Functional Analysis

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Introduction

Chapter 1

Metric and Topological Spaces

In this first and preliminary chapter we recall some of the basic notions from metric spaces and topological spaces as they should be familiar from previous courses in undergraduate analysis. As there are many textbooks on this subject, we will be rather brief and omit several proofs, details, and examples. The aim of this chapter is to arrive at the first important statement, the Baire theorem, which will play a fundamental role throughout functional analysis.

1.1 Metric Spaces

Metric spaces generalize subsets of $\mathbb{R}^n$ with the usual Euclidean distance by axiomatizing the important features of "distance". This way, we can speak about open and closed balls, convergence, and continuity in a much more general context. In this preliminary section we will recap the well-known basic features of metric spaces. Proofs should be known from calculus courses, so that we can be rather brief here.

1.1.1 Definition of Metric Spaces and their Open Subsets

For points $x, y \in \mathbb{R}^n$ one has their Euclidean distance defined by

$$\mathrm{d}(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}. \tag{1.1.1}$$

The well-known properties of this distance are now captured by the following general definition:

Definition 1.1.1 (Metric space) *A metric space is a pair (M, d) of a non-empty set M together with a map*

$$\mathrm{d} \colon M \times M \longrightarrow \mathbb{R}, \tag{1.1.2}$$

called the distance or metric, satisfying the following properties:

i.) One has $\mathrm{d}(p, q) \geq 0$ for all $p, q \in M$ and $\mathrm{d}(p, q) = 0$ iff $p = q$.

ii.) One has the symmetry $\mathrm{d}(p, q) = \mathrm{d}(q, p)$ for all $p, q \in M$.

iii.) The triangle inequality

$$\mathrm{d}(p, q) \leq \mathrm{d}(p, z) + \mathrm{d}(z, q) \tag{1.1.3}$$

holds for all $p, q, z \in M$.

A map $\Phi \colon (M_1, \mathrm{d}_1) \longrightarrow (M_2, \mathrm{d}_2)$ between metric spaces is called *isometry* if

$$\mathrm{d}_2(\Phi(p), \Phi(q)) = \mathrm{d}_1(p, q) \tag{1.1.4}$$

for all $p, q \in M_1$. Clearly, the Euclidean metric (1.1.1) satisfies these requirements, turning $\mathbb{R}^n$ into a metric space. Metric spaces are in fact omnipresent in analysis, a few more exotic examples can be found in Exercise **??**, while Exercise **??** contains some standard constructions of metric spaces.

It is a routine check that the composition of isometries is again an isometry. Moreover, isometries are necessarily injective. One can use isometries to define the category of metric spaces by taking isometries as morphisms. While in principle this is a meaningful definition, it turns out to be rather limited: there are typically not many isometries between metric spaces, making it hard to compare and relate them. We will see more interesting maps soon which will yield more interesting notions of morphisms. In any case, we will call two metric spaces (M, d_M) and (N, d_N) *isometrically isomorphic* if there exists a bijective isometry between them. This implements an equivalence relation for metric spaces.

As in the case of the Euclidean metric of $\mathbb{R}^n$ we can define open and closed balls in a metric space:

Definition 1.1.2 (Open and closed balls) *Let (M, d) be a metric space and $r > 0$.*

i.) The open ball $\mathrm{B}_r(p)$ around $p \in M$ with radius r is defined by

$$\mathrm{B}_r(p) = \big\{ q \in M \;\big|\; \mathrm{d}(p, q) < r \big\}. \tag{1.1.5}$$

ii.) The closed ball $\mathrm{B}_r(p)^{\mathrm{cl}}$ around $p \in M$ with radius r is defined by

$$\mathrm{B}_r(p)^{\mathrm{cl}} = \big\{ q \in M \;\big|\; \mathrm{d}(p, q) \leq r \big\}. \tag{1.1.6}$$

Sometimes it is useful to think of the single point set $\{p\}$ as a closed ball of radius zero. Indeed, for $r = 0$ the condition (1.1.6) gives just $\{p\}$.

Metric balls capture the geometric information in a metric space very well. As a first step we use them to define general open and closed subsets in a metric space.

Definition 1.1.3 (Open and closed subsets) *Let (M, d) be a metric space.*

i.) A subset $O \subseteq M$ is called open if for every $p \in O$ there exists a radius $r > 0$ with

$$\mathrm{B}_r(p) \subseteq O. \tag{1.1.7}$$

ii.) A subset $A \subseteq M$ is called closed if $M \setminus A$ is open.

The following properties of open and closed subsets are familiar from the case of the Euclidean metric and follow with the triangle inequality right away, see also Exercise 1.5.1:

Proposition 1.1.4 *Let (M, d) be a metric space.*

i.) All open ball $\mathrm{B}_r(p)$ for $r > 0$ and $p \in M$ are open.

ii.) The empty set $\emptyset$ as well as M are open and closed.

iii.) If $\{O_i\}_{i \in I}$ is a set of open subsets $O_i \subseteq M$ then the union $\bigcup_{i \in I} O_i \subseteq M$ is open.

iv.) If $O_1, \ldots, O_n \subseteq M$ are finitely many open subsets then $O_1 \cap \cdots \cap O_n \subseteq M$ is open.

v.) All closed balls $\mathrm{B}_r(p)^{\mathrm{cl}}$ for $r \geq 0$ and $p \in M$ are closed.

vi.) If $\{A_i\}_{i \in I}$ is a set of closed subsets $A_i \subseteq M$ then the intersection $\bigcap_{i \in I} A_i \subseteq M$ is closed.

vii.) If $A_1, \ldots, A_n \subseteq M$ are finitely many closed subsets then $A_1 \cup \cdots \cup A_n \subseteq M$ is open.

We can use open balls also to define neighbourhoods. One should note that with our convention neighbourhoods need not be open subsets:

Definition 1.1.5 (Neighbourhood) *Let (M, d) be a metric space and let $p \in M$. Then a subset $U \subseteq M$ is called neighbourhood of p if there exists a radius $r > 0$ with*

$$\mathrm{B}_r(p) \subseteq U. \tag{1.1.8}$$

In particular, a neighbourhood of p contains p. We collect some first properties of neighbourhoods:

Proposition 1.1.6 *Let (M, d) be a metric space.*

i.) *A subset $U \subseteq M$ is a neighbourhood of $p \in M$ iff it contains an open subset $O \subseteq U$ with $p \in O$.*

ii.) *A neighbourhood of p contains p.*

iii.) *If $U \subseteq M$ is a neighbourhood of $p \in M$ and $U \subseteq U' \subseteq M$ then U' is a neighbourhood of p, too.*

iv.) *If $U_1, \ldots, U_n \subseteq M$ are neighbourhoods of $p \in M$ then $U_1 \cap \cdots \cap U_n \subseteq M$ is a neighbourhood of p, too.*

v.) *If $U \subseteq M$ is a neighbourhood of $p \in M$ then there exists a neighbourhood $V \subseteq M$ of p with $V \subseteq U$ such that V is a neighbourhood of all $q \in V$.*

vi.) *A subset $O \subseteq M$ is open iff it is a neighbourhood of all points $p \in O$.*

Again, the proof requires a simple verification similar to the more familiar situation of the Euclidean metric for $\mathbb{R}^n$, see Exercise 1.5.1. The last part is interesting in so far as it allows to reconstruct the open subsets from the collection of all neighbourhoods.

1.1.2 Convergence and Completeness

As already in elementary calculus, we are interested in convergence properties of sequences, now for sequences in metric spaces. The familiar definition from calculus can be copied almost literally:

Definition 1.1.7 (Convergent sequence and Cauchy sequence) *Let (M, d) be a metric space.*

i.) *A sequence $(p_n)_{n\in\mathbb{N}}$ of points $p_n \in M$ is called convergent to $p \in M$ if for all $\varepsilon > 0$ one finds an $N \in \mathbb{N}$ such that*

$$\mathrm{d}(p_n, p) < \varepsilon \tag{1.1.9}$$

for all $n \geq N$. In this case, p is called the limit of the sequence $(p_n)_{n\in\mathbb{N}}$ and we write

$$p = \lim_{n\to\infty} p_n \tag{1.1.10}$$

or $p_n \longrightarrow p$ for short.

ii.) *A sequence $(p_n)_{n\in\mathbb{N}}$ of points $p_n \in M$ is called a Cauchy sequence if for all $\varepsilon > 0$ one finds an $N \in \mathbb{N}$ such that*

$$\mathrm{d}(p_n, p_m) < \varepsilon \tag{1.1.11}$$

for all $n, m \geq N$.

The abstract properties of a metric are now sufficient to recover the basic features of convergent sequences as known from elementary calculus. We collect some of them:

Proposition 1.1.8 *Let (M, d) be a metric space.*

i.) *If a sequence $(p_n)_{n\in\mathbb{N}}$ in M converges then its limit is unique.*

ii.) *A sequence $(p_n)_{n\in\mathbb{N}}$ in M converges to $p \in M$ iff every neighbourhood of p contains all but finitely many elements of the sequence.*

iii.) *A convergent sequence is a Cauchy sequence.*

iv.) *A Cauchy sequence $(p_n)_{n\in\mathbb{N}}$ in M is bounded, i.e. there exists a $p_0 \in M$ and a radius $r > 0$ with*

$$p_n \in \mathrm{B}_r(p_0) \tag{1.1.12}$$

for all $n \in \mathbb{N}$.

