show that $V_t = \exp(\mathrm{i}tA)$ coincides with U_t. But this is simple since for $\phi \in \operatorname{dom} A$ we have by Theorem 5.4.24 also $V_t\phi \in \operatorname{dom} A$ and

$$
\frac{\mathrm{d}}{\mathrm{d}t} V_t \phi = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} V_s V_t \phi = \mathrm{i} A V_t \phi
$$

and by ($\star\star$) also $U_t\phi$ satisfies this differential equation. Thus we can argue as in Remark 5.4.25 by noting that the difference $\psi(t) = V_t\phi - U_t\phi$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \mathrm{i}A\psi(t)$ with initial condition $\psi(0) = 0$. Then differentiating $\|\psi(t)\|^2$ gives the result at once. Since both are unitary and $\operatorname{dom} A \subseteq \mathfrak{H}$ is dense this is sufficient to complete the proof. □

As already in the norm-continuous case we call $(\operatorname{dom} A, A)$ the *infinitesimal generator* of the one-parameter group U_t.

Remark 5.4.27 (Time evolution in quantum mechanics) One way to interpret the Stone's Theorem is that as soon as we want a quantum mechanical time evolution by means of a unitary one-parameter group subject to strong or even weak continuity (by Corollary 5.4.20), we already have a self-adjoint Hamiltonian generating it via the Schrödinger equation. In particular, a symmetric but not self-adjoint Hamiltonian would *not* suffice.

As a first application we use Stone's Theorem to construct self-adjoint operators: indeed, it is sometimes easier to show that a unitary one-parameter group is strongly continuous rather than showing the self-adjointness of some symmetric operator. We start with the following result:

Proposition 5.4.28 *Let U_t be a strongly continuous unitary one-parameter group with infinitesimal generator* $(\operatorname{dom} A, A)$. *Suppose $\mathfrak{D} \subseteq \mathfrak{H}$ is a dense subspace such that*

i.) one has $U_t\mathfrak{D} \subseteq \mathfrak{D}$ for all $t \in \mathbb{R}$,

ii.) for all $\phi \in \mathfrak{D}$ the map $t \mapsto U_t\phi$ is differentiable at $t = 0$.

Then $\mathfrak{D} \subseteq \operatorname{dom} A$ and $\mathfrak{D}$ is a core for A.

PROOF: By Theorem 5.4.24, *iii.)*, it is clear that $\mathfrak{D} \subseteq \operatorname{dom} A$, this follows directly from the property *ii.)*. Now let $(\operatorname{dom} B = \mathfrak{D}, B = A\big|_{\mathfrak{D}})$ and $\phi_\pm \in \operatorname{dom} B^*$ with $B^*\phi_\pm = \pm\mathrm{i}\phi_\pm$ be given. Then for $\psi \in \operatorname{dom} B$ we have $U_t\psi \in \operatorname{dom} B$ by property *i.)* and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}U_t\psi = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}U_sU_t\psi = \mathrm{i}AU_t\psi = \mathrm{i}BU_t\psi$$

by property *ii.)* and $B = A\big|_{\mathfrak{D}}$ as well as Theorem 5.4.24, *iii.)*. From here we can proceed as in the proof of Stone's Theorem when showing $n_\pm(A) = 0$ by looking at $\langle U_t\psi, \phi_\pm\rangle$. We conclude that $\phi_\pm = 0$ and hence B^* has no $\pm\mathrm{i}$ eigenvectors. Since $(B^{\mathrm{cl}})^* = B^*$ as well, this shows that B is essentially self-adjoint since B^{cl} is closed, symmetric and has $n_\pm(B^{\mathrm{cl}}) = 0$, i.e. B^{cl} is self-adjoint. Since A is also a self-adjoint extension of B by construction we must have $B^{\mathrm{cl}} = A$. □

With other words, the infinitesimal generator $(\operatorname{dom} A, A)$ of a strongly continuous unitary one-parameter group U_t is essentially self-adjoint on every dense U_t-invariant subspace of $\operatorname{dom} A$. There are two major examples of this:

Corollary 5.4.29 *Let* $(\operatorname{dom} A, A)$ *be a self-adjoint operator in $\mathfrak{H}$ and $U_t = \exp(\mathrm{i}tA)$ the corresponding one-parameter group. Then*

$$\mathfrak{D} = \operatorname{span}_{\mathbb{C}}\left\{\phi_f = \int f(t)U_t\phi \,\mathrm{d}t \;\middle|\; f \in \mathscr{C}_0^\infty(\mathbb{R}), \phi \in \mathfrak{H}\right\} \tag{5.4.55}$$

as well as

$$\mathfrak{D}' = \operatorname{span}_{\mathbb{C}}\left\{\int_0^t U_s\phi \,\mathrm{d}s \;\middle|\; \phi \in \mathfrak{H}, t \in \mathbb{R}\right\} \tag{5.4.56}$$

are cores for $(\operatorname{dom} A, A)$.

PROOF: In the proof of Stone's Theorem we have already seen that $\mathfrak{D}$ is dense, invariant under U_t, and contained in $\operatorname{dom} A$. In the same proof we showed that also $\mathfrak{D}' \subseteq \operatorname{dom} A$. Clearly, $\mathfrak{D}'$ is invariant under U_t and dense since $\lim_{t\longrightarrow 0} \frac{1}{t}\int_0^t U_\sigma\phi \,\mathrm{d}\sigma = \phi$. Hence for both subspaces the proposition applies.□

As a first concrete example we consider the quantum mechanical momentum operator as infinitesimal generator of the translations:

Proposition 5.4.30 (Momentum operator) *Consider the Hilbert space* $\mathfrak{H} = \mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$.

i.) For $\varphi \in \mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$

$$(U_t\varphi)(x) = \varphi(x + t) \tag{5.4.57}$$

defines a strongly continuous unitary one-parameter group $U_t \colon \mathrm{L}^2(\mathbb{R}), \mathrm{d}x) \longrightarrow \mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$.

ii.) The subspaces $\mathscr{C}_0^\infty(\mathbb{R})$, $\mathscr{C}_0^k(\mathbb{R})$, $\mathscr{S}(\mathbb{R})$ for $k \geq 1$ are dense, U_t-invariant subspaces in the domain $\operatorname{dom} P$ *of the infinitesimal generator P of U_t.*

iii.) On these subspaces one have $P = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$.

PROOF: Here we first define the above maps in the space of square integrable functions and see that they pass to well-defined linear maps on the quotient $\mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$. It is clear from the invariance under translations of the Lebesgue measure that U_t is a unitary one-parameter group. Now let $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$ be given then

$$\|U_t\varphi - \varphi\|_{\mathrm{L}^2}^2 = \int_{-\infty}^{\infty} |\varphi(x+t) - \varphi(x)|^2 \, \mathrm{d}x \longrightarrow 0$$

for $t \longrightarrow 0$ since $U_t\varphi \longrightarrow \varphi$ even uniformly. Thus on the dense subspace $\mathscr{C}_0^\infty(\mathbb{R}) \subseteq \mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$ we have strong continuity. But this suffices to have strong continuity on the whole Hilbert space by Exercise **??**. Hence the first part is shown. For the second part, clearly all the space are dense and U_t-invariant. We have to show that U_t is differentiable on them. We consider first $\varphi \in \mathscr{C}_0^1(\mathbb{R})$. Then for all $x \in \mathbb{R}$ we have

$$\left| \frac{\varphi(x+t) - \varphi(x)}{t} - \varphi'(x) \right| = \left| \int_0^1 (\varphi'(x+\tau t) - \varphi'(x)) \, \mathrm{d}\tau \right| \le \int_0^1 |\varphi'(x+\tau t) - \varphi'(x)| \, \mathrm{d}\tau. \qquad (*)$$

Since φ' has compact support as well it is even uniformly continuous. Thus for $\varepsilon > 0$ we find a $\delta > 0$ with $|\varphi'(x) - \varphi'(y)| < \varepsilon$ whenever $|x - y| < \delta$. Thus let $|t| < \delta$ then the right hand side of $(*)$ becomes less than ε. We conclude that

$$\frac{U_t\varphi - \varphi}{t} \longrightarrow \varphi'$$

with respect to the supremum norm. Since uniform convergence implies L^2- convergence we conclude that

$$\lim_{t \longrightarrow 0} \frac{1}{\mathrm{i}} \frac{U_t\varphi - \varphi}{t} = \frac{1}{\mathrm{i}}\varphi'$$

in the L^2-sense, too. Hence $\varphi \in \operatorname{dom} P$ and $P\varphi = -\mathrm{i}\varphi'$ for $\varphi \in \mathscr{C}_0^1(\mathbb{R})$. It follows that also $\mathscr{C}_0^\infty(\mathbb{R})$ and $\mathscr{C}_0^k(\mathbb{R})$ for $k \ge 2$ are contained in $\operatorname{dom} P$. The argument for the Schwartz space is similar. For $\varphi \in \mathscr{S}(\mathbb{R})$ we have $\varphi(x+t) - \varphi(x) - t\varphi'(x) = \frac{t^2}{2}\varphi''(c)$ for some $c \in [x, x+t]$ by the mean value theorem. Thus

$$\left\| \frac{\varphi(x+t) - \varphi(x)}{t} - \varphi'(x) \right\|_\infty \le \frac{t}{2}\|\varphi''\|_\infty$$

follows at once and we can proceed as above. The third part is clear from our computation. □

Remark 5.4.31 (Momentum operator) Even though it will be not so easy to characterize the full domain

$$\operatorname{dom} P = \left\{ \phi \in \mathrm{L}^2(\mathbb{R}, \mathrm{d}x) \;\middle|\; \lim_{t \longrightarrow 0} \frac{U_t\phi - \phi}{t} \text{ exists} \right\} \qquad (5.4.58)$$

of the momentum operator explicitly, we have found many nice function spaces on which P is already essentially self-adjoint. One can characterize $\operatorname{dom} P$ by means of absolutely continuous L^2-functions with L^2 derivative, or using the Fourier transform. However we shall not delve into the details here.

5.4.4 Analytic and Smooth Vectors

In this subsection we continue our discussion of particular subspaces of the domain $\operatorname{dom} A$ of a self-adjoint operator A. In particular, we are interested in vectors φ such that the time evolution $U_t = \exp(\mathrm{i}tA)$ of A is indeed given by a convergent exponential series and not just a spectral integral. In order to be able to study the convergence of the series $\sum_{n=0}^\infty \frac{(\mathrm{i}tA)^n}{n!}\phi$ we have to require that ϕ is in all the domains of the operators A^n. This will motivate the following notions which we can make for an arbitrary operator in $\mathfrak{H}$:

Definition 5.4.32 (Smooth vectors) *Let* $(\operatorname{dom} A, A)$ *be an operator in* $\mathfrak{H}$ *and* $\varphi \in \operatorname{dom} A$.

i.) *The vector* φ *is called* $\mathscr{C}^k$ *if* $\varphi \in \operatorname{dom} A^n$ *for all* $n = 1, \ldots, k$. *The set of* $\mathscr{C}^k$*-vectors is denoted by*

$$\operatorname{dom} A^k = \bigcap_{n=1}^{k} \operatorname{dom} A^k. \tag{5.4.59}$$

ii.) *The vector* φ *is called* $\mathscr{C}^\infty$ *or smooth if* φ *is* $\mathscr{C}^k$ *for all* k. *The set of smooth vectors is denoted by*

$$\operatorname{dom} A^\infty = \bigcap_{n=1}^{\infty} \operatorname{dom} A^n. \tag{5.4.60}$$

Definition 5.4.33 (Analytic vectors) *Let* $(\operatorname{dom} A, A)$ *be an operator in* $\mathfrak{H}$ *and* $\varphi \in \operatorname{dom} A^\infty$.

i.) *The vector* φ *is called analytic if there exists an* $M \geq 0$ *with* $\|A^n\varphi\| \leq M^n n!$. *The set of analytic vectors is denoted by*

$$\operatorname{dom} A^\omega = \left\{\varphi \in \operatorname{dom} A^\infty \;\middle|\; \varphi \textit{ is analytic}\right\}. \tag{5.4.61}$$

ii.) *For* $t > 0$ *one defines*

$$\operatorname{dom} A^{\omega,t} = \left\{\varphi \in \operatorname{dom} A^\infty \;\middle|\; \sum_{n=0}^{\infty} \frac{\|A^n\varphi\|}{n!} z^n \textit{ has radius of convergence at least } t\right\}. \tag{5.4.62}$$

It is clear that for $t \leq t'$ we have

$$\operatorname{dom} A^{\omega,t'} \subseteq \operatorname{dom} A^{\omega,t}, \tag{5.4.63}$$

and every $\varphi \in \operatorname{dom} A^{\omega,t}$ is analytic since the convergence of $\sum_{n=0}^{\infty} \frac{\|A^n\varphi\|}{n!} z^n$ implies $\|A^n\varphi\| \leq \frac{1}{|z|^n} n!$. Conversely, if φ is analytic with $\|A^n\varphi\| \leq M^n n!$ then $\varphi \in \operatorname{dom} A^{\omega,t}$ for e.g. $t = \frac{1}{2M}$. Thus we have

$$\operatorname{dom} A^\omega = \bigcup_{t>0} \operatorname{dom} A^{\omega,t}. \tag{5.4.64}$$

Even more specific are the holomorphic and bounded vectors.

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Definition 5.4.34 (Holomorphic and bounded vectors) *Let* $(\operatorname{dom} A, A)$ *be an operator in* $\mathfrak{H}$ *and* $\varphi \in \operatorname{dom} A^\infty$.

i.) *The vector* φ *is called (entire) holomorphic if* $\varphi \in \operatorname{dom} A^{\omega,t}$ *for all* $t > 0$. *The set of holomorphic vectors is denoted by*

$$\operatorname{dom} A^{\mathrm{hol}} = \bigcap_{t>0} \operatorname{dom} A^{\omega,t}. \tag{5.4.65}$$

ii.) *The vector* φ *is called bounded if there are* $c, M \geq 0$ *such that* $\|A^n\varphi\| \leq cM^n$. *One defines*

$$\operatorname{dom} A^{\mathrm{b},M} = \left\{\varphi \in \operatorname{dom} A^\infty \;\middle|\; \|A^n\varphi\| \leq cM^n \textit{ for some } c \geq 0\right\} \tag{5.4.66}$$

and

$$\operatorname{dom} A^{\mathrm{b}} = \bigcup_{M \geq 0} \operatorname{dom} A^{\mathrm{b},M}. \tag{5.4.67}$$

Again, it is clear that for $M \leq M'$ we have

$$\operatorname{dom} A^{\mathrm{b},M} \subseteq \operatorname{dom} A^{\mathrm{b},M'}. \tag{5.4.68}$$

Proposition 5.4.35 *Let* $(\operatorname{dom} A, A)$ *be an operator in* $\mathfrak{H}$ *and* $k \in \mathbb{N}$.

i.) All the sets $\operatorname{dom} A^k, \operatorname{dom} A^\infty, \operatorname{dom} A^{\omega,t}, \operatorname{dom} A^\omega, \operatorname{dom} A^{\mathrm{hol}}, \operatorname{dom} A^{\mathrm{b},M}$ *and* $\operatorname{dom} A^{\mathrm{b}}$ *are subspaces.*

ii.) The operator A *maps* $\operatorname{dom} A^{k+1}$ *into* $\operatorname{dom} A^k$.

iii.) The subspaces $\operatorname{dom} A^\infty, \operatorname{dom} A^{\omega,t}, \operatorname{dom} A^\omega, \operatorname{dom} A^{\mathrm{hol}}$ *and* $\operatorname{dom} A^{\mathrm{b}}$ *are invariant under* A.

iv.) We have the inclusions

$$\operatorname{dom} A^{\mathrm{b}} \subseteq \operatorname{dom} A^{\mathrm{hol}} \subseteq \operatorname{dom} A^\omega \subseteq \operatorname{dom} A^\infty \subseteq \operatorname{dom} A^k \subseteq \operatorname{dom} A. \tag{5.4.69}$$

PROOF: Being intersection of subspaces, $\operatorname{dom} A^k$ and $\operatorname{dom} A^\infty$ are clearly subspaces themselves. The convergence condition for $\operatorname{dom} A^{\omega,t}$ also leads to a subspace by a simple estimate. Thanks to the filtration property (5.4.63) also $\operatorname{dom} A^\omega$ is a subspace, which can also be seen directly. For $\operatorname{dom} A^{\mathrm{hol}}$ we have an intersection of subspaces again and a simple estimate shows that also $\operatorname{dom} A^{\mathrm{b},M}$ and $\operatorname{dom} A^{\mathrm{b}}$ are subspaces. The second part is clear by definition. Then it is clear that $\operatorname{dom} A^\infty$ is invariant under A. Let $\varphi \in \operatorname{dom} A^{\omega,t}$ and set $a_n = \|A^n\varphi\|$. Then

$$\sum_{n=0}^{\infty} \frac{\|A^n A\varphi\|}{n!} z^n = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} z^n = \frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

converges for $|z| < t$ since a convergent power series is differentiable on its open disc of convergence and can be differentiated termwise. Thus $A\varphi \in \operatorname{dom} A^{\omega,t}$ again. From this the invariance of $\operatorname{dom} A^\omega$ and $\operatorname{dom} A^{\mathrm{hol}}$ follows at once. Finally, let $\varphi \in \operatorname{dom} A^{\mathrm{b},M}$ then $\|A^n A\varphi\| \leq cM^{n+1} = (cM)M^n$ shows $A\varphi \in \operatorname{dom} A^{\mathrm{b},M}$. This gives the invariance $\operatorname{dom} A^{\mathrm{b}}$. The last part is now trivial from what we said already. □

In general, all the above subspaces (except of course $\operatorname{dom} A^1 = \operatorname{dom} A$) may be trivial. However, if $(\operatorname{dom} A, A)$ is self-adjoint then we can determine these subspaces more explicitly and show that they are even dense. We start with the smallest subspace, the bounded vectors:

Proposition 5.4.36 *Let* $(\operatorname{dom} A, A)$ *be a self-adjoint operator with spectral measure* E.

i.) For $M \geq 0$ *we have*

$$\operatorname{dom} A^{\mathrm{b},M} = \operatorname{im} \chi_{[-M,M]}(A) = \operatorname{im} E_{[-M,M]} \tag{5.4.70}$$

ii.) The subspace $\operatorname{dom} A^{\mathrm{b}} \subseteq \mathfrak{H}$ *is dense.*

PROOF: Note that we have some abuse of notation here and should better write $E_{[-M,M]\cap \operatorname{spec}(A)}$ but we avoid this clumsy notation. Let $\varphi \in \operatorname{im} E_{[-M,M]}$ be given then by spectral calculus we have

$$\begin{aligned}
\|A^n\varphi\|^2 &= \left\|A^n \chi_{[-M,M]}(A)\varphi\right\|^2 \\
&= \int_{\operatorname{spec}(A)} \lambda^{2n} \chi_{[-M,M]}(\lambda)\, \mathrm{d}\langle \varphi, E\varphi\rangle \\
&\leq M^{2n} \int_{\operatorname{spec}(A)} \mathrm{d}\langle \varphi, E\varphi\rangle \\
&= M^{2n}\|\varphi\|^2
\end{aligned}$$

showing $\varphi \in \operatorname{dom} A^{\mathrm{b},M}$ as claimed. Conversely, let $\varphi \in \operatorname{dom} A^{\mathrm{b},M}$ then assume $\|\varphi\| = 1$ without restriction. We have

$$\int \lambda^{2n}\, \mathrm{d}\langle \varphi, E\varphi\rangle = \|A^n\varphi\|^2 \leq M^{2n}.$$

Assume φ is not in the image of $E_{[-M,M]}$. Then $\varphi = E_{[-M,M]}\varphi + E_{\operatorname{spec} A\setminus[-M,M]}\varphi$ with the second term being non-trivial. Since $\operatorname{spec}(A) \setminus [-M, M]$ is open in $\operatorname{spec}(A)$ and non-empty, there is a $\delta > 0$ such that also $E_{\operatorname{spec}(A)\setminus[-M-\delta,M+\delta]}\varphi \neq 0$. Indeed, assume the converse which would give

$$E_{\operatorname{spec}(A)\setminus[-M,M]}\varphi = \lim_{\delta\longrightarrow 0} E_{\operatorname{spec}(A)\setminus[-M-\delta,M+\delta]}\varphi = 0$$

by the strong σ-additivity of a projection-valued measure, see also Exercise 5.5.17. Then we pick a λ_0 in $\operatorname{spec}(A) \setminus [-M-\delta, M+\delta]$, i.e. $|\lambda_0| > M$, for which we get

$$M^{2n} \geq \int \lambda^{2n}\, \mathrm{d}\langle\varphi, E\varphi\rangle \geq \lambda_0^{2n} \left\| E_{\operatorname{spec}(A)\setminus[-M-\delta,M+\delta]}\varphi \right\|^2,$$

which is not possible for all n since $|\lambda_0| > M$. Thus the reverse implication holds as well and we have shown the first part. From this the second part is clear by the fact that the intervals $[-n, n]$ exhaust $\mathbb{R}$ and E is a projection-valued measure. $\square$

In particular, all the subspaces $\operatorname{dom} A^{\mathrm{hol}}, \operatorname{dom} A^{\omega}, \operatorname{dom} A^{\omega,t}, \operatorname{dom} A^{\infty}$, and $\operatorname{dom} A^k$ are dense in $\mathfrak{H}$ by Proposition 5.4.35, *iv.)*. Moreover, all these subspaces are invariant under the time evolution generated by A:

Proposition 5.4.37 *Let* $(\operatorname{dom} A, A)$ *be self-adjoint. The subspaces* $\operatorname{dom} A^{\mathrm{b},M}$*,* $\operatorname{dom} A^{\mathrm{b}}$*,* $\operatorname{dom} A^{\mathrm{hol}}$*,* $\operatorname{dom} A^{\omega,s}$*,* $\operatorname{dom} A^{\omega}$*, and* $\operatorname{dom} A^k$ *for* $M > 0, s > 0$*, and* $k \in \mathbb{N} \cup \{+\infty\}$ *are invariant under* $U_t = \exp(\mathrm{i}tA)$*.*

PROOF: We know $U_t \operatorname{dom} A \subseteq \operatorname{dom} A$ by Theorem 5.4.24, *i.)*, and $A\mathrm{e}^{\mathrm{i}tA}\phi = \mathrm{e}^{\mathrm{i}tA}A\phi$ for all $\phi \in \operatorname{dom} A$ by the functional calculus, see Corollary 5.4.3. Thus $A\phi \in \operatorname{dom} A$ iff $A\mathrm{e}^{\mathrm{i}tA}\phi \in \operatorname{dom} A$. By induction we get $\phi \in \operatorname{dom} A^k$ iff $\mathrm{e}^{\mathrm{i}tA}\phi \in \operatorname{dom} A^k$ for all $k \in \mathbb{N}$ and hence also for $k = +\infty$. Since $\mathrm{e}^{\mathrm{i}tA}$ is unitary, we have $\|A^k\phi\| = \|A^k\mathrm{e}^{\mathrm{i}tA}\phi\|$ for all $\phi \in \operatorname{dom} A^k$. Thus every estimate for $\|A^k\phi\|$ holds for $\|A^k\mathrm{e}^{\mathrm{i}tA}\phi\|$ as well. From this the remaining claims follow at once. $\square$

The next theorem will now justify the names $\mathscr{C}^k$, smooth, analytic, and entire vectors: these are precisely the properties of the *function* $U\colon t \mapsto U_t\phi$:

Theorem 5.4.38 (Smooth, analytic, and holomorphic vectors) *Let* $(\operatorname{dom} A, A)$ *be a self-adjoint operator and let* $U_t = \exp(\mathrm{i}tA)$ *be its unitary one-parameter group.*

i.) Let $k \in \mathbb{N} \cup \{+\infty\}$ *and* $\phi \in \mathfrak{H}$*. Then* $\phi \in \operatorname{dom} A^k$ *iff the map* $t \mapsto U_t\phi$ *is* $\mathscr{C}^k$*. In this case we have for* $n \leq k$

$$\left.\frac{\mathrm{d}^n}{\mathrm{d}t^n}\right|_{t=0} U_t\phi = (\mathrm{i}A)^n\phi. \tag{5.4.71}$$

ii.) Let $s > 0$ *and* $\phi \in \mathfrak{H}$*. Then* $\phi \in \operatorname{dom} A^{\omega,s}$ *iff the map* $t \mapsto U_t\phi$ *is real analytic at* $t = 0$ *with radius of convergence at least* s*. In this case,* $t \mapsto U_t\phi$ *extends to a holomorphic map in the open strip* $\{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < s\}$*, we have* $\phi \in \operatorname{dom}\exp(\mathrm{i}zA)$ *for those* z*, and*

$$\exp(\mathrm{i}zA)\phi = \sum_{n=0}^{\infty} \frac{(\mathrm{i}zA)^n}{n!}\phi \tag{5.4.72}$$

converges absolutely for $|z| < s$*.*

iii.) Let $\phi \in \mathfrak{H}$*. Then* $\phi \in \operatorname{dom} A^{\mathrm{hol}}$ *iff the map* $t \mapsto U_t\phi$ *extends to an entire map. In this case,* $\phi \in \operatorname{dom}\exp(\mathrm{i}zA)$ *for all* $z \in \mathbb{C}$ *and*

$$\exp(\mathrm{i}zA)\phi = \sum_{n=0}^{\infty} \frac{(\mathrm{i}zA)^n}{n!}\phi \tag{5.4.73}$$

converges absolutely for all $z \in \mathbb{C}$*.*

In particular, $\operatorname{dom} A^{\mathrm{b}}$*,* $\operatorname{dom} A^{\mathrm{hol}}$*,* $\operatorname{dom} A^{\omega}$*,* $\operatorname{dom} A^{\omega,s}$*, and* $\operatorname{dom} A^k$ *for* $k \geq 1$ *are cores of* A*.*

PROOF: Assume first that $\phi \in \operatorname{dom} A^k$, i.e. $\phi \in \operatorname{dom} A^n$ for all $n \leq k$. Then we have by $\phi \in \operatorname{dom} A$ that $t \mapsto U_t\phi$ is differentiable at zero with derivative given by $\mathrm{i}A\phi$. Moreover, we have seen in Remark 5.4.25 that $t \mapsto U_t\phi$ is also differentiable at all other times t with derivative $\mathrm{i}AU_t\phi = U_t\mathrm{i}A\phi$. This is precisely Schrödinger's equation and Corollary 5.4.3 applied to the bounded function $\mathrm{e}^{\mathrm{i}\lambda t}$. Since U_t is strongly continuous, $t \mapsto U_t\phi$ is $\mathscr{C}^1$ with continuous derivative $t \mapsto U_t\mathrm{i}A\phi = \mathrm{i}AU_t\phi$. But since $\mathrm{i}A\phi \in \operatorname{dom} A$ again, we can repeat this argument by induction to conclude that $t \mapsto U_t\phi$ is $\mathscr{C}^\ell$ as long as $(\mathrm{i}A)^{\ell-1} \in \operatorname{dom} A$. Hence in total, the map is $\mathscr{C}^k$ with derivatives given by

$$\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} U_t\phi = U_t(\mathrm{i}A)^\ell\phi = (\mathrm{i}A)^\ell U_t\phi$$

for all $\ell \leq k$. Conversely, assume that $t \mapsto U_t\phi$ is $\mathscr{C}^k$. Then differentiating at $t = 0$ gives $\phi \in \operatorname{dom} A$ with $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} U_t\phi = \mathrm{i}A\phi$ by Theorem 5.4.24, *iii.)*. Hence for all t we have

$$\frac{\mathrm{d}}{\mathrm{d}t} U_t\phi = \mathrm{i}AU_t\phi = U_t\mathrm{i}A\phi.$$

By induction we can proceed differentiating to show $\phi \in \operatorname{dom} A, \mathrm{i}A\phi \in \operatorname{dom} A, \ldots, (\mathrm{i}A)^{k-1}\phi \in \operatorname{dom} A$ which means $\phi \in \operatorname{dom} A^k$ as claimed. This shows the first part. For the second part, let $\phi \in \operatorname{dom} A^{\omega,s}$ be given. Since then $\sum_{n=0}^\infty \frac{\|A^n\phi\|}{n!}|z|^n$ converges absolutely for $|z| < s$ also the series

$$\phi(z) = \sum_{n=0}^\infty \frac{(\mathrm{i}A)^n\phi}{n!} z^n \in \mathfrak{H} \tag{$*$}$$

converges absolutely for all $z \in \mathrm{B}_s(0) \subseteq \mathbb{C}$, thereby defining a holomorphic function with values in $\mathfrak{H}$. Now also the series

$$\sum_{n=0}^\infty \frac{\|A^n\phi\|}{n!} n z^{n-1} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\sum_{n=0}^\infty \frac{\|A^n\phi\|}{n!} z^n\right)$$

converges. Hence for $|t| < s$ we have the absolute convergence of

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=0}^\infty \frac{(\mathrm{i}tA)^n}{n!}\phi = \mathrm{i}\sum_{n=0}^\infty \frac{(\mathrm{i}tA)^n}{n!} A\phi.$$

Now on one hand

$$\sum_{n=0}^N \frac{(\mathrm{i}tA)^n}{n!}\phi \longrightarrow \sum_{n=0}^\infty \frac{(\mathrm{i}tA)^n}{n!}\phi,$$

and on the other hand

$$\mathrm{i}A\sum_{n=0}^N \frac{(\mathrm{i}tA)^n}{n!}\phi = \mathrm{i}\sum_{n=0}^N \frac{(\mathrm{i}tA)^n}{n!}A\phi \longrightarrow \mathrm{i}\sum_{n=0}^\infty \frac{(\mathrm{i}tA)^n}{n!}A\phi.$$

Since the operator A is closed we conclude that

$$A\sum_{n=0}^\infty \frac{(\mathrm{i}tA)^n}{n!}\phi = \sum_{n=0}^\infty \frac{(\mathrm{i}tA)^n}{n!}A\phi.$$

This shows that the maps $t \mapsto \phi(t)$ fulfills the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = \sum_{n=0}^\infty \frac{(\mathrm{i}tA)^n}{n!}\mathrm{i}A\phi = \mathrm{i}A\sum_{n=0}^\infty \frac{(\mathrm{i}tA)^n}{n!}\phi = \mathrm{i}A\phi(t)$$

for $|t| < s$ with the obvious initial condition $\phi(0) = \phi$. By the uniqueness of the solution to the Schrödinger equation we get $\phi(t) = \exp(\mathrm{i}tA)\phi = U_t\phi$ for $|t| < s$. Since $\phi(t)$ as defined in $(*)$ is clearly real analytic around $t = 0$, also $t \mapsto U_t\phi$ is real analytic as claimed. Now let t be arbitrary. Then by the one-parameter group property of U_t and the invariance of $\operatorname{dom} A^{\omega,s}$ under U_t we see that

$$U_t\phi = U_\tau U_{t-\tau}\phi = U_\tau \sum_{n=0}^{\infty} \frac{(\mathrm{i}(t-\tau)A)^n\phi}{n!} = \sum_{n=0}^{\infty} \frac{(\mathrm{i}A)^n U_\tau\phi}{n!}(t-\tau)^n$$

as soon as $|t - \tau| < s$. This shows the analyticity also for arbitrary t. The holomorphic extension to the strip of $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| < s$ is now clear from the formulas. Let us now show that $\phi \in \operatorname{dom}\exp(\mathrm{i}zA)$ for those z with $|\mathrm{Im}(z)| < s$. First we note that for all $N \in \mathbb{N}$

$$\begin{aligned}\int_{\operatorname{spec}(A)} \left(\sum_{n=0}^{N} \frac{(\mathrm{Im}(z)\lambda)^n}{n!}\right)^2 \mathrm{d}\langle\phi, E\phi\rangle &= \left\|\sum_{n=0}^{N} \frac{(\mathrm{Im}(z)A)^n}{n!}\phi\right\|^2 \\ &\le \left(\sum_{n=0}^{N} \frac{s^n}{n!}\|A^n\phi\|\right)^2 \\ &\le \left(\sum_{n=0}^{\infty} \frac{s^n}{n!}\|A^n\phi\|\right)^2 = C \qquad (\star)\end{aligned}$$

and $C < \infty$ by $\phi \in \operatorname{dom} A^{\omega,s}$. Thus the functions $f_N(\lambda) = \sum_{n=0}^{N} \frac{(\mathrm{Im}(z)\lambda)^n}{n!}$ are all square integrable with $\|f_N\|^2_{2,\langle\phi,E\phi\rangle} \le C$ for all N. This allows to apply the dominated convergence in (a) to get

$$\begin{aligned}\int_{\operatorname{spec}(A)} \left|\mathrm{e}^{\mathrm{i}z\lambda}\right|^2 \mathrm{d}\langle\phi, E\phi\rangle &= \int_{\operatorname{spec}(A)} \mathrm{e}^{2\,\mathrm{Im}(z)\lambda}\, \mathrm{d}\langle\phi, E\phi\rangle \\ &= \int_{\operatorname{spec}(A)} \left(\sum_{n=0}^{\infty} \frac{(\mathrm{Im}(z)\lambda)^n}{n!}\right)^2 \mathrm{d}\langle\phi, E\phi\rangle \\ &\overset{(a)}{=} \lim_{N\longrightarrow\infty} \int_{\operatorname{spec}(A)} \left(\sum_{n=0}^{N} \frac{(\mathrm{Im}(z)\lambda)^n}{n!}\right)^2 \mathrm{d}\langle\phi, E\phi\rangle \\ &= \lim_{N\longrightarrow\infty} \left\|\sum_{n=0}^{N} \frac{(\mathrm{Im}(z)A)^n}{n!}\phi\right\|^2 \\ &= \left\|\sum_{n=0}^{\infty} \frac{(\mathrm{Im}(z)A)^n}{n!}\phi\right\|^2.\end{aligned}$$

In particular, $\mathrm{e}^{\mathrm{i}z\lambda}$ is square integrable with respect to $\langle\phi, E\phi\rangle$ which means $\phi \in \operatorname{dom}\exp(\mathrm{i}zA)$ as claimed. Moreover, the above estimate $(\star)$ also shows that the functions f_N are a L^2-Cauchy sequence and hence convergent to a L^2-function given by $|\mathrm{e}^{\mathrm{i}z\lambda}|^2$ as they clearly converge to this function pointwise everywhere. This allows to apply Theorem 5.4.2, *ii.)*, to conclude that for $|z| < s$

$$\int_{\operatorname{spec}(A)} \mathrm{e}^{\mathrm{i}z\lambda}\, \mathrm{d}E\phi = \lim_{N\longrightarrow\infty} \int_{\operatorname{spec}(A)} \sum_{n=0}^{N} \frac{(\mathrm{i}z\lambda)^n}{n!}\, \mathrm{d}E\phi = \lim_{N\longrightarrow\infty} \sum_{n=0}^{N} \frac{(\mathrm{i}zA)^n}{n!}\phi = \sum_{n=0}^{\infty} \frac{(\mathrm{i}zA)^n}{n!}\phi,$$

since in this case also $|\mathrm{Im}(z)| < s$. If now $z \in \mathbb{C}$ has a larger absolute value $|z|$ but still $|\mathrm{Im}(z)| < s$ then we can use the one-parameter group property of both side for real $t \in \mathbb{R}$ to move things into the region where $|z| < s$. Since $\operatorname{dom} A^{\omega,s}$ is invariant under U_t this is possible. Now let $t \mapsto U_t\phi$ be real analytic in $t = 0$ with radius of convergence at least s. Then

$$U_t\phi = \sum_{n=0}^{\infty} \phi_n t^n$$

for $|t| < s$ with some vectors $\phi_n \in \mathfrak{H}$. Since the right hand side is an absolutely convergent power series, it is smooth with n-th derivatives given by $n!\phi_n$ at $t = 0$. Thus also the right hand side is smooth with derivatives given by $n!\phi_n$. However, if $U_t\phi$ is smooth in t then its n-th derivatives are given by $(\mathrm{i}A)^n\phi$ at $t = 0$. Thus $\phi_n = \frac{1}{n!}(\mathrm{i}A)^n\phi$. This shows the absolute convergence of $\sum_{n=0}^{\infty} \frac{s^n\|A^n\phi\|}{n!}$ and hence $\phi \in \operatorname{dom} A^{\omega,s}$. The last part is now easy since $\phi \in \operatorname{dom} A^{\mathrm{hol}}$ iff $\phi \in \operatorname{dom} A^{\omega,s}$ for all $s \geq 0$. Thus the claim follows from the second part at once. The fact that all the subspaces are cores follows now from Proposition 5.4.28. □

Since we know that $\operatorname{dom} A^{\mathrm{hol}}$ is a dense subspace, we can think of the definition of U_t as exponential *series* in a meaningful way at least on this subspace. However, in general, U_t is only be definable via the spectral *integrals* and not by the series.

The following theorem of Nelson closes our current discussion on special vectors in $\operatorname{dom} A$: it gives yet another criterion for self-adjointness which is sometimes very useful:

Theorem 5.4.39 (Nelson) *Let* $(\operatorname{dom} A, A)$ *be a symmetric operator with a dense subspace of analytic vectors* $\operatorname{dom} A^{\omega} \subseteq \mathfrak{H}$*. Then* $(\operatorname{dom} A, A)$ *is essentially self-adjoint*

PROOF: Clearly, the analytic vectors of A will still be analytic for its closure A^{cl}. First, we assume that the deficiency indices are equal, $n_+(A^{\mathrm{cl}}) = n_-(A^{\mathrm{cl}})$. Thus A has at least one self-adjoint extension by Corollary 5.3.16, *i.)*. Let $(\operatorname{dom} B, B)$ be such a self-adjoint extension and let $\psi \in \mathfrak{K}_+(A^{\mathrm{cl}}) = \ker((A^{\mathrm{cl}})^* - \mathrm{i})$ be a $+\mathrm{i}$ eigenvector of $A^* = (A^{\mathrm{cl}})^*$. Since $\phi \in \operatorname{dom} A^{\omega}$ is still analytic for B, we have $\phi \in \operatorname{dom} B^{\omega,s}$ for some $s > 0$. Thus $\exp(\mathrm{i}zB)\phi = \sum_{n=0}^{\infty} \frac{(\mathrm{i}zB)^n}{n!}\phi$ is convergent for $|z| < s$ and holomorphic on the strip $|\mathrm{Im}(z)| < s$. Hence also $f(z) = \langle \psi, \exp(\mathrm{i}zB)\phi\rangle$ is holomorphic in z for $|\mathrm{Im}(z)| < s$ and for $|z| < s$ we have

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} \frac{(\mathrm{i}z)^n}{n!}\langle \psi, B^n\phi\rangle \\
&= \sum_{n=0}^{\infty} \frac{(\mathrm{i}z)^n}{n!}\langle \psi, A^n\phi\rangle \\
&= \sum_{n=0}^{\infty} \frac{(\mathrm{i}z)^n}{n!}\langle A^*\psi, A^{n-1}\phi\rangle \\
&= \sum_{n=0}^{\infty} \frac{(\mathrm{i}z)^n}{n!}\langle \mathrm{i}\psi, A^{n-1}\phi\rangle \\
&= \sum_{n=0}^{\infty} \frac{(\mathrm{i}z)^n(-\mathrm{i})^n}{n!}\langle \psi, \phi\rangle \\
&= \mathrm{e}^z\langle \psi, \phi\rangle.
\end{aligned}$$

Since f is holomorphic, we have $f(z) = \mathrm{e}^z\langle \psi, \phi\rangle$ for all $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| < s$. In particular, for $z = t \in \mathbb{R}$ we have on the one hand

$$f(t) = \mathrm{e}^t|\langle \psi, \phi\rangle|,$$

and on the other hand, by the unitarity of $\exp(\mathrm{i}tB)$,

$$|f(t)| \leq \|\psi\|\|\phi\|,$$

from which we deduce $\langle \psi, \phi \rangle = 0$. Since the space of analytic vectors is assumed to be dense, we conclude $\psi = 0$ and hence $n_+(A^{\mathrm{cl}}) = 0$. Thus A^{cl} is self-adjoint which means that A is essentially self-adjoint. Now assume that $n_+(A^{\mathrm{cl}})$ and $n_-(A^{\mathrm{cl}})$ are not necessarily equal. Then we consider the symmetric operator $(\operatorname{dom} \tilde{A}, \tilde{A})$ on $\mathfrak{H} \oplus \mathfrak{H}$ defined by

$$\operatorname{dom} \tilde{A} = \operatorname{dom} A \oplus \operatorname{dom} A \quad \text{and} \quad \tilde{A} = A \oplus (-A).$$

Thanks to the direct orthogonal sum we have

$$\operatorname{dom} \tilde{A}^* = \operatorname{dom} A^* \oplus \operatorname{dom} A^* \quad \text{with} \quad \tilde{A}^* = A^* \oplus (-A)^*.$$

Finally, it is clear that $\operatorname{dom} A^\omega \oplus \operatorname{dom} A^\omega \subseteq \operatorname{dom} \tilde{A}^\omega$ and hence $\tilde{A}$ still has a dense subspace of analytic vectors. Now assume $\Psi \in \ker(\tilde{A}^* - \mathrm{i})$ with $\Psi = \psi_1 + \psi_2$ according to the decomposition $\mathfrak{H} \oplus \mathfrak{H}$. Then $\mathrm{i}\psi_1 + \mathrm{i}\psi_2 = \mathrm{i}\Psi = \tilde{A}^*\Psi = A^*\psi_1 - A^*\psi_2$. Thanks to the direct sum, this means $\psi_1 \in \ker(A^* - \mathrm{i})$ while $\psi_2 \in \ker(A^* + \mathrm{i})$. Being orthogonal in $\mathfrak{H} \oplus \mathfrak{H}$ this implies immediately $\mathfrak{K}_+(\tilde{A}^{\mathrm{cl}}) = \mathfrak{K}_+(A^{\mathrm{cl}}) \oplus \mathfrak{K}_-(A^{\mathrm{cl}})$. Analogously, we get $\mathfrak{K}_-(\tilde{A}^{\mathrm{cl}}) = \mathfrak{K}_-(A^{\mathrm{cl}}) \oplus \mathfrak{K}_+(A^{\mathrm{cl}})$ and hence

$$n_+(\tilde{A}^{\mathrm{cl}}) = n_+(A^{\mathrm{cl}}) + n_-(A^{\mathrm{cl}}) = n_-(\tilde{A}^{\mathrm{cl}}).$$

Thus our previous argument applies and gives an essentially self-adjoint operator $(\operatorname{dom} \tilde{A}, \tilde{A})$. But $n_\pm(\tilde{A}^{\mathrm{cl}}) = 0$ clearly implies $n_\pm(A^{\mathrm{cl}}) = 0$ as dimensions are always non-negative. □

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5.4.5 Positive Operators and the Polar Decomposition

5.5 Exercises

Exercise 5.5.1 (Topologies on $\mathfrak{B}(\mathfrak{H})$) Consider a finite-dimensional Hilbert space $\mathfrak{H}$ and give an elementary and direct proof that the σ-strong*, the σ-strong, the σ-weak, the strong*, the strong, and the weak operator topology coincide with the norm topology.

Exercise 5.5.2 (The strong and σ-strong topology) Let $\mathfrak{H}$ be a Hilbert space and consider the direct sum $\mathfrak{H}^N = \widehat{\bigoplus}_{n=0}^N \mathfrak{H}_n$ with $\mathfrak{H}_n = \mathfrak{H}$, i.e. N copies of $\mathfrak{H}$. Here either $N \in \mathbb{N}$ or N is countably infinite. Note that for N finite, the algebraic direct sum is already complete.

i.) Show that one can identify canonically $\mathfrak{H}^N = \mathfrak{H} \mathbin{\hat{\otimes}} \mathbb{C}^N$ for $N \in \mathbb{N}$ and $\mathfrak{H}^\infty = \mathfrak{H} \mathbin{\hat{\otimes}} \ell^2(\mathbb{N})$ where we used the completed tensor product in the sense of Hilbert spaces.

Hint: Use the canonical Hilbert bases of $\mathbb{C}^N$ and ℓ^2, respectively.

ii.) Let $A \in \mathfrak{B}(\mathfrak{H})$. Then we can act with A block-diagonally on $\mathfrak{H}^N$. Show that this corresponds to the operator $A \otimes \mathrm{id}_N$ where id_N is the identity on $\mathbb{C}^N$ for $N \in \mathbb{N}$ or on $\ell^2(\mathbb{N})$.

iii.) Show that the strong topology for $\mathfrak{B}(\mathfrak{H})$ can be obtained alternatively from the seminorms

$$\|A\|_\Phi = \|(A \otimes \mathrm{id}_N)\Phi\|_{\mathfrak{H} \hat{\otimes} \mathbb{C}^N}, \tag{5.5.1}$$

where $\Phi \in \mathfrak{H} \mathbin{\hat{\otimes}} \mathbb{C}^N$ and $N \in \mathbb{N}$.

iv.) Show that the seminorms (5.1.4) from the σ-strong topology are given by

$$\|A\|_\Phi = \left\| \left(A \otimes \mathrm{id}_{\ell^2(\mathbb{N})}\right) \Phi \right\|_{\mathfrak{H} \hat{\otimes} \ell^2(\mathbb{N})}, \tag{5.5.2}$$

where $\Phi \in \mathfrak{H} \mathbin{\hat{\otimes}} \ell^2(\mathbb{N})$.

v.) Proceed analogously for the strong* and σ-strong* topology.

Exercise 5.5.3 (Operator topologies on normal elements) Let $\mathfrak{H}$ be a Hilbert space.

i.) Prove that on normal elements of $\mathfrak{B}(\mathfrak{H})$ the strong* topology coincides with the strong topology.

ii.) Prove that on normal elements of $\mathfrak{B}(\mathfrak{H})$ the σ-strong* topology coincides with the σ-strong topology.

Hint: Proposition 5.1.13, *ii.)*.

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Exercise 5.5.4 (The unitary group is a topological group)

Exercise 5.5.5 (Closedness of the ordering relation) Let $\mathfrak{H}$ be a Hilbert space. Show that the ordering relation of the Hermitian elements in $\mathfrak{B}(\mathfrak{H})$ is not only closed in the norm-topology, see Exercise 4.5.44, but also in the σ-strong*, the σ-strong, the σ-weak, the strong*, the strong, and the weak topology.

Hint: Make use of Theorem 5.1.10.

Exercise 5.5.6 (Accumulation point of eigenvalues) Consider the Hilbert space $\mathfrak{H} = \ell^2(\mathbb{N})$ with its usual Hilbert basis e_n. Moreover, let $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ be a bounded sequence. Define a linear map

$$A_\lambda \colon \operatorname{span}_{\mathbb{C}}\{\mathrm{e}_n\}_{n\in\mathbb{N}} \longrightarrow \operatorname{span}_{\mathbb{C}}\{\mathrm{e}_n\}_{n\in\mathbb{N}} \tag{5.5.3}$$

by specifying $A_\lambda \mathrm{e}_n = \lambda_n \mathrm{e}_\lambda$.

i.) Show that the vectors e_n are eigenvectors to the eigenvalues λ_n.

ii.) Show that the only eigenvalues are the λ_n with eigenspaces

$$E_{\{\lambda_n\}} = \left(\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_m\}_{\lambda_m=\lambda_n}\right)^{\mathrm{cl}}. \tag{5.5.4}$$

Give an example where the closure is necessary.

iii.) Show that the spectrum of A_λ is given by the closure

$$\operatorname{spec}(A_\lambda) = \left\{\lambda_n \,\middle|\, n \in \mathbb{N}\right\}^{\mathrm{cl}} \tag{5.5.5}$$

of the set of all eigenvalues.

iv.) Give an example such that $\operatorname{spec}(A_\lambda) = [0, 1]$.

v.) Give an example such that A_λ is injective, has dense image, but is not surjective.

Exercise 5.5.7 (Properties of projection-valued measures) Let E be a projection-valued measure on a measurable space $(X, \mathfrak{a})$.

i.) Show that the map

$$E \colon \mathfrak{H} \times \mathfrak{H} \ni (\phi, \psi) \;\mapsto\; \mu_{\phi,\psi} = \langle \phi, E\psi \rangle \in \operatorname{Meas}(X, \mathfrak{a}) \tag{5.5.6}$$

is a sesquilinear map.

ii.) Show that for all $\phi, \psi \in \mathfrak{H}$ one has

$$\|\langle \phi, E\psi \rangle\| \le \|\phi\| \|\psi\|, \tag{5.5.7}$$

where the left hand side is the variational norm of the complex measure $\langle \phi, E\psi \rangle$.

iii.) Show that the variational norm (5.5.6) is a continuous Banach-space valued map.

Exercise 5.5.8 (Approximate eigenvalues of the position operator) Consider the Hilbert space $\mathfrak{H} = \mathrm{L}^2([0, 1], \mathrm{d}x)$ of square-integrable functions on the unit interval.

i.) Prove that the linear map

$$Q\colon \psi \mapsto (x \mapsto x\psi(x)) \tag{5.5.8}$$

for $\psi \in \mathscr{L}^2([0,1], \mathrm{d}x)$ defines a bounded linear endomorphism which descends to a well-defined Hermitian operator on $\mathrm{L}^2([0,1], \mathrm{d}x)$, still denoted by Q. Show that $\|Q\| = 1$.

ii.) Show that Q is injective.

iii.) Show that the spectrum of Q is the closed interval $[0,1]$ by constructing for each $\lambda \in [0,1]$ a sequence of approximate eigenvectors ϕ_n as in Corollary 5.1.14.

iv.) Describe the behaviour of the approximate eigenvectors for $n \longrightarrow \infty$ quantitatively.

Exercise 5.5.9 (Bounded operators between Hilbert spaces) Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces.

i.) Show that

$$\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{K}) = \mathfrak{B}(\mathfrak{H}) \oplus \mathfrak{B}(\mathfrak{H}, \mathfrak{K}) \oplus \mathfrak{B}(\mathfrak{K}, \mathfrak{H}) \oplus \mathfrak{B}(\mathfrak{K}) \tag{5.5.9}$$

by writing all endomorphisms of the direct sum in block form. Formulate and show that this decomposition is compatible with compositions and taking adjoints.

ii.) Use the identification (5.5.9) to reduce the question on the positivity of A^*A for $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$ to the case of positive elements in a C^*-algebra and give thus an alternative proof of the statement of Lemma 5.1.40, *i.)*.

iii.) Show that $|A|$ in the sense of Lemma 5.1.40 and $|A|$ in the sense of Proposition 4.4.4, *ii.)*, coincide under the identification (5.5.9).

Exercise 5.5.10 (Polar decomposition) Let $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$ be a bounded operator between Hilbert spaces. Show that the partial isometry U from the polar decomposition $A = U|A|$ satisfies

$$\operatorname{im} U = (\operatorname{im} A)^{\mathrm{cl}}. \tag{5.5.10}$$

Exercise 5.5.11 (The groupoid structure of partial isometries)

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Exercise 5.5.12 (Very unbounded operator) Consider the unbounded operator $(\operatorname{dom} A, A)$ from Example 5.2.9 and show that $\operatorname{dom} A^* = \{0\}$.

Hint: Consider $\psi = \sum_{n,m} \psi_{nm} \mathrm{f}_{nm} \in \operatorname{dom} A$ with only finitely many ψ_{nm} different from zero and $\phi = \sum_n \phi_n \mathrm{e}_n \in \mathfrak{H}$ arbitrary. Compute $\langle \phi, A\psi \rangle$ explicitly and show that for $\psi = \mathrm{f}_{nm}$ and fixed n one obtains an unbounded sequence. Conclude $\operatorname{dom} A^* = \{0\}$ from this.

Exercise 5.5.13 (Operator norm for a multiplication operator) Consider a non-empty open subset $U \subseteq \mathbb{R}^n$ and a continuous function $f \in \mathscr{C}(U)$. Then on $\mathscr{L}^2(U, \mathrm{d}^n x)$ one defines the multiplication operator

$$M_f\colon \psi \ \mapsto \ (x \mapsto f(x)\psi(x)), \tag{5.5.11}$$

with values in the measurable functions.

i.) Show that for a bounded function $f \in \mathscr{C}_b(U)$ the operator M_f passes to a well-defined operator on $\mathrm{L}^2(U, \mathrm{d}^n x)$ with operator norm given by $\|M_f\| = \|f\|_\infty$.

ii.) Find a condition on the zeros of f to make M_f injective.

iii.) Use the techniques from Exercise 5.5.8 to show that the spectrum of M_f is given by the closure $f(U)^{\mathrm{cl}} \subseteq \mathbb{C}$.

Exercise 5.5.14 (The operators J and τ) Let $\mathfrak{H}$ be a Hilbert space and let $J, \tau\colon \mathfrak{H} \oplus \mathfrak{H} \longrightarrow \mathfrak{H} \oplus \mathfrak{H}$ be defined by

$$J(\phi, \psi) = (-\psi, \phi) \quad \text{and} \quad \tau(\phi, \psi) = (\psi, \phi). \tag{5.5.12}$$

i.) Show that J is unitary and satisfies $J^2 = -\operatorname{id}$. Decompose $\mathfrak{H} \oplus \mathfrak{H}$ into the eigenspaces of J.

ii.) Show that τ is unitary and satisfies $\tau^2 = \operatorname{id}$. Decompose $\mathfrak{H} \oplus \mathfrak{H}$ into the eigenspaces of τ.

iii.) Compute $J\tau$ and τJ.

Exercise 5.5.15 (Properties of unbounded operators) Let $(\operatorname{dom} A, A)$ be a densely defined operator on a Hilbert space $\mathfrak{H}$.

i.) Let $U\colon \mathfrak{H} \longrightarrow \mathfrak{K}$ be a unitary map and define $\operatorname{dom} B = U(\operatorname{dom} A)$ and $B = UAU^{-1}$ as operator with domain $\operatorname{dom} B$. Formulate and show that the notions of dense domain, extension, closability, closedness, adjoint, core, resolvent set, spectrum, and resolvent behave well under unitary conjugations.

ii.) Show that A is closed iff $z - A$ is closed for one $z \in \mathbb{C}$ iff $z - A$ is closed for all $z \in \mathbb{C}$.

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Exercise 5.5.16 (Direct sum of self-adjoint operators)

Exercise 5.5.17 (A continuity property of spectral measures) Let $(\operatorname{dom} A, A)$ be a self-adjoint operator in $\mathfrak{H}$ with spectral measure E on $\operatorname{spec}(A)$. Let $M, \epsilon > 0$. Show that

$$E_{\operatorname{spec}(A)\setminus[-M,M]} = \lim_{\epsilon \longrightarrow 0} E_{\operatorname{spec}(A)\setminus[-M-\epsilon,M+\epsilon]} \tag{5.5.13}$$

in the strong operator topology.

Exercise 5.5.18 (Spectral projection from multiplication operator)

Exercise 5.5.19 (Essential range of continuous functions)

Exercise 5.5.20 (A bound on the exponential function) Show that for all $t \in \mathbb{R}$ and all $\lambda \in \mathbb{C}$ one has

$$\left| \frac{\mathrm{e}^{\mathrm{i}\lambda t} - 1}{t} \right| \le |\lambda|. \tag{5.5.14}$$

Exercise 5.5.21 (Strong continuity on a dense subspace) Let U_t be a one-parameter group of unitary operators on a Hilbert space $\mathfrak{H}$. Let $\operatorname{dom} \subseteq \mathfrak{H}$ be a dense subspace invariant under all the maps U_t. Show that U_t is strongly continuous iff U_t is strongly continuous on dom.

Hint: Use a standard $\frac{\epsilon}{3}$-argument.

Exercise 5.5.22 (A diagonal self-adjoint operator) Let $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ be a Hilbert basis of a Hilbert space $\mathfrak{H}$ and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Define on $\operatorname{dom} = \operatorname{span}_{\mathbb{C}}\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ the linear map $A\colon \operatorname{dom} \longrightarrow \mathfrak{H}$ by

$$A\mathrm{e}_n = \lambda_n e_n. \tag{5.5.15}$$

Prove that (dom, A) is an essentially self-adjoint operator.

Hint: There are many ways to show this but Nelson's Theorem is probably the easiest.

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Chapter 6

Ideals in $\mathfrak{B}(\mathfrak{H})$

In finite dimensions the algebra of endomorphisms of a vector space has no ideals: it is a simple algebra. In infinite dimensions, this changes drastically. We will investigate in this chapter the rich ideal structure of the continuous linear operators on a Banach space and, more specifically, on a Hilbert space. It turns out that one has at least one ideal of these endomorphisms which is closed in the operator norm, the *compact operators*. Quite contrary to the compact operators behaves the class of *Fredholm operators*: they are very far from being compact but closer to being invertible. One of the many characterizations will show that they are invertible modulo compact operators. For Fredholm operators we will define the *index* which will provide a first classification result for them. In the case of a Hilbert space, the compact operators will be the smallest ideal while in the Banach space case things are more complicated. Depending on the dimension of the Hilbert space there might be more closed ideals than the compact operators. But we are mainly interested in other, non-closed ideals being contained in the compact operators. The smallest ideal will always be the ideal of *finite-rank operators* which can also be understood as the algebraic tensor product of the topological dual of the space with the space itself. In the case of a Hilbert space, we find a wealth of ideals between the finite-rank operators and the compact operators, the *Schatten ideals*. All of them will be characterized by spectral properties: elements A of the p-th Schatten class will have an absolute value $|A|$ with a spectrum consisting of isolated finitely degenerate eigenvalues λ_n with zero being the only accumulation point. For $p = 1$ such operators have a summable sequence of eigenvalues which will allow to define a *trace* for them. This operator trace is a major tool for many further constructions in the theory of operator algebras. For statistical physics we can use the trace to define density matrices and hence mixed states analogously to the mixed states of $\mathrm{M}_n(\mathbb{C})$. These states will be the *normal states* of $\mathfrak{B}(\mathfrak{H})$. We will find several independent characterizations of such states and prepare the grounds for the theory of von Neumann algebras to be developed in the following chapter.

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6.1 Compact Operators and the Index

In this section we discuss the notion of compact operators and give first properties for Fredholm operators and their index. The whole theory works well for Banach spaces and since many applications prefer this slightly more general setting, we stick to the Banach situation for a while. For general compact operators between Banach spaces we formulate and prove a spectral theorem generalizing the Jordan normal form from finite dimensions. In fact, the compactness allows to proceed in a very similar way as in the finite-dimensional case. In the Hilbert space case we can specialize the spectral theorem to normal compact operators which will essentially behave like normal matrices. The polar decomposition will yield a singular value decomposition as already for finite dimensions.

6.1.1 Compact Operators between Banach Spaces

In the following we denote by V and W Banach spaces and consider linear maps from V to W.

Definition 6.1.1 (Compact operator) *Let $A\colon V \longrightarrow W$ be a linear map between Banach spaces. Then A is called compact if it maps bounded subsets of V into compact subsets of W. The set of compact operators from V to W is denoted by $\mathfrak{K}(V, W)$.*

Remark 6.1.2 Let $A\colon V \longrightarrow W$ be a linear map between Banach spaces.

i.) Clearly, A is compact iff the unit ball $\mathrm{B}_1(0) \subseteq V$ is mapped into a compact subset of W since every bounded subset $B \subseteq V$ can be rescaled to $\lambda B \subseteq \mathrm{B}_1(0)$ for a suitable $\lambda > 0$ and $A(\lambda B) = \lambda A(B)$ by the linearity of A. Note that the image $A(\mathrm{B}_1(0))$ needs not to be compact itself but only its closure: A is compact iff $A(\mathrm{B}_1(0))^{\mathrm{cl}} \subseteq W$ is compact.

ii.) Compact subsets of W are bounded themselves since the norm is continuous and therefore takes a maximum on a compact subset. Thus for a compact operator A we have $A(\mathrm{B}_1(0)) \subseteq \mathrm{B}_R(0)^{\mathrm{cl}}$ for some suitable $R > 0$. This shows that $\|v\| \leq 1$ implies $\|Av\| \leq R$ and hence

$$\|A\| = \sup_{\substack{v \neq 0 \\ \|v\| \leq 1}} \frac{\|Av\|}{\|v\|} \leq R. \tag{6.1.1}$$

It follows that A is continuous and hence we have

$$\mathfrak{K}(V, W) \subseteq \mathrm{L}(V, W). \tag{6.1.2}$$

iii.) Since W is metrizable, the compact subsets of W are precisely the sequentially compact subsets. Thus we get another criterion for compactness: A is compact iff A maps every bounded sequence $v_n \in V$ to a sequence $(Av_n)_{n\in\mathbb{N}}$ in W which has a convergent subsequence. Indeed, suppose A is compact. Then $(Av_n)_{n\in\mathbb{N}}$ is contained in a compact subset and hence has a convergent subsequence for every bounded sequence $(v_n)_{n\in\mathbb{N}}$. Conversely, if $B \subseteq V$ is bounded and $Av_n \in A(B)$ has a convergent subsequence for every sequence $v_n \in B$, then $A(B)^{\mathrm{cl}}$ is sequentially compact and thus compact.

The next example gives a large class of compact operators:

Example 6.1.3 (Finite rank operators) Denote by $\mathfrak{F}(V, W)$ those continuous operators $A\colon V \longrightarrow W$ which have a finite-dimensional image, i.e.

$$\mathfrak{F}(V, W) = \left\{ A \in \mathrm{L}(V, W) \;\middle|\; \dim \operatorname{im} A < \infty \right\}. \tag{6.1.3}$$

Then the image of $\mathrm{B}_1(0) \subseteq V$ under $A \in \mathfrak{F}(V, W)$ is bounded by $\|A\|$ according to the continuity of A and contained in $\operatorname{im} A$. But on a finite-dimensional subspace $\operatorname{im} A$ all norms are equivalent and a bounded subset is clearly contained in a compact subset in $\operatorname{im} A$. Since in addition the inclusion map $\operatorname{im} A \longrightarrow W$ is norm preserving, it is continuous and hence maps compact subsets to compact subsets. Thus $A(\mathrm{B}_1(0)) \subseteq W$ is contained in a compact subset. We conclude that the finite-rank operators are necessarily compact

$$\mathfrak{F}(V, W) \subseteq \mathfrak{K}(V, W). \tag{6.1.4}$$

Having finite rank and being continuous, it is straightforward to see that the finite-rank operators can be identified with the *algebraic* tensor product

$$\mathfrak{F}(V, W) \cong W \otimes V' \tag{6.1.5}$$

of the target space with the topological dual of the source space, see also Exercise 6.4.2 for this and further properties of finite-rank operators.

Example 6.1.4 Let $A\colon V \longrightarrow W$ be a compact operator and let $U \subseteq V$ be a closed subspace. Then $A\big|_U\colon U \longrightarrow W$ is also compact. Indeed, bounded subsets in U are still bounded in V and hence mapped into compact subsets of W by $A\big|_U$.

The following proposition shows that the unit ball $\mathrm{B}_1(0) \subseteq V$ will, in the infinite-dimensional case, not be contained in a compact subset. Hence the condition of being compact is nontrivial for a continuous map $A\colon V \longrightarrow W$ as soon as the Banach spaces are infinite-dimensional.

Proposition 6.1.5 *Let V be a Banach space. Then $\mathrm{B}_1(0)^{\mathrm{cl}}$ is compact iff $\dim V < \infty$.*

Proof: From elementary calculus we know that on $\mathbb{R}^n$ all norms are equivalent and $\mathrm{B}_1(0)^{\mathrm{cl}}$ is compact for e.g. the Euclidean norm and hence for all norms. This shows one direction. Suppose now that $\mathrm{B}_1(0)^{\mathrm{cl}}$ is compact. Then there are finitely many $v_1, \ldots, v_n \in V$ such that $\mathrm{B}_1(0)^{\mathrm{cl}}$ is covered by $\mathrm{B}_{1/2}(v_1), \ldots \mathrm{B}_{1/2}(v_n)$ by compactness, i.e.

$$\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq \bigcup_{i=1}^{n} \mathrm{B}_{1/2}(v_i). \tag{$*$}$$

We consider the span of these vectors, $U = \operatorname{span}_{\mathbb{C}}\{v_1, \ldots, v_n\}$, which is a closed subspace of V since it is finite-dimensional, see also Exercise 2.5.9, *iv.)*. For $v \notin U$ we have $\alpha = \inf_{u \in U} \|v - u\| > 0$. In fact, $\alpha = \|[v]\|$ is the quotient norm of $0 \neq [v] \in V/U$. We fix some $u \in U$ with $\alpha \leq \|v - u\| \leq \frac{3}{2}\alpha$ and consider $w = \frac{1}{\|v-u\|}(v - u) \in \mathrm{B}_1(0)^{\mathrm{cl}}$. By assumption $(*)$ we have some v_i with $\|w - v_i\| < \frac{1}{2}$ and hence

$$v = \underbrace{u + \|v - u\| v_i}_{\in U} + \|v - u\|(w - v_i).$$

From this we get

$$\|v - u\| \|w - v_i\| = \|v - u + \|v - u\| v_i\| \geq \inf_{\tilde{u} \in U} \|v - \tilde{u}\| = \alpha.$$

But $\|w - v_i\| < \frac{1}{2}$ and hence $\|v - u\| \geq 2\alpha$ follows which is in contradiction to the choice of u. Thus there is no such vector v and $V = U$ follows. ☐

In fact, this proposition can also be obtained from the more general fact that a Hausdorff topological vector space which is locally compact has to be finite dimensional, see Theorem 2.4.17 and its Corollary 2.4.18. However, though being very close from the idea, the above proof for the Banach space situation is less involved than the proof of Theorem 2.4.17.

Corollary 6.1.6 *Let V be a Banach space. Then $\mathrm{id}_V \in \mathrm{L}(V)$ is compact iff $\dim V < \infty$.*

Proof: If V is finite-dimensional then every linear map $A\colon V \longrightarrow V$ is compact by Example 6.1.3. If V is infinite-dimensional then $\mathrm{B}_1(0)^{\mathrm{cl}}$ is not compact and hence $\mathrm{B}_1(0)$ can not be contained in a compact subset by Proposition 6.1.5. ☐

The next proposition shows that the compact operators have a nice algebraic and topological structure themselves:

Proposition 6.1.7 *Let V, W, and U be Banach spaces.*

i.) The compact operators $\mathfrak{K}(V, W) \subseteq \mathrm{L}(V, W)$ constitute a closed subspace with respect to the operator norm topology.

ii.) *One has*

$$\mathfrak{K}(V, W) \circ \mathrm{L}(U, V) \subseteq \mathfrak{K}(U, W), \tag{6.1.6}$$

and

$$\mathrm{L}(V, W) \circ \mathfrak{K}(U, V) \subseteq \mathfrak{K}(U, W). \tag{6.1.7}$$

iii.) *The compact operators $\mathfrak{K}(V) \subseteq \mathrm{L}(V)$ on V form a closed ideal which is proper iff $\dim V = \infty$.*

iv.) *If $A \in \mathfrak{K}(V, W)$ and $B \in \mathrm{L}(V, W)$ satisfies*

$$\|Bv\| \leq \|Av\| \tag{6.1.8}$$

for all $v \in V$, then $B \in \mathfrak{K}(V, W)$, too.

PROOF: We use the characterization of compactness via sequences according to Remark 6.1.2, *iii.)*. Thus let $A, B \in \mathfrak{K}(V, W)$ and $z, w \in \mathbb{C}$ be given. If $(v_n)_{n\in\mathbb{N}}$ is a bounded sequence in V then $(Av_n)_{n\in\mathbb{N}}$ has a convergent subsequence, say $(Av_{n_k})_{k\in\mathbb{N}}$. Since $(v_{n_k})_{k\in\mathbb{N}}$ is still bounded, $(Bv_{n_k})_{k\in\mathbb{N}}$ has a convergent subsequence, say $(Bv_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ for which $(Av_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ still converges. Thus $(zAv_{n_{k_\ell}} + wBv_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ converges, too. This shows that $\mathfrak{K}(V, W)$ is a subspace. Now let $A \in \mathfrak{K}(V, W)^{\mathrm{cl}}$ be in the closure of the compact operators with respect to the norm topology. Then we can fix a sequence $A_n \in \mathfrak{K}(V, W)$ with $A_n \longrightarrow A$ in the norm sense. Let $(v_n)_{n\in\mathbb{N}}$ be a bounded sequence in V where we can safely assume that $\|v_n\| < 1$ for all $n \in \mathbb{N}$. As A_1 is compact, we get a subsequence $v_k^{(1)} = v_{n_k}$ of this sequence such that $(A_1 v_n^{(1)})_{n\in\mathbb{N}}$ converges. Within this subsequence we find another subsequence $v_k^{(2)} = v_{n_k}^{(1)}$ such that now $(A_2 v_n^{(2)})_{n\in\mathbb{N}}$ converges. By induction we obtain successive subsequences $(v_n^{(k)})_{n\in\mathbb{N}}$ for all $k \in \mathbb{N}$ such that $(A_k v_n^{(k)})_{n\in\mathbb{N}}$ converges. By this construction it is clear that we still have convergence of $(A_i v_n^{(k)})_{n\in\mathbb{N}}$ for $i = 1, \ldots, k$. In a next step we consider the diagonal sequence

$$(w_n)_{n\in\mathbb{N}} = (v_n^{(n)})_{n\in\mathbb{N}}.$$

Since for $k \geq k'$ the vectors $(v_n^{(k)})_{n\in\mathbb{N}}$ appear also in the sequence $(v_n^{(k')})_{n\in\mathbb{N}}$ it is clear that the sequence $(w_n)_{n\in\mathbb{N}}$ is, except for finitely many elements, a subsequence of $(v_n^{(k)})_{n\in\mathbb{N}}$ for *all* $k \in \mathbb{N}$. Thus all the sequences $(A_k w_n)_{n\in\mathbb{N}}$ converge. Now let $\epsilon > 0$ and let k be such that $\|A - A_k\| < \epsilon$. Moreover, since $(A_k w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, we have

$$\|A_k w_n - A_k w_m\| < \epsilon$$

for $n, m \geq N$ with some suitable N. Then we get for those $n, m \geq N$

$$\|Aw_n - Aw_m\| \leq \|Aw_n - A_k w_n\| + \|A_k w_n - A_k w_m\| + \|A_k w_m - Aw_m\| < \epsilon + \epsilon + \epsilon,$$

showing that $(Aw_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence convergent in W. Thus A is compact. The second part is easier: if $A \in \mathfrak{K}(V, W)$ and $B \in \mathrm{L}(U, V)$ then $B(\mathrm{B}_1(0)) \subseteq \mathrm{B}_{\|B\|}(0)$ is still bounded and hence mapped into some compact subset by A. Thus $A \circ B \in \mathfrak{K}(U, W)$. Analogously, if $A\colon V \longrightarrow W$ is continuous and $B\colon U \longrightarrow W$ is compact then $B(\mathrm{B}_1(0))^{\mathrm{cl}}$ is compact and mapped to a compact subset under the continuous map A. Thus the composition is compact in this case, too. The third part is clear from the first and the second part as well as from Corollary 6.1.6. Finally, assume B satisfies (6.1.8) then for a bounded sequence $(v_n)_{n\in\mathbb{N}}$ we have $\|Bv_n - Bv_m\| \leq \|Av_n - Av_m\|$. Since the sequence $(Av_n)_{n\in\mathbb{N}}$ has a convergent subsequence we see that this subsequence $(v_{n_k})_{k\in\mathbb{N}}$ yields a Cauchy sequence $(Bv_{n_k})_{k\in\mathbb{N}}$. As W is a Banach space, it converges, showing that B is compact. $\square$

The next result on compact operators requires the following lemma which is also of independent interest. In fact, it is a straightforward consequence of the *Arzéla-Ascoli Theorem*:

Lemma 6.1.8 *Let V be a Banach space and $K \subseteq V$ a compact subset. Then every sequence $\phi_n \in V'$ with $\|\phi_n\| \leq 1$ contains a subsequence which converges uniformly on K.*

Proof: We consider the subset

$$\mathcal{U} = \left\{ \phi\big|_K \;\middle|\; \phi \in V' \text{ and } \|\phi\| \leq 1 \right\} \subseteq \mathscr{C}(K)$$

of all the restrictions of continuous linear functionals with norm at most 1, viewed as particular continuous functions on the compact Hausdorff space K. Then $\mathcal{U}$ is bounded in the supremum norm since for $v \in K$ we have $\|v\| \leq C$ with some $C \geq 0$ and hence

$$|\phi(v)| \leq \|\phi\| \|v\| \leq C.$$

This shows $\|\phi\|_\infty \leq C$ for all $\phi \in \mathcal{U}$. Moreover, $\mathcal{U}$ is equicontinuous. Indeed, let $\epsilon > 0$ and $v \in K$ be given then for all $w \in K$ with $\|v - w\| < \epsilon$ we have

$$|\phi(v) - \phi(w)| = |\phi(v - w)| < \epsilon$$

by $\|\phi\| \leq 1$. By the Arzéla-Ascoli Theorem in form of Corollary **??** the sequence has a convergent subsequence with respect to the supremum norm on K. □

We use this lemma now to get the following theorem of Schauder:

Theorem 6.1.9 (Schauder) *Let $A\colon V \longrightarrow W$ be a continuous linear map. Then A is compact iff $A'\colon W' \longrightarrow V'$ is compact.*

Proof: First assume that A is compact and let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{B}_1(0) \subseteq W'$. For the compact subset $A(\mathrm{B}_1(0))^{\mathrm{cl}} \subseteq W$ we can apply Lemma 6.1.8 and find a subsequence $(\phi_{n_k})_{k\in\mathbb{N}}$ converging uniformly on $A(\mathrm{B}_1(0))^{\mathrm{cl}}$, i.e. for a given $\epsilon > 0$ there is a N with

$$\left|\phi_{n_k}(w) - \phi_{n_\ell}(w)\right| < \epsilon$$

for all $k, \ell \geq N$ and all $w \in A(\mathrm{B}_1(0))^{\mathrm{cl}}$. In particular, we can consider w of the form $w = Av$ with some $v \in \mathrm{B}_1(0)$. Hence

$$\left|(A'\phi_{n_k})(v) - (A'\phi_{n_\ell})(v)\right| = \left|\phi_{n_k}(Av) - \phi_{n_\ell}(Av)\right| < \epsilon$$

holds for all $v \in \mathrm{B}_1(0)$. Thus we have $\|A'\phi_{n_k} - A'\phi_{n_\ell}\| \leq \epsilon$ for those k, ℓ, showing that $(A'\phi_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence. By completeness of V' this sequence converges and thus A' is compact. The converse can be shown along the very same lines or with the following argument involving the double duals: if A' is compact we can apply the already shown case and conclude that $A''\colon V'' \longrightarrow W''$ is compact, too. Denote by $\iota_V\colon V \longrightarrow V''$ and $\iota_W\colon W \longrightarrow W''$ the canonical isometric inclusions. Then we have

$$\iota_W \circ A = A'' \circ \iota_V.$$

By the ideal property from Proposition 6.1.7, *ii.)*, we conclude that $A'' \circ \iota_V$ and thus also $\iota_W \circ A$ is compact. Since ι_W is isometric, the sequence criterion for compactness shows immediately that A itself is compact, see Exercise 6.4.5. □

We can apply this to Hilbert spaces by noting that the dual map is just the adjoint in disguise:

Corollary 6.1.10 *Let $A\colon \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$ be a continuous linear map between Hilbert spaces. Then A is compact iff A^* is compact.*

PROOF: If $\jmath_{1/2} \colon \mathfrak{H}_{1/2} \longrightarrow \mathfrak{H}'_{1/2}$ denotes the antilinear isometric isomorphism from Riesz' Theorem 3.2.11, we have

$$\jmath_1^{-1} \circ A' \circ \jmath_2 = A^*.$$

Even though the maps $\jmath_{1/2}$ are antilinear, we can use Schauder's Theorem 6.1.9 and e.g. the sequence criterion to get the claim. ☐

We conclude this section with more examples which were historically some of the starting points and main motivations to develop functional analysis:

Example 6.1.11 (Integral operators I) Let X, Y be compact metric spaces and μ a finite Borel measure on Y. For $k \in \mathscr{C}(X \times Y)$ the linear operator $K \colon \mathscr{C}(Y) \longrightarrow \mathscr{C}(X)$ defined by

$$(Kf)(x) = \int_Y k(x, y) f(y) \,\mathrm{d}\mu \tag{6.1.9}$$

is a compact operator with operator norm bounded by

$$\|K\| \leq \|k\|_\infty \operatorname{vol}(Y). \tag{6.1.10}$$

Here we view $\mathscr{C}(X)$ and $\mathscr{C}(Y)$ as Banach spaces with respect to the supremum norm as usual. To show the claim we first note that K is continuous and maps into $\mathscr{C}(X)$ at all. Indeed, if $x_n \longrightarrow x$ is a convergent sequence in X then $k(x_n, y) \longrightarrow k(x, y)$ pointwise in $y \in Y$. Since $|k(x, y)| \leq \|k\|_\infty$ for all $(x, y) \in X \times Y$ and since μ is finite, we can apply Lebesgue's Theorem of dominated convergence to deduce that $(Kf)(x_n) \longrightarrow (Kf)(x)$. Since X is first countable, sequential continuity is enough to conclude continuity of Kf. Thus $K \colon \mathscr{C}(Y) \longrightarrow \mathscr{C}(X)$ as wanted. Moreover, for all $x \in X$ we have

$$|(Kf)(x)| \leq \int_Y |k(x, y)||f(y)| \,\mathrm{d}\mu \leq \|k\|_\infty \|f\|_\infty \operatorname{vol}(Y), \tag{6.1.11}$$

showing the continuity of the operator K and (6.1.10). Now let $f_n \in \mathscr{C}(Y)$ be a bounded sequence, i.e. $\|f_n\|_\infty \leq c$. Then for all $x \in X$ we have

$$|(Kf_n)(x)| \leq \|k\|_\infty \|f_n\|_\infty \operatorname{vol}(Y) \leq c\|k\|_\infty \operatorname{vol}(Y). \tag{6.1.12}$$

Hence $(Kf_n)_{n\in\mathbb{N}}$ is *pointwise bounded* in $\mathscr{C}(X)$. Moreover, for $\epsilon > 0$ we find a $\delta > 0$ such that $d((x, y), (x', y')) < \delta$ implies $|k(x, y) - k(x', y')| < \epsilon$ by the uniform continuity of k on the compact metric space $X \times Y$. Thus for $d(x, x') < \delta$ we see that

$$|(Kf_n)(x) - (Kf_n)(x')| \leq \int_Y |k(x, y) - k(x', y)||f_n(y)| \,\mathrm{d}\mu \leq \epsilon c \operatorname{vol}(Y), \tag{6.1.13}$$

which means that the sequence $(Kf_n)_{n\in\mathbb{N}}$ is *equicontinuous*. By the Arzéla-Ascoli Theorem in form of Corollary **??** we find a convergent subsequence of $(Kf_n)_{n\in\mathbb{N}}$. Hence K is compact.

Example 6.1.12 (Finite-rank operators II) Consider two Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Then we already know that the finite-rank operators $\mathfrak{F}(\mathfrak{H}_1, \mathfrak{H}_2)$ are compact. As discussed in Exercise 6.4.2, it is easy to see that $A \in \mathfrak{F}(\mathfrak{H}_1, \mathfrak{H}_2)$ is of the form

$$A = \sum_{i=1}^{n} \Theta_{\phi_i, \psi_i} \tag{6.1.14}$$

for $\phi_i \in \mathfrak{H}_2$ and $\psi_i \in \mathfrak{H}_1$, where in general we define the *rank-one operator*

$$\Theta_{\phi,\psi}(\chi) = \phi\langle\psi, \chi\rangle_1. \tag{6.1.15}$$

Denote by $\overline{\mathfrak{H}}_1$ the complex conjugate Hilbert space, see Exercise 1.5.8. Then (6.1.15) can be considered as a linear map

$$\Theta\colon \mathfrak{H}_2 \otimes \overline{\mathfrak{H}}_1 \longrightarrow \mathfrak{F}(\mathfrak{H}_1, \mathfrak{H}_2), \tag{6.1.16}$$

by extending $\phi \otimes \overline{\psi} \mapsto \Theta_{\phi,\psi}$ linearly. Clearly, the resulting map Θ is a linear bijection. Now let $\{\mathrm{e}_i\}_{i\in I}$ and $\{\mathrm{f}_j\}_{j\in J}$ be Hilbert bases of $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Then $\{\overline{\mathrm{e}}_i\}_{i\in I}$ is a Hilbert basis of $\overline{\mathfrak{H}}_1$ and we know that $\{\mathrm{f}_j \otimes \overline{\mathrm{e}}_i\}_{(i,j)\in I\times J}$ is a Hilbert basis of the Hilbert space tensor product $\mathfrak{H}_2 \hat{\otimes} \overline{\mathfrak{H}}_1$. Now let $\Psi \in \mathfrak{H}_2 \hat{\otimes} \overline{\mathfrak{H}}_1$ be written with respect to this basis as

$$\Psi = \sum_{(i,j)\in I\times J} \Psi_{ji} \mathrm{f}_j \otimes \overline{\mathrm{e}}_i, \tag{6.1.17}$$

with $\Psi_{ji} = \langle \mathrm{f}_j \otimes \overline{\mathrm{e}}_i, \Psi\rangle$ as usual. For $\chi \in \mathfrak{H}_1$ we estimate

$$\begin{aligned}
\Big\|\sum\nolimits_{(i,j)\in I\times J} \Psi_{ji}\mathrm{f}_j \langle \mathrm{e}_i, \chi\rangle\Big\| &= \Big\|\sum\nolimits_{j\in J} \mathrm{f}_j \Big\langle \sum\nolimits_{i\in I} \overline{\Psi_{ji}}\mathrm{e}_i, \chi\Big\rangle\Big\| \\
&= \sum\nolimits_{j\in J} \Big|\Big\langle \sum\nolimits_{i\in I} \overline{\Psi_{ji}}\mathrm{e}_i, \chi\Big\rangle\Big|^2 \\
&= \sum\nolimits_{j\in J} \Big|\sum\nolimits_{i\in I} \Psi_{ji}\langle \mathrm{e}_i, \chi\rangle\Big|^2 \\
&\le \sum\nolimits_{j\in J} \Big(\sum\nolimits_{i\in I} |\Psi_{ji}|^2\Big)\Big(\sum\nolimits_{i'\in I} |\langle \mathrm{e}_i, \chi\rangle|^2\Big) \\
&= \|\Psi\|^2\|\chi\|^2.
\end{aligned}$$

From this we conclude that Θ is continuous with respect to the operator norm of $\mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ and the Hilbert space norm of $\mathfrak{H}_2 \hat{\otimes} \overline{\mathfrak{H}}_1$, respectively. Hence it extends to a continuous linear map

$$\Theta\colon \mathfrak{H}_2 \hat{\otimes} \overline{\mathfrak{H}}_1 \longrightarrow \mathfrak{F}(\mathfrak{H}_1, \mathfrak{H}_2)^{\mathrm{cl}} \tag{6.1.18}$$

into the completion of the finite-rank operators with respect to the operator norm. In fact, for $\Psi \in \mathfrak{H}_2 \hat{\otimes} \overline{\mathfrak{H}_1}$ as above we have the convergent series

$$\Theta_\Phi(\chi) = \sum_{(i,j)\in I\times J} \Psi_{ji}\mathrm{f}_j\langle \mathrm{e}_i, \chi\rangle. \tag{6.1.19}$$

From this formula it is clear that Θ is still injective and we have

$$\|\Theta_\Psi\| \le \|\Psi\|. \tag{6.1.20}$$

Now we know from Example 6.1.3 that $\mathfrak{F}(\mathfrak{H}_1, \mathfrak{H}_2) \subseteq \mathfrak{K}(\mathfrak{H}_1, \mathfrak{H}_2)$ and from Proposition 6.1.7, *i.)*, that $\mathfrak{K}(\mathfrak{H}_1, \mathfrak{H}_2)$ is closed for the operator norm. Thus we have $\mathfrak{F}(\mathfrak{H}_1, \mathfrak{H}_2)^{\mathrm{cl}} \subseteq \mathfrak{K}(\mathfrak{H}_1, \mathfrak{H}_2)$ and hence

$$\Theta\colon \mathfrak{H}_2 \hat{\otimes} \overline{\mathfrak{H}}_1 \longrightarrow \mathfrak{K}(\mathfrak{H}_1, \mathfrak{H}_2). \tag{6.1.21}$$

Later, we shall see that Θ is *not* surjective but its image will be the Hilbert-Schmidt operators only, see Exercise **??**.

Example 6.1.13 (Integral operators II) Let $(X, \mathfrak{a}, \mu)$ and $(Y, \mathfrak{b}, \nu)$ be measure spaces. For $\psi \in \mathrm{L}^2(Y, \mathfrak{b}, \nu)$ and $\phi \in \mathrm{L}^2(X, \mathfrak{a}, \mu)$ the operator $\Theta_{\psi,\phi}$ from Example 6.1.12 is given by

$$\Theta_{\psi,\phi}(\chi)(y) = \phi(y)\int_X \overline{\phi(x)}\chi(x)\,\mathrm{d}\mu = \int_X (\psi \otimes \overline{\phi})(y,x)\chi(x)\,\mathrm{d}\mu, \tag{6.1.22}$$

with the usual abuse of notation that we deal with representatives of the classes of L^2-functions in order to make sense out of the evaluation at points. It is now rather easy to see that passing to the completion $\mathrm{L}^2(Y\mathfrak{b},\nu)\,\hat{\otimes}\,\mathrm{L}^2(X,\mathfrak{a},\mu)$ yields just $\mathrm{L}^2(Y\times X,\mathfrak{b}\otimes\mathfrak{a},\nu\times\mu)$ and hence the map Θ becomes

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$$\Theta_\Psi(\chi)(y)=\int_X \Psi(y,x)\chi(x)\,\mathrm{d}\mu, \tag{6.1.23}$$

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where the integral exists almost for every $y\in Y$. It follows that such integral operators are compact, too.

6.1.2 Fredholm Operators and the Index

As preparatory material we collect some basic results on finite-dimensional subspaces and quotients of Banach spaces. Recall that the (algebraic) *codimension* of a subspace $U\subseteq V$ is defined by

$$\operatorname{codim} U=\dim V\big/U. \tag{6.1.24}$$

Dealing with infinite-dimensional Banach spaces, the algebraic dimension of $V\big/U$ or U does not yield any interesting quantity unless it is *finite*. Thus we consider now subspaces with either finite dimension or finite codimension, i.e. either very small ones or very large ones. Unlike as for Hilbert spaces, closed subspaces do not have complementary closed subspaces in a Banach space in general. Thus the following lemmas will be a sort of replacement for this under the additional assumption of finite dimension or codimension. In the particular case of Hilbert spaces, the following statements become trivial once we use our results on complementary subspaces from Theorem 3.2.1.

Lemma 6.1.14 *Let V be a Banach space.*

i.) Let $U\subseteq V$ be a finite-dimensional subspace. Then there exists a closed subspace $W\subseteq V$ with $U\oplus W=V$.

ii.) If $W\subseteq V$ is a subspace with finite codimension then there exists a finite-dimensional subspace $U\subseteq V$ with $U\oplus W=V$.

PROOF: For the first part, let $\mathrm{e}_1,\ldots,\mathrm{e}_n\in U$ be a basis with dual basis $\mathrm{e}^1,\ldots,\mathrm{e}^n\in U^*$. Then the Hahn-Banach Theorem gives us extensions $\varphi_1,\ldots,\varphi_n\in V'$ of the $\mathrm{e}^1,\ldots,\mathrm{e}^n$, since on the finite dimensional U all linear functionals are continuous. Define now

$$P(v)=\sum_{i=1}^n \mathrm{e}_i\varphi_i(v)$$

for $v\in V$. This gives a continuous projection $P\colon V\longrightarrow V$ with image $\operatorname{im}P=U$. Thus by linear algebra we have $V=\operatorname{im}P\oplus\ker P$. Since P is continuous, $\ker P\subseteq V$ is closed and $W=\ker P$ will do the job. The second part is even easier and no analytic argument is needed: we fix a basis $\mathrm{e}_1,\ldots,\mathrm{e}_n\in V\big/W$ and take preimages $u_1,\ldots,u_n\in V$. Then $U=\operatorname{span}_{\mathbb{C}}\{u_1,\ldots,u_n\}$ is a complement. □

The last part is clearly valid in any vector space and does not refer to any topological structure, we just included it here for completeness. Note that it may well happen that a subspace of finite codimension is *not* closed: we will meet at least two typical situations where this happens. The first possibility is to chose a linear functional $\varphi\in V^*$ which is not continuous. Then $W=\ker\varphi$ has codimension $\operatorname{codim}W=1$ but is not closed. In fact, this is an alternative characterization of the (dis-)continuity of φ, see also Exercise 2.5.5. The second arises from a dense proper subspace $\operatorname{dom}D\subseteq V$. This can e.g. be the domain of some unbounded densely defined operator. Now we choose a vector space basis $\{\mathrm{e}_i\}_{i\in I}$ of $\operatorname{dom}D$ and extend it to a vector space basis $\{\mathrm{e}_i\}_{i\in I}\cup\{\mathrm{f}_j\}_{j\in J}$ of V by the usual argument from linear algebra. Now consider $W=\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_i\}_{i\in I}\cup\{\mathrm{f}_j\}_{j\in J\setminus\{j_1,\ldots,j_n\}}$ for some indices $j_1,\ldots,j_n$. Then clearly, W has finite codimension $\operatorname{codim}W=n$ with a complement given by e.g. $U=\operatorname{span}_{\mathbb{C}}\{\mathrm{f}_{j_1},\ldots,\mathrm{f}_{j_n}\}$. However, $\operatorname{dom}D\subseteq W$ shows that W is still dense in V. It follows that W is not closed. Surprisingly enough, the first part of the lemma gives another subspace $\tilde{W} \subseteq V$ complementary to U which is now closed. The direct sum $W \oplus U = V = \tilde{W} \oplus U$ then gives immediately a linear isomorphism $\iota\colon W \longrightarrow \tilde{W}$ which, for the induced topologies, can not be continuous.

For the next lemma we recall the definition of the annihilator of a subspace $U \subseteq V$. In linear algebra, the *annihilator* of U consists of those linear functionals φ with $U \subseteq \ker \varphi$, i.e. $\varphi\big|_U = 0$. In our context, we require continuity of the linear functionals in addition and set

$$U^{\mathrm{ann}} = \big\{\varphi \in V' \;\big|\; \varphi\big|_U = 0\big\}. \tag{6.1.25}$$

Lemma 6.1.15 *Let $U \subseteq V$ be a closed subspace of a Banach space V.*

i.) *The canonical map*
$$\big(V\big/U\big)' \ni \psi \;\mapsto\; (v \;\mapsto\; \psi([v])) \in U^{\mathrm{ann}} \tag{6.1.26}$$
is an isometric isomorphism.

ii.) *The annihilator $U^{\mathrm{ann}} \subseteq V'$ is a closed subspace.*

iii.) *The canonical map*
$$V'\big/U^{\mathrm{ann}} \ni [\varphi] \;\mapsto\; (u \mapsto \varphi(u)) \in U' \tag{6.1.27}$$
is an isometric isomorphism.

PROOF: For the first part, we note that $\tilde{\psi}\colon v \mapsto \psi([v])$ is clearly continuous and annihilates U. Hence (6.1.26) is well-defined. Since the quotient norm on $V\big/U$ is given by

$$\|[v]\| = \inf\big\{\|v + u\| \;\big|\; u \in U\big\}$$

we get the estimate

$$\|\psi([v])\| \leq \|\psi\|\|[v]\| = \|\psi\| \inf\big\{\|v + u\| \;\big|\; u \in U\big\} \leq \|\psi\|\|v\|,$$

which shows $\|\tilde{\psi}\| \leq \|\psi\|$. We consider now the map $U^{\mathrm{ann}} \ni \varphi \mapsto \hat{\varphi} = ([v] \mapsto \varphi(v)) \in \big(V\big/U\big)'$, which is again well-defined and yields a continuous linear functional on $V\big/U$ since

$$|\varphi(v)| = |\varphi(v + u)| \leq \|\varphi\|\|v + u\|$$

holds for all $u \in U$. Taking the infimum over all $u \in U$ gives the estimate $|\varphi(v)| \leq \|\varphi\|\|[v]\|$ and hence the continuity of $\hat{\varphi}$. Moreover, we have $\|\hat{\varphi}\| \leq \|\varphi\|$. For $\varphi \in U^{\mathrm{ann}}$ we compute $\tilde{\hat{\varphi}}(v) = \hat{\varphi}([v]) = \varphi(v)$ and hence $\tilde{\hat{\varphi}} = \varphi$. Conversely, for $\psi \in \big(V\big/U\big)'$ we compute $\hat{\tilde{\psi}}([v]) = \tilde{\psi}(v) = \psi([v])$, and thus also $\hat{\tilde{\psi}} = \psi$. This proves that the two continuous maps are inverse to each other. Moreover, both are norm-decreasing, and thus necessarily norm-preserving, i.e. isometric. The second part is clear, see also Exercise 6.4.3. For the third part, we first note that (6.1.27) is well-defined indeed, since any $\varphi \in U^{\mathrm{ann}}$ vanishes on U by the very definition. Moreover, (6.1.27) is linear and results in a continuous linear functional on U. For $u \in U$ we have for all $\psi \in U^{\mathrm{ann}}$

$$|\varphi(u)| = |\varphi(u) + \psi(u)| \leq \|\varphi + \psi\|\|u\|.$$

Hence taking the infimum over those $\psi \in U^{\mathrm{ann}}$ gives $|\varphi(u)| \leq \|[\varphi]\|\|u\|$ showing that (6.1.27) is continuous and norm-decreasing. Now let $\varphi \in U'$ be given. By the Hahn-Banach Theorem 2.3.10 we find an extension $\Phi \in V'$ with $\Phi\big|_U = \varphi$ (and $\|\Phi\| = \|\varphi\|$). We choose such a (non-unique) extension Φ and define the map $U' \ni \varphi \mapsto [\Phi] \in V'\big/U^{\mathrm{ann}}$. We claim that this is well-defined, linear, and continuous. Indeed, choosing another such extension $\tilde{\Phi}$ we get $\Phi - \tilde{\Phi} \in U^{\mathrm{ann}}$ and hence $[\Phi] = [\tilde{\Phi}]$. The linearity is clear as if Φ and Ψ are extensions of φ and ψ, then $z\Phi + w\Psi$ is an extension of $z\varphi + w\psi$. For the continuity we use that we find at least one extension Φ_0 of φ with $\|\Phi_0\| = \|\varphi\|$. Thus

$$\|[\Phi_0]\| = \inf\{\|\Phi_0 + \Psi\| \bigm| \Psi \in U^{\mathrm{ann}}\} \leq \|\Phi_0\| = \|\varphi\|$$

shows continuity. Finally, this map is clearly the inverse to (6.1.27) and again norm-decreasing. It follows that both maps are necessarily norm-preserving, completing the proof. □

Even though a finite-codimensional subspace needs not to be closed in general, this turns out to be true if it is the image of a continuous operator:

Lemma 6.1.16 *Let V and W be Banach spaces and $A \in \mathrm{L}(V, W)$. If $\operatorname{codim} \operatorname{im} A < \infty$ then $\operatorname{im} A \subseteq W$ is closed.*

PROOF: By passing to the Banach quotient $V\big/\ker A$ if necessary, we can assume that A is injective from the beginning. Now choose vectors $w_1, \ldots, w_n \in W$ such that $[w_1], \ldots, [w_n] \in W\big/\operatorname{im} A$ form a basis. We consider the map

$$\tilde{A}\colon \mathbb{C}^n \oplus V \ni (\vec{z}, v) \ \mapsto\ A(v) + \sum_{i=1}^n z_i w_i \in W.$$

Clearly, $\tilde{A}$ is linear and still continuous, where on $\mathbb{C}^n \oplus V$ we take e.g. the maximum norm of the canonical norm on $\mathbb{C}^n$ and the norm on V: the resulting topology is then the Cartesian product topology. Since the w_i are a basis of a linear complement of $\operatorname{im} A$, this extended map is still injective, but now also surjective. By the Open Mapping Theorem for Banach spaces, see Theorem 2.3.19, we conclude that $\tilde{A}$ is an isomorphism with continuous inverse $\tilde{A}^{-1}\colon W \longrightarrow \mathbb{C}^n \oplus V$. Hence also $\mathrm{pr}_1 \circ \tilde{A}^{-1}\colon W \longrightarrow \mathbb{C}^n$ is continuous. Since

$$\mathrm{pr}_1 \circ \tilde{A}^{-1}\left(A(v) + \sum_{i=1}^n z_i w_i\right) = \vec{z},$$

we have $\ker \mathrm{pr}_1 \tilde{A}^{-1} = \operatorname{im} A$ and thus $\operatorname{im} A$ is closed. □

After this preparatory lemmas we can now formulate the central definition of this subsection:

Definition 6.1.17 (Fredholm operator) *A continuous linear map $A\colon V \longrightarrow W$ between Banach spaces is called Fredholm operator if*

$$\dim \ker A < \infty \quad \textit{and} \quad \operatorname{codim} \operatorname{im} A < \infty. \tag{6.1.28}$$

The set of Fredholm operators will be denoted by $\mathrm{Fredholm}(V, W)$.

By the previous lemma we know that $\operatorname{im} A$ is a closed subspace, the kernel $\ker A$ is trivially closed by continuity. Note that for $z \in \mathbb{C} \setminus \{0\}$ and $A \in \mathrm{Fredholm}(V, W)$ also $zA \in \mathrm{Fredholm}(V, W)$.

If both Banach spaces are finite-dimensional then the dimension formula will tell use that

$$\dim \ker A - \operatorname{codim} \operatorname{im} A = \dim V - \dim W \tag{6.1.29}$$

is independent of A at all and just given by the difference of the finite dimensions of V and W. In the infinite-dimensional case the right hand side of (6.1.29) is no longer meaningful, the left hand side will become a characteristic quantity of A:

Definition 6.1.18 (Fredholm index) *Let $A\colon V \longrightarrow W$ be a Fredholm operator. Then the Fredholm index of A is defined by*

$$\operatorname{ind}(A) = \dim\ker A - \operatorname{codim}\operatorname{im} A. \tag{6.1.30}$$

Remark 6.1.19 Equivalently, the definition of a Fredholm operator and its index can be given as follows: A is Fredholm iff $\operatorname{im} A$ is closed and both, $\ker A$ and $\ker A'$ are finite-dimensional. Here $A'\colon W' \longrightarrow V'$ is the transposed operator as usual. Indeed, if $\operatorname{im} A$ is closed then Lemma 6.1.15, *i.)*, shows that $(\operatorname{im} A)^{\mathrm{ann}} \cong \big(V/\operatorname{im} A\big)'$ and $(\operatorname{im} A)^{\mathrm{ann}} = \ker A'$ holds in general. Thus

$$\ker A' \cong \big(V/\operatorname{im} A\big)'. \tag{6.1.31}$$

Now the finite-dimensionality of $\ker A'$ implies that $(V/\operatorname{im} A)'$ and hence also $V/\operatorname{im} A$ are finite-dimensional with dimension given by

$$\dim\ker A' = \dim\big(V/\operatorname{im} A\big)' = \dim V/\operatorname{im} A = \operatorname{codim}\operatorname{im} A. \tag{6.1.32}$$

Thus A is Fredholm. The converse is clear by the same line of arguments. Moreover, we see that the index is given by

$$\operatorname{ind}(A) = \dim\ker A - \dim\ker A'. \tag{6.1.33}$$

In the case of Hilbert spaces this leads to the formula

$$\operatorname{ind}(A) = \dim\ker A - \dim\ker A^* \tag{6.1.34}$$

for the index, avoiding thereby the notion of codimension. Note however, that the requirement of $\operatorname{im} A$ being closed is not superfluous, see Exercise 6.4.9.

Example 6.1.20 (The shift operator) We consider again the shift operator $U_k\colon \ell^2 \longrightarrow \ell^2$ from Example 5.3.27. Being an isometry, we have $\ker U_k = \{0\}$. However, we see that

$$\operatorname{im} U_k = \left(\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_0, \ldots, \mathrm{e}_{k-1}\}\right)^{\perp} \tag{6.1.35}$$

is closed and has finite codimension given by k. It follows that U_k is a Fredholm operator and its index is given by

$$\operatorname{ind}(U_k) = -k. \tag{6.1.36}$$

Moreover, it is easy to see that the adjoint of U_k has $\operatorname{span}_{\mathbb{C}}\{\mathrm{e}_0, \ldots, \mathrm{e}_{k-1}\}$ as its kernel and is surjective. Thus also U_k^* is a Fredholm operator with index given by

$$\operatorname{ind}(U_k^*) = k. \tag{6.1.37}$$

This example shows that unlike in the finite-dimensional case, the index now depends on the Fredholm operator.

The following theorem of Riesz relates Fredholm operators to compact operators:

Theorem 6.1.21 (Riesz) *Let V be a Banach space and $K \in \mathfrak{K}(V)$. Then $\mathbb{1} - K$ is a Fredholm operator with*

$$\operatorname{ind}(\mathbb{1} - K) = 0. \tag{6.1.38}$$

Proof: Set $A = \mathbb{1} - K$ for abbreviation. On $\ker A$ we have $\mathbb{1}\big|_{\ker A} = K\big|_{\ker A}$ and since $K\big|_{\ker A}$ is still compact by Example 6.1.4, we have $\dim\ker A < \infty$ by Corollary 6.1.6. Next, we have to show that $\operatorname{im} A$ is closed. To this end, we choose a closed complement $W \subseteq V$ of $\ker A$ such that $V = \ker A \oplus W$, which is possible according to Lemma 6.1.14, *i.)*. Then $A\big|_W\colon W \longrightarrow \operatorname{im} A$ is bijective. We want to show that $(A\big|_W)^{-1}\colon \operatorname{im} A \longrightarrow W$ is continuous. Note that we can not just apply the Open Mapping Theorem yet, since we do not know whether $\operatorname{im} A$ is complete or, equivalently, a closed subspace of V. Thus we assume that $(A\big|_W)^{-1}$ is not continuous. Then there is a bounded sequence $Aw_n \in \operatorname{im} A$ with unbounded $w_n \in W$. Hence we can arrange things such that $\|w_n\| \geq n$ while $\|Aw_n\| \leq 1$. By rescaling, this gives us a sequence $w_n \in W$ with $\|w_n\| = 1$ but $\|Aw_n\| \leq \frac{1}{n}$. Since K is compact and the w_n's are bounded, we find a convergent subsequence $Kw_{n_k} \longrightarrow w$. Since $Aw_n = w_n - Kw_n \longrightarrow 0$ by continuity of A, we conclude that $w_{n_k} \longrightarrow w$, too. Since W was closed, $w \in W$ and from $\|w_n\| = 1$ we get $\|w\| = 1$, too. The continuity of A then gives

$$Aw = \lim_k Aw_{n_k} = \lim_k w_{n_k} - \lim_k Kw_{n_k} = 0.$$

Thus $w \in \ker A \cap W = \{0\}$ by the choice of W. This is a contradiction to $\|w\| = 1$. Thus $\operatorname{im} A$ is closed. Now we use

$$\ker A' = (\operatorname{im} A)^{\mathrm{ann}},$$

which holds for every continuous linear map, see also Exercise 6.4.4. On the one hand we know $A' = \mathbb{1} - K'$ and on the other hand, by Schauder's Theorem 6.1.9, we know that K' is again compact. Thus $\ker A'$ is finite-dimensional by the above argument. By Lemma 6.1.15, *ii.)*, and the closedness of $\operatorname{im} A$, we know that also $V\big/\operatorname{im} A$ is finite-dimensional with $\dim\ker A' = \dim(\operatorname{im} A)^{\mathrm{ann}} = \dim(V\big/\operatorname{im} A)'$. But for the finite-dimensional Banach space $(V\big/\operatorname{im} A)'$ we get a finite-dimensional dual $(V\big/\operatorname{im} A)''$ in which the Banach space $V\big/\operatorname{im} A$ is injectively embedded by Proposition 2.3.12. Thus also $V\big/\operatorname{im} A$ is finite-dimensional with $\dim V\big/\operatorname{im} A = \dim(V\big/\operatorname{im} A)'$. But this shows that $\operatorname{codim}\operatorname{im} A = \dim(V\big/\operatorname{im} A) < \infty$. It remains to show $\dim\ker A = \operatorname{codim}\operatorname{im} A$. We postpone the proof till Corollary 6.1.25, where we will have a more general result from which $\operatorname{ind}(A) = 0$ will follow. □

After this first and important example we continue with our discussion of the index of a Fredholm operator.

Theorem 6.1.22 (Fredholm operators and the index) *Let V, W, and U be Banach spaces.*

i.) The set of Fredholm operators $\operatorname{Fredholm}(V, W) \subseteq \mathrm{L}(V, W)$ *is open in the norm topology.*

ii.) The index

$$\operatorname{ind}\colon \operatorname{Fredholm}(V, W) \longrightarrow \mathbb{Z} \tag{6.1.39}$$

is a continuous map and hence constant on the connected components of $\operatorname{Fredholm}(V, W)$.

iii.) If $A \in \operatorname{Fredholm}(V, W)$ *then* $A' \in \operatorname{Fredholm}(W', V')$ *and*

$$\operatorname{ind}(A) = -\operatorname{ind}(A'). \tag{6.1.40}$$

iv.) If $A \in \operatorname{Fredholm}(V, W)$ *and* $B \in \operatorname{Fredholm}(W, U)$ *then* $AB \in \operatorname{Fredholm}(V, U)$ *and*

$$\operatorname{ind}(BA) = \operatorname{ind}(A) + \operatorname{ind}(B). \tag{6.1.41}$$

v.) $A \in \mathrm{L}(V, W)$ *is a Fredholm operator iff there are operators* $B_1, B_2 \in \mathrm{L}(W, V)$ *such that*

$$\mathbb{1} - B_1 A \in \mathfrak{K}(V) \quad \textit{and} \quad \mathbb{1} - AB_2 \in \mathfrak{K}(W). \tag{6.1.42}$$

In this case we can even choose $B_1 = B_2$ with this property.

Proof: Since $\dim\ker A$ and $\operatorname{codim}\operatorname{im} A$ are finite, we find closed subspaces $X \subseteq V$ and $Y \subseteq W$ with

$$\ker A \oplus X = V \quad \text{and} \quad \operatorname{im} A \oplus Y = W,$$

where $\dim Y = \operatorname{codim} \operatorname{im} A$ is finite. After fixing such a choice we define the map

$$\mathrm{L}(V, W) \ni B \;\mapsto\; \tilde{B} \in \mathrm{L}(X \oplus Y, W) \tag{$*$}$$

by $\tilde{B}(x + y) = Bx + y$. Indeed, $\tilde{B}$ is again a continuous linear map. The map $(*)$ is continuous itself, though not linear but only affine. Applied to A we see that $\tilde{A}$ is now injective: indeed, if $0 = \tilde{A}(x+y) = A(x)+y$ then this implies $y = 0$ since $A(x) \in \operatorname{im} A$ is in a complementary space to Y. Moreover, $A(x) = 0$ shows $x \in \ker A \cap X = \{0\}$. In addition, $\tilde{A}$ is surjective, again by the property of Y being a complementary subspace to $\operatorname{im} A$. Being continuous, $\tilde{A}$ has a continuous inverse, too, by the Open Mapping Theorem. Now we know that the invertible elements of $\mathrm{L}(X \oplus Y, W)$ are an open subset, see Exercise 6.4.10. Let $O \subseteq \mathrm{L}(X \oplus Y, W)$ be an open neighbourhood of invertible elements close to $\tilde{A}$. Then the preimage of O under $(*)$ consists of those continuous operators B close to A with $\tilde{B}$ being in O and hence $\tilde{B}$ is invertible. We claim that such a B is a Fredholm operator, too. Since $\tilde{B}$ has a continuous inverse it is clear that $\tilde{B}(X) \subseteq W$ is a closed subspace and $\tilde{B}(X) = B(X)$. Moreover, $\dim W\big/\tilde{B}(X) = \dim Y < \infty$ since $\tilde{B}$ is an isomorphism. This shows that $\operatorname{im} B \supseteq B(X)$ has finite codimension

$$\operatorname{codim} \operatorname{im} B = \dim\bigl(W\big/ \operatorname{im} B\bigr) \leq \dim\bigl(W\big/B(X)\bigr) = \dim Y < \infty,$$

and in particular

$$\operatorname{codim} \operatorname{im} B \leq \operatorname{codim} \operatorname{im} A. \tag{$\star$}$$

Moreover, since $\tilde{B}$ is injective, we see that $\ker B \cap X = \{0\}$ since $\tilde{B}(x, 0) = Bx$ for $x \in X$. Thus $\ker B \subseteq \ker A$ and hence

$$\dim \ker B \leq \dim \ker A < \infty \tag{$\star\star$}$$

is finite as well. This shows the first part, since the set of such B was an open neighbourhood of A by construction. For the second part, consider an operator B as above. In addition to $(\star)$ and $(\star\star)$ we have $\ker B \cap X = \{0\}$ and hence there is a finite-dimensional subspace $Z \subseteq \ker A$ such that $\ker A = \ker B \oplus Z$, i.e. we have the decomposition

$$\ker A \oplus X = V = \ker B \oplus Z \oplus X.$$

It follows that $\dim \ker B = \dim \ker A - \dim Z$. We have to compute $\operatorname{codim} \operatorname{im} B = \dim(W\big/ \operatorname{im} B)$. Since B is injective on Z we have $\dim \operatorname{im}(B\big|_Z) = \dim Z$. Moreover, B is injective on $Z \oplus X$ and hence $B(Z \oplus X) = B(Z) \oplus B(X)$ is a direct sum. Thus

$$\begin{aligned}
\dim\bigl(W\big/ \operatorname{im} B\bigr) &= \dim\bigl(W\big/B(Z) \oplus B(X)\bigr) \\
&= \dim\bigl(W\big/B(X)\bigr) - \dim B(Z) \\
&= \dim Y - \dim Z \\
&= \operatorname{codim} A - \dim Z.
\end{aligned}$$

But this shows that also the codimension of $\operatorname{im} B$ shrinks by the same amount $\dim Z$. Hence we see that $\operatorname{ind} A = \operatorname{ind} B$ for all the B in this open neighbourhood of A. Thus the index is locally constant and hence continuous. For the third part, we note that for any continuous operator $A\colon V \longrightarrow W$ we have

$$\ker A' = (\operatorname{im} A)^{\mathrm{ann}} \cong \bigl(W\big/ \operatorname{im} A\bigr)'$$

by Lemma 6.1.15, *i.)*. Since the topological dual of a finite-dimensional vector space is just its algebraic dual, we conclude that for a Fredholm operator we have $\dim \ker A' = \operatorname{codim} \operatorname{im} A$. On the other hand, we know $(\ker A)^{\mathrm{ann}} = \operatorname{im} A'$ for an operator with closed image of the dual, see also

Exercise 6.4.11. Since A is a Fredholm operator we know by Lemma 6.1.16 that $\operatorname{im} A$ is closed indeed and hence we can apply $(\ker A)^{\mathrm{ann}} = \operatorname{im} A'$. Then by Lemma 6.1.15, *iii.)*, we get

$$\dim\bigl(V' / \operatorname{im} A'\bigr) = \dim\bigl(V'/(\ker A)^{\mathrm{ann}}\bigr) = \dim(\ker A)' = \dim \ker A,$$

since $\ker A$ is finite-dimensional. This shows that A' is Fredholm and $\operatorname{ind} A' = \dim\ker A' - \operatorname{codim}\operatorname{im} A' = \operatorname{codim}\operatorname{im} A - \dim\ker A = -\operatorname{ind} A$. For the fourth part, let A and B be Fredholm operators and fix closed subspaces $X \subseteq V$, $M, Y \subseteq W$, and $N \subset U$ such that $V = \ker A \oplus X$, $W = \ker B \oplus M = \operatorname{im} A \oplus Y$, and $U = \operatorname{im} B \oplus N$. Moreover, we have

$$\dim Y = \operatorname{codim}\operatorname{im} A = \dim\bigl(W / \operatorname{im} A\bigr) < \infty$$

as well as

$$\dim N = \operatorname{codim}\operatorname{im} B = \dim\bigl(U / \operatorname{im} B\bigr) < \infty.$$

From this we get the decompositions

$$V = \ker A \oplus A^{-1}(\ker B \cap \operatorname{im} A) \oplus A^{-1}(M \cap \operatorname{im} A),$$

$$W = (\ker B \cap \operatorname{im} A) \oplus (\ker B \cap Y) \oplus (M \cap \operatorname{im} A) \oplus (M \cap Y),$$

and

$$U = B(M \cap \operatorname{im} A) \oplus B(M \cap Y) \oplus N,$$

since $B\big|_M$ is injective and bijective onto $\operatorname{im} B$ as well as since $A\big|_X$ is injective and bijective onto $\operatorname{im} A$. Now

$$\ker(BA) = \ker A \oplus A^{-1}(\ker B \cap \operatorname{im} A) \quad \text{and} \quad \operatorname{im}(BA) = B(M \cap \operatorname{im} A).$$

Since A and B are injective on X and on M, respectively, we see that

$$\dim B \circ A = \dim\ker A + \dim(\ker B \cap \operatorname{im} A) < \infty$$

as well as

$$\operatorname{codim}\operatorname{im} B \circ A = \operatorname{codim} B(M \cap \operatorname{im} A) = \dim B(M \cap Y) \oplus N = \dim M \cap Y + \dim N < \infty,$$

since $\ker A$, $\ker B$, Y, and N are finite-dimensional. Thus BA is a Fredholm operator as claimed. For the index we first note that

$$\dim\ker B = \dim(\ker B \cap \operatorname{im} A) + \dim(\ker B \cap Y)$$

and

$$\operatorname{codim}\operatorname{im} A = \dim(\ker B \cap Y) + \dim(M \cap Y).$$

Using this and $\operatorname{codim}\operatorname{im} B = \dim N$, the formula (6.1.41) is a straightforward computation. For the last part, let $A \in \mathrm{L}(V, W)$ be given and assume that we have a $B_1 \in \mathrm{L}(W, V)$ with $\mathbb{1} - B_1 A$ compact. By Riesz' Theorem 6.1.21 we conclude that $B_1 A$ is Fredholm and hence $\ker A \subseteq \ker B_1 A$ is finite-dimensional. Moreover, if we find a $B_2 \in \mathrm{L}(W, V)$ such that $\mathbb{1} - AB_2$ is compact then $\operatorname{im}(AB_2) \subseteq \operatorname{im} A$ has finite codimension. But then $\operatorname{im} A$ has finite codimension, too. Together, we conclude that A is Fredholm in this case. For the converse implication, assume that A is Fredholm and choose again closed subspaces $X \subseteq V$ and $Y \subseteq W$ with $V = \ker A \oplus V$ and $W = \operatorname{im} A \oplus Y$ as before. Since X and $\operatorname{im} A$ are closed and hence Banach spaces, we get for the continuous bijection

$$A\big|_X \colon X \longrightarrow \operatorname{im} A$$

a continuous inverse $(A\big|_X)^{-1}\colon \operatorname{im} A \longrightarrow X$ by the Open Mapping Theorem 2.3.19 as usual. Thus we can define a continuous map $B\colon W \longrightarrow V$ by $B\big|_{\operatorname{im} A} = (A\big|_X)^{-1}$ and $B\big|_Y = 0$. Then we have $AB\big|_Y = 0$ and $AB\big|_{\operatorname{im} A} = \mathrm{id}_{\operatorname{im} A}$ while $BA\big|_X = \mathrm{id}_X$ and $BA\big|_{\ker A} = 0$. Thus AB is the continuous projection onto $\operatorname{im} A$ with respect to the splitting $W = \operatorname{im} A \oplus Y$ and BA is the continuous projection onto X with respect to the splitting $V = \ker A \oplus X$. It follows that $\mathbb{1} - AB$ is the continuous projection onto Y and $\mathbb{1} - BA$ is the continuous projection onto $\ker A$. They have both finite-dimensional image and are continuous. Thus by Example 6.1.3 they are compact operators. □

We have now a few simple conclusions concerning the relations between Fredholm operators and compact operators.

Corollary 6.1.23 *Let $A \in \mathrm{L}(V, W)$. Then A is Fredholm iff there is a Fredholm operator $B \in \mathrm{Fredholm}(W, V)$ with*

$$\mathbb{1} - AB \in \mathfrak{K}(V) \quad \textit{and} \quad \mathbb{1} - BA \in \mathfrak{K}(W). \tag{6.1.43}$$

Proof: If we find such a B then A is Fredholm by Theorem 6.1.22, *v.)*. Conversely, if A is Fredholm, then we find a $B \in \mathrm{L}(W, V)$ with (6.1.43), which is Fredholm by applying Theorem 6.1.22, *v.)*, to B. □

In general, B is not uniquely determined: we can add an arbitrary compact operator $K \in \mathfrak{K}(W, V)$ to B. Then we still have (6.1.43) for $B' = B + K$ by Proposition 6.1.7, *ii.)*. However, the B we constructed in the proof of the Theorem had the following additional property:

Corollary 6.1.24 *Let $A \in \mathrm{L}(V, W)$. Then A is Fredholm iff there is a Fredholm operator $B \in \mathrm{L}(W, V)$ such that $\mathbb{1} - AB$ and $\mathbb{1} - BA$ are finite-rank projections onto a complement to $\operatorname{im} A$ and onto $\ker A$, respectively.*

Note that this additional feature does still not fix B since we had to chose the complements to $\operatorname{im} A$ and $\ker A$.

Finally, we can now complete the proof of Riesz' Theorem 6.1.21, where we still have to show $\operatorname{ind}(\mathbb{1} - K) = 0$ for a compact operator. This follows now easily from the slightly more general corollary:

Corollary 6.1.25 *Let $A \in \mathrm{Fredholm}(V, W)$ and $K \in \mathfrak{K}(V, W)$. Then $A + K \in \mathrm{Fredholm}(V, W)$ and*

$$\operatorname{ind}(A + K) = \operatorname{ind} A. \tag{6.1.44}$$

Proof: By Theorem 6.1.22, *v.)*, we find an operator $B \in \mathrm{L}(V, W)$ with $\mathbb{1} - AB \in \mathfrak{K}(W)$ and $\mathbb{1} - BA \in \mathfrak{K}(V)$. By Proposition 6.1.7, *ii.)*, we know that $BK \in \mathfrak{K}(V)$ and $KB \in \mathfrak{K}(W)$. Thus we see that $\mathbb{1} - (A+K)B = \mathbb{1} - AB - KB \in \mathfrak{K}(W)$ as well as $\mathbb{1} - B(A+K) = \mathbb{1} - BA - BK \in \mathfrak{K}(V)$. Again by Theorem 6.1.22, *v.)*, the operator B applies to $A + K$ as well and hence $A + K \in \mathrm{Fredholm}(V, W)$. Now $t \mapsto A + tK$ is a continuous curve of Fredholm operators for all $t \in \mathbb{R}$. Since the index is locally constant we conclude (6.1.44). □

We arrive at the picture that Fredholm operators are those continuous operators which are *invertible up to compact operators*. The corresponding "inverse" is then again a Fredholm operator. In many applications such an "inverse" is almost as good as an honest inverse. This leads to the following definition:

Definition 6.1.26 (Parametrix) *Let $A \in \mathrm{Fredholm}(V, W)$. An operator $B \in \mathrm{Fredholm}(W, V)$ with $\mathbb{1} - AB \in \mathfrak{K}(W)$ and $\mathbb{1} - BA \in \mathfrak{K}(V)$ is called a parametrix for A.*

Since the compact operators enjoy the ideal properties from Proposition 6.1.7, *ii.)*, we can define the following quotient spaces:

Definition 6.1.27 (Calkin algebra) *Let V and W be Banach spaces. Then one defines*

$$\mathfrak{C}(V, W) = \mathrm{L}(V, W)/\mathfrak{K}(V, W). \tag{6.1.45}$$

The quotient algebra $\mathfrak{C}(V) = \mathfrak{C}(V, V)$ is called the Calkin algebra of V.

Since the compact operators are a closed subspace the quotient $\mathfrak{C}(V, W)$ inherits the operator norm and becomes a Banach space itself. We will always equip $\mathfrak{C}(V, W)$ with this norm.

We can now rephrase our results on compact and Fredholm operators as follows:

Theorem 6.1.28 (Calkin algebra) *Let V, W, and U be Banach spaces.*

i.) *There is a well-defined continuous bilinear composition*

$$\circ\colon \mathfrak{C}(W, U) \times \mathfrak{C}(V, W) \longrightarrow \mathfrak{C}(V, U) \tag{6.1.46}$$

defined by $[B] \circ [A] = [BA]$ for $B \in \mathrm{L}(W, U)$ and $A \in \mathrm{L}(V, W)$. The composition $\circ$ enjoys the usual associativity relations and $[\mathbb{1}_V]$ are the unit elements.

ii.) *$[A] \in \mathfrak{C}(V, W)$ is invertible iff it has a representative $A \in$ Fredholm(V, W). In this case, every representative of $[A]$ is Fredholm.*

iii.) *On the groupoid of invertible elements in $\mathfrak{C}(\,\cdot\,,\,\cdot\,)$ one has a groupoid morphism*

$$\mathrm{ind}\colon \mathrm{GL}(\mathfrak{C}(\,\cdot\,,\,\cdot\,)) \longrightarrow \mathbb{Z} \tag{6.1.47}$$

defined by $\mathrm{ind}([A]) = \mathrm{ind}\, A$.

iv.) *The Calkin algebra $\mathfrak{C}(V)$ is a Banach algebra with unit and* $\mathrm{ind}\colon \mathrm{GL}(\mathfrak{C}(V)) \longrightarrow \mathbb{Z}$ *is a group morphism.*

PROOF: First it is clear from Proposition 6.1.7, *ii.)*, that (6.1.46) is well-defined on the quotients $\mathfrak{C}(V, W)$ and $\mathfrak{C}(W, U)$. We have already observed that $\mathfrak{C}(V, W)$ is a Banach space itself. For the continuity of $\circ$ we can apply the argument from Proposition 4.1.18 also to this situation to show $\|[B] \circ [A]\| \le \|[B]\|\|[A]\|$, even though $[A]$ and $[B]$ are not in the same Banach space. The associativity properties and the identity features of $[\mathbb{1}_V]$ can then be checked on representatives. For the second part, we note that $[A]$ is invertible iff for some representative A we find a $B, \tilde{B} \in \mathrm{L}(V, W)$ such that $\mathbb{1} - AB$ and $\mathbb{1} - \tilde{B}A$ are compact. Thus A is Fredholm by Theorem 6.1.22, *v.)*, as well. Conversely, if we have some representative A with this property, then any other representative differs from A by a compact operator and is thus Fredholm again. The third part is now clear by $\mathrm{ind}(\mathbb{1}_V) = 0$, Theorem 6.1.22, *iv.)*, and Corollary 6.1.25. The last part is a consequence of the third part. □

Remark 6.1.29 (Index Theorems) Just as a remark it should be mentioned that the stability of the index with respect to compact perturbations is the main ingredient for various *index theorems*, e.g. in differential geometry. Here one has a Fredholm operator A, typically constructed as a differential operator depending on geometric choices like a Riemannian metric etc., such that different choices lead to different Fredholm operators. But the differences can then be shown to be compact and thus the index of A is an invariant property, often a topological invariant of the underlying manifold. Note however, that the details of the index theorems are much more involved and require, among others, an extension of the above technology to unbounded Fredholm operators. For further details confer e.g. [7].

6.1.3 Spectral Theorem for Compact Operators

It turns out that for compact operators between Banach spaces there is a good spectral theory which is in many aspects much simpler than the spectral theory of bounded operators between Hilbert spaces. In fact, compact operators behave very much like matrices in finite dimensions. In this subsection we sketch the general Banach space situation and pass on to the Hilbert space situation in the next subsection.

We start with the following theorem known as Fredholm's alternative:

Theorem 6.1.30 (Fredholm's alternative) *Let $A\colon V \longrightarrow W$ be a Fredholm operator between Banach spaces with $\operatorname{ind}(A) = 0$. Then either*

i.) the equation $Av = w$ has a solution for every $w \in W$. In this case, A is even an isomorphism.

or

ii.) the equation $Av = w$ is not solvable for all $w \in W$. In this case, the solutions (if any) for a particular $w_0 \in W$ form a finite-dimensional affine subspace of V.

PROOF: Suppose $Av = w$ has a solution for every $w \in W$, i.e. $\operatorname{im} A = W$. Then by $\operatorname{ind}(A) = 0$ we have $\ker A = \{0\}$. Thus A is a continuous bijection which, by the Open Mapping Theorem 2.3.19 is an isomorphism. Conversely, assume that $Av = w$ is not solvable for all $w \in W$, i.e. A is not surjective. Since $\operatorname{ind}(A) = 0$ it is neither surjective nor injective. If $w_0 \in W$ is in the image, i.e. if we have solutions $Av_0 = w_0$, then also $v_0 + v$ is a solution iff $v \in \ker A$, which is a nontrivial finite-dimensional subspace. Thus all solutions are given by $v_0 + \ker A$. ☐

Corollary 6.1.31 *Let $K \in \mathfrak{K}(V)$ be a compact operator. Then every nonzero spectral value is an eigenvalue and the corresponding eigenspace is finite-dimensional.*

PROOF: Let $\lambda \neq 0$. Then $\lambda - K = \lambda(\mathbb{1} - \frac{1}{\lambda}K)$ is a Fredholm operator of index 0 by Theorem 6.1.21. If λ is a spectral value, then $\lambda - K$ is not invertible. By $\operatorname{ind}(\lambda - K) = 0$ this can only happen if $\ker(\lambda - K) \neq \{0\}$. Moreover, $\dim\ker(\lambda - K) < \infty$ as this holds for every Fredholm operator. ☐

Note however that 0 may be a spectral value without being an eigenvalue and if 0 is an eigenvalue it may be infinitely degenerate, take e.g. $K = 0$. In fact, we will see soon that 0 is always a spectral value of a compact operator. The corollary also suggest that the spectral theory for compact operators is very simple even in the case of a general Banach space. We will need now the following lemma of Riesz:

Lemma 6.1.32 (Riesz) *Let V be a normed vector space and $U \subseteq V$ a closed and proper subspace. For a given $0 < \delta < 1$ there exists a vector $v_\delta \in V$ with $\|v_\delta\| = 1$ and*

$$\|v_\delta - u\| \geq 1 - \delta \tag{6.1.48}$$

for all $u \in U$.

PROOF: Let $v \notin U$ and consider $d = \inf_{u\in U}\|v - u\|$ which is strictly positive since U is closed and not equal to V. Since $1 - \delta < 1$ we find a $u_\delta \in U$ with

$$d \leq \|v - u_\delta\| < \frac{d}{1-\delta}.$$

We claim that $v_\delta = \frac{v-u_\delta}{\|v-u_\delta\|}$ will do the job. Indeed, for $u \in U$ we have

$$\|v_\delta - u\| = \frac{1}{\|v-u_\delta\|}\big\|v - (u_\delta + \|v-u_\delta\|u)\big\| \geq \frac{d}{\|v-u_\delta\|} > 1 - \delta.$$

☐

Already in finite dimensions, where every operator is compact, we can not diagonalize every endomorphism: the nilpotent part of it prevents this. In order to understand this phenomenon we have to investigate not only K but also all powers K^n. To this end, we need the following lemma, the finite-dimensional case of which is well-known from linear algebra:

Lemma 6.1.33 *Let $K \in \mathfrak{K}(V)$ and set $A = \mathbb{1} - K$. Then we have:*

i.) For all $n \in \mathbb{N}$ one has

$$\ker A^n \subseteq \ker A^{n+1} \quad \textit{and} \quad \operatorname{im} A^n \supseteq \operatorname{im} A^{n+1}. \tag{6.1.49}$$

ii.) There exists a minimal $p \in \mathbb{N}$ such that for all $n \geq p$ one has

$$\ker A^n = \ker A^p. \tag{6.1.50}$$

For this p one also has for all $n \geq p$

$$\operatorname{im} A^n = \operatorname{im} A^p. \tag{6.1.51}$$

iii.) For the minimal $p \in \mathbb{N}$ with (6.1.50) the subspaces $\ker A^p$ and $\operatorname{im} A^p$ are closed complementary subspaces, i.e.

$$\ker A^p \oplus \operatorname{im} A^p = V. \tag{6.1.52}$$

iv.) For the minimal $p \in \mathbb{N}$ with (6.1.50) the restriction

$$A\big|_{\operatorname{im} A^p} \colon \operatorname{im} A^p \longrightarrow \operatorname{im} A^p \tag{6.1.53}$$

is an isomorphism.

PROOF: The first part is known from linear algebra not using compactness at all; we just included it for convenience. For the second part, we know that A^n is a Fredholm operator of index 0 by Theorem 6.1.22, *iv.)*, and Riesz' Theorem 6.1.21. Thus $\ker A^n$ is finite-dimensional for all $n \in \mathbb{N}$ and in particular closed. Now we assume that the sequence of kernels (6.1.49) does not stabilize. Then we can find vectors $v_n \in \ker A^n$ such that $\|v_n\| = 1$ and

$$\operatorname{dist}(v_n, \ker A^{n-1}) > \frac{1}{2} \tag{$*$}$$

for all $n \in \mathbb{N}$, by applying Riesz' Lemma 6.1.32 to $\ker A^{n-1} \subsetneq \ker A^n$. Now for $n > m > 1$ we have $A^{n-1}(Av_n - v_m + Av_m) = A^n v_n - A^{n-1} v_m + A^n v_m = 0$ since $v_n \in \ker A^n$ and $v_m \in \ker A^{n-1}$. Thus we get from $(*)$

$$\|Kv_n - Kv_m\| = \|(\mathbb{1} - A)(v_n - v_m)\| = \|v_n - (Av_n - v_m + Av_m)\| > \frac{1}{2}.$$

It follows that the sequence $(Kv_n)_{n\in\mathbb{N}}$ can not have any convergent subsequence. However, $\|v_n\| = 1$ and K was assumed to be compact. Hence we have reached a contradiction and (6.1.50) has to stabilize at some minimal $p \in \mathbb{N}$. Now $\operatorname{ind}(A^n) = 0$ and hence we have $\operatorname{codim} \operatorname{im} A^n = \dim \ker A^n = \dim \ker A^p = \operatorname{codim} \operatorname{im} A^p$ for all $n \geq p$. Together with the trivial inclusion (6.1.49) this implies (6.1.51). For the third part we know already that $\ker A^n$ and $\operatorname{im} A^n$ are closed subspaces for all $n \in \mathbb{N}$ by the Fredholm property of A. Thus let $v \in \ker A^p \cap \operatorname{im} A^p$ be given, i.e. $A^p v = 0$ and $v = A^p w$ for some $w \in V$. But then $A^{2p} w = 0$ and hence, by (6.1.50), we get $A^p w = 0$. This shows $v = 0$ and consequently the sum in (6.1.52) is direct. Since $\operatorname{codim} \operatorname{im} A^p = \dim \ker A^p$ we have already a complementary pair of closed subspaces. For the last part, $A\big|_{\operatorname{im} A^p}$ is injective by the direct sum decomposition (6.1.52). Conversely, $\operatorname{im} A^{p+1} = \operatorname{im} A^p$ and hence $v = A^p w \in \operatorname{im} A^p$ is of the form $v = A^{p+1} u = AA^p u$ proving the surjectivity of (6.1.53). The continuity of the inverse follows as usual from the Open Mapping Theorem 2.3.19 since $\operatorname{im} A^p$ is a Banach space for its own. ☐

We can now formulate the spectral theorem for compact operators on Banach space:

Theorem 6.1.34 (Spectral theorem for compact operators) *Let V be a Banach space and let $K \in \mathfrak{K}(V)$ be a compact operator on V.*

i.) *If $\dim V = \infty$ then $0 \in \operatorname{spec}(K)$.*

ii.) *The spectrum of K is at most countable and 0 is the only possible accumulation point.*

iii.) *Every spectral value $\lambda \in \operatorname{spec}(K) \setminus \{0\}$ is an eigenvalue and $\ker(\lambda - K)$ is finite-dimensional.*

iv.) *For every $\lambda \in \operatorname{spec}(K) \setminus \{0\}$ there are unique closed complementary subspaces $\mathcal{N}(\lambda)$ and $\mathcal{R}(\lambda)$ of V such that*

  i.) *The subspace $\mathcal{N}(\lambda)$ is finite-dimensional.*

  ii.) *One has $K\mathcal{N}(\lambda) \subseteq \mathcal{N}(\lambda)$ and there exists a minimal $p_\lambda \in \mathbb{N}$ such that $\lambda - K\big|_{\mathcal{N}(\lambda)}$ is nilpotent of order p_λ, i.e.*

$$\left(\lambda - K\big|_{\mathcal{N}(\lambda)}\right)^{p_\lambda} = 0. \tag{6.1.54}$$

  iii.) *One has $K\mathcal{R}(\lambda) \subseteq \mathcal{R}(\lambda)$ and $\lambda - K\big|_{\mathcal{R}(\lambda)}$ is an isomorphism onto $\mathcal{R}(\lambda)$.*

v.) *For all $\lambda \in \operatorname{spec}(K) \setminus \{0\}$ one has $\ker(\lambda - K) \subseteq \mathcal{N}(\lambda)$.*

vi.) *If $\lambda \neq \mu$ are nonzero spectral values of K then $\mathcal{N}(\lambda) \subseteq \mathcal{R}(\mu)$.*

PROOF: The first part is clear by Proposition 6.1.7, *iii.)*. The third part was already shown in Corollary 6.1.31. Thus we consider the second part. We want to show that for every $\epsilon > 0$ there are only finitely many spectral values and hence eigenvalues of K with $|\lambda| \geq \epsilon$. Assume that this is not the case. Then there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of eigenvalues with corresponding eigenvectors $v_n \neq 0$ such that

$$|\lambda_n| \geq \epsilon, \quad Kv_n = \lambda_n v_n, \quad \text{and} \quad \lambda_n \neq \lambda_m$$

for $n \neq m$. Being eigenvectors to pairwise distinct eigenvalues, the set $\{v_1, v_2, \ldots\}$ is linearly independent. We consider the finite-dimensional and hence closed subspaces

$$E_n = \operatorname{span}_{\mathbb{C}}\{v_1, \ldots, v_n\} \subseteq V.$$

Then $E_n \subsetneq E_{n+1}$ is a proper subspace and $KE_n \subseteq E_n$. By Riesz' Lemma 6.1.32 we get a sequence $w_n \in E_n$ with $\|w_n\| = 1$ and

$$\operatorname{dist}(w_n, E_{n-1}) > \frac{1}{2}. \tag{$*$}$$

Now we can express w_n in terms of the basis $v_1, \ldots, v_n$ giving the linear combination

$$w_n = \sum_{i=1}^{n} w_{ni} v_i$$

with uniquely determined $w_{ni} \in \mathbb{C}$. We get

$$(\lambda_n - K)w_n = \sum_{i=1}^{n} (\lambda_n - \lambda_i) w_{ni} v_i = \sum_{i=1}^{n-1} (\lambda_n - \lambda_i) w_{ni} v_i \in E_{n-1}$$

and hence $Kw_m + \lambda_n w_n - Kw_n \in E_{n-1}$ for $n > m$. By ($*$) this implies

$$\|Kw_n - Kw_m\| = \|\lambda_n w_n - (Kw_m + \lambda_n w_n - .Kw_n)\| \geq |\lambda_n| \operatorname{dist}(w_n, E_{n-1}) \geq \frac{\epsilon}{2}.$$

Thus the sequence $(Kw_n)_{n\in\mathbb{N}}$ can not have any convergent subsequence in contradiction to the compactness of K. Thus the second part follows. For the fourth part, we consider $\lambda \neq 0$ and apply

Lemma 6.1.33 to $A_\lambda = \mathbb{1} - \frac{1}{\lambda}K$. This gives a minimal $p_\lambda \in \mathbb{N}$ for which $\ker(\lambda - K)^n = \ker(\lambda - K)^{p_\lambda}$ for all $n \geq p_\lambda$. We take now $\mathcal{N}(\lambda) = \ker(\lambda - K)^{p_\lambda}$ and $\mathcal{R}(\lambda) = \operatorname{im}(\lambda - K)^{p_\lambda}$. Then $K(\lambda - K)^{p_\lambda} = (\lambda - K)^{p_\lambda} K$ shows $K\mathcal{N}(\lambda) \subseteq \mathcal{N}(\lambda)$ and $K\mathcal{R}(\lambda) \subseteq \mathcal{R}(\lambda)$. The remaining properties of $\mathcal{N}(\lambda)$ and $\mathcal{R}(\lambda)$ follow from Lemma 6.1.33. This shows the existence. For uniqueness suppose that $\tilde{\mathcal{N}}$, $\tilde{\mathcal{R}}$ and $\tilde{p}$ are a different choice. Let $v \in \tilde{\mathcal{N}}$, i.e. $(\lambda - K)^{\tilde{p}} v = 0$. Then $v \in \ker(\lambda - K)^{\tilde{p}} \subseteq \ker(\lambda - K)^{p_\lambda} = \mathcal{N}(\lambda)$ by Lemma 6.1.33, *i.)* and *ii.)*. Hence $\tilde{\mathcal{N}} \subseteq \mathcal{N}(\lambda)$ follows. Now let $v \in \tilde{\mathcal{R}}$ then $v = u + w$ with $u \in \mathcal{N}(\lambda)$ and $w \in \mathcal{R}(\lambda)$ according to the direct sum decomposition $\mathcal{N}(\lambda) \oplus \mathcal{R}(\lambda) = V$. Hence $(\lambda - K)^{p_\lambda} v = (\lambda - K)^{p_\lambda} w \in \mathcal{R}(\lambda)$ and $(\lambda - K)^{p_\lambda} \tilde{\mathcal{R}} \subseteq \mathcal{R}(\lambda)$ follows. However, by the third part of *iv.)* for $\tilde{\mathcal{R}}$ we also have $(\lambda - K)^{p_\lambda} \colon \tilde{\mathcal{R}} \longrightarrow \tilde{\mathcal{R}}$ being an isomorphism. Hence $\tilde{\mathcal{R}} \subseteq \mathcal{R}(\lambda)$ follows. Since $\tilde{\mathcal{N}} \oplus \tilde{\mathcal{R}} = V$, too, this can only be possible with $\tilde{\mathcal{N}} = \mathcal{N}(\lambda)$ and $\tilde{\mathcal{R}} = \mathcal{R}(\lambda)$. But then $\tilde{p} = p_\lambda$ is clear. The fifth part is clear by Lemma 6.1.33, *i.)*. For the last part, consider $\lambda, \mu \in \operatorname{spec}(K) \setminus \{0\}$ with $\lambda \neq \mu$. On the finite-dimensional subspace $\mathcal{N}(\lambda)$ we have

$$(\mu - K)\big|_{\mathcal{N}(\lambda)} = (\mu - \lambda)\mathbb{1}\big|_{\mathcal{N}(\lambda)} + (\lambda - K)\big|_{\mathcal{N}(\lambda)},$$

and $(\mu - \lambda)\mathbb{1}\big|_{\mathcal{N}(\lambda)}$ is clearly invertible. Since $(\lambda - K)\big|_{\mathcal{N}(\lambda)}$ is nilpotent, $(\mu - K)\big|_{\mathcal{N}(\lambda)} \colon \mathcal{N}(\lambda) \longrightarrow \mathcal{N}(\lambda)$ is invertible by a geometric series, which even terminates after p_λ many steps. Thus $(\mu - K)(\mathcal{N}(\lambda)) = \mathcal{N}(\lambda)$ and hence also $(\mu - K)^n \mathcal{N}(\lambda) = \mathcal{N}(\lambda)$ for all $n \in \mathbb{N}$. By Lemma 6.1.33, *i.)* and *ii.)*, we conclude that $\mathcal{N}(\lambda) \subseteq \mathcal{R}(\mu)$. ☐

Remark 6.1.35 (Jordan decomposition) This theorem is in some sense the Banach space generalization of the Jordan decomposition in finite dimensions. Indeed, if $V = \mathbb{C}^n$ is finite-dimensional then any operator is compact and the Jordan decomposition and the Jordan normal form can be recovered easily from this theorem. The spaces $\mathcal{N}(\lambda)$ correspond to the *generalized eigenspaces* on which $\lambda - K$ is not zero but at least nilpotent. Since these spaces are finite-dimensional one can find basis of $\mathcal{N}(\lambda)$ such that $\lambda - K$ consists of the usual Jordan blocks in this basis. If $p_\lambda = 1$ then $\mathcal{N}(\lambda) = \ker(\lambda - K)$ consists of eigenvectors only. Thus one defines as in linear algebra $\dim \mathcal{N}(\lambda)$ to be the *algebraic multiplicity* of λ while $\dim \ker(\lambda - K)$ is called the *geometric multiplicity*. It is clear that Theorem 6.1.34 is the best we can hope for since already in the finite-dimensional spaces we can not diagonalize every endomorphism.

6.1.4 Spectral Theory of Normal Compact Operators

We shall now specialize our investigations of compact operators to the case of Hilbert spaces. We first note the following result for compact and Fredholm operators in this case:

Proposition 6.1.36 *Let $\mathfrak{H}_1$, $\mathfrak{H}_2$, and $\mathfrak{H}$ be Hilbert spaces.*

i.) *For $A \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ one has $A \in \mathfrak{K}(\mathfrak{H}_1, \mathfrak{H}_2)$ iff $A^* \in \mathfrak{K}(\mathfrak{H}_2, \mathfrak{H}_1)$.*

ii.) *The adjoint descends to a well-defined antilinear map*

$$^* \colon \mathfrak{C}(\mathfrak{H}_1, \mathfrak{H}_2) \longrightarrow \mathfrak{C}(\mathfrak{H}_2, \mathfrak{H}_1), \tag{6.1.55}$$

obeying $[AB]^* = [B]^*[A]^*$ and $([A]^*)^* = [A]$.

iii.) *The compact operators $\mathfrak{K}(\mathfrak{H}) \subseteq \mathfrak{B}(\mathfrak{H})$ form a closed *-ideal which is proper iff $\dim \mathfrak{H} = \infty$.*

iv.) *For $A \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ one has $A \in \operatorname{Fredholm}(\mathfrak{H}_1, \mathfrak{H}_2)$ iff $A^* \in \operatorname{Fredholm}(\mathfrak{H}_2, \mathfrak{H}_1)$ and in this case*

$$\operatorname{ind}(A^*) = -\operatorname{ind}(A). \tag{6.1.56}$$

v.) *The Calkin algebra $\mathfrak{C}(\mathfrak{H})$ is a unital C^*-algebra.*

vi.) *An element* $[A] \in \mathfrak{C}(\mathfrak{H})$ *is invertible iff one and hence all representatives* A *are Fredholm. The index gives a group homomorphism*

$$\operatorname{ind}\colon \operatorname{GL}(\mathfrak{C}(\mathfrak{H})) \longrightarrow \mathbb{Z}. \tag{6.1.57}$$

All statements are clear from Corollary 6.1.10, Theorem 6.1.22, and Theorem 6.1.28. We do not repeat the details which are easily filled in. From this point of view, the results of Example 6.1.20 become now more transparent. In particular, the shift operators U_k become honest unitaries in $\mathfrak{C}(\ell^2)$ with nontrivial index $\operatorname{ind}(U_k) = -k$.

