Homework for the Lecture

Functional Analysis

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Homework Sheet No 2

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Homework 2-1: Finite Dimensional Topological Vector Spaces

The goal of this exercise is to show that every finite dimensional vector space carries a unique Hausdorff topology. Let V be a finite dimensional topological vector space of dimension $n \in \mathbb{N}$.

i.) **(2 Points)** Use the continuity of the scalar multiplication to show that every open neighborhood U of zero contains an open balanced neighborhood U_0 of zero, that is $zU_0 \subseteq U_0$ for all $z \in \mathbb{K}$ with $|z| \leq 1$.

ii.) **(4 Points)** Given a basis $(e_1, \ldots, e_n)$ of $\mathbb{K}^n$ and a basis $(v_1, \ldots, v_n)$ of V, we define the map $\phi \colon \mathbb{K}^n \to V$ as the $\mathbb{K}$-linear extension of the map $e_i \mapsto v_i$. Recall that ϕ is an isomorphism of vector spaces. Show that ϕ is continuous if $\mathbb{K}^n$ is endowed with the standard topology.

iii.) **(4 Points)** Let V be Hausdorff. Show that $0 \in \phi(\mathrm{B}_r(0))^\circ$ for every $r > 0$.
Hint: Consider the subset $V \setminus \phi(\mathbb{S}^{n-1})$.

iv.) **(1 Point)** Conclude that also ϕ^{-1} is continuous.

Homework 2-2: An Application of Baire's Theorem

Let $(M, \mathscr{M})$ be a topological space and $(f_n)_{n\in\mathbb{N}} \subset C(M, \mathbb{K})$ be a sequence of continuous functions that converges pointwise to a (not necessarily continuous!) function f. For $\varepsilon > 0$ and $n \in \mathbb{N}$ we define

$$C_n(\varepsilon) := \{p \in M : |f_n(p) - f(p)| \leq \varepsilon\} \tag{2.1}$$

and set

$$C(\varepsilon) := \bigcup_{n=1}^{\infty} C_n(\varepsilon)^\circ \tag{2.2}$$

and

$$C := \bigcap_{n=1}^{\infty} C\Big(\frac{1}{n}\Big). \tag{2.3}$$

i.) **(2 Points)** Show that f is continuous at $p \in M$ iff $p \in C$.

ii.) **(2 Points)** Consider the set

$$A_n(\varepsilon) := \{p \in M : |f_n(p) - f_k(p)| \leq \varepsilon \text{ for all } k \geq n\}. \tag{2.4}$$

Show that the boundary of $A_n(\varepsilon)$ is nowhere dense.

iii.) **(4 Points)** Show that the discontinuities of f form a meager subset of M.

iv.) **(5 Points)** Prove the following statement:
There is no differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative equals the function

$$g : \mathbb{R} \ni x \mapsto g(x) := \begin{cases} 1 & x \in (\mathbb{R}\backslash(0,1)) \cup (\mathbb{Q} \cap (0,1)) \\ 0 & x \in (\mathbb{R}\backslash\mathbb{Q}) \cap (0,1) \end{cases}. \tag{2.5}$$