Again, the familiar proofs from calculus transfer easily also to this more general framework. The last observation motivates the following definition of a bounded subset:

Definition 1.1.9 (Bounded subset) *A subset $B \subseteq M$ of a metric space (M, d) is called bounded if there exists an open ball $\mathrm{B}_r(p)$ such that*

$$B \subseteq \mathrm{B}_r(p). \tag{1.1.13}$$

In particular, the points in a Cauchy sequence and hence also the points in a convergent sequence form a bounded subset. The converse needs not to be true, not every bounded sequence is convergent. This is already the case for the Euclidean metric in $\mathbb{R}^n$.

In view of Proposition 1.1.8, *iii.)*, the question arises whether a Cauchy sequence is convergent, i.e. whether we have equivalence instead of a one-sided implication. In general, this is not the case.

Definition 1.1.10 (Complete metric space) *A metric space (M, d) is called complete if every Cauchy sequence in M converges.*

Not every metric space is complete. The standard example is the non-complete $\mathbb{Q}$ with the Euclidean metric in contrast to the complete $\mathbb{R}$. More examples can be found in Exercise **??**.

Once we have completeness it is inherited to closed subsets. This gives an important construction of new complete metric spaces:

Proposition 1.1.11 *Let (M, d) be a metric space.*

i.) *A subset $A \subseteq M$ is closed iff for every convergent sequence $(p_n)_{n\in\mathbb{N}}$ in M with $p_n \in A$ for all $n \in \mathbb{N}$ one has $\lim_{n\to\infty} p_n \in A$.*

ii.) *If (M, d) is complete then a subset $A \subseteq M$ is itself complete with respect to the restricted metric $\mathrm{d}_A = \mathrm{d}\big|_{A\times A}$ iff A is closed.*

1.1.3 Continuous and Lipschitz Continuous Maps

We can now copy the usual $\varepsilon\delta$-continuity and state the following definition:

Definition 1.1.12 ($\varepsilon\delta$-continuity) *Let $f\colon (M, \mathrm{d}_M) \longrightarrow (N, \mathrm{d}_N)$ be a map between metric spaces.*

i.) *The map f is called continuous at $p \in M$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\mathrm{d}_N(f(p), f(q)) < \varepsilon \quad \textit{whenever} \quad \mathrm{d}_M(p, q) < \delta. \tag{1.1.14}$$

ii.) *The map f is called continuous if it is continuous at every point $p \in M$.*

An immediate reformulation of (1.1.14) is that for all $\varepsilon > 0$ there is a $\delta > 0$ with

$$\mathrm{B}_\delta(p) \subseteq f^{-1}(\mathrm{B}_\varepsilon(f(p))). \tag{1.1.15}$$

This leads to the following equivalent descriptions of continuity:

Proposition 1.1.13 *Let $f\colon (M, \mathrm{d}_M) \longrightarrow (N, \mathrm{d}_N)$ be a map between metric spaces. Then the following statements are equivalent:*

i.) *The map f is continuous at $p \in M$.*

ii.) *For every convergent sequence $(p_n)_{n\in\mathbb{N}}$ in M with $\lim_{n\to\infty} p_n = p$ the sequence $(f(p_n))_{n\in\mathbb{N}}$ is convergent in N with*

$$\lim_{n\to\infty} f(p_n) = f(p). \tag{1.1.16}$$

iii.) *The preimage of every neighbourhood of $f(p)$ is a neighbourhood of p.*

The equivalence of *i.)* and *iii.)* is quite straightforward, while the equivalence to *ii.)* requires the choice of suitable sequences. The property in *ii.)* is also referred to as *sequential continuity* at p. If we have continuity everywhere, Proposition 1.1.13 yields directly the following characterization:

Proposition 1.1.14 *Let $f\colon (M, \mathrm{d}_M) \longrightarrow (N, \mathrm{d}_N)$ be a map between metric spaces. Then the following statements are equivalent:*

i.) *The map f is continuous.*

ii.) *The preimage of every open subset in N is open in M.*

iii.) *The preimage of every closed subset in N is closed in M.*

The equivalence *ii.)* $\Leftrightarrow$ *iii.)* is trivial since taking preimages is compatible with taking complements, i.e. one has

$$f^{-1}(N \setminus X) = M \setminus f^{-1}(X) \tag{1.1.17}$$

for all maps $f\colon M \longrightarrow N$ between sets and all subsets $X \subseteq N$. The interesting equivalence relies on Proposition 1.1.6, *vi.)*. We will see drastic generalizations of these two propositions later on. A first consequence of Proposition 1.1.13, *iii.)*, and Proposition 1.1.14, *ii.)*, is the following:

Corollary 1.1.15 *Let $f\colon (M, \mathrm{d}_M) \longrightarrow (N, \mathrm{d}_N)$ and $g\colon (N, \mathrm{d}_N) \longrightarrow (K, \mathrm{d}_K)$ be maps between metric spaces.*

i.) *If f is continuous at $p \in M$ and g is continuous at $f(p) \in N$ then $g \circ f$ is continuous at p.*

ii.) *If f and g are continuous then $g \circ f$ is continuous, too.*

Since the identity map $\mathrm{id}_M\colon (M, \mathrm{d}_M) \longrightarrow (M, \mathrm{d}_M)$ is obviously continuous, we arrive at a good categorical framework: taking continuous maps between metric spaces as morphisms yields a category. This is already much more interesting than taking just isometries, since we typically have much more continuous maps.

More specific to the metric situation are *Lipschitz continuous maps.* Here we explicitly use estimates between the metrics and not just open subsets:

Definition 1.1.16 (Lipschitz continuous maps) *Let $f\colon (M, \mathrm{d}_M) \longrightarrow (N, \mathrm{d}_N)$ be a map between metric spaces.*

i.) *The map f is called Lipschitz continuous if there exists a constant $\lambda \geq 0$ with*

$$\mathrm{d}_N(f(p), f(q)) \leq \lambda \, \mathrm{d}_M(p, q) \tag{1.1.18}$$

for all $p, q \in M$. In this case, λ is called a Lipschitz constant for f.

ii.) *The map f is called locally Lipschitz continuous if for every $p \in M$ there is a neighbourhood $U \subseteq M$ of p such that $f\big|_U$ is Lipschitz continuous.*

If there is a Lipschitz constant $\lambda < 1$ then f is also called a *contraction.* Lipschitz continuous maps are particular continuous maps. More precisely, we have the following properties:

Proposition 1.1.17

i.) *The composition of (locally) Lipschitz continuous maps is again (locally) Lipschitz continuous.*

ii.) *The identity map is Lipschitz continuous with Lipschitz constant $\lambda = 1$.*

iii.) *An isometry is Lipschitz continuous with Lipschitz constant $\lambda = 1$.*

iv.) *A Lipschitz continuous map is locally Lipschitz continuous.*

v.) *A locally Lipschitz continuous map is continuous.*

The first two parts show that we get again a good categorical framework if we take Lipschitz continuous maps or locally Lipschitz continuous maps as morphisms. The implications in *iv.)* and *v.)* can not be reversed in general, as simple examples show.

1.1.4 Completion of Metric Spaces

Completeness is, not only in analysis, a very desirable feature of a metric space. Since one has easy examples of metric spaces which are not complete but still relevant for various reasons, one wants a construction to add ideal points, turning them into complete spaces. This is the well-known idea underlying the construction of the reals $\mathbb{R}$ out of the rationals $\mathbb{Q}$. Note, however, that the completion of $\mathbb{Q}$ to $\mathbb{R}$ is slightly more complicated, since here we do not yet have the reals at hand where the Euclidean metric should take its values. Thus one has to proceed in two steps here.

For a metric space (M, d) we can of course already make use of the real numbers by the very definition of a metric d. Before we can define a completion, we need the notion of closures and dense subsets:

Definition 1.1.18 (Closure) *Let (M, d) be a metric space with a subset $A \subseteq M$. Then the closure A^{cl} of A is the smallest closed subset $A^{\mathrm{cl}} \subseteq M$ containing A.*

This is well-defined since M is closed and $A \subseteq M$, so first of all there are closed subsets containing A. Second, the intersection of closed subsets stays closed by Proposition 1.1.4, *vi.)*. Hence

$$A^{\mathrm{cl}} = \bigcap_{\substack{C \subseteq M \text{ closed} \\ A \subseteq C}} C \tag{1.1.19}$$

is closed and clearly the smallest closed subset containing A. We have the following alternative characterizations:

Proposition 1.1.19 *Let (M, d) be a metric space with a subset $A \subseteq M$. For a point $p \in M$ the following statements are equivalent:*

i.) *One has $p \in A^{\mathrm{cl}}$.*

ii.) *For every neighbourhood $U \subseteq M$ of p one has $U \cap A \neq \emptyset$.*

iii.) *For every $r > 0$ one has $\mathrm{B}_r(p) \cap A \neq \emptyset$.*

iv.) *There exists a sequence $(p_n)_{n \in \mathbb{N}}$ of points $p_n \in A$ with $\lim_{n \to \infty} p_n = p$.*

The important observation is that the equivalence *i.)* $\iff$ *ii.)* follows solely using the properties of closed subsets from Proposition 1.1.4, *ii.)*, *vi.)*, and *vii.)*, as well as the properties of neighbourhoods from Proposition 1.1.6. The equivalence of *ii.)* and *iii.)* follows from the observation that every neighbourhood of p contains an open ball $\mathrm{B}_r(p)$ with some appropriate $r > 0$ and conversely, every such ball $\mathrm{B}_r(p)$ is a neighbourhood. Finally, the equivalence to *iv.)* is obtained by choosing points $p_n \in \mathrm{B}_{\frac{1}{n}}(p) \cap A$.

