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Problem Sheet 4
for the tutorial on May 30th, 2025
Quantum Mechanics II
Summer term 2025

Sheet handed out on May 20th, 2025; to be handed in on May 27th, 2025 until 2 pm

Exercise 4.1: The dipole approximation and the $\vec{r} \cdot \vec{E}$ Hamiltonian [10 P.]

We now examine the problem of an electron bound by a central potential $V(r)$ to a nucleus located at $\vec{r}_0$. Let us start from the minimal coupling Hamiltonian (1)

$$\left\{-\frac{\hbar^2}{2m}\left[\nabla - i\frac{e}{\hbar}\vec{A}(\vec{r},t)\right]^2 + e\Phi(\vec{r},t)\right\}\Psi = i\hbar\frac{\partial\Psi}{\partial t}\,, \tag{1}$$

considered in the Coulomb gauge and apply the dipole approximation $\vec{A}(\vec{r},t) \simeq \vec{A}(\vec{r}_0,t)$. By performing a wave function transformation

$$\Psi(\vec{r},t) = \exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]\tilde{\Psi}(\vec{r},t) \tag{2}$$

show that the total Hamiltonian can be separated in the unperturbed Hamiltonian of the electron and the interaction term given by $\mathcal{H}_{int} = -e\vec{r}\cdot\vec{E}(\vec{r}_0,t)$.

Exercise 4.2: Phenomenological spontaneous decay in semi-classical theory [5 P.]

As long as the electromagnetic field is treated classically, the spontaneous decay of the system can only be implemented phenomenologically. Let us consider now once more the simple case of a two-level system with states $|1\rangle$ and $|2\rangle$ interacting with a classical electromagnetic field. The finite lifetime of the atomic levels can be described very well by adding phenomenological decay terms to the density operator equation. The decay rates can be incorporated by a relaxation matrix Γ defined as $\langle n|\Gamma|m\rangle = \gamma_n\delta_{nm}$, where $\{n,m\} \in \{1,2\}$. With this addition, we can write the density matrix equation of motion as

$$\dot{\rho} = -\frac{i}{\hbar}[\mathcal{H},\rho] - \frac{1}{2}\{\Gamma,\rho\}\,, \tag{3}$$

where

$$\mathcal{H} = \hbar\Delta\,|2\rangle\langle 2| - \hbar\Omega\,(|2\rangle\langle 1| + |1\rangle\langle 2|) \tag{4}$$

is the Hamiltonian for a two-level system and $\{\Gamma,\rho\} = \Gamma\rho + \rho\Gamma$. Write down the Bloch equations resulting from Eq. (3).

Exercise 4.3: **Hadamard lemma and interaction picture of the JC model** [5+5 P.]

Consider the Hamiltonian of the quantum mechanical harmonic oscillator

$$\mathcal{H}_O = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) \tag{5}$$

with the ladder operators a, $a^\dagger$ satisfying $[a, a^\dagger] = 1$, and $[a, a] = [a^\dagger, a^\dagger] = 0$. In the lecture, we will also find this to be the Hamiltonian for a single mode of the quantized electromagnetic field.

a) Using the time evolution operator $U = e^{-\frac{i}{\hbar}\mathcal{H}_O t}$, calculate $U^\dagger\, a\, U$ and $U^\dagger\, a^\dagger\, U$.

   Hint: Use that according to the Hadamard lemma for two operators A and B

$$e^A\, B\, e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, , B]]] + \dots\,. \tag{6}$$

b) Use $\mathcal{H}_T = \mathcal{H}_O + \hbar x_g\, A_{gg} + \hbar x_e\, A_{ee}$ to transform the Jaynes-Cummings model $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ with

$$\mathcal{H}_0 = \mathcal{H}_O + \hbar\omega_g\, A_{gg} + \hbar\omega_e\, A_{ee}\,, \tag{7}$$

$$\mathcal{H}_I = \hbar \left(g\, A_{eg}\, a + g^*\, a^\dagger A_{ge} \right)\,, \tag{8}$$

   to an interaction picture Hamiltonian without explicit time dependence. Here, $A_{ij} = |i\rangle\,\langle j|$ and g is the atom-field mode coupling constant.