We can now specialize the spectral analysis of compact operators from Theorem 6.1.34 to the case of normal operators on a Hilbert space $\mathfrak{H}$. Here we get the following statement:

Theorem 6.1.37 (Spectral theorem for compact normal operators) *Let* $K \in \mathfrak{K}(\mathfrak{H})$ *be a compact normal operator on a Hilbert space* $\mathfrak{H}$.

i.) *The spectrum* $\operatorname{spec}(K)$ *is countable and has* 0 *as only possible accumulation point. If* $\dim \mathfrak{H} = \infty$ *then necessarily* $0 \in \operatorname{spec}(K)$. *All spectral values different from* 0 *are eigenvalues.*

ii.) *The Hilbert space* $\mathfrak{H}$ *decomposes as (countable) direct orthogonal sum*

$$\mathfrak{H} = \widehat{\bigoplus_{\lambda \in \operatorname{spec}(K)}} \ker(\lambda - K) \tag{6.1.58}$$

of eigenspaces and all eigenspaces $\ker(\lambda - K)$ *with* $\lambda \neq 0$ *are finite-dimensional. Note that* $\ker K = \{0\}$ *may happen.*

iii.) *One has*

$$K = \sum_{\lambda \in \operatorname{spec}(K)} \lambda P_{\ker(\lambda - K)} \tag{6.1.59}$$

as convergent series in the operator norm and

$$\|K\| = \sup\bigl\{|\lambda| \bigm| \lambda \in \operatorname{spec}(K)\bigr\} = \max\bigl\{|\lambda| \bigm| \lambda \in \operatorname{spec}(K)\bigr\}. \tag{6.1.60}$$

iv.) *The spectral measure* E *of* K *is given by*

$$E_U = P_{\widehat{\bigoplus}_{\lambda \in \operatorname{spec}(K) \cap U} \ker(\lambda - K)}, \tag{6.1.61}$$

and in particular $E_{\{\lambda\}} = P_{\ker(\lambda - K)}$ *for* $\lambda \in \operatorname{spec}(K)$.

v.) *The bounded measurable calculus of* K *is given by*

$$f(K) = \sum_{\lambda \in \operatorname{spec}(K)} f(\lambda) P_{\ker(\lambda - K)} \tag{6.1.62}$$

for $f \in \mathscr{BM}(\operatorname{spec}(K))$ *in the sense of the strong operator convergence.*

vi.) *If* $f \in \mathscr{C}(\operatorname{spec}(K))$ *then* $f(K)$ *is compact if* $f(0) = 0$ *and not compact if* $f(0) \neq 0$ *and* $\dim \mathfrak{H} = \infty$. *In the case* $f(0) = 0$ *the series* (6.1.62) *is convergent in the operator norm.*

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PROOF: The first part is Theorem 6.1.34, repeated just for convenience. Since the spectrum is countable, we can now rely on Theorem 5.1.34: since we know already that all $\lambda \neq 0$ are eigenvalues, we have $E_{\{\lambda\}} = P_{\ker(\lambda - K)}$ for $\lambda \neq 0$ according to Theorem 5.1.34, *iii.)*. In fact, all spectral values $\lambda \neq 0$ are isolated and hence also part *v.)* of that theorem applies. Now *vi.)* of that theorem gives (6.1.58); the finite-dimensionality was shown already in Theorem 6.1.34 in general. We know that the bounded measurable calculus of K is given by the strongly convergent expression (6.1.62) according to Theorem 5.1.34, *vi.)*. In particular, (6.1.59) holds as unconditionally convergent series in the strong operator topology. But now in addition we have only 0 as possible accumulation point of $\operatorname{spec}(K)$. Thus the finite rank operator $\sum_{\delta \le |\lambda| \le \epsilon} \lambda P_{\{\ker(\lambda - K)\}}$, where $\epsilon > \delta > 0$, has spectrum given by those finitely many $\lambda \in \operatorname{spec}(K)$ with $\delta \le |\lambda| \le \epsilon$ and possibly 0 in the infinite-dimensional case. Since it is normal, we have for the operator norm

$$\left\| \sum\nolimits_{\delta \le |\lambda| \le \epsilon} \lambda P_{\{\ker(\lambda - K)\}} \right\| = \sup\big\{ |\lambda| \bigm| \lambda \in \operatorname{spec}(K), \delta \le |\lambda| \le \epsilon \big\} \le \epsilon.$$

This shows that the series (6.1.59) is convergent even in the operator norm topology. Note however, that it is not absolutely convergent as $\|P_{\ker(\lambda - K)}\| = 1$ and the $\lambda \in \operatorname{spec}(K)$ only form a zero sequence but not a summable sequence in general. The equality (6.1.60) is clear as there are only finitely many $\lambda \in \operatorname{spec}(K)$ with $|\lambda| \ge \epsilon$ for every $\epsilon > 0$. The fourth part is clear and also the fifth part is a particular case of Theorem 5.1.34, *vii.)*. For the last part, note that $f \in \mathscr{C}(\operatorname{spec}(K))$ simply means $f(\lambda) \longrightarrow f(0)$ for $\lambda \longrightarrow 0$ since away from 0 the spectrum consists of isolated points only. If now in addition $f(0) = 0$ then Corollary 4.3.28 shows that $f(K) \in \mathfrak{K}(\mathfrak{H})$ since the compact operators are a C^*-algebra without unit according to Proposition 6.1.7, *iii.)*. Moreover, in this case the same argument as for the norm convergence of (6.1.59) applies. If $\dim \mathfrak{H} = \infty$ and $f(0) \ne 0$ then we choose a sequence of eigenvectors ψ_n with $K\psi_n = \lambda_n \psi_n$ and $\langle \psi_n, \psi_m \rangle = \delta_{nm}$, which is possible thanks to the second part. Then $f(K)\psi_n = f(\lambda_n)\psi_n$ and by the orthogonality we get

$$\begin{aligned} \|f(K)\psi_n - f(K)\psi_m\|^2 &= \langle f(\lambda_n)\psi_n - f(\lambda_m)\psi_m, f(\lambda_n)\psi_n - f(\lambda_m)\psi_m \rangle \\ &= |f(\lambda_n)|^2 + |f(\lambda_m)|^2 \\ &\longrightarrow 2|f(0)|^2 > 0 \end{aligned}$$

for $n \ne m$ and $n, m \longrightarrow \infty$. Thus $f(\lambda_n)\psi_n$ does not contain a convergent subsequence and hence $f(K)$ can not be compact. □

Remark 6.1.38 (Alternative proof of Theorem 6.1.37) There exists a more direct proof of the spectral theorem for compact normal operators not relying on the general spectral theorem which is rather elementary: one uses Theorem 6.1.34 to show that $\operatorname{spec}(K)$ is countable with finite-dimensional $\ker(\lambda - K)$ for $\lambda \ne 0$. Then a proof by induction allows to show (6.1.58) and (6.1.59). The details are discussed in Exercise **??**.

We collect now some easy but yet useful corollaries of the characterization of compact operators on a Hilbert space:

Corollary 6.1.39 *Let $K \in \mathfrak{K}(\mathfrak{H})$ be a normal compact operator. Then K is unitarily diagonalizable, i.e. K has a Hilbert basis of eigenvectors.*

Corollary 6.1.40 *Let $K \in \mathfrak{K}(\mathfrak{H})$. Then $|K|^\alpha = \sqrt{K^*K}^\alpha \in \mathfrak{K}(\mathfrak{H})$ for all $\alpha > 0$.*

PROOF: This is more a corollary to the fact that $\mathfrak{K}(\mathfrak{H}) \subseteq \mathfrak{B}(\mathfrak{H})$ is a (non-unital) C^*-algebra using Corollary 4.3.28. □

We can also take other continuous functions of compact operators. As an example one considers a positive compact $K \ge 0$ with $\|K\| \le 1$. Then the operator

$$-K \log(K) \ge 0 \tag{6.1.63}$$

is again a compact positive operator since the function $x \mapsto -x \log(x)$ is positive and continuous on $[0, 1]$. This will be important when defining the von Neumann entropy of a density matrix later.

We can now directly characterize compact operators by their spectral properties: this is a converse statement to the first part of Theorem 6.1.37:

Proposition 6.1.41 *Let $K \in \mathfrak{B}(\mathfrak{H})$ be a normal operator. Then K is compact iff $\operatorname{spec}(K) \setminus \{0\}$ consists of isolated points only and $\ker(\lambda - K)$ is finite-dimensional for all $\lambda \in \operatorname{spec}(K) \setminus \{0\}$.*

PROOF: If K is compact then Theorem 6.1.37, *i.)*, gives us the result immediately. Thus assume that $\operatorname{spec}(K) \setminus \{0\}$ consists of isolated points only. By the compactness of $\operatorname{spec}(K)$ we have only finitely many $\lambda \in \operatorname{spec}(K) \setminus \{0\}$ with $|\lambda| \geq \delta$ for every $\delta > 0$. By Theorem 5.1.34 we know that $\mathfrak{H} = \widehat{\bigoplus}_{\lambda \in \operatorname{spec}(K)} \ker(\lambda - K)$ and $K = \sum_{\lambda \in \operatorname{spec}(K)} \lambda P_{\ker(\lambda - K)}$ converges in the strong operator topology. With the same argument as in the proof of Theorem 6.1.37, *iii.)*, we see that the series actually converges in the operator norm topology. Since $\dim \ker(\lambda - K) < \infty$ for all $\lambda \neq 0$ we see that K is the norm limit of finite-rank operators. By $\mathfrak{F}(\mathfrak{H}) \subseteq \mathfrak{K}(K)$ according to Example 6.1.3 and by the norm closedness of $\mathfrak{K}(\mathfrak{H})$ according to Proposition 6.1.7 we conclude $K \in \mathfrak{K}(\mathfrak{H})$. ☐

Together with the polar decomposition, the normal case allows to give a canonical form for an arbitrary compact operator between two Hilbert spaces. Recall that for every bounded operator $A \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ there exists a unique partial isometry $U \colon \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$ such that

$$A = U|A| \quad \text{with} \quad |A| = \sqrt{A^* A} \quad \text{and} \quad \ker U = \ker A = \ker |A|, \tag{6.1.64}$$

see Theorem 5.1.42. In the case of a compact operator, this gives the following normal form theorem:

Theorem 6.1.42 (Normal form of compact operators) *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces and $K \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. Then the following statements are equivalent:*

i.) The operator K is compact.

ii.) There are at most countably infinitely many positive numbers

$$s_0(K) \geq s_1(K) \geq \cdots > 0 \tag{6.1.65}$$

and corresponding orthonormal systems $\mathrm{e}_0, \mathrm{e}_1, \ldots \in \mathfrak{H}_1$ and $\mathrm{f}_0, \mathrm{f}_1, \ldots, \in \mathfrak{H}_2$ with

$$K = \sum_{k \geq 0} s_k(K) \Theta_{\mathrm{f}_k, \mathrm{e}_k}, \tag{6.1.66}$$

converging in the strong operator topology.

In this case, the numbers $s_k(K)^2$ are the eigenvalues of $K^* K$ ordered by size and counted by multiplicity and (6.1.66) *converges in the operator norm topology.*

PROOF: Assume that K is compact. Then $K^* K$ is compact by Proposition 6.1.7, *ii.)*, and $|K| = \sqrt{K^* K}$ is compact by Theorem 6.1.37, *vi.)*. Thus we can find a Hilbert basis $\{\mathrm{e}_k\}_k \cup \{\mathrm{e}_i\}_{i \in I}$ for $\mathfrak{H}_1$ of eigenvectors of $|K|$ where at most countably infinitely many eigenvectors e_k correspond to positive eigenvalues $s_k > 0$, and the remaining ones $\{\mathrm{e}_i\}_{i \in I}$ form a Hilbert basis of $\ker |K| = \ker K$. Without restriction, we order them as $s_0(K) \geq s_1(K) \geq \cdots > 0$ and repeat each eigenvalue according to its (finite) multiplicity. Either, there are only finitely many $s_k(K)$ then also the set of eigenvectors $\{\mathrm{e}_k\}_k$ is finite, or we have a sequence $s_k(K) \longrightarrow 0$. In any case, we have as *norm* convergent series $|K| = \sum_k s_k \Theta_{\mathrm{e}_k, \mathrm{e}_k}$. Using the polar decomposition $K = U|K|$ we get

$$K = U \sum_k s_k(K) \Theta_{\mathrm{e}_k, \mathrm{e}_k} = \sum_k s_k(K) \Theta_{U\mathrm{e}_k, \mathrm{e}_k}$$

by continuity of the operator product in the norm topology. Thus with $\mathrm{f}_k = U\mathrm{e}_k$ we have (6.1.66). Indeed, since $\ker |K| = \ker K$ and hence $\mathrm{e}_k \in (\ker |K|)^\perp = (\ker U)^\perp$ shows that the f_k are still orthonormal. This completes the proof of *i.)* $\implies$ *ii.)*. For the converse, assume *ii.)* holds. In the case where we have only have finitely many $s_k(K)$ different from zero, there is nothing to show. Otherwise, we consider for $m \geq n$ the finite-rank operators

$$K_{n,m} = \sum_{k=n}^{m} s_k(K)\Theta_{\mathrm{f}_k,\mathrm{e}_k},$$

with adjoints $K_{n,m}^* = \sum_{k=n}^{m} s_k(K)\Theta_{\mathrm{e}_k,\mathrm{f}_k}$. Since $\langle \mathrm{f}_k, \mathrm{f}_\ell \rangle = \delta_{k\ell}$ this gives

$$K_{n,m}^* K_{n,m} = \sum_{k=n}^{m} s_k^2(K)\Theta_{\mathrm{e}_k,\mathrm{e}_k}.$$

The operator $K_{n,m}^* K_{n,m}$ is already diagonal and has spectrum $\{s_n^2(K), \ldots, s_m^2(K)\}$. Moreover, we have $\|K_{n,m}\|^2 = \|K_{n,m}^* K_{n,m}\| = s_n^2(K) \longrightarrow 0$ for $n \longrightarrow \infty$. This shows that the right hand side in (6.1.66) is necessarily norm convergent. Thus it defines a *compact* operator K since limits of finite-rank operators in the operator norm are compact. □

Note that if at least one of the Hilbert spaces is finite-dimensional, the statement becomes the usual *singular value decomposition* from linear algebra, see e.g. [62, Sect. 7.9]. Thus the numbers appearing in the theorem play a particular role for compact operators and are named as in the finite-dimensional situation:

Definition 6.1.43 (Singular values) *Let $K \in \mathfrak{K}(\mathfrak{H}_1, \mathfrak{H}_2)$. Then the non-zero eigenvalues of $|K| = \sqrt{K^*K}$, counted by multiplicity, are denoted by $s_0(K) \geq s_1(K) \geq \cdots > 0$. They are called the singular values of K.*

Sometimes it will be advantageous to have a sequence of singular values even if K^*K has only finitely many non-zero eigenvalues or if the dimensions of the Hilbert spaces are finite. Thus we will set $s_k(K) = 0$ for all $k \in \mathbb{N}_0$ where $k > \dim \operatorname{im} K$.

As an immediate consequence of this normal form we note that the finite-rank operators are norm-dense in the compact ones. This is true for Hilbert spaces but may fail for general Banach spaces, see e.g. the discussion in [26, Sect. 18.4].

Corollary 6.1.44 *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces. Then the finite-rank operators $\mathfrak{F}(\mathfrak{H}_1, \mathfrak{H}_2)$ are norm dense in the compact operators $\mathfrak{K}(\mathfrak{H}_1, \mathfrak{H}_2)$.*

The next proposition gives yet another criterion for compactness in terms of orthonormal systems. One should note the similarity between these statements and the corresponding relations of the sequence spaces $c_{\circ\circ} \subseteq c_\circ \subseteq c \subseteq \ell^\infty$ in Exercise 2.5.37. Later, we will also find analogs of the sequence spaces ℓ^p for all $p \in [1, \infty)$ thus arriving at a *noncommutative* version of all these sequence spaces. To make things interesting, we assume that the Hilbert spaces are both infinite-dimensional.

Proposition 6.1.45 *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be (infinite-dimensional) Hilbert spaces and let $A \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. Then the following statements are equivalent:*

i.) The operator A is compact.

ii.) For all countable orthonormal systems $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ in $\mathfrak{H}_1$ we have

$$\lim_{n\longrightarrow\infty} \|A\mathrm{e}_n\| = 0. \tag{6.1.67}$$

iii.) For all countable orthonormal systems $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ in $\mathfrak{H}_1$ and $\{\mathrm{f}_n\}_{n\in\mathbb{N}}$ in $\mathfrak{H}_2$ we have

$$\lim_{n\longrightarrow\infty} \langle \mathrm{f}_n, A\mathrm{e}_n \rangle = 0. \tag{6.1.68}$$

PROOF: Suppose first that A is compact and let $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ be an orthonormal system. Since $\|\mathrm{e}_n\| = 1$ the sequence $A\mathrm{e}_n$ contains a norm convergent subsequence $A\mathrm{e}_{n_k}$ by compactness. Now for $\phi \in \mathfrak{H}_2$ we have $\langle \phi, A\mathrm{e}_{n_k}\rangle = \langle A^*\phi, \mathrm{e}_{n_k}\rangle \longrightarrow 0$ by the fact that every orthonormal sequence is a weak zero sequence. Thus the norm-convergence of $A\mathrm{e}_{n_k}$ for $k \longrightarrow \infty$ implies $A\mathrm{e}_{n_k} \longrightarrow 0$. Now suppose that (6.1.67) does not hold. Then there is a subsequence of the e_n such that $\|A\mathrm{e}_{n_\ell}\| \geq \epsilon > 0$ for some suitable $\epsilon > 0$. But then we can apply our above argument to this subsequence to find yet another subsequence $\mathrm{e}_{n_{\ell_k}}$ of it for which $A\mathrm{e}_{n_{\ell_k}} \longrightarrow 0$, which is a contradiction. Thus *i.)* $\implies$ *ii.)* follows. Suppose now *ii.)* and let $\{\mathrm{f}_n\}_{n\in\mathbb{N}}$ any orthonormal system in $\mathfrak{H}_2$. Then $|\langle \mathrm{f}_n, A\mathrm{e}_n\rangle| \leq \|\mathrm{f}_n\|\|A\mathrm{e}_n\| = \|A\mathrm{e}_n\| \longrightarrow 0$ shows (6.1.68) and hence *iii.)* follows. Finally, assume *iii.)* and assume that A is not compact. Hence, by Corollary 6.1.44, there is a $\epsilon > 0$ such that $\|A - F\| \geq \epsilon$ for all $F \in \mathfrak{F}(\mathfrak{H}_1, \mathfrak{H}_2)$. For all orthonormal systems $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ in $\mathfrak{H}_1$ and $\{\mathrm{f}_n\}_{n\in\mathbb{N}}$ in $\mathfrak{H}_2$ we know that only finitely many terms $|\langle \mathrm{f}_n, A_n\mathrm{e}_n\rangle|$ can be $\geq \epsilon$. Suppose we have found $\mathrm{e}_1, \ldots, \mathrm{e}_n$ and $\mathrm{f}_1, \ldots, \mathrm{f}_n$ with $|\langle \mathrm{f}_k, A\mathrm{e}_k\rangle| \geq \epsilon$. Then we consider the projections

$$P = \sum_{k=1}^{n} \Theta_{\mathrm{e}_k,\mathrm{e}_k} \quad \text{and} \quad Q = \sum_{k=1}^{n} \Theta_{\mathrm{f}_k,\mathrm{f}_k},$$

which have rank n. Thus the operator $F = QA + AP - QAP \in \mathfrak{F}(\mathfrak{H}_1, \mathfrak{H}_2)$ has still finite rank. By assumption, $\|A - F\| \geq \epsilon$ and hence there are vectors $\phi \in \mathfrak{H}_2$ and $\psi \in \mathfrak{H}_1$ with

$$\epsilon\|\phi\|\|\psi\| \leq |\langle \phi, (A - F)\psi\rangle| = |\langle \phi, (\mathbb{1} - P)A(\mathbb{1} - Q)\psi\rangle| = |\langle (\mathbb{1} - P)\phi, A(\mathbb{1} - Q)\psi\rangle|. \tag{$*$}$$

In particular, $(\mathbb{1} - P)\phi \neq 0$ as well as $(\mathbb{1} - Q)\psi \neq 0$. Hence we can define

$$\mathrm{e}_{n+1} = \frac{(\mathbb{1} - Q)\psi}{\|(\mathbb{1} - Q)\psi\|} \quad \text{and} \quad \mathrm{f}_{n+1} = \frac{(\mathbb{1} - P)\phi}{\|(\mathbb{1} - P)\phi\|}.$$

Then $\mathrm{e}_1, \ldots, \mathrm{e}_{n+1}$ and $\mathrm{f}_1, \ldots, \mathrm{f}_{n+1}$ are still orthonormal by definition of P and Q. Moreover, $(*)$ shows that $|\langle \mathrm{f}_{n+1}, A\mathrm{e}_{n+1}\rangle| \geq \epsilon$ still holds. Thus we can repeat the construction and end up with countably infinite orthogonal systems contradicting (6.1.68). Thus $A \in \mathfrak{K}(\mathfrak{H}_1, \mathfrak{H}_2)$ was compact. □

6.1.5 Singular Values and Approximation Numbers

We shall now establish another interpretation of the singular values $s_n(A)$ of a compact operator on a Hilbert space. Still in the general case of Banach spaces one can ask how well a certain operator can be approximated by finite-rank operators. This leads to the following definition:

Definition 6.1.46 (Approximation numbers) *Let $A\colon V \longrightarrow W$ be a continuous linear map between Banach spaces. Then the n-th approximation number $a_n(A)$ is defined by*

$$a_n(A) = \inf\big\{\|A - F\| \;\big|\; F \in \mathfrak{F}(V, W) \text{ with } \dim\operatorname{im} F \leq n\big\}. \tag{6.1.69}$$

We are now interested in the collection of all the approximation numbers $(a_n(A))_{n\in\mathbb{N}_0}$. We want to understand how the behaviour of this sequence is characterizing for the operator and vice versa. We clearly have a monotonously decreasing sequence

$$\|A\| = a_0(A) \geq a_1(A) \geq \cdots \geq 0. \tag{6.1.70}$$

Example 6.1.47 Let V be an infinite-dimensional Banach space and let $F \in \mathfrak{F}(V)$ be a finite-rank operator. Then $\ker F$ is an infinite-dimensional closed subspace. Thus for $v \in \ker F$ we have $\|v - Fv\| = \|v\|$ and hence $\|\mathrm{id}_V - F\| = 1$ follows. Hence for all $n \in \mathbb{N}_0$ one has

$$a_n(\mathrm{id}_V) = 1. \tag{6.1.71}$$

If V is finite-dimensional, say $\dim V = N$, then $\dim \ker F = N - \dim \operatorname{im} F$. Hence for a finite-rank operator $F \in \mathfrak{F}(V) = \operatorname{End}(V)$ with $\dim \operatorname{im} F \leq n$ we still get vectors in $\ker F$ as long as $n < N$. Thus we have

$$a_n(\mathrm{id}_V) = \begin{cases} 1 & \text{for } n < N \\ 0 & \text{for } n \geq N. \end{cases} \tag{6.1.72}$$

As a reformulation of Corollary 6.1.44 we get the following statement for the particular case of Hilbert spaces:

Corollary 6.1.48 *Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be Hilbert spaces and let $A \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. Then $A \in \mathfrak{K}(\mathfrak{H}_1, \mathfrak{H}_2)$ iff*

$$\lim_{n \longrightarrow \infty} a_n(A) = 0. \tag{6.1.73}$$

In the general Banach space situation one has only one direction: if $\lim_{n\longrightarrow\infty} a_n(A) = 0$ then A is compact since finite rank operators are in the closed subspace of compact operators.

We collect some further properties of the approximation numbers in the general Banach space case:

se: continuity tion numbers pect to norm topology

Proposition 6.1.49 *Let V, W, and U be Banach spaces and let $n, m \geq 0$.*

i.) For all $A, B \in \mathrm{L}(V, W)$ one has

$$a_{n+m}(A + B) \leq a_n(A) + a_m(B). \tag{6.1.74}$$

ii.) For all $A, B \in \mathrm{L}(V, W)$ one has

$$|a_n(A) - a_n(B)| \leq \|A - B\|. \tag{6.1.75}$$

iii.) For all $A \in \mathrm{L}(V, W)$ and $B \in \mathrm{L}(W, U)$ one has

$$a_{n+m}(B \circ A) \leq a_n(A) a_m(B). \tag{6.1.76}$$

iv.) For all $A \in \mathrm{L}(V, W)$ one has

$$a_n(A) = 0 \quad \textit{iff} \quad \dim \operatorname{im} A \leq n. \tag{6.1.77}$$

Proof: Let $F, G \in \mathfrak{F}(V, W)$ with $\dim \operatorname{im} F \leq n$ and $\dim \operatorname{im} G \leq m$ be given. Then $\dim \operatorname{im}(F + G) \leq n + m$. Thus we have

$$a_{n+m}(A + B) \leq \|A + B - F - G\| \leq \|A - F\| + \|B + G\|.$$

Taking the infimum over all such F and G gives the first part. Since $a_0(A) = \|A\|$ we get with the first part

$$a_n(A) = a_{n+0}(A - B + B) \leq a_n(B) + a_0(A - B) = a_n(B) + \|A - B\|,$$

and hence $a_n(A) - a_n(B) \leq \|A - B\|$. Exchanging the role of A and B gives the second part. For the third part consider $F \in \mathfrak{F}(V, W)$ and $G \in \mathfrak{F}(W, U)$ with $\dim \operatorname{im} F \leq n$ and $\dim \operatorname{im} G \leq m$. Since the finite rank operators enjoy a similar ideal property as the compact ones we see that the operator $G \circ A + (B - G) \circ F$ has a finite-dimensional image with dimension at most $n + m$ by standard linear algebra arguments. Hence we have

$$a_{n+m}(B \circ A) \leq \|B \circ A - G \circ A - (B - G) \circ F\| = \|(B - G) \circ (A - F)\| \leq \|B - G\| \|A - F\|.$$

Taking the infimum over all these F and G results in the third part. For the fourth part, $\dim \operatorname{im} A \leq n$ clearly implies that A can be approximated by a rank n operator *exactly*, namely by A itself. For the converse, assume that $\dim \operatorname{im} A > n$. Then we find $v_1, \ldots, v_{n+1} \in V$ such that the vectors $Av_1, \ldots, Av_{n+1}$ are linearly independent in W. We consider now the Banach space $\mathbb{C}^{n+1}$. By the Hahn-Banach Theorem 2.3.10 we find continuous linear functionals $\varphi^1, \ldots, \varphi^{n+1} \in W'$ such that $\varphi^i(Av_j) = \delta^i_j$. This allows to consider the (obviously continuous) operators

$$B\colon \mathbb{C}^{n+1} \ni z \ \mapsto \ \sum\nolimits_{i=1}^{n+1} z^i v_i \in V$$

and

$$C\colon W \in w \ \mapsto \ \sum\nolimits_{i=1}^{n+1} \varphi^i(w)\mathrm{e}_i \in \mathbb{C}^{n+1},$$

where $\mathrm{e}_1, \ldots, \mathrm{e}_{n+1} \in \mathbb{C}^{n+1}$ is the canonical basis and $z^i = \mathrm{e}^i(z)$ as usual. By construction we get $CAB = \mathrm{id}_{\mathbb{C}^{n+1}}$. From Example 6.1.47 we know $a_n(\mathrm{id}_{\mathbb{C}^{n+1}}) = 1$ and hence by the third part

$$1 = a_n(CAB) \leq a_0(C)a_n(A)a_0(B),$$

which implies $a_n(A) > 0$, completing the proof. □

Remark 6.1.50 (p-Approximable operators) Let $1 \leq p \leq \infty$. Then one defines the *p-approximable operators* by

$$\mathcal{S}^p(V, W) = \left\{ A \in \mathrm{L}(V, W) \;\middle|\; (a_n(A)) \in \ell^p \right\} \tag{6.1.78}$$

and sets

$$\|A\|_{\mathcal{S}^p} = \|(a_n(A))_{n \in \mathbb{N}_0}\|_p. \tag{6.1.79}$$

It can be shown that $\mathcal{S}^p(V, W) \in \mathfrak{K}(V, W)$ is a subspace and $\|\cdot\|_{\mathcal{S}^p}$ is a complete norm for it. Moreover, $\mathcal{S}^p(V, W)$ enjoys similar ideal properties as $\mathfrak{K}(V, W)$ and $\mathfrak{F}(V, W)$. By definition, $\mathcal{S}^p(V, W)$ can be thought of as the $\|\cdot\|_{\mathcal{S}^p}$-completion of the finite rank operators $\mathfrak{F}(V, W)$. For a further discussion of these p-approximable operators see e.g. [26, Sect. 19.8].

While for general Banach spaces, the situation is quite complicated, the case of Hilbert spaces allows a nice characterization of the approximation numbers in terms of spectral data:

Theorem 6.1.51 (Singular values) *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces and let $K \in \mathfrak{K}(\mathfrak{H}_1, \mathfrak{H}_2)$ be a compact operator. Then for all $n \in \mathbb{N}_0$ one has*

$$s_n(K) = a_n(K). \tag{6.1.80}$$

PROOF: By Theorem 6.1.42 we can write A as

$$K = \sum_{k \geq 0} a_k(K) \Theta_{\mathrm{f}_k, \mathrm{e}_k}$$

with some orthonormal sets $\{\mathrm{e}_k\}$ in $\mathfrak{H}_1$ and $\{\mathrm{f}_k\} \in \mathfrak{H}_2$. If K has only a finite-dimensional image we set $s_k(K) = 0$ for $k > \dim \operatorname{im} K$, according to our previous convention. In this case, we chose the corresponding vectors e_k and f_k to be 0, so that they still form orthogonal sets (though no longer orthonormal). In any case, the interesting part of the theorem is when $\dim \operatorname{im} K = \infty$. We consider for $n \in \mathbb{N}_0$ the finite rank operator

$$K_n = \sum_{k=0}^{n-1} s_k(K) \Theta_{\mathrm{f}_k, \mathrm{e}_k},$$

for which we have $\dim \operatorname{im} K_n \leq n$. Moreover,

$$\|K - K_n\| = \left\| \sum\nolimits_{k \geq n} s_k(K) \Theta_{\mathrm{f}_k, \mathrm{e}_k} \right\| = s_n(K), \tag{$*$}$$

by an analogous argument as in the proof of Theorem 6.1.42, using $s_k(K) \geq s_{k+1}(K)$ for all $k \in \mathbb{N}_0$. Since K_n has at most an n-dimensional image, we get $a_n(K) \leq s_n(K)$ from $(*)$. Conversely, assume $F \in \mathfrak{F}(\mathfrak{H}_1, \mathfrak{H}_2)$ has $\dim \operatorname{im} F \leq n$. Then there is at least one vector $\phi \in \operatorname{span}_{\mathbb{C}}\{\mathrm{e}_0, \ldots, \mathrm{e}_n\}$ with $\|\phi\| = 1$ and $F\phi = 0$ for dimensional reasons. Then

$$
\begin{aligned}
\|K - F\| &\geq \|(K - F)\phi\| \\
&= \left\| \sum\nolimits_{k \geq 0} s_k(K) \Theta_{\mathrm{f}_k, \mathrm{e}_k} \phi - 0 \right\| \\
&= \left\| \sum\nolimits_{k=0}^{n} s_k(K) \mathrm{f}_k \langle \mathrm{e}_k, \phi \rangle \right\| \\
&= \sqrt{\sum\nolimits_{k=0}^{n} s_k(K)^2 |\langle \mathrm{e}_k, \phi \rangle|^2} \\
&\geq s_n(K) \sqrt{\sum\nolimits_{k=0}^{n} |\langle \mathrm{e}_k, \phi \rangle|^2} \\
&= s_n(K),
\end{aligned}
$$

since the sequence of singular values decreases monotonously and ϕ is in the linear span of the $\mathrm{e}_0, \ldots, \mathrm{e}_n$. Taking now the infimum over all such F we get $a_n(K) \geq s_n(K)$. □

Corollary 6.1.52 *Let $K \in \mathfrak{K}(\mathfrak{H})$ be a compact positive operator on a Hilbert space $\mathfrak{H}$. Then for all $p > 0$ and $n \in \mathbb{N}_0$ we have*

$$
a_n(K^p) = (a_n(K))^p. \tag{6.1.81}
$$

PROOF: Indeed, the map $x \mapsto x^p$ is continuous on $[0, \infty)$ and maps 0 to 0. Thus $K^p \in \mathfrak{K}(\mathfrak{H})$ is again compact. Moreover, by the Spectral Mapping Theorem we know that the n-th eigenvalue (counted with multiplicity) of K^p is the p-th power of the n-th eigenvalue of K. Hence $s_n(K^p) = (s_n(K))^p$ by the very definition of the singular values and $K = |K|$. Thus Theorem 6.1.51 gives the result. □

6.2 The Schatten Classes and the Trace

In this section we focus now exclusively on the Hilbert space case even though certain aspects can also be done in the more general Banach space case, see e.g. [26, Chap. 19]. The main objective is to understand the various ideals and their topologies arising from the trace of positive operators on a Hilbert space. Eventually, we will meet some noncommutative versions of the classical L^p- or ℓ^p-spaces and their dualities as discussed in Appendix C.3.

6.2.1 Closed Ideals in $\mathfrak{B}(\mathfrak{H})$

For a finite-dimensional Hilbert space $\mathfrak{H} \cong \mathbb{C}^n$ we know that the algebra $\mathfrak{B}(\mathfrak{H}) \cong \mathrm{M}_n(\mathbb{C})$ is *simple*: there are no other ideals in $\mathrm{M}_n(\mathbb{C})$ than $\{0\}$ and $\mathrm{M}_n(\mathbb{C})$. This changes now drastically in infinite dimensions: we have already seen that $\mathfrak{F}(\mathfrak{H})$ is a proper *-ideal whose norm closure is the proper *-ideal $\mathfrak{K}(\mathfrak{H})$. Note that for $\mathfrak{B}(\mathfrak{H})$ being a C^*-algebra, every closed two-sided ideal is necessarily a *-ideal by Corollary 4.4.16. Of course, for $\mathfrak{K}(\mathfrak{H})$ this is clear.

Theorem 6.2.1 (Ideals of $\mathfrak{B}(\mathfrak{H})$) *Let $\mathfrak{H}$ be a Hilbert space and let $\mathscr{J} \subseteq \mathfrak{B}(\mathfrak{H})$ be an ideal.*

i.) If $\mathscr{J} \neq \{0\}$ then $\mathfrak{F}(\mathfrak{H}) \subseteq \mathscr{J}$.

ii.) If $\{0\} \neq \mathscr{J} = \mathscr{J}^{\mathrm{cl}}$ then $\mathfrak{K}(\mathfrak{H}) \subseteq \mathscr{J}$.

iii.) If $\mathfrak{H}$ is separable, then the only non-zero closed ideal of $\mathfrak{B}(\mathfrak{H})$ is $\mathfrak{K}(\mathfrak{H})$. It is proper iff $\mathfrak{H}$ is infinite-dimensional.

PROOF: We show that the rank-one operators $\Theta_{\phi,\psi} \in \mathfrak{F}(\mathfrak{H})$ are contained in $\mathscr{J}$. Indeed, let $A \in \mathscr{J}$ be non-zero and fix vectors $\phi_1, \phi_2 \in \mathfrak{H}$ with $A\phi_1 = \phi_2 \neq 0$. Then we compute

$$\Theta_{\phi, \frac{\phi_2}{\|\phi_2\|^2}} A \Theta_{\phi_1,\psi}(\chi) = \Theta_{\phi, \frac{\phi_2}{\|\phi_2\|^2}} (A\phi_1 \langle \psi, \chi \rangle) = \phi \Big\langle \frac{\phi_2}{\|\phi_2\|^2}, \phi_2 \Big\rangle \langle \psi, \chi \rangle = \Theta_{\phi,\psi}(\chi).$$

Hence the ideal property of $\mathscr{J}$ shows $\Theta_{\phi,\psi} \in \mathscr{J}$. This is sufficient to conclude the first part. Since by Corollary 6.1.44 we have $\mathfrak{F}(\mathfrak{H})^{\mathrm{cl}} = \mathfrak{K}(\mathfrak{H})$, the second part is clear. Now assume that $\mathfrak{H}$ is separable. If $\dim \mathfrak{H} < \infty$, then the statement is clear from linear algebra since in this case $\mathfrak{K}(\mathfrak{H}) = \mathfrak{B}(\mathfrak{H}) = \mathrm{End}(\mathfrak{H}) = \mathfrak{F}(\mathfrak{H})$. Thus we can assume $\dim \mathfrak{H} = \infty$. We know that a non-zero closed ideal contains $\mathfrak{K}(\mathfrak{H})$, thus let us assume that $\mathscr{J}$ is strictly bigger. For $A \in \mathscr{J}$ with $A \notin \mathfrak{K}(\mathfrak{H})$ we have $A^*A \in \mathscr{J}$. If $A^*A \in \mathfrak{K}(\mathfrak{H})$ then also $\sqrt{A^*A} \in \mathfrak{K}(\mathfrak{H})$ by Corollary 6.1.40. Hence, by polar decomposition, $A = U\sqrt{A^*A} \in \mathfrak{K}(\mathfrak{H})$. Thus we must have $A^*A \notin \mathfrak{K}(\mathfrak{H})$. Since $B = A^*A$ is positive, we have a spectral measure E of B on $\mathrm{spec}(B) \subseteq [0, \|B\|]$ with $\|B\| = \|A\|^2$. We claim that there is an $\epsilon > 0$ such that $E_{[\epsilon,\|B\|]}$ has already an infinite-dimensional image. Indeed, the functions $f_\epsilon(x) = x\chi_{[\epsilon,\|B\|]}$ converge uniformly to $f(x) = x$ on $\mathrm{spec}(B)$. Hence by the spectral calculus we get

$$BE_{[\epsilon,\|B\|]} = \int_{\mathrm{spec}(B)} f_\epsilon \, \mathrm{d}E \longrightarrow \int_{\mathrm{spec}\, B} f \, \mathrm{d}E = B$$

in the sense of operator-norm convergence, see Theorem 5.1.32, *ii.)*. So if $E_{[\epsilon,\|B\|]} \in \mathfrak{F}(\mathfrak{H})$ for all $\epsilon > 0$ then $B \in \mathfrak{F}(\mathfrak{H})^{\mathrm{cl}} = \mathfrak{K}(\mathfrak{H})$ would follow, which can not be the case. Hence there is an $\epsilon > 0$ such that the projection $E_{[\epsilon,\|B\|]}$ has an infinite-dimensional image as claimed. Now $\chi_{[\epsilon,\|B\|]} = 0$ on $[0, \epsilon)$ and hence the function $g_\epsilon(x) = \frac{1}{x}\chi_{[\epsilon(x),\|B\|]}$ is bounded on $\mathrm{spec}(B)$. Thus by functional calculus $g_\epsilon(B)B = \chi_{[\epsilon(x),\|B\|]}(B) = E_{[\epsilon,\|B\|]} \in \mathscr{J}$ since $B \in \mathscr{J}$. In conclusion, we see that $\mathscr{J}$ contains a projection $P = E_{[\epsilon,\|B\|]}$ with $\dim \mathrm{im}\, P = \infty$. Since $\mathfrak{H}$ is separable, $\mathrm{im}\, P$ is separable as well and hence isometrically isomorphic to $\mathfrak{H}$. Thus there exists a partial isometry $U \in \mathfrak{B}(\mathfrak{H})$ with $U\big|_{(\mathrm{im}\, P)^\perp} = 0$ and $U\big|_{\mathrm{im}\, P}$ being a unitary map from $\mathrm{im}\, P$ to $\mathfrak{H}$. Analogously, we have an isometry $V \in \mathfrak{B}(\mathfrak{H})$ with $\mathrm{im}\, V = \mathrm{im}\, P$ onto $\mathrm{im}\, P$. But then $C = UPV$ is injective and surjective. Thus C is invertible and $C \in \mathscr{J}$. It follows that $\mathscr{J} = \mathfrak{B}(\mathfrak{H})$. ☐

Beyond the separable case there are more closed ideals in $\mathfrak{B}(\mathfrak{H})$. Note however, that for physical applications a non-separable Hilbert space is only of limited interest. Nevertheless, one has the following result:

Proposition 6.2.2 *Let $\mathfrak{H}$ be a non-separable Hilbert space. Then*

$$\mathfrak{B}_{\mathrm{sep}}(\mathfrak{H}) = \big\{ A \in \mathfrak{B}(\mathfrak{H}) \;\big|\; \mathrm{im}\, A \text{ is separable} \big\} \tag{6.2.1}$$

is a proper closed *-ideal with $\mathfrak{K}(\mathfrak{H}) \subsetneq \mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$. In fact, $\mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$ is sequentially closed in the strong operator topology.

PROOF: If $A, B \in \mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$ then also the image of $zA + wB$ is separable since it is contained in the separable subspace $\mathrm{im}\, A + \mathrm{im}\, B$. Moreover, for $A\mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$ and $B \in \mathfrak{B}(\mathfrak{H})$ the image of BA is the continuous image of a separable subspace and hence again separable. Trivially, $\mathrm{im}\, AB \subseteq \mathrm{im}\, A$ is separable as well. Thus $\mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$ is a two-sided ideal, which is proper since $\mathrm{id}_{\mathfrak{H}}$ has non-separable image by assumption. Now let $A_n \in \mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$ with $A_n \longrightarrow A$ in the strong sense be given. We consider the subspace

$$U = \left(\sum_{n=1}^{\infty} \mathrm{im}\, A_n \right)^{\mathrm{cl}} \subseteq \mathfrak{H},$$

which is separable since each $\operatorname{im} A_n$ is separable and the sum is countable. Note that the closure of a separable subspace is separable, too. If $\phi \in \mathfrak{H}$ then

$$A\phi = \lim_{n \longrightarrow \infty} A_n \phi \in U,$$

since each $A_n\phi \in U$ and U is closed. Thus $\operatorname{im} A \subseteq U$ implying thereby $A \in \mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$. This shows that $\mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$ is sequentially closed in the strong operator topology. Since norm convergence implies strong convergence, $\mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$ is also sequentially closed in the norm topology, which is metric and hence sequential closure coincides with closure. Since in a C^*-algebra like $\mathfrak{B}(\mathfrak{H})$ every closed ideal is a *-ideal according to Corollary 4.4.16, we see that $\mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$ is a *-ideal. Of course, in this case this can also be seen directly. Finally, there are clearly operators in $\mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$ which are not compact, like e.g. every projection onto a separable infinite-dimensional subspace of $\mathfrak{H}$. $\square$

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Of course, one can continue with constructing closed *-ideals by considering operators with images whose closures having a Hilbert basis of a certain cardinality strictly smaller than $\dim \mathfrak{H}$ but larger than countable. However, this does not lead to anything interesting for us.

6.2.2 The Trace of Positive Operators

For the construction of the trace we want to proceed naively as in the finite-dimensional case: the trace of an operator is the sum of the diagonal elements in one of its (equivalent) matrix representations. Of course, in infinite dimensions we face a serious problem here. Even using a Hilbert basis $\{\mathrm{e}_i\}_{i \in I}$ the sum over the diagonal terms $\langle \mathrm{e}_i, A\mathrm{e}_i \rangle$ needs not to converge at all, or, if it converges, the result may depend on the choice of the Hilbert basis. Thus we expect that a reasonable definition of an operator trace can only work for a limited class of operators. In this section we shall construct a trace for positive operators with values in $[0, \infty]$. The whole procedure will be very much analogous to the construction of the Lebesgue integral, a point of view to which we shall come back from time to time.

The following technical lemma shows under which conditions we can expect to have at most countably many $\langle \mathrm{e}_i, A\mathrm{e}_i \rangle$ different from zero:

Lemma 6.2.3 *Let $A \in \mathfrak{B}(\mathfrak{H})$ be a positive operator. Then the following statements are equivalent:*

i.) The operator A has separable image, i.e. $A \in \mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$.

ii.) There is a Hilbert basis $\{\mathrm{e}_i\}_{i \in I}$ with $\langle \mathrm{e}_i, A\mathrm{e}_i \rangle = 0$ for all except for countably many $i \in I$.

iii.) For all Hilbert basis $\{\mathrm{e}_i\}_{i \in I}$ we have $\langle \mathrm{e}_i, A\mathrm{e}_i \rangle = 0$ for all except for countably many $i \in I$.

Proof: Assume $A \in \mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$. Since $A = A^*$ we have $\ker A = (\operatorname{im} A)^\perp$ and thus $\mathfrak{H} = (\operatorname{im} A)^{\mathrm{cl}} \oplus \ker A$. We choose an at most countable Hilbert basis of $(\operatorname{im} A)^{\mathrm{cl}}$ and an arbitrary one of $\ker A$. The resulting Hilbert basis of $\mathfrak{H}$ will do the job for *ii.)*. Now assume $\{\mathrm{e}_i\}_{i \in I}$ is a Hilbert basis with *ii.)* and let $\{\mathrm{f}_i\}_{i \in J}$ be an arbitrary one. Let $I_0 \subseteq I$ be the countable subset with $\langle \mathrm{e}_i, A\mathrm{e}_i \rangle \neq 0$ for $i \in I_0$. For $j \in J$ we denote by $I_{\mathrm{f}_j} \subseteq I$ the countable subset such that

$$\mathrm{f}_j = \sum_{i \in I_{\mathrm{f}_j}} \mathrm{e}_i \langle \mathrm{e}_i, \mathrm{f}_j \rangle,$$

and analogously $J_{\mathrm{e}_i} \subseteq J$ denotes the countable subset with

$$\mathrm{e}_i = \sum_{j \in J_{\mathrm{e}_i}} \mathrm{f}_j \langle \mathrm{f}_j, \mathrm{e}_i \rangle.$$

Now $|\langle \mathrm{e}_i, A\mathrm{e}_{i'} \rangle| \leq \left\| \sqrt{A}\mathrm{e}_i \right\| \left\| \sqrt{A}\mathrm{e}_{i'} \right\|$ and hence $\langle \mathrm{e}_i, A\mathrm{e}_{i'} \rangle = 0$ unless $i, i' \in I_0$. Thus by continuity of A we have

$$\langle \mathrm{f}_j, A\mathrm{f}_j \rangle = \sum_{i \in I_{\mathrm{f}_j}} \sum_{i' \in I_{\mathrm{f}_j}} \overline{\langle \mathrm{e}_i, \mathrm{f}_j \rangle} \langle \mathrm{e}_i, A\mathrm{e}_{i'} \rangle \langle \mathrm{e}_{i'}, \mathrm{f}_j \rangle = \sum_{i, i' \in I_{\mathrm{f}_j} \cap I_0} \overline{\langle \mathrm{e}_i, \mathrm{f}_j \rangle} \langle \mathrm{e}_i, A\mathrm{e}_{i'} \rangle \langle \mathrm{e}_{i'}, \mathrm{f}_j \rangle.$$

Thus if $I_{\mathrm{f}_j} \cap I_0 = \emptyset$ then necessarily $\langle \mathrm{f}_j, A\mathrm{f}_j \rangle = 0$. Since for $i \in I_0$ the set J_{e_i} is at most countable, the set of those $j \in J$ with $\langle \mathrm{e}_i, \mathrm{f}_j \rangle \neq 0$ for some $i \in I_0$ is given by $\bigcup_{i \in I_0} J_{\mathrm{e}_i}$ and again at most countable. With other words, there are only countably many $j \in J$ with $\langle \mathrm{e}_i, \mathrm{f}_j \rangle \neq 0$ for some $i \in I_0$. But then $I_{\mathrm{f}_j} \cap I_0 \neq \emptyset$ holds for at most countably many $j \in J$, showing *iii.)*. Finally, *iii.)* implies $\|\sqrt{A}\mathrm{e}_i\| = 0$ for all except countably many $i \in I$. Hence all e_i are in the kernel of $\sqrt{A}$ and hence in the kernel of A, except for countably many $i \in I$. Since $(\operatorname{im} A)^{\mathrm{cl}} = (\ker A)^{\perp}$ by $A = A^*$ it follows that $(\operatorname{im} A)^{\mathrm{cl}}$ is the closure of the $\mathbb{C}$-span of the countably many e_i with $A\mathrm{e}_i \neq 0$. Thus $\operatorname{im} A$ is separable. ☐

For a positive operator we can now define the trace taking values in $[0, \infty]$ as follows:

Definition 6.2.4 (Trace of a positive operator) *Let $A \in \mathfrak{B}(\mathfrak{H})$ be a positive operator and let $\{\mathrm{e}_i\}_{i \in I}$ be a Hilbert basis of $\mathfrak{H}$. Then*

$$\operatorname{tr} A = \sum_{i \in I} \langle \mathrm{e}_i, A\mathrm{e}_i \rangle \in [0, \infty] \tag{6.2.2}$$

is called the trace of A.

Equivalently, we can write

$$\operatorname{tr} A = \sum_{i \in I} \left\| \sqrt{A}\mathrm{e}_i \right\|^2, \tag{6.2.3}$$

since A is positive. Note that it may well happen that $\operatorname{tr} A = \infty$. By the last lemma this is necessarily the case whenever $A \notin \mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$, no matter which Hilbert basis we choose. The following theorem gives now some first properties of the trace, showing in particular that it is well-defined at all:

Theorem 6.2.5 (Trace of positive operators) *Let $A, B \in \mathfrak{B}(\mathfrak{H})$ be positive operators.*

i.) *The trace* $\operatorname{tr} A$ *is well-defined and* $\operatorname{tr} A < \infty$ *implies* $A \in \mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$.

ii.) *One has* $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$.

iii.) *One has* $\operatorname{tr}(\lambda A) = \lambda \operatorname{tr} A$ *for all* $\lambda \geq 0$.

iv.) *One has* $\operatorname{tr}(UAU^{-1}) = \operatorname{tr} A$ *for all unitary* $U \in \mathfrak{B}(\mathfrak{H})$.

v.) *If* $0 \leq A \leq B$ *then* $\operatorname{tr} A \leq \operatorname{tr} B$.

vi.) *One has* $\operatorname{tr} \mathbb{1} < \infty$ *iff* $\dim \mathfrak{H} < \infty$. *In this case* $\operatorname{tr} \mathbb{1} = \dim \mathfrak{H}$.

PROOF: First it is clear by Lemma 6.2.3 that for $A \notin \mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$ we have in any Hilbert basis uncountably many $i \in I$ with $\langle \mathrm{e}_i, A\mathrm{e}_i \rangle \neq 0$. Thus the series $\operatorname{tr} A$ diverges necessarily to $+\infty$ for all Hilbert bases. This shows the well-definedness in this case. Now let $A \in \mathfrak{B}_{\mathrm{sep}}(\mathfrak{H})$ and let $\{\mathrm{e}_i\}_{i \in I}$ as well as $\{\mathrm{f}_j\}_{j \in J}$ be two Hilbert bases. Denote by $I_0 \subseteq I$ and $J_0 \subseteq J$ the at most countable subsets with $\langle \mathrm{e}_i, A\mathrm{e}_i \rangle \neq 0$ and $\langle \mathrm{f}_j, A\mathrm{f}_j \rangle \neq 0$ for $i \in I_0$ and $j \in J_0$, respectively. Then we have by Parseval's equality

$$\begin{aligned}
\sum_{i \in I_0} \langle \mathrm{e}_i, A\mathrm{e}_i \rangle &= \sum_{i \in I} \left\langle \sqrt{A}\mathrm{e}_i, \sqrt{A}\mathrm{e}_i \right\rangle \\
&= \sum_{i \in I} \sum_{j \in J} \left| \left\langle \mathrm{f}_j, \sqrt{A}\mathrm{e}_i \right\rangle \right|^2 \\
&= \sum_{j \in J} \sum_{i \in I} \left| \left\langle \mathrm{e}_i, \sqrt{A}\mathrm{f}_j \right\rangle \right|^2 \\
&= \sum_{j \in J} \left\langle \sqrt{A}\mathrm{f}_j, \sqrt{A}\mathrm{f}_j \right\rangle \\
&= \sum_{j \in J_0} \langle \mathrm{f}_j, A\mathrm{f}_j \rangle,
\end{aligned}$$

where we have used that $\sqrt{A}^* = \sqrt{A}$ and that all series have at most countably many non-zero (and even non-negative) terms. This shows that $\operatorname{tr} A$ is well-defined also in this case since we have either absolute convergence to some value in $[0, \infty)$ or absolute divergence to $+\infty$ for all Hilbert bases. Thus the first part is shown. The second and the third part are clear. For the fourth part we note that a unitary map U^{-1} maps a Hilbert basis $\{\mathrm{e}_i\}_{i\in I}$ to a Hilbert basis $\{U^{-1}\mathrm{e}_i\}_{i\in I}$. Moreover, UAU^{-1} is still positive. Hence

$$\operatorname{tr}(UAU^{-1}) = \sum_{i\in I} \langle \mathrm{e}_i, UAU^{-1}\mathrm{e}_i \rangle = \sum_{i\in I} \langle U^{-1}\mathrm{e}_i, AU^{-1}\mathrm{e}_i \rangle = \operatorname{tr} A$$

by the first part. The fifth part is clear since $A \le B$ implies $\langle \mathrm{e}_i, A\mathrm{e}_i \rangle \le \langle \mathrm{e}_i, B\mathrm{e}_i \rangle$ for all $i \in I$. The last part is trivial. □

: UAU^{-1} for
t and unitary
again positive
fan: Exercise:
$P = \dim \operatorname{im} P$

If a positive operator has a finite trace then it has a separable image. The next proposition shows that it is even a compact operator:

Proposition 6.2.6 *Let $A \in \mathfrak{B}(\mathfrak{H})$ be positive with $\operatorname{tr} A < \infty$. Then $A \in \mathfrak{K}(\mathfrak{H})$.*

PROOF: If $\operatorname{tr} A < \infty$ is finite, then $\langle \mathrm{e}_i, A\mathrm{e}_i \rangle = \left\| \sqrt{A}\mathrm{e}_i \right\|^2$ must be a zero sequence. By Proposition 6.1.45 we see that this implies the compactness of $\sqrt{A}$. By the *-ideal property, also $A \in \mathfrak{K}(\mathfrak{H})$.□

Note however, not all positive compact operators have a finite trace. In fact, let $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ be a countable orthonormal system then

$$A = \sum_{n=1}^{\infty} \frac{1}{n} \Theta_{\mathrm{e}_n, \mathrm{e}_n} \tag{6.2.4}$$

is clearly a positive compact operator with singular values given by $s_{n-1}(A) = \frac{1}{n}$. However, since the harmonic series diverges we have $\operatorname{tr} A = \infty$. Note also that for this operator its square

$$\operatorname{tr} A^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty \tag{6.2.5}$$

does have a finite trace. Thus it may well happen that a positive operator has a finite trace while its square root does not.

6.2.3 The Schatten Classes

Up to now we have the trace only defined for positive operators where it can assume values in $[0, \infty]$. Of special interest are of course those operators with finite trace. In analogy to the usual "commutative" L^p-spaces in measure theory, see Appendix C.3.2, the following definition is well motivated. As usual, we write $|A| = \sqrt{A^*A}$.

Definition 6.2.7 (Schatten classes, trace class, and Hilbert-Schmidt operators) *Let $\mathfrak{H}$ be a Hilbert space and let $p \ge 1$. Then the p-th Schatten class is defined by*

$$\mathfrak{L}^p(\mathfrak{H}) = \left\{ A \in \mathfrak{B}(\mathfrak{H}) \;\middle|\; \operatorname{tr}(|A|^p) < \infty \right\}, \tag{6.2.6}$$

and the p-th Schatten norm is defined by

$$\|A\|_p = \sqrt[p]{\operatorname{tr}(|A|^p)}. \tag{6.2.7}$$

For $p = 1$ one calls $\mathfrak{L}^1(\mathfrak{H})$ also the trace class operators while for $p = 2$ the operators in $\mathfrak{L}^2(\mathfrak{H})$ are called Hilbert-Schmidt operators.

We shall use the symbol $\|A\|_p$ for all operators $A \in \mathfrak{B}(\mathfrak{H})$: here it can take values in $[0, \infty]$. Then the p-th Schatten class consists of those operators for which the quantity $\|A\|_p$ is actually finite. In the case $p = 1$ we call $\|\cdot\|_1$ also the *trace norm*. For $p = 2$ one calls $\|\cdot\|_2$ the *Hilbert-Schmidt norm*.

From Proposition 6.2.6 in combination with Corollary 6.1.40 we note the following simple statement:

Proposition 6.2.8 *Let $p \geq 1$. Then one has*

$$\mathfrak{L}^p(\mathfrak{H}) \subseteq \mathfrak{K}(\mathfrak{H}). \tag{6.2.8}$$

Proof: Indeed, for $A \in \mathfrak{L}^p(\mathfrak{H})$ we know that $|A|^p \in \mathfrak{K}(\mathfrak{H})$ and hence $|A| \in \mathfrak{K}(\mathfrak{H})$, too. By the polar decomposition $A = U|A|$ and the ideal property of the compact operators, we get $A \in \mathfrak{K}(\mathfrak{H})$. □

This suggests that one can use the singular values $s_n(A)$ of $A \in \mathfrak{K}(\mathfrak{H})$ to characterize $\|A\|_p$ since by definition $s_n(A) = s_n(|A|)$ depends only on $|A|$. This is indeed possible. Moreover, we will have other characterizations involving Hilbert bases in many ways. For practical use, the characterizations via $|A|$ are rather tricky as the explicit computation of $|A|$ from A is usually quite involved. Thus it will be desirable to have formulations to get $\|A\|_p$ from A directly instead of relying on $|A|$.

Theorem 6.2.9 (Schatten norm) *Let $p, q, r \geq 1$.*

i.) For all $A \in \mathfrak{B}(\mathfrak{H})$ one has

$$\|A\|_p = \big\||A|\big\|_p \tag{6.2.9}$$

$$= \big\|(a_n(A))_{n\in\mathbb{N}_0}\big\|_p \tag{6.2.10}$$

$$= \sup_{\{\mathrm{e}_n\}_{n\in\mathbb{N}_0}, \{\mathrm{f}_n\}_{n\in\mathbb{N}_0}} \sqrt[p]{\sum\nolimits_{n=0}^{\infty} |\langle \mathrm{f}_n, A\mathrm{e}_n\rangle|^p} \tag{6.2.11}$$

$$= \sup_{\{\mathrm{e}_n\}_{n\in\mathbb{N}_0}} \sqrt[p]{\sum\nolimits_{n=0}^{\infty} \|A\mathrm{e}_n\|^p}, \tag{6.2.12}$$

where the suprema are taken over all (pairs of) at most countable orthonormal systems.

ii.) For all $A, B \in \mathfrak{B}(\mathfrak{H})$ one has

$$\|A + B\|_p \leq \|A\|_p + \|B\|_p. \tag{6.2.13}$$

iii.) For all $A, B \in \mathfrak{B}(\mathfrak{H})$ one has

$$\|AB\|_p \leq \|A\|\|B\|_p \tag{6.2.14}$$

and

$$\|A^*\|_p = \|A\|_p. \tag{6.2.15}$$

iv.) For all $A \in \mathfrak{B}(\mathfrak{H})$ and all $p \leq q$ one has

$$\|A\| \leq \|A\|_q \leq \|A\|_p. \tag{6.2.16}$$

v.) If $A \geq 0$ and $\alpha > 0$ with $p\alpha \geq 1$ then

$$\|A^\alpha\|_p = \|A\|_{p\alpha}^\alpha. \tag{6.2.17}$$

vi.) If $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ then for all $A, B \in \mathfrak{B}(\mathfrak{H})$ one has

$$\|AB\|_r \leq \|A\|_p\|B\|_q. \tag{6.2.18}$$

All the (in-) equalities between the Schatten norms are understood in $[0, \infty]$.

PROOF: Since $\|A\|_p = \sqrt[p]{\operatorname{tr}|A|^p}$ the first equality (6.2.9) is obvious. Thus we can focus on $|A| \geq 0$ in the first place. If now $|A|$ and hence $|A|^p$ is not compact, then the approximation numbers do not converge to zero, see Corollary 6.1.48. But then the sequence $(a_n(A))_{n\in\mathbb{N}_0}$ can not be p-summable. Hence we have $\|(a_n(A))_{n\in\mathbb{N}_0}\|_p = +\infty$ as well as $\operatorname{tr}|A|^p = +\infty$, showing the equality (6.2.10) in the case of a non-compact A. Thus we can assume that A is compact. Then $a_n(A) = s_n(A)$ by Theorem 6.1.51 and hence there is an orthonormal system of eigenvectors $\{\mathrm{e}_n\}_{n\in I}$ with $I \subseteq \mathbb{N}_0$ such that

$$|A|^p = \sum\nolimits_{n\in I} s_n(A)^p \Theta_{\mathrm{e}_n,\mathrm{e}_n}.$$

It follows from the definition of the trace that we can use the $\{\mathrm{e}_n\}_{n\in I}$ and extend them to a Hilbert basis to conclude

$$\operatorname{tr}|A|^p = \sum\nolimits_{n\in I} s_n(|A|^p) = \sum\nolimits_{n\in I} s_n(|A|)^p = \sum\nolimits_{n\in I} s_n(A)^p,$$

by the Spectral Mapping Theorem and $s_n(A) = s_n(|A|)$. Since again $a_n(A) = s_n(A)$ we get the equality (6.2.10) also for compact A. For the next equality, we note that if A is non-compact then there are countable orthonormal systems $\{\mathrm{e}_n\}_{n\in\mathbb{N}_0}$ and $\{\mathrm{f}_n\}_{n\in\mathbb{N}_0}$ such that $(\langle \mathrm{f}_n, A\mathrm{e}_n\rangle)_{n\in\mathbb{N}_0}$ is not a zero sequence by Proposition 6.1.45. Thus the supremum is clearly $+\infty$ in this case. Conversely, assume that A is compact then we use again the normal form

$$A = \sum\nolimits_{n\in I} s_n(A) \Theta_{\tilde{\mathrm{f}}_n,\tilde{\mathrm{e}}_n} \tag{$\star$}$$

with orthonormal systems $\{\tilde{\mathrm{e}}_n\}_{n\in I}$ and $\{\tilde{\mathrm{f}}_n\}_{n\in I}$ where $I \subseteq \mathbb{N}_0$ according to Theorem 6.1.42. Using this particular orthonormal system gives

$$\sum\nolimits_{n\in I} |\langle \tilde{\mathrm{f}}_n, A\tilde{\mathrm{e}}_n\rangle|^p = \sum\nolimits_{n\in I} \left|\left\langle \tilde{\mathrm{f}}_n, \sum\nolimits_m s_m(A)\Theta_{\tilde{\mathrm{f}}_m,\tilde{\mathrm{e}}_m}\tilde{\mathrm{e}}_n\right\rangle\right|^p = \sum\nolimits_{n\in I} s_n(A)^p = \operatorname{tr}(|A|^p)$$

by (6.2.10). Thus there exists at least one choice of orthonormal systems producing an equality in (6.2.11). Now let $\{\mathrm{e}_n\}_{n\in J}$ and $\{\mathrm{f}_n\}_{n\in J}$ be arbitrary orthonormal systems with $J \subseteq \mathbb{N}_0$. Then for $p > 1$ we get by Hölder's inequality

$$\begin{aligned}
\sum\nolimits_{n\in I} s_n(A)|\langle \mathrm{e}_k, \tilde{\mathrm{e}}_n\rangle|^2 &= \sum_{n\in I} s_n(A)|\langle \mathrm{e}_k, \tilde{\mathrm{e}}_n\rangle|^{\frac{2}{p}}|\langle \mathrm{e}_k, \tilde{\mathrm{e}}_n\rangle|^{2-\frac{2}{p}} \\
&\leq \left(\sum\nolimits_{n\in I} s_n(A)^p|\langle \mathrm{e}_k, \tilde{\mathrm{e}}_n\rangle|^2\right)^{\frac{1}{p}} \left(\sum\nolimits_{n\in I} |\langle \mathrm{e}_k, \tilde{\mathrm{e}}_n\rangle|^{\left(2-\frac{2}{p}\right)\left(\frac{p}{p-1}\right)}\right)^{\frac{p-1}{p}} \\
&= \left(\sum\nolimits_{n\in I} s_n(A)^p|\langle \mathrm{e}_k, \tilde{\mathrm{e}}_n\rangle|^2\right)^{\frac{1}{p}} \left(\sum\nolimits_{n\in I} |\langle \mathrm{e}_k, \tilde{\mathrm{e}}_n\rangle|^2\right)^{\frac{p-1}{p}} \\
&\leq \left(\sum\nolimits_{n\in I} s_n(A)^p|\langle \mathrm{e}_k, \tilde{\mathrm{e}}_n\rangle|^2\right)^{\frac{1}{p}},
\end{aligned}$$

since by Bessel's inequality, the sum in the second brackets is less or equal to $\|\mathrm{e}_k\|^2 = 1$. This gives the estimate

$$\left(\sum\nolimits_{n\in I} s_n(A)|\langle \mathrm{e}_k, \tilde{\mathrm{e}}_n\rangle|^2\right)^{\frac{p}{2}} \leq \sqrt{\sum\nolimits_{n\in I} s_n(A)^p|\langle \mathrm{e}_k, \tilde{\mathrm{e}}_n\rangle|^2} \tag{$*$}$$

and analogously

$$\left(\sum\nolimits_{n\in I} s_n(A)|\langle \tilde{\mathrm{f}}_n, \mathrm{f}_k\rangle|^2\right)^{\frac{p}{2}} \leq \sqrt{\sum\nolimits_{n\in I} s_n(A)^p|\langle \tilde{\mathrm{f}}_n, \mathrm{f}_k\rangle|^2}. \tag{$**$}$$

Clearly, these two estimates $(*)$ and $(**)$ trivially hold in the case $p = 1$ as well. By the Cauchy-Schwarz inequality we have

$$|\langle \mathrm{f}_k, A\mathrm{e}_k\rangle| \leq \sum\nolimits_{n\in I} s_n(A)|\langle \mathrm{f}_k, \tilde{\mathrm{f}}_n\rangle||\langle \tilde{\mathrm{e}}_n, \mathrm{e}_k\rangle| \leq \sqrt{\sum\nolimits_{n\in I} s_n(A)|\langle \mathrm{f}_k, \tilde{\mathrm{f}}_n\rangle|^2}\sqrt{\sum\nolimits_{n\in I} s_n(A)|\langle \tilde{\mathrm{e}}_n, \mathrm{e}_k\rangle|^2}. \tag{$***$}$$

Putting things together, we get

$$
\begin{aligned}
\sum_{k\in J}|\langle \mathrm{f}_k, A\mathrm{e}_k\rangle|^p &\overset{(***)}{\le} \sum_{k\in J}\Big(\sum_{n\in I} s_n(A)|\langle \mathrm{f}_k, \tilde{\mathrm{f}}_n\rangle|^2\Big)^{\frac{p}{2}}\Big(\sum_{n\in I} s_n(A)|\langle \tilde{\mathrm{e}}_n, \mathrm{e}_k\rangle|^2\Big)^{\frac{p}{2}} \\
&\overset{(*),(**)}{\le} \sum_{k\in J}\sqrt{\sum_{n\in I} s_n(A)^p|\langle \tilde{\mathrm{f}}_n, \mathrm{f}_k\rangle|^2}\sqrt{\sum_{n\in I} s_n(A)^p|\langle \mathrm{e}_k, \tilde{\mathrm{e}}_n\rangle|^2} \\
&\overset{(a)}{\le} \sqrt{\Big(\sum_{k\in J, n\in I} s_n(A)^p|\langle \tilde{\mathrm{f}}_n, \mathrm{f}_k\rangle|^2\Big)\Big(\sum_{k\in J, n\in I} s_n(A)^p|\langle \mathrm{e}_k, \tilde{\mathrm{e}}_n\rangle|^2\Big)} \\
&\overset{(b)}{\le} \sqrt{\sum_{n\in I} s_n(A)^p}\sqrt{\sum_{n\in I} s_n(A)^p} \\
&= \sum_{n\in I} s_n(A)^p,
\end{aligned}
$$

where we have used the Cauchy-Schwarz inequality once more in (a) and Bessel's inequality in (b) for the summation over k. Note that the convergence/divergence is necessarily absolute since the terms in the (double) series are all non-negative. This shows that the supremum over all such countable orthonormal systems is always less or equal to the p-th Schatten norm $\|A\|_p$ of A. Since we have found a particular choice where equality holds, we get (6.2.11). For the last equation we assume first that A is non-compact. Then there is an orthonormal system $\{\mathrm{e}_n\}_{nq\in\mathbb{N}_0}$ for which $\|A\mathrm{e}_n\|$ is not converging to zero, again by Proposition 6.1.45. Thus the supremum on the right hand side of (6.2.12) is $+\infty$ and equals $\sqrt[p]{\operatorname{tr}|A|^p}$. Thus, let A be compact again and given in the normal form $(\star)$. Then

$$
\|A\tilde{\mathrm{e}}_k\| = \Big\|\sum_{n\in I} s_n(A)\Theta_{\tilde{\mathrm{f}}_n, \tilde{\mathrm{e}}_n}\tilde{\mathrm{e}}_k\Big\| = \|s_k(A)\tilde{\mathrm{f}}_k\| = s_k(A).
$$

Hence, for the particular orthonormal system $\{\tilde{\mathrm{e}}_n\}_{n\in I}$ we get the equality

$$
\operatorname{tr}|A|^p = \sum_{n\in I}\|A\tilde{\mathrm{e}}_n\|^p
$$

by (6.2.10) and $s_n(A) = a_n(A)$. On the other hand, we have

$$
\begin{aligned}
\sup_{\{\mathrm{e}_n\}_{n\in J}} \sqrt[p]{\sum_{n\in J}\|A\mathrm{e}_n\|^p} &= \sup_{\{\mathrm{e}_n\}_{n\in J}} \sqrt[p]{\sum_{n\in J}\sqrt{\langle \mathrm{e}_n, A^*A\mathrm{e}_n\rangle}^p} \\
&\le \sup_{\{\mathrm{e}_n\}_{n\in J}, \{\mathrm{f}_n\}_{n\in J}} \sqrt[p]{\sum_{n\in J}|\langle \mathrm{f}_n, A^*A\mathrm{e}_n\rangle|^{\frac{p}{2}}} \\
&= \sqrt{\|A^*A\|_{\frac{p}{2}}} \\
&= \Big(\sum_{n\in I} s_n(A^*A)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \\
&= \|A\|_p,
\end{aligned}
$$

since $s_n(A^*A) = s_n(|A|^2) = s_n(A)^2$. From this estimate and the above computation we finally arrive at (6.2.12) and hence the first part is shown. For the second part, we use Minkowski's inequality to get

$$
\sqrt[p]{\sum_{n\in J}|\langle \mathrm{f}_n, (A+B)\mathrm{e}_n\rangle|^p} \le \sqrt[p]{\sum_{n\in J}|\langle \mathrm{f}_n, A\mathrm{e}_n\rangle|^p} + \sqrt[p]{\sum_{n\in J}|\langle \mathrm{f}_n, A\mathrm{e}_n\rangle|^p}.
$$

Taking the supremum over the countable orthonormal systems gives (6.2.13) by (6.2.11) from the first part. The third part is clear since first $\|AB\mathrm{e}_n\|^p \le \|A\|^p\|B\mathrm{e}_n\|^p$ and then (6.2.12) gives (6.2.14) right away. Second, $|\langle \mathrm{f}_n, A\mathrm{e}_n\rangle|^p = |\langle \mathrm{e}_n, A^*\mathrm{f}_n\rangle|^p$ and hence (6.2.11) implies (6.2.15). For the fourth part we note that $a_0(A) = \|A\|$ by the very definition of the approximation numbers. Thus (6.2.10) gives $\|A\| \leq \|A\|_p$ for all $p \geq 1$. For $1 \leq q \leq p$ we have by Jensen's inequality, see Exercise 6.4.1,

$$\|A\|_p = \sqrt[p]{\sum\nolimits_n a_n(A)^p} \leq \sqrt[q]{\sum\nolimits_n a_n(A)^q} = \|A\|_q.$$

For the fifth part we know that a positive operator $A \in \mathfrak{B}(\mathfrak{H})$ is compact iff A^α is compact for all $\alpha > 0$ by Corollary 6.1.40. Thus only the case $A \in \mathfrak{K}(\mathfrak{H})$ is interesting for (6.2.17) since otherwise both sides are $+\infty$. For a compact $A \geq 0$ we have $a_n(A^\alpha) = a_n(A)^\alpha$ by Corollary 6.1.52. Then

$$\|A^\alpha\|_p = \sqrt[p]{\sum\nolimits_n a_n(A)^{p\alpha}} = \left(\sqrt[p\alpha]{\sum\nolimits_n a_n(A)^{p\alpha}}\right)^\alpha = \|A\|_{p\alpha}^\alpha$$

shows (6.2.17). For the last part, we have

$$\begin{aligned}
\sqrt[r]{\sum\nolimits_n |\langle \mathrm{f}_n, AB\mathrm{e}_n\rangle|^r} &= \sqrt[r]{\sum\nolimits_n |\langle A^*\mathrm{f}_n, B\mathrm{e}_n\rangle|^r} \\
&\leq \sqrt[r]{\sum\nolimits_n \|A^*\mathrm{f}_n\|^r \|B\mathrm{e}_n\|^r} \\
&\leq \sqrt[p]{\sum\nolimits_n \|A^*\mathrm{f}_n\|^p} \sqrt[q]{\sum\nolimits_n \|B\mathrm{e}_n\|^q},
\end{aligned}$$

by the Cauchy-Schwarz inequality and Hölder's inequality applied to the conjugate indices $\frac{p}{r}$ and $\frac{q}{r}$. Using the two characterizations (6.2.11) and (6.2.12) we can now take suprema to conclude (6.2.18).□

Since obviously $\|zA\|_p = |z|\|A\|_p$ we finally arrive at the statement that $\mathfrak{L}^p(\mathfrak{H})$ is a subspace of $\mathfrak{K}(\mathfrak{H})$ and $\|\cdot\|_p$ is a norm on it. Some first basic properties of this normed space are now summarized in the following theorem:

Theorem 6.2.10 (Schatten classes) *Let $p, q \geq 1$.*

i.) The Schatten class $\mathfrak{L}^p(\mathfrak{H}) \subseteq \mathfrak{B}(\mathfrak{H})$ is a *-ideal.

ii.) The Schatten class $\mathfrak{L}^p(\mathfrak{H})$ becomes a Banach *-algebra with respect to the Schatten norm $\|\cdot\|_p$.

iii.) The finite rank operators $\mathfrak{F}(\mathfrak{H})$ are dense in $\mathfrak{L}^p(\mathfrak{H})$ with respect to the Schatten norm $\|\cdot\|_p$. More precisely, for $A \in \mathfrak{L}^p(\mathfrak{H})$ the normal form

$$A = \sum_n s_n(A)\Theta_{\mathrm{f}_n,\mathrm{e}_n} \tag{6.2.19}$$

converges unconditionally with respect to the Schatten norm $\|\cdot\|_p$ and even absolutely for the case $p = 1$.

iv.) For $p \geq q$ we have the continuous inclusions

$$\mathfrak{L}^1(\mathfrak{H}) \longrightarrow \mathfrak{L}^q(\mathfrak{H}) \longrightarrow \mathfrak{L}^p(\mathfrak{H}) \longrightarrow \mathfrak{K}(\mathfrak{H}) \tag{6.2.20}$$

with dense images where the compact operators are equipped with the operator norm topology.

v.) The operator product gives a continuous bilinear map

$$\mathfrak{L}^p(\mathfrak{H}) \times \mathfrak{L}^q(\mathfrak{H}) \longrightarrow \mathfrak{L}^r(\mathfrak{H}) \tag{6.2.21}$$

for $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

PROOF: First note that the second part of Theorem 6.2.9 together with $\|zA\|_p = |z|\|A\|_p$ shows that $\mathfrak{L}^p(\mathfrak{H})$ is a subspace and $\|\cdot\|_p$ is a seminorm on it. From *iv.)* of that Theorem we see that $\|\cdot\|_p$ is indeed a norm inducing a finer topology than the operator norm. Finally, Theorem 6.2.9, *iii.)*, shows that $\mathfrak{L}^p(\mathfrak{H})$ is a *-ideal in $\mathfrak{B}(\mathfrak{H})$. Moreover, with $\|AB\|_p \le \|A\|\|B\|_p \le \|A\|_p\|B\|_p$ Theorem 6.2.9, *iii.)*, shows that $\mathfrak{L}^p(\mathfrak{H})$ is a normed *-algebra. We have to check completeness: let $A_n \in \mathfrak{L}^p(\mathfrak{H})$ be a Cauchy sequence with respect to $\|\cdot\|_p$. Since $\|\cdot\| \le \|\cdot\|_p$, the A_n are also a Cauchy sequence with respect to the operator norm and hence converging to some $A \in \mathfrak{B}(\mathfrak{H})$ with respect to $\|\cdot\|$. For the approximation numbers we get from Proposition 6.1.49, *ii.)*,

$$|a_k(A - A_n) - a_k(A_m - A_n)| \le \|A - A_n + A_n - A_m\| = \|A - A_n\|,$$

implying for all n and k

$$\lim_{m \longrightarrow \infty} a_k(A_m - A_n) = a_k(A - A_n). \tag{$*$}$$

Now let $\epsilon > 0$ be given and let $N \in \mathbb{N}$ be such that for $n, m \ge N$ we have $\|A_n - A_m\|_p < \epsilon$. Then

$$\begin{aligned}
\sqrt[p]{\sum\nolimits_{k=0}^{K} a_k(A - A_n)^p} &\overset{(*)}{\le} \lim_{m \longrightarrow \infty} \sqrt[p]{\sum\nolimits_{k=0}^{K} a_k(A_m - A_n)^p} \\
&\le \limsup_{m \longrightarrow \infty} \sqrt[p]{\sum\nolimits_{k=0}^{\infty} a_k(A_m - A_n)^p} \\
&\le \limsup_{m \longrightarrow \infty} \|A_m - A_n\|_p \\
&\le \epsilon,
\end{aligned}$$

independently of K. Thus also $\|A - A_n\|_p \le \epsilon$ showing $A \in \mathfrak{L}^p(\mathfrak{H})$ and $A_n \longrightarrow A$ with respect to $\|\cdot\|_p$ by (6.2.13). Hence the p-th Schatten class is complete. For the third part, it is clear that $\mathfrak{F}(\mathfrak{H}) \subseteq \mathfrak{L}^p(\mathfrak{H})$ since for $A \in \mathfrak{F}(\mathfrak{H})$ also $|A| \in \mathfrak{F}(\mathfrak{H})$ and clearly $\operatorname{tr}|A|^p < \infty$ for a finite rank operator $|A|$. We show the convergence of (6.2.19): to this end we first compute

$$\|\Theta_{\mathrm{f}_n,\mathrm{e}_n}\|_p = \sqrt[p]{\operatorname{tr}\left|\Theta_{\mathrm{f}_n,\mathrm{e}_n}\right|^p} = \sqrt[p]{\operatorname{tr}\left|\Theta^*_{\mathrm{f}_n,\mathrm{e}_n}\Theta_{\mathrm{f}_n,\mathrm{e}_n}\right|^{\frac{p}{2}}} = \sqrt[p]{\operatorname{tr}\left|\Theta_{\mathrm{e}_n,\mathrm{e}_n}\right|^{\frac{p}{2}}} = 1,$$

since the projection $\Theta_{\mathrm{e}_n,\mathrm{e}_n}$ coincides with all its positive powers $|\Theta_{\mathrm{e}_n,\mathrm{e}_n}|^{\frac{p}{2}}$ and clearly has trace one. For $p = 1$ we have

$$\sum\nolimits_n s_n(A)\|\Theta_{\mathrm{f}_n,\mathrm{e}_n}\|_1 = \sum\nolimits_n s_n(A) = \|A\|_1,$$

showing the absolute convergence of (6.2.19) with respect to $\|\cdot\|_1$. For $p > 1$ the convergence needs not to be absolute anymore. Instead, let $I \subseteq \mathbb{N}_0$ be a finite subset and consider

$$A_I = \sum_{n \in I} s_n(A)\Theta_{\mathrm{f}_n,\mathrm{e}_n}.$$

Then

$$\|A - A_I\|_p^p = \sum_{n \in \mathbb{N}_0 \setminus I} s_n(A)^p \longrightarrow 0$$

for $I \longrightarrow \mathbb{N}_0$ since for a sequence in ℓ^p like $(s_n(A))_{n \in \mathbb{N}_0}$ the series $\sum_{n=0}^{\infty} s_n(A)^p$ converges absolutely and hence unconditionally. This shows *iii.)* also for the case $p > 1$. The continuous inclusions in (6.2.20) are a consequence of the norm estimates (6.2.16). Since $\mathfrak{F}(\mathfrak{H})$ is dense in $\mathfrak{L}^p(\mathfrak{H})$ and in $\mathfrak{K}(\mathfrak{H})$, the fourth part follows. Finally, Theorem 6.2.9, *vi.)*, gives directly the last part. ☐

One should note at this stage that much of the statements can be transferred to the general case of Banach spaces and the p-approximable operators as discussed in Remark 6.1.50. In particular, the p-approximable operators on a Hilbert space are precisely given by the p-th Schatten class $\mathfrak{L}^p(\mathfrak{H})$. For a further discussion we refer to [26, Chap. 19 and Chap. 20].

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Remark 6.2.11 (*-Ideals in $\mathfrak{B}(\mathfrak{H})$) For a Hilbert space $\mathfrak{H}$ we arrive at the following sequence of *-ideals

$$\mathfrak{F}(\mathfrak{H}) \subseteq \mathfrak{L}^1(\mathfrak{H}) \subseteq \cdots \subseteq \mathfrak{L}^p(\mathfrak{H}) \subseteq \mathfrak{L}^q(\mathfrak{H}) \subseteq \cdots \subseteq \mathfrak{K}(\mathfrak{H}) \subseteq \mathfrak{B}_{\text{sep}}(\mathfrak{H}) \subseteq \mathfrak{B}(\mathfrak{H}), \tag{6.2.22}$$

where $p \leq q$. In the finite-dimensional case, all the spaces coincide while in the infinite-dimensional case, all inclusion except for the last are proper. The last is proper only for non-separable Hilbert spaces. Moreover, the inclusion maps between the Schatten classes and the compact operators are continuous with dense image with respect to the corresponding Schatten norms and the operator norm, respectively. Finally, none of the *-ideals $\mathfrak{F}(\mathfrak{H})$ or $\mathfrak{L}^p(\mathfrak{H})$ is closed with respect to the operator norm, only $\mathfrak{K}(\mathfrak{H})$ and $\mathfrak{B}_{\text{sep}}(\mathfrak{H})$ are closed *-ideals with respect to the operator norm in the infinite-dimensional case.

Theorem 6.2.10, *v.)*, shows that the product of $A \in \mathfrak{L}^p(\mathfrak{H})$ and $B \in \mathfrak{L}^p(\mathfrak{H})$ gives an operator $AB \in \mathfrak{L}^r(\mathfrak{H})$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. The following statement shows the converse: every operator in $\mathfrak{L}^r(\mathfrak{H})$ can be factorized like this and the norm estimate (6.2.18) can be made an equality:

Proposition 6.2.12 *Let $p, q, r \geq 1$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ be given. Then for every $C \in \mathfrak{L}^r(\mathfrak{H})$ there exist operators $A \in \mathfrak{L}^p(\mathfrak{H})$ and $B \in \mathfrak{L}^q(\mathfrak{H})$ with*

$$C = AB \quad \textit{and} \quad \|C\|_r = \|A\|_p \|B\|_q. \tag{6.2.23}$$

PROOF: Let $C = \sum_n s_n(C)\Theta_{\mathrm{f}_n,\mathrm{e}_n}$ be given in its normal form with $(s_n(C))_{n\in\mathbb{N}_0} \in \ell^r$. Then we claim that

$$A = \sum\nolimits_n s_n(C)^{\frac{r}{p}}\Theta_{\mathrm{f}_n,\mathrm{e}_n} \quad \text{and} \quad B = \sum\nolimits_n s_n(C)^{\frac{r}{q}}\Theta_{\mathrm{e}_n,\mathrm{e}_n}$$

will do the job. Since $(s_n(C))_{n\in\mathbb{N}_0} \in \ell^r$ we see that $(s_n(C)^{\frac{r}{p}})_{n\in\mathbb{N}_0} \in \ell^p$ as well as $(s_n(C)^{\frac{r}{q}})_{n\in\mathbb{N}_0} \in \ell^q$. Hence A and B are in the correct Schatten classes as claimed. Moreover, $\|A\|_p^p = \sum_n (s_n(A)^{\frac{r}{p}})^p = \|C\|_r^r$ and $\|B\|_q^q = \|C\|_r^r$, too. This shows the norm equality $\|C\|_r = \|A\|_p\|B\|_q$. Finally, $C = AB$ is clear from the construction. □

Of course, the operators A and B with this factorization property are typically far from being unique. Note that in addition our B is a positive operator, $B \geq 0$.

6.2.4 The Trace and Density Matrices

We are now in the position to define the operator trace for trace class operators:

Definition 6.2.13 (Trace) *Let $A \in \mathfrak{L}^1(\mathfrak{H})$ be a trace class operator. Then the trace of A is defined by*

$$\operatorname{tr}(A) = \sum_{i\in I} \langle \mathrm{e}_i, A\mathrm{e}_i \rangle, \tag{6.2.24}$$

where $\{\mathrm{e}_i\}_{i\in I}$ is a Hilbert basis of $\mathfrak{H}$.

Theorem 6.2.14 (Trace) *Let $A \in \mathfrak{L}^1(\mathfrak{H})$.*

i.) The trace $\operatorname{tr} A$ of A is well-defined and converges absolutely.

ii.) The trace is the unique continuous linear extension of the trace of finite-rank operators to $\mathfrak{L}^1(\mathfrak{H})$. One has

$$|\operatorname{tr} A| \leq \operatorname{tr}|A| = \|A\|_1, \tag{6.2.25}$$

and the norm of the functional $\operatorname{tr}\colon \mathfrak{L}^1(\mathfrak{H}) \longrightarrow \mathbb{C}$ is

$$\|\operatorname{tr}\| = 1. \tag{6.2.26}$$

iii.) The trace functional is positive and real, i.e.

$$\operatorname{tr}(A^*A) \geq 0 \quad \textit{and} \quad \operatorname{tr} A^* = \overline{\operatorname{tr} A}. \tag{6.2.27}$$

iv.) The trace vanishes on commutators, i.e. for all $B \in \mathfrak{B}(\mathfrak{H})$ we have

$$\operatorname{tr}(AB) = \operatorname{tr}(BA). \tag{6.2.28}$$

PROOF: We write A in its normal form

$$A = \sum\nolimits_n s_n(A)\Theta_{\tilde{\mathrm{f}}_n, \tilde{\mathrm{e}}_n}$$

as usual. By a simple application of the Cauchy-Schwarz inequality and Parseval's equation we get

$$\begin{aligned}
\left| \sum\nolimits_{i\in I} \sum\nolimits_n s_n(A) \langle \mathrm{e}_i, \Theta_{\tilde{\mathrm{f}}_n, \tilde{\mathrm{e}}_n} \mathrm{e}_i \rangle \right| &\leq \sum\nolimits_{i\in I} \sum\nolimits_n s_n(A) |\langle \mathrm{e}_i, \tilde{\mathrm{f}}_n\rangle| |\langle \tilde{\mathrm{e}}_n, \mathrm{e}_i\rangle| \\
&\leq \sum s_n(A) \sqrt{\sum\nolimits_{i\in I} |\langle \mathrm{e}_i, \tilde{\mathrm{f}}_n\rangle|^2} \sqrt{\sum\nolimits_{i\in I} |\langle \tilde{\mathrm{e}}_n, \mathrm{e}_i\rangle|^2} \\
&= \sum\nolimits_n s_n(A) \\
&= \|A\|_1.
\end{aligned} \tag{$*$}$$

Note that the index set I might be uncountable but only countably many $i \in I$ contribute in $\langle \mathrm{e}_i, \tilde{\mathrm{f}}_n\rangle$ as well as in $\langle \tilde{\mathrm{e}}_n, \mathrm{e}_i\rangle$. This shows the absolute convergence of the series defining the trace. We have to check whether or not it is well-defined, i.e. independent of the chosen Hilbert basis $\{\mathrm{e}_i\}_{i\in I}$. Thus let $\{\mathrm{f}_j\}_{j\in J}$ be another Hilbert basis. Since $\langle \phi, \psi\rangle = \sum_{j\in J} \langle \phi, \mathrm{f}_j\rangle \langle \mathrm{f}_j, \psi\rangle$ converges absolutely for any Hilbert basis, see Theorem 3.3.7, *v.)*, we get by the continuity of A

$$\begin{aligned}
\sum_{i\in I} \langle \mathrm{e}_i, A\mathrm{e}_i\rangle &= \sum_{i\in I} \sum_{j\in J} \langle \mathrm{e}_i, \mathrm{f}_j\rangle \langle \mathrm{f}_j, A\mathrm{e}_i\rangle \\
&= \sum_{i\in I} \sum_{j\in J} \sum_{k\in J} \langle \mathrm{e}_i, \mathrm{f}_j\rangle \langle \mathrm{f}_j, A\mathrm{f}_k\rangle \langle \mathrm{f}_k, \mathrm{e}_i\rangle \\
&= \sum_{j,k\in J} \langle \mathrm{f}_j, A\mathrm{f}_k\rangle \sum_{i\in I} \langle \mathrm{f}_k, \mathrm{e}_i\rangle \langle \mathrm{e}_i, \mathrm{f}_k\rangle \\
&= \sum_{j\in J} \langle \mathrm{f}_j, A\mathrm{f}_j\rangle,
\end{aligned}$$

since $\langle \mathrm{f}_j, \mathrm{f}_k\rangle = \delta_{jk}$ and since the order of the summations can be exchanged thanks to absolute convergence. As alternative one can use that for finite rank operators $A \in \mathfrak{F}(\mathfrak{H})$, the definition of the trace in (6.2.24) coincides with the linear-algebraic definition of the trace which is known to be independent of the chosen Hilbert basis, see also Exercise 6.4.2. Then the continuity statement (**??**) and the density of $\mathfrak{F}(\mathfrak{H}) \subseteq \mathfrak{L}^1(\mathfrak{H})$ shows that (6.2.24) is the unique continuous extension of the trace functional on $\mathfrak{F}(\mathfrak{H})$ to $\mathfrak{L}^1(\mathfrak{H})$. This shows the first part. The second part is clear from ($*$) and the fact that for positive trace class operators A we have the equality $|\operatorname{tr}(A)| = \operatorname{tr}(A) = \|A\|_1$. Since the finite rank operators are dense in $\mathfrak{L}^1(\mathfrak{H})$ by Theorem 6.2.10, *iii.)*, the continuous linear functional tr is uniquely determined by its values on $\mathfrak{F}(\mathfrak{H})$. Here it is a simple check that (6.2.24) is indeed that canonical trace on finite rank operators as known from linear algebra. The third part is clear. For the last part we know that by the independence of (6.2.24) on the chosen Hilbert basis we have $\operatorname{tr}(UAU^*) = \operatorname{tr}(A)$ for any unitary map U. Hence (6.2.28) holds for B being unitary. But in every unital C^*-algebra any element is a linear combination of at most four unitary elements, see Exercise 4.5.42. □

Remark 6.2.15 Needless to say, the definition of the trace of a positive operator as in Definition 6.2.4 coincides with Definition 6.2.13 if the positive operator is actually trace class. Moreover, if $A \in \mathfrak{F}(\mathfrak{H})$ is a finite-rank operator, then $\operatorname{tr}(A)$ is the usual trace from linear algebra, see Exercise 6.4.2 as one can define this also beyond Hilbert spaces.

Remark 6.2.16 For a normal trace class operator $A \in \mathfrak{L}^1(\mathfrak{H})$ we can choose the Hilbert basis to consist of eigenvectors of A according to Corollary 6.1.39. Thus we can write

$$A = \sum_{\lambda \in \operatorname{spec}(A) \setminus \{0\}} \lambda P_{\ker(\lambda - A)} \tag{6.2.29}$$

with the corresponding spectral projections. For the trace we get

$$\operatorname{tr}(A) = \sum_{\lambda \in \operatorname{spec}(A) \setminus \{0\}} \lambda \dim \ker(\lambda - A), \tag{6.2.30}$$

which is the sum of all the eigenvalues, counted by multiplicity. Note that for a projection P_U we clearly have

$$\operatorname{tr}(P_U) = \dim \operatorname{im} P_U = \dim U, \tag{6.2.31}$$

where $U \subseteq \mathfrak{H}$ is the closed subspace onto which P_U projects. This follows directly from Theorem 6.2.5, *vi.)*. Note also that $s_n(A) = |\lambda_n|$ for a normal operator, where the eigenvalues are ordered decreasingly and repeated according to their multiplicities. Thus we see again that the series (6.2.24) converges absolutely since (6.2.30) converges absolutely by the characterization (6.2.10) of the trace norm.

Definition 6.2.17 (Density matrix) *A positive trace class operator ϱ with $\operatorname{tr}(\varrho) = 1$ is called density matrix or statistical operator.*

The importance of density matrices is that they yield states for the C^*-algebra $\mathfrak{B}(\mathfrak{H})$:

Proposition 6.2.18 *Let $\varrho \in \mathfrak{L}^1(\mathfrak{H})$ be a density matrix. Then*

$$\omega_\varrho \colon \mathfrak{B}(\mathfrak{H}) \ni A \;\mapsto\; \operatorname{tr}(\varrho A) \in \mathbb{C} \tag{6.2.32}$$

defines a state on $\mathfrak{B}(\mathfrak{H})$.

PROOF: First we note that $\varrho A \in \mathfrak{L}^1(\mathfrak{H})$ and hence ω_ϱ is a well-defined linear functional. Since $\omega_\varrho(A^*A) = \operatorname{tr}(\varrho A^* A) = \operatorname{tr}(A \varrho A^*)$ and $A \varrho A^*$ is a positive operator, ω_ϱ is indeed a positive functional. Finally, $\omega_\varrho(\mathbb{1}) = \operatorname{tr}(\varrho) = 1$ is precise the normalization condition for a density matrix. ☐

Example 6.2.19 (Density matrices) The following simple examples show that many states in quantum mechanics are described by density matrices:

i.) If $\phi \in \mathfrak{H}$ is a unit vector then the orthogonal projection $P_\phi = \Theta_{\phi,\phi}$ is a density matrix with

$$\omega_{P_\phi}(A) = \operatorname{tr}(P_\phi A) = \langle \phi, A\phi \rangle, \tag{6.2.33}$$

since we can always extend the vector ϕ to a Hilbert basis and use such a Hilbert basis to evaluate the trace. Thus all vector states are of the form (6.2.33).

ii.) Let $\{\phi_n\}_{n\in\mathbb{N}}$ be unit vectors, not necessarily orthonormal and let $(\lambda_n)_{n\in\mathbb{N}} \in \ell^1$ be a summable sequence with $\lambda_n \geq 0$ and $\sum_{n=1}^\infty \lambda_n = 1$. Then

$$\varrho = \sum_{n=1}^{\infty} \lambda_n P_{\phi_n} \tag{6.2.34}$$

is a density matrix. Indeed, each term of the sum is a positive rank-one operator and hence trace class. Moreover, the series converges absolutely in the trace norm since $\|P_{\phi_n}\|_1 = 1$ and $(\lambda_n)_{n\in\mathbb{N}} \in \ell^1$. Thus it also converges absolutely in the operator norm to some $\varrho \in \mathfrak{B}(\mathfrak{H})$ which is still a trace class operator by the completeness of $\mathfrak{L}^1(\mathfrak{H})$ with respect to $\|\cdot\|_1$. Clearly, $\varrho \geq 0$ and $\operatorname{tr}(\varrho) = \sum_{n=1}^{\infty} \lambda_n = 1$ by the continuity and linearity of the trace and $\operatorname{tr}(P_{\phi_n}) = 1$. Now for this ϱ we have

$$\varrho A = \left(\sum_{n=1}^{\infty} \lambda_n P_{\phi_n}\right) A = \sum_{n=1}^{\infty} \lambda_n P_{\phi_n} A \tag{6.2.35}$$

by the continuity of the operator product with respect to the trace norm and the operator norm according to Theorem 6.2.9, *iii.)*, Since $\|P_{\phi_n} A\|_1 \leq \|A\|$ by that theorem, the right hand side of (6.2.35) still converges (even absolutely) in the trace norm. We conclude by continuity of tr that

$$\omega_\varrho(A) = \sum_{n=1}^{\infty} \lambda_n \langle \phi_n, A\phi_n \rangle. \tag{6.2.36}$$

Thus ω_ϱ is the mixed state corresponding to a *countable mixture* of the vector states ϕ_n with weights λ_n. Setting all but finitely many of the weights λ_n to be zero gives us ordinary convex combinations of vector states.

iii.) A particular case can be obtained in finite dimensions: here $\mathbb{1}$ is a trace class operator with $\operatorname{tr}(\mathbb{1}) = \dim \mathfrak{H}$ by Theorem 6.2.5, *vi.)*. Hence

$$\varrho = \frac{1}{\dim \mathfrak{H}} \mathbb{1} \tag{6.2.37}$$

is a density matrix. The interpretation is that ϱ is *maximally mixed*.

In statistical mechanics one considers the density matrix of the canonical ensemble: one fixes an inverse temperature

$$\beta = \frac{1}{kT} > 0 \tag{6.2.38}$$

where k is Boltzmann's constant and T denotes the temperature. Then for a self-adjoint Hamiltonian $(\operatorname{dom} H, H)$ of the system under consideration one considers $\mathrm{e}^{-\beta H}$. By our spectral calculus, we know that if $\operatorname{spec}(H)$ is bounded from below, then $\mathrm{e}^{-\beta H} \in \mathfrak{B}(\mathfrak{H})$ is indeed a bounded operator. The next proposition gives a criterion whether $\mathrm{e}^{-\beta H}$ is a trace class operator:

Proposition 6.2.20 *Let* $(\operatorname{dom} H, H)$ *be self-adjoint and consider* $(\operatorname{dom} \mathrm{e}^{-\beta H}, \mathrm{e}^{-\beta H})$ *for* $\beta > 0$ *as self-adjoint operator in* $\mathfrak{H}$. *Then the following statements are equivalent:*

i.) *There exists a* $\beta_0 > 0$ *such that* $\mathrm{e}^{-\beta_0 H} \in \mathfrak{L}^1(\mathfrak{H})$.

ii.) *There exists a* $\beta_0 > 0$ *such that for all* $\beta \geq \beta_0 > 0$ *one has* $\mathrm{e}^{-\beta H} \in \mathfrak{L}^1(\mathfrak{H})$.

iii.) *The spectrum of* H *is bounded from below, countable without accumulation points*

$$\operatorname{spec}(H) = \{\lambda_1 \leq \lambda_2 \leq \cdots\} \subseteq \mathbb{R}, \tag{6.2.39}$$

and one has

$$\sum_n \mathrm{e}^{-\beta_0 \lambda_n} \dim \ker(\lambda_n - H) < \infty. \tag{6.2.40}$$

In particular, $\dim \ker(\lambda_n - H) < \infty$ *for all* n, 0 *is never an eigenvalue of* $\mathrm{e}^{-\beta H}$, *and if* $\dim H = \infty$ *then* H *is necessarily unbounded.*

PROOF: We consider the functions $g(x) = \mathrm{e}^{-\beta x}$ and $f(x) = \mathrm{e}^{\beta x}$ then $fg = 1$ which shows that the operator $f(H)g(H)$ is defined on $\operatorname{dom} g$ by Theorem 5.4.2, *iv.)*. Thus for $\phi \in \operatorname{dom} g = \operatorname{dom} \mathrm{e}^{-\beta H}$ we have by that theorem

$$\int_{\operatorname{spec}(H)} f \,\mathrm{d}E \int_{\operatorname{spec}(H)} g \,\mathrm{d}E\phi = \int_{\operatorname{spec}(H)} fg \,\mathrm{d}E\phi = \int_{\operatorname{spec}(H)} 1 \,\mathrm{d}E\phi = \phi,$$

which shows that $g(H) = \mathrm{e}^{-\beta H}$ is injective. Thus 0 can not be an eigenvalue of $\mathrm{e}^{-\beta H}$ no matter what $\beta \in \mathbb{R}$ is. Denote by $\tilde{E}$ the spectral measure of $\mathrm{e}^{-\beta H}$ which is defined on

$$\operatorname{spec}\left(\mathrm{e}^{-\beta H}\right) = g(\operatorname{spec}(H))^{\mathrm{cl}},$$

according to Remark 5.4.14, *iii.)*. Since $g\colon \mathbb{R} \longrightarrow \mathbb{R}^+$ is a bijection with inverse $g^{-1}(y) = \frac{1}{\beta} \ln(y)$ we can push forward E to $\operatorname{spec}(\mathrm{e}^{-\beta H})$ via g, once we have eliminated the point $\{0\}$ from $\operatorname{spec}(\mathrm{e}^{-\beta H})$: this will be no problem as we have just seen that $\tilde{E}_{\{0\}} = 0$ thanks to Theorem 5.4.11, *ii.)*. Thus we have

$$\mathrm{e}^{-\beta H} = \int_{\operatorname{spec}(H)} g(\lambda) \,\mathrm{d}E = \int_{g(\operatorname{spec}(H))} g \circ g^{-1}(\mu) \,\mathrm{d}g_*E = \int_{g(\operatorname{spec}(H))} \mu \,\mathrm{d}g_*E.$$

If we extend now g_*E to $\{0\}$ by declaring $g_*E_{\{0\}} = 0$ we have represented the operator $\mathrm{e}^{-\beta H}$ via

$$\mathrm{e}^{-\beta H} = \int_{\operatorname{spec}(\mathrm{e}^{-\beta H})} \mu \,\mathrm{d}g_*E.$$

Hence by the uniqueness of the spectral measure $\tilde{E}$ of $\mathrm{e}^{-\beta H}$ according to Theorem 5.4.13 we get $g_*E = \tilde{E}$. Up to now, this holds for every self-adjoint operator $(\operatorname{dom} H, H)$. The equivalence of *i.)* and *ii.)* follows since for $\mathrm{e}^{-\beta_0 H}$ being trace class we have $(s_n(\mathrm{e}^{-\beta_0 H}))_{n\in\mathbb{N}} \in \ell^1$. If now $\beta \geq \beta_0$ then $\frac{\beta}{\beta_0} \geq 1$ and hence

$$s_n\left(\mathrm{e}^{-\beta H}\right) = s_n\left(\left(\mathrm{e}^{-\beta_0 H}\right)^{\frac{\beta}{\beta_0}}\right) = s_n\left(\mathrm{e}^{-\beta_0 H}\right)^{\frac{\beta}{\beta_0}}$$

decreases even faster except for the finitely many values $s_n(\mathrm{e}^{-\beta H}) \geq 1$. Since *i.)* implies *ii.)*, the converse is trivial. Now suppose *i.)* holds then $\operatorname{spec}(\mathrm{e}^{-\beta_0 H}) = \{\mu_1 \geq \mu_2 \geq \cdots\}$ is countable with only possible accumulation point 0 (necessarily for $\dim \mathfrak{H} = \infty$). Thus

$$\tilde{E}_{\{\mu_n\}} = P_{\ker(\mu_n - \mathrm{e}^{-\beta_0 H})}$$

determines $\tilde{E}$ completely since we already know $\tilde{E}_{\{0\}} = 0$. Since g is bijective, we get

$$E_{\{\lambda_n\}} = \tilde{E}_{\{g(\lambda_n)\}} = P_{\ker(\mu_n - \mathrm{e}^{-\beta_0 H})},$$

where $\lambda_n = -\frac{1}{\beta_0} \ln(\mu_n) \in \operatorname{spec}(H)$. In particular, $\operatorname{spec}(H) = \{\lambda_1 \leq \lambda_2 \leq \dots\}$ is countable and bounded from below without accumulation points. Since $\ker(\lambda_n - H) = \operatorname{im} E_{\{\lambda_n\}}$ by Theorem 5.4.11, *i.)*, we conclude

$$\ker(\lambda_n - H) = \ker\left(\mu_n - \mathrm{e}^{-\beta_0 H}\right).$$

Thus for the distinct eigenvalues λ_n we get

$$\sum\nolimits_n \mathrm{e}^{-\beta_0 \lambda_n} \dim\ker(\lambda_n - H) = \sum\nolimits_n \mu_n \dim\ker\left(\mu_n - \mathrm{e}^{-\beta_0 H}\right) = \operatorname{tr}\left(\mathrm{e}^{-\beta_0 H}\right),$$

and hence (6.2.40) follows. The converse is done along the same lines and the remaining statements are now clear. $\square$

Note that in general, there are self-adjoint operators with spectrum growing too slowly such that $\mathrm{e}^{-\beta H}$ fails to be trace class for all β but $\mathrm{e}^{-\beta H} \in \mathfrak{L}^1(\mathfrak{H})$ only for some $\beta \geq \beta_0 > 0$. On the other hand, there are self-adjoint operators such that $\mathrm{e}^{-\beta H}$ is trace class for *all* $\beta > 0$. If $(\operatorname{dom} H, H)$ satisfies now the conditions from the proposition for some β then

: Exercises!!!

$$Z = \operatorname{tr} \mathrm{e}^{-\beta H} \tag{6.2.41}$$

is called the *canonical partition function* and the density matrix

$$\varrho = \frac{1}{Z} E^{-\beta H} \tag{6.2.42}$$

describes the *canonical ensemble* with inverse temperature $\beta = \frac{1}{kT}$ and the Hamiltonian H.

For a density matrix one characteristic quantity needed in statistical (quantum) physics is its entropy: Since $\varrho \in \mathscr{L}^1(\mathfrak{H})$ is compact with $\|\varrho\| \leq \|\varrho\|_1 = \operatorname{tr} \varrho = 1$ we can use the continuous positive function

$$[0, 1] \ni x \;\mapsto\; -x \ln x \in \mathbb{R}_0^+ \tag{6.2.43}$$

to form a compact operator $-\varrho \ln \varrho \in \mathfrak{K}(\mathfrak{H})$ by the continuous calculus in the non-unital C^*-algebra $\mathfrak{K}(\mathfrak{H})$, see Corollary 4.3.28, since by continuity $-0 \ln 0 = 0$, see also Exercise 6.4.13.

Definition 6.2.21 (von Neumann entropy) *For a density matrix $\varrho \in \mathscr{L}^1(\mathfrak{H})$ the quantity*

$$\sigma(\varrho) = \operatorname{tr}(-\varrho \ln \varrho) \in [0, +\infty] \tag{6.2.44}$$

is called the von Neumann entropy of ϱ. For $\alpha > 1$ the α-entropy of ϱ is defined by

$$\sigma_\alpha(\varrho) = \frac{1}{1-\alpha} \ln\big(\operatorname{tr} \varrho^\alpha\big). \tag{6.2.45}$$

We collect now some first simple continuity properties of the entropy, for a more detailed discussion of its convexity properties we refer to [57, Part II, Sect. 2.2].

Theorem 6.2.22 (Discontinuity of von Neumann Entropy) *Let $\alpha > 1$.*

i.) The α-entropy σ_α is continuous with respect to the trace norm.

ii.) For a fixed density matrix ϱ we have

$$\sigma(\varrho) = \sup_{\alpha > 1} \sigma_\alpha(\varrho). \tag{6.2.46}$$

iii.) The von Neumann entropy is lower semicontinuous but discontinuous for $\dim \mathfrak{H} = \infty$: In fact, for $\dim \mathfrak{H} = \infty$ and a density matrix ϱ one finds for every $\varepsilon > 0$ a density matrix ϱ_ε with

$$\|\varrho - \varrho_\varepsilon\|_1 < \epsilon \quad \textit{and} \quad \sigma(\varrho_\varepsilon) = +\infty. \tag{6.2.47}$$

iv.) The set of density matrices ϱ with $\sigma(\varrho) \leq n$ is closed and nowhere dense.

Proof: First we note that $\sigma_\alpha(\varrho) = \frac{1}{1-\alpha} \ln \operatorname{tr}(\varrho^\alpha) = \frac{1}{1-\alpha} \ln \|\varrho\|_\alpha^\alpha$. Since for $\alpha > 1$ the α-th Schatten-norm $\|\cdot\|_\alpha$ can be estimated by the trace norm, it is a continuous norm with respect to $\|\cdot\|_1$. Thus σ_α is continuous as claimed. Now we use the convexity of the logarithm function to get

$$\ln\left(\sum_{n=0}^{\infty} \lambda_n x_n\right) \geq \sum_{n=0}^{\infty} \lambda_n \ln x_n, \tag{$*$}$$

for $\lambda_n > 0$ and $\sum_{n=0}^\infty \lambda_n = 1$, where the x_n are a sequence in $\mathbb{R}^+$. Indeed, for a finite sum this is just the convexity of the logarithm, for the infinite series one takes the limits and interpretes $(*)$ as inequality in $[-\infty, +\infty]$. Thus we get for $\alpha > 1$ and a density matrix ϱ with the singular values $\lambda_n = s_n(\varrho)$

$$
\begin{aligned}
\frac{1}{1-\alpha} \ln \operatorname{tr} \varrho^\alpha &= \frac{1}{1-\alpha} \ln\left(\sum_{n=0}^{\infty} \lambda_n^\alpha\right) \\
&= \frac{1}{1-\alpha} \ln\left(\sum_{n=0}^{\infty} \lambda_n \lambda_n^{\alpha-1}\right) \\
&\leq \frac{1}{1-\alpha} \sum_{n=0}^{\infty} \lambda_n \ln \lambda_n^{\alpha-1} \\
&= -\sum_{n=0}^{\infty} \lambda_n \ln \lambda_n \\
&= \sigma(\varrho).
\end{aligned}
$$

Hence the von Neumann entropy is always larger than the α-entropies for $\alpha > 1$. We claim that for a fixed ϱ the function $\alpha \mapsto \sigma_\alpha(\varrho)$ is differentiable in α and we can differentiate termwise. Clearly $\alpha \mapsto \lambda_n^\alpha$ is even smooth with derivative $\lambda_n^\alpha \ln \lambda_n$. We claim that the series $\sum_{n=0}^{\infty} \lambda_n^\alpha \ln \lambda_n$ converges locally uniformly in α. Indeed, if $\alpha \geq 1 + \varepsilon > 1$ we have

$$
\sum_{n=0}^{\infty} |\lambda_n^\alpha \ln \lambda_n| \leq \sum_{n=0}^{\infty} \lambda_n^{1+\varepsilon} \ln \lambda_n \leq c_\varepsilon \sum_{n=0}^{\infty} \lambda_n < \infty,
$$

since the function $x \mapsto x^\varepsilon \ln x$ is continuous on $[0, \infty)$ for $\varepsilon > 0$ and on $[0, 1]$ bounded by some c_ε. Thus we have uniform and absolute convergence for $\alpha \geq 1+\varepsilon$ for every $\varepsilon > 0$. Analogously, one shows that also $\sum_{n=0}^{\infty} \lambda_n^\alpha$ converges uniformly for $\alpha \geq 1 + \varepsilon > 1$ for every $\varepsilon > 0$. Thus we can differentiate into the sum which gives a $\mathscr{C}^1$-function $\alpha \mapsto \sigma_\alpha(\varrho)$ for all $\alpha > 1$. By l'Hospital's rule we get

$$
\lim_{\alpha \longrightarrow 1} \frac{1}{1-\alpha} \ln\left(\sum_{n=0}^{\infty} \lambda_n^\alpha\right) = \frac{\lim_{\alpha \longrightarrow 1} \frac{1}{\sum_{n=0}^{\infty} \lambda_n^\alpha} \sum_{n=0}^{\infty} \lambda_n^\alpha \ln \lambda_n}{\lim_{\alpha \longrightarrow 1} -1} = -\sum_{n=0}^{\infty} \lambda_n \ln \lambda_n = \sigma(\varrho),
$$

as we can differentiate termwise and the resulting series are either convergent for $\alpha = 1$ or monotonic increasing for $\alpha \longrightarrow 1$. Note however, that the limit may be $+\infty$. In any case, this shows the second part. The third part is now obtained as follows: as a supremum of continuous maps, σ is lower semicontinuous but not necessarily continuous. If $\dim \mathfrak{H} = +\infty$ than we actually have in every neighbourhood of ϱ another density matrix with infinite entropy, hence σ can not be continuous: let ϱ be given with normal form

$$
\varrho = \sum_{n=1}^{\infty} \lambda_n \theta_{\mathrm{e}_n, \mathrm{e}_n},
$$

with the (repeated according to multiplicity) eigenvalues λ_n of ϱ. We also consider the density matrix

$$
\tilde{\varrho} = \frac{1}{c} \sum_{n=1}^{\infty} \frac{1}{n(\ln(n))^2} \theta_{\mathrm{e}_n, \mathrm{e}_n},
$$

where $c = \sum_{n=1}^{\infty} \frac{1}{n(\ln(n))^2} < \infty$. Since this series converges we get indeed a density matrix. Now we compute the entropy of $\tilde{\varrho}$ which is

$$
\sigma(\tilde{\varrho}) = -\frac{1}{c} \sum_{n=1}^{\infty} \frac{1}{n(\ln(n))^2} \ln\left(\frac{1}{cn(\ln(n))^2}\right) = \frac{1}{c} \sum_{n=1}^{\infty} \frac{\ln c + \ln n + 2\ln(\ln(n))}{n(\ln(n))^2} = +\infty, \tag{$**$}
$$

since already the series $\sum_{n=1}^{\infty} \frac{1}{n\ln(n)} = +\infty$ diverges. Now we construct a density matrix ϱ_N close to ϱ with $\sigma(\varrho_N) = +\infty$ for all N by defining

$$\varrho_N = \frac{1}{c_N}\left(\sum_{n=1}^{N} \lambda_n \theta_{\mathrm{e}_n,\mathrm{e}_n} + \sum_{n=N+1}^{\infty} \frac{1}{n(\ln(n))^2}\theta_{\mathrm{e}_n,\mathrm{e}_n}\right),$$

where

$$c_N = \sum_{n=1}^{N} \lambda_n + \sum_{n=N+1}^{\infty} \frac{1}{n(\ln(n))^2} < \infty$$

is the normalization such that $\operatorname{tr}(\varrho_N) = 1$. Since the divergence in $\sigma(\varrho_N)$ is determined by the tail of the series, we can ignore the first N terms and use the argument $(**)$ also for ϱ_N to show $\sigma(\varrho_N) = +\infty$ for all N. Now

$$\varrho_N - \varrho = \sum_{n=1}^{N}\left(\frac{1}{c_N}\lambda_n - \lambda_n\right)\theta_{\mathrm{e}_n,\mathrm{e}_n} + \sum_{n=N+1}^{\infty}\left(\frac{1}{c_N}(\frac{1}{n\ln(n)^2} - \lambda_n\right)\theta_{\mathrm{e}_n,\mathrm{e}_n},$$

and thus

$$\operatorname{tr}(|\varrho_n - \varrho|) = \left|\frac{1}{c_N} - 1\right| \sum_{n=1}^{N} \lambda_n + \sum_{n=N+1}^{\infty} \left|\frac{1}{c_N}\frac{1}{n\ln(n)^2} - \lambda_n\right|.$$

We claim that this converges to zero for $N \longrightarrow \infty$. Indeed, the first contribution converges to zero as $\sum_{n=1}^{N} \lambda_n \longrightarrow 1$ and $c_N \longrightarrow 1$ as well. For the second contribution, we note that,

$$\sum_{n=N+1}^{\infty} \left|\frac{1}{c_N}\frac{1}{n\ln(n)^2} - \lambda_n\right| \le \frac{1}{c_N} \sum_{n=N+1}^{\infty} \frac{1}{n\ln(n)^2} + \sum_{n=N+1}^{\infty} \lambda_n,$$

and since the numbers $\frac{1}{c_N}$ are bounded and both ϱ and $\tilde{\varrho}$ are trace class, also this contribution converges to 0. Thus we conclude $\|\varrho - \varrho_N\|_1 \longrightarrow 0$ for $N \longrightarrow \infty$, showing the third part. Since σ is lower semicontinuous, the subset of density matrices with $\sigma(\varrho) \le n$ is closed. Since by the third part it can not contain an open subset of density matrices, it is nowhere dense. ☐

Remark 6.2.23 (Entropy) The entropy is, despite its unpleasant discontinuity in infinite dimensions, one of the major tools in the understanding and formulation of statistical quantum physics, quantum information theory, and also in entanglement theory. Beside the above features it enjoys many interesting convexity properties and exhibits good behaviour with respect to tensor product and hence compound systems. The last part of the theorem can be understood, in the light of Baire's Theorem, as a statement that the set of density matrices with finite entropy is actually very small: being a countable union of closed nowhere dense subsets it is a meager subset of the Banach space of trace class operators. Note that while the inequalities involving the entropy are very useful, the numerical value of $\sigma(\varrho)$ for a given ϱ has only limited physical significance due to the discontinuity properties. For further properties of σ and σ_α we refer again to the textbook [57, Part. II, Sect. 2.2] as well as to the exercises.

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6.3 Dualities and Normal States

In analogy to the sequence spaces ℓ^p, $c_\circ$ and ℓ^∞ we shall now describe the dualities between the various Schatten classes, the compact operators, and all bounded operators. This way we will arrive at the normal states of $\mathfrak{B}(\mathfrak{H})$ which are the physically most relevant ones.

6.3.1 Dualities between the Schatten Classes

As warming up we start by re-investigating Example 6.1.12 where we considered the map

$$\Theta\colon \mathfrak{H} \hat{\otimes} \overline{\mathfrak{H}} \longrightarrow \mathfrak{K}(\mathfrak{H}), \tag{6.3.1}$$

sending elementary tensors $\phi \otimes \psi$ to the rank one operator $\Theta_{\phi,\psi}$. We want to give now a precise interpretation of its image. First we note that the trace of a finite rank operator can be expressed explicitly as

$$\operatorname{tr} \sum_{j=1}^{N} z_j \Theta_{\phi_j,\psi_j} = \sum_{j=1}^{N} z_j \sum_{i\in I} \langle \mathrm{e}_i, \phi_j\rangle \langle \psi_j, \mathrm{e}_i\rangle = \sum_{j=1}^{N} z_j \langle \psi_j, \phi_j\rangle \tag{6.3.2}$$

by Theorem 3.3.7, *v.)*, using a Hilbert basis $\{\mathrm{e}_i\}_{i\in I}$. This will give us the following statement:

Proposition 6.3.1 (Hilbert-Schmidt operators) *Let $\mathfrak{H}$ be a Hilbert space.*

i.) The Hilbert-Schmidt operators $\mathfrak{L}^2(\mathfrak{H})$ become canonically a Hilbert space via the scalar product

$$\langle A, B\rangle_{\mathrm{HS}} = \operatorname{tr}(A^* B) \tag{6.3.3}$$

for $A, B \in \mathfrak{L}^2(\mathfrak{H})$ such that the Hilbert-Schmidt norm is the Hilbert space norm from $\langle \,\cdot\,, \,\cdot\,\rangle_{\mathrm{HS}}$.

ii.) The canonical map Θ is an unitary isomorphism

$$\Theta\colon \mathfrak{H} \hat{\otimes} \overline{\mathfrak{H}} \longrightarrow \mathfrak{L}^2(\mathfrak{H}). \tag{6.3.4}$$

PROOF: First we note that $A^*B \in \mathfrak{L}^1(\mathfrak{H})$ for $A, B \in \mathfrak{L}^2(\mathfrak{H})$ by Theorem 6.2.10, *iv.)*. Hence $\langle \,\cdot\,, \,\cdot\,\rangle_{\mathrm{HS}}$ is well-defined. Clearly, $\langle \,\cdot\,, \,\cdot\,\rangle_{\mathrm{HS}}$ is linear in the second argument, enjoys $\langle A, B\rangle_{\mathrm{HS}} = \overline{\langle B, A\rangle_{\mathrm{HS}}}$, and satisfies $\langle A, A\rangle_{\mathrm{HS}} = \operatorname{tr}(A^*A) = \|A\|_2^2 > 0$. Hence $\mathfrak{L}^2(\mathfrak{H})$ indeed becomes a pre-Hilbert space with respect to $\langle \,\cdot\,, \,\cdot\,\rangle_{\mathrm{HS}}$ such that $\|\cdot\|_2$ is the corresponding norm. Since we already know by Theorem 6.2.10, *ii.)*, that all the Schatten classes are Banach spaces, $\|\cdot\|_2$ is complete and hence $\mathfrak{L}^2(\mathfrak{H})$ is a Hilbert space as claimed. Now for $\phi, \psi, \chi, \eta \in \mathfrak{H}$ we have

$$\Theta_{\phi,\psi}\Theta_{\chi,\eta} = \langle \psi, \chi\rangle \Theta_{\phi,\eta}$$

and hence

$$\langle \Theta_{\phi,\psi}, \Theta_{\chi,\eta}\rangle_{\mathrm{HS}} = \operatorname{tr}(\Theta_{\phi,\psi}^*\Theta_{\chi,\eta}) = \operatorname{tr}(\Theta_{\psi,\phi}\Theta_{\chi,\eta}) = \operatorname{tr}(\langle \phi, \chi\rangle \Theta_{\psi,\eta}) = \langle \phi, \chi\rangle \overline{\langle \psi, \eta\rangle} = \langle \phi \otimes \overline{\psi}, \chi \otimes \overline{\eta}\rangle_{\mathfrak{H}\otimes\overline{\mathfrak{H}}}.$$

Thus on $\mathfrak{H} \otimes \overline{\mathfrak{H}}$ the map is isometric and hence also on $\mathfrak{H} \hat{\otimes} \overline{\mathfrak{H}}$. Since the image of $\mathfrak{H} \otimes \overline{\mathfrak{H}}$ are precisely the finite rank operators and completion of $\mathfrak{H} \otimes \overline{\mathfrak{H}}$ gives $\mathfrak{H} \hat{\otimes} \overline{\mathfrak{H}}$, the image of the isometric map Θ is the completion of $\mathfrak{F}(\mathfrak{H})$ in the norm of $\langle \,\cdot\,, \,\cdot\,\rangle_{\mathrm{HS}}$, i.e. the Hilbert-Schmidt norm. By Theorem 6.2.10, *iii.)*, this is $\mathfrak{L}^2(\mathfrak{H})$. Hence Θ is even unitary. □

In particular, if $\dim \mathfrak{H} = \infty$ then there are compact operators on $\mathfrak{H}$ which are not Hilbert-Schmidt. Thus the map Θ from Example 6.1.12 is *not* surjective onto $\mathfrak{K}(\mathfrak{H})$ and the integral operators in Example 6.1.11 are even Hilbert-Schmidt with Hilbert-Schmidt norm given by the L^2-norm of the L^2-kernel function.

n: Exercises?

We can now use the sesquilinear pairing induced by the trace also for other Schatten classes. We choose here the version without operator adjoint in order to get a *linear* map into the duals. Then we have the following result:

Theorem 6.3.2 (Dual pairing between Schatten classes) *Let $1 < p < +\infty$ and let q be the conjugate index. Then the map*

$$\ell\colon \mathfrak{L}^p(\mathfrak{H}) \ni A \;\mapsto\; \ell_A \in \mathfrak{L}^p(\mathfrak{H})' \tag{6.3.5}$$

with $\ell_A(B) = \operatorname{tr}(AB)$ is an isometric isomorphism.

PROOF: First we check that ℓ_A is indeed a continuous linear functional on $\mathfrak{L}^p(\mathfrak{H})$. By Theorem 6.2.10, *iv.)*, and the estimate (6.2.18) as well as by the estimate (6.2.25) from Theorem 6.2.14 we get

$$|\ell_A(B)| = |\mathrm{tr}(AB)| \leq \|AB\|_1 \leq \|A\|_p \|B\|_q,$$

and hence ℓ_A is continuous with $\|\ell_A\| \leq \|A\|_p$. We check now that the norm is in fact equal to $\|A\|_p$. Thus let A be given in normal form

$$A = \sum_{n=0}^{\infty} s_n(A) \Theta_{\mathrm{f}_n, \mathrm{e}_n}, \tag{$*$}$$

and define

$$B = \sum_{n=0}^{\infty} s_n(A)^{p-1} \Theta_{\mathrm{e}_n, \mathrm{f}_n}. \tag{$**$}$$

We have

$$\big\|(s_n(A)^{p-1})\big\|_q^q = \sum_{n=0}^{\infty} s_n(A)^{(p-1)q} = \sum_{n=0}^{\infty} s_n(A)^p = \big\|(s_n(A))\big\|_p^p,$$

and hence $B \in \mathfrak{L}^p(\mathfrak{H})$ with $\|B\|_q^q = \|A\|_p^p$. For this B we explicitly compute

$$\begin{aligned}
\mathrm{tr}(AB) &= \mathrm{tr}\left(\sum_{n=0}^{\infty} s_n(A)\Theta_{\mathrm{f}_n,\mathrm{e}_n} \sum_{m=0}^{\infty} s_m(A)^{p-1}\Theta_{\mathrm{e}_m,\mathrm{f}_m}\right) \\
&\overset{(a)}{=} \mathrm{tr}\left(\sum_{n=0}^{\infty} s_n(A)s_n(A)^{p-1}\Theta_{\mathrm{f}_n,\mathrm{f}_n}\right) \\
&= \sum_{n=0}^{\infty} s_n(A)^p \\
&= \|A\|_p^p \\
&= \|A\|_p \|B\|_q,
\end{aligned}$$

where in (a) we used the fact that the operator product is continuous and the series $(*)$ and $(**)$ are norm convergent. This shows that $\|\ell_A\| = \|A\|_p$. It remains to check the surjectivity of the map ℓ. For a rank one operator $\Theta_{\phi,\psi} \in \mathfrak{F}(\mathfrak{H})$ we know $\|\Theta_{\phi,\psi}\|_q = \|\phi\|\|\psi\|$ for all $q \geq 1$, see Exercise 6.4.7. Now fix a continuous linear functional $a \in \mathfrak{L}^p(\mathfrak{H})'$ then this gives us by the continuity of a the estimate

$$|a(\Theta_{\phi,\psi})| \leq \|a\|\|\phi\|\|\psi\|.$$

This shows that the sesquilinear map

$$\mathfrak{H} \times \mathfrak{H} \ni (\phi, \psi) \mapsto a(\Theta_{\phi,\psi}) \in \mathbb{C}$$

is continuous. By the Lax-Milgram Theorem, see Exercise 3.6.20, this gives us an operator $A \in \mathfrak{B}(\mathfrak{H})$ such that

$$a(\Theta_{\phi,\psi}) = \langle \psi, A\phi \rangle = \mathrm{tr}(A\Theta_{\phi,\psi}).$$

By linearity we have then for all $B \in \mathfrak{F}(\mathfrak{H})$

$$a(B) = \mathrm{tr}(AB). \tag{$\star$}$$

To conclude equality on all of $\mathfrak{L}^p(\mathfrak{H})$ we have to show that the right hand side is continuous in the q-th Schatten norm. Thus let $(\lambda_n)_{n\in\mathbb{N}_0} \in \ell^q$ be arbitrary and $B = \sum_{n=0}^{\infty} \lambda_n \Theta_{\mathrm{f}_n,\mathrm{e}_n}$. Then

$$a(B) = a\left(\sum_{n=0}^{\infty} \lambda_n \Theta_{\mathrm{f}_n,\mathrm{e}_n}\right) = \sum_{n=0}^{\infty} \lambda_n a(\Theta_{\mathrm{f}_n,\mathrm{e}_n}) = \sum_{n=0}^{\infty} \lambda_n \langle \mathrm{e}_n, A\mathrm{f}_n\rangle,$$

since a is continuous with respect to $\|\cdot\|_q$ and the series converges (unconditionally) in the Schatten class $\mathfrak{L}^p(\mathfrak{H})$. Moreover, we have, up to reordering the λ_n's in decreasing order, $s_n(B) = |\lambda_n|$ and hence by the continuity of a

$$\left|\sum_{n=0}^{\infty} \lambda_n \langle \mathrm{e}_n, A\mathrm{f}_n\rangle\right| \le \|a\| \big\|(\lambda_n)_{n\in\mathbb{N}_0}\big\|_q,$$

since $\|B\|_q = \|(\lambda_n)_{n\in\mathbb{N}_0}\|_q$ by Theorem 6.2.9, *i.)*. This shows that the linear map

$$\ell^q \ni (\lambda_n)_{n\in\mathbb{N}_0} \;\mapsto\; \sum_{n=0}^{\infty} \lambda_n \langle \mathrm{e}_n, \mathrm{f}_n\rangle \in \mathbb{C}$$

is a continuous linear functional on ℓ^q. Hence, by the duality of the sequence space $(\ell^q)' = \ell^p$ according to the general statement of Theorem C.3.55 or Exercise 2.5.41 for a more elementary proof in the present case of sequence spaces, we see that the sequence $(\langle \mathrm{e}_n, A\mathrm{f}_n\rangle)_{n\in\mathbb{N}_0}$ is an element in ℓ^p with ℓ^p-norm bounded by $\|a\|$, independently on the choice of the orthonormal systems. Applying again Theorem 6.2.9, *i.)*, shows $A \in \mathfrak{L}^p(\mathfrak{H})$ with $\|A\|_p \le \|a\|$. But then $a = \ell_A$ follows from $(\star)$ and continuity. □

Remark 6.3.3 Here we see again the close analogy of (commutative) $\mathscr{L}^p$-spaces over measure spaces and the (non-commutative) Schatten ideals $\mathfrak{L}^p(\mathfrak{H})$.

Corollary 6.3.4 *Let $1 < p < +\infty$. Then $\mathfrak{L}^p(\mathfrak{H})$ is reflexive.*

6.3.2 Normal States of $\mathfrak{B}(\mathfrak{H})$

For $p = 1$, i.e. for trace class operators, the dualities are slightly more complicated: from the analogy with $\mathscr{L}^p$-spaces we expect this, see also the discussion in Example C.3.57 and Exercise 2.5.41. Here we have the following two dualities, where again $\ell_A(B) = \mathrm{tr}(AB)$ whenever the trace exists.

Theorem 6.3.5 (Duals of $\mathfrak{K}(\mathfrak{H})$ and $\mathfrak{L}^1(\mathfrak{H})$) *The map*

$$\ell\colon \mathfrak{L}^1(\mathfrak{H}) \ni A \;\mapsto\; \ell_A \in \mathfrak{K}(\mathfrak{H})' \tag{6.3.6}$$

as well as the map

$$\ell\colon \mathfrak{B}(\mathfrak{H}) \ni A \;\mapsto\; \ell_A \in \mathfrak{L}^1(\mathfrak{H})' \tag{6.3.7}$$

are isometric isomorphisms.

Proof: We proceed very much along the lines of the proof of Theorem 6.3.2. For a trace class operator A we have $AB \in \mathfrak{L}^1(\mathfrak{H})$ for all $B \in \mathfrak{K}(\mathfrak{H})$ and

$$|\mathrm{tr}(AB)| \le \|AB\|_1 \le \|A\|_1 \|B\|,$$

showing $\ell_A \in \mathfrak{K}(\mathfrak{H})'$ with $\|\ell_A\| \le \|A\|_1$. The same estimate shows that $A \in \mathfrak{B}(\mathfrak{H})$ gives $\ell_A \in \mathfrak{L}^1(\mathfrak{H})'$ with $\|\ell_A\| \le \|A\|$. Now let $A = \sum_{n=0}^{\infty} s_n(A)\Theta_{\mathrm{f}_n,\mathrm{e}_n} \in \mathfrak{L}^1(\mathfrak{H})$ be given in its normal form and let $B_N = \sum_{n=0}^{N} \Theta_{\mathrm{f}_n,\mathrm{e}_n} \in \mathfrak{F}(\mathfrak{H}) \subseteq \mathfrak{K}(\mathfrak{H})$. Since the $\{\mathrm{e}_n\}_{n\in\mathbb{N}_0}$ as well as the $\{\mathrm{f}\}_{n\in\mathbb{N}_0}$ are orthonormal, we see that B_N is a unitary map

$$B_N\colon \mathrm{span}_{\mathbb{C}}\{\mathrm{e}_n\}_{n=0,\ldots,N} \longrightarrow \mathrm{span}_{\mathbb{C}}\{\mathrm{f}_n\}_{n=0,\ldots,N},$$

and zero only on the orthogonal complement. Hence $\|B_N\| = 1$. Moreover,

$$\mathrm{tr}(AB_N) = \mathrm{tr}\left(\sum_{n=0}^{N} s_n(A)\Theta_{\mathrm{f}_n,\mathrm{f}_n}\right) = \sum_{n=0}^{N} s_n(A) \longrightarrow \|A\|_1$$

for $N \longrightarrow \infty$ shows that $\|\ell_A\| = \|A\|_1$. For the other case of $A \in \mathfrak{B}(\mathfrak{H})$ we consider vectors $\phi_n, \psi_n \in \mathfrak{H}$ with $\|\phi_n\| = 1 = \|\psi_n\|$ and $\langle \psi_n, A\phi_n \rangle \longrightarrow \|A\|$, which is clearly possible by Exercise 3.6.18, *ii.)*. Then $\Theta_{\phi_n,\psi_n} \in \mathfrak{F}(\mathfrak{H})$ is trace class with trace norm given by $\|\Theta_{\phi_n,\psi_n}\| = \|\phi_n\|\|\psi_n\| = 1$ for all $n \in \mathbb{N}$. Since

$$\operatorname{tr}(A\Theta_{\phi_n,\psi_n}) = \langle \psi_n, A\phi_n \rangle \longrightarrow \|A\|,$$

we get $\|\ell_A\| = \|A\|$ also in this case. Thus both maps (6.3.6) and (6.3.7) are isometric. It remains to check the surjectivity. Since $\Theta_{\phi,\psi} \in \mathfrak{F}(\mathfrak{H})$ is trace class and compact with $\|\Theta_{\phi,\psi}\| = \|\phi\|\|\psi\| = \|\Theta_{\phi,\psi}\|_1$, we get for $a \in \mathfrak{K}(\mathfrak{H})'$ an operator $A \in \mathfrak{B}(\mathfrak{H})$ with

$$a(\Theta_{\phi,\psi}) = \langle \psi, A\phi \rangle$$

as well as for $a \in \mathfrak{L}^1(\mathfrak{H})'$, by the Lax-Milgram Theorem as in the proof of Theorem 6.3.2. Thus the surjectivity of the second version follows right away by continuity. For $a \in \mathfrak{K}(\mathfrak{H})'$ we consider a zero sequence $(\lambda_n)_{n\in\mathbb{N}_0}$ and orthonormal systems $\{\mathrm{e}_n\}_{n\in\mathbb{N}_0}$, $\{\mathrm{f}_n\}_{n\in\mathbb{N}_0}$ with the operator

$$B = \sum_{n=0}^{\infty} \lambda_n \Theta_{\mathrm{f}_n,\mathrm{e}_n} \in \mathfrak{K}(\mathfrak{H}), \tag{$*$}$$

where we know that the series converges in the operator norm. Moreover, it is easy to see that the singular values of B are given by the absolute values of the λ_n's after rearranging them in decreasing order. Finally, we note that $(*)$ converges unconditionally. Now by the continuity of a

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$$a(B) = \sum_{n=0}^{\infty} \lambda_n a_n(\Theta_{\mathrm{f}_n,\mathrm{e}_n}) = \sum_{n=0}^{\infty} \lambda_n \langle \mathrm{e}_n, A\mathrm{f}_n \rangle \tag{$\star$}$$

is unconditionally convergent. Moreover, since $\|B\|$ is given by $\max_{n\in\mathbb{N}_0}|\lambda_n|$ we see that the right hand side can be estimated by $|a(B)| \le \|a\| \max_{n\in\mathbb{N}_0}|\lambda_n|$. Thus it defines a continuous linear functional on the sequence space $c_\circ$ of zero sequences $(\lambda_n)_{n\in\mathbb{N}_0}$. By the duality $(c_\circ)' = \ell^1$ this implies $(\langle \mathrm{e}_n, A\mathrm{f}_n \rangle)_{n\in\mathbb{N}_0} \in \ell^1$. Hence Theorem 6.2.9, *i.)*, shows that $A \in \mathfrak{L}^1(\mathfrak{H})$ as claimed. Hence also the first map (6.3.6) is surjective. □

Corollary 6.3.6 *The Banach space $\mathfrak{K}(\mathfrak{H})$ is not reflexive if $\dim \mathfrak{H} = \infty$.*

Proof: By the theorem we have $\mathfrak{K}(\mathfrak{H})'' = \mathfrak{L}^1(\mathfrak{H})' = \mathfrak{B}(\mathfrak{H})$ which is strictly bigger than $\mathfrak{K}(\mathfrak{H})$ in infinite dimensions. □

Also $\mathfrak{L}^1(\mathfrak{H})$ is not reflexive. We formulate this in a slightly more specific way:

Proposition 6.3.7 *Let $\dim \mathfrak{H} = \infty$. Then there are states on $\mathfrak{B}(\mathfrak{H})$ which are not of the form ω_ϱ with a density matrix $\varrho \in \mathfrak{L}^1(\mathfrak{H})$.*

Proof: We consider the Calkin algebra $\mathfrak{C}(\mathfrak{H}) = \mathfrak{B}(\mathfrak{H}) / \mathfrak{K}(\mathfrak{H})$ which is a unital C^*-algebra by Proposition 6.1.36. Moreover, the quotient map $\pi\colon \mathfrak{B}(\mathfrak{H}) \longrightarrow \mathfrak{C}(\mathfrak{H})$ is a unital *-homomorphism. According to Theorem 4.4.21 we have states $\Omega\colon \mathfrak{C}(\mathfrak{H}) \longrightarrow \mathbb{C}$. But then $\omega = \Omega \circ \pi\colon \mathfrak{B}(\mathfrak{H}) \longrightarrow \mathbb{C}$ is again a state with $\omega(K) = 0$ for all compact operators K. Thus ω can not be of the form ω_ϱ. □

It follows that $\mathfrak{L}^1(\mathfrak{H})'' \cong \mathfrak{B}(\mathfrak{H})'$ is strictly bigger than $\mathfrak{L}^1(\mathfrak{H})$ and hence $\mathfrak{L}^1(\mathfrak{H})$ is not reflexive, too. Since $\mathfrak{B}(\mathfrak{H})$ is now shown to be a dual space we can apply all the results from Section 2.3.3. In particular, we have a weak* topology on $\mathfrak{B}(\mathfrak{H}) \cong \mathfrak{L}^1(\mathfrak{H})'$. This topology can be determined explicitly:

Theorem 6.3.8 (Weak* topology of $\mathfrak{B}(\mathfrak{H})$) *Under the identification of $\mathfrak{B}(\mathfrak{H})$ with $\mathfrak{L}^1(\mathfrak{H})'$, the weak* topology of $\mathfrak{L}^1(\mathfrak{H})'$ corresponds to the σ-weak topology of $\mathfrak{B}(\mathfrak{H})$.*

PROOF: We show that the seminorm systems of the σ-weak topology according to Definition 5.1.7 and the weak* topology coincide. The latter is given by all seminorms of the form $\|A\|_\varrho = |\operatorname{tr}(A\varrho)|$ with $\varrho \in \mathfrak{L}^1(\mathfrak{H})$. Indeed, let $\varrho \in \mathfrak{L}^1(\mathfrak{H})$ be given in normal form as usual. Then

$$\varrho = \sum_{n=0}^{\infty} s_n(\varrho)\Theta_{\mathrm{f}_n,\mathrm{e}_n} = \sum_{n=0}^{\infty} \Theta_{\sqrt{s_n(\varrho)}\mathrm{f}_n,\sqrt{s_n(\varrho)}\mathrm{e}_n}, \tag{$*$}$$

and $\phi_n = \sqrt{s_n(\varrho)}\mathrm{f}_n$ as well as $\psi_n = \sqrt{s_n(\varrho)}\mathrm{e}_n$ satisfy

$$\sum_{n=0}^{\infty}\|\phi_n\|^2 = \sum_{n=0}^{\infty}\|\psi_n\|^2 = \sum_{n=0}^{\infty} s_n(\varrho) = \|\varrho\|_1 < \infty.$$

Since the normal form $(*)$ converges in the trace norm we have

$$\operatorname{tr}(A\varrho) = \sum_{n=0}^{\infty}\operatorname{tr}(s_n(\varrho)A\Theta_{\mathrm{f}_n,\mathrm{e}_n}) = \sum_{n=0}^{\infty} s_n(\varrho)\langle \mathrm{e}_n, A\mathrm{f}_n\rangle = \sum_{n=0}^{\infty}\langle\psi_n, A\phi_n\rangle,$$

and hence $|\operatorname{tr}(A\varrho)| = \|A\|_{\{\psi_n\}_{n\in\mathbb{N}_0},\{\phi_n\}_{n\in\mathbb{N}_0}}$ given as in Definition 5.1.7. This shows that every defining seminorm of the weak* topology is contained in the defining system of seminorms of the σ-weak topology. Conversely, let $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ be sequences of vectors in $\mathfrak{H}$ such that

$$\sum_{n=0}^{\infty}\|\phi_n\|^2 \quad\text{and}\quad \sum_{n=0}^{\infty}\|\psi_n\|^2 < \infty.$$

Then we claim that $\varrho = \sum_{n=0}^{\infty}\Theta_{\psi_n,\phi_n}$ converges in the trace norm. Indeed,

$$\left\|\sum_{n=0}^{\infty}\Theta_{\psi_n,\phi_n}\right\|_1 \le \sum_{n=0}^{\infty}\|\Theta_{\psi_n,\phi_n}\|_1 = \sum_{n=0}^{\infty}\|\psi_n\|\|\phi_n\| \le \sqrt{\sum_{n=0}^{\infty}\|\phi_n\|^2}\sqrt{\sum_{n=0}^{\infty}\|\psi_n\|^2}$$

shows even absolute convergence. Thus $\varrho \in \mathfrak{L}^1(\mathfrak{H})$ and

$$|\operatorname{tr}(\varrho A)| = \left|\sum_{n=0}^{\infty}\langle\phi_n, A\psi_n\rangle\right| = \|A\|_{\{\phi_n\},\{\psi_n\}}$$

follows at once. This shows that the defining system of seminorms simply coincide. □

One of the first consequences is that we can apply the Banach-Alaoglu theorem to the σ-weak topology:

Corollary 6.3.9 *The closed unit ball* $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq \mathfrak{B}(\mathfrak{H})$ *is compact in the* σ*-weak and in the weak topology.*

PROOF: The compactness in the σ-weak, i.e. the weak* topology of $\mathfrak{B}(\mathfrak{H}) = \mathfrak{L}^1(\mathfrak{H})'$ follows from the Banach-Alaoglu Theorem in the form of Corollary 2.3.34. The compactness in the weak topology follows from Theorem 5.1.10, *ii.)*: the coarser topology has at least as many compact sets as the finer. □

Remark 6.3.10 Note, however, that $\mathfrak{B}(\mathfrak{H})$ is not separable in the case $\dim\mathfrak{H} = \infty$ and hence the nice features of separable Banach spaces can not be applied: to see this we fix a countable orthonormal system $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ and define for $I \subseteq \mathbb{N}$ a partial isometry by

$$U_I\mathrm{e}_n = \begin{cases}\mathrm{e}_n & n \in I\\ 0 & n \notin I\end{cases} \tag{6.3.8}$$

with the usual extension to all of $\mathfrak{H}$. Then $\|U_I - U_J\| = \delta_{IJ}$. Since we can do this for every subset of $\mathbb{N}$ we have $\#2^{\mathbb{N}} = \#\mathbb{R}$ many elements with norm distance 1 in $\mathfrak{B}(\mathfrak{H})$. Thus $\mathfrak{B}(\mathfrak{H})$ can not be separable.

heck: (Hence $\subseteq \mathfrak{B}(\mathfrak{H})$ is not lly compact.)