The last statement can be interpreted as an approximation property: points in the closure of A can be approached by convergent sequences of points in A. This idea of approximating points in A^{cl} by points in A is the motivation for the following definition:

Definition 1.1.20 (Dense subset) *Let (M, d) be a metric space and let $X, Y \subseteq M$ be subsets. Then X is called dense in Y if*

$$X \subseteq Y \subseteq X^{\mathrm{cl}}. \tag{1.1.20}$$

We say X is dense if X is dense in M.

Since clearly $M = M^{\mathrm{cl}}$ and every closure $X^{\mathrm{cl}} \subseteq M$ still is a subset of M, a subset X is dense iff

$$X^{\mathrm{cl}} = M. \tag{1.1.21}$$

In this sense, X is rich enough to approximate all points in M, even though $X \neq M$ might be a proper subset.

One of the important features of dense subsets is that continuous maps are determined by their values on them:

Proposition 1.1.21 *Let $f, g \colon (M, \mathrm{d}_M) \longrightarrow (N, \mathrm{d}_N)$ be continuous maps between metric spaces and let $X \subseteq M$ be dense. If $f\big|_X = g\big|_X$ then $f = g$.*

The easiest way to check this is using Proposition 1.1.19, *iv.)*. Note that the converse is far less trivial. Given a function $f \colon X \longrightarrow N$ which is continuous with respect to the restricted metric $\mathrm{d}_X = \mathrm{d}\big|_{X \times X}$, it is not clear and in fact not true in general that this f has an extension to a continuous function on M. The proposition only tells that if such an extension exists, it is necessarily unique. Again, this supports the idea that dense subsets are considered to be large.

We can now state the concept of a completion. The idea is to find a complete metric space $(\widehat{M}, \widehat{\mathrm{d}})$ for the given (M, d) into which (M, d) is isometrically embedded. This alone is not yet interesting as $\widehat{M}$ could be unnecessarily large. Thus M should also be dense in $\widehat{M}$:

Definition 1.1.22 (Completion of metric space) *Let (M, d) be a metric space. A metric space $(\widehat{M}, \widehat{\mathrm{d}})$ together with a map $\iota \colon M \longrightarrow \widehat{M}$ is called a completion of (M, d) if the following conditions are satisfied:*

i.) *The metric space $(\widehat{M}, \widehat{\mathrm{d}})$ is complete.*

ii.) *The map $\iota \colon (M, \mathrm{d}) \longrightarrow (\widehat{M}, \widehat{\mathrm{d}})$ is isometric.*

iii.) *One has $\iota(M)^{\mathrm{cl}} = \widehat{M}$.*

It turns out that there always exists a completion of a given metric space. The idea is to use equivalence classes of Cauchy sequences as new ideal points in $\widehat{M}$:

Proposition 1.1.23 *Let (M, d) be a metric space. Then there exists a completion of (M, d).*

1.2 Topological Spaces

1.3 Convergence and Compactness

1.4 Baire Spaces

1.5 Exercises

Exercise 1.5.1 (Topology of a metric space) Let (M, d) be a metric space.

i.) Provide a detailed proof of Proposition 1.1.4.

Hint: The argument is entirely parallel to the case of open subsets of $\mathbb{R}$ as known from calculus. One has to replace the usage of the absolute value $|\cdot|$ by the metric everywhere.

ii.) Provide a detailed proof of Proposition 1.1.6.

Exercise 1.5.2 (Subbases and comparing topologies)

Exercise 1.5.3 (Subbasis of the subspace topology) Let $(M, \mathscr{M})$ be a topological space with a subset $N \subseteq M$, equipped with the subspace topology $\mathscr{N} = \mathscr{M}\big|_N$. Suppose that $\mathscr{S} \subseteq \mathscr{M}$ is a subbasis of the topology. Show that

$$\mathscr{S}\Big|_N = \big\{ N \cap S \bigm| S \in \mathscr{S} \big\} \tag{1.5.1}$$

is a subbasis of $\mathscr{N}$. Is $\mathscr{S}\big|_N$ a basis of $\mathscr{N}$ for a basis $\mathscr{S}$ of $\mathscr{M}$?

Exercise 1.5.4 (Continuity of the diagonal) Let $(M, \mathscr{M})$ be a topological space and consider the diagonal map

$$\Delta \colon M \ni p \;\mapsto\; \Delta(p) = (p, p) \in M \times M. \tag{1.5.2}$$

i.) Show that for $X, Y \subseteq M$ one has $\Delta^{-1}(X \times Y) = X \cap Y$.

ii.) Show that the diagonal map is an injective continuous map.

iii.) Show that the diagonal map is an embedding.

Hint: Here one has to determine the subspace topology of $\Delta(M) \subseteq M \times M$ explicitly.

iv.) Show that the image $\Delta(M) \subseteq M \times M$ is closed iff M is Hausdorff. This gives yet another interpretation of the Hausdorff property.

Exercise 1.5.5 (Cartesian product of continuous maps) Let $(M_i, \mathscr{M}_i)$ and $(N_i, \mathscr{N}_i)$ be topological spaces for $i \in I$ with some index set I. Moreover, let $\phi_i \colon M_i \longrightarrow N_i$ be maps for $i \in I$.

i.) Show that the product map

$$\phi = \prod_{i \in I} \phi_i \colon M = \prod_{i \in I} M_i \ni (p_i)_{i \in I} \mapsto (\phi_i(p_i))_{i \in I} \in N = \prod_{i \in I} N_i \tag{1.5.3}$$

is continuous with respect to the Cartesian product topologies iff ϕ_i is continuous for all $i \in I$.

Hint: Here it is useful to consider *sections* $s_i \colon M_i \longrightarrow M$ defined for given points $p_j \in M_j$ with $j \neq i$ by

$$(s_i(p_i))_j = \begin{cases} p_i & i = j \\ p_j & \text{otherwise} \end{cases}. \tag{1.5.4}$$

Visualize the geometric meaning for such sections for a Cartesian product of two factors and show that s_i is continuous in general.

ii.) Formulate and prove that the Cartesian product is functorial.

Exercise 1.5.6 (Neighbourhood basis in Cartesian product)

Chapter 2

Banach Spaces

Banach spaces beyond $\mathbb{R}^n$ are the first and probably most important class of spaces one is investigating in functional analysis. In this chapter we will give a first look at the analytic and geometric features of Banach spaces. On the one hand, they are complete metric spaces so that all results on metric spaces can be applied. On the other hand, they are vector spaces making them subject to techniques from linear algebra. Since the interesting Banach spaces will be infinite dimensional, linear algebra alone will not be very efficient and typically not answering the interesting questions. However, the combination of linear-algebraic and analytic concepts turns out the be the right framework.

Banach spaces constitute a class of particular topological vector spaces which we will investigate first to set up the context properly. This will not only clarify the notions of convergence, completeness and continuity from a general perspective but becomes important also for Banach spaces as soon as we will investigate coarser but still important topologies on them. After a general discussion of the passage from normed to Banach spaces we investigate several standard constructions known from linear algebra but now with analytic enhancements. Here we will see many examples from various areas of analysis, most prominently the function spaces of continuous functions, bounded functions or integrable functions. But also sequence spaces will provide interesting examples.

Here and in the following, all vector spaces will be either real or complex. To handle both situations simultaneously, we write

$$\mathbb{K} \text{ for either } \mathbb{R} \text{ or } \mathbb{C}, \tag{2.0.1}$$

and consider vector spaces over $\mathbb{K}$ with $\mathbb{K}$-linear maps between them. Only if the distinction between $\mathbb{R}$ and $\mathbb{C}$ becomes important we emphasize this by explicitly mentioning the underlying field of scalars. The field $\mathbb{K}$ is always equipped with its canonical topology induced by the metric coming from the absolute value $|\cdot|$ on $\mathbb{K}$.

