Homework for the Lecture

Algebra and Dynamics of Quantum Systems

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Homework Sheet No 1

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(24 Points. Submission deadline 24. 10. 2023)

Homework 1-1: The commutator

Let $\mathscr{A}$ be an associative algebra over $\mathbb{C}$. We define the commutator

$$[a, b] = ab - ba \tag{1.1}$$

for $a, b \in \mathscr{A}$ as usual. Furthermore, we write $\operatorname{ad}(a)\colon b \mapsto [a, b]$.

i.) Prove that $[\,\cdot\,,\,\cdot\,]$ turns $\mathscr{A}$ into a Lie algebra. **(1 Point)**

ii.) Let $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ be an algebra morphism into another associative algebra $\mathscr{B}$. Show that Φ is also a Lie algebra morphism with respect to the commutator Lie brackets. Conclude that this yields a functor from the category of associative algebras $\mathsf{alg}_{\mathbb{C}}$ into the category of Lie algebras over $\mathbb{C}$. **(2 Points)**

iii.) Consider the left and right multiplications

$$\mathsf{L}_a, \mathsf{R}_b\colon \mathscr{A} \longrightarrow \mathscr{A} \tag{1.2}$$

for a fixed algebra element $a \in \mathscr{A}$, i.e. $\mathsf{L}_a(b) = ab$ as well as $\mathsf{R}_a(b) = ba$. Show $[\mathsf{L}_a, \mathsf{R}_b] = 0$ as well as $\operatorname{ad}(a) = \mathsf{L}_a - \mathsf{R}_a$ for all $a, b \in \mathscr{A}$. **(1 Point)**

iv.) Let $\mathfrak{g}$ be a Lie algebra. Prove that $\operatorname{ad}\colon \mathfrak{g} \ni \xi \mapsto (\eta \mapsto \operatorname{ad}(\xi)\eta = [\xi, \eta]) \in \operatorname{End}(\mathfrak{g})$ yields a homomorphism of Lie algebras

$$\operatorname{ad}\colon \mathfrak{g} \longrightarrow \operatorname{End}(\mathfrak{g}), \tag{1.3}$$

where we equip $\operatorname{End}(\mathfrak{g})$ with the commutator as Lie bracket. **(2 Points)** *v.)* Prove that the map $\mathrm{ad}(a)$ is a derivation of the associative product for $a \in \mathscr{A}$. Furthermore show that the set of derivations of $\mathscr{A}$ constitutes a Lie subalgebra $\mathrm{Der}(\mathscr{A}) \subseteq \mathrm{End}(\mathscr{A})$ of all endomorphisms of $\mathscr{A}$. Finally, prove that $\mathrm{ad}\colon \mathscr{A} \longrightarrow \mathrm{Der}(\mathscr{A})$ is a Lie algebra homomorphism. **(3 Points)**

vi.) Derivations of the form $\mathrm{ad}(a)$ are called *inner derivations*, whose set we denote by $\mathrm{InnDer}(\mathscr{A})$. Show first that $\mathrm{InnDer}(\mathscr{A})$ is a subspace of $\mathrm{End}(\mathscr{A})$. Furthermore prove

$$[D, \mathrm{ad}(a)] = \mathrm{ad}(Da) \tag{1.4}$$

for every derivation $D \in \mathrm{Der}(\mathscr{A})$ and every algebra element $a \in \mathscr{A}$. Conclude that the quotient $\mathrm{OutDer}(\mathscr{A}) = \mathrm{Der}(\mathscr{A}) \big/ \mathrm{InnDer}(\mathscr{A})$ carries a Lie algebra structure. The elements of $\mathrm{OutDer}(\mathscr{A})$ are called *outer derivations* of $\mathscr{A}$. **(3 Points)**

vii.) Let now $\mathscr{A}$ be a *-algebra. Compute $[a, b]^*$ for $a, b \in \mathscr{A}$. Use this to characterize the elements $a \in \mathscr{A}$, for which $\mathrm{ad}(a)$ is a *-derivation. **(1 Point)**

Homework 1-2: A positive quadratic polynomial

Consider complex numbers $a, b, b', c \in \mathbb{C}$ with

$$p(z, w) = a\overline{z}z + bz\overline{w} + b'\overline{z}w + cw\overline{w} \geq 0 \tag{1.5}$$

for all $z, w \in \mathbb{C}$. Show that this implies $a \geq 0$, $c \geq 0$, $\overline{b} = b'$ and $ac \geq b\overline{b}$. **(2 Points)**

Homework 1-3: The polynomial calculus I

Let $\mathscr{A}$ be a unital associative algebra over some field $\mathbb{k}$ and let $a \in \mathscr{A}$ be a fixed element. For a polynomial $p \in \mathbb{k}[x]$ one defines $p(a) \in \mathscr{A}$ as usual by substituting the variable x by the algebra element a. If $\mathscr{A}$ is not unital, then this is only possible for polynomials $p \in x\mathbb{k}[x]$ with vanishing constant part.

i.) **(2 Points)** Show that the map

$$\mathbb{k}[x] \ni p \ \mapsto \ p(a) \in \mathscr{A} \tag{1.6}$$

is a unital algebra homomorphism.

ii.) **(1 Point)** Show that if $\Phi\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a unital homomorphism into some other unital associative algebra $\mathscr{B}$ over $\mathbb{k}$, then

$$\Phi(p(a)) = p(\Phi(a)) \tag{1.7}$$

for all $a \in \mathscr{A}$ and $p \in \mathbb{k}[x]$. In which sense does this still hold in the non-unital situation?

Homework 1-4: The polynomial calculus II

Assume that $\mathscr{A}$ is a unital *-algebra over $\mathbb{C}$ and let $a \in \mathscr{A}$ be a normal element. Consider polynomials $\mathbb{C}[z, \overline{z}]$ in two variables.

i.) **(2 Points)** Show that the algebra $\mathbb{C}[z, \overline{z}]$ becomes a *-algebra if one defines $z^* = \overline{z}$ for the generators, thereby explaining the notation.

ii.) **(1 Point)** Define for $p \in \mathbb{C}[z, \overline{z}]$ the algebra element $p(a, a^*) \in \mathscr{A}$ by substituting z by a and $\overline{z}$ by a^*. Show that this is well-defined by using the fact that a is normal.

iii.) **(2 Points)** Show that the map

$$\mathbb{C}[z, \overline{z}] \ni p \ \mapsto \ p(a, a^*) \in \mathscr{A} \tag{1.8}$$

is a unital *-homomorphism.

iv.) **(1 Point)** Formulate and prove an analogous statement for the case where $\mathscr{A}$ is non-unital.