Geometric Analysis Exam Presentation Outline

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I. INTRODUCTION

1. **Define** Lie groups
2. **State** example of $\mathrm{GL}(n, \mathbb{R})$ and **prove** that it is a Lie group.
3. **State** that Lie groups provide a way to move between elements (group multiplication)
4. **Prove** left translation is diffeo
   (a) Invertible ($L_{g^{-1}}$)
   (b) Smooth by definition
5. **Prove** Lie group homos have constant rank
   (a) Compare to rank at e
   (b) Consider $F(L_{g_0}(g)) = L_{F(g_0)}(F(g))$
   (c) Take differential at $g = e$
6. **Prove** that open subgroups are closed.
   (a) Consider cosets
7. **Prove** identity component is only connected open subgroup, all connected components are diffeo to identity component
   (a) Connected subsets generate connected subgroups
   (b) Consider elements that can be expressed as a product of k elements of the set.
   (c) Because they share 1 element, the union is connected.
   (d) Consider subgroup generated by identity component.
   (e) Use previous result (open subgroups are closed) to prove uniqueness
8. **Draw** picture corresponding to previous proof

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II. GROUP ACTIONS

9. **Define** a smooth group action of G on M as assigning a smooth map $G \times M \to M$.
10. **Prove** that smooth actions are diffeos (ext. smooth inv)
11. **Define** what it means for a smooth function to intertwine actions. If G is a lie group acting on manifolds M and N with actions θ and φ respectively, then $F : M \to N$ intertwines actions if the following diagram commutes for all g:

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \big\downarrow{\scriptstyle \theta_g} & & \big\downarrow{\scriptstyle \varphi_g} \\ M & \xrightarrow{F} & N \end{array}$$

12. **Prove:** If group action on M, N is transitive on M and F intertwines actions, F has constant rank.

$$\begin{array}{ccc} T_pM & \xrightarrow{\mathrm{d}F_p} & T_{F(p)}N \\ \big\downarrow{\scriptstyle \mathrm{d}(\theta_g)_p} & & \big\downarrow{\scriptstyle \mathrm{d}(\varphi_g)_{F(p)}} \\ T_qM & \xrightarrow{\mathrm{d}F_q} & T_{F(q)}N \end{array}$$

13. **Prove** orbit map $G \to M$ (fixed p) is constant rank.
    (a) Orbit map is equivariant wrt the action

III. LIE ALGEBRAS

14. **State** commutator properties
    (a) Bilinearity
    (b) Anticommutativity
    (c) Jacobi Identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

15. **Define** a lie algebra
16. **Define** left invariant vector fields

    Invariant under all left translations, so

$$\mathrm{d}(L_g)_{g'}(X_{g'}) = X_{gg'}$$

    forall $g, g' \in G$

17. **Prove** that left invariant vector fields are closed under the commutator

    (a) Y is F-related to X iff for every $f : N \to \mathbb{R}$

    $$X(f \circ F) = (Yf) \circ F$$

    Proof:

    $$X(f \circ F)(p) = X_p(f \circ F) = \mathrm{d}F_p\,(X_p)(f)$$
    $$(Yf) \circ F(p) = (Yf)(F(p)) = Y_{F(p)}f$$

    (b) Show that $\mathrm{d}F\,[X, Y] = [\mathrm{d}F\,X, \mathrm{d}F\,Y]$.

    Do this by showing

    $$XY(f \circ F) = (\mathrm{d}F\,(X)\,\mathrm{d}F\,(Y)f) \circ F$$

    Idea: We "bring Y in", then bring X in.

    Then we apply this to the explicit expression for the commutator.

    (c) Apply this result to a left translation and use the invariance under left translator.

18. **Define** Lie algebra as the vector space of left invariant vector fields under the commutator

19. **Prove** that $\dim(\mathrm{Lie}(G)) = \dim(G)$ by showing that the evaluation map at the identity is an isomorphism.

    (a) We show injectivity by assuming the vector field is 0 at e and translating this away.

    (b) We show surjectivity by defining a rough vector field using $X_p = \mathrm{d}(L_p)_e\,(v)$. It remains to show it is smooth. To do this, we show that Xf is smooth for all smooth functions f.

    We choose a path $\gamma : (-\delta, \delta) \to M$, $\gamma(0) = e$, $\gamma'(0) = v$. Then

    $$(Xf)(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ L_g \circ \gamma)(t)$$

    (Draw picture of path through e being translated to path through g)

    Define $\varphi : (-\delta, \delta) \to M$ as $\varphi(t, g) = f \circ L_g \circ \gamma(t)$. Then this is a smooth map, and hence the vector field is smooth.

(c) We show left invariance by using the composition property of translations

$$\mathrm{d}(L_h)_g\,(X_g) = X_{hg}$$

20. **Deduce** as a corollary that all left invariant vector fields on a lie group are smooth.

A. Matrix Lie Group & Algebra

21. **State** that $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathfrak{gl}(n, \mathbb{R})$.
22. **Prove** that $\mathrm{GL}(n, \mathbb{R}) \cong T_{I_n}\mathrm{GL}(n, \mathbb{R}) \cong \mathfrak{gl}(n, \mathbb{R})$