2.1 Topological Vector Spaces

Finite-dimensional vector spaces can be well and fully understood by techniques from linear algebra. However, in many examples and applications in all branches of mathematics and the sciences, vector spaces of infinite dimension show up naturally. With linear-algebraic methods one can say certain general things also in this case. But ultimately, the setting remains too weak to obtain something interesting, useful and non-trivial. This changes if one endows the vector spaces with topologies and requires linear maps to be continuous in addition. Of course, the topology should be suitably compatible with the linear structure for the mutual benefit. In this section we establish this compatibility and give some first definitions related to the notion of a topological vector space. It will turn out that this setting is still too general to admit many non-trivial statements. This is the reason to move on to Banach spaces in the next section.

2.1.1 Topological Vector Spaces and Convergence

The idea with topological vector spaces is that we want a topology such that the vector space operations are continuous maps:

Definition 2.1.1 (Topological vector space) *A topological vector space $(V, \mathcal{V})$ is a $\mathbb{K}$-vector space V together with a topology $\mathcal{V}$ for V such that the addition of vectors*

$$+ \colon V \times V \ni (v, w) \;\mapsto\; v + w \in V \tag{2.1.1}$$

and the multiplication with scalars

$$\cdot \colon \mathbb{K} \times V \ni (z, v) \;\mapsto\; zv \in V \tag{2.1.2}$$

are continuous.

Here the Cartesian products $V \times V$ and $\mathbb{K} \times V$ are equipped with the corresponding product topologies. If the context is clear we will write just V instead of $(V, \mathcal{V})$.

Proposition 2.1.2 *Let V be a topological vector space.*

i.) For every $v \in V$ the translation

$$\tau_v \colon V \ni w \;\mapsto\; \tau_v(w) = w + v \in V \tag{2.1.3}$$

is a homeomorphism, thus defining a continuous group action of the additive group $(V, +)$ on V.

ii.) For every $z \in \mathbb{K}^\times$ the scaling

$$V \ni v \;\mapsto\; zv \in V \tag{2.1.4}$$

is a homeomorphism, thus defining a continuous group action of the multiplicative group $\mathbb{K}^\times$ on V.

PROOF: Fixing one argument does not spoil the continuity of (2.1.1) and (2.1.2), respectively. Hence (2.1.3) and (2.1.4) are continuous. Since the inverse of τ_v is τ_{-v}, we have a homeomorphism. The same argument holds for (2.1.4) since the scaling with $\frac{1}{z}$ is continuous again. The statement about the group actions is entirely algebraic and holds for vector spaces in general. □

Note that the *reflection*

$$V \ni v \;\mapsto\; -v \in V \tag{2.1.5}$$

is a particular case of (2.1.4) and thus an involutive homeomorphism of V. Since homeomorphisms preserve openness and neighbourhoodness, we get the following useful consequence:

Corollary 2.1.3 *Let V be a topological vector space with $v \in V$.*

i.) For every (open) neighbourhood $U \subseteq V$ of 0 also $-U \subseteq V$ is an (open) neighbourhood of 0.

ii.) For every (open) neighbourhood $U \subseteq V$ of 0 the subset $\tau_v(U) = U + v$ is an (open) neighbourhood of v.

iii.) Let $\{B_i\}_{i \in I}$ be a basis of (open) neighbourhoods of $0 \in V$. Then $\{\tau_v(B_i)\}_{i \in I}$ as well as $\{\tau_v(-B_i)\}_{i \in I}$ are both (open) neighbourhoods of v.

Since a topology can be reconstructed from its systems of neighbourhoods and since in a topological vector space the neighbourhoods of $0 \in V$ determine all other neighbourhood systems by Corollary 2.1.3, *ii.)*, this single neighbourhood system is enough to fix the topology. This is the translation invariance of the topology of a topological vector space.

The translation invariance of the topology also simplifies convergence of nets drastically: Everything can be decided with a basis of zero neighbourhoods:

Corollary 2.1.4 *Let V be a topological vector space and let $(v_i)_{i \in I}$ be a net in V. Then the following statements are equivalent:*

i.) *The net $(v_i)_{i \in I}$ converges to $v \in V$.*

ii.) *The net $(v_i - v)_{i \in I}$ converges to 0.*

iii.) *For all neighbourhoods $U \subseteq V$ of zero one has an index $i \in I$ such that*

$$v_j - v \in U \tag{2.1.6}$$

for $j \succcurlyeq i$.

iv.) *There exists a basis of neighbourhoods of zero such that iii.) holds for the neighbourhoods of this basis.*

This is a trivial consequence of the continuity of τ_{-v} and τ_v as well as the observation that $\tau_v(0) = v$. We will use this characterization quite frequently.

Neighbourhoods of zero can also be used to exhaust the vector space V by scaling. Since for a given vector $v \in V$ the sequence $(\frac{1}{n}v)_{n \in \mathbb{N}}$ is a zero sequence according to the continuity of the multiplication with scalars, we conclude that for every neighbourhood $U \subseteq V$ of 0 we have

$$V = \bigcup_{n=1}^{\infty} nU. \tag{2.1.7}$$

The translation invariance also simplifies the question of first countability:

Corollary 2.1.5 *A topological vector space V is first countable iff $0 \in V$ has a countable basis of neighbourhoods.*

The continuity of the vector space operations shows that taking closures of subspaces yields again subspaces:

Proposition 2.1.6 *Let $W \subseteq V$ be a subspace in a topological vector space. Then the topological closure W^{cl} of W is still a subspace.*

PROOF: We use the characterization of the closure W^{cl} by convergent nets according to **??**. For $v, w \in W^{\mathrm{cl}}$ we find nets $(v_i)_{i \in I}$ and $(w_i)_{i \in I}$ with the same index set I such that $v_i \to v$ and $w_i \to w$. Indeed, as index sets one can take a basis of neighbourhoods and thanks to Corollary 2.1.3, *iii.)*, they look the same for all points. Then the continuity of the vector space operations give the convergence

$$\lambda v_i + \eta w_i \longrightarrow \lambda v + \eta w,$$

showing $\lambda v + \eta w \in W^{\mathrm{cl}}$ for all $\lambda, \eta \in \mathbb{K}$. □

We will endow subspaces of a topological vector space with the subspace topology. This turns them into topological vector spaces themselves, since the restrictions of the continuous maps (2.1.1) and (2.1.2) stay continuous.

We mention now the following result without proof, as we will see an almost trivial proof later for the more particular case of normed spaces. Nevertheless, a guided proof can be found in Exercise 2.5.1.

Proposition 2.1.7 *A topological vector space is a T_3-space. In fact, for every compact $K \subseteq V$ and every closed $A \subseteq V$ with $K \cap A = \emptyset$ we find an open neighbourhood $U \subseteq V$ of zero such that*

$$(K + U) \cap (A + U) = \emptyset. \tag{2.1.8}$$

Remark 2.1.8 (Hausdorff topological vector spaces) Note that we have not required a topological vector space V to be Hausdorff. In most cases this will be satisfied and in the literature this is sometimes even part of the definition. However, there is at least one naturally appearing class of examples where one starts with non-Hausdorff topologies as we will see in Examples 2.2.9 and Examples 2.2.10. In view of the last proposition, we conclude the equivalence of the following properties:

i.) V is a T_1-space.

ii.) V is Hausdorff.

iii.) V is regular, i.e. a T_1- and a T_3-space.

Indeed, one has the implications *iii.)* $\implies$ *ii.)* $\implies$ *i.)* in general. Since topological vector spaces are T_3 automatically by Proposition2.1.7, the equivalence follows.

2.1.2 Continuity of Linear Maps

Since for topological vector spaces we have two structures, the topology and the linear structure, we want morphisms between them to respect both. This gives the following definition:

Definition 2.1.9 (Continuous linear maps and topological dual) *Let V and W be topological vector spaces over $\mathbb{K}$.*

i.) A morphism $\phi\colon V \longrightarrow W$ is a continuous linear map.

ii.) The set of continuous linear maps from V to W is denoted by

$$\mathrm{L}(V, W) = \big\{\phi\colon V \longrightarrow W \;\big|\; \phi \textit{ is a continuous linear map}\big\}. \tag{2.1.9}$$

We set $\mathrm{L}(V) = \mathrm{L}(V, V)$.

iii.) The topological dual of V is defined by

$$V' = \mathrm{L}(V, \mathbb{K}). \tag{2.1.10}$$

The set of all linear maps is denoted by

$$\mathrm{Hom}_{\mathbb{K}}(V, W) = \big\{\phi\colon V \longrightarrow W \;\big|\; \phi \text{ is } \mathbb{K}\text{-linear}\big\}, \tag{2.1.11}$$

and we set

$$\mathrm{End}_{\mathbb{K}}(V) = \mathrm{Hom}_{\mathbb{K}}(V, V) \tag{2.1.12}$$

for abbreviation as usual. Here we completely forget about the topologies and treat V and W just as vector spaces. The *algebraic dual* is then

$$V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}). \tag{2.1.13}$$

One should be careful with the notation for the topological and the algebraic dual. This is not uniform in the literature and there are textbooks where V^* denotes the topological dual while V' is the algebraic one.