We concluded this subsection now with a characterization of those linear functionals on $\mathfrak{B}(\mathfrak{H})$ which are in the image of the map $\ell\colon \mathfrak{L}^1(\mathfrak{H}) \longrightarrow \mathfrak{B}(\mathfrak{H})$:

Theorem 6.3.11 (σ-Weakly continuous functionals on $\mathfrak{B}(\mathfrak{H})$) *Let $\omega\colon \mathfrak{B}(\mathfrak{H}) \longrightarrow \mathbb{C}$ be a linear functional. Then the following statements are equivalent:*

i.) The functional ω is σ-weakly continuous

ii.) The functional ω is σ-strongly continuous.

iii.) The functional ω is σ-strongly* continuous.

iv.) There exists a unique trace class operator $\varrho \in \mathfrak{L}^1(\mathfrak{H})$ with $\omega(A) = \operatorname{tr}(\varrho A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$.

PROOF: According to the comparison of the topologies in Theorem 5.1.10, *ii.)*, the implications *i.)* $\Longrightarrow$ *ii.)* $\Longrightarrow$ *iii.)* are clear. The equivalence of *i.)* and *iv.)* is clear by the fact that the σ-weak topology coincides with the weak* topology: a functional on a dual Banach space is weak* continuous iff it is of the form $i(v) \in V''$ with $v \in V$, see Theorem 2.3.25. Thus it remains to show *iii.)* $\Longrightarrow$ *iv.)*. Let $\omega\colon \mathfrak{B}(\mathfrak{H}) \longrightarrow \mathbb{C}$ be σ-strongly* continuous. Hence we have a seminorm $\|\cdot\|_{(\phi_n)_{n\in\mathbb{N}}}$ of the σ-strong* topology, where $\phi_n \in \mathfrak{H}$ are such that $\sum_{n=1}^\infty \|\phi_n\|^2 < \infty$, with the estimate

$$|\omega(A)| \le \|A\|^*_{(\phi_n)_{n\in\mathbb{N}}} = \sqrt{\sum_{n=1}^\infty \|A\phi_n\|^2 + \sum_{n=1}^\infty \|A^*\phi_n\|^2}. \tag{$*$}$$

We consider now the following Hilbert space

$$\widetilde{\mathfrak{H}} = \hat{\bigoplus_{n=1}^{\infty}} \mathfrak{H}_n \mathbin{\hat{\oplus}} \hat{\bigoplus_{n=1}^{\infty}} \overline{\mathfrak{H}}_n$$

with $\mathfrak{H}_n = \mathfrak{H}$, where $\overline{\mathfrak{H}}$ denotes the complex conjugate Hilbert space as usual. The elements of $\widetilde{\mathfrak{H}}$ can thus be written as pairs of sequences $(\chi_n, \overline{\psi_n})_{n\in\mathbb{N}}$ with $\chi_n, \psi_n \in \mathfrak{H}$ such that

$$\sum_{n=1}^\infty \|\chi_n\|^2 + \sum_{n=1}^\infty \|\psi_n\|^2 < \infty.$$

For $A \in \mathfrak{B}(\mathfrak{H})$ we define now an operator $\widetilde{A} \in \mathfrak{B}(\widetilde{\mathfrak{H}})$ by

$$\widetilde{A}(\chi_n, \overline{\psi_n})_{n\in\mathbb{N}} = (A\chi_n, \overline{A^*\psi_n})_{n\in\mathbb{N}}.$$

It is now easy to see that

$$\begin{aligned}
\big\|\widetilde{A}(\chi_n, \overline{\psi_n})_{n\in\mathbb{N}}\big\|^2_{\widetilde{\mathfrak{H}}} &= \sum_{n=1}^\infty \|A\chi_n\|^2 + \sum_{n=1}^\infty \|\overline{A^*\psi_n}\|^2 \\
&= \sum_{n=1}^\infty \|A\chi_n\|^2 + \sum_{n=1}^\infty \|A^*\psi_n\|^2 \\
&\le \|A\| \left(\sum_{n=1}^\infty \|\chi_n\|^2 + \sum_{n=1}^\infty \|\psi_n\|^2\right) \\
&= \|A\| \big\|(\chi_n, \overline{\psi_n})_{n\in\mathbb{N}}\big\|^2_{\widetilde{\mathfrak{H}}},
\end{aligned}$$

since $\|A^*\| = \|A\|$. The reverse estimate being trivial we get $\|\widetilde{A}\| = \|A\|$ and hence indeed $\widetilde{A} \in \mathfrak{B}(\widetilde{\mathfrak{H}})$. Note that $A \mapsto \widetilde{A}$ is a linear map. Now let ϕ_n be as in $(*)$ and form the vector $\Phi = (\phi_n, \overline{\phi_n})_{n\in\mathbb{N}} \in \widetilde{\mathfrak{H}}$. Moreover, let

$$U = \left\{ \widetilde{A}\Phi \;\middle|\; A \in \mathfrak{B}(\mathfrak{H}) \right\} \subseteq \widetilde{\mathfrak{H}}$$

be the subspace generated by applying all the operators of the form $\widetilde{A}$ to this vector Φ. Then we have

$$\|\widetilde{A}\Phi\|^2_{\widetilde{\mathfrak{H}}} = \sum_{n=1}^{\infty} \|A\phi_n\|^2 + \sum_{n=1}^{\infty} \|A^*\phi_n\|^2 = \left(\|A\|^*_{(\phi_n)_{n\in\mathbb{N}}} \right)^2,$$

and hence $|\omega(A)| \leq \left\|\widetilde{A}\Phi\right\|^2_{\widetilde{\mathfrak{H}}}$. This shows that

$$\Omega\colon U \ni \widetilde{A}\Phi \;\mapsto\; \omega(A) \in \mathbb{C}$$

is a well-defined linear continuous functional on the subspace $U \subseteq \widetilde{\mathfrak{H}}$. Hence Ω extends to the completion $U^{\mathrm{cl}} \subseteq \widetilde{\mathfrak{H}}$ in a unique way. By Riesz' Theorem 3.2.11 we find a vector $\Psi \in U^{\mathrm{cl}}$ such that $\Omega(\widetilde{A}\Phi) = \langle \Psi, \widetilde{A}\Phi \rangle$ since U^{cl} is a Hilbert space for its own. Let $\Psi = (\chi_n, \overline{\psi_n})_{n\in\mathbb{N}}$ then we compute

$$\begin{aligned}
\omega(A) &= \sum_{n=1}^{\infty} \langle \chi_n, A\phi_n \rangle_{\mathfrak{H}} + \sum_{n=1}^{\infty} \langle \overline{\psi_n}, \overline{A^*\phi_n} \rangle_{\overline{\mathfrak{H}}} \\
&= \sum_{n=1}^{\infty} \langle \chi_n, A\phi_n \rangle_{\mathfrak{H}} + \sum_{n=1}^{\infty} \overline{\langle \psi_n, A^*\phi_n \rangle_{\mathfrak{H}}} \\
&= \sum_{n=1}^{\infty} (\langle \chi_n, A\phi_n \rangle_{\mathfrak{H}} + \langle \phi_n, A^*\psi_n \rangle_{\mathfrak{H}}).
\end{aligned}$$

Since $\Psi \in \widetilde{\mathfrak{H}}$ we have

$$\|\Psi\|^2_{\widetilde{\mathfrak{H}}} = \sum_{n=1}^{\infty} \|\chi_n\|^2_{\mathfrak{H}} + \sum_{n=1}^{\infty} \|\psi_n\|^2_{\mathfrak{H}} < \infty.$$

This allows to define the operator

$$\varrho = \sum_{n=1}^{\infty} \Theta_{\phi_n,\psi_n} + \sum_{n=1}^{\infty} \Theta_{\chi_n,\phi_n}$$

as a convergent series in the trace norm $\|\cdot\|_1$. Hence $\varrho \in \mathfrak{L}^1(\mathfrak{H})$ and $\operatorname{tr}(\rho A) = \omega(A)$ follows easily. □

The states of $\mathfrak{B}(\mathfrak{H})$ satisfying the above four equivalent properties are the normal states:

Definition 6.3.12 (Normal states) *A state ω of $\mathfrak{B}(\mathfrak{H})$ is called normal if $\omega = \omega_\varrho$ with a density matrix $\varrho \in \mathfrak{L}^1(\mathfrak{H})$.*

As we have seen in Proposition 6.3.7 the C^*-algebra $\mathfrak{B}(\mathfrak{H})$ has states which are not of the form ω_ϱ, i.e. which are not normal, as soon as $\dim \mathfrak{H} = \infty$. The proof of that proposition gives also the following characterization of non-normal states:

Corollary 6.3.13 *Let $\omega\colon \mathfrak{B}(\mathfrak{H}) \longrightarrow \mathbb{C}$ be a state. Then the following statements are equivalent:*

i.) One has $\omega(K) = 0$ for all $K \in \mathfrak{K}(\mathfrak{H})$.

ii.) One has $\omega = \Omega \circ \pi$ with a state $\Omega\colon \mathfrak{C}(\mathfrak{H}) \longrightarrow \mathbb{C}$ on the Calkin algebra.

In this case, ω is not a normal state.

PROOF: Clearly, *i.)* and *ii.)* are equivalent. In this case, $\omega = \omega_\varrho$ would contradict *i.)* since $\omega_\varrho(\varrho) = \operatorname{tr}(\varrho^2) > 0$ for a density matrix $\varrho \in \mathfrak{L}^1(\mathfrak{H})$. $\square$

Corollary 6.3.14 *A state ω_ϱ of $\mathfrak{K}(\mathfrak{H})$ is pure iff $\varrho = P$ is projection.*

PROOF: First we note that a density matrix ρ which is a projection, $\rho^2 = \rho$, is necessarily a projection onto a *one-dimensional* subspace. Hence such a ω_ϱ is a vector state. Now assume ω_ϱ is pure but $\varrho^2 \neq \varrho$. In the normal form

$$\rho = \sum_{n=0}^{\infty} s_n(\varrho)\Theta_{\mathrm{e}_n,\mathrm{e}_n}$$

this can only happen if there are at least two of the singular values different from zero. But then $\varrho = s_0(\varrho)\Theta_{\mathrm{e}_0,\mathrm{e}_0} + \sum_{n=1}^{\infty} s_n(\varrho)\Theta_{\mathrm{e}_n,\mathrm{e}_n}$ is a non-trivial convex decomposition into the density matrix $\Theta_{\mathrm{e}_0,\mathrm{e}_0}$ and the density $\frac{1}{1-s_0(\varrho)}\sum_{n=0}^{\infty} s_n(\varrho)\Theta_{\mathrm{e}_n,\mathrm{e}_n}$ with non-trivial weights $s_0(\varrho)$ and $1 - s_0(\varrho)$. Thus ω_ϱ is not pure. Conversely, if $\varrho = P$ and $\varrho = \lambda\varrho' + (1-\lambda)\varrho''$ with density matrices $\varrho' \neq \varrho''$ and $\lambda \in (0,1)$ then

$$\varrho = \varrho^2 = \lambda^2(\varrho')^2 + \lambda(1-\lambda)(\varrho'\varrho'' + \varrho''\varrho') + (1-\lambda)(\varrho'')^2. \tag{$*$}$$

We have the following inequalities: if ϱ' is not a projection (onto a one-dimensional subspace) then

$$\operatorname{tr}(\varrho')^2 = \sum_{n=1}^{\infty} s_n(\varrho'^2) = \sum_{n=1}^{\infty} s_n(\varrho')^2 < \sum_{n=1}^{\infty} s_n(\varrho') = 1,$$

since not all $s_n(\varrho')$ are different from 0 or 1 but all are in $[0,1]$. Moreover for general density matrices we have

$$|\operatorname{tr}(\varrho'\varrho'')| \leq \|\varrho'\varrho''\|_1 \leq \|\varrho'\|\|\varrho''\|_1 \leq 1$$

by Theorem 6.2.9, *iii.)*. Finally, if ϱ' and ϱ'' are projections, say $\varrho' = \Theta_{\phi,\phi}$ and $\varrho'' = \Theta_{\psi,\psi}$ with unit vectors $\phi, \psi \in \mathfrak{H}$ then

$$\operatorname{tr}(\varrho'\varrho'') = \operatorname{tr}\left(\Theta_{\phi,\phi}\Theta_{\psi,\psi}\right) = |\langle\phi,\psi\rangle|^2.$$

Thus $|\operatorname{tr}(\varrho'\varrho'')| < 1$ iff ϕ and ψ are linearly independent iff $\varrho' \neq \varrho''$. Putting these together we have two cases: either ϱ', ϱ'' are not both projection, then at least one of the two terms $\operatorname{tr}(\varrho')^2$ or $\operatorname{tr}(\varrho'')^2$ in $(*)$ is strictly less than 1 and hence $\operatorname{tr}(\varrho^2) < 1$. Or both ϱ' and ϱ'' are projections with $\varrho' \neq \varrho''$ then $\operatorname{tr}(\varrho'\varrho'') = \operatorname{tr}(\varrho''\varrho') < 1$ and hence $\operatorname{tr}(\varrho^2) < 1$. In both cases we have a contradiction to $\operatorname{tr}(\varrho^2) = \operatorname{tr}(\rho) = 1$. $\square$

For the sake of completeness we also consider the relation between the strongly*, the strongly, and the weakly continuous linear functionals. Here one gets the following result:

Theorem 6.3.15 (Weakly continuous functionals) *Let $\omega\colon \mathfrak{B}(\mathfrak{H}) \longrightarrow \mathbb{C}$ be a linear functional. Then the following statements are equivalent:*

i.) The functional ω is weakly continuous.

ii.) The functional ω is strongly continuous.

iii.) The functional ω is strongly* continuous.

iv.) There exists a finite-rank operator $\varrho \in \mathfrak{F}(\mathfrak{H})$ such that for all $A \in \mathfrak{B}(\mathfrak{H})$

$$\omega(A) = \operatorname{tr}(\varrho A). \tag{6.3.9}$$

PROOF: We clearly have *i.)* $\implies$ *ii.)* $\implies$ *iii.)* according to Theorem 5.1.10, *ii.)*. Thus assume that ω is strongly* continuous. Hence there are vectors $\phi_1, \ldots, \phi_N \in \mathfrak{H}$ with

$$|\omega(A)| \leq \|A\phi_1\| + \|A^*\phi_1\| + \cdots + \|A\phi_N\| + \|A^*\phi_N\|$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. Recall that the seminorms $\|\cdot\|_\phi^*$ of the strong* topology as in Definition 5.1.1 do not yet form a filtrating system. Hence we have to use finitely many of them. To make this continuity property look more like the σ-strong* case we can equivalently find vectors, still denoted by $\phi_1, \ldots, \phi_N$, such that

$$|\omega(A)| \leq \sqrt{\|A\phi_1\|^2 + \|A^*\phi_1\|^2 + \cdots + \|A\phi_N\|^2 + \|A^*\phi_N\|^2},$$

since on $\mathbb{C}^N$ all norms are equivalent, see also Exercise 5.5.2. Now consider analogously to the proof of Theorem 6.3.11 the *finite* direct sum

$$\widetilde{\mathfrak{H}} = \bigoplus_{n=1}^{N} \mathfrak{H}_n \oplus \bigoplus_{n=1}^{N} \overline{\mathfrak{H}_n} \quad \text{with} \quad \mathfrak{H}_n = \mathfrak{H}.$$

Again, we can define the operator $\widetilde{A} \in \mathfrak{B}(\widetilde{\mathfrak{H}})$ for $A \in \mathfrak{B}(\mathfrak{H})$ block-diagonally by

$$\widetilde{A}\big(\chi_n, \overline{\psi_n}\big)_{n=1,\ldots,N} = \big(A\chi_n, \overline{A^*\psi_n}\big)_{n=1,\ldots,N}.$$

From here we can literally proceed as in the proof of Theorem 6.3.11 with the only difference that the series are now finite sums over $n = 1, \ldots, N$. We conclude that we find a vector $\Psi \in \widetilde{\mathfrak{H}}$ with components $(\chi_n, \psi_n)_{n=1,\ldots,N}$ such that

$$\begin{aligned}
\omega(A) &= \sum_{n=1}^{N} \langle \chi_n, A\phi_n \rangle + \overline{\langle \psi_n, A^*\phi_n \rangle} \\
&= \sum_{n=1}^{N} \langle \chi_n, A\phi_n \rangle + \langle A^*\phi_n, \psi_n \rangle \\
&= \sum_{n=1}^{N} \langle \chi_n, A\phi_n \rangle + \langle \phi_n, A\psi_n \rangle \\
&= \operatorname{tr}(\varrho A),
\end{aligned}$$

where ϱ is the finite-rank operator

$$\varrho = \sum_{n=1}^{N} \Theta_{\phi_n,\chi_n} + \Theta_{\psi_n,\phi_n}.$$

Thus we get *iii.)* $\implies$ *iv.)*. Finally, *iv.)* $\implies$ *i.)* is trivial since for $\varrho = \sum_{n=1}^{N} \Theta_{\phi_n,\psi_n}$ we have

$$|\operatorname{tr}(\varrho A)| = \left| \sum_{n=1}^{N} \langle \psi_n, A\phi_n \rangle \right| \leq \sum_{n=1}^{N} \|A\|_{\psi_n,\phi_n},$$

which is precisely the condition for weak continuity. ☐

6.4 Exercises

Exercise 6.4.1 (Jensen's inequality I) Consider a convex function $f \colon I \longrightarrow \mathbb{R}$ defined on an open interval $I \subseteq \mathbb{R}$.

i.) Show that for $\lambda_1, \ldots, \lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$ one has Jensen's inequality

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) \tag{6.4.1}$$

for all $x_1, \ldots, x_n \in I$.

ii.) Let now $\lambda_n \geq 0$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and $x_n \in I$ be given. Assume that $\sum_{n=1}^{\infty} \lambda_n x_n \in I$ and $\sum_{n=1}^{\infty} \lambda_n f(x_n)$ converge. Show that also in this case

$$f\left(\sum_{n=1}^{\infty} \lambda_n x_n\right) \leq \sum_{n=1}^{\infty} \lambda_n f(x_n). \tag{6.4.2}$$

iii.) Consider now the function $f(x) = x^r$ for $r \geq 1$. Apply the second version to this function with $x_n = 1$ for all $n \in \mathbb{N}$ to show

$$\left(\sum_{n=1}^{\infty} \lambda_n\right)^r \leq \sum_{n=1}^{\infty} \lambda_n^r. \tag{6.4.3}$$

iv.) Consider now $1 \leq q \leq p$ and use (6.4.3) with $r = \frac{p}{q}$ to get the inequality

$$\|a\|_p \leq \|a\|_q \tag{6.4.4}$$

for all complex sequences $a = (a_n)_{n \in \mathbb{N}}$.

Hint: It suffices to assume $\|a\|_q = 1$ and $a_n \geq 0$.

Exercise 6.4.2 (Finite-rank operators) Let V and W be Banach spaces.

i.) Show that a linear map $A\colon V \longrightarrow W$ is a finite-rank operator $A \in \mathfrak{F}(V, W)$ iff there are continuous linear functionals $\varphi_1, \ldots, \varphi_n \in V'$ and vectors $w_1, \ldots, w_n \in W$ such that for all $v \in V$

$$Av = \sum_{i=1}^{n} w_i \varphi_i(v). \tag{6.4.5}$$

ii.) Conclude that $\mathfrak{F}(V, W) \cong W \otimes V'$ under the canonical inclusion map of $W \otimes V^*$ into all the linear maps from V to W.

iii.) Show that $\mathfrak{F}(V) = \mathfrak{F}(V, V)$ are the smallest ideal inside all continuous linear endomorphisms $\mathrm{L}(V)$.

Hint: Use *i.)*.

iv.) Show that the finite rank operators $\mathfrak{F}(V) = \mathfrak{F}(V, V)$ have a canonical trace functional, i.e. a linear functional vanishing on commutators. Find an expression for the trace using a vector space basis of V and the corresponding linear coefficient functionals.

v.) How can one write (6.4.5) if both Banach spaces are even Hilbert spaces? How can one write the trace?

Exercise 6.4.3 (The annihilator is closed) Let V be a locally convex vector space and let $U \subseteq V$ be a closed subspace.

i.) Show that the annihilator U^{ann} of U is a closed subspace in V' with respect to the weak*-topology.

Hint: Let $(\varphi_i)_{i \in I}$ be a net in U^{ann} converging to $\varphi \in V'$ in the weak* topology. What is $\lim_{i \in I} \varphi_i(u)$ for $u \in U$?

ii.) Assume in addition that V is a Banach space. Conclude that U^{ann} is also closed in the norm topology of V'.

Exercise 6.4.4 (The annihilator of $\operatorname{im} A$) Let $A\colon V \longrightarrow W$ be a continuous linear map between Banach spaces. Show that for the dual map $A'\colon W' \longrightarrow V'$ one has

$$\ker A' = (\operatorname{im} A)^{\mathrm{ann}}. \tag{6.4.6}$$

Exercise 6.4.5 (Compact operators and isometries) Let U, V, and W be Banach spaces and let $\iota \colon V \longrightarrow W$ be isometric. Show that a linear map $A \colon U \longrightarrow V$ is compact iff $\iota \circ A$ is compact.

Exercise 6.4.6 (Singular values are norm-continuous) Consider Banach spaces V and W.

i.) Show that for $n \in \mathbb{N}_0$ the n-th approximation number

$$a_n \colon \mathrm{L}(V, W) \longrightarrow \mathbb{R} \tag{6.4.7}$$

is a continuous map.

Next one considers a Hilbert space $\mathfrak{H}$.

ii.) Show that the n-th eigenvalue of A^*A for $A \in \mathfrak{K}(\mathfrak{H})$, ordered by size and counted with multiplicity, depends continuously on $A \in\in \mathfrak{K}(\mathfrak{H})$.

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Exercise 6.4.7 (Trace of rank-one operators) Let $\mathfrak{H}$ be a Hilbert space and consider the rank-one operator $\Theta_{\phi,\psi} \in \mathfrak{F}(\mathfrak{H})$ for $\phi, \psi \in \mathfrak{H}$.

i.) Show that $\operatorname{tr}(\Theta_{\phi,\psi}) = \langle \phi, \psi \rangle$.

ii.) Show that $\|\Theta_{\phi,\psi}\| = \|\phi\| \|\psi\|$ for the operator norm of $\Theta_{\phi,\psi}$.

iii.) Show that the approximation numbers $a_n(\Theta_{\phi,\psi})$ of $\Theta_{\phi,\psi}$ vanish for $n \geq 1$.

iv.) Show that the p-th Schatten norm of $\Theta_{\phi,\psi}$ is given by $\|\Theta_{\phi,\psi}\|_p = \|\phi\| \|\psi\|$, where $p \geq 1$.

Exercise 6.4.8 (The trace for finite-rank operators II) Compare the construction of the finite-rank operators on general locally convex spaces and their trace functional as in Exercise 4.5.8 with the construction in the case of a Hilbert space.

Exercise 6.4.9 (Cayley transform is not Fredholm)

Exercise 6.4.10 (The group $\mathrm{GL}(V)$) Let V and W be Banach spaces.

i.) Show that the invertible elements in $\mathrm{L}(V, W)$ form a (possibly empty) open subset. Show that the inversion $A \mapsto A^{-1}$ is continuous with respect to the operator norms.

ii.) Show that $\mathrm{GL}(V)$ forms a topological group with respect to the operator norm.

Exercise 6.4.11 (The image of the dual operator) Let $A \colon V \longrightarrow W$ be a continuous operator between Banach spaces and consider its dual operator $A' \colon W' \longrightarrow V'$. Assume that $\operatorname{im} A$ is closed. Show that in this case

$$(\ker A)^{\mathrm{ann}} = \operatorname{im} A'. \tag{6.4.8}$$

Hint: The inclusion $\supseteq$ is easy. For the reverse inclusion, use that $A \colon V \big/ \ker A \longrightarrow \operatorname{im} A$ is a continuous bijection between Banach spaces, hence it has a continuous inverse. The use the Hahn-Banach Theorem.

Exercise 6.4.12 (Alternative proof for Theorem 6.1.37) The aim of this exercise is to give a more direct proof of the spectral theorem for compact normal operators not relying on the general results for Banach space.

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Exercise 6.4.13 (The function $x \mapsto -x \ln x$) Consider the function

$$f \colon x \mapsto \begin{cases} 0 & \text{for } x = 0 \\ -x \ln x & \text{for } x > 0. \end{cases} \tag{6.4.9}$$

Show that this defines a continuous function $f \colon \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ which is non-negative on $[0, 1]$. Compute the zeros and show that there is a unique maximum of f in $[0, 1]$. Compute the derivative of f on $\mathbb{R}^+$ and discuss the asymptotic behaviour of the derivative for $x \longrightarrow 0$ and $x \longrightarrow \infty$.

Chapter 7

*-Representation Theory of C^*-Algebras

As already argued, for a physically reasonable model we do not only need the observable algebra to be somehow and abstractly given but also represented on a Hilbert space by operators allowing for a good spectral theory. For a C^*-algebra $\mathscr{A}$, being the most simple type of an observable algebra, we have seen in Section 4.4 that we indeed can represent $\mathscr{A}$ by bounded operators in a faithful way. In this chapter we shall continue our investigation of the *-representation theory of C^*-algebras.

We will focus on the questions of constructing and characterizing irreducible representations. Here the commutant of the representation will play the crucial role. Since it is a von Neumann algebra we will discuss several general facts about von Neumann algebras like the double commutant theorem and Kaplansky's density theorem. Since the commutant of a *-representation is a von Neumann algebra one tries to characterize the *-representation mainly by this von Neumann algebra: e.g. the question of irreducibility can be rephrased this way. If the commutant is nontrivial the *-representation is not irreducible and hence allows for a decomposition. This decomposition theory makes heavy use of the classification results on von Neumann algebras. We present here only the rather rough characterization of different types I, II, and III of von Neumann algebras without going into the details of further classifying within the types. To this end we need to understand the role of projections in von Neumann algebras in some more detail. Among many other results, this will ultimately lead to a better understanding of quasi-equivalences of *-representations.

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7.1 First Properties of *-Representations

In this motivating and preliminary section we collect a few general remarks on the *-representation theory of C^*-algebra to establish some new vocabulary.

7.1.1 The Representation Theory of a C^*-Algebra, Revisited

In Chapter 1 we defined the *-representation theory of an arbitrary *-algebra $\mathscr{A}$ as the category of *-representations of $\mathscr{A}$ on pre-Hilbert spaces with adjointable intertwiners as morphisms. For a C^*-algebra we know that any *-representation on a pre-Hilbert space extends to the Hilbert space completion and yields a *-representation on it. However, adjointable intertwiners will, in general, not extend to this completion unless they are *bounded*.

Example 7.1.1 Let $\mathfrak{H}$ be a pre-Hilbert space and $\mathscr{A} = \mathbb{C}$. Then $\phi\colon z \mapsto z\,\mathrm{id}_{\mathfrak{H}}$ defines a *-representation of $\mathscr{A}$ on $\mathfrak{H}$. In this case, all adjointable operators $\mathfrak{B}(\mathfrak{H})$ are self-intertwiners but only those pass to the completion $\widehat{\mathfrak{H}}$ which are bounded. Moreover, for the completions, $\mathfrak{B}(\widehat{\mathfrak{H}})$ will be the self-intertwiners. It is clear that not every continuous operator $A\colon \widehat{\mathfrak{H}} \longrightarrow \widehat{\mathfrak{H}}$ is the completion of a bounded operator $B\colon \mathfrak{H} \longrightarrow \mathfrak{H}$, i.e. $A = \widehat{B}$. In fact, this is the case iff $A(\mathfrak{H}) \subseteq \mathfrak{H}$.

This example suggests to refine our notions of *-representations for the case of C^*-algebras as follows. Inside $^*\text{-}\mathsf{rep}(\mathscr{A})$ we have the following subcategory

$$^*\text{-}\mathsf{rep}_{\mathrm{bounded}}(\mathscr{A}) \subseteq {}^*\text{-}\mathsf{rep}(\mathscr{A}), \tag{7.1.1}$$

where we take again *-representations on pre-Hilbert spaces as objects but now only bounded, i.e. continuous, adjointable intertwiners as morphisms.

Note that for pre-Hilbert spaces the notions of bounded and adjointable operators do *not* coincide as this is the case for Hilbert spaces according to the Hellinger-Toeplitz Theorem, see Theorem 3.5.1. For two pre-Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2$ there may be an adjointable operator $A\colon \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$ which is not bounded. Conversely, there may be a bounded operator A which is not adjointable: of course, the continuous extension $\widehat{A}\colon \widehat{\mathfrak{H}_1} \longrightarrow \widehat{\mathfrak{H}_2}$ to the completions has an adjoint $\widehat{A}^*$ but it may happen that $\widehat{A}^*$ maps $\mathfrak{H}_2$ *not* into $\mathfrak{H}_1$ but only into $\widehat{\mathfrak{H}_1}$. As a simple example consider the inclusion map $\mathrm{id}\colon \mathscr{C}^\infty_0(\mathbb{R}) \longrightarrow \mathrm{L}^2(\mathbb{R})$ from the dense subspace into its completion.

Moreover, we can define the *-representation theory of $\mathscr{A}$ with *-representations on *Hilbert spaces* and adjointable intertwiners, which are now automatically bounded. We denote this category by $^*\text{-}\mathfrak{rep}(\mathscr{A})$. We have the obvious inclusion

$$^*\text{-}\mathfrak{rep}(\mathscr{A}) \subseteq {}^*\text{-}\mathsf{rep}_{\mathrm{bounded}}(\mathscr{A}), \tag{7.1.2}$$

which is now a *full* subcategory. The relation between the three versions is now clarified in the following proposition:

Proposition 7.1.2 *Let $\mathscr{A}$ be a C^*-algebra.*

*i.) Completing a pre-Hilbert space to a Hilbert space and extending a *-representation of a the C^*-algebra $\mathscr{A}$ to the completion gives a functor*

$$\widehat{\ }\colon {}^*\text{-}\mathsf{rep}_{\mathrm{bounded}}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathfrak{rep}(\mathscr{A}). \tag{7.1.3}$$

ii.) On $^*\text{-}\mathfrak{rep}(\mathscr{A})$ the completion is the identity.

iii.) The completion functor is faithful but not full in general.

PROOF: Let $(\mathfrak{H}, \pi)$ be a *-representation on a pre-Hilbert space then the canonical extension $\widehat{\pi}$ of π to the completion $\widehat{\mathfrak{H}}$ of $\mathfrak{H}$ is still a *-representation by our results from Corollary 4.4.17. If $A\colon (\mathfrak{H}, \pi) \longrightarrow (\mathfrak{H}', \pi')$ is a bounded adjointable intertwiner then the canonical extension $\widehat{A}\colon \widehat{\mathfrak{H}} \longrightarrow \widehat{\mathfrak{H}}'$ is still an intertwiner since for $\phi \in \mathfrak{H}$ we have for all $a \in A$

$$\widehat{A}\widehat{\pi}(a)\phi = \widehat{A}\pi(a)\phi = A\pi(a)\phi = \pi'(a)A\phi = \widehat{\pi}'(a)A\phi = \widehat{\pi}'(a)\widehat{A}\phi.$$

Thus the intertwiner property holds on the dense subspace $\mathfrak{H} \subseteq \widehat{\mathfrak{H}}$ and, by continuity of $\widehat{A}$ and $\widehat{\pi}(a)$, on all of $\widehat{\mathfrak{H}}$. This shows that the completion is also well-defined on intertwiners. Clearly $\widehat{(AB)} = \widehat{A}\widehat{B}$ and $\widehat{\mathrm{id}_{\mathfrak{H}}} = \mathrm{id}_{\widehat{\mathfrak{H}}}$ for completions which gives the functoriality of (7.1.3). The second part is obvious. For the third part it is clear that for two intertwiners $A, B\colon (\mathfrak{H}, \pi) \longrightarrow (\mathfrak{H}', \pi')$ with $\widehat{A} = \widehat{B}$ we have $A = B$ since the extension by continuity is unique. However we have seen in Example 7.1.1 that not all intertwiners in $^*\text{-}\mathfrak{rep}(\mathscr{A})$ may be obtained from completions. Thus $\widehat{\ }$ is faithful but not full in general. ☐

Remark 7.1.3 It follows that completion is not an equivalence of the categories $^*\text{-}\mathfrak{rep}(\mathscr{A})$ and $^*\text{-}\mathsf{rep}_{\mathrm{bounded}}(\mathscr{A})$: in some sense the category $^*\text{-}\mathsf{rep}_{\mathrm{bounded}}(\mathscr{A})$ carries simply more information, namely about the dense and invariant subspace $\mathfrak{H} \subseteq \widehat{\mathfrak{H}}$. Nevertheless, this information is mainly unimportant and hence we will focus on $^*\text{-}\mathfrak{rep}(\mathscr{A})$ for C^*-algebras. Note that also for general *-algebras $\mathscr{A}$ the subcategory $^*\text{-}\mathsf{rep}_{\mathrm{bounded}}(\mathscr{A}) \subseteq {}^*\text{-}\mathsf{rep}(\mathscr{A})$ can be defined and studied, even though in general there will be no completion functor.

The next proposition clarifies the notions of equivalence of *-representation in the two categories $^*\text{-rep}_{\text{bounded}}(\mathscr{A})$ and $^*\text{-}\mathfrak{rep}(\mathscr{A})$:

Proposition 7.1.4 *Let $\mathscr{A}$ be a C^*-algebra and $(\mathfrak{H}, \pi)$ and $(\mathfrak{H}', \pi')$ be two *-representations on pre-Hilbert spaces.*

i.) If $A\colon (\mathfrak{H}, \pi) \longrightarrow (\mathfrak{H}', \pi')$ is an invertible bounded intertwiner then $\widehat{A}\colon (\widehat{\mathfrak{H}}, \widehat{\pi}) \longrightarrow (\widehat{\mathfrak{H}}', \widehat{\pi}')$ is still invertible.

ii.) If there exists an invertible intertwiner between $(\widehat{\mathfrak{H}}, \widehat{\pi})$ and $(\widehat{\mathfrak{H}}', \widehat{\pi}')$ then there is also a unitary intertwiner.

PROOF: The first part is clear by the functoriality of the completion according to Proposition 7.1.2, *i.)*. For the second, let $A \in \mathfrak{B}(\widehat{\mathfrak{H}}, \widehat{\mathfrak{H}}')$ be an invertible intertwiner with polar decomposition $A = U|A|$ as in Theorem 5.1.42. Note that for invertible A the partial isometry U is in fact unitary. Then by Proposition 5.1.45 the unitary U satisfies $U\pi(a) = \pi(a)U$ for all Hermitian $a \in \mathscr{A}$. Since every element $a \in \mathscr{A}$ is a linear combination of Hermitian ones it follows that U is an intertwiner. □

Thus the notions of *equivalence* and *unitary equivalence* of *-representations lead to the same equivalence classes in the category $^*\text{-}\mathfrak{rep}(\mathscr{A})$ which is one of its main advantages. In the following we shall mainly consider $^*\text{-}\mathfrak{rep}(\mathscr{A})$ and use the above result on equivalence throughout.

We shall now specialize our notions of *-representations even further. Recall that even for unital *-algebras a *-homomorphism (and hence a *-representation) was not required to be unital by convention. Thus for every Hilbert space $\mathfrak{H}$ we obtain a trivial *-representation by setting $\pi(a) = 0$ for all $a \in \mathscr{A}$. To exclude such non-interesting *-representations one considers the following notion:

Definition 7.1.5 (Non-degenerate *-representation) *A *-representation $(\mathfrak{H}, \pi)$ of a C^*-algebra $\mathscr{A}$ is called non-degenerate if $\pi(a)\phi = 0$ for all $a \in \mathscr{A}$ implies $\phi = 0$.*

An arbitrary *-representation $(\mathfrak{H}, \pi)$ can now canonically be split into a non-degenerate one and the trivial one:

Theorem 7.1.6 (Non-degenerate *-representation) *Let $\mathscr{A}$ be a C^*-algebra and let $(\mathfrak{H}, \pi)$ be a *-representation of $\mathscr{A}$ on a Hilbert space.*

i.) The set

$$\mathfrak{H}_{\text{trivial}} = \left\{ \phi \in \mathfrak{H} \;\middle|\; \pi(a)\phi = 0 \textit{ for all } a \in \mathscr{A} \right\} \tag{7.1.4}$$

is a closed subspace of $\mathfrak{H}$ invariant under all the operators $\pi(a)$.

ii.) The closed subspace

$$\mathfrak{H}_{\text{nondeg}} = \mathfrak{H}_{\textit{trivial}}^{\perp} \tag{7.1.5}$$

is invariant under all the operators $\pi(a)$, too, and the restriction $\pi_{\text{nondeg}} = \pi\big|_{\mathfrak{H}_{\text{nondeg}}}$ is non-degenerate.

iii.) For every $\phi \in \mathfrak{H}_{\text{nondeg}}$ and $\epsilon > 0$ there exists a $\psi \in \mathfrak{H}_{\text{nondeg}}$ and $a \in \mathscr{A}^+$ such that

$$\phi = \pi(a)\psi \textit{ and } \|\phi - \psi\| \le \epsilon. \tag{7.1.6}$$

iv.) If $\{\mathrm{e}_i\}_{i\in I}$ is an approximate identity then in the strong operator topology one has

$$\lim_{i\in I} \pi(\mathrm{e}_i) = P_{\mathfrak{H}_{\text{nondeg}}}. \tag{7.1.7}$$

PROOF: Since $\mathfrak{H}_{\text{trivial}} = \bigcap_{a\in\mathscr{A}} \ker \pi(a)$ and all $\pi(a)$ are continuous, it is clear that $\mathfrak{H}_{\text{trivial}}$ is a closed subspace. Obviously, it is invariant under all $\pi(a)$. Now let ϕ be orthogonal to $\mathfrak{H}_{\text{trivial}}$ and $\psi \in \mathfrak{H}_{\text{trivial}}$. Then $\langle \pi(a)\phi, \psi \rangle = \langle \phi, \pi(a^*)\psi \rangle = 0$, since $\pi(a^*)\psi = 0$, shows that also $\mathfrak{H}_{\text{nondeg}}$ is invariant under all

$\pi(a)$. Suppose $\phi \in \mathfrak{H}_{\mathrm{nondeg}}$ satisfies $\pi(a)\phi = 0$ for all $a \in \mathscr{A}$. Then $\phi \in \mathfrak{H}_{\mathrm{trivial}}$ by definition and $\mathfrak{H}_{\mathrm{nondeg}} = \mathfrak{H}_{\mathrm{trivial}}^{\perp}$ gives $\phi = 0$. This shows the second part. For the third part we adjoin a unit if necessary and hence pass to $\widetilde{\mathscr{A}}$ if $\mathscr{A}$ is non-unital. Moreover, we choose an approximate identity $\{\mathrm{e}_i\}_{i\in I}$ in $\mathscr{A}$. Then consider the subspace

$$K = \pi(\mathscr{A})\mathfrak{H}_{\mathrm{nondeg}} = \left\{ \sum\nolimits_n \pi(a_n)\phi_n \;\middle|\; a_n \in \mathscr{A}, \phi_n \in \mathfrak{H}_{\mathrm{nondeg}} \right\} \subseteq \mathfrak{H}_{\mathrm{nondeg}}.$$

We first claim that $K^{\mathrm{cl}} = \mathfrak{H}_{\mathrm{nondeg}}$. Indeed, let $\phi \in \mathfrak{H}_{\mathrm{nondeg}}$ be orthogonal to all vectors $\psi \in K$ then in particular $\langle \phi, \pi(a)\phi' \rangle = 0$ for all $\phi' \in \mathfrak{H}_{\mathrm{nondeg}}$ and $a \in \mathscr{A}$. But this shows $\pi(a^*)\phi \in \mathfrak{H}_{\mathrm{trivial}}$ for all $a \in \mathscr{A}$ and hence $\pi(a)\pi(b)\phi = \pi(ab)\phi = 0$ for all $a, b \in \mathscr{A}$. Thus $K^{\perp} = \{0\}$ and $K^{\mathrm{cl}} = \mathfrak{H}_{\mathrm{nondeg}}$ follows. Since $\mathscr{A}$ is idempotent by Corollary 4.4.7 this gives $\pi(a)\phi = 0$ for all $a \in \mathscr{A}$ and thus $\phi \in \mathfrak{H}_{\mathrm{trivial}}$. Hence $\phi = 0$ follows. Thus let $\phi \in \mathfrak{H}_{\mathrm{nondeg}}$ be given. We claim that

$$\lim_{i\in I} \pi(\mathrm{e}_i)\phi = \phi. \tag{$*$}$$

Indeed, for $\phi = \pi(a)\psi$ this is easy since $\pi(\mathrm{e}_i)\phi = \pi(\mathrm{e}_i)\pi(a)\phi = \pi(\mathrm{e}_i a)\phi \longrightarrow \pi(a)\phi = \psi$ since $\mathrm{e}_i a \longrightarrow a$ in the norm topology of $\mathscr{A}$ by definition of an approximate identity and since π is continuous. If $\phi \in \mathfrak{H}_{\mathrm{nondeg}}$ is arbitrary we find for any given $\epsilon > 0$ a $\psi \in K$ with $\|\phi - \psi\| < \epsilon$. Moreover, we find an index $i \in I$ with $\|\pi(\mathrm{e}_j)\psi - \psi\| < \epsilon$ for all $j \geq i$. Thus for those j we have

$$\begin{aligned}
\|\pi(\mathrm{e}_j)\phi - \phi\| &= \|\pi(\mathrm{e}_j)(\phi - \psi) + \pi(\mathrm{e}_j)\psi - \psi + \psi - \phi\| \\
&\leq \|\phi - \psi\| + \|\pi(\mathrm{e}_j)\psi - \psi\| + \|\phi - \psi\| \\
&< 3\epsilon,
\end{aligned}$$

where we have used $\|\pi(\mathrm{e}_i)\| \leq \|\mathrm{e}_i\| \leq 1$. This shows $(*)$. Now let $\phi \in \mathfrak{H}_{\mathrm{nondeg}}$ be given and set $\phi_0 = \phi$ as well as $a_0 = \mathbb{1} \in \widetilde{\mathscr{A}}$. Moreover, fix $\epsilon > 0$. We want to construct inductively a sequence $\phi_n \in \mathfrak{H}_{\mathrm{nondeg}}$ converging to the vector ψ and a sequence $a_n \in \widetilde{\mathscr{A}}$ converging to the element a we are looking for. So suppose we have already constructed $\phi_0, \ldots, \phi_{n-1}$. Then we fix $i_n \in I$ with

$$\|\phi_{n-1} - \pi(\mathrm{e}_{i_n})\phi_{i_n}\| < \frac{\epsilon}{2^n}.$$

Moreover, set $a_n = a_{n-1} - \frac{1}{2^n}(\mathbb{1} - \mathrm{e}_{i_n}) \in \tilde{A}$. Since the e_i are positive elements we have

$$a_n = \mathbb{1} - \frac{1}{2}(\mathbb{1} - \mathrm{e}_{i_1}) - \frac{1}{4}(\mathbb{1} - \mathrm{e}_{i_2}) - \cdots - \frac{1}{2^n}(\mathbb{1} - \mathrm{e}_{i_n}) = \frac{1}{2^n}\mathbb{1} + \sum_{n=1}^{n} \frac{1}{2^n}\mathrm{e}_{i_k} \geq \frac{1}{2^n}\mathbb{1}.$$

In particular, a_n is invertible in $\tilde{\mathscr{A}}$ with $\|a_n^{-1}\| \leq 2^n$. If $\mathscr{A}$ is non-unital, then π extends uniquely to a unital *-homomorphism $\widetilde{\pi}\colon \widetilde{\mathscr{A}} \longrightarrow \mathfrak{B}(\mathfrak{H}_{\mathrm{nondeg}})$ by setting $\widetilde{\pi}(\mathbb{1}) = \mathrm{id}$, see also Exercise 4.5.22. This allows to consider the vector

$$\phi_n = \widetilde{\pi}(a_n^{-1})\phi \in \mathfrak{H}_{\mathrm{nondeg}}.$$

We get

$$\begin{aligned}
\phi_n - \phi_{n-1} &= (\widetilde{\pi}(a_n^{-1}) - \widetilde{\pi}(a_{n-1}^{-1}))\phi \\
&= \widetilde{\pi}(a_n^{-1})(\widetilde{\pi}(a_{n-1}) - \widetilde{\pi}(a_n))\widetilde{\pi}(a_{n-1}^{-1})\phi \\
&= \widetilde{\pi}(a_n^{-1})\widetilde{\pi}\left(\frac{1}{2^n}(\mathbb{1} - \mathrm{e}_{i_n})\right)\phi_{n-1}.
\end{aligned}$$

From $(*)$ and $\|\widetilde{\pi}(a_n^{-1})\| \le \|a_n^{-1}\| \le 2^n$ we obtain $\|\phi_n - \phi_{n-1}\| < \frac{\epsilon}{2^n}$. By the usual telescope argument the sequence $(\phi_n)_{n\in\mathbb{N}_0}$ is a Cauchy sequence with a limit $\psi \in \mathfrak{H}_{\mathrm{nondeg}}$. Since also the sequence $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence we get a limit $a \in \widetilde{\mathscr{A}}$ as well, explicitly given by

$$a = \lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty}\left(\frac{1}{2^n}\mathbb{1} + \sum\nolimits_{k=1}^{n} \frac{1}{2^k}\mathrm{e}_{i_k}\right) = \sum\nolimits_{k=1}^{\infty} \frac{1}{2^k}\mathrm{e}_{i_k}.$$

Thus $a \in \mathscr{A}$ and, being a limit of positive elements, also $a \in \mathscr{A}^+$. We have from continuity

$$\phi = \pi(a_n)\phi_n = \lim_{n\longrightarrow\infty} \pi(a_n)\phi_n = \pi(a)\psi,$$

as well as

$$\|\phi - \phi_n\| = \|\phi_0 - \phi_1 + \phi_1 - \cdots + \phi_{n-1} - \phi_n\| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \cdots + \frac{\epsilon}{2^n} < \epsilon$$

for all n. Thus also $\|\phi - \psi\| \le \epsilon$ as claimed. This shows the third part. The fourth is clear as we have already shown $\pi(\mathrm{e}_i)\phi \longrightarrow \phi$ for $\phi \in \mathfrak{H}_{\mathrm{nondeg}}$ and $\pi(\mathrm{e}_i)\phi = 0$ for $\phi \in \mathfrak{H}_{\mathrm{trivial}}$ holds anyway. ☐

Corollary 7.1.7 *Let $\mathscr{A}$ be a unital C^*-algebra. Then a *-representation $(\mathfrak{H}, \pi)$ is non-degenerate iff $\pi(\mathbb{1}) = \mathrm{id}_{\mathfrak{H}}$.*

The decomposition of a *-representation into a trivial and a non-degenerate part is a first example of subrepresentations:

Definition 7.1.8 (Subrepresentation) *Let $\mathscr{A}$ be a C^*-algebra with a *-representation $(\mathfrak{H}, \pi)$ on a Hilbert space. Then a closed subspace $\mathfrak{K} \subseteq \mathfrak{H}$ is called a subrepresentation of $(\mathfrak{H}, \pi)$ if $\mathfrak{K}$ is invariant under π, i.e. for all $a \in \mathscr{A}$ one has*

$$\pi(a)K \subseteq K. \tag{7.1.8}$$

For a unital C^*-algebra a *-representation is non-degenerate iff $\pi(\mathbb{1}) = \mathrm{id}$ and hence $\phi = \pi(\mathbb{1})\phi$ for all $\phi \in \mathfrak{H}$. The third part of the theorem gives a replacement for this trivial observation in the non-unital case. Moreover, every *-representation on a Hilbert space splits into a trivial and a non-degenerate subrepresentation in a canonical way. This suggests to consider only non-degenerate *-representations from the beginning.

Definition 7.1.9 (The *-representation theory of a C^*-algebra) *Let $\mathscr{A}$ be a C^*-algebra. Then the full subcategory of *-$\mathfrak{rep}(\mathscr{A})$ of non-degenerate *-representations of $\mathscr{A}$ on Hilbert spaces is denoted by *-$\mathfrak{Rep}(\mathscr{A})$ and called the *-representation theory of the C^*-algebra $\mathscr{A}$.*

Corollary 7.1.10 *Let $\mathscr{A}$ be a C^*-algebra. Passing from an arbitrary *-representation $(\mathfrak{H}, \pi)$ on a Hilbert space to its non-degenerate part $(\mathfrak{H}_{\mathrm{nondeg}}, \pi_{\mathrm{nondeg}})$ yields a full functor*

$$^*\text{-}\mathfrak{rep}(\mathscr{A}) \longrightarrow {}^*\text{-}\mathfrak{Rep}(\mathscr{A}), \tag{7.1.9}$$

being the identity on *-$\mathfrak{Rep}(\mathscr{A}) \subseteq {}^*$-$\mathfrak{rep}(\mathscr{A})$.

Proof: By the previous theorem the construction indeed gives a non-degenerate *-representation of $\mathscr{A}$. To check functoriality, let $A\colon (\mathfrak{H}, \pi) \longrightarrow (\mathfrak{H}', \pi')$ be an intertwiner. Then for $\phi \in \mathfrak{H}_{\mathrm{trivial}}$ we have for all $a \in \mathscr{A}$

$$\pi'(a)A\phi = A\pi(a)\phi = 0,$$

and thus $A\phi \in \mathfrak{H}'_{\mathrm{trivial}}$. Conversely, let $\phi \in \mathfrak{H}_{\mathrm{nondeg}}$ and $\psi \in \mathfrak{H}'_{\mathrm{trivial}}$ then $\langle\psi, A\phi\rangle = \langle A^*\psi, \phi\rangle = 0$ since A^* is an intertwiner as well and hence $A^*\psi \in \mathfrak{H}_{\mathrm{trivial}} = \mathfrak{H}_{\mathrm{nondeg}}^{\perp}$. This shows that any intertwiner is block-diagonal with respect to the splittings into the trivial and non-degenerate parts. Thus the definition

$$A_{\mathrm{nondeg}} = A\big|_{\mathfrak{H}_{\mathrm{nondeg}}} : \mathfrak{H}_{\mathrm{nondeg}} \longrightarrow \mathfrak{H}'_{\mathrm{nondeg}}$$

is well-defined and yields the functorial behaviour on morphisms at once. If now $B\colon (\mathfrak{H}_{\mathrm{nondeg}}, \pi_{\mathrm{nondeg}}) \longrightarrow (\mathfrak{H}'_{\mathrm{nondeg}}, \pi'_{\mathrm{nondeg}})$ is an intertwiner then we can extend B to $\mathfrak{H}$ by $B\big|_{\mathfrak{H}_{\mathrm{trivial}}} = 0$ and obtain an intertwiner $B\colon (\mathfrak{H}, \pi) \longrightarrow (\mathfrak{H}', \pi')$. This shows that (7.1.9) is full. $\square$

Thus the corollary justifies to concentrate on $^*\text{-}\mathfrak{Rep}(\mathscr{A})$, we do not loose anything important when passing to the non-degenerate part of a *-representation.

7.1.2 Cyclic *-Representations and Direct Sums

In this subsection we shall consider the GNS construction once more. First we need the behaviour of the GNS representations under *-homomorphisms. Thus let $\mathscr{A}$ and $\mathscr{B}$ be two C^*-algebras and let

$$\Phi\colon \mathscr{A} \longrightarrow \mathscr{B} \tag{7.1.10}$$

be a *-homomorphism, which we know to be continuous by Proposition 4.4.27. Moreover, let $\omega\colon \mathscr{B} \longrightarrow \mathbb{C}$ be a state. Then $\Phi^*\omega = \omega \circ \Phi\colon \mathscr{A} \longrightarrow \mathbb{C}$ is at least a positive functional. Note that it may happen that $\Phi^*\omega = 0$ even though Φ is non-trivial. If $\mathscr{A}$ and $\mathscr{B}$ are unital and Φ is a unital *-homomorphism then $\Phi^*\omega$ is of course again a state. The following lemma is purely algebraic and compares the GNS representations with respect to ω and $\Phi^*\omega$:

cise: exercise
$Y, f^* = Phi$

Lemma 7.1.11 *Let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a *-homomorphism between *-algebras and let $\omega\colon \mathscr{B} \longrightarrow \mathbb{C}$ be a positive linear functional.*

i.) The Gel'fand ideal of the positive linear functional $\Phi^*\omega$ of $\mathscr{A}$ is given by

$$\mathscr{J}_{\Phi^*\omega} = \Phi^{-1}(\mathscr{J}_\omega). \tag{7.1.11}$$

ii.) The map

$$U_\Phi\colon \mathscr{A}\big/\mathscr{J}_{\Phi^*\omega} \ni \psi_a \;\mapsto\; \psi_{\Phi(a)} \in \mathscr{B}\big/\mathscr{J}_\omega \tag{7.1.12}$$

is well-defined and isometric. Moreover, for the GNS representation one has

$$U_\Phi \pi_{\Phi^*\omega}(a) = \pi_\omega(\Phi(a)) U_\Phi. \tag{7.1.13}$$

iii.) If Φ is surjective then U_Φ is unitary.

Proof: Clearly, $\Phi^*\omega$ is positive again and $(\Phi^*\omega)(a^*a) = \omega(\Phi(a)^*\Phi(a))$ vanishes iff $\Phi(a) \in \mathscr{J}_\omega$. This shows (7.1.11). This implies that Φ passes to the quotients and yields a well-defined linear map U_Φ as described by (7.1.12). We have for $a, b \in \mathscr{A}$

$$\left\langle U_\Phi\psi_a, U_\Phi\psi_b \right\rangle_\omega = \left\langle \psi_{\Phi(a)}, \psi_{\Phi(b)} \right\rangle_\omega = \omega(\Phi(a)^*\Phi(b)) = \omega(\Phi(a^*b)) = (\Phi^*\omega)(a^*b) = \left\langle \psi_a, \psi_b \right\rangle_{\Phi^*\omega},$$

showing that U_Φ is isometric. Finally, for $a, b \in \mathscr{A}$ we get

$$U_\Phi \pi_{\Phi^*\omega}(a)\psi_a = U_\Phi\psi_{ab} = \psi_{\Phi(ab)} = \psi_{\Phi(a)\Phi(b)} = \pi_\omega(\Phi(a))\psi_{\Phi(b)} = \pi_\omega(\Phi(a))U_\Phi\psi_b,$$

and hence (7.1.13). The last statement is clear since U_Φ is certainly surjective in this case. $\square$

In the case of C^*-algebras we can pass to the completions and get an isometric intertwiner along Φ which is unitary in the case where Φ is surjective. The following particular situation shows that, after completion, U_Φ might be surjective even though Φ is not surjective:

Proposition 7.1.12 *Let $\mathscr{A}$ be a non-unital C^*-algebra and let $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a state. Moreover, let $\widetilde{\mathscr{A}}$ be the unitization of $\mathscr{A}$ and $\widetilde{\omega}\colon \widetilde{\mathscr{A}} \longrightarrow \mathbb{C}$ the canonical extension of ω to a state of $\widetilde{\mathscr{A}}$. Then the canonical map*

$$U\colon \mathscr{A}\big/\mathscr{J}_\omega \ni \psi_a \mapsto \widetilde{\psi_a} \in \widetilde{\mathscr{A}}\big/\mathscr{J}_{\widetilde{\omega}} \tag{7.1.14}$$

extends to a unitary map $U\colon \mathfrak{H}_\omega \longrightarrow \mathfrak{H}_{\widetilde{\omega}}$ between the completions such that for $a \in \mathscr{A}$

$$U\pi_\omega(a) = \pi_{\widetilde{\omega}}(a)U. \tag{7.1.15}$$

PROOF: Since the inclusion $\iota\colon \mathscr{A} \longrightarrow \widetilde{\mathscr{A}}$ is a *-homomorphism and $\widetilde{\omega}$ restricts to $\omega = \iota^*\widetilde{\omega}$ we can apply the algebraic Lemma 7.1.11 to get the isometric map U. Identifying a with $\iota(a)$ in $\widetilde{\mathscr{A}}$ we get (7.1.15) on $\mathscr{A}\big/\mathscr{J}_\omega$ by (7.1.13) and hence also on the completion by continuity. It remains to show that U is surjective. In general, it might not be surjective on the level of pre-Hilbert spaces. However, choose an approximate identity $\{\mathrm{e}_i\}_{i\in I}$ in $\mathscr{A}$. We claim that $\widetilde{\psi}_{\mathrm{e}_i} \in \widetilde{\mathscr{A}}\big/\mathscr{J}_{\widetilde{\omega}}$ converges to $\widetilde{\psi}_{\mathbb{1}}$. Indeed, with $e_i = e_i^*$ we get

$$\left\|\widetilde{\psi}_{\mathbb{1}} - \widetilde{\psi}_{\mathrm{e}_i}\right\|_{\widetilde{\omega}}^2 = \widetilde{\omega}((\mathbb{1} - \mathrm{e}_i)^*(\mathbb{1} - \mathrm{e}_i)) = \widetilde{\omega}\big(\mathbb{1} - \mathrm{e}_i - \mathrm{e}_i + \mathrm{e}_i^2\big) = \widetilde{\omega}(\mathbb{1} - \mathrm{e}_i) + \widetilde{\omega}(\mathrm{e}_i^2 - \mathrm{e}_i).$$

From Proposition 4.4.19, *ii.)*, we know that the first term converges to 0. Moreover, by (4.4.27) in the same proposition we see that also the second contribution converges to 0 showing $\widetilde{\psi}_{\mathrm{e}_i} \longrightarrow \widetilde{\psi}_{\mathbb{1}}$. Since $\widetilde{\mathscr{A}} = \mathscr{A} \oplus \mathbb{C}\mathbb{1}$ we conclude that the image of U is dense in $\widetilde{\mathscr{A}}\big/\mathscr{J}_{\widetilde{\omega}}$. Hence after completion the image of the isometric map $U\colon \mathfrak{H}_\omega \longrightarrow \mathfrak{H}_{\widetilde{\omega}}$ is still dense. But for Hilbert spaces isometric maps have closed images and hence $\operatorname{im} U = \mathfrak{H}_{\widetilde{\omega}}$. ☐

With other words, the GNS representation of $\widetilde{\mathscr{A}}$ with respect to $\widetilde{\omega}$ is just the usual extension of the GNS representation of $\mathscr{A}$ to the unitization by setting $\pi_\omega(\mathbb{1}) = \mathrm{id}$. Hence, also for the GNS construction we can safely assume that $\mathscr{A}$ is unital from the beginning. Note however that the completion is necessary to conclude that U is unitary.

Definition 7.1.13 (Cyclic *-representation) *Let $(\mathfrak{H}, \pi)$ be a *-representation of a C^*-algebra $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$. Then a vector $\Omega \in \mathfrak{H}$ is called cyclic vector if*

$$\pi(\mathscr{A})\Omega = \big\{\pi(a)\Omega \bigm| a \in \mathscr{A}\big\} \subseteq \mathfrak{H} \tag{7.1.16}$$

is dense. If $(\mathfrak{H}, \pi)$ has a cyclic vector then it is called a cyclic *-representation.

We can now use this to characterize the GNS representation as follows:

Theorem 7.1.14 (Cyclic representations) *Let $\mathscr{A}$ be a C^*-algebra and $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ a state.*

i.) The GNS representation $(\mathfrak{H}_\omega, \pi_\omega)$ of $\mathscr{A}$ with respect to ω has a cyclic vector $\Omega_\omega \in \mathfrak{H}_\omega$ such that

$$\omega(a) = \langle \Omega_\omega, \pi_\omega(a)\Omega_\omega \rangle, \tag{7.1.17}$$

explicitly given by $\Omega_\omega = \psi_{\mathbb{1}}$ if $\mathscr{A}$ is unital.

ii.) If $(\mathfrak{H}, \pi, \Omega)$ is another cyclic *-representation of $\mathscr{A}$ with $\omega(a) = \langle \Omega, \pi(a)\Omega \rangle$ then it is unitarily equivalent to the GNS representation via

$$U\colon \mathfrak{H} \ni \pi(a)\Omega \mapsto \pi_\omega(a)\Omega_\omega \in \mathfrak{H}_\omega. \tag{7.1.18}$$

iii.) The GNS representation $(\mathfrak{H}_\omega, \pi_\omega)$ is non-degenerate.

PROOF: In the unital case we know (7.1.17) from purely algebraic consideration in Section 1.3, even for general *-algebras. With Proposition 7.1.12 we can also handle the case of a non-unital $\mathscr{A}$ by taking $\Omega_\omega = \psi_{\mathbb{1}}$ in $\widetilde{\mathscr{A}}/\mathscr{J}_{\widetilde{\omega}}$ under the identification (7.1.14). Then (7.1.17) holds for $a \in \widetilde{\mathscr{A}}$ and hence in particular for $a \in \mathscr{A}$. Moreover, Ω_ω is cyclic since for all $a \in \mathscr{A}$ we have $\pi_{\widetilde{\omega}}(a)\widetilde{\psi}_{\mathbb{1}} = \widetilde{\psi}_a$. Thus $\pi_\omega(\mathscr{A})\Omega_\omega \subseteq \mathfrak{H}_\omega$ is dense since the vectors of the form $\psi_a \in \mathfrak{H}_\omega$ are dense. For the second part, let $(\mathfrak{H}, \pi, \Omega)$ be given. First we claim that U is well-defined. Indeed, if $\pi(a)\Omega = 0$ then $\omega(a^*a) = \langle \pi(a)\Omega, \pi(a)\Omega \rangle = 0$ shows $a \in \mathscr{J}_\omega$. Hence $\pi_\omega(a)\Omega_\omega = 0$ as well. Clearly, U is defined on a dense subspace and maps it onto a dense subspace by cyclicity. Finally,

$$
\begin{aligned}
\big\langle U\pi(a)\Omega, U\pi(b)\Omega \big\rangle_{\mathfrak{H}_\omega} &= \big\langle \pi_\omega(a)\Omega_\omega, \pi_\omega(b)\Omega_\omega \big\rangle_{\mathfrak{H}_\omega} \\
&= \big\langle \Omega_\omega, \pi_\omega(a^*b)\Omega_\omega \big\rangle_{\mathfrak{H}_\omega} \\
&= \omega(a^*b) \\
&= \big\langle \Omega, \pi(a^*b)\Omega \big\rangle_{\mathfrak{H}} \\
&= \big\langle \pi(a)\Omega, \pi(b)\Omega \big\rangle_{\mathfrak{H}}
\end{aligned}
$$

shows that U is isometric. Hence U extends to a unitary map $U\colon \mathfrak{H} \longrightarrow \mathfrak{H}_\omega$. Now

$$
\begin{aligned}
U\pi(a)\pi(b)\Omega &= U\pi(ab)\Omega \\
&= \pi_\omega(ab)\Omega_\omega \\
&= \pi_\omega(a)\pi_\omega(b)\Omega_\omega \\
&= \pi_\omega(a)U\pi(b)\Omega
\end{aligned}
$$

shows that U is an intertwiner on the dense subspace of all the vectors of the form $\pi(b)\Omega$. By continuity it is an intertwiner on the whole space $\mathfrak{H}$. Now let $\phi \in \mathfrak{H}_\omega$ with $\pi_\omega(a)\phi = 0$ for all a be given. Then $0 = \langle \pi_\omega(a)\phi, \Omega_\omega \rangle = \langle \phi, \pi_\omega(a)\Omega_\omega \rangle$ shows that ϕ is orthogonal to the vectors $\pi_\omega(a)\Omega_\omega$ for all $a \in \mathscr{A}$. Since Ω_ω is cyclic these vectors are dense and hence have a trivial orthogonal complement. It follows that $\phi = 0$. □

Since GNS representations are in some sense easy to handle as everything is encoded in one functional, the theorem suggests to focus on the *cyclic *-representations* of a C^*-algebra. To this end, we introduce now the concept of a direct orthogonal sum of *-representations:

Proposition 7.1.15 (Direct orthogonal sum of *-representation) *Let $\mathscr{A}$ be a C^*-algebra and $\{(\mathfrak{H}_i, \pi_i)\}_{i \in I}$ a family of *-representations.*

i.) Then for $a \in \mathscr{A}$

$$
\pi(a) = \widehat{\bigoplus_{i \in I}} \pi_i(a) \tag{7.1.19}
$$

defines a *-representation of $\mathscr{A}$ on $\mathfrak{H} = \widehat{\bigoplus}_{i \in I} \mathfrak{H}_i$.

ii.) For every $i \in I$ the subspace $\mathfrak{H}_i \subseteq \mathfrak{H}$ is subrepresentation and π restricts to π_i on $\mathfrak{H}_i$.

iii.) The *-representation $(\mathfrak{H}, \pi)$ is non-degenerate iff $(\mathfrak{H}_i, \pi_i)$ is non-degenerate for all $i \in I$.

iv.) Conversely, if $(\mathfrak{H}, \pi)$ is a *-representation and $P \in \mathfrak{B}(\mathfrak{H})$ a projection commuting with all $\pi(a)$ for $a \in \mathscr{A}$ then the $(\mathfrak{H}, \pi)$ decomposes into

$$
(\mathfrak{H}, \pi) = (\operatorname{im} P, \pi\big|_{\operatorname{im} P}) \oplus (\ker P, \pi\big|_{\ker P}). \tag{7.1.20}
$$

v.) If $(\mathfrak{H}, \pi)$ is a *-representation and $\mathfrak{K} \subseteq \mathfrak{H}$ an invariant closed subspace, i.e. $(\mathfrak{K}, \pi\big|_{\mathfrak{K}})$ is a subrepresentation, then the projection $P_{\mathfrak{K}}$ onto $\mathfrak{K}$ commutes with all $\pi(a)$ and

$$
(\mathfrak{H}, \pi) = (\mathfrak{K}, \pi\big|_{\mathfrak{K}}) \oplus (\mathfrak{K}^\perp, \pi\big|_{\mathfrak{K}^\perp}). \tag{7.1.21}
$$

PROOF: The statements *i.)* – *iii.)* are just a straightforward verification. For *iv.)* it is clear that $\mathfrak{H} = \operatorname{im} P \oplus \ker P$ as P is a projection. We have to show that $\pi(a)$ is block-diagonal with respect to this decomposition. If $\phi \in \ker P$ then $P\pi(a)\phi = \pi(a)P\phi = 0$ shows that π maps $\ker P$ into $\ker P$. If $\phi = P\psi$ then also $\pi(a)\phi = P\pi(a)\psi$ is in the image. Thus (7.1.20) follows. For the fifth part we only have to show that $P_{\mathfrak{K}}$ commutes with all $\pi(a)$ then part *iv.)* will give (7.1.32). Thus let $\phi = \phi_{\parallel} + \phi_{\perp} \in \mathfrak{H}$ be decomposed according to $\mathfrak{H} = \mathfrak{K} \oplus \mathfrak{K}^{\perp}$. Then by assumption $P_{\mathfrak{K}}\pi(a)\phi_{\parallel} = \pi(a)\phi_{\parallel} = \pi(a)P_{\mathfrak{K}}\phi_{\parallel}$. If now $\psi \in \mathfrak{H}$ is arbitrary then

$$\langle \psi, P_{\mathfrak{K}}\pi(a)\phi_{\perp} \rangle = \langle P_{\mathfrak{K}}\psi, \pi(a)\phi_{\perp} \rangle = \langle \pi(a^*)P_{\mathfrak{K}}\psi, \phi_{\perp} \rangle = \langle P_{\mathfrak{K}}\pi(a^*)\psi, \phi_{\perp} \rangle = 0,$$

which shows $P_{\mathfrak{K}}\pi(a)\phi_{\perp} = 0 = \pi(a)P_{\mathfrak{K}}\phi_{\perp}$. Thus we get $P_{\mathfrak{K}}\pi(a) = \pi(a)P_{\mathfrak{K}}$ on the whole space $\mathfrak{H}$. □

Note that the decomposition in (7.1.20) can also be written as

$$\pi(a) = P\pi(a)P + (\mathbb{1} - P)\pi(a)(\mathbb{1} - P) \tag{7.1.22}$$

for all $a \in \mathfrak{H}$. The off-diagonal blocks $P\pi(a)(\mathbb{1} - P)$ and $(\mathbb{1} - P)\pi(a)P$ are missing in this decomposition. Occasionally, we denote the restricted *-representation $\pi\big|_{\operatorname{im} P}$ simply by

$$\pi_P = \pi\big|_{\operatorname{im} P}. \tag{7.1.23}$$

Definition 7.1.16 (Direct sum of *-representations) *Let $\mathscr{A}$ be a C^*-algebra and $\{(\mathfrak{H}_i, \pi_i)\}_{i \in I}$ a family of *-representations. Then the *-representation π on $\mathfrak{H}$ according to Proposition 7.1.15 is called the direct sum of the $\{(\mathfrak{H}_i, \pi_i)\}_{i \in I}$.*

Thus we anticipate already here that the projections P which commute with all $\pi(a)$ will play a crucial role in decomposing π into a direct sum of subrepresentations.

We come now to the main result of this subsection: every non-degenerate *-representation of a C^*-algebra is the direct sum of cyclic ones and hence the direct sum of GNS representations:

Theorem 7.1.17 (Decomposition into cyclic *-representations) *Let $\mathscr{A}$ be a C^*-algebra and let $(\mathfrak{H}, \pi)$ be a non-degenerate *-representation of $\mathscr{A}$ on a Hilbert space. Then $(\mathfrak{H}, \pi)$ is unitarily equivalent to a direct sum of cyclic *-representations.*

PROOF: Note that it is necessary to assume $(\mathfrak{H}, \pi)$ to be non-degenerate since cyclic *-representations are non-degenerate by Theorem 7.1.14, *iii.)*, and the direct sum remains non-degenerate by Proposition 7.1.15, *iii.)*. We consider now the following set

$$\Omega = \left\{ \Psi \subseteq \mathfrak{H} \setminus \{0\} \;\middle|\; \text{for all } \psi, \phi \in \Psi \text{ with } \psi \neq \phi \text{ we have } \langle \pi(a)\psi, \pi(b)\phi \rangle = 0 \text{ for all } a, b \in \mathscr{A} \right\}. \tag{$*$}$$

Indeed, Ω is non-empty as the set $\{\psi\}$ consisting of a single non-zero vector is clearly in Ω. We endow Ω with a partial order by set-theoretic inclusion, i.e. $\Psi \preccurlyeq \Phi$ if $\Psi \subseteq \Phi$. Let now $\{\Psi_\alpha\}_{\alpha \in J}$ be a totally ordered subset of Ω. Then we define $\Psi = \bigcup_{\alpha \in J} \Psi_\alpha$ and claim that this is an upper bound for all the Ψ_α. Indeed, if $\psi \neq \psi' \in \Psi$ are two vectors then there is a single $\alpha \in J$ with $\psi, \psi' \in \Psi_\alpha$ since we had a total ordering. But then $\langle \pi(a)\psi, \pi(b)\psi' \rangle = 0$ for all $a, b \in \mathscr{A}$ according to $\Psi_\alpha \in \Omega$. Thus we are in the position to apply Zorn's Lemma: there is a maximal set $\Psi_{\max}$ of vectors with respect to the defining property $(*)$. We claim that these vectors give the desired form. It is clear that for all $\psi \in \Psi_{\max}$ the subspace

$$\mathfrak{H}_\psi = \left\{ \pi(a)\psi \;\middle|\; a \in \mathscr{A} \right\}^{\mathrm{cl}} \subseteq \mathfrak{H}$$

is a closed subspace, invariant under the *-representation π, and $\mathfrak{H}_\psi \perp \mathfrak{H}_\phi$ for $\psi \neq \phi$ according to the definition $(*)$ of Ω. It remains to show that $\mathfrak{H}$ coincides with $\widehat{\bigoplus}_{\psi \in \Psi_{\max}} \mathfrak{H}_\psi$, i.e. we have enough cyclic vectors. Suppose this is not the case then

$$\mathfrak{K} = \widehat{\bigoplus_{\psi \in \Psi_{\max}}} \mathfrak{H}_\psi \subseteq \mathfrak{H}$$

is a proper closed subspace of $\mathfrak{H}$. Clearly, $\mathfrak{K}$ is invariant under the *-representation π since every $\mathfrak{H}_\psi$ is invariant. Now let $\phi \in \mathfrak{K}^\perp$ then for all $\chi \in \mathfrak{K}$ we have

$$\langle \pi(a)\phi, \chi \rangle = \langle \phi, \pi(a^*)\chi \rangle = 0,$$

showing that $\pi(a)\phi \in \mathfrak{K}^\perp$ again. Thus $\mathfrak{K}^\perp$ is a non-trivial closed invariant subspace and we can fix a non-zero vector $\phi \in \mathfrak{K}^\perp$. Then

$$\mathfrak{H}_0 = \left\{ \pi(a)\phi \;\middle|\; a \in \mathscr{A} \right\}^{\mathrm{cl}} \subseteq \mathfrak{K}^\perp$$

gives yet another orthogonal cyclic piece and hence $\Psi_{\max} \cup \{\phi\} \in \Omega$, contradicting the maximality of $\Psi_{\max}$. Thus we already had $\mathfrak{H} = \widehat{\bigoplus}_{\psi \in \Psi_{\max}} \mathfrak{H}_i$. □

Remark 7.1.18 The theorem justifies that for a C^*-algebra we only have to care about cyclic *-representations which are just, up to unitary equivalence, the GNS representations. Thus the study of *-$\mathfrak{Rep}(\mathscr{A})$ is essentially traced back to the study of positive functionals of $\mathscr{A}$.

Clearly, the theorem is very inexplicit: it just guarantees that there are cyclic subrepresentations whose direct sum gives the whole *-representation. In general, it will be very hard to explicitly find the cyclic vectors and, even worse, one does not have any sort of uniqueness statements.

Example 7.1.19 Consider $\mathfrak{H} = \mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$ and $\mathscr{A} = \mathscr{C}_{\mathrm{b}}(\mathbb{R})$ as commutative unital C^*-algebra. Then for $f \in \mathrm{L}^\infty(\mathbb{R}, \mathrm{d}x)$ the multiplication operator

$$(\pi(f)\psi)(x) = f(x)\psi(x) \tag{7.1.24}$$

determines a *-representation of $\mathscr{A}$ on $\mathfrak{H}$. Since $\pi(\mathbb{1}) = \mathrm{id}$, it is clearly non-degenerate. Now let $U_n = [n, n+1) \subseteq \mathbb{R}$ for $n \in \mathbb{Z}$ and consider the subspaces $\mathrm{L}^2(U_n, \mathrm{d}x) \subseteq \mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$ then clearly

$$\mathrm{L}^2(\mathbb{R}, \mathrm{d}x) = \widehat{\bigoplus_{n \in \mathbb{Z}}} \mathrm{L}^2(U_n, \mathrm{d}x). \tag{7.1.25}$$

Moreover, all the U_n are invariant under π and the vectors $\psi_n = \chi_{U_n}$ are cyclic: indeed taking an arbitrary continuous function f on $\mathbb{R}$ gives, via $\pi(f)\psi_n = f\big|_{U_n} \in \mathscr{C}_{\mathrm{b}}(U_n)$ all bounded continuous functions on U_n. These are known to be dense in $\mathrm{L}^2(U_n, \mathrm{d}x)$. Thus we have found a decomposition into cyclic representations. However, there is also another possibility. We can take

$$\psi(x) = \mathrm{e}^{-x^2} \tag{7.1.26}$$

and have a single cyclic vector for π: Indeed, inside $\{\pi(f)\psi \mid f \in \mathscr{C}_{\mathrm{b}}(\mathbb{R})\}$ we have the subspace $\mathscr{C}_0^\infty(\mathbb{R})$ which is dense in $\mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$, too. Thus we see that in this case we have a very non-unique decomposition in cyclic *-representations.

7.1.3 Amplification and Quasi-Equivalence

We introduce now some more notation and establish the following construction: Suppose $(\mathfrak{H}, \pi)$ is a *-representation of a C^*-algebra $\mathscr{A}$ on a Hilbert space and let $\mathfrak{K}$ be another Hilbert space. We want to define a *-representation on the tensor product $\mathfrak{H} \mathbin{\hat{\otimes}} \mathfrak{K}$ originating from π. To this end we need the following simple lemma, see also Exercise 5.5.2.

Lemma 7.1.20 *If $\mathfrak{H}$, $\mathfrak{K}$ are Hilbert spaces and $\mathfrak{K} \cong \ell^2(I)$ via the choice of a Hilbert basis $\{\mathrm{e}_i\}_{i\in I}$ then for $\phi \in \mathfrak{H}$ and $\psi \in \mathfrak{K}$*

$$U(\phi \otimes \psi) = (\phi\langle \mathrm{e}_i, \psi\rangle)_{i\in I} \tag{7.1.27}$$

with $\mathfrak{H}_i = \mathfrak{H}$ for all $i \in I$ defines an isometric isomorphism

$$U\colon \mathfrak{H} \mathbin{\hat{\otimes}} \mathfrak{K} \longrightarrow \widehat{\bigoplus_{i\in I}} \mathfrak{H}_i. \tag{7.1.28}$$

PROOF: First it is clear that U is well-defined as linear map

$$U\colon \mathfrak{H} \otimes \mathfrak{K} \longrightarrow \prod_{i\in I} \mathfrak{H}_i.$$

Since for a fixed ψ only countably many $\langle \mathrm{e}_i, \psi\rangle$ are different from 0 we can compute the norm of the right hand side to be

$$\sum_{i\in I} \|\phi\langle \mathrm{e}_i, \psi\rangle\|^2_{\mathfrak{H}} = \sum_{i\in I} \|\phi\|^2_{\mathfrak{H}} |\langle \mathrm{e}_i, \psi\rangle|^2 = \|\phi\|^2_{\mathfrak{H}} \|\psi\|^2_{\mathfrak{K}} = \|\phi \otimes \psi\|^2_{\mathfrak{H}\otimes\mathfrak{K}}$$

by Parseval's equation. Thus U maps into $\widehat{\bigoplus}_{i\in I} \mathfrak{H}_i$ and is isometric. Moreover, it is clear that U restricted to $\mathfrak{H} \otimes \operatorname{span}_{\mathbb{C}}\{\mathrm{e}_i\}_{i\in I}$ is already surjective onto the algebraic direct sum $\bigoplus_{i\in I} \mathfrak{H}_i$. Thus U has dense image and gives a unitary map. □

In fact, the inverse of U can explicitly be describes by setting

$$U^{-1}(\phi_i)_{i\in I} = \sum_{i\in I} \phi_i \otimes \mathrm{e}_i. \tag{7.1.29}$$

Note that the isomorphism (7.1.28) depends on the choice of the Hilbert basis $\{\mathrm{e}_i\}_{i\in I}$ and is *not* canonical. However, we do not need to specify a Hilbert basis for $\mathfrak{H}$. We can use this construction now to *amplify* a given *-representation of $\mathscr{A}$:

Definition 7.1.21 (Amplification) *Let π be a *-representation of $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$ and let $\mathfrak{K}$ be another Hilbert space. Then the amplification of π by $\mathfrak{K}$ is the *-representation*

$$\pi \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}}\colon \mathscr{A} \ni a \;\mapsto\; \pi(a) \mathbin{\widehat{\otimes}} \mathrm{id}_{\mathfrak{K}} \in \mathfrak{B}(\mathfrak{H} \mathbin{\hat{\otimes}} \mathfrak{K}) \tag{7.1.30}$$

on the tensor product $\mathfrak{H} \mathbin{\hat{\otimes}} \mathfrak{K}$.

This turns out to be a *-representation indeed. The following proposition also explains the name amplification.

Proposition 7.1.22 *Let π be a *-representation of $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$ and let $\mathfrak{K} \neq \{0\}$ be another Hilbert space.*

i.) The amplification $\pi \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}}$ is a *-representation of $\mathscr{A}$ on $\mathfrak{H} \mathbin{\hat{\otimes}} \mathfrak{K}$.

ii.) *The amplification* $\pi \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}}$ *is unitarily equivalent to the direct sum of* $\dim \mathfrak{K}$ *copies of* π. *More explicitly, if* $\{\mathrm{e}_i\}_{i\in I}$ *is a Hilbert basis of* $\mathfrak{K}$ *then* U *defined as in* (7.1.28) *gives the unitary intertwiner to* $\widehat{\bigoplus}_{i\in I} \pi_i$ *on* $\widehat{\bigoplus}_{i\in I} \mathfrak{H}_i$ *where* $(\mathfrak{H}_i, \pi_i) = (\mathfrak{H}, \pi)$ *for all* $i \in I$.

iii.) *The amplification* $\pi \otimes \mathrm{id}_{\mathfrak{K}}$ *is non-degenerate iff* π *is non-degenerate.*

PROOF: Since $\mathfrak{B}(\mathfrak{H}) \ni A \mapsto A \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}} \in \mathfrak{B}(\mathfrak{H} \mathbin{\hat{\otimes}} \mathfrak{K})$ is a unital *-homomorphism by Corollary 3.5.12, it is clear that $\pi \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}}$ is again a *-representation. Now let $a \in \mathscr{A}$ and let $\phi \otimes \psi \in \mathfrak{H} \otimes \mathfrak{K}$ be an elementary tensor. Then

$$
\begin{aligned}
U\big((\pi \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}})(a)(\phi \otimes \psi)\big) &= U\pi(a)\phi \otimes \psi \\
&= (\pi(a)\phi\langle \mathrm{e}_i, \psi\rangle)_{i\in I} \\
&= \widehat{\bigoplus_{i\in I}} \pi(a)(\phi\langle \mathrm{e}_i, \psi\rangle)_{i\in I} \\
&= \left(\widehat{\bigoplus_{i\in I}} \pi(a)\right) U(\phi \otimes \psi).
\end{aligned}
$$

By linearity we get that U is an intertwiner. Since U is unitary by Lemma 7.1.20, the second part follows. The last part is clear by Proposition 7.1.15, *iii.)*. ☐

The representations π and its amplification $\pi \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}}$ are typically not unitarily equivalent. However, they differ only by *multiplicity*. To handle this situation one introduces the following notation:

Definition 7.1.23 (Disjoint and quasi-equivalent *-representations) *Let* $(\mathfrak{H}, \pi)$ *and* $(\mathfrak{H}', \pi')$ *be non-degenerate *-representations of* $\mathscr{A}$ *on Hilbert spaces.*

i.) *The *-representations* $(\mathfrak{H}, \pi)$ *and* $(\mathfrak{H}', \pi')$ *are called disjoint if the only intertwiner from* $(\mathfrak{H}, \pi)$ *to* $(\mathfrak{H}', \pi')$ *is* 0. *In this case one writes* $(\mathfrak{H}, \pi) \perp (\mathfrak{H}', \pi')$.

ii.) *The *-representation* $(\mathfrak{H}, \pi)$ *is called subordinate to* $(\mathfrak{H}', \pi')$ *if every nonzero subrepresentation of* $(\mathfrak{H}, \pi)$ *contains a nonzero subrepresentation which is equivalent to a nonzero subrepresentation of* $(\mathfrak{H}', \pi')$. *In this case we write* $(\mathfrak{H}, \pi) \le (\mathfrak{H}', \pi')$.

iii.) *The *-representations* $(\mathfrak{H}, \pi)$ *and* $(\mathfrak{H}', \pi')$ *are called quasi-equivalent if* $(\mathfrak{H}, \pi) \le (\mathfrak{H}', \pi')$ *and* $(\mathfrak{H}', \pi') \le (\mathfrak{H}, \pi)$. *In this case we write* $(\mathfrak{H}, \pi) \sim (\mathfrak{H}', \pi')$.

The following proposition gives a first step towards a clarification of the relations between the above notions:

Proposition 7.1.24 *Let* $(\mathfrak{H}, \pi)$ *and* $(\mathfrak{H}', \pi')$ *be non-degenerate *-representations of a* C^**-algebra* $\mathscr{A}$ *on Hilbert spaces.*

i.) *Disjointness of *-representations is a symmetric relation.*

ii.) *Being subordinate is transitive and reflexive.*

iii.) *Quasi-equivalence of *-representations is an equivalence relation.*

iv.) *The *-representation* $(\mathfrak{H}, \pi)$ *is disjoint from* $(\mathfrak{H}', \pi')$ *iff no nonzero subrepresentation of* $(\mathfrak{H}, \pi)$ *is equivalent to a subrepresentation of* $(\mathfrak{H}', \pi')$.

v.) *Equivalent representations are quasi-equivalent.*

vi.) *The *-representation* $(\mathfrak{H}, \pi)$ *is quasi-equivalent to any of its amplifications.*

PROOF: Since $A \in \mathfrak{B}(\mathfrak{H}, \mathfrak{H}')$ is an intertwiner from π to π' iff A^* is an intertwiner from π' to π, the first part is clear. For the second part we note that $(\mathfrak{H}, \pi)$ is clearly subordinate to itself. Also the transitivity is clear. Then the third part is simple consequence: we have made the relation symmetric by definition. The fourth part is more tricky: first assume that $A\colon \mathfrak{H} \longrightarrow \mathfrak{H}'$ is a non-zero intertwiner, i.e. we have $A\pi(a) = \pi'(a)A$ for all $a \in \mathscr{A}$. From the polar decomposition we know that the unique partial isometry $U\colon \mathfrak{H} \longrightarrow \mathfrak{H}'$ with $A = U|A|$ and $\ker U = \ker|A|$ intertwines the Hermitian $\pi(a)$, i.e. we have

$$U\pi(a) = \pi'(a)U \tag{$*$}$$

for $a = a^* \in \mathscr{A}$, see again Proposition 5.1.45. Since $(*)$ is clearly linear in a, it still holds for arbitrary $a \in \mathscr{A}$. Thus U is a nonzero intertwiner, too. It follows that the projections

$$P_{\operatorname{im} U^*} = U^*U \in \mathfrak{B}(\mathfrak{H})$$

$$Q_{\operatorname{im} U} = UU^* \in \mathfrak{B}(\mathfrak{H}')$$

are still non-zero self-intertwiners of π and π'. By Proposition 7.1.15 this gives nonzero subrepresentations $\pi_{P_{\operatorname{im} U^*}}$ of π and $\pi'_{Q_{\operatorname{im} U}}$ of π'. Since U is an intertwiner for π and π' it is still an intertwiner for these subrepresentations since U maps into $\operatorname{im} U$. But now U is a unitary map

$$U\colon \operatorname{im} U^* = (\ker U)^\perp \longrightarrow \operatorname{im} U$$

by Proposition 5.1.41, *iii.)*. Hence $\pi_{P_{\operatorname{im} U^*}}$ and $\pi'_{Q_{\operatorname{im} U}}$ are equivalent. Conversely, assume that there is a non-zero subrepresentation of π which is equivalent to a non-zero subrepresentation of π'. From Proposition 7.1.15, *v.)*, we know that there are non-zero projections $P \in \mathfrak{B}(\mathfrak{H})$ and $Q \in \mathfrak{B}(\mathfrak{H}')$ such that the subrepresentations are of the form π_P and π'_Q, respectively. Hence let $U\colon \operatorname{im} P \longrightarrow \operatorname{im} Q$ be the corresponding unitary intertwiner. We extend U to a partial isometry $U \in \mathfrak{B}(\mathfrak{H}, \mathfrak{H}')$ by setting

$$U\Big|_{(\operatorname{im} P)^\perp = \ker P} = 0.$$

Since π and π' are block-diagonal with respect to the splittings $\mathfrak{H} = \operatorname{im} P \oplus \ker P$ and $\mathfrak{H}' = \operatorname{im} Q \oplus \ker Q$, respectively, see Proposition 7.1.15, *v.)*, we see that U is a non-zero intertwiner from π to π'. Hence π is not disjoint from π', proving *iv.)*. Now let $(\mathfrak{H}, \pi)$ and $(\mathfrak{H}', \pi')$ be equivalent via a unitary U. Then U maps every subrepresentation of $(\mathfrak{H}, \pi)$ unitarily to a unique subrepresentation of $(\mathfrak{H}', \pi')$ and implements an equivalence between them. Thus part *v.)* is clear. Finally, let $\mathfrak{K} \neq \{0\}$ be a Hilbert space and consider the amplification $\pi \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}}$. By Proposition 7.1.22 we know that $\pi \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}}$ is equivalent to a suitable direct sum $\widehat{\bigoplus}_{i \in I} \pi_i$ of copies of $\pi_i = \pi$. Thus ever nonzero subrepresentation π' of π appears as nonzero subrepresentation π'_{i_0} of $\pi \mathbin{\hat{\otimes}} \mathrm{id}_{\mathfrak{K}}$ for a chosen $i_0 \in I$. Now let $(\mathfrak{H}', \pi') \subseteq (\mathfrak{H} \otimes \mathfrak{K}, \pi \otimes \mathrm{id}_{\mathfrak{K}})$ be a subrepresentation. Since all projections $P_{\mathfrak{H}_i} \in \mathfrak{B}(\widehat{\bigoplus}_{i \in I} \mathfrak{H}_i)$ are self-intertwiners of $\pi \otimes \mathrm{id}_{\mathfrak{K}}$ we see that for all $i \in I$ the components $P_{\mathfrak{H}_i}\mathfrak{H}' \subseteq \mathfrak{H}_i$ give subrepresentations of $(\mathfrak{H} \otimes \mathfrak{K}, \pi \otimes \mathrm{id}_{\mathfrak{K}})$ with

$$P_{\mathfrak{H}_i}\colon \mathfrak{H}' \longrightarrow \mathfrak{H}_i$$

being an intertwiner. Here we view $P_{\mathfrak{H}_i}$ as operator $P_{\mathfrak{H}_i} \in \mathfrak{B}(\mathfrak{H}, \mathfrak{H}_i)$. Indeed, if $a \in \mathscr{A}$ then

$$(\pi \otimes \mathrm{id})(a) P_{\mathfrak{H}_i}\phi = P_{\mathfrak{H}_i}(\pi \otimes \mathrm{id})(a)\phi = \pi(a)\phi_i$$

for all $\phi \in \mathfrak{H}'$ shows that $P_{\mathfrak{H}_i}\mathfrak{H}'$ is a subrepresentation, now inside $\mathfrak{H}_i \cong \mathfrak{H}$. Since not all $P_{\mathfrak{H}_i}\mathfrak{H}'$ can be trivial, we see that at least one $P_{\mathfrak{H}_i} \in \mathfrak{B}(\mathfrak{H}', \mathfrak{H}_i)$ is a non-zero intertwiner out of which we can construct unitarily equivalent subrepresentations of $\mathfrak{H}'$ and $\mathfrak{H}_i \cong \mathfrak{H}$ as in part *iv.)*. Thus also $\pi \otimes \mathrm{id}_{\mathfrak{K}}$ is subordinate to π. $\square$

Once we have developed substantially more technology we will see that quasi-equivalence is indeed just equivalence up to multiplicity, see Theorem 7.4.18.

7.1.4 Irreducible *-Representations

One main idea of *-representation theory is to decompose a given *-representation into smaller and hopefully easier ones. We have seen this already at several instances now: first we can split an arbitrary *-representation $(\mathfrak{H}, \pi)$ into its trivial part $(\mathfrak{H}_{\mathrm{trivial}}, \pi_{\mathrm{trivial}} = 0)$ and its non-degenerate part $(\mathfrak{H}_{\mathrm{nondeg}}, \pi_{\mathrm{nondeg}})$ by Theorem 7.1.6. We have also seen that amplifications split into direct sums of the original *-representation.

We take this now as a motivation for the definition of a (topologically) irreducible *-representation. They can be considered as the *atoms* of representation theory: they are not decomposable in a non-trivial way. It remains to be investigated whether all *-representations are then *molecules*, i.e. build out of irreducible ones as direct sums.

Definition 7.1.25 (Irreducible *-representation) *Let π be a non-zero *-representation of $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$. Then $(\mathfrak{H}, \pi)$ is called irreducible if the only closed subspaces of $\mathfrak{H}$ which are invariant under all the operators $\{\pi(a)\}_{a \in \mathscr{A}}$ are given by $\mathfrak{H}$ and $\{0\}$.*

Strictly speaking, one should call such a *-representation *topologically irreducible.* For an *algebraically irreducible* *-representation we would ask for no nontrivial subspace of $\mathfrak{H}$, whether closed or not. Hence we have the implication

$$\text{algebraic irreducible} \implies \text{topologically irreducible}. \tag{7.1.31}$$

However, for C^*-algebras we are clearly interested in a topological context. Hence the more interesting notion is the topological irreducibility. It is common to refer to this as just *irreducibility.* In any case, we will come back to the reverse implication in (7.1.31) in Subsection 7.2.3.

Since a closed invariant subspace is the same thing as a subrepresentation, a *-representation $(\mathfrak{H}, \pi)$ is irreducible iff it has no non-trivial subrepresentations, i.e. others than $\{0\}$ and $\mathfrak{H}$. The following theorem gives a first characterization of irreducible *-representations. It is the Hilbert space generalization of the finite-dimensional Schur lemma:

Theorem 7.1.26 (Schur's lemma) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a nonempty set of bounded operators such that $A \in \mathscr{A}$ iff $A^* \in \mathscr{A}$. Then the following statements are equivalent:*

i.) The only closed subspaces $\mathfrak{K} \subseteq \mathfrak{H}$ which are invariant under all $A \in \mathscr{A}$ are $\mathfrak{K} = \{0\}$ and $\mathfrak{H}$.

ii.) The only operators $B \in \mathfrak{B}(\mathfrak{H})$ commuting with all $A \in \mathscr{A}$ are $B = z \, \mathrm{id}_{\mathfrak{H}}$ with $z \in \mathbb{C}$.

iii.) There is no non-trivial orthogonal decomposition

$$\mathfrak{H} = \bigoplus_{i \in I} \mathfrak{H}_i \tag{7.1.32}$$

into $\mathscr{A}$-invariant closed subspaces $\mathfrak{H}_i \subseteq \mathfrak{H}$.

iv.) For every $\phi \neq 0$ the subspace $\operatorname{span}_{\mathbb{C}}\{A_1 \cdots A_n \phi\}_{A_1, \ldots, A_n \in \mathscr{A}, n \in \mathbb{N}}$ is either dense in $\mathfrak{H}$ or $\mathscr{A} = \{0\}$ and $\mathfrak{H} \cong \mathbb{C}$.

PROOF: We prove *i.)* $\implies$ *ii.)* $\implies$ *iii.)* $\implies$ *i.)* $\iff$ *iv.)*. Thus assume *i.)* and let $B \in \mathfrak{B}(\mathfrak{H})$ satisfy $BA = AB$ for all $A \in \mathscr{A}$. Then also B^* has this property by $A^* \in \mathscr{A}$ iff $A \in \mathscr{A}$. Now assume that B is not a multiple of the identity. It follows that $\operatorname{Re}(B)$ and $\operatorname{Im}(B)$ can not be both multiples of the identity. Thus we have found a *Hermitian* operator not being a multiple of the identity which commutes with all $A \in \mathscr{A}$. By the spectral theorem for bounded normal operators, see Theorem 5.1.32, *i.)*, we find at least one spectral projection E_U for some measurable subset of the spectrum which is different from 0 and $\mathbb{1}$. Indeed, this follows from the uniqueness of the spectral measure at once. But then E_U still commutes with all $A \in \mathscr{A}$ by Theorem 5.1.32, *iii.)*. This shows that $\operatorname{im} E_U$ is a non-trivial closed subspace mapped into itself by all $A \in \mathscr{A}$, contradicting *i.)*. Hence *i.)* $\implies$ *ii.)* follows. Now assume *ii.)* and assume that a non-trivial decomposition (7.1.32) into $\mathscr{A}$-invariant subspaces can be found. Then $P_{\mathfrak{H}_i}$ commutes with all $A \in \mathscr{A}$ contradicting *ii.)*, thereby showing *ii.)* $\implies$ *iii.)*. The implication *iii.)* $\implies$ *i.)* is trivial. Finally, assume *i.)*. If $A\phi = 0$ for all $A \in \mathscr{A}$ the $\operatorname{span}_{\mathbb{C}}\{\phi\} \neq \{0\}$ is closed and invariant and hence by *i.)* we have $\operatorname{span}_{\mathbb{C}}\{\phi\} = \mathfrak{H}$. Thus $\mathfrak{H}$ was one-dimensional and thus necessarily all $A = 0$. This is one case. Thus assume $A\phi \neq 0$ for at least one A. But then $\operatorname{span}_{\mathbb{C}}\{A_1 \dots A_n\phi\}_{A_i \in \mathscr{A}, n \in \mathbb{N}}$ is at least one-dimensional so the invariant closed subspace $(\operatorname{span}_{\mathbb{C}}\{A_1 \dots A_n\phi\}_{A_i \in \mathscr{A}, n \in \mathbb{N}})^{\mathrm{cl}}$ has to be $\mathfrak{H}$ by *i.)*, proving the density. Conversely, assume *iv.)*. Since the case $\mathscr{A} = \{0\}$ and $\mathfrak{H} \cong \mathbb{C}$ clearly gives *i.)* we pass to the interesting case where $\operatorname{span}_{\mathbb{C}}\{A\phi\}_{A \in \mathscr{A}}$ is dense for every $\phi \neq 0$. Suppose now $\mathfrak{K} \subseteq \mathfrak{H}$ is a closed invariant subspace with $\mathfrak{K} \neq \{0\}$. Then pick $\phi \in \mathfrak{K}$ with $\phi \neq 0$. By invariance $\operatorname{span}_{\mathbb{C}}\{A\phi\} \subseteq \mathfrak{K}$ and by closedness of $\mathfrak{K}$ also $\mathfrak{H} = (\operatorname{span}_{\mathbb{C}}\{A\phi\})^{\mathrm{cl}} \subseteq \mathfrak{K}$ which shows $\mathfrak{K} = \mathfrak{H}$ and hence *i.)*. □

From the proof we get a few further statements about invariant subspaces:

Proposition 7.1.27 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a subset of operators with $A \in \mathscr{A}$ iff $A^* \in \mathscr{A}$, and let $\mathfrak{K} \subseteq \mathfrak{H}$ be a closed subspace. Then the following statements are equivalent:*

i.) The subspace $\mathfrak{K}$ is invariant under all $A \in \mathscr{A}$.

ii.) The orthogonal complement $\mathfrak{K}^\perp$ is invariant under all $A \in \mathscr{A}$.

iii.) The projection $P_{\mathfrak{K}}$ commutes with all $A \in \mathscr{A}$.

PROOF: Indeed, let $\phi \in \mathfrak{K}$ and $\psi \in \mathfrak{K}^\perp$ and assume *i.)*. Then $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle = 0$ since $A^*\phi \in \mathfrak{K}$. Thus $A\psi \in \mathfrak{K}^\perp$ follows which is *ii.)*. Since $\mathfrak{K} = \mathfrak{K}^{\perp\perp}$ we get *ii.)* $\implies$ *i.)* as well. This also means that A is block-diagonal with respect to $\mathfrak{H} = \mathfrak{K} \oplus \mathfrak{K}^\perp$ and hence *iii.)* follows. The implication *iii.)* $\implies$ *i.)* is clear. □

Note that up to now $\mathscr{A}$ was just a subset stable under the *-involution. This will allow to use the above statements not only for *-representations of C^*-algebras but also in much wider contexts.

Example 7.1.28 (Unitary group representation) Let G be a group and denote the unitary group of $\mathfrak{H}$ by $\mathrm{U}(\mathfrak{H})$. Then a unitary group representation U of G on $\mathfrak{H}$ is a group homomorphism

$$U \colon G \longrightarrow \mathfrak{U}(\mathfrak{H}), \tag{7.1.33}$$

i.e. for $g, h \in G$ we have

$$U_g U_h = U_{gh} \text{ and } U_e = \mathrm{id}_{\mathfrak{H}}. \tag{7.1.34}$$

It follows from the unitarity that

$$U_{g^{-1}} = U_g^{-1} = U_g^* \tag{7.1.35}$$

and hence the set of operator $\{U_g\}_{g \in G}$ satisfies the hypotheses of Theorem 7.1.26 and Proposition 7.1.27. Thus we can start discussing the notions of irreducibility and invariant subspaces of the unitary representation U of G by the very same techniques provided in Theorem 7.1.26 and Proposition 7.1.27. We will come back to these questions in Section . bastian: sec (

Applied to a *-representation of a C^*-algebra $\mathscr{A}$ we see that also $\pi(\mathscr{A}) \subseteq \mathfrak{B}(\mathfrak{H})$ satisfies the hypotheses of Theorem 7.1.26 and Proposition 7.1.27. This gives the following corollaries:

Corollary 7.1.29 *Let π be a non-zero *-representation of a C^*-algebra $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

i.) The *-representation $(\mathfrak{H}, \pi)$ is irreducible.

ii.) The only operators $B \in \mathfrak{B}(\mathfrak{H})$ commuting with all $\pi(a)$ with $a \in \mathscr{A}$ are $B = z\,\mathrm{id}_{\mathfrak{H}}$ with $z \in \mathbb{C}$.

iii.) *There is no nontrivial direct sum decomposition of $(\mathfrak{H}, \pi)$ into subrepresentations.*

iv.) *Every vector $\phi \neq 0$ is cyclic.*

Corollary 7.1.30 *Let π be a *-representation of a C^*-algebra $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$ and let $\mathfrak{K} \subseteq \mathfrak{H}$ be a closed subspace. Then the following statements are equivalent:*

i.) *The restriction $(\mathfrak{K}, \pi\big|_{\mathfrak{K}})$ is a subrepresentation.*

ii.) *The restriction $(\mathfrak{K}^\perp, \pi\big|_{\mathfrak{K}^\perp})$ is a subrepresentation.*

iii.) *The projection $P_{\mathfrak{K}}$ commutes with all $\pi(a)$ for $a \in \mathscr{A}$.*

iv.) *The *-representation $(\mathfrak{H}, \pi)$ is a direct sum*

$$(\mathfrak{H}, \pi) = \left(\mathfrak{K}, \pi\big|_{\mathfrak{K}}\right) \oplus \left(\mathfrak{K}^\perp, \pi\big|_{\mathfrak{K}^\perp}\right). \tag{7.1.36}$$

In particular, an irreducible *-representation (different from the trivial *-representation $\pi = 0$ on either $\mathfrak{H} = \{0\}$ or $\mathfrak{H} = \mathbb{C}$) is necessarily *cyclic* and hence a *GNS representation* by Theorem 7.1.14, *ii.)*. The corresponding positive functional is e.g.

$$\omega(a) = \langle \phi, \pi(a)\phi \rangle, \tag{7.1.37}$$

where ϕ is any non-zero vector. Thus the study of irreducible *-representation can be traced back to the study of *certain* positive linear functionals: it remains to be investigated which ω actually have an irreducible GNS representation, a clarification that has to wait till Section 7.4.1.

The following statement gives our first conclusion on irreducible *-representations:

Theorem 7.1.31 (Disjointness of irreducible *-representations) *Let π and π' be two (non-zero) irreducible *-representation of a C^*-algebra $\mathscr{A}$ on Hilbert spaces $\mathfrak{H}$ and $\mathfrak{H}'$, respectively. Then either the *-representations are disjoint or equivalent.*

Proof: Suppose they are not disjoint. Then there is a non-zero intertwiner $A\colon (\mathfrak{H}, \pi) \longrightarrow (\mathfrak{H}', \pi')$. But then A^*A is a self-intertwiner of $(\mathfrak{H}, \pi)$ and hence a non-zero multiple of the identity by Corollary 7.1.29, *ii.)*. Analogously, AA^* is a nonzero self-intertwiner of $(\mathfrak{H}', \pi')$ and thus a non-zero multiple of the identity as well. It follows that A is invertible and hence the *-representations are equivalent.□

By Proposition 7.1.4 we know that in this case we even find a unitary equivalence.

Example 7.1.32 Let $\mathscr{A} = \mathfrak{K}(\mathfrak{H})$ be the C^*-algebra of compact operators on $\mathfrak{H}$ acting on $\mathfrak{H}$ as usual. This is the defining *-representation.

i.) The defining *-representation is clearly irreducible: we can e.g. use the criterion *iv.)* from Corollary 7.1.29: if $\phi \neq 0$ and ψ is any other vector then we have

$$\psi = \Theta_{\psi, \frac{\phi}{\langle \phi, \phi \rangle}}(\phi), \tag{7.1.38}$$

showing that ϕ is cyclic since $\mathfrak{F}(\mathfrak{H}) \subseteq \mathfrak{K}(\mathfrak{H})$.

ii.) The functional $\omega_\phi\colon A \mapsto \langle \phi, A\phi \rangle$ is clearly positive for every $\phi \neq 0$. The corresponding GNS representation of $\mathfrak{K}(\mathfrak{H})$ with respect to ω_ϕ is, by the uniqueness statement from Theorem 7.1.14, *ii.)*, and the fact that ϕ is cyclic, given, up to unitary equivalence, by the defining *-representation again. This can of course also be checked directly.

n: Exercies!!!

Analogous statements hold for $\mathfrak{B}(\mathfrak{H})$ instead of $\mathfrak{K}(\mathfrak{H})$ the C^*-algebra of bounded operators $\mathfrak{B}(\mathfrak{H})$ acts irreducibly on $\mathfrak{H}$ and every non-zero vector $\phi \in \mathfrak{H}$ is cyclic.

7.2 Basic Theory of von Neumann Algebras

We have seen that the self-intertwiners of a *-representation play a crucial role in understanding the irreducibility of it. Thus it will be important to understand the structure of self-intertwiners in more detail. It turns out that they enjoy remarkable properties leading to the notion of a von Neumann algebra. The theory of von Neumann algebras is far too rich to be discussed here even in an only approximatively complete way. For a general discussion we refer to [28, 29, 54–56]. Instead, we will focus here just on basic aspects of the theory.

7.2.1 Commutants

We start with some basic and purely algebraic consideration concerning operators *which commute with something.* In fact, we are not restricted to operators at all.

We consider the following situation. Suppose $\mathscr{B}$ is an associative unital algebra and $\mathscr{A} \subseteq \mathscr{B}$ is some subset. Then we are interested in all those elements of $\mathscr{B}$ which commute with all elements from $\mathscr{A}$:

Definition 7.2.1 (Commutant) *Let $\mathscr{B}$ be a unital associative algebra and $\mathscr{A} \subseteq \mathscr{B}$ a non-empty subset. Then the commutant $\mathscr{A}'$ of $\mathscr{A}$ in $\mathscr{B}$ is defined by*

$$\mathscr{A}' = \left\{ b \in \mathscr{B} \;\middle|\; ab = ba \textit{ for all } a \in \mathscr{A} \right\} \subseteq \mathscr{B}. \tag{7.2.1}$$

Note that it is crucial to fix $\mathscr{B}$ in order to define the commutant of $\mathscr{A}$. If $\mathscr{B} \subseteq \mathscr{C}$ is in some other unital algebra then the commutants of $\mathscr{A}$ with respect to $\mathscr{B}$ and with respect to $\mathscr{C}$ can be very different. However, from the context it is usually clear inside which algebra $\mathscr{B}$ we consider commutants. Hence we will suppress this dependence in our notion.

The commutant of a subset automatically enjoys nice algebraic properties:

Proposition 7.2.2 *Let $\mathscr{B}$ be a unital associative algebra and $\mathscr{A} \subseteq \mathscr{B}$ a non-empty subset.*

i.) The commutant $\mathscr{A}'$ is a unital subalgebra of $\mathscr{B}$.

ii.) If $\mathscr{A} \subseteq \mathscr{C}$ then $\mathscr{C}' \subseteq \mathscr{A}'$.

iii.) One has $\mathscr{A} \subseteq \mathscr{A}''$.

PROOF: Let $a \in \mathscr{A}$ and $b, c \in \mathscr{A}'$. Then for $z, w \in \mathbb{C}$ we have $(zb+wc)a = zba+wca = a(zb)+a(wc) = a(zb+wc)$. Hence $zb+wc \in \mathscr{A}'$. Moreover $\mathbb{1} \in \mathscr{A}'$ is clear since $\mathbb{1}a = a = a\mathbb{1}$ for any element a. Finally, $(bc)a = bac = abc$ shows $bc \in \mathscr{A}'$ and thus the first part follows. The second part is clear since for a larger set $\mathscr{C}$ we ask for more conditions on its commutant $\mathscr{C}'$. Finally, $a \in \mathscr{A}$ commutes with every $b \in \mathscr{A}'$ by definition. Hence $a \in \mathscr{A}''$. □

Formally, taking the commutant behaves very much like taking the orthogonal complement in a (pre-) Hilbert space. Thus we get analogously to Remark 3.1.8 the property

$$\mathscr{A}' = \mathscr{A}''' = \mathscr{A}''''' = \cdots \tag{7.2.2}$$

as well as

$$\mathscr{A}'' = \mathscr{A}'''' = \cdots \tag{7.2.3}$$

for any subset $\mathscr{A} \subseteq \mathscr{B}$. However, unlike for subspaces in a (pre-) Hilbert space the intersection $\mathscr{A}' \cap \mathscr{A}''$ can be quite arbitrary: e.g. for a commutative $\mathscr{B}$ we always have $\mathscr{A}' = \mathscr{B}$ and hence also $\mathscr{A}' \cap \mathscr{A}'' = \mathscr{B}$ for every subset $\mathscr{A} \subseteq \mathscr{B}$.

Proposition 7.2.3 *Let $\mathscr{B}$ be a unital associative algebra and consider*

$$\mathcal{L}(\mathscr{B}) = \left\{ \mathscr{A} \subseteq \mathscr{B} \;\middle|\; \mathscr{A} = \mathscr{A}'' \right\}. \tag{7.2.4}$$

Then $\mathcal{L}(\mathscr{B})$ becomes a lattice via the operations

$$\mathscr{A}_1 \vee \mathscr{A}_2 = (\mathscr{A}_1 \cup \mathscr{A}_2)'' \tag{7.2.5}$$

and

$$\mathscr{A}_1 \wedge \mathscr{A}_2 = \mathscr{A}_1 \cap \mathscr{A}_2. \tag{7.2.6}$$

Each $\mathscr{A} \in \mathcal{L}(\mathscr{B})$ is a unital subalgebra and $\mathcal{L}(\mathscr{B})$ has the minimal element $\mathscr{Z}(\mathscr{B}) = \{\mathbb{1}\}''$ and the maximal element $\mathscr{B}$. The lattice relation "$\leq$" is just the set-theoretic "$\subseteq$".

PROOF: First it is clear that $\mathscr{A}_1 \vee \mathscr{A}_2 \in \mathcal{L}(\mathscr{B})$. For $\mathscr{A}_1 \wedge \mathscr{A}_2$ we first note that $\mathscr{A}_1 \cap \mathscr{A}_2 \subseteq (\mathscr{A}_1 \cap \mathscr{A}_2)''$. Since $\mathscr{A}_1 \cap \mathscr{A}_2 \subseteq \mathscr{A}_1$ we get by twice applying Proposition 7.2.2, *ii.)*, the inclusion $(\mathscr{A}_1 \cap \mathscr{A}_2)'' \subseteq \mathscr{A}_1''$ which gives $(\mathscr{A}_1 \cap \mathscr{A}_2)'' \subseteq \mathscr{A}_1$ by definition of $\mathcal{L}(\mathscr{B})$. Analogously we get $(\mathscr{A}_1 \cap \mathscr{A}_2)'' \subseteq \mathscr{A}_2$ and hence $(\mathscr{A}_1 \cap \mathscr{A}_2)'' = \mathscr{A}_1 \cap \mathscr{A}_2$ follows. Thus $\wedge$ and $\vee$ are well-defined maps $\wedge, \vee \colon \mathcal{L}(\mathscr{B}) \times \mathcal{L}(\mathscr{B}) \longrightarrow \mathcal{L}(\mathscr{B})$. Now $\vee$ clearly is associative and commutative and satisfies $\mathscr{A} \vee \mathscr{A} = \mathscr{A}$. For $\vee$ we have first

$$\mathscr{A}_1, \mathscr{A}_2 \subseteq (\mathscr{A}_1 \cup \mathscr{A}_2)'' = \mathscr{A}_1 \vee \mathscr{A}_2$$

by Proposition 7.2.2, *iii.)*. On the other hand, $\mathscr{A}_1 \vee \mathscr{A}_2$ is the smallest subset of $\mathscr{B}$ in $\mathcal{L}(\mathscr{B})$ which contains both $\mathscr{A}_1$ and $\mathscr{A}_2$: Indeed, if $\mathscr{C} \in \mathcal{L}(\mathscr{B})$ satisfies $\mathscr{A}_1 \cup \mathscr{A}_2 \subseteq \mathscr{C}$ then twice applying Proposition 7.2.2, *iii.)* gives $(\mathscr{A}_1 \cup \mathscr{A}_2)'' \subseteq \mathscr{C}'' = \mathscr{C}$. Now it is clear that

$$\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3 \subseteq \mathscr{A}_1 \vee (\mathscr{A}_2 \vee \mathscr{A}_3),$$

and hence $\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3 \subseteq \mathscr{A}_1 \vee (\mathscr{A}_2 \vee \mathscr{A}_3)$ leading to

$$(\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3)'' \subseteq \mathscr{A}_1 \vee (\mathscr{A}_2 \vee \mathscr{A}_3).$$

Since $\mathscr{A}_2 \cup \mathscr{A}_3 \subseteq \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$ we get

$$\mathscr{A}_2 \vee \mathscr{A}_3 = (\mathscr{A}_2 \cup \mathscr{A}_3)'' \subseteq (\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3)'',$$

as well as

$$\mathscr{A}_1 \subseteq (\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3)''.$$

Thus also

$$\mathscr{A}_1 \cup (\mathscr{A}_2 \vee \mathscr{A}_3) \subseteq (\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3)''.$$

Again, this gives

$$\mathscr{A}_1 \vee (\mathscr{A}_2 \vee \mathscr{A}_3) = (\mathscr{A}_1 \cup (\mathscr{A}_2 \vee \mathscr{A}_3))'' \subseteq (\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3)''.$$

Thus we conclude

$$\mathscr{A}_1 \vee (\mathscr{A}_2 \vee \mathscr{A}_3) = (\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3)'',$$

and analogously we have $(\mathscr{A}_1 \vee \mathscr{A}_2) \vee \mathscr{A}_3 = (\mathscr{A}_1 \cup \mathscr{A}_2 \mathscr{A}_3)''$. Thus also $\vee$ satisfies associativity as well as commutativity. Finally, we check

$$(\mathscr{A}_1 \vee \mathscr{A}_2) \wedge \mathscr{A}_1 = (\mathscr{A}_1 \cup \mathscr{A}_2)'' \cap \mathscr{A}_1 \subseteq \mathscr{A}_1, \tag{$*$}$$

and since $\mathscr{A}_1 \subseteq \mathscr{A}_1 \cup \mathscr{A}_2$ we have $\mathscr{A}_1 \subseteq (\mathscr{A}_1 \cup \mathscr{A}_2)''$. Similarly,

$$(\mathscr{A}_1 \wedge \mathscr{A}_2) \vee \mathscr{A}_1 = ((\mathscr{A}_1 \cap \mathscr{A}_2) \cup \mathscr{A}_1)'' = \mathscr{A}_1'' = \mathscr{A}_1,$$

which establishes that $\mathcal{L}(\mathscr{B})$ becomes a lattice via $\wedge$ and $\vee$. Now $\mathscr{A}_1 \leq \mathscr{A}_2$ iff $\mathscr{A}_1 \cap \mathscr{A}_2 = \mathscr{A}_1$ by Lemma 3.2.4. But $\mathscr{A}_1 \cap \mathscr{A}_2 = \mathscr{A}_1 \wedge \mathscr{A}_2$ then shows that $\leq$ is just $\subseteq$. Clearly $\mathscr{B} \in \mathcal{L}(\mathscr{B})$ is the maximal element. Since $\mathbb{1}$ commutes with everything we have $\mathbb{1} \in \mathscr{A}'' = \mathscr{A}$ for any $\mathscr{A}$. Thus also $\{\mathbb{1}\}'' \subseteq \mathscr{A}$ follows making $\{\mathbb{1}\}''$ the smallest element in $\mathcal{L}(\mathscr{B})$. But $\{\mathbb{1}\}' = \mathscr{B}$ and thus $\{\mathbb{1}\}$ consists of those elements in $\mathscr{B}$ which commute with everything, i.e. the central elements. Thus $\mathscr{Z}(\mathscr{B}) = \{\mathbb{1}\}''$. □

Remark 7.2.4 (Galois correspondence) Notably, the lattice properties follow directly from the features *ii.)* and *iii.)* in Proposition 7.2.2 and are hence applicable also in other contexts. Sometimes such a map is called a *Galois correspondence* as it also occurs in the Galois theory of field extensions. However, in general the commutant $\mathscr{A} \mapsto \mathscr{A}'$ does not yield an *orthocomplemented* lattice: even though we have the properties

$$\mathscr{A}_1 \le \mathscr{A}_2 \implies \mathscr{A}_2' \le \mathscr{A}_1' \tag{7.2.7}$$

and

$$\mathscr{A}'' = \mathscr{A} \tag{7.2.8}$$

for $\mathscr{A}, \mathscr{A}_1, \mathscr{A}_2 \in \mathcal{L}(\mathscr{B})$, we only get

$$\mathscr{A} \wedge \mathscr{A}' = \mathscr{Z}(\mathscr{A}) \supseteq \mathscr{Z}(\mathscr{B}), \tag{7.2.9}$$

where $\mathscr{Z}(\mathscr{A})$ is the *center* of $\mathscr{A}$. This can be strictly larger than $\mathscr{Z}(\mathscr{B})$. Examples are easy to find. Every *commutative* subalgebra $\mathscr{A} = \mathscr{A}''$ of an algebra $\mathscr{B}$ with trivial center $\mathscr{Z}(\mathscr{B}) = \mathbb{C}\mathbb{1}$ will do the job, see also Exercise **??**. Exercise: Exe

We conclude our discussion of the algebraic features of commutants with the case of a *-algebra:

Proposition 7.2.5 *Let $\mathscr{B}$ be a unital *-algebra.*

i.) If $\mathscr{A} \subseteq \mathscr{B}$ is a subset with $a \in \mathscr{A}$ iff $a^* \in \mathscr{A}$ then $\mathscr{A}' \subseteq \mathscr{B}$ is a unital *-subalgebra.

ii.) The set

$$\mathrm{vN}(\mathscr{B}) = \left\{ \mathscr{A} \in \mathcal{L}(\mathscr{B}) \;\middle|\; \mathscr{A} \textit{ is stable under } ^* \right\} \subseteq \mathcal{L}(\mathscr{B}) \tag{7.2.10}$$

is a sublattice containing the minimal and the maximal element of $\mathcal{L}(\mathscr{B})$.

PROOF: Let $a \in \mathscr{A}$ and $b \in \mathscr{A}'$ then $b^*a = (a^*b) = (ba^*) = ab^*$ shows $b^* \in \mathscr{A}'$ as well. The second part is easy since if $\mathscr{A}_1, \mathscr{A}_2 \in \mathrm{vN}(\mathscr{B})$ then their intersection $\mathscr{A}_1 \cap \mathscr{A}_2 = \mathscr{A}_1 \wedge \mathscr{A}_2$ is clearly again stable under the *-involution. Since also $\mathscr{A}_1 \cup \mathscr{A}_2$ is stable, we see that $\mathscr{A}_1 \vee \mathscr{A}_2 \in \mathrm{vN}(\mathscr{B})$, too. Clearly, the center of a *-algebra is always a *-subalgebra. Thus $\mathscr{Z}(\mathscr{B}) \in \mathrm{vN}(\mathscr{B})$ and $\mathscr{B} \in \mathrm{vN}(\mathscr{B})$ is trivial. □

7.2.2 The Bicommutant Theorem

We specialize now our discussion again to the case of the bounded operators $\mathfrak{B}(\mathfrak{H})$ on a Hilbert space $\mathfrak{H}$ as environment for considering commutants.

In the following, let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a (necessarily) unital *-subalgebra of all bounded operators. We want to understand now the algebraic property $\mathscr{A} = \mathscr{A}''$ in terms of topological features of $\mathscr{A}$.

Lemma 7.2.6 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a unital *-subalgebra of the bounded operators on a Hilbert space $\mathfrak{H}$ such that*

$$\mathscr{A} = \mathscr{A}''. \tag{7.2.11}$$

Then $\mathscr{A}$ is weakly closed.

PROOF: Let $(A_i)_{i\in I}$ be a net in $\mathscr{A}$ such that $A_i \longrightarrow A \in \mathfrak{B}(\mathfrak{H})$ converges in the weak operator topology, i.e.

$$\langle \phi, A_i\psi\rangle \longrightarrow \langle \phi, A\psi\rangle$$

for all ϕ, ψ. Moreover, let $B \in \mathscr{A}'$ commute with every element of $\mathscr{A}$. Then

$$\langle \phi, AB\psi\rangle = \lim_{i\in I}\langle \phi, A_iB\psi\rangle = \lim_{i\in I}\langle \phi, BA_i\psi\rangle = \lim_{i\in I}\langle B^*\phi, A_i\psi\rangle = \langle B^*\phi, A\psi\rangle = \langle \phi, BA\psi\rangle$$

for all $\phi, \psi \in \mathfrak{H}$. This implies that $AB = BA$ and thus $A \in \mathscr{A}'' = \mathscr{A}$. □

Since the weak topology of $\mathfrak{B}(\mathfrak{H})$ is the coarsest topology from the zoo in Section 5.1.1 according to Theorem 5.1.10, *ii.)*, we get the following corollary by general arguments:

Corollary 7.2.7 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a unital *-subalgebra with $\mathscr{A} = \mathscr{A}''$. Then $\mathscr{A}$ is closed with respect to*

i.) the norm topology, i.e. $\mathscr{A}$ is a C^*-subalgebra,

ii.) the σ-strong* topology,

iii.) the σ-strong topology,

iv.) the σ-weak topology,

v.) the strong* topology,

vi.) the strong topology,

vii.) the weak topology.

PROOF: Indeed, by Theorem 5.1.10, *ii.)* the statement *vii.)* implies all the others. □

It is fairly easy to construct C^*-subalgebras of $\mathfrak{B}(\mathfrak{H})$ which are not closed in any of the other, coarser topologies than the norm topology:

Example 7.2.8 Let $\mathfrak{H}$ be an infinite-dimensional Hilbert space. For every $A \in \mathfrak{B}(\mathfrak{H})$ and $\phi_1, \ldots, \phi_n \in \mathfrak{H}$ we consider the following finite rank operator

$$F = AP_{\operatorname{span}_{\mathbb{C}}\{\phi_1,\ldots,\phi_n\}} \in \mathfrak{F}(\mathfrak{H}) \tag{7.2.12}$$

with rank $\leq n$. Thus we have in the strong topology for this F the property

$$\|A - F\|_{\phi_1} = \cdots = \|A - F\|_{\phi_n} = 0. \tag{7.2.13}$$

Since the open balls with respect to the seminorms $\|\cdot\|_{\phi_n}$ constitute a subbasis of the strong topology, F is in such an open neighbourhood of A in the strong topology. Since the $\phi_1, \ldots, \phi_n$ are arbitrary, we get an operator in $\mathfrak{F}(\mathfrak{H})$ in every strong open neighbourhood of A. Thus the strong closure of $\mathfrak{F}(\mathfrak{H})$ is $\mathfrak{B}(\mathfrak{H})$. Hence also the strong closure of $\mathfrak{K}(\mathfrak{H})$ is $\mathfrak{B}(\mathfrak{H})$ even though $\mathfrak{K}(\mathfrak{H})$ was norm-closed.

In view of this example it is remarkable that, as it will turn out, the closedness in the σ-strong* topology already does imply the closedness in the weak topology. Hence all conditions of Corollary 7.2.7 *except the first* turn out to be in fact equivalent. Moreover, they are equivalent to $\mathscr{A} = \mathscr{A}''$. In order to prove this fundamental statement we need some preparations.

Knowing the precise form of the continuous linear functionals in the various operator topologies of $\mathfrak{B}(\mathfrak{H})$ according to Theorem 6.3.11 and Theorem 6.3.15 we can easily show the following properties. They also follow in larger generality from more general arguments but for the time being the following will suffice:

Proposition 7.2.9 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a subspace.*

i.) The closures of $\mathscr{A}$ in the σ-strong*, the σ-strong, and the σ-weak topology coincide.

ii.) The closures of $\mathscr{A}$ in the strong*, the strong, and the weak topology coincide.

PROOF: By Theorem 5.1.10, *ii.)* we have the following inclusions of the closures

$$\mathscr{A} \subseteq \mathscr{A}^{\mathrm{cl}_{\sigma\text{-strong}^*}} \subseteq \mathscr{A}^{\mathrm{cl}_{\sigma\text{-strong}}} \subseteq \mathscr{A}^{\mathrm{cl}_{\sigma\text{-weak}}}.$$

Now suppose $A \in \mathscr{A}^{\mathrm{cl}_{\sigma\text{-weak}}}$ is not in $\mathscr{A}^{\mathrm{cl}_{\sigma\text{-strong}^*}}$. By Corollary 2.2.44 we find a σ-strongly continuous functional $\omega\colon \mathfrak{B}(\mathfrak{H}) \longrightarrow \mathbb{C}$ such that $\omega(A) \neq 0$ but $\omega\big|_{\mathscr{A}^{\mathrm{cl}_{\sigma\text{-strong}^*}}} = 0$. By Theorem 6.3.11, ω is also σ-weakly continuous. Since $\mathscr{A}$ and hence also $\mathscr{A}^{\mathrm{cl}_{\sigma\text{-strongly}^*}}$ is dense in $\mathscr{A}^{\mathrm{cl}_{\sigma\text{-weak}}}$ in the σ-weak topology ω has to vanish on $\mathscr{A}^{\mathrm{cl}_{\sigma\text{-weak}}}$ which is a contradiction. Thus all closures coincide. The second part is analogous using Theorem 6.3.15 instead. □

Now let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a *-subalgebra then we consider the amplification of $\mathscr{A}$ by $\mathbb{N}$, i.e. the Hilbert space

$$\widetilde{\mathfrak{H}} = \widehat{\bigoplus_{n=1}^{\infty}} \mathfrak{H}_n \quad \text{with} \quad \mathfrak{H}_n = \mathfrak{H}, \tag{7.2.14}$$

which is just $\mathfrak{H} \mathbin{\hat{\otimes}} \ell^2$. On $\widetilde{\mathfrak{H}}$ the operators from $\mathscr{A}$ act like

$$\widetilde{A}(\phi_n)_{n\in\mathbb{N}} = (A\phi_n)_{n\in\mathbb{N}}, \tag{7.2.15}$$

which is just $\widetilde{A} = A \otimes \mathrm{id}_{\ell^2}$ according to Proposition 7.1.22, *ii.)*. Clearly, $A \mapsto \widetilde{A}$ is a *-homomorphism. We are now interested in the relation between the commutants of $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ and $\widetilde{\mathscr{A}} \subseteq \mathfrak{B}(\widetilde{\mathfrak{H}})$.

Lemma 7.2.10 *Let $B \in \mathfrak{B}(\widetilde{\mathfrak{H}})$. Then we have $B \in (\widetilde{\mathscr{A}})'$ iff for all components $B_{nm} \in \mathscr{A}'$.*

PROOF: Recall that every $B \in \mathfrak{B}(\widetilde{\mathfrak{H}})$ is an infinite block matrix $B = (B_{nm})_{n,m\in\mathbb{N}}$ with certain $B_{nm} \in \mathfrak{B}(\mathfrak{H})$ according to Proposition 3.5.11, *i.)*. Suppose B commutes with every $\widetilde{A}$ for $A \in \mathscr{A}$. Then applied to $\Phi = \phi_k \in \mathfrak{H}_k \subseteq \widehat{\bigoplus}_{n=0}^{\infty} \mathfrak{H}_n$ this means

$$0 = B\widetilde{A}\Phi - \widetilde{A}B\Phi = BA\phi_k - \widetilde{A}(B_{nk}\phi_k)_{n\in\mathbb{N}} = (B_{nk}A\phi_k - AB_{nk}\phi_k)_{n\in\mathbb{N}}.$$

Hence $B_{nk} \in \mathscr{A}'$ for all n, k follows. The converse is clear as well. $\square$

Lemma 7.2.11 *We have $\widetilde{\mathscr{A}''} = \widetilde{\mathscr{A}}''$.*

PROOF: Suppose C is in $\widetilde{\mathscr{A}}''$ and hence commutes with every $B \in \mathfrak{B}(\widetilde{\mathfrak{H}})$ with $B_{nm} \in \mathscr{A}'$ by Lemma 7.2.10. First it is clear that C has to be diagonal

$$C = (C_{nn})_{n\in\mathbb{N}} \text{ with } C_{nn} = D \in \mathfrak{B}(\mathfrak{H}),$$

being the same for all n. Otherwise, C would not even commute with all B with B_{nm} being multiples of the identity. But then the diagonal element D has still to commute with a diagonal $B = (B_{nn})_{n\in\mathbb{N}}$ with $B_{nn} \in \mathscr{A}'$. This implies $D \in \mathscr{A}''$ and hence $C \in \widetilde{\mathscr{A}''}$. The reverse inclusion is trivial. $\square$

Lemma 7.2.12 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a unital *-subalgebra. Then for every $B \in \mathscr{A}''$, every $\epsilon > 0$ and every $\phi \in \mathfrak{H}$ there is a $A \in \mathscr{A}$ with*

$$\|A\phi - B\phi\| < \epsilon. \tag{7.2.16}$$

PROOF: We consider the subspace generated from ϕ

$$U = (\operatorname{span}_{\mathbb{C}}\{A\phi\}_{A\in\mathscr{A}})^{\mathrm{cl}}.$$

Then we clearly have $AU \subseteq U$ for every $A \in \mathscr{A}$ by the continuity of A. Thus for the orthogonal projection P_U onto U we get for all $A \in \mathscr{A}$

$$(\mathbb{1} - P_U)AP_U = 0.$$

Since $\mathscr{A}$ is a *-subalgebra, we also get

$$P_U A(\mathbb{1} - P_U) = 0$$

for all $A \in \mathscr{A}$ and thus P_U commutes with every $A \in \mathscr{A}$, i.e. we have $P_U \in \mathscr{A}'$. Thus also $P_U B = BP_U$ for $B \in \mathscr{A}''$. Since $\mathscr{A}$ is unital, we have $\phi \in U$ and thus $P_U\phi = \phi$. This gives $P_U B\phi = B\phi$ and hence $B\phi \in U$ as well. Thus we find a $A \in \mathscr{A}$ with $\|B\phi - A\phi\| < \epsilon$ by the definition of U. $\square$

Using these lemmas it is now easy to prove the following theorem of von Neumann on the bicommutant:

Theorem 7.2.13 (Von Neumann's Bicommutant theorem) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a unital *-subalgebra. Then the following statements are equivalent:*

i.) One has $\mathscr{A} = \mathscr{A}''$.

ii.) The *-subalgebra $\mathscr{A}$ is closed in the σ-strong* topology.

iii.) The *-subalgebra $\mathscr{A}$ is closed in the σ-strong topology.

iv.) The *-subalgebra $\mathscr{A}$ is closed in the σ-weak topology.

v.) The *-subalgebra $\mathscr{A}$ is closed in the strong* topology.

vi.) The *-subalgebra $\mathscr{A}$ is closed in the strong topology.

vii.) The *-subalgebra $\mathscr{A}$ is closed in the weak topology.

PROOF: From Corollary 7.2.7 we know already that *i.)* implies all other statements. From Proposition 7.2.9 we know that *ii.)*, *iii.)*, and *iv.)* as well as *v.)*, *vi.)* and *vii.)* are equivalent. Finally, from Theorem 5.1.10, *ii.)*, we know that the weak topology is the coarsest and hence *vii.)* implies *ii.)* – *vi.)*. We show *iii.)* $\implies$ *i.)* to complete the proof: assume that $\mathscr{A}$ is σ-strongly closed and $B \in \mathscr{A}''$ is given. Moreover, let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence of vectors in $\mathfrak{H}$ with $\sum_{n=1}^{\infty}\|\phi_n\|^2 < \infty$ and $\epsilon > 0$. Consider $\Phi = (\phi_n)_{n\in\mathbb{N}} \in \widetilde{\mathfrak{H}}$ and $\widetilde{B} \in \widetilde{\mathscr{A}''} = \widetilde{\mathscr{A}}''$ according to Lemma 7.2.11. From Lemma 7.2.12 we get an operator $\widetilde{A} \in \widetilde{\mathscr{A}}$ with $\|\widetilde{B}\Phi - \widetilde{A}\Phi\|_{\widetilde{\mathfrak{H}}} < \epsilon$. But this means

$$\big\|\widetilde{B}\Phi - \widetilde{A}\Phi\big\|_{\widetilde{\mathfrak{H}}} = \sqrt{\sum_{n=1}^{\infty}\big\|B\phi_n - A\phi_n\big\|_{\mathfrak{H}}^2} = \big\|B - A\big\|_{\{\phi_n\}_{n\in\mathbb{N}}} < \epsilon,$$

showing that in every σ-strong neighbourhood of B we find an operator $A \in \mathscr{A}$. Thus B is in the σ-strong closure of $\mathscr{A}$ which is $\mathscr{A}$ by assumption. Thus *i.)* follows. □

Definition 7.2.14 (Von Neumann algebra) *A unital *-subalgebra $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ is called von Neumann algebra if $\mathscr{A} = \mathscr{A}''$.*

Remark 7.2.15 (Von Neumann algebra) Equivalently, by the Bicommutant Theorem, a von Neumann algebra is a unital *-subalgebra $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ which is closed in one and hence all of the six operator topologies: the σ-strong*, the σ-strong, the σ-weak, the strong*, the strong, or the weak topology. As a consequence, a von Neumann algebra is always a *C^*-subalgebra* of $\mathfrak{B}(\mathfrak{H})$. However, the converse is not true, see Example 7.2.8.

Remark 7.2.16 (Lattice of von Neumann algebras) The von Neumann algebras in $\mathfrak{B}(\mathfrak{H})$ from a lattice according to Proposition 7.2.5. In particular, for two von Neumann algebras $\mathscr{A}, \mathscr{B} \subseteq \mathfrak{B}(\mathfrak{H})$ also $\mathscr{A} \cap \mathscr{B}$ is a von Neumann algebra. We also see that $\mathscr{A}'$ is a von Neumann algebra, too, and the center

$$\mathscr{Z}(\mathscr{A}) = \mathscr{A} \cap \mathscr{A}' \tag{7.2.17}$$

of a von Neumann algebra is again a (commutative) von Neumann algebra. Clearly, $\mathscr{A}$ is commutative if

$$\mathscr{A} \subseteq \mathscr{A}'. \tag{7.2.18}$$

As a first application of the Bicommutant Theorem we get von Neumann's Density Theorem:

Theorem 7.2.17 (Von Neumann's Density Theorem) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a unital *-subalgebra. Then $\mathscr{A}$ is dense in $\mathscr{A}''$ with respect to the σ-strong*, the σ-strong, the σ-weak, the strong*, the strong, and the weak topology.*

PROOF: Denote by $\mathscr{A}^{\mathrm{cl}}$ the closure of $\mathscr{A}$ in any of the above six topologies. Since $\mathscr{A} \subseteq \mathscr{A}^{\mathrm{cl}}$ we have $(\mathscr{A}^{\mathrm{cl}})' \subseteq \mathscr{A}'$ by Proposition 7.2.2, *ii.)*. We claim that we actually have equality. Since the weak closure is the largest, the commutant of the weak closure is the smallest. Thus let $B \in \mathscr{A}'$ and A in the weak closure of $\mathscr{A}$ with a net $(A_i)_{i\in I}$ in $\mathscr{A}$ converging weakly to A. By the same argument as in the proof of Lemma 7.2.6 we see that $BA = AB$ and thus $B \in (\mathscr{A}^{\mathrm{cl}})'$. But this shows $(\mathscr{A}^{\mathrm{cl}})' = \mathscr{A}'$ for all the six topologies. Hence $\mathscr{A}'' = (\mathscr{A}^{\mathrm{cl}})''$ follows. But $\mathscr{A}^{\mathrm{cl}}$ is already closed, so by the Bicommutant Theorem we have $(\mathscr{A}^{\mathrm{cl}})'' = \mathscr{A}^{\mathrm{cl}}$. This shows $\mathscr{A}^{\mathrm{cl}} = \mathscr{A}''$. ☐

There is a word of caution necessary at this stage. All the six topologies are usually *not* first countable and thus taking a closure involves *nets* and not just sequences. As a first application we note that the following corollary of von Neumann's Density Theorem:

Corollary 7.2.18 *Let $A \in \mathfrak{B}(\mathfrak{H})$. Then the smallest von Neumann algebra containing A is the weak closure of the polynomials in A and A^*.*

PROOF: First we note that the weak closure $\mathscr{A}$ of the polynomials in A and A^* is the weak closure of a *-subalgebra and hence a von Neumann algebra by Theorem 7.2.13. It is the bicommutant of these polynomials. Since it clearly contains A, the smallest von Neumann algebra containing A is contained in $\mathscr{A}$. Conversely, if $\mathscr{B}$ is a von Neumann algebra containing A then it also contains A^* and thus every polynomial in A and A^*. The commutant $\mathscr{B}'$ is thus contained in $\mathscr{A}'$ and hence $\mathscr{A}'' \subseteq \mathscr{B}''$ follows. This shows that $\mathscr{A}$ was minimal already. ☐

Corollary 7.2.19 *Let $A \in \mathfrak{B}(\mathfrak{H})$ be a normal operator with spectral measure E. Then for every measurable subset $U \subseteq \operatorname{spec}(A)$ the projection E_U is in the σ-strong* closure of $\mathbb{C}[A, A^*] \subseteq \mathfrak{B}(\mathfrak{H})$.*

PROOF: From the Spectral Theorem 5.1.32 we know that $B \in \mathfrak{B}(\mathfrak{H})$ commutes with A iff B commutes with every E_U. Thus E_U is in the bicommutant of the *-subalgebra $\mathbb{C}[A, A^*]$. By Theorem 7.2.17 we can approximate E_U σ-strongly* by polynomials in A and A^*. ☐

Of course, we can also approximate E_U in the weaker topologies, too. Again it should be noted that sequences will not suffice in general.

Slightly more general is the following approximation of the bounded measurable functions via polynomials:

Corollary 7.2.20 *Let $A \in \mathfrak{B}(\mathfrak{H})$ be a normal operator and let $f \in \mathscr{B}\mathscr{M}(\operatorname{spec}(A))$ be a bounded measurable function. Then $f(A) \in \mathfrak{B}(\mathfrak{H})$ can be approximated by polynomials in A and A^* with respect to the σ-strong* topology.*

PROOF: From Theorem 5.1.32, *iii.)*, we know that $B \in \mathfrak{B}(\mathfrak{H})$ commutes with A iff it commutes with every spectral projection E_U of A and hence with all polynomials $\mathbb{C}[A, A^*]$. By Theorem 5.1.28, *iii.)*, we know that in this case B commutes with all $f(A)$ as well. Thus $f(A)$ is in the bicommutant of the polynomials $\mathbb{C}[A, A^*]$ and can therefore be approximated in the σ-strong* topology by polynomials by Theorem 7.2.17. ☐

Clearly, Corollary 7.2.19 is a special case for $f = \chi_U$ being a characteristic function. Nevertheless, this case is important enough to state it as a separate Corollary. Again, the corollary is not very constructive as we typically will need nets instead of sequences of polynomials.

In Theorem 5.1.42 we introduced the polar decomposition of a bounded map $A\colon \mathfrak{H} \longrightarrow \mathfrak{K}$ between Hilbert spaces using a unique partial isometry $U\colon \mathfrak{H} \longrightarrow \mathfrak{K}$ with $\ker U = \ker A$ such that $A = U|A|$. In the subsequent discussion we found certain commutation relations in Corollary 5.1.46 which we can now interpret as follows:

Proposition 7.2.21 *Let $A \in \mathfrak{B}(\mathfrak{H})$ be an operator with polar decomposition $A = U|A|$ where the partial isometry U is the unique one with $\ker U = \ker A$. Then U and $|A|$ are contained in the smallest von Neumann algebra containing A.*

PROOF: For $|A|$ this is clear as $|A| = \sqrt{A^*A}$ is contained already in the smallest C^*-subalgebra containing A which is therefore contained in the smallest von Neumann algebra $\mathscr{A}$ containing A. By Corollary 7.2.18 we know that $\mathscr{A}$ is the weak closure of the polynomials in A and A^*. Suppose that $B \in \mathscr{A}'$ then also B^* and thus $\operatorname{Re}(B)$ and $\operatorname{Im}(B)$ are in $\mathscr{A}'$. Since these Hermitian elements commute with A, Corollary **??** shows that U commutes with $\operatorname{Re}(B)$ and with $\operatorname{Im}(B)$, hence with B as well. Thus $U \in \mathscr{A}'' = \mathscr{A}$ follows. □

As yet another reformulation we see that a von Neumann algebra is necessarily closed under the bounded measurable calculus (and not just under the continuous calculus as every C^*-subalgebra):

Corollary 7.2.22 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra and let $A \in \mathscr{A}$ be normal. Then for every $f \in \mathscr{BM}(\operatorname{spec}(A))$ we have $f(A) \in \mathscr{A}$.*

PROOF: Again by the spectral theorem we know that for $B \in \mathscr{A}$ we have $[B, f(A)] = 0$ and thus $f(A) \in \mathscr{A}'' = \mathscr{A}$. □

The next application of the density theorem shows that a von Neumann algebra has a lot of projections:

Proposition 7.2.23 *Let $\mathscr{A} = \mathscr{A}'' \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra. Then the projections in $\mathscr{A}$ span a norm-dense subspace of $\mathscr{A}$.*

PROOF: Let $A \in \mathscr{A}$ be given. Since also $A^* \in \mathscr{A}$ we can assume that A is Hermitian, otherwise consider $\operatorname{Re}(A)$ and $\operatorname{Im}(A)$ separately. Then we know from Corollary 7.2.19 that every spectral projection E_U of A for all measurable $U \subseteq \operatorname{spec}(A)$ belongs to $\mathscr{A}$. Since we can uniformly approximate the identity function $f(x) = x$ on the compact subset $\operatorname{spec}(A)$ by linear combinations of characteristic functions, see Proposition C.1.25, *ii.)*, the measurable calculus from Theorem 5.1.19 shows that we can approximate $A = f(A)$ in the norm topology by linear combinations of the spectral projections E_U of A. Hence A is in the norm closure of the span of its spectral projections. □

This is a remarkable feature of von Neumann algebras compared to general C^*-algebras or C^*-subalgebras of $\mathfrak{B}(\mathfrak{H})$: for such there need not to be any non-trivial projections different from 0 and $\mathbb{1}$, see e.g. Example 5.1.12.