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show that the total Hamiltonian can be separated in the unperturbed Hamiltonian of the electron and the interaction term given by $\mathcal{H}_{int} = -e\vec{r}\cdot\vec{E}(\vec{r}_0,t)$.

$$\left[\frac{\hbar^2}{2m}\left[\nabla - i\frac{e}{\hbar}\vec{A}(\vec{r},t)\right]^2 + e\phi(\vec{r},t)\right]\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$$\vec{A}(\vec{r},t) = \vec{A}(\vec{r}_0,t), \quad \Psi(\vec{r},t) = \exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]\hat{\Psi}(\vec{r},t)$$

$$\left[\frac{\hbar^2}{2m}\left[\nabla - \frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\right]^2 + e\phi(\vec{r},t)\right]\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]\hat{\Psi} = i\hbar\frac{\partial}{\partial t}\left[\left(\exp\left(\frac{ie}{\hbar}\vec{A}\cdot\vec{r}\right)\right)\hat{\Psi}\right]$$

$$= i\hbar\left(\frac{ie}{\hbar}\right)\left(\frac{\partial\vec{A}}{\partial t}\cdot\vec{r}\right)\exp\left(\frac{ie}{\hbar}\vec{A}\cdot\vec{r}\right)\hat{\Psi} + i\hbar\exp\left(\frac{ie}{\hbar}\vec{A}\cdot\vec{r}\right)\frac{\partial\hat{\Psi}}{\partial t}$$

$$= -e\left(\frac{\partial\vec{A}}{\partial t}\cdot\vec{r}\right)\exp\left(\frac{ie}{\hbar}\vec{A}\cdot\vec{r}\right)\hat{\Psi} + i\hbar\exp\left(\frac{ie}{\hbar}\vec{A}\cdot\vec{r}\right)\frac{\partial\hat{\Psi}}{\partial t}$$

$$\left[\nabla - \frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\right]^2\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]\hat{\Psi}$$

$$= \left[\nabla^2 - \frac{2ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\nabla - \left(\frac{e}{\hbar}\right)^2 A^2(\vec{r}_0,t)\right]\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]\hat{\Psi}$$

$$\nabla\left(\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]\right) = \frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]$$

$$\nabla^2\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right] = \frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\nabla\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right] = -\frac{e^2}{\hbar^2}A^2(\vec{r}_0,t)\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]$$

$$\nabla^2\left[\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]\hat{\Psi}\right] = \exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]\nabla^2\hat{\Psi} + 2\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]\right]\cdot\nabla\hat{\Psi}$$
$$- \frac{e^2}{\hbar^2}A^2(\vec{r}_0,t)\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]\hat{\Psi}$$

$$\frac{\hbar^2}{2m}\cancel{\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]}\left[\nabla^2\hat{\Psi} + \frac{2ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\nabla\hat{\Psi} - \frac{e^2}{\hbar^2}A^2(\vec{r}_0,t)\hat{\Psi}\right] + e\phi\cancel{\exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right]}\hat{\Psi}$$

$$= -e\left(\frac{\partial\vec{A}}{\partial t}\cdot\vec{r}\right)\cancel{\exp\left(\frac{ie}{\hbar}\vec{A}\cdot\vec{r}\right)}\hat{\Psi} + i\hbar\cancel{\exp\left(\frac{ie}{\hbar}\vec{A}\cdot\vec{r}\right)}\frac{\partial\hat{\Psi}}{\partial t}$$

$$\underbrace{-\frac{\hbar^2}{2m}\left[\nabla - \frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\right]^2\hat{\Psi}(r,t) + e\phi\hat{\Psi}}_{\text{unperturbed Hamiltonian}} + \underbrace{e\left(\overbrace{\frac{\partial\vec{A}}{\partial t}}^{-\vec{E}}\cdot\vec{r}\right)}_{-e\vec{E}\cdot\vec{r}\ \hat{\Psi}}\hat{\Psi} = i\hbar\frac{\partial\hat{\Psi}}{\partial t}$$

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is the Hamiltonian for a two-level system and $\{\Gamma, \rho\} = \Gamma\rho + \rho\Gamma$. Write down the Bloch equations resulting from Eq. (3).

$$\dot{\rho} = -\frac{i}{\hbar}\left[\begin{pmatrix} 0 & -\hbar\Omega \\ -\hbar\Omega & \hbar\Delta \end{pmatrix}, \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\right] - \frac{1}{2}\left\{\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\right\}$$

By direct computation

$$\dot{\rho}_{11} = -\gamma_1\rho_{11} - i(\rho_{12} - \rho_{21})\Omega$$

$$\dot{\rho}_{12} = \frac{1}{2}\rho_{12}(-\gamma_1 - \gamma_2 + 2i\Omega) - i\rho_{11}\Omega + i\rho_{22}\Omega$$

$$\dot{\rho}_{21} = \frac{1}{2}i(\rho_{21}(i\gamma_1 + i\gamma_2 - 2\Omega) + 2\rho_{11}\Omega - 2\rho_{22}\Omega)$$

$$\dot{\rho}_{22} = i(i\gamma_2\rho_{22} + \rho_{12}\Omega - \rho_{21}\Omega)$$

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to an interaction picture Hamiltonian without explicit time dependence. Here, $A_{ij} = |i\rangle\langle j|$ and g is the atom-field mode coupling constant.