Another way to phrase the definition of the morphism space is

$$\mathrm{L}(V, W) = \mathrm{Hom}_{\mathbb{K}}(V, W) \cap \mathscr{C}(V, W) \tag{2.1.14}$$

and

$$V' = V^* \cap \mathscr{C}(V, \mathbb{K}). \tag{2.1.15}$$

We collect a few first properties thereby justifying the notion "morphism":

Proposition 2.1.10 *Let V, W, and X be topological vector spaces.*

i.) *One has* $\mathrm{id}_V \in \mathrm{L}(V)$.

ii.) *For* $\phi \in \mathrm{L}(V, W)$ *and* $\psi \in \mathrm{L}(W, X)$ *one has* $\psi \circ \phi \in \mathrm{L}(V, X)$.

iii.) *For* $\phi, \psi \in \mathrm{L}(V, W)$ *and* $\lambda, \mu \in \mathbb{K}$ *one has* $\lambda\phi + \mu\psi \in \mathrm{L}(V, W)$ *as well as* $0 \in \mathrm{L}(V, W)$.

PROOF: The first statement is clear. Since the composition of linear maps stays linear and the composition of continuous maps stays continuous, the second follows at once. For the third statement, the constant map $0\colon V \longrightarrow W$ is clearly continuous and linear. Moreover, $\lambda\phi + \mu\psi$ stays linear. To prove continuity, one can most easily use convergent nets in V and the characterization of continuity by nets, see Exercise 2.5.2. We take the opportunity to present a more conceptual and diagrammatic proof showing some further insight. First, the diagonal map

$$\Delta\colon V \ni v \;\mapsto\; (v, v) \in V \times V$$

is continuous in general for arbitrary topological spaces, see Exercise 1.5.4. Also the Cartesian product of continuous maps

$$\phi \times \psi\colon V \times V \ni (v, v') \;\mapsto\; (\phi(v), \psi(v')) \in W \times W$$

is continuous in general, see Exercise 1.5.5. Finally, the continuity of the vector space operations in W shows that the map

$$\ell_{\lambda,\mu}\colon W \times W \ni (w, w') \;\mapsto\; \lambda w + \mu w' \in W$$

is continuous for every $\lambda, \mu \in \mathbb{K}$. Indeed, it is a composition of the scalings with λ and μ, respectively, followed by $+$. Then the composition

$$\lambda\phi + \mu\psi = \ell_{\lambda,\mu} \circ (\phi \times \psi) \circ \Delta$$

is continuous, too. Note that this proof only uses the topological vector space structure of W, but not that of V. The same proof applies to any topological space V and the set of W-valued maps on V, see also Exercise 2.5.3. ☐

Corollary 2.1.11 *The topological vector spaces over* $\mathbb{K}$ *form a category with respect to continuous linear maps as morphisms.*

Corollary 2.1.12 *The continuous linear maps* $\mathrm{L}(V, W)$ *form a subspace of* $\mathrm{Hom}_{\mathbb{K}}(V, W)$. *In particular, the topological dual* V' *is a vector space again.*

Before investigating the properties of the space of continuous linear maps further, we need the definition of uniformly continuous maps extended to topological vector spaces. Note that we do not have any metric involved yet and hence the definition from the theory of metric spaces does not apply directly. Nevertheless, the idea is quite familiar:

Definition 2.1.13 (Uniformly continuous maps) *Let* $\phi\colon V \longrightarrow W$ *be a map between topological vector spaces. Then* ϕ *is called uniformly continuous if for all neighbourhoods* $U \subseteq W$ *of zero there exists a neighbourhood* $Z \subseteq V$ *of zero such that*

$$\phi(v) - \phi(v') \in U, \tag{2.1.16}$$

whenever $v - v' \in Z$.

It is fairly easy to see that uniformly continuous maps are continuous. The idea of uniform continuity is that in (2.1.16) *one* neighbourhood Z is enough to control the continuity at *all* points uniformly. To make sense of this one needs the translation invariance of the topology, see also Exercise **??** for some further properties. Note that for a general topological space one can not define uniform continuity intrinsically. The important observation for us is that for linear maps continuity and uniform continuity coincides:

Proposition 2.1.14 *Let $\phi\colon V \longrightarrow W$ be a linear map between topological vector spaces. Then the following statements are equivalent:*

i.) The map ϕ is uniformly continuous.

ii.) The map ϕ is continuous.

iii.) The map ϕ is continuous at some point $v \in V$.

iv.) The map ϕ is continuous at zero.

Proof: We have *i.)* $\implies$ *ii.)* $\implies$ *iii.)* in general. Suppose *iii.)* holds and let $U \subseteq W$ be a neighbourhood of zero. Then $\tau_{\phi(v)}(U)$ is a neighbourhood of $\phi(v)$. Thus $\phi^{-1}(\tau_{\phi(v)}(U))$ is a neighbourhood of v by continuity at v. Finally, $\tau_{-v}(\phi^{-1}(\tau_{\phi(v)}(U))) \subseteq V$ is a neighbourhood of $0 \in V$, again using the translation invariance of the topologies of V and W, respectively. Since ϕ is linear we have

$$\phi = \tau_{\phi(v)} \circ \phi \circ \tau_{-v},$$

showing that $\phi^{-1}(U) = \tau_{-v}(\phi^{-1}(\tau_{\phi(v)}(U)))$ and thus proving that ϕ is continuous at zero. It remains to show *iv.)* $\implies$ *i.)*. Thus let again $U \subseteq W$ be a neighbourhood of zero and choose a neighbourhood $Z \subseteq V$ of zero with $Z \subseteq \phi^{-1}(U)$, e.g. $Z = \phi^{-1}(U)$. Then for $v - v' \in Z$ we have

$$\phi(v) - \phi(v') = \phi(v - v') \in U$$

by the linearity of ϕ. ☐

The main structure behind this observation is that a topological vector space has a *uniform structure* in a canonical way. The notion of *uniform spaces* will not be needed in the sequel and therefore we refrain from giving precise definitions here. One can find more information on uniform spaces, uniform continuity and their completions e.g. in the classical textbook [3, Chap 6].

2.1.3 Cauchy Nets and Completeness

The uniform structure of topological vector spaces automatically yields a notion of Cauchy nets, completeness and completions. We give the definition directly:

Definition 2.1.15 (Cauchy net) *Let V be a topological vector space. A net $(v_i)_{i\in I}$ of vectors $v_i \in V$ is called a Cauchy net in V if for every neighbourhood $U \subseteq V$ of zero there is an index $i \in I$ with*

$$v_j - v_{j'} \in U \tag{2.1.17}$$

whenever $j \succcurlyeq i$ and $j' \succcurlyeq i$. If $I = \mathbb{N}$ then $(v_i)_{i\in\mathbb{N}}$ is called a Cauchy sequence.

Again, this definition does not refer to a metric but serves the same idea to capture the Cauchy feature.

Proposition 2.1.16 *Let V and W be topological vector spaces.*

i.) If $\phi\colon V \longrightarrow W$ is uniformly continuous and if $(v_i)_{i\in I}$ is a Cauchy net in V, then $(\phi(v_i))_{i\in I}$ is a Cauchy net in W.

ii.) A convergent net is a Cauchy net.

iii.) A Cauchy net is convergent iff it has a convergent subnet.

Proof: Let $(v_i)_{i\in I}$ be a Cauchy net in V and let $U \subseteq W$ be a neighbourhood of zero. We find a zero neighbourhood $Z \subseteq V$ with $\phi(v) - \phi(v') \in U$ whenever $v - v' \in Z$ by the uniform continuity of ϕ. Moreover, we find $i \in I$ with $v_j - v_{j'} \in Z$ for all later indices $j, j' \succcurlyeq i$. Hence $\phi(v_j) - \phi(v_{j'}) \in U$ for these indices showing that the image net is a Cauchy net, too. For the second statement, let $(v_i)_{i\in I}$ be a convergent net with limit $v \in V$. Let $U \subseteq V$ be a zero neighbourhood. Since the vector space operations are continuous, we find another zero neighbourhood $Z \subseteq V$ with

$$Z - Z = \{u - w \mid u, w \in Z\} \subseteq U.$$

Since $\tau_v(Z) = Z + v$ is a neighbourhood of v, we find an index $i \in I$ with $v_j \in Z + v$ whenever $j \succcurlyeq i$. Then $v_j - v_{j'} \in Z - Z \subseteq U$ for $j, j' \succcurlyeq i$ shows that $(v_i)_{i\in I}$ is a Cauchy net. Finally suppose $(v_i)_{i\in I}$ is a Cauchy net. If it converges then every subnet converges, too. This holds in general for convergent nets in topological spaces by Proposition **??**. Hence suppose that $\Phi\colon J \longrightarrow I$ is a morphism of directed sets such that $(v_{\Phi(j)})_{j\in J}$ is a convergent subnet of the Cauchy net $(v_i)_{i\in I}$, converging to $v \in V$. Let $U \subseteq V$ be a zero neighbourhood and fix another zero neighbourhood $Z \subseteq V$ with