Up to now it is not really clear whether we have interesting von Neumann algebras at all: it could well be that the weak closures are always trivially large and hence equal to $\mathfrak{B}(\mathfrak{H})$. However, this is not the case: as a first example of a von Neumann algebra different from $\mathfrak{B}(\mathfrak{H})$ we consider the following situation. Let $(X, \mathfrak{a}, \mu)$ be a σ-finite measure space and consider the C^*-algebra $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ as well as the Hilbert space $\mathrm{L}^2(X, \mathfrak{a}, \mu)$. We want to view the essentially bounded functions as multiplication operators on $\mathrm{L}^2(X, \mathfrak{a}, \mu)$:

Theorem 7.2.24 (The von Neumann algebra $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$) *Let $(X, \mathfrak{a}, \mu)$ be a σ-finite measure space.*

i.) *The C^*-algebra of essentially bounded functions $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ can be identified as unital C^*-subalgebra of $\mathfrak{B}(\mathrm{L}^2(X, \mathfrak{a}, \mu))$ via the *-representation*

$$\mathrm{L}^\infty(X, \mathfrak{a}, \mu) \ni f \mapsto (\phi \mapsto f\phi) \in \mathfrak{B}(\mathrm{L}^2(X, \mathfrak{a}, \mu)). \tag{7.2.19}$$

ii.) *The *-representation (7.2.19) of $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ is cyclic.*

iii.) *The image of* $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ *under* (7.2.19) *coincides with its commutant.*

iv.) *The image of* $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ *under* (7.2.19) *is a maximally commutative von Neumann algebra in* $\mathfrak{B}(\mathrm{L}^2(X, \mathfrak{a}, \mu))$.

PROOF: First we show that (7.2.19) is a *-homomorphism. Thus let $\phi \in \mathrm{L}^2(X, \mathfrak{a}, \mu)$ and $f \in \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$. Then we get analogously to the considerations in Theorem 5.2.5 the estimate

$$
\begin{aligned}
\|f\phi\|^2_{\mathrm{L}^2} &= \int_X |f(x)|^2 |\phi(x)|^2 \,\mathrm{d}\mu(x) \\
&= \int_{X\setminus N} |f(x)|^2 |\phi(x)|^2 \,\mathrm{d}\mu(x) \\
&\le \|f\|^2_{\mu,\infty} \int_{X\setminus N} |\phi(x)|^2 \,\mathrm{d}\mu(x) \\
&= \|f\|^2_{\mu,\infty} \|\phi\|^2_{\mathrm{L}^2},
\end{aligned}
$$

where $N \subseteq X$ is the subset of measure zero where $|f|$ is not smaller than its essential supremum $\|f\|_{\mu,\infty}$, see Lemma C.2.17, *iii.)*. This shows that the multiplication operator by f is bounded with operator norm at most $\|f\|_{\mu,\infty}$. In particular, (7.2.19) is indeed well-defined on the quotient $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ as well as on $\mathrm{L}^2(X, \mathfrak{a}, \mu)$. Clearly (7.2.19) is linear and a *-homomorphism. Up to here we have not yet used that $(X, \mathfrak{a}, \mu)$ is σ-finite. We need this to show that (7.2.19) is actually isometric. Thus let $\epsilon > 0$ and consider the measurable subset $U \subseteq X$ with $|f(x)| \ge \|f\|_{\mu,\infty} - \epsilon$ for $x \in U$. In general, it may happen that U has infinite measure. Nevertheless, on a σ-finite measure space we know that there is an integrable function $\chi \in \mathrm{L}^1(X, \mathfrak{a}, \mu)$ with $0 < \chi(X) < 1$ for all $x \in X$, see Example C.3.17. Hence $\sqrt{\chi} \in \mathrm{L}^2(X, \mathfrak{a}, \mu)$ and thus also $\sqrt{\chi}\chi_U \in \mathrm{L}^2(X, \mathfrak{a}, \mu)$. Since U has non-zero measure by definition of the essential supremum, and since $\sqrt{\chi} > 0$ everywhere, we conclude that $\|\sqrt{\chi}\chi_U\|_{\mathrm{L}^2} > 0$ by the non-degeneracy of the integral. Now we have

$$
\begin{aligned}
\left\|f\sqrt{\chi}\chi_U\right\|_{\mathrm{L}^2} &= \int_X |f|^2 \chi\chi_U \,\mathrm{d}\mu \\
&= \int_U |f|^2 \chi \,\mathrm{d}\mu \\
&\ge \left(\|f\|_{\mu,\infty} - \epsilon\right)^2 \int_U \chi \,\mathrm{d}\mu \\
&= \left(\|f\|_{\mu,\infty} - \epsilon\right)^2 \int_X \chi\chi_U \,\mathrm{d}\mu \\
&= \left(\|f\|_{\mu,\infty} - \epsilon\right)^2 \left\|\sqrt{\chi}\chi_U\right\|_{\mathrm{L}^2},
\end{aligned}
$$

showing that the operator norm of the multiplication operator is at least $\|f\|_{\mu,\infty} - \epsilon$ for all $\epsilon > 0$. Thus (7.2.19) is isometric and hence injective, allowing to identify $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ with the multiplication operators. To prove that (7.2.19) is cyclic we take a sequence of measurable subsets $A_n \subseteq X$ with $\bigcup_{n=1}^\infty A_n = X$ and $\mu(A_n) < \infty$ according to the assumption that X is σ-finite. Then we consider again the function χ from Example C.3.17, i.e.

$$
\chi = \sum_{n=1}^\infty \frac{1}{2^n} \frac{1}{1 + \mu(A_n)} \chi_{A_n} \in \mathrm{L}^2(X, \mathfrak{a}, \mu).
$$

Clearly, we can arrange the A_n to be pairwise disjoint. If $U \subseteq X$ is now any measurable subset with finite measure $\mu(U) < \infty$ then we have the pointwise convergence of $\sum_{n=1}^\infty \chi_U\chi_{A_n} = \sum_{n=1}^\infty \chi_{U\cap A_n}$ to χ_U which gives, by the dominated convergence, the L^2-convergence as well as $\chi_{U\cap A_n}\chi_{U\cap A_m} =$

$\chi_{U\cap A_n}\delta_{nm}$. Now let $f_N \in \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ be defined by

$$f_N = \sum_{n=1}^{N} 2^n(1 + \mu(A_n))\chi_{A_n\cap U},$$

then we have

$$f_N\chi = \sum_{n=1}^{N} 2^n(1 + \mu(A_n))\chi_{A_n\cap U} \sum_{m=1}^{\infty} \frac{1}{2^m}\frac{1}{1 + \mu(A_m)}\chi_{A_m} = \sum_{n=1}^{N} \chi_{A_n\cap U},$$

again by $\chi_{A_n}\chi_{A_m} = \chi_{A_n}\delta_{nm}$. This shows that

$$\|f_n\chi - \chi_U\|_{\mathrm{L}^2} = \left\| \sum_{n=N+1}^{\infty} \chi_{A_n\cap U} \right\|_{\mathrm{L}^2} \longrightarrow 0,$$

and hence $f_N\chi \longrightarrow \chi_U$. Thus we can approximate every characteristic function χ_U of a subset $U \in \mathfrak{a}$ with *finite* measure by vectors of the form $f\chi$ with $f \in \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$. Since these characteristic functions span a dense subspace of $\mathrm{L}^2(X, \mathfrak{a}, \mu)$ by Proposition C.3.28, *iii.)*, we conclude that χ is a cyclic vector. Now let $A \in \mathfrak{B}(\mathrm{L}^2(X, \mathfrak{a}, \mu))$ be a bounded operator commuting with all the multiplication operators and let χ be as above. We consider now the function $A(\chi) \in \mathrm{L}^2(X, \mathfrak{a}, \mu)$ and claim that

$$f = \frac{1}{\chi}A(\chi) \in \mathrm{L}^\infty(X, \mathfrak{a}, \mu).$$

To show this we consider again the dense subspace $\mathfrak{H}_{\mathrm{finite}} \subseteq \mathrm{L}^2(X, \mathfrak{a}, \mu)$ of finite linear combinations of characteristic functions χ_U with $\mu(U) < \infty$. Since $\mathfrak{H}_{\mathrm{finite}} \subseteq \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ as well, we know for $g \in \mathfrak{H}_{\mathrm{finite}}$

$$A(\chi)g\chi = gA(\chi)\chi = (gA\chi)\chi = (A(g\chi))\chi = (\chi A(g))\chi, \tag{$*$}$$

since A commutes with the multiplications operator by g as well as by χ. Of course, $(*)$ has to be interpreted as an equation which holds pointwise almost everywhere. Thus we get pointwise everywhere

$$\frac{1}{\chi}A(\chi)g = A(g),$$

and hence the multiplication operator by the a priori unbounded function $f = \frac{1}{\chi}A(\chi)$ defined on $\mathfrak{H}_{\mathrm{finite}}$ coincides with A, restricted to $\mathfrak{H}_{\mathrm{finite}}$. Since A is bounded, the multiplication operator by f is bounded on $\mathfrak{H}_{\mathrm{finite}}$, too. By an argument analogous to the proof of the isometry features of (7.2.19) this is enough to show that $f \in \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$. Indeed, suppose that $\|\frac{1}{\chi}A(\chi)\|_{\mu,\infty} = +\infty$ then for every $n \in \mathbb{N}$ we get a measurable subset U_N with $\mu(U_N) > 0$ and $|f| \geq N$ on U_N. Thus the function $|f|$ is also $\geq N$ on every $A_n \cap U_N$ and thus

$$\|f\chi_{A_n\cap U_N}\|_{\mathrm{L}^2} \geq N^2\|\chi_{A_n\cap U_n}\|_{\mathrm{L}^2}$$

follows at once, in contradiction to the boundedness of the multiplication operator on $\mathfrak{H}_{\mathrm{finite}}$. Thus $f \in \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ and $f = A$ by continuity and the density of $\mathfrak{H}_{\mathrm{finite}}$. The last statement is now easy; first it is clear that $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ is a commutative von Neumann algebra since it is a commutant (of its own). Moreover, if $\mathrm{L}^\infty(X, \mathfrak{a}, \mu) \subseteq \mathscr{A}$ with a commutative von Neumann algebra $\mathscr{A}$ then on one hand $\mathscr{A}' \subseteq \mathrm{L}^\infty(X, \mathfrak{a}, \mu)' = \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ by *iii.)* and on the other hand $\mathscr{A} \subseteq \mathscr{A}'$ since $\mathscr{A}$ is commutative. Thus $\mathscr{A} = \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ follows. □

Beyond the σ-finite case already the first statement may fail: on a measure space with $\mu(A) = +\infty$ for all $A \in \mathfrak{A}$ except $\emptyset$ we have $\mathrm{L}^2(X, \mathfrak{a}, \mu) = \{0\}$ while $\mathrm{L}^\infty(X, \mathfrak{a}, \mu) = \mathscr{BM}(X, \mathfrak{a})$ needs not to be trivial at all. In the σ-finite case we shall always identify $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ as the von Neumann algebra of multiplication operators. On the other hand, in the case of a finite measure μ, the proof can be simplified in so far as now the constant function $1 \in \mathrm{L}^2(X, \mathfrak{a}, \mu)$ is a cyclic vector for trivial reasons.

7.2.3 Kaplansky's Density Theorem and Irreducibility

The next fundamental theorem on von Neumann algebras is Kaplansky's Density Theorem. In some sense it is a refinement of von Neumann's Densitiy Theorem.

As preparation we need to show some continuity properties of the continuous calculus. We know already that for a normal element a in a C^*-algebra $\mathscr{A}$ the map $\mathscr{C}(\operatorname{spec}(a)) \ni f \mapsto f(A) \in \mathscr{A}$ is a *-homomorphism and hence continuous. Now we are interested in the continuity properties when *fixing* the function f but *varying* the normal elements. Here we obtain the following result:

Proposition 7.2.25 *Let f be a continuous complex-valued function on $\mathbb{R}$ or $\mathbb{C}$, respectively, and consider the map $f\colon A \mapsto f(A)$ for Hermitian or normal operators $A \in \mathfrak{B}(\mathfrak{H})$, respectively.*

i.) The map f is σ-strongly* continuous on norm-bounded subsets of Hermitian or normal operators, respectively.

ii.) The Cayley transform $A \mapsto U_A = \frac{A-\mathrm{i}}{A+\mathrm{i}}$ is σ-strongly* continuous on all Hermitian operators.

iii.) If $f \in \mathscr{C}_\infty(\mathbb{R}, \mathbb{R})$ is a continuous function vanishing at infinity then f is σ-strongly* continuous on all Hermitian operators.

PROOF: The subtlety arising here is that a σ-strong* neighbourhood of a given operator A can contain operators of arbitrarily large operator norm. Thus the extra assumption in the first part is not superfluous. Now let $R > 0$ be given and consider Hermitian or normal operators A with $\|A\| < R$. First we note that on $\mathrm{B}_R(\mathfrak{B}(\mathfrak{H}))$ any polynomial $p\colon \mathrm{B}_R(\mathfrak{B}(\mathfrak{H})) \longrightarrow \mathfrak{B}(\mathfrak{H})$ is σ-strongly* continuous thanks to the continuity of the operator product with respect to the σ-strong* topology according to Theorem 5.1.10, *v.)*. Second, we pass again to the $\mathbb{N}$-fold amplification $\widetilde{\mathfrak{H}}$ and use the unital *-homomorphism $\mathfrak{B}(\mathfrak{H}) \ni A \mapsto \widetilde{A} = A \otimes \mathrm{id}_{\ell^2} \in \mathfrak{B}(\widetilde{\mathfrak{H}})$ as before. Then we know from the covariance of the continuous calculus under unital *-homomorphisms that $\widetilde{f(A)} = f(\widetilde{A})$, see Remark 4.3.26, eq:ContinuousCalculusCovariant. Now fix a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ in $\mathfrak{H}$ with $\sum_{n=1}^\infty \|\phi_n\|^2 < \infty$ and consider the σ-strong* seminorm

$$\|A\|^*_{\{\phi_n\}} = \sqrt{\sum_{n=1}^\infty \|A\phi_n\|^2_{\mathfrak{H}} + \sum_{n=1}^\infty \|A^*\phi_n\|^2_{\mathfrak{H}}} = \sqrt{\big\|\widetilde{A}\Phi\big\|^2 + \big\|\widetilde{A}^*\Phi\big\|^2},$$

where $\Phi = (\phi_n)_{n\in\mathbb{N}} \in \widetilde{\mathfrak{H}}$. Since we are considering Hermitian or normal operators anyway, we have $\|A^*\phi\| = \|A\phi\|$ by Proposition 5.1.13, *ii.)*, for all $\phi \in \mathfrak{H}$. Thus $\|A\|^2_{\{\phi_n\}} = \sqrt{2}\|\tilde{A}\Phi\|$ follows and it suffices to consider the σ-strong topology since this topology coincides with the σ-strong* topology on normal operators, see also Exercise 5.5.3, *ii.)*. Now finally, let $\epsilon > 0$ and choose a polynomial $p \in \mathbb{C}[z, \overline{z}]$ with $\|f - p\|_{\mathrm{B}_R(0)^{\mathrm{cl}}} < \frac{\epsilon}{3\|\Phi\|}$ by the Stone-Weierstraß Theorem. Then for a fixed $A_0 \in \mathrm{B}_R(0) \subseteq \mathfrak{B}(\mathfrak{H})$ we first get a $\Psi \in \widetilde{\mathfrak{H}}$ such that

$$\|\widetilde{A} - \widetilde{A}_0\|_\Psi = \|A - A_0\|_{\{\psi_n\}} < 1$$

for a normal $A \in \mathrm{B}_R(0)$ implies

$$\|p(A) - p(A_0)\|_{\{\phi_n\}} = \|\widetilde{(p(A))} - \widetilde{p(A_0)}\|_\Phi < \frac{\epsilon}{3} \tag{$*$}$$

by the continuity of the operator product on norm-bounded subsets. Then for such A

$$\begin{aligned}
\big\|f(A) - f(A_0)\big\|_{\{\phi_n\}_{n\in\mathbb{N}}} &= \big\|\widetilde{f(A)} - \widetilde{f(A_0)}\big\|_\Phi \\
&= \big\|f(\widetilde{A}) - f(\widetilde{A_0})\big\|_\Phi \\
&\le \big\|f(\widetilde{A}) - p(\widetilde{A})\big\|_\Phi + \big\|p(\widetilde{A}) - p(\widetilde{A_0})\big\|_\Phi + \big\|p(\widetilde{A_0}) - f(\widetilde{A_0})\big\|_\Phi
\end{aligned}$$

$$\leq \varepsilon,$$

by the continuity of the continuous calculus with respect to the maximum norm on $\operatorname{spec}(A) \subseteq \mathrm{B}_R(0)^{\mathrm{cl}}$ and $\operatorname{spec}(A_0) \subseteq \mathrm{B}_R(0)^{\mathrm{cl}}$, respectively, as well as $(*)$. This shows the continuity at A_0 and hence the first part. For the second part we first note that $\|(A+\mathrm{i})^{-1}\| \leq 1$ by the trivial estimate $|\frac{1}{x+\mathrm{i}}| \leq 1$ for all $x \in \mathbb{R}$ and the continuity of the continuous calculus in the supremum norm. Thus for Hermitian A and A_0 we get

$$\begin{aligned}
\|U_A - U_{A_0}\|_{\{\phi_n\}_{n\in\mathbb{N}}} &= \|\widetilde{U_A} - \widetilde{U_{A_0}}\|_{\Phi} \\
&= \|U_{\widetilde{A}} - U_{\widetilde{A_0}}\|_{\Phi} \\
&= \left\| \frac{1}{\widetilde{A}+\mathrm{i}}(\widetilde{A}-\mathrm{i}) - (\widetilde{A_0}-\mathrm{i})\frac{1}{\widetilde{A_0}+\mathrm{i}} \right\|_{\Phi} \\
&= \left\| \frac{1}{\widetilde{A}+\mathrm{i}}\Big((\widetilde{A}-\mathrm{i})(\widetilde{A_0}+\mathrm{i}) - (\widetilde{A}+\mathrm{i})(\widetilde{A_0}-\mathrm{i})\Big)\frac{1}{\widetilde{A_0}+\mathrm{i}} \right\|_{\Phi} \\
&= \left\| \frac{1}{\widetilde{A}+\mathrm{i}}(\widetilde{A}\widetilde{A_0} - \mathrm{i}\widetilde{A_0} + \mathrm{i}\widetilde{A} + 1 - \widetilde{A}\widetilde{A_0} + \mathrm{i}\widetilde{A} - \mathrm{i}\widetilde{A_0} - 1)\frac{1}{\widetilde{A_0}+\mathrm{i}} \right\|_{\Phi} \\
&= \left\| \frac{2\mathrm{i}}{\widetilde{A}+\mathrm{i}}(\widetilde{A} - \widetilde{A_0})\frac{1}{\widetilde{A_0}+\mathrm{i}} \right\|_{\Phi} \\
&\leq 2\left\| (\widetilde{A} - \widetilde{A_0})\frac{1}{\widetilde{A_0}+\mathrm{i}} \right\|_{\Phi} \\
&= 2\|\widetilde{A} - \widetilde{A_0}\|_{\frac{1}{\widetilde{A_0}+\mathrm{i}}\Phi}.
\end{aligned}$$

From this estimate we immediately get the σ-strong and thus also the σ-strong* continuity of the Cayley transform, now on *all* Hermitian operators. Finally, assume that f vanishes at infinity. From our previous discussion on the Cayley back-transformation we know that for a unitary map U with $1 \notin \operatorname{spec}(U)$ the back-transformation $A_U = \mathrm{i}\frac{1+U}{1-U}$ yields a Hermitian operator $A_U \in \mathfrak{B}(\mathfrak{H})$, see Section 5.3.3. Moreover, if f vanishes at infinity we see that the map

$$F\colon z \mapsto f\left(\mathrm{i}\frac{1+z}{1-z}\right) \quad \text{for} \quad |z| = 1, z \neq 1$$

extends to a continuous map on the unit circle if we set $F(1) = 0$. Thus we can write $f(A) = F(U_A)$. By the second part we know that $A \mapsto U_A$ is σ-strongly* continuous while F is σ-strongly* continuous on the (clearly norm-bounded) subset of unitaries: in principle we have shown this only for continuous maps defined on all of $\mathbb{C}$, but clearly we can extend F from $S^1 \subseteq \mathbb{C}$ to all of $\mathbb{C}$ in a continuous way. Thus we conclude that f itself is σ-strongly* continuous. □

Remark 7.2.26 Since on the normal elements the σ-strong* and the σ-strong topology coincide we can equally well write σ-strong everywhere in Proposition 7.2.25. By an analogous argument one shows that all three statements also hold for the strong* topology (being equal to the strong topology on normal elements), see Exercise 5.5.3.

We use this continuity now for a very particular continuous function in order to show the following density theorem of Kaplansky[1]

Theorem 7.2.27 (Kaplansky's Density Theorem) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a unital *-subalgebra. Then the closed unit ball of $\mathscr{A}$ is σ-strongly* dense in the closed unit ball of $\mathscr{A}''$.*

[1] According to [41, Sect 2.3.4]: *The density theorem is Kaplansky's great gift to mankind. It can be used every day, and twice on Sundays.*

PROOF: First we note that the closed unit ball of $\mathscr{A}$ is even norm-dense in the closed unit ball of the norm closure $\mathscr{A}^{\mathrm{cl}} \subseteq \mathscr{A}''$. Since norm convergence implies σ-strong* convergence it suffices to find a net in the unit ball of the norm closure of $\mathscr{A}$ σ-strongly* converging to a given point in the unit ball of $\mathscr{A}''$. Hence, without restriction, we can assume that $\mathscr{A} = \mathscr{A}^{\mathrm{cl}}$ is already norm closed and thus a C^*-subalgebra of $\mathfrak{B}(\mathfrak{H})$. We know that $\mathscr{A}''$ is the σ-strong* closure of $\mathscr{A}$ by von Neumann's Density Theorem 7.2.17. Hence let $A \in \mathscr{A}''$ with $\|A\| \leq 1$ be given and assume first that $A = A^*$ is Hermitian. Then we get a net $\{A_i\}_{i \in I}$ with $A_i \in \mathscr{A}$ and $A_i \longrightarrow A$ in the σ-strong* topology. Since the *-involution is continuous in the σ-strong* topology, we also have $A_i^* \longrightarrow A^* = A$ and hence $\mathrm{Re}(A_i) \longrightarrow A$ as well. Thus we get a net of *Hermitian* elements in $\mathscr{A}$ converging to A in the σ-strong* topology. The crucial point is that we can not say anything about $\|A_i\|$ in general. Now consider the continuous function

$$f(x) = \begin{cases} x & \text{for } x \in [-1, 1] \\ \frac{1}{x} & \text{for } x \in \mathbb{R} \setminus [-1, 1] \end{cases}$$

on the real axis, which clearly vanishes at infinity. Thus f is σ-strongly* continuous by Proposition 7.2.25, *iii.)*, on the Hermitian operators. Now $\mathrm{spec}(A) \subseteq [-1, 1]$ and thus by the continuous calculus we get $f(A) = A$. Moreover, f maps $\mathbb{R}$ into $[-1, 1]$ and thus $\mathrm{spec}(f(A_i)) \subseteq [-1, 1]$ by the Spectral Mapping Theorem 4.3.29, resulting in $\|f(A_i)\| \leq 1$. Since we assumed without restriction that $\mathscr{A}$ is a C^*-subalgebra, $f(A_i) \in \mathscr{A}$ again, now in the closed unit ball. Hence we conclude that

$$\lim_{i \in I} f(A_i) = f(A) = A$$

in the σ-strong* topology, yielding a net as claimed for the case that the limit point was Hermitian. Finally, assume that $A \in \mathscr{A}''$ with $\|A\| \leq 1$ is arbitrary. Then consider the twofold amplification $\tilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}$ with

$$\tilde{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix},$$

which is now a Hermitian operator in $\mathfrak{B}(\tilde{\mathfrak{H}}) = \mathrm{M}_2(\mathfrak{B}(\mathfrak{H}))$. Moreover, for $\Phi = \binom{\phi_1}{\phi_2}$ with $\phi_1, \phi_2 \in \mathfrak{H}$ we have

$$\|\tilde{A}\Phi\|^2 = \|A\phi_2\|^2 + \|A^*\phi_1\|^2 \leq \|\phi_2\|^2 + \|\phi_1\|^2 = \|\Phi\|^2.$$

Thus $\|\tilde{A}\| \leq 1$ again. Finally, we clearly have $\tilde{A} \in \mathrm{M}_2(\mathscr{A}'')$ by definition. Since $\mathrm{M}_2(\mathscr{A}'') = \mathrm{M}_2(\mathscr{A})''$ as e.g. the weak closure can be taken componentwise, we can apply our above result to $\mathrm{M}_2(\mathscr{A})$ instead of $\mathscr{A}$. Thus we get a net of Hermitian 2×2-matrices

$$A_i = A_i^* = \begin{pmatrix} A_i^{11} & A_i^{12} \\ A_i^{21} & A_i^{22} \end{pmatrix} \in \mathrm{M}_2(\mathscr{A})$$

with $\|A_i\| \leq 1$ and with σ-strong* convergence $A_i \longrightarrow \tilde{A}$. Again, this is just σ-strong* convergence componentwise and thus $A_i^{12} \longrightarrow A$ in the σ-strong* topology. From $\|A_i\| \leq 1$ we clearly get $\|A_i^{12}\| \leq 1$ and thus we have found the desired net also in this case. □

Corollary 7.2.28 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a unital *-subalgebra. Then the closed unit ball of $\mathscr{A}$ is dense in the closed unit ball of $\mathscr{A}''$ with respect to the σ-strong, the σ-weak, the strong*, the strong, and the weak topology.*

PROOF: Indeed, σ-strong* convergence implies convergence in all the coarser topologies. □

Corollary 7.2.29 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a unital *-subalgebra and $A \in \mathscr{A}''$ be a Hermitian (positive) operator with $\|A\| \leq 1$. Then A can be approximated by Hermitian (positive) operators from the closed unit ball of $\mathscr{A}$ in the σ-strong* topology.*

PROOF: The Hermitian case was already shown in the proof of the density theorem. If now $A \geq 0$ then $A = \sqrt{A}^2$ with $\sqrt{A} \in \mathscr{A}''$, since $\mathscr{A}''$ is necessarily a C^*-subalgebra. Now we only need to approximate $\sqrt{A}$ and use the fact that the operator product is σ-strongly* continuous on bounded subsets. Note that $\|\sqrt{A}\| \leq 1$ as well. ☐

Corollary 7.2.30 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a unital C^*-subalgebra and $U \in \mathscr{A}''$ a unitary operator. Then U is a σ-strong* limit of unitary operators from $\mathscr{A}$.*

PROOF: By the bounded measurable calculus we can write $U = \exp(\mathrm{i}A)$ with some Hermitian operator $A = A^* \in \mathfrak{B}(\mathfrak{H})$ where we take any discontinuous but measurable version of the logarithm as inverse of exp. From Corollary 7.2.22 we conclude that $A \in \mathscr{A}''$ again. In general, we can arrange things such that $\|A\| \leq 2\pi$ and it is clear that we get an analogous statement of Kaplansky's Density Theorem for every norm closed ball, not just the unit ball. Thus we find a net $\{A_i\}_{i\in I}$ in $\mathscr{A}$ of Hermitian operators with $\|A_i\| \leq 2\pi$ and $A_i \longrightarrow A$ σ-strongly*. Now $x \mapsto \mathrm{e}^{\mathrm{i}x}$ is continuous and thus by Proposition 7.2.25, *i.)*, we get the σ-strong* convergence

$$U_i = \exp(\mathrm{i}A_i) \longrightarrow \exp(\mathrm{i}A) = U.$$

Clearly, the U_i are unitary and, since $\mathscr{A}$ was assumed to be a C^*-subalgebra, $U_i \in \mathscr{A}$. ☐

We use now Kaplansky's Density Theorem to show that algebraic irreducibility is actually the same as topological irreducibility. To this end, we just mention the following re-formulation of Schur's Lemma, now using commutants:

Remark 7.2.31 (Schur's Lemma) Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a subset of operators with $A \in \mathscr{A}$ iff $A^* \in \mathscr{A}$. Then $\mathscr{A}$ acts topologically irreducibly on $\mathfrak{H}$, i.e. there are no non-trivial closed invariant subspaces, iff $\mathscr{A}' = \mathbb{C}\,\mathrm{id}_{\mathfrak{H}}$ iff $\mathscr{A}'' = \mathfrak{B}(\mathfrak{H})$. Indeed, this is just a reformulation of Theorem 7.2.27.

Now let $\mathscr{A}$ be a C^*-algebra acting irreducibly on $\mathfrak{H}$ via π. Clearly, we can safely assume that $\mathscr{A}$ is unital, otherwise we extend π to the unitization by $\pi(\mathbb{1}) = \mathrm{id}$ as usual, keeping of course the irreducibility. We will need the following technical lemma which is also of independent interest:

Lemma 7.2.32 *Let $\{\mathrm{e}_1, \ldots, \mathrm{e}_n\} \subseteq \mathfrak{H}$ be a system of orthonormal vectors and let $\phi_1, \ldots, \phi_n \in \mathrm{B}_R(0)^{\mathrm{cl}}$ for some $R > 0$.*

i.) *Then there exists an operator $B \in \mathfrak{F}(\mathfrak{H})$ with $\|B\| \leq \sqrt{2n}R$ such that $B\mathrm{e}_i = \phi_i$ for $i = 1, \ldots, n$.*

ii.) *If, in addition, there is a Hermitian operator A with $A\mathrm{e}_i = \phi_i$ for $i = 1, \ldots, n$ then the operator B as in i.) can be chosen to be Hermitian as well.*

PROOF: For the first part we consider the operator

$$B = \sum_{i=1}^{n} \Theta_{\phi_i, \mathrm{e}_i},$$

which clearly maps e_i to ϕ_i. For the operator norm we get the estimate

$$\begin{aligned}
\|B\phi\| &= \left\| \sum_{i=1}^{n} \phi_i \langle \mathrm{e}_i, \phi \rangle \right\| \\
&\leq R \sum_{i=1}^{n} |\langle \mathrm{e}_i, \phi \rangle| \\
&\leq R \sqrt{\sum_{i=1}^{n} |\langle \mathrm{e}_i, \phi \rangle|^2} \sqrt{\sum_{i=1}^{n} 1}
\end{aligned}$$

$$\leq R\|\phi\|\sqrt{n},$$

and hence $\|B\| \leq \sqrt{n}R$. Clearly $B \in \mathfrak{F}(\mathfrak{H})$. For the second case we assume that a Hermitian or positive A with $A\mathrm{e}_i = \phi_i$ exists. Denote by P the projection onto the subspace spanned by the $\mathrm{e}_1, \ldots, \mathrm{e}_n$, i.e.

$$P = \sum_{i=1}^{n} \Theta_{\mathrm{e}_i, \mathrm{e}_i}.$$

Then we compute

$$\begin{aligned} PAP\phi &= PA \sum_{i=1}^{n} \mathrm{e}_i \langle \mathrm{e}_i, \phi \rangle \\ &= P \sum_{i=1}^{n} \phi_i \langle \mathrm{e}_i, \phi \rangle \\ &= P \sum_{i=1}^{n} \Theta_{\phi_i, \mathrm{e}_i} \phi, \end{aligned} \tag{$*$}$$

showing that the right hand side defines a Hermitian operator since PAP is Hermitian. Now we define

$$\begin{aligned} B &= \sum_{i=1}^{n} \Theta_{\phi_i, \mathrm{e}_i} + \sum_{i=1}^{n} \Theta_{\mathrm{e}_i, \phi_i} (\mathbb{1} - P) \\ &= \sum_{i=1}^{n} (\Theta_{\phi_i, \mathrm{e}_i} + \Theta_{\mathrm{e}_i, \phi_i}) - \sum_{i=1}^{n} \Theta_{\mathrm{e}_i, \phi_i} P \\ &= \sum_{i=1}^{n} (\Theta_{\phi_i, \mathrm{e}_i} + \Theta_{\mathrm{e}_i, \phi_i}) - P \sum_{i=1}^{n} \Theta_{\phi_i, \mathrm{e}_i}, \end{aligned}$$

which is Hermitian thanks to $(*)$ and in $\mathfrak{F}(\mathfrak{H})$, too. We have $B\mathrm{e}_i = \phi_i$ since $(\mathbb{1} - P)$ vanishes on e_i for all $i = 1, \ldots, n$. Finally, we have

$$\begin{aligned} BB^* &= \left(\sum_{i=1}^{n} \Theta_{\phi_i, \mathrm{e}_i} + \sum_{i=1}^{n} \Theta_{\mathrm{e}_i, \phi_i} (\mathbb{1} - P) \right) \left(\sum_{i=1}^{n} \Theta_{\mathrm{e}_i, \phi_i} + (\mathbb{1} - P) \sum_{i=1}^{n} \Theta_{\phi_i, \mathrm{e}_i} \right) \\ &= \left(\sum_{i=1}^{n} \Theta_{\phi_i, \mathrm{e}_i} \right) \left(\sum_{i=1}^{n} \Theta_{\mathrm{e}_i, \phi_i} \right) + \sum_{i=1}^{n} \Theta_{\mathrm{e}_i, \phi_i} (\mathbb{1} - P) \sum_{i=1}^{n} \Theta_{\phi_i, \mathrm{e}_i}, \end{aligned}$$

since $(\mathbb{1} - P)$ vanishes on the e_i. By the C^*-property and the estimate from the first part we get $\|B\|^2 = \|BB^*\| \leq R^2 n + R^2 n = 2nR^2$ since $\|\mathbb{1} - P\| \leq 1$. Thus the second part is shown. □

Using this lemma and Kaplansky's Density Theorem we can now prove the following statement:

Theorem 7.2.33 (Transitivity of irreducible *-representations) *Let $\mathscr{A}$ be a unital C^*-algebra and let π be a topologically irreducible *representation of $\mathscr{A}$ on $\mathfrak{H}$. Let $U \subseteq \mathfrak{H}$ be a finite-dimensional subspace and $B \in \mathfrak{B}(\mathfrak{H})$. Then there is an element $a \in \mathscr{A}$ with*

$$\pi(a)\big|_U = B\big|_U. \tag{7.2.20}$$

If $B = B^*$ then we can choose $a = a^*$.

PROOF: By Remark 7.2.31 we know that $\pi(\mathscr{A})'' = \mathfrak{B}(\mathfrak{H})$ and thus we can apply the approximation results from Kaplansky's Density Theorem to any operator in $\mathfrak{B}(\mathfrak{H})$ and get σ-strong* approximation by operators in $\pi(\mathscr{A})$ with the *same* bounds on their operator norm. We consider $B = B^*$ first. Set $B_0 = B$ and fix an orthonormal basis $\mathrm{e}_1, \ldots, \mathrm{e}_n \in U$. Then $\|B\mathrm{e}_i\| \leq \|B\|$ for all $i = 1, \ldots, n$. By Corollary 7.2.28 we find an operator $\pi(a_0) \in \pi(\mathscr{A})$ such that $\|\pi(a_0)\| \leq \|B\|$ and

$$\|\pi(a_0)\mathrm{e}_i - B_0\mathrm{e}_i\| = \|\pi(a_0) - B_0\|_{\mathrm{e}_i} < \frac{1}{2\sqrt{2n}}, \tag{$*$}$$

since this is precisely an approximation in the strong topology. Now unless π is injective we can not say much about the norm of a_0 itself. Thus consider the quotient $\mathscr{A}/\ker\pi$ which is again a C^*-algebra since $\ker\pi$ is a closed *-ideal. On $\mathscr{A}/\ker\pi$ the *-representation π is injective and hence norm preserving by Proposition 4.4.30. Hence we know $\|\pi(a_0)\| = \|[a_0]\|$ is the quotient C^*-norm of the class of a_0 in $\mathscr{A}/\ker\pi$. By Proposition 4.4.18 we can find a (Hermitian) representative $a_0 \in \mathscr{A}$ realizing the quotient norm, i.e. $\|a_0\| = \|[a_0]\| = \|\pi(a_0)\|$. Thus let a_0 be such a representative. Consider the vectors $-\phi_i = \pi(a_0)\mathrm{e}_i - B_0\mathrm{e}_i \in \mathrm{B}_{\frac{1}{2\sqrt{2n}}}(0)^{\mathrm{cl}}$ according to ($*$). By Lemma 7.2.32 we find a $B_1 \in \mathfrak{B}(\mathfrak{H})$ with $\|B_1\| \leq \sqrt{2n}\frac{1}{2\sqrt{2n}} = \frac{1}{2}$ such that $B_1\mathrm{e}_i = \phi_i = B_0\mathrm{e}_i - \pi(a_0)\mathrm{e}_i$. Since $\pi(a_0) - B_0$ is Hermitian if B and hence a_0 by our choice was Hermitian, we can choose B_1 to be Hermitian in this case. Now we proceed and get a $a_1 \in \mathscr{A}$ with $\|\pi(a_1)\mathrm{e}_i - B_1\mathrm{e}_i\| \leq \frac{1}{4\sqrt{2n}}$ and $\|\pi(a_1)\| \leq \frac{1}{2}$ and hence also $\|a_1\| \leq \frac{1}{2}$ by repeating the above argument. Inductively, this gives a sequence $a_0, a_1, \ldots \in \mathscr{A}$ and a sequence $B_0 = B, B_1, \ldots \in \mathfrak{B}(\mathfrak{H})$ with the properties

$$\|B_k\| \leq \frac{1}{2^k} \quad \text{and} \quad \|a_k\| \leq \frac{1}{2^k},$$

$$B_k\mathrm{e}_i = B_0\mathrm{e}_i - \pi(a_0)\mathrm{e}_i - \cdots - \pi(a_{k-1})\mathrm{e}_i,$$

and

$$\|\pi(a_k)\mathrm{e}_i - B_k\mathrm{e}_i\| \leq \frac{1}{2^{k+1}\sqrt{2n}}.$$

Since B was Hermitian then we can arrange things such that all elements $a_0, a_1, \ldots$ and $B_1, B_2, \ldots$ are Hermitian as well. Define now $a = \sum_{k=1}^{\infty} a_k$ which clearly converges absolutely. Then

$$B\mathrm{e}_i - \pi(a)\mathrm{e}_i = \lim_{k\longrightarrow\infty}(B_0\mathrm{e}_i - \pi(a_0)\mathrm{e}_i - \cdots - \pi(a_k)\mathrm{e}_i) = \lim_{k\longrightarrow\infty} B_{k+1}\mathrm{e}_i = 0.$$

The case of a general B now follows by constructing a_1 and a_2 for the real and imaginary part of B.□

We have now two important corollaries of this theorem, the first is the Kadison Transitivity [27]:

Corollary 7.2.34 (Kadison Transitivity) *Let π be an irreducible *-representation of a unital C^*-algebra $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$. Then for any linearly independent vectors $\{\psi_1, \ldots, \psi_n\}$ and any vectors $\{\phi_1, \ldots, \phi_n\}$ we get an algebra element $a \in \mathscr{A}$ with $\pi(a)\psi_i = \phi_i$ for $i = 1, \ldots, n$.*

PROOF: Since we clearly have a finite rank operator $B \in \mathfrak{F}(\mathfrak{H})$ moving the ψ_i to the ϕ_i we can use the last theorem applied to B and $U = \operatorname{span}_{\mathbb{C}}\{\psi_1, \ldots, \psi_n\}$. □

Note that this is a very strong from of transitivity which could be called *complete transitivity* as it is *n-transitivity* for all $n \in \mathbb{N}$ where 1-transitivity is the usual transitivity that we can move any non-zero vector to any given other vector. This result implies immediately the following statement, already announced in Subsection 7.1.4:

Corollary 7.2.35 (Algebraic irreducibility) *Let $\mathscr{A}$ be a unital C^*-algebra. Then every topologically irreducible *-representation is even algebraically irreducible.*

Indeed, this would already follow from 1-transitivity. Note that for unitary group representations we do not have 1-transitivity as, by the very definition, the norms are preserved.

Remark 7.2.36 Kaplansky's Density Theorem stays valid also in the non-unital case if one replaces $\mathscr{A}''$ with the weak closure (which, for a unital *-subalgebra is the same). Then also all conclusions drawn in this subsection have their non-unital counterparts. Details can e.g. be found in [28, Sect. 5.3].

7.2.4 The Predual of a von Neumann Algebra

In this subsection we give another, and more conceptual, characterizing property of a von Neumann algebra by showing that von Neumann algebras have a *predual*.

Definition 7.2.37 (Predual) *Let V be a Banach space. Then a Banach space W is called a predual of V if $W' \cong V$ as Banach spaces.*

We know already some examples. The trace class operators $\mathfrak{L}^1(\mathfrak{H})$ are a predual of $\mathfrak{B}(\mathfrak{H})$ by Theorem 6.3.5 while the compact operators $\mathfrak{K}(\mathfrak{H})$ are a predual for $\mathfrak{L}^1(\mathfrak{H})$. Also, every Hilbert space $\mathfrak{H}$ is its own predual via the Riesz Theorem 3.2.11 (up to anti-linearity). However, there are Banach spaces which do not allow for a predual. We will see later that $\mathfrak{K}(\mathfrak{H})$ for an infinite-dimensional Hilbert space does not have a predual.

We consider now a von Neumann algebra $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$. Using the extension of continuous linear functionals via the Hahn-Banach Theorem and the characterization from Theorem 6.3.11 we obtain the following property of σ-weakly continuous functionals of $\mathscr{A}$:

Proposition 7.2.38 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra and $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ a linear functional. Then the following statements are equivalent:*

i.) The functional ω is σ-weakly continuous.

ii.) The functional ω is σ-strongly continuous.

iii.) The functional ω is σ-strongly* continuous.

iv.) There exists a not necessarily unique trace class operator $\rho \in \mathfrak{L}^1(\mathfrak{H})$ with $\omega(A) = \operatorname{tr}(\rho A)$ for all $A \in \mathscr{A}$.

PROOF: If ω is a linear functional satisfying one of the first three conditions it can be extended to a linear functional $\Omega\colon \mathfrak{B}(\mathfrak{H}) \longrightarrow \mathbb{C}$ obeying the same continuity property by the Hahn-Banach Theorem in from of Corollary 2.2.19. Then by Theorem 6.3.11 we know that Ω satisfies all three continuity properties and is of the form $\Omega(A) = \operatorname{tr}(\rho A)$ with a unique $\rho \in \mathfrak{L}^1(\mathfrak{H})$. Restricting this back to $\mathscr{A}$ gives the equivalence of *i.)*, *ii.)*, *iii.)*, and the implication *i.)* $\implies$ *iv.)*. The reverse implication is trivial. ☐

Note that due to the usage of the Hahn-Banach argument the uniqueness of ρ in the fourth part can not be guaranteed. As for $\mathfrak{B}(\mathfrak{H})$ we call a linear functional on $\mathscr{A}$ satisfying one of the above equivalent conditions *normal*.

Corollary 7.2.39 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra. Then every normal linear functional is a linear combination of 4 normal states.*

PROOF: Indeed, let $\omega(A) = \operatorname{tr}(\rho A)$ for $A \in \mathscr{A}$ be a normal linear functional. Then we know that $\operatorname{Re}(\rho), \operatorname{Im}(\rho) \in \mathfrak{L}^1(\mathfrak{H})$ since $\mathfrak{L}^1(\mathfrak{H})$ is a *-ideal. Moreover $\operatorname{Re}(\rho)_\pm, \operatorname{Im}(\rho)_\pm \in \mathfrak{L}^1(\mathfrak{H})$ as $|\operatorname{Re}(\rho)|, |\operatorname{Im}(\rho)| \in \mathfrak{L}^1(\mathfrak{H})$ by the very definition of the trace class operators. Thus we can write

$$\omega(A) = \operatorname{tr}(\operatorname{Re}(\rho)_+ A) - \operatorname{tr}(\operatorname{Re}(\rho)_- A) + \mathrm{i}\operatorname{tr}(\operatorname{Im}(\rho)_+ A) - \mathrm{i}\operatorname{tr}(\operatorname{Im}(\rho)_- A)$$

with four normal positive linear functionals $\operatorname{tr}(\operatorname{Re}(\rho)_\pm \,\cdot\,)$ and $\operatorname{tr}(\operatorname{Im}(\rho)_\pm \,\cdot\,)$. Normalizing to states gives the result. ☐

Theorem 7.2.40 (Predual of a von Neumann algebra) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra. Then $\mathscr{A}$ has a predual. More precisely, denote by $\mathscr{A}^{\mathrm{ann}} \subseteq \mathfrak{L}^1(\mathfrak{H})$ the annihilator space of $\mathscr{A}$, i.e. those trace class operators ρ with $\operatorname{tr}(\rho A) = 0$ for all $A \in \mathscr{A}$. Then $\mathscr{A}^{\mathrm{ann}}$ is a closed subspace of $\mathfrak{L}^1(\mathfrak{H})$ and $\mathfrak{L}^1(\mathfrak{H}) \big/ \mathscr{A}^{\mathrm{ann}}$ is a predual of $\mathscr{A}$.*

Proof: Since every bounded operator $A \in \mathfrak{B}(\mathfrak{H})$ gives a continuous linear functional $\ell_A \colon \mathfrak{L}^1(\mathfrak{H}) \longrightarrow \mathbb{C}$ with respect to the trace norm according to Theorem 6.3.5, the space

$$\mathscr{A}^{\mathrm{ann}} = \bigcap_{A \in \mathscr{A}} \ker \ell_A$$

is closed and thus $\mathfrak{L}^1(\mathfrak{H}) \big/ \mathscr{A}^{\mathrm{ann}}$ is a Banach space. Since we even have $\mathfrak{L}^1(\mathfrak{H})' \cong \mathfrak{B}(\mathfrak{H})$ via ℓ by Theorem 6.3.5 we can consider the annihilator space $(\mathscr{A}^{\mathrm{ann}})^{\mathrm{ann}} \subseteq \mathfrak{B}(\mathfrak{H})$ of $\mathscr{A}^{\mathrm{ann}}$, i.e. those linear continuous functionals on $\mathfrak{L}^1(\mathfrak{H})$ which vanish on $\mathscr{A}^{\mathrm{ann}}$. It is clear that $(\mathscr{A}^{\mathrm{ann}})^{\mathrm{ann}}$ is a weak* closed subspace of $\mathfrak{L}^1(\mathfrak{H})'$. Since the weak* topology of $(\mathfrak{L}^1(\mathfrak{H}))' \cong \mathfrak{B}(\mathfrak{H})$ coincides with the σ-weak topology of $\mathfrak{B}(\mathfrak{H})$ by Theorem 6.3.8, the annihilator $(\mathscr{A}^{\mathrm{ann}})^{\mathrm{ann}}$ is a σ-weakly closed subspace of $\mathfrak{B}(\mathfrak{H})$. We know that $\mathscr{A} \subseteq (\mathscr{A}^{\mathrm{ann}})^{\mathrm{ann}}$. Since $\mathscr{A}$ is also σ-weakly closed we have a σ-weakly closed subspace $\mathscr{A} \subseteq (\mathscr{A}^{\mathrm{ann}})^{\mathrm{ann}}$. Thus if they do not agree and $B \in (\mathscr{A}^{\mathrm{ann}})^{\mathrm{ann}}$ is not in $\mathscr{A}$ we find a σ-weakly continuous linear functional $\varphi \colon \mathfrak{B}(\mathfrak{H}) \longrightarrow \mathbb{C}$ with $\varphi(B) = 1$ but $\varphi(A) = 0$ for all $A \in \mathscr{A}$ by the Hahn-Banach Theorem in form of Corollary 2.2.44. But every σ-weakly continuous functional of $\mathfrak{B}(\mathfrak{H})$ is of the form $\varphi(A) = \operatorname{tr}(\rho A)$ with a unique $\rho \in \mathfrak{L}^1(\mathfrak{H})$ according to Theorem 6.3.11. Thus this φ gives a $\rho \in \mathscr{A}^{\mathrm{ann}}$ and hence also $\operatorname{tr}(\rho B) = 0$ for all $B \in (\mathscr{A}^{\mathrm{ann}})^{\mathrm{ann}}$ which is a contradiction. Thus $\mathscr{A} = (\mathscr{A}^{\mathrm{ann}})^{\mathrm{ann}}$. By the general Lemma 6.1.15, *i.)*, we know that $(\mathfrak{L}^1(\mathfrak{H}) \big/ \mathscr{A}^{\mathrm{ann}})' \cong (\mathscr{A}^{\mathrm{ann}})^{\mathrm{ann}}$ as Banach spaces, completing the proof. □

Equivalently and more intrinsically, we can view the quotient $\mathfrak{L}^1(\mathfrak{H}) \big/ \mathscr{A}^{\mathrm{ann}}$ as the subspace of those continuous linear functionals on $\mathscr{A}$ which are in addition even σ-weakly continuous (or σ-strongly or σ-strongly* continuous) by Proposition 7.2.38.

Note also that the weak* topology induced by this predual is precisely the σ-weak topology on $\mathscr{A}$. This is clear by the same arguments as for $\mathfrak{B}(\mathfrak{H})$ in Theorem 6.3.8.

Note that is was crucial in the above proof to use the σ-weak closedness of $\mathscr{A}$. In fact, the above theorem even characterizes von Neumann algebras completely. We do not prove this result but just state the following remark:

Remark 7.2.41 (W^*-Algebras and Sakai's Theorem) Let $\mathscr{A}$ be a unital C^*-algebra. One may wonder whether $\mathscr{A}$ is *-isomorphic to a von Neumann algebra on some Hilbert space $\mathfrak{H}$. We know that $\mathscr{A}$ has faithful *-representations and hence is *-isomorphic to a unital C^*-subalgebra of $\mathfrak{B}(\mathfrak{H})$. However, in general we will not obtain a von Neumann algebra as e.g. $\mathscr{A}$ needs not to have non-trivial projections in contrast to a von Neumann algebra. It is now a theorem of Sakai that a unital C^*-algebra is *-isomorphic to a von Neumann algebra iff it has a predual. This leads to the more conceptual definition of a *W^*-algebra* as a C^*-algebra with a predual, which is the abstract version of a von Neumann algebra instead of the more concretely realized von Neumann algebra as subalgebra of $\mathfrak{B}(\mathfrak{H})$. For further details on W^*-algebras we refer to the classical textbook of Sakai [50] as well as to [28, 29, 54–56].

We still need another characterization of the normal states of $\mathfrak{B}(\mathfrak{H})$ and of a von Neumann algebra $\mathscr{A}$. Up to now, all the characterizations in Proposition 7.2.38 refer to the underlying Hilbert space and the induced coarser topologies of the ambient $\mathfrak{B}(\mathfrak{H})$. The following characterization is now more algebraic as it uses only the positive cone of $\mathscr{A}$: for preparation we need the following statement which is also of independent interest.

Proposition 7.2.42 *Let $\{A_i\}_{i\in I}$ be an increasing (decreasing) net of Hermitian operators in $\mathfrak{B}(\mathfrak{H})$, i.e. $A_i \leq A_j$ ($A_j \leq A_i$) for $i \preccurlyeq j$. If $\{A_i\}_{i\in I}$ is bounded then there exists a unique supremum*

$$A = \sup_{i\in I} A_i \in \mathfrak{B}(\mathfrak{H}) \tag{7.2.21}$$

or infimum

$$A = \inf_{i\in I} A_i \in \mathfrak{B}(\mathfrak{H}), \tag{7.2.22}$$

respectively, such that in the σ-strong* topology

$$\lim_{i\in I} A_i = A. \tag{7.2.23}$$

PROOF: We only have to consider the increasing case as for an decreasing net we can pass to $\{-A_i\}_{i\in I}$ which is now increasing. Note that for Hermitian operators $\{A_i\}_{i\in I}$ the boundedness with respect to the operator norm, i.e. $\|A_i\| \leq c$, is the same as boundedness with respect to the ordering relation $\leq$, i.e. $B_1 \leq A_i \leq B_2$ for some Hermitian B_1, B_2 (Proposition 4.4.9, *v.)*). Thus the A_i are all contained in some norm-closed ball $\mathrm{B}_R(0)^{\mathrm{cl}} \subseteq \mathfrak{B}(\mathfrak{H})$ of some radius $R > 0$. Since by the Banach-Alaoglu Theorem in form of Corollary 6.3.9 we know that $\mathrm{B}_R(0)^{\mathrm{cl}}$ is σ-weakly and hence also weakly compact, we get a subnet $\{A_{\phi(j)}\}_{j\in J}$ of $\{A_i\}_{i\in I}$ which converges weakly to some

$$A = \lim_{j\in J} A_{\phi(j)}.$$

Recall that here J is another directed set and $\phi\colon J \longrightarrow I$ has the property that for all $i \in I$ there is a $j \in J$ with $\phi(j') \succcurlyeq i$ for $j' \succcurlyeq j$. The order relation "$\leq$" is clearly compatible with the weak topology in the sense that a weak limit of operators $\leq B$ also is $\leq B$, see also . Hence for every $i \in I$ we have $A_{\phi(j')} \geq A_i$ for $j' \geq i$ and thus also $A \geq A_i$. This shows that A is an upper bound for $\{A_i\}_{i\in I}$. The weak convergence of $A_{\phi(j)} \longrightarrow A$ then gives the following estimates: fix $\epsilon > 0$ and $\psi \in \mathfrak{H}$. Then we have some index $j \in J$ with $0 \leq \langle \phi, (A - A_{\phi(j')})\phi\rangle < \epsilon$ for all $j' \geq j$. Since for $i \succcurlyeq \phi(j)$ we have $A_i \geq A_{\phi(j)}$ this gives by positivity bastian: exer

$$\begin{aligned}
0 &\leq \langle \psi, (A - A_i)\psi\rangle \\
&= \langle \psi, (A - A_{\phi(j)} + A_{\phi(j)} - A_i)\psi\rangle \\
&= \langle \psi, (A - A_{\phi(j)})\psi\rangle - \langle \psi, (A_i - A_{\phi(j)})\psi\rangle \\
&< \epsilon - \langle \psi, (A_i - A_{\phi(j)})\psi\rangle \\
&\leq \epsilon.
\end{aligned}$$

Hence for all $i \succcurlyeq \phi(j)$ we have $|\langle \psi, (A - A_i)\psi\rangle| < \epsilon$. By Remark 5.1.9 this is enough to conclude the weak convergence $A_i \longrightarrow A$ of the *original* net. From this we immediately conclude that A is the least upper bound of the net $\{A_i\}_{i\in I}$: indeed, if $A_i \leq B$ for all $i \in I$ then also the weak limit of the A_i satisfies $\lim_{i\in I} A_i = A \leq B$. It remains to show the σ-strong* convergence. Since $A - A_i$ is positive for all $i \in I$ we have for every $\psi \in \mathfrak{H}$

$$\begin{aligned}
\|(A - A_i)\psi\|^2 &= \left\| \sqrt{A - A_i}\sqrt{A - A_i}\psi \right\|^2 \\
&\leq \|A - A_i\| \left\| \sqrt{A - A_i}\psi \right\|^2 \\
&= \|A - A_i\| \langle \psi, (A - A_i)\psi\rangle \\
&\leq C\langle \psi, (A - A_i)\psi\rangle,
\end{aligned}$$

since the $\{A_i\}_{i\in I}$ are assumed to be bounded and hence also $\|A - A_i\|$ is bounded by some $C > 0$ independently of $i \in I$. Thus $\|(A - A_i)\phi\| \longrightarrow 0$ by the weak convergence of $A_i \longrightarrow A$, showing strong convergence. In a last step we note that for a *bounded* net strong and σ-strong convergence is the same thing: if $\{\phi_n\}_{n\in\mathbb{N}}$ with $\sum_{n=1}^{\infty}\|\phi_n\|^2 < \infty$ and $\epsilon > 0$ is given then let $N \in \mathbb{N}$ be large enough such that $\sum_{N+1}^{\infty}\|\phi_n\|^2 < \frac{\epsilon^2}{C^2}$, with $C > 0$ as above. Then

stefan: Once such stuff do on norm bou strong and w comparing...

$$\|A - A_i\|^2_{\{\phi_n\}_{n\in\mathbb{N}}} \leq \sum_{n=1}^{N}\|(A - A_i)\phi_n\|^2 + \sum_{n=N+1}^{\infty} C^2\|\phi_n\|^2 = \sum_{n=1}^{N}\|(A - A_i)\phi_n\|^2 + \varepsilon^2$$

shows that the strong convergence also implies $\|A - A_i\|_{\{\phi_n\}_{n\in\mathbb{N}}} \longrightarrow 0$. Finally, on Hermitian elements the σ-strong and the σ-strong* topologies coincide. □

Corollary 7.2.43 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra and $\{A_i\}_{i\in I}$ a bounded increasing (decreasing) net of Hermitian operators in $\mathscr{A}$. Then the supremum (infimum) of $\{A_i\}_{i\in I}$ exists, belongs to $\mathscr{A}$, and the net $\{A_i\}_{i\in I}$ converges σ-strongly* to it.*

PROOF: The von Neumann algebra $\mathscr{A}$ is σ-strongly* closed. Hence the statement follows immediately from the proposition. □

The next theorem of Dixmier [] gives now another characterization of a normal state of a von Neumann algebra.

Theorem 7.2.44 (Dixmier) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra and $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ a state. Then ω is normal iff for every increasing bounded net $\{A_i\}_{i\in I}$ of positive operators $A_i \in \mathscr{A}^+$ we have*

$$\omega\left(\sup_{i\in I} A_i\right) = \sup_{i\in I} \omega(A_i). \tag{7.2.24}$$

PROOF: We essentially follow [9, Thm. 2.4.21]. Suppose that ω is normal, i.e. σ-strong* continuous by Proposition 7.2.38. If $\{A_i\}_{i\in I}$ is an increasing bounded net of positive operators in $\mathscr{A}$ then by Proposition 7.2.42 we have $\sup_{i\in I} A_i = \lim_{i\in I} A_i$ in the σ-strong* topology. Hence

$$\omega(\sup_{i\in I} A_i) = \omega(\lim_{i\in I} A_i) = \lim_{i\in I} \omega(A_i) = \sup_{i\in I} \omega(A_i)$$

by the continuity of ω and the fact that $\omega(A_i) \leq \omega(A_j)$ for $A_i \leq A_j$ which is the case for $i \preccurlyeq j$. Hence the limit of the $\omega(A_i)$ exists and coincides with the supremum. Now assume the converse, i.e. ω fulfills (7.2.24). We consider now the following operators

$$\mathscr{B} = \left\{B \in \mathscr{A}^+ \;\middle|\; \|B\| \leq 1 \text{ and } A \mapsto \omega(AB) \text{ is } \sigma\text{-strongly continuous for } A \in \mathscr{A}\right\}.$$

Clearly $0 \in \mathscr{B}$ showing $\mathscr{B} \neq \emptyset$. We claim that $\mathscr{B}$ has a maximal element with respect to the ordering of Hermitian operators. To prove this, let $\{B_i\}_{i\in I}$ be an increasing chain. Then we claim that

$$B = \sup_{i\in I} B_i = \lim_{i\in I} B_i \in \mathscr{A}^+$$

is again in $\mathscr{B}$. We know $0 \leq B_i \leq B \leq \mathbb{1}$ and hence

$$\begin{aligned}
|\omega(AB - AB_i)|^2 &= \left|\omega\left(A\sqrt{B - B_i}\sqrt{B - B_i}\right)\right|^2 \\
&\leq \omega\left(A\sqrt{B - B_i}\sqrt{B - B_i}A^*\right)\omega\left(\sqrt{B - B_i}\sqrt{B - B_i}\right) \\
&= \omega(A(B - B_i)A^*)\omega(B - B_i) \\
&\overset{(a)}{\leq} \omega(A^*A)\omega(B - B_i)
\end{aligned}$$

$$\overset{(b)}{\leq} \|A\|^2 \omega(B - B_i), \tag{$*$}$$

by the Cauchy-Schwarz inequality for ω, Proposition 4.4.9, *ii.)*, in (a), as well as $\|\omega\| = 1$ in (b). Since ω satisfies (7.2.24) we see that $\omega(B - B_i) \longrightarrow 0$ which implies that the linear functionals $\omega(\,\cdot\,B_i)$ converge to $\omega(\,\cdot\,B)$ in norm by $(*)$. By assumption, the functionals $\omega(\,\cdot\,B_i)$ are all σ-strongly continuous and hence in the predual of $\mathscr{A}$ according to Theorem 7.2.40 and Proposition 7.2.38. Since the predual is a Banach space and a closed subspace of the dual of $\mathscr{A}$, we conclude that the norm limit $\omega(\,\cdot\,B)$ of the $\omega(\,\cdot\,B_i)$ is again in the predual. Thus $\omega(\,\cdot\,B)$ is σ-strongly continuous and hence $B \in \mathscr{B}$. This shows that every increasing chain in $\mathscr{B}$ has a supremum. Thus we can apply Zorn's Lemma to conclude that $\mathscr{B}$ has maximal elements, say $P \in \mathscr{B}$. We want to show $P = \mathbb{1}$ since then ω itself is σ-strongly continuous. Thus assume $P \neq \mathbb{1}$. Since $0 \leq P \leq \mathbb{1}$ we know $\mathbb{1} - P \geq 0$ and $\mathbb{1} - P \neq 0$. Hence we have a vector $\phi \in \mathfrak{H}$ with $\omega(\mathbb{1} - P) < \langle \phi, (\mathbb{1} - P)\phi \rangle$. We consider now the following set of operators

$$\mathscr{B}_P = \Big\{ B \in \mathscr{A}^+ \;\Big|\; B \leq \mathbb{1} - P \text{ and } \omega(B) \geq \langle \phi, B\phi \rangle \Big\}.$$

Again $0 \in \mathscr{B}_P$ shows $\mathscr{B}_P \neq \emptyset$. Let $\{B_i\}_{i \in I}$ be again an increasing chain, now in $\mathscr{B}_P$ and consider $B = \sup_{i \in I} B_i = \lim_{i \in I} B_i$ (in the σ-strong topology). Then the closedness of the order relation implies $0 \leq B \leq \mathbb{1} - P$. Since ω satisfies (7.2.24) we get

$$\omega(B) = \sup_{i \in I} \omega(B_i) \geq \sup_{i \in I} \langle \phi, B_i \phi \rangle = \lim_{i \in I} \langle \phi, B_i \phi \rangle = \langle \phi, B\phi \rangle,$$

since a vector state is clearly σ-strongly continuous. Hence $B \in \mathscr{B}_P$ again and we get maximal elements of $\mathscr{B}_P$ by Zorn's Lemma. Let $B \in \mathscr{B}_P$ be maximal. Then we know that $0 \leq B \leq \mathbb{1} - P$ but $B \neq \mathbb{1} - P$ since $\langle \phi, (\mathbb{1} - P)\phi \rangle > \omega(\mathbb{1} - P)$ while for B we have the reverse inequality according to $B \in \mathscr{B}_P$. Thus we can consider $Q = \mathbb{1} - P - B \geq 0$ which is non-zero. If $A \in \mathscr{A}^+$ satisfies $A \leq Q$ and $A \neq 0$ then necessarily

$$\omega(A) < \langle \phi, A\phi \rangle, \tag{☺☺}$$

since otherwise $A + B$ would be in $\mathscr{B}_P$ contradicting the maximality of B. By Proposition 4.4.9, *ix.)* and *vi.)*, we get for all $A \in \mathscr{A}$ the inequality

$$0 \leq QA^*AQ \leq \|A\|^2 Q^2 \leq \|A\|^2 \|Q\| Q$$

in the sense of positive operators. This shows that for $A \neq 0$ we have

$$0 \leq \frac{QA^*AQ}{\|A\|^2 \|Q\|} \leq Q. \tag{$*$}$$

Now either $QA^*AQ = 0$ which is the case iff $AQ = 0$. Then we clearly have $|\omega(AQ)|^2 = 0 \leq \|AQ\phi\|^2 = 0$. Or we have $QA^*AQ \neq 0$ then $(*)$ gives us by (☺☺) the estimate bastian: meh verwendung

$$\omega\left(\frac{QA^*AQ}{\|A\|^2\|Q\|} \right) < \left\langle \phi, \frac{QA^*AQ}{\|A\|^2\|Q\|} \phi \right\rangle.$$

This estimate leads to

$$|\omega(AQ)|^2 \leq \omega(\mathbb{1})\omega(QA^*AQ) < \langle \phi, QA^*AQ\phi \rangle = \|AQ\phi\|^2,$$

and thus in both cases we arrive at the estimate

$$|\omega(AQ)| \leq \|AQ\phi\| = \|A\|_{Q\phi}$$

This shows that the functional $A \mapsto \omega(AQ)$ is strongly continuous and hence also σ-strongly continuous. Thus $0 \neq Q \in \mathscr{B}$. Moreover, $P + Q = \mathbb{1} - B \leq \mathbb{1}$ and hence $P + Q \in \mathscr{B}$ as well. But this contradicts the maximality of P and thus the assumption $P \neq \mathbb{1}$ was wrong. It follows that ω is σ-strongly continuous, i.e. normal. □

Remark 7.2.45 The theorem is important in so far as it says that we can characterize the normal states of a von Neumann algebra $\mathscr{A}$ by means of its *-*algebraic properties* alone. In particular, the reference to the underlying Hilbert space is not needed.

Theorem 7.2.46 (Ideals in von Neumann algebras) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra and let $\mathscr{J} \subseteq \mathscr{A}$ be a left (or right) ideal. Then the following statements are equivalent:*

i.) The ideal $\mathscr{J}$ is σ-weakly closed.

ii.) The ideal $\mathscr{J}$ is σ-strongly closed.

iii.) The ideal $\mathscr{J}$ is σ-strongly closed*

iv.) There exists a unique projection $P \in \mathscr{A}$ with $\mathscr{J} = \mathscr{A}P$ (or $\mathscr{J} = P\mathscr{A}$).

PROOF: We consider the case of a left ideal, the right ideal case is analogous. Clearly *i.)* $\implies$ *ii.)* $\implies$ *iii.)* as usual by Theorem 5.1.10, *ii.)*. We choose a right approximate identity $\{\mathrm{e}_i\}_{i\in I}$ for $\mathscr{J}$ by Proposition 4.4.9, i.e. an increasing net of positive elements $0 \leq \mathrm{e}_i \leq \mathbb{1}$ with $\lim_{i\in I} A\mathrm{e}_i = A$ for all $A \in \mathscr{J}$. Thus we have a σ-strong* limit $P = \lim_{i\in I} \mathrm{e}_i = \sup_{i\in I} \mathrm{e}_i \in \mathscr{J}$ by Corollary 7.2.43 and the σ-strong* closedness of $\mathscr{J}$. Now let $A \in \mathscr{J}$ then by the σ-strong* continuity of the multiplication by the fixed operator A we have

$$AP = A \lim_{i\in I} \mathrm{e}_i = \lim_{i\in I} A\mathrm{e}_i = A, \tag{$*$}$$

where the first equation is the definition of P and the last the defining property of a right approximate identity where we even have norm convergence and hence clearly σ-strong* convergence as well. Since $P \in \mathscr{J}$ we have $\mathscr{A}P \subseteq \mathscr{J}$. But $(*)$ shows that every $A \in \mathscr{J}$ is also in $\mathscr{A}P$ and thus $\mathscr{A}P = \mathscr{J}$. Finally, P is Hermitian as a σ-strong* limit of Hermitian elements and by the same continuity of the multiplication with P

$$P^2 = P(\lim_{i\in I} \mathrm{e}_i) \stackrel{(☺☺)}{=} \lim_{i\in I} \mathrm{e}_i = P$$

Stefan: Labels? For (☺☺) suppose $Q \in \mathscr{A}$ is another projection with $\mathscr{A}P = \mathscr{A}Q$. Since $Q \in \mathscr{A}P$ we have $Q = AP$ for some $A \in \mathscr{A}$ and hence $QP = AP^2 = AP = Q$ as well as $PQ = (QP)^* = Q^* = Q$. By Theorem 3.5.8 this implies $Q \leq P$. Since we have the same argument for the roles of P and Q exchanged we see $Q = P$. Hence P is a projection as claimed, showing *iii.)* $\implies$ *iv.)*. Now assume *iv.)* and let $\{A_i\}_{i\in I}$ be a net in $\mathscr{A}P$ converging to some $A \in \mathscr{A}$ in the σ-weakly topology. Then $A_i = A_iP \longrightarrow AP \in \mathscr{A}P$ still converges σ-weakly since the right multiplication with a fixed operator P is σ-weakly continuous. But this shows $A = AP \in \mathscr{A}P$ and hence $\mathscr{A}P$ is σ-weakly closed. □

Corollary 7.2.47 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra and let $\mathscr{J} \subseteq \mathscr{A}$ be a two-sided ideal. Then the following statements are equivalent:*

i.) The ideal $\mathscr{J}$ is σ-weakly closed.

ii.) The ideal $\mathscr{J}$ is σ-strongly closed.

iii.) The ideal $\mathscr{J}$ is σ-strongly closed.*

iv.) There exists a unique central projection $P \in \mathscr{Z}(\mathscr{A})$ with $\mathscr{J} = \mathscr{A}P = P\mathscr{A}$.

*In this case, $\mathscr{J}$ is a norm-closed *-ideal as well.*

PROOF: Since $\mathscr{J}$ is both, a left and a right ideal we have by the theorem two unique projections P, Q with $\mathscr{J} = \mathscr{A}P = Q\mathscr{A}$ if one of the first three condition is satisfied. We only have to check $P = Q$. Since $P \in Q\mathscr{A}$ we get $QP = P$ and hence $P \leq Q$. Since $Q \in \mathscr{A}P$ we get $Q = QP$ and thus $Q \leq P$ showing $P = Q$. For $A \in \mathscr{A}$ we have $AP \in \mathscr{J}$ as well as $PA \in \mathscr{J}$. Hence $AP = PAP$ as well as $PA = PAP$. This shows $PA = AP$ and hence P is central. It follows that $\mathscr{J}$ is a *-ideal. □

The second application is the following automatic continuity theorem for von Neumann algebras:

Theorem 7.2.48 (Continuity properties of *-isomorphisms) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H}_1)$ and $\mathscr{B} \subseteq \mathfrak{B}(\mathfrak{H}_2)$ be von Neumann algebras and let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a *-isomorphism. Then Φ is continuous with respect to the σ-weak, the σ-strong, and the σ-strong* topologies, respectively.*

PROOF: Being a *-isomorphism, Φ induces an isomorphism of the partially ordered sets $(\mathscr{A}^+, \leq)$ as well as $(\mathscr{B}^+, \leq)$. Thus it is clear that for every bounded increasing net $\{A_i\}_{i\in I}$ of positive operators in $\mathscr{A}$ we have

$$\sup_{i\in I} \Phi(A_i) = \Phi(\sup_{i\in I} A_i). \tag{$*$}$$

If now $\omega\colon \mathscr{B} \longrightarrow \mathbb{C}$ is a normal linear functional then ω is a linear combination of normal states by Corollary 7.2.39. But for a normal state ω on $\mathscr{B}$ we get from Dixmier's Theorem 7.2.44

$$\omega(\Phi(\sup_{i\in I}(A_i))) \stackrel{(*)}{=} \omega(\sup_{i\in I} \Phi(A_i)) = \sup_{i\in I} \omega(\Phi(A_i))$$

Hence $\omega \circ \Phi\colon \mathscr{A} \longrightarrow \mathbb{C}$ is a normal state as well. Thus for all normal linear functionals ω on $\mathscr{B}$ also $\Phi^*\omega = \omega \circ \Phi$ is a normal linear functional on $\mathscr{A}$ and hence σ-weakly continuous on $\mathscr{A}$. Since the σ-weak topology on $\mathscr{B}$ is the weak* topology and hence the initial topology with respect to all the normal functionals, by the general characterization of initial topologies Φ is continuous since every $\omega \circ \Phi$ is continuous. This shows the case where both von Neumann algebras are equipped with the σ-weak topology. Now suppose $A_i \longrightarrow 0$ is a σ-strongly convergent net in $\mathscr{A}$. We claim that in this case $A_i^* A_i \longrightarrow 0$ in the σ-weak topology. Indeed, if $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{H}$ with $\sum_{n=0}^\infty \|\phi_n\|^2 < \infty$ then

$$\|A_i\|^2_{\{\phi_n\}} = \sum_{n=0}^\infty \|A_i\phi_n\|^2 = \sum_{n=0}^\infty \langle \phi_n, A_i^* A_i \phi_n\rangle = \operatorname{tr}(\rho A_i^* A_i)$$

with $\rho = \sum_{n=0}^\infty \Theta_{\phi_n,\phi_n} \in \mathfrak{L}^1(\mathfrak{H})$. Thus $\|A_i\|^2_{\{\phi_n\}} \longrightarrow 0$ is equivalent to $|\operatorname{tr} \rho A_i^* A_i| \longrightarrow 0$. By polarization, this is also equivalent to $|\sum_{n=0}^\infty \langle \psi_n, A_i^* A_i \phi_n\rangle| \longrightarrow 0$ for all $\{\phi_n\}_{n\in\mathbb{N}}, \{\psi_n\}_{n\in\mathbb{N}}$, which is the σ-weak convergence of $A_i^* A_i \longrightarrow 0$. Thus we see that in the σ-weak topology $\Phi(A_i)^*\Phi(A_i) = \Phi(A_i^* A_i) \longrightarrow 0$ since Φ is σ-weakly continuous. Reversing the above equivalence gives $\Phi(A_i) \longrightarrow 0$ σ-strongly and hence Φ is σ-strongly continuous at 0, hence everywhere. Finally, let $A_i \longrightarrow 0$ σ-strongly* which means $A_i \longrightarrow 0$ and $A_i^* \longrightarrow 0$ σ-strongly. Then we get $\Phi(A_i) \longrightarrow 0$ and $\Phi(A_i^*) \longrightarrow 0$ in the σ-strong topology. Since Φ is a *-isomorphism, also $\Phi(A_i)^* = \Phi(A_i^*) \longrightarrow 0$ showing that $\Phi(A_i) \longrightarrow 0$ σ-strongly*. □

Thus this theorem guarantees that von Neumann algebras enjoy good categorical properties, determined already by their *-algebra structure.

Exercise: Cat properties in

7.3 Decomposition and Factors

We will continue our basic investigation of von Neumann algebras by considering the possibilities to decompose a given von Neumann algebra into a direct sum of simpler ones. The main idea comes from Theorem 7.2.46: if $\mathfrak{J} \subseteq \mathscr{A}$ is a weakly closed ideal (and hence a *-ideal) then it is generated by a *central* projection $P \in \mathscr{Z}(\mathscr{A})$. But then $\mathbb{1} - P$ is also central and hence

$$\mathscr{A} = \mathscr{A}P \oplus \mathscr{A}(\mathbb{1} - P) \tag{7.3.1}$$

decomposes into a direct sum of two complementary ideals. Moreover, having a central projection P we get a direct orthogonal sum decomposition

$$\mathfrak{H} = P\mathfrak{H} \oplus (\mathbb{1} - P)\mathfrak{H}, \tag{7.3.2}$$

which is invariant under $\mathscr{A}$. In fact, $\mathscr{A}P$ is the part which acts on $P\mathfrak{H}$ while $\mathscr{A}(\mathbb{1}-P)$ is the part acting on $(\mathbb{1}-P)\mathfrak{H}$ and the cross terms are trivial. This motivates to study projections and in particular central projections of $\mathscr{A}$ in detail.

The main application we have in mind is of course still the case where we have a non-degenerate *-representation π of a C^*-algebra $\mathscr{A}$: then the von Neumann algebra we want to decompose is $\pi(\mathscr{A})''$.

7.3.1 Projections and K_0

In this preliminary subsection we consider first the purely algebraic situation of a unital *-algebra $\mathscr{A}$. In fact, most definition and statements can be transferred rather easily to the non-unital case as well. We refer to [2, 47] for more details on K-theory.

The first task will be to establish a way of comparing projections and, more general, idempotent elements of $\mathscr{A}$. A first idea is to compare two idempotent elements $e, f \in \mathscr{A}$ in a unital algebra by requiring an invertible element $u \in \mathscr{A}$ with

$$ueu^{-1} = f. \tag{7.3.3}$$

In the case of a unital *-algebra and projections $p, q \in \mathscr{A}$ instead of arbitrary idempotents it would be more natural to require a unitary element u with

$$upu^* = q. \tag{7.3.4}$$

It is easy to see that in both cases this gives an equivalence relation on the set of idempotents and projections, respectively. However, it turns out that this equivalence relation might be too restrictive to be useful. The following definition is slightly more general:

Definition 7.3.1 (Equivalence of projections) *Let $\mathscr{A}$ be a (unital) algebra. Then two idempotent elements $e, f \in \mathscr{A}$ are called equivalent if there are elements $u, v \in \mathscr{A}$ with*

$$e = uv \quad \textit{and} \quad f = vu. \tag{7.3.5}$$

If $\mathscr{A}$ is a (unital) *-algebra and $p, q \in \mathscr{A}$ are projections then they are called equivalent if there is a $u \in \mathscr{A}$ with

$$p = uu^* \quad \textit{and} \quad q = u^*u. \tag{7.3.6}$$

The following proposition shows that we get equivalence relations which are not very far from our first attempts (7.3.3) and (7.3.4). Note also that u with (7.3.6) is a partial isometry.

Proposition 7.3.2 *Let $\mathscr{A}$ be an associative algebra.*

i.) Equivalence of idempotents is an equivalence relation.

ii.) If $\mathscr{A}$ is unital then conjugate idempotents in the sense of (7.3.3) *are equivalent.*

iii.) If $\mathscr{A}$ is unital then equivalent idempotents are conjugate in $\mathrm{M}_2(\mathscr{A})$. In detail, one has

$$\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -u & 1-e \\ 1-f & v \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -v & 1-f \\ 1-e & u \end{pmatrix} \tag{7.3.7}$$

whenever u satisfies (7.3.5).

iv.) If $\mathscr{A}$ is a *-algebra then equivalence of projections is an equivalence relation.

v.) If $\mathscr{A}$ is a unital *-algebra then unitarily conjugate projections in the sense of (7.3.4) *are equivalent.*

vi.) If $\mathscr{A}$ is a unital **-algebra then equivalent projections are unitarily conjugate in* $\mathrm{M}_2(\mathscr{A})$*. One has*

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -u & 1-p \\ 1-q & u^* \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -u^* & 1-q \\ 1-p & u \end{pmatrix} \tag{7.3.8}$$

for every u satisfying (7.3.6).

Proof: Reflexivity is clear by taking $u = v = e$ and $u = u^* = p$, respectively. Symmetry is also clear by exchanging the roles of u and v as well as u and u^*, respectively. For transitivity, let $e = uv$ and $f = vu$ as well as $f = ab$ and $g = ba$. Then we have

$$(ua)(bv) = ufv = uvuv = e^2 = e$$

and

$$bvua = bfa = baba = g^2 = g$$

showing the equivalence of e and g. The *-case is analogous, establishing *i.)* and *iv.)*. The second and fifth part is trivial since if $f = aea^{-1}$ then $e = a^{-1}fa$ and thus $u = a^{-1}f$ and $v = a$ will do the job. In the *-case, $f = aea^*$ with $a^* = a^{-1}$ gives $e = a^*fa$ and $u = a^*f$ will do the job as $u^* = f^*a^* = fa^*$ and $f^2 = f$. For the third and sixth part, it is a straight forward computation to verify first (7.3.7) and (7.3.8), and second to show that the matrices are indeed invertible and unitary, respectively, i.e.

$$\begin{pmatrix} -u & 1-f \\ 1-f & v \end{pmatrix}^{-1} = \begin{pmatrix} -v & 1-f \\ 1-e & u \end{pmatrix}$$

and

$$\begin{pmatrix} -u & 1-f \\ 1-q & u^* \end{pmatrix}^{-1} = \begin{pmatrix} -u^* & 1-q \\ 1-p & u \end{pmatrix}. \qquad \square$$

It may actually happen that in $\mathscr{A}$ two idempotents (or projections) are equivalent but not conjugate. Only after passing to $\mathrm{M}_2(\mathscr{A})$ this can be achieved in general. This motivates also the point of view to consider not only $\mathscr{A}$ but also *all* matrix algebras $\mathrm{M}_n(\mathscr{A})$ at once and study idempotents or projections in there. Thus we consider the algebra $\mathrm{M}_\infty(\mathscr{A})$ of infinite matrices with entries in $\mathscr{A}$ such that only finitely many entries are different from zero. Clearly, we can view $\mathrm{M}_n(\mathscr{A})$ as subalgebra of $\mathrm{M}_\infty(\mathscr{A})$ sitting in the upper left corner for every n. With this identification we have

$$\bigcup_{n=1}^{\infty} \mathrm{M}_n(\mathscr{A}) = \mathrm{M}_\infty(\mathscr{A}), \tag{7.3.9}$$

and the embedding $\mathrm{M}_n(\mathscr{A}) \hookrightarrow \mathrm{M}_\infty(\mathscr{A})$ is compatible with the embeddings $\mathrm{M}_n(\mathscr{A}) \hookrightarrow \mathrm{M}_m(\mathscr{A})$ for $n \le m$, again in the upper left corner. With some more highbrow language, $\mathrm{M}_\infty(\mathscr{A})$ is the inductive limit of $\{\mathrm{M}_n(\mathscr{A})\}_{n\in\mathbb{N}}$ with respect to the embeddings $\mathrm{M}_n(\mathscr{A}) \hookrightarrow \mathrm{M}_m(\mathscr{A})$ as above. It is clear that $\mathrm{M}_\infty(\mathscr{A})$ is again an associative algebra, though no longer unital even if $\mathscr{A}$ was unital. If $\mathscr{A}$ is a *-algebra, $\mathrm{M}_\infty(\mathscr{A})$ is a *-algebra as well. The main advantage is now that for every $n \in \mathbb{N}$ we have

$$\mathrm{M}_\infty(\mathrm{M}_n(\mathscr{A})) \cong \mathrm{M}_\infty(\mathscr{A}). \tag{7.3.10}$$

Thus the algebra and all its (finite) matrix algebras appear on the same footing inside $\mathrm{M}_\infty(\mathscr{A})$.

Definition 7.3.3 ($\mathsf{Idemp}(\mathscr{A})$ and $\mathsf{Proj}(\mathscr{A})$) *Let $\mathscr{A}$ be an associative algebra then we define*

$$\underline{\mathsf{Idemp}}(\mathscr{A}) = \left\{ e \in \mathrm{M}_\infty(\mathscr{A}) \;\middle|\; e^2 = e \right\}, \tag{7.3.11}$$

and set

$$\mathsf{Idemp}(\mathscr{A}) = \underline{\mathsf{Idemp}}(\mathscr{A})/\sim. \tag{7.3.12}$$

For a **-algebra* $\mathscr{A}$ *we define*

$$\underline{\mathsf{Proj}}(\mathscr{A}) = \left\{p \in \mathrm{M}_\infty(\mathscr{A}) \;\middle|\; p = p^2 = p^*\right\}, \tag{7.3.13}$$

and set

$$\mathsf{Proj}(\mathscr{A}) = \underline{\mathsf{Proj}}/\sim. \tag{7.3.14}$$

Of course in (7.3.14) we use the more restrictive equivalence relation (7.3.6) for projections. Spelling out the details again this means that a representative of $[e] \in \mathsf{Idemp}(\mathscr{A})$ consists of an idempotent $e \in \mathrm{M}_n(\mathscr{A})$ for some $n \in \mathbb{N}$. Two representatives $e \in \mathrm{M}_n(\mathscr{A})$ and $f \in \mathrm{M}_m(\mathscr{A})$ yield the same class if, after adding enough zeros, $\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)$ become equivalent in some $\mathrm{M}_N(\mathscr{A})$ with $N \geq n, m$. In fact, $N = \max(n, m)$ will suffice.

Proposition 7.3.4 *Let* $\mathscr{A}$ *be an associative algebra. Then for* $[e], [f] \in \mathsf{Idemp}(\mathscr{A})$ *the definition*

$$[e] \oplus [f] = \left[\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right] \tag{7.3.15}$$

yields a well-defined abelian semigroup structure on $\mathsf{Idemp}(\mathscr{A})$*. With the analogous definition,* $\mathsf{Proj}(\mathscr{A})$ *becomes an abelian semigroup, too.*

Proof: We have to check that (7.3.15) is well-defined. First it is clear that $\left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right) \in \mathrm{M}_{n+m}(\mathscr{A})$ is again an idempotent (a projection) whenever $e \in \mathrm{M}_n(\mathscr{A})$ and $f \in \mathrm{M}_m(\mathscr{A})$ are idempotents (projections). Let e' and f' be other representations of $[e]$ and $[f]$ then we can assume $e' \in \mathrm{M}_n(\mathscr{A})$ and $f' \in \mathrm{M}_m(\mathscr{A})$ for the same n and m by adding zeros if necessary. In $\mathrm{M}_{2n}(\mathscr{A})$ the idempotents (projections) e' and e are then conjugated by some invertible (unitary) $u \in \mathrm{M}_{2n}(\mathscr{A})$ while f and f' are conjugated by some invertible (unitary) $v \in \mathrm{M}_{2m}(\mathscr{A})$. But then $\left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right) \in \mathrm{M}_{n+m}(\mathscr{A})$ is conjugated inside $\mathrm{M}_{2n+2m}(\mathscr{A})$ by $\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)$. This shows that $\oplus$ is well-defined. But then it is clear that we get the structure of an abelian semigroup as this can be checked on representatives. □

Example 7.3.5 For $\mathscr{A} = \mathbb{C}$ we have as abelian semigroup

$$\mathsf{Idemp}(\mathbb{C}) = \mathsf{Proj}(\mathbb{C}) = \mathbb{N}_0, \tag{7.3.16}$$

since every idempotent (projection) $e \in \mathrm{M}_n(\mathbb{C})$ is uniquely characterized up to equivalence (here even conjugation) by its trace. Indeed, for $e \in \mathrm{M}_n(\mathbb{C})$ with $e^2 = e$ the number $\operatorname{tr}(e) \in \mathbb{N}_0$ is the dimension of the image of e which is the only invariant e has.

Remark 7.3.6 (Serre-Swan Theorem) A slightly more "geometric" interpretation of $\mathsf{Idemp}(\mathscr{A})$ is the following. Recall that a right $\mathscr{A}$-module $\mathscr{E}_{\mathscr{A}}$ is called finitely generated and projective if there is an idempotent $e = e^2 \in \mathrm{M}_n(\mathscr{A})$ such that $\mathscr{E}_{\mathscr{A}} \cong e\mathscr{A}^n \subseteq \mathscr{A}^n$ as right $\mathscr{A}$-modules. The idempotent e is not uniquely determined by $\mathscr{E}_{\mathscr{A}}$ but any two are equivalent in the sense of Definition 7.3.1 inside a large enough $\mathrm{M}_m(\mathscr{A})$ or better: directly inside $\mathrm{M}_\infty(\mathscr{A})$. Moreover, isomorphic right modules give raise to equivalent idempotents. Thus $\mathsf{Idemp}(\mathscr{A})$ classifies finitely generated projective modules over $\mathscr{A}$ up to isomorphism. The semigroup structure $\oplus$ corresponds then just to the direct sum of finitely generated and projective modules. We call this a more geometric interpretation as for $\mathscr{A} = \mathscr{C}(X)$ with a compact Hausdorff space one can show that a finitely generated and projective module is isomorphic to the continuous sections of a vector bundle over X: this is the famous Serre-Swan Theorem. Since we do not want to enter the geometric discussion here, instead we refer to for further reading.

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We are now able to define the K_0-group of an associative algebra $\mathscr{A}$. In the *unital* case we simply take the *Grothendieck group* of the abelian semigroup $Idemp(\mathscr{A})$: recall that for an abelian semigroup S on can construct an abelian group $\mathrm{G}(S)$ by taking equivalence classes of formal differences, i.e.

$$\mathrm{G}(S) = \{(a, b) \in S \times S \mid a, b \in S\}/\sim, \tag{7.3.17}$$

where $(a, b) \sim (a', b')$ if there is a $c \in S$ such that $a + b' + c = a' + b + c$. The idea is that (a, b) corresponds to the difference "$a - b$" and the equivalence relation implements the usual computational rules. It is now easy to see that $\mathrm{G}(S)$ is an abelian group and $S \ni a \mapsto [(a, 0)] \in \mathrm{G}(S)$ is a semigroup homomorphism. Applying this construction, which is in fact functorial, to $S = \mathbb{N}_0$ we get $\mathrm{G}(S) = \mathbb{Z}$, a fact known from primary school, see also Exercise 7.5.3.

Definition 7.3.7 (K_0-Theory) *Let $\mathscr{A}$ be an associative algebra.*

i.) If $\mathscr{A}$ is unital one defines the K_0-group $\mathsf{K}_0(\mathscr{A})$ of $\mathscr{A}$ to be the Grothendieck group of $\mathsf{Idemp}(\mathscr{A})$.

ii.) If $\mathscr{A}$ is a unital *-algebra, one defines the Hermitian K_0-group $\mathsf{K}_0^*(\mathscr{A})$ of $\mathscr{A}$ to be the Grothendieck group of $\mathsf{Proj}(\mathscr{A})$.

iii.) If $\mathscr{A}$ is non-unital, then one defines the K_0-group $\mathsf{K}_0(\mathscr{A})$ of $\mathscr{A}$ to be the kernel of the canonical group homomorphism

$$\mathsf{K}_0(\widetilde{\mathscr{A}}) \longrightarrow \mathsf{K}_0(\mathbb{C}), \tag{7.3.18}$$

where $\widetilde{\mathscr{A}}$ is the unitization of $\mathscr{A}$ as usual.

iv.) If $\mathscr{A}$ is a non-unital *-algebra then one defines the Hermitian K_0-group $\mathsf{K}_0^*(\mathscr{A})$ to be the kernel of the canonical group homomorphism

$$\mathsf{K}_0^*(\widetilde{\mathscr{A}}) \longrightarrow \mathsf{K}_0^*(\mathbb{C}). \tag{7.3.19}$$

Remark 7.3.8 In the non-unital case, we consider the unitization $\widetilde{\mathscr{A}} = \mathscr{A} \oplus \mathbb{C}\mathbb{1}$. By forgetting the components in $\mathscr{A}$ we get an induced map $\widetilde{\mathscr{A}} \longrightarrow \mathbb{C}$ and hence also $\mathrm{M}_\infty(\widetilde{\mathscr{A}}) \longrightarrow \mathrm{M}_\infty(\mathbb{C})$. Clearly, it maps idempotents (projections) to idempotents (projections). Thus we get an induced map

$$\underline{\mathsf{Idemp}}(\widetilde{\mathscr{A}}) \longrightarrow \underline{\mathsf{Idemp}}(\mathbb{C}) \tag{7.3.20}$$

or

$$\underline{\mathsf{Proj}}(\widetilde{\mathscr{A}}) \longrightarrow \underline{\mathsf{Proj}}(\mathbb{C}), \tag{7.3.21}$$

respectively. It is now easy to check that this is compatible with the equivalence relation and thus drops to a morphism of semigroups on the level of Idemp or Proj, respectively. Since the Grothendieck construction is functorial, this gives the maps (7.3.20) and (7.3.21), respectively. The reason for this somehow more complicated definition is a better functorial behaviour of K_0.

The K_0-group of $\mathscr{A}$ is now one of the very important and fundamental invariants. In particular for C^*-algebra it proved to be a crucial tool in many aspects. We will not discuss the *K-theory* in much detail but conclude this subsection with two first results:

Proposition 7.3.9 (Functoriality of K_0) *Building* $\mathsf{Idemp}(\mathscr{A})$ *yields a functor*

$$\mathsf{Idemp}\colon \mathsf{alg} \longrightarrow \mathsf{Semigroups} \tag{7.3.22}$$

and also

$$\mathsf{K}_0\colon \mathsf{alg} \longrightarrow \mathsf{Ab}. \tag{7.3.23}$$

Analogously, we get functors

$$\mathsf{Proj}\colon {}^*\text{-}\mathsf{alg} \longrightarrow \mathsf{Semigroups} \tag{7.3.24}$$

$$\mathsf{K}_0^*\colon {}^*\text{-}\mathsf{alg} \longrightarrow \mathsf{Ab}. \tag{7.3.25}$$

PROOF: Let $\phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be an algebra homomorphism then also $\phi^{(n)}\colon \mathrm{M}_n(\mathscr{A}) \longrightarrow \mathrm{M}_n(\mathscr{B})$ is an algebra homomorphism. It clearly maps idempotents to idempotents. Moreover, if $e \sim f$ via u and v then $\phi(e) \sim \phi(f)$ via $\phi(u)$ and $\phi(v)$. This gives a well-defined map $\phi\colon \mathsf{Idemp}(\mathscr{A}) \longrightarrow \mathsf{Idemp}(\mathscr{B})$ which clearly respects $\oplus$. From this the functoriality of (7.3.22) easily follows. In the unital case, (7.3.23) is functorial as well since the Grothendieck construction is. For the non-unital case(s) one first notes that we can extend ϕ to an algebra morphism between the unitizations of $\mathscr{A}$ and/or $\mathscr{B}$ such that

$$\widetilde{\phi}\colon \widetilde{\mathscr{A}} \longrightarrow \widetilde{\mathscr{B}} \text{ is unital,}$$

$$\widetilde{\phi}\colon \widetilde{\mathscr{A}} \longrightarrow \mathscr{B} \text{ is unital,}$$

or

$$\phi\colon \mathscr{A} \longrightarrow \mathscr{B} \text{ is non-unital}$$

for the possible cases of $\mathscr{A}$ and/or $\mathscr{B}$ being non-unital or unital. Any case gives a well-defined map on $\mathsf{Idemp}(\,\cdot\,)$ of the unitizations. Moreover, the passage to unitization is functorial again which in the end results in functorial behaviour of K_0, too, also for this case. Finally, the *-algebra case is completely analogous. ☐

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The next theorem explains that in a C^*-algebraic case we do not have to care about the difference between projections and idempotents at all [30, Thm. 26].

Theorem 7.3.10 (K_0 for C^*-algebras) *Let $\mathscr{A}$ be a C^*-algebra.*

i.) The matrix algebra $\mathrm{M}_n(\mathscr{A})$ is again a C^*-algebra.

ii.) Every idempotent $e \in \mathrm{M}_n(\mathscr{A})$ is equivalent to a projection.

iii.) Two projections $p, q \in \mathrm{M}_n(\mathscr{A})$ are equivalent as projections iff they are equivalent as idempotents.

iv.) One has $\mathsf{Idemp}(\mathscr{A}) = \mathsf{Proj}(\mathscr{A})$.

v.) One has $\mathsf{K}_0(\mathscr{A}) = \mathsf{K}_0^*(\mathscr{A})$.

PROOF: For the first part, let $\pi\colon \mathscr{A} \longrightarrow \mathfrak{B}(\mathfrak{H})$ be a faithful *-representation which we know to exist thanks to Theorem 4.4.33. Then also $\pi^{(n)}\colon \mathrm{M}_n(\mathscr{A}) \longrightarrow \mathfrak{B}(\mathfrak{H}^n)$ is a faithful *-representation with respect to the usual *-algebra structure of $\mathrm{M}_n(\mathscr{A})$ and $\pi^{(n)}$ being "matrix-multiplication" using π on each component. Since $\pi(\mathscr{A}) \subseteq \mathfrak{B}(\mathfrak{H})$ is a norm-closed subalgebra it easily follows that also $\pi^{(n)}(\mathrm{M}_n(\mathscr{A})) \subseteq \mathfrak{B}(\mathfrak{H}^n)$ is norm-closed, the convergence being just the componentwise convergence in $\mathfrak{B}(\mathfrak{H}^n) \cong \mathrm{M}_n(\mathfrak{B}(\mathfrak{H}))$. Thus $\mathrm{M}_n(\mathscr{A})$ is *-isomorphic to a C^*-algebra $\pi^{(n)}(\mathrm{M}_n(\mathscr{A}))$ which allows to pull-back the operator norm. This makes $\mathrm{M}_n(\mathscr{A})$ a C^*-algebra, too. Note that the C^*-norm is uniquely determined by the algebraic relations via the spectra, see Theorem 4.3.9. It was only necessary to show that there is a C^*-norm at all. Hence the above C^*-norm does not depend on the chosen faithful representation π a posteriori. Now let $e \in \mathscr{A}$ be an idempotent, where it suffices to consider the case $n = 1$ thanks to *i.)*. Then define

$$z = \mathbb{1} + (e - e^*)(e^* - e) = z^*$$

(as an element of the unitization $\widetilde{\mathscr{A}}$ if $\mathscr{A}$ is non-unital). Since $(e - e^*)(e^* - e) \geq 0$ we know that z is invertible. Then

$$\begin{aligned} ez &= e + e(e - e^*)(e^* - e) \\ &= e + (e - ee^*)(e^* - e) \\ &= e + ee^* - ee^*e^* - e + ee^*e \\ &= ee^*e, \end{aligned}$$

since $e^2 = e$ and hence also $(e^*)^2 = e^*$. Analogously, one computes $ze = ee^*e$ showing that z and e commute. But $z = z^*$ and hence also z and e^* commute. Finally, also z^{-1} commutes with e and e^*. We define now

$$p = ee^*z^{-1} = z^{-1}ee^* = ez^{-1}e^*.$$

Then

$$p^2 = ee^*z^{-1}ee^*z^{-1} = z^{-1}ee^*ee^*z^{-1} = z^{-1}zee^*z^{-1} = ee^*z^{-1} = p,$$

and $p^* = (ez^{-1}e^*)^* = ez^{-1}e^*$ since z^{-1} is Hermitian. This shows that p is a projection. Finally

$$pe = z^{-1}ee^*e = z^{-1}ze = e$$

and

$$ep = eee^*z^{-1} = ee^*z^{-1} = p$$

shows that p and e are equivalent via p and e. Note that in the non-unital case we have $p \in \mathscr{A}$ since $\mathscr{A} \subseteq \widetilde{\mathscr{A}}$ is a *-ideal. This shows the second part. For the third part it is clear that $p \sim q$ in the sense of projections implies $p \sim q$ in the sense of idempotents. Thus assume the converse and let $p = uv$ and $q = vu$ with $u, v \in \mathscr{A}$. By passing to matrices if necessary we can assume that $p = aqa^{-1}$ with some invertible a according to Proposition 7.3.2, *iii.)*. Thus we have from $q = q^*$ the relation

$$a^*aq = a^*pa = (a^*pa)^* = qa^*a$$

and thus also $|a|q = q|a|$. Let $a = u|a|$ be the polar decomposition of the invertible element a according to Theorem 5.1.36 with $u^* = u^{-1}$ being unitary. Then

$$p = aqa^{-1} = u|a|q(u|a|)^{-1} = uqu^{-1} = uqu^*$$

shows that p and q are unitarily conjugate and hence equivalent as projections. This proves *iii.)*. But then *iv.)* and *v.)* are clear. □

Remark 7.3.11 Thanks to this result we can safely speak of *the* K_0-group of a C^*-algebra without being too precise whether we mean the Hermitian version or the ring-theoretic. Note that for the second part, which essentially means that the forgetting map

$$\mathsf{Idemp}(\mathscr{A}) \longrightarrow \mathsf{Proj}(\mathscr{A}) \tag{7.3.26}$$

is surjective, we only need that $\mathbb{1} + a^*a$ is invertible for all $a \in \mathrm{M}_n(\mathscr{A})$. This is of course a much weaker requirement than having a C^*-algebra.

7.3.2 Subprojections and Comparison

We shall now give a more refined classification of projections in a von Neumann algebra. We start with the following result still valid in general C^*-algebras and generalizing our results from Theorem 3.5.8:

Proposition 7.3.12 *Let $\mathscr{A}$ be a C^*-algebra and let $p, q \in \mathscr{A}$ be projections. Then the following statements are equivalent:*

i.) $p \leq q$.

ii.) $q - p$ *is a projection.*

iii.) $pq = qp = p$.

PROOF: Here $p \leq q$ is of course to be understood in sense of positive elements of $\mathscr{A}$. We assume *i.)*. Then from Proposition 4.4.9, *vii.)*, we get

$$0 \leq p \leq pqp, \tag{$*$}$$

since $p^2 = p = p^*$. Now consider the positive element

$$\begin{aligned} 0 &\leq (p - qp)^*(p - qp) \\ &= (p - qp)(p - qp) \\ &= p^2 - pqp - pqp + pq^2p \\ &= p - pqp \\ &\leq 0 \end{aligned}$$

by $(*)$. This means that $(p - qp)^*(p - qp)$ is both a positive and a negative element of $\mathscr{A}$. By Proposition 4.4.9, *ii.)*, we get $(p - qp)^*(p - qp) = 0$ and by the C^*-property we have $p - qp = 0$ and hence $p = qp$. Playing the same trick with $qp - p$ instead gives $p = pq$ (or just use $p^* = p = p^*q^* = pq$) and hence *iii.)*. If *iii.)* holds then *ii.)* is just a computation

$$(q - p)(q - p) = q^2 - pq - qp + p^2 = q - p - p + p = q - p,$$

and $(q - p)^* = q - p$ is obvious. Thus *iii.)* $\implies$ *ii.)* follows. Finally, a projection is clearly positive and thus *ii.)* $\implies$ *i.)* holds as well. $\square$

Note that heavy C^*-machinery is needed and the equivalence of the three statements is not true in a *-algebra in general. Thus the following definition will only be stated for C^*-algebras as for general *-algebras one would need to chose between the three possibilities.

Definition 7.3.13 (Subprojection) *Let $p, q \in \mathscr{A}$ be projections in a C^*-algebra. Then p is called a subprojection of q if one of the (equivalent) conditions in Proposition 7.3.12 holds.*

More useful is the notion of a subprojection *up to equivalence* which is formulated as follows:

Definition 7.3.14 (Subordinate projection) *Let $p, q \in \mathscr{A}$ be projections in a C^*-algebra. Then p is called subordinate to q if p is equivalent to a subprojection of q. One writes $p \lesssim q$ in this case.*

Remark 7.3.15 (Subordinate projections)

i.) The relation $\lesssim$ is clearly reflexive. Moreover, it is easy to see that it is transitive: let $p \lesssim q \lesssim r$ and hence we have a subprojection $r' \leq r$ with $q = u^*u$ and $r' = uu^*$. Moreover, we have a subprojection $q' \leq q$ with $q' = v^*v$ and $p = vv^*$. Then we define $r'' = uq'(uq')^* = uq'u^*$. Clearly $r'' \leq r' \leq r$ is again a subprojection of r and $q' \sim r''$ since $(uq')^*(uq') = q'u^*uq' = q'qq' = q'$ by Proposition 7.3.12. Hence $p \sim q' \sim r''$ gives $p \lesssim r$ as claimed. On the other hand, there are examples of C^*-algebras $\mathscr{A}$ with projections $p \lesssim q$ and $q \lesssim p$ but p is not equivalent to q in this case, see e.g. [8, II.3.3.3].

ii.) We will often encounter the situation of a von Neumann algebra $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ inside which we want to study projections. Then the notion of subordinate projections always refers to the subalgebra $\mathscr{A}$ and not to the ambient algebra $\mathfrak{B}(\mathfrak{H})$, even though we will not stress this in our notation.

Even though we have a partial order $\leq$ on the projections of a C^*-algebra $\mathscr{A}$ this does not yet give the structure of a lattice as we know this from $\mathfrak{B}(\mathfrak{H})$, see Theorem 3.5.8. The reason is that in general two projections $p, q \in \mathscr{A}$ do not have a supremum or infimum in $\mathscr{A}$, as needed for a lattice by Lemma 3.2.4.

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If $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ is a von Neumann algebra then things change and the projections of $\mathscr{A}$ become a sublattice of those in $\mathfrak{B}(\mathfrak{H})$:

Theorem 7.3.16 (Lattice of projections) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra. Then the projections in $\mathscr{A}$ form a sublattice of the lattice of all projections in $\mathfrak{B}(\mathfrak{H})$. In fact, arbitrary suprema and infima of projections in $\mathscr{A}$ are again in $\mathscr{A}$ and the lattice relation "$\leq$" coincides with the order relation "$\leq$".*

PROOF: Let $\{P_i\}_{i\in I}$ be an arbitrary non-empty collection of projections in $\mathscr{A}$. Then in the lattice of all projections in $\mathfrak{B}(\mathfrak{H})$ the projections

$$P = \bigwedge_{i\in I} P_i = \inf_{i\in I} P_i = P_{\bigcap_{i\in I} \operatorname{im} P_i} \in \mathfrak{B}(\mathfrak{H})$$

$$Q = \bigvee_{i\in I} P_i = \sup_{i\in I} P_i = P_{(\sum_{i\in I} \operatorname{im} P_i)^{\perp\perp}} \in \mathfrak{B}(\mathfrak{H})$$

are the infimum and the supremum of the P_i constructed in the ambient lattice of $\mathfrak{B}(\mathfrak{H})$. We have to show $P, Q \in \mathscr{A}$. Suppose $U \in \mathscr{A}'$ is unitary, then for each $i \in I$ we have $UP_i = P_iU$ and thus $U \operatorname{im} P_i \subseteq \operatorname{im} P_i$. Hence also

$$U \operatorname{im} P = U \bigcap_{i\in I} \operatorname{im} P_i \subseteq \bigcap_{i\in I} \operatorname{im} P_i = \operatorname{im} P,$$

and thus $UP\phi = P\phi$ for all $\phi \in \mathfrak{H}$. Since U is unitary and $U^* = U^{-1} \in \mathscr{A}'$ as well, we also have $U^{-1}P = P$ and hence $P = PU^{**} = PU$ which shows that P commutes with U. Since every element in $\mathscr{A}'$ is a linear combination of 4 unitaries in $\mathscr{A}'$, see Exercise 4.5.42, this shows that $P \in \mathscr{A}'' = \mathscr{A}$. For Q we have in $\mathfrak{B}(\mathfrak{H})$

$$Q = \bigvee_{i\in I} P_i = \bigwedge_{i\in I} (\mathbb{1} - P_i),$$

and thus the same argument can be used by replacing P_i with $\mathbb{1} - P_i \in \mathscr{A}$. Thus also $Q \in \mathscr{A}$. Finally, the lattice relation $P \leq Q$ means $PQ = P$ (or $QP = P$) and hence $P \leq Q$ in the sense of positive operators, see Theorem 3.5.8 as well as Proposition 7.3.12. $\square$

A particular case of this general statement is obtained for a family of pairwise orthogonal projections $\{P_i\}_{i\in I}$:

Proposition 7.3.17 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra and let $\{P_i\}_{i\in I}$ and $\{Q_i\}_{i\in I}$ be families of pairwise orthogonal projections in $\mathscr{A}$.*

i.) The net of finite sums of the P_i is σ-strongly* convergent to

$$P = \sum_{i\in I} P_i = \sup_{i\in I} P_i. \tag{7.3.27}$$

ii.) If for all $i \in I$ one has $P_i \sim Q_i$ via $U_i \in \mathscr{A}$ then also $P \sim Q = \bigvee_{i\in I} Q_i$ via the σ-strongly* convergent net of finite sums

$$U = \sum_{i\in I} U_i. \tag{7.3.28}$$

iii.) If for all $i \in I$ one has $P_i \precsim Q_i$ then also $P \precsim Q$.

PROOF: First we note that since $P_iP_j = \delta_{ij}P_i$ we get projections $P_{i_1} + \cdots + P_{i_N}$ for every $i_1, \ldots, i_N \in I$ again. Then it is clear that the net of finite sums is increasing and bounded, since all members are projections and hence $\leq \mathbb{1}$. By Corollary 7.2.43 we have a limit in the σ-strong* topology in $\mathscr{A}$ which we denote by $P = \sum_{i\in I} P_i$. Moreover, the limit is the supremum showing that $P \leq \bigvee_{i\in I} P_i$ as positive elements. Since the operator multiplication is σ-strongly* continuous on bounded subsets by Theorem 5.1.10, *v.)*, and since all members of the net are bounded we have

$$P^2 = \left(\sum_{i\in I} P_i\right)\left(\sum_{j\in I} P_j\right) = \sum_{i,j\in I} P_i P_j = \sum_{i\in I} P_i = P,$$

showing that P is a projection. Since $\bigvee_{i\in I} P_i$ is the smallest projection in $\mathfrak{B}(\mathfrak{H})$ larger or equal than every P_i we conclude $P = \bigvee_{i\in I} P_i$. For the second part, we first show that (7.3.28) converges σ-strongly*. We know $P_i = U_i U_i^*$ while $Q_i = U_i^* U_i$, and thus $U_i \colon \operatorname{im} Q_i \longrightarrow \operatorname{im} P_i$ is unitary while $U_i\big|_{(\operatorname{im} Q_i)^\perp} = 0$ for all $i \in I$, see again Proposition 5.1.41. Since the Q_i are pairwise orthogonal the sum of the U_i is blockdiagonal and defines a partial isometry U with $UU^* = P$ and $Q = U^*U$ according to the block structure. We only have to show the σ-strong* convergence of $\sum_{i\in I} U_i = U$. Since all the finite sums $U_{i_1} + \cdots + U_{i_n}$ are partial isometries and hence bounded in norm by 1 we know that σ-strong convergence to U is the same as strong convergence. Since we have the same arguments for U_i and U_i^* we see that σ-strong convergence is the same as σ-strong* convergence in this case. Now let $\phi \in \mathfrak{H}$ be given then

$$\begin{aligned}
\|(U_{i_1} + \cdots + U_{i_n} - U)\phi\| &= \left\| \sum_{i\in I\setminus\{i_1,\ldots,i_n\}} U_i\phi \right\| \\
&= \left\| \sum_{i\in I\setminus\{i_1,\ldots,\beta_n\}} U_i Q_i \phi \right\| \\
&= \left\| \left(\sum_{i\in I\setminus\{i_1,\ldots,i_n\}} U_i \right)\left(\sum_{j\in I\setminus\{i_1,\ldots,i_n\}} Q_j\phi \right) \right\| \\
&= \left\| \sum_{j\in I\setminus\{i_1,\ldots,i_n\}} Q_j\phi \right\|, \qquad (*)
\end{aligned}$$

since on the image of this sum of the Q_j's the partial isometry is already isometric and on the complement it is zero. But then the strong convergence of the Q_i gives that $(*)$ converges to zero for large enough index sets $\{i_1, \ldots, i_n\}$ which is (7.3.28). Finally, assume $P_i \precsim Q_i$ then there are projections $R_i \le Q_i$ in $\mathscr{A}$ with $P_i \sim R_i$ via some partial isometries $U_i \in \mathscr{A}$. First we note that $R = \sum_{i\in I} R_i \le Q$ gives a subprojection of Q in $\mathscr{A}$ as clearly the R_i are pairwise orthogonal, too. Then $P \sim R$ via U as in part *ii.)* and $P \precsim Q$ follows. □

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The following statement is similar to the set-theoretic Schröder-Bernstein Theorem, see .

Proposition 7.3.18 *Let $P, Q \in \mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be projections in a von Neumann algebra. Then $P \precsim Q$ and $Q \precsim P$ implies $P \sim Q$.*

PROOF: Let $U, V \in \mathscr{A}$ be partial isometries with $P = U^*U$ and $Q_1 = UU^* \le Q$ as well as $Q = V^*V$ and $P_1 = VV^* \le P$. Then we take this as a starting point to define recursively

$$P_{n+1} = VQ_nV^* \text{ and } Q_{n+1} = UP_nU^* \qquad (*)$$

for $n = 1, 2, \ldots$. We first claim that this gives two decreasing sequences of projections: Indeed, unwinding the recursive definition gives first

$$P_{n+1} = VUP_{n-1}(VU)^*,$$

and thus, with $P_0 = P$,

$$P_{2n} = (VU)^n P_0 (VU)^{n*} \quad \text{and} \quad P_{2n+1} = (VU)^n P_1 (VU)^{n*}.$$

Now $P_2 = VUU^*V^* \leq \|UU^*\| VV^* = P_1$ and clearly $P_1 \leq P_0$. Thus conjugating these two inequalities with $(VU)^n$ gives

$$P_{2n+2} = (VU)^n P_2 (VU)^{n*} \leq (VU)^n P_1 (VU)^n = P_{2n+1}$$

and

$$P_{2n+1} = (VU)^n P_1 (VU)^{n*} \leq (VU)^n P_0 (VU)^{n*} = P_{2n},$$

see also Proposition 4.4.9, *vii.)*. Hence we indeed have $P_{n+1} \leq P_n$ for all $n \in \mathbb{N}_0$ and analogously for Q_n. Next we compute

$$P_{n+1}P_{n+1} = VQ_nV^*VQ_nV^* = VQ_nQ_0Q_nV^*.$$

Since Q_1 is a projection $\leq Q_0$ we can assume by induction that also Q_n is a projection, necessarily $\leq Q_0$ as we have already seen this. Then $Q_nQ_0 = Q_n$ and thus $P_{n+1}P_{n+1} = P_{n+1}$ follows. By the analogous induction we get $Q_{n+1}Q_{n+1} = Q_{n+1}$ completing our inductive proof that all the P_n and Q_n are projections. Being a decreasing sequence we have σ-strong* convergence to limiting projections

$$P_\infty = \lim_{n\longrightarrow\infty} P_n \quad \text{and} \quad Q_\infty = \lim_{n\longrightarrow\infty} Q_n,$$

respectively. By the recursive construction we have

$$U(P_n - P_{n+1})U^* = Q_{n+1} - Q_{n+2},$$

and hence $P_n - P_{n+1}$ is equivalent to $Q_{n+1} - Q_{n+2}$ via $V_n = U(P_n - P_{n+1})$ since indeed

$$\begin{aligned} V_n^*V_n &= (P_n - P_{n+1})U^*U(P_n - P_{n+1}) \\ &= (P_n - P_{n+1})P_0(P_n - P_{n+1}) \\ &= P_n - P_{n+1}, \end{aligned}$$

by $P_0 \geq P_n - P_{n+1}$. Analogously, $Q_n - Q_{n+1}$ is equivalent to $P_{n+1} - P_{n+2}$, needed to cover the case $Q_0 - Q_1$. Finally, taking the σ-strong* limit of the defining relation $(*)$ gives

$$P_\infty = VQ_\infty V^*,$$

as the multiplication with fixed operators is σ-strongly* continuous. Since again $V^*V = Q \geq Q_\infty$ we see that $P_\infty \sim Q_\infty$ via VQ_∞. Now we have by the usual telescope sum argument

$$P_0 = P_n + \sum_{k=1}^{n-1} (P_k - P_{k+1})$$

for all n. Since the σ-strong* limit of each term exists by the decreasing features of $(P_n - P_{n+1}) \geq (P_{n+1} - P_{n+2})$ and $P_n \geq P_{n+1}$ we get

$$P_0 = P_\infty + \sum_{n=0}^{\infty} (P_n - P_{n+1})$$

and analogously for Q_0. Moreover, we can also split the series in even and odd n yielding

$$P_0 = P_\infty + \sum_{n=0}^{\infty} (P_{2n} - P_{2n+1}) + \sum_{n=0}^{\infty} (P_{2n+1} - P_{2n+1})$$

as well as

$$Q_0 = Q_\infty + \sum_{n=0}^{\infty} (Q_{2n} - Q_{2n+1}) + \sum_{n=0}^{\infty} (Q_{2n+1} - Q_{2n+2}).$$

Now $P_{2n} - P_{2n+1} \sim Q_{2n+1} - Q_{2n+2}$ for all $n = 0, 1, \ldots$ and $Q_{2n} - Q_{2n+1} \sim P_{2n+1} - P_{2n+2}$. Since also $P_\infty \sim Q_\infty$ we can apply Proposition 7.3.17, *ii.)*, since clearly all the projections $P_{2n} - P_{2n+1}, P_{2n+1} - P_{2n+2}, P_\infty$ are pairwise orthogonal thanks to $P_n \geq P_{n+1} \geq P_\infty$ and analogously for the Q_n's. Hence $P_0 \sim Q_0$ follows. ☐

In the following we shall frequently define projections in a von Neumann algebra $\mathscr{A}$ by extreme properties like "the largest projection with ...". Since arbitrary *non-empty* subsets of projections have suprema and infima in $\mathscr{A}$ this will make sense.

Definition 7.3.19 (Support projections) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra.*

i.) Let $A \in \mathscr{A}$ then the left support $s_{\mathrm{left}}(A)$ (right support $s_{\mathrm{right}}(A)$) of A is the smallest projection in $\mathscr{A}$ with

$$s_{\mathrm{left}}(A)A = A \quad \textit{and} \quad As_{\mathrm{right}}(A) = A. \tag{7.3.29}$$

ii.) If $P \in \mathscr{A}$ is a projection then the central support $Z(P)$ is the smallest central projection in $\mathscr{A}$ with $P \leq Z(P)$.

$P \leq Q$ implies $Z(P) \leq Z(Q)$

Again, as already stressed in Remark 7.3.15, *ii.)*, the notions of supports of projections refer to a specific von Neumann algebra $\mathscr{A}$ inside $\mathfrak{B}(\mathfrak{H})$: we will (typically) not be interested in the supports of a projection with respect to the ambient $\mathfrak{B}(\mathfrak{H})$.

Proposition 7.3.20 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra.*

i.) For every $A \in \mathscr{A}$ the left and right support is well-defined and explicitly given by

$$s_{\mathrm{left}}(A) = UU^* \quad \textit{and} \quad s_{\mathrm{right}}(A) = U^*U, \tag{7.3.30}$$

where $A = U|A|$ is the polar decomposition of A in $\mathscr{A}$. Equivalently,

$$s_{\mathrm{left}}(A) = P_{(\operatorname{im} A)^{\perp\perp}} \quad \textit{and} \quad s_{\mathrm{right}}(A) = P_{(\ker A)^{\perp}}. \tag{7.3.31}$$

ii.) For every $A \in \mathscr{A}$ one has

$$s_{\mathrm{left}}(A) \sim s_{\mathrm{right}}(A). \tag{7.3.32}$$

iii.) If $A \in \mathscr{A}$ is normal then

$$s_{\mathrm{left}}(A) = s_{\mathrm{right}}(A), \tag{7.3.33}$$

which is then simply called the support $s(A)$ of A.

iv.) For two projections $P, Q \in \mathscr{A}$ one has

$$s_{\mathrm{left}}(Q(\mathbb{1} - P)) = Q - Q \wedge P \tag{7.3.34}$$

and

$$s_{\mathrm{right}}(Q(\mathbb{1} - P)) = P \vee P - P, \tag{7.3.35}$$

and hence

$$(P \vee Q) - P \sim Q - (P \wedge Q). \tag{7.3.36}$$

PROOF: For the first part we note that the projection $\mathbb{1} \in \mathscr{A}$ clearly satisfies $\mathbb{1}A = A$ as well as $A\mathbb{1} = A$. Hence the infimum needed in the definition of the left and right support is well-defined and exists by Theorem 7.3.16. Now let $A = U|A|$ with $|A| = \sqrt{A^*A}$ as usual be the polar decomposition of A which gives $U, |A| \in \mathscr{A}$, see Proposition 7.2.21. We know that U is a partial isometry and hence $P = U^*U$ and $Q = UU^*$ are projections. Moreover, P is the projection onto $\operatorname{im} U^* = (\ker U)^{\perp}$ while Q is the projection onto $\operatorname{im} U = (\ker U^*)^{\perp}$ according to Proposition 5.1.41. Finally, we know that $\ker U = \ker A$ and $\operatorname{im} U = (\operatorname{im} A)^{\perp\perp}$, by Theorem 5.1.42. For any partial isometry we have

$$UP = UU^*U = QU,$$

and since Q projects onto $\operatorname{im} U$ we get

$$UP = U.$$

Analogously, we have $U^*Q = U^*$ or $U = QU$. Using this we get

$$QA = QU|A| = U|A| = A.$$

Moreover, since P is the projection onto $\ker U^\perp = (\ker A)^\perp$ we get $AP = A$ immediately. This shows that $s_{\mathrm{left}}(A) \leq Q$ and $s_{\mathrm{right}}(A) \leq P$. We have to show the minimality of P and Q. Now if $\tilde{Q}A = A$ then $\tilde{Q}$ is the identity on $\operatorname{im} A$ and thus also on $(\operatorname{im} A)^{\perp\perp} = \operatorname{im} U = \operatorname{im} Q$. This shows $\tilde{Q}Q = Q$ and $\tilde{Q} \geq Q$. If $A\tilde{P} = A$ then $\tilde{P}A^* = A^*$ and thus $\tilde{P}$ is the identity on $(\operatorname{im} A^*)^{\perp\perp} = (\ker A)^\perp = \operatorname{im} P$. Again, we get $\tilde{P}P = P$ and thus $\tilde{P} \geq P$ showing (7.3.30). Then the second part is clear by the very definition of equivalence. If A is normal then the smallest von Neumann algebra containing A (and hence A^*) is the bicommutant of $\mathbb{C}[A, A^*]$ which is abelian. Hence also this von Neumann algebra is abelian since $\mathscr{A} \subseteq \mathscr{A}'$ implies $\mathscr{A}'' \subseteq \mathscr{A}'$. Thus, since $|A|$ and U are in this subalgebra, they all commute, in particular $UU^* = U^*U$, see also Proposition 7.2.21. Hence the third part follows. Now let $P, Q \in \mathscr{A}$ be projections and consider $A = Q(\mathbb{1} - P)$. We know from the first part that $s_{\mathrm{left}}(A)$ is the projection onto the closure of the image of A and $s_{\mathrm{right}}(A)$ is the projection onto the orthogonal space of the kernel of A, i.e. $(\ker A)^\perp$. Now $Q \wedge P$ is the projection onto $\operatorname{im} Q \cap \operatorname{im} P$ and hence $Q - Q \wedge P + Q \wedge P = Q$ gives the orthogonal decomposition

$$\operatorname{im} Q = (\operatorname{im} P \cap \operatorname{im} Q) \oplus (\operatorname{im} P \cap \operatorname{im} Q)^\perp,$$

where now the complement is taken in $\operatorname{im} Q$ only. Thus $Q - Q \wedge P$ projects onto the complement of $\operatorname{im} P \cap \operatorname{im} Q$ inside $\operatorname{im} Q$. In particular,

$$\ker(Q - Q \wedge P) = \bigl\{\phi \bigm| \phi \in \ker Q \oplus (\operatorname{im} P \cap \operatorname{im} P)\bigr\}.$$

Analogously, we see that the kernel of $A^* = (\mathbb{1} - P)Q$ is given by linear combinations of ϕ with either $\phi \in \ker Q$ or, if $\phi \in \ker Q^\perp = \operatorname{im} Q$ then $\phi \in \operatorname{im} P$, i.e.

$$\ker A^* = \bigl\{\phi \bigm| \phi \in \ker Q \oplus (\operatorname{im} P \cap \operatorname{im} Q)\bigr\} = \ker(Q - Q \wedge P).$$

Since $(\ker A^*)^\perp = (\operatorname{im} A)^{\perp\perp}$ we see that $Q - Q \wedge P$ is the projection onto the closure of the image of A, i.e. $Q - Q \wedge P = s_{\mathrm{left}}(A)$. By an analogous argument we see that

$$s_{\mathrm{left}}((\mathbb{1} - P)Q) = (\mathbb{1} - P) - (\mathbb{1} - P) \wedge (\mathbb{1} - Q),$$

by exchanging the roles of Q and $\mathbb{1} - P$. Now by the usual lattice manipulations we have $(\mathbb{1} - P) \wedge (\mathbb{1} - Q) = \mathbb{1} - (P \vee Q)$ and thus

$$s_{\mathrm{left}}((\mathbb{1} - P)Q) = P \vee Q - P$$

is the projection onto the closure of the image of $(\mathbb{1} - P)Q$, i.e. onto $(\operatorname{im} A^*)^{\perp\perp}$. But $(\operatorname{im} A^*)^{\perp\perp} = (\ker A)^\perp$ and thus $P \vee Q - P$ is the projection onto the orthogonal space of $\ker A$, i.e. $s_{\mathrm{right}}(A)$ as before. This shows (7.3.35). But then (7.3.36) is a consequence of the second part. ☐

In a next step we investigate the central support $Z(P)$ of a projection $P \in \mathscr{A}$. Clearly $Z(P)$ is well-defined as $P \leq \mathbb{1} \in \mathscr{Z}(\mathscr{A})$. We first note the following preparatory lemma:

Lemma 7.3.21 *Let $P, Q \in \mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be projection in a von Neumann algebra. Then the following statements are equivalent:*

i.) The central supports $Z(P)$ and $Z(Q)$ are not orthogonal.

ii.) One has $P\mathscr{A}Q = \{PAQ \mid A \in \mathscr{A}\} \neq \{0\}$.

iii.) There exist non-zero subprojections $P_1 \leq P$ and $Q_1 \leq Q$ in $\mathscr{A}$ with $P_1 \sim Q_1$.

PROOF: We show *i.)* $\implies$ *ii.)* $\implies$ *iii.)* $\implies$ *i.)*. Thus suppose *i.)* holds and assume $PAQ = 0$ for all $A \in \mathscr{A}$. Then consider

$$\mathscr{J} = \{B \in \mathscr{A} \mid PAB = 0 \text{ for all } A \in \mathscr{A}\}.$$

This is clearly a two-sided ideal and σ-weakly closed as the multiplication by fixed elements PA is σ-weakly continuous. Thus by Corollary 7.2.47 we get a unique central projections $Z \in \mathscr{J}$ with $\mathscr{J} = \mathscr{A}Z$. Since $Q \in \mathscr{J}$ we have $QZ = Q$ and thus $Q \leq Z$. Thus also $Z(Q) \leq Z$. Moreover, $PZ = 0$ and hence also $PZ(Q) = 0$. Thus $P \leq \mathbb{1} - Z(Q)$. But this means that $Z(P)Z(Q) = 0$, i.e. they are orthogonal in contradiction to *i.)*. Thus *i.)* $\implies$ *ii.)* follows. Now assume *ii.)* and let $A \in \mathscr{A}$ with $B = PAQ \neq 0$ be given. Then $PB = B = BQ$ and hence $P_1 = s_{\mathrm{left}}(B) \leq P$ while $Q_1 = s_{\mathrm{right}}(B) \leq Q$. Since $B \neq 0$ we have $P_1 \neq 0 \neq Q_1$ and by Proposition 7.3.20, *ii.)*, we get $P_1 \sim Q_1$, showing *ii.)* $\implies$ *iii.)*. Finally, assume *iii.)* and fix some non-zero $P_1 = U^*U \leq P \leq Z(P)$ and $Q_1 = UU^* \leq Q \leq Z(Q)$. Then

$$\begin{aligned}
Z(P)Z(Q)Q_1 &= Z(P)Q_1 \\
&= Z(P)UU^* \\
&= UZ(P)U^* \\
&= UP_1U^* \\
&\overset{UP_1=U}{=} UU^* \\
&= Q_1,
\end{aligned}$$

shows that $Z(P)Z(Q) \neq 0$. ☐

We can now use this lemma to get the following comparison theorem. Note that in general one may have projections which can not be compared via the relation $\lesssim$, we only can expect a partial order. Thus the following theorem says that we get comparability up to central projections:

Theorem 7.3.22 (Generalized comparability) *Let $P, Q \in \mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be projections in a von Neumann algebra. Then there exists a central projection $Z \in \mathscr{Z}(\mathscr{A})$ such that*

$$PZ \gtrsim QZ \quad \text{and} \quad Q(\mathbb{1} - Z) \gtrsim P(\mathbb{1} - Z). \tag{7.3.37}$$

PROOF: We consider the following set of pairs of projections

$$\mathcal{P} = \{\{(P_i, Q_i)\}_{i\in I} \mid P_i \leq P, Q_i \leq Q, P_i \sim Q_i, P_iP_j = \delta_{ij}P, \text{ and } Q_iQ_j = \delta_{ij}Q_i\}.$$

Clearly $\{(0,0)\} \in \mathcal{P}$ which is therefore non-empty. We order $\mathcal{P}$ by inclusion. Thus if we have an increasing chain $\{\{(P_i^{(j)}, Q_i^{(j)})\}_{i\in I_j}\}_{j\in J}$ in $\mathcal{P}$ then also the union $\bigcup_{j\in J}\{\{(P_i^{(j)}, Q_i^{(j)})_{i\in I_j}\}$ is still in $\mathcal{P}$. Hence, by Zorn's Lemma, we get maximal elements in $\mathcal{P}$. Let $\{(P_i, Q_i)\}_{i\in I}$ be such a maximal collection and set

$$\tilde{P} = \sum_{i\in I} P_i \quad \text{and} \quad \tilde{Q} = \sum_{i\in I} Q_i.$$

Since the P_i as well as the Q_i are pairwise orthogonal we can apply Proposition 7.3.17 to conclude that $\tilde{P}, \tilde{Q} \in \mathscr{A}$ are well-defined and satisfy $\tilde{P} \sim \tilde{Q}$. Moreover, $\tilde{P} \leq P$ and $\tilde{Q} \leq Q$ and there are no other projections $\hat{P}, \hat{Q} \in \mathscr{A}$ with $\tilde{P} \leq \hat{P} \leq P$ and $\tilde{Q} \leq \hat{Q} \leq Q$ still satisfying $\hat{P} \sim \hat{Q}$ by the maximality of the above chosen collection: otherwise we could add $\hat{P} - \tilde{P}$ and $\hat{Q} - \tilde{Q}$ to it. Hence $P - \tilde{P}$ and $Q - \tilde{Q}$ have no non-trivial equivalent subprojections. By the previous lemma, $Z(P - \tilde{P})Z(Q - \tilde{Q}) = 0$ and thus there is a central projection Z with

$$P - \tilde{P} \leq Z \text{ and } Q - \tilde{Q} \leq \mathbb{1} - Z.$$

We conclude that $(P - \tilde{P})Z = P - \tilde{P}$ and $(Q - \tilde{Q})Z = 0$ giving $QZ = \tilde{Q}Z \sim \tilde{P}Z \leq PZ$ as the multiplication by a *central* projection Z preserves the equivalence $\sim$ as well as the relation $\leq$. Similarly, we get

$$P(\mathbb{1} - Z) = \tilde{P}(\mathbb{1} - Z) \sim \tilde{Q}(\mathbb{1} - Z) \leq Q(\mathbb{1} - Z),$$

which proves the theorem. □

Finally, we mention the following notion: a projection $P \in \mathscr{A}$ in a von Neumann algebra is called *countably decomposable* if it has at most countably many pairwise orthogonal subprojections, i.e. if $P = \sum_{i \in I} P_i$ with $P_i P_j = \delta_{ij} P_i$ and $P_i \neq 0$ for all $i \in I$ then $\#I \leq \#\mathbb{N}$. A von Neumann algebra is called *countably decomposable* if $\mathbb{1} \in \mathscr{A}$ is countably decomposable. Clearly, if $\mathfrak{H}$ is separable then any von Neumann algebra $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ is countably decomposable as there are at most countably many pairwise orthogonal non-zero projections in $\mathfrak{B}(\mathfrak{H})$. Thus this definition controls that a von Neumann algebra does not get "too big". Note, however, that there are von Neumann algebras $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ which are countably decomposable but require a Hilbert space of uncountable dimension, see e.g. the discussion in [8, III 1.3.1].

Exercise: In

7.3.3 Factors and Types

In the last subsection we have seen that the center $\mathscr{Z}(\mathscr{A})$ of a von Neumann algebra $\mathscr{A}$ and in particular its central projections control the behaviour of the lattice of all projections in $\mathscr{A}$ very much. Moreover, as discussed in the beginning of Section 7.3 a central projection allows to decompose a von Neumann algebra into two smaller von Neumann algebras, each acting on an invariant subspace. Thus the following definition of a *factor* provides those von Neumann algebras which can no longer be decomposed:

Definition 7.3.23 (Factor) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra. Then $\mathscr{A}$ is called a factor if $\mathscr{Z}(\mathscr{A}) = \mathbb{C}\mathbb{1}$.*

Equivalently, this means that

$$\mathscr{A} \cap \mathscr{A}' = \mathbb{C}\mathbb{1}. \tag{7.3.38}$$

Hence we see that $\mathscr{A}$ is a factor iff $\mathscr{A}'$ is a factor as well. It is clear that $\mathfrak{B}(\mathfrak{H})$, having trivial center, is a factor for each Hilbert space. Moreover, the only commutative factor is $\mathbb{C}$, up to *-isomorphisms.

The name factor comes now from the following factorization of the bounded operators on a tensor product:

Proposition 7.3.24 *Let $\mathfrak{H}_1, \mathfrak{H}_2$ be Hilbert spaces and consider $\mathfrak{H} = \mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2$. Viewing $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$ as *-subalgebras of $\mathfrak{B}(\mathfrak{H})$ according to Proposition 3.5.11 we get*

bastian: Folg
Lemma 6.2.1
$\mathfrak{H} \cong \bigoplus_{i \in I_2} \mathfrak{H}$
Hilbertbasis

$$\mathfrak{B}(\mathfrak{H}_1)' = \mathfrak{B}(\mathfrak{H}_2) \tag{7.3.39}$$

and hence

$$(\mathfrak{B}(\mathfrak{H}_1) \otimes \mathfrak{B}(\mathfrak{H}_2))'' = \mathfrak{B}(\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2). \tag{7.3.40}$$

PROOF: Consider an operator $A \in \mathfrak{B}(\mathfrak{H})$ which commutes with all operators of the from $A_1 \otimes \mathbb{1}_{\mathfrak{H}_2}$ for $A_1 \in \mathfrak{B}(\mathfrak{H}_1)$. Let $\psi \in \mathfrak{H}_2$ be fixed and consider for a given non-zero vector $\phi \in \mathfrak{H}_1$ the orthogonal projection P_ϕ onto ϕ. Then we have

$$(P_\phi \otimes \mathbb{1})(\phi \otimes \psi) = \phi \otimes \psi.$$

Conversely, if $\Psi \in \mathfrak{H}$ is a vector with

$$(P_\phi \otimes \mathbb{1})\Psi = \Psi,$$

then necessarily $\Psi = \phi \otimes \psi$ with a unique $\psi \in \mathfrak{H}_2$: indeed, we can extend $\frac{\phi}{\|\phi\|}$ to a Hilbert basis $\{\mathrm{e}_i\}_{i\in I}$ and choose a Hilbert basis $\{\mathrm{f}_j\}_{j\in J}$ of $\mathfrak{H}_2$ then $\{\mathrm{e}_i \otimes \mathrm{f}_j\}_{i\in I, j\in J}$ is a Hilbert basis of $\mathfrak{H}$ and we have

$$\Psi = \sum_{i,j} \psi_{ij} \mathrm{e}_i \otimes \mathrm{f}_j$$

with $\Psi_{ij} = \langle \mathrm{e}_i \otimes \mathrm{f}_j, \Psi \rangle$ as usual. Now

$$\begin{aligned}
\Psi &= P_{\mathrm{e}_{i_0}} \Psi \\
&= \sum_j \Psi_{i_0 j} \mathrm{e}_{i_0} \otimes \mathrm{f}_j \\
&= \mathrm{e}_{i_0} \otimes \left(\sum_{j\in J} \Psi_{i_0 j} \right) \\
&= \mathrm{e}_{i_0} \otimes \psi
\end{aligned}$$

shows the claim. Thus $P_\phi \Psi = \Psi$ iff $\Psi = \phi \otimes \psi$ for some unique $\psi \in \mathfrak{H}_2$. For A this means

$$P_\phi A(\phi \otimes \psi) = A P_\phi(\phi \otimes \psi) = A(\phi \otimes \psi),$$

and thus

$$A(\phi \otimes \psi) = \phi \otimes A_\phi(\psi)$$

with some linear map $\psi \mapsto A_\phi(\psi)$. Clearly $A_\phi \in \mathfrak{B}(\mathfrak{H}_2)$ is bounded as for factorizing tensors:

$$\|\phi\| \|A_\phi(\psi)\| = \|\phi \otimes A_\phi(\psi)\| \leq \|A\| \|\phi\| \|\psi\|.$$

Hence $\|A_\phi\| \leq \|A\|$. Now let ϕ_1, ϕ_2 be linearly independent, e.g. orthogonal, and consider $P_{\phi_1+\phi_2}$. Then we get on one hand

$$A((\phi_1 + \phi_2) \otimes \psi) = (\phi_1 + \phi_2) \otimes A_{\phi_1+\phi_2}(\psi) = \phi_1 \otimes A_{\phi_1+\phi_2}(\psi) + \phi_2 \otimes A_{\phi_1+\phi_2}(\psi)$$

but on the other hand

$$A((\phi_1 + \phi_2) \otimes \psi) = A(\phi_1 \otimes \psi) + A(\phi_2 \otimes \psi) = \phi_1 \otimes A_{\phi_1}(\psi) + \phi_2 \otimes A_{\phi_2}(\psi).$$

Applying again the argument with the Hilbert basis gives immediately the equality

$$A_{\phi_1}(\psi) = A_{\phi_2}(\psi) = A_{\phi_1+\phi_2}(\psi).$$

Thus the map A_ϕ does not depend on ϕ at all and defines a map $A_2 = A_\phi \in \mathfrak{B}(\mathfrak{H}_2)$ with

$$A(\phi \otimes \psi) = \phi \otimes A_2(\psi).$$

By linearity and the usual continuity and density argument we see $A = \mathbb{1} \hat{\otimes} A_2$. This shows that $(\mathfrak{B}(\mathfrak{H}_1) \otimes \mathbb{1})' \subseteq \mathbb{1} \otimes \mathfrak{B}(\mathfrak{H}_2)$. The converse inclusion is trivial according to Corollary 3.5.12. This shows (7.3.39). The algebraic tensor product $\mathfrak{B}(\mathfrak{H}_1) \otimes \mathfrak{B}(\mathfrak{H}_2)$ has now a trivial commutant as every operator commuting with it has to commute in particular with each factor. Thus (7.3.40) follows. □

Remark 7.3.25 In general, if $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ is a factor then

$$\mathfrak{B}(\mathfrak{H}) = (\mathscr{A}\mathscr{A}')'', \tag{7.3.41}$$

since every operator commuting with $\mathscr{A}'$ has to be in $\mathscr{A}$ and thus if it also commutes with $\mathscr{A}$ it has to be a multiple of the identity. Hence $(\mathscr{A}\mathscr{A}')' = \mathbb{C}\mathbb{1}$ and (7.3.41) follows. This way a factor "factorizes" $\mathfrak{B}(\mathfrak{H})$ up to weak closure.

If the center is trivial, i.e. if $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ is a factor, we get the following refinement of the generalized comparability:

Corollary 7.3.26 *Let $P, Q \in \mathscr{A}$ be projections in a factor. Then either $P \lesssim Q$ or $Q \lesssim P$.*

PROOF: This is now easy: by Theorem 7.3.22 we have a central projection $Z \in \mathscr{A}$ with $PZ \gtrsim QZ$ and $Q(\mathbb{1} - Z) \gtrsim P(\mathbb{1} - Z)$. So either $Z = 0$ or $Z = \mathbb{1}$, since $\mathscr{A}$ is a factor. Hence the claim follows.□

If $P, Q \in \mathfrak{B}(\mathfrak{H})$ are projections with $P \sim Q$ then this is equivalent to the fact that

$$\dim \operatorname{im} P = \dim \operatorname{im} Q, \tag{7.3.42}$$

since if $P = UU^*$ and $Q = U^*U$ then $\operatorname{im} P = \operatorname{im} U$ and $\operatorname{im} Q = \operatorname{im} U^*$ with

$$U \colon \operatorname{im} U^* = (\ker U)^\perp \longrightarrow \operatorname{im} U = (\ker U^*)^\perp \tag{7.3.43}$$

being a unitary isomorphism between the images, implying (7.3.42). Conversely, (7.3.42) clearly allows to find a partial isometry $U \in \mathfrak{B}(\mathfrak{H})$ implementing the equivalence $P \sim Q$ by fixing a unitary $\operatorname{im} Q \longrightarrow \operatorname{im} P$ and extending it trivially to $(\operatorname{im} Q)^\perp$. Now $\operatorname{im} P$ is finite-dimensional iff for any subspace $V \subseteq \operatorname{im} P$ with V being isometrically isomorphic to $\operatorname{im} P$ we have $V = \operatorname{im} P$. This allows to detect the finite-dimensionality of $\operatorname{im} P$ in terms of the equivalence relation of projections as $V \cong \operatorname{im} P$ clearly means $P \sim P_V \leq P$. Hence the following definition is now well-motivated.

Definition 7.3.27 (Finite and infinite projections) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra and $P \in \mathscr{A}$ a projection.*

i.) The projection P is called finite if $P \sim Q \leq P$ for some $Q \in \mathscr{A}$ implies $P = Q$. Otherwise P is called infinite.

ii.) The projection P is called purely infinite if there is no non-zero finite projection $Q \leq P$ in $\mathscr{A}$.

iii.) The projection P is called properly infinite if for every central projection $Z \in \mathscr{A}$ with $ZP \neq 0$ the projection ZP is infinite.

iv.) The projection P is called abelian if $P\mathscr{A}P$ is abelian.

Remark 7.3.28 It is important to stress that the above notions of projection always refer to a specified von Neumann algebra $\mathscr{A}$. This von Neumann algebra will always be clear from the context and is therefore not indicated in the notation. To make sure which von Neumann algebra is used one should perhaps say $\mathscr{A}$-finite, $\mathscr{A}$-infinite etc. but we try to avoid this clumsy notation, see also Remark 7.3.15, *ii.)*.

We have the following immediate properties and relations between these notions:

Lemma 7.3.29 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra and let $P, Q \in \mathscr{A}$ be projections.*

i.) If $Q \leq P$ and P is finite (abelian) then Q is finite (abelian) as well.

ii.) Any abelian projection is finite.

iii.) The zero projection 0 is finite, purely infinite and properly infinite.

iv.) If P is finite (or infinite, purely infinite, properly infinite, abelian) and $Q \sim P$ then Q is finite (infinite, purely infinite, properly infinite, abelian), too.

PROOF: Let $Q \leq P$ with P finite and assume that $Q \sim R \leq Q$. Then $P \sim P - Q + R$ since obviously $P = P - Q + Q$ and hence we can apply Proposition 7.3.17, *ii.)* as $R \leq Q \leq P$ shows that R is orthogonal to $P - Q$. Since P is finite and since $P - Q + R \leq P$ we get $P = P - Q + R$ and thus $Q = R$. Therefore Q is finite. Now suppose P is abelian and $Q \leq P$. Then we first have $Q\mathscr{A}Q \subseteq P\mathscr{A}P$ since

$PQ = Q = QP$ and $A \in P\mathscr{A}P$ iff $PA = A = AP$. Since $P\mathscr{A}P$ is abelian, the subalgebra $Q\mathscr{A}Q$ inherits this feature, thereby completing the proof of the first part. For the second part, let P be abelian and $P \sim Q \leq P$ via some partial isometry $U \in \mathscr{A}$, i.e. $P = U^*U$ and $Q = UU^*$. Then

$$PU^*Q = U^*UU^*UU^* = PPU^* = U^*,$$

since $P = P_{\operatorname{im} U^*}$ as usual. But clearly $PU^*Q \in P\mathscr{A}P$ since $PU^*QP = PU^*Q$ by $Q \leq P$. Hence $U^* \in P\mathscr{A}P$ and thus also $U \in P\mathscr{A}P$ since $P\mathscr{A}P$ is a *-subalgebra. Now $P\mathscr{A}P$ is abelian and hence $P = U^*U = UU^* = Q$ shows that P is finite. The third part is clear and turns out to be a useful convention. Finally, let $P \sim Q$ via U and assume that P is finite. Consider now the situation $Q \sim Q_1 \leq Q$ for some Q_1. Then $U_1 = Q_1U$ implements an equivalence between $P_1 = U_1^*U_1 = U^*Q_1U \leq U^*QU = P$ and Q_1. Since $P \sim Q \sim Q_1$ we get $P \sim P_1$ by transitivity. Since P is finite, $P_1 = P$ follows. Hence $U^*Q_1U = U^*QU$. But this gives

$$Q_1 = QQ_1Q = UU^*Q_1UU^* = UU^*QUU^* = QQQ = Q,$$

and hence Q is finite, too. Thus for $P \sim Q$ the projection P is finite *iff* Q is finite by symmetry. Hence also P is infinite iff Q is infinite for $P \sim Q$. Now suppose $P \sim Q$ via U with P purely infinite and let $Q_1 \leq Q$. Define again $P_1 = U_1^*U_1$ with $U_1 = Q_1U$ such that we get $P_1 \leq P$ and $P_1 \sim Q_1$ via U_1. By assumption $P_1 = 0$ and thus $Q_1 = 0$ as well, showing that Q is purely infinite, too. Next, assume P is properly infinite and let $Z \in \mathscr{A}$ be a central projection with $ZUU^* = ZQ \neq 0$ be given. Then

$$ZP = ZU^*U = U^*ZU = U^*ZZU$$

has the operator norm

$$\|ZP\| = \|U^*ZZU\| = \|ZU\|^2,$$

and analogously

$$\|ZQ\| = \|ZUU^*\| = \|UZZU^*\| = \|ZU^*\|^2 = \|UZ\|^2 = \|ZU\|^2,$$

by using the C^*-property and the fact that $Z = Z^*$ is central. Hence $ZP \neq 0$, too. By assumption, ZP is infinite. But it is clear that $ZP \sim ZQ$ via ZU since Z is central. Hence we have an infinite ZQ, too, showing that Q is properly infinite again. Finally, let P be abelian and $P \sim Q$ via U. Then it is easy to see that

$$\Phi\colon P\mathscr{A}P \ni PAP \mapsto UPAPU^* = QUAU^*Q \in Q\mathscr{A}Q$$

is a *-isomorphism. Hence $Q\mathscr{A}Q$ is abelian, too. □

Thus the different characterizations of projections are compatible with our notion of equivalence. In particular, a combination of *i.)* and *iv.)* of the lemma gives that for a finite (abelian) projection P and $Q \precsim P$ we have that also Q is finite (abelian).

