Notes in

Field Theory

By

Jun Wei Tan

Julius-Maximilians-Universität Würzburg

jun-wei.tan@stud-mail.uni-wuerzburg.de

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1.2.1 Introduction

Because every operator can be written in the formalism of second quantisation as a product of creation and annihilation operators, coherent states turn these operators into scalars, which are then easier to deal with. We define a fermionic coherent state by the usual equation

$$a_k \left|\phi\right\rangle = \phi_k \left|\phi\right\rangle .$$

Because annihilation operators for different k anticommute rather than commute, we must have

$$\phi_i \phi_j = -\phi_j \phi_i .$$

Thus, the ϕ_is cannot be part of a field, because they must anticommute rather than commute! We define the Grassman algebra to be generated by n generators ξ_i, with the basis coming from all products $\xi_i \xi_j$ etc. We will assume that there is an even number of generators, and to each generator ξ_i we assign an inversion $(\xi_i)^* = \xi_j$ such that the inversion satisfies $(\xi^*)^* = \xi$ and $(\xi_i \xi_j)^* = \xi_j^* \xi_i^*$.

Because of the anticommutativity, we have $\xi^2 = -\xi^2 = 0$ for all Grassman numbers ξ. Explicitly, we can construct the Grassman algebra as the exterior algebra on some differential forms. Thus, all analytic functions can be expressed in terms of their Taylor series

$$f(\xi) = f_0 + f_1 \xi .$$

All operators are then bilinear:

$$A(\xi^*, \xi) = a_0 + a_1 \xi + a_2 \xi^* + a_{12} \xi^* \xi .$$

We define the derivatives to be equal to the integral

$$\frac{\partial}{\partial \xi} f(\xi) = f_1 = \int \mathrm{d}\xi \, f(\xi).$$

Notably, we work in spirit analogously to the Wirtinger derivatives, and let ξ and ξ^* be independent. For reasons of anticommutativity, we require that the derivative be next to ξ in order to act on it, for example

$$\frac{\partial}{\partial \xi}(\xi^* \xi) = \frac{\partial}{\partial \xi}(-\xi \xi^*) = -\xi^* .$$

Next, we seek to deal with Gaussian integrals. We will see how they pop up later; for now, it suffices to say that the partition function is the integral of an exponential. After substituting in the fermionic coherent states, we will get something that looks like a Gaussian integral. The desired result is

Gaussian Integrals

$$\int \pi_\alpha \, \mathrm{d}\xi_\alpha^* \, \mathrm{d}\xi_\alpha \exp\left[-\sum_{\alpha,\beta} \xi_\alpha^* M_{\alpha,\beta} \xi_\alpha + \sum_\alpha (J_\alpha^* \xi_\alpha + \xi_\alpha^* J_\alpha)\right] = \det(M) \exp\left(\sum_{\alpha,\beta} J_\alpha^* (M^{-1})_{\alpha,\beta} J_\beta\right)$$

where the Js are Grassman variables and M is Hermitian.

We show this by diagonalising $\lambda = (\lambda_i)_{ii} = UMU^\dagger$. Then,

$$\begin{aligned} -\xi^\dagger M \xi + J^\dagger \xi + \xi^\dagger J &= -\xi^\dagger U^\dagger \lambda U \xi + J^\dagger U^\dagger U \xi + \xi^\dagger U^\dagger U J \\ &= \sum_\alpha (-\lambda_\alpha \eta_\alpha^* \eta_\alpha + \tilde{J}_\alpha^\dagger + \eta_\alpha + \eta_\alpha^\dagger \tilde{J}_\alpha \end{aligned}$$

and hence the integral simplifies to

$$\begin{aligned} &\int \prod_\alpha \mathrm{d}\eta_\alpha^\dagger \, \mathrm{d}\eta_\alpha \exp\left[\sum_\alpha -\lambda_\alpha \eta_\alpha^\dagger \eta_\alpha + \tilde{J}_\alpha^* \eta_\alpha + \eta_\alpha^* \tilde{J}_\alpha\right] \\ =& \prod_\alpha \int \mathrm{d}\eta_\alpha^\dagger \, \mathrm{d}\eta_\alpha \exp\left[-\lambda \eta_\alpha^\dagger \eta_\alpha\right] \exp\left[J_\alpha^* \eta_\alpha + \eta_\alpha^* J_\alpha\right] \\ =& \det(M) \exp\big(J^\dagger M^{-1} J\big) \end{aligned}$$

1.2.2 Wick's Theorem

Now, we are in a good position to prove Wick's theorem, the statement of which is

Wick's Theorem

$$\frac{\int \prod_\alpha \mathrm{d}\psi_\alpha^* \, \mathrm{d}\psi_\alpha \, e^{-\sum_{i,j} \psi_i^* M_{ij} \psi_j + \sum_i (J_i^* \psi_i + \psi_i^* J_i)}}{\int \prod_\alpha \mathrm{d}\psi_\alpha^* \, \mathrm{d}\psi_\alpha \, e^{-\sum_{ij} \psi_i^* M_{ij} \psi_j}} = \sum_P \zeta^P M^{-1}_{i_{P_n}, j_n} \dots M^{-1}_{i_{P_1}, j_1}.$$

To do so, we consider the generating function

$$G(J^*, J) = \frac{\int \prod_\alpha \mathrm{d}\psi_\alpha^* \, \mathrm{d}\psi_\alpha \, e^{-\sum_{i,j} \psi_i^* M_{ij} \psi_j + \sum_i (J_i^* \psi_i + \psi_i^* J_i)}}{\int \prod_\alpha \mathrm{d}\psi_\alpha^* \, \mathrm{d}\psi_\alpha \, e^{-\sum_{ij} \psi_i^* M_{ij} \psi_j}} = e^{\sum_{ij} J_i^* (M^{-1})_{ij} J_j}$$

(note that the action of dividing is to take away the $\det M$). We differentiate the first line