$$Z + Z = \{u + w \mid u, w \in Z\} \subseteq U,$$

which is again possible by the continuity of the vector space operations of V. Let again $i_Z \in I$ be an appropriate index such that $i, i' \succcurlyeq i_Z$ implies $v_i - v_{i'} \in Z$ by the Cauchy condition. Thanks to convergence of the subnet we find an index $j_1 \in J$ with $v_{\Phi(j)} \in Z + v$ whenever $j \succcurlyeq j_1$. Being a subnet means that we find an index $j_2 \in J$ with $\Phi(j) \succcurlyeq i_Z$ whenever $j \succcurlyeq j_2$. Taking now $j_Z \in J$ with $j_Z \succcurlyeq j_1$ and $j_Z \succcurlyeq j_1$, which is possible as J is directed, too, we have $\Phi(j_Z) \succcurlyeq i_Z$ and $v_{\Phi(j_Z)} - v \in Z$. For $i \succcurlyeq i_Z$ we thus get

$$v_i - v = v_i - v_{\Phi(j_Z)} + v_{\Phi(j_Z)} - v \in Z + Z \subseteq U,$$

showing the convergence $v_i \to v$. □

Convergence of a net in a topological space can always be encoded using a neighbourhood basis: we have seen that a neighbourhood basis is a directed set and a convergent net has a subnet indexed by the neighbourhood basis by choosing a point of the net in the given neighbourhood. With the same argument it is sufficient to consider Cauchy nets indexed by a neighbourhood basis. It follows that in a first-countable topological vector space a Cauchy net converges iff it has a convergent *subsequence*, a situation which we will encounter most of the time.

Since for general metric spaces we already know that a Cauchy sequence does not need to converge, we state completeness of topological vector spaces in complete analogy to metric spaces:

Definition 2.1.17 (Complete topological vector space) *Let V be a topological vector space.*

i.) If every Cauchy net in V converges then V is called complete.

ii.) If every Cauchy sequence in V converges then V is called sequentially complete.

Most of the time we assume to have Hausdorff spaces. The reason is that we want uniqueness of limits, a useful feature also when dealing with completeness.

Of course, completeness implies sequential completeness. The converse needs not to be true, and we will see important examples of this soon. The reason is the possible failure of first countability:

Proposition 2.1.18 *Let V be a first-countable topological vector space. Then V is complete iff V is sequentially complete.*

Proof: Let $(v_i)_{i\in I}$ be a Cauchy net and fix a countable basis $\{U_n\}_{n\in\mathbb{N}}$ of zero neighbourhoods. Then we find zero neighbourhoods $Z_n \subseteq V$ with $Z_n + Z_n \subseteq U_n$ for all $n \in \mathbb{N}$ as we have seen already before. Fix now indices $i_n \in I$ with $v_j - v_{j'} \in Z_n$ whenever $j, j' \succcurlyeq i_n$ by the Cauchy property. Inductively, we can assume without restriction to have $i_n \preccurlyeq i_{n+1}$ for all $n \in \mathbb{N}$, since I is directed. This defines a sequence $(v_{i_n})_{n\in\mathbb{N}}$ which is a Cauchy sequence. Indeed, if $n \geq m$ then $v_{i_n} - v_{i_m} \in Z_m \subseteq U_m$ since $i_n \succcurlyeq i_m$ and we can apply the Cauchy condition of $(v_i)_{i\in I}$. If V is sequentially complete we have a limit $v \in V$ with $v_{i_n} \to v$. We claim that $(v_i)_{i\in I}$ converges to v as well. To see this, let $k_n \geq n \in \mathbb{N}$ be chosen such that

$$v_{i_\ell} - v \in Z_n,$$

whenever $\ell \geq k_n$ according to the convergence of $(v_{i_n})_{n\in\mathbb{N}}$ to v. Let $i \succcurlyeq i_{k_n}$, then

$$v_i - v = v_i - v_{i_{k_n}} + v_{i_{k_n}} - v \in Z_n + Z_n \subseteq U_n.$$

This shows the convergence of the net $(v_i)_{i\in I}$ to v. The opposite direction is clear. □

Completeness is inherited to closed subspaces as we expect this from metric spaces, at least in the Hausdorff situation:

Proposition 2.1.19 *Let V be a complete Hausdorff topological vector space. A subspace $W \subseteq V$ is complete iff $W = W^{\mathrm{cl}}$ is closed.*

Proof: Assume W is complete and let $v \in W^{\mathrm{cl}}$ be a point in the closure. We find a net $(w_i)_{i\in I}$ of points $w_i \in W$ with $w_i \to v$. Being convergent in V the net $(w_i)_{i\in I}$ is a Cauchy net, now also in W by the definition of the subspace topology. By completeness of W we have $w_i \to w$ for some $w \in W$. Using the Hausdorff property we conclude $w = v$ and hence $v \in W$. Conversely, suppose $W = W^{\mathrm{cl}}$ and let $(w_i)_{i\in I}$ be a Cauchy net in W. Then $(w_i)_{i\in I}$ is a Cauchy net in V, too, and hence convergent to some $v \in V$. From the general properties of closures as in Proposition **??** it follows that $v \in W^{\mathrm{cl}} = W$ showing the completeness of W. □

As for metric spaces we can now try to enhance a non-complete V to become complete by adding ideal points, the missing limits of Cauchy nets. This is possible, even along the same line of arguments as for metric spaces with some minor modifications. One needs to replace the indexing natural numbers $\mathbb{N}$ once and for all by a suitable basis of zero neighbourhoods $\{U_i\}_{i\in I}$ viewed as a directed set via inclusions as already done before. Then the space of all Cauchy nets indexed by $\{U_i\}_{i\in I}$ modulo zero sequences provides such a completion, which is unique up to a unique isomorphism in the Hausdorff case. We do not need this result in generality and therefore refrain from formulating the details, see e.g. [2, Sect. 3.3]. Later on, we will see a very easy construction of the completion in the case of normed spaces.

2.2 From Normed Spaces to Banach Spaces

Having fixed the topological requirements we are now interested in actually constructing examples of topological vector spaces. The most important and simplest class of interest is given by the normed spaces and their complete versions, the Banach spaces. In this section we will see the basic constructions and first examples together with a detailed investigation of the continuous linear maps between them. Here the surprising result is that for normed spaces V and W also $\mathrm{L}(V, W)$ is a normed space in a canonical way. This is very different from the general situation of topological vector spaces, where $\mathrm{L}(V, W)$ does not even carry a canonical topology, let alone a norm.

2.2.1 Normed Vector Spaces: Examples

In the bachelor courses on calculus one has seen normed spaces and, in particular, various norms for $\mathbb{R}^n$. In this section we will give a more systematic discussion of their basic properties.

Definition 2.2.1 (Normed space) *Let V be a vector space over $\mathbb{K}$.*

i.) *A seminorm* p *on* V *is a map*

$$\mathrm{p}\colon V \longrightarrow \mathbb{R}_0^+, \tag{2.2.1}$$

such that

$$\mathrm{p}(zv) = |z|\mathrm{p}(v) \tag{2.2.2}$$

and

$$\mathrm{p}(v+w) \le \mathrm{p}(v) + \mathrm{p}(w) \tag{2.2.3}$$

for all $z \in \mathbb{K}$ *and* $v, w \in V$*.*

ii.) *A norm* $\|\cdot\|$ *on* V *is a seminorm with the additional property that*

$$\|v\| > 0 \quad \textit{for } v \in V \setminus \{0\}. \tag{2.2.4}$$

iii.) *A normed space is a pair* $(V, \|\cdot\|)$ *of a vector space with a norm for it.*

Remark 2.2.2 (Seminorms and norms) Let V be a vector space over $\mathbb{K}$.

i.) For any seminorm p on V we have

$$\mathrm{p}(0) = 0, \tag{2.2.5}$$

as it follows from (2.2.2) by setting $z = 0$.

ii.) The condition (2.2.3) is also called the *triangle inequality*.

iii.) If $\mathrm{p}_1, \ldots, \mathrm{p}_n$ are seminorms on V and $\lambda_1, \ldots, \lambda_n > 0$ then

$$\mathrm{q} = \lambda_1\mathrm{p}_1 + \cdots + \lambda_n\mathrm{p}_n \tag{2.2.6}$$

as well as

$$\tilde{\mathrm{q}} = \max\{\lambda_1\mathrm{p}_1, \ldots, \lambda_n\mathrm{p}_n\} \tag{2.2.7}$$

are seminorms on V. In particular, the set of seminorms is a convex cone inside all maps $\mathrm{Map}(V, \mathbb{R})$.

iv.) The set of seminorms is directed by defining $\mathrm{p} \le \mathrm{q}$ if we have pointwise $\mathrm{p}(v) \le \mathrm{q}(v)$ for all $v \in V$. This gives also a partial order but not any two seminorms can be compared in general.

v.) If $\phi\colon V \longrightarrow W$ is a linear map and q is a seminorm on W then $\mathrm{p} = \mathrm{q} \circ \phi$ is a seminorm on V. We call this also the *pull-back* $\mathrm{p} = \phi^*\mathrm{q}$ of q. Pull-backs of seminorms are compatible with convex combinations (2.2.6), maxima (2.2.7), and the direction $\le$ from *iv.)*. In particular, for every linear functional $\varphi \in V^*$ we get a seminorm

$$\mathrm{p}_\varphi(v) = |\varphi(v)|, \tag{2.2.8}$$

which can be seen as the pull-back of the canonical norm $|\cdot|$ on $\mathbb{K}$. If $W \subseteq V$ is a subspace and p a seminorm on V, then the restriction

$$\mathrm{p}\big|_W = i^*\mathrm{p} \tag{2.2.9}$$

is a seminorm on W, which we can also interpret as the pull-back of p with the inclusion map $i\colon W \longrightarrow V$. Note that $\mathrm{p}\big|_W$ is still a norm if p is a norm

vi.) The *kernel*

$$\ker \mathrm{p} = \big\{ v \in V \;\big|\; \mathrm{p}(v) = 0 \big\} \subseteq V \tag{2.2.10}$$

of a seminorm turns out to be a subspace of V even though p is not a linear map. One has $\ker \mathrm{p} = \{0\}$ iff p is a norm. On the quotient vector space

$$V_{\mathrm{p}} = V \big/ \ker \mathrm{p} \tag{2.2.11}$$

the seminorm is still well-defined and yields a norm, see also Exercise **??** for these and a few other elementary statements on seminorms.