Example 7.3.30 Consider $\mathscr{A} = \mathfrak{B}(\mathfrak{H})$ and $P \in \mathfrak{B}(\mathfrak{H})$. Then P is finite iff $\dim \operatorname{im} P < \infty$ and abelian if $\dim \operatorname{im} P = 1$ or 0. In fact, we always have $P\mathfrak{B}(\mathfrak{H})P \cong \mathfrak{B}(\operatorname{im} P)$ for any projection.

The infinite projections enjoy now the following property, again justifying the name infinite.

Lemma 7.3.31 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra with a projection $P \in \mathscr{A}$. Then the following statements are equivalent:*

i.) The projection P is infinite.

ii.) There exists a sequence of pairwise orthogonal and equivalent non-zero subprojections of P.

PROOF: First let $P = P_0 \in \mathscr{A}$ be infinite. Thus there is a $P_1 \leq P_0$ with $P_0 \sim P_1$ via some U_1 but $P_1 \neq P_0$. Set $Q_1 = P_0 - P_1 \leq P_0$ then $Q_1 \neq 0$ by assumption. Next we define

$$Q_2 = U_1 Q_1 U_1^* = U_1(P_0 - P_1)U^*,$$

and get from $P_0 - P_1 \leq P$ the relation

$$Q_2 = U_1(P_0 - P_1)U_1^* \leq U_1 P_0 U_1^* = U_1 U_1^* U_1 U_1^* = P_1 P_1 = P_1 \leq P_0.$$

Moreover, since $U_1^* U_1 = P_0 \geq P_0 - P_1$ it follows that $Q_2^2 = Q_2$ is a projection and hence a subprojection of P. Next, $U_1^* U_1 = P_0$ gives

$$Q_1 U_1^* U_1 Q_1 = Q_1 P_0 Q_1 = Q_1,$$

and thus $Q_1 \sim Q_2$ via $U_1 Q_1$. Finally, since $Q_2 \leq P_1$ and $Q_1 = P_0 - P_1$ we see that Q_2 is orthogonal to P_1. Now P_1 is an infinite projection itself since $P_1 \sim P_0$, see Lemma 7.3.29, *iv.)*. Since $Q_2 \neq 0$ the projection $P_2 = P_1 - Q_2 \leq P_1$ is different from P_1. However, with $U_2 = U_1 P_1$ we get

$$U_2^* U_2 = P_1 U_1^* U_1 P_1 = P_1 P_0 P_1 = P_1$$

and

$$U_2 U_2^* = U_1 P_1 U_1^* = U_1(P_1 - P_0 + P_0)U_1^* = -U_1\underbrace{(P_0 - P_1)}_{Q_1}U_1^* + U_1 U_1^* U_1 U_1^* = -Q_2 + P_1 = P_2$$

and thus $P_2 \sim P_1$. Thus we are in the same situation as before and can define $Q_3 = U_2 Q_2 U_2^* \leq P_2 \leq P_1$ and $Q_3 \sim Q_2$. Since $P_2 = P_1 - Q_2$ we have $Q_3 Q_2 = 0$ and since $P_1 = P_0 - Q_1$ we still have $Q_3 Q_1 = 0$. Equivalence being transitive gives also $Q_3 \sim Q_1$. It follows by induction that we can repeat this construction to get a sequence Q_n of subprojections of P with $Q_n Q_m = \delta_{nm} Q_n$, all being non-zero, and $Q_n \sim Q_m$ for all $n, m \in \mathbb{N}$. This shows *i.)* $\implies$ *ii.)*. Conversely, assume we have such a sequence of projections. Then

$$Q = \sum_{n=1}^{\infty} Q_n$$

is a well-defined projection in $\mathscr{A}$ with $Q \leq P$. Since $Q_n \sim Q_{n+1}$ we have by Proposition 7.3.17, *ii.)* the equivalence

$$Q = \sum_{n=1}^{\infty} Q_n \sim \sum_{n=1}^{\infty} Q_{n+1} = \sum_{n=2}^{\infty} Q_n = Q - Q_1.$$

Since $Q_1 \neq 0$ this gives

$$P = Q + (P - Q) \sim Q - Q_1 + (P - Q) = P - Q_1 \leq P$$

with $P - Q_1 \neq P$. Thus P is infinite. □

Lemma 7.3.32 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra and let $P \in \mathscr{A}$ be a projection. If there are $P_1, P_2 \in \mathscr{A}$ with $P \sim P_1 \sim P_2$ and $P_1 \perp P_2$ as well as $P_1, P_2 \leq P$ then P is properly infinite.*

PROOF: Let $Z \in \mathscr{A}$ be a central projection and assume $ZP \neq 0$ since otherwise nothing is to be shown. Since Z is central we still have $ZP \sim ZP_1 \sim ZP_2$ as well as $ZP_1 \perp ZP_2$ and $ZP_1, ZP_2 \leq ZP$. Since $ZP \neq 0$ also ZP_1, ZP_2 are non-zero. Thus $ZP \sim ZP_1 \leq ZP - ZP_2 \leq ZP$ and since $ZP - ZP_2 \neq ZP$ we see that ZP is equivalent to a non-trivial subprojection of ZP. Hence ZP is infinite. □

Remark 7.3.33 Remarkably, the reverse is true as well. Even more, for a (non-zero) properly infinite projection P one can find orthogonal subprojections $P_1, P_2 \leq P$ such that

$$P \sim P_1 \sim P_2 \quad \text{and} \quad P = P_1 + P_2. \tag{7.3.44}$$

There are even countably many $P_n \leq P$ for $n \in \mathbb{N}$ with $P_n P_m = \delta_{nm} P_n$ such that

$$P \sim P_n \text{ for all } n \in \mathbb{N} \text{ and } P = \sum_{n=1}^{\infty} P_n, \tag{7.3.45}$$

i.e. a properly infinite projection can be "halved" or even divided by "$\mathbb{N}$". For details and a proof see e.g. [8, III.1.3.4] or [29, Sect. 6.3].

We need now the following lemma on centrally orthogonal projections: here P, Q are called *centrally orthogonal* if their central supports $Z(P)$ and $Z(Q)$ are orthogonal. This is a refinement of orthogonality and by Lemma 7.3.21 we note that P and Q are centrally orthogonal iff they have no non-zero equivalent subprojections. In this situation one has the following statement:

Lemma 7.3.34 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra. If $\{P_i\}_{i\in I}$ are pairwise centrally orthogonal abelian (finite) projections then also their sum $\sum_{i\in I} P_i$ is abelian (finite).*

PROOF: First consider the case of abelian P_i and let $P = \sum_{i\in I} P_i$ according to Proposition 7.3.17, *i.)*. Since $Z(P_i)Z(P_j) = 0$ we get by Lemma 7.3.21 that

$$P_i \mathscr{A} P_j = \{0\} \quad \text{for} \quad i \neq j, \tag{$*$}$$

and hence it follows that $P\mathscr{A}P$ has the block-diagonal form $P\mathscr{A}P \cong \bigoplus_{i\in I} P_i \mathscr{A} P_i$ as there are no cross-terms by $(*)$. Hence if $P_i\mathscr{A}P_i$ is commutative for all $i \in I$ then also $P\mathscr{A}P$ is commutative, showing that P is abelian. Next, assume all P_i are finite and let Q satisfy $P \sim Q \leq P$. Since equivalence and "$\leq$" are compatible with multiplication by *central* projections we get

$$P_i \sim Z(P_i)Q \leq P_i$$

for all $i \in I$. Since P_i is finite, this implies $Z(P_i)Q = P_i$. Since clearly $Z(P_i)Q \leq Q$ we get also $P = \sum_{i\in I} Z(P_i)Q \leq Q$ as the $Z(P_i)Q$ are pairwise orthogonal. Thus we have $P = Q$ and hence P is finite, too. $\square$

The identity $\mathbb{1} \in \mathscr{A}$ is always the largest projection in a von Neumann algebra. Its type determines the structure of $\mathscr{A}$ very much: if e.g. $\mathbb{1}$ is purely infinite then $\mathscr{A}$ does not contain any finite projections beside zero at all. This leads to the following definition:

Definition 7.3.35 (Finite and infinite von Neumann algebra) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra. Then $\mathscr{A}$ is called finite, infinite, purely infinite, or properly infinite if $\mathbb{1} \in \mathscr{A}$ is finite, infinite, purely infinite, or properly infinite, respectively.*

Definition 7.3.36 (Type of von Neumann algebra) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra.*

i.) The von Neumann algebra $\mathscr{A}$ is called to be of type I if every non-zero central projection majorizes a nonzero abelian projection.

ii.) The von Neumann algebra $\mathscr{A}$ is called to be of type II if $\mathscr{A}$ has no nonzero abelian projection and if every nonzero central projection majorizes a nonzero finite projection.

iii.) The von Neumann algebra $\mathscr{A}$ is called to be of type II_1 if $\mathscr{A}$ is finite and of type II.

iv.) *The von Neumann algebra $\mathscr{A}$ is called to be of type II_∞ if $\mathscr{A}$ has no nonzero central finite projections and is of type II.*

v.) *The von Neumann algebra $\mathscr{A}$ is called to be of type III if $\mathscr{A}$ has no non-zero finite projection.*

We first note that, unless $\mathfrak{H}$ is zero-dimensional, the above types are exclusive: indeed a type I algebra can not be type II or type III as $0 \neq \mathbb{1} \in \mathscr{A}$ is always a central projection. Analogously, a type II algebra can not be type III, again using $\mathbb{1} \in \mathscr{A}$ which majorizes some non-zero finite projection in the type II case. Finally in the type II_1 case $\mathbb{1}$ is finite while II_∞ means that $\mathbb{1}$ is purely infinite. Note also that type III simply means that $\mathscr{A}$ is purely infinite. The following main result of this subsection now shows that the above types are not only exhaustive but every von Neumann algebra canonically decomposes into pieces of fixed type:

Theorem 7.3.37 (Decomposition of a von Neumann algebra) *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a von Neumann algebra. Then $\mathscr{A}$ decomposes uniquely into a direct sum of von Neumann algebras of type I, II_1, II_∞ and III, i.e.*

$$\mathscr{A} = \mathscr{A}_I \oplus \mathscr{A}_{II_1} \oplus \mathscr{A}_{II_\infty} \oplus \mathscr{A}_{III}, \tag{7.3.46}$$

with a corresponding orthogonal decomposition of $\mathfrak{H}$ into

$$\mathfrak{H} = \mathfrak{H}_I \oplus \mathfrak{H}_{II_1} \oplus \mathfrak{H}_{II_\infty} \oplus \mathfrak{H}_{III}. \tag{7.3.47}$$

PROOF: The idea is to decompose the identity orthogonally into central projection $Z_I, Z_{II_1}, Z_{II_\infty}$ and Z_{III} such that the corresponding non-unital subalgebras $\mathscr{A}Z_I, \mathscr{A}Z_{II}, \mathscr{A}Z_{II_\infty}$ and $\mathscr{A}Z_{III}$, viewed as von Neumann algebras acting on $Z_I\mathfrak{H}, Z_{II_1}\mathfrak{H}, Z_{II_\infty}\mathfrak{H}$ and $Z_{III}\mathfrak{H}$, respectively, have the above types. We consider the set of all families of centrally orthogonal abelian projections. Since 0 is abelian, this set is non-empty and a routine application of Zorn's Lemma yields a maximal set $\{P_\ell\}_{\ell\in L}$ of centrally orthogonal abelian projections. By Lemma 7.3.34 the projection $P = \sum_{\ell\in L} P_\ell$ is again abelian. We set

$$Z_I = Z(P),$$

and have by definition $Z_I P = P$ with Z_I being the smallest central projection with this property. Suppose $P \neq 0$ and hence $Z(P) \neq 0$, otherwise the following is trivial. Now let Z be any nonzero central projection majorized by Z_I, i.e. $Z \leq Z_I$. Then $ZP \neq 0$ since otherwise $Z_I - Z$ would still be a central projection with $P(Z_I - Z) = P$. But this shows that $ZP \leq Z$ gives a nonzero abelian projection majorized by Z. Since inside the von Neumann subalgebra $\mathscr{A}_I = \mathscr{A}Z_I$ the projection Z_I plays the role of the identity and $Z \leq Z_I$ simply means $Z \in \mathscr{A}_I$ we see that $\mathscr{A}_I$ is of type I. We have $\mathscr{A} = \mathscr{A}_I \oplus (\mathbb{1} - Z_I)\mathscr{A}$. Inside $(\mathbb{1} - Z_I)\mathscr{A}$ there is no nonzero abelian projection: indeed suppose $Q \in (\mathbb{1} - Z_I)\mathscr{A}$ would be abelian then $Z(Q) \leq (\mathbb{1} - Z_I)$ and thus Q is centrally orthogonal to the $\{P_\ell\}_{\ell\in L}$ contradicting their maximality. Hence $(\mathbb{1} - Z_I)\mathscr{A}$ can not have any direct summand of type *I* different from $\{0\}$. Inside $(\mathbb{1} - Z_I)\mathscr{A}$ we find, again by Zorn's Lemma, a maximal family $\{Q_j\}_{j\in J}$ of centrally orthogonal finite projections. Then $Q = \sum_{j\in J} Q_j \in (\mathbb{1} - Z_I)\mathscr{A}$ is again finite by Lemma 7.3.34, again possibly zero. We set now $Z_{II} = Z(Q)$ and $\mathscr{A}_{II} = Z_{II}\mathscr{A} = Z_{II}(\mathbb{1} - Z_I)\mathscr{A}$. As said already, in $\mathscr{A}_{II}$ there are no nonzero abelian projections. If $Z \in \mathscr{A}_{II}$ is a non-zero central projection then $ZQ \neq 0$ again by the maximality. Hence Z majorizes a finite projection ZQ showing that $\mathscr{A}_{II}$ is of type II. Finally, set $Z_{III} = \mathbb{1} - Z_I - Z_{II}$. By the maximality of the $\{Q_j\}_{j\in J}$ this central projection is not majorizing any finite projection. Hence in $\mathscr{A}_{III} = Z_{III}\mathscr{A}$ we have no finite projections making it type III. Up to now we have $\mathscr{A} = \mathscr{A}_I \oplus \mathscr{A}_{II} \oplus \mathscr{A}_{III}$ via $\mathbb{1} = Z_I + Z_{II} + Z_{III}$. Inside $\mathscr{A}_{II}$ we can decompose further: let $\{Z_k\}_{k\in K}$ be a maximal family of central orthogonal finite projections in $\mathscr{A}_{II}$ and set $Z_{II_1} = \sum_{k\in K} Z_k$ which is then again central and finite as central orthogonal projections are automatically centrally orthogonal. Clearly $Z_{II_1} \leq Z_{II}$ and hence $Z_{II_\infty} = Z_{II} - Z_{II_1}$ gives an orthogonal decomposition of Z_{II}. Clearly $\mathscr{A}_{II_1} = Z_{II_1}\mathscr{A}$ is now finite and hence of type

II_1. Conversely, by the maximal choice of the Z_k there is no nonzero central finite projection in $\mathscr{A}_{II_\infty} = Z_{II_\infty}\mathscr{A}$ left so this piece is of type II_∞. This shows the existence of the decomposition (7.3.46) and (7.3.47). Suppose now we have a different such decomposition yielding $\mathbb{1} = Z'_I + Z'_{II_1} + Z'_{II_\infty} + Z'_{III}$. Then $\mathbb{1} - Z_I$ majorizes no nonzero abelian projection thus $Z'_I(\mathbb{1} - Z_I) = 0$ because $Z'_I(\mathbb{1} - Z_I)$ is a central projection in $\mathscr{A}Z'_I$ therefore majorizing an abelian projection if nonzero. Hence $Z'_I \leq Z_I$. Exchanging the role of Z_I and Z'_I gives $Z'_I = Z_I$. For the other projections one argues analogously yielding the uniqueness. □

Of course, it may happen that some of the above direct summands are trivial. In fact, for a factor we get immediately the following:

Corollary 7.3.38 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a factor with $\dim \mathfrak{H} > 0$. Then $\mathscr{A}$ is either of type I, of type II_1, of type II_∞, or of type III.*

Since in a commutative von Neumann algebra all projections are abelian one gets the following result:

Corollary 7.3.39 *Let $\mathscr{A} \subseteq \mathfrak{B}(\mathfrak{H})$ be a commutative von Neumann algebra. Then $\mathscr{A}$ is of type I.*

Corollary 7.3.40 *The von Neumann algebra $\mathfrak{B}(\mathfrak{H})$ is of type I and finite iff $\dim \mathfrak{H} < \infty$.*

Proof: From Example 7.3.30 we know that $0 \neq P$ is abelian if $\dim \operatorname{im} P = 1$. Hence every (central or not) nonzero projection P majorizes an abelian projection: just pick any projection onto a one-dimensional subspace of $\operatorname{im} P$. Thus $\mathfrak{B}(\mathfrak{H})$ is of type I. Again by Example 7.3.30 we see that $\mathbb{1}$ is finite iff $\dim \mathfrak{H} < \infty$. □

This last corollary suggest that one can decompose the type I case even further: one says that $\mathscr{A}$ is of *type* I_n, where $n \leq \dim \mathfrak{H}$ is a cardinal, if $\mathbb{1} \in \mathscr{A}$ is the orthogonal sum of n equivalent abelian projections. It is clear that $\mathfrak{B}(\mathfrak{H})$ is of type $\mathrm{I}_{\dim \mathfrak{H}}$ as we always have a Hilbert basis yielding the corresponding abelian projections. Since in a commutative von Neumann algebra $\mathbb{1}$ is already an abelian projection, a commutative von Neumann algebra is of type I_1. By a slightly refined construction, see e.g. [29, Thm. 6.5.2], one can now show that the type I part $\mathscr{A}_I$ of $\mathscr{A}$ decomposes uniquely into a direct sum of type I_n parts $\mathscr{A}_{I_n}$ for $1 \leq n \leq \dim \mathfrak{H}$. Hence a factor of type I is necessarily of type I_n for one n. It turns out that up to *-isomorphism the only type I_n factor is $\mathfrak{B}(\mathfrak{H})$ with $\dim \mathfrak{H} = n$. The structure of factors of other types is much more complicated. Already for a Hilbert space with countable infinite Hilbert basis, i.e. for $\mathfrak{H} \cong \ell^2(\mathbb{N})$, there are uncountably many non-isomorphic factors of type II_1, and their classification is still not known. Surprisingly, there is a classification of the type III factors (under the extra assumption of being "hyperfinite") by Connes: there is a refined invariant leading to non-isomorphic factors of type III_λ for every $\lambda \in [0, 1]$. We shall not enter the discussion any further but refer to [8, 28, 29, 54–56] for further reading.

Exercises???

7.4 Types of *-Representations

With our von Neumann algebra technology at hand we can now re-investigate the C^*-algebra representation theory of a C^*-algebra $\mathscr{A}$ on Hilbert spaces.

7.4.1 Pure States and Irreducible *-Representations

The aim of this subsection is to establish the fact that every irreducible *-representation is the GNS representation of a pure state and vice versa. We have already seen that every irreducible *-representation is cyclic (even every non-zero vector is cyclic) and hence a GNS representation, see Corollary 7.1.29. The first step in this direction is the following non-commutative version of the Radon-Nikodym Theorem:

Theorem 7.4.1 (Noncommutative Radon-Nikodym-Theorem) *Let $\mathscr{A}$ be a C^*-algebra and let $\omega_1, \omega_2 \colon \mathscr{A} \longrightarrow \mathbb{C}$ be positive linear functionals. If $\omega_1 \leq \omega_2$ (meaning $\omega_1(a^*a) \leq \omega_2(a^*a)$ for all $a \in \mathscr{A}$) then there is a unique positive operator $\varrho \in \mathfrak{B}(\widehat{\mathfrak{H}}_{\omega_2})$ with*

i.) $0 \leq \varrho \leq \mathbb{1}$,

ii.) $\rho \in \pi_{\omega_2}(\mathscr{A})'$,

iii.) $\omega_1(a) = \langle \psi_{\mathbb{1}}, \varrho \pi_{\omega_2}(a) \psi_{\mathbb{1}} \rangle$ *for all $a \in \mathscr{A}$ with the cyclic vector $\psi_{\mathbb{1}}$ of the GNS representation of ω_2.*

PROOF: The proof is now surprisingly simple. Let $a, b \in \mathscr{A}$ and consider the corresponding vectors $\psi_a, \psi_b \in \widehat{\mathfrak{H}}_{\omega_2}$ in the GNS Hilbert space of ω_2, i.e. $\psi_a = \pi_{\omega_2}(a)\psi_{\mathbb{1}}$ and $\psi_b = \pi_{\omega_2}(b)\psi_{\mathbb{1}}$ where $\psi_{\mathbb{1}}$ is the cyclic vector given either by the equivalence class of $\mathbb{1}$ if $\mathscr{A}$ is unital or via Theorem 7.1.14, *i.)*, if $\mathscr{A}$ is non-unital. Then we define a new sesquilinear form on the dense subspace $\mathfrak{H}_{\omega_2} \subseteq \widehat{\mathfrak{H}}_{\omega_2}$ via

$$(\psi_a, \psi_b) = \omega_1(a^*b). \tag{$*$}$$

This is well-defined since we have $\mathscr{J}_{\omega_2} \subseteq \mathscr{J}_{\omega_1}$ according to $\omega_1 \leq \omega_2$ and hence for $b \in \mathscr{J}_{\omega_2}$ we have $\omega_1(a^*b) = 0$ for all ψ_a. Clearly, $(*)$ is positive semidefinite and sesquilinear. Moreover, we have

$$|(\psi_a, \psi_n)|^2 = |\omega_1(a^*b)|^2 \leq \omega_1(a^*a)\omega_1(b^*b) \leq \omega_2(a^*a)\omega_2(b^*b) = \|\psi_a\|^2_{\omega_2} \|\psi_b\|^2_{\omega_2},$$

showing that $(\,\cdot\,,\,\cdot\,)$ is a continuous sesquilinear form on $\mathfrak{H}_{\omega_2}$. Therefore it extends to a continuous positive semidefinite sesquilinear form $(\,\cdot\,,\,\cdot\,)$ on $\widehat{\mathfrak{H}}_{\omega_2}$. By the Lax-Milgram Theorem, see Exercise 3.6.20, there is a unique operator $\varrho \in \mathfrak{B}(\widehat{\mathfrak{H}}_{\omega_2})$ such that

$$(\phi, \psi) = \langle \phi, \varrho\psi \rangle_{\omega_2}$$

for all $\phi, \psi \in \widehat{\mathfrak{H}}_{\omega_2}$. Since $(\phi, \phi) \geq 0$ for all $\phi \in \widehat{\mathfrak{H}}_{\omega_2}$ we have $\varrho \geq 0$. Since on the dense subspace $\mathfrak{H}_{\omega_2}$ we have

$$(\psi_a, \psi_a) = \omega_1(a^*a) \leq \omega_2(a^*a) = \langle \psi_a, \psi_2 \rangle_{\omega_2},$$

this translates into $\varrho \leq \mathbb{1}$. For $a, b, c \in \mathscr{A}$ we get

$$\begin{aligned}
\langle \psi_b, \varrho \pi_{\omega_2}(a)\psi_c \rangle_{\omega_2} &= (\psi_b, \pi_{\omega_2}(a)\psi_c) \\
&= (\psi_b, \psi_{ac}) \\
&= \omega_1(b^*(ac)) \\
&= \omega_1((a^*b)^*c) \\
&= (\psi_{a^*b}, \psi_c) \\
&= \langle \psi_{a^*b}, \varrho\psi_c \rangle_{\omega_2} \\
&= \langle \pi_{\omega_2}(a^*)\psi_b, \varrho\psi_c \rangle_{\omega_2} \\
&= \langle \psi_b, \pi_{\omega_2}(a)\varrho\psi_c \rangle_{\omega_2},
\end{aligned}$$

and hence $\varrho\pi_{\omega_2}(a) = \pi_{\omega_2}(a)\varrho$ as this holds on the dense subspace $\mathfrak{H}_{\omega_2}$. Finally, we have

$$\omega_1(a) = \omega_1(\mathbb{1}a) = (\psi_{\mathbb{1}}, \psi_a) = \langle \psi_{\mathbb{1}}, \varrho\psi_a \rangle_{\omega_2} = \langle \psi_{\mathbb{1}}, \varrho\pi_{\omega_2}(a)\psi_{\mathbb{1}} \rangle_{\omega_2}.$$ ☐

Another way to phrase this is that whenever ω_1 and ω_2 are two positive linear functionals on $\mathscr{A}$ with a constant $c > 0$ such that

$$\omega_1 \leq c\omega_2, \tag{7.4.1}$$

then there exists a vector $\phi \in \widehat{\mathfrak{H}}_{\omega_2}$ in the GNS Hilbert space of ω_2 such that

$$\omega_1(a) = \langle \phi, \pi_{\omega_2}(a)\phi \rangle. \tag{7.4.2}$$

Indeed, first we can rescale ω_1 to $\frac{1}{c}\omega_1$ in order to be in the situation of Theorem 7.4.1. Then $\phi = \sqrt{\varrho}\psi_{\mathbb{1}}$ will do the job.

Remark 7.4.2 (Noncommutative Radon-Nikodym Theorem) The theorem is indeed a generalization of the usual Radon-Nikodym Theorem C.3.44 in the following sense: Suppose we have a finite measure space $(X, \mathfrak{a}, \mu)$ and we consider $\mathscr{A} = \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$. Then the functional

$$\omega_2 \colon \mathrm{L}^\infty(X, \mathfrak{a}, \mu) \ni f \;\mapsto\; \int_X f \, \mathrm{d}\mu \tag{7.4.3}$$

is easily shown to be positive and the GNS representation of ω_2 is the usual action of $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ on $\mathrm{L}^2(X, \mathfrak{a}, \mu)$ as in Theorem 7.2.24. Now $1 \in \mathrm{L}^2(X, \mathfrak{a}, \mu)$ is the cyclic vector corresponding to ω_2. Suppose now $\omega_1 \leq \omega_2$ is another positive functional on $\mathscr{A}$. Then

$$\nu(A) = \omega_1(\chi_A) \tag{7.4.4}$$

for $A \in \mathfrak{a}$ turns out to define a finite measure with $\nu \leq \mu$ and hence $\nu \ll \mu$ by using monotone convergence. It follows that

$$\omega_1(f) = \int_X f \, \mathrm{d}\nu. \tag{7.4.5}$$

Now Theorem 7.4.1 gives a positive operator ϱ in the commutant of $\mathscr{A}$ which, by Theorem 7.2.24, *iii.)*, coincides with $\mathscr{A}$. Thus ϱ is the multiplication operator with some f. Moreover, $\varrho \geq 0$ simply means $f \geq 0$ almost everywhere. Thus

$$\int_x g \, \mathrm{d}\nu = \omega_1(g) = \langle 1, \varrho \pi_{\omega_2}(g) 1 \rangle = \int_X f g \, \mathrm{d}\mu, \tag{7.4.6}$$

or $\mathrm{d}\nu = f \, \mathrm{d}\mu$, which is the Radon-Nikodym Theorem for a finite measure space and two positive measures.

σ-finite case?

We use this theorem now to determine irreducible *-representations. Since every irreducible *-representation is cyclic we only have to care about GNS representations.

Theorem 7.4.3 (Irreducible *-representations) *Let $\mathscr{A}$ be a C^*-algebra and let $\omega \colon \mathscr{A} \longrightarrow \mathbb{C}$ be a state. Then the GNS representation π_ω of ω is irreducible iff ω is pure and any irreducible *-representation of $\mathscr{A}$ arises this way up to unitary equivalence.*

PROOF: As already recalled, every irreducible *-representation is cyclic and hence unitarily equivalent to a GNS representation, see Corollary 7.1.29 and Theorem 7.1.14, *ii.)*. Thus let ω be given and suppose that π_ω is irreducible. Suppose that $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ with states ω_1, ω_2 and $\lambda \in (0, 1)$. Then we have $\lambda\omega_1 \leq \omega$ and hence we find an operator $\varrho \in \mathfrak{B}(\widehat{\mathfrak{H}}_\omega)$ with

$$\lambda\omega_1(a) = \langle \psi_{\mathbb{1}}, \varrho \pi_\omega(a) \psi_{\mathbb{1}} \rangle_\omega,$$

and $\varrho \in \pi_\omega(\mathscr{A})'$ by Theorem 7.4.1. Since π_ω is irreducible we have $\varrho = \tilde{\lambda}\,\mathrm{id}$ and thus $\lambda\omega_1(a) = \tilde{\lambda}\langle \psi_{\mathbb{1}}, \pi_\omega(a)\psi_{\mathbb{1}} \rangle = \tilde{\lambda}\omega(a)$ giving $\omega_1 = \omega$ by evaluating this for $a = \mathbb{1}$, after adjoining a unit if necessary. Analogously, $\omega_2 = \omega$ follows and the decomposition of ω is necessarily trivial. Hence ω is pure. Conversely, suppose ω is pure and let $P \in \mathfrak{B}(\widehat{\mathfrak{H}}_\omega)$ be a non-trivial projection in the commutant of $\pi_\omega(\mathscr{A})$, i.e. π_ω is reducible. First we claim that the two orthogonal components $P\psi_{\mathbb{1}}$ and $(1-P)\psi_{\mathbb{1}}$ of $\psi_{\mathbb{1}}$ are both non-trivial. Indeed, suppose $P\psi_{\mathbb{1}} = 0$ then $0 = \pi_\omega(a)P\psi_{\mathbb{1}} = P\pi_\omega(a)\psi_{\mathbb{1}} = P\psi_a$. Since the ψ_a span the dense subspace $\mathfrak{H}_\omega \subseteq \widehat{\mathfrak{H}}_\omega$ this would mean $P = 0$, a contradiction. Analogously, $(1-P)\psi_{\mathbb{1}} \neq 0$ follows. Now we consider the positive functionals

$$\omega_1(a) = \langle P\psi_{\mathbb{1}}, \pi_\omega(a) P\psi_{\mathbb{1}} \rangle$$

and

$$\omega_2(a) = \langle (1-P)\psi_{\mathbb{1}}, \pi_\omega(a)(1-P)\psi_{\mathbb{1}} \rangle.$$

From the block-diagonal form

$$\pi_\omega(a) = P\pi_\omega(a)P + (1-P)\pi_\omega(a)(1-P)$$

we get the decomposition $\omega = \omega_1 + \omega_2$. Since (after adjoining a unit if necessary)

$$\|\omega_1\| = \omega_1(\mathbb{1}) = \|P\psi_{\mathbb{1}}\|^2 > 0 \quad \text{and} \quad \|\omega_2\| = \omega_2(\mathbb{1}) = \|(1-P)\psi_{\mathbb{1}}\|^2 > 0,$$

we see that actually $\lambda \in (0,1)$. Since ω is assumed to be pure we have

$$\omega_1 = \lambda\omega \quad \text{and} \quad \omega_2 = (1-\lambda)\omega$$

for some $\lambda \in (0,1)$. We fix now a sequence $a_n \in \mathscr{A}$ with $\psi_{a_n} \longrightarrow P\psi_{\mathbb{1}}$ which is possible as $\mathfrak{H}_\omega \subseteq \widehat{\mathfrak{H}}_\omega$ is dense. Then $\lambda = \|P\psi_{\mathbb{1}}\|^2$ gives the estimate

$$\begin{aligned}
\lambda &= \lim_{n\longrightarrow\infty} |\langle P\psi_{\mathbb{1}}, \psi_{a_n}\rangle| \\
&= \lim_{n\longrightarrow\infty} |\omega_1(a_n)| \\
&= \lim_{n\longrightarrow\infty} \lambda|\omega(a_n)| \\
&\leq \lambda \lim_{n\longrightarrow\infty} \omega(a_n^* a_n) \\
&= \lambda\|P\psi_{\mathbb{1}}\|^2 \\
&= \lambda^2,
\end{aligned}$$

and thus $\lambda \leq \lambda^2$ which is impossible for $\lambda \in (0,1)$. Hence we have a contradiction to P being different from 0 and $\mathbb{1}$, finally proving that π_ω is irreducible. □

This characterization of irreducible *-representations has now a remarkable consequence based on Kaplansky's Density Theorem:

Corollary 7.4.4 *Let $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a pure state of a unital C^*-algebra $\mathscr{A}$. Then the GNS pre-Hilbert space $\mathfrak{H}_\omega = \mathscr{A}\big/\mathscr{J}_\omega$ is already complete, i.e. $\mathfrak{H}_\omega = \widehat{\mathfrak{H}}_\omega$.*

PROOF: If $\mathfrak{H}_\omega \subsetneq \widehat{\mathfrak{H}}_\omega$ would not be complete then it yields an invariant dense subspace since by the very definition $\pi_\omega(a)\psi_b = \psi_{ab} \in \mathfrak{H}_\omega$ for all $\psi_b \in \mathfrak{H}_\omega$. By Corollary 7.2.35, π_ω is algebraically irreducible and hence this gives a contradiction. □

Example 7.4.5 Let $\mathscr{A} = \mathscr{C}(X)$ be the C^*-algebra of continuous functions on a compact Hausdorff space as usual. Using Riesz's Representation Theorem **??** for positive functionals on $\mathscr{C}(X)$ it is easy to see that the pure states are precisely the δ-functionals. Conversely, we can argue with Theorem 7.4.3, which is slightly more elementary: Given a state ω its Gel'fand ideal $\mathscr{J}_\omega$ is of the form $\mathscr{J}_A$ for some closed subset $A \subseteq X$, as $\mathscr{J}_\omega$ is clearly a closed ideal of $\mathscr{C}(X)$, see Theorem 4.3.10, *ii.)*. Then $\mathscr{C}(X)\big/\mathscr{J}_A \cong \mathscr{C}(A)$ as $\mathscr{C}(X)$-module where the module structure is simply given by (left) multiplication of the restriction to A. If A consists of more than one point, we find a function $f \in \mathscr{C}(X)$ which is non-constant on A by the usual Urysohn argument. But then the operator corresponding to f is not a multiple of the identity and, since $\mathscr{A}$ is commutative, in the commutant. Hence the GNS representation can not be irreducible and thus, by Theorem 7.4.3, ω was not pure. Combining this with our previous results from Section 4.3, we arrive at the following one-to-one correspondences for a unital commutative C^*-algebra $\mathscr{A}$ of

i.) pure states of $\mathscr{A}$,

ii.) points in $\operatorname{Spec}(\mathscr{A})$, *iii.)* maximal ideals (maximal *-ideals) in $\mathscr{A}$,

iv.) irreducible *-representations of $\mathscr{A}$,

v.) unital *-representations on $\mathbb{C}$ of $\mathscr{A}$,

vi.) characters of $\mathscr{A}$.

This list shows once more the intimate link between algebraic features of $\mathscr{C}(X)$ on the one hand and geometric properties of X on the other hand.

In particular, for a commutative unital C^*-algebra $\mathscr{A}$ we have many pure states: they allow to separate the elements in $\mathscr{A}$. In general, this is not obvious. Nevertheless, we have the following statement:

Theorem 7.4.6 (Existence of pure states) *Let $\mathscr{A}$ be a unital C^*-algebra.*

i.) The set of states of $\mathscr{A}$ is a convex and weak* compact subset of the dual $\mathscr{A}'$.

ii.) The set of pure states of $\mathscr{A}$ is non-empty and convex combinations of pure states approximate every other state in the weak* topology.

iii.) If $\mathscr{B} \subseteq \mathscr{A}$ is a unital C^*-subalgebra and $\omega\colon \mathscr{B} \longrightarrow \mathbb{C}$ is a pure state then ω extends to a pure state of $\mathscr{A}$. If the extension of ω as pure states is unique then there are no non-pure extensions.

PROOF: Clearly, the conditions $\omega(\mathbb{1}) = 1$ and $\omega(a^*a) \geq 0$ for all $a \in \mathscr{A}$ are convex and yield functionals in the closed unit ball $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq \mathscr{A}'$ as $\omega(\mathbb{1}) = \|\omega\|$ for a positive functional by Proposition 4.4.19. Moreover, if $\{\omega_i\}_{i\in I}$ is a net of positive functionals converging weakly* to some ω then $\omega(a^*a) = \lim_{i\in I} \omega_i(a^*a) \geq 0$ for all $a \in \mathscr{A}$ by the very definition of weak* convergence. Hence ω is again positive. In conclusion, we get the first part by the compactness of $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq \mathscr{A}'$ in the weak* topology according to the Banach-Alaoglu Theorem in form of Corollary 2.3.34. Since we already know that the set of states is non-empty, see Theorem 4.4.21, we have a convex and weak* closed non-empty set of all states to which we can apply the Krein-Milman Theorem 2.4.24. This gives directly the second statement. For the third, let $\omega\colon \mathscr{B} \longrightarrow \mathbb{C}$ be a given pure state on $\mathscr{B}$ and consider the set of all states on $\mathscr{A}$ which restricts to ω on $\mathscr{B}$, i.e.

$$\mathcal{S}_\omega = \left\{\Omega\colon \mathscr{A} \longrightarrow \mathbb{C} \;\middle|\; \Omega \text{ is a state with } \Omega\big|_{\mathscr{B}} = \omega\right\}.$$

First we note that any Hahn-Banach extension of ω to $\mathscr{A}$ with the same norm is again a state since $\Omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ with $\|\Omega\| = \|\omega\|$ satisfies $\Omega(\mathbb{1}) = \omega(\mathbb{1}) = \|\omega\| = \|\Omega\|$ yielding the positivity of Ω by Proposition 4.4.19. Thus $\mathcal{S}_\omega$ is non-empty. Moreover, $\mathcal{S}_\omega$ is clearly convex. Finally, we note that the conditions for a state Ω to be in $\mathcal{S}_\omega$ is closed in the weak* topology as we fix the values on some subset $\mathscr{B} \subseteq \mathscr{A}$. Thus $\mathcal{S}_\omega$ is a closed subset of the set of all states and hence compact in the weak* topology. Up to now we have not yet used that ω is pure. By the Krein-Milman Theorem 2.4.24 we have extreme points $\Omega \in \mathcal{S}_\omega$. We claim that Ω is a pure state of $\mathscr{A}$. Suppose $\Omega = \lambda\Omega_1 + (1-\lambda)\Omega_2$ with $\lambda \in (0,1)$. Then consider $\omega_i = \Omega_i\big|_{\mathscr{B}}$ which are still states, now on $\mathscr{B}$. We have $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$. Since ω was pure this implies $\omega = \omega_1 = \omega_2$ and hence $\Omega_1, \Omega_2 \in \mathcal{S}_\omega$ extend ω. But inside $\mathcal{S}_\omega$ the state Ω was extreme and hence $\Omega = \Omega_1 = \Omega_2$ follows, showing that Ω is pure as a state of $\mathscr{A}$. This shows that we have extensions of ω as pure state. Finally, if the extension as a pure state is unique then $\mathcal{S}_\omega$ has only one extreme point. Again by the Krein-Milman Theorem, $\mathcal{S}_\omega$ has to coincide with this point, see also . □

e: exercise ...

The theorem shows that we have many pure states on a unital C^*-algebra. In fact, we can obtain some more specific statements as follows:

Corollary 7.4.7 *Let $\mathscr{A}$ be a unital C^*-algebra and let $a \in \mathscr{A}$ be normal. Then for every $\lambda \in \operatorname{spec}(a)$ there is a pure state $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ with*

$$\omega(a) = \lambda. \tag{7.4.7}$$

Proof: As already in Theorem 4.4.21 we consider the unital commutative C^*-subalgebra $\mathscr{C}(\operatorname{spec}(a)) \cong \mathsf{C}^*\langle a\rangle \subseteq \mathscr{A}$ on which the δ-functional δ_λ is a well-defined pure state, see Example 7.4.5. We have $\delta_\lambda(a) = \lambda$ and hence any pure extension of δ_λ to a pure state ω of $\mathscr{A}$ will do the job. $\square$

Corollary 7.4.8 *Let $\mathscr{A}$ be a unital C^*-algebra and let $a \in \mathscr{A}$. Then $a = 0$ iff for every pure state ω on $\mathscr{A}$ we have $\omega(a) = 0$.*

Corollary 7.4.9 *Let $\mathscr{A}$ be a unital C^*-algebra and let $a \in \mathscr{A}$. Then $a \in \mathscr{A}^+$ iff $\omega(a) \geq 0$ for all pure states ω of $\mathscr{A}$.*

Proof: From Corollary 7.4.8 we first see that $a - a^*$ is annihilated by all pure states and hence $a = a^*$. Then Corollary 7.4.7 gives $\operatorname{spec}(a) \subseteq \mathbb{R}_0^+$. $\square$

Thus we can sharpen the results from Subsection 4.4.3 by only using pure states instead of all states. In fact, for $\mathscr{A} = \mathscr{C}(X)$ this becomes the statement that a function f is a positive algebra element iff it is pointwise a positive function.

Also the existence of a faithful *-representation as we know it from Theorem 4.4.33 can be refined by using only pure states:

Corollary 7.4.10 *Let $\mathscr{A}$ be a unital C^*-algebra. Then there exists a faithful *-representation which is the direct sum of irreducible *-representations. More precisely,*

$$\pi = \widehat{\bigoplus_{\omega \text{ pure state}}} \pi_\omega \quad \text{on} \quad \mathfrak{H} = \widehat{\bigoplus_{\omega \text{ pure state}}} \mathfrak{H}_\omega, \tag{7.4.8}$$

is faithful.

The proof is literally the same as for Theorem 4.4.33 now using Corollary 7.4.7 instead of Theorem 4.4.21. We note that an irreducible *-representation needs not to be faithful in general: for the commutative case the irreducible *-representations are faithful iff $X = \{\mathrm{pt}\}$ consists of a single point.

7.4.2 Types of *-Representations

Having a general (non-degenerate) *-representation π of a unital C^*-algebra we can decompose $\pi(\mathscr{A})''$ according to Theorem 7.3.37. This yields the following statement:

Theorem 7.4.11 (Type decomposition of a *-representation) *Let $\mathscr{A}$ be a unital C^*-algebra and let π be a non-degenerate *-representation on a Hilbert space $\mathfrak{H}$. Then there is a unique orthogonal decomposition*

$$\mathfrak{H} = \mathfrak{H}_I \oplus \mathfrak{H}_{II_1} \oplus \mathfrak{H}_{II_\infty} \oplus \mathfrak{H}_{III}, \tag{7.4.9}$$

for which π is block-diagonal, i.e.

$$\pi = \pi_I \oplus \pi_{II_1} \oplus \pi_{II_\infty} \oplus \pi_{III}, \tag{7.4.10}$$

such that $\pi_I(\mathscr{A})''$ is of type I, $\pi_{II_1}(\mathscr{A})''$ is of type II_1, $\pi_{II_\infty}(\mathscr{A})''$ is of type II_∞, and $\pi_{III}(\mathscr{A})''$ is of type III.

Proof: This is direct consequence of Theorem 7.3.37 applied to the von Neumann algebra $\pi(\mathscr{A})''$. $\square$

Definition 7.4.12 (Type of *-representation) *A non-degenerate *-representation π of a unital C^*-algebra $\mathscr{A}$ is called of type I, of type II_1, of type II_∞, or of type III if the von Neumann algebra $\pi(\mathscr{A})''$ if of type I, of type II_1, type II_∞, or of type III, respectively. Moreover, π is called a factor (or primary) representation if $\pi(\mathscr{A})''$ is a factor.*

In addition, we call a state a *factor state* or *primary state* if the corresponding GNS representation is a factor representation. Factor representations are thus a natural generalization of irreducible representations.

Remark 7.4.13 A C^*-algebra $\mathscr{A}$ has always many type I *-representations: any irreducible *-representation is of type I. Since direct sums respect the type, see Exercise 7.5.2, also direct sums of irreducible *-representations are again of type I. Since for an irreducible *-representation we have $\pi(\mathscr{A})'' = \mathfrak{B}(\mathfrak{H})$ it is also a factor representation. However, it may well happen that a given C^*-algebra has only type I representations.

We use now the commutant of the representations in order to study the subrepresentations. Recall that, if π is a non-degenerate *-representation of $\mathscr{A}$ on $\mathfrak{H}$ and $P \in \pi(\mathscr{A})'$ is a projections in the commutant then

$$\pi_P(a) = P\pi(a)P \tag{7.4.11}$$

and

$$\pi_{(\mathbb{1}-P)}(a) = (1-P)\pi(a)(1-P) \tag{7.4.12}$$

provide subrepresentations of $\mathscr{A}$ acting non-trivially only on $P\mathfrak{H}$ and $(\mathbb{1}-P)\mathfrak{H}$, respectively, such that $\pi = \pi_P \oplus \pi_{(\mathbb{1}-P)}$ is block-diagonal. Conversely, every subrepresentation arises this way, see our discussion in Proposition 7.1.15.

The following theorem clarifies now the role of von Neumann technology for understanding (quasi-) equivalence and disjointness:

Theorem 7.4.14 (Disjointness and quasi-equivalence) *Let π be a non-degenerate *-representation of a unital C^*-algebra $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$. Moreover, let $P, Q \in \pi(\mathscr{A})'$ be projections in the commutant. For the subrepresentations π_P and π_Q one has:*

i.) The space of intertwiners from $(P\mathfrak{H}, \pi_P)$ to $(Q\mathfrak{H}, \pi_Q)$ is given by $Q\pi(\mathscr{A})'P$.

*ii.) The *-representations π_P and π_Q are disjoint iff $Z(P)Z(Q) = 0$.*

*iii.) The *-representation π_P is subordinate to π_Q iff $Z(P) \le Z(Q)$.*

*iv.) The *-representation π_P is quasi-equivalent to π_Q iff $Z(P) = Z(Q)$.*

PROOF: Here the central supports of P and Q is meant to be with respect to the von Neumann algebra $\pi(\mathscr{A})'$. For the first part, let $A\colon P\mathfrak{H} \longrightarrow Q\mathfrak{H}$ be an intertwiner from π_P to π_Q, i.e. $A\pi_P(a) = \pi_Q(a)A$ for all $a \in \mathscr{A}$. Then extending A by 0 on $(\mathbb{1}-P)\mathfrak{H}$ to an operator $A \in \mathfrak{B}(\mathfrak{H})$ yields an operator in $Q\pi(\mathscr{A})'P$ since

$$\begin{aligned}
A\pi(a) &= AP\pi(a) + A(\mathbb{1}-P)\pi(a) \\
&= A\pi_P(a) + 0 \\
&= \pi_Q(a)A \\
&= \pi(a)QA \\
&= \pi(a)A,
\end{aligned}$$

as A vanishes on $\operatorname{im}(\mathbb{1}-P)$ and has image in $\operatorname{im} Q$. Conversely, given any $A \in Q\pi(\mathscr{A})'P$ it is a straightforward computation that $A\big|_{P\mathfrak{H}}\colon P\mathfrak{H} \longrightarrow Q\mathfrak{H} \subseteq \mathfrak{H}$ is an intertwiner from π_P to π_Q. Finally, it is easy to see that the two constructions are really inverse to each other since $A \in Q\pi(\mathscr{A})'P$ has an image in $Q\mathfrak{H}$ and vanishes on $(\mathbb{1}-P)\mathfrak{H}$. Hence the first part is shown. For the second part we note that $Q\pi(\mathscr{A})'P = \{0\}$ is equivalent to $Z(P)Z(Q) = 0$ by Lemma 7.3.21 and hence the first part implies the second. For the third part, we first note that the subrepresentations of π_P are in one-to-one correspondence with projections $P' \le P$ in $\pi(\mathscr{A})'$. This is clear by applying the construction

$\pi \rightsquigarrow \pi_P$ to π_P itself. Now suppose $Z(P) \le Z(Q)$ then also $Z(P') \le Z(P) \le Z(Q)$ for all $P' \le P$ in $\pi(\mathscr{A})'$. Hence if $P' \ne 0$ then also $Z(P') \ne 0$ and we have $Z(P')Z(Q) = Z(P')$ by Lemma 7.3.21. This means $Q\pi(\mathscr{A})'P' \ne \{0\}$ and hence there is a non-trivial intertwiner $A \in Q\pi(\mathscr{A})'P'$ from $\pi_{P'}$ to π_Q by the first part. This intertwiner has a polar decomposition $A = U|A|$ with $U, |A| \in \pi(\mathscr{A})'$ as usual and we get $\operatorname{im} U = (\operatorname{im} A)^{\mathrm{cl}} \subseteq \operatorname{im} Q$ while $\ker U = \ker|A| \supseteq \ker P'$. Thus $0 \ne U^*U \le P'$ and $0 \ne UU^* \le Q$ yields equivalent projections U^*U and UU^* with corresponding equivalent subprojections

$$\pi_{U^*U} \text{ of } \pi_{P'} \quad \text{and} \quad \pi_{UU^*} \text{ of } \pi_Q,$$

where the equivalence is implemented by U and U^*, restricted to the corresponding domain of π_{U^*U} and π_{UU^*}, respectively. This shows that every non-zero subrepresentation $\pi_{P'}$ of π_P has a non-zero subrepresentation π_{U^*U} being equivalent to a subrepresentation π_{UU^*} of π_Q. Thus $\pi_P \le \pi_Q$ is established. The next claim still works in a general von Neumann algebra. We have

$$Q \precsim P \implies Z(Q) \le Z(P). \tag{$**$}$$

Indeed, let $Q = U^*U \sim UU^* \le P$. Then $QU^* = U^*$ and $Z(Q)U^* = U^*$ follows from $Q \le Z(Q)$. Thus $Z(Q)UU^* = UZ(Q)U^* = UU^*$ shows that $UU^* \le Z(Q)$. Exchanging the role of U^*U and UU^* shows $Z(Q) = Z(UU^*)$. But then $Z(UU^*) \le Z(P)$ is clear as $UU^* \le P$. Now back to the third part we assume that π_P is subordinate to π_Q. Hence for every non-zero subrepresentation of π_P given as π_{P_1} with $P_1 \le P$ we find a non-zero subrepresentation π_{U^*U} with $U^*U \le P_1$ such that π_{UU^*} is a subrepresentation of π_Q, i.e. $UU^* \le Q$. We consider now all the subprojections $0 \ne P' \le P$ for which $Z(P') \le Z(Q)$. By assumption, the set of those is non-empty and hence

$$R = \sup\bigl\{0 \ne P' \le P \bigm| Z(P') \le Z(Q)\bigr\}$$

defines a non-zero subprojection of P. Suppose $R \ne P$ then $P - R$ is a non-zero subprojection of P leading to a non-zero subrepresentation π_{P-R} of π_P. By $\pi_P \le \pi_Q$ we find a non-zero subrepresentation of π_{P-Q} being equivalent to a subrepresentation of π_Q, hence we get a $0 \ne U \in \pi(\mathscr{A})'$ with $U^*U \le P - Q$ and $UU^* \le Q$, the corresponding equivalent subrepresentations are π_{U^*U} and π_{UU^*}, respectively. From $(**)$ we know that $Z(R) \le Z(Q)$ as well as $Z(U^*U) \le Z(Q)$, i.e. $Z(Q)R = R$ and $Z(Q)U^*U = U^*U$. Since R and U^*U are orthogonal, $R + U^*U \le P$ is again a subprojection and we have $Z(Q)(R + U^*U) = R + U^*U$. This shows $Z(R + U^*U) \le Z(Q)$ contradicting the maximality of R. Hence $R = P$ and the third part follows. Since quasi-equivalence means $\pi_P \le \pi_Q$ and $\pi_Q \le \pi_P$ the fourth part is clear by the statement of the third part. □

Since the quasi-equivalence of subrepresentations is governed by the central projections in $\pi(\mathscr{A})'$ this gives the following property of factor representations. While irreducible *-representations simply have no non-trivial subrepresentations, the more general factor representations have not many different ones:

Corollary 7.4.15 *Let π be a factor representation of a unital C^*-algebra on a Hilbert space $\mathfrak{H}$. Then any two non-zero subrepresentations of π are quasi-equivalent and hence quasi-equivalent to π itself.*

Proof: The representation π being a factor representation means $\pi(\mathscr{A})'$ is a factor. Thus the only nonzero central projections in $\pi(\mathscr{A})'$ is $Z = \mathbb{1}$. Hence the statement follows from Theorem 7.4.14, *iv.)*, at once. □

The importance of Theorem 7.4.14 lies also in the fact that studying the relations between two *-representations is essentially the same thing as studying subrepresentations of one *-representation: indeed if $(\mathfrak{H}_1, \pi_1)$ and $(\mathfrak{H}_2, \pi_2)$ are, say non-degenerate, *-representations then $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ with $\pi = \pi_1 \oplus \pi_2$ gives a non-degenerate *-representation with π_1 and π_2 being subrepresentations of π.

An intertwiner $A\colon (\mathfrak{H}_1, \pi_1) \longrightarrow (\mathfrak{H}_2, \pi_2)$ can now be viewed as a self-intertwiner $A \in \mathfrak{B}(\mathfrak{H})$ sitting in the upper block above the diagonal while the intertwiners in the opposite direction are located in the lower block under the diagonal, all with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Note that the projections $P_1, P_2 \in \mathfrak{B}(\mathfrak{H})$ onto $\mathfrak{H}_1, \mathfrak{H}_2$, respectively, give back the representations π_1, π_2 as

$$\pi_1 = \pi_{P_1} \quad \text{and} \quad \pi_2 = \pi_{P_2}, \tag{7.4.13}$$

but in general they are not central projections in $\pi(\mathscr{A})'$: this happens precisely iff there is no nonzero intertwiner from π_1 to π_2, i.e. the *-representations are disjoint. We use these ideas now in the following proposition:

Sebastian: why?

Proposition 7.4.16 *Let $(\mathfrak{H}_1, \pi_1)$ and $(\mathfrak{H}_2, \pi_2)$ be non-degenerate *-representations of a unital C^*-algebra $\mathscr{A}$. Then there exist unique (central) projections $P_1 \in \pi_1(\mathscr{A})'$ and $P_2 \in \pi_2(\mathscr{A})'$, respectively, such that*

i.) $(\pi_1)_{P_1} \sim (\pi_2)_{P_2}$,

ii.) $(\pi_1)_{(\mathbb{1}-P_1)} \perp \pi_2$,

iii.) $\pi_1 \perp (\pi_2)_{(1-P_2)}$.

Proof: First we consider $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and $\pi = \pi_1 \oplus \pi_2$ to which we apply the above arguments. Then we consider pairs of projections (Q_1, Q_2) in $\pi(\mathscr{A})'$ with $Q_1 \leq \mathrm{pr}_{\mathfrak{H}_1}$ and $Q_2 \leq \mathrm{pr}_{\mathfrak{H}_2}$ such that $Z(Q_1) = Z(Q_2)$ and define

$$P_1 = \sup Q_1 \quad \text{and} \quad P_2 = \sup Q_2, \tag{$*$}$$

where the supremum is taken over all such pairs. Clearly $P_1 \leq \mathrm{pr}_{\mathfrak{H}_i}$ and $P_i \in \pi(\mathscr{A})'$ giving subrepresentations $\pi_{P_1} \leq \pi_1$ and $\pi_{P_2} \leq \pi_2$. Now for a general set of projections $\{Q_i\}_{i \in I}$ we have

$$\sup_{i \in I} Z(Q_i) = Z(\sup\nolimits_{i \in I} Q_i), \tag{$**$}$$

since on one hand we have $Q_i \leq \sup_{i \in I} Q_i = Q$ and hence $Z(Q_i) \leq Z(Q)$ by Exercise**??** for all $i \in I$ leading to "$\leq$" in $(**)$. On the other hand, $\lim_{i \in I} Q_i = \sup_{i \in I} Q_i$ in the σ-strongly* topology for which the multiplication by a fixed operator $Z(Q)$ is continuous. Hence

$$\begin{aligned}
\sup_{i \in I} Z(Q_i) Q &= \sup_{i \in I} Z(Q_i) \lim_{j \in I} Q_j \\
&= \lim_{j \in I} (\sup\nolimits_{i \in I} Z(Q_i)) Q_j \\
&\geq \lim_{j \in I} Z(Q_j) Q_j \\
&= \lim_{j \in I} Q_j \\
&= Q
\end{aligned}$$

shows "$\geq$" in $(**)$ and therefore $(**)$ holds. We conclude that $Z(P_1) = Z(P_2)$ for the above projections $P_1, P_2 \in \pi(\mathscr{A})'$ and thus $\pi_{P_1} \sim \pi_{P_2}$ establishing the first property by Theorem 7.4.14, *iv.)*. Now consider $\mathrm{pr}_1 - P_1$ and the corresponding subrepresentation $\pi_{\mathrm{pr}_1 - P_1} \subseteq \pi_1$. Either $\mathrm{pr}_1 - P_1 = 0$, i.e. $P_1 = \mathrm{pr}_1$. Then we do not have to show anything as the zero representation is disjoint from any other. Or, $\mathrm{pr}_1 - P_1 \neq 0$. In this case we assume that $\pi_{\mathrm{pr}_1 - P_1}$ and π_2 are not disjoint. Thus there is a non-zero intertwiner and hence there is an nonzero subrepresentation π_{Q_1} of $\pi_{\mathrm{pr}_1 - P_1}$ and π_{Q_2} of π_{pr_2} with π_{Q_1} and π_{Q_2} being equivalent. But then π_Q and π_{Q_2} are also quasi-equivalent and we have $Z(Q_1) = Z(Q_2)$ by Theorem 7.4.14, *iv.)*. This show that the pair (Q_1, Q_2) belongs to the above set and hence $P_1 \geq Q_1$ contradicting $Q_1 \leq (\mathrm{pr}_1 - P_1)$. This shows the property *ii.)* and the property *iii.)* is analogous. This shows the existence. Now suppose that we have found another pair of projections $\tilde{P}_1$ and $\tilde{P}_2$ with the above properties. Then clearly $\tilde{P}_1 \leq P_1$ and $\tilde{P}_2 \leq P_2$ by the maximality in $(*)$. Assume $Q_1 = P_1 - \tilde{P}_1 \neq 0$. Then Q_1 is a subprojection of $\mathrm{pr}_1 - \tilde{P}_1$ and hence π_{Q_1} is a subrepresentation of $\pi_{\mathrm{pr}_1 - \tilde{P}_1}$ which is disjoint from π_2 by assumption. However, it is also a subprojection of P_1 and hence π_{Q_1} is a subrepresentation of π_{P_1}. Hence it contains a non-zero subrepresentation being equivalent to a subrepresentation of π_{P_2} and hence of π_2. Thus π_{Q_1} can not be disjoint from π_2 which contradicts the property *ii.)* for $\tilde{P}_1$. Hence $P_1 = \tilde{P}_1$ and analogously one shows $P_2 = \tilde{P}_2$. ☐

In a final step we shall now clarify the relations between the notion of quasi-equivalence and (unitary) equivalence.

Definition 7.4.17 (π-normal states) *Let π be a non-degenerate *-representation of a unital C^*-algebra $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$. Then a state ω is called π-normal is there exists a normal state of $\pi(\mathscr{A})''$ such that ω is the pull-back of this state via π.*

Since the normal states of a von Neumann algebra are determined by the four equivalent characterizations in Proposition 7.2.38 we can rephrase this by saying that ω is π-normal iff there is a density matrix $\varrho \in \mathfrak{L}^1(\mathfrak{H})$ with

$$\omega(a) = \operatorname{tr}(\varrho\pi(a)). \tag{7.4.14}$$

Recall that, in general, the density matrix need not to be unique: this is precisely the case for $\pi(\mathscr{A})'' = \mathfrak{B}(\mathfrak{H})$ and hence for an irreducible *-representation.

Theorem 7.4.18 (Quasi-equivalence) *Let $\mathscr{A}$ be a unital C^*-algebra and let $(\mathfrak{H}_1, \pi_1)$ and $(\mathfrak{H}_2, \pi_2)$ be two non-degenerate *-representations of $\mathscr{A}$. Then the following statements are equivalent:*

i.) *The *-representations π_1 and π_2 are quasi-equivalent.*

ii.) *The *-representation π_1 is unitarily equivalent to a subrepresentation of a suitable amplification of π_2 and vice versa.*

iii.) *There exists an index set I such that one has the unitary equivalence*

$$\pi_1 \mathbin{\hat{\otimes}} \mathrm{id}_{\ell^2(I)} \approx \pi_2 \mathbin{\hat{\otimes}} \mathrm{id}_{\ell^2(I)}, \tag{7.4.15}$$

i.e. π_1 and π_2 are unitarily equivalent up to multiplicity.

iv.) *There exists a *-isomorphism*

$$\Phi\colon \pi_1(\mathscr{A})'' \longrightarrow \pi_2(\mathscr{A})'' \tag{7.4.16}$$

with $\Phi(\pi_1(a)) = \pi_2(a)$ for all $a \in \mathscr{A}$.

v.) *All π_1-normal states are π_2-normal states and vice versa.*

PROOF: First we show *i.)* $\implies$ *ii.)* $\implies$ *iii.)* $\implies$ *i.)*. Thus assume π_1 is subordinate to π_2, i.e. $\pi_1 \leq \pi_2$. Then we consider the set $\mathcal{P}$ of all families $\{P_i\}_{i\in I}$ of nonzero pairwise orthogonal projections $P_i \in \pi_1(\mathscr{A})'$ such that $(\pi_1)_{P_i}$ is unitarily equivalent to a subrepresentation of π_2. We know that there is a non-zero subrepresentation of π_1 being unitarily equivalent to a subrepresentation of π_2 hence such families exist and $\mathcal{P} \neq \emptyset$. As usual, we can order $\mathcal{P}$ by inclusion and for a linearly ordered set in $\mathcal{P}$ the union is again in $\mathcal{P}$ and gives a supremum. Thus by Zorn's Lemma there is a maximal family $\{P_i\}_{i\in I}$ with the above property. We claim that $P = \sum_{i\in I} P_i = \mathbb{1}$. Indeed, otherwise $0 \neq \mathbb{1} - P$ gives a nonzero subrepresentation $(\pi_1)_{\mathbb{1}-P}$ which contains a non-zero subrepresentation $(\pi_1)_Q$ unitarily equivalent to a subrepresentation of π_2 by $\pi_1 \leq \pi_2$. Thus $\{P_i\}_{i\in I} \cup \{Q\}$ is again in $\mathcal{P}$ contradicting the maximality. Thus we have $\pi_1 \approx \widehat{\bigoplus}_{i\in I} (\pi_1)_{P_i}$ and $(\pi_1)_{P_i} \approx (\pi_2)_{Q_i}$ for some $Q_i \in \pi_2(\mathscr{A})''$ and $i \in I$. This allows to consider the amplification $\pi_2 \mathbin{\hat{\otimes}} \mathrm{id}_{\ell^2(I)}$ and to use the projection $Q = \widehat{\bigoplus}_{i\in I} Q_i$ on $\mathfrak{H}_2 \mathbin{\hat{\otimes}} \ell^2(I)$. Clearly, we now have

$$\pi_1 \approx \pi_Q \leq (\pi_2 \mathbin{\hat{\otimes}} \mathrm{id}_{\ell^2(I)}),$$

showing that π_1 is unitarily equivalent to a subprojection of $\pi_2 \hat{\otimes} \mathrm{id}_{\ell^2(I)}$. Note that we have only used $\pi_1 \leq \pi_2$ up to here. Exchanging the role of π_1 and π_2 gives immediately *i.)* $\implies$ *ii.)*. Next assume *ii.)* and let I_1, I_2 be index sets such that π_1 is unitarily equivalent to a subprojection of $\pi_2 \hat{\otimes} \mathrm{id}_{\ell^2(I_2)}$ and vice versa. Without restriction, we can assume that both set I_1, I_2 are infinite. Then the product $I = I_1 \times I_2$ satisfies $\#I = \max\{\#I_1, \#I_2\}$ and taking further products with I_1, I_2 or I just reproduce $\#I$. Thus we get that $\pi_1 \hat{\otimes} \mathrm{id}_{\ell^2(I)}$ is equivalent to a subrepresentation of $\pi_2 \hat{\otimes} \mathrm{id}_{\ell^2(I)}$ and vice versa. Now we consider $\pi = (\pi_1 \hat{\otimes} \mathrm{id}_{\ell^2(I)}) \oplus (\pi_2 \hat{\otimes} \mathrm{id}_{\ell^2(I)})$ on $\mathfrak{H} = (\mathfrak{H}_1 \otimes \ell^2(I)) \otimes (\mathfrak{H}_2 \otimes \ell^2(I))$ as usual. We have projections $P_1, P_2 \in \pi(\mathscr{A})'$ with

$$\pi_{P_k} = \pi_k \hat{\otimes} \mathrm{id}_{\ell^2(I)} \quad \text{for} \quad k = 1, 2,$$

and thus the above relations yield projections $Q_k \in \pi(\mathscr{A})'$ with

$$P_1 \sim Q_1 \leq P_2 \quad \text{and} \quad P_2 \sim Q_2 \leq P_1.$$

But this just means $P_1 \precsim P_2$ and $P_2 \precsim P_1$ from which we deduce $P_1 \sim P_2$ by Proposition 7.3.18. Translating this back to the representations gives us the unitary equivalence of $\pi_1 \hat{\otimes} \mathrm{id}_{\ell^2(I)}$ and $\pi_2 \hat{\otimes} \mathrm{id}_{\ell^2(I)}$. Hence *ii.)* $\implies$ *iii.)* follows. Now assume *iii.)*. Then we know from Proposition 7.1.24, *vi.)*, that $\pi_1 \sim \pi_1 \hat{\otimes} \mathrm{id}_{\ell^2(I)}$ and $\pi_2 \sim \pi_2 \hat{\otimes} \mathrm{id}_{\ell^2(I)}$. From *v.)* of the same proposition we know that $\pi_1 \hat{\otimes} \mathrm{id}_{\ell^2(I)} \sim \pi_2 \hat{\otimes} \mathrm{id}_{\ell^2(I)}$ and hence the transitivity of $\sim$ which comes from *iii.)* of that proposition gives $\pi_1 \sim \pi_2$. Thus *iii.)* $\implies$ *i.)*. Next we prove *iii.)* $\implies$ *iv.)* $\implies$ *v.)* $\implies$ *ii.)*. If *iii.)* holds we first note that for any amplification we have

$$\pi(\mathscr{A})'' \hat{\otimes} \mathrm{id}_{\ell^2(I)} = (\pi \otimes \mathrm{id}_{\ell^2(I)})(\mathscr{A})''. \tag{$*$}$$

Indeed, this follows along the same lines of the proof of Lemma 7.2.11 once using the fact that $\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2 \cong \bigoplus_{\in \in I_2} \mathfrak{H}_1$ with I_2 being the index set of a Hilbert basis of $\mathfrak{H}_2$, see Lemma 7.1.20. Now any unitary intertwiner

$$U \colon \mathfrak{H}_1 \hat{\otimes} \mathrm{id}_{\ell^2(I)} \longrightarrow \mathfrak{H}_2 \hat{\otimes} \mathrm{id}_{\ell^2(I)}$$

gives a *-isomorphism

$$\tilde{\Phi} \colon (\pi_1 \hat{\otimes} \mathrm{id}_{\ell^2(I)})(\mathscr{A})'' \ni A \mapsto UAU^* \in (\pi_2 \hat{\otimes} \mathrm{id}_{\ell^2(I)})(\mathscr{A})''$$

between the bicommutants satisfying by the intertwiner property

$$\tilde{\Phi}(\pi_1(a) \hat{\otimes} \mathrm{id}_{\ell^2(I)}) = \pi_2(a) \hat{\otimes} \mathrm{id}_{\ell^2(I)}\,.$$

Since $\tilde{\Phi}$ is obviously continuous in any topology of the bounded operators, we indeed end up again in the bicommutant. Combining this with the *-isomorphism $\pi_k(\mathscr{A})'' \cong (\pi_k \hat{\otimes} \mathrm{id}_{\ell^2(I)})(\mathscr{A})''$ coming from $(*)$ for $k = 1, 2$ we get Φ as wanted. Now assume that we have Φ as in *iv.)*. By Theorem 7.2.48 we know that Φ is continuous in, say, the σ-weak topologies. But then Φ pulls back σ-weakly continuous functionals on $\pi_2(\mathscr{A})''$ to σ-weakly continuous functionals of $\pi_2(\mathscr{A})''$. Since the $\pi_{1/2}$-normal states are precisely those which have a σ-weakly continuous extension to $\pi_{1/2}(\mathscr{A})''$, see Proposition 7.2.38, we get the implication *iv.)* $\implies$ *v.)* immediately, since Φ is bijective. Finally, assume *v.)*. We know that π_1 is unitarily equivalent to a direct sum of cyclic *-representations by Theorem 7.1.17 and every cyclic *-representation is unitarily equivalent to a GNS representation by Theorem 7.1.14. Hence we find vectors $\phi_i \in \mathfrak{H}_1$ such that

$$\pi_1 \cong \bigoplus_{i \in I} \pi_{\omega_i} \quad \text{with} \quad \omega_i(a) = \langle \phi_i, \pi_1(a)\phi_i \rangle_{\mathfrak{H}_1}. \tag{☺}$$

Now, by construction, ω_i is π_1-normal and hence there are density matrices $\varrho_i \in \mathfrak{B}(\mathfrak{H}_2)$ with

$$\omega_i(a) = \mathrm{tr}(\varrho_i \pi_2(a)) \tag{$*$}$$

by assumption. Any density matrix ϱ_i can now be written as convergent series in the trace norm

$$\varrho_i = \sum_{n=0}^{\infty} \Theta_{\phi_n^i, \phi_n^i}$$

with $\sum_{n=0}^{\infty} \|\phi_n^i\|^2 = 1$. Thus we get from $(*)$

$$\omega_i(a) = \sum_{n=0}^{\infty} \langle \phi_n^i, \pi_2(a)\phi_n^i \rangle = \left\langle \Phi_i, \left(\pi_2 \mathbin{\hat{\otimes}} \mathrm{id}_{\ell^2(\mathbb{N}_0)}\right)(a)\Phi_i \right\rangle_{\mathfrak{H}_2 \hat{\otimes} \ell^2(\mathbb{N}_0)},$$

where we use as Φ_i the vector in $\mathfrak{H}_2 \mathbin{\hat{\otimes}} \ell^2(\mathbb{N}_0)$ whose n-th component with respect to the $\ell^2(\mathbb{N}_0)$ part is given by $\phi_n^i \in \mathfrak{H}_2$. The summability $\sum_{n=0}^{\infty} \|\phi_n^i\|^2 = 1$ shows that ϕ_i is a well-defined vector indeed. By uniqueness of the GNS representation, the GNS representation of ω_i can also be realized as the cyclic subrepresentation of $\pi_2 \mathbin{\hat{\otimes}} \mathrm{id}_{\ell^2(\mathbb{N}_0)}$ with the cyclic vector Φ_i, i.e. $\pi_2 \mathbin{\hat{\otimes}} \mathrm{id}_{\ell^2(\mathbb{N}_0)}$ restricted to the invariant subspace

$$\mathfrak{H}_2^{(i)} = \left((\pi_2(\mathscr{A}) \mathbin{\hat{\otimes}} \mathrm{id}_{\ell^2(\mathbb{N}_0)})\Phi_i\right)^{\mathrm{cl}} \subseteq \mathfrak{H}_2 \mathbin{\hat{\otimes}} \ell^2(\mathbb{N}_0).$$

Thus π_{ω_i} is unitarily equivalent to a subrepresentation of $\pi_2 \mathbin{\hat{\otimes}} \mathrm{id}_{\ell^2(\mathbb{N}_0)}$. By (☺) we see that π_1 is unitarily equivalent to a subrepresentation of $\widehat{\bigoplus}_{i \in I} \pi_2 \mathbin{\hat{\otimes}} \mathrm{id}_{\ell^2(\mathbb{N}_0)} \cong \pi_2 \mathbin{\hat{\otimes}} \mathrm{id}_{\ell^2(I \times \mathbb{N}_0)}$. Exchanging once more the roles of π_1 and π_2 gives *v.)* $\implies$ *i.)* and hence the proof is complete. □

7.5 Exercises

Exercise 7.5.1 (Direct sum of von Neumann algebras) Let $\{\mathscr{A}_i\}_{i \in I}$ be a family of von Neumann algebras acting on Hilbert spaces $\{\mathfrak{H}_i\}_{i \in I}$.

i.) Show that the algebraic direct sum $\bigoplus_{i \in I} \mathscr{A}_i$ acts block-diagonally on $\mathcal{H} = \widehat{\bigoplus}_{i \in I} \mathfrak{H}_i$.

ii.) Show that this action extends to a faithful *-representation of the C^*-algebraic direct sum $\mathscr{A} = \widehat{\bigoplus}_{i \in I} \mathscr{A}_i$ on $\mathfrak{H}$ as defined in Example 4.3.7, *vii.)*.

iii.) Identify $\mathscr{A}$ with its image in $\mathfrak{B}(\mathfrak{H})$ and show that $\mathscr{A}$ is a von Neumann algebra, called the direct sum von Neumann algebra of the $\{\mathscr{A}_i\}_{i \in I}$.

Exercise 7.5.2 (Type of a direct sum) Let $\{\mathscr{A}_i\}_{i \in I}$ be a family of von Neumann algebras acting on Hilbert spaces $\{\mathfrak{H}_i\}_{i \in I}$ and consider their von Neumann algebraic direct sum $\mathscr{A} = \widehat{\bigoplus}_{i \in I} \mathscr{A}_i$ acting on $\mathfrak{H} = \widehat{\bigoplus}_{i \in I} \mathfrak{H}_i$.

i.) Denote by $Z_i \in \mathfrak{B}(\mathfrak{H})$ the projection onto $\mathfrak{H}_i$ for all $i \in I$. Show that the $Z_i \in \mathscr{A}$ are central and pairwise orthogonal projections and $\mathbb{1} = \sum_{i \in I} Z_i$.

ii.) Let $P \in \mathscr{A}$ be a non-zero projection. Show that there is at least one $i \in I$ with $Z_i P \neq 0$.

iii.) Suppose each $\mathscr{A}_i$ is of type I (or type II, or type II_1, or type II_∞, or type III, respectively). Show that in this case $\mathscr{A}$ is also of type I (or type II, or type II_1, or type II_∞, or type III, respectively).

Hint: Use Lemma **??**.

Exercise 7.5.3 (The Grothendieck group)

Exercise: fini
purely infinit
projections a
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Appendix A

Notions from General Topology

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A.1 Topological Spaces and Continuous Maps

Proposition A.1.1 *Let $f: M \longrightarrow N$ be a continuous map between topological spaces with N being Hausdorff. Then* $\operatorname{graph}(f) \subseteq M \times N$ *is closed.*

Proof: We show that $M \times N \setminus \operatorname{graph}(f)$ is open: let $(p, q) \in M \times N \setminus \operatorname{graph}(f)$, i.e $f(p) \neq q$. Since N is Hausdorff there are two open subsets $U, V \subseteq N$ with $f(p) \in U$ and $q \in V$ but $U \cap V = \emptyset$. Since f is continuous, there is a neighbourhood $W \subseteq M$ of p with $f(W) \subseteq U$. Thus the neighbourhood $W \times V$ of (p, q) in $M \times N$ is actually contained in the complement of the graph. This implies the claim. □

Lemma A.1.2 (Homeomorphisms) *Let $(M, \mathcal{M})$ and $(N, \mathcal{N})$ be topological spaces and let $f: M \longrightarrow N$ be a bijection. Then the following statements are equivalent:*

i.) *f is a homeomorphism.*

ii.) *f is continuous and open.*

iii.) *f is continuous and closed.*

A.2 Properties of Topological Spaces

Lemma A.2.1 *Let $(M, \mathcal{M})$ and $(N, \mathcal{N})$ be topological spaces with $(N, \mathcal{N})$ being Hausdorff. Moreover, let $f, g: M \longrightarrow N$ be continuous maps.*

i.) *The coincidence set $\{x \in M \mid f(x) = g(x)\}$ is closed in M.*

ii.) *If $U \subseteq M$ is dense and $f\big|_U = g\big|_U$ then $f = g$.*

Theorem A.2.2 (Urysohn Lemma)

Theorem A.2.3 (Stone Weierstrass)

A.3 Constructions of Topological Spaces

Lemma A.3.1 *i.)*

If all the spaces $(M_i, \mathcal{M}_i)$ happen to be compact then their Cartesian product is compact as well. This is the famous Theorem of Tikhonov:

Theorem A.3.2 (Tikhonov) *Let $(M_i, \mathcal{M}_i)_{i \in I}$ be an arbitrary set of compact topological spaces. Then their Cartesian product is compact again.*

This slightly contra-intuitive statement (recall that the index set I can be very large) is in fact highly nontrivial as it uses in a crucial way the axiom of choice. In fact, it is equivalent to it.

Conversely, it is clear that if the Cartesian product $(M = \prod_{i \in I} M_i, \mathcal{M})$ with its product topology is compact, then each $(M_i, \mathcal{M}_i)$ has to be compact: M_i is the continuous image of the compact M under the continuous projection pr_i.

Lemma A.3.3 *i.)*

ii.)

Definition A.3.4 (Initial topology)

Lemma A.3.5

Definition A.3.6 (Final topology)

A.4 Convergence

Lemma A.4.1

A.5 Metric Spaces and Baire's Theorem

Theorem A.5.1 (Baire's Theorem)

Corollary A.5.2 (Uniform boundedness)

A.6 Locally Compact Groups

A.7 Exercises

Exercise A.7.1 (Cartesian product of continuous maps)

Appendix B

Vector-Valued Functions

The purpose of this appendix is to introduce and transfer notions from elementary calculus to the case of functions on $\mathbb{R}^n$ which take values in a reasonable topological vector space V. While most ideas are straightforward generalizations there are some subtle details which one has to take care of, making this appendix a good playground to elaborate on the general techniques as acquired in the course of Chapters 2, 3, and also 4. In exchange, this appendix will provide not only important tools needed in various constructions throughout the main text but it will also yield more sophisticated classes of examples for locally convex spaces and (mostly commutative) locally convex algebras of all sorts.

We start by investigating continuous and locally bounded functions on a reasonable topological space. Since we want to control locally uniform convergence by means of seminorms reasonable will mean to have a basis of neighbourhoods consisting of compact subsets. Thus we focus on vector-valued functions on locally compact Hausdorff spaces in Section B.1. In a next step we pass to functions defined on compact or open subsets in $\mathbb{R}^n$ and discuss integration of vector-valued functions in Section B.2. While the Lebesgue integral usually is considered to be superior to the Riemann integral, it will be the Riemann integral which provides a greater generality here: we only need a very mild assumption on the target space V to make the theory work. Sequential completeness will be sufficient for continuous functions to be Riemann integrable. Within this framework we develop the basic properties of the Riemann integral with particular emphasize to continuity properties. In a next step, differentiability and smoothness of vector-valued functions is discussed in detail in the Sections B.3 and B.4. Here we meet several important classes of test function spaces, now allowing for values in a sequentially complete Hausdorff locally convex space. A first non-trivial application is provided by the classical Borel Lemma which we formulate for the case when the target space is a Fréchet space in Section B.5. In Section B.6 we consider holomorphic functions with values in V. Here the main result is that several reasonable definitions, of what holomorphic should be, actually coincide: in particular, weakly holomorphic functions are holomorphic. In the last Section B.7 we give yet another, more technical, application to symbol spaces and oscillatory integrals for vector-valued functions. Application of these oscillatory integrals are manifold and will allow to produce many interesting examples of noncommutative locally convex algebras from commutative ones.

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B.1 Continuous and Locally Bounded Functions

In the following we consider a Hausdorff locally convex space V which we require to be either complete or, less restrictive, sequentially complete. For many applications we will be interested in, V is even a Banach space. The domain of the functions we consider will be either an open or compact subset $U \subseteq \mathbb{R}^n$ or, more generally, a locally compact Hausdorff space X.

B.1.1 First Properties of Continuous and Locally Bounded Functions

We start wit the following standard definition of continuous functions with values in a locally convex space V:

Definition B.1.1 (Continuous functions) *Let X be a locally compact Hausdorff space. Then the space of continuous functions from X to V is denoted by $\mathscr{C}^0(X, V)$ or simply by $\mathscr{C}(X, V)$.*

Clearly, $\mathscr{C}(X, V)$ is a vector space by the pointwise operations.

Since the composition of continuous maps is again continuous, the map $x \mapsto \mathrm{q}(f(x))$ is continuous for every continuous seminorm q on V and every $f \in \mathscr{C}(X, V)$. This allows to define

$$\mathrm{p}_{K,0,\mathrm{q}}(f) = \sup_{x \in K} \mathrm{q}(f(x)) = \max_{x \in K} \mathrm{q}(f(x)) \tag{B.1.1}$$

for a compact subset $K \subseteq X$. Clearly, it gives us a seminorm on $\mathscr{C}(X, V)$. The reason for this notation will become clear in Section B.4. Alternatively, we shall also use the notation

$$\mathrm{q}_K(f) = \mathrm{p}_{K,0,\mathrm{q}}(f). \tag{B.1.2}$$

Finally, recall that in the scalar case we denote the supremum seminorm over a compact subset $K \subseteq X$ simply by $\mathrm{p}_{K,0}$ or by $\|\cdot\|_K$ as in Exercise 2.5.29.

Definition B.1.2 ($\mathscr{C}$-Topology) *Let X be a locally compact Hausdorff space and let V be a Hausdorff locally convex space. The $\mathscr{C}$-topology (or $\mathscr{C}^0$-topology) on $\mathscr{C}(X, V)$ is the locally convex topology determined by all the seminorms $\mathrm{p}_{K,0,\mathrm{q}}$ with $K \subseteq X$ compact and q a continuous seminorm on V.*

We will also need another space of functions on X with values in V. In view of the definition of the seminorm in (B.1.1) it is interesting to consider all those functions for which (B.1.1) is finite, not just the continuous ones. These functions will be called the locally bounded ones:

Definition B.1.3 (Locally bounded functions) *Let X be a locally compact Hausdorff space. A function $f\colon X \longrightarrow V$ is called locally bounded if for all compact subsets $K \subseteq X$ and all continuous seminorms q on V we have $\mathrm{p}_{K,0,\mathrm{q}}(f) < \infty$. The space of all locally bounded functions is denoted by $\mathscr{B}_{\mathrm{loc}}(X, V)$. The locally convex topology on it determined by all the seminorms $\mathrm{p}_{K,0,\mathrm{q}}$ is called the $\mathscr{B}_{\mathrm{loc}}$-topology.*

More generally, one can define the locally bounded functions on an arbitrary topological space as those functions for which for every point there is neighbourhood of this point on which the function is bounded. Since for a locally compact Hausdorff space we have a compact neighbourhood basis, the two definitions agree in this case, see also Exercise B.8.1. In all relevant applications we encounter only locally compact Hausdorff spaces later on. Moreover, note that $\mathscr{C}(X, V) \subseteq \mathscr{B}_{\mathrm{loc}}(X, V)$ and the induced topology on $\mathscr{C}(X, V)$ is just the $\mathscr{C}$-topology.

One main feature of vector-valued functions is that they can be multiplied by scalar functions. This is a continuous operation in the following sense:

Proposition B.1.4 *The vector-valued locally bounded functions $\mathscr{B}_{\mathrm{loc}}(X, V)$ are a topological module over the locally multiplicatively convex algebra of scalar locally bounded functions $\mathscr{B}_{\mathrm{loc}}(X) = \mathscr{B}_{\mathrm{loc}}(X, \mathbb{C})$. More precisely, for every continuous seminorm q on V and every compact subset $K \subseteq X$ we have*

$$\mathrm{p}_{K,0,\mathrm{q}}(fg) \le \|f\|_K \mathrm{p}_{K,0,\mathrm{q}}(g) \tag{B.1.3}$$

for every $f \in \mathscr{B}_{\mathrm{loc}}(X)$ and $g \in \mathscr{B}_{\mathrm{loc}}(X, V)$. Also, $\mathscr{C}(X, V)$ is a module over $\mathscr{C}(X) = \mathscr{C}(X, \mathbb{C})$.

Proof: Let $f \in \mathscr{B}_{\mathrm{loc}}(X)$ and $g \in \mathscr{B}_{\mathrm{loc}}(X, V)$ be given. Then we have the estimate

$$\sup_{x \in K} \mathrm{q}(f(x)g(x)) = \sup_{x \in K} |f(x)| \mathrm{q}(g(x)) \leq \sup_{x \in K} |f(x)| \sup_{x \in K} \mathrm{q}(g(x)),$$

which gives (B.1.3) right away. But this implies immediately that the module multiplication yields again a locally bounded vector-valued function. Moreover, viewed as a bilinear map

$$\mathscr{B}_{\mathrm{loc}}(X) \times \mathscr{B}_{\mathrm{loc}}(X, V) \longrightarrow \mathscr{B}_{\mathrm{loc}}(X, V)$$

it is continuous. Taking $V = \mathbb{C}$ we see that $\mathscr{B}_{\mathrm{loc}}(X)$ is a locally multiplicatively convex algebra. If in addition the functions f and g are continuous, then the continuity of fg is clear by the continuity of the vector space operations in V. Thus the second statement follows at once. □

Remark B.1.5 Note that the seminorms $\mathrm{p}_{K,0}$ on $\mathscr{B}_{\mathrm{loc}}(X)$ satisfy the C^*-property, see also Exercise B.8.3. A simple argument shows that the scalar locally bounded functions $\mathscr{B}_{\mathrm{loc}}(X)$ are indeed complete with respect to this topology. Hence $\mathscr{B}_{\mathrm{loc}}(X)$ is a pro-C^* algebra. The completeness of $\mathscr{B}_{\mathrm{loc}}(X)$ also follows from the next theorem which deals with a much more general situation.

Theorem B.1.6 (The spaces $\mathscr{B}_{\mathrm{loc}}(X, V)$ and $\mathscr{C}(X, V)$) *Let X be a locally compact Hausdorff space and let V be a Hausdorff (sequentially) complete locally convex space. Then $\mathscr{B}_{\mathrm{loc}}(X, V)$ is a Hausdorff (sequentially) complete locally convex space with respect to the $\mathscr{B}_{\mathrm{loc}}$-topology and $\mathscr{C}(X, V)$ is a closed subspace, hence (sequentially) complete itself.*

Proof: The Hausdorff property is clear since for $f \neq 0$ we have at least one $x \in X$ with $f(x) \neq 0$ and hence there is at least one continuous seminorm q with $\mathrm{q}(f(x)) > 0$ since we assume V to be Hausdorff. Then $\mathrm{p}_{\{x\},0,\mathrm{q}}(f) = \mathrm{q}(f(x)) > 0$. This also shows that the $\mathscr{B}_{\mathrm{loc}}$-topology is finer than the topology of pointwise convergence. To show (sequential) completeness, let $(f_i)_{i \in I}$ be a Cauchy net (Cauchy sequence) in $\mathscr{B}_{\mathrm{loc}}(X, V)$ with respect to the $\mathscr{B}_{\mathrm{loc}}$-topology. Then for each $x \in X$ the net (sequence) $(f_i(x))_{i \in I}$ is a Cauchy net (Cauchy sequence) in V and hence convergent to a unique limit which we denote by $f(x) = \lim_{i \in I} f_i(x)$. This defines a candidate f for the limit of $(f_i)_{i \in I}$. Let now $\epsilon > 0$. We know that for all compact subsets $K \subseteq X$ and all continuous seminorms q on V we have $\mathrm{p}_{K,0,\mathrm{q}}(f_i - f_j) < \epsilon$ for $i, j \in I$ sufficiently late in I. This means $\sup_{x \in K} \mathrm{q}(f_i(x) - f_j(x)) < \epsilon$. Taking now the limit over i of this inequality gives by continuity of q

$$\mathrm{q}(f(x) - f_j(x)) = \lim_{i \in I} \mathrm{q}(f_i(x) - f_j(x)) \leq \epsilon,$$

valid for all $x \in K$. Thus $\mathrm{p}_{K,0,\mathrm{q}}(f - f_j) \leq \epsilon$ follows implying $\mathrm{p}_{K,0,\mathrm{q}}(f) \leq \mathrm{p}_{K,0,\mathrm{q}}(f - f_j) + \mathrm{p}_{K,0,\mathrm{q}}(f_j) < \infty$. First, this shows $f \in \mathscr{B}_{\mathrm{loc}}(X, V)$. Knowing this, it is clear that $f_j \longrightarrow f$ in the $\mathscr{B}_{\mathrm{loc}}$-topology. This shows (sequential) completeness of $\mathscr{B}_{\mathrm{loc}}(X, V)$. To show that $\mathscr{C}(X, V)$ is a closed subspace we assume that $f_i \longrightarrow f$ is a convergent net in $\mathscr{B}_{\mathrm{loc}}(X, V)$ with $f_i \in \mathscr{C}(X, V)$. Then we have to show $f \in \mathscr{C}(X, V)$. But this can be done with the usual $\frac{\epsilon}{3}$-argument: let $\epsilon > 0$ and fix a continuous seminorm q on V. For a fixed $x \in X$ we choose an open neighbourhood of x with compact closure K whose existence is guaranteed by the local compactness of X. By convergence, we choose $i \in I$ such that

$$\sup_{y \in K} \mathrm{q}(f(y) - f_i(y)) = \mathrm{p}_{K,0,\mathrm{q}}(f - f_i) < \frac{\epsilon}{3}. \tag{$*$}$$

Since f_i is continuous we find a smaller open neighbourhood $O \subseteq K$ of x with $\mathrm{q}(f_i(x) - f_i(y)) < \frac{\epsilon}{3}$ for all $y \in O$. Thus we have

$$\mathrm{q}(f(x) - f(y)) \leq \mathrm{q}(f(x) - f_i(x)) + \mathrm{q}(f_i(x) - f_i(y)) + \mathrm{q}(f_i(y) - f(y)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

since for the first and the third term we can apply ($*$) and for the middle term we use the continuity of f_i. This shows that f is continuous at $x \in X$ and hence $f \in \mathscr{C}(X, V)$ follows. □

Remark B.1.7 (Locally bounded functions) Let V be a locally convex space.

i.) The proof actually shows that $\mathscr{C}(X, V) \subseteq \mathscr{B}_{\mathrm{loc}}(X, V)$ is always a closed subspace, though of course in general only (sequentially) complete if V is (sequentially) complete.

ii.) It will be sufficient to consider only those seminorms which arise from an exhausting set of compact subsets and a defining system of seminorms for V, see Exercise B.8.1, *v.)*.

iii.) Assume $U \subseteq \mathbb{R}^n$ is open and V is even a Fréchet space. Then also $\mathscr{C}(U, V)$ and $\mathscr{B}_{\mathrm{loc}}(U, V)$ are Fréchet spaces as countably many compact subsets can exhaust U. However, if V is a Banach space, $\mathscr{C}(U, V)$ and $\mathscr{B}_{\mathrm{loc}}(U, V)$ will still only be Fréchet spaces but not Banach spaces in general. Again, we can replace U by a σ-compact locally compact space, see Exercise **??**,

iv.) LocallyBoundedFrechet.

v.) Clearly, for $V = \mathbb{C}$ we are back to the usual characterizations of locally bounded and continuous functions.

B.1.2 Linear Maps

As application and for later use we consider the space of all linear maps from $\mathbb{R}^n$ to V.

Lemma B.1.8 *A linear map $f\colon \mathbb{R}^n \longrightarrow V$ is continuous and*

$$\mathrm{L}(\mathbb{R}^n, V) = \mathrm{Hom}(\mathbb{R}^n, V) \subseteq \mathscr{C}(\mathbb{R}^n, V) \tag{B.1.4}$$

is a closed subspace.

PROOF: Here we need that $\mathbb{R}^n$ is finite-dimensional in an essential way. Let $\mathrm{e}_1, \ldots, \mathrm{e}_n \in \mathbb{R}^n$ be the standard basis. Then $f(x) = \sum_{i=1}^n x^i f(\mathrm{e}_i)$ with fixed vectors $f(\mathrm{e}_i) \in V$ and the coordinate functions $x \mapsto x^i$. Since these coordinate functions are continuous scalar functions and since the constant functions $f(\mathrm{e}_i)$ are continuous as well, the continuity of f follows at once from Proposition B.1.4. The closedness of $\mathrm{Hom}(\mathbb{R}^n, V)$ is now clear since the linearity is certainly preserved already under pointwise limits. $\square$

Since $\mathrm{Hom}(\mathbb{R}^n, V)$ is closed it is a Hausdorff locally convex space by its own with respect to the seminorms $\mathrm{p}_{K,0,\mathrm{q}}$ which is (sequentially) complete whenever V is (sequentially) complete. Explicitly, for a linear map $f \in \mathrm{Hom}(\mathbb{R}^n, V)$ we have for $K \subseteq \mathrm{B}_r(c)^{\mathrm{cl}}$ and a continuous seminorm q

$$\mathrm{p}_{K,0,\mathrm{q}}(f) \leq r \max_{i=1,\ldots,n} \mathrm{q}(f(\mathrm{e}_i)). \tag{B.1.5}$$

This suggests to consider the seminorms

$$\|f\|_{\mathrm{q},\infty} = \max_{i=1,\ldots,n} \mathrm{q}(f(\mathrm{e}_i)) \tag{B.1.6}$$

on $\mathrm{Hom}(\mathbb{R}^n, V)$ as well. Yet another system of seminorms would be

$$\|f\|_{\mathrm{q},\ell}(f) = \sqrt[\ell]{\sum\nolimits_{i=1}^n \mathrm{q}(f(\mathrm{e}_i))^\ell} \tag{B.1.7}$$

for $\ell \geq 1$. Since the maximum in (B.1.6) is taken over a finite set of indices as well as the summation in (B.1.7) is finite, it is easy to see that the two seminorms are in fact equivalent. Finally, there is an "operator norm" like seminorm defined by

$$\|f\|_{\mathrm{q}} = \sup_{x \neq 0} \frac{\mathrm{q}(f(x))}{\|x\|}, \tag{B.1.8}$$

where we use some auxiliary norm $\|\cdot\|$ on $\mathbb{R}^n$. It is now an easy check that all these systems of seminorms lead to the same topology:

Lemma B.1.9 *Let $\ell \geq 1$. The locally convex topologies on* $\operatorname{Hom}(\mathbb{R}^n, V)$ *induced by the systems of seminorms* $\{\|\cdot\|_{,\infty}\}$, $\{\|\cdot\|_{\mathrm{q},\ell}\}$, *and* $\{\|\cdot\|_{\mathrm{q}}\}$, *where q runs through the continuous seminorms on V, all coincide with the $\mathscr{C}$-topology inherited from $\mathscr{C}(\mathbb{R}^n, V)$.*

The proof relies of course crucially on the fact that $\mathbb{R}^n$ is finite-dimensional, see Exercise B.8.2. To characterize the $\mathscr{C}$-topology on $\operatorname{Hom}(\mathbb{R}^n, V)$ it will sometimes be convenient to pass to another system of seminorms than the usual one from (B.1.1).

B.2 Vector-Valued Riemann Integration

In this section we will outline a method of integration for vector-valued functions. Since in many applications the target space V is only *sequentially* complete but not complete, like e.g. duals of Fréchet spaces with the weak* topology, we need a theory working with sequentially complete spaces. On the other hand, we are focusing here on functions defined either on the real line or slightly more general, on nice subsets of $\mathbb{R}^n$. Moreover, the functions we want to integrate will typically be at least continuous. This will simplify things drastically compared to integration theory on general measure spaces with values in locally convex spaces. It turns out that the Riemann integral will be an appropriate choice for the time being. In fact, it will even have certain advantages compared to Lebesgue-like integrals. Nevertheless, with some more effort one can extend the notion of Lebesgue's integration theory also to functions with values in not too bad locally convex space, see e.g. [3] for a recent account on this as well as the classical approach of Bochner and Gel'fand-Pettis [49, ???].

B.2.1 Riemann Integrable Functions

We consider again a Hausdorff locally convex space V as target. The maps we want to integrate will be defined on an n-dimensional compact interval with non-empty open interior

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n] \tag{B.2.1}$$

in $\mathbb{R}^n$ where $a_1 < b_1$, …, $a_n < b_n$. With some slight abuse of notation we shall refer to I just as interval. For such an interval I we have the usual Euclidean volume denoted by

$$\operatorname{vol}(I) = \prod_{k=1}^{n} (b_k - a_k). \tag{B.2.2}$$

On such a compact interval, $\mathscr{B}_{\mathrm{loc}}(I, V)$ coincides with the bounded functions $\mathscr{B}(I, V)$. This will simplify things and the seminorms used for $\mathscr{B}_{\mathrm{loc}}(I, V) = \mathscr{B}(I, V)$ reduce to the seminorms

$$\mathrm{q}_\infty(f) = \mathrm{q}_I(f) = \sup_{x \in I} \mathrm{q}(f(x)) = \mathrm{p}_{I,0,\mathrm{q}}(f) \tag{B.2.3}$$

for $f \in \mathscr{B}(I, V)$. By the compactness of I we observe that a continuous map $f\colon I \longrightarrow V$ is necessarily uniformly continuous as this is known from the scalar situation:

Lemma B.2.1 *Let $f \in \mathscr{C}(I, V)$. Then f is uniformly continuous, i.e. for every continuous seminorm* q *on V and every $\epsilon > 0$ we have a $\delta > 0$ such that $\|x - x'\| < \delta$ implies* $\mathrm{q}(f(x) - f(x')) < \epsilon$.

PROOF: The proof is completely analogous to the scalar case and relies on the sequential compactness of I. Suppose f is not uniformly continuous. Then we have a seminorm q and some $\epsilon > 0$ such that for all $\delta > 0$ we have points $x, x' \in I$ with $\|x - x'\| < \delta$ but $\mathrm{q}(f(x) - f(x')) \geq \epsilon$. Choose such points x_n, x'_n with $\|x_n - x'_n\| < \frac{1}{n}$. By (sequential) compactness of I we find subsequences x_{n_k} and x'_{n_k} which converge to some x, x', respectively. Then $f(x_{n_k}) \longrightarrow f(x)$ and $f(x'_{n_k}) \longrightarrow f(x')$ by continuity. But $\|x_{n_k} - x'_{n_k}\| < \frac{1}{n_k}$ implies that they converge towards the same limit $x = x'$. Hence $f(x) = f(x')$ and the convergence contradicts $\mathrm{q}(f(x_{n_k}) - f(x'_{n_k})) \geq \epsilon$ for all n_k. ◻

Clearly, the same argument works e.g. for every compact metric space instead of the compact interval I as for metric spaces all notions of compactness coincide.

We construct now the Riemann integral essentially as known from the scalar situation. Recall that a *partition* of I consists of finitely many compact subintervals $\mathcal{I} = \{I_1, \ldots, I_N\}$ of I with $I_j^\circ \cap I_{j'}^\circ = \emptyset$ for $j \neq j'$ and

$$I = \bigcup_{j=1}^{N} I_j. \tag{B.2.4}$$

The maximum of the lengths of the edges of the I_j will be denoted by

$$\Delta\mathcal{I} = \max_{j=1}^{N} \max_{k=1}^{n} (b_k^j - a_k^j), \tag{B.2.5}$$

where $I_j = [a_1^j, b_1^j] \times \cdots \times [a_n^j, b_n^j]$. For such a partition $\mathcal{I}$ of I we choose points $\xi_1 \in I_1, \ldots, \xi_N \in I_N$ the collection of which we denote by $\Xi = \{\xi_1, \ldots, \xi_N\}$. Then the *Riemann sum* corresponding to the partition $\mathcal{I}$ and the points Ξ is defined by

$$\textstyle\sum_{\mathcal{I},\Xi}(f) = \displaystyle\sum_{j=1}^{N} f(\xi_j)\mathrm{vol}(I_j). \tag{B.2.6}$$

The idea is now that the Riemann integral is the limit of the Riemann sums for finer and finer partitions. To make this precise we interpret this as a net limit. Indeed, the set of all pairs $(\mathcal{I}, \Xi)$ of partitions $\mathcal{I}$ of I and corresponding points Ξ is ordered by inclusion: we say $(\mathcal{I}, \Xi) \preccurlyeq (\mathcal{I}', \Xi')$ if every subinterval I_j' of $\mathcal{I}'$ is contained in some (necessarily unique) subinterval I_j of $\mathcal{I}$. Thus this direction encodes that the intervals should become smaller and smaller. It is clear that this makes the set of all such pairs a directed set. In particular, two given partitions clearly have a common refinement. The remaining properties of a directed set are trivial.

Definition B.2.2 (Riemann integral) *Let V be a Hausdorff locally convex space and let $f\colon I \longrightarrow V$ be a map. Then f is called Riemann integrable if the net limit of its Riemann sums*

$$\int_I f(x)\,\mathrm{d}^n x = \lim_{(\mathcal{I},\Xi)} \textstyle\sum_{(\mathcal{I},\Xi)}(f) \tag{B.2.7}$$

exists. In this case the limit is called the Riemann integral of f over I. The set of Riemann integrable functions on I is denoted by $\mathscr{R}(I, V)$.

Clearly, it is reasonable to restrict to *Hausdorff* locally convex spaces as we want to assign a unique value to the integral of f over I if it exists. The following simple proposition collects a few basic properties of the Riemann integral which follow directly from the definition:

Proposition B.2.3 *Let $I \subseteq \mathbb{R}^n$ be a compact interval and let V and W be Hausdorff locally convex spaces.*

i.) The constant functions are Riemann integrable and for all $v \in V$ we have

$$\int_I v\,\mathrm{d}^n x = v\mathrm{vol}(I). \tag{B.2.8}$$

ii.) The Riemann integrable functions $\mathscr{R}(I, V)$ form a vector space.

iii.) Let $A\colon V \longrightarrow W$ be a continuous linear map. Then $A \circ f \in \mathscr{R}(I, W)$ for all $f \in \mathscr{R}(I, V)$ and

$$A \int_I f(x)\,\mathrm{d}^n x = \int_I (A \circ f)(x)\,\mathrm{d}^n x. \tag{B.2.9}$$

PROOF: For the first part it is clear that for every Riemann sum of the constant function $f(x) = v$ we have

$$\textstyle\sum_{(\mathcal{J},\Xi)}(f) = \sum_{j=1}^{N} f(\xi_j)\operatorname{vol}(I_j) = \sum_{j=1}^{N} v\operatorname{vol}(I_j) = v\operatorname{vol}(I),$$

implying that the limit exists and is given by (B.2.8). Since the vector space operations are continuous and since the Riemann *sums* depend linearly of f the second part is clear as well. Since A is continuous and linear we have

$$A\int_I f(x)\,\mathrm{d}^n x = A\lim_{(\mathcal{I},\Xi)}\sum_{j=1}^{N} f(\xi_j)\operatorname{vol}(I_j) = \lim_{(\mathcal{I},\Xi)} A\sum_{j=1}^{N} f(\xi_j)\operatorname{vol}(I_j) = \lim_{(\mathcal{I},\Xi)}\sum_{j=1}^{N}(A\circ f)(\xi_j)\operatorname{vol}(I_j),$$

with all limits existing. But the last is precisely the Riemann integral of $A \circ f$. ☐

Corollary B.2.4 *Let $f \in \mathscr{R}(I, V)$. Then for every continuous linear functional $\varphi \in V'$ the function $\varphi \circ f\colon I \longrightarrow \mathbb{C}$ is Riemann integrable and, in particular, $\varphi \circ f \in \mathscr{B}(I)$.*

PROOF: Indeed, the third part of the proposition gives $\varphi\circ f \in \mathscr{R}(I, \mathbb{C})$. For scalar functions, Riemann integrability implies boundedness, see Exercise B.8.4. ☐

In other words, Riemann integrable functions are weakly bounded, i.e. with respect to the weak topology. In general, weakly bounded subsets of V are also bounded with respect to the original topology according to Theorem 2.4.20. Hence the above corollary implies that a Riemann integrable function is *bounded* and we have

$$\mathscr{R}(I, V) \subseteq \mathscr{B}(I, V), \tag{B.2.10}$$

see Exercise B.8.4. However, we shall not need this fact in the following.

B.2.2 Riemann Integral of Continuous Functions

In general, not much can be said about the class of Riemann integrable functions due to the possible lack of completeness of the target space V. However, if V is at least sequentially complete then continuous functions are Riemann integrable.

Proposition B.2.5 *If V is sequentially complete then $\mathscr{C}(I, V) \subseteq \mathscr{R}(I, V)$.*

PROOF: We have to show that the net (!) of Riemann sums for a continuous function $f\colon I \longrightarrow V$ converges. To this end, we first prove that the Riemann sums are a Cauchy net. Thus let a continuous seminorm q of V as well as $\epsilon > 0$ be given. By Lemma B.2.1 we find a $\delta > 0$ such that $\|x - x'\|_\infty < \delta$ for $x, x' \in I$ implies $\mathrm{q}(f(x) - f(x')) < \epsilon$. Now let $\mathcal{I} = \{I_1, \ldots, I_N\}$ be a partition of I such that $\Delta\mathcal{I} < \delta$. If $\mathcal{I}'$ is any refinement of $\mathcal{I}$, i.e. $\mathcal{I} \preccurlyeq \mathcal{I}'$, then we have for a subinterval I_j from $\mathcal{I}$ unique $I'_{k_1}, \ldots, I'_{k_r} \in \mathcal{I}'$ such that $I_j = I'_{k_1} \cup \cdots \cup I'_{k_r}$. For arbitrary choices Ξ and Ξ' we get

$$\begin{aligned}\mathrm{q}\left(f(\xi_j)\operatorname{vol}(I_j) - \sum_{i=1}^{r} f(\xi'_{k_i})\operatorname{vol}(I'_{k_i})\right) &= \mathrm{q}\left(\sum_{i=1}^{r}\big(f(\xi_j) - f(\xi'_{k_i})\big)\operatorname{vol}(I'_{k_i})\right)\\ &\le \sum_{i=1}^{r}\mathrm{q}\big(f(\xi_j) - f(\xi'_{k_i})\big)\operatorname{vol}(I'_{k_i})\\ &< \epsilon,\end{aligned}$$

since all the points $\xi_j, \xi_{k_1}, \ldots, \xi_{k_r}$ are in I_j and hence have distances bounded by δ. Repeating this for every j and taking the sum over all j we get the important statement

$$\mathrm{q}\Big(\textstyle\sum_{\mathcal{I},\Xi}(f) - \sum_{\mathcal{I}',\Xi'}(f)\Big) < \epsilon\operatorname{vol}(I), \tag{$*$}$$

since in the sum over all j also all subintervals I'_k occur *once*. Now if we fix some (clearly existing) partition $\mathcal{I}_0$ with $\Delta\mathcal{I}_0 < \delta$. Then for all partitions $\mathcal{I}, \mathcal{I}' \succcurlyeq \mathcal{I}_0$ we have

$$\mathrm{q}\Big(\sum\nolimits_{\mathcal{I},\Xi}(f) - \sum\nolimits_{\mathcal{I}',\Xi'}(f)\Big) \leq \mathrm{q}\Big(\sum\nolimits_{\mathcal{I},\Xi}(f) - \sum\nolimits_{\mathcal{I}_0,\Xi_0}(f)\Big) + \mathrm{q}\Big(\sum\nolimits_{\mathcal{I}_0,\Xi_0}(f) - \sum\nolimits_{\mathcal{I}',\Xi'}(f)\Big) < 2\epsilon\operatorname{vol}(I), \quad (**)$$

by applying $(*)$ twice. This shows that the net of Riemann sums is a Cauchy net. If V were complete, we would have shown the claim. If V is only sequentially complete we have to be slightly more careful here: The crucial point is that we can choose a *sequence* of partitions $\mathcal{I}_m$ by dividing each one-dimensional interval $[a_1, b_1], \ldots, [a_n, b_n]$ into 2^m subintervals of equal length and take all their Cartesian products to obtain a partition of $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Then $\mathcal{I}_m$ consists of 2^{mn} small subintervals with equal volume. By this construction, $\mathcal{I}_{m+1}$ is a refinement of $\mathcal{I}_m$ for all $m \geq 1$ and $\Delta\mathcal{I}_m \longrightarrow 0$. Thus there is a m_0 with $\Delta\mathcal{I}_{m_0} < \delta$ and we can apply $(**)$ for the $\mathcal{I}_m$ to see that their Riemann sums provides a Cauchy *sequence*. If V is sequentially complete this gives us a limit $v = \lim_m \sum_{\mathcal{I}_m,\Xi_m}(f)$. We claim that this v is also the limit of the whole Cauchy net. Thus let m_1 be such that $\mathrm{q}(\sum_{\mathcal{I}_{m_1},\Xi_{m_1}}(f) - v) < \epsilon$ and such that $\Delta\mathcal{I}_{m_1} < \delta$. If $\mathcal{I}$ is an arbitrary partition with $\mathcal{I} \succcurlyeq \mathcal{I}_{m_1}$ then we can apply $(*)$ to conclude that

$$\mathrm{q}\Big(\sum\nolimits_{\mathcal{I},\Xi}(f) - v\Big) \leq \mathrm{q}\Big(\sum\nolimits_{\mathcal{I},\Xi}(f) - \sum\nolimits_{\mathcal{I}_{m_1},\Xi_{m_1}}(f)\Big) + \mathrm{q}\Big(\sum\nolimits_{\mathcal{I}_{m_1},\Xi_{m_1}}(f) - v\Big) < \epsilon\operatorname{vol}(I) + \epsilon.$$

This shows that for all such partitions their Riemann sum is close to v proving the convergence in the sense of a net. In Exercise B.8.5 this last part of the proof is put into a larger perspective. ☐

We focus now on the case where f is continuous and V is sequentially complete. First we will establish some continuity properties of the integral:

Proposition B.2.6 *Let $f \in \mathscr{C}(I, V)$ with a sequentially complete locally convex space V and let q be a continuous seminorm on V.*

i.) One has

$$\mathrm{q}\left(\int_I f(x)\,\mathrm{d}^n x\right) \leq \int_I (\mathrm{q} \circ f)(x)\,\mathrm{d}^n x \leq \operatorname{vol}(I)\mathrm{q}_I(f). \tag{B.2.11}$$

ii.) The integral is a continuous linear functional

$$\int_I \cdot\ \mathrm{d}^n x \colon \mathscr{C}(I, V) \longrightarrow \mathbb{C}. \tag{B.2.12}$$

Proof: Clearly, for every Riemann sum we have

$$\mathrm{q}\left(\sum_{j=1}^N f(\xi_j)\operatorname{vol}(I_j)\right) \leq \sum_{j=1}^N (\mathrm{q} \circ f)(\xi_j)\operatorname{vol}(I_j).$$

Since the right hand side is precisely the Riemann sum of the (continuous) function $\mathrm{q} \circ F$ corresponding to the same partition and the same choices of points, we can take the existing limits on both sides as q is a continuous seminorm. This gives the first estimate in (B.2.11). The second estimate is then clear from the scalar Riemann integral theory. It also gives the second part at once. ☐

The continuity statement is in fact very useful as it allows to exchange integration and limits with respect to the $\mathscr{C}$-topology. We will frequently use that type of exchanging of limits.

B.2.3 Fubini's Theorem for Riemann Integrals

Next we discuss whether we can exchange the orders of integration. To formulate this version of Fubini's theorem we need the following preparatory lemma which is also of independent interest:

Lemma B.2.7 *Let $U = U_1 \times U_2 \subseteq \mathbb{R}^{n_1+n_2}$ with open subsets $U_k \subseteq \mathbb{R}^{n_k}$ for $k = 1, 2$. Let $f \in \mathscr{C}(U, V)$. For every convergent sequence $x_1^j \longrightarrow x_1$ in U_1 the restrictions $f(x_1^j, \cdot\,) \in \mathscr{C}(U_2, V)$ converge to $f(x_1, \cdot\,)$ in the $\mathscr{C}$-topology.*

PROOF: First we note that the sequence with its limit constitute a compact subset $K_1 \subseteq U_1$. Now let $K_2 \subseteq U_2$ be a compact subset with respect to the second variables. Then also $K = K_1 \times K_2 \subseteq U$ is compact. On K the function f is *uniformly* continuous according to Lemma B.2.1: for a continuous seminorm q on V and $\epsilon > 0$ we find a $\delta > 0$ such that $\|x - x'\|_\infty < \delta$ implies $\mathrm{q}(f(x) - f(x')) < \epsilon$ for all $x, x' \in K$. We apply this to $x = (x_1^j, x_2)$ and $x' = (x_1, x_2)$ where $x_2 \in K_2$. For $j \geq j_0$ with j_0 large enough we get $\|x - x'\|_\infty < \delta$ and hence $\mathrm{q}\Big(f(x_1^j, x_2) - f(x_1, x_2)\Big) < \epsilon$ for all $x_2 \in K_2$. Taking the supremum over $x_2 \in K_2$ gives

$$\mathrm{p}_{K_2,0,\mathrm{q}}\Big(f(x_1^j, \cdot\,) - f(x_1, \cdot\,)\Big) \leq \epsilon$$

for $j \geq j_0$. But these seminorms fix the $\mathscr{C}$-topology of $\mathscr{C}(U_2, V)$ thereby finishing the proof. □

Proposition B.2.8 (Fubini) *Let V be sequentially complete and $I = I_1 \times I_2 \subseteq \mathbb{R}^{n_1+n_2}$ with compact intervals $I_1 \subseteq \mathbb{R}^{n_1}$ and $I_2 \subseteq \mathbb{R}^{n_2}$. For all $f \in \mathscr{C}(I, V)$ one has:*

i.) The functions

$$x_1 \mapsto \int_{I_2} f(x_1, x_2)\, \mathrm{d}^{n_2} x_2 \quad \text{and} \quad x_2 \mapsto \int_{I_1} f(x_1, x_2)\, \mathrm{d}^{n_1} x_1 \tag{B.2.13}$$

are continuous functions on I_1 and I_2, respectively.

ii.) The integrals yield continuous linear maps

$$\int_{I_2} \cdot\, \mathrm{d}^{n_2} x_2 \colon \mathscr{C}(I, V) \longrightarrow \mathscr{C}(I_1, V) \quad \text{and} \quad \int_{I_1} \cdot\, \mathrm{d}^{n_1} x_1 \colon \mathscr{C}(I, V) \longrightarrow \mathscr{C}(I_2, V). \tag{B.2.14}$$

iii.) One has

$$\int_I f(x)\, \mathrm{d}^{n_1+n_2} x = \int_{I_1} \left(\int_{I_2} f(x_1, x_2)\, \mathrm{d}^{n_2} x_2 \right) \mathrm{d}^{n_1} x_1 = \int_{I_2} \left(\int_{I_1} f(x_1, x_2)\, \mathrm{d}^{n_1} x_1 \right) \mathrm{d}^{n_2} x_2. \tag{B.2.15}$$

PROOF: First we note that for fixed x_1 the function $f(x_1, x_2)$ is continuous in x_2 and hence Riemann integrable by Proposition B.2.5. Thus the first function in (B.2.13) is well-defined. By symmetry the same holds for the other one. To check continuity we consider a convergent sequence $x_1^j \longrightarrow x_1$ in I_1. By Lemma B.2.7 the restricted functions $f(x_1^j, \cdot\,)$ converge in the $\mathscr{C}$-topology to $f(x_1, \cdot\,)$. By the continuity of the integral $\int_{I_2} \mathrm{d}^{n_2} x_2$ according to Proposition B.2.6 we are allowed to exchange the limit with the integral. Hence $\int_{I_2} f(x_1^j, x_2)\, \mathrm{d}^{n_2} x_2$ converges to $\int_{I_2} f(x_1, x_2)\, \mathrm{d}^{n_2} x_2$ which establishes the continuity of the first function in (B.2.13). By symmetry, the same holds for the second as well, proving the first part. For the second part, let $\mathcal{I}_2$ be a partition of I_2 with a choice of corresponding points Ξ_2. Then for the corresponding Riemann sum and a continuous seminorm q on V we have

$$\mathrm{q}\left(\sum_{j=1}^{N} f(x_1, \xi_2^j) \mathrm{vol}(I_2^j) \right) \leq \sum_{j=1}^{N} \mathrm{q}\Big(f(x_1, \xi_2^j)\Big) \mathrm{vol}(I_2^j).$$

Taking the limit over all Riemann sums preserves this inequality. The left hand side converges, by continuity of q, to $\mathrm{q}\Big(\int_{I_2} f(x_1, x_2)\, \mathrm{d}^{n_2}x_2\Big)$. The right hand side converges to $\int_{I_2}(\mathrm{q} \circ f)(x_1, x_2)\, \mathrm{d}^{n_2}x_2$ which we can estimate further by $\mathrm{p}_{I_1\times I_2,0,\mathrm{q}}(f)\mathrm{vol}(I_2)$. Taking now the supremum over x_1 gives the estimate

$$\mathrm{p}_{I_1,0,\mathrm{q}}\left(\int_{I_2} f(\,\cdot\,, x_2)\, \mathrm{d}^{n_2}x_2\right) \leq \mathrm{p}_{I_1\times I_2,0,\mathrm{q}}(f)\mathrm{vol}(I_2), \tag{$*$}$$

which is a generalization of (B.2.11), now even uniformly in $x_1 \in I_1$. Clearly, $(*)$ is the continuity of the integral $\int_{I_2} \cdot\, \mathrm{d}^{n_2}x_2$. The other continuity claimed in (B.2.14) is again clear by symmetry. For the last part we use the general statement from Proposition B.2.3, *iii.)*, to conclude that the second equation holds: we have established the needed continuity of the integrals in the second part. To show that the iterated integrals coincide with the integral over $I = I_1 \times I_2$ we first note that any partition $\mathcal{I}$ of I has a refinement of the form $\mathcal{I}_1 \times \mathcal{I}_2$ in the sense that there is a partition $\mathcal{I}_1$ of I_1 and $\mathcal{I}_2$ of I_2 such that $\mathcal{I}_1 \times \mathcal{I}_2$ consists of all Cartesian products of subintervals from $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. We conclude that it will be sufficient to consider those more particular Riemann sums for the integral $\int_I f(x)\, \mathrm{d}^{n_1+n_2}x$. Now we are in the position to apply Lemma **??**: the integral $\int_I f(x)\, \mathrm{d}^{n_1+n_2}x$ corresponds to a net limit with respect to the Cartesian product net and the iterated integrals correspond to the iterated limit of one of the nets for fixed index of the other. Since we know that all limits exist (in particular for the left hand side of (B.2.15) by continuity of f), Lemma **??** can indeed be applied. In fact, we could have used that lemma directly also to show the equality between the two iterated integrals. However, the statements in (B.2.14) as well as the estimate $(*)$ are of independent interest. □

This proposition allows us to reduce the problem of integration of *continuous* functions to the one-dimensional case. Note however, already in the scalar situation it may happen that for general Riemann integrable functions the statement (B.2.15) is false.

The last remark in this section is about how the Riemann integral behaves under dividing I into smaller subintervals:

Proposition B.2.9 *Let $\mathcal{I} = \{I_1, \ldots, I_N\}$ be a fixed partition of I and let $f \in \mathscr{C}(I, V)$ with V being sequentially complete. Then*

$$\int_I f(x)\, \mathrm{d}^n x = \sum_{j=1}^{N} \int_{I_j} f\big|_{I_j}(x)\, \mathrm{d}^n x. \tag{B.2.16}$$

PROOF: First it is clear that $f\big|_{I_j} \in \mathscr{C}(I_j, V)$ and hence $f\big|_{I_j}$ is integrable over I_j by Proposition B.2.5 for all $j = 1, \ldots, N$. Thus the right hand side makes sense at all. Since we know that all involved integrals exist we can evaluate the net limits on particular cofinal subnets. Thus we consider only those partitions which are refinements of $\mathcal{I}$. They provide partitions of all the $I_1, \ldots, I_N$ and from the definition it is clear that the Riemann sum of f with respect to such a partition of I is the sum over the Riemann sums of the $f\big|_{I_j}$ with respect to the induced partitions of the I_j. Taking the limit immediately gives the result. □

We conclude this section with the special case of $n = 1$. For $I = [a, b] \subseteq \mathbb{R}$ one traditionally writes

$$\int_a^b f(x)\, \mathrm{d}x = \int_I f(x)\, \mathrm{d}x. \tag{B.2.17}$$

Then the last proposition says that for every $c \in (a, b)$ we have

$$\int_a^b f(x)\, \mathrm{d}x = \int_a^c f(x)\, \mathrm{d}x + \int_c^b f(x)\, \mathrm{d}x \tag{B.2.18}$$

for all $f \in \mathscr{C}([a, b], V)$. It is consistent with this property to define

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x. \tag{B.2.19}$$

Then we have (B.2.18) for any three points a, b, c in the domain of the definition of a continuous function f, no matter whether $a < c < b$ holds or not. This will be very convenient and avoids discussing separate cases at many places.

B.3 Differentiable Functions

Let us now recall the notion of differentiability and how this relates to integration. As already in the scalar case, for differentiation it will be advantageous to consider an open domain of definition $X \subseteq \mathbb{R}^n$ for our vector-valued functions.

B.3.1 Differentiability

As in the finite-dimensional case one defines the *directional derivative* of $f\colon X \longrightarrow V$ in direction of a (unit) vector $u \in \mathbb{R}^n$ at the point $x \in X$ by

$$\mathrm{D}_u f(x) = \lim_{t \longrightarrow 0} \frac{f(x + tu) - f(x)}{t}, \tag{B.3.1}$$

provided this limit exists. Of course, the limit procedure always refers to the locally convex topology of the target space V. In principle, (B.3.1) is a short-hand notation for a net limit for the net indexed by those $t \neq 0$ for which $x + tu \in U$. Since we have again zero *sequences* in $X \subseteq \mathbb{R}^n$ replacing the net limit $t \longrightarrow 0$ we anticipate that sequential completeness of V will be already enough to guarantee good behaviour. We will come back to this later in detail. In any case, we will assume that V is Hausdorff as before.

The particular case $u = \mathrm{e}_k$ with $k = 1, \ldots, n$ being a vector from the standard basis gives us the *partial derivative* in the k-th direction which we shall denote by

$$\frac{\partial f}{\partial x^k}(x) = D_{\mathrm{e}_k} f(x), \tag{B.3.2}$$

whenever the limit exists.

More important than the partial derivatives and the directional derivatives is the notion of the (total) derivative and differentiability:

Definition B.3.1 (Differentiability) *A function $f\colon X \longrightarrow V$ is called differentiable at $x \in X$ if there exists a linear map $\mathrm{D}f(x) \in \operatorname{Hom}(\mathbb{R}^n, V)$ and a map r_x defined for small vectors such that*

$$f(x + h) - f(x) = \mathrm{D}f(x)h + r_x(h) \tag{B.3.3}$$

for all $x + h$ in an open neighbourhood of x in X such that

$$\lim_{h \longrightarrow 0} \frac{r_x(h)}{\|h\|} = 0. \tag{B.3.4}$$

In this case $\mathrm{D}f(x)$ is called the derivative (or the differential) of f at x.

For (B.3.4) we need some auxiliary norm on $\mathbb{R}^n$ and clearly the property (B.3.4) of r_x does not depend on the choice of it. Since X is assumed to be open throughout this section we find some open ball $\mathrm{B}_\delta(x) \subseteq X$ with $\delta > 0$. Then it will be sufficient for evaluating $h \longrightarrow 0$ to consider $h \in \mathrm{B}_\delta(0)$

only. In particular, the actual size of $\delta > 0$ is not important. This shows that differentiability is a local concept which only needs information of f in a small open neighbourhood of x. By convention, we sometimes write just r for the remainder term r_x if the reference to the point x is clear from the context.

Proposition B.3.2 *Let $f, g\colon X \longrightarrow V$ be differentiable at $x \in X$. Then one has:*

i.) The linear map $\mathrm{D}f(x)$ is uniquely determined.

ii.) The map f is continuous at x.

iii.) All directional derivatives of f at x exist and are given by

$$\mathrm{D}_u f(x) = \mathrm{D}f(x)u. \tag{B.3.5}$$

iv.) For all $z, w \in \mathbb{C}$ also $zf + wg$ is differentiable at x and

$$\mathrm{D}(zf + wg)(x) = z\mathrm{D}f(x) + w\mathrm{D}g(x). \tag{B.3.6}$$

v.) Let $A\colon V \longrightarrow W$ be a continuous linear map into another locally convex space. Then $A \circ f$ is differentiable at x, too, and

$$\mathrm{D}(A \circ f)(x) = A(\mathrm{D}f(x)). \tag{B.3.7}$$

PROOF: Let u be a unit vector and $h = tu$ with $t \neq 0$ small enough such that $x + tu \in X$. Then

$$\mathrm{D}f(x)u = \frac{f(x + tu) - f(x)}{t} + \frac{r_x(tu)}{t},$$

if $\mathrm{D}f(x)$ is a linear map satisfying (B.3.3). If in addition the limit (B.3.4) exists then we see that the limit $t \longrightarrow 0$ for the right hand side exists and is given by

$$\mathrm{D}f(x)u = \lim_{t \longrightarrow 0} \frac{f(x + tu) - f(x)}{t},$$

since the left hand side is independent of t. This shows that $\mathrm{D}f(x)u$ is uniquely determined and hence $\mathrm{D}f(x)$ is unique by the Hausdorff property of V. Moreover, it shows (B.3.5). Since in the defining equation (B.3.3) the limit $h \longrightarrow 0$ of the right hand side exists and is equal to zero we conclude that f is continuous at x. Moreover, if r_x and $\tilde{r}_x$ are the remainder terms for f and g then it is easy to see that $zr_x + w\tilde{r}_x$ will do the job for $zf + wg$. The last part follows since with the remainder term r_x for f the map $h \mapsto A(r_x(h))$ will do the job for $A \circ f$. Indeed, since A is linear we have $\frac{1}{\|h\|}A(r_x(h)) = A\Big(\frac{1}{\|h\|}r_x(h)\Big)$ and since A is continuous we have

$$\lim_{h \longrightarrow 0} A\left(\frac{1}{\|h\|} r_x(h)\right) = A\left(\lim_{h \longrightarrow 0} \frac{1}{\|h\|} r_x(h)\right) = A(0) = 0.$$

Then the argument is completed by the observation that $h \mapsto A \circ (\mathrm{D}f(x)h)$ is still linear. ☐

We consider now the one-dimensional situation in some more detail. Here we adopt the convention to denote the derivative simply by

$$f'(x) = \mathrm{D}f(x)1 = \lim_{t \longrightarrow 0} \frac{f(x + t) - f(x)}{t}, \tag{B.3.8}$$

identifying the linear map $\mathrm{D}f(x)\colon \mathbb{R} \longrightarrow V$ with its value on $1 \in \mathbb{R}$. Clearly, the existence of the limit (B.3.8) is equivalent to differentiability in the one-dimensional case.

B.3.2 Fundamental Theorem of Calculus and Applications

The first major result is the fundamental theorem of calculus which we formulate for a continuous integrand for simplicity:

Theorem B.3.3 (Fundamental Theorem of Calculus) *Let V be a sequentially complete locally convex space and let $[a, b] \subseteq \mathbb{R}$ be a compact interval. Let $f \in \mathscr{C}([a, b], V)$.*

i.) The function

$$F(x) = \int_a^x f(t) \, \mathrm{d}t \tag{B.3.9}$$

is continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) = f(x)$.

ii.) For any $x, y \in [a, b]$ and every continuous seminorm q on V the function F satisfies

$$\mathrm{q}(F(x) - F(y)) \leq \mathrm{q}_\infty(f)|x - y|. \tag{B.3.10}$$

iii.) If $\tilde{F}$ is another continuous function on $[a, b]$ with $\tilde{F}' = f$ on (a, b) then $F - \tilde{F} = const$.

PROOF: For the first part let q be a continuous seminorm on V and $\epsilon > 0$. Then we fix $\delta > 0$ for the uniformly continuous function f according to Lemma B.2.1, i.e. for $|x - y| < \delta$ we have $\mathrm{q}(f(x) - f(y)) < \epsilon$. Now consider $x \in (a, b)$ and $0 < h < \delta$ such that $x + h \in (a, b)$. We have

$$\begin{aligned} \mathrm{q}\left(\frac{F(x+h) - F(x)}{h} - f(x)\right) &= \mathrm{q}\left(\frac{1}{h}\int_x^{x+h} f(t) \, \mathrm{d}t - f(x)\right) \\ &= \mathrm{q}\left(\frac{1}{h}\int_x^{x+h} (f(t) - f(x)) \, \mathrm{d}t\right) \\ &\overset{\text{(B.2.11)}}{\leq} \frac{1}{h}\int_x^{x+h} \mathrm{q}(f(t) - f(x)) \, \mathrm{d}t \\ &< \epsilon, \end{aligned}$$

since $|t - x| < \delta$ for all t between x and $x + h$ by our choice of h. The case $-\delta < h < 0$ is analogous. This shows that F is indeed differentiable with derivative given by f. The second part is now easy as $F(x) - F(y) = \int_x^y f(t) \, \mathrm{d}t$ by (B.2.18). Hence Proposition B.2.6, *i.)*, can be applied. For the last part, let $\tilde{F}$ be another such function then $F - \tilde{F}$ is differentiable with derivative identically zero on (a, b). We claim that a differentiable function $g\colon (a, b) \longrightarrow V$ with vanishing derivative is necessarily constant. Indeed, by Proposition B.3.2, *v.)*, we see that for every continuous linear functional $\varphi \in V'$ the scalar function $x \mapsto \varphi(g(x))$ is differentiable with derivative identically zero. By elementary calculus we have $\varphi(g(x)) = \varphi(g(y))$ for all $x, y \in (a, b)$ for all such φ. Since the continuous linear functionals separate points by Corollary 2.2.21 we conclude that g is constant indeed. Applying this to $\tilde{F} - F$ and using the continuity on $[a, b]$ gives the result. □

Actually, the proof even shows that F is differentiable from one side on the whole interval: the limit of the difference quotient at $x = a$ exists for $h \longrightarrow 0^+$ and is $f(a)$ while the limit of the difference quotient at $x = b$ exists for $h \longrightarrow 0^-$ and yields $f(b)$. As in the scalar situation, we call a function F with $F' = f$ a *primitive* of f. Then Theorem B.3.3 shows that for a continuous f there is a primitive given by (B.3.9) and any two primitives differ by a constant.

Remark B.3.4 As familiar from the scalar theory this theorem is most useful for computing integrals. In combination with Fubini's theorem as formulated in Proposition B.2.8 one has an efficient tool to compute also the higher-dimensional integrals.

The next feature of differentiability is the chain rule which we formulate as follows:

Proposition B.3.5 (Chain rule) *Let $X \subseteq \mathbb{R}^n$ and $X' \subseteq \mathbb{R}^m$ be open and let $g\colon X \longrightarrow X'$ and $f\colon X' \longrightarrow V$ with a Hausdorff locally convex space V. If g is differentiable at $x \in X$ and f is differentiable at $g(x) \in X'$ then $f \circ g$ is differentiable at x with derivative given by*

$$\mathrm{D}(f \circ g)(x) = \mathrm{D}f(g(x)) \circ \mathrm{D}g(x). \tag{B.3.11}$$

Proof: Here the proof is again as in the usual calculus situation, the vector values of f do not really cause any further problems. Let $y = g(x)$ and let

$$f(y + h') = f(y) + \mathrm{D}f(x)h' + r'_y(h') \tag{$*$}$$

as well as

$$g(x + h) = g(x) + \mathrm{D}g(x)h + r_x(h) \tag{$**$}$$

with remainder terms r'_y and r_x satisfying (B.3.4), i.e.

$$\lim_{h' \longrightarrow 0} \frac{r'_y(h')}{\|h'\|} = 0 \quad \text{and} \quad \lim_{h \longrightarrow 0} \frac{r_x(h)}{\|h\|} = 0. \tag{☺}$$

Inserting ($**$) into ($*$) gives

$$\begin{aligned} f(g(x+h)) - f(g(x)) &= f(g(x) + \mathrm{D}g(x)h + r_x(h)) - f(g(x)) \\ &= f(g(x)) + \mathrm{D}f(g(x))(\mathrm{D}g(x)h + r_x(h)) + r'_y(\mathrm{D}g(x)h + r_x(h)) - f(g(x)) \\ &= (\mathrm{D}f(g(x)) \circ \mathrm{D}g(x))h + \mathrm{D}f(g(x))r_x(h) + r'_y(\mathrm{D}g(x)h + r_x(h)). \end{aligned}$$

Hence the relevant remainder term is given by $\varrho(h) = \mathrm{D}f(g(x))r_x(h) + r'_y(\mathrm{D}g(x)h + r_x(h))$. By the linearity of $\mathrm{D}f(g(x))$ and by (☺) we have for $h \neq 0$

$$\lim_{h \longrightarrow 0} \frac{1}{\|h\|}\mathrm{D}f(g(x))r_x(h) = \lim_{h \longrightarrow 0} \mathrm{D}f(g(x))\frac{r_x(h)}{\|h\|} = \mathrm{D}f(g(x)) \lim_{h \longrightarrow 0} \frac{r_x(h)}{\|h\|} = 0,$$

since every linear map $\mathrm{D}f(g(x))\colon \mathbb{R}^n \longrightarrow V$ is continuous by Lemma B.1.8. The other contribution requires some more work. Let q be a continuous seminorm on V and $\epsilon > 0$. Then (☺) means that for $0 < \|h'\| < \delta$ with an appropriate $\delta > 0$, depending on q and ϵ, we have $\frac{1}{\|h'\|}\mathrm{q}(r'_y(h')) < \epsilon$. Similarly, let $\delta' > 0$ be such that $0 < \|h\| < \delta'$ implies $\frac{1}{\|h\|}\|r_x(h)\| < \epsilon$. Then for $h \neq 0$ with $\|h\| < \min\Big(\frac{\delta}{\|\mathrm{D}g(x)\| + \epsilon}, \delta'\Big)$ we get the combined estimate: first we have

$$\|\mathrm{D}g(x)h + r_x(h)\| \leq \|\mathrm{D}g(x)\|\|h\| + \epsilon\|h\| < \delta,$$

and, second,

$$\mathrm{q}\big(r'_y(\mathrm{D}g(x)h + r_x(h))\big) < \epsilon\|\mathrm{D}g(x)h + r_x(h)\| \leq \epsilon(\|\mathrm{D}g(x)\| + \epsilon)\|h\|,$$

from which we deduce $\lim_{h \longrightarrow 0} \frac{1}{\|h\|} r'_y(\mathrm{D}g(x)h + r_x(h)) = 0$. Thus the statement follows. □

Later on, we will use the chain rule for reparametrizations of curves in V. The simplest case of such a curve is needed in the proof of the mean value theorem. Unlike in the scalar situation we only get a seminorm estimate:

Proposition B.3.6 (Mean Value Theorem) *Let $X \subseteq \mathbb{R}^n$ be open and let $f\colon X \subseteq \mathbb{R}^n \longrightarrow V$ be a differentiable function such that the differential $\mathrm{D}f\colon X \longrightarrow \mathrm{Hom}(\mathbb{R}^n, V)$ is continuous. Moreover, let V be sequentially complete and $x, y \in X$ such that the connecting line segment from x to y is entirely in X.*

i.) In this situation one has

$$f(x) - f(y) = \int_0^1 \mathrm{D}f(y + t(x-y))(x-y)\,\mathrm{d}t. \tag{B.3.12}$$

ii.) For every continuous seminorm q *on* V *one has*

$$\mathrm{q}(f(x) - f(y)) \le \int_0^1 \|\mathrm{D}f(y+t(x-y))\|_{\mathrm{q}} \|x-y\|\,\mathrm{d}t \le \sup_{t\in[0,1]} \|\mathrm{D}f(y+t(x-y))\|_{\mathrm{q}}\|x-y\|, \tag{B.3.13}$$

where $\|\cdot\|_{\mathrm{q}}$ *is the operator seminorm build out of* q *as in* (B.1.8).

PROOF: Since the straight line $g(t) = y + t(x-y)$ is a continuously differentiable map with $g'(t) = x-y$ for all t, we have a curve $f \circ g$ which is defined for some small open interval containing $[0,1]$ as X is open. By the chain rule, $f \circ g$ is differentiable with derivative

$$(f \circ g)'(t) = \mathrm{D}f(g(t))g'(t) = \mathrm{D}f(y + t(x-y))(x-y)$$

for all such t. By assumption, $\mathrm{D}f$ is continuous and hence also $(f \circ g)'$ is continuous. Applying the Fundamental Theorem of Calculus to the continuous function $(f \circ g)'\colon [0,1] \longrightarrow \mathrm{Hom}(\mathbb{R}^n, V) \circ \mathrm{Hom}(\mathbb{R}, \mathbb{R}^n) = \mathrm{Hom}(\mathbb{R}, V)$ gives the primitive

$$F(T) = \int_0^T (f \circ g)'(t)\,\mathrm{d}t = \int_0^T \mathrm{D}f(y + t(x-y))(x-y)\,\mathrm{d}t.$$

On the other hand, $f \circ g$ is clearly a primitive as well. Hence they coincide up to a constant vector by Theorem B.3.3, *iii.)*, which drops out in the difference needed for (B.3.12). Thus $F(1) - F(0)$ coincides with $f(x) - f(y)$. The estimates in (B.3.13) are then just the ones from Proposition B.2.6, *i.)*. □

Remark B.3.7 Without the assumption that $\mathrm{D}f$ is a continuous map itself, one can still show the second (and weaker) estimate in (B.3.13), thereby avoiding the integral. However, this needs a slightly more careful and technical investigation. For our purposes later on, we are mainly concerned with functions having even nicer features than a continuous derivative. Thus we stick to the above simpler approach to (B.3.13).

Exercise: Exe notes

A last tool for evaluating integrals is the change of variables. We also formulate this for the simplest situation of one-dimensional integrals only:

Proposition B.3.8 (Change of variables) *Let* $f \in \mathscr{C}([a,b], V)$ *with a sequentially complete locally convex space* V*. Let* $g\colon [c,d] \longrightarrow [a,b]$ *be a continuous map with a continuous derivative* g' *on* (c,d) *extending continuously to* $[c,d]$*. Then one has*

$$\int_c^d f(g(x))g'(x)\,\mathrm{d}x = \int_{g(c)}^{g(d)} f(y)\,\mathrm{d}y. \tag{B.3.14}$$

PROOF: By Theorem B.3.3 we have a primitive F for f which is differentiable on (a,b) and continuous on $[a,b]$. The function $F \circ g$ is therefore continuous on $[c,d]$ and by Proposition B.3.5 differentiable on (c,d) with derivative $(F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$. Integrating this identity gives by the Fundamental Theorem of Calculus

$$\int_c^d f(g(x))g'(x)\,\mathrm{d}x = \int_c^d (F \circ g)'(x)\,\mathrm{d}x = (F \circ g)(d) - (F \circ g)(c) = \int_{g(c)}^{g(d)} f(y)\,\mathrm{d}y. \qquad \square$$

We conclude this section with a criterion for differentiability based on the partial derivatives. In general, the existence of the partial derivative does not imply differentiability at all. However, if one adds continuity one has the following statement:

Proposition B.3.9 *Let V be a sequentially complete locally convex space and let $X \subseteq \mathbb{R}^n$ be open. For a map $f\colon X \longrightarrow V$ the following statements are equivalent:*

i.) The map f is differentiable on X and $\mathrm{D}f\colon X \longrightarrow \operatorname{Hom}(\mathbb{R}^n, V)$ is continuous.

ii.) All directional derivatives $\mathrm{D}_u f(x)$ exist for all $x \in X$ and give continuous functions $\mathrm{D}_u f\colon X \longrightarrow V$.

iii.) All partial derivatives $\frac{\partial f}{\partial x^i}(x)$ exist for all $x \in X$ and give continuous functions $\frac{\partial f}{\partial x^i}\colon X \longrightarrow V$.

PROOF: Clearly, we have *i.)* $\implies$ *ii.)* $\implies$ *iii.)* since the evaluation of $\mathrm{D}f(x)$ on a (unit) vector $u \in \mathbb{R}^n$ gives $\mathrm{D}_u f(x)$, preserving the continuous dependence on x. Thus we are left with *iii.)* $\implies$ *i.)*. The candidate for the linear map $\mathrm{D}f(x)$ will be the map

$$A(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x)\mathrm{e}^i \in \operatorname{Hom}(\mathbb{R}^n, V),$$

where $\mathrm{e}^1, \ldots, \mathrm{e}^n \in (\mathbb{R}^n)^*$ denotes the dual basis of the canonical basis $\mathrm{e}_1, \ldots, \mathrm{e}_n$ of $\mathbb{R}^n$ as usual. By *iii.)* we already know that $A\colon x \mapsto A(x)$ is continuous. Consider a small vector $h \neq 0$ and the points $x_0 = x$ and $x_k = x + \sum_{i=1}^{k} h^i \mathrm{e}_i$ where $h^i = \mathrm{e}^i(h)$ are the components of h. For a small enough h all the points $x_0, \ldots, x_n$ are in X including their convex combinations. We have by Proposition B.3.6, *i.)*,

$$f(x_k) - f(x_{k-1}) = \int_0^1 \frac{\partial f}{\partial x^k}(x_{k-1} + th^k \mathrm{e}_k)\,\mathrm{d}t,$$

where we interpret this as a function of the k-th variable alone. With the usual telescope sum we have

$$\begin{aligned} f(x+h) - f(x) - A(x)h &= \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) - \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(x)h^k \\ &= \sum_{k=1}^{n} \left(\int_0^1 \frac{\partial f}{\partial x^k}(x + th^k \mathrm{e}_k)h^k \,\mathrm{d}t - \frac{\partial f}{\partial x^k}(x)h^k \right) \\ &= \sum_{k=1}^{n} h^k \int_0^1 \left(\frac{\partial f}{\partial x^k}(x + th^k \mathrm{e}_k) - \frac{\partial f}{\partial x^k}(x) \right) \mathrm{d}t. \end{aligned}$$

We fix a continuous seminorm q on V and $\epsilon > 0$. For convenience, we use the maximum norm $\|\cdot\|_\infty$ on $\mathbb{R}^n$ with respect to the canonical basis to estimate the size of h. Since $\frac{\partial f}{\partial x^k}(x + th^k \mathrm{e}_k)$ is a continuous function of t we have for $|h^k|$ small enough

$$\mathrm{q}\left(\frac{\partial f}{\partial x^k}(x + th^k \mathrm{e}_k) - \frac{\partial f}{\partial x^k}(x) \right) < \frac{\epsilon}{n}$$

for all $t \in [0, 1]$. Now $\|h\|_\infty$ small implies that all $|h^k| \leq \|h\|_\infty$ are small. Hence by (B.3.13)

$$\begin{aligned} \mathrm{q}(f(x+h) - f(x) - A(x)h) &\leq \sum_{k=1}^{n} |h^k| \int_0^1 \mathrm{q}\left(\frac{\partial f}{\partial x^k}(x + th^k \mathrm{e}_k) - \frac{\partial f}{\partial x^k}(x) \right) \mathrm{d}t \\ &\leq n\|h\|_\infty \frac{\epsilon}{n} \\ &= \epsilon\|h\|_\infty \end{aligned}$$

for small h. But this is the desired estimate to conclude that f is differentiable at x with derivative given by $\mathrm{D}f(x) = A(x)$. Since we already know that $A(x)$ depends continuously on x we get *i.)*. $\square$

Clearly, yet another equivalent formulation is that all the directional derivatives for a *basis* $u_1, \ldots, u_n$ at all points $x \in X$ exist and are continuous. There is nothing special about the canonical basis in the above proof.

