Geometric Analysis Exam Presentation Outline

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I. INTRODUCTION

1. **Define** Lie groups
2. **State** example of $\mathrm{GL}(n, \mathbb{R})$ and **prove** that it is a Lie group.
3. **State** that Lie groups provide a way to move between elements (group multiplication)
4. **Prove** left translation is diffeo
   (a) Invertible ($L_{g^{-1}}$)
   (b) Smooth by definition
5. **Prove** Lie group homos have constant rank
   (a) Compare to rank at e
   (b) Consider $F(L_{g_0}(g)) = L_{F(g_0)}(F(g))$
   (c) Take differential at $g = e$
6. **Prove** that open subgroups are closed.
   (a) Consider cosets
7. **Prove** identity component is only connected open subgroup, all connected components are diffeo to identity component
   (a) Connected subsets generate connected subgroups
   (b) Consider elements that can be expressed as a product of k elements of the set.
   (c) Because they share 1 element, the union is connected.
   (d) Consider subgroup generated by identity component.
   (e) Use previous result (open subgroups are closed) to prove uniqueness
8. **Draw** picture corresponding to previous proof

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II. GROUP ACTIONS

9. Define a smooth group action of G on M as assigning a smooth map $M \to M$ to each G.

10. **Define** what it means for a smooth function to intertwine actions. If G is a lie group acting on manifolds M and N with actions θ and φ respectively, then $F : M \to N$ intertwines actions if the following diagram commutes for all g:

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\theta_g"] & N \arrow[d, "\varphi_g"] \\ M \arrow[r, "F"] & N \end{tikzcd}$$

11. **Prove:** If group action on M, N is transitive on M and F intertwines actions, F has constant rank.

$$\begin{tikzcd} T_pM \arrow[r, "\mathrm{d}F_p"] \arrow[d, "\theta_g"] & T_{F(p)}N \arrow[d, "\mathrm{d}(\varphi_g)_{F(p)}"] \\ T_qM \arrow[r, "\mathrm{d}F_q"] & T_{F(q)}N \end{tikzcd}$$

12. **Prove** orbit map $G \to M$ (fixed p) is constant rank.
    (a) Orbit map is equivariant wrt the action

III. LIE ALGEBRAS

13. **State** commutator properties
14. **Define** a lie algebra
15. **Define** left invariant vector fields
16. **Prove** that left invariant vector fields are closed under the commutator
17. **Prove** that $\dim(\mathrm{Lie}(G)) = \dim(G)$ by showing that the evaluation map is an isomorphism.
18. **Deduce** as a corollary that all left invariant vector fields on a lie group are smooth.

A. Matrix Lie Group & Algebra

19. **State** that $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathfrak{gl}(n, \mathbb{R})$.

20. **Prove** that $\mathrm{GL}(n,\mathbb{R}) \cong T_{I_n}\mathrm{GL}(n,\mathbb{R}) \cong \mathfrak{gl}(n,\mathbb{R})$

B. Lie Algebra Homomorphisms