Quantum field theory in the solid state, Exercise sheet 5

Corrections: Week of June 3rd

Coherent state path integral for bosons.

Consider the one-dimensional Harmonic oscillator

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 \tag{1}$$

with $\left[\hat{P}, \hat{X}\right] = \frac{\hbar}{i}$.

(a) Show that

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{2}$$

with

$$\hat{a}^\dagger = \frac{\omega m\hat{X} + i\hat{P}}{\sqrt{2m\omega\hbar}} \quad \text{and} \quad \left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{3}$$

(b) Show how to build eigenstates satisfying:

$$\hat{a}^\dagger\hat{a}\ |n\rangle = n|n\rangle \tag{4}$$

Here is a hint. Find the ground state of the above operator and show that $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

(c) Show that for $\alpha \in \mathbb{C}$

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{5}$$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad \text{and} \quad \langle\alpha|\alpha\rangle = 1 \tag{6}$$

(d) Show that

$$\langle\alpha|\beta\rangle = e^{-|\alpha|^2/2-|\beta|^2/2+\beta\overline{\alpha}} \tag{7}$$

(e) Show that

$$\frac{1}{\pi}\int_{\mathbb{C}} d\alpha|\alpha\rangle\langle\alpha| = \hat{1} \tag{8}$$

(f) Show that

$$\mathrm{Tr}\,\hat{O} = \frac{1}{\pi}\int_{\mathbb{C}} d\alpha\langle\alpha|\hat{O}|\alpha\rangle \tag{9}$$

(g) With the above, compute the path integral representation for the propagator:

$$\langle\alpha_b|e^{-it\hat{H}}|\alpha_a\rangle \propto \int D\left\{\alpha(t)\right\} e^{i\int_{t_a}^{t_b} dtL(\alpha,\dot{\alpha},t)} \tag{10}$$

with

$$L(\alpha, \dot{\alpha}, t) = \frac{i}{2}\left(\overline{\alpha}\dot{\alpha} - \alpha\dot{\overline{\alpha}}\right) - \omega\overline{\alpha}\alpha \tag{11}$$

(h) Solve the saddle point equations:

$$\delta \int_{t_a}^{t_b} dt L(\alpha, \dot{\alpha}, t) = 0 \tag{12}$$

and show that they produce the classical equation of motion of the harmonic oscillator.