B.4 Smooth Functions

In this subsection we will turn to higher derivatives and investigate their properties. Eventually, this will lead to new locally convex spaces of $\mathscr{C}^k$-functions and $\mathscr{C}^\infty$-functions.

B.4.1 Smoothness

We have seen that it will be interesting to consider functions $f\colon X \longrightarrow V$ which are differentiable on X and which have a continuous derivative $\mathrm{D}f\colon X \longrightarrow \mathrm{Hom}(\mathbb{R}^n, V)$. Note again that this is non-ambiguous: since $\mathbb{R}^n$ is finite-dimensional we have a canonical locally convex topology on $\mathrm{Hom}(\mathbb{R}^n, V)$ inherited from V by the various systems of seminorms discussed in Lemma B.1.9. Thus we do not have to specify a new topology on $\mathrm{Hom}(\mathbb{R}^n, V)$ but can always rely on this one to make sense out of continuity statements about $\mathrm{D}f$. In particular, we can iterate this procedure which leads to the following definition:

Definition B.4.1 (Smooth functions) *Let $X \subseteq \mathbb{R}^n$ be open and let V be a Hausdorff locally convex space. Then $f\colon X \longrightarrow V$ is called $\mathscr{C}^1$ if f is differentiable with continuous derivative $\mathrm{D}f$ on X. For $k > 1$ one defines inductively that f is $\mathscr{C}^k$ if $\mathrm{D}f$ is $\mathscr{C}^{k-1}$. Finally, f is called smooth or $\mathscr{C}^\infty$ if f is $\mathscr{C}^k$ for all $k \in \mathbb{N}$. The corresponding spaces of $\mathscr{C}^k$ and smooth functions will be denoted by $\mathscr{C}^k(X, V)$ and $\mathscr{C}^\infty(X, V)$, respectively.*

With other words, f is $\mathscr{C}^k$ if f is k-times differentiable with continuous derivatives. We will now investigate the properties of these higher derivatives $\mathrm{D}^2 f$, $\mathrm{D}^3 f$, etc. First we note that canonically we have

$$\mathrm{Hom}(\mathbb{R}^n, V) \cong (\mathbb{R}^n)^* \otimes V, \tag{B.4.1}$$

$$\mathrm{Hom}(\mathbb{R}^n, \mathrm{Hom}(\mathbb{R}^n, V)) \cong \mathrm{Hom}(\otimes^2 \mathbb{R}^n, V) \cong \otimes^2 (\mathbb{R}^n)^* \otimes V, \tag{B.4.2}$$

and for general k

$$\mathrm{Hom}(\mathbb{R}^n, \ldots \mathrm{Hom}(\mathbb{R}^n, V) \cdots) \cong \mathrm{Hom}(\otimes^k \mathbb{R}^n, V) \cong \otimes^k (\mathbb{R}^n)^* \otimes V, \tag{B.4.3}$$

since $\mathbb{R}^n$ is *finite-dimensional*. Thus we can view the k-th iterated derivative $\mathrm{D}^k f$ as a map

$$\mathrm{D}^k f\colon X \longrightarrow \mathrm{Hom}(\otimes^k \mathbb{R}^n, V) \quad \text{or} \quad \mathrm{D}^k f\colon X \longrightarrow \otimes^k (\mathbb{R}^n)^* \otimes V. \tag{B.4.4}$$

In the following, we shall frequently change from one to the other point of view without further comments. In fact, the above isomorphisms are even topological isomorphisms if we take the π-topology for the tensor products:

Lemma B.4.2 *Let U be a finite-dimensional vector space and let V be a locally convex space. Then the canonical topology on $\mathrm{L}(U, V)$ according to Lemma B.1.9 corresponds to the π-topology on $U^* \otimes V$ under the linear isomorphism $\mathrm{Hom}(U, V) \cong U^* \otimes V$.*

PROOF: Here the finite-dimensionality is of course crucial. We fix a basis $\mathrm{e}_1, \ldots, \mathrm{e}_n$ of U with dual basis $\mathrm{e}^1, \ldots, \mathrm{e}^n$ of U^*. For the locally convex topology of $\mathrm{Hom}(U, V)$ we choose the seminorms $\|\cdot\|_{\mathrm{q},1}$ from (B.1.7), where q ranges over the continuous seminorms of V. To describe the π-topology on $U^* \otimes V$ we also take the ℓ^1-norm $\|\cdot\|_1$ with respect to the basis $\mathrm{e}^1, \ldots, \mathrm{e}^n$ on U^* and again all continuous seminorms on V. Then we known from Theorem 4.1.5 that the seminorms $\|\cdot\|_1 \otimes \mathrm{q}$

specify the π-topology on $U^* \otimes V$. Now let $\phi \in \operatorname{Hom}(U, V)$ be given. Then its image in $U^* \otimes V$ is given by $f = \sum_{i=1}^n \mathrm{e}^i \otimes \phi(\mathrm{e}_i)$. Taking an arbitrary representation $f = \sum_k \alpha_k \otimes v_k$ we get

$$\begin{aligned} \sum\nolimits_k \|\alpha_k\|_1 \mathrm{q}(v_k) &= \sum\nolimits_k \left\| \sum_{i=1}^n \alpha_{ki} \mathrm{e}^i \right\|_1 \mathrm{q}(v_k) \\ &= \sum\nolimits_k \sum_{i=1}^n |\alpha_{ki}| \mathrm{q}(v_k) \\ &\geq \sum_{i=1}^n \mathrm{q}\Big(\sum\nolimits_k \alpha_{ki} v_k\Big) \\ &= \sum_{i=1}^n \mathrm{q}(\phi(\mathrm{e}_i)), \end{aligned} \tag{$*$}$$

since by the uniqueness of the coefficients of f with respect to the basis $\mathrm{e}^1, \ldots, \mathrm{e}^n$ we have

$$\phi(\mathrm{e}_i) = \mathrm{e}_i \left(\sum_{j=1}^n \mathrm{e}^j \otimes \phi(\mathrm{e}_j) \right) = \mathrm{e}_i \left(\sum_k \alpha_k \otimes v_k \right) = \sum_k \alpha_{ki} v_k.$$

By definition of $\|\cdot\|_{\mathrm{q},1}$ the last term in $(*)$ equals $\|\phi\|_{\mathrm{q},1}$. Since the seminorm $\|\cdot\|_1 \otimes \mathrm{q}$ is the infimum of the left hand side of $(*)$ over all the possible decompositions of f we get the estimate

$$(\|\cdot\|_1 \otimes \mathrm{q})(f) \geq \|\phi\|_{\mathrm{q},1}.$$

Now the converse estimate is trivial since the right hand side of $(*)$ is one particular decomposition of $f = \mathrm{e}^i \otimes \phi(\mathrm{e}_i)$. Thus we conclude that $\|\cdot\|_1 \otimes \mathrm{q}$ coincides with $\|\cdot\|_{\mathrm{q},1}$ under the canonical identification of $U^* \otimes V$ with $\operatorname{Hom}(U, V)$. Since on both sides we have defining systems of seminorms the claim follows. ☐

In fact, from a more sophisticated point of view the lemma is not very much surprising as on the tensor product with finite-dimensional vector spaces there is essentially only one reasonable locally convex topology.

We can now apply this lemma to conclude that for all $k \in \mathbb{N}$ we have

$$\operatorname{Hom}(\otimes^k \mathbb{R}^n, V) \cong \otimes^k (\mathbb{R}^n)^* \otimes_\pi V \tag{B.4.5}$$

also in the topological sense. Since we know from Lemma B.1.8 that $\operatorname{Hom}(\otimes^k \mathbb{R}^n, V) \subseteq \mathscr{C}(\otimes^k \mathbb{R}^n, V)$ is a closed subspace we obtain that both spaces in (B.4.5) are (sequentially) complete as soon as V is (sequentially) complete. In particular, if V is complete the tensor product $\otimes_\pi$ coincides with the completed tensor product $\hat{\otimes}_\pi$. Of course, the finite-dimensionality of $\otimes^k (\mathbb{R}^n)^*$ is important for this statement.

The following proposition will be crucial at many places and states that the higher derivatives $\mathrm{D}^\ell f$ of a $\mathscr{C}^k$-function are all in the *symmetric* tensor powers rather than general tensors:

Proposition B.4.3 *Let $X \subseteq \mathbb{R}^n$ be open and let V be a sequentially complete locally convex space. Let $f \in \mathscr{C}^k(X, V)$ with $k \geq 2$. Then for every $\ell \leq k$ and $x \in X$ we have $\mathrm{D}^\ell f(x) \in \mathrm{S}^\ell (\mathbb{R}^n)^* \otimes V$.*

Proof: We consider first the case $k = 2$. Let $u_1, u_2 \in \mathbb{R}^n$ be two unit vectors and let $x \in X$. Then for small $t, s \neq 0$ we consider the vector

$$F_{st} = \frac{1}{st}(f(x + su_1 + tu_2) - f(x + su_1) - f(x + tu_2) + f(x)),$$

which is well-defined as long as s, t are small enough to guarantee that all points are in X. We fix now small enough $S, T > 0$ and consider $0 < |s| \leq S$ as well as $0 < |t| \leq T$ only. This gives us a net $\{F_{st}\}$ with values in V indexed by the pairs (s, t) where the direction is "towards $(0, 0)$". Even though this is an honest net we have *sequences* $(s_n, t_n) \longrightarrow (0, 0)$ which are cofinal, i.e. for every index (s, t) we have some $n \in \mathbb{N}$ such that $(s_n, t_n) \succcurlyeq (s, t)$. Hence the *sequential* completeness of V will be sufficient to show the convergence of the *net* once we know that it is a Cauchy net. The argument is completely analogous to our discussion for the Riemann sums in the proof of Proposition B.2.5, see also Exercise B.8.5. To show that the net is Cauchy we apply Proposition **??**: in fact, in this proposition the convergence is shown under the stronger assumption of a complete target space V but with the above discussion in mind sequential completeness will be sufficient for this particular net. To see that we can apply Proposition **??** we first note that for fixed t we have

$$\begin{aligned}\lim_{s \longrightarrow 0} F_{st} &= \lim_{s \longrightarrow 0} \frac{1}{st}\big(f(x + tu_2 + su_1) - f(x + tu_2)\big) + \lim_{s \longrightarrow 0} \frac{1}{st}\big(f(x + su_1) - f(x)\big) \\ &= \frac{1}{t}\big(\mathrm{D}f(x + tu_2)u_1 - \mathrm{D}f(x)u_1\big), \end{aligned} \tag{$*$}$$

since f is differentiable at all points in X. Now we can take the limit $t \longrightarrow 0$ of $(*)$ which gives

$$\lim_{t \longrightarrow 0} \lim_{s \longrightarrow 0} F_{st} = \lim_{t \longrightarrow 0} \frac{1}{t}\big(\mathrm{D}f(x + tu_2)u_1 - \mathrm{D}f(x)u_1\big) = \big(\mathrm{D}(\mathrm{D}f)(x)u_2\big)u_1 = \mathrm{D}^2 f(x)(u_2, u_1), \tag{☺}$$

since $\mathrm{D}f$ is still differentiable at all points in X by assumption. Analogously, we see that the other order of the limits gives $\mathrm{D}^2 f(x)(u_1, u_2)$ instead. Hence we are done if we can show that the choice of the orders does not matter at all. So the only thing to do is to apply Proposition **??** which guarantees that even the net limit of $\{F_{st}\}$ exists and hence coincides with any order of separate limits in s and t. We have to show that the limit in $(*)$ is uniform in t. To show this we consider

$$\begin{aligned} F_{st} - &\frac{1}{t}\big(\mathrm{D}f(x + tu_2)u_1 - \mathrm{D}f(x)u_1\big) \\ &= \frac{1}{t}\left(\frac{1}{s}\big(f(x + tu_2 + su_1) - f(x + tu_2)\big) - \mathrm{D}f(x + tu_2)u_1\right) \\ &\qquad - \frac{1}{t}\left(\frac{1}{s}\big(f(x + su_1) - f(x)\big) - \mathrm{D}f(x)u_1\right) \\ &= \frac{1}{t}\left(\int_0^1 \mathrm{D}f(x + tu_2 + \sigma su_1)u_1 \,\mathrm{d}\sigma - \mathrm{D}f(x + tu_2)u_1\right) \\ &\qquad - \frac{1}{t}\left(\int_0^1 \mathrm{D}f(x + \sigma su_1)u_1 \,\mathrm{d}\sigma - \mathrm{D}f(x)u_1\right) \\ &= \frac{1}{t}\int_0^1 \big(\mathrm{D}f(x + tu_2 + \sigma su_1)u_1 - \mathrm{D}f(x + tu_2)u_1\big) \,\mathrm{d}\sigma \\ &\qquad - \frac{1}{t}\int_0^1 \big(\mathrm{D}f(x + \sigma su_1) - \mathrm{D}f(x)u_1\big) \,\mathrm{d}\sigma \\ &= \int_0^1 \int_0^1 \big(\mathrm{D}^2 f(x + \tau tu_2 + \sigma su_1)(u_2, u_1) - \mathrm{D}^2 f(x + \tau tu_2)(u_2, u_1)\big) \,\mathrm{d}\tau \,\mathrm{d}\sigma, \end{aligned} \tag{$**$}$$

by using the Fundamental Theorem of Calculus several times, in the form of the Mean Value Theorem (B.3.12). Now let q be a continuous seminorm on V then $(**)$ gives the estimate

$$\begin{aligned} \mathrm{q}&\left(F_{st} - \frac{1}{t}(\mathrm{D}f(x + tu_2)u_1 - \mathrm{D}f(x)u_1)\right) \\ &\leq \int_0^1 \int_0^1 \mathrm{q}\big(\mathrm{D}^2 f(x + \tau tu_2 + \sigma su_1)(u_2, u_1) - \mathrm{D}^2 f(x + \tau tu_2)(u_2, u_1)\big) \,\mathrm{d}\tau \,\mathrm{d}\sigma, \end{aligned}$$

by (B.2.11) and the continuity of $\mathrm{D}^2 f$. Since $\mathrm{D}^2 f$ is continuous by assumption, the integrand, viewed as a function of s and t, is uniformly continuous on $[-S, S] \times [-T, T]$. Thus taking S small enough the integrand becomes less than ϵ for all $t \in [-T, T]$. This shows the uniform convergence in t of the net F_{st} for $s \longrightarrow 0$. As in Proposition **??** we conclude that F_{st} is a Cauchy net which by our previous discussion converges according to the sequential completeness of V. But then the net limit coincides with the iterated limit (☺) which gives the symmetry of $\mathrm{D}^2 f$ at once. This shows the case $k = 2$. For higher derivatives one proceeds inductively: for $k = 3$ we have a $\mathscr{C}^2$-derivative $\mathrm{D}f$. This implies that $\mathrm{D}^2(\mathrm{D}f)$ is symmetric in the first two arguments. On the other hand, $\mathrm{D}^2 f$ is symmetric and hence $\mathrm{D}(\mathrm{D}^2 f)$ is symmetric in the last two arguments. Since $\mathrm{D}^2(\mathrm{D}f) = \mathrm{D}^3 f = \mathrm{D}(\mathrm{D}^2 f)$ we conclude that $\mathrm{D}^3 f$ is totally symmetric. Repeating this argument gives the result for all $\ell \leq k$ and all $k \geq 2$. □

Remark B.4.4 Actually, we have shown more than just the symmetry of $\mathrm{D}^\ell f$. Since the net limit of F_{st} also gives the second derivative at x we can approach this limit in various other ways. In particular, taking $s = t$ and $t \longrightarrow 0$ gives the sometimes useful formula

$$\mathrm{D}^2 f(x)(u_1, u_2) = \lim_{t \longrightarrow 0} \frac{f(x + tu_1 + tu_2) - f(x + tu_1) - f(x + tu_2) + f(x)}{t^2} \tag{B.4.6}$$

for a $\mathscr{C}^2$-function f. Analogously, one gets formulas for the higher derivatives.

In the following, we will always deal with $\mathscr{C}^k$-functions so we can be sloppy in our notation for the higher derivatives: we do not have to take care concerning the orders of the arguments $u_1, \ldots, u_\ell$.

Applying this proposition to the basis vectors $\mathrm{e}_1, \ldots, \mathrm{e}_n$ of the canonical basis of $\mathbb{R}^n$ gives us the following result:

Corollary B.4.5 *Let V be sequentially complete and let $f \in \mathscr{C}^k(X, V)$. Then for all $\ell \leq k$ one has*

$$\mathrm{D}^\ell f(x) = \sum_{i_1, \ldots, i_\ell = 1}^{n} \frac{\partial^\ell f}{\partial x^{i_1} \cdots \partial x^{i_\ell}}(x) \mathrm{e}^{i_1} \otimes \cdots \otimes \mathrm{e}^{i_\ell} = \sum_{i_1, \ldots, i_\ell = 1}^{n} \frac{\partial^\ell f}{\partial x^{i_1} \cdots \partial x^{i_\ell}}(x) \mathrm{e}^{i_1} \vee \cdots \vee \mathrm{e}^{i_\ell}, \tag{B.4.7}$$

with the symmetrized tensor product $\vee$ as in (3.4.40).

B.4.2 The $\mathscr{C}^k$- and $\mathscr{C}^\infty$-Topology

After these preparations it is now easy to equip $\mathscr{C}^k(X, V)$ with a finer topology than the $\mathscr{C}^0$-topology in order to make it a complete locally convex space by its own. The idea is very simple: we want the process of differentiation to be continuous itself.

We take a continuous seminorm q on V and build a corresponding seminorm $\|\cdot\|_{\mathrm{q}}$ on $\otimes^\ell(\mathbb{R}^n)^* \otimes V$ by tensoring with some auxiliary norm on $\otimes^\ell(\mathbb{R}^n)^*$ as we did before. Then we take the supremum of these seminorms of the ℓ-th derivative of f over the points x in some compact subset $K \subseteq X$. When the derivative is continuous this supremum is indeed a maximum. This results in the seminorm

$$\mathrm{p}_{K,\ell,\mathrm{q}}(f) = \sup_{\substack{x \in K \\ \ell' \leq \ell}} \left\| \mathrm{D}^{\ell'} f(x) \right\|_{\mathrm{q}}, \tag{B.4.8}$$

generalizing the scalar case from Exercise 2.5.29. Clearly, a different choice of the auxiliary norm on $\otimes^\ell(\mathbb{R}^n)^*$ leads to a different but equivalent seminorm. Specifying the canonical basis $\mathrm{e}^1, \ldots, \mathrm{e}^n$ of $(\mathbb{R}^n)^*$ and using the maximum norm with respect to the corresponding basis on the tensor powers, we get from Corollary B.4.5 the following more explicit form

$$\mathrm{p}_{K,\ell,\mathrm{q}}(f) = \sup_{\substack{x \in K \\ |\alpha| \leq \ell}} \mathrm{q}\left(\frac{\partial^{|\alpha|} f}{\partial (x^1)^{\alpha_1} \cdots \partial (x^n)^{\alpha_n}} \right) \tag{B.4.9}$$

of the seminorm. As in the scalar theory we abbreviate this collection of partial derivatives by

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial (x^1)^{\alpha_1} \cdots \partial (x^n)^{\alpha_n}}(x) \tag{B.4.10}$$

for any multiindex $\alpha \in \mathbb{N}_0^n$. Using these seminorms we can now define the $\mathscr{C}^k$-topology:

Definition B.4.6 ($\mathscr{C}^k$-Topology) *Let $X \subseteq \mathbb{R}^n$ be open, V a sequentially complete locally convex space, and $k \in \mathbb{N}_0 \cup \{+\infty\}$. Then the $\mathscr{C}^k$-topology on $\mathscr{C}^k(X, V)$ is the locally convex topology determined by all the seminorms $\{\mathrm{p}_{K,\ell,\mathrm{q}}\}$, where $K \subseteq X$ runs through all compact subsets of X, $\ell \in \mathbb{N}_0$ satisfies $\ell \leq k$, and q runs through all continuous seminorms on V.*

We will always equip $\mathscr{C}^k(X, V)$ with this Hausdorff locally convex topology. Note that $k = +\infty$ is explicitly allowed. The $\mathscr{C}^\infty$-topology will also be called the *smooth topology.* Note also that we have the trivial estimates

$$\mathrm{p}_{K,\ell,\mathrm{q}}(f) \leq \mathrm{p}_{K',\ell',\mathrm{q}'}(f) \tag{B.4.11}$$

for $K \subseteq K'$, $\ell \leq \ell'$ and $\mathrm{q} \leq \mathrm{q}'$. Thus we conclude that an exhausting sequence K_n of compact subsets and seminorms q from a collection which determines the topology of V are already enough to determine the $\mathscr{C}^k$-topology. We collect now the basic properties of the $\mathscr{C}^k$-topology:

Theorem B.4.7 ($\mathscr{C}^k$-Topology) *Let $X \subseteq \mathbb{R}^n$ be open, let V be a sequentially complete locally convex space, and let $k \in \mathbb{N}_0 \cup \{+\infty\}$.*

i.) The inclusion map

$$\mathscr{C}^{k+1}(X, V) \longrightarrow \mathscr{C}^k(X, V) \tag{B.4.12}$$

is continuous. Hence the $\mathscr{C}^{k+1}$-topology is finer than the $\mathscr{C}^k$-Topology. Also the inclusion map

$$\mathscr{C}^\infty(X, V) \longrightarrow \mathscr{C}^k(X, V) \tag{B.4.13}$$

is continuous. Similarly, the restriction maps to a smaller open subset $X' \subseteq X$ are continuous linear maps

$$\mathscr{C}^k(X, V) \ni f \;\mapsto\; f\big|_{X'} \in \mathscr{C}^k(X', V). \tag{B.4.14}$$

ii.) The derivative is a continuous linear map

$$\mathrm{D} \colon \mathscr{C}^{k+1}(X, V) \longrightarrow \mathscr{C}^k(X, \operatorname{Hom}(\mathbb{R}^n, V)). \tag{B.4.15}$$

iii.) For every unit vector $u \in \mathbb{R}^n$ the directional derivative is a continuous linear map

$$\mathrm{D}_u \colon \mathscr{C}^{k+1}(X, V) \longrightarrow \mathscr{C}^k(X, V). \tag{B.4.16}$$

iv.) For every $h \in \mathbb{R}^n$ and $f \in \mathscr{C}^{k+1}(X, V)$ one has

$$\mathrm{D}f(\,\cdot\,)h = \lim_{t \longrightarrow 0} \frac{f(\,\cdot\, + th) - f(\,\cdot\,)}{t} \tag{B.4.17}$$

as a limit in the $\mathscr{C}^k$-topology. More precisely, for every compact subset $K \subseteq X$ and t sufficiently small such that $x + th \in X$ for all $x \in K$ we have

$$\mathrm{p}_{K,\ell,\mathrm{q}}\left(\frac{f(\,\cdot\, + th) - f(\,\cdot\,)}{t} - \mathrm{D}f(\,\cdot\,)h\right) \longrightarrow 0 \tag{B.4.18}$$

for all $\ell \in \mathbb{N}_0$ with $\ell \leq k$ and all continuous seminorms q on V.

v.) The locally convex space $\mathscr{C}^k(X, V)$ is sequentially complete.

vi.) *If V is complete then $\mathscr{C}^k(X, V)$ is complete, too.*

vii.) *If $A\colon V \longrightarrow W$ is a continuous linear map into some other sequentially complete locally convex space then*

$$A\colon \mathscr{C}^k(X, V) \ni f \;\mapsto\; A \circ f \in \mathscr{C}^k(X, W) \tag{B.4.19}$$

is a well-defined continuous linear map with respect to the $\mathscr{C}^k$-topologies.

PROOF: The first part is clear from (B.4.11). For the second part we observe that for $\ell \in \mathbb{N}_0$ with $\ell \le k$

$$\mathrm{p}_{K,\ell,\mathrm{q}}(\mathrm{D}f) = \sup_{\substack{x\in K\\ \ell'\le \ell}} \left\| \mathrm{D}^{\ell'}(\mathrm{D}f)(x) \right\|_{\mathrm{q}} = \sup_{\substack{x\in K\\ \ell'\le \ell}} \left\| \mathrm{D}^{\ell'+1}f(x) \right\|_{\mathrm{q}} \le \sup_{\substack{x\in K\\ \ell'\le \ell+1}} \left\| \mathrm{D}^{\ell'}f(x) \right\|_{\mathrm{q}} = \mathrm{p}_{K,\ell+1,\mathrm{q}}(f)$$

for any compact subset $K \subseteq X$ and any continuous seminorm q on V. Thus (B.4.15) follows. The third part follows at once since D_u is the composition of the continuous map D by (B.4.15) with the evaluation on u. The latter is clearly continuous as well, implying (B.4.16). For the fourth part we need some more care. First we note that if X is a proper subset of $\mathbb{R}^n$ then the shifted function $f(\,\cdot + th)$ is, in general, no longer defined on all of X making (B.4.17) somewhat meaningless. However, the way to interpret (B.4.17) is by (B.4.18): for a fixed compact subset $K \subseteq X$ there is a $T > 0$ with $x \mapsto f(x + th)$ being still defined for all $x \in K$ and $|t| \le T$. Thus the statement (B.4.18) makes sense. To prove that (B.4.18) is actually true we fix h and consider only those t with $|t| \le T$ and T as above. Then for $t \ne 0$ and $\ell \le k$ we have

$$\begin{aligned} \left\| \frac{1}{t}\Big(\mathrm{D}^\ell f(x+th) - \mathrm{D}^\ell f(x)\Big) - \mathrm{D}^{\ell+1}f(x)h \right\|_{\mathrm{q}} &= \left\| \int_0^1 \Big(\mathrm{D}^{\ell+1}f(x+\tau th) - \mathrm{D}^{\ell+1}f(x)\Big) h \,\mathrm{d}\tau \right\|_{\mathrm{q}} \\ &\le \int_0^1 \left\| \mathrm{D}^{\ell+1}f(x+\tau th) - \mathrm{D}^{\ell+1}f(x) \right\|_{\mathrm{q}} \|h\| \,\mathrm{d}\tau, \qquad (*) \end{aligned}$$

where we have used an appropriate operator norm like seminorm $\|\cdot\|_{\mathrm{q}}$ build out of q and the fact that we do not have to take care in which argument of $\mathrm{D}^{\ell+1}f$ we have to insert h by the symmetry of $\mathrm{D}^{\ell+1}f$. For $|t| \le T$ we know that only points in $K + \mathrm{B}_{T\|h\|}(0)^{\mathrm{cl}}$ enter in $(*)$. But this is still a compact subset of X and thus $\mathrm{D}^{\ell+1}f$ is uniformly continuous on it: we find for $\epsilon > 0$ a $\delta > 0$ with the property that $\|x - y\| < \delta$ implies $\left\|\mathrm{D}^{\ell+1}f(x) - \mathrm{D}^{\ell+1}f(y)\right\|_{\mathrm{q}} < \epsilon$. Choosing t small enough brings us into this situation. Hence the integrand in $(*)$ is less than ϵ and thus

$$\left\| \frac{1}{t}\Big(\mathrm{D}^\ell f(x+th) - \mathrm{D}^\ell f(x)\Big) - \mathrm{D}^{\ell+1}f(x)h \right\|_{\mathrm{q}} < \epsilon\|h\|$$

for those t. Taking now the minimum over all the δ's for $0 \le \ell' \le \ell$ gives (B.4.18) thereby proving the fourth part. For the fifth part, let $(f_i)_{i\in\mathbb{N}_0}$ be a Cauchy sequence with respect to the $\mathscr{C}^k$-topology. Let $\alpha \in \mathbb{N}_0^n$ be a multiindex with $|\alpha| \le k$. Then the $\mathscr{C}^k$-seminorms show that each $(\partial^\alpha f_i)_{i\in I}$ is a Cauchy sequence with respect to the $\mathscr{C}^0$-topology. Hence by sequential completeness of V and Theorem B.1.6 we have $\mathscr{C}^0$-convergence to some $g_\alpha \in \mathscr{C}^0(X, V)$. In particular, $f_i \longrightarrow f \in \mathscr{C}^0(X, V)$. We claim that f is actually $\mathscr{C}^k$ and $\partial^\alpha f = g_\alpha$. Let $j = 1, \ldots, n$ then $\frac{\partial f_i}{\partial x^j} \longrightarrow g_j$ in the $\mathscr{C}^0$-topology implies that for a fixed $x \in X$ and all small enough t we have

$$\begin{aligned} f(x + t\mathrm{e}_j) - f(x) &= \lim_{i\longrightarrow\infty} (f_i(x + t\mathrm{e}_j) - f_i(x)) \\ &= \lim_{i\longrightarrow\infty} \int_0^t \frac{\partial f}{\partial x^j}(x + \tau\mathrm{e}_j)\,\mathrm{d}\tau \\ &= \int_0^t \lim_{i\longrightarrow\infty} \frac{\partial f}{\partial x^j}(x + \tau\mathrm{e}_j)\,\mathrm{d}\tau \end{aligned}$$

$$= \int_0^t g_j(x + \tau \mathrm{e}_j) \,\mathrm{d}\tau,$$

since the partial derivatives converge in the $\mathscr{C}^0$-topology and the integral is continuous with respect to that topology according to Proposition B.2.6, *ii.)*. But now g_j is a continuous function and for such we can apply the Fundamental Theorem of Calculus to see that $t \mapsto f(x + t\mathrm{e}_j)$ is differentiable at $t = 0$ with derivative given by $g_j(x)$. But this means that f has g_j as its j-th partial derivative, i.e. we showed that

$$\frac{\partial f}{\partial x^j}(x) = g_j(x)$$

for all $x \in X$ and $j = 1, \ldots, n$. Since all g_j are continuous we conclude that f is differentiable by Proposition B.3.9. By definition of the g_j we see that

$$\mathrm{D}f(x) = \lim_{i \longrightarrow \infty} \mathrm{D}f_i(x).$$

Repeating this argument shows that f is $\mathscr{C}^k$ with derivatives given as the limit of the corresponding derivatives of the f_i. A last check ensures that for all derivatives this limit is uniform on compact subsets. Hence $f_i \longrightarrow f$ in the $\mathscr{C}^k$-topology. The next part is completely analogous with the only modification that the Cauchy nets $(\partial^\alpha f_i)_{i \in I}$ now converge locally uniformly to some continuous g_α by completeness of V according to Theorem B.1.6. For the last part, we first note that $A \circ f$ is again $\mathscr{C}^k$. This follows inductively from Proposition B.3.2, *v.)*, and the fact that the composition of continuous maps is continuous. To show the continuity of (B.4.19) let q be a continuous seminorm on W. Then we find a continuous seminorm p on V such that $\mathrm{q}(Av) \le \mathrm{p}(v)$ for all $v \in V$ by the continuity of A. We have now for a compact subset $K \subseteq X$ and $\ell \in \mathbb{N}_0$ with $\ell \le k$

$$\begin{aligned}
\mathrm{p}_{K,\ell,\mathrm{q}}(A \circ f) &= \sup_{\substack{x \in K \\ |\alpha| \le \ell}} \mathrm{q}\left(\frac{\partial^{|\alpha|}(A \circ f)}{\partial (x^1)^{\alpha_1} \cdots \partial (x^n)^{\alpha_n}}(x) \right) \\
&= \sup_{\substack{x \in K \\ |\alpha| \le \ell}} \mathrm{q}\left(A \frac{\partial^{|\alpha|} f}{\partial (x^1)^{\alpha_1} \cdots \partial (x^n)^{\alpha_n}}(x) \right) \\
&\le \sup_{\substack{x \in K \\ |\alpha| \le \ell}} \mathrm{p}\left(\frac{\partial^{|\alpha|} f}{\partial (x^1)^{\alpha_1} \cdots \partial (x^n)^{\alpha_n}}(x) \right) \\
&= \mathrm{p}_{K,\ell,\mathrm{p}}(f),
\end{aligned}$$

by using Proposition B.3.2, *v.)*, several times. But this estimate shows immediately the continuity as claimed. □

Remark B.4.8 Since open subsets $X \subseteq \mathbb{R}^n$ allow for an exhausting *sequence* of compact subsets we see that for a Fréchet space V also $\mathscr{C}^k(X, V)$ is Fréchet for all $k \in \mathbb{N}_0 \cup \{+\infty\}$. However, for a Banach space V the functions $\mathscr{C}^k(X, V)$ will not be Banach anymore but only Fréchet.

B.4.3 The Leibniz Rule

The last feature of the $\mathscr{C}^k$-topology we want to discuss is the Leibniz rule for products. To have a rather flexible formulation we consider the following situation. Suppose we have a bilinear map

$$\mu \colon V \times W \longrightarrow U, \tag{B.4.20}$$

where V, W, and U are locally convex spaces. To be within reach of our present technology we require them all to be Hausdorff and sequentially complete. Instead of a continuous bilinear map μ

it will be sufficient to assume that μ is *sequentially continuous*, i.e. for convergent sequences $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ in V and W, respectively, we have

$$\lim_{n\longrightarrow\infty} \mu(v_n, w_n) = \mu\Big(\lim_{n\longrightarrow\infty} v_n, \lim_{n\longrightarrow\infty} w_n\Big). \tag{B.4.21}$$

This is precisely the sequential continuity with respect to the Cartesian product topology on $V \times W$. Clearly, any continuous μ is also sequentially continuous but the converse needs not to be true in general. Note also that for the following a *separate* continuity or even a separate sequential continuity would not be sufficient.

Lemma B.4.9 (Leibniz rule) *Let V, W, and U be sequentially complete locally convex spaces and let $\mu\colon V \times W \longrightarrow U$ be a sequentially continuous bilinear map. Moreover, let $X \subseteq \mathbb{R}^n$ be open and let $k \in \mathbb{N}_0 \cup \{+\infty\}$.*

i.) For every $f \in \mathscr{C}^k(X, V)$ and $g \in \mathscr{C}^k(X, W)$ we have $\mu(f, g) \in \mathscr{C}^k(X, U)$.

ii.) For every multiindex $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ we have the Leibniz rule

$$\partial^\alpha(\mu(f, g)) = \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \mu\Big(\partial^\beta f, \partial^{\alpha-\beta} g\Big). \tag{B.4.22}$$

PROOF: Let f and g be $\mathscr{C}^k$. We first note that the map $x \mapsto \mu(f(x)g(x))$ is sequentially continuous and hence a continuous map $\mu(f, g) \in \mathscr{C}^0(X, V)$ since X is first countable. This shows the case $k = 0$. Now let $k \geq 1$ and $j = 1, \ldots, n$. Then for small $t \neq 0$ we have

$$\begin{aligned}
&\frac{1}{t}\big(\mu(f, g)(x + t\mathrm{e}_j) - \mu(f, g)(x)\big) \\
&\quad= \frac{1}{t}\big(\mu(f(x + t\mathrm{e}_j), g(x + t\mathrm{e}_j)) - \mu(f(x), g(x))\big) \\
&\quad= \mu\left(\frac{f(x + t\mathrm{e}_j) - f(x)}{t}, g(x + t\mathrm{e}_j)\right) + \mu\left(f(x), \frac{g(x + t\mathrm{e}_j) - g(x)}{t}\right).
\end{aligned}$$

By sequential continuity of μ we see that the first term converges to $\mu\Big(\frac{\partial f}{\partial x^j}(x), g(x)\Big)$. Note that separate continuity would not be enough here. The second term converges to $\mu\Big(f(x), \frac{\partial g}{\partial x^j}(x)\Big)$. Since f and g have continuous (partial) derivatives we conclude by the case $k = 0$ that $\mu(f, g)$ has continuous partial derivatives obeying the Leibniz rule (B.4.22). Hence $\mu(f, g) \in \mathscr{C}^1(X, U)$ follows by Proposition B.3.9. A standard induction gives now the result for all derivatives $|\alpha| \leq k$. □

We have now several applications of the Leibniz rule. First we specialize to the situation where μ is even continuous:

Proposition B.4.10 *Let V, W, and U be sequentially complete locally convex spaces and let $\mu\colon V \times W \longrightarrow W$ be a continuous bilinear map. Moreover, let $X \subseteq \mathbb{R}^n$ be open and $k \in \mathbb{N}_0 \cup \{+\infty\}$. Then the bilinear map*

$$\mu\colon \mathscr{C}^k(X, V) \times \mathscr{C}^k(X, W) \longrightarrow \mathscr{C}^k(X, U) \tag{B.4.23}$$

is continuous. More precisely, if for a continuous seminorm r on U we have $\mathrm{r}(\mu(v, w)) \leq \mathrm{p}(v)\mathrm{q}(w)$ for suitable continuous seminorms p and q on V and W, respectively, then

$$\mathrm{p}_{K,\ell,\mathrm{r}}(\mu(f, g)) \leq 2^\ell \mathrm{p}_{K,\ell,\mathrm{p}}(f)\mathrm{p}_{K,\ell,\mathrm{q}}(g) \tag{B.4.24}$$

for all $\ell \in \mathbb{N}_0$ with $\ell \leq k$ and all compact subsets $K \subseteq X$.

PROOF: Clearly, it suffices to check (B.4.24). We estimate

$$
\begin{aligned}
\mathrm{p}_{K,\ell,\mathrm{r}}(\mu(f,g)) &= \sup_{\substack{x\in K\\ |\alpha|\le\ell}} \mathrm{r}\big(\partial^\alpha \mu(f,g)(x)\big)\\
&= \sup_{\substack{x\in K\\ |\alpha|\le\ell}} \mathrm{r}\left(\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\mu\big(\partial^\beta f(x), \partial^{\alpha-\beta} g(x)\big)\right)\\
&\le \sup_{\substack{x\in K\\ |\alpha|\le\ell}} \sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\mathrm{p}\big(\partial^\beta f(x)\big)\mathrm{q}\big(\partial^{\alpha-\beta} g(x)\big)\\
&\le 2^\ell \sup_{\substack{x\in K\\ |\alpha|\le\ell}} \mathrm{p}\big(\partial^\alpha f(x)\big) \sup_{\substack{y\in K\\ |\beta|\le\ell}} \mathrm{q}\big(\partial^\beta g(y)\big)\\
&= 2^\ell \mathrm{p}_{K,\ell,\mathrm{p}}(f)\mathrm{p}_{K,\ell,\mathrm{q}}(g),
\end{aligned}
$$

since $\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta} = 2^{|\alpha|} \le 2^\ell$. Thus (B.4.24) is shown. □

Corollary B.4.11 *Let $\mathscr{A}$ be a (sequentially) complete locally convex algebra. Then for every open $X \subseteq \mathbb{R}^n$ and for every $k \in \mathbb{N}_0 \cup \{+\infty\}$ the functions $\mathscr{C}^k(X, \mathscr{A})$ are a (sequentially) complete locally convex algebra, too.*

Corollary B.4.12 *Let $\mathscr{A}$ be a (sequentially) complete locally multiplicatively convex algebra. Then for every open $X \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}_0 \cup \{+\infty\}$ the functions $\mathscr{C}^k(X, \mathscr{A})$ are a (sequentially) complete locally multiplicatively convex algebra again.*

PROOF: Indeed, this follows from (B.4.24) by rescaling as in (4.1.33). □

Corollary B.4.13 *Let $\mathscr{A}$ be a (sequentially) complete locally convex algebra and $\mathscr{M}$ be a (sequentially) complete locally convex space with a sequentially continuous left $\mathscr{A}$-module structure. Then for every open $X \subseteq \mathbb{R}^n$ and every $k \in \mathbb{N}_0 \cup \{+\infty\}$ the (sequentially) complete locally convex space $\mathscr{C}^k(X, \mathscr{M})$ becomes a sequentially continuous left $\mathscr{C}^k(X, \mathscr{A})$-module. If the $\mathscr{A}$-module structure was even continuous then also $\mathscr{C}^k(X, \mathscr{M})$ is a continuous $\mathscr{C}^k(X, \mathscr{A})$-module.*

Example B.4.14 We can apply this to $\mathscr{A} = \mathbb{C}$ and $\mathscr{M} = V$ with an arbitrary (sequentially) complete locally convex space. Then $\mathscr{C}^k(X, \mathbb{C}) = \mathscr{C}^k(X)$ is a commutative associative complete locally multiplicatively convex unital algebra and $\mathscr{C}^k(X, V)$ becomes a (sequentially) complete $\mathscr{C}^k(X)$-module with continuous module structure since the multiplication with scalars is continuous for a topological vector space.

B.4.4 The space $\mathscr{C}^\infty_0(X, V)$

stefan: Defin… and $\mathscr{C}^\infty_K(X, V$… subset $K \subseteq X$…

B.5 Asymptotic Expansions and the Borel Lemma

B.5.1 Taylor's Theorem and the Borel Lemma

B.5.2 Asymptotic Expansions

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B.6 Holomorphic Functions

Let us now pass to holomorphic functions with values in a complex locally convex space V. We will establish some first properties of such functions under some very mild assumptions on the target space V, which essentially will allow to reduce everything to the scalar case.

B.6.1 Parameterized Curves and Integrals

In this preliminary subsection we recall some basic concepts from the theory of scalar holomorphic functions to fix our notation. More details can be found in any textbook on function theory like e.g. [14, 44, 45, 48] to name just a few.

closed curve

B.7 Symbols and Oscillatory Integrals

In this section we describe a first non-trivial application of the theory of Riemann integrals developed in Section B.2. The aim is to define integrals of functions $f\colon \mathbb{R}^n \longrightarrow V$ which may be very unbounded but oscillate "very fast" such that the averaging effect of the improper Riemann integral overcompensates the unboundedness. The results are based on [33], see also the classical texts [17, 22, 23] for the scalar case. Beyond the extremely useful scalar case also the vector-valued case is of importance as many recent constructions in deformation theory rely on these kind of oscillatory integrals, most notably the deformation of C^*-algebras according to Rieffel [46].

aybe in later chapter?

B.7.1 Vector-Valued Symbols

Though not literally the same, the class of functions we want to integrate is very much analogous to Hörmander's symbol classes, see [23, Section 7.8] as well as [17, 22]. In this subsection, let V be a sequentially complete Hausdorff locally convex space.

Definition B.7.1 (Symbol seminorms) *Let V be a sequentially complete locally convex space and let $f \in \mathscr{C}^k(\mathbb{R}^n, V)$ where $k \in \mathbb{N}_0 \cup \{+\infty\}$. Then for every continuous seminorm q on V, every multiindex $\mu \in \mathbb{N}_0^n$ with $|\mu| \leq k$ and $m, \rho \in \mathbb{R}$ one defines*

$$\|f\|^{m,\rho}_{\mathrm{q},\mu} = \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(m - \rho|\mu|)} \mathrm{q}(\partial^\mu f(x)) \in [0, +\infty]. \tag{B.7.1}$$

Here and in the following we will write $\partial^\mu f$ instead of $\frac{\partial^{|\mu|} f}{\partial x^\mu}$ for a multiindex $\mu \in \mathbb{N}_0^n$ as before. The quantity $\|f\|^{m,\rho}_{\mathrm{q},\mu}$ controls how the μ-th derivative of f grows at infinity with respect to the seminorm q compared to a polynomial of order m.

Example B.7.2 Let $f \in V[x^1, \ldots, x^n] \subseteq \mathscr{C}^\infty(\mathbb{R}^n, V)$ be a polynomial of degree $m \in \mathbb{N}_0$. Then for every continuous seminorm q on V and all $\mu \in \mathbb{N}_0^n$ we have

$$\|f\|^{m,1}_{\mathrm{q},\mu} < \infty. \tag{B.7.2}$$

Indeed, the μ-th derivative of f is a polynomial of degree $m - |\mu|$ and hence $\mathrm{q}(\partial^\mu f(x))$ is bounded by a polynomial of degree $m - \mu$. Thus the prefactor ensures that (B.7.2) is finite. Hence the parameter m encodes the polynomial growth at infinity while the parameter ρ is responsible for possible deviations of this polynomial behaviour concerning the derivatives. Clearly, $\rho = 1$ can be taken for polynomials, $\rho > 1$ says that the derivatives grow slower while $\rho < 1$ tells that the derivatives grow faster compared to those of a polynomial.

The next obvious lemma shows that the quantities $\|\cdot\|^{m,\rho}_{\mathrm{q},\mu}$ can be used as seminorms once they are finite:

Lemma B.7.3 *Let $m, \rho \in \mathbb{R}$ and let $\mu \in \mathbb{N}_0^n$ with $|\mu| \leq k \in \mathbb{N}_0 \cup \{+\infty\}$. Then for every continuous seminorm q on V and $f, g \in \mathscr{C}^k(\mathbb{R}^n, V)$ one has*

$$\|zf\|^{m,\rho}_{\mathrm{q},\mu} = |z| \|f\|^{m,\rho}_{\mathrm{q},\mu} \quad \textit{for} \quad z \in \mathbb{C} \tag{B.7.3}$$

and

$$\|f+g\|^{m,\rho}_{\mathrm{q},\mu} \le \|f\|^{m,\rho}_{\mathrm{q},\mu} + \|g\|^{m,\rho}_{\mathrm{q},\mu} \tag{B.7.4}$$

in the sense of (in-) equalities in $[0,+\infty]$.

In order to define the symbol classes we are now fixing a defining system $\mathcal{Q}$ of continuous seminorms on V. The canonical choice is of course to take all continuous seminorms while sometimes it will be advantageous to take only a small and manageable system. The following definitions will formally depend on this choice but the effect is only minor. Later we will see that the oscillatory integrals will not depend on the particular choice of $\mathcal{Q}$.

We will now assign to every $\mathrm{q} \in \mathcal{Q}$ real numbers $\boldsymbol{m}(q)$ and $\boldsymbol{\rho}(q)$. The corresponding map

$$\boldsymbol{m}\colon \mathcal{Q} \ni \mathrm{q} \;\mapsto\; \boldsymbol{m}(\mathrm{q}) \in \mathbb{R} \tag{B.7.5}$$

will be called the *order* and the map

$$\boldsymbol{\rho}\colon \mathcal{Q} \ni \mathrm{q} \;\mapsto\; \boldsymbol{\rho}(\mathrm{q}) \in \mathbb{R} \tag{B.7.6}$$

is referred to as the *type*. It will be important to allow *different* values $\boldsymbol{m}(\mathrm{q})$ for the different seminorms in the following applications.

The natural ordering of $\mathbb{R}$ induces one for the set of all orders as well as for the set of all types. For two orders $\boldsymbol{m}$, $\boldsymbol{m}'$ we write

$$\boldsymbol{m} \le \boldsymbol{m}' \quad \text{if} \quad \boldsymbol{m}(\mathrm{q}) \le \boldsymbol{m}'(\mathrm{q}) \tag{B.7.7}$$

for all continuous seminorms $\mathrm{q} \in \mathcal{Q}$. Then "$\le$" makes the set of all orders a directed set. Finally, we write

$$\boldsymbol{m} < \boldsymbol{m}' \quad \text{if} \quad \boldsymbol{m}(\mathrm{q}) < \boldsymbol{m}'(\mathrm{q}) \tag{B.7.8}$$

for all continuous seminorms $\mathrm{q} \in \mathcal{Q}$. Clearly, $\boldsymbol{m} < \boldsymbol{m}'$ implies $\boldsymbol{m} \le \boldsymbol{m}'$ and $<$ is transitive. Note that for two orders $\boldsymbol{m}$ and $\boldsymbol{m}'$ we can find another order $\boldsymbol{m}''$ with $\boldsymbol{m}, \boldsymbol{m}'' < \boldsymbol{m}''$, e.g. by setting

$$\boldsymbol{m}''(\mathrm{q}) = \boldsymbol{m}(\mathrm{q}) + \boldsymbol{m}'(\mathrm{q}) + \boldsymbol{\epsilon}(\mathrm{q}) \tag{B.7.9}$$

where $\boldsymbol{\epsilon}(\mathrm{q}) > 0$ may depend on $\mathrm{q} \in \mathcal{Q}$. If both orders satisfy $\boldsymbol{m}, \boldsymbol{m}' < \alpha$ then $\boldsymbol{m}''$ can be chosen to satisfy $\boldsymbol{m}, \boldsymbol{m}' < \boldsymbol{m}'' < \alpha$.

If we set $\boldsymbol{m}(\mathrm{q}) = m \in \mathbb{R}$ for all $\mathrm{q} \in \mathcal{Q}$ we get an order, called the *constant order*. The same applies for the constant type $\boldsymbol{\rho}(\mathrm{q}) = \rho \in \mathbb{R}$. It will turn out that this is usually too restrictive and we need more freedom in choosing an order and a type. More general, an order $\boldsymbol{m}$ is called *bounded* from above or from below by some number α or β if for all continuous seminorms $\mathrm{q} \in \mathcal{Q}$ one has $\boldsymbol{m}(\mathrm{q}) \le \alpha$ or $\boldsymbol{m}(\mathrm{q}) \ge \beta$, respectively.

In the following it will be reasonable to ask for the condition

$$\boldsymbol{m}(c\mathrm{q}) = \boldsymbol{m}(\mathrm{q}) \tag{B.7.10}$$

whenever $\mathrm{q}, c\mathrm{q} \in \mathcal{Q}$ for a constant $c > 0$. Assume V allows for a countable defining system of seminorms $\{\mathrm{q}_1, \mathrm{q}_2, \ldots\}$, i.e. V is a Fréchet space. Fixing any sequence $m_1, m_2, \ldots \in \mathbb{R}$ will determine an order by setting $\boldsymbol{m}(\mathrm{q}_n) = m_n$. In this situation we will be interested in having *unbounded* orders. For a Banach space V we usually take only the constant orders by specifying their value on the norm. Thus in this situation we will have *bounded* orders.

We are now able to define the space of symbols of order $\boldsymbol{m}$ and type $\boldsymbol{\rho}$ as a subspace of the smooth functions $\mathscr{C}^\infty(\mathbb{R}^n, V)$.

Definition B.7.4 (Symbols) *Let V be a sequentially complete locally convex space and $\mathcal{Q}$ a defining system of seminorms. Moreover, let $\boldsymbol{m}$ and $\boldsymbol{\rho}$ be an order and a type with respect to $\mathcal{Q}$.*

i.) *A function* $f \in \mathscr{C}^\infty(\mathbb{R}^n, V)$ *is called a symbol of order* $\boldsymbol{m}$ *and type* $\boldsymbol{\rho}$ *if for all continuous seminorms* q *and* $\mu \in \mathbb{N}_0^n$ *one has*

$$\|f\|_{\mathrm{q},\mu}^{\boldsymbol{m},\boldsymbol{\rho}} = \|f\|_{\mathrm{q},\mu}^{\boldsymbol{m}(\mathrm{q}),\boldsymbol{\rho}(\mathrm{q})} < \infty. \tag{B.7.11}$$

ii.) *The set of all symbols of order* $\boldsymbol{m}$ *and type* $\boldsymbol{\rho}$ *is denoted by* $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$.

Note that $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ still depends on the choice of $\mathcal{Q}$, even though we do not emphasize this in our notation.

From Lemma B.7.3 we see that the $\|\cdot\|_{\mathrm{q},\mu}^{\boldsymbol{m},\boldsymbol{\rho}}$ define a system of seminorms on $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$. We will endow the space of symbols with the corresponding locally convex topology, called the $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$*-topology* in the following. It is clear that this makes $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ a *Hausdorff* locally convex space since V is Hausdorff and the prefactor $(1 + \|x\|^2)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)}$ is nowhere vanishing.

Proposition B.7.5 *Let* V *be a sequentially complete locally convex space with a defining system of seminorms* $\mathcal{Q}$ *and let* $\boldsymbol{m}$ *and* $\boldsymbol{\rho}$ *be an order and a type with respect to* $\mathcal{Q}$.

i.) *We have continuous inclusions*

$$\mathscr{C}_0^\infty(\mathbb{R}^n, V) \longrightarrow \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow \mathscr{C}^\infty(\mathbb{R}^n, V). \tag{B.7.12}$$

ii.) *The symbols* $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ *are dense in* $\mathscr{C}^\infty(\mathbb{R}^n, V)$.

iii.) *The symbols* $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ *are sequentially complete and even complete if* V *is complete.*

iv.) *For* $\boldsymbol{m} \le \boldsymbol{m}'$ *and* $\boldsymbol{\rho} \ge \boldsymbol{\rho}'$ *we have the continuous inclusion*

$$\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow \mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}(\mathbb{R}^n, V). \tag{B.7.13}$$

More precisely, we have for all $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$, *all* $\mathrm{q} \in \mathcal{Q}$, *and all* $\mu \in \mathbb{N}_0^n$

$$\|f\|_{\mathrm{q},\mu}^{\boldsymbol{m}',\boldsymbol{\rho}'} \le \|f\|_{\mathrm{q},\mu}^{\boldsymbol{m},\boldsymbol{\rho}}. \tag{B.7.14}$$

PROOF: Clearly, we have a set-theoretic inclusion in (B.7.12) as compactly supported smooth functions decay fast enough to have finite symbol norms (B.7.1) for any choices of the orders or types. More precisely, let $K \subseteq \mathbb{R}^n$ be a compact subset and $f \in \mathscr{C}_K^\infty(\mathbb{R}^n, V)$. Then we get

$$\begin{aligned}
\|f\|_{\mathrm{q},\mu}^{\boldsymbol{m},\boldsymbol{\rho}} &= \sup_{x\in\mathbb{R}^n} \big(1 + \|x\|^2\big)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)} \mathrm{q}(\partial^\mu f(x)) \\
&= \max_{x\in K} \big(1 + \|x\|^2\big)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)} \mathrm{q}(\partial^\mu f(x)) \\
&\le \underbrace{\max_{x\in K} \big(1 + \|x\|^2\big)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)}}_{c} \max_{\substack{x\in K\\ |\nu|\le|\mu|}} \mathrm{q}(\partial^\nu f(x)) \\
&= c\,\mathrm{p}_{K,|\mu|,\mathrm{q}}(f),
\end{aligned}$$

with the seminorms $\mathrm{p}_{K,|\mu|,\mathrm{q}}(f)$ from the $\mathscr{C}^\infty$-topology as in Definition B.4.6. This shows that for every compact subset the inclusion

$$\mathscr{C}_K^\infty(\mathbb{R}^n, V) \longrightarrow \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$$

is continuous. By the universal property of the inductive limit topology of $\mathscr{C}_0^\infty(\mathbb{R}^n, V)$, see **??**, this is equivalent to the continuity of the first inclusion in (B.7.12). For the second inclusion we fix a compact subset $K \subseteq \mathbb{R}^n$ as well as $\ell \in \mathbb{N}_0$ and $\mathrm{q} \in \mathcal{Q}$. Then for a symbol $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ we have

$$\mathrm{p}_{K,\ell,\mathrm{q}}(f) = \max_{\substack{x\in K\\ |\mu|\le\ell}} \mathrm{q}(\partial^\mu f(x))$$

$$
\begin{aligned}
&\leq \underbrace{\max_{\substack{x\in K\\ |\mu|\leq \ell}} \bigl(1+\|x\|^2\bigr)^{\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)}}_{c} \sup_{\substack{x\in\mathbb{R}^n\\ |\mu|\leq\ell}} \bigl(1+\|x\|^2\bigr)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)} \mathrm{q}(\partial^\mu f(x))\\
&= c \max_{|\mu|\leq\ell} \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu},
\end{aligned}
$$

where we use the fact that the function $x \mapsto (1+\|x\|^2)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)}$ is nowhere vanishing and hence has a locally bounded inverse. This shows the continuity of the second inclusion in (B.7.12). But then the second part is clear since already $\mathscr{C}^\infty_0(\mathbb{R}^n, V)$ is dense in $\mathscr{C}^\infty(\mathbb{R}^n, V)$. In order to show sequential completeness let $(f_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$. Since the $\mathscr{C}^\infty$-topology is coarser than the $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$-topology by the first part, and since $\mathscr{C}^\infty(\mathbb{R}^n, V)$ is sequentially complete by Theorem B.4.7, *v.)*, we get a convergence $f_i \longrightarrow f$ to some smooth function $f \in \mathscr{C}^\infty(\mathbb{R}^n, V)$ in the $\mathscr{C}^\infty$-topology. We have to show that $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ and $f_i \longrightarrow f$ in the $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$-topology. Thus let $\epsilon > 0$, $\mathrm{q} \in \mathcal{Q}$, and $\mu \in \mathbb{N}_0^n$ be given. Then fix $N \in \mathbb{N}_0$ such that $i, j \geq N$ gives $\|f_i - f_j\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu} < \epsilon$ by the Cauchy condition. For a point $x \in \mathbb{R}^n$ we have by the pointwise convergence $\partial^\mu f_j(x) \longrightarrow \partial^\mu f(x)$ an $N' \geq N$ depending on x such that

$$
\bigl(1+\|x\|^2\bigr)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)} \mathrm{q}\bigl(\partial^\mu f_j(x) - \partial^\mu f(x)\bigr) < \epsilon \tag{$*$}
$$

for all $j \geq N'$. Thus for $i \geq N$ we get

$$
\begin{aligned}
&\bigl(1+\|x\|^2\bigr)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)} \mathrm{q}\bigl(\partial^\mu (f-f_i)(x)\bigr)\\
&\quad\leq \bigl(1+\|x\|^2\bigr)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)} \Bigl(\mathrm{q}\bigl(\partial^\mu (f-f_j)(x)\bigr) + \mathrm{q}\bigl(\partial^\mu (f_j-f_i)(x)\bigr)\Bigr)\\
&\quad\overset{(*)}{\leq} \epsilon + \|f_i - f_j\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu}\\
&\quad\leq 2\epsilon.
\end{aligned}
$$

Since this estimate can be done for all $x \in \mathbb{R}^n$ we can take the supremum over all $x \in \mathbb{R}^n$ and get $\|f - f_i\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu} \leq 2\epsilon$. Hence $f - f_i \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ for $i \geq N$ and thus also $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$. Moreover, we conclude that $f_i \longrightarrow f$ in the $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$-topology. Clearly, if V is even complete we can repeat the argument with nets instead of sequences thanks to Theorem B.4.7, *vi.)*. For the last part, it is clearly sufficient to show the estimate (B.7.14). Since for $\boldsymbol{m}(\mathrm{q}) \leq \boldsymbol{m}'(\mathrm{q})$ and $\boldsymbol{\rho}(\mathrm{q}) \geq \boldsymbol{\rho}'(\mathrm{q})$ we have

$$
\bigl(1+\|x\|^2\bigr)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)} \geq \bigl(1+\|x\|^2\bigr)^{-\frac{1}{2}(\boldsymbol{m}'(\mathrm{q})-\boldsymbol{\rho}'(\mathrm{q})|\mu|)}
$$

for every point $x \in \mathbb{R}^n$ and every $\mu \in \mathbb{N}_0^n$, the estimate (B.7.14) is clear. □

Remark B.7.6 (Banach and Fréchet space valued symbols) Recall that if V is a Banach space we choose the norm in order to define the space of symbols. In this case, the order $\boldsymbol{m} = m$ and the type $\boldsymbol{\rho} = \rho$ are just single numbers. However, $\mathrm{S}^{m,\rho}(\mathbb{R}^n, V)$ is no longer a Banach space but a Fréchet space since we still have to take care of countably many differentiations. For a Fréchet space V, we take a countable defining system of seminorms and hence an order is determined by fixing countably many numbers $\boldsymbol{m}(\mathrm{q}_n)$. Thus, in this situation the symbols are again a Fréchet space.

The topologies of the symbol spaces allow for a good behaviour under differentiation. More precisely, we have the following result:

Proposition B.7.7 *Let V be a sequentially complete locally convex space with a defining system of seminorms $\mathcal{Q}$. Then the partial derivatives are continuous linear maps*

$$
\frac{\partial^{|\nu|}}{\partial x^\nu} \colon \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow \mathrm{S}^{\boldsymbol{m}-\boldsymbol{\rho}|\nu|,\boldsymbol{\rho}}(\mathbb{R}^n, V). \tag{B.7.15}
$$

More precisely, we have for all $\mu \in \mathbb{N}_0^n$ *and* $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$

$$\|\partial^\nu f\|^{\boldsymbol{m}-\boldsymbol{\rho}|\nu|,\boldsymbol{\rho}}_{\mathrm{q},\mu} = \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu+\nu}. \tag{B.7.16}$$

PROOF: Clearly, we only have to show the second statement (B.7.16). We compute

$$\begin{aligned}
\|\partial^\nu f\|^{\boldsymbol{m}-\boldsymbol{\rho}|\nu|,\boldsymbol{\rho}}_{\mathrm{q},\mu} &= \sup_{x\in\mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\nu|-\boldsymbol{\rho}(\mathrm{q})|\mu|)} \mathrm{q}(\partial^\mu\partial^\nu f(x)) \\
&= \sup_{x\in\mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu+\nu|)} \mathrm{q}(\partial^{\mu+\nu} f(x)) \\
&= \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu+\nu}.
\end{aligned}$$

□

In a next step we consider the pointwise multiplication of symbols. To this end we consider three sequentially complete locally convex spaces V, W, and U together with a bilinear map

$$m\colon V \times W \longrightarrow U. \tag{B.7.17}$$

For simplicity, we require that m is continuous and not just separately continuous or sequentially continuous. For many applications this will be the case. Now we fix a defining system $\mathcal{R}$ of seminorms on U and filtrating defining systems of seminorms $\mathcal{Q}$ and $\mathcal{Q}'$ on V and W. Then by continuity of m we get for every $\mathrm{r} \in \mathcal{R}$ seminorms $\mathrm{q} \in \mathcal{Q}$ and $\mathrm{q}' \in \mathcal{Q}'$ such that

$$\mathrm{r}(m(v, w)) \le \mathrm{q}(v)\mathrm{q}'(w) \tag{B.7.18}$$

For two orders $\boldsymbol{m}$ and $\boldsymbol{m}'$ on V and W we consider an order $\boldsymbol{m}''$ on U such that for all $\mathrm{r} \in \mathcal{R}$ we have $\mathrm{q} \in \mathcal{Q}$ and $\mathrm{q}' \in \mathcal{Q}'$ such that (B.7.18) holds *and*

$$\boldsymbol{m}''(\mathrm{r}) \ge \boldsymbol{m}(\mathrm{q}) + \boldsymbol{m}'(\mathrm{q}'). \tag{B.7.19}$$

In this case, we symbolically write $\boldsymbol{m}'' \ge \boldsymbol{m} + \boldsymbol{m}'$ by some slight abuse of notation. Note that we relate here orders on *different* sets of seminorms (even on different spaces). Clearly, for given orders $\boldsymbol{m}$ and $\boldsymbol{m}'$ we can construct an order $\boldsymbol{m}''$ with this property by fixing a choice of seminorms $\mathrm{q}(\mathrm{r})$ and $\mathrm{q}'(\mathrm{r})$ satisfying (B.7.18) and setting

$$\boldsymbol{m}''(\mathrm{r}) = \boldsymbol{m}(\mathrm{q}(\mathrm{r})) + \boldsymbol{m}''(\mathrm{q}'(\mathrm{r})) \tag{B.7.20}$$

for all $\mathrm{r} \in \mathcal{R}$. In the same spirit we write $\boldsymbol{\rho}'' \le \min(\boldsymbol{\rho}, \boldsymbol{\rho}')$ for types $\boldsymbol{\rho}$ on V, $\boldsymbol{\rho}'$ on W, and $\boldsymbol{\rho}''$ on U, again with respect to the continuous bilinear map m. With these conventions in mind we can prove now the following statement:

Proposition B.7.8 *Let V, W, and U be sequentially complete locally convex spaces and let $m\colon V \times W \longrightarrow U$ be a continuous bilinear map. For a defining system $\mathcal{R}$ of seminorms on U and filtrating defining systems $\mathcal{Q}$ and $\mathcal{Q}'$ on V and W, respectively, we get a continuous bilinear map*

$$m\colon \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \times \mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}(\mathbb{R}^n, W) \longrightarrow \mathrm{S}^{\boldsymbol{m}'',\boldsymbol{\rho}''}(\mathbb{R}^n, U) \tag{B.7.21}$$

whenever $\boldsymbol{m}'' \ge \boldsymbol{m} + \boldsymbol{m}'$ and $\boldsymbol{\rho}'' \le \min(\boldsymbol{\rho}, \boldsymbol{\rho}')$ with respect to m. More precisely, for $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ and $g \in \mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}(\mathbb{R}^n, W)$ we get

$$\|m(f, g)\|^{\boldsymbol{m}'',\boldsymbol{\rho}''}_{\mathrm{r},\mu} \le 2^{|\mu|} \max_{\nu\le\mu} \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\nu} \max_{\nu'\le\mu} \|g\|^{\boldsymbol{m}',\boldsymbol{\rho}'}_{\mathrm{q}',\nu'} \tag{B.7.22}$$

whenever r, q, *and* q′ *satisfy the continuity property* (B.7.18) *of* m.

Proof: Even though the formulation looks rather technical, this is essentially just the Leibniz rule as shown in Lemma B.4.9. Let r be a continuous seminorm from $\mathcal{R}$ and choose corresponding seminorms q and q' with (B.7.18). Then we have

$$
\begin{aligned}
\|m(f,g)\|^{\boldsymbol{m}'',\boldsymbol{\rho}''}_{\mathrm{r},\mu} &= \sup_{x\in\mathbb{R}^n} \left(1+\|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}''(\mathrm{r})-\boldsymbol{\rho}''(\mathrm{r})|\mu|)} \mathrm{r}\left((\partial^\mu m(f,g))(x)\right) \\
&= \sup_{x\in\mathbb{R}^n} \left(1+\|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}''(\mathrm{r})-\boldsymbol{\rho}''(\mathrm{r})|\mu|)} \mathrm{r}\left(\sum_{\nu+\nu'=\mu} \binom{\mu}{\nu} m\left(\partial^\nu f(x), \partial^{\nu'} g(x)\right)\right) \\
&\le \sum_{\nu+\nu'=\mu} \binom{\mu}{\nu} \sup_{x\in\mathbb{R}^n} \left(1+\|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\nu|)} \mathrm{q}(\partial^\nu f(x)) \\
&\qquad \sup_{x\in\mathbb{R}^n} \left(1+\|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}'(\mathrm{q}')-\boldsymbol{\rho}'(\mathrm{q}')|\nu'|)} \mathrm{q}'\left(\partial^{\nu'} g(x)\right) \\
&\le 2^{|\mu|} \max_{\nu\le\mu} \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\nu} \max_{\nu'\le\mu} \|g\|^{\boldsymbol{m}',\boldsymbol{\rho}'}_{\mathrm{q}',\nu'},
\end{aligned}
$$

since $|\mu| = |\nu| + |\nu'|$. This shows (B.7.22) which implies the continuity statement (B.7.21). □

Then main application will be to multiply vector-valued symbols with scalar symbols: choosing one vector space just to be $\mathbb{C}$ and choosing as seminorms on $\mathbb{C}$ just the absolute value we get for every order $m \in \mathbb{R}$ and every type $\rho \in \mathbb{R}$ the space of scalar symbols

Exercise: Mo proof for non systems of se

$$
\mathrm{S}^{m,\rho}(\mathbb{R}^n) = \mathrm{S}^{m,\rho}(\mathbb{R}^n, \mathbb{C}). \tag{B.7.23}
$$

Note that now the order and the type are indeed just single numbers and we are thus back in the situation of [24].

Corollary B.7.9 *Let V be a sequentially complete locally convex space with a defining system of seminorms $\mathcal{Q}$.*

i.) For all orders $\boldsymbol{m}$ and types $\boldsymbol{\rho}$ and all $m, \rho \in \mathbb{R}$, the pointwise multiplication gives a continuous bilinear map

$$
\mathrm{S}^{m,\rho}(\mathbb{R}^n, \mathbb{C}) \times \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow \mathrm{S}^{\boldsymbol{m}+m,\min(\boldsymbol{\rho},\rho)}(\mathbb{R}^n, V). \tag{B.7.24}
$$

ii.) In particular, if the type $\boldsymbol{\rho}$ is bounded from above by $\rho \in \mathbb{R}$ then

$$
\mathrm{S}^{m,\rho}(\mathbb{R}^n, \mathbb{C}) \times \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow \mathrm{S}^{\boldsymbol{m}+m,\boldsymbol{\rho}}(\mathbb{R}^n, V). \tag{B.7.25}
$$

iii.) If $m \le 0$ then $\mathrm{S}^{m,\rho}(\mathbb{R}^n, \mathbb{C})$ is a Fréchet algebra and $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ is a continuous module over it for all types $\boldsymbol{\rho}$ with $\boldsymbol{\rho} \le \rho$.

Proof: Everything is immediate from the last proposition and the fact that the multiplication by scalars is continuous for every locally convex space. □

Corollary B.7.10 *Let $\mathscr{A}$ be a sequentially complete locally convex algebra with a defining system of seminorms $\mathcal{Q}$. Then the multiplication induces a continuous product*

$$
\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, \mathscr{A}) \times \mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}}(\mathbb{R}^n, \mathscr{A}) \longrightarrow \mathrm{S}^{\boldsymbol{m}+\boldsymbol{m}',\boldsymbol{\rho}}(\mathbb{R}^n, \mathscr{A}). \tag{B.7.26}
$$

In particular, for $\boldsymbol{m} \le 0$ the symbols $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, \mathscr{A})$ form a sequentially complete locally convex algebra themselves and any $\mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}}(\mathbb{R}^n, \mathscr{A})$ is a sequentially complete locally convex continuous module over them.

We come now to the important fact that the compactly supported functions are sequentially dense in the symbols with respect to the topology given by a strictly larger order:

Proposition B.7.11 *Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ be a compactly supported smooth function with*

$$\chi_{\mathrm{B}_r(0)} = 1 \quad \textit{and} \quad \operatorname{supp}\chi \subseteq \mathrm{B}_R(0), \tag{B.7.27}$$

taking values between 0 and 1, where $0 < r < R$. Moreover, let $\chi_\epsilon \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ be defined by $\chi_\epsilon(x) = \chi(\epsilon x)$ for $\epsilon > 0$.

i.) One has $\chi_\epsilon - 1 \in \mathrm{S}^{0,\rho}(\mathbb{R}^n, \mathbb{C})$ for all $\rho \in \mathbb{R}$.

ii.) One has

$$\lim_{\epsilon \longrightarrow 0} \chi_\epsilon = 1 \tag{B.7.28}$$

in the $\mathrm{S}^{m,\rho}$-topology for all $m > 0$ and $\rho \leq 1$.

iii.) For all $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ we have

$$\lim_{\epsilon \longrightarrow 0} \chi_\epsilon f = f \tag{B.7.29}$$

in the $\mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}$-topology for all $\boldsymbol{m}' > \boldsymbol{m}$ and $\boldsymbol{\rho}' \leq \min(1, \boldsymbol{\rho})$.

iv.) For all $\boldsymbol{m}' > \boldsymbol{m}$ and $\boldsymbol{\rho}' \leq \min(1, \boldsymbol{\rho})$ the compactly supported smooth functions $\mathscr{C}_0^\infty(\mathbb{R}^n, V)$ are sequentially dense in $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ with respect to the $\mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}$-topology.

PROOF: Clearly, $1 \in \mathrm{S}^{0,\rho}(\mathbb{R}^n, \mathbb{C})$ for all $\rho \in \mathbb{R}$ and $\chi_\epsilon \in \mathrm{S}^{m,\rho}(\mathbb{R}^n, \mathbb{C})$ for all $m, \rho \in \mathbb{R}$ by Proposition B.7.5, *i.)*. For the second part, we clearly have pointwise convergence and even convergence in the $\mathscr{C}^\infty$-topology. For $\mu = 0$ we have

$$\begin{aligned}
\|\chi_\epsilon - 1\|_0^{m,\rho} &= \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}m} |\chi_\epsilon(x) - 1| \\
&= \sup_{\|x\| \geq \frac{r}{\epsilon}} \left(1 + \|x\|^2\right)^{-\frac{1}{2}m} |\chi_\epsilon(x) - 1| \\
&\leq \sup_{\|x\| \geq \frac{r}{\epsilon}} \left(1 + \|x\|^2\right)^{-\frac{1}{2}m} \\
&= \left(\frac{r^2 + \epsilon^2}{\epsilon^2}\right)^{-\frac{1}{2}m},
\end{aligned}$$

since $\|\chi_\epsilon - 1\|_\infty = 1$ by the properties χ_ϵ. This clearly converges to zero since $m > 0$. For $\mu \neq 0$ we have $\partial^\mu \chi_\epsilon(x) = \epsilon^{|\mu|}(\partial^\mu \chi)(\epsilon x)$ and hence

$$\begin{aligned}
\|\chi_\epsilon - 1\|_\mu^{m,\rho} &= \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(m - \rho|\mu|)} \left|\partial^\mu \chi_\epsilon(x)\right| \\
&\leq \sup_{\frac{r}{\epsilon} \leq \|x\| \leq \frac{R}{\epsilon}} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(m - \rho|\mu|)} \epsilon^{|\mu|} c_\mu,
\end{aligned}$$

where $c_\mu = \|\partial^\mu \chi(x)\|_\infty < \infty$, again thanks to the compact support. Now either $m - \rho|\mu| \geq 0$ then the supremum is taken at the smallest possible $\|x\| = \frac{r}{\epsilon}$ or $m - \rho|\mu| < 0$ then the supremum is taken at the largest possible $\|x\| = \frac{R}{\epsilon}$. Thus we get in the first case for $\epsilon \leq 1$

$$\|\chi_\epsilon - 1\|_\mu^{m,\rho} \leq c_\mu \epsilon^{|\mu|} \left(\frac{r^2 + \epsilon^2}{\epsilon^2}\right)^{-\frac{1}{2}(m - \rho|\mu|)} \leq c'_\mu \epsilon^{m + (1-\rho)|\mu|},$$

and in the second case we get the same estimate with a different numerical constant c''_μ instead of c_μ. For the behaviour under $\epsilon \longrightarrow 0$ these factors do not play any role but the sign of $m + (1 - \rho)|\mu|$

does: If $\rho > 1$ then for large enough $|\mu|$ we get divergence and hence $\|\chi_\epsilon - 1\|_\mu^{m,\rho}$ does not converge to zero. If, on the other hand $\rho \leq 1$ then $m + (1-\rho)|\mu|$ is always strictly positive. In this case we have convergence $\|\chi_\epsilon - 1\|_\mu^{m,\rho} \longrightarrow 0$ for all μ. This explains the condition $\rho \leq 1$ and proves the second part. For the third part we rely on the estimates proved in Proposition B.7.8: for $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ and a fixed seminorm q from the defining system $\mathcal{Q}$ we get the estimate

$$\|(\chi_\epsilon - 1)f\|_{\mathrm{q},\mu}^{\boldsymbol{m}',\boldsymbol{\rho}'} \leq 2^{|\mu|} \max_{\nu \leq \mu} \|\chi_\epsilon - 1\|_\nu^{m,\rho} \max_{\nu' \leq \mu} \|f\|_{\mathrm{q},\nu'}^{\boldsymbol{m},\boldsymbol{\rho}}$$

for every m and $\boldsymbol{m}'$ provided $\boldsymbol{m}'(\mathrm{q}) \geq \boldsymbol{m}(\mathrm{q}) + m$, and every ρ and $\boldsymbol{\rho}'$ provided $\boldsymbol{\rho}'(\mathrm{q}) \leq \min(\rho, \boldsymbol{\rho}(\mathrm{q}))$. Now from the second part we know that $\|\chi_\epsilon - 1\|_\nu^{m,\rho}$ converges to zero whenever $\rho \leq 1$ and $m > 0$. This means that for the fixed seminorm q we get $\|(\chi_\epsilon - 1)f\|_{\mathrm{q},\mu}^{\boldsymbol{m}',\boldsymbol{\rho}'} \longrightarrow 0$ whenever $\boldsymbol{m}'(\mathrm{q}) > \boldsymbol{m}(\mathrm{q})$ and $\boldsymbol{\rho}'(\mathrm{q}) \leq \min(1, \boldsymbol{\rho}(\mathrm{q}))$. Since this is the condition for every $\mathrm{q} \in \mathcal{Q}$ we get the third part. Note that we are allowed to make the parameter m depend on q as long as we have $m > 0$. Thus $\boldsymbol{m}'(\mathrm{q}) > \boldsymbol{m}(\mathrm{q})$ does *not* have to be uniformly satisfied. The last part is now clear as it suffices to take $\epsilon = \frac{1}{n}$ as usual. □

We shall now discuss how our definition of the symbol spaces depends on the choice of the defining system of seminorms $\mathcal{Q}$. To this end, we shall proceed in two steps: first we show how one can pass from an arbitrary system to a filtrating one, then we compare two filtrating systems.

Now suppose $\mathcal{Q}$ is an arbitrary defining system of continuous seminorms for V. Then we consider the larger system

$$\tilde{\mathcal{Q}} = \left\{ \mathrm{q} = c \max\{\mathrm{q}_1, \ldots, \mathrm{q}_n\} \;\middle|\; n \in \mathbb{N},\ c > 0, \text{ and } \mathrm{q}_1, \ldots, \mathrm{q}_n \in \mathcal{Q} \right\} \tag{B.7.30}$$

which is now filtrating. Suppose now that $\boldsymbol{m}$ is an order with respect to $\mathcal{Q}$. Then we want to extend $\boldsymbol{m}$ to an order on $\tilde{\mathcal{Q}}$ as follows. We define for all $c > 0$

$$\boldsymbol{m}_{\max}(c \max\{\mathrm{q}_1, \ldots, \mathrm{q}_n\}) = \max\{\boldsymbol{m}(\mathrm{q}_1), \ldots, \boldsymbol{m}(\mathrm{q}_n)\} \tag{B.7.31}$$

in accordance with our convention (B.7.10). Clearly, this gives an order on $\tilde{\mathrm{Q}}$ which extends $\boldsymbol{m}$. Analogously, for a type $\boldsymbol{\rho}$ with respect to $\mathcal{Q}$ we define a type $\boldsymbol{\rho}_{\min}$ with respect to $\tilde{\mathcal{Q}}$ extending $\boldsymbol{\rho}$ by taking the minimum of the types $\boldsymbol{\rho}(\mathrm{q}_i)$ instead of the maximum.

Proposition B.7.12 *Let $\mathcal{Q}$ be a defining system of continuous seminorms on V and $\tilde{\mathcal{Q}}$ the corresponding filtrating system of finite maxima. Then for every order $\boldsymbol{m}$ and every type $\boldsymbol{\rho}$ with respect to $\mathcal{Q}$ and their corresponding extensions $\boldsymbol{m}_{\max}$ and $\boldsymbol{\rho}_{\min}$ to $\tilde{\mathcal{Q}}$ we have*

$$\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) = \mathrm{S}^{\boldsymbol{m}_{\max},\boldsymbol{\rho}_{\min}}(\mathbb{R}^n, V) \tag{B.7.32}$$

as locally convex spaces.

PROOF: First let $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ and let $\mathrm{q}_1, \ldots, \mathrm{q}_n \in \mathcal{Q}$ and $c > 0$ be given. We set $\mathrm{q} = c \max\{\mathrm{q}_1, \ldots, \mathrm{q}_n\}$. For $\mu \in \mathbb{N}_0^n$ we have the estimate

$$\begin{aligned}
\|f\|_{\mathrm{q},\mu}^{\boldsymbol{m}_{\max},\boldsymbol{\rho}_{\min}} &= \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}_{\max}(\mathrm{q}) - \boldsymbol{\rho}_{\min}(\mathrm{q})|\mu|)} \mathrm{q}(\partial^\mu f(x)) \\
&\leq c \sum_{i=1}^n \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}_{\max}(\mathrm{q}) - \boldsymbol{\rho}_{\min}(\mathrm{q})|\mu|)} \mathrm{q}_i(\partial^\mu f(x)) \\
&\leq c \sum_{i=1}^n \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q}_i) - \boldsymbol{\rho}(\mathrm{q}_i)|\mu|)} \mathrm{q}_i(\partial^\mu f(x))
\end{aligned}$$

$$= c\sum_{i=1}^{n} \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q}_i,\mu},$$

since $\boldsymbol{m}_{\max}(\mathrm{q}) - \boldsymbol{\rho}_{\min}(\mathrm{q})|\mu| \geq \boldsymbol{m}(\mathrm{q}_i) - \boldsymbol{\rho}(\mathrm{q}_i)|\mu|$ This shows $f \in \mathrm{S}^{\boldsymbol{m}_{\max},\boldsymbol{\rho}_{\min}}(\mathbb{R}^n, V)$ as well as the continuity of the inclusion map

$$\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow \mathrm{S}^{\boldsymbol{m}_{\max},\boldsymbol{\rho}_{\min}}(\mathbb{R}^n, V).$$

Conversely, let $f \in \mathrm{S}^{\boldsymbol{m}_{\max},\boldsymbol{\rho}_{\min}}(\mathbb{R}^n, V)$ be given. Then clearly $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ since all the seminorms $\|\cdot\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu}$ of the $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$-topology appear also as seminorms of the $\mathrm{S}^{\boldsymbol{m}_{\max},\boldsymbol{\rho}_{\min}}$-topology, since $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}$ and the order and type are extended to the larger system of seminorms. With respect to these seminorms $\|\cdot\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu}$, the reverse inclusion

$$\mathrm{S}^{\boldsymbol{m}_{\max},\boldsymbol{\rho}_{\min}}(\mathbb{R}^n, V) \longrightarrow \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$$

is even isometric and hence continuous, too. Thus we have mutually inverse continuous inclusions proving the claim. □

Next we consider two defining systems of seminorms $\mathcal{Q}$ and $\mathcal{Q}'$ on V where we can assume that they are already filtrating. Thus for every $\mathrm{q} \in \mathcal{Q}$ we find a $\mathrm{q}' \in \mathcal{Q}'$ with $\mathrm{q} \leq \mathrm{q}'$, and vice versa. In this situation we have the following statement:

Proposition B.7.13 *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be defining systems of seminorms for V with $\mathcal{Q}'$ being filtrating. Moreover, let $\boldsymbol{m}$, $\boldsymbol{m}'$ be orders and $\boldsymbol{\rho}, \boldsymbol{\rho}'$ be types with respect to $\mathcal{Q}$ and $\mathcal{Q}'$, respectively. If for every $\mathrm{q} \in \mathcal{Q}$ there exists a $\mathrm{q}' \in \mathcal{Q}'$ such that*

$$\mathrm{q} \leq \mathrm{q}', \quad \boldsymbol{m}(\mathrm{q}) \geq \boldsymbol{m}'(\mathrm{q}'), \quad \textit{and} \quad \boldsymbol{\rho}(\mathrm{q}) \leq \boldsymbol{\rho}'(\mathrm{q}'), \tag{B.7.33}$$

then one has a continuous inclusion

$$\mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}(\mathbb{R}^n, V) \subseteq \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V). \tag{B.7.34}$$

PROOF: Let $\mathrm{q} \in \mathcal{Q}$ be given and choose q' according to (B.7.33). Then for every $\mu \in \mathbb{N}_0^n$ we have

$$\begin{aligned}
\|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu} &= \sup_{x\in\mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)} \mathrm{q}(\partial^\mu f(x)) \\
&\leq \sup_{x\in\mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}'(\mathrm{q}')-\boldsymbol{\rho}'(\mathrm{q}')|\mu|)} \mathrm{q}'(\partial^\mu f(x)) \\
&= \|f\|^{\boldsymbol{m}',\boldsymbol{\rho}'}_{\mathrm{q}',\mu},
\end{aligned}$$

since $\boldsymbol{m}(\mathrm{q}) - \boldsymbol{\rho}(\mathrm{q})|\mu| \geq \boldsymbol{m}'(\mathrm{q}') - \boldsymbol{\rho}'(\mathrm{q}')|\mu|$ which shows that claim. □

Corollary B.7.14 *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be defining systems of continuous seminorms. Moreover, let $\boldsymbol{m}'$ and $\boldsymbol{\rho}'$ be an order and a type for $\mathcal{Q}'$. Then there exists an order $\boldsymbol{m}$ and a type $\boldsymbol{\rho}$ with respect to $\mathcal{Q}$ such that*

$$\mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}(\mathbb{R}^n, V) \subseteq \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \tag{B.7.35}$$

is continuously included. If in addition $\boldsymbol{m}'$ or $\boldsymbol{\rho}'$ are bounded then $\boldsymbol{m}$ and $\boldsymbol{\rho}$ can be chosen to satisfy the same bounds, respectively.

PROOF: By Proposition B.7.12 we can pass to a filtrating system without changing the symbol space on the left hand side. Thus we can assume that $\mathcal{Q}'$ is already filtrating without restriction. Let $\mathrm{q} \in \mathcal{Q}$. Then we fix a particular choice $\mathrm{q}'(\mathrm{q}) \in \mathcal{Q}'$ with $\mathrm{q} \leq \mathrm{q}'$. This defines a map $\mathcal{Q} \longrightarrow \mathcal{Q}'$, existing thanks to the fact that $\mathcal{Q}'$ is filtrating. Now we define $\boldsymbol{m}(\mathrm{q}) = \boldsymbol{m}'(\mathrm{q}'(\mathrm{q}))$ and $\boldsymbol{\rho}(\mathrm{q}) = \boldsymbol{\rho}(\mathrm{q}'(\mathrm{q}))$. Then clearly the condition (B.7.33) from Proposition B.7.13 is satisfied and we get (B.7.35). The statements about the bounds is then clear. □

Corollary B.7.15 *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be two defining systems of seminorms for V and let $f \in \mathscr{C}^\infty(\mathbb{R}^n, V)$ be a smooth function. Then f is a symbol of some (bounded) order $\boldsymbol{m}$ and some (bounded) type $\boldsymbol{\rho}$ with respect to $\mathcal{Q}$ iff f is a symbol of some (bounded) order $\boldsymbol{m}'$ and some (bounded) type $\boldsymbol{\rho}'$ with respect to $\mathcal{Q}'$ (and the same bounds).*

Proof: By Proposition B.7.12 we can assume to have filtrating systems from the beginning. Since the extension of the order and the type according to that proposition clearly preserves the bounds, Corollary B.7.14 can be applied in both directions. ☐

We can now use the last corollaries to speak about the space of *all* symbols: there are two options whether or not we allow for bounded orders only:

Definition B.7.16 (Spaces of all symbols) *Let V be a sequentially complete locally convex space. Then we define for a given type $\boldsymbol{\rho}$ with respect to a defining system of seminorms $\mathcal{Q}$*

$$\mathrm{S}^{\infty,\boldsymbol{\rho}}(\mathbb{R}^n, V) = \bigcup_{\boldsymbol{m} \text{ bounded}} \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \tag{B.7.36}$$

and

$$\mathrm{S}^{\infty+,\boldsymbol{\rho}}(\mathbb{R}^n, V) = \bigcup_{\boldsymbol{m}} \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V). \tag{B.7.37}$$

Moreover, we set

$$\mathrm{S}^{\infty}(\mathbb{R}^n, V) = \mathrm{S}^{\infty,1}(\mathbb{R}^n, V) \quad \textit{and} \quad \mathrm{S}^{\infty+}(\mathbb{R}^n, V) = \mathrm{S}^{\infty+,1}(\mathbb{R}^n, V). \tag{B.7.38}$$

It follows that for another defining system of seminorms $\mathcal{Q}'$ we get the same spaces $\mathrm{S}^\infty(\mathbb{R}^n, V)$ and $\mathrm{S}^{\infty+}(\mathbb{R}^n, V)$, which are therefore intrinsically defined. Note that for a Banach space V with $\mathcal{Q}$ consisting just of the norm itself we have $\mathrm{S}^{\infty,\rho}(\mathbb{R}^n, V) = \mathrm{S}^{\infty+,\rho}(\mathbb{R}^n, V)$ for all types $\rho \in \mathbb{R}$. However, in general we have a proper inclusion

$$\mathrm{S}^{\infty,\boldsymbol{\rho}}(\mathbb{R}^n, V) \subsetneq \mathrm{S}^{\infty+,\boldsymbol{\rho}}(\mathbb{R}^n, V). \tag{B.7.39}$$

Also the intersection of all the symbol spaces is of interest: here we get an analog of the usual Schwartz space. First we note the following simple facts:

Lemma B.7.17 *Let V be a sequentially complete locally convex space and $f \in \mathscr{C}^\infty(\mathbb{R}^n, V)$. Then the following statements are equivalent:*

i.) For all continuous seminorms q of a defining system $\mathcal{Q}$, for all $\mu \in \mathbb{N}_0^n$ and all $m \in \mathbb{N}_0$ one has

$$\mathrm{q}_{m,\mu}(f) = \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{\frac{m}{2}} \mathrm{q}(\partial^\mu f(x)) < \infty. \tag{B.7.40}$$

ii.) For all orders $\boldsymbol{m}$ and all types $\boldsymbol{\rho}$ with respect to a given defining system $\mathcal{Q}$ of continuous seminorms one has $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$.

iii.) For all orders $\boldsymbol{m}$ and one type $\boldsymbol{\rho}$ with respect to a given defining system $\mathcal{Q}$ of continuous seminorms one has $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$.

Proof: First we note that if *i.)* holds for a defining system of seminorms $\mathcal{Q}$ then it also holds for *all* continuous seminorms of V. This is clear. Thus assume *i.)* and let $\mathcal{Q}$ be a defining system of seminorms. Moreover, fix an order $\boldsymbol{m}$ and a type $\boldsymbol{\rho}$ for this system $\mathcal{Q}$. Then for $\mu \in \mathbb{N}_0^n$ we have

$$\|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu} = \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q}) - \boldsymbol{\rho}(\mathrm{q})|\mu|)} \mathrm{q}(\partial^\mu f(x)) \leq \mathrm{q}_{m,\mu}(f),$$

with m being any integer larger than $-\boldsymbol{m}(\mathrm{q}) + \boldsymbol{\rho}(\mathrm{q})|\mu|$. This shows that $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$. The implication *ii.)* $\implies$ *iii.)* is trivial. Thus assume *iii.)* and hence let $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ for all orders $\boldsymbol{m}$ and a fixed type $\boldsymbol{\rho}$. Then let $m \in \mathbb{N}_0$ and $\mu \in \mathbb{N}_0^n$ be given. We have

$$\mathrm{q}_{m,\mu}(f) = \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{\frac{m}{2}} \mathrm{q}(\partial^\mu f(x)) \leq \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q}) - \boldsymbol{\rho}(\mathrm{q})|\mu|)} \mathrm{q}(\partial^\mu f(x)) = \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu},$$

where we have to choose an order $\boldsymbol{m}$ such that $\boldsymbol{m}(\mathrm{q}) - \boldsymbol{\rho}(\mathrm{q})|\mu| \leq -m$. This is clearly possible as we can e.g. take the constant order with $\boldsymbol{m} = -m + \boldsymbol{\rho}(\mathrm{q})|\mu|$. Thus *i.)* follows. ☐

Hence for the intersection of all symbol spaces the type $\boldsymbol{\rho}$ does not play any role any more. Also the dependence on the chosen system of seminorms $\mathcal{Q}$ disappears. This motivates the following definition:

Definition B.7.18 (Vector-valued Schwartz space) *Let V be a sequentially complete locally convex space. Then we define the symbols of order $-\infty$ by*

$$\mathrm{S}^{-\infty}(\mathbb{R}^n, V) = \bigcap_{\boldsymbol{m},\boldsymbol{\rho}} \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V). \tag{B.7.41}$$

We also use the notation

$$\mathscr{S}(\mathbb{R}^n, V) = \mathrm{S}^{-\infty}(\mathbb{R}^n, V) \tag{B.7.42}$$

and call $\mathscr{S}(\mathbb{R}^n, V)$ the space of V-valued Schwartz functions.

Clearly, the V-valued Schwartz functions are a straightforward generalization of the scalar case as discussed in Exercise 2.5.32. The Schwartz space $\mathscr{S}(\mathbb{R}^n, V)$ will always be endowed with the locally convex topology determined by the seminorms $\mathrm{q}_{m,\mu}$ as in (B.7.40). We call this the $\mathrm{S}^{-\infty}$- or the Schwartz topology of $\mathscr{S}(\mathbb{R}^n, V)$. We collect now some easy properties of the Schwartz space:

Proposition B.7.19 *Let V be a sequentially complete locally convex space with a defining system of seminorms $\mathcal{Q}$.*

i.) *The Schwartz space $\mathscr{S}(\mathbb{R}^n, V)$ is sequentially complete and complete whenever V is complete.*

ii.) *We have continuous inclusions*

$$\mathscr{C}_0^\infty(\mathbb{R}^n, V) \longrightarrow \mathscr{S}(\mathbb{R}^n, V) \longrightarrow \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \tag{B.7.43}$$

for all orders $\boldsymbol{m}$ and all types $\boldsymbol{\rho}$ with respect to $\mathcal{Q}$.

iii.) *The space $\mathscr{C}_0^\infty(\mathbb{R}^n, V)$ is sequentially dense in $\mathscr{S}(\mathbb{R}^n, V)$ and $\mathscr{S}(\mathbb{R}^n, V)$ is sequentially dense in $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ in the $\mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}$-topology wherever $\boldsymbol{m}' > \boldsymbol{m}$ and $\boldsymbol{\rho}' \leq \min(1, \boldsymbol{\rho})$.*

iv.) *The partial derivatives are continuous linear maps*

$$\frac{\partial^{|\nu|}}{\partial x^\nu} \colon \mathscr{S}(\mathbb{R}^n, V) \longrightarrow \mathscr{S}(\mathbb{R}^n, V), \tag{B.7.44}$$

satisfying the estimate (equality)

$$\mathrm{q}_{m,\mu}(\partial^\nu f) = \mathrm{q}_{m,\mu+\nu}(f). \tag{B.7.45}$$

v.) *If W and U are two other sequentially complete locally convex spaces and $m \colon V \times W \longrightarrow U$ is a continuous bilinear map then it induces continuous bilinear maps*

$$m \colon \mathscr{S}(\mathbb{R}^n, V) \times \mathscr{S}(\mathbb{R}^n, W) \longrightarrow \mathscr{S}(\mathbb{R}^n, U), \tag{B.7.46}$$

$$m \colon \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \times \mathscr{S}(\mathbb{R}^n, W) \longrightarrow \mathscr{S}(\mathbb{R}^n, U), \tag{B.7.47}$$

and

$$m \colon \mathscr{S}(\mathbb{R}^n, V) \times \mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}(\mathbb{R}^n, W) \longrightarrow \mathscr{S}(\mathbb{R}^n, U) \tag{B.7.48}$$

for all orders $\boldsymbol{m}$ and types $\boldsymbol{\rho}$ with respect to $\mathcal{Q}$ and all orders $\boldsymbol{m}'$ and types $\boldsymbol{\rho}'$ with respect to some defining system of seminorms $\mathcal{Q}'$ for W.

vi.) For all orders $m \in \mathbb{R}$ and types $\rho \in \mathbb{R}$ the pointwise multiplication

$$\mathrm{S}^{m,\rho}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n, V) \longrightarrow \mathscr{S}(\mathbb{R}^n, V) \tag{B.7.49}$$

is a continuous bilinear map.

PROOF: The first statement can most easily be checked using the explicit seminorms $\mathrm{q}_{m,\mu}$ in the same spirit as the proof of Proposition B.7.5, *iii.)*. Then also the second part is clear since we get a continuous inclusion of $\mathscr{C}^\infty_K(\mathbb{R}^n, V)$ into $\mathscr{S}(\mathbb{R}^n, V)$ with estimates like

$$\mathrm{q}_{m,\mu}(f) = \|f\|^{m,0}_{\mathrm{q},\mu} \leq c_K \mathrm{p}_{K,|\mu|,\mathrm{q}}(f)$$

as in the proof of Proposition B.7.5, *i.)*, for every compact subset $K \subseteq \mathbb{R}^n$. The second inclusion is continuous thanks to the estimate $\|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu}(f) \leq \mathrm{q}_{m,\mu}(f)$ for m an integer being at least $-\boldsymbol{m}(\mathrm{q}) + \boldsymbol{\rho}(\mathrm{q})|\mu|$, which we have established in the proof of Lemma B.7.17. The density of $\mathscr{C}^\infty_0(\mathbb{R}^n, V)$ in $\mathscr{S}(\mathbb{R}^n, V)$ is checked directly as in the scalar case, see Exercise 2.5.32. The second statement of part *iii.)* is clear as $\mathscr{C}^\infty_0(\mathbb{R}^n, V)$ has this property by Proposition B.7.11, *iv.)*. The fourth part is clear since the estimate (B.7.45) is obvious by definition. For part *v.)* we first consider the case (B.7.47). Thus let $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ and $g \in \mathscr{S}(\mathbb{R}^n, W)$ be given. The continuity of m means that for a continuous seminorm r on U we find a $\mathrm{q} \in \mathcal{Q}$ and a continuous seminorm q' on W such that

$$\mathrm{r}(m(v, w)) \leq \mathrm{q}(v)\mathrm{q}'(w)$$

for all $v \in V$ and $w \in W$ since we can assume that the defining system $\mathcal{Q}$ on V is already filtrating by Proposition B.7.12. Then for $m \in \mathbb{N}_0$ and $\mu \in \mathbb{N}_0^n$ we estimate using the Leibniz rule from Lemma B.4.9, *ii.)*,

$$\begin{aligned}
\mathrm{r}_{m,\mu}(m(f,g)) &= \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{\frac{m}{2}} \mathrm{r}\left((\partial^\mu m(f,g))(x)\right) \\
&= \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{\frac{m}{2}} \mathrm{r}\left(\sum_{\nu+\nu'=\mu} \binom{\mu}{\nu} m\left(\partial^\nu f(x), \partial^{\nu'} g(x)\right)\right) \\
&\leq 2^{|\mu|} \sum_{\nu+\nu'=\mu} \left(1 + \|x\|^2\right)^{\frac{m}{2}} \mathrm{q}\left(\partial^\nu f(x)\right)\mathrm{q}'\left(\partial^{\nu'} g(x)\right) \\
&= 2^{|\mu|} \sum_{\nu+\nu'=\mu} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q}) - \boldsymbol{\rho}(\mathrm{q})|\nu|)} \mathrm{q}\left(\partial^\nu f(x)\right) \\
&\qquad \left(1 + \|x\|^2\right)^{\frac{m}{2} + \frac{1}{2}(\boldsymbol{m}(\mathrm{q}) - \boldsymbol{\rho}(\mathrm{q})|\nu|)} \mathrm{q}'\left(\partial^{\nu'} g(x)\right) \\
&\leq 2^{|\mu|} \sum_{\nu+\nu'=\mu} \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu} \mathrm{q}'_{m',\nu'}(g),
\end{aligned}$$

with $m' = m + \boldsymbol{m}(\mathrm{q}) - \boldsymbol{\rho}|\mu|$. This shows the continuity of (B.7.47) and (B.7.48) is analogous. But then (B.7.46) follows as well since $\mathscr{S}(\mathbb{R}^n, V) \longrightarrow \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ is continuous. In fact, one can also estimate the bilinear expression directly. The last part is then clear. □

Corollary B.7.20 *Let $\mathscr{A}$ be a (sequentially) complete locally convex algebra and let $\mathscr{M}$ be a (sequentially) complete locally convex topological module over $\mathscr{A}$. Then $\mathscr{S}(\mathbb{R}^n, \mathscr{A})$ is a (sequentially) complete locally convex algebra and $\mathscr{S}(\mathbb{R}^n, \mathscr{M})$ is a (sequentially) complete locally convex topological module over $\mathscr{S}(\mathbb{R}^n, \mathscr{A})$.*

This follows directly from Proposition B.7.19, *v.)*. It is an easy exercise to give a direct proof with explicit estimates, see also Exercise B.8.7.

B.7.2 Further Properties of Symbols

In this subsection we investigate some further properties of symbols. First we show that the affine group of $\mathbb{R}^n$ acts on the symbol spaces by pull-backs. Denote by $A^* f$ the function $(A^* f)(x) = f(Ax)$ whenever $A \in \mathrm{GL}_n(\mathbb{R})$. Moreover, the translation by $y \in \mathbb{R}^n$ is denoted by $(\tau_y^* f)(x) = f(x+y)$ as usual. We start with the following basic observations:

Lemma B.7.21 *Let* q *be a continuous seminorm on* V *and* $m, \rho \in \mathbb{R}$*. Then for* $f \in \mathscr{C}^\infty(\mathbb{R}^n, V)$ *we have for all* $\mu \in \mathbb{N}_0^n$ *and all* $A \in \mathrm{GL}_n(\mathbb{R})$

$$\|A^* f\|_{\mathrm{q},\mu}^{m,\rho} \le c_\mu^{m,\rho}(A) \|f\|_{\mathrm{q},\mu}^{m,\rho} \tag{B.7.50}$$

with some $c_\mu^{m,\rho}(A) > 0$ *depending continuously on* A *and satisfying* $c_\mu^{m,\rho}(\mathbb{1}) = 1$.

PROOF: As usual, this is to be understood as an inequality in $[0, +\infty]$. First we note that with the operator norm of A we have

$$\mathrm{q}((\partial^\mu (A^* f))(x)) \le \|A\|^{|\mu|} \mathrm{q}(\partial^\mu f(Ax))$$

for all $x \in \mathbb{R}^n$ by the chain rule. Next we recall that

$$\frac{1}{\|A\|} \|y\| \le \|A^{-1} y\| \le \|A^{-1}\| \|y\|$$

for an invertible $A \in \mathrm{GL}_n(\mathbb{R})$. We have to distinguish a few cases. Depending on the sign of $m - \rho|\mu|$ we first get

$$\big(1 + \|A^{-1}y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \le \begin{cases} \big(1 + \|A^{-1}\|^2 \|y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} & \text{for } m - \rho|\mu| < 0 \\ \big(1 + \|A\|^{-2} \|y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} & \text{for } m - \rho|\mu| \ge 0. \end{cases}$$

In the case $\|A^{-1}\| \le 1$ and hence $\|A\| \ge 1$ we can continue the estimate by

$$\big(1 + \|A^{-1}y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \le \begin{cases} \big(1 + \|y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} & \text{for } m - \rho|\mu| < 0 \\ \|A\|^{m-\rho|\mu|} \big(1 + \|y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} & \text{for } m - \rho|\mu| \ge 0. \end{cases} \tag{$*$}$$

Conversely, in the case $\|A^{-1}\| \ge 1$ we get

$$\big(1 + \|A^{-1}y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \le \|A^{-1}\|^{-m+\rho|\mu|} \big(1 + \|y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \tag{$**$}$$

for the case $m - \rho|\mu| < 0$. For $m - \rho|\mu| \ge 0$ we still have to distinguish the two possibilities $\|A\| \le 1$ and $\|A\| \ge 1$. For $\|A\| \le 1$ we have

$$\big(1 + \|A^{-1}y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \le \big(1 + \|y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)}, \tag{$\star$}$$

and in the case $\|A\| > 1$ we finally get

$$\big(1 + \|A^{-1}y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \le \|A\|^{m-\rho|\mu|} \big(1 + \|y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)}. \tag{$\star\star$}$$

Combining the four cases $(*)$, $(**)$, $(\star)$, and $(\star\star)$ we get the estimate

$$\|A\|^{|\mu|} \big(1 + \|A^{-1}y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \le c_\mu^{m,\rho}(A) \big(1 + \|y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)}$$

where

$$c_\mu^{m,\rho}(A) = \|A\|^{|\mu|} \max\Big\{1, \|A\|^{m-\rho|\mu|}, \|A^{-1}\|^{-m+\rho|\mu|}\Big\}.$$

Since $A \mapsto A^{-1}$ is continuous and since the operator norm is continuous as well this constant depends continuously on A and clearly satisfies $c_\mu^{m,\rho}(\mathbb{1}) = 1$. It is now easy to see that we get the estimate (B.7.50). $\square$

Lemma B.7.22 *Let* q *be a continuous seminorm on* V *and* $m, \rho \in \mathbb{R}$. *Then for* $f \in \mathscr{C}^\infty(\mathbb{R}^n, V)$ *we have for all* $\mu \in \mathbb{N}_0^n$ *and all* $y \in \mathbb{R}^n$

$$\|\tau_y^* f\|_{\mathrm{q},\mu}^{m,\rho} \le c_\mu^{m,\rho}(y) \|f\|_{\mathrm{q},\mu}^{m,\rho}, \tag{B.7.51}$$

with some positive $c_\mu^{m,\rho}(y) > 0$ *begin a scalar symbol* $c_\mu^{m,\rho} \in \mathrm{S}^{\left|m-\rho|\mu|\right|,1}(\mathbb{R}^n)$.

Proof: We proceed similar as in the previous lemma. First it is clear that

$$\mathrm{q}\big((\partial^\mu(\tau_y^* f))(x)\big) = \mathrm{q}(\partial^\mu f(x+y)) \tag{$*$}$$

by the chain rule. For the prefactor in the definition of the seminorm we first consider the following elementary estimates from Exercise B.8.8: there is a constant $c > 0$ such that for all $x, y \geq 0$ we have

$$\frac{1}{1+(x-y)^2} \le c \frac{1+y^2}{1+x^2}. \tag{$**$}$$

We use this now to consider first the case $m - \rho|\mu| \geq 0$. There we have

$$\big(1+\|x-y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \le \frac{1}{\Big(1+\big|\|x\|-\|y\|\big|^2\Big)^{\frac{1}{2}(m-\rho|\mu|)}} \overset{(**)}{\le} c^{\frac{1}{2}(m-\rho|\mu|)} \left(\frac{1+\|y\|^2}{1+\|x\|^2}\right)^{\frac{1}{2}(m-\rho|\mu|)}.$$

This gives the estimate

$$\begin{aligned}
\|\tau_y^* f\|_{\mathrm{q},\mu}^{m,\rho} &\overset{(*)}{=} \sup_{x\in\mathbb{R}^n} \big(1+\|x\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \mathrm{q}(\partial^\mu f(x+y)) \\
&= \sup_{x\in\mathbb{R}^n} \big(1+\|x-y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \mathrm{q}(\partial^\mu f(x)) \\
&\le c^{\frac{1}{2}(m-\rho|\mu|)} \sup_{x\in\mathbb{R}^n} \left(\frac{1+\|y\|^2}{1+\|x\|^2}\right)^{\frac{1}{2}(m-\rho|\mu|)} \mathrm{q}(\partial^\mu f(x)) \\
&= c^{\frac{1}{2}(m-\rho|\mu|)} \big(1+\|y\|^2\big)^{\frac{1}{2}(m-\rho|\mu|)} \|f\|_{\mathrm{q},\mu}^{m,\rho}.
\end{aligned}$$

The case $m - \rho|\mu| < 0$ is even simpler. Here we have $-\frac{1}{2}(m-\rho|\mu|) \geq 0$ and hence

$$\begin{aligned}
\big(1+\|x-y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} &\le \big(1+2\|x\|^2+2\|y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \\
&\le 2^{-\frac{1}{2}(m-\rho|\mu|)} \big(1+\|x\|^2+\|y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \\
&\le 2^{-\frac{1}{2}(m-\rho|\mu|)} \big(1+\|x\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \big(1+\|y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)}.
\end{aligned}$$

By an analogous argument as for the previous case this results in the estimate

$$\|\tau_y^* f\|_{\mathrm{q},\mu}^{m,\rho} \le 2^{-\frac{1}{2}(m-\rho|\mu|)} \big(1+\|y\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)} \|f\|_{\mathrm{q},\mu}^{m,\rho}.$$

It remains to show that the function $g(y) = (1+\|y\|^2)^{\frac{m}{2}}$ is a scalar symbol of order m and type 1. In fact, $y \mapsto 1+\|y\|^2$ is clearly in $\mathrm{S}^{2,1}(\mathbb{R}^n, \mathbb{C})$ and thus the next lemma applies. ☐

Lemma B.7.23 *Let* $f \in \mathrm{S}^{m,\rho}(\mathbb{R}^n, \mathbb{C})$ *be a scalar symbol of order* m *and type* ρ *with* $f(x) \in \mathbb{C} \setminus [0, -\infty)$ *for all* $x \in \mathbb{R}^n$. *Then for all* $\alpha \in \mathbb{C}$ *with* $\mathrm{Re}(\alpha) > 0$ *we have* $f^\alpha \in \mathrm{S}^{\mathrm{Re}(\alpha)m,\rho}(\mathbb{R}^n, \mathbb{C})$.

PROOF: Since $f(x)$ takes values only in the complex plane without the closed negative real half axis, we can use the (smooth) principal branch of the logarithm to define the powers $f^\alpha = \exp(\alpha \log(f)) \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{C})$ for all $\alpha \in \mathbb{C}$. Note that we do not have to take care of the values $\alpha \in \mathbb{N}_0$ as these particular cases are already settled by Corollary B.7.10 by induction. By the chain rule we get

$$\partial^\mu f^\alpha = \sum_{\substack{1 \le k \le |\mu| \\ \nu_1 + \cdots + \nu_k = \mu}} C^k_{\nu_1 \ldots \nu_k} \alpha(\alpha - 1) \cdots (\alpha - k + 1) f^{\alpha - k} \, \partial^{\nu_1} f \cdots \partial^{\nu_k} f$$

with some universal coefficients $C^k_{\nu_1 \ldots \nu_k} \ge 0$, see also Exercise B.8.6. We first note that for f^α without derivatives we get the estimate

$$\begin{aligned}
\|f^\alpha\|_0^{\mathrm{Re}(\alpha)m, \rho} &= \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2} \mathrm{Re}(\alpha) m} |f^\alpha(x)| \\
&\le \mathrm{e}^{\pi |\mathrm{Im}(\alpha)|} \Big(\left(1 + \|x\|^2\right)^{-\frac{m}{2}} |f(x)| \Big)^{\mathrm{Re}(\alpha)} \\
&= \mathrm{e}^{\pi |\mathrm{Im}(\alpha)|} \left(\|f\|_0^{m, \rho} \right)^{\mathrm{Re}(\alpha)},
\end{aligned}$$

since for any complex number $z \in \mathbb{C} \setminus [0, -\infty)$ we have $|z^\alpha| \le |z|^{\mathrm{Re}(\alpha)} \mathrm{e}^{\pi |\mathrm{Im}(\alpha)|}$. Thus we need the prefactor $(1 + \|x\|^2)^{-\frac{1}{2} \mathrm{Re}(\alpha) m}$ to compensate the growth of f^α. To estimate the derivatives, we get

$$\begin{aligned}
\|f^\alpha\|_\mu^{\mathrm{Re}(\alpha)m, \rho} &= \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\mathrm{Re}(\alpha) m - \rho |\mu|)} \left|\partial^\mu f^\alpha(x)\right| \\
&\le \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\mathrm{Re}(\alpha) m - \rho |\mu|)} \\
&\qquad \sum_{\substack{1 \le k \le |\mu| \\ \nu_1 + \cdots + \nu_k = \mu}} C^k_{\nu_1 \ldots \nu_k} \left|\alpha(\alpha - 1) \cdots (\alpha - k + 1)\right| \left|f^{\alpha - k}(x)\right| \left|\partial^{\nu_1} f(x)\right| \cdots \left|\partial^{\nu_k} f(x)\right| \\
&\le \sum_{\substack{1 \le k \le |\mu| \\ \nu_1 + \cdots + \nu_k = \mu}} C^k_{\nu_1 \ldots \nu_k} \left|\alpha(\alpha - 1) \cdots (\alpha - k + 1)\right| \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(\mathrm{Re}(\alpha) m - k m)} \left|f^{\alpha - k}(x)\right| \\
&\qquad \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(m - \rho |\nu_1|)} \left|\partial^{\nu_1} f(x)\right| \cdots \sup_{x \in \mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(m - \rho |\nu_k|)} \left|\partial^{\nu_k} f(x)\right| \\
&\le \sum_{\substack{1 \le k \le |\mu| \\ \nu_1 + \cdots + \nu_k = \mu}} C^k_{\nu_1 \ldots \nu_k} \left|\alpha(\alpha - 1) \cdots (\alpha - k + 1)\right| \left(\|f\|_0^{m, \rho}\right)^{\mathrm{Re}(\alpha) - k} \|f\|_{\nu_1}^{m, \rho} \cdots \|f\|_{\nu_k}^{m, \rho} \\
&< \infty,
\end{aligned}$$

which proves that $f^\alpha \in \mathrm{S}^{\mathrm{Re}(\alpha)m, \rho}(\mathbb{R}^n, \mathbb{C})$ as claimed. □

Remark B.7.24 Note that for the translations τ_y the pre-factor $c_\mu^{m, \rho}(y)$ in (B.7.51) is always a symbol of *non-negative* order, even if $m - \rho|\mu|$ was negative. Thus the bounds in (B.7.51) typically *grow* with y and are also growing with increasing differentiations μ unless $\rho = 0$.

As an easy consequence of the two lemmas we get the affine invariance of the symbol spaces:

Proposition B.7.25 *Let V be a sequentially complete locally convex space and $\mathcal{Q}$ a defining system of seminorms. Moreover, let $\boldsymbol{m}$ and $\boldsymbol{\rho}$ be an order and a type with respect to $\mathcal{Q}$. Then the affine group $\mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ of $\mathbb{R}^n$ acts on the symbols $\mathrm{S}^{\boldsymbol{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$ via pull-backs by continuous endomorphisms.*

PROOF: The pull-backs with $A \in \mathrm{GL}_n(\mathbb{R})$ or with a translation by $y \in \mathbb{R}^n$ map $\mathrm{S}^{\boldsymbol{m}, \boldsymbol{\rho}}(\mathbb{R}^n, V)$ continuously into itself according to Lemma B.7.21 and Lemma B.7.22, respectively. The fact that this gives a (right) group action is clear. □

In a next step we want to refine this statement for the translations: we want to show that the map $y \mapsto \tau_y^* f$ is actually smooth. We begin with the following observation:

Lemma B.7.26 *Let $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$. Then the map*

$$\mathbb{R}^n \ni y \;\mapsto\; \tau_y^* f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \tag{B.7.52}$$

is continuous at zero provided $\boldsymbol{\rho} \geq 0$.

PROOF: We have to show that $\tau_y^* f \longrightarrow f$ in the $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$-topology for $y \longrightarrow 0$. Let $x \in \mathbb{R}^n$ be given. Then we have for $y \in \mathbb{R}^n$ and $\mathrm{q} \in \mathfrak{Q}$

$$\begin{aligned} \mathrm{q}\big((\partial^\mu(\tau_y^* f))(x) - \partial^\mu f(x)\big) &= \mathrm{q}\left(\int_0^1 \left(\frac{\partial}{\partial x^i}\frac{\partial^{|\mu|} f}{\partial x^\mu}\right)(x + ty)\,\mathrm{d}ty^i\right) \\ &\leq \sup_{\substack{t\in[0,1]\\ i=1,\ldots,n}} \mathrm{q}\left(\left(\frac{\partial}{\partial x^i}\frac{\partial^{|\mu|} f}{\partial x^\mu}\right)(x + ty)\right)\|y\|, \end{aligned} \tag{$*$}$$

according to Proposition B.3.6, *ii.)*. Now we use the fact that f is a symbol. This means that the $(\mu + e_i)$-th derivative of f satisfies

$$\mathrm{q}\left(\left(\frac{\partial}{\partial x^i}\frac{\partial^{|\mu|} f}{\partial x^\mu}\right)(x + ty)\right) \leq \big(1 + \|x + ty\|^2\big)^{\frac{1}{2}(m-\rho|\mu|-\rho)}\|f\|^{m,\rho}_{\mathrm{q},\mu+e_i}, \tag{$**$}$$

where for abbreviation we set $m = \boldsymbol{m}(\mathrm{q})$ and $\rho = \boldsymbol{\rho}(\mathrm{q})$. Then we get

$$\begin{aligned} \|\tau_y^* f - f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu} &= \sup_{x\in\mathbb{R}^n}\big(1 + \|x\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)}\mathrm{q}\left(\frac{\partial^{|\mu|}(\tau_y^* f)}{\partial x^\mu}(x) - \frac{\partial^{|\mu|} f}{\partial x^\mu}(x)\right) \\ &\overset{(*),(**)}{\leq} \|y\| \sup_{\substack{x\in\mathbb{R}^n\\ t\in[0,1]\\ i=1,\ldots,n}} \big(1 + \|x\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)}\big(1 + \|x + ty\|^2\big)^{\frac{1}{2}(m-\rho|\mu|-\rho)}\|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu+e_i} \\ &= \|y\| \sup_{\substack{x\in\mathbb{R}^n\\ t\in[0,1]\\ i=1,\ldots,n}} \big(1 + \|x - ty\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)}\big(1 + \|x\|^2\big)^{\frac{1}{2}(m-\rho|\mu|-\rho)}\|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu+e_i}. \end{aligned}$$

We can again estimate the first factor by the same techniques as in the Lemma B.7.22: we get a constant c (depending on m, ρ, and μ but not on ty or x) such that we can continue our estimate and get

$$\begin{aligned} \|\tau_y^* f - f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu} &\leq c\|y\| \sup_{\substack{x\in\mathbb{R}^n\\ t\in[0,1]\\ i=1,\ldots,n}} \big(1 + \|ty\|^2\big)^{\frac{1}{2}\left|m-\rho|\mu|\right|}\big(1 + \|x\|^2\big)^{-\frac{1}{2}(m-\rho|\mu|)+\frac{1}{2}(m-\rho|\mu|-\rho)}\|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu+e_i} \\ &\leq c\|y\|\big(1 + \|y\|^2\big)^{\frac{1}{2}\left|m-\rho|\mu|\right|} \sup_{x\in\mathbb{R}^n}\big(1 + \|x\|^2\big)^{-\frac{\rho}{2}} \max_{i=1,\ldots,n}\|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu+e_i}. \end{aligned}$$

Now if $\rho \geq 0$ then the supremum over all $x \in \mathbb{R}^n$ exists and hence we get an estimate of the form

$$\|\tau_y^* f - f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu} \leq c'\|y\|\big(1 + \|y\|^2\big)^{\frac{1}{2}\left|m-\rho|\mu|\right|},$$

from which the continuity at $y = 0$ follows. $\square$

Lemma B.7.27 *Let $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ be given with $\boldsymbol{\rho} \geq 0$. Then we have*

$$\lim_{\epsilon \longrightarrow 0} \frac{\tau^*_{\epsilon e_i} f - f}{\epsilon} = \frac{\partial f}{\partial x^i} \tag{B.7.53}$$

in the $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$-topology for all $i = 1, \ldots, n$.

PROOF: We proceed analogously to the continuity statement. Let again $\mathrm{q} \in \mathfrak{Q}$ and set $m = \boldsymbol{m}(\mathrm{q})$ and $\rho = \boldsymbol{\rho}(\mathrm{q})$ for abbreviation. Moreover, let $\epsilon \neq 0$. First we note that for all $x \in \mathbb{R}^n$ and $\mu \in \mathbb{N}_0^n$ we have by repeated use of Proposition B.3.6, *ii.)*

$$\begin{aligned}
&\mathrm{q}\left(\frac{1}{\epsilon}\left(\frac{\partial^{|\mu|}(\tau^*_{\epsilon e_i} f)}{\partial x^\mu}(x) - \frac{\partial^{|\mu|} f}{\partial x^\mu}(x)\right) - \frac{\partial}{\partial x^i}\frac{\partial^{|\mu|} f}{\partial x^\mu}(x)\right) \\
&\quad = \mathrm{q}\left(\frac{1}{\epsilon}\left(\frac{\partial^{|\mu|} f}{\partial x^\mu}(x + \epsilon e_i) - \frac{\partial^{|\mu|} f}{\partial x^\mu}(x)\right) - \frac{\partial}{\partial x^i}\frac{\partial^{|\mu|} f}{\partial x^\mu}(x)\right) \\
&\quad = \mathrm{q}\left(\int_0^1 \frac{\partial}{\partial x^i}\frac{\partial^{|\mu|} f}{\partial x^\mu}(x + t\epsilon e_i)\,\mathrm{d}t - \frac{\partial}{\partial x^i}\frac{\partial^{|\mu|} f}{\partial x^\mu}(x)\right) \\
&\quad = \mathrm{q}\left(\int_0^1 \int_0^1 \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}\frac{\partial^{|\mu|} f}{\partial x^\mu}(x + ts\epsilon e_i) t\epsilon \,\mathrm{d}s\,\mathrm{d}t\right) \\
&\quad \leq \epsilon \sup_{s\in[0,1]} \mathrm{q}\left(\frac{\partial^2}{\partial (x^i)^2}\frac{\partial^{|\mu|} f}{\partial x^\mu}(x + s\epsilon e_i)\right) \\
&\quad \leq \epsilon \sup_{s\in[0,1]} \left(1 + \|x + s\epsilon e_i\|^2\right)^{\frac{1}{2}(m - \rho|\mu| - 2\rho)} \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu+2e_i},
\end{aligned}$$

since by assumption $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$. Thus we get

$$\begin{aligned}
&\left\|\frac{1}{\epsilon}\left(\tau^*_{\epsilon e_i} f - f\right) - \frac{\partial f}{\partial x^i}\right\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu} \\
&\quad = \sup_{x\in\mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(m - \rho|\mu|)} \mathrm{q}\left(\frac{1}{\epsilon}\left(\frac{\partial^{|\mu|}(\tau^*_{\epsilon e_i} f)}{\partial x^\mu}(x) - \frac{\partial^{|\mu|} f}{\partial x^\mu}(x)\right) - \frac{\partial^{|\mu|}}{\partial x^\mu}\frac{\partial f}{\partial x^i}(x)\right) \\
&\quad \leq \epsilon \sup_{\substack{x\in\mathbb{R}^n \\ s\in[0,1]}} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(m - \rho|\mu|)} \left(1 + \|x + s\epsilon e_i\|^2\right)^{\frac{1}{2}(m - \rho|\mu| - 2\rho)} \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu+2e_i} \\
&\quad = \epsilon \sup_{\substack{x\in\mathbb{R}^n \\ s\in[0,1]}} \left(1 + \|x - s\epsilon e_i\|^2\right)^{-\frac{1}{2}(m - \rho|\mu|)} \left(1 + \|x\|^2\right)^{\frac{1}{2}(m - \rho|\mu| - 2\rho)} \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu+2e_i} \\
&\quad \leq c\epsilon \sup_{\substack{x\in\mathbb{R}^n \\ s\in[0,1]}} \left(1 + \|s\epsilon e_i\|^2\right)^{\frac{1}{2}\left|m - \rho|\mu|\right|} \left(1 + \|x\|^2\right)^{-\frac{1}{2}(m - \rho|\mu|) + \frac{1}{2}(m - \rho|\mu| - 2\rho)} \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu+2e_i} \\
&\quad = c\epsilon(1 + \epsilon^2)^{\frac{1}{2}|m - \rho|\mu||} \sup_{x\in\mathbb{R}^n} \left(1 + \|x\|^2\right)^{-\rho} \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu+2e_i}.
\end{aligned}$$

Using again $\rho \geq 0$ shows that the remaining supremum is finite. Thanks to the pre-factor ϵ we get the desired limit (B.7.53). □

These two lemmas are now enough to conclude the following smoothness statement of the action of the translations:

Proposition B.7.28 *Let V be a sequentially complete locally convex space and let $\mathcal{Q}$ be a defining system of seminorms for V. Let $\boldsymbol{m}$ and $\boldsymbol{\rho}$ be an order and a type with respect to $\mathcal{Q}$ and assume $\boldsymbol{\rho} \geq 0$. Then the action τ of $\mathbb{R}^n$ on $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ by translations is smooth, i.e. for every $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ the map $\tau(f)\colon y \mapsto \tau_y^* f$ is a smooth map. The derivatives are explicitly given by*

$$\frac{\partial^{|\mu|}}{\partial y^\mu} \tau_y^* f = \tau_y^* \frac{\partial^{|\mu|} f}{\partial x^\mu}. \tag{B.7.54}$$

PROOF: This is a general argument about group actions of Lie groups: We know already that $\tau(f)$ is continuous at $y = 0$ by Lemma B.7.26. Moreover, every map τ_y^* is continuous by Lemma B.7.22. Thus we have by the group action property

$$\lim_{y \longrightarrow y'} \tau_y^* f = \lim_{y \longrightarrow 0} \tau_{y'+y}^* f = \lim_{y \longrightarrow 0} \tau_y^* \tau_{y'}^* f = \tau_{y'}^* f$$

in the $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$-topology since we have continuity at zero. This shows continuity everywhere. Moreover, by the same argument

$$\lim_{\epsilon \longrightarrow 0} \frac{\tau_{y+\epsilon e_i}^* f - \tau_y^* f}{\epsilon} = \lim_{\epsilon \longrightarrow 0} \tau_y^* \frac{\tau_{\epsilon e_i}^* f - f}{\epsilon} = \tau_y^* \lim_{\epsilon \longrightarrow 0} \frac{\tau_{\epsilon e_i}^* f - f}{\epsilon} = \tau_y^* \frac{\partial f}{\partial x^i},$$

using Lemma B.7.27 and the continuity of τ_y^*. This shows that $\tau(f)$ has first partial derivatives everywhere given as in (B.7.54). Now $\frac{\partial f}{\partial x^i} \in \mathrm{S}^{\boldsymbol{m}-\boldsymbol{\rho},\boldsymbol{\rho}}(\mathbb{R}^n, V) \subseteq \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ thanks to $\boldsymbol{\rho} \geq 0$ and Proposition B.7.5, *iv.)*, as well as Proposition B.7.7. Thus the partial derivatives $\frac{\partial}{\partial y^i}\tau(f) = \tau(\frac{\partial f}{\partial x^i})$ are again of the form as we started with. Hence they are continuous as function of y and thus $\tau(f)$ is $\mathscr{C}^1$ by the general criterion from Proposition B.3.9. This allows to iterate the above argument finishing the proof. □

As a first application of the affine invariance of the spaces $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ we get the following generalization of the approximation from Proposition B.7.11, *iii.)*.

Corollary B.7.29 *Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ satisfy $\chi\big|_{\mathrm{B}_r(0)} = 1$ for some $r > 0$. Consider $\tau_y^* \chi_\epsilon$ for $\epsilon > 0$ and $y \in \mathbb{R}^n$ where as usual $\chi_\epsilon(x) = \chi(\epsilon x)$. Then for every $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ we have*

$$\lim_{\epsilon \longrightarrow 0} (\tau_y^* \chi_\epsilon) f = f \tag{B.7.55}$$

in the $\mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}(\mathbb{R}^n, V)$-topology provided $\boldsymbol{\rho}' \leq \min(1, \boldsymbol{\rho})$ and $\boldsymbol{m}' > \boldsymbol{m}$.

PROOF: We have $(\tau_y^* \chi_\epsilon) f = \tau_y^*(\chi_\epsilon \tau_{-y} f)$ and then the continuity of τ_y^* according to Proposition B.7.25 allows to exchange τ_y^* with the limit. The result follows from Proposition B.7.11, *iii.)*. □

The last feature of symbols we want to discuss here is the behaviour under continuous linear maps. As we have already discussed the bilinear case, the linear situation is no surprise:

Proposition B.7.30 *Let $A\colon V \longrightarrow W$ be a continuous linear map between sequentially complete locally convex spaces. Let $\mathcal{Q}$ and $\mathcal{Q}'$ be defining systems of continuous seminorms on V and W, respectively, such that $\mathcal{Q}$ is filtrating. Moreover, let orders $\boldsymbol{m}$ and $\boldsymbol{m}'$ and types $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ with respect to $\mathcal{Q}$ and $\mathcal{Q}'$ be given. Suppose for every seminorm $\mathrm{q}' \in \mathcal{Q}'$ we find a seminorm $\mathrm{q} \in \mathcal{Q}$ such that*

$$\mathrm{q}'(Av) \leq \mathrm{q}(v), \quad \boldsymbol{m}(\mathrm{q}) \leq \boldsymbol{m}'(\mathrm{q}'), \quad \textit{and} \quad \boldsymbol{\rho}(\mathrm{q}) \geq \boldsymbol{\rho}'(\mathrm{q}') \tag{B.7.56}$$

for all $v \in V$. Then the induced linear map $A\colon \mathscr{C}^\infty(\mathbb{R}^n, V) \longrightarrow \mathscr{C}^\infty(\mathbb{R}^n, W)$ restricts to a continuous linear map

$$A\colon \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \longrightarrow \mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}(\mathbb{R}^n, W). \tag{B.7.57}$$

More precisely, for every $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ and every $\mu \in \mathbb{N}_0^n$ we have

$$\|Af\|_{\mathrm{q}',\mu}^{\boldsymbol{m}',\boldsymbol{\rho}'} \leq \|f\|_{\mathrm{q},\mu}^{\boldsymbol{m},\boldsymbol{\rho}}. \tag{B.7.58}$$

PROOF: Note that the first condition $\mathrm{q}'(Av) \leq \mathrm{q}(v)$ can always be satisfied since $\mathcal{Q}$ was assumed to be filtrating and A is continuous. Thus assume that the other two requirements in (B.7.56) are fulfilled as well. Then we have for $f \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V)$ by Proposition B.3.2, *v.)*,

$$\begin{aligned}
\|Af\|^{\boldsymbol{m}',\boldsymbol{\rho}'}_{\mathrm{q}',\mu} &= \sup_{x\in\mathbb{R}^n} \left(1+\|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}'(\mathrm{q}')-\boldsymbol{\rho}'(\mathrm{q}')|\mu|)} \mathrm{q}'((\partial^\mu Af)(x)) \\
&= \sup_{x\in\mathbb{R}^n} \left(1+\|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}'(\mathrm{q}')-\boldsymbol{\rho}'(\mathrm{q}')|\mu|)} \mathrm{q}'(A\partial^\mu f(x)) \\
&\leq \sup_{x\in\mathbb{R}^n} \left(1+\|x\|^2\right)^{-\frac{1}{2}(\boldsymbol{m}(\mathrm{q})-\boldsymbol{\rho}(\mathrm{q})|\mu|)} \mathrm{q}(\partial^\mu f(x)) \\
&= \|f\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu}.
\end{aligned}$$

Since the seminorms $\|\cdot\|^{\boldsymbol{m}',\boldsymbol{\rho}'}_{\mathrm{q}',\mu}$ determine the $\mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}$-topology this gives the continuity of (B.7.57). □

B.7.3 Oscillatory Integrals

We can now define the oscillatory integrals of symbols following [33]. Again, we proceed very much analogously to the scalar case, see [23, Sect. 7.8] as well as [22]. The essential idea is to use the Riemann integral for compactly supported smooth functions as developed in Section B.2 and show that it enjoys a remarkable continuity property with respect to the symbol topologies.

In the following, we are interested not in the most general case where oscillatory integrals are used to define maps from test function spaces to distributions as discussed in [23, Sect. 7.8]. Instead we are just interested in the values of the oscillatory integrals per se. To this end we use a scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathbb{R}^n$.

Then we consider the integral with an *oscillatory phase* for a compactly supported function $F \in \mathscr{C}_0(\mathbb{R}^n, V)$

$$I_0(F) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \mathrm{e}^{\mathrm{i}\langle p,x\rangle} F(x,p)\,\mathrm{d}^n x\,\mathrm{d}^n p, \tag{B.7.59}$$

which is a well-defined Riemann integral thanks to the continuity of the integrand and the compact support of F. Thus the results of Section B.2 and in particular Proposition B.2.5 apply. We get a linear map

$$I_0 \colon \mathscr{C}_0(\mathbb{R}^{2n}, V) \longrightarrow V, \tag{B.7.60}$$

which is continuous in the $\mathscr{C}_0$-topology. Indeed, we have for every compact subset $K \subseteq \mathbb{R}^{2n}$, every continuous seminorm q on V, and $F \in \mathscr{C}_0(K, V)$ the estimate

$$\mathrm{q}(I_0(F)) \leq \frac{1}{(2\pi)^n}\mathrm{vol}(I)\mathrm{q}_\infty(F) = \frac{1}{(2\pi)^n}\mathrm{vol}(I)\mathrm{q}_K(F), \tag{B.7.61}$$

where $I \subseteq \mathbb{R}^{2n}$ is any compact interval containing K to define the Riemann integral and q_K is the usual supremum over K. Thus I_0 is continuous as a linear map $I_0 \colon \mathscr{C}_K(\mathbb{R}^{2n}, V) \longrightarrow V$ for every compact K. By the universal property of the inductive limit topology, (B.7.60) is continuous, too. Since the $\mathscr{C}_0$-topology is coarser than every $\mathscr{C}^k_0$-topology for $k \in \mathbb{N}_0 \cup \{+\infty\}$ we see that for all k we have a continuous linear map

$$I_0 \colon \mathscr{C}^k_0(\mathbb{R}^{2n}, V) \longrightarrow V. \tag{B.7.62}$$

Up to now, we have not used any particular properties of the phase function beside its continuity. It turns out that the continuity with respect to the $\mathscr{C}^k_0$-topologies is not the right one to extend I_0 to the symbol spaces.

Instead we have to show the continuity of I_0 with respect to some appropriate $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$-topology. This will make use of more specific properties of the phase function. We begin with the following preparations. We consider the polynomial

$$P(x) = (\mathrm{i} + x^1) \cdots (\mathrm{i} + x^n) \tag{B.7.63}$$

on $\mathbb{R}^n$ which is clearly a (non-homogeneous) polynomial of degree n. Hence P is a scalar symbol of order n and type 1 by Example B.7.2, i.e.

$$P \in \mathrm{S}^{n,1}(\mathbb{R}^n, \mathbb{C}). \tag{B.7.64}$$

Since each factor $(\mathrm{i} + x_k)$ takes values in the upper half plane, we can define arbitrary powers $(\mathrm{i} + x_k)^s$ for $s \in \mathbb{C}$ and obtain symbols of order $\mathrm{Re}(s)$. Their product is then by definition P^s which is a symbol of order $n\,\mathrm{Re}(s)$ according to Corollary B.7.10, see also Exercise B.8.9.

Lemma B.7.31 *Let $F \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^{2n}, V)$ with $\boldsymbol{\rho} \leq 1$. With the above definition of P^s the function*

$$\mathbb{R}^{2n} \ni (x, p) \;\mapsto\; P^s(x) P^s(p) F(x, p) \in V \tag{B.7.65}$$

is a symbol of order $\boldsymbol{m} + 2\,\mathrm{Re}(s) n$ and type $\boldsymbol{\rho}$ fro any $s \in \mathbb{C}$.

PROOF: Here we interpret $2\,\mathrm{Re}(s)n$ as a constant order as usual. Since $(x, p) \mapsto P^s(x)P^s(p)$ is a symbol of order $2\,\mathrm{Re}(s)n$ and type 1 according to the above discussion, we can apply Corollary B.7.9, *i.)*, since $\boldsymbol{\rho} \leq 1$ by assumption. ☐

The following technical lemma is well-known and occurs in many variations in the literature. We formulate it here in a way most suited for the construction of the oscillatory integrals.

Lemma B.7.32 *For every $s \in \mathbb{N}_0$ there exists a differential operator*

$$Q_s = \sum_{|\mu|,|\nu| \leq s} a_s^{\mu\nu} \frac{\partial^{|\mu|}}{\partial x^\mu} \frac{\partial^{|\nu|}}{\partial p^\nu} \tag{B.7.66}$$

with constant coefficients $a_s^{\mu\nu} \in \mathbb{C}$ such that

$$Q_s \mathrm{e}^{\mathrm{i}\langle p, x\rangle} = P^s(x) P^s(p) \mathrm{e}^{\mathrm{i}\langle p, x\rangle}. \tag{B.7.67}$$

PROOF: Since the exponentials as well as the polynomials $P(x)$ and $P(p)$ factorize, it is sufficient to consider the case $n = 1$ and multiply things together to get the higher-dimensional case. We prove the lemma by induction on s. Clearly, for $s = 0$ the identity operator $Q_0 = \mathrm{id}$ will do the job. Suppose now that we have found

$$Q_s = \sum_{0 \leq \mu, \nu \leq s} a_s^{\mu\nu} \frac{\partial^\mu}{\partial x^\mu} \frac{\partial^\nu}{\partial p^\nu}.$$

Then we have the commutation rule

$$\begin{aligned}
(\mathrm{i} + p) Q_s &= (\mathrm{i} + p) \sum_{0 \leq \mu, \nu \leq s} a_s^{\mu\nu} \frac{\partial^\mu}{\partial x^\mu} \frac{\partial^\nu}{\partial p^\nu} \\
&= \sum_{0 \leq \mu, \nu \leq s} a_s^{\mu\nu} (-\nu) \frac{\partial^\mu}{\partial x^\mu} \frac{\partial^{\nu-1}}{\partial p^{\nu-1}} + \sum_{0 \leq \mu, \nu \leq s} a_s^{\mu\nu} \frac{\partial^\mu}{\partial x^\mu} \frac{\partial^\nu}{\partial p^\nu} (\mathrm{i} + p) \\
&= Q_s' + Q_s (\mathrm{i} + p),
\end{aligned} \tag{$*$}$$

where Q_s' is of order s in the x-variables but only of order $s - 1$ in the p-variables. This gives

$$P^{s+1}(x) P^{s+1}(p) \mathrm{e}^{\mathrm{i}px} = (\mathrm{i} + x)(\mathrm{i} + p) Q_s \mathrm{e}^{\mathrm{i}px}$$

$$
\begin{aligned}
&= (\mathrm{i}+x)Q_s'\mathrm{e}^{\mathrm{i}px} + (\mathrm{i}+x)Q_s(\mathrm{i}+p)\mathrm{e}^{\mathrm{i}px} \\
&= (\mathrm{i}+x)Q_s'\mathrm{e}^{\mathrm{i}px} + (\mathrm{i}+x)Q_s\left(\mathrm{i} - \mathrm{i}\frac{\partial}{\partial x}\right)\mathrm{e}^{\mathrm{i}px} \\
&= (\mathrm{i}+x)\tilde{Q}_s\mathrm{e}^{\mathrm{i}px},
\end{aligned}
$$

where the differential operator $\tilde{Q}_s$ has constant coefficients, at most s derivatives in p-direction and at most $s+1$ derivatives in x-direction. Using now $(*)$ for x and p exchanged gives

$$
(\mathrm{i}+x)\tilde{Q}_s\mathrm{e}^{\mathrm{i}px} = \left(\tilde{Q}_s' + \tilde{Q}_s(\mathrm{i}+x)\right)\mathrm{e}^{\mathrm{i}px} = \left(\tilde{Q}_s' + \tilde{Q}_s\left(\mathrm{i} - \mathrm{i}\frac{\partial}{\partial p}\right)\right)\mathrm{e}^{\mathrm{i}px} = Q_{s+1}\mathrm{e}^{\mathrm{i}px},
$$

with a differential operator Q_{s+1} having constant coefficients and at most $s+1$ derivatives with respect to each variable x and p. This shows the lemma. For an alternative proof using Fourier transformations see Exercise B.8.9. ☐

The next lemma contains now the crucial estimate of the integral I_0 with respect to the symbol topologies: we only have to make a restriction to the type but not to the order.