We collect now some first examples of seminorms and norms which we will see and use frequently:

Example 2.2.3 (The sequence spaces c, $c_\circ$, and $c_{\circ\circ}$) Let

$$c = \left\{ (a_n)_{n\in\mathbb{N}} \in \operatorname{Map}(\mathbb{N}, \mathbb{K}) \;\middle|\; \lim_{n\to\infty} a_n \text{ exists} \right\} \tag{2.2.12}$$

be the space of *convergent sequences*. Clearly, $c \subseteq \operatorname{Map}(\mathbb{N}, \mathbb{K})$ is a subspace of the vector space of all sequences. On c one defines

$$\|(a_n)_{n\in\mathbb{N}}\|_\infty = \sup_{n\in\mathbb{N}} |a_n|. \tag{2.2.13}$$

Since convergent sequences are bounded, (2.2.13) is finite for all sequences $(a_n)_{n\in\mathbb{N}} \in c$. It is then easily verified that $\|\cdot\|_\infty$ is a norm. We will always equip c with this supremum norm. In c we have two interesting subspaces. The space of *zero sequences*

$$c_\circ = \left\{ (a_n)_{n\in\mathbb{N}} \in c \;\middle|\; \lim_{n\to\infty} a_n = 0 \right\} \tag{2.2.14}$$

and the space of *finite sequences*

$$c_{\circ\circ} = \left\{ (a_n)_{n\in\mathbb{N}} \in \operatorname{Map}(\mathbb{N}, \mathbb{C}) \;\middle|\; a_n = 0 \text{ for all except finitely many } n \right\}. \tag{2.2.15}$$

We have

$$c_\circ = \ker \lim, \tag{2.2.16}$$

where

$$\lim\colon c \longrightarrow \mathbb{C} \tag{2.2.17}$$

is the linear functional defined by taking the limit. In particular, $c_\circ$ is a subspace of c as claimed. Also $c_{\circ\circ}$ is a subspace and we have

$$c_{\circ\circ} \subseteq c_\circ \subseteq c. \tag{2.2.18}$$

For $c_{\circ\circ}$ we have a vector space basis $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ given by the sequences $\mathrm{e}_n \in c_{\circ\circ}$ with

$$\mathrm{e}_n(m) = \delta_{nm}. \tag{2.2.19}$$

Then indeed we have $c_{\circ\circ} = \operatorname{span}_{\mathbb{K}}\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ and for $(a_n)_{n\in\mathbb{N}} \in c_{\circ\circ}$ we get

$$\sum_{n\in\mathbb{N}} a_n \mathrm{e}_n = (a_n)_{n\in\mathbb{N}}. \tag{2.2.20}$$

Note that this series is in fact a finite sum since all but finitely many coefficients a_n are zero by the definition of $c_{\circ\circ}$. We will see later that neither c nor $c_\circ$ have a countable basis.

Example 2.2.4 (The sequence space ℓ^∞) Generalizing the convergent sequences, we consider the space of *bounded sequences*

$$\ell^\infty = \left\{ (a_n)_{n\in\mathbb{N}} \in \operatorname{Map}(\mathbb{N}, \mathbb{C}) \;\middle|\; \sup_{n\in\mathbb{N}} |a_n| < \infty \right\}. \tag{2.2.21}$$

Again, one verifies immediately that $\ell^\infty \subseteq \operatorname{Map}(\mathbb{N}, \mathbb{C})$ is a subspace and

$$\|(a_n)_{n\in\mathbb{N}}\|_\infty = \sup_{n\in\mathbb{N}} |a_n| \tag{2.2.22}$$

is a norm on ℓ^∞. Note that we have the inclusion

$$c \subseteq \ell^\infty. \tag{2.2.23}$$

In Exercise 2.5.7 one finds some more properties of ℓ^∞ in comparison to c.

Example 2.2.5 (The sequence spaces ℓ^p) Let $p \in [1, \infty)$ be a parameter. We define the *p-summable sequences* by

$$\ell^p = \left\{(a_n)_{n\in\mathbb{N}} \in \operatorname{Map}(\mathbb{N}, \mathbb{C}) \;\middle|\; \|(a_n)_{n\in\mathbb{N}}\|_p < \infty\right\}, \tag{2.2.24}$$

where

$$\|(a_n)_{n\in\mathbb{N}}\|_p = \sqrt[p]{\sum_{n\in\mathbb{N}} |a_n|^p}. \tag{2.2.25}$$

First we note that (2.2.25) contains a series of non-negative terms only. Hence the convergence is necessarily absolute, maybe to $+\infty$. In particular, the convergence is unconditional, i.e. the order of summation will not play a role. With the Minkowski inequality for series one verifies now that, first, ℓ^p is a subspace of $\operatorname{Map}(\mathbb{N}, \mathbb{C})$ and, second, $\|\cdot\|_p$ defines a norm on it. Suppose $p \leq q$, then for all sequences we have

$$\|(a_n)_{n\in\mathbb{N}}\|_q \leq \|(a_n)_{n\in\mathbb{N}}\|_p \tag{2.2.26}$$

as an inequality in $[0, \infty]$. It follows that

$$\ell^p \subseteq \ell^q. \tag{2.2.27}$$

Since a p-summable sequence is necessarily a zero sequence, we have the inclusions

$$\ell^p \subseteq c_\circ \tag{2.2.28}$$

for all $p \in [1, \infty)$. In fact, we have the estimate

$$\|(a_n)_{n\in\mathbb{N}}\|_\infty \leq \|(a_n)_{n\in\mathbb{N}}\|_p \tag{2.2.29}$$

for all $(a_n)_{n\in\mathbb{N}} \in \operatorname{Map}(\mathbb{N}, \mathbb{C})$, again as an inequality in $[0, \infty]$. Since a finite sequence is clearly p-summable for all p because the series in (2.2.25) is then just a finite sum, we have

$$c_{\circ\circ} \subseteq \ell^p \tag{2.2.30}$$

for all $p \in [1, \infty)$. In total we obtain the inclusions

$$c_{\circ\circ} \subseteq \ell^1 \subseteq \cdots \subseteq \ell^p \subseteq \ell^q \subseteq \cdots \subseteq c_\circ \subseteq c \subseteq \ell^\infty \tag{2.2.31}$$

for $q \geq p \geq 1$.

There are several other sequence spaces with norms to be found in the exercises, see in particular Exercise 2.5.8.

Sequences are maps defined on the natural numbers as domain. Replacing $\mathbb{N}$ by other sets with more interesting structure gives new options for functional spaces and norms.

Example 2.2.6 (Bounded functions) Let X be a (non-empty) set and define the space of *bounded functions* by

$$\mathscr{B}(X) = \left\{ f \in \operatorname{Map}(X, \mathbb{K}) \;\middle|\; \sup_{x\in X} |f(x)| < \infty \right\}. \tag{2.2.32}$$

Again, a simple argument shows that $\mathscr{B}(X)$ is a subspace of the vector space $\operatorname{Map}(X, \mathbb{K})$ of all functions and

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \tag{2.2.33}$$

is a norm on $\mathscr{B}(X)$. For $X = \mathbb{N}$ this reproduces

$$\mathscr{B}(\mathbb{N}) = \ell^\infty. \tag{2.2.34}$$

Slightly more interesting examples are obtained on measurable spaces $(X, \mathfrak{a})$. Recall that a *measurable space* is a pair of a set X with a *σ-algebra*

$$\mathfrak{a} \subseteq 2^X \tag{2.2.35}$$

on it. Here a σ-algebra is a collection of subsets which contains $\emptyset$ and is stable under taking countable unions and complements. Then a function

$$f\colon X \longrightarrow \mathbb{K} \tag{2.2.36}$$

is called *measurable* if the preimage of every Borel-measurable subset of $\mathbb{K}$ is $\mathfrak{a}$-measurable, i.e.

$$f^{-1}(B) \in \mathfrak{a} \quad \text{for all} \quad B \in \mathfrak{a}_{\mathrm{Borel}}(\mathbb{K}). \tag{2.2.37}$$

Note that the definition of measurable functions always refers to the *Borel σ-algebra* $\mathfrak{a}_{\mathrm{Borel}}(\mathbb{K})$ and not to the *Lebesgue σ-algebra*. For details on measure theory we refer to e.g. [4]. The set

$$\mathscr{M}(X, \mathfrak{a}) = \left\{ f \in \operatorname{Map}(X, \mathbb{K}) \;\middle|\; f \text{ is } \mathfrak{a}\text{-measurable} \right\} \tag{2.2.38}$$

of $\mathfrak{a}$-measurable functions is a subspace of all functions.

