Universität Würzburg
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Prof. Dr. Björn Trauzettel
Aisel Shiralieva

6. Open quantum systems

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Throughout this exercise sheet, we adopt the convention $\boxed{\hbar = 1}$.

Exercise 1 *Dynamics at and away from exceptional points* **4 P.**

Consider the same two-level system as in the previous exercise sheet, with the following system Hamiltonian and Lindblad operators:

$$H_S = \omega\sigma_x, \qquad L_1 = \sqrt{\gamma}\,\sigma_-, \qquad L_2 = \sqrt{\gamma}\,\sigma_+,$$

where $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ are the lowering and raising operators.

Let $\mathcal{L}$ be the Lindbladian superoperator acting on the vectorized density matrix $|\rho(t)\rangle$, such that:

$$\frac{d}{dt}|\rho(t)\rangle = \mathcal{L}\,|\rho(t)\rangle\,.$$

Using the vectorization rule, the explicit form of the Lindbladian superoperator $\mathcal{L} \in \mathbb{C}^{4\times 4}$ is:

$$\mathcal{L} = \begin{pmatrix} -\gamma & i\omega & -i\omega & \gamma \\ i\omega & -\gamma & 0 & -i\omega \\ -i\omega & 0 & -\gamma & i\omega \\ \gamma & -i\omega & i\omega & -\gamma \end{pmatrix}.$$

For this system, the exceptional point occurs when:

$$\boxed{\gamma^2 = 16\omega^2}.$$

a) At the EP ($\gamma^2 = 16\omega^2$), the Lindbladian is no longer diagonalizable. In this case, assume $\mathcal{L} = PJP^{-1}$, where J is a Jordan matrix. Show that:

$$|\rho(t)\rangle = Pe^{Jt}P^{-1}\,|\rho(0)\rangle\,,$$

and explain the general structure of e^{Jt}, particularly how polynomial prefactors appear in the time evolution (e.g., terms like $te^{\lambda t}$).

Note: The initial density matrix is:

$$\rho_0 = \rho(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad |\rho(0)\rangle = (1, 0, 0, 0)^T.$$

You may use `Wolfram Mathematica` (or another symbolic computation tool) to compute the eigenvalues, eigenvectors, and matrix exponential.

b) For $\gamma^2 \neq 16\omega^2$, the Lindbladian is diagonalizable. Express the time evolution of the density matrix.

Note: The initial density matrix is again ρ_0.

c) Compare the long-time behavior of the system in both regimes: at the EP and away from it. What features distinguish the dynamics at the EP? Are there steady states in the system in both cases?

Exercise 2 *Non-Markovian open quantum systems* **7 P.**

Let $\mathcal{E}(t)$ be a known dynamical map describing the evolution of a reduced density matrix in vectorized form:

$$\bar{\rho}(t) = \mathcal{E}_{t,0}\, \bar{\rho}(0),$$

where $\bar{\rho}(t)$ is the vectorized density matrix at time t.

Assume that the map is differentiable and invertible at all times, and that

$$\mathcal{E}_{t+\Delta t,t} = \mathcal{E}_{t+\Delta t,0}\, \mathcal{E}_{t,0}^{-1}.$$

a) Find the relation between the time-local generator $\mathcal{L}(t)$ and the dynamical map $\mathcal{E}_{t+\Delta t,t}$, up to first order in Δt.

Note: Use the definition of dynamical maps in terms of the time-ordered exponential, which relates them to the Liouvillian:

$$\mathcal{E}_{t+\Delta t,t} = \mathcal{T} \exp\left(\int_t^{t+\Delta t} \mathcal{L}(\tau)\, d\tau \right).$$

Hint: Use the Magnus expansion in combination with the Taylor series of the unknown function $\mathcal{L}(t)$.

Assume the system evolves under a Lindblad master equation in superoperator form:

$$\mathcal{L} = -\frac{i}{\hbar} \tilde{H} \otimes \mathbb{1} + \frac{i}{\hbar} \mathbb{1} \otimes \tilde{H}^* + \sum_j L_j \otimes L_j^*, \quad \tilde{H} = \left(H - i\frac{\hbar}{2} \sum L_j^\dagger L_j \right).,$$

where the quantum jump operators L_j and the Hamiltonian H can be chosen to be traceless. We define the non-Hermitian Hamiltonian as $\tilde{H}$.

We denote $\mathrm{Tr}_{\mathrm{bw}}$ as the partial trace over only the "backward" subspace of the squared Hilbert space, such that $\mathrm{Tr}_{\mathrm{bw}}(A \otimes B) = A\,\mathrm{Tr}(B)$.

b) Derive the following identities, assuming the Lindblad structure above and that H and L_j can be chosen as traceless matrices:

- $\mathrm{Tr}\{\tilde{H}\} = -\,\mathrm{Tr}\left\{ \tilde{H}^* \right\}$
- $\mathrm{Tr}\{\mathcal{L}\} = 2d\frac{i}{\hbar}\,\mathrm{Tr}\left\{ \tilde{H}^* \right\}$

- $\mathrm{Tr}_{bw}\{\mathcal{L}\} = -\frac{i}{\hbar} d\tilde{H} + \frac{1}{2d}\,\mathrm{Tr}\{\mathcal{L}\}\mathbb{1}$

  The dependence on the dimension d of the system Hilbert space arises from the trace over the identity matrix: $\mathrm{Tr}(\mathbb{1}) = d$.

c) Use the results from (b) to show that the non-Hermitian Hamiltonian is given by:

$$\tilde{H} = \frac{i}{\hbar d}\left(\mathrm{Tr}_{\mathrm{bw}}\{\mathcal{L}\} - \frac{1}{2d}\mathrm{Tr}\{\mathcal{L}\}\,\mathbb{1}\right).$$

d) Write down how to extract the Hermitian Hamiltonian H that appears in the Lindblad master equation from $\tilde{H}$.