Lemma B.7.33 *Let V be a sequentially complete locally convex space and $\mathcal{Q}$ a defining system of seminorms. Moreover, let $\boldsymbol{m}$ be an order and $\boldsymbol{\rho}$ a type with respect to $\mathcal{Q}$ such that $-1 < \boldsymbol{\rho} \leq 1$. Then for every $\mathrm{q} \in \mathcal{Q}$ there exists a constant $c > 0$ and a $N \in \mathbb{N}_0$ such that for all $F \in \mathscr{C}_0^\infty(\mathbb{R}^{2n}, V)$ we have*

$$
\mathrm{q}(I_0(F)) \leq c \sum_{|\mu| \leq N} \|F\|_{\mathrm{q},\mu}^{\boldsymbol{m},\boldsymbol{\rho}}. \tag{B.7.68}
$$

Proof: Let $F \in \mathscr{C}_0^\infty(\mathbb{R}^{2n}, V)$ have compact support in a compact interval $K \subseteq \mathbb{R}^{2n}$. Then we compute by Lemma B.7.32

$$
\begin{aligned}
I_0(F) &= \frac{1}{(2\pi)^n}\int_K \mathrm{e}^{\mathrm{i}\langle p,x\rangle}F(x,p)\,\mathrm{d}^n x\,\mathrm{d}^n p \\
&\overset{(\mathrm{B.7.67})}{=} \frac{1}{(2\pi)^n}\int_K \frac{1}{P^s(x)P^s(p)}\left(Q_s\mathrm{e}^{\mathrm{i}\langle p,x\rangle}\right)F(x,p)\,\mathrm{d}^n x\,\mathrm{d}^n p \\
&= \frac{1}{(2\pi)^n}\int_K \mathrm{e}^{\mathrm{i}\langle p,x\rangle}Q_s^{\mathrm{T}}\frac{F(x,p)}{P^s(x)P^s(p)}\,\mathrm{d}^n x\,\mathrm{d}^n p,
\end{aligned} \tag{$*$}
$$

where Q_s^{T} denotes the transposed differential operator and $s \in \mathbb{N}_0$ is arbitrary. Indeed, the integration by parts is possible since F has compact support inside the interval K. Thus we can apply Fubini's Theorem in form of Proposition B.2.8 as well as Theorem B.3.3 for each of the variables separately. Explicitly, this yields the transposed operator

$$
Q_s^{\mathrm{T}} = \sum_{0 \leq |\mu|,|\nu| \leq s} (-1)^{|\mu|+|\nu|} a_s^{\mu\nu}\frac{\partial^{|\mu|}}{\partial x^\mu}\frac{\partial^{|\nu|}}{\partial p^\nu}
$$

as usual. Since $(*)$ is valid for all $s \in \mathbb{N}_0$ the idea is to use a large enough s which produces under the integral an integrable symbol on the right hand side, independent of K. Since F has compact support we can view it as a symbol for any order $\boldsymbol{m}$ and any type $\boldsymbol{\rho}$. Thus also the function $(x,p) \mapsto \frac{F(x,p)}{P^s(x)P^s(p)}$ is a symbol, now of order $\boldsymbol{m} - 2sn$ and type $\boldsymbol{\rho}$. Thus for all $\mu, \nu \in \mathbb{N}_0^n$ we have the estimate

$$
\mathrm{q}\left(\frac{\partial^{|\mu|}}{\partial x^\mu}\frac{\partial^{|\nu|}}{\partial p^\nu}\frac{F(x,p)}{P^s(x)P^s(p)}\right) \leq \left(1 + \|(x,p)\|^2\right)^{\frac{1}{2}(\boldsymbol{m}(\mathrm{q}) - 2sn - \boldsymbol{\rho}(\mathrm{q})|\mu\oplus\nu|)}\left\|\frac{F(\,\cdot\,,\,\cdot\,)}{P^s(\,\cdot\,)P^s(\,\cdot\,)}\right\|_{\mathrm{q},\mu\oplus\nu}^{\boldsymbol{m}-2sn,\boldsymbol{\rho}}
$$

for all $s \in \mathbb{N}_0$. We know that $|\mu \oplus \nu| = |\mu| + |\nu| \le 2sn$ as the operator Q_s is of order $2sn$ only. Hence the condition $\boldsymbol{\rho}(\mathrm{q}) > -1$ shows that there is a $s \in \mathbb{N}_0$ such that for all $|\mu|, |\nu| \le sn$ we have

$$\boldsymbol{m}(\mathrm{q}) - 2sn - \boldsymbol{\rho}(\mathrm{q})|\mu \oplus \nu| < -2(n+1). \tag{$**$}$$

In fact, we get the left hand side as negative as we want by taking large enough s. Finally, by Proposition B.7.8 we get the estimate

$$\left\| \frac{F}{P^s(\,\cdot\,)P^s(\,\cdot\,)} \right\|^{\boldsymbol{m}-2sn,\boldsymbol{\rho}}_{\mathrm{q},\mu\oplus\nu} \le 2^{|\mu|+|\nu|} \max_{\mu'\oplus\nu' \le \mu\oplus\nu} \left\| \frac{1}{P^s(\,\cdot\,)P^s(\,\cdot\,)} \right\|^{-2sn,1}_{\mu'\oplus\nu'} \max_{\mu''\oplus\nu''\le\mu\oplus\nu} \|F\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu''\oplus\nu''},$$

since we have $\boldsymbol{\rho} \le 1$ and $\frac{1}{P^s(\,\cdot\,)P^s(\,\cdot\,)}$ is a symbol of order $-2sn$ and type 1 according to Lemma B.7.31. Taking now s large enough so that $(**)$ is satisfied we get by the standard estimate from Proposition B.2.6, *i.)*, the estimate

$$\begin{aligned}
\mathrm{q}(I_0(F)) &= \mathrm{q}\left(\frac{1}{(2\pi)^n} \int_K \mathrm{e}^{\mathrm{i}\langle p,x\rangle} Q_s^{\mathrm{T}} \frac{F(x,p)}{P^s(x)P^s(p)} \,\mathrm{d}^n x \,\mathrm{d}^n p \right) \\
&\le \frac{1}{(2\pi)^n} \int_K \mathrm{q}\left(\sum_{0\le\mu,\nu\le s} a_s^{\mu\nu} (-1)^{|\mu|+|\nu|} \frac{\partial^{|\mu|}}{\partial x^\mu} \frac{\partial^{|\nu|}}{\partial p^\nu} \frac{F(x,p)}{P^s(x)P^s(p)} \right) \mathrm{d}^n x \,\mathrm{d}^n p \\
&\le \frac{1}{(2\pi)^n} \sum_{0\le\mu,\nu\le s} |a_s^{\mu\nu}| \int_K \left(1 + \|(x,p)\|^2\right)^{-(n+1)} \mathrm{d}^n x \,\mathrm{d}^n p \left\| \frac{F}{P^s(\,\cdot\,)P^s(\,\cdot\,)} \right\|^{\boldsymbol{m}-2sn,\boldsymbol{\rho}}_{\mathrm{q},\mu\oplus\nu} \\
&\le c \sum_{0\le|\mu|,|\nu|\le s} \|F\|^{\boldsymbol{m},\boldsymbol{\rho}}_{\mathrm{q},\mu\oplus\nu},
\end{aligned}$$

with the constant

$$c = \frac{2^{2sn}}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \left(1 + \|(x,p)\|^2\right)^{-(n+1)} \mathrm{d}^n x \,\mathrm{d}^n p \max_{0\le|\mu|,|\nu|\le s} |a_s^{\mu\nu}| \max_{0\le|\mu'|,|\nu'|\le s} \left\| \frac{1}{P^s(\,\cdot\,)P^s(\,\cdot\,)} \right\|^{-2sn,1}_{\mu'\oplus\nu'} < \infty.$$

Note that the integral is finite indeed as we were able to make the exponent $(**)$ negative enough such that the dependence on the compact interval K disappears. □

Remark B.7.34 A posteriori we see that the choices of the polynomials P^s and the differential operators Q_s are not essential for the continuity statement (B.7.68), only the numerical value of the constant as well as the needed order N depend on these choices.

B.8 Exercises

Exercise B.8.1 (Locally bounded functions) Let X be a locally compact Hausdorff space and let V be a Hausdorff locally convex vector space.

i.) Show that if X is compact then $\mathscr{B}_{\mathrm{loc}}(X, V) = \mathscr{B}(X, V)$.

ii.) Show that a function $f\colon X \longrightarrow V$ is locally bounded in the sense of Definition B.1.3 iff for every point $p \in X$ there is an open neighbourhood $U \subseteq X$ of p such that $f\big|_U$ is bounded, i.e. for all continuous seminorms p of V the scalar function $\mathrm{p} \circ f$ is bounded on U.

iii.) Show that on X the continuous functions with values in V are locally bounded.

iv.) Give an example of a Hausdorff space on which the two notions of locally bounded differ.

Hint: Use the fact that a compact subset of a Banach space has empty open interior in infinite dimensions, see Proposition 6.1.5.

v.) Now let $\{K_i\}_{i\in I}$ be a set of compact subsets of X with $X = \bigcup_{i\in I} K_i^\circ$. Note that such systems of compact subsets always exist for a locally compact space. Furthermore, let $\mathfrak{Q}$ be a defining subset of continuous seminorms of V. Show that the seminorms $\mathrm{p}_{K_i,0,\mathrm{q}}$ with $i \in I$ and $\mathrm{q} \in \mathfrak{Q}$ determine the $\mathscr{B}_{\mathrm{loc}}$-topology already.

vi.) Show that for a σ-compact locally compact space X and a Fréchet space V the locally bounded functions $\mathscr{B}_{\mathrm{loc}}(X, V)$ are a Fréchet space again.

Hint: Use *v.)*.

vii.) Suppose now that X carries the discrete topology. Show that every scalar function is locally bounded and prove that the $\mathscr{B}_{\mathrm{loc}}$-topology is the topology of pointwise convergence, i.e. the topology of the Cartesian product.

Exercise B.8.2 (Topologies on $\mathrm{Hom}(\mathbb{R}^n, V)$) Prove Lemma B.1.9 by finding explicit estimates between the systems of seminorms.

Exercise B.8.3 (Algebra-valued locally bounded functions) Let $\mathscr{A}$ be a locally convex algebra and X a locally compact Hausdorff space. Consider $\mathscr{B}_{\mathrm{loc}}(X, \mathscr{A})$ endowed with the $\mathscr{B}_{\mathrm{loc}}$-topology.

i.) Show that $\mathscr{B}_{\mathrm{loc}}(X, \mathscr{A})$ becomes a locally convex algebra with respect to the pointwise multiplication.

ii.) Show that $\mathscr{B}_{\mathrm{loc}}(X, \mathscr{A})$ is again locally multiplicatively convex if $\mathscr{A}$ is locally multiplicatively convex.

iii.) Show that $\mathscr{B}_{\mathrm{loc}}(X, \mathscr{A})$ is a locally convex *-algebra with respect to the pointwise *-involution if $\mathscr{A}$ is a locally convex *-algebra.

iv.) Show that $\mathscr{B}_{\mathrm{loc}}(X, \mathscr{A})$ is unital if $\mathscr{A}$ is unital. Show that in this case, the center contains $\mathscr{B}_{\mathrm{loc}}(X)$.

v.) Show that $\mathscr{B}_{\mathrm{loc}}(X, \mathscr{A})$ is a pro C^*-algebra if $\mathscr{A}$ is a pro C^*-algebra. Is $\mathscr{B}_{\mathrm{loc}}(X, \mathscr{A})$ even a C^*-algebra for a C^*-algebra $\mathscr{A}$?

vi.) Discuss the functoriality of $\mathscr{B}_{\mathrm{loc}}(X, \mathscr{A})$ with respect to X and with respect to $\mathscr{A}$.

Exercise B.8.4 (Riemann integrable functions are bounded) Let $I \subseteq \mathbb{R}^n$ be a compact interval and let V be a Hausdorff locally convex space.

i.) Consider first the case of a scalar function $f\colon I \longrightarrow \mathbb{C}$. Show that if f is Riemann integrable then f is bounded.

ii.) Now let $f\colon I \longrightarrow V$ be Riemann integrable. Use Theorem 2.4.20 to show that f is bounded.

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Exercise B.8.5 (Convergence of Cauchy nets) Let V be a sequentially complete Hausdorff locally convex space which is not necessarily complete. Moreover, let I be a directed set and $(v_i)_{i\in I}$ a Cauchy net in V. Suppose that $(v_i)_{i\in I}$ has a cofinal subsequence, i.e. suppose that there is a sequence $i_n \in I$ such that for every index i there is a $n \in \mathbb{N}$ with $i_n \succcurlyeq i$. Prove that $(v_i)_{i\in I}$ converges. This is the general mechanism which makes the Riemann integral nice already for sequentially complete spaces even though it is defined by means of Cauchy nets.

Exercise B.8.6 (Chain rule) Let V be a sequentially complete locally convex space and $f \in \mathscr{C}^k(U, V)$ as well as $g \in \mathscr{C}^k(W, U)$ where $W \subseteq \mathbb{R}^m$ and $U \subseteq \mathbb{R}^n$ are non-empty open subsets.

i.) Prove by induction on $|\mu| \le k$ that $f \circ g\colon W \longrightarrow V$ has a continuous μ-th partial derivative given by

$$\frac{\partial^{|\mu|}(f \circ g)}{\partial y^\mu}(y) = \sum_{\substack{r \le |\mu| \\ \mu_1 + \cdots \mu_r = \mu \\ |\nu| = r}} C^\mu_{\nu_1 \ldots \nu_r} \frac{\partial^\nu f}{\partial x^\nu}(g(y)) \frac{\partial^{\mu_1} g}{\partial y^{\mu_1}}(y) \cdots \frac{\partial^{\mu_r} g}{\partial y^{\mu_r}}(y) \tag{B.8.1}$$

with some universal constants $C^{\nu}_{\mu_1\ldots\mu_r} \in \mathbb{R}$. Give a recursion formula for the $C^{\nu}_{\mu_1\ldots\mu_r}$.

ii.) Show that $f \circ g \in \mathscr{C}^k(W, V)$.

Exercise B.8.7 (Algebra-valued Schwartz functions) Let $\mathscr{A}$ be a sequentially complete locally convex algebra with multiplication m and consider $\mathscr{S}(\mathbb{R}^n, \mathscr{A})$.

i.) Find explicit estimates for the continuity of the pointwise product of $\mathscr{S}(\mathbb{R}^n, \mathscr{A})$.

ii.) Suppose that $\mathscr{A}$ is locally multiplicatively convex. Show that also $\mathscr{S}(\mathbb{R}^n, \mathscr{A})$ is locally multiplicatively convex, too.

iii.) Suppose $\mathscr{A}$ is unital. Is $\mathscr{S}(\mathbb{R}^n, \mathscr{A})$ unital, too?

Exercise B.8.8 (Some elementary estimates) Let $x, y \geq 0$ and $a > 0$.

i.) Show that

$$\frac{1}{1 + (x - y)^2} \leq \left(\frac{y + 2}{x + 1}\right)^2, \tag{B.8.2}$$

by considering the cases $x \geq y + 1$ and $x < y + 1$ separately.

ii.) Show that there are constants $0 < c_1 \leq c_2$ depending on a but independent on x such that

$$c_1(1 + x^2) \leq (a + x)^2 \leq c_2(1 + x^2). \tag{B.8.3}$$

iii.) Show that there is a constant $c > 0$ independent on x and y such that

$$\frac{1}{1 + (x - y)^2} \leq c\,\frac{1 + y^2}{1 + x^2}. \tag{B.8.4}$$

Exercise B.8.9 (The symbols $\mathrm{i} + x_k$)

Appendix C

Notions from Measure Theory

In this appendix we collect some basic definitions and fundamental results from measure theory. We refer to e.g. [13, 48] for a more detailed treatment of this subject. Here we will focus on relations to C^*-algebra theory. The way we present the theory will show that the abstract properties of measures and integration is fairly easy. In some sense, the most difficult part of measure theory is to actually construct interesting measures like the Lebesgue measure on $\mathbb{R}^n$. This will be postponed till a very late section.

In Section C.1 in a first step, we define and study σ-algebras, measurable spaces, and measurable maps between them leading already to interesting classes of functions: the measurable and bounded functions. In Section C.2 we consider now measures on a measurable space. Here two types of measures are investigated in detail: First, the positive measures for which we define completions and essentially bounded functions. Second, the complex measures which form a Banach space with respect to the variational norm. The usual constructions like push-forwards are studied for both types of measures. Only after having defined measures in general we can pass to integration theory in Section C.3. First we define the integral with respect to a positive measure and prove the fundamental theorems of measure theory: the theorem of monotonous convergence, Fatou's Lemma, and Lebesgue's Theorem of dominated convergence. In some sense the are fairly easy as the definition of the integral is designed in a way to make them work. In a next step we consider the Banach spaces of p-integrable functions. Here we establish various natural pairings and show their continuity properties. Finally, we pass to integration with respect to complex measures. Here the most important step will be the Lebesgue-Radon-Nikodym Theorem, which we discuss in detail. As applications we obtain the Lebesgue decomposition of a complex measure, the Hahn and Jordan decompositions of a real measure, as well as the polar decomposition which is the basis for the definition of the integration with respect to a complex measure. Finally, we establish the dualities between the L^p-spaces.

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C.1 Measurable Spaces and Maps

Measurable spaces and measurable maps between them are introduced very much parallel to the concepts of topological spaces and continuous maps. In this preparatory section we collect some of their fundamental properties and discuss first examples and constructions.

C.1.1 Measurable Spaces

The notion of a σ-algebra is used to determine those subsets of a given set X which are designated to have a measure: later on we want to assign a (positive) number $\mu(A)$ to any measurable subset A subject to some natural conditions resembling the naive ideas about viewing $\mu(A)$ as an area or a volume of A. It will turn out that not all subsets of X will be suitable in general, hence we have to single out a subset of the power set 2^X for which we will have a reasonable notion of volume later on. We recall the general definition:

Definition C.1.1 (σ-Algebra) *Let X be a set and let $\mathfrak{a} \subseteq 2^X$ be a subset of the power set of X. Then $\mathfrak{a}$ is called σ-algebra if*

i.) *one has $\emptyset \in \mathfrak{a}$,*

ii.) *for $A \in \mathfrak{a}$ also $X \setminus A \in \mathfrak{a}$,*

iii.) *for a sequence $A_n \in \mathfrak{a}$ also $\bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{a}$.*

The pair $(X, \mathfrak{a})$ is then called a measurable space. The subsets $A \subseteq X$ in $\mathfrak{a}$ are called measurable subsets of X.

As it easily follows from the axioms of a σ-algebra the following subsets are also measurable in a measurable space $(X, \mathfrak{a})$. First we have $X \in \mathfrak{a}$. Moreover, if $A_n \in \mathfrak{a}$ for $n \in \mathbb{N}$ then also the countable intersection $\bigcap_{n\in\mathbb{N}} A_n \in \mathfrak{a}$ is measurable. Finally, taking all but finitely many of the A_n to be $\emptyset$ or X, respectively, shows that also finite unions and finite intersections of measurable subsets are measurable. To avoid trivialities it is convenient to assume that a measurable space X is non-empty.

Lemma C.1.2 *Let $(\mathfrak{a}_i)_{i\in I}$ be a non-empty family of σ-algebras on X. Then $\mathfrak{a} = \bigcap_{i\in I} \mathfrak{a}_i$ is a σ-algebra, too.*

PROOF: Clearly $\emptyset \in \mathfrak{a}$. If $A \in \mathfrak{a}$ then $A \in \mathfrak{a}_i$ for all $i \in I$ and hence $X \setminus A \in \mathfrak{a}_i$ for all $i \in I$ since the $\mathfrak{a}_i$ are σ-algebras. Thus $X \setminus A \in \mathfrak{a}$ follows. Analogously, one verifies that the countable union of elements in $\mathfrak{a}$ is again in $\mathfrak{a}$. □

We have at least two σ-algebras on a (non-empty) set X: the whole power set 2^X is clearly a σ-algebra, the largest possible one. On the other extreme, $\mathfrak{a} = \{\emptyset, X\}$ is also a σ-algebra, the smallest possible one. Using this, the lemma has an immediate corollary:

Corollary C.1.3 *Let X be a non-empty set and let $\mathcal{B} \subseteq 2^X$ be an arbitrary non-empty collection of subsets of X. Then there is a unique σ-algebra $\mathfrak{a}(\mathcal{B})$ containing $\mathcal{B}$ such that $\mathfrak{a}(\mathcal{B})$ is contained in every other σ-algebra which contains $\mathcal{B}$.*

PROOF: Since $\mathcal{B}$ is contained in a σ-algebra, namely in 2^X, the set of σ-algebras containing $\mathcal{B}$ is non-empty. Taking the intersection of all of them yields a σ-algebra $\mathfrak{a}(\mathcal{B})$ by Lemma C.1.2, which has the desired properties. □

We call $\mathfrak{a}(\mathcal{B})$ the σ-algebra *generated* by the collection $\mathcal{B}$. Conversely, $\mathcal{B}$ is called a *basis* or *generating set* of $\mathfrak{a}(\mathcal{B})$. With other words, $\mathfrak{a}(\mathcal{B})$ is the smallest σ-algebra containing $\mathcal{B}$.

Remark C.1.4 Let X be a non-empty set and let $\mathcal{B} \subseteq 2^X$ be non-empty.

i.) The explicit form of the sets belonging to $\mathfrak{a}(\mathcal{B})$ can be rather difficult to describe, even if $\mathcal{B}$ consists of very nice (or few) sets. We will see this when dealing with the Borel σ-algebra of $\mathbb{R}^n$. We also note that generating a topology by some sets is usually much easier and more explicit.

ii.) If $\mathcal{C} \subseteq \mathcal{B}$ is a smaller collection of subsets then it is clear that

$$\mathfrak{a}(\mathcal{C}) \subseteq \mathfrak{a}(\mathcal{B}). \tag{C.1.1}$$

It is also clear that

$$\mathfrak{a}(\mathfrak{a}(\mathcal{B})) = \mathfrak{a}(\mathcal{B}). \tag{C.1.2}$$

We have several basic constructions of σ-algebras out of given ones. We list some of the most important ones:

Example C.1.5 (Push-forward) Let $f\colon X \longrightarrow Y$ be a map and let $\mathfrak{a}$ be a σ-algebra on X. Then

$$f_*\mathfrak{a} = \big\{B \subseteq Y \;\big|\; f^{-1}(B) \in \mathfrak{a}\big\} \tag{C.1.3}$$

is easily shown to be a σ-algebra. The reason is that the inverse image map $f^{-1}\colon 2^Y \longrightarrow 2^X$ is compatible with all set operations, in particular with complements and (even arbitrary) unions. This σ-algebra is called the *push-forward* of $\mathfrak{a}$ via f, see also Exercise C.6.1.

Example C.1.6 (Inverse image) Let again $f\colon X \longrightarrow Y$ be a map and let $\mathfrak{b}$ be a σ-algebra on Y. Then

$$f^{-1}(\mathfrak{b}) = \big\{A \subseteq X \;\big|\; \text{there is a } B \in \mathfrak{b} \text{ with } f^{-1}(B) = A\big\} \tag{C.1.4}$$

is a σ-algebra as well, called the *inverse image* or *pull-back* of $\mathfrak{b}$ via f, see again Exercise C.6.1.

Example C.1.7 (Product) Let $(X_1, \mathfrak{a}_1)$ and $(X_2, \mathfrak{a}_2)$ be measurable spaces. Then their Cartesian product $X_1 \times X_2$ becomes a measurable space by taking

$$\mathfrak{a}_1 \otimes \mathfrak{a}_2 = \mathfrak{a}(\mathfrak{a}_1 \times \mathfrak{a}_2), \tag{C.1.5}$$

where $\mathfrak{a}_1 \times \mathfrak{a}_2 \subseteq 2^{X \times Y}$ denotes all those subsets of $X_1 \times X_2$ which are Cartesian products $A_1 \times A_2$ of subsets $A_1 \in \mathfrak{a}_1$ and $A_2 \in \mathfrak{a}_2$, i.e. the “rectangles”. This σ-algebra is called the (Cartesian) *product* of $\mathfrak{a}_1$ and $\mathfrak{a}_2$. Here we can of course take also more that two factors , see once more Exercise C.6.1.

For the product σ-algebra the following lemma is sometimes very useful: it allows to reformulate the construction in terms of generating sets:

Lemma C.1.8 *Let $(X_1, \mathfrak{a}_1)$ and $(X_2, \mathfrak{a}_2)$ be measurable spaces and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be generating sets of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ with $X_1 \in \mathcal{B}_1$ and $X_2 \in \mathcal{B}_2$, respectively. Then*

$$\mathfrak{a}_1 \otimes \mathfrak{a}_2 = \mathfrak{a}(\mathcal{B}_1 \times \mathcal{B}_2). \tag{C.1.6}$$

PROOF: Since $\mathcal{B}_1 \times \mathcal{B}_2 \subseteq \mathfrak{a}_1 \times \mathfrak{a}_2$ it follows from (C.1.1) that $\mathfrak{a}(\mathcal{B}_1 \times \mathcal{B}_2) \subseteq \mathfrak{a}_1 \otimes \mathfrak{a}_2$. For the other inclusion we consider the push-forwards $\tilde{\mathfrak{a}}_i = (\mathrm{pr}_i)_*(\mathfrak{a}(\mathcal{B}_1 \times \mathcal{B}_2))$ for $i = 1, 2$ where $\mathrm{pr}_i\colon X_1 \times X_2 \longrightarrow X_i$ are the canonical projections. Since by assumption $X_i \in \mathcal{B}_i$ we see that for all $B_1 \in \mathcal{B}_1$ also $B_1 = \mathrm{pr}_1(B_1 \times X_2) \in \tilde{\mathfrak{a}}_1$ and analogously for $B_2 \in \mathcal{B}_2$. Hence $\mathcal{B}_i \subseteq \tilde{\mathfrak{a}}_i$ follows which gives $\mathfrak{a}_i \subseteq \tilde{\mathfrak{a}}_i$ since $\mathfrak{a}_i$ is generated by $\mathcal{B}_i$. Now let $A_i \in \mathfrak{a}_i$ be arbitrary then $A_1 \times X_2 = (\mathrm{pr}_1)^{-1}(A_1)$ as well as $X_1 \times A_2 = (\mathrm{pr}_2)^{-1}(A_2)$ shows that $A_1 \times X_2$, $X_1 \times A_2$ are elements of $\mathfrak{a}(\mathcal{B}_1 \times \mathcal{B}_2)$ by the definition of the push-forward σ-algebras $\tilde{\mathfrak{a}}_i$. But this shows that $A_1 \times A_2 = (A_1 \times X_2) \cap (X_1 \times A_2)$ is also contained in $\mathfrak{a}(\mathcal{B}_1 \times \mathcal{B}_2)$. Hence $\mathfrak{a}_1 \times \mathfrak{a}_2 \subseteq \mathfrak{a}(\mathcal{B}_1 \times \mathcal{B}_2)$ is shown from which the remaining inclusion in (C.1.6) follows. $\square$

Having defined measurable spaces as new mathematical objects we should also take care of the corresponding structure compatible maps. This motivates the following definition:

Definition C.1.9 (Measurable map) *Let $(X, \mathfrak{a})$ and $(Y, \mathfrak{b})$ be measurable spaces. Then a map $f\colon X \longrightarrow Y$ is called measurable if $f^{-1}(B) \in \mathfrak{a}$ for all $B \in \mathfrak{b}$.*

Remark C.1.10 (The category Mess)

i.) It is immediate that $\mathrm{id}_X\colon X \longrightarrow X$ is measurable for any measurable space $(X, \mathfrak{a})$. Moreover, the composition of measurable maps is again measurable. Hence we obtain a category Mess of measurable spaces as objects and measurable maps between them as morphisms.

ii.) It should be clear from the above presentation that the concept of measurable spaces and maps is very much parallel to the concept of topological spaces and continuous maps. We will see more and even deeper relations between the two theories in the sequel. It will turn out that they are more intimately linked than just the above structural similarity.

The following lemma is useful to decide whether a map is measurable or not:

Lemma C.1.11 *Let $(X, \mathfrak{a})$ and $(Y, \mathfrak{b})$ be measurable spaces and $f\colon X \longrightarrow Y$ a map. Let furthermore $\mathcal{B} \subseteq 2^Y$ be a generating set for $\mathfrak{b}$. Then f is measurable iff $f^{-1}(B) \in \mathfrak{a}$ for all $B \in \mathcal{B}$.*

PROOF: One direction is trivial since $\mathcal{B} \subseteq \mathfrak{b}$. Thus we assume that $f^{-1}(\mathcal{B}) \subseteq \mathfrak{a}$ holds. We consider the σ-algebra $\mathfrak{a}(f^{-1}(\mathcal{B}))$ generated by $f^{-1}(\mathcal{B})$. Clearly, we have $\mathfrak{a}(f^{-1}(\mathcal{B})) \subseteq \mathfrak{a}$. The push-forward $f_*(\mathfrak{a}(f^{-1}(\mathcal{B})))$ gives then a σ-algebra on Y with $\mathcal{B} \subseteq f_*(\mathfrak{a}(f^{-1}(\mathcal{B})))$. Indeed, this is trivial since for $B \in \mathcal{B}$ we have $f^{-1}(B) \in f^{-1}(\mathcal{B}) \subseteq \mathfrak{a}(f^{-1}(\mathcal{B}))$ and hence $B \in f_*(\mathfrak{a}(f^{-1}(\mathcal{B})))$ by the definition of the push-forward. Thus by minimality of $\mathfrak{b}$ we have $\mathfrak{b} \subseteq f_*(\mathfrak{a}(f^{-1}(\mathcal{B})))$. But this means that for $B \in \mathfrak{b}$ we have $f^{-1}(B) \in \mathfrak{a}(f^{-1}(\mathcal{B})) \subseteq \mathfrak{a}$. Hence f is measurable. $\square$

In fact, the proof shows that the σ-algebra $f^{-1}(\mathfrak{b})$ on X actually coincides with the σ-algebra generated by $f^{-1}(\mathcal{B})$ for a generating set $\mathcal{B}$ of $\mathfrak{b}$: indeed, $\mathfrak{a}(f^{-1}(\mathcal{B})) \subseteq f^{-1}(\mathfrak{b})$ follows from (C.1.1) since $f^{-1}(\mathcal{B}) \subseteq f^{-1}(\mathfrak{b})$. Then the proof shows that $f^{-1}(B) \in \mathfrak{a}(f^{-1}(\mathcal{B}))$ for all $B \in \mathfrak{b}$ which is the opposite inclusion. Hence we have in general

$$f^{-1}(\mathfrak{a}(\mathcal{B})) = \mathfrak{a}(f^{-1}(\mathcal{B})), \tag{C.1.7}$$

i.e. generating a σ-algebra commutes with inverse images.

Remark C.1.12 Let $f\colon X \longrightarrow Y$ be a map. If $\mathfrak{a}$ is a σ-algebra on X then $f_*\mathfrak{a}$ is the largest σ-algebra on Y such that f is measurable. Conversely, for a σ-algebra $\mathfrak{b}$ on Y the inverse image $f^{-1}(\mathfrak{b})$ is the smallest σ-algebra on X such that f is measurable. This is clear from the definitions. One should compare this with the construction of the initial and final topologies.

Comparison, perties, more details here?

While general measurable spaces and measurable maps are of limited interest for us, the following construction will provide us an important class of measurable spaces:

Proposition C.1.13 (Borel σ-algebra) *Let $(M, \mathcal{M})$ be a topological space.*

i.) *There exists a unique smallest σ-algebra $\mathfrak{a}(\mathcal{M})$ on M containing the topology $\mathcal{M}$, called the Borel σ-algebra of $(M, \mathcal{M})$.*

ii.) *A continuous map $f\colon (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ into another topological space is measurable with respect to the Borel σ-algebras $\mathfrak{a}(\mathcal{M})$ and $\mathfrak{a}(\mathcal{N})$, respectively.*

iii.) *The assignment $(M, \mathcal{M}) \mapsto (M, \mathfrak{a}(\mathcal{M}))$ for objects and the identity on morphisms yields a functor*

$$\mathsf{Borel}\colon \mathsf{Top} \longrightarrow \mathsf{Mess}. \tag{C.1.8}$$

PROOF: The first part is clear by Corollary C.1.3. Since for a continuous map f we have $f^{-1}(O) \in \mathcal{M} \subseteq \mathfrak{a}(\mathcal{M})$ for all open subsets $O \in \mathcal{N}$, the second part follows from the characterization in Lemma C.1.11. Then the third part is clear as continuous functions are automatically measurable. $\square$

In the following, a topological space will always be equipped with its Borel σ-algebra if not stated otherwise. In particular, all open subsets of M as well as all closed subsets of M are measurable. From the properties of a σ-algebra it follows that also countable intersections of open subsets are measurable. Such subsets are also called G_δ *subsets*. Taking complements shows that countable unions of closed subsets are measurable, too, called F_σ *subsets* of M. However, in general there are more measurable subsets than G_δ and F_σ subsets as we can take again and again countable unions, countable intersections and complements. This makes it very complicated to characterize the measurable subsets in $\mathfrak{a}(\mathscr{M})$ in an efficient way.

In general, a basis $\mathcal{B}$ of the topology $\mathscr{M}$ of M may generate a *strictly* smaller σ-algebra $\mathfrak{a}(\mathcal{B})$ than the Borel σ-algebra $\mathfrak{a}(\mathscr{M})$ since we are allowed to take arbitrary unions of elements in $\mathcal{B}$ to generate all elements of $\mathscr{M}$ instead of just countable ones for $\mathfrak{a}(\mathcal{B})$. However, if countable unions are sufficient then we have the following statement:

Lemma C.1.14 *Let $(M, \mathscr{M})$ be a topological space and let $\mathcal{B} \subseteq \mathscr{M}$ be a basis of the topology such that countable unions of elements of $\mathcal{B}$ yield already all elements of $\mathscr{M}$. Then $\mathfrak{a}(\mathcal{B}) = \mathfrak{a}(\mathscr{M})$.*

PROOF: Clearly $\mathfrak{a}(\mathcal{B}) \subseteq \mathfrak{a}(\mathscr{M})$ by (C.1.1). Since in $\mathfrak{a}(\mathcal{B})$ we are allowed to take countable unions and since $\mathcal{B} \subseteq \mathfrak{a}(\mathcal{B})$ we conclude $\mathscr{M} \subseteq \mathfrak{a}(\mathcal{B})$ by assumption. Then (C.1.1) and (C.1.2) complete the proof. □

If the reference to the topology is clear we shall also write $\mathfrak{a}_M = \mathfrak{a}(\mathscr{M})$ for simplicity or we even omit to mention the Borel σ-algebra at all since for topological spaces we will (almost) exclusively use the Borel σ-algebra.

We give now several equivalent characterizations of the Borel σ-algebra of the topological space $\mathbb{R}^n$:

Theorem C.1.15 (Borel σ-algebra of $\mathbb{R}^n$) *The following σ-algebras on $\mathbb{R}^n$ coincide:*

i.) The Borel σ-algebra $\mathfrak{a}_{\mathbb{R}^n}$.

ii.) The σ-algebra $\mathfrak{a}_2$ generated by all the open intervals $(a_1, b_1) \times \cdots \times (a_n, b_n) \subseteq \mathbb{R}^n$ where $a_1 < b_1$, $\ldots$, $a_n < b_n$.

iii.) The σ-algebra $\mathfrak{a}_3$ generated by all the compact intervals $[a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ where again $a_1 < b_1$, $\ldots$, $a_n < b_n$.

iv.) The σ-algebra $\mathfrak{a}_4$ generated by all the open half spaces $H_k(a)$ with $k = 1, \ldots, n$ and $a \in \mathbb{R}$ where

$$H_k(a) = \Big\{ x \in \mathbb{R}^n \;\Big|\; x^k > a \Big\}. \tag{C.1.9}$$

v.) The σ-algebra $\mathfrak{a}_5$ generated by all the closed half spaces $H_k^{\mathrm{cl}}(a)$ with $k = 1, \ldots, n$ and $a \in \mathbb{R}$.

vi.) The σ-algebra $\mathfrak{a}_6$ generated by all the open balls $\mathrm{B}_r(x)$ for $r > 0$ and $x \in \mathbb{R}^n$.

vii.) The σ-algebras $\mathfrak{a}'_2$, $\mathfrak{a}'_3$, $\mathfrak{a}'_4$, $\mathfrak{a}'_5$, $\mathfrak{a}'_6$ obtained from ii.) – vi.) by restricting the relevant parameters to rational numbers.

PROOF: Clearly, $\mathfrak{a}'_i \subseteq \mathfrak{a}_i$ for $i = 2, \ldots, 6$ and all the σ-algebras are contained in $\mathfrak{a}_{\mathbb{R}^n}$ as the generating sets consist either of closed or open subsets of $\mathbb{R}^n$. It is well-known that every open subset of $\mathbb{R}^n$ is a *countable* union of either open balls with rational radii and rational centers or of open intervals with rational endpoints. This shows that $\mathfrak{a}_2 = \mathfrak{a}'_2 = \mathfrak{a}_6 = \mathfrak{a}'_6 = \mathfrak{a}_{\mathbb{R}^n}$ by Lemma C.1.14 and by the already mentioned obvious inclusions. A countable union of compact intervals $[a_1^k, b_1^k] \times \cdots \times [a_n^k, b_n^k]$ with rational endpoints $a_i^k, b_i^k \in \mathbb{Q}$ with $i = 1, \ldots, n$ and $k \in \mathbb{N}$ can be used to get an arbitrary open interval $(a_1, b_1) \times \cdots \times (a_n, b_n)$. Indeed, taking monotonously convergent sequences of rational numbers with $a_i^k \searrow a_i$ and $b_i^k \nearrow b_i$ for $i = 1, \ldots, n$ will do the job. Thus $(a_1, b_1) \times \cdots \times (a_n, b_n) \in \mathfrak{a}'_3 \subseteq \mathfrak{a}_3$ follows. Hence $\mathfrak{a}_{\mathbb{R}^n} = \mathfrak{a}_2 \subseteq \mathfrak{a}'_3$ which is the non-trivial inclusion to prove $\mathfrak{a}'_3 = \mathfrak{a}_3 = \mathfrak{a}_{\mathbb{R}^n}$. For the open half spaces we first observe that $\mathbb{R}^n \setminus H_k(a) = \{x \in \mathbb{R}^n \mid x^k \le a\}$ is a closed half space oriented the opposite direction. Taking a countable union of $\mathbb{R}^n \setminus H_k(b^\ell)$ with rational $b^\ell \nearrow b$ gives the open subset $\{x \in \mathbb{R}^n \mid x^k < b\}$. Taking finally the intersection of $\{x \in \mathbb{R}^n \mid x^k < b_k\}$ with $H_k(a_k)$ over all $k = 1, \ldots, n$ again with rational parameters gives the open interval $(a_1, b_1) \times \cdots \times (a_n, b_n)$. This shows that $\mathfrak{a}_2 \subseteq \mathfrak{a}'_4$ and hence $\mathfrak{a}'_4 = \mathfrak{a}_4 = \mathfrak{a}_{\mathbb{R}^n}$ follows. The case of closed half spaces is treated analogously. □

There are many more slight modifications of the above characterizations of the Borel σ-algebra of $\mathbb{R}^n$, everyone should find his personal favorite.

Corollary C.1.16 *The product of the Borel σ-algebras of $\mathbb{R}^n$ and $\mathbb{R}^m$ coincides with the Borel σ-algebra of $\mathbb{R}^{n+m}$, i.e.*

$$\mathfrak{a}_{\mathbb{R}^n} \otimes \mathfrak{a}_{\mathbb{R}^m} = \mathfrak{a}_{\mathbb{R}^{n+m}}. \tag{C.1.10}$$

PROOF: This is clear, e.g. by the second part of Theorem C.1.15. □

Note however, that for general topological spaces M, N we only have

$$\mathfrak{a}_M \otimes \mathfrak{a}_N \subseteq \mathfrak{a}_{M\times N}, \tag{C.1.11}$$

but the inclusion may be strict.

n: Example?

Proposition C.1.17 *Let $(X, \mathfrak{a})$ be a measurable space and $f\colon X \longrightarrow \mathbb{R}^n$. Then f is measurable iff each component $f_k = \mathrm{pr}_k \circ f\colon X \longrightarrow \mathbb{R}$ is measurable.*

PROOF: First note that the projection $\mathrm{pr}_k\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ onto the k-th component is measurable since it is even continuous. Thus f_k is measurable if f is measurable by Remark C.1.10, *i.)*. For the reverse assume that f_k is measurable for all k. Then $f_k^{-1}(a_k, b_k) \subseteq X$ is a measurable subset for all $a_k < b_k$ and $k = 1, \ldots, n$. It follows that

$$f^{-1}\big((a_1, b_1) \times \cdots \times (a_n, b_n)\big) = f_1^{-1}(a_1, b_1) \cap \cdots \cap f_n^{-1}(a_n, b_n)$$

is measurable, too. Since the open intervals generate the Borel σ-algebra of $\mathbb{R}^n$ by Theorem C.1.15, we can now apply Lemma C.1.11 to conclude that f is measurable. □

A last construction is the restriction of a σ-algebra to a subset:

Proposition C.1.18 *Let $(X, \mathfrak{a})$ be a measurable space and $Y \subseteq X$ a subset. Then*

$$\mathfrak{a}\big|_Y = \{A \cap Y \mid A \in \mathfrak{a}\} \subseteq 2^Y \tag{C.1.12}$$

is a σ-algebra on Y, the restricted σ-algebra from X. If X is in addition a topological space then the restriction of the Borel σ-algebra $\mathfrak{a}_X$ to Y gives the Borel σ-algebra $\mathfrak{a}_Y$, i.e. we have

$$\mathfrak{a}_X\big|_Y = \mathfrak{a}_Y. \tag{C.1.13}$$

PROOF: The axioms of a σ-algebra are easily verified for $\mathfrak{a}\big|_Y$. In fact, it coincides with the inverse image $\iota^{-1}(\mathfrak{a})$ with respect to the canonical inclusion map $\iota\colon Y \longrightarrow X$, see also Exercise C.6.1, *ii.)*. For the second statement we first recall that the induced topology on Y has precisely the subsets $O \cap Y$ with $O \subseteq X$ open as open subsets. In particular, $\iota\colon Y \longrightarrow X$ is now continuous. This implies that $\iota^{-1}(\mathfrak{a}_X) \subseteq \mathfrak{a}_Y$. However, the σ-algebra $\mathfrak{a}_Y$ is generated by the open subsets $O \cap Y$ which are in $\mathfrak{a}_X\big|_Y$ since $O \in \mathfrak{a}_X$. Thus we have the opposite inclusion $\mathfrak{a}_Y \subseteq \iota^{-1}(\mathfrak{a}_X) = \mathfrak{a}_X\big|_Y$, showing (C.1.13). □

Subsets of measurable spaces will always be equipped with this σ-algebra. Then the inclusion map becomes measurable and the σ-algebra on the subset is the smallest with this property by Remark C.1.12.

An important case is where the subset $Y \subseteq X$ is *measurable* itself. Then $A \cap Y$ is measurable, too, for every $A \in \mathfrak{a}$. Hence in this case

$$\mathfrak{a}\big|_Y = \big\{A \subseteq Y \bigm| A \in \mathfrak{a}\big\} \tag{C.1.14}$$

consists of those measurable subsets of X which happen to lie entirely in Y already.

C.1.2 Spaces of Measurable Functions

We will now introduce the bounded measurable functions and other related spaces of measurable functions. Proposition C.1.17 allows to consider the following space of measurable functions on a measurable space:

Definition C.1.19 (Measurable functions) *Let $(X, \mathfrak{a})$ be a measurable space. Then we define*

$$\mathscr{M}(X, \mathfrak{a}) = \bigl\{ f\colon X \longrightarrow \mathbb{C} \bigm| f \text{ is measurable} \bigr\}, \tag{C.1.15}$$

where $\mathbb{C} \cong \mathbb{R}^2$ is endowed with the Borel σ-algebra $\mathfrak{a}_{\mathbb{R}^2}$. If the reference to the σ-algebra is clear from the context we simply write $\mathscr{M}(X) = \mathscr{M}(X, \mathfrak{a})$.

Proposition C.1.20 *Let $(X, \mathfrak{a})$ be a measurable space.*

i.) The set $\mathscr{M}(X, \mathfrak{a})$ is a unital commutative *-algebra with respect to the pointwise operations.

ii.) If X is a topological space and $\mathfrak{a}$ the Borel σ-algebra then $\mathscr{C}(X) \subseteq \mathscr{M}(X, \mathfrak{a})$.

PROOF: Since the addition and the multiplication are continuous and hence measurable maps $\mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ the functions $f + g = + \circ (f, g)$ and $fg = \cdot \circ (f, g)$ are compositions of measurable maps and hence measurable by Proposition C.1.17 and Remark C.1.10, *i.)*. Also $\overline{f}$ is measurable since the complex conjugation is a continuous map $\mathbb{C} \longrightarrow \mathbb{C}$ and hence measurable, too. The constant functions are clearly measurable since the pre-images of measurable subsets $A \subseteq \mathbb{C}$ are either $\emptyset$ or X depending on whether A contains the value or not. The second part is clear by Proposition C.1.13.□

stefan: $f \times g\colon X \times X$ measurable, r arguments

In particular, $|f| = \sqrt{\overline{f}f}$ is measurable for $f \in \mathscr{M}(X)$ since $\overline{f}f \in \mathscr{M}(X)$ and $\sqrt{\cdot}$ is continuous. It follows that for $f = \overline{f}$ also the positive part f_+ and the negative part f_- are measurable. Also a function is measurable iff its real and imaginary part are measurable.

The next remarkable feature is that the measurable functions behave well under pointwise limits: this is indeed the first non-trivial class of functions where this is true. For continuous functions (or even $\mathscr{C}^k$-functions) or only bounded functions pointwise limits usually destroy all nice properties.

Theorem C.1.21 (Measurable functions) *Let $(X, \mathfrak{a})$ be a measurable space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex-valued measurable functions on X.*

i.) If $f_n = \overline{f}_n$ are real functions for all $n \in \mathbb{N}$ then the pointwise operations $\sup_{n\in\mathbb{N}} f_n$, $\inf_{n\in\mathbb{N}} f_n$ $\max_{1\le k\le n} f_k$, $\min_{1\le k\le n} f_k$, $\liminf_{n\in\mathbb{N}} f_n$, and $\limsup_{n\in\mathbb{N}} f_n$ yield again measurable functions if they are functions at all.

ii.) If $f_n(x) \longrightarrow f(x)$ converges pointwise for all $x \in X$ then the limit f is again measurable.

PROOF: We consider the (pointwise) supremum first which we assume to exist. If $f = \overline{f}$ is real-valued the f is measurable iff for all (rational) $a \in \mathbb{R}$ the subset $f^{-1}((a, \infty)) \subseteq X$ is measurable. This follows from the characterization of the Borel σ-algebra $\mathfrak{a}_{\mathbb{R}}$ via open half spaces according to Theorem C.1.15, *iv.)*, and Lemma C.1.11 applied to these generating subsets. Now let $f = \sup_{n\in\mathbb{N}} f_n$ be the pointwise supremum which we assume to exist pointwise. Then we have the implication that $f_n(x) > a$ for some n implies $x \in \bigcup_{m\in\mathbb{N}} f_m^{-1}((a, \infty))$. Now $\sup_{n\in\mathbb{N}} f_n(x) > a$ implies that there is at least one n with $f_n(x) > a$ hence $x \in \bigcup_{m\in\mathbb{N}} f_m^{-1}((a, \infty))$ also in this case. Conversely, if $x \in \bigcup_{m\in\mathbb{N}} f_m^{-1}((a, \infty))$ then there is at least one n with $f_n(x) > a$. But then also $\sup_{n\in\mathbb{N}} f_n(x) > a$. We conclude that

$$f^{-1}((a, \infty)) = \bigcup_{m\in\mathbb{N}} f_m^{-1}((a, \infty))$$

for all $a \in \mathbb{R}$. Since the right hand side is a measurable subset as the countable union of measurable subsets, the function f satisfies the above criterion. This shows that f is measurable. Since

$\inf_{n\in\mathbb{N}} f_n = -\sup_{n\in\mathbb{N}}(-f_n)$ we also get that $\inf_{n\in\mathbb{N}} f_n$ is measurable. Taking only finitely many different members of the sequence gives then that the finite maximum and minimum are measurable. Finally, recall that

$$\limsup_{n\in\mathbb{N}} f_n(x) = \inf_{n\in\mathbb{N}} \sup_{k\geq n} f_k(x) \quad \text{and} \quad \liminf_{n\in\mathbb{N}} f_n(x) = \sup_{n\in\mathbb{N}} \inf_{k\geq n} f_k(x),$$

e exercises to p and lim inf

see Exercise ??. Hence a combination of sup and inf gives the measurability also of $\limsup_{n\in\mathbb{N}} f_n$ and $\liminf_{n\in\mathbb{N}} f_n$. This proves the first part. For the second part we first notice that we can treat the real and the imaginary part separately. Then the limit is in particular a lim sup and hence we can apply the first part. ☐

n: Glueing of ps in App. A

Since by Proposition C.1.18 measurable subsets of a measurable space carry a canonical σ-algebra, we can now glue measurable maps with similar techniques like gluing continuous maps in topology. We have the following result:

Proposition C.1.22 *Let $(X,\mathfrak{a})$ and $(Y,\mathfrak{b})$ be measurable spaces.*

i.) If $f\colon X \longrightarrow Y$ is measurable then $f\big|_A\colon A \longrightarrow Y$ is measurable for any subset $A \subseteq X$ with respect to $\mathfrak{a}\big|_A$.

ii.) Suppose $A_n \in \mathfrak{a}$ is a sequence of measurable subsets with $X = \bigcup_{n\in\mathbb{N}} A_n$. Let $f_n\colon A_n \longrightarrow Y$ be measurable maps with

$$f_n\big|_{A_n\cap A_m} = f_m\big|_{A_n\cap A_m} \tag{C.1.16}$$

whenever $A_n\cap A_m \neq \emptyset$. Then there exists a unique measurable map $f\colon X \longrightarrow Y$ with $f\big|_{A_n} = f_n$.

iii.) Let $f\colon X \longrightarrow Y$ be a map and let $A_n \in \mathfrak{a}$ be measurable with $\bigcup_{n\in\mathbb{N}} A_n = X$. Then f is measurable iff $f\big|_{A_n}\colon A_n \longrightarrow Y$ is measurable for all $n \in \mathbb{N}$.

Proof: Let $B \subseteq Y$ be any subset then we have

$$\left(f\big|_A\right)^{-1}(B) = f^{-1}(B) \cap A, \tag{$*$}$$

which is a simple set-theoretic consideration. Now if f was measurable and $B \in \mathfrak{b}$ then $f^{-1}(B) \in \mathfrak{a}$ and hence $f^{-1}(B)\cap A \in \mathfrak{a}\big|_A$ by the very definition of $\mathfrak{a}\big|_A$. But then $(*)$ shows that $f\big|_A$ is measurable. For the second part we first observe that there is a unique map $f\colon X \longrightarrow Y$ with $f\big|_{A_n} = f_n$ since the compatibility (C.1.16) shows that $f(x) = f_n(x)$ for $x \in A_n$ yields a well-defined map f on $\bigcup_{n\in\mathbb{N}} A_n = X$. We have to show that f is measurable. Thus let $B \in \mathfrak{b}$ be given. Again by purely set-theoretic considerations we have

$$f^{-1}(B) = \bigcup_{n\in\mathbb{N}} f_n^{-1}(B). \tag{$**$}$$

Since the f_n are measurable we have $f_n^{-1}(B) \in \mathfrak{a}\big|_{A_n}$. Since the A_n are measurable we know from (C.1.14) that $f_n^{-1}(B) \in \mathfrak{a}$. Thus $(**)$ gives us $f^{-1}(B) \in \mathfrak{a}$. Note that without the assumption that A_n is measurable not much can be said about the measurability of $f_n^{-1}(B) \subseteq X$ with respect to $\mathfrak{a}$. For the last part we know that if f is measurable then $f\big|_{A_n}$ is measurable by the first part, no matter what A_n is. Conversely, if $f_n = f\big|_{A_n}$ is measurable we automatically have the compatibility (C.1.16) for the f_n. Thus by the second part the corresponding glued map, which is of course again f, is measurable. ☐

A typical application is that we cut X into at most countably many *disjoint* $A_n \in \mathfrak{a}$. Then we can specify maps $f_n\colon A_n \longrightarrow Y$ without restriction since the gluing condition (C.1.16) is void. The combined map f is then measurable iff the f_n are measurable for all n. Even more specifically, suppose

$A \subseteq X$ is measurable and $f\colon A \longrightarrow Y$ is already a measurable map. Then we can fix any value $y \in Y$ and set

$$f\big|_{X\setminus A} = y \tag{C.1.17}$$

to be the *constant* map on the (measurable) complement of A. Since the constant map is clearly measurable, the combined map $f\colon X \longrightarrow Y$ is measurable. Thus we can *always* extend a measurable map from a measurable subset to the whole measurable space simply by setting it to be constant outside. This makes measurable spaces and maps quite different from topological spaces and continuous maps where such an extension usually is obstructed and even if it is not obstructed, this requires usually a non-trivial argument like Tietze's Theorem **??**.

We introduce now a particular class of measurable functions, the simple functions and the characteristic functions:

Definition C.1.23 (Simple functions) *Let $(X, \mathfrak{a})$ be a measurable space. Then a measurable function $f\colon X \longrightarrow \mathbb{C}$ is called simple if it has only finitely many different values. If a measurable function f has only values in $\{0, 1\}$ then f is called the characteristic function of the subset $A = f^{-1}(\{1\})$. We write $f = \chi_A$ in this case.*

If $f = \chi_A$ is a characteristic function then A is a measurable subset since $\{1\} \subseteq \mathbb{C}$ is measurable. Conversely, if A is measurable, then

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{C.1.18}$$

is measurable and hence a characteristic function. With other words, the characteristic functions are in one-to-one correspondence to the elements of the σ-algebra $\mathfrak{a}$. By convention, we use the symbol χ_A for the function (C.1.18) for *all* subsets $A \subseteq X$, whether A is measurable or not.

If f is a simple function and $z_1, \ldots, z_n \in \mathbb{C}$ are the distinct values of f then we have

$$f = \sum_{i=1}^{n} z_i \chi_{A_i} \tag{C.1.19}$$

with $A_i = f^{-1}(\{z_i\})$. As before, $A_i \in \mathfrak{a}$ is measurable for all i and hence the χ_{A_i} are characteristic functions. Moreover, we clearly have $A_i \cap A_j = \emptyset$ for $i \neq j$ and $X = \bigcup_{i=1}^{n} A_i$. Since we required all the z_i to be different, it is now clear that the form (C.1.19) is unique: it is called the *normal form* of a simple function.

Conversely, since the measurable functions form a complex vector space by Proposition C.1.20 we see that a linear combination of characteristic functions like in (C.1.19) results in a simple function, which, however, needs not to be in its normal form. The reason is that even if we use pairwise distinct values $z_1, \ldots, z_n$ in (C.1.19) the characteristic functions are not linearly independent. In fact, we have relations like

$$\chi_A + \chi_B - \chi_{A\cap B} - \chi_{A\cup B} = 0 \tag{C.1.20}$$

for $A, B \in \mathfrak{a}$. Thus the characteristic functions span the subspace of all simple functions but they do not form a vector space basis.

Exercise: Otl

While $\mathscr{M}(X, \mathfrak{a})$ consists of all measurable functions on X, the simple functions are in addition bounded. This motivates to consider the bounded measurable functions in general:

Definition C.1.24 (Bounded measurable functions) *Let $(X, \mathfrak{a})$ be a measurable space. Then we denote the space of bounded measurable functions on X by*

$$\mathscr{B}\mathscr{M}(X, \mathfrak{a}) = \mathscr{B}(X) \cap \mathscr{M}(X, \mathfrak{a}), \tag{C.1.21}$$

and endow $\mathscr{B}\mathscr{M}(X, \mathfrak{a})$ with the supremum norm $\|\cdot\|_\infty$ inherited from $\mathscr{B}(X)$.

If the reference to the σ-algebra is clear from the context, like e.g. on a topological space, then we simply write $\mathscr{BM}(X)$ instead of $\mathscr{BM}(X, \mathfrak{a})$, Note that we always consider *complex-valued* functions. The next proposition clarifies the structure of $\mathscr{BM}(X)$ and illustrates the relevance of the simple functions:

Proposition C.1.25 (Bounded measurable functions) *Let $(X, \mathfrak{a})$ be a measurable space.*

i.) The bounded measurable functions $\mathscr{BM}(X)$ form a unital C^*-subalgebra of the unital C^*-algebra $\mathscr{B}(X)$.

ii.) The simple functions are dense in $\mathscr{BM}(X)$.

Proof: First we note that the uniform limit of bounded functions is again bounded. From this trivial observation it easily follows that $\mathscr{B}(X)$ is indeed complete with respect to the supremum norm $\|\cdot\|_\infty$. Then the properties of a unital C^*-algebra are easily verified, we only check the C^*-property of $\|\cdot\|_\infty$: let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X with $|f(x_n)| \longrightarrow \sup_{x\in X}|f(x)| = \|f\|_\infty = \|\overline{f}\|_\infty$. Then also $|f(x_n)|^2 \longrightarrow \sup_{x\in X}|f(x)|^2 = \|\overline{f}f\|_\infty$. From this we conclude $\|\overline{f}f\|_\infty = \|f\|_\infty^2$ as wanted. The remaining properties are easily checked. Now $\mathscr{BM}(X) = \mathscr{B}(X) \cap \mathscr{M}(X)$ is a subspace and a unital *-subalgebra by Proposition C.1.20, *i.)*. We have to show that $\mathscr{BM}(X)$ is closed with respect to $\|\cdot\|_\infty$. But this is clear from Theorem C.1.21, *ii.)*, since uniform limits are in particular pointwise limits. This shows the first claim. For the second part, let $f \in \mathscr{BM}(X)$ be given. Since then $f = \mathrm{Re}(f)_+ - \mathrm{Re}(f)_- + \mathrm{i}\,\mathrm{Im}(f)_+ - \mathrm{i}\,\mathrm{Im}(f)_-$ is a linear combination of four non-negative functions which are all measurable themselves we can assume that $f \geq 0$ from the beginning. We consider now the following subsets

$$A_{i,n} = \left\{x \in X \;\middle|\; i2^{-n} \leq f(x) < (i+1)2^{-n}\right\}$$

for $i = 0, \ldots, n2^n - 1$. With other words, $A_{i,n} = f^{-1}\big([i2^{-n}, (i+1)2^{-n})\big)$. Since f is measurable it follows that $A_{i,n}$ is a measurable subset for all i and n. Moreover, we note that all $A_{i,n}$ are pairwise disjoint. We fix some $n > \|f\|_\infty$. Then every point $x \in X$ is contained in some $A_{i,n}$ since $f\big|_{A_{i,n}} \subseteq [i2^{-n}, (i+1)2^{-n})$ and

$$\bigcup_{i=0}^{n2^n-1} \big[i2^{-n}, (i+1)2^{-n}\big) = [0, n),$$

because $f \geq 0$ by assumption. Now we consider the simple functions

$$g_n = \sum_{i=0}^{n2^n-1} i2^{-n}\chi_{A_{i,n}}.$$

Note that this is already the normal form of g_n. For $x \in A_{i,n}$ we have the estimate

$$g_n(x) = i2^{-n} \leq f(x) < (i+1)2^{-n} = g_n(x) + 2^{-n}.$$

Since the $A_{i,n}$ cover all of X we conclude that

$$g_n(x) \leq f(x) < g_n(x) + 2^{-n}$$

for *all* $x \in X$ and hence $\|g_n - f\|_\infty < 2^{-n}$. This completes the proof of the second part. □

In the case of a topological Hausdorff space X, the bounded continuous functions

$$\mathscr{C}_{\mathrm{b}}(X) = \left\{f \in \mathscr{C}(X) \;\middle|\; \|f\|_\infty < \infty\right\} \tag{C.1.22}$$

are bounded by definition and measurable by Proposition C.1.20, *ii.)*. Thus we end up with a unital *-subalgebra

$$\mathscr{C}_{\mathrm{b}}(X) \subseteq \mathscr{BM}(X). \tag{C.1.23}$$

If X is in addition *compact* then $\mathscr{C}(X) = \mathscr{C}_{\mathrm{b}}(X)$ is a unital C^*-algebra by its own and hence a unital C^*-subalgebra $\mathscr{C}(X) \subseteq \mathscr{BM}(X)$. If X is only *locally compact* then $\mathscr{C}_{\mathrm{b}}(X)$ is still complete and hence a unital C^*-subalgebra of $\mathscr{BM}(X)$. However, the space of all continuous functions $\mathscr{C}(X)$ will only be a pro-C^*-algebra not contained in $\mathscr{BM}(X)$ in general, see also Exercise C.6.2.

Remark C.1.26 (The C^*-algebra $\mathscr{BM}(X, \mathfrak{a})$) Let $(X, \mathfrak{a})$ be a measurable space.

i.) The positive elements of $\mathscr{BM}(X, \mathfrak{a})$ can be obtained as follows: First we note that the δ-functionals

$$\delta_x \colon \mathscr{BM}(X, \mathfrak{a}) \ni f \;\mapsto\; f(x) \in \mathbb{C} \tag{C.1.24}$$

are positive linear functionals (even states) for every $x \in X$. Thus for $f \in \mathscr{BM}(X, \mathfrak{a})^+$ it is necessary that $f(x) \geq 0$ for all $x \in X$. It is also sufficient since if $f(x) \geq 0$ for every $x \in X$ then $\overline{\sqrt{f}} = \sqrt{f} \in \mathscr{BM}(X, \mathfrak{a})$ is measurable by the continuity of $\sqrt{\cdot}$. Thus $f = (\sqrt{f})^2$ is a square and hence positive.

ii.) For $f \in \mathscr{BM}(X, \mathfrak{a})$ we want to determine the spectrum. Clearly, every value $z \in f(X) \subseteq \mathbb{C}$ is in the spectrum as $z - f$ is not invertible at all. Next consider $z \in \mathbb{C} \setminus f(X)$ then the inverse of $z - f$ exists pointwise and is measurable again: this follows as the inversion is a continuous map from $\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$ and hence measurable. But the inverse of $z - f$ is only bounded if z is not in the *closure* of $f(X)$. Thus we have

$$\operatorname{spec}(f) = f(X)^{\mathrm{cl}} \subseteq \mathbb{C}. \tag{C.1.25}$$

Note that $\operatorname{spec}(f)$ is indeed compact as it should be for an element f in a C^*-algebra.

iii.) Let $\mathfrak{a} = 2^X$ be the whole power set. Then $\mathscr{BM}(X, \mathfrak{a}) = \mathscr{B}(X)$ coincides with all the bounded functions. Hence $\mathscr{B}(X)$ is a particular case of bounded measurable functions.

C.2 Measures

We come now to the notion of a measure on a measurable space $(X, \mathfrak{a})$. Here we will meet two versions: positive measures and complex measures.

C.2.1 Positive Measures

We begin with the discussion of positive measures. The following definition formalizes the idea that we want to assign a measure (a volume, an area) to every measurable subset:

Definition C.2.1 (Positive measure) *Let $(X, \mathfrak{a})$ be a measurable space. Then a positive measure μ on $(X, \mathfrak{a})$ is a map*

$$\mu \colon \mathfrak{a} \longrightarrow [0, +\infty], \tag{C.2.1}$$

which is σ-additive. This means that for a sequence of pairwise disjoint measurable subsets $A_n \in \mathfrak{a}$ we have

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n). \tag{C.2.2}$$

Moreover, we require that there is at least one measurable subset $A \in \mathfrak{a}$ with $\mu(A) < \infty$. A measure space is then a measurable space equipped with a positive measure.

If no confusion with complex measures is in sight we call a positive measure simply a *measure*. Later on, we will need to be more careful.

Remark C.2.2 (The interval $[0,+\infty]$) In the definition of a measure it is explicitly allowed that for certain $A \in \mathfrak{a}$ we may have $\mu(A) = +\infty$. In this sense we also have to understand the σ-additivity: it may well happen that the right hand side of (C.2.2) diverges. This simply means that the union of such A_n has infinite measure. Note that since $\mu(A_n) \in [0,+\infty]$ the convergence or divergence of (C.2.2) is necessarily *absolute* and hence unconditional. For later use, we mention some conventions concerning the "number" $+\infty \in [0,+\infty]$. We set for $a \in [0,+\infty]$

$$a \cdot (+\infty) = \begin{cases} +\infty & \text{if } a \neq 0 \\ 0 & \text{if } a = 0. \end{cases} \tag{C.2.3}$$

Moreover, one defines $a + \infty = +\infty$ for all $a \in [0,+\infty]$. Analogously, we set $\frac{1}{+\infty} = 0$. These conventions will help to formulate the principal results of measure theory in a neater way instead of making artificial case decisions later on. Note that in $[0,+\infty]$ every subset has a supremum which, in the unbounded case, is $+\infty$. Moreover, we can view $[0,+\infty]$ as a topological space homeomorphic to the unit interval $[0,1]$: the intervals $(a,+\infty]$ constitute a basis of open neighbourhoods of $+\infty$.

Example C.2.3 (Trivial and infinite measure) Let X be a non-empty set. For the following two examples we can take $\mathfrak{a} = 2^X$ as σ-algebra.

i.) The *trivial measure* or *zero measure* assigns to every subset $A \subseteq X$ the measure 0, i.e. we have

$$\mu_{\mathrm{trivial}}(A) = 0 \tag{C.2.4}$$

for all $A \in \mathfrak{a}$. This is clearly a positive measure.

ii.) The other canonical example is to assign the value ∞ to every non-empty subset: we define

$$\mu_{\mathrm{infinite}}(A) = \begin{cases} \infty & A \neq \emptyset \\ 0 & A = \emptyset, \end{cases} \tag{C.2.5}$$

which is again easily verified to be a measure, called the *infinite measure*.

Both examples are somehow pretty useless for practical purposes but serve as valid (counter-) examples in many situations.

Remark C.2.4 (Ordering of positive measures) Sometimes we are given two positive measures μ and ν on the same measurable space $(X,\mathfrak{a})$. Then we can compare their values pointwise on the σ-algebra. Since the interval $[0,+\infty]$ is still ordered, we define an ordering for the measures by setting $\mu \leq \nu$ iff

$$\mu(A) \leq \nu(A) \tag{C.2.6}$$

holds for all $A \in \mathfrak{a}$. Clearly, this is a partial ordering on the set of all positive measures.

There are now some simple properties of a positive measure which follow directly from the definition. We collect them in the following proposition:

Proposition C.2.5 *Let $(X,\mathfrak{a},\mu)$ be a measure space.*

i.) One has $\mu(\emptyset) = 0$.

ii.) The measure μ is additive, i.e. for pairwise disjoint $A_1, \ldots, A_n \in \mathfrak{a}$ we have

$$\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n). \tag{C.2.7}$$

iii.) The measure μ is monotonous, i.e. for $A, B \in \mathfrak{a}$ with $A \subseteq B$ we have

$$\mu(A) \leq \mu(B). \tag{C.2.8}$$

Proof: For the first part we need the assumption that there is at least one $A \in \mathfrak{a}$ with $\mu(A) < \infty$. Then taking the pairwise disjoint sequence $A, \emptyset, \emptyset, \ldots$ gives

$$\mu(A) = \mu(A \cup \emptyset \cup \emptyset \cup \cdots) = \mu(A) + \mu(\emptyset) + \mu(\emptyset) + \cdots$$

from (C.2.2). This is only possible for $\mu(\emptyset) = 0$. Then the second part is clear by filling the $A_1, \ldots, A_n$ to a sequence adding infinitely many $\emptyset$. For the third part we have $X \setminus A \in \mathfrak{a}$ and hence $B \cap (X \setminus A) = B \setminus A \in \mathfrak{a}$. But $B = (B \setminus A) \cup A$ gives $\mu(B) = \mu(B \setminus A) + \mu(A) \geq \mu(A)$ by the second part since clearly $A \cap (B \setminus A) = \emptyset$. □

Note that some authors require $\mu(\emptyset) = 0$ directly in the definition of a positive measure. From the proof we note the important relation

$$\mu(B) = \mu(B \setminus A) + \mu(A) \tag{C.2.9}$$

for $A \subseteq B$ both being measurable. Clearly, these features of a measure formalize the heuristic ideas of a volume or area measure for the subsets of X.

Proposition C.2.6 *Let $(X, \mathfrak{a}, \mu)$ be a measure space and let $A_1, A_2, \ldots \in \mathfrak{a}$ be a sequence of measurable subsets.*

i.) If $A_1 \subseteq A_2 \subseteq \cdots$ then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \longrightarrow \infty} \mu(A_n). \tag{C.2.10}$$

ii.) If $A_1 \supseteq A_2 \supseteq \cdots$ and if $\mu(A_1) < \infty$ then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \longrightarrow \infty} \mu(A_n). \tag{C.2.11}$$

Proof: We define inductively $B_1 = A_1$ and $B_n = A_n \setminus B_{n-1}$ for $n \geq 2$. Then all the sets B_n are again measurable by the argument used already in the proof of Proposition C.2.5, *iii.)*, and clearly pairwise disjoint. Moreover, for all $n \in \mathbb{N}$ we have

$$A_n = B_1 \cup \cdots \cup B_n,$$

and hence $\mu(A_n) = \mu(B_1) + \cdots + \mu(B_n)$ by additivity of the measure μ. Since $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ we get (C.2.10) by σ-additivity. For the second part, we set analogously $B_n = A_1 \setminus A_n$. Again $B_n \in \mathfrak{a}$ and now we have $\emptyset = B_1 \subseteq B_2 \subseteq \cdots$. Hence $\mu(B_n) = \mu(A_1) - \mu(A_n)$ thanks to (C.2.9) gives by the first part

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n \longrightarrow \infty} \mu(B_n) = \mu(A_1) - \lim_{n \longrightarrow \infty} \mu(A_n).$$

On the other hand, $A_1 \setminus \bigcap_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$. Thus

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) = \mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right),$$

from which we obtain (C.2.11) since $\mu(A_1) < \infty$ was assumed to be finite. Note that this is crucial and there are simple counterexamples showing that the assumption $\mu(A_1) < \infty$ is *not* superfluous for *ii.)*, see Exercise C.6.6. □

The properties discussed in this proposition are also called the *continuity* (from outside and inside) of the measure μ. One should note that the σ-additivity of a measure is crucial for its continuity, a mere "additivity" would not give (C.2.10) or (C.2.11).

We introduce some more terminology: a measure μ on a measurable space $(X, \mathfrak{a})$ is called *σ-finite* if there are countably many measurable subsets $A_n \in \mathfrak{a}$ with

$$\mu(A_n) < \infty \quad \text{and} \quad \bigcup_{n=1}^{\infty} A_n = X. \tag{C.2.12}$$

Even more restrictive, a measure μ is called *finite* if

$$\mu(X) < \infty. \tag{C.2.13}$$

In this case, $X = A \cup (X \setminus A)$ shows that the measure $\mu(A)$ is finite for every measurable subset $A \in \mathfrak{a}$, too. In the case of a σ-finite measure $\mu(A) = +\infty$ may well happen for some $A \in \mathfrak{a}$.

Up to now, we do not yet have interesting examples of measures beside trivial ones from Example C.2.3. In fact, the construction of interesting measures like e.g. the Lebesgue measure on $\mathbb{R}^n$ which assigns to every compact interval $[a_1, b_a] \times \cdots \times [a_n, b_n]$ the Euclidean volume $(b_1 - a_1) \cdots (b_n - a_n)$ is a quite non-trivial task which we postpone until Section **??**. The following examples are much more naive, yet still important:

Example C.2.7 (Counting and Dirac measures) Let X be a set and take $\mathfrak{a} = 2^X$ as σ-algebra.

i.) For $A \subseteq X$ we define $\mu_{\text{count}}(A) = +\infty$ if $\#A$ is not finite and $\mu_{\text{count}}(A) = \#A$ for a finite subset A. This gives a measure defined on the whole power set 2^X, the *counting measure* on X.

ii.) Pick a point $x_0 \in X$ then we define $\delta_{x_0}(A) = 1$ if $x_0 \in A$ and $\delta_{x_0}(A) = 0$ otherwise. This again gives a measure, called the *point measure* or *Dirac measure* at x_0.

The following examples provide constructions we can perform on given measures to obtain new ones.

Example C.2.8 (Restriction of a measure) Let $Y \subseteq X$ be a measurable subset of a measure space $(X, \mathfrak{a}, \mu)$. Then the *restriction* of μ to Y is defined by

$$\mu\big|_Y(A) = \mu(A) \tag{C.2.14}$$

for all $A \in \mathfrak{a}\big|_Y$. Here it is crucial that for a measurable subset Y the σ-algebra $\mathfrak{a}\big|_Y$ consists just of those $A \in \mathfrak{a}$ which are in Y. Thus the right hand side of (C.2.14) is defined, which would not be the case for a general subset Y. It is now easy to see that $\mu\big|_Y$ is indeed a measure on $(Y, \mathfrak{a}\big|_Y)$.

Example C.2.9 (Push-forward measure) If $f\colon (X, \mathfrak{a}) \longrightarrow (Y, \mathfrak{b})$ is a measurable map between measurable spaces and μ is a measure on X then one defines

$$(f_*\mu)(B) = \mu(f^{-1}(B)) \tag{C.2.15}$$

for $B \in \mathfrak{b}$. Since $f^{-1}(B) \in \mathfrak{a}$ this is a well-defined map $f_*\mu\colon \mathfrak{b} \longrightarrow [0, \infty]$. Moreover, since $f^{-1}\colon 2^Y \longrightarrow 2^X$ is compatible with all set operations it is clear that $f_*\mu$ is again a measure, called the *push-forward* of μ. In particular, if Y is just a set, we can take $\mathfrak{b} = f_*\mathfrak{a}$ to be the push-forward σ-algebra and then push forward the measure μ, too. We note that measures behave covariantly (i.e. they push forward) while continuous functions behave contravariantly (i.e. they pull back) with respect to morphisms in Mess and Top, respectively. Thus we can anticipate already here that on a topological space measures and functions are "dual" to each other.

Example C.2.10 (Convex combination) Let $(X, \mathfrak{a})$ be a measurable space and let $\mu_1, \ldots, \mu_N$ be positive measures on $(X, \mathfrak{a})$. Then for every $\lambda_1, \ldots \lambda_N \geq 0$ the convex combination

$$\mu = \lambda_1 \mu_1 + \cdots + \lambda_N \mu_N \tag{C.2.16}$$

is again a positive measure on $(X, \mathfrak{a})$. This is clear since the σ-additivity is clearly compatible with finite convex combinations. In fact, since we are dealing with values in $[0, +\infty]$ anyway, the relevant series converge all absolutely or diverge absolutely. Hence also an infinite convex combination would still give a measure, see Exercise C.6.7. An important application are the Dirac measures located at several points with possibly different weights.

The subsets $A \in \mathfrak{a}$ in a measure space $(X, \mathfrak{a}, \mu)$ with measure zero, i.e. $\mu(A) = 0$, play a crucial role in the whole theory. Since the idea of a measure is to assign a "volume" to a subset $A \in \mathfrak{a}$, *any* subset B of a set $A \in \mathfrak{a}$ with $\mu(A) = 0$ should also have volume zero. If $B \in \mathfrak{a}$ then the monotonicity of μ according to Proposition C.2.5, *iii.)*, guarantees precisely this feature. The problem is that the subsets of such a set A need not to be in the σ-algebra at all. Hence we can not assign a measure to them by μ at all. This motivates the following definition:

Definition C.2.11 (Zero sets and complete measure) *Let $(X, \mathfrak{a}, \mu)$ be a measure space.*

i.) A measurable subset $A \in \mathfrak{a}$ is called zero set (or null set) if $\mu(A) = 0$.

ii.) The measure μ is called complete and $(X, \mathfrak{a}, \mu)$ is called a complete measure space if every subset of a zero set is measurable.

Clearly, it is a very desirable feature to have a *complete* measure. Note that this feature involves both, the σ-algebra and the measure.

Lemma C.2.12 *Let $(X, \mathfrak{a}, \mu)$ be a measurable space and $A_n \in \mathfrak{a}$ a sequence of measurable subsets. Then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n). \tag{C.2.17}$$

In particular, the countable union of zero sets is again a zero set.

PROOF: Note that if the A_n are pairwise disjoint we have equality by the σ-additivity of μ. We define $B_1 = A_1$ and $B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right)$ which are all measurable. Then on the one hand, $B_n \subseteq A_n$ and hence $\mu(B_n) \leq \mu(A_n)$. On the other hand, the B_n are pairwise disjoint with the same union as the A_n. Hence

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

The statement about zero sets is clear from this. □

The following construction shows that we can always complete a measure space to a complete measure space by enlarging the σ-algebra if necessary:

Exercise: abs of σ-ideals ar properties

Theorem C.2.13 (Completion of a measure space) *Let $(X, \mathfrak{a}, \mu)$ be a measure space. Then*

$$\hat{\mathfrak{a}} = \left\{ A \cup N \;\middle|\; A \in \mathfrak{a} \textit{ and there exists a } B \in \mathfrak{a} \textit{ with } \mu(B) = 0 \textit{ and } N \subseteq B \right\} \tag{C.2.18}$$

is a σ-algebra containing $\mathfrak{a}$. Moreover, defining $\hat{\mu} \colon \hat{\mathfrak{a}} \longrightarrow [0, +\infty]$ by

$$\hat{\mu}(A \cup N) = \mu(A) \tag{C.2.19}$$

provides a complete positive measure on $(X, \hat{\mathfrak{a}})$ such that $\hat{\mu}\big|_{\mathfrak{a}} = \mu$. Finally, if $(X, \mathfrak{b}, \nu)$ is another complete measure space with $\mathfrak{b} \supseteq \mathfrak{a}$ and $\nu\big|_{\mathfrak{a}} = \mu$ then $\mathfrak{b} \supseteq \hat{\mathfrak{a}}$ and $\nu\big|_{\hat{\mathfrak{a}}} = \hat{\mu}$.

PROOF: In the proof, N will always be a subset of a measurable set $B \in \mathfrak{a}$ with measure $\mu(B) = 0$. Since $N = \emptyset$ is allowed we have $\mathfrak{a} \subseteq \hat{\mathfrak{a}}$ and in particular $\emptyset \in \hat{\mathfrak{a}}$. Let $A \cup N \in \hat{\mathfrak{a}}$ with $A \in \mathfrak{a}$ be given. First we note that from $N \subseteq B$ we get

$$X \setminus N = (X \setminus B) \cup (B \setminus N) = (X \setminus B) \cup (B \cap (X \setminus N)).$$

Hence

$$\begin{aligned} X \setminus (A \cup N) &= (X \setminus A) \cap (X \setminus N) = (X \setminus A) \cap ((X \setminus B) \cup (B \cap (X \setminus N))) \\ &= ((X \setminus A) \cap (X \setminus B)) \cup ((X \setminus A) \cap B \cap (X \setminus N)). \end{aligned}$$

Since $\mathfrak{a}$ is a σ-algebra and $A, B \in \mathfrak{a}$ are measurable we have $X \setminus A, X \setminus B \in \mathfrak{a}$. Moreover, since B is a zero set, the triple $(X \setminus A) \cap B \cap (X \setminus N)$ is a subset of a zero set showing that $X \setminus (A \cup N)$ is again a union of a member of $\mathfrak{a}$ and a subset of a zero set. Thus $X \setminus (A \cup N) \in \hat{\mathfrak{a}}$ and hence $\hat{\mathfrak{a}}$ is closed under taking complements. For the countable union we take $A_n \in \mathfrak{a}$ and $N_n \subseteq B_n \in \mathfrak{a}$ with $\mu(B_n) = 0$. By Lemma C.2.12 the union $B = \bigcup_{n=1}^{\infty} B_n$ is a zero set again and hence $N = \bigcup_{n=1}^{\infty} N_n$ is contained in a zero set. Thus $\bigcup_{n=1}^{\infty}(A_n \cup N_n) = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} N_n = A \cup N$ is again of the desired form. This shows that $\hat{\mathfrak{a}}$ is a σ-algebra containing $\mathfrak{a}$. In a next step we show that $\hat{\mu}$ is a well-defined positive measure on $\hat{\mathfrak{a}}$. Thus assume that $A_1, A_2 \in \mathfrak{a}$ and $N_1 \subseteq B_1$ and $N_2 \subseteq B_2$ are subsets of zero sets $B_1, B_2 \in \mathfrak{a}$ such that $A_1 \cup N_1 = A_2 \cup N_2$. Then $A_1 \subseteq A_1 \cup N_1 = A_2 \cup N_2 \subseteq A_2 \cup B_2$. Hence by Lemma C.2.12, applied to a finite union, we get

$$\mu(A_1) \leq \mu(A_2 \cup B_2) \leq \mu(A_2) + \mu(B_2) = \mu(A_2),$$

since $\mu(B_2) = 0$. Thus $\mu(A_1) \leq \mu(A_2)$ and by symmetry we conclude that $\mu(A_1) = \mu(A_2)$. This shows that $\hat{\mu}$ is well-defined. Clearly, $\hat{\mu}(\emptyset) = 0$ and if $A_n \cup N_n$ is a sequence of pairwise disjoint elements of $\hat{\mathfrak{a}}$ then the A_n are pairwise disjoint as well. Thus the σ-additivity of $\hat{\mu}$ is inherited from the one of μ. We conclude that $\hat{\mu}$ is a positive measure. Since $\emptyset$ is a necessarily a zero set, $\hat{\mu}(A) = \hat{\mu}(A \cup \emptyset) = \mu(A)$ shows that $\hat{\mu}$ extends μ. Next we show that $\hat{\mu}$ is complete. Assume that $A \cup N$ is a zero set with respect to $\hat{\mu}$. This is the case iff $\mu(A) = 0$ and hence A is a zero set with respect to μ. Since $N \subseteq B$ is contained in a zero set with respect to μ we see that $A \cup N \subseteq A \cup B$ is in a zero set of μ. From this we conclude that the zero sets with respect to $\hat{\mu}$ are precisely all the subsets of zero sets with respect to μ. But then $\hat{\mu}$ is obviously complete. Finally, we show that $\hat{\mu}$ is the minimal extension of μ with respect to this feature. Thus let $\mathfrak{b} \supseteq \mathfrak{a}$ be another σ-algebra with another measure ν extending μ such that ν is complete. Then every zero set with respect to μ is also a zero set with respect to ν. Hence $N \subseteq B$ for a zero set B with respect to μ is also contained in a zero set with respect to ν. By completeness we have $N \in \mathfrak{b}$ and hence $\hat{\mathfrak{a}} \subseteq \mathfrak{b}$. Moreover, $A \cup N \in \mathfrak{b}$ and $\nu(A \cup N) = \nu(A) + \nu(N) - \nu(A \cap N) = \nu(A) = \mu(A) = \hat{\mu}(A \cup N)$. This shows that ν extends also $\hat{\mu}$. □

Definition C.2.14 (Completion of a measure) *Let $(X, \mathfrak{a}, \mu)$ be a measure space. Then the measure space $(X, \hat{\mathfrak{a}}, \hat{\mu})$ as in Theorem C.2.13 is called the completion of $(X, \mathfrak{a}, \mu)$.*

Remark C.2.15 (Completion of a measure)

i.) The completion of measure spaces does not quite enjoy the nice functorial properties one might expect: if $(X, \mathfrak{a}, \mu)$ and $(Y, \mathfrak{b}, \nu)$ are measure spaces and $f\colon (X, \mathfrak{a}) \longrightarrow (Y, \mathfrak{b})$ is a measurable map then the same map might *not* be measurable viewed as map $f\colon (X, \hat{\mathfrak{a}}) \longrightarrow (Y, \hat{\mathfrak{b}})$. It might happen that the pre-image of a subset $B \in \mathfrak{b}$ of measure zero, $\nu(B) = 0$, does not have measure zero, i.e. $\mu(f^{-1}(B)) > 0$. Then any subset $N \subseteq B$ will be measurable with respect to the completion $\hat{\mathfrak{b}}$ but $f^{-1}(N)$ needs not to be measurable with respect to $\hat{\mathfrak{a}}$. The reason is that measurability depends on the σ-algebras alone while the notion of completion depends on the notion of zero sets and hence on the chosen measures. However, if we only complete $(X, \mathfrak{a})$ to $(X, \hat{\mathfrak{a}})$ then trivially f stays measurable.

nter-example
for this?

ii.) Since the completion of μ only makes use of the zero sets but does not depend on the precise values of the measure on other sets, two measures on $\mathfrak{a}$ having the same zero sets will result in the same completed σ-algebra $\hat{\mathfrak{a}}$. Thus this already indicates that measures with the same zero sets are closely related. We will come back to this when discussing the Theorem of Radon-Nikodym.

C.2.2 Essentially Bounded Functions

Since for integration theory the behaviour of a function on a zero set will not play any role we can ignore zero sets in the following. It is common usage to say that a certain property of points in a measure space $(X, \mathfrak{a}, \mu)$ holds *almost everywhere* on X if there is a zero set N such that the property holds on $X \setminus N$. In particular, we consider maps f which are defined on $(X, \mathfrak{a}, \mu)$ only almost everywhere, i.e. there is a zero set $N \subseteq X$ such that f is defined on $X \setminus N$. We call a map f on $(X, \mathfrak{a}, \mu)$ *almost everywhere measurable* if $f\big|_{X\setminus N}$ is measurable for some suitable zero set $N \subseteq X$. If the measure space is even complete then an almost everywhere measurable map is actually measurable. This follows from the gluing statements of Proposition C.1.22. Moreover, we say that f coincides with g almost everywhere if $f\big|_{X\setminus N} = g\big|_{X\setminus N}$ for some zero set N. In general, the gluing statements allow us to modify an almost everywhere measurable map f by setting it equal to a constant on the zero set N where it is not measurable. Then the newly glued function is measurable on $X \setminus N$ where it coincides with f and it is measurable on N as it is a constant. By Proposition C.1.22, *iii.)*, it is measurable on X. When dealing with functions this procedure will result in a measurable function which has the same properties concerning integration than f. To rephrase this more precisely, in the following we will mainly deal with equivalence classes of functions modulo functions which vanish almost everywhere. However, by some slight abuse of notation common in measure theory we will not indicate this in our notation and simply write f for the equivalence class of f as well.

We introduce now the following space of essentially bounded (and measurable) functions:

Definition C.2.16 (Essentially bounded functions) *Let $(X, \mathfrak{a}, \mu)$ be a measure space and let $f\colon X \longrightarrow \mathbb{C}$ be measurable (or almost everywhere measurable).*

i.) *The essential range of f are the points*

$$\operatorname{ess\,range}(f) = \left\{ z \in \mathbb{C} \;\middle|\; \mu(f^{-1}(B_\epsilon(z))) > 0 \textit{ for all } \epsilon > 0 \right\} \subseteq \mathbb{C}. \tag{C.2.20}$$

ii.) *The essential supremum of f is*

$$\operatorname*{ess\,sup}_{x \in X} |f(x)| = \sup\left\{ |z| \;\middle|\; z \in \operatorname{ess\,range}(f) \right\}. \tag{C.2.21}$$

iii.) *The function f is called essentially bounded if $\operatorname{ess\,sup}_{x\in X}|f(x)| < \infty$. In this case we define*

$$\|f\|_{\mu,\infty} = \operatorname*{ess\,sup}_{x \in X} |f(x)|. \tag{C.2.22}$$

iv.) *The set of all essentially bounded measurable functions is denoted by $\mathscr{L}^\infty(X, \mathfrak{a}, \mu)$.*

Of course, the essential range as well as the essential supremum depends on the choice of the measure μ even though we do not indicate this in our notation. More precisely, it depends on the choice of what is called a zero set in $(X, \mathfrak{a})$. The next technical lemma collects some basic properties of the essential range and the essential supremum:

Lemma C.2.17 *Let $(X, \mathfrak{a}, \mu)$ be a measure space and let $f, g\colon X \longrightarrow \mathbb{C}$ be measurable functions.*

i.) *If $f = g$ almost everywhere then*

$$\operatorname{ess\,range}(f) = \operatorname{ess\,range}(g), \tag{C.2.23}$$

and $\operatorname{ess\,range}(f) \subseteq \mathbb{C}$ is a closed subset.

ii.) For the essential supremum one has

$$\operatorname*{ess\,sup}_{x\in X}|f(x)| = \inf\bigl\{a \geq 0 \bigm| \mu(|f|^{-1}([a,\infty))) = 0\bigr\}. \tag{C.2.24}$$

iii.) One has

$$\operatorname{ess\,range}(f) \subseteq \operatorname{range}(f)^{\mathrm{cl}}, \tag{C.2.25}$$

$$\operatorname*{ess\,sup}_{x\in X}|f(x)| \leq \sup_{x\in X}|f(x)|, \tag{C.2.26}$$

and for almost all $y \in X$ we have

$$|f(y)| \leq \operatorname*{ess\,sup}_{x\in X}|f(x)|. \tag{C.2.27}$$

Proof: Suppose $N \subseteq X$ is a zero set such that $f\big|_{X\setminus N} = g\big|_{X\setminus N}$ and let $z \in \mathbb{C} \setminus \operatorname{ess\,range}(f)$. Then there is a $\epsilon_0 > 0$ with $\mu(f^{-1}(\mathrm{B}_{\epsilon_0}(z))) = 0$. Now

$$g^{-1}(\mathrm{B}_{\epsilon_0}(z)) \cap (X \setminus N) = \Bigl(g\big|_{X\setminus N}\Bigr)^{-1}(\mathrm{B}_{\epsilon_0}(z)) = \Bigl(f\big|_{X\setminus N}\Bigr)^{-1}(\mathrm{B}_{\epsilon_0}(z)) = f^{-1}(\mathrm{B}_{\epsilon_0}(z)) \cap (X \setminus N)$$

is a measurable set contained in a zero set, hence a zero set itself. Since

$$g^{-1}(\mathrm{B}_{\epsilon_0}(z)) \subseteq \bigl(g^{-1}(\mathrm{B}_{\epsilon_0}(z)) \cap (X \setminus N)\bigr) \cup N,$$

we conclude that $g^{-1}(\mathrm{B}_{\epsilon_0}(z))$ is contained in the union of two zero sets and is thus a zero set itself. Hence we have $\mu(g^{-1}(\mathrm{B}_{\epsilon_0}(z))) = 0$. By symmetry, (C.2.23) follows. Suppose now that $z \notin \operatorname{ess\,range}(f)$ then there is a $\epsilon > 0$ with $\mu(f^{-1}(\mathrm{B}_\epsilon(z))) = 0$. For $w \in \mathrm{B}_\epsilon(z)$ we have a $\epsilon' > 0$ with $\mathrm{B}_{\epsilon'}(w) \subseteq \mathrm{B}_\epsilon(z)$ and hence $f^{-1}(\mathrm{B}_{\epsilon'}(w)) \subseteq f^{-1}(\mathrm{B}_\epsilon(z))$. By the monotonicity of a positive measure we conclude that $\mu(f^{-1}(\mathrm{B}_{\epsilon'}(w))) = 0$ as well. This shows that $\mathbb{C} \setminus \operatorname{ess\,range}(f)$ is open and the first part follows. For the second, let first $z \in \operatorname{ess\,range}(f)$. Then $\mu(f^{-1}(\mathrm{B}_\epsilon(z))) > 0$ for all $\epsilon > 0$. Since

$$f^{-1}(\mathrm{B}_\epsilon(z)) \subseteq |f|^{-1}\bigl(\bigl[|z| - \epsilon, \infty\bigr)\bigr),$$

we conclude that $\mu\bigl(|f|^{-1}\bigl(\bigl[|z| - \epsilon, \infty\bigr)\bigr)\bigr) > 0$ by monotonicity. Hence $|z| - \epsilon$ is less or equal to the infimum on the right hand side of (C.2.24). Since this is true for all $\epsilon > 0$ the same holds for $|z|$. Taking now the supremum over all those $z \in \operatorname{ess\,range}(f)$ gives the inequality "$\leq$" in (C.2.24). For the converse, assume that the essential supremum is strictly less than the infimum α on the right hand side of (C.2.24). Then for $z \in \mathbb{C}$ with $|z| = \alpha$ we have $|z| > \|f\|_{\mu,\infty}$ and hence there exists a $\epsilon_z > 0$ with $\mu(f^{-1}(\mathrm{B}_{\epsilon_z}(z))) = 0$. Now the *compact* sphere of radius α is covered by finitely many of these open balls $\mathrm{B}_{\epsilon_{z_k}}(z_k)$ with $k = 1, \ldots, n$. A simple geometric consideration with the covering of the circle by open discs shows that for a suitable $\epsilon > 0$ the closed ring $\mathrm{B}_{\alpha+\epsilon}(0)^{\mathrm{cl}} \setminus \mathrm{B}_{\alpha-\epsilon}(0)$ is still covered by the balls $\mathrm{B}_{\epsilon_1}(z_1) \cup \cdots \cup \mathrm{B}_{\epsilon_n}(z_n)$. The precise value of ϵ depends of course on the position and sizes of the z_n and ϵ_n. We conclude that the set $|f|^{-1}([\alpha - \epsilon, \alpha + \epsilon])$ is contained in the finite union of these sets of measure zero. Hence we also have $\mu(f^{-1}([\alpha-\epsilon, \alpha+\epsilon])) = 0$. Moreover, we know $\mu(|f|^{-1}([\alpha+\epsilon, \infty))) = 0$ by definition of α. This shows that $\mu(f^{-1}([\alpha-\epsilon, \infty))) = 0$, too, contradicting the definition of α. The the equality in (C.2.24) follows. Finally, the third part is clear. ☐

Note that the first part of this lemma justifies to identify functions which differ only on a zero set. In particular, if one of them is measurable the other may not be measurable as the measure space may not be complete but this will not alter the behaviour. The following proposition clarifies the structure of the essentially bounded measurable functions.

Proposition C.2.18 (Essentially bounded measurable functions) *Let $(X, \mathfrak{a}, \mu)$ be a measurable space.*

i.) *One has* $\mathscr{BM}(X, \mathfrak{a}) \subseteq \mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ *and for all* $f \in \mathscr{BM}(X, \mathfrak{a})$ *one has*

$$\|f\|_{\mu,\infty} \le \|f\|_\infty. \tag{C.2.28}$$

ii.) *The set* $\mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ *is a unital commutative* **-algebra and* $\|\cdot\|_{\mu,\infty}$ *is a submultiplicative seminorm.*

iii.) *One has*

$$\ker\|\cdot\|_{\mu,\infty} = \big\{f \in \mathscr{L}^\infty(X, \mathfrak{a}, \mu) \bigm| f = 0 \text{ almost everywhere}\big\}. \tag{C.2.29}$$

iv.) *The quotient*

$$\mathrm{L}^\infty(X, \mathfrak{a}, \mu) = \mathscr{L}^\infty(X, \mathfrak{a}, \mu) \big/ \ker\|\cdot\|_{\mu,\infty} \tag{C.2.30}$$

is a commutative unital C^**-algebra and the induced map*

$$\mathscr{BM}(X, \mathfrak{a}) \longrightarrow \mathrm{L}^\infty(X, \mathfrak{a}, \mu) \tag{C.2.31}$$

is a unital surjective **-homomorphism.*

PROOF: The first part is clear by Lemma C.2.17, *iii.)*. Note however, that $\|f\|_{\mu,\infty} = 0$ may well happen for $f \in \mathscr{BM}(X, \mathfrak{a})$ even though $f \ne 0$. Now let $f, g \in \mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ and $z, w \in \mathbb{C}$. Then there are zero sets $N, M \in \mathfrak{a}$ such that $|f(x)| \le \|f\|_{\mu,\infty}$ for all $x \in X \setminus N$ and $|g(x)| \le \|g\|_{\mu,\infty}$ for all $x \in X \setminus M$. Since $N \cup M$ is still a zero set we get for all $x \in X \setminus (M \cup N)$

$$|zf(x) + wg(x)| \le |z|\|f\|_{\mu,\infty} + |w|\|g\|_{\mu,\infty}. \tag{$*$}$$

From the characterization (C.2.24) of the essential supremum we see that

$$\operatorname*{ess\,sup}_{x\in X} |zf(x) + wg(x)| \le |z|\|f\|_{\mu,\infty} + |w|\|g\|_{\mu,\infty}$$

holds as well since $(*)$ holds almost everywhere. This shows $\|zf + wg\|_{\mu,\infty} \le |z|\|f\|_{\mu,\infty} + |w|\|g\|_{\mu,\infty}$. Note also that $\|zf\|_{\mu,\infty} = |z|\|f\|_{\mu,\infty}$. This shows that $\|\cdot\|_{\mu,\infty}$ is a seminorm on $\mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ and the latter is a complex vector space. Clearly, the constant function 1 has $\|1\|_{\mu,\infty} = 1$ as soon as $\mu(X) > 0$. Finally, for $x \in X \setminus (N \cup M)$ we have

$$|f(x)g(x)| = |f(x)||g(x)| \le \|f\|_{\mu,\infty}\|g\|_{\mu,\infty},$$

from which we obtain the second part. The third part is clear by definition. For the last part we first note that the C^*-property holds already on the level of $\mathscr{L}^\infty(X, \mathfrak{a}, \mu)$. Indeed, we have $|f(x)| > a$ iff $|f(x)|^2 > a^2$ from which we immediately get

$$\operatorname*{ess\,sup}_{x\in X} \big|\overline{f(x)}f(x)\big| = \Big(\operatorname*{ess\,sup}_{x\in X} |f(x)|\Big)^2.$$

But this gives $\|\overline{f}f\|_{\mu,\infty} = \|f\|^2_{\mu,\infty}$ right away. Thus the quotient $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ is a pre C^*-algebra and we have to show that it is complete. Thus let $f_n \in \mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ be a Cauchy sequence with respect to $\|\cdot\|_{\mu,\infty}$. Define

$$A_n = \big\{x \in X \bigm| |f_n(x)| > \|f_n\|_{\mu,\infty}\big\},$$

and

$$B_{n,m} = \big\{x \in X \bigm| |f_n(x) - f_m(x)| > \|f_n - f_m\|_{\mu,\infty}\big\},$$

which are all zero sets with respect to μ by Lemma C.2.17, *iii.)*. Since the countable union of zero sets is again a zero set by Lemma C.2.12, the subset $N = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n,m=1}^{\infty} B_{n,m}$ is a zero set. For $x \in X \setminus N$ we have

$$|f_n(x)| \leq \|f_n\|_{\mu,\infty} \quad \text{and} \quad |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{\mu,\infty}. \tag{$**$}$$

This shows that on $X \setminus N$ the restricted functions $f_n\big|_{X\setminus N}$ are bounded with $\big\|f\big|_{X\setminus N}\big\|_\infty \leq \|f\|_{\mu,\infty}$ and form a Cauchy sequence with respect to the supremum norm $\|\cdot\|_\infty$ on $X \setminus N$. Thus, by completeness of $\mathscr{BM}\big(X \setminus N, \mathfrak{a}\big|_{X\setminus N}\big)$ according to Proposition C.1.25, there is a bounded measurable function $f \in \mathscr{BM}\big(X, \mathfrak{a}\big|_{X\setminus N}\big)$ such that $f_n\big|_{X\setminus N} \longrightarrow f$ uniformly. By setting $f = 0$ on N we can extend f to a measurable and even bounded function on X according Proposition C.1.22. This gives indeed a measurable bounded function $f \in \mathscr{BM}(X, \mathfrak{a})$ for which $f_n \longrightarrow f$ almost everywhere uniformly, namely on $X \setminus N$. Hence on the level of equivalence classes in $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ we have shown the completeness. Finally, it is clear that (C.2.31) is a unital *-homomorphism. Ignoring zero sets shows immediately that it is surjective. $\square$

Remark C.2.19 (The C^*-algebra $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$) Let $(X, \mathfrak{a}, \mu)$ be a measure space.

i.) As a unital commutative C^*-algebra, $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ is a rather weird object: we know that there is a compact Hausdorff space Y such that

$$\mathrm{L}^\infty(X, \mathfrak{a}, \mu) \cong \mathscr{C}(Y) \tag{C.2.32}$$

by the Gel'fand transform. But now Y is a rather weird topological space since in $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ we typically have a lot of projections: every characteristic function $\chi_A \in \mathscr{BM}(X, \mathfrak{a})$ gives a non-trivial element $\chi_A \in \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ iff $\mu(A) > 0$. Clearly, $\chi_A = \chi_A^2 = \overline{\chi_A}$ already on the level of representatives. Hence we conclude that Y is typically very *disconnected.*

letails: Stone A?, Exercise?

ii.) An element $f \in \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ is positive iff there is a representative in $\mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ (or even in $\mathscr{BM}(X, \mathfrak{a})$) which is almost everywhere ≥ 0. This can be shown by a slight modification of the argument in Remark C.1.26, *i.)*, see also Exercise C.6.3, *ii.)*.

iii.) For $f \in \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ any representative has the same essential range, i.e. $\operatorname{ess\,range}(f)$ is well-defined on $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ by Lemma C.2.17, *i.)*. It is now easy to see that

$$\operatorname{ess\,range}(f) = \operatorname{spec}(f). \tag{C.2.33}$$

Again, this can be shown analogously to the statement of Remark C.1.26, *ii.)*, see also Exercise C.6.3, *i.)*.

iv.) It is sometimes useful to note that the images of simple functions in $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ are dense with respect to $\|\cdot\|_{\mu,\infty}$. This follows immediately from the fact that the simple functions are dense in $\mathscr{BM}(X, \mathfrak{a})$ and that $\mathscr{BM}(X, \mathfrak{a})$ continuously surjects onto $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$.

Remark C.2.20 If the reference to the set X, the measure μ, or to the σ-algebra $\mathfrak{a}$ is clear it is common to write $\mathrm{L}^\infty(X) = \mathrm{L}^\infty(X, \mathfrak{a}) = \mathrm{L}^\infty(X, \mu) = \mathrm{L}^\infty(\mu)$ instead of $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$.

Remark C.2.21 The definition of the essential supremum ess sup and hence the definition of $\mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ and $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ only depends on the zero sets. Thus the spaces $\mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ and $\mathscr{L}^\infty(X, \mathfrak{a}, \nu)$ coincide provided the zero sets of the two measures coincide. In fact, we do not even have to know that the zero sets actually come from a measure. It will be enough to have a subset $\mathfrak{n} \subseteq \mathfrak{a}$ containing $\emptyset$ and which is stable under countable unions and taking subsets, see also Exercise **??**.

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C.2.3 Complex Measures

We shall now pass to complex measures. Here the definition is slightly different as we have no distinguished way of divergence to $+\infty$ in $\mathbb{C}$. Hence we require convergence inside the complex numbers in the σ-additivity. In particular, infinite measures are now excluded.

Definition C.2.22 (Complex measure) *Let $(X, \mathfrak{a})$ be a measurable space. Then a complex measure μ on $(X, \mathfrak{a})$ is a map*

$$\mu\colon \mathfrak{a} \longrightarrow \mathbb{C}, \tag{C.2.34}$$

such that μ is σ-additive, i.e. for a sequence of pairwise disjoint measurable subsets $A_n \in \mathfrak{a}$ we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n). \tag{C.2.35}$$

The set of complex measures on $(X, \mathfrak{a})$ is denoted by $\operatorname{Meas}(X, \mathfrak{a})$.

Note that now the right hand side of (C.2.35) is required to converge in $\mathbb{C}$. Since the order of the A_n in the union $\bigcup_{n=1}^{\infty} A_n$ does not matter, the definition requires that the right hand side converges *unconditionally* and hence *absolutely.*

Proposition C.2.23 *Let $(X, \mathfrak{a})$ be a measurable space.*

i.) *We have $\mu(\emptyset) = 0$ and*

$$\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n) \tag{C.2.36}$$

for every pairwise disjoint $A_1, \ldots, A_n \in \mathfrak{a}$ and every complex measure $\mu \in \operatorname{Meas}(X, \mathfrak{a})$.

ii.) *For $\mu, \nu \in \operatorname{Meas}(X, \mathfrak{a})$, $z, w \in \mathbb{C}$, and $A \in \mathfrak{a}$*

$$(z\mu + w\nu)(A) = z\mu(A) + w\nu(A) \tag{C.2.37}$$

defines a complex measure $z\mu + w\nu \in \operatorname{Meas}(X, \mathfrak{a})$, making $\operatorname{Meas}(X, \mathfrak{a})$ a complex vector space.

iii.) *For $\mu \in \operatorname{Meas}(X, \mathfrak{a})$ also $\overline{\mu}$ defined by $\overline{\mu}(A) = \overline{\mu(A)}$ for $A \in \mathfrak{a}$ is a complex measure on $(X, \mathfrak{a})$.*

iv.) *If $A_1 \subseteq A_2 \subseteq \cdots \in \mathfrak{a}$ and $\mu \in \operatorname{Meas}(X, \mathfrak{a})$ then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \longrightarrow \infty} \mu(A_n). \tag{C.2.38}$$

v.) *If $A_1 \supseteq A_2 \supseteq \cdots \in \mathfrak{a}$ and $\mu \in \operatorname{Meas}(X, \mathfrak{a})$ then*

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \longrightarrow \infty} \mu(A_n). \tag{C.2.39}$$

vi.) *If $\Phi\colon (X, \mathfrak{a}) \longrightarrow (Y, \mathfrak{b})$ is a measurable map into another measurable space and $\mu \in \operatorname{Meas}(X, \mathfrak{a})$ then $\Phi_*\mu(A) = \mu(\Phi^{-1}(A))$ defines a complex measure $\Phi_*\mu \in \operatorname{Meas}(Y, \mathfrak{b})$, the push-forward of μ by Φ.*

PROOF: The first part is done as in Proposition C.2.5, *i.)* and *ii.)*. The second part is clear as the σ-additivity of μ and ν immediately translates to the required σ-additivity of $z\mu + w\nu$. The zero vector is the trivial measure. Also the third part is clear. For part *iv.)* and *v.)* we note that in the proof of Proposition C.2.6 the positivity of the values $\mu(A)$ was not used but only the σ-additivity. Note that $\mu(A_1) \in \mathbb{C}$ is automatically finite for a complex measure and therefore it has not to be added to the assumptions in contrast to Proposition C.2.6, *ii.)*. Finally, the σ-additivity of $\Phi_*\mu$ is clear. □

Remark C.2.24 (Real measure) A complex measure $\mu \in \operatorname{Meas}(X, \mathfrak{a})$ is called a *real measure* or *signed measure* if $\mu = \overline{\mu}$, i.e. μ takes only real values. A positive measure μ in the sense of Definition C.2.1 is a complex measure with values in $\mathbb{R}_0^+$ iff μ is finite.

We prove now that for every complex measure μ on $(X, \mathfrak{a})$ there is a finite positive measure $|\mu|$ which dominates μ in the sense that $|\mu(A)| \leq |\mu|(A)$ for all $A \in \mathfrak{a}$ and which is minimal with respect to this property. To this end, consider a countable union $A = \bigcup_{n=1}^{\infty} A_n$ of pairwise disjoint $A_n \in \mathfrak{a}$, which we call also a (countable) *partition* of A. Then we have by σ-additivity

$$|\mu(A)| = \left|\sum_{n=1}^{\infty} \mu(A_n)\right| \leq \sum_{n=1}^{\infty} |\mu(A_n)|, \tag{C.2.40}$$

where the right hand side converges since (C.2.35) converges unconditionally and hence absolutely. Now suppose ν is a positive (not necessarily finite) measure with $|\mu(A)| \leq \nu(A)$ for all $A \in \mathfrak{a}$. Then the above estimate gives

$$\sum_{n=1}^{\infty} |\mu(A_n)| \leq \sum_{n=1}^{\infty} \nu(A_n) = \nu(A), \tag{C.2.41}$$

by the σ-additivity of ν. Thus, if we want $|\mu(A)| \leq \nu(A)$ then $\nu(A)$ has to be at least the *supremum* over all possible right hand sides of (C.2.40), i.e. over all partitions of A. In fact, this will already do the job. We define

$$|\mu|(A) = \sup \sum_{n=1}^{\infty} |\mu(A_n)| \tag{C.2.42}$$

for $A \in \mathfrak{a}$ where the supremum is taken over all possible countable partitions $A = \bigcup_{n=1}^{\infty} A_n$ of A with $A_n \in \mathfrak{a}$.

Proposition C.2.25 *Let* $(X, \mathfrak{a})$ *be a measurable space and* $\mu \in \operatorname{Meas}(X, \mathfrak{a})$*. For* $|\mu| \colon \mathfrak{a} \longrightarrow [0, +\infty]$ *defined by* (C.2.42) *one has:*

i.) The map $|\mu|$ *is a positive measure.*

ii.) One has $|\mu|(A) \geq |\mu(A)|$ *for all* $A \in \mathfrak{a}$*.*

iii.) The measure $|\mu|$ *is the smallest positive measure with the property ii.), i.e. if* ν *is another positive measure with* $\nu(A) \geq |\mu(A)|$ *for all* $A \in \mathfrak{a}$ *then* $\nu(A) \geq |\mu|(A)$*.*

iv.) The measure $|\mu|$ *is finite.*

v.) One has $|\overline{\mu}| = |\mu|$*.*

vi.) If μ *was already positive then* $|\mu| = \mu$*.*

PROOF: Since the only (countable) partitions of $\emptyset$ are given by copies of $\emptyset$ we have immediately $|\mu|(\emptyset) = 0$. Thus we have to show the σ-additivity of $|\mu|$. Let $A_n \in \mathfrak{a}$ be a sequence of pairwise disjoint measurable subsets and let $A = \bigcup_{n=1}^{\infty} A_n$. Let furthermore $A_{nm} \in \mathfrak{a}$ be a partition of A_n, i.e. $A_n = \bigcup_{m=1}^{\infty} A_{nm}$ with pairwise disjoint A_{nm}. Then the collection of all A_{nm} provides a partition of A as well. Hence

$$|\mu|(A) \geq \sum_{n,m} |\mu(A_{nm})| = \sum_n \sum_m |\mu(A_{nm})|.$$

Since this holds for all partitions we can take the supremum over all the partitions of the A_n's and still have the inequality. We get

$$|\mu|(A) \geq \sum_n \sup \sum_m |\mu(A_{nm})| = \sum_n |\mu|(A_n),$$

which shows the inequality $|\mu(A)| \geq \sum_n |\mu|(A_n)$ for all partitions of A. Conversely, let $\{B_m\}_{m\in\mathbb{N}}$ be another partition of A. Then the intersections $B_m \cap A_n$ provide a partition of B_m for every m. Thus we get by σ-additivity of μ

$$\begin{aligned}\sum_m |\mu(B_m)| &= \sum_m \left|\sum_n \mu(B_m \cap A_n)\right| \\ &\leq \sum_m \sum_n |\mu(B_m \cap A_n)| \\ &= \sum_n \sum_m |\mu(B_m \cap A_n)| \\ &\leq \sum_n |\mu|(A_n),\end{aligned}$$

where we have used the fact that for non-negative terms in a double series we have either divergence to $+\infty$ or convergence to the same value for both orders of the summation. In fact, we shall see a proof of this well-known and elementary fact later on in Example C.3.11, *i.)*. In the last step we used the definition of $|\mu|(A_n)$ being the supremum over all partitions since $\{B_m \cap A_n\}_{m\in\mathbb{N}}$ is one of them. Since the right hand side is independent of the partition $\{B_m\}_{m\in\mathbb{N}}$ of A we can take the supremum over all of them and conclude

$$|\mu|(A) = \sup \sum_m |\mu(B_m)| \leq \sum_n |\mu(A_n)|,$$

which finally shows the σ-additivity needed for the first part. The second part is clear by construction since A, $\emptyset$, $\emptyset$, ... is a particular partition of A yielding the estimate right away. For the third part we have already argued that $|\mu|$ is the smallest positive measure with $|\mu|(A) \geq |\mu(A)|$ for all $A \in \mathfrak{a}$. The fourth part is probably surprising and relies on a nice geometric consideration in the complex plane: if we have complex numbers with absolute values summing up to a certain large value, already a subset of them sums up to a complex number with absolute value not much smaller. The idea is that many of the complex numbers necessarily point in more or less the same direction. In detail, we have the following statement, see [48, Lem. 6.3]: For $z_1, \ldots, z_n \in \mathbb{C}$ there is a subset $\{z_k\}_{k\in S}$ such that

$$\left|\sum_{k\in S} z_k\right| \geq \frac{1}{\pi} \sum_{k=1}^N |z_k|. \tag{$*$}$$

We prove this claim: write $z_k = \mathrm{e}^{\mathrm{i}\theta_k}|z_k|$ with $\theta_k \in [-\pi, \pi]$ and fix a direction $\theta \in [-\pi, \pi]$. Let $S(\theta)$ be those indexes for which $\cos(\theta_k - \theta) > 0$. Geometrically, this means that z_k points in a direction close to the fixed one. Then

$$\begin{aligned}\left|\sum_{k\in S(\theta)} z_k\right| &= \left|\sum_{k\in S(\theta)} \mathrm{e}^{-\mathrm{i}\theta} z_k\right| \\ &\geq \operatorname{Re} \sum_{k\in S(\theta)} \mathrm{e}^{-\mathrm{i}\theta+\mathrm{i}\theta_k}|z_k| \\ &= \sum_{k\in S(\theta)} \cos(\theta_k - \theta)|z_k| \\ &= \sum_{k=1}^N (\cos(\theta_k - \theta))_+ |z_k|,\end{aligned} \tag{$**$}$$

where $(\cos(\theta_k - \theta))_+$ denotes the positive part as usual. Varying over a compact subset there is a θ_0 for which this last sum becomes largest. By the mean value theorem, the maximum of a non-negative continuous function $f(\theta)$ is at least the integral over f divided by the interval length. In our case we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\theta_k-\theta)_+\,\mathrm{d}\theta=\frac{1}{\pi}$$

for all values of θ_k. Hence we conclude that for this particular θ_0 the sum $(**)$ is at least the right hand side of $(*)$ and the claim follows. Now we apply this to the values of μ. Suppose that $|\mu|$ is *not* finite. Then there is a measurable subset A with $|\mu|(A)=+\infty$ and hence there is a partition $\{A_n\}_{n\in\mathbb{N}}$ of A for which $\sum_{n=1}^{N}|\mu(A_n)|>\pi(1+|\mu(A)|)$. By the above argument $(*)$ we find a collection $\{A_k\}_{k\in S}$ with S a subset of $\{1,\ldots,N\}$ with $|\sum_{k\in S}\mu(A_k)|>1+|\mu(A)|$. But then the σ-additivity of μ gives us for the set $B=\bigcup_{k\in S}A_k\subseteq A$ the measure $|\mu(B)|>1+|\mu(A)|$. The complement $A\setminus B$ has the measure $|\mu(A\setminus B)|=|\mu(A)-\mu(B)|\geq|\mu(B)|-|\mu(A)|>1$. Thus we have split the subset A into B and $A\setminus B$ where both subsets have $|\mu(B)|,|\mu(A\setminus B)|>1$. Since $|\mu|(A)=+\infty$ at least one of these subsets B and $A\setminus B$ has to have infinite measure with respect to $|\mu|$ since $|\mu|$ is additive. This allows to start an induction: if $|\mu|(X)=+\infty$ we find two disjoint subsets $A_1\cup B_1=X$ with $|\mu(A_1)|>1$ and, say $|\mu|(B_1)=+\infty$. We repeat this with B_1, B_2, ...and obtain by induction pairwise disjoint subsets A_1, A_2, ...with $|\mu(A_n)|>1$. Since μ is σ-additive we have

$$\mu\left(\bigcup_{n=1}^{\infty}A_n\right)=\sum_{n=1}^{\infty}\mu(A_n).$$

But with $|\mu(A_n)|>1$, the right hand side can not converge at all, which is a contradiction. Hence $|\mu|(X)<\infty$ was finite, showing the fourth part. Now suppose μ was already a (finite) positive measure. Then for any partition $\{A_n\}_{n\in\mathbb{N}}$ of $A\in\mathfrak{a}$ we have

$$\sum_{n=1}^{\infty}|\mu(A_n)|=\sum_{n=1}^{\infty}\mu(A_n)=\mu(A)$$

by the σ-additivity of μ and $\mu(A_n)\geq 0$. Thus $|\mu|=\mu$ follows by taking the supremum of the left hand side of this equation. The last two parts are clear. ☐

Remark C.2.26 (Push-forward of complex measures) If $\Phi\colon(X,\mathfrak{a})\longrightarrow(Y,\mathfrak{b})$ is a measurable map and $\mu\in\operatorname{Meas}(X,\mathfrak{a})$ a complex measure, then passing to $|\mu|$ does not work well together with push-forward in general: since every partition of $B\in\mathfrak{b}$ induces a partition of $\Phi^{-1}(B)$, it is easy to obtain the estimate

$$|\Phi_*\mu|\leq\Phi_*|\mu|. \tag{C.2.43}$$

However, in general the two positive measures do not coincide. A simple example is given by a two point set $X=\{p_1,p_2\}$ and $\mu=\delta_{p_1}-\delta_{p_2}$ while $Y=\{p\}$ consists only of one point. Thus the unique map sends both points of X to the single point of Y and it is easy to see that $\Phi_*\mu=0$. Thus we get a strict inequality in (C.2.43) for this case.

The last result of this section is that the space of complex measures on $(X,\mathfrak{a})$ has a natural Banach space structure:

Proposition C.2.27 (The Banach space $\operatorname{Meas}(X,\mathfrak{a})$**)** *Let* $(X,\mathfrak{a})$ *be a measurable space. For* $\mu\in\operatorname{Meas}(X,\mathfrak{a})$

$$\|\mu\|=|\mu|(X) \tag{C.2.44}$$

defines a norm on $\operatorname{Meas}(X,\mathfrak{a})$ *making it a Banach space.*

PROOF: We first show that $\|\cdot\|$ is indeed a seminorm. By Proposition C.2.25, *iv.)*, we have $\|\mu\| < \infty$ for all $\mu \in \mathrm{Meas}(X, \mathfrak{a})$. For complex measures μ, ν we have for all partitions $\{A_n\}_{n\in\mathbb{N}}$ of a subset $A \in \mathfrak{a}$

$$\sum_n |(\mu+\nu)(A_n)| \leq \sum_n |\mu(A_n)| + \sum_n |\nu(A_n)| \leq |\mu(A)| + |\nu|(A), \tag{$*$}$$

from which we deduce that

$$|\mu+\nu|(A) \leq |\mu|(A) + |\nu|(A)$$

by taking the supremum over all partitions of the left hand side of $(*)$. Thus we have a triangle inequality even pointwise on the σ-algebra $\mathfrak{a}$, i.e. for all measurable subsets $A \in \mathfrak{a}$. Then the triangle inequality of $\|\cdot\|$ follows by taking $X = A$. More elementary, we have $|z\mu|(A) = |z||\mu|(A)$ for all $z \in \mathbb{C}$, $\mu \in \mathrm{Meas}(X, \mathfrak{a})$ and $A \in \mathfrak{a}$. From this we conclude that $\|\cdot\|$ is a seminorm. Now assume that $\|\mu\| = 0$ then $|\mu|(X) = 0$ and hence $|\mu|(A) = 0$ for all $A \in \mathfrak{a}$. Since $|\mu|(A)$ is the supremum over all partitions of A in (C.2.42) and since $A, \emptyset, \emptyset, \ldots,$ is a particular one, we see that necessarily $\mu(A) = 0$ for all $A \in \mathfrak{a}$. This shows $\mu = 0$ and $\|\cdot\|$ is a norm. It remains to show completeness: let $\mu_k \in \mathrm{Meas}(X, \mathfrak{a})$ be a Cauchy sequence with respect to $\|\cdot\|$. Since clearly $|\mu(A)| \leq \|\mu\|$ for all $\mu \in \mathrm{Meas}(X, \mathfrak{a})$ and $A \in \mathfrak{a}$ we see that for every fixed $A \in \mathfrak{a}$ the complex numbers $\mu_k(A)$ form a Cauchy sequence. Hence we can define a map $\mu\colon \mathfrak{a} \longrightarrow \mathbb{C}$ pointwise by

$$\mu(A) = \lim_{k\longrightarrow\infty} \mu_k(A) \tag{$**$}$$

to get a candidate for the limit of the Cauchy sequence μ_k. We have to show that μ is σ-additive. To this end, we first show that $(**)$ holds even uniformly in A. Indeed, if $\epsilon > 0$ is given and $N \in \mathbb{N}$ is chosen such that $k, \ell \geq N$ implies $\|\mu_k - \mu_\ell\| < \epsilon$ then we have for all $A \in \mathfrak{a}$

$$|\mu_k(A) - \mu_\ell(A)| = |(\mu_k - \mu_\ell)(A)| \leq |\mu_k - \mu_\ell|(A) \leq |\mu_k - \mu_\ell|(X) = \|\mu_k - \mu_\ell\| < \epsilon.$$

Taking the limit $\ell \longrightarrow \infty$ of this inequality gives the inequality

$$|\mu(A) - \mu_k(A)| \leq \epsilon$$

for $k \geq N$ and *all* $A \in \mathfrak{a}$. Now let $\{A_n\}_{n\in\mathbb{N}}$ be a partition of A. Then for $k \geq N$ we have

$$\begin{aligned} &\left|\mu(A) - \mu\Big(\bigcup\nolimits_{n=1}^M A_n\Big)\right| \\ &\quad\leq |\mu(A) - \mu_k(A)| + \left|\mu_k(A) - \mu_k\Big(\bigcup\nolimits_{n=1}^M A_n\Big)\right| + \left|\mu_k\Big(\bigcup\nolimits_{n=1}^M A_n\Big) - \mu\Big(\bigcup\nolimits_{n=1}^M A_n\Big)\right| \\ &\quad\leq \epsilon + \left|\mu_k\Big(\bigcup\nolimits_{n=M+1}^\infty A_n\Big)\right| + \epsilon. \end{aligned}$$

Since the measure μ_k is σ-additive, the middle term becomes smaller than ϵ for large enough M. Thus we conclude that μ is σ-additive, too, and hence $\mu \in \mathrm{Meas}(X, \mathfrak{a})$. It finally remains to show that $\mu_k \longrightarrow \mu$ in the sense of the norm (C.2.44). Thus let $\epsilon > 0$ and N as above. Fix a partition $\{A_n\}_{n\in\mathbb{N}}$ of X and let $K \in \mathbb{N}$. Then there is a L (depending on K) such that $\ell \geq L$ implies

$$\sum_{n=1}^K |\mu(A_n) - \mu_\ell(A_n)| < \epsilon$$

by $(**)$. Now let $\ell \geq L, N$ then

$$\sum_{n=1}^K |\mu(A_n) - \mu_k(A_n)| \leq \sum_{n=1}^K |\mu(A_n) - \mu_\ell(A_n)| + \sum_{n=1}^K |\mu_\ell(A_n) - \mu_k(A_n)|$$

$$
\begin{aligned}
&< \epsilon + \sum_{n=1}^{\infty} |\mu_\ell(A_n) - \mu_k(A_n)| \\
&< \epsilon + \epsilon, && (☺)
\end{aligned}
$$

since $\|\mu_\ell - \mu_k\| < \epsilon$ implies that the second contribution in (☺) is $< \epsilon$ for *all* partitions. Since (☺) is valid for all K we get for all partitions

$$
\sum_{n=1}^{\infty} |\mu(A_n) - \mu_k(A_n)| < 2\epsilon
$$

for $k \geq N$. As the partition was arbitrary and N was not depending on this choice of the partition we can take the supremum over all partitions on the left hand side which results in $\|\mu - \mu_k\| \leq 2\epsilon$, completing the proof. □

Definition C.2.28 (Total variation) *Let $(X, \mathfrak{a})$ be a measurable space and let $\mu \in \mathrm{Meas}(X, \mathfrak{a})$. Then the measure $|\mu|$ is called the total variation of μ and $\|\mu\|$ is called the variational norm of μ.*

In the literature, sometimes the quantity $\|\mu\| = |\mu|(X)$ is called the total variation of μ. Hence $|\mu|$ should probably better be called the *total variation measure* of μ. Note that

$$
\|\overline{\mu}\| = \|\mu\|, \tag{C.2.45}
$$

and for a finite positive measure $\mu \in \mathrm{Meas}(X, \mathfrak{a})$ one has

$$
\|\mu\| = \mu(X), \tag{C.2.46}
$$

by Proposition C.2.25, *vi.)* and *v.)*, respectively. From Remark C.2.26 we immediately obtain the following corollary:

Corollary C.2.29 *Let $\Phi\colon (X, \mathfrak{a}) \longrightarrow (Y, \mathfrak{b})$ be a measurable map. Then the push-forward map*

$$
\Phi_*\colon \mathrm{Meas}(X, \mathfrak{a}) \longrightarrow \mathrm{Meas}(Y, \mathfrak{b}) \tag{C.2.47}
$$

is continuous with respect to the variational norms and we have for all $\mu \in \mathrm{Meas}(X, \mathfrak{a})$

$$
\|\Phi_*\mu\| \leq \|\mu\|. \tag{C.2.48}
$$

We conclude this subsection with some standard constructions arising from the relation of $|\mu|$ and μ. For a complex measure $\mu \in \mathrm{Meas}(X, \mathfrak{a})$ one defines the new measures

$$
\mathrm{Re}(\mu) = \frac{1}{2}(\mu + \overline{\mu}) \quad \text{and} \quad \mathrm{Im}(\mu) = \frac{1}{2\mathrm{i}}(\mu - \overline{\mu}), \tag{C.2.49}
$$

which are now easily seen to be real measures. As usual, one calls $\mathrm{Re}(\mu)$ and $\mathrm{Im}(\mu)$ the *real* and *imaginary part* of μ, respectively. For any measurable $A \in \mathfrak{a}$ one has $|\mathrm{Re}(\mu)(A)| \leq \frac{1}{2}(|\mu|(A) + |\overline{\mu}|(A)) = |\mu|(A)$ since $|\overline{\mu}| = |\mu|$ according to Proposition C.2.25, *v.)*. The analogous estimate holds for the imaginary part and hence we conclude

$$
|\mathrm{Re}(\mu)|, |\mathrm{Im}(\mu)| \leq |\mu|, \tag{C.2.50}
$$

by applying the above estimates to all partitions of some $A \in \mathfrak{a}$. If $\mu \in \mathrm{Meas}(X, \mathfrak{a})$ is a real measure, $\overline{\mu} = \mu$, then the property $|\mu(A)| \leq |\mu|(A)$ for all $A \in \mathfrak{a}$ shows that

$$
\mu_\pm = \frac{1}{2}(|\mu| \pm \mu) \tag{C.2.51}
$$

are (still finite) positive measures $\mu_\pm \in \mathrm{Meas}(X, \mathfrak{a})$. We call them the *positive* and *negative part* of μ, respectively. For $A \in \mathfrak{a}$ we have $|\mu_\pm(A)| \leq \frac{1}{2}(|\mu|(A) + |\mu(A)|) \leq |\mu|(A)$ from which we deduce

$$\mu_\pm = |\mu_\pm| \leq |\mu|. \tag{C.2.52}$$

We summarize this in the following proposition:

Proposition C.2.30 *Let $\mu \in \mathrm{Meas}(X, \mathfrak{a})$ be a complex measure. Then there exist unique real measures $\mathrm{Re}(\mu), \mathrm{Im}(\mu) \in \mathrm{Meas}(X, \mathfrak{a})$ and finite positive measures $\mathrm{Re}(\mu)_\pm$, $\mathrm{Im}(\mu)_\pm \in \mathrm{Meas}(X, \mathfrak{a})$ such that*

$$\mu = \mathrm{Re}(\mu) + \mathrm{i}\,\mathrm{Im}(\mu), \tag{C.2.53}$$

and

$$\mathrm{Re}(\mu) = \mathrm{Re}(\mu)_+ - \mathrm{Re}(\mu)_- \quad \textit{and} \quad \mathrm{Im}(\mu) = \mathrm{Im}(\mu)_+ - \mathrm{Im}(\mu)_- \tag{C.2.54}$$

such that

$$|\mathrm{Re}(\mu)|, |\mathrm{Im}(\mu)| \leq |\mu| \tag{C.2.55}$$

as well as

$$\mathrm{Re}(\mu)_\pm \leq |\mathrm{Re}(\mu)| \quad \textit{and} \quad \mathrm{Im}(\mu)_\pm \leq |\mathrm{Im}(\mu)|. \tag{C.2.56}$$

While the real and imaginary part are clearly unique, the positive and negative parts of a real measure are not yet uniquely determined by $\mu = \mu_+ - \mu_-$ as here we can add an arbitrary positive measure ν to both, the positive and the negative part without changing their difference. Later we will see how the particular decomposition (C.2.51) can be characterized in a unique way.

Remark C.2.31 Denoting the positive finite measures on $(X, \mathfrak{a})$ by $\mathrm{Meas}^+(X, \mathfrak{a})$ we see that the subset $\mathrm{Meas}^+(X, \mathfrak{a})$ is a convex cone. Moreover, the proposition shows that every element in $\mathrm{Meas}(X, \mathfrak{a})$ can be written as a complex linear combination of four positive ones. The usage of this convex cone allows to establish a partial order on the real measures by setting

$$\mu \leq \nu \quad \text{if} \quad \nu - \mu \in \mathrm{Meas}^+(X, \mathfrak{a}). \tag{C.2.57}$$

Clearly, this is equivalent to the condition $\mu(A) \leq \nu(A)$ for all $A \in \mathfrak{a}$. Thus it extends the partial order of positive measures to real ones. For real measures we then have $\mu \leq |\mu|$. Note also that according to the proof of Proposition C.2.27 we have for $\mu, \nu \in \mathrm{Meas}(X, \mathfrak{a})$ and $z, w \in \mathbb{C}$

$$|z\mu + w\nu| \leq |z||\mu| + |w||\nu|. \tag{C.2.58}$$

Again, we have to admit that at the present stage we do not have many interesting examples of complex measures. This will change in Section C.3.4.

C.3 Integration and L^p-Spaces

We are now in the position to define integration with respect to a measure. Again we will discuss several different versions for positive and complex measures.

C.3.1 Integration with Respect to a Positive Measure

First we consider functions which may take values in $[0, +\infty]$ where $+\infty$ is explicitly allowed, with the conventions about arithmetic as in Remark C.2.2. One reason for considering such values for a function is that now every family of such functions has a well-defined pointwise supremum, it may be $+\infty$ which is again in the set of allowed values. Viewing $[0, +\infty]$ as topological space homeomorphic to

$[0, 1]$ we obtain a Borel σ-algebra and hence a concept of measurable functions $f : (X, \mathfrak{a}) \longrightarrow [0, +\infty]$. In complete analogy to Theorem C.1.15 using Lemma C.1.11 we obtain the following criterion

$$f \text{ is measurable} \quad \text{iff} \quad f^{-1}((a, +\infty]) \in \mathfrak{a} \text{ for all } a \geq 0 \tag{C.3.1}$$

for measurability. A closer look at the proof of Proposition C.1.25, *ii.)*, shows the following statement:

Proposition C.3.1 *Let $(X, \mathfrak{a})$ be a measurable space and let $f\colon X \longrightarrow [0, +\infty]$ be a measurable function. Then there are simple functions g_n on $(X, \mathfrak{a})$ with*

$$0 \leq g_1 \leq g_2 \leq \cdots \leq f, \tag{C.3.2}$$

such that for all $x \in X$

$$\lim_{n \longrightarrow \infty} g_n(x) = f(x). \tag{C.3.3}$$

Proof: Indeed, with the notation from the proof of Proposition C.1.25 we consider again the subsets

$$A_{i,n} = f^{-1}\big([i2^{-n}, (i+1)2^{-n})\big)$$

of X for $i = 0, \ldots, n2^n - 1$. On the union $A_n = \bigcup_{i=0}^{n2^n-1} A_{i,n}$ of the pairwise disjoint $A_{i,n}$ the function has values less than n. Outside of A_n the function f is at least n. Thus we consider the simple function

$$g_n(x) = \sum_{i=0}^{n2^n-1} i2^{-n}\chi_{A_{i,n}} + n\chi_{X\setminus A_n}.$$

For $x \in A_n$ we have $g_n(x) \leq f(x) < g_n(x) + 2^{-n}$ as before while for $x \in X \setminus A_n$ we get $g_n(x) = n \leq f(x)$. Now for a fixed $x \in X$ we have either $f(x) = +\infty$ then $g_n(x) = n$ grows monotonously and diverges to $+\infty$. Otherwise $x \in f^{-1}([0, +\infty))$ and hence there is a n_0 with $f(x) < n_0$. For $n \geq n_0$ we have $g_n(x) \longrightarrow f(x)$ by the estimate above. Thus $g_n(x) \longrightarrow f(x)$ in both cases which is the desired pointwise convergence. It remains to show that $g_n(x) \leq g_{n+1}(x)$ for all $x \in X$ and $n \in \mathbb{N}$. But this is an elementary consideration for the explicitly given functions g_n. □

We can now define the integral of a simple function taking values in the non-negative real numbers:

Definition C.3.2 (Integral of simple functions) *Let $(X, \mathfrak{a}, \mu)$ be a measure space and let $f\colon X \longrightarrow$ $[0, +\infty)$ be a simple function written as*

$$f = \sum_{i=1}^{n} a_i\chi_{A_i}, \tag{C.3.4}$$

where $a_1, \ldots, a_n$ are the distinct values of f and $A_i = f^{-1}(\{a_i\}) \in \mathfrak{a}$. Then the integral of f over X with respect to μ is defined by

$$\int_X f \,\mathrm{d}\mu = \sum_{i=1}^{n} a_i\mu(A_i) \in [0, +\infty]. \tag{C.3.5}$$

There are some comments necessary here: first we remind on the convention $0 \cdot (+\infty) = 0$ since it might happen that the measure of some of the subsets A_i might be infinite, $\mu(A_i) = +\infty$. It also may happen that the integral is infinite, namely if $a_i \neq 0$ and $\mu(A_i) = +\infty$ for some i. Finally, it is clear that by the same formula we can also define the integral of a simple function with arbitrary complex values, provided all occurring measures $\mu(A_i) < +\infty$ are finite. Currently, we shall only need the non-negative ones.

We want to extend the integral of non-negative simple functions to arbitrary non-negative measurable functions by an appropriate continuity argument. Here we have two options: first, if we deal with a finite measure and if we deal with bounded measurable functions then one can easily check that the definition (C.3.5) gives a linear functional on all simple functions (not only the non-negative ones) which is even continuous with respect to the supremum norm. Thus Proposition C.1.25, *ii.)*, gives us an immediate extension of the integral to $\mathscr{BM}(X, \mathfrak{a})$ in this case, preserving the continuity properties. If however, the measure of X is infinite or if we are interested in unbounded functions as well, then we have to proceed differently: we first deal with non-negative measurable functions and define their integral, possibly taking the value $+\infty$. Arbitrary complex-valued measurable functions are dealt with by investigating the positive and negative parts of their real and imaginary parts separately. The passage from simple to arbitrary measurable non-negative functions is then obtained by using Proposition C.3.1.

We start with some simple observations about the integral of simple non-negative functions:

Lemma C.3.3 *Let $(X, \mathfrak{a}, \mu)$ be a measure space and let $f, g\colon X \longrightarrow [0, +\infty)$ be simple functions.*

i.) If $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ with a finite partition $\{A_i\}_{i=1,\ldots,n}$ of X then

$$\int_X f \,\mathrm{d}\mu = \sum_{i=1}^{n} a_i \mu(A_i). \tag{C.3.6}$$

ii.) For $c \geq 0$ one has

$$\int_X (cf + g) \,\mathrm{d}\mu = c \int_X f \,\mathrm{d}\mu + \int_X g \,\mathrm{d}\mu. \tag{C.3.7}$$

Proof: Since the A_i form a partition we have $f\big|_{A_i} = a_i$ and hence $A_i \subseteq f^{-1}(\{a_i\})$. Thus the form is very close to the normal form of a simple function. If $f = \sum_{j=1}^{m} b_j \chi_{B_j}$ is the normal form we know that for every j there are indices $i_1, \ldots, i_{k_j}$ with $b_j = a_{i_1} = \cdots = a_{i_{k_j}}$ and $B_j = A_{i_1} \cup \cdots \cup A_{i_j}$. With other words, $\{A_i\}_{i=1,\ldots,n}$ is a refinement of the partition from the normal form of f. Then $\mu(B_j) = \mu(A_{i_1}) + \cdots + \mu(A_{i_j})$ shows that after taking the sum over j in $b_j \mu(B_j)$ we obtain (C.3.6). For the second part, let $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ and $g = \sum_{j=1}^{m} b_j \chi_{B_j}$ be given in their normal forms. Then $C_{ij} = A_i \cap B_j$ yields a partition of X with the property that f, g, as well as $cf + g$ are constant on each C_{ij}. Hence we can apply the first part for

$$\int_X (cf + g) \,\mathrm{d}\mu = \sum_{i,j} (ca_i + b_j) \mu(C_{ij}) = c \sum_i a_i \mu(A_i) + \sum_j b_j \mu(B_j) = c \int_X f \,\mathrm{d}\mu + \int_X g \,\mathrm{d}\mu,$$

where we used $\mu(A_i) = \mu(C_{i1}) + \cdots + \mu(C_{im})$ as well as $\mu(B_j) = \mu(C_{1j}) + \cdots + \mu(C_{nj})$. □

Since we can approximate every measurable function $f\colon X \longrightarrow [0, +\infty]$ by simple functions monotonously from below by Proposition C.3.1, the following definition makes sense:

Definition C.3.4 (Lebesgue integral) *The integral of a measurable function $f\colon X \longrightarrow [0, +\infty]$ is defined by*

$$\int_X f \,\mathrm{d}\mu = \sup \int_X g \,\mathrm{d}\mu, \tag{C.3.8}$$

where the supremum is taken over all simple functions $g\colon X \longrightarrow [0, \infty)$ such that $0 \leq g \leq f$.

Remark C.3.5 We list some first properties of the integral (C.3.8): let $f, g\colon X \longrightarrow [0, +\infty]$ be measurable.

i.) If $f \leq g$ pointwise then

$$\int_X f \,\mathrm{d}\mu \leq \int_X g \,\mathrm{d}\mu. \tag{C.3.9}$$

ii.) For a simple function $f\colon X \longrightarrow [0, +\infty)$ the definition (C.3.8) coincides with the previous definition in (C.3.5). This is clear by the first part and the fact that the largest simple function less or equal to f is f itself.

iii.) If $A \subseteq X$ is a measurable subset then we can restrict not only every measurable function from X to A and get a measurable function again, but also the measure μ restricts to a measure $\mu\big|_A$ on A. To avoid clumsy notation we shall omit the restriction symbol $\big|_A$ and simply write $\int_A f \, \mathrm{d}\mu$ for the integral of $f\big|_A$ over A with respect to $\mu\big|_A$. With this convention it is clear that

$$\int_A f \, \mathrm{d}\mu \leq \int_B f \, \mathrm{d}\mu \tag{C.3.10}$$

for every measurable subsets $A \subseteq B \subseteq X$.

iv.) If $\Phi\colon (X, \mathfrak{a}) \longrightarrow (Y, \mathfrak{b})$ is a measurable map and $f\colon Y \longrightarrow [0, +\infty]$ a measurable function then also the pull-back $\Phi^* f\colon X \longrightarrow [0, +\infty]$ is measurable and

$$\int_Y f \, \mathrm{d}\Phi_*\mu = \int_X \Phi^* f \, \mathrm{d}\mu \tag{C.3.11}$$

holds for any positive measure μ of $(X, \mathfrak{a})$. Indeed, for characteristic and hence simple functions this is trivial since

$$\Phi^* \chi_B = \chi_{\Phi^{-1}(B)} \tag{C.3.12}$$

for any characteristic function and $B \in \mathfrak{b}$. Moreover, it is easy to see that the supremum needed for the two integrals is taken over the same set of simple functions. Hence the result follows at once.

We are now in the position to formulate the first non-trivial result of integration theory:

Theorem C.3.6 (Monotonous convergence) *Let $(X, \mathfrak{a}, \mu)$ be a measure space and let $f_n\colon X \longrightarrow [0, +\infty]$ be a sequence of measurable functions with the property the $0 \leq f_1(x) \leq f_2(x) \leq \cdots$ for all $x \in X$. Then the pointwise limit $f(x) = \lim_{n\longrightarrow\infty} f_n(x)$ exists and gives a measurable function $f\colon X \longrightarrow [0, +\infty]$ with*

$$\int_X f \, \mathrm{d}\mu = \lim_{n\longrightarrow\infty} \int_X f_n \, \mathrm{d}\mu. \tag{C.3.13}$$

PROOF: Since we require the sequence $f_n(x)$ to be monotonously increasing the limit always exists as an element in $[0, +\infty]$: either the sequence $f_n(x)$ converges to some finite value in $[0, +\infty)$ or it is unbounded in which case we say that it converges to $+\infty$ as usual. Thus f is well-defined. The measurability is shown analogously to Theorem C.1.21: in fact, that theorem can be formulated for measurable functions taking values in $[0, +\infty]$ in precisely the same way with all statements remaining valid. Since $f_n \leq f_{n+1}$ holds pointwise, we can apply (C.3.9) to conclude that

$$\int_X f_n \, \mathrm{d}\mu \leq \int_X f_{n+1} \, \mathrm{d}\mu$$

for all $n \in \mathbb{N}$. Hence the sequence of integrals is monotonously increasing, too. Therefore it converges to some limit in $[0, +\infty]$. Again, from $f_n \leq f$ it is clear that

$$\lim_{n\longrightarrow\infty} \int_X f_n \, \mathrm{d}\mu \leq \int_X f \, \mathrm{d}\mu,$$

and it remains to show the opposite inequality. We fix a simple function $g\colon X \longrightarrow [0, \infty)$ with $0 \leq g \leq f$ pointwise and a constant $0 < c < 1$. Depending on these choices we consider the subsets

$$A_n = \left\{ x \in X \;\middle|\; f_n(x) \geq cg(x) \right\} = (f_n - cg)^{-1}([0, \infty]),$$

which is again measurable since $f_n - cg$ is measurable. Note that $f_n - cg$ takes values in $(-\infty, +\infty]$ and one shows the measurability analogously to Proposition C.1.20. Since $f_n \leq f_{n+1}$ we conclude that $A_n \subseteq A_{n+1}$. Moreover, for $x \in X$ we have either $f(x) = 0$ and then $f_n(x) = g(x) = 0$ for all n. In this case $x \in A_1$. Otherwise, $f(x) > 0$ and then $cg(x) < f(x)$ by $c < 1$. But then the pointwise convergence $f_n(x) \longrightarrow f(x)$ shows that $x \in A_n$ for some n. Hence we conclude that $X = \bigcup_{n=1}^{\infty} A_n$. Applying (C.3.10), (C.3.9), and (C.3.7) gives

$$\int_X f_n \, \mathrm{d}\mu \geq \int_{A_n} f_n \, \mathrm{d}\mu \geq \int_{A_n} cg \, \mathrm{d}\mu = c \int_{A_n} g \, \mathrm{d}\mu.$$

To examine the right hand side further we first note that for a characteristic function χ_A we have

$$\lim_{n \longrightarrow \infty} \int_{A_n} \chi_A \, \mathrm{d}\mu = \lim_{n \longrightarrow \infty} \mu(A \cap A_n) = \mu(A) = \int_X \chi_A \, \mathrm{d}\mu$$

by $\bigcup_{n=1}^{\infty} (A \cap A_n) = A$ and $A_1 \subseteq A_2 \subseteq \cdots$ and the statement from Proposition C.2.6, *i.)*. Finitely many additions and multiplications with positive real numbers give then

$$\lim_{n \longrightarrow \infty} \int_{A_n} g \, \mathrm{d}\mu = \int_X g \, \mathrm{d}\mu,$$

thanks to Lemma C.3.3, *ii.)*. Hence

$$\lim_{n \longrightarrow \infty} \int_X f_n \, \mathrm{d}\mu \geq c \int_X g \, \mathrm{d}\mu.$$

Now we can take the limit $c \longrightarrow 1$ and then the supremum over all the g without disturbing this inequality. This gives then the remaining estimate proving (C.3.13). ☐

Remark C.3.7 In particular, the theorem shows that for a measurable function $f \colon X \longrightarrow [0, +\infty]$ and any choice of simple functions $0 \leq g_1 \leq g_2 \leq \cdots \leq f$ with $\lim_{n \longrightarrow \infty} g_n(x) = f(x)$ for all $x \in X$ we have

$$\int_X f \, \mathrm{d}\mu = \lim_{n \longrightarrow \infty} \int_X g_n \, \mathrm{d}\mu. \tag{C.3.14}$$

Since by Proposition C.3.1 we always have such simple functions and since the integral of simple functions is easy to define and to compute by (C.3.5), this result gives an alternative definition of the integral for an arbitrary f. For explicit computations, (C.3.14) seems to be more accessible than the supremum in (C.3.8). We can also view this as an exchange of limits

$$\int_X \lim_{n \longrightarrow \infty} g_n \, \mathrm{d}\mu = \lim_{n \longrightarrow \infty} \int_X g_n \, \mathrm{d}\mu, \tag{C.3.15}$$

which is essentially build into the very definition of the integral. This makes the Lebesgue integral such a powerful theory of integration as e.g. in the Riemann theory such a simple exchange of limits will not be possible in general.

The next result shows that the integral is σ-additive with respect to the addition rules of $[0, +\infty]$.