$$U^\dagger a U = e^{\frac{iH_O t}{\hbar}}\, a\, e^{-\frac{iH_O t}{\hbar}}$$

$$= a + \left[\frac{iH_O t}{\hbar}, a\right] + \frac{1}{2!}\left[\frac{iH_O t}{\hbar}, \left[\frac{iH_O t}{\hbar}, a\right]\right] + \dots$$

$$\left[\frac{iH_O t}{\hbar}, a\right] = \frac{it}{\hbar}\left[H_O, a\right] = \frac{it}{\hbar}(\hbar\omega)\left[a^\dagger a + \frac{1}{2}, a\right]$$

$$= -it\omega\left[a, a^\dagger a\right]$$

$$= -it\omega\left\{\underbrace{[a, a^\dagger]}_{1} a + a^\dagger \underbrace{[a, a]}_{0}\right\}$$

$$= -it\omega\, a$$

$$\left[\frac{iH_O t}{\hbar}, \left[\frac{iH_O t}{\hbar}, a\right]\right] = -it\omega\left[\frac{iH_O t}{\hbar}, a\right]$$

$$= (-it\omega)^n a$$

$$U^\dagger a U = \sum_{n=0}^{\infty} \frac{1}{n!}(-it\omega)^n a = e^{-i\omega t} a$$

By adjointing

$$(U^\dagger a U)^\dagger = U^\dagger a^\dagger U = \left(e^{-i\omega t} a\right)^\dagger = e^{i\omega t} a^\dagger$$

which makes sense, because the canonical commutation relations must be preserved

b) Use $\mathcal{H}_T = \mathcal{H}_O + \hbar x_g\, A_{gg} + \hbar x_e\, A_{ee}$ to transform the Jaynes-Cummings model $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ with

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to an interaction picture Hamiltonian without explicit time dependence. Here, $A_{ij} = |i\rangle\langle j|$ and g is the atom-field mode coupling constant.

$$\mathcal{H} = \underbrace{\hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right) + \hbar W_g |g\rangle\langle g| + \hbar W_e |e\rangle\langle e|}_{\mathcal{H}_0}$$

$$\underbrace{+\hbar\left(g|e\rangle\langle g|a + g^\dagger a^\dagger |g\rangle\langle e|\right)}_{\mathcal{H}_I}$$

1. Determine Propagator from $\mathcal{H}_0$

$\mathcal{H}_0$ and the rest commute since they act on different spaces so

$$U_0 = \underbrace{U}_{\text{from }(a)} \otimes \left[e^{iW_g t}|g\rangle\langle g| + e^{iW_e t}|e\rangle\langle e|\right]$$

2. Transform $H_I' = U_0^\dagger H_I U_0$

$$= \left[U^\dagger \otimes \left(e^{-iW_g t}|g\rangle\langle g| + e^{-iW_e t}|e\rangle\langle e|\right)\right]\left[\hbar\left(g|e\rangle\langle g|a + g^\dagger a^\dagger|g\rangle\langle e|\right)\right]$$

$$\left[U \otimes \left[e^{iW_g t}|g\rangle\langle g| + e^{iW_e t}|e\rangle\langle e|\right]\right]$$

Then compute $\left(e^{-iW_g t}|g\rangle\langle g| + e^{-iW_e t}|e\rangle\langle e|\right)|e\rangle\langle g|\left(e^{iW_g t}|g\rangle\langle g| + e^{iW_e t}|e\rangle\langle e|\right) = |e\rangle\langle g|\, e^{i(W_g - W_e)t}$

$\left(e^{-iW_g t}|g\rangle\langle g| + e^{-iW_e t}|e\rangle\langle e|\right)|g\rangle\langle e|\left(e^{iW_g t}|g\rangle\langle g| + e^{iW_e t}|e\rangle\langle e|\right) = |g\rangle\langle e|\, e^{i(W_e - W_g)t}$

$$H_I' = \hbar g\, e^{i(W_g - W_e)t}\, e^{-i\omega t}\, |e\rangle\langle g|\, a + \hbar g^\dagger\, e^{i(W_e - W_g)t}\, e^{i\omega t}\, |g\rangle\langle e|\, a^\dagger$$