Proposition C.3.8 *Let $(X, \mathfrak{a}, \mu)$ be a measure space and let $f_n \colon X \longrightarrow [0, +\infty]$ be measurable functions for $n \in \mathbb{N}$. Then*

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_X f_n \, \mathrm{d}\mu. \tag{C.3.16}$$

PROOF: Note first that $\sum_{n=1}^{\infty} f_n$, defined pointwise, will result in a well-defined map $X \longrightarrow [0, +\infty]$, which, as a limit of measurable maps, is again measurable. Hence the left hand side in (C.3.16) is defined. From Remark C.3.7 and the additivity of the integral according to Lemma C.3.3, *ii.)*, we see that we have additivity in general, i.e.

$$\int_X \left(\sum_{n=1}^{N} f_n \right) \mathrm{d}\mu = \sum_{n=1}^{N} \int_X f_n \,\mathrm{d}\mu,$$

by the continuity of the addition in $[0, +\infty]$. Since $f_n \geq 0$ we have for $g_N = \sum_{n=1}^{N} f_n$ the property $g_N \leq g_{N+1}$. Hence the sequence g_N satisfies the requirements of Theorem C.3.6. □

Proposition C.3.9 *Let $(X, \mathfrak{a}, \mu)$ be a measure space.*

i.) For $f, g\colon X \longrightarrow [0, +\infty]$ measurable and $c \geq 0$ we have

$$\int_X (cf + g) \,\mathrm{d}\mu = c \int_X f \,\mathrm{d}\mu + \int_X g \,\mathrm{d}\mu. \tag{C.3.17}$$

ii.) If $A, B \in \mathfrak{a}$ are disjoint then for every measurable $f\colon X \longrightarrow [0, +\infty]$

$$\int_A f \,\mathrm{d}\mu + \int_B f \,\mathrm{d}\mu = \int_{A\cup B} f \,\mathrm{d}\mu. \tag{C.3.18}$$

iii.) For $A \in \mathfrak{a}$ and a measurable $f\colon X \longrightarrow [0, +\infty]$ one has

$$\int_A f \,\mathrm{d}\mu = \int_X \chi_A f \,\mathrm{d}\mu. \tag{C.3.19}$$

iv.) For a zero set $N \in \mathfrak{a}$ and a measurable function $f\colon X \longrightarrow [0, +\infty]$ one has

$$\int_N f \,\mathrm{d}\mu = 0 \quad \textit{and} \quad \int_X f \,\mathrm{d}\mu = \int_{X\setminus N} f \,\mathrm{d}\mu. \tag{C.3.20}$$

v.) If $A_1 \subseteq A_2 \subseteq \cdots$ are measurable subsets and $f\colon X \longrightarrow [0, +\infty]$ is measurable then for $A = \bigcup_{n=1}^{\infty} A_n$ one has

$$\lim_{n\longrightarrow\infty} \int_{A_n} f \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu. \tag{C.3.21}$$

PROOF: The first part is clear by Proposition C.3.8 and the obvious relation $\int_X cf \,\mathrm{d}\mu = c\int_X f \,\mathrm{d}\mu$. This generalizes Lemma C.3.3, *ii.)*, to arbitrary measurable functions with values in $[0, +\infty]$. For the second part we note that (C.3.18) is fulfilled for a characteristic function χ_C since $\chi_C\big|_A = \chi_{C\cap A}$ and hence

$$\int_A \chi_C \,\mathrm{d}\mu = \mu(C \cap A) \quad \text{as well as} \quad \int_B \chi_C \,\mathrm{d}\mu = \mu(C \cap B).$$

But $\mu(C\cap A) + \mu(C\cap B) = \mu(C \cap (A\cup B))$ since $C\cap A$ and $C\cap B$ are disjoint and μ is additive. By Lemma C.3.3, *ii.)*, we conclude that (C.3.18) holds for arbitrary simple functions. Approximating a measurable function $f\colon X \longrightarrow [0, +\infty]$ by simple functions $g_n \nearrow f$ according to Proposition C.3.1 allows to apply Remark C.3.7 for all the three integrals over A, B, and $A\cup B$, respectively. Since (C.3.18) holds for each term of the sequence it holds for the limit as well which gives (C.3.18) for f. For the third part we argue analogously: clearly, (C.3.19) holds for a characteristic function and hence for a simple function. Then we can again use Remark C.3.7 in combination with Proposition C.3.1 to get (C.3.19) for arbitrary f. Now let N be a zero set. Then

$$\int_N g \,\mathrm{d}\mu = 0 \tag{$*$}$$

for every simple function by the fact that $\mu(A \cap N) = 0$ for every measurable subset A. Again, this is sufficient to conclude $(*)$ for every measurable $f\colon X \longrightarrow [0, +\infty]$. Hence (C.3.20) follows from (C.3.18). Finally, let $A_1 \subseteq A_2 \subseteq \cdots$ be measurable subsets with $A = \bigcup_{n=1}^{\infty} A_n$. Then

$$\int_{A_n} f \, \mathrm{d}\mu = \int_X \chi_{A_n} f \, \mathrm{d}\mu$$

by the third part. Now we have $\chi_{A_n} f = f$ on A_n and $\chi_{A_n} f = 0$ elsewhere. Since $A_n \subseteq A_{n+1}$ this shows that $\chi_{A_n} f \le \chi_{A_{n+1}} f$ and $\chi_{A_n} f \longrightarrow \chi_A f$ pointwise. Applying Theorem C.3.6 to this sequence gives

$$\lim_{n \longrightarrow \infty} \int_{A_n} f \, \mathrm{d}\mu = \lim_{n \longrightarrow \infty} \int_X \chi_{A_n} f \, \mathrm{d}\mu = \int_X \chi_A f \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu. \qquad \square$$

The fourth part of the Proposition shows that zero sets do not play any role for integration. In particular, we can modify f on N to any fixed value we want and still get the same integral. If the measure space is even complete, we can modify f in N even in an arbitrary point-dependent way and still get a measurable function (here we need the completeness) with the same integral.

A variation of the last part is that we take a partition $\{A_n\}_{n \in \mathbb{N}}$ of X with measurable subsets A_n and consider $B_n = A_1 \cup \cdots \cup A_n$. Then the B_n satisfy the assumption of the last part. By (C.3.18) we have

$$\sum_{n=1}^{\infty} \int_{A_n} f \, \mathrm{d}\mu = \lim_{n \longrightarrow \infty} \int_{B_n} f \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu. \tag{C.3.22}$$

Thus we can evaluate the integral locally on each A_n and sum up the resulting series afterwards.

Theorem C.3.10 (Fatou's Lemma) *Let $(X, \mathfrak{a}, \mu)$ be a measure space and let $f_n\colon X \longrightarrow [0, +\infty]$ be a sequence of measurable functions. Then*

$$\int_X \Big(\liminf_{n \longrightarrow \infty} f_n \Big) \, \mathrm{d}\mu \le \liminf_{n \longrightarrow \infty} \int_X f_n \, \mathrm{d}\mu. \tag{C.3.23}$$

PROOF: First we recall that the limes inferior of measurable functions yields again a measurable function with values in $[0, +\infty]$. Indeed, as already mentioned, we can adapt Theorem C.1.21 easily to this situation. We first consider the measurable functions defined pointwise by $g_n(x) = \inf_{k \ge n} f_k(x)$. Then $\liminf_{n \longrightarrow \infty} f_n(x) = \sup_{n \in \mathbb{N}} g_n(x)$. But taking the infimum over fewer functions yields an increasing sequence $0 \le g_1(x) \le g_2(x) \le \cdots$ for every $x \in X$. Hence the pointwise supremum is actually a limit in $[0, +\infty]$ which is approached from below. By Theorem C.3.6 we have

$$\int_X \Big(\liminf_{n \longrightarrow \infty} f_n \Big) \, \mathrm{d}\mu = \int_X \Big(\lim_{n \longrightarrow \infty} g_n \Big) \, \mathrm{d}\mu = \lim_{n \longrightarrow \infty} \int_X g_n \, \mathrm{d}\mu \le \liminf_{n \longrightarrow \infty} \int_X f_n \, \mathrm{d}\mu,$$

since clearly $g_n \le f_n$ pointwise and hence Remark C.3.5, *i.)*, applies, yielding the last estimate. $\square$

One should be aware of the fact that in Fatou's Lemma strict inequality may occur. Currently, we are not in the possession of many interesting measure spaces but the following simple example illustrates Fatou's Lemma quite drastically:

Example C.3.11 Let $X = \mathbb{N}$ be equipped with the σ-algebra $\mathfrak{a} = 2^{\mathbb{N}}$ and the counting measure μ_{count}. Then a measurable function $f\colon \mathbb{N} \longrightarrow [0, +\infty]$ is just a sequence $a_n = f(n)$ of numbers in $[0, +\infty]$. The integral $\int_{\mathbb{N}} f \, \mathrm{d}\mu_{\mathrm{count}}$ is then nothing else than the series $\sum_{n=1}^{\infty} a_n$. Note that from $a_n \in [0, +\infty]$ the series converges absolutely in $[0, +\infty]$.

i.) Let $a_{nm} \in [0, +\infty]$ be a double sequence. Then Proposition C.3.8 gives

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{nm}, \tag{C.3.24}$$

where the common value may be $+\infty$. This is of course also known from elementary calculus.

ii.) Consider $a_{nm} = \delta_{nm}$. Then we have $\liminf_{n \longrightarrow \infty} a_{nm} = 0$ for all m since there is only one exception in the values. However, $\sum_{m=1}^{\infty} a_{nm} = 1$ for all n. Hence Fatou's Lemma gives

$$0 = \sum_{m=1}^{\infty} \left(\liminf_{n \longrightarrow \infty} a_{nm} \right) < \liminf_{n \longrightarrow \infty} \sum_{m=1}^{\infty} a_{nm} = 1. \tag{C.3.25}$$

This shows that a strict inequality can occur already in quite simple situations.

The somehow weird value $+\infty$ which we had to allow in order to have a simple formulation of the theorem of monotonous convergence and Fatou's Lemma can, a posteriori, be avoided: either the function has $+\infty$ in its essential range, i.e. the subset $f^{-1}(\{+\infty\})$ has non-zero measure. Then $\int_X f \, \mathrm{d}\mu = +\infty$ does usually not yield anything interesting. Or, $f^{-1}(\{+\infty\})$ has measure zero. Then we can re-define f to be, say, 42 on $f^{-1}(\{+\infty\})$ without changing its behaviour concerning the integration thanks to Proposition C.3.9, *iv.)*. Nevertheless, the integral itself still may have the value $+\infty$ even for a measurable function $f \colon X \longrightarrow [0, +\infty)$.

We can now extend the integration from functions taking their values in $[0, +\infty]$ to complex-valued functions. For complex-valued measurable functions $f \colon X \longrightarrow \mathbb{C}$ we have a measurable function $|f| \colon X \longrightarrow [0, +\infty)$ to which we can apply our integration techniques. This motivates the following definition:

Definition C.3.12 (Integrable functions) *Let $(X, \mathfrak{a}, \mu)$ be a measure space. Then a measurable complex-valued function $f \in \mathscr{M}(X, \mathfrak{a})$ is called integrable if*

$$\|f\|_{\mu,1} = \int_X |f| \, \mathrm{d}\mu < \infty. \tag{C.3.26}$$

The set of all integrable functions is denoted by $\mathscr{L}^1(X, \mathfrak{a}, \mu)$.

Again, if the reference to either X, $\mathfrak{a}$, or μ is clear from the context we occasionally write $\mathscr{L}^1(X) = \mathscr{L}^1(X, \mathfrak{a}) = \mathscr{L}^1(X, \mu) = \mathscr{L}^1(\mu)$ for $\mathscr{L}^1(X, \mathfrak{a}, \mu)$. Note that

$$\mathscr{L}^1(X, \mathfrak{a}, \mu) \subseteq \mathscr{M}(X, \mathfrak{a}), \tag{C.3.27}$$

but $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ depends on the additional choice of the measure while $\mathscr{M}(X, \mathfrak{a})$ depends only on X and $\mathfrak{a}$. Moreover, we shall sometimes write just

$$\|f\|_1 = \|f\|_{\mu,1}, \tag{C.3.28}$$

if the reference to μ is clear. If $f \ni \mathscr{L}^1(X, \mathfrak{a}, \mu)$ then we can decompose f into the positive and negative parts of its real and imaginary part, i.e.

$$f = \operatorname{Re}(f)_+ - \operatorname{Re}(f)_- + \mathrm{i} \operatorname{Im}(f)_+ - \mathrm{i} \operatorname{Im}(f)_-, \tag{C.3.29}$$

which are all measurable, see Proposition C.1.20. Moreover, we have

$$\operatorname{Re}(f)_\pm, \operatorname{Im}(f)_\pm \le |f| \tag{C.3.30}$$

pointwise on X. This shows by Remark C.3.5, *i.)*, that $\operatorname{Re}(f)_\pm, \operatorname{Im}(f)_\pm \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$, too. Thus each part has a finite integral with a value in $[0, +\infty)$. Conversely, if $\operatorname{Re}(f)_\pm, \operatorname{Im}(f)_\pm$ have a finite integral then also $|f|$ has a finite integral as we can estimate $|f|$ by $\operatorname{Re}(f)_\pm$ and $\operatorname{Im}(f)_\pm$. These observations motivate the following definition:

Definition C.3.13 (Lebesgue integral) *Let $(X, \mathfrak{a}, \mu)$ be a measure space and let $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$. Then one defines the Lebesgue integral of f over X with respect to μ by*

$$\int_X f \,\mathrm{d}\mu = \int_X \mathrm{Re}(f)_+ \,\mathrm{d}\mu - \int_X \mathrm{Re}(f)_- \,\mathrm{d}\mu + \mathrm{i} \int_X \mathrm{Im}(f)_+ \,\mathrm{d}\mu - \mathrm{i} \int_X \mathrm{Im}(f)_- \,\mathrm{d}\mu. \tag{C.3.31}$$

The following proposition shows that this is indeed a reasonable definition:

Proposition C.3.14 *Let $(X, \mathfrak{a}, \mu)$ be a measurable space.*

i.) The set of measurable functions $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ is a complex vector space and $\|\cdot\|_{\mu,1}$ is a seminorm on $\mathscr{L}^1(X, \mathfrak{a}, \mu)$.

ii.) The kernel of the seminorm $\|\cdot\|_{\mu,1}$ is given by

$$\ker\|\cdot\|_{\mu,1} = \big\{ f \in \mathscr{M}(X, \mathfrak{a}) \;\big|\; f = 0 \textit{ almost everywhere} \big\}. \tag{C.3.32}$$

iii.) The integral is a well-defined linear functional

$$\int_X \cdot \,\mathrm{d}\mu \colon \mathscr{L}^1(X, \mathfrak{a}, \mu) \longrightarrow \mathbb{C}, \tag{C.3.33}$$

which is continuous with respect to $\|\cdot\|_{\mu,1}$. More precisely, we have for $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$

$$\left| \int_X f \,\mathrm{d}\mu \right| \le \|f\|_{\mu,1} = \int_X |f| \,\mathrm{d}\mu. \tag{C.3.34}$$

PROOF: For the first part, let $z, w \in \mathbb{C}$ and $f, g \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$. Then pointwise on X we have $|zf + wg| \le |z||f| + |w||g|$. Since $|f|$ and $|g|$ have finite integrals, Proposition C.3.9, *i.)*, shows that also $|z||f| + |w||g|$ has a finite integral. By Remark C.3.5, *i.)*, this holds also for $|zf + wg|$. Hence $zf + wg \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ showing that $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ is a complex subspace of $\mathscr{M}(X, \mathfrak{a})$. Clearly, $\|f\|_{\mu,1} \ge 0$ and

$$\|zf\|_{\mu,1} = \int_X |zf| \,\mathrm{d}\mu = |z| \int_X |f| \,\mathrm{d}\mu = |z| \|f\|_{\mu,1}$$

by (C.3.17). Finally, one has

$$\|f + g\|_{\mu,1} = \int_X |f + g| \,\mathrm{d}\mu \le \int_X (|f| + |g|) \,\mathrm{d}\mu = \int_X |f| \,\mathrm{d}\mu + \int_X |g| \,\mathrm{d}\mu = \|f\|_{\mu,1} + \|g\|_{\mu,1}$$

by the pointwise inequality $|f+g| \le |f|+|g|$, the inequality (C.3.9), and again the additivity (C.3.17). This shows the first part. For the second part it is clear that a function with $f = 0$ almost everywhere has $\|f\|_{\mu,1} = 0$. Indeed, in this case $|f| = 0$ almost everywhere and hence Proposition C.3.9, *iv.)*, gives the result. Conversely, assume that f is not almost everywhere zero. Then also $|f|$ is not almost everywhere zero and hence there is a constant $c > 0$ and a measurable subset $A \subseteq X$ with $|f| \ge c$ on A and with $\mu(A) > 0$. But then

$$\int_X |f| \,\mathrm{d}\mu \overset{\text{(C.3.10)}}{\ge} \int_A |f| \,\mathrm{d}\mu \overset{\text{(C.3.9)}}{\ge} \int_A c \,\mathrm{d}\mu = c\mu(A) > 0$$

shows $\|f\|_{\mu,1} \ge c\mu(A) > 0$. This proves the second part. For the last part we first note that $\int_X f \,\mathrm{d}\mu \in \mathbb{C}$ is well-defined since $\mathrm{Re}(f)_\pm$ and $\mathrm{Im}(f)_\pm$ are measurable functions with values in $[0, +\infty)$ and with finite integrals by (C.3.30). To check the linearity one has to decompose $zf + wg$ into the positive and negative parts of the real and imaginary parts and unwind the definition. Then everything boils down to use Proposition C.3.9, *i.)*, several times. We omit the tedious details. Now let $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ be given. Then there is a complex phase $\mathrm{e}^{\mathrm{i}\varphi}$ such that $\mathrm{e}^{\mathrm{i}\varphi} \int_X f \,\mathrm{d}\mu \in [0, +\infty)$. Using this reality feature we get

$$\begin{aligned}
\left| \int_X f \,\mathrm{d}\mu \right| &= \mathrm{e}^{\mathrm{i}\varphi} \int_X f \,\mathrm{d}\mu \\
&= \int_X \mathrm{e}^{\mathrm{i}\varphi} f \,\mathrm{d}\mu \\
&= \int_X \mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\varphi} f\big) \,\mathrm{d}\mu + \mathrm{i} \int_X \mathrm{Im}\big(\mathrm{e}^{\mathrm{i}\varphi} f\big) \,\mathrm{d}\mu \\
&= \int_X \mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\varphi} f\big)_+ \,\mathrm{d}\mu - \int_X \mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\varphi} f\big)_- \,\mathrm{d}\mu + 0 \\
&\le \int_X \mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\varphi} f\big)_+ \,\mathrm{d}\mu \\
&\le \int_X |f| \,\mathrm{d}\mu,
\end{aligned}$$

since clearly $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\varphi} f)_+ \le |\mathrm{e}^{\mathrm{i}\varphi} f| = |f|$ and hence (C.3.9) can be applied for the last estimate. This completes the proof. □

Concerning subsets we have the following statements which follow essentially from Proposition C.3.9 in a straightforward manner:

Proposition C.3.15 *Let $(X, \mathfrak{a}, \mu)$ be a measure space.*

i.) For every measurable $A \subseteq X$ the restriction map gives a continuous linear map

$$\mathscr{L}^1(X, \mathfrak{a}, \mu) \longrightarrow \mathscr{L}^1\big(A, \mathfrak{a}\big|_A, \mu\big|_A\big) \tag{C.3.35}$$

with respect to the seminorms $\|\cdot\|_{\mu,1}$ and $\|\cdot\|_{\mu|_A,1}$, respectively. More precisely, for every $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ one has

$$\|f\big|_A\|_{\mu|_A,1} \le \|f\|_{\mu,1}. \tag{C.3.36}$$

Conversely, $\mathscr{L}^1\big(A, \mathfrak{a}\big|_A, \mu\big|_A\big)$ is isometrically (and hence continuously) embedded into $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ by extending a function on A by 0 on $X \setminus A$. The composition of first restricting and then extending is the multiplication with χ_A.

ii.) If $A, B \in \mathfrak{a}$ are disjoint then for every $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ we have

$$\int_A f \,\mathrm{d}\mu + \int_B f \,\mathrm{d}\mu = \int_{A \cup B} f \,\mathrm{d}\mu. \tag{C.3.37}$$

iii.) For $A \in \mathfrak{a}$ and every $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ we have

$$\int_A f \,\mathrm{d}\mu = \int_X \chi_A f \,\mathrm{d}\mu. \tag{C.3.38}$$

iv.) For a zero set $N \in \mathfrak{a}$ and $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ we have

$$\int_N f \,\mathrm{d}\mu = 0 \quad \textit{and} \quad \int_X f \,\mathrm{d}\mu = \int_{X \setminus N} f \,\mathrm{d}\mu. \tag{C.3.39}$$

v.) If $A_1 \subseteq A_2 \subseteq \cdots$ are measurable subsets and $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ then for $A = \bigcup_{n=1}^\infty A_n$ we have

$$\lim_{n \longrightarrow \infty} \int_{A_n} f \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu. \tag{C.3.40}$$

PROOF: Let $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ then clearly $\operatorname{Re}(f)_\pm\big|_A = \operatorname{Re}\big(f\big|_A\big)_\pm$ etc. and $\big|f\big|_A\big| = |f|\big|_A$. By (C.3.9) we see that $|f|\big|_A$ has a finite integral. Thus $f\big|_A \in \mathscr{L}^1\big(A, \mathfrak{a}\big|_A, \mu\big|_A\big)$. Moreover, (C.3.9) gives the estimate (C.3.36). The statements about the embedding are trivial. The parts *ii.)*, *iii.)*, *iv.)*, and *v.)* follow directly from the corresponding statements of Proposition C.3.9 by evaluating the integrals of $\operatorname{Re}(f)_\pm$ and $\operatorname{Im}(f)_\pm$ separately. □

We come now to one of the probably most important theorem in integration theory, Lebesgue's Theorem of dominated convergence:

Theorem C.3.16 (Lebesgue's dominated convergence) *Let $(X, \mathfrak{a}, \mu)$ be a measure space. Let $f_n \in \mathscr{M}(X, \mathfrak{a})$ be a sequence of measurable, complex-valued functions such that*

$$f(x) = \lim_{n \longrightarrow \infty} f_n(x) \tag{C.3.41}$$

converges pointwise for $x \in X$. If there exists an integrable function $0 \le g \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ such that for all $n \in \mathbb{N}$ one has

$$|f_n(x)| \le g(x), \tag{C.3.42}$$

and hence $f_n \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$, too, then the limit function f is integrable and

$$f_n \longrightarrow f \tag{C.3.43}$$

with respect to $\|\cdot\|_{\mu,1}$. In particular,

$$\lim_{n \longrightarrow \infty} \int_X f_n \,\mathrm{d}\mu = \int_X \Big(\lim_{n \longrightarrow \infty} f_n\Big) \,\mathrm{d}\mu. \tag{C.3.44}$$

PROOF: First we note that the pointwise limit f is measurable by Theorem C.1.21, *i.)*. Since $|f_n(x)| \le g(x)$ and $g \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ we also have $f_n \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$, as usual by (C.3.9). Moreover,

$$|f(x)| = \lim_{n \longrightarrow \infty} |f_n(x)| \le g(x) \tag{$*$}$$

shows that also $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$. We have to show the convergence $f_n \longrightarrow f$ in the sense of the seminorm $\|\cdot\|_{\mu,1}$. From $(*)$ and the assumption (C.3.42) we get $|f_n - f| \le 2g$ pointwise on X. Hence we can apply Fatou's Lemma to the function $2g - |f_n - f| \ge 0$. This gives

$$\begin{aligned}
\int_X 2g \,\mathrm{d}\mu &\overset{\text{(C.3.41)}}{=} \int_X \lim_{n \longrightarrow \infty} \big(2g - |f_n - f|\big) \,\mathrm{d}\mu \\
&\overset{\text{Fatou}}{\le} \liminf_{n \longrightarrow \infty} \int_X \big(2g - |f_n - f|\big) \,\mathrm{d}\mu \\
&= \int_X 2g \,\mathrm{d}\mu + \liminf_{n \longrightarrow \infty} \int_X (-|f_n - f|) \,\mathrm{d}\mu \\
&= \int_X 2g \,\mathrm{d}\mu - \limsup_{n \longrightarrow \infty} \int_X |f_n - f| \,\mathrm{d}\mu.
\end{aligned}$$

Since $\int_X 2g \,\mathrm{d}\mu < \infty$ we conclude that

$$0 \le \limsup_{n \longrightarrow \infty} \int_X |f_n - f| \,\mathrm{d}\mu \le 0,$$

and hence $\limsup_{n \longrightarrow \infty} \int_X |f_n - f| \,\mathrm{d}\mu = 0$. Since the integral gives a non-negative value anyway, we see that the limes superior is actually a limit and hence $\|f_n - f\|_{\mu,1} \longrightarrow 0$ as claimed. Now the continuity of the integral according to Proposition C.3.14, *iii.)*, gives (C.3.44). □

In fact, the exchange of the limit (C.3.44) with the integral is a very powerful tool and the conditions (C.3.41) and (C.3.42) for its validity are usually both very mild and easy to check.

Up to now, without additional assumptions about the positive measure μ, it may well happen that the class of integrable functions $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ is very small. Indeed, assume as extreme case that $\mu(A) = +\infty$ for all $A \in \mathfrak{a}$ except for $A = \emptyset$. Then f is integrable iff $f = 0$. The following example shows that we get interesting integrable functions on σ-finite measure spaces:

Example C.3.17 Let $(X, \mathfrak{a}, \mu)$ be a σ-finite measure space. Then there exists a function $\chi \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ with $\chi(x) > 0$ for all $x \in X$. Indeed, let $A_n \in \mathfrak{a}$ be a sequence of measurable subsets for which $\mu(A_n) < \infty$ and $X = \bigcup_{n=1}^{\infty} A_n$. Then

$$\chi = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{1 + \mu(A_n)} \chi_{A_n} \tag{C.3.45}$$

converges pointwise to a measurable function bounded by 1. Since every point is in some A_n we conclude $0 < \chi < 1$. Then Proposition C.3.8 shows that

$$\int_X \chi \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mu(A_n)}{1 + \mu(A_n)} < 1, \tag{C.3.46}$$

and hence $\chi \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ as wanted.

To conclude this subsection we again consider the behaviour of the integral under measurable maps:

Proposition C.3.18 *Let $\Phi\colon (X, \mathfrak{a}) \longrightarrow (Y, \mathfrak{b})$ be a measurable map and μ a positive measure on $(X, \mathfrak{a})$.*

i.) For $f \in \mathscr{M}(Y, \mathfrak{b})$ we have $f \in \mathscr{L}^1(Y, \mathfrak{b}, \Phi_*\mu)$ iff $\Phi^* f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ and

$$\Phi^*\colon \mathscr{L}^1(Y, \mathfrak{b}, \Phi_*\mu) \longrightarrow \mathscr{L}^1(X, \mathfrak{a}, \mu) \tag{C.3.47}$$

is a seminorm-preserving linear map.

ii.) For $f \in \mathscr{L}^1(X, \mathfrak{b}, \Phi_*\mu)$ we have

$$\int_Y f \, \mathrm{d}\Phi_*\mu = \int_X \Phi^* f \, \mathrm{d}\mu. \tag{C.3.48}$$

Proof: First we note that $\Phi^*|f| = |\Phi^* f|$ and hence the first part is clear by Remark C.3.5, *iv.)*. Since also $\mathrm{Re}(\Phi^* f)_\pm = \Phi^*(\mathrm{Re}(f)_\pm)$ and $\mathrm{Im}(\Phi^* f)_\pm = \Phi^*(\mathrm{Im}(f)_\pm)$ the second part follows at once from the definition (C.3.31) of the integral. □

C.3.2 The L^p-Spaces

The space $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ lacks to have an honest norm: we only have a seminorm $\|\cdot\|_{\mu,1}$ and hence $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ is not Hausdorff (unless the only zero set with respect to μ is the empty set $\emptyset$). Of course, we know how to handle this situation. We just have to divide by the kernel of the seminorm to get a normed and hence Hausdorff quotient. By the characterization (C.3.32) we have to divide by the functions which vanish almost everywhere. Since they do not play any interesting role in integration anyway, this quotient procedure is very desirable and matches the one needed for the passage from $\mathscr{BM}(X, \mathfrak{a})$ to $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$. We will now show that we even get a Banach space by this procedure. Moreover, $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ has some cousins where we control the integrability of $|f|^p$ for $p \in (1, \infty)$ instead of just $|f|$. We start with the following preparatory results:

Proposition C.3.19 (Hölder and Minkowski inequality) *Let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and let $(X, \mathfrak{a}, \mu)$ be a measure space. Let $f, g\colon X \longrightarrow [0, +\infty]$ be measurable functions.*

i.) One has Hölder's inequality

$$\int_X fg \,\mathrm{d}\mu \leq \left(\int_X f^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_X g^q \,\mathrm{d}\mu\right)^{\frac{1}{q}}. \tag{C.3.49}$$

ii.) One has Minkowski's inequality

$$\left(\int_X (f+g)^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \leq \left(\int_X f^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X g^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}. \tag{C.3.50}$$

PROOF: First we emphasize that the inequalities are understood to include the possible value $+\infty$ of the integrals. For $\alpha, \beta \in [0, +\infty]$ one has the inequality

$$\alpha^{\frac{1}{p}} \beta^{\frac{1}{q}} \leq \frac{\alpha}{p} + \frac{\beta}{q}, \tag{$*$}$$

where we use the convention $(+\infty)^{\frac{1}{p}} = +\infty$. In fact, this inequality is known from elementary calculus and follows from the convexity properties of the logarithm, see Exercise 2.5.39, *i.)*. To avoid trivialities we can assume that both factors of the right hand side (C.3.49) are different from 0 and $+\infty$: indeed, if one is zero, then the corresponding function is almost everywhere 0 and hence also the product fg is almost everywhere 0. Thus (C.3.49) is satisfied. If both are non-zero and one is $+\infty$ then (C.3.49) trivially holds as well. Thus consider the rescaled functions

$$F(x) = \frac{1}{\sqrt[p]{\int_X f^p \,\mathrm{d}\mu}} f(x) \quad \text{and} \quad G(x) = \frac{1}{\sqrt[q]{\int_X g^q \,\mathrm{d}\mu}} g(x),$$

which are now measurable functions such that

$$\int_X F^p \,\mathrm{d}\mu = 1 = \int_X G^q \,\mathrm{d}\mu. \tag{$**$}$$

Pointwise, we have $F(x)G(x) \leq \frac{1}{p}F^p(x) + \frac{1}{q}G^q(x)$ by $(*)$ for those $x \in X$ where both functions are different from $+\infty$. Note that a finite right hand side implies that the set N of points where either F or G are $+\infty$ have necessarily measure zero and hence can be ignored in the following estimate

$$\begin{aligned}
\int_X FG \,\mathrm{d}\mu &= \int_{X\setminus N} FG \,\mathrm{d}\mu \\
&\leq \int_{X\setminus N} \frac{1}{p} F^p \,\mathrm{d}\mu + \int_{X\setminus N} \frac{1}{q} G^q \,\mathrm{d}\mu \\
&= \frac{1}{p} \int_X F^p \,\mathrm{d}\mu + \frac{1}{q} \int_X G^q \,\mathrm{d}\mu \\
&\overset{(**)}{=} 1.
\end{aligned}$$

But this implies (C.3.49) for f and g. For Minkowski's inequality we first have

$$(f+g)^p = f(f+g)^{p-1} + g(f+g)^{p-1}$$

Then Hölder's inequality gives

$$\int_X f(f+g)^{p-1} \,\mathrm{d}\mu \leq \left(\int_X f^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_X (f+g)^{q(p-1)} \,\mathrm{d}\mu\right)^{\frac{1}{q}}$$

and

$$\int_X g(f+g)^{p-1}\,\mathrm{d}\mu \leq \left(\int_X g^p\,\mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_X (f+g)^{q(p-1)}\,\mathrm{d}\mu\right)^{\frac{1}{q}}.$$

Now $q(p-1) = p$ from which we get

$$\int_X (f+g)^p\,\mathrm{d}\mu \leq \left(\int_X (f+g)^p\,\mathrm{d}\mu\right)^{\frac{1}{q}}\left(\left(\int_X f^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X g^p\,\mathrm{d}\mu\right)^{\frac{1}{p}}\right). \tag{☺}$$

We have to distinguish several cases. If the left hand side of (☺) is zero then (C.3.50) holds. Thus we can assume that the left hand side of (☺) is in $(0, +\infty]$. Since in general $\left(\frac{f+g}{2}\right)^p \leq \frac{1}{2}(f^p + g^p)$ we conclude that if the left hand side is $+\infty$ then at least one of the integrals over f^p or g^p has to be $+\infty$ as well, which shows (C.3.50) also in this case. Thus the remaining case is that the left hand side of (☺) is in $(0, \infty)$. But then we can divide by $\left(\int_X (f+g)^p\,\mathrm{d}\mu\right)^{\frac{1}{q}}$ and get (C.3.50) since $1 - \frac{1}{q} = \frac{1}{p}$. ☐

Remark C.3.20 For $p = 1$ Hölder's inequality still holds if we take $q = +\infty$ in the sense of the essential supremum. In fact, we have $|f(x)g(x)| \leq |f(x)| \operatorname{ess\,sup}_{y\in X}|g(y)|$ almost everywhere since $|g(x)| \leq \operatorname{ess\,sup}_{y\in X}|g(y)|$ almost everywhere according to (C.2.27). Integrating this inequality gives then

$$\int_X fg\,\mathrm{d}\mu \leq \int_X f\,\mathrm{d}\mu \ \operatorname*{ess\,sup}_{x\in X}|g(x)|, \tag{C.3.51}$$

since we can ignore the zero set where the pointwise estimate fails in the integration. In this sense we can use Hölder's Inequality also for $p = 1$ and $q = +\infty$. Minkowski's inequality for $p = 1$ was already shown when proving that $\|\cdot\|_{\mu,1}$ is a seminorm. Minkowski's inequality for $p = +\infty$ is a property of the essential supremum and was used when showing that $\|\cdot\|_{\mu,\infty}$ is a seminorm.

As a generalization of the spaces $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ and $\mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ we can now define the *p-integrable functions* as follows:

Definition C.3.21 ($\mathscr{L}^p$-Space) *Let $(X, \mathfrak{a}, \mu)$ be a measure space and $1 \leq p < +\infty$. Then a complex-valued measurable function $f \in \mathscr{M}(X, \mathfrak{a})$ is called p-integrable (or of class $\mathscr{L}^p$) if*

$$\|f\|_{\mu,p} = \left(\int_X |f|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} < +\infty. \tag{C.3.52}$$

The set of all p-integrable functions is denoted by $\mathscr{L}^p(X, \mathfrak{a}, \mu)$.

Obviously, the case $p = 1$ matches our previous definition of integrable functions in Definition C.3.12. Thus we refer to the case $p = 1$ simply as integrable functions. The case $p = 2$ is also of particular importance and the functions will be called the *square-integrable functions.*

Proposition C.3.22 *Let $(X, \mathfrak{a}, \mu)$ be a measure space and $1 \leq p < +\infty$.*

i.) The set $\mathscr{L}^p(X, \mathfrak{a}, \mu)$ is a complex subspace of $\mathscr{M}(X, \mathfrak{a})$ and $\|\cdot\|_{\mu,p}$ is a seminorm on it.

ii.) The kernel of the seminorm $\|\cdot\|_{\mu,p}$ is given by

$$\ker\|\cdot\|_{\mu,p} = \left\{f \in \mathscr{M}(X, \mathfrak{a}) \;\middle|\; f = 0 \textit{ almost everywhere}\right\}. \tag{C.3.53}$$

iii.) For all $f \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$ one has $\overline{f} \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$, too, and

$$\|f\|_{\mu,p} = \|\overline{f}\|_{\mu,p}. \tag{C.3.54}$$

PROOF: We only have to check the new cases $1 < p < \infty$. For $z \in \mathbb{C}$ we have $\|zf\|_{\mu,p} = |z|\|f\|_{\mu,p}$, whether this quantity is finite or $+\infty$. Moreover, we have the estimate $\|f+g\|_{\mu,p} \le \|f\|_{\mu,p} + \|g\|_{\mu,p}$ which is precisely the Minkowski inequality (C.3.50) for $|f|$ and $|g|$. Thus it follows at once that those measurable functions for which (C.3.52) is finite form a subspace of $\mathscr{M}(X, \mathfrak{a})$ and $\|\cdot\|_{\mu,p}$ is a seminorm in it. Clearly, $f(x) = 0$ iff $|f(x)|^p = 0$. Therefore $f = 0$ almost everywhere iff $|f|^p = 0$ almost everywhere. Then the second part follows from Proposition C.3.14, *ii.)*. The last part is obvious. □

For all cases (including $p = 1$ and $p = +\infty$) we call the corresponding $q \in [1, +\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ the *conjugate exponent*. For conjugate exponents we can use Hölder's inequality to construct a natural pairing between $\mathscr{L}^p(X, \mathfrak{a}, \mu)$ and $\mathscr{L}^q(X, \mathfrak{a}, \mu)$. To this end, we need the following lemma:

Lemma C.3.23 (Polar decomposition) *Let $(X, \mathfrak{a})$ be a measurable space and let $f \in \mathscr{M}(X, \mathfrak{a})$ be measurable function. Then one defines*

$$u(x) = \begin{cases} \frac{f(x)}{|f(x)|} & \text{for } f(x) \neq 0 \\ 1 & \text{for } f(x) = 0, \end{cases} \tag{C.3.55}$$

and obtains a function $u \in \mathscr{B}\mathscr{M}(X, \mathfrak{a})$ with $u\overline{u} = 1$ and $\|u\|_\infty = 1$ as well as

$$f = u|f|. \tag{C.3.56}$$

PROOF: First it is clear that u is well-defined and $|u| = 1$ everywhere. Let $A = f^{-1}(\{0\})$ be the measurable subset where f is zero. Since on $X \setminus A$ the function $f\big|_{X\setminus A}$ is still measurable and non-zero and since $z \mapsto \frac{z}{|z|}$ is continuous on $\mathbb{C} \setminus \{0\}$ we see that $u\big|_{X\setminus A} = \frac{f|_{X\setminus A}}{|f||_{X\setminus A}}$ is measurable as the composition of a measurable and a continuous map. Since $u\big|_A = 1$ is measurable as well, the gluing statement from Proposition C.1.22, *iii.)*, shows that u is measurable on X. Clearly, (C.3.56) holds pointwise on X by construction. □

Definition C.3.24 (Pairing of $\mathscr{L}^p$ and $\mathscr{L}^q$) *Let $(X, \mathfrak{a}, \mu)$ be a measure space and $1 \le p \le \infty$ with conjugate exponent q. For $f \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$ and $g \in \mathscr{L}^q(X, \mathfrak{a}, \mu)$ one defines the pairing*

$$\langle f, g \rangle = \int_X \overline{f} g \, \mathrm{d}\mu. \tag{C.3.57}$$

Note that we could also define a bilinear pairing with f instead of $\overline{f}$ on the right hand side. The sesquilinear choice (C.3.57) will match well the Hilbert space features of square-integrable functions later. In view of Proposition C.3.22, *iii.)*, the analysis for the bilinear version is identical. So we do not need to spell out the details.

Proposition C.3.25 *Let $(X, \mathfrak{a}, \mu)$ be a measure space and $1 \le p \le +\infty$ with conjugate exponent q.*

i.) The pairing $\langle \cdot\,, \cdot \rangle \colon \mathscr{L}^p(X, \mathfrak{a}, \mu) \times \mathscr{L}^q(X, \mathfrak{a}, \mu) \longrightarrow \mathbb{C}$ is well-defined and sesquilinear.

ii.) The pairing (C.3.57) *is continuous. More precisely,*

$$|\langle f, g \rangle| \le \|f\|_{\mu,p} \|g\|_{\mu,q} \tag{C.3.58}$$

for all $f \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$ and $g \in \mathscr{L}^q(X, \mathfrak{a}, \mu)$.

iii.) For for $1 \le p < +\infty$ the pairing (C.3.57) *is non-degenerate in the first argument modulo functions vanishing almost everywhere. More precisely, for every $f \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$ there is a $g \in \mathscr{L}^q(X, \mathfrak{a}, \mu)$ such that in* (C.3.58) *we have equality.*

Proof: We first consider the case $p = 1$ and hence $q = +\infty$. In this case, (C.3.51) shows that $\overline{f}g$ is still integrable. Hence the pairing is well-defined and clearly sesquilinear. From the same Hölder estimate (C.3.51) we get also the estimate (C.3.58). The case $p = +\infty$ and $q = 1$ follows analogously. For $1 < p < +\infty$ we use Hölder's inequality (C.3.49) and $\|\overline{f}\|_{\mu,p} = \|f\|_{\mu,p}$ to conclude that $\overline{f}g \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$. Again, this gives a well-defined complex number when integrating as in (C.3.57). The sesquilinearity is clear also in this case. Moreover, Hölder's inequality applied to $|\overline{f}|$ and $|g|$ gives immediately (C.3.58). For the last part, let first $p = 1$ and $f = u|f|$ according to Lemma C.3.23. Then we can take $g = u$ and get immediately $\langle f, g\rangle = \|f\|_{\mu,1}$. Note that $\|u\|_{\mu,\infty} = 1$ hence we have $\langle f, g\rangle = \|f\|_{\mu,1}\|g\|_{\mu,\infty}$. Now let $1 < p < +\infty$ and write $f \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$ again in its polar decomposition $f = u|f|$. Then $|f|^{\frac{p}{q}} \in \mathscr{L}^q(X, \mathfrak{a}, \mu)$ and hence $g = u|f|^{\frac{p}{q}}$ is still q-integrable since $|g|^q = |f|^p \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ as $|u| = 1$. Thus we compute

$$\int_X \overline{f}g \,\mathrm{d}\mu = \int_X \overline{u}|f|u|f|^{\frac{p}{q}} \,\mathrm{d}\mu = \int_X |f|^{1+\frac{p}{q}} \,\mathrm{d}\mu = \int_X |f|^p \,\mathrm{d}\mu = \|f\|^p_{\mu,p}.$$

Now we have $g(x) = u(x)|f(x)|^{\frac{p}{q}} = 0$ iff $f(x) = 0$. Hence $\int_X |g|^q \,\mathrm{d}\mu = \int_X |f|^p \,\mathrm{d}\mu$ follows. But the relation $1 = \frac{1}{p} + \frac{1}{q}$ shows that

$$\|f\|^p_{\mu,p} = \|f\|_{\mu,p}\|f\|^{p-1}_{\mu,p} = \|f\|_{\mu,p}\left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{p-1}{p}} = \|f\|_{\mu,p}\left(\int_X |g|^q \,\mathrm{d}\mu\right)^{\frac{p-1}{p}} = \|f\|_{\mu,p}\|g\|_{\mu,q}.$$

Hence also in this case we have found a g such that $\langle f, g\rangle = \|f\|_{\mu,p}\|g\|_{\mu,q}$. □

Remark C.3.26 The non-degeneracy in the case $p = +\infty$ is slightly more complicated: we assume in addition that $(X, \mathfrak{a}, \mu)$ is at least *σ-finite*. Then for $f \in \mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ there is a sequence $g_n \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ with $\|g_n\|_{\mu,1} = 1$ such that $|\langle f, g_n\rangle| \longrightarrow \|f\|_{\mu,\infty}$. Indeed, assume $\|f\|_{\mu,\infty} > 0$ to avoid trivialities and let $z \in \operatorname{ess\,range}(f)$ be given. Then let $A_\delta = f^{-1}(\mathrm{B}_\delta(z)) \subseteq X$ which has positive measure $\mu(A_\delta) > 0$ for all $\delta > 0$. However, it might happen that $\mu(A_\delta) = +\infty$. Since we assume that X is σ-finite there is another measurable subset $\tilde{A}_\delta \subseteq A_\delta$ with non-zero finite measure $0 < \mu(\tilde{A}_\delta) < +\infty$. Using the polar decomposition $f = u|f|$ we define now $g_{\delta,z} = \frac{1}{\mu(\tilde{A}_\delta)}\chi_{\tilde{A}_\delta}u \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$. Then we get

$$\int_X \overline{f}g_{\delta,z} \,\mathrm{d}\mu = \int_{\tilde{A}_\delta} \frac{1}{\mu(\tilde{A}_\delta)}|f| \,\mathrm{d}\mu \geq |z| - \delta. \tag{C.3.59}$$

Thus for every $z \in \operatorname{ess\,range}(f)$ there is a $g_{\delta,z} \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ with $|\langle f, g_{\delta,z}\rangle| \geq |z| - \delta$ and $\|g_{\delta,z}\|_{\mu,1} = 1$ for $|z| - \delta > 0$. Taking now a sequence $z_n \in \operatorname{ess\,range}(f)$ with $|z_n| \longrightarrow \|f\|_{\mu,\infty}$ and taking $\delta = \frac{1}{n}$ gives (for large enough n) a sequence g_n meeting the desired criteria. Note however that this construction also shows why the pairing may be highly degenerate: for the extreme case of a measure μ which is $+\infty$ on all non-empty measurable subsets, the space $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ is simply $\{0\}$ while $\mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ generically is non-trivial. Thus there can not be any non-degenerate pairing.

The next result is the completeness of the spaces $\mathscr{L}^p(\mathfrak{a}, \mu)$. Here we first pass to the Hausdorffization by dividing by the almost everywhere vanishing functions. Note that they constitute the kernel of the seminorm $\|\cdot\|_{\mu,p}$ by Proposition C.3.22, *ii.)*. Thus we can define the following normed spaces analogously to $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$:

Definition C.3.27 (L^p-spaces) *Let $(X, \mathfrak{a}, \mu)$ be a measurable space. Then one defines*

$$\mathrm{L}^p(X, \mathfrak{a}, \mu) = \mathscr{L}^p(X, \mathfrak{a}, \mu)\big/\ker\|\cdot\|_{\mu,p} \tag{C.3.60}$$

for all $1 \leq p < +\infty$ and endows $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ with the induced quotient norm, still denoted by $\|\cdot\|_{\mu,p}$.

Proposition C.3.28 *Let $(X, \mathfrak{a}, \mu)$ be a measure space and $1 \leq p < +\infty$.*

i.) The space $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ is a Banach space.

ii.) A simple function $f \in \mathscr{M}(X, \mathfrak{a})$ is in $\mathscr{L}^p(X, \mathfrak{a}, \mu)$ iff for every value $z \in \mathbb{C} \setminus \{0\}$ of f one has $\mu(f^{-1}(\{z\})) < +\infty$.

iii.) The subspace of all equivalence classes of p-integrable simple functions is dense in $\mathrm{L}^p(X, \mathfrak{a}, \mu)$.

iv.) The pairing (C.3.57) *descends to a sesquilinear and continuous pairing*

$$\langle \cdot\,, \cdot \rangle \colon \mathrm{L}^p(X, \mathfrak{a}, \mu) \times \mathrm{L}^q(X, \mathfrak{a}, \mu) \longrightarrow \mathbb{C}, \tag{C.3.61}$$

such that for all $f \in \mathrm{L}^p(X, \mathfrak{a}, \mu)$ and $g \in \mathrm{L}^q(X, \mathfrak{a}, \mu)$ one has

$$|\langle f, g\rangle| \leq \|f\|_{\mu,p}\|g\|_{\mu,q}. \tag{C.3.62}$$

In the first argument, it is non-degenerate for $1 \leq p < +\infty$ and for $p = \infty$ in the case of a σ-finite measure space and analogously for the second argument.

v.) The space $\mathrm{L}^2(X, \mathfrak{a}, \mu)$ is a Hilbert space via (C.3.61) *and $\|\cdot\|_{\mu,2}$ is the Hilbert norm corresponding to the scalar product $\langle \cdot\,, \cdot \rangle$.*

Proof: By construction, $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ is a normed vector space, we have to show completeness. Thus let $f_n \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$ be representatives of a Cauchy sequence in $\mathrm{L}^p(X, \mathfrak{a}, \mu)$. This means that for $\epsilon > 0$ we have a N with $\|f_n - f_m\|_{\mu,p} < \epsilon$ for all $n, m \geq N$. To get a candidate for the limit we fix a subsequence f_{n_k} with $\|f_{n_{k+1}} - f_{n_k}\| < 2^{-k}$ for $k \in \mathbb{N}$. This allows to consider the functions

$$g_\ell = \sum_{k=1}^{\ell} \big|f_{n_{k+1}} - f_{n_k}\big| \quad \text{and} \quad g = \sum_{k=1}^{\infty} \big|f_{n_{k+1}} - f_{n_k}\big|,$$

defined pointwise on X as usual. Clearly, $g_\ell \colon X \longrightarrow [0, \infty)$ and $g_\ell \leq g_{\ell+1}$ with $g_\ell(x) \longrightarrow g(x) \in [0, +\infty]$ for all $x \in X$. Now,

$$\|g_\ell\|_{\mu,p} \leq \sum_{k=1}^{\ell} \big\|\big|f_{n_{k+1}} - f_{n_k}\big|\big\|_{\mu,p} = \sum_{k=1}^{\ell} \big\|f_{n_{k+1}} - f_{n_k}\big\|_{\mu,p} < 1$$

by the choice of the subsequence. By Fatou's Lemma we have for the function g

$$\int_X |g|^p \,\mathrm{d}\mu = \int_X \lim_{\ell \longrightarrow \infty} |g_\ell|^p \,\mathrm{d}\mu = \int_X \liminf_{\ell \longrightarrow \infty} |g_\ell|^p \,\mathrm{d}\mu \leq \liminf_{\ell \longrightarrow \infty} \int_X |g_\ell|^p \,\mathrm{d}\mu \leq 1. \tag{$*$}$$

In particular, the value $+\infty$, which may occur for $g(x)$ for some x, can only occur on a subset of measure zero. Otherwise the integral $(*)$ would be infinite. Hence $g < +\infty$ almost everywhere. But this means that the series $f_{n_1} + \sum_{k=1}^{\infty}(f_{n_{k+1}} - f_{n_k})$ converges at almost all $x \in X$ absolutely to a finite value. We define now

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} \big(f_{n_{k+1}}(x) - f_{n_k}(x)\big)$$

for those $x \in X$ where we have absolute convergence. For the remaining $x \in X$ we set $f(x) = 0$, this is a set of measure zero anyway. We obtain a measurable function by the usual gluing arguments. By the usual telescope sum we have

$$f_{n_1}(x) + \sum_{k=1}^{\ell} \big(f_{n_{k+1}}(x) - f_{n_k}(x)\big) = f_{n_{\ell+1}}(x) \longrightarrow f(x)$$

for almost all $x \in X$. Hence the subsequence converges pointwise to f almost everywhere. We want to show that this f gives a candidate for the limit. Thus let $\epsilon > 0$ and N as above. By Fatou's Lemma we get for $m \geq N$

$$
\begin{aligned}
\int_X |f - f_m|^p \, \mathrm{d}\mu &= \int_{X \setminus Z} |f - f_m|^p \, \mathrm{d}\mu \\
&= \int_{X \setminus Z} \lim_{k \longrightarrow \infty} |f_{n_k} - f_m|^p \, \mathrm{d}\mu \\
&\leq \liminf_{k \longrightarrow \infty} \int_{X \setminus Z} |f_{n_k} - f_m|^p \, \mathrm{d}\mu \\
&= \liminf_{k \longrightarrow \infty} \int_X |f_{n_k} - f_m|^p \, \mathrm{d}\mu \\
&= \liminf_{k \longrightarrow \infty} \|f_{n_k} - f_m\|_{\mu,p}^p \\
&\leq \epsilon^p,
\end{aligned}
$$

where Z denotes the zero set on which we can say nothing about the pointwise convergence. The values of the integrals are, as usual, not sensitive to that. But this shows that $f - f_m \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$ and hence also $f \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$. Moreover, $f_m \longrightarrow f$ in the sense of the seminorm $\|\cdot\|_{\mu,p}$. Passing to equivalence classes in $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ gives the completeness and hence the first part. For the second part, let $f = \sum_{i=1}^n z_i \chi_{A_i}$ be a simple function in its normal form. Then the requirement in part *ii.)* means $\mu(A_i) < \infty$ for $z_i \neq 0$. We have $\|\chi_{A_i}\|_{\mu,p} = \sqrt[p]{\mu(A_i)}$ for every $A_i \in \mathfrak{a}$ and hence $\|f\|_{\mu,p} \leq \sum_{i=1}^n |z_i| \sqrt[p]{\mu(A_i)} < \infty$ as claimed. For the third part, let $f \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$ be given. Since with f also $\mathrm{Re}(f)_\pm$ and $\mathrm{Im}(f)_\pm$ are p-integrable we can assume that $f \geq 0$ from the beginning and take a finite linear combination afterwards. By Proposition C.3.1 we can choose a sequence of simple functions $0 \leq f_1 \leq f_2 \leq \cdots \leq f$ with $f_n(x) \longrightarrow f(x)$ everywhere. Then $|f_n|^p \leq |f|^p$ shows that $f_n \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$. Since we already know that $|f|^p$ is integrable and $|f - f_n|^p \leq |f|^p$ we can apply Theorem C.3.16 to the pointwise convergent sequence $|f - f_n|^p \longrightarrow 0$ and get $\||f - f_n|^p\|_{\mu,1} = \|f - f_n\|_{\mu,p}^p \longrightarrow 0$ thereby proving the third part. The fourth part is now an easy consequence of Proposition C.3.25 and Remark C.3.26. Finally, for $p = 2$ we have $\langle f, f \rangle > 0$ for $f \neq 0$ in $\mathrm{L}^2(X, \mathfrak{a}, \mu)$ and $\langle f, f \rangle = \|f\|_{\mu,2}^2$. Thus $\mathrm{L}^2(X, \mathfrak{a}, \mu)$ is a Hilbert space with scalar product $\langle \cdot\,, \cdot \rangle$ and induced Hilbert norm $\|\cdot\|_{\mu,2}$. □

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Remark C.3.29 The relation between the $\mathscr{L}^p$-spaces and hence between the L^p-spaces for different p is, in general, quite complicated. From part *iii.)* we see that the intersection $\mathscr{L}^p(X, \mathfrak{a}, \mu) \cap \mathscr{L}^{p'}(X, \mathfrak{a}, \mu)$ is always non-trivial and even dense in both. However, in general one has functions $f \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$ which are not in $\mathscr{L}^{p'}(X, \mathfrak{a}, \mu)$ and vice versa. In the cases of the Lebesgue measure on $\mathbb{R}^n$ we will see examples later on.

If however, the measure μ is finite things simplify in the following way:

Proposition C.3.30 *Let $(X, \mathfrak{a}, \mu)$ be a finite measure space and $1 \leq p < p' \leq +\infty$. Then*

$$
\mathscr{L}^{p'}(X, \mathfrak{a}, \mu) \subseteq \mathscr{L}^p(X, \mathfrak{a}, \mu), \tag{C.3.63}
$$

and for $f \in \mathscr{L}^{p'}(X, \mathfrak{a}, \mu)$ we have

$$
\|f\|_{\mu,p} \leq \mu(X)^{\frac{1}{p} - \frac{1}{p'}} \|f\|_{\mu,p'}. \tag{C.3.64}
$$

Proof: First let p' be different from $+\infty$ and let $f \in \mathscr{L}^{p'}(X, \mathfrak{a}, \mu)$ be given. Then $|f|^{\frac{p'}{p}}$ is in $\mathscr{L}^p(X, \mathfrak{a}, \mu)$ since

$$\left\| |f|^{\frac{p'}{p}} \right\|_{\mu,p} = \left(\int_X \left(|f|^{\frac{p'}{p}} \right)^p \mathrm{d}\mu \right)^{\frac{1}{p}} = \left(\int_X |f|^{p'} \mathrm{d}\mu \right)^{\frac{1}{p}} = \left(\|f\|_{\mu,p'} \right)^{\frac{p'}{p}} < \infty.$$

The conjugate exponent to $\frac{p'}{p}$ is $\frac{1}{1-\frac{p}{p'}} = \frac{p'}{p'-p}$. Since the constant function $\chi_X = 1$ is in every $\mathscr{L}^p(X, \mathfrak{a}, \mu)$ with $\|\chi_X\|_{\mu,p} = \sqrt[p]{\mu(X)}$ we can apply Hölder's inequality to get

$$\begin{aligned}
\int_X |f|^p \chi_X \, \mathrm{d}\mu &\le \left(\int_X (|f|^p)^{\frac{p'}{p}} \mathrm{d}\mu \right)^{\frac{p}{p'}} \left(\int_X \chi_X^{\frac{p'}{p'-p}} \mathrm{d}\mu \right)^{\frac{p'-p}{p'}} \\
&= \left(\int_X |f|^{p'} \mathrm{d}\mu \right)^{\frac{p}{p'}} \mu(X)^{\frac{p'-p}{p'}} \\
&= \left(\|f\|_{\mu,p'} \right)^{p' \frac{p}{p'}} \mu(X)^{\frac{p'-p}{p'}},
\end{aligned}$$

which means $\|f\|_{\mu,p} \le \|f\|_{\mu,p'} \mu(X)^{\frac{p'-p}{pp'}}$ and hence (C.3.64). The case $p' = +\infty$ is simpler, we have

$$\|f\|_{\mu,p} = \int_X |f|^p \, \mathrm{d}\mu \le \int_X \|f\|_{\mu,\infty}^p \, \mathrm{d}\mu = \|f\|_{\mu,\infty}^p \mu(X).$$

□

Corollary C.3.31 *Let $1 \le p < p' \le +\infty$ and let $(X, \mathfrak{a}, \mu)$ be a finite measure space. Then we have a continuous inclusion*

$$\mathrm{L}^{p'}(X, \mathfrak{a}, \mu) \longrightarrow \mathrm{L}^p(X, \mathfrak{a}, \mu) \tag{C.3.65}$$

with operator norm equal to $\mu(X)^{\frac{1}{p} - \frac{1}{p'}}$.

Proof: Clearly, the functoriality of the Hausdorffization shows that (C.3.63) yields a well-defined inclusion map on the quotients and (C.3.64) holds for the quotient norms as well. Since we divide by the same space of functions vanishing almost everywhere, the induced map is still injective. Finally, taking $\chi_X = 1$ shows that we have equality in the estimate (C.3.64) which, therefore, was already optimal. □

Another application of Hölder's inequality is that we can multiply p- and q-integrable functions to get a new integrable function:

Proposition C.3.32 *Let $(X, \mathfrak{a}, \mu)$ be a measure space and let $1 \le p, q \le +\infty$. Then the pointwise product gives a continuous bilinear map*

$$\mathscr{L}^p(X, \mathfrak{a}, \mu) \times \mathscr{L}^q(X, \mathfrak{a}, \mu) \longrightarrow \mathscr{L}^r(X, \mathfrak{a}, \mu), \tag{C.3.66}$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, such that

$$\|fg\|_{\mu,r} \le \|f\|_{\mu,p} \|g\|_{\mu,q}. \tag{C.3.67}$$

Proof: Let first $1 < p, q < \infty$. Then Hölder's inequality will be applied to the conjugate exponents $\frac{p}{r}$ and $\frac{q}{r}$. Indeed, note that $\frac{r}{p} + \frac{r}{q} = 1$ and $1 < \frac{r}{p}, \frac{r}{q} < \infty$. Thus we have

$$\int_X |fg|^r \, \mathrm{d}\mu \le \left(\int_X (|f|^r)^{\frac{p}{r}} \mathrm{d}\mu \right)^{\frac{r}{p}} \left(\int_X (|g|^r)^{\frac{q}{r}} \mathrm{d}\mu \right)^{\frac{r}{p}} = \|f\|_{\mu,p}^r \|g\|_{\mu,q}^r,$$

which is (C.3.67). The remaining cases are now obtained analogously. In particular, $p = +\infty = q$ is the statement that $\|\cdot\|_{\mu,\infty}$ is a submultiplicative seminorm, as established already in Proposition C.2.18, *ii.)*. □

Corollary C.3.33 *Let $(X, \mathfrak{a}, \mu)$ be a measure space and let $1 \leq p, q \leq \infty$. For $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ the pointwise product induces a continuous bilinear map*

$$\mathrm{L}^p(X, \mathfrak{a}, \mu) \times \mathrm{L}^q(X, \mathfrak{a}, \mu) \longrightarrow \mathrm{L}^r(X, \mathfrak{a}, \mu), \tag{C.3.68}$$

obeying the estimate (C.3.67).

In fact, one can also use this continuity statement and the obvious continuity of the complex conjugation to obtain the results from Proposition C.3.25 and Proposition C.3.28, *iv.)*, respectively: the pairing discussed there is the above product followed by the continuous integration.

C.3.3 Integration with Respect to Complex Measures

Having introduced complex measures one may wonder if there is also an integration theory with respect to such measures. This is indeed the case. On the one hand, for a fixed complex measure μ one can define $\mathscr{L}^1$- and also $\mathscr{L}^p$-spaces essentially by means of the *finite* positive measure $|\mu|$ with the help of the Radon-Nikodym Theorem. We shall come back to this option in course of the next subsection. On the other hand, one can restrict the integration theory to bounded measurable functions $\mathscr{BM}(X, \mathfrak{a})$. Since μ and also $|\mu|$ are finite measures, this seems to be reasonable in view of Proposition C.3.30. In particular, $\mathscr{BM}(X, \mathfrak{a})$ will be a common domain on which *all* integrations with respect to complex measures will be defined simultaneously, while the $\mathscr{L}^p$-spaces would depend on one particular, chosen complex measure μ. In this subsection we discuss this second approach.

Let $f \in \mathscr{BM}(X, \mathfrak{a})$ be a complex-valued simple function and $\mu \in \mathrm{Meas}(X, \mathfrak{a})$ a complex measure. We write $f = \sum_{i=1}^n z_i \chi_{A_i}$ in its unique normal form and define

$$\int_X f \,\mathrm{d}\mu = \sum_{i=1}^n z_i \mu(A_i), \tag{C.3.69}$$

which gives a well-defined complex number. The following lemma is then the crucial observation:

Lemma C.3.34 *Let $(X, \mathfrak{a})$ a measurable space. Then the integral* (C.3.69) *defines a bilinear pairing between the complex-valued simple functions and the complex measures on $(X, \mathfrak{a})$. One has*

$$\left| \int_X f \,\mathrm{d}\mu \right| \leq \|f\|_\infty \|\mu\|. \tag{C.3.70}$$

PROOF: Let $z, w \in \mathbb{C}$ and $\mu, \nu \in \mathrm{Meas}(X, \mathfrak{a})$. Then for a simple function f we have

$$\int_X f \,\mathrm{d}(z\mu + w\nu) = \sum_{i=1}^n z_i (z\mu + w\nu)(A_i) = z \sum_{i=1}^n z_i \mu(A_i) + w \sum_{i=1}^n z_i \nu(A_i) = z \int_X f \,\mathrm{d}\mu + w \int_X f \,\mathrm{d}\nu,$$

showing the linearity in the second argument. The linearity in the function argument is shown completely analogously to Lemma C.3.3, *ii.)*, and will be omitted here. It remains to show the estimate (C.3.70). We have

$$\left| \int_X f \,\mathrm{d}\mu \right| \leq \sum_{i=1}^n |z_i| |\mu(A_i)| \leq \sum_{i=1}^n \|f\|_\infty |\mu|(A_i) = \|f\|_\infty |\mu| \left(\bigcup_{i=1}^n A_i \right) = \|f\|_\infty \|\mu\|,$$

since the union of the A_i's in the normal form of f is disjoint and equal to X as well as $\|\mu\| = |\mu|(X)$ by definition. ☐

Using the results from Proposition C.1.25, *ii.)*, and the usual abstract nonsense on dense subspaces and continuous extensions we can immediately extend the integral (C.3.69) from simple functions to arbitrary bounded measurable functions:

Proposition C.3.35 *Let $(X, \mathfrak{a})$ be a measurable space. Then the integral* (C.3.69) *extends uniquely to a continuous bilinear pairing*

$$\int_X : \mathscr{BM}(X, \mathfrak{a}) \times \operatorname{Meas}(X, \mathfrak{a}) \longrightarrow \mathbb{C}, \tag{C.3.71}$$

obeying the estimate

$$\left| \int_X f \, \mathrm{d}\mu \right| \leq \|f\|_\infty \|\mu\|. \tag{C.3.72}$$

Concerning measurable maps the above integration behaves well. We have the following result as already for positive measures:

Proposition C.3.36 *Let $\Phi \colon (X, \mathfrak{a}) \longrightarrow (Y, \mathfrak{b})$ be a measurable map.*

i.) The pull-back $\Phi^* \colon \mathscr{BM}(Y, \mathfrak{b}) \longrightarrow \mathscr{BM}(X, \mathfrak{a})$ is continuous with respect to the supremum norms and we have for every $f \in \mathscr{BM}(Y, \mathfrak{b})$

$$\|\Phi^* f\|_\infty \leq \|f\|_\infty. \tag{C.3.73}$$

ii.) For $\mu \in \operatorname{Meas}(X, \mathfrak{a})$ and $f \in \mathscr{BM}(Y, \mathfrak{b})$ we have

$$\int_X \Phi^* f \, \mathrm{d}\mu = \int_Y f \, \mathrm{d}\Phi_* \mu. \tag{C.3.74}$$

PROOF: For the first part we note that Φ^* maps bounded measurable functions to bounded measurable functions since Φ is measurable. The estimate (C.3.73) is trivial. For the second part, we notice that for characteristic functions and hence for simple functions the equality clearly holds. Now the integrals are continuous with respect to the supremum norms and the variational norms, respectively. Since Φ^* and Φ_* are continuous maps in these topologies, see Corollary C.2.29, we conclude (C.3.74) by the usual continuity and density argument. □

We discuss now the non-degeneracy of the above pairing: if f is a fixed non-zero bounded measurable function we have at least one point $x \in X$ with $f(x) \neq 0$. This allows to choose the δ-measure at x for which we get $\int_X f \, \mathrm{d}\delta_x = f(x) \neq 0$. Conversely, let μ be a non-zero complex measure and $A \in \mathfrak{a}$ a measurable subset with $\mu(A) \neq 0$. Then the simple function $\chi_A \in \mathscr{BM}(X, \mathfrak{a})$ has integral $\int_X \chi_A \, \mathrm{d}\mu = \mu(A) \neq 0$, showing that the pairing is non-degenerate also in the second argument. These considerations also show that the estimate (C.3.72) is optimal: we have equality for $f = \chi_A$ and $\mu = \delta_x$ with some measurable non-empty $A \in \mathfrak{a}$ and $x \in A$.

Remark C.3.37 In the language of Subsection 2.3.3 the bounded measurable functions and the complex measures on $(X, \mathfrak{a})$ form a dual pair with respect to integration. Thus we get the corresponding weak topologies on each of the two spaces coming from this dual pairing.

For a *fixed* complex measure $\mu \in \operatorname{Meas}(X, \mathfrak{a})$ the linear map

$$\int_X \cdot \, \mathrm{d}\mu \colon \mathscr{BM}(X, \mathfrak{a}) \longrightarrow \mathbb{C} \tag{C.3.75}$$

will have a certain degeneracy space and the continuity estimate (C.3.72) might be improved. To this end, we first note that for $A \in \mathfrak{a}$ we know that

$$|\mu|(A) = 0 \quad \text{implies} \quad \mu(A) = 0 \tag{C.3.76}$$

by Proposition C.2.25, *ii.)*. The converse, however, needs not to be true at all. Taking two δ-measures at different points $x \neq y$ the measure $\mu = \delta_x - \delta_y$ provides an easy counterexample: we have $\mu(X) = 0$ while $|\mu|(X) = 2$.

This allows now to define the essential range, the essential supremum and the $\|\cdot\|_{\mu,\infty}$-seminorm with respect to a complex measure by using the positive measure $|\mu|$ and the previous definitions from Definition C.2.16. We set for $f \in \mathscr{BM}(X, \mathfrak{a})$

$$\operatorname{ess\,range}(f) = \left\{ z \in \mathbb{C} \;\middle|\; |\mu|(f^{-1}(\mathrm{B}_\epsilon(z))) > 0 \text{ for all } \epsilon > 0 \right\}, \tag{C.3.77}$$

$$\operatorname*{ess\,sup}_{x \in X} |f(x)| = \sup_{z \in \operatorname{ess\,range}(f)} |z|, \tag{C.3.78}$$

and

$$\|f\|_{\mu,\infty} = \|f\|_{|\mu|,\infty}. \tag{C.3.79}$$

Then we revisit the estimate (C.3.70) for a simple function $f = \sum_{i=1}^n z_i \chi_{A_i}$. We have

$$\left| \int_X f \, \mathrm{d}\mu \right| \leq \sum_{i=1}^n |z_i| |\mu|(A_i) = \sum\nolimits' |z_i| |\mu|(A_i) \leq \|f\|_{|\mu|,\infty} \|\mu\|, \tag{C.3.80}$$

where in the sum $\sum'$ only those i contribute for which $|\mu|(A_i) \neq 0$. Note that for those we indeed have $|z_i| \leq \|f\|_{|\mu|,\infty}$. By the usual density and continuity argument we observe the following statement:

Proposition C.3.38 *Let $(X, \mathfrak{a})$ be measurable space and let $\mu \in \mathrm{Meas}(X, \mathfrak{a})$ be a complex measure on it.*

i.) For all $f \in \mathscr{BM}(X, \mathfrak{a})$ we have

$$\left| \int_X f \, \mathrm{d}\mu \right| \leq \|f\|_{\mu,\infty} \|\mu\|. \tag{C.3.81}$$

ii.) The integral gives a well-defined linear functional

$$\int_X \cdot \, \mathrm{d}\mu \colon \mathrm{L}^\infty(X, \mathfrak{a}, |\mu|) \longrightarrow \mathbb{C}, \tag{C.3.82}$$

obeying the continuity condition

$$\left| \int_X f \, \mathrm{d}\mu \right| \leq \|f\|_{\mu,\infty} \|\mu\|. \tag{C.3.83}$$

PROOF: The first part is clear as (C.3.81) holds on the dense subspace of simple functions. This also shows that if $f = 0$ almost everywhere with respect to $|\mu|$ then $\int_X f \, \mathrm{d}\mu = 0$. Thus the integral is also well-defined on the quotient $\mathrm{L}^\infty(X, \mathfrak{a}, |\mu|)$ and still obeys (C.3.83). Recall that the simple functions form a dense subspace of $\mathrm{L}^\infty(X, \mathfrak{a}, |\mu|)$. ☐

C.3.4 The Lebesgue-Radon-Nikodym Theorem

We are now interested in comparing two measures by investigating their zero sets. To this end, we fix a positive measure μ on a measurable space $(X, \mathfrak{a})$ and consider either another positive measure or a complex measure.

Definition C.3.39 (Absolute continuity and mutual singularity) *Let $(X, \mathfrak{a})$ be a measurable space.*

i.) A positive or complex measure ν on $(X, \mathfrak{a})$ is called absolutely continuous with respect to a positive measure μ if every μ-zero set is also a ν-zero set. In this case we write $\nu \ll \mu$.

ii.) *A positive or complex measure μ is concentrated on $A \in \mathfrak{a}$ if for every $B \in \mathfrak{a}$ we have $\mu(B) = \mu(A \cap B)$.*

iii.) *Let μ, ν be two (positive or complex) measures on $(X, \mathfrak{a})$. Then μ and ν are called mutually singular if there are disjoint subsets $A, B \in \mathfrak{a}$ such that μ is concentrated on A and ν is concentrated on B. In this case we write $\mu \perp \nu$.*

Remark C.3.40 Let $(X, \mathfrak{a})$ be a measurable space. If μ is a positive measure then μ is concentrated on A iff $\mu(X \setminus A) = 0$. Indeed, by monotonicity we have $\mu(B) = \mu(B \cap (X \setminus A)) + \mu(B \cap A) = \mu(B \cap A)$ in the case where $\mu(X \setminus A) = 0$. The converse is clear. Note however, that for a complex or real measure the statement $\mu(X \setminus A) = 0$ does not imply that μ is concentrated on A. In particular, a complex measure with $\mu(X) = 0$ but $\mu \neq 0$ provides a simple counterexample. Equivalently to the definition we can say that μ is concentrated on A iff for all $B \in \mathfrak{a}$ with $A \cap B = \emptyset$ we have $\mu(B) = 0$. Note also that the set A on which a measure might be concentrated will in general not be unique: the δ-measure at $0 \in \mathbb{R}$ is concentrated on e.g. every interval $(-\epsilon, \epsilon)$ for $\epsilon > 0$ as well as on $\{0\}$.

The following propositions clarify how the notion of mutually singular measures behaves with respect to the operations we can perform on (complex) measures:

Proposition C.3.41 *Let $\mu_1, \mu_2, \mu \in \operatorname{Meas}(X, \mathfrak{a})$ be complex measures on a measurable space $(X, \mathfrak{a})$ and let $z, w \in \mathbb{C}$.*

i.) *The zero measure is concentrated on $\emptyset$ and hence mutually singular to any other measure as well as absolutely continuous with respect to any other positive measure.*

ii.) *If μ_i is concentrated on A_i for $i = 1, 2$ then $z\mu_1 + w\mu_2$ is concentrated on $A_1 \cup A_2$.*

iii.) *If μ is concentrated on A then $\overline{\mu}$, $\operatorname{Re}(\mu)$, $\operatorname{Im}(\mu)$, $|\mu|$ as well as $\operatorname{Re}(\mu)_\pm$ and $\operatorname{Im}(\mu)_\pm$ are concentrated on A, too. Conversely, if $|\mu|$ is concentrated on A then so is μ.*

iv.) *If $\mu_1 \perp \mu$ and $\mu_2 \perp \mu$ then $z\mu_1 + w\mu_2 \perp \mu$.*

v.) *If $\mu_1 \perp \mu_2$ then all the measures μ_1, $\overline{\mu_1}$, $\operatorname{Re}(\mu_1)$, $\operatorname{Im}(\mu_1)$, $|\mu_1|$, $\operatorname{Re}(\mu_1)_\pm$, $\operatorname{Im}(\mu_1)_\pm$ are mutually singular to all the measures μ_2, $\overline{\mu_2}$, $\operatorname{Re}(\mu_2)$, $\operatorname{Im}(\mu_2)$, $|\mu_2|$, $\operatorname{Re}(\mu_2)_\pm$, $\operatorname{Im}(\mu_2)_\pm$, respectively.*

Proof: The first part is clear since for the zero measure we have $0(A) = 0$ for all $A \in \mathfrak{a}$. For the second part, let μ_i be concentrated on A_i. Then for every $B \in \mathfrak{a}$ with $B \cap (A_1 \cup A_2) = \emptyset$ we have $B \cap A_i = \emptyset$ and hence $(z\mu_1 + w\mu_2)(B) = z\mu_1(B) + w\mu_2(B) = 0$ by the formulation according to Remark C.3.40. Note that $z\mu_1 + w\mu_2$ can indeed be concentrated on the whole union $A_1 \cup A_2$. For the third part it is clear that $\overline{\mu}$ is concentrated on the same subset A since $\mu(B) = \overline{\overline{\mu}(B)}$ for all $B \in \mathfrak{a}$. Then the second part shows that $\operatorname{Re}(\mu) = \frac{1}{2}(\mu + \overline{\mu})$ as well as $\operatorname{Im}(\mu) = \frac{1}{2\mathrm{i}}(\mu - \overline{\mu})$ are still concentrated on A. Now suppose $B \in \mathfrak{a}$ satisfies $A \cap B = \emptyset$ and $B = \bigcup_{n=1}^\infty B_n$ is a partition. Then first we see that $A \cap B_n = \emptyset$ for all n. Hence $\mu(B_n) = 0$ and thus also $\sum_{n=1}^\infty |\mu(B_n)| = 0$. Since this holds for all partitions of B we conclude that $|\mu|(B) = 0$. Thus $|\mu|$ is concentrated on A, too. But then (C.2.51) gives the results for $\operatorname{Re}(\mu)_\pm$ and $\operatorname{Im}(\mu)_\pm$. Conversely, if $|\mu|$ is concentrated on A and $A \cap B = \emptyset$ then we have $|\mu(B)| \leq |\mu|(B) = 0$. This implies that μ is concentrated on A, too. For the fourth part, let A_i, B_i be disjoint measurable subsets such that μ_i is concentrated on A_i while μ is concentrated on B_i for $i = 1, 2$. Clearly, μ is concentrated on $B_1 \cap B_2$ in this case and then $z\mu_1 + w\mu_2$ is concentrated on $A_1 \cup A_2$ by the second part. Since $(A_1 \cup A_2) \cap (B_1 \cap B_2) = \emptyset$ the claim follows. The last part is then obtained from the third and fourth part. □

Iterating the fourth part of the proposition for $\mu_1, \mu_2 \perp \nu_1, \nu_2$ gives

$$z\mu_1 + w\mu_2 \perp u\nu_1 + v\nu_2 \tag{C.3.84}$$

for all $z, w, u, v \in \mathbb{C}$.

Also the absolute continuity behaves well with respect to the other operations on measures. Here we have the following statements:

Proposition C.3.42 *Let ν be a positive measure on a measurable space $(X, \mathfrak{a})$.*

i.) If $\mu_1 \ll \nu$ and $\mu_2 \ll \nu$ for $\mu_1, \mu_2 \in \operatorname{Meas}(X, \mathfrak{a})$ then $z\mu_1 + w\mu_2 \ll \nu$ for all $z, w \in \mathbb{C}$. Analogously, for positive measures one has $\lambda_1\mu_1 + \lambda_2\mu_2 \ll \nu$ if $\mu_1, \mu_2 \ll \nu$ for all $\lambda_1, \lambda_2 \in \mathbb{R}_0^+$.

ii.) If $\mu \ll \nu$ for $\mu \in \operatorname{Meas}(X, \mathfrak{a})$ then also

$$\overline{\mu}, \operatorname{Re}(\mu), \operatorname{Im}(\mu), |\mu|, \operatorname{Re}(\mu)_\pm, \operatorname{Im}(\mu)_\pm \ll \nu. \tag{C.3.85}$$

iii.) If μ is a positive measure with $\mu \leq \nu$ then $\mu \ll \nu$.

iv.) If μ_1, μ_2 are positive or complex measures with $\mu_1 \ll \nu$ and $\mu_2 \perp \nu$ then $\mu_1 \perp \mu_2$.

v.) If μ is a positive or complex measure with $\mu \ll \nu$ and $\mu \perp \nu$ then $\mu = 0$.

PROOF: Let $A \in \mathfrak{a}$ be a zero set with respect to ν. Then, by assumption, we have $\mu_1(A) = 0 = \mu_2(A)$ and thus $(z\mu_1 + w\mu_2)(A) = 0$, too. For the second part it is clear that $\overline{\mu}, \operatorname{Re}(\mu), \operatorname{Im}(\mu) \ll \nu$. Thus let $A \in \mathfrak{a}$ be a zero set of ν and $A = \bigcup_{n=1}^\infty A_n$ a partition. Since ν is positive also $\nu(A_n) = 0$ for all n. Thus $\sum_{n=1}^\infty |\mu(A_n)| = 0$ follows. Since this holds for all partitions of A we conclude $|\mu|(A) = 0$, showing $|\mu| \ll \nu$. Then $\operatorname{Re}(\mu)_\pm, \operatorname{Im}(\mu)_\pm \ll \nu$ follows from the previous results. The third part is obvious as $0 \leq \mu()A) \leq \nu(A)$ for all $A \in \mathfrak{a}$. For the fourth part, let μ_2 be concentrated on $A \in \mathfrak{a}$ and let ν be concentrated on $B \in \mathfrak{a}$ with $A \cap B = \emptyset$. In particular, $\nu(C) = 0$ for all $C \in \mathfrak{a}$ with $C \subseteq A$. Thus $\mu_1(C) = 0$, too, and hence μ_1 is concentrated on $X \setminus A$. This gives $\mu_1 \perp \mu_2$ as claimed. The last part is then clear since $\mu \perp \mu$ is only possible for $\mu = 0$. ☐

We come now to a construction of measures out of a given positive measure μ which are all absolutely continuous with respect to μ. We have two versions of this construction:

Proposition C.3.43 *Let $(X, \mathfrak{a}, \mu)$ be a measure space with positive measure μ.*

i.) Let $f\colon X \longrightarrow [0, +\infty]$ be a measurable function. Then

$$\nu(A) = \int_A f \, \mathrm{d}\mu \tag{C.3.86}$$

for $A \in \mathfrak{a}$ defines a positive measure ν on $(X, \mathfrak{a})$ with $\nu \ll \mu$. Moreover, for every measurable function $g\colon X \longrightarrow [0, +\infty]$ we have

$$\int_X g \, \mathrm{d}\nu = \int_X gf \, \mathrm{d}\mu. \tag{C.3.87}$$

ii.) Let $f \in \mathrm{L}^1(X, \mathfrak{a}, \mu)$. Then

$$\nu(A) = \int_A f \, \mathrm{d}\mu \tag{C.3.88}$$

for $A \in \mathfrak{a}$ defines a complex measure $\nu \in \operatorname{Meas}(X, \mathfrak{a})$ with $\nu \ll \mu$ such that for $g \in \mathscr{BM}(X, \mathfrak{a})$ we have

$$\int_X g \, \mathrm{d}\nu = \int_X gf \, \mathrm{d}\mu. \tag{C.3.89}$$

PROOF: Let $A_1, A_2, \ldots \in \mathfrak{a}$ be pairwise disjoint. Then pointwise in X we have the convergence

$$\chi_A f = \sum_{n=1}^\infty \chi_{A_n} f \quad \text{with} \quad A = \bigcup_{n=1}^\infty A_n. \tag{$*$}$$

Thus by Proposition C.3.8 we get

$$\int_X \chi_A f \, \mathrm{d}\mu = \int_X \left(\sum_{n=1}^\infty \chi_{A_n} f \right) \mathrm{d}\mu = \sum_{n=1}^\infty \int_X \chi_{A_n} f \, \mathrm{d}\mu = \sum_{n=1}^\infty \nu(A_n).$$

Since the left hand side is by definition $\nu(A)$ the map ν is σ-additive. Clearly, $\nu(\emptyset) = 0$ and hence ν is a positive measure. Now if A is a zero set for μ then $\int_A f \,\mathrm{d}\mu = 0$ by Proposition C.3.9, *iv.)*. Hence $\mu(A) = 0$ implies $\nu(A) = 0$ which is $\nu \ll \mu$. Finally, the claim (C.3.87) clearly holds for every characteristic function $g = \chi_A$ by the very definition of ν. Hence by convexity, it holds for every non-negative simple function. But then it holds for every measurable function $f\colon X \longrightarrow [0, +\infty]$ by monotonous convergence, see Theorem C.3.6, since we can approximate g by simple functions monotonously from below. The second part is essentially analogous: to show the σ-additivity we have to use the dominated convergence of $(*)$ for $f \in \mathrm{L}^1(X, \mathfrak{a}, \mu)$, see Theorem C.3.16, where we use the obvious fact that $\chi_A f$ is dominated by $|f|$ for all $A \in \mathfrak{a}$. Again, $\nu \ll \mu$ is clear. For (C.3.89) we observe that the claim holds for characteristic functions and hence for simple functions by linearity. Then the left hand side is a continuous linear functional on $\mathscr{BM}(X, \mathfrak{a})$ by Proposition C.3.35 while the right hand side is a continuous linear functional even on $\mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ by Proposition C.3.25, *ii.)*. Since $\|g\|_{\infty,\mu} \leq \|g\|_\infty$ we have the same continuity for both sides and hence equality according to the density statement from Proposition C.1.25, *ii.)*. Alternatively, we can again argue with dominated convergence to conclude (C.3.89). Finally, we note that ν only depends on the equivalence class $f \in \mathrm{L}^1(X, \mathfrak{a}, \mu)$ but not on the representative in $\mathscr{L}^1(X, \mathfrak{a}, \mu)$. □

In the above situation we sometimes write symbolically

$$\mathrm{d}\nu = f \,\mathrm{d}\mu \tag{C.3.90}$$

for the relation between ν and μ. The fundamental theorem of Lebesgue-Radon-Nikodym gives now a partial converse of the above statement: as soon as $\nu \ll \mu$ the measure ν is of the form (C.3.88) with some suitable f, up to the following assumptions:

Theorem C.3.44 (Lebesgue-Radon-Nikodym) *Let $(X, \mathfrak{a}, \mu)$ be a σ-finite measure space and $\nu \in \mathrm{Meas}(X, \mathfrak{a})$.*

i.) There exist unique complex measures ν_{ac} and $\nu_{\mathrm{sing}} \in \mathrm{Meas}(X, \mathfrak{a})$ such that

$$\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{sing}}, \quad \nu_{\mathrm{ac}} \ll \mu, \quad \textit{and} \quad \nu_{\mathrm{sing}} \perp \mu. \tag{C.3.91}$$

Moreover, if $\nu = \overline{\nu}$ then also ν_{ac} and ν_{sing} are real measures. If $\nu \geq 0$ is a positive measure then also ν_{ac} and ν_{sing} are positive.

ii.) There exists a unique $f \in \mathrm{L}^1(X, \mathfrak{a}, \mu)$ such that for all $A \in \mathfrak{a}$ one has

$$\nu_{\mathrm{ac}}(A) = \int_A f \,\mathrm{d}\mu. \tag{C.3.92}$$

Proof: Suppose there would be another such decomposition $\tilde{\nu}_{\mathrm{ac}} + \tilde{\nu}_{\mathrm{sing}} = \nu$ with (C.3.91). Then $\lambda = \tilde{\nu}_{\mathrm{ac}} - \nu_{\mathrm{ac}} = \nu_{\mathrm{sing}} - \tilde{\nu}_{\mathrm{sing}}$ as well as $\tilde{\nu}_{\mathrm{ac}} - \nu_{\mathrm{ac}} \ll \mu$ and $\nu_{\mathrm{sing}} - \tilde{\nu}_{\mathrm{sing}} \perp \mu$ by Proposition C.3.42, *i.)*, and Proposition C.3.41, *iv.)*, respectively. But then Proposition C.3.42, *v.)*, gives $\lambda = 0$ and the uniqueness statement follows. Also the uniqueness of f in the second part is clear: the integral with respect to ν_{ac} gives for all $g \in \mathscr{BM}(X, \mathfrak{a})$ the equality

$$\int_X g \,\mathrm{d}\nu_{\mathrm{ac}} = \int_X gf \,\mathrm{d}\mu$$

as we have see this in Proposition C.3.43. Then the non-degeneracy of the pairing in Proposition C.3.25, *iii.)*, determines f up to zero functions, hence $f \in \mathrm{L}^1(X, \mathfrak{a}, \mu)$ is unique.

Thus it is the existence in both cases which is the non-trivial part. To show the existence we rely on an argument of von Neumann [39, Thm. VII] using Hilbert space techniques, see also [48, Thm. 6.10] for this approach.

First we use a function $\chi \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ such that $0 < \chi(x) < 1$ for all $x \in X$, whose existence is guaranteed by Example C.3.17. Since χ is strictly positive everywhere we see that the positive measure $\tilde{\mu}$ with $\mathrm{d}\tilde{\mu} = \chi \,\mathrm{d}\mu$ according to Proposition C.3.43, *i.)*, has the *same* zero sets as μ: indeed, $\tilde{\mu} \ll \mu$ is part of that proposition. Conversely, let $A \in \mathfrak{a}$ with $\tilde{\mu}(A) = 0$ be given, i.e. $\int_A \chi \,\mathrm{d}\mu = 0$. We conclude $\mu(A) = 0$ by Proposition C.3.14, *ii.)*, applied to the measure space $(A, \mathfrak{a}\big|_A, \mu\big|_A)$. Thus also $\mu \ll \tilde{\mu}$. Note that the positive measure $\tilde{\mu}$ is now a *finite* measure with

$$\tilde{\mu}(X) = \int_X \chi \,\mathrm{d}\mu = \|\chi\|_{\mu,1} < \infty.$$

Now we first assume that ν is even a positive finite measure $\nu \in \mathrm{Meas}^+(X, \mathfrak{a})$. Then $\lambda = \nu + \tilde{\mu}$ is still a finite positive measure with $\nu \le \lambda$. Thus for every $\psi \in \mathrm{L}^1(X, \mathfrak{a}, \lambda)$ we have $\psi \in \mathrm{L}^1(X, \mathfrak{a}, \nu)$ and

$$\left| \int_X \psi \,\mathrm{d}\nu \right| \le \int_X |\psi| \,\mathrm{d}\nu \le \int_X |\psi| \,\mathrm{d}\lambda = \|\psi\|_{\lambda,1}.$$

This shows that

$$\psi \mapsto \int_X \psi \,\mathrm{d}\nu \tag{$*$}$$

is a continuous linear functional on $\mathrm{L}^1(X, \mathfrak{a}, \lambda)$. Since λ is a *finite* positive measure, the Hilbert space $\mathrm{L}^2(X, \mathfrak{a}, \lambda)$ is continuously included into $\mathrm{L}^1(X, \mathfrak{a}, \lambda)$ according to Corollary C.3.31. Hence, the restriction of $(*)$ to $\mathrm{L}^2(X, \mathfrak{a}, \lambda)$ is still well-defined and continuous. Thus there is a function $\overline{\phi} \in \mathrm{L}^2(X, \mathfrak{a}, \lambda)$ with

$$\int_X \psi \,\mathrm{d}\nu = \langle \overline{\phi}, \psi \rangle_{\mathrm{L}^2} = \int_X \phi\psi \,\mathrm{d}\lambda$$

according to the Riesz Representation Theorem 3.2.11. As a function on X, this ϕ is only determined up to a zero set with respect to λ. Since $\int_X \overline{\psi} \,\mathrm{d}\nu = \overline{\int_X \psi \,\mathrm{d}\nu}$ we can choose $\phi = \overline{\phi}$ to be real. Moreover, taking $\psi = \chi_A \in \mathrm{L}^2(X, \mathfrak{a}, \lambda)$ we get

$$\nu(A) = \int_X \chi_A \,\mathrm{d}\nu = \int_X \chi_A \phi \,\mathrm{d}\lambda = \int_A \phi \,\mathrm{d}\lambda. \tag{$**$}$$

We choose a representative ϕ and consider $A_n = \{x \in X \mid \phi(x) \ge 1 + \frac{1}{n}\} \in \mathfrak{a}$. Then $(**)$ gives $\nu(A_n) = \int_{A_n} \phi \,\mathrm{d}\lambda \ge (1 + \frac{1}{n})\lambda(A_n)$. But $\nu \le \lambda$, hence $\nu(A_n) = \lambda(A_n) = 0$ follows. This shows that $A = \bigcup_{n=1}^\infty A_n = \{x \in X \mid \phi(x) > 1\}$ is a set of measure zero with respect to λ (and also with respect to ν). Thus we have $\phi \le 1$ almost everywhere. Analogously, we consider the measurable subsets $B_n = \{x \in X \mid \phi(x) < -\frac{1}{n}\}$ and get

$$0 \le \nu(B_n) = \int_{B_n} \phi \,\mathrm{d}\lambda \le -\frac{1}{n}\lambda(B_n),$$

which gives $\lambda(B_n) = 0 = \nu(B_n)$. Hence the B_n as well as their union are zero sets with respect to λ. Thus, without restriction, we can choose the representative ϕ in such a way that $\phi\colon X \longrightarrow [0, 1]$. Thus we have achieved that $\mathrm{d}\nu = \phi \,\mathrm{d}\lambda = \phi \,\mathrm{d}\nu + \phi\chi \,\mathrm{d}\mu$ or

$$(1 - \phi) \,\mathrm{d}\nu = \phi\chi \,\mathrm{d}\mu \tag{☺}$$

in the sense of the abbreviation (C.3.90).

The idea is now to divide by $(1 - \phi)$ to get the function f. Of course, $\phi(x) \in [0, 1]$ but $\phi(x) = 1$ is actually possible. This suggest to treat those points separately. We define

$$X_{\mathrm{ac}} = \big\{x \in X \bigm| 0 \le \phi(x) < 1\big\} \quad \text{and} \quad X_{\mathrm{sing}} = \big\{x \in X \bigm| \phi(x) = 1\big\} = X \setminus X_{\mathrm{ac}}.$$

We can now define measures ν_{ac} and ν_{sing} by setting

$$\nu_{\mathrm{ac}}(A) = \nu(X_{\mathrm{ac}} \cap A) \quad \text{and} \quad \nu_{\mathrm{sing}}(A) = \nu(X_{\mathrm{sing}} \cap A).$$

Indeed, it is easy to see that these are finite positive measures with $\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{sing}}$ since $X = X_{\mathrm{ac}} \cup X_{\mathrm{sing}}$ is a disjoint union. From (☺) we see that $0 = \chi_{X_{\mathrm{sing}}}(1-\phi)$ gives

$$0 = \int_X \chi_{X_{\mathrm{sing}}}(1-\phi)\,\mathrm{d}\nu = \int_X \chi_{X_{\mathrm{sing}}}\phi\chi\,\mathrm{d}\mu = \int_{X_{\mathrm{sing}}} \chi\,\mathrm{d}\mu.$$

Since $\chi > 0$ everywhere we conclude, by the same argument as before, that X_{sing} is a zero set with respect to μ. On the other hand, it is clear by construction that ν_{sing} is concentrated on X_{sing} and hence $\nu_{\mathrm{sing}} \perp \mu$ follows. On X_{ac} we define now $f = \frac{\phi\chi}{1-\phi}$ and extend f to all of X by setting e.g. $f\big|_{X_{\mathrm{sing}}} = 0$. This gives a measurable function and we claim that f is integrable with respect to μ such that (C.3.92) holds. Using the geometric series we get pointwise monotonous convergence

$$f = \phi\chi\sum_{n=0}^{\infty}\phi^n$$

since $0 \le \phi < 1$ on X_{ac}. Now we integrate (☺) for the finite sum and a characteristic function χ_A. This gives

$$\int_X \chi_A(1+\phi+\cdots+\phi^n)(1-\phi)\,\mathrm{d}\nu = \int_X \chi_A(1+\phi+\cdots+\phi^n)\phi\chi\,\mathrm{d}\mu. \tag{☺☺}$$

Since $(1-\phi) = 0$ on X_{sing}, only $A \cap X_{\mathrm{ac}}$ contributes to the left hand side which is then $\int_{X_{\mathrm{ac}}\cap A}(1-\phi^{n+1})\,\mathrm{d}\nu$. The integrand converges monotonously to 1 and hence by monotonous convergence we get

$$\lim_{n\longrightarrow\infty}\int_{X_{\mathrm{ac}}\cap A}(1-\phi^{n+1})\,\mathrm{d}\nu = \int_{X_{\mathrm{ac}}\cap A}\mathrm{d}\nu = \nu_{\mathrm{ac}}(A).$$

On the other hand, the right hand side of (☺☺) has a monotonously increasing integrand converging pointwise to f on $X_{\mathrm{ac}} \cap A$ and diverging on $X_{\mathrm{sing}} \cap A$. Since X_{sing} is a set of measure zero with respect to μ, the right hand side converges, too, by monotonous convergence, to

$$\lim_{n\longrightarrow\infty}\int_X \chi_A(1+\phi+\cdots+\phi^n)\phi\chi\,\mathrm{d}\mu = \lim_{n\longrightarrow\infty}\int_{X_{\mathrm{ac}}} \chi_A(1+\phi+\cdots+\phi^n)\phi\chi\,\mathrm{d}\mu = \int_{X_{\mathrm{ac}}\cap A} f\,\mathrm{d}\mu = \int_A f\,\mathrm{d}\mu.$$

This establishes (C.3.92). Moreover, by taking $A = X$ we get $\nu_{\mathrm{ac}}(X) = \int_X f\,\mathrm{d}\mu = \|f\|_{\mu,1}$. Since ν_{ac} was a finite positive measure, $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ follows. Moreover, $\nu_{\mathrm{ac}} \ll \mu$ is clear by Proposition C.3.43. This shows the theorem for a finite positive measure ν. The general case is obtained by applying the previous construction for the positive and negative part of the real and imaginary part of ν. This way, the additional statements are build into the construction. □

Definition C.3.45 (Lebesgue decomposition) *The decomposition of* $\nu \in \mathrm{Meas}(X, \mathfrak{a})$ *into* $\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{sing}}$ *with respect to* μ *is called the Lebesgue decomposition of* ν*.*

Remark C.3.46 (Lebesgue-Radon-Nikodym Theorem) Let μ be a σ-finite positive measure on $(X, \mathfrak{a})$.

i.) The first part is referred to as the Lebesgue Theorem, the second part of Theorem C.3.44 is referred to as the Radon-Nikodym Theorem: for $\nu \in \mathrm{Meas}(X, \mathfrak{a})$ with $\nu \ll \mu$, in which case we have $\nu_{\mathrm{sing}} = 0$ and $\nu = \nu_{\mathrm{ac}}$, there is a *density* $f \in \mathrm{L}^1(X, \mathfrak{a}, \mu)$ for ν with respect to μ, i.e.

$$\nu(A) = \int_A f\,\mathrm{d}\mu \quad \text{for all} \quad A \in \mathfrak{a}. \tag{C.3.93}$$

In this case one also writes symbolically

$$f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}, \tag{C.3.94}$$

and calls f the *Radon-Nikodym derivative* of ν with respect to μ.

ii.) If ν is a positive but no longer finite measure then one still obtains an analogous statement if ν is at least *σ-finite*: if $X = \bigcup_{n=1}^{\infty} A_n$ with $\nu(A_n) < +\infty$ we can apply Theorem C.3.44 to each $\nu\big|_{A_n}$ and $\mu\big|_{A_n}$. This gives positive measures $\nu_{\mathrm{ac}}^{(n)}$ and $\nu_{\mathrm{sing}}^{(n)}$ with $\nu_{\mathrm{ac}}^{(n)} \ll \mu\big|_{A_n}$ and $\nu_{\mathrm{sing}}^{(n)} \perp \mu\big|_{A_n}$ and functions $0 \le f_n \in \mathrm{L}^1\Big(A_n, \mathfrak{a}\big|_{A_n}, \mu\big|_{A_n}\Big)$ such that

$$\nu\big|_{A_n} = \nu_{\mathrm{ac}}^{(n)} + \nu_{\mathrm{sing}}^{(n)} \quad \text{and} \quad \nu_{\mathrm{ac}}^{(n)} = f_n \mu\big|_{A_n}. \tag{C.3.95}$$

By uniqueness it is easy to see that the local measures glue together to ν_{ac} and ν_{sing}. Moreover, one has $f_n\big|_{A_n \cap A_m} = f_m\big|_{A_n \cap A_m}$ which yields a globally defined function $0 \le f \in \mathscr{M}(X, \mathfrak{a})$ with

$$\mathrm{d}\nu_{\mathrm{ac}} = f\, \mathrm{d}\mu. \tag{C.3.96}$$

Thus the only point which may (and in general will) fail in the statement is that f is integrable: this is already the case for $\nu = \mu$ and an infinite measure $\mu(X) = +\infty$. In this case $f = 1$, which is not integrable.

iii.) There are counterexamples showing that in general the Lebesgue-Radon-Nikodym Theorem can not be extended beyond the σ-finite case. Indeed, the infinite measure μ_{infinite} on a non-empty set $(X, \mathfrak{a})$ provides a quick counter-example: every other measure on X is absolutely continuous since the only zero set of μ_{infinite} is the empty set. Moreover, the only integrable function is $f = 0$. Thus we the only complex measure ν on X by which we can recover by (C.3.92) is the zero measure $\nu = 0$.

iv.) For a complex measure $\nu \in \mathrm{Meas}(X, \mathfrak{a})$ one has

$$\overline{\nu}_{\mathrm{ac/sing}} = \overline{\nu_{\mathrm{ac/sing}}},\ \mathrm{Re}(\nu)_{\mathrm{ac/sing}} = \mathrm{Re}(\nu_{\mathrm{ac/sing}}),\ \text{and}\ \mathrm{Im}(\nu)_{\mathrm{ac/sing}} = \mathrm{Im}(\nu_{\mathrm{ac/sing}}), \tag{C.3.97}$$

as well as

$$|\nu|_{\mathrm{ac/sing}} = |\nu_{\mathrm{ac/sing}}|,\ (\mathrm{Re}(\nu)_{\pm})_{\mathrm{ac/sing}} = \mathrm{Re}(\nu_{\mathrm{ac/sing}})_{\pm},\ \text{and}\ (\mathrm{Im}(\nu)_{\pm})_{\mathrm{ac/sing}} = \mathrm{Im}(\nu_{\mathrm{ac/sing}})_{\pm}. \tag{C.3.98}$$

In fact, this was part of the construction of $\nu_{\mathrm{ac/sing}}$ for complex measures in the proof of the theorem.

v.) Suppose $\nu_1, \nu_2 \in \mathrm{Meas}(X, \mathfrak{a})$ are both absolutely continuous with respect to a positive measure μ. Then also $z\nu_1 + w\nu_2$ is absolutely continuous according to Proposition C.3.42, *i.)*. It follows directly from the uniqueness of the Radon-Nikodym derivative that we have

$$\frac{\mathrm{d}(z\nu_1 + w\nu_2)}{\mathrm{d}\mu} = z\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} + w\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}. \tag{C.3.99}$$

We come now to some first applications of the Lebesgue-Radon-Nikodym Theorem. First, we get the polar decomposition of a complex measure:

Theorem C.3.47 (Polar decomposition) *Let $\mu \in \mathrm{Meas}(X, \mathfrak{a})$ be a complex measure on $(X, \mathfrak{a})$. Then there exists a function $u \in \mathscr{L}^1(X, \mathfrak{a}, |\mu|)$ with $u\overline{u} = 1$ everywhere such that*

$$\mathrm{d}\mu = u\, \mathrm{d}|\mu|. \tag{C.3.100}$$

Proof: First recall that $|\mu|$ is a finite positive measure and hence σ-finite. From the defining properties of $|\mu|$ it is clear that $\mu \ll |\mu|$, see (C.3.76). Thus Theorem C.3.44 gives $\mu = \mu_{\mathrm{ac}}$ with respect to $|\mu|$ and a unique equivalence class $u \in \mathrm{L}^1(X, \mathfrak{a}, |\mu|)$ with $\mathrm{d}\mu = u\,\mathrm{d}|\mu|$. We have to show that we have a representative with $|u| = 1$. First we consider

$$A_n = \left\{ x \in X \;\middle|\; |u(x)| \leq 1 - \tfrac{1}{n} \right\}.$$

Suppose $A_n = \bigcup_{m=1}^{\infty} B_{nm}$ is partition of A_n then

$$\sum_{m=1}^{\infty} |\mu(B_{nm})| = \sum_{m=1}^{\infty} \left| \int_{B_{nm}} u\,\mathrm{d}|\mu| \right| \leq \sum_{m=1}^{\infty} \int_{B_{nm}} \left(1 - \frac{1}{n}\right) \mathrm{d}|\mu| = \left(1 - \frac{1}{n}\right) |\mu|(A).$$

The supremum of the left hand side over all such partitions is $|\mu|(A)$. Hence we conclude that $|\mu|(A_n) = 0$ and thus A_n is a zero set with respect to $|\mu|$. This shows that the points with $|u(x)| < 1$ form a zero set as well. For the points with $|u(x)| > 1$ we use the following lemma which is also of independent interest:

Lemma C.3.48 *Let $(X, \mathfrak{a}, \mu)$ be a measure space and $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$. Then*

$$\left| \int_A f\,\mathrm{d}\mu \right| \leq \mu(A) \quad \textit{for all} \quad A \in \mathfrak{a} \tag{C.3.101}$$

iff $|f| \leq 1$ almost everywhere.

Proof: First we assume (C.3.101). Fix a point $z \in \mathbb{C}$ with $|z| > 1$ and $\epsilon > 0$ such that $|z| - \epsilon > 1$. Then we consider $f^{-1}(\mathrm{B}_\epsilon(z)^{\mathrm{cl}})$, i.e. those points $x \in X$ with $f(x) \in \mathrm{B}_\epsilon(z)^{\mathrm{cl}}$. We claim that this is a zero set. Indeed, we have

$$\begin{aligned}
\left| \int_{f^{-1}(\mathrm{B}_\epsilon(z)^{\mathrm{cl}})} f\,\mathrm{d}\mu - |z| \mu\big(f^{-1}\big(\mathrm{B}_\epsilon(z)^{\mathrm{cl}}\big)\big) \right| &= \left| \int_{f^{-1}(\mathrm{B}_\epsilon(z)^{\mathrm{cl}})} (f - z)\,\mathrm{d}\mu \right| \\
&\leq \int_{f^{-1}(\mathrm{B}_\epsilon(z)^{\mathrm{cl}})} |f - z|\,\mathrm{d}\mu \\
&\leq \epsilon \mu\big(f^{-1}\big(\mathrm{B}_\epsilon(z)^{\mathrm{cl}}\big)\big).
\end{aligned}$$

On the other hand, by (C.3.101) we see that the integral of f over $f^{-1}(\mathrm{B}_\epsilon(z)^{\mathrm{cl}})$ has an absolute value $\leq \mu(f^{-1}(\mathrm{B}_\epsilon(z)^{\mathrm{cl}}))$. Hence the left hand side is at least $(|z| - 1)\mu(f^{-1}(\mathrm{B}_\epsilon(z)^{\mathrm{cl}}))$ since $|z| > 1$ by assumption. Then we get a contradiction to $|z| - \epsilon > 1$ unless $\mu(f^{-1}(\mathrm{B}_\epsilon(z)^{\mathrm{cl}})) = 0$. Now the set of points where $|f(x)| > 1$ can be covered by countable many sets of the form $f^{-1}(\mathrm{B}_\epsilon(z)^{\mathrm{cl}})$. Therefore it is also a set of measure zero showing $|f| \leq 1$ almost everywhere. The converse is trivial. ▽

Back to the proof of the polar decomposition we see that for all $A \in \mathfrak{a}$ we have

$$\left| \int_A u\,\mathrm{d}|\mu| \right| = |\mu(A)| \leq |\mu|(A),$$

and hence $|u| \leq 1$ almost everywhere with respect to $|\mu|$ according to Lemma C.3.48. Thus the points where $|u| \neq 1$ form a subset of measure 0 and we can change u to 1 on this subset without disturbing (C.3.100). □

Corollary C.3.49 *Let $(X, \mathfrak{a}, \mu)$ be a measure space and $f \in \mathrm{L}^1(X, \mathfrak{a}, \mu)$. For the complex measure $\nu \in \mathrm{Meas}(X, \mathfrak{a})$ with $\mathrm{d}\nu = f\,\mathrm{d}\mu$ as in Proposition C.3.43, ii.), one has*

$$\mathrm{d}|\nu| = |f|\,\mathrm{d}\mu. \tag{C.3.102}$$

PROOF: Let $u \in \mathscr{L}^1(X, \mathfrak{a}, |\nu|)$ be the phase function in the polar decomposition $\mathrm{d}\nu = u \, \mathrm{d}|\nu|$ of ν according to Theorem C.3.47. Since $u\overline{u} = 1$ we have $u \in \mathscr{BM}(X, \mathfrak{a})$ and hence by Proposition C.3.43, *ii.)*,

$$|\nu(A)| = \int_A \mathrm{d}|\nu| = \int_A \overline{u}u \, \mathrm{d}|\nu| = \int_A \overline{u} \, \mathrm{d}\nu = \int_A \overline{u}f \, \mathrm{d}\mu$$

for all $A \in \mathfrak{a}$. Thus $\int_A \overline{u}f \, \mathrm{d}\mu \geq 0$ for all $A \in \mathfrak{a}$ which implies that $\overline{u}f \geq 0$ almost everywhere with respect to μ by the usual decomposition of $\overline{u}f$ into positive and negative parts of its real and imaginary part. Thus $\overline{u}f = |f|$ almost everywhere which shows that claim. ☐

While for a general complex measure μ the total variation $|\mu|$ is usually hard to compute this corollary simplifies things drastically for complex measures of the form $\mathrm{d}\nu = f \, \mathrm{d}\mu$ with $f \in \mathrm{L}^1(X, \mathfrak{a}, \mu)$ and a positive measure μ.

One of the important applications of the polar decomposition is that we can now define integration with respect to complex measures for more than just bounded measurable functions as we did that before in Subsection C.3.3.

Definition C.3.50 (Integral with complex measure) *Let $\mu \in \mathrm{Meas}(X, \mathfrak{a})$ be a complex measure on $(X, \mathfrak{a})$. Then we define the integrable functions with respect to μ by $\mathscr{L}^1(X, \mathfrak{a}, \mu) = \mathscr{L}^1(X, \mathfrak{a}, |\mu|)$ and set*

$$\int_X f \, \mathrm{d}\mu = \int_X f u \, \mathrm{d}|\mu| \tag{C.3.103}$$

for $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ where u is the phase function from the polar decomposition $\mathrm{d}\mu = u \, \mathrm{d}|\mu|$ of μ. Moreover, $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ is equipped with the seminorm $\|\cdot\|_{\mu,1}$ defined for $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ by

$$\|f\|_{\mu,1} = \|f\|_{|\mu|,1}. \tag{C.3.104}$$

Proposition C.3.51 *Let $\mu \in \mathrm{Meas}(X, \mathfrak{a})$ be a complex measure on $(X, \mathfrak{a})$.*

i.) The integral (C.3.103) is a well-defined linear functional on $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ such that

$$\left| \int_X f \, \mathrm{d}\mu \right| \leq \|f\|_{\mu,1}. \tag{C.3.105}$$

ii.) We have the continuous inclusion

$$\mathscr{BM}(X, \mathfrak{a}) \longrightarrow \mathscr{L}^1(x, \mathfrak{a}, \mu), \tag{C.3.106}$$

and (C.3.103) extends the integral as defined in Proposition C.3.35.

iii.) We have

$$\mathscr{L}^1(X, \mathfrak{a}, \mu) = \mathscr{L}^1(X, \mathfrak{a}, \mathrm{Re}(\mu)_+) \cap \mathscr{L}^1(X, \mathfrak{a}, \mathrm{Re}(\mu)_-) \cap \mathscr{L}^1(X, \mathfrak{a}, \mathrm{Im}(\mu)_+) \cap \mathscr{L}^1(X, \mathfrak{a}, \mathrm{Im}(\mu)_-). \tag{C.3.107}$$

iv.) For $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ one has

$$\int_X f \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\,\mathrm{Re}(\mu)_+ - \int_X f \, \mathrm{d}\,\mathrm{Re}(\mu)_- + \mathrm{i} \int_X f \, \mathrm{d}\,\mathrm{Im}(\mu)_+ - \mathrm{i} \int_X f \, \mathrm{d}\,\mathrm{Im}(\mu)_-. \tag{C.3.108}$$

v.) Let $\Phi \colon (X, \mathfrak{a}) \longrightarrow (Y, \mathfrak{b})$ be a measurable map and $f \in \mathscr{M}(Y, \mathfrak{b})$. Then $\Phi^* f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ implies $f \in \mathscr{L}^1(Y, \mathfrak{b}, \Phi_* \mu)$ and

$$\int_X \Phi^* f \, \mathrm{d}\mu = \int_Y f \, \mathrm{d}\Phi_* \mu. \tag{C.3.109}$$

PROOF: Since $\overline{u}u = 1$ we have $|f| = |uf|$. Hence $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ iff $uf \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ and thus the right hand side of (C.3.103) is a well-defined integral. Clearly, $f \mapsto uf$ is a linear functional on $\mathscr{L}^1(X, \mathfrak{a}, |\mu|)$ obeying the estimate (C.3.105). For the second part we note that $|\mu|$ is a finite measure and hence $\mathscr{BM}(X, \mathfrak{a})$ is continuously included into $\mathscr{L}^1(X, \mathfrak{a}, |\mu|)$ with the estimate

$$\left|\int_X uf \,\mathrm{d}|\mu|\right| \leq \int_X |f| \,\mathrm{d}|\mu| \leq \|f\|_\infty |\mu|(X).$$

On characteristic and hence on simple functions the two definitions of the integral (C.3.103) and (C.3.69) clearly coincide. Then the continuity with respect to $\|\cdot\|_\infty$ shows that they coincide on all of $\mathscr{BM}(X, \mathfrak{a})$. For the third part we observe that by Proposition C.2.30 we have $\mathrm{Re}(\mu)_\pm, \mathrm{Im}(\mu)_\pm \leq |\mu|$. Thus for every characteristic and hence also for every non-negative simple function g we have

$$\int_X g \,\mathrm{d}\,\mathrm{Re}(\mu)_\pm, \int_X g \,\mathrm{d}\,\mathrm{Im}(\mu)_\pm \;\leq\; \int_X g \,\mathrm{d}|\mu|.$$

Taking the corresponding supremum over all non-negative simple functions g with $g \leq |f|$ gives

$$\int_X |f| \,\mathrm{d}\,\mathrm{Re}(\mu)_\pm, \int_X |f| \,\mathrm{d}\,\mathrm{Im}(\mu)_\pm \;\leq\; \int_X |f| \,\mathrm{d}|\mu|. \tag{$*$}$$

Hence $f \in \mathscr{L}^1(X, \mathfrak{a}, |\mu|)$ implies that f is in the intersection on the right hand side of (C.3.107). Conversely, for $A \in \mathfrak{a}$ we have

$$\mathrm{Re}(\mu)_+(A) + \mathrm{Re}(\mu)_-(A) + \mathrm{Im}(\mu)_+(A) + \mathrm{Im}(\mu)_-(A) \geq |\mathrm{Re}(\mu)(A)| + |\mathrm{Im}(\mu)(A)| \geq |\mu(A)|.$$

By the minimality property of $|\mu|$ according to Proposition C.2.25, *iii.)*, we obtain

$$|\mu|(A) \leq \mathrm{Re}(\mu)_+(A) + \mathrm{Re}(\mu)_-(A) + \mathrm{Im}(\mu)_+(A) + \mathrm{Im}(\mu)_-(A).$$

From this we get with an analogous argument the opposite inclusion in (C.3.107) and hence equality. Part *iv.)* is now simple since for a characteristic function and hence a simple function (C.3.108) is true by the linearity of $\int_X$ in the measure argument according to Proposition C.3.35 and $\mu = \mathrm{Re}(\mu)_+ - \mathrm{Re}(\mu)_- + \mathrm{i}\,\mathrm{Im}(\mu)_+ - \mathrm{i}\,\mathrm{Im}(\mu)_-$. Now $(*)$ shows that all the four integrals on the right hand side are continuous linear functionals with respect to $\|\cdot\|_{|\mu|,1}$. Thus the usual continuity and density argument based on Proposition C.3.28, *iii.)*, gives the equality of (C.3.108) for all integrable functions. Finally, consider $\Phi^* f$. By the estimate (C.2.43) from Remark C.2.26 and the equality (C.3.48) from Proposition C.3.18, *ii.)*, we get for the positive measures $|\mu|$, $|\Phi_*\mu|$, and $\Phi_*|\mu|$ the (in-)equalities

$$\|\Phi^* f\|_{\mu,1} = \int_X |\Phi^* f| \,\mathrm{d}|\mu| = \int_X \Phi^*|f| \,\mathrm{d}|\mu| = \int_Y |f| \,\mathrm{d}\Phi_*|\mu| \geq \int_Y |f| \,\mathrm{d}|\Phi_*\mu| = \|f\|_{\Phi_*\mu,1} \tag{$**$}$$

in $[0, +\infty]$. Thus if $\|\Phi^* f\|_{1,\mu}$ is finite then $\|f\|_{1,\Phi_*\mu}$ is also finite. This shows the first statement of *v.)*. The integrals (C.3.109) coincide for $f \in \mathscr{BM}(Y, \mathfrak{b})$ according to Proposition C.3.36, but unfortunately the estimate $(**)$ does not give the correct continuity property right away to conclude (C.3.109) for general f. Suppose that $f \in \mathscr{L}^1(Y, \mathfrak{b}, \Phi_*\mu)$ satisfies $\Phi^* f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$. Then we know that $f \in \mathscr{L}^1(Y, \mathfrak{b}, \Phi_*|\mu|)$ by $(**)$ and hence we can approximate f by $f_n \in \mathscr{BM}(Y, \mathfrak{b})$ with respect to $\|\cdot\|_{\Phi_*|\mu|,1}$. From $(**)$ we know that $f_n \longrightarrow f$ also with respect to $\|\cdot\|_{\Phi_*\mu,1}$. Conversely, we know that $\|f\|_{\Phi_*|\mu|,1} = \|\Phi^* f\|_{|\mu|,1}$ by Proposition C.3.18, *ii.)*, applied to the *positive* measure $|\mu|$ and its push-forward. Then we conclude that both sides of (C.3.109) allow the necessary estimates with respect to the seminorms $\|\cdot\|_{|\mu|,1}$ and $\|\cdot\|_{\Phi_*|\mu|,1}$ to conclude the claimed equality by passing from the f_n to f. □

Analogously, one can define p-integrable functions with respect to a complex measure and transfer all the results from Subsection C.3.2 also to this situation. Also, we can pass from one complex measure μ to another one using an integrable function $f \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ by setting

$$\nu(A) = \int_A f \, \mathrm{d}\mu \tag{C.3.110}$$

analogous to the case where μ is positive, see Proposition C.3.43, *ii.)*. Since with $\mathrm{d}\mu = u \, \mathrm{d}|\mu|$ we get $\mathrm{d}\nu = fu \, \mathrm{d}|\mu|$ we can transfer all properties from the case where μ is positive also to this situation.

A next application of the Lebesgue-Radon-Nikodym Theorem is the following characterization of the decomposition

$$\mu = \mu_+ - \mu_- \tag{C.3.111}$$

of a real measure into its positive and negative part as in Proposition C.2.30.

Theorem C.3.52 (Hahn and Jordan decomposition) *Let $\mu \in \mathrm{Meas}(X, \mathfrak{a})$ be a real measure.*

i.) There exist measurable subsets $X_+, X_- \in \mathfrak{a}$ such that

$$X = X_+ \cup X_- \quad \textit{and} \quad X_+ \cap X_- = \emptyset, \tag{C.3.112}$$

such that $\mu_\pm$ is concentrated on $X_\pm$. Hence $\mu_+ \perp \mu_-$ and, moreover, for all $A \in \mathfrak{a}$

$$\mu_\pm(A) = \pm\mu(X_\pm \cap A). \tag{C.3.113}$$

ii.) If $\lambda_\pm \in \mathrm{Meas}(X, \mathfrak{a})$ are positive with $\mu = \lambda_+ - \lambda_-$ then $\lambda_\pm \geq \mu_\pm$.

PROOF: We use the polar decomposition $\mu = u \, \mathrm{d}|\mu|$ according to Theorem C.3.47. Since $\mu = \overline{\mu}$ is assumed to be real it follows from the construction of u as in Theorem C.3.44 that $u = \overline{u}$ almost everywhere. Redefining u if necessary on a subset of measure zero we get $u = \overline{u}$ and $\overline{u}u = 1$ everywhere. Thus $u(x) = \pm 1$ are the only possible values. We set $X_\pm = u^{-1}(\{\pm 1\})$. With other words, $u_\pm = \frac{1}{2}(1 \pm u) = \chi_{X_\pm}$. Then (C.3.112) is obvious. Now let $A \in \mathfrak{a}$ be given then

$$\begin{aligned}
\mu_\pm(A) &= \frac{1}{2}(|\mu|(A) \pm \mu(A)) \\
&= \frac{1}{2} \int_X \chi_A \, \mathrm{d}|\mu| \pm \frac{1}{2} \int_X \chi_A u \, \mathrm{d}|\mu| \\
&= \int_X \chi_A \chi_{X_\pm} \, \mathrm{d}|\mu| \\
&= \int_{X_\pm} \chi_A \, \mathrm{d}|\mu| \\
&= \pm \int_{X_\pm} \chi_A u \, \mathrm{d}|\mu| \\
&= \pm\mu(X_\pm \cap A),
\end{aligned}$$

since on $X_\pm$ the function u is constant ± 1. On the one hand, this shows (C.3.113). On the other hand, it follows that $\mu_\pm$ is concentrated on $X_\pm$ which are disjoint. Hence $\mu_+ \perp \mu_-$ follows as well. For the second part let $\lambda_\pm$ be another choice of positive measures with $\mu = \lambda_+ - \lambda_-$. Then $\pm\mu \leq \lambda_\pm$ and hence

$$\mu_\pm(A) = \pm\mu(X_\pm \cap A) \leq \lambda_\pm(X_\pm \cap A) \leq \lambda_\pm(A)$$

shows $\mu_\pm \leq \lambda_\pm$. ☐

Remark C.3.53 Let $\mu = \overline{\mu} \in \operatorname{Meas}(X, \mathfrak{a})$ be a real measure. The decomposition $X = X_+ \cup X_-$ is called the *Hahn decomposition*. As one easily verifies, the subsets $X_\pm$ are uniquely determined by the properties in Theorem C.3.52, *i.)*, up to zero sets with respect to $|\mu|$. The decomposition $\mu = \mu_+ - \mu_-$ is called the *Jordan decomposition*.

The last application of the Lebesgue-Radon-Nikodym Theorem is the computation of the dual spaces of the Banach spaces $\mathrm{L}^p(X, \mathfrak{a}, \mu)$. Here we first observe the following lemma:

Lemma C.3.54 *Let $(X, \mathfrak{a}, \mu)$ be a σ-finite measure space and let $\chi \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ satisfy $\chi > 0$. Moreover, let $\tilde{\mu} \in \operatorname{Meas}^+(X, \mathfrak{a})$ be the finite positive measure with $\mathrm{d}\tilde{\mu} = \chi \, \mathrm{d}\mu$. Then for $1 \leq p < \infty$ the map*

$$\mathscr{L}^p(X, \mathfrak{a}, \tilde{\mu}) \ni f \;\mapsto\; \chi^{\frac{1}{p}} f \in \mathscr{L}^p(X, \mathfrak{a}, \mu) \tag{C.3.114}$$

is a well-defined bijective linear map inducing an isometric isomorphism of Banach spaces

$$\mathrm{L}^p(X, \mathfrak{a}, \tilde{\mu}) \longrightarrow \mathrm{L}^p(X, \mathfrak{a}, \mu). \tag{C.3.115}$$

Proof: First let $f \in \mathscr{L}^p(X, \mathfrak{a}, \tilde{\mu})$ be given. Then

$$\left\|\chi^{\frac{1}{p}} f\right\|_{\mu,p}^p = \int_X \left|\chi^{\frac{1}{p}} f\right|^p \mathrm{d}\mu = \int_X |f|^p \chi \, \mathrm{d}\mu = \int_X |f|^p \, \mathrm{d}\tilde{\mu} = \|f\|_{\tilde{\mu},p}^p \tag{$*$}$$

shows that $\chi^{\frac{1}{p}} f \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$ as wanted. Clearly, (C.3.114) is linear and by $(*)$ seminorm preserving. Finally, the inverse map is given by $g \mapsto \chi^{-\frac{1}{p}} g$ which is again well-defined since $\chi > 0$ everywhere. It is an easy computation analogously to $(*)$ that this maps into $\mathscr{L}^p(X, \mathfrak{a}, \tilde{\mu})$ and is seminorm preserving, too. Thus everything descends to the quotients (C.3.115). $\square$

Theorem C.3.55 (Dual space of $\mathrm{L}^p(X, \mathfrak{a}, \mu)$) *Let $(X, \mathfrak{a}, \mu)$ be a measure space.*

i.) For $1 < p < \infty$ the pairing of $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ and $\mathrm{L}^q(X, \mathfrak{a}, \mu)$ induces an isometric isomorphism

$$\mathrm{L}^q(X, \mathfrak{a}, \mu) \cong \mathrm{L}^p(X, \mathfrak{a}, \mu)'. \tag{C.3.116}$$

ii.) If μ is σ-finite then also

$$\mathrm{L}^\infty(X, \mathfrak{a}, \mu) \cong \mathrm{L}^1(X, \mathfrak{a}, \mu)'. \tag{C.3.117}$$

Proof: We proceed in three steps: the proof for a finite, a σ-finite, and a general measure space. The first two steps will be carried out for $1 \leq p < \infty$, the third will require $p > 1$. From Proposition C.3.28, *iv.)*, we already know that the linear map

$$\mathrm{L}^q(X, \mathfrak{a}, \mu) \ni g \;\mapsto\; \left(f \;\mapsto\; \int_X fg \, \mathrm{d}\mu\right) \in \left(\mathrm{L}^p(X, \mathfrak{a}, \mu)\right)' \tag{$*$}$$

is a norm preserving linear map for $1 < p < +\infty$ and also for $p = 1$ in the σ-finite case, see Proposition C.3.25 and Remark C.3.26 for the statement that this map is norm-preserving. Thus the only thing left to do is to show surjectivity of $(*)$.

As a first step we consider a finite measure space, $\mu(X) < +\infty$, and a continuous linear functional

$$\varphi \colon \mathrm{L}^p(X, \mathfrak{a}, \mu) \longrightarrow \mathbb{C}.$$

In the following we shall write $f \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$ also for the representative of some equivalence class $f \in \mathrm{L}^p(X, \mathfrak{a}, \mu)$ as usual. For a measurable subset $A \in \mathfrak{a}$ one defines now

$$\nu(A) = \varphi(\chi_A).$$

Here it is crucial that *every* χ_A is actually p-integrable since we assume μ to be a finite measure. We claim that this is a complex measure on $(X, \mathfrak{a})$. Indeed, let $A_1, A_2, \ldots \in \mathfrak{a}$ be a sequence of pairwise disjoint measurable subsets. Then we have for $A = \bigcup_{n=1}^{\infty} A_n$

$$\chi_A = \sum_{n=1}^{\infty} \chi_{A_n},$$

which is convergent in the sense of $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ since

$$\left\| \chi_A - \sum_{n=1}^{N} \chi_{A_n} \right\|_{\mu,p}^{p} = \int_X \left| \chi_A - \sum_{n=1}^{N} \chi_{A_n} \right|^p \mathrm{d}\mu = \int_X \sum_{n=N+1}^{\infty} \chi_{A_n} \,\mathrm{d}\mu = \sum_{n=N+1}^{\infty} \mu(A_n) \longrightarrow 0$$

by Proposition C.3.8 and the σ-additivity of μ. Thus by continuity of φ we have

$$\nu\left(\bigcup_{n=1}^{\infty} A_n \right) = \varphi\left(\sum_{n=1}^{\infty} \chi_{A_n} \right) = \sum_{n=1}^{\infty} \varphi(\chi_{A_n}) = \sum_{n=1}^{\infty} \nu(A_n),$$

which is the desired σ-additivity for ν. Moreover, if $A \in \mathfrak{a}$ is a zero set with respect to μ then χ_A vanishes almost everywhere and hence $\chi_A = 0$ in the quotient $\mathrm{L}^p(X, \mathfrak{a}, \mu)$. Thus $\nu(A) = 0$ follows. This shows $\nu \ll \mu$. Thus Theorem C.3.44 gives us a function $g \in \mathscr{L}^1(X, \mathfrak{a}, \mu)$ such that $\mathrm{d}\nu = g \,\mathrm{d}\mu$, i.e.

$$\varphi(\chi_A) = \nu(A) = \int_X \chi_A g \,\mathrm{d}\mu$$

for all $A \in \mathfrak{a}$. Since this holds for all characteristic functions and since φ is linear, we have

$$\varphi(f) = \int_X f g \,\mathrm{d}\mu \tag{$**$}$$

for all (equivalence classes of) simple functions f. Moreover, since uniform convergence implies convergence in the sense of $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ on a *finite* measure space, see Proposition C.3.30, we see that $(**)$ holds for all $f \in \mathscr{B}\mathscr{M}(X, \mathfrak{a})$ or better, for all $f \in \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$. The task is now to show $g \in \mathrm{L}^q(X, \mathfrak{a}, \mu)$ and to prove that $(**)$ holds for all $f \in \mathrm{L}^p(X, \mathfrak{a}, \mu)$.

Suppose first $p = 1$. Then the continuity of φ with respect to $\|\cdot\|_{\mu,1}$ shows

$$\left| \int_X \chi_A g \,\mathrm{d}\mu \right| = |\varphi(\chi_A)| \le \|\varphi\| \|\chi_A\|_{\mu,1}.$$

Since $\|\varphi_A\|_{\mu,1} = \mu(A)$ we can apply Lemma C.3.48 (after rescaling with $\|\varphi\|$) to conclude that $|g| \le \|\varphi\|$ almost everywhere with respect to μ. Thus $g \in \mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ with $\|g\|_{\mu,\infty} \le \|\varphi\|$ follows. But g being essentially bounded means that the integral on the right hand side in $(**)$ is continuous in f with respect to the norm $\|\cdot\|_{\mu,1}$. Thus $(**)$ holds for all $f \in \mathrm{L}^1(X, \mathfrak{a}, \mu)$ by the usual density argument.

Second, we consider $1 < p < +\infty$. Here we use the polar decomposition $g = u|g|$ with $\overline{u}u = 1$ defined as in Lemma C.3.23. Then we consider $A_n = \{x \in X \mid |g(x)| \le n\}$ for $n \ge 1$ and set

$$f_n = \chi_{A_n} \overline{u} |g|^{q-1}.$$

Clearly, $\|f_n\|_\infty \le n^{q-1}$. Moreover, $|f_n|^p = |g|^{(q-1)p} = |g|^q = f_n g$ on A_n. Since f_n is bounded we can use $(**)$ to get

$$\int_{A_n} |g|^q \,\mathrm{d}\mu = \int_{A_n} f_n g \,\mathrm{d}\mu$$

$$
\begin{aligned}
&= \int_X f_n g \, \mathrm{d}\mu \\
&\overset{(**)}{=} \varphi(f_n) \\
&= |\varphi(f_n)| \\
&\overset{(a)}{\le} \|\varphi\| \|f_n\|_{\mu,p} \\
&= \|\varphi\| \bigg(\int_{A_n} (|g|^{q-1})^p \, \mathrm{d}\mu \bigg)^{\frac{1}{p}},
\end{aligned}
$$

using the continuity of φ with respect to $\|\cdot\|_{\mu,p}$ in (a). This gives immediately the estimate

$$
\bigg(\int_{A_n} |g|^q \, \mathrm{d}\mu \bigg)^{\frac{1}{q}} \le \|\varphi\|
$$

for all n. Since $X = \bigcup_{n=1}^\infty A_n$ we get by monotonous convergence in the form of Proposition C.3.15, *v.)*, the result

$$
\|g\|_{\mu,q} = \lim_{n \longrightarrow \infty} \bigg(\int_{A_n} |g|^q \, \mathrm{d}\mu \bigg)^{\frac{1}{q}} \le \|\varphi\|,
$$

which shows $g \in \mathrm{L}^q(X, \mathfrak{a}, \mu)$. But then the right hand side of $(**)$ is also continuous with respect to the norm $\|\cdot\|_{\mu,p}$ in f and hence, by the density of simple functions in $\mathrm{L}^p(X, \mathfrak{a}, \mu)$, we conclude that $(**)$ holds for all p-integrable functions. This completes the proof for finite measure spaces.

The case of σ-finite measure spaces is obtained from Lemma C.3.54: for the rescaling isomorphism $\Phi_p \colon f \mapsto \chi^{\frac{1}{p}} f$ we get for a continuous linear functional φ on $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ that $\varphi \circ \Phi_p$ is a continuous linear functional on $\mathrm{L}^p(X, \mathfrak{a}, \tilde{\mu})$ with the same functional norm. Thus for $f \in \mathrm{L}^p(X, \mathfrak{a}, \mu)$ we have

$$
\varphi(f) = (\varphi \circ \Phi_p)\big(\Phi_p^{-1}(f)\big) = \int_X \Phi_p^{-1}(f) \tilde{g} \, \mathrm{d}\tilde{\mu} = \int_X f g \, \mathrm{d}\mu,
$$

where we apply the above result for the finite measure $\tilde{\mu}$ and set $g = \chi^{-\frac{1}{p}} \chi \tilde{g} = \chi^{\frac{1}{q}} \tilde{g} = \Phi_q(\tilde{g}) \in \mathrm{L}^q(X, \mathfrak{a}, \mu)$. Thus also this case is shown.

In the last step we consider $1 < p < +\infty$ and hence $1 < q < +\infty$ for a general measure space. We will frequently make use of the fact that extending a function on a measurable subset $A \subseteq X$ by 0 to all of X gives a norm-preserving continuous inclusion $\mathrm{L}^p\big(A, \mathfrak{a}\big|_A, \mu\big|_A\big) \longrightarrow \mathrm{L}^p(X, \mathfrak{a}, \mu)$, see also Exercise **??**. This allows to restrict the continuous linear functional φ to a continuous linear functional $\varphi_A \colon \mathrm{L}^p\big(A, \mathfrak{a}\big|_A, \mu\big|_A\big) \longrightarrow \mathbb{C}$ with functional norm $\|\varphi_A\| \le \|\varphi\|$. Now we consider the following subset of measurable subsets of X

$$
\mathfrak{a}_{\sigma\text{-finite}} = \big\{ A \in \mathfrak{a} \bigm| \mu\big|_A \text{ is } \sigma\text{-finite} \big\}.
$$

This is, in general, no longer a σ-algebra on X as e.g. $X \notin \mathfrak{a}_{\sigma\text{-finite}}$ unless $(X, \mathfrak{a}, \mu)$ is σ-finite itself. However, it is easy to see that $\mathfrak{a}_{\sigma\text{-finite}}$ is still closed under countable unions. For each $A \in \mathfrak{a}_{\sigma\text{-finite}}$ we can apply the already proved σ-finite version of the theorem to obtain $g_A \in \mathrm{L}^q\big(A, \mathfrak{a}\big|_A, \mu\big|_A\big)$ with

$$
\varphi_A(f) = \int_A f g_A \, \mathrm{d}\mu\big|_A
$$

for all $f \in \mathrm{L}^p\big(A, \mathfrak{a}\big|_A, \mu\big|_A\big)$. Since the functionals φ_A determine g_A uniquely as elements in the space $\mathrm{L}^q\big(A, \mathfrak{a}\big|_A, \mu\big|_A\big)$ and since the further restrictions of φ_A and φ_B to $A \cap B$ yield the same functional

$$
\varphi_A\big|_{\mathrm{L}^p(A \cap B, \mathfrak{a}|_A|_B, \mu|_A|_B)} = \varphi_{A \cap B} = \varphi_B\big|_{\mathrm{L}^p(A \cap B, \mathfrak{a}|_B|_A, \mu|_B|_A)},
$$

it follows that

$$g_A\big|_{A\cap B} = g_{A\cap B} = g_B\big|_{A\cap B}$$

for any two $A, B \in \mathfrak{a}_{\sigma\text{-finite}}$. Since we know $\|\varphi_A\| = \|g_A\|_{\mu|_A,q} = \|g_A\|_{\mu,q}$ by the isometric isomorphism (C.3.116) for the σ-finite case, we get for disjoint $A, B \in \mathfrak{a}_{\sigma\text{-finite}}$

$$\|\varphi_{A\cup B}\|^q = \|g_{A\cup B}\|^q_{\mu,q} = \|g_A + g_B\|^q_{\mu,q} = \|g_A\|^q_{\mu,q} + \|g_B\|^q_{\mu,q} = \|\varphi_A\|^q + \|\varphi_B\|^q. \tag{☺}$$

Note that at this stage we make use of the additional assumption $q < +\infty$ by $p > 1$. Now we consider

$$c = \sup\bigl\{\|\varphi_A\| \bigm| A \in \mathfrak{a}_{\sigma\text{-finite}}\bigr\} \le \|\varphi\|,$$

and choose a *sequence* $A_n \in \mathfrak{a}_{\sigma\text{-finite}}$ to approach this supremum, i.e. $\|\varphi_{A_n}\| \longrightarrow c$. Then $A_\infty = \bigcup_{n=1}^\infty A_n \in \mathfrak{a}_{\sigma\text{-finite}}$, too, and since clearly $\|\varphi_{A_n}\| \le \|\varphi_{A_\infty}\|$ for all n we conclude that

$$\|\varphi_{A_\infty}\| = c,$$

as c was the supremum over all $A \in \mathfrak{a}_{\sigma\text{-finite}}$. Now if $B \in \mathfrak{a}_{\sigma\text{-finite}}$ then we see that

$$c \ge \|\varphi_{A_\infty\cup B}\|^q \overset{(☺)}{=} \|\varphi_{A_\infty}\|^q + \|\varphi_{B\setminus A_\infty}\|^q = c + \|\varphi_{B\setminus A_\infty}\|^q,$$

and hence for all B

$$\|\varphi_{B\setminus A_\infty}\| = 0. \tag{☹}$$

The next crucial step, which is of interest by its own, is to see that the points where $f \in \mathscr{L}^p(X, \mathfrak{a}, \mu)$ is non-zero form a set in $\mathfrak{a}_{\sigma\text{-finite}}$. Indeed, let $B_n = |f|^{-1}\bigl(\bigl[\frac{1}{n}, +\infty)\bigr) \subseteq \mathfrak{a}$ be those points where $f \ge \frac{1}{n}$. Then $B = \bigcup_{n=1}^\infty B_n = f^{-1}(\mathbb{C} \setminus \{0\})$. On the other hand

$$\mu(B_n) = \int_X \chi_{B_n}\,\mathrm{d}\mu \le \int_X n^p |f|^p\,\mathrm{d}\mu = n^p \|f\|^p_{\mu,p}$$

shows that $\mu(B_n) < \infty$. Since $\mathfrak{a}_{\sigma\text{-finite}}$ is closed under countable unions we see that $B \in \mathfrak{a}_{\sigma\text{-finite}}$. But this shows that $f \in \mathscr{L}^p\bigl(B, \mathfrak{a}\big|_B, \mu\big|_B\bigr)$ and hence

$$\begin{aligned}
\varphi(f) &= \varphi_B(f) \\
&= \varphi_B(\chi_{A_\infty} f + (1 - \chi_{A_\infty}) f) \\
&= \varphi_{A_\infty\cap B}(\chi_{A_\infty} f) + \varphi_{B\setminus A_\infty}((1 - \chi_{A_\infty}) f) \\
&\overset{(☹)}{=} \varphi_{A_\infty}(\chi_{A_\infty} f) \\
&= \int_{A_\infty} f g_{A_\infty}\,\mathrm{d}\mu \\
&= \int_X f g_{A_\infty}\,\mathrm{d}\mu,
\end{aligned}$$

where in the last step we have used that g_{A_∞} vanishes outside of A_∞. Thus we have found the function $g = g_{A_\infty} \in \mathscr{L}^q(X, \mathfrak{a}, \mu)$ we were looking for. □

Corollary C.3.56 *Let $(X, \mathfrak{a}, \mu)$ be a measure space. Then the Banach spaces $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ are reflexive for $1 < p < +\infty$.*

Example C.3.57 (Sequence spaces) Already for the σ-finite space $\mathbb{N}$ with the counting measure the dual space of $\ell^\infty = \mathrm{L}^\infty(\mathbb{N}, 2^{\mathbb{N}}, \mu_{\mathrm{count}})$ is strictly larger than $\ell^1 = \mathrm{L}^1(\mathbb{N}, 2^{\mathbb{N}}, \mu_{\mathrm{count}})$. This follows from the results of the Exercises 2.5.37, 2.5.38, 2.5.40, and 2.5.41.

Remark C.3.58 The assumption of having a σ-finite measure is not superfluous for the second part of Theorem C.3.55. Indeed, take the measure with $\mu(A) = +\infty$ for all $A \neq \emptyset$ then $\mathrm{L}^1(X, \mathfrak{a}, \mu) = \{0\}$. However, $\mathrm{L}^\infty(X, \mathfrak{a}, \mu) = \mathscr{BM}(X, \mathfrak{a})$ is non-trivial and hence larger than the dual space of $\mathrm{L}^1(X, \mathfrak{a}, \mu)$. Nevertheless, there are examples and slightly less restrictive assumptions than σ-finiteness which make the theorem still work. In particular, the counting measure on a uncountable measurable space easily provides an example: in this case the dualities can be check by hand relying on the Schauder bases as in Exercise 2.5.37, 2.5.38, 2.5.40, and 2.5.41.

C.4 Borel Measures and the Riesz Representation Theorem

C.5 The Haar Measure

C.5.1 Haar Measure on Locally Compact Groups

C.5.2 The Haar Measure on Lie Groups

C.6 Exercises

Exercise C.6.1 (Constructions with σ-algebras) Let $(X, \mathfrak{a})$ be a measurable space.

i.) Show that for maps $f\colon Z \longrightarrow Y$ and $g\colon Y \longrightarrow X$ one has $(g \circ f)^{-1}(\mathfrak{a}) = f^{-1}(g^{-1}(\mathfrak{a}))$.

ii.) Prove that for a subset $Y \subseteq X$ the restriction $\mathfrak{a}\big|_Y$ coincides with the inverse image $\iota^{-1}(\mathfrak{a})$ under the inclusion map $\iota\colon Y \longrightarrow X$.

iii.) Let $Z \subseteq Y \subseteq X$ be subsets. Show that $\mathfrak{a}\big|_Z = (\mathfrak{a}\big|_Y)\big|_Z$.

Exercise C.6.2 (The pro-C^*-algebra $\mathscr{B}_{\mathrm{loc}}$) Let X be a locally compact topological space with its canonical Borel σ-algebra $\mathfrak{a}_X$.

i.) Show that the locally bounded measurable functions

$$\mathscr{BM}_{\mathrm{loc}}(X) = \mathscr{B}_{\mathrm{loc}}(X) \cap \mathscr{M}(X, \mathfrak{a}_X) \tag{C.6.1}$$

form a unital pro-C^*-algebra with respect to the pointwise operations and the seminorms p_K for all compact subsets $K \subseteq X$.

ii.) Show that $\mathscr{C}(X) \subseteq \mathscr{BM}_{\mathrm{loc}}(X)$ is a unital closed pro-C^*-subalgebra.

iii.) Discuss in particular the case where X carries the discrete topology and compare the function spaces $\mathscr{B}_{\mathrm{loc}}(X)$, $\mathscr{M}(X)$, $\mathscr{BM}_{\mathrm{loc}}(X)$, and $\mathscr{C}(X)$ in this case.

Exercise C.6.3 (The C^*-algebra $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$) Consider a measurable space $(X, \mathfrak{a})$ with a positive measure μ and the resulting C^*-algebra $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$.

i.) Show that $\operatorname{spec}(f) = \operatorname{ess\,range}(f)$ for $f \in \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$.

ii.) Show that $f \in \mathrm{L}^\infty(X, \mathfrak{a}, \mu)$ is a positive element iff $f(x) \geq 0$ almost everywhere.

Exercise C.6.4 .

Exercise: BM some funny n

Exercise: Me version of Jen inequality aft elementary o Exercise ??

Exercise C.6.5 (Inverse image and push forward)

Exercise C.6.6 (An example for Proposition C.2.6) Consider $\mathbb{N}$ with the whole power set as σ-algebra and the counting measure on it. Moreover, let $A_n = \{n, n+1, \ldots\}$. Show that this provides an example that the condition $\mu(A_1) < \infty$ in Proposition C.2.6, *ii.)*, is not superfluous.

Exercise C.6.7 (Convex combination of measures) Let $(X, \mathfrak{a})$ be a measurable space and let μ_i be positive measures on $(X, \mathfrak{a})$ for $i \in I$. Moreover, let $\lambda_i > 0$ be given positive numbers. For $A \in \mathfrak{a}$ one defines $\mu(A)$ by

$$\mu(A) = \sum_{i \in I} \lambda_i \mu_i(A). \tag{C.6.2}$$

Since all occurring terms in this series are non-negative, we have either absolute convergence or absolute divergence to $+\infty$. Show that this defines a positive measure μ on $(X, \mathfrak{a})$.

: Exercise for easures to set ne new point ise: compute omplex linear of δ-measures

Exercise C.6.8 (Construction of complex measures) Let $(X, \mathfrak{a}, \mu)$ be a measure space with positive measure. For $f \in \mathrm{L}^1(X, \mathfrak{a}, \mu)$ we denote by ν_f the complex measure on X such that $\mathrm{d}\nu_f = f\,\mathrm{d}\mu$. Show that this yields a linear map

$$\mathrm{L}^1(X, \mathfrak{a}, \mu) \ni f \;\mapsto\; \nu_f \in \operatorname{Meas}(X, \mathfrak{a}), \tag{C.6.3}$$

such that $\overline{\nu_f} = \nu_{\overline{f}}$.

e: Gluing and easures with int unions as don-Nikodym etc n: Dominated lso works for asures, all the tes as well. . . : Reparieren!

Exercise C.6.9 (Dominated convergence with complex measures)

Exercise C.6.10 (Extreme points of convex subsets II) *i.)* Consider $\mathscr{A} = \mathscr{C}([0, 1])$ and show that there are states of $\mathscr{A}$ which are *not* convex combinations of pure states. Describe the states which are obtained as convex combinations of pure ones explicitly.

ii.) Use part *i.)* and Riesz' Representation Theorem as well as the Krein-Milman Theorem to formulate and prove an approximation theorem for finite positive Borel measures on $[0, 1]$.

Exercise C.6.11 (The functor Meas) Formulate and show that taking complex measures on measurable spaces yields a covariant functor from Mess to Banach.

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