Example 2.2.7 (Bounded measurable functions) Let $(X, \mathfrak{a})$ be a measurable space. Then the *bounded measurable functions* on X are defined by

$$\mathscr{B}\mathscr{M}(X, \mathfrak{a}) = \mathscr{B}(X) \cap \mathscr{M}(X, \mathfrak{a}), \tag{2.2.39}$$

and hence constitute a subspace of $\mathscr{B}(X)$. As such, we restrict the supremum norm $\|\cdot\|_\infty$ to them and obtain a normed space. Inside $\mathscr{B}\mathscr{M}(X, \mathfrak{a})$ we have an interesting subspace given by the simple functions. Recall that a measurable function is called *simple* if it takes only finitely many different values. It follows that for a simple function $f \in \mathscr{M}(X, \mathfrak{a})$ one has unique distinct numbers $z_1, \ldots, z_n \in \mathbb{K}$ and pairwise disjoint measurable sets $A_1, \ldots, A_n \in \mathfrak{a}$ with $X = A_1 \cup \cdots \cup A_n$ and

$$f = \sum_{i=1}^{n} z_i \chi_{A_i}, \tag{2.2.40}$$

where

$$\chi_A(p) = \begin{cases} 1 & \text{if } p \in A \\ 0 & \text{if } p \notin A \end{cases} \tag{2.2.41}$$

is the *characteristic function* of a subset $A \subseteq X$ as usual. They form a subspace in $\mathscr{B}\mathscr{M}(X, \mathfrak{a})$, namely the linear span of all characteristic functions of measurable subsets of X. Note, however, that characteristic functions to different measurable sets are not linearly independent: one has relations like

$$\chi_A + \chi_B = \chi_{A \cup B} + \chi_{A \cap B}. \tag{2.2.42}$$

Still in the context of measurable functions one finds the essentially bounded functions. Here we need more than just a measurable space $(X, \mathfrak{a})$ for the definition. In addition, we need to distinguish measurable subsets of measure zero. This can be done by means of an actual positive measure on $(X, \mathfrak{a})$. However, we do not need the full information of a measure to say what measure zero means. A specific *σ-ideal* $\mathfrak{n} \subseteq \mathfrak{a}$ is sufficient. In fact, this slightly more general point of view will become very helpful when discussing projection-valued measures in **??**.

Definition 2.2.8 (Essential range and supremum) *Let $(X, \mathfrak{a})$ be a measurable space with a σ-ideal $\mathfrak{n} \subseteq \mathfrak{a}$. Moreover, let $f \in \mathscr{M}(X, \mathfrak{a})$ be a measurable function.*

i.) The essential range of f with respect to $\mathfrak{n}$ is defined by

$$\operatorname{ess\,range}(f) = \{z \in \mathbb{K} \mid f^{-1}(\mathrm{B}_\varepsilon(z)) \notin \mathfrak{n} \textit{ for all } \varepsilon > 0\}. \tag{2.2.43}$$

ii.) The essential supremum of $|f|$ with respect to $\mathfrak{n}$ is defined by

$$\|f\|_{\mathfrak{n},\infty} = \operatorname{ess\,sup}(|f|) = \sup\{|z| \mid z \in \operatorname{ess\,range}(f)\}. \tag{2.2.44}$$

Some first properties and reformulations can be found in Exercise **??**. If μ is a positive measure on $(X, \mathfrak{a})$ then

$$\mathfrak{n} = \{A \in \mathfrak{a} \mid \mu(A) = 0\} \tag{2.2.45}$$

is a σ-ideal, the *zero sets* with respect to μ. This is how σ-ideals typically arise and the notion of essential range and supremum typically refers to this more specific situation. In this case one also writes μ-ess range, μ-ess sup and $\|\cdot\|_{\mu,\infty}$ to emphasize the dependence on μ. But the essential range and the essential supremum only depend on $\mathfrak{n}$ from (2.2.45) and not the whole information contained in μ.

Example 2.2.9 (Essentially bounded functions) Let $(X, \mathfrak{a}, \mathfrak{n})$ be a measurable space with a σ-ideal. Then the set

$$\mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n}) = \{f \in \mathscr{M}(X, \mathfrak{a}) \mid \|f\|_{\mathfrak{n},\infty} < \infty\} \tag{2.2.46}$$

is called the space of *essentially bounded functions*. The basic features of the essential supremum then show that $\mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$ is a vector space and $\|\cdot\|_{\mathfrak{n},\infty}$ is a seminorm on it. Suppose $f \in \mathscr{M}(X, \mathfrak{a})$. Then $\operatorname{ess\,range}(f) = \{0\}$ iff there is a zero set $A \in \mathfrak{n}$ with

$$f\Big|_{X\setminus A} = 0. \tag{2.2.47}$$

Thus

$$\ker\|\cdot\|_{\mathfrak{n},\infty} = \Big\{f \in \mathscr{M}(X, \mathfrak{a}) \;\Big|\; f\big|_{X\setminus A} = 0 \text{ for some } A \in \mathfrak{n}\Big\}. \tag{2.2.48}$$

Functions in $\ker\|\cdot\|_{\mathfrak{n},\infty}$ are also called *zero functions* or *null functions*. It depends now on the choice of $\mathfrak{n}$ if there are non-trivial functions in the kernel of $\|\cdot\|_{\mathfrak{n},\infty}$, i.e. whether it is a norm or a mere seminorm. In fact, for a characteristic function of a subset $N \subseteq M$ we have $\chi_N \in \ker\|\cdot\|_{\mathfrak{n},\infty}$ iff $N \in \mathfrak{n}$. In any case, we can pass to a suitable quotient to obtain an honest norm as we have seen this in Remark 2.2.2, *vi.)*. One defines

$$\mathrm{L}^\infty(X, \mathfrak{a}, \mathfrak{n}) = \mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n}) \big/ \ker\|\cdot\|_{\mathfrak{n},\infty}, \tag{2.2.49}$$

and uses $\|\cdot\|_{\mathfrak{n},\infty}$ also for the equivalence classes, i.e.

$$\|[f]\|_{\mathfrak{n},\infty} = \|f\|_{\mathfrak{n},\infty} \tag{2.2.50}$$

for every $[f] \in \mathrm{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$ with representative $f \in [f]$. This is well-defined and yields then a normed space, also called the space of *essentially bounded functions* by some mild abuse of notation. Again, if $\mathfrak{n}$ is the σ-ideal of zero sets with respect to some measure μ, then we write $\mathscr{L}^\infty(X, \mathfrak{a}, \mu)$ as well as $\mathrm{L}^\infty(X, \mathfrak{a}, \mu)$.

Still in a measure-theoretic context, we get the following examples of p-integrable functions:

Example 2.2.10 (p-Integrable functions) Let $(X, \mathfrak{a}, \mu)$ be a measure space, i.e. a measurable space $(X, \mathfrak{a})$ together with a positive measure μ on it. For a measurable function $f \in \mathscr{M}(X, \mathfrak{a})$ one defines

$$\|f\|_{\mu,p} = \sqrt[p]{\int_X |f|^p \,\mathrm{d}\mu}, \tag{2.2.51}$$

where $p \in [1, \infty)$ is a parameter and we have values in $[0, \infty]$. The functions where (2.2.51) is finite are then called the *p-integrable functions*

$$\mathscr{L}^p(X, \mathfrak{a}, \mu) = \big\{ f \in \mathscr{M}(X, \mathfrak{a}) \;\big|\; \|f\|_{\mu,p} < \infty \big\}. \tag{2.2.52}$$

As for the sequence spaces ℓ^p we use the Minkowski inequality for integrals to see that $\mathscr{L}^p(X, \mathfrak{a}, \mu)$ is a vector space and $\|\cdot\|_{\mu,p}$ is a seminorm for it. As in the case of essentially bounded functions, the kernel is again given by the same zero-functions

$$\ker\|\cdot\|_{\mu,p} = \Big\{ f \in \mathscr{M}(X, \mathfrak{a}) \;\Big|\; f\big|_{X\setminus A} = 0 \text{ for some } A \in \mathfrak{a} \text{ with } \mu(A) = 0 \Big\}, \tag{2.2.53}$$

where now the σ-ideal consists of the zero sets with respect to μ. To get an honest norm instead of a seminorm we pass again to the quotient

$$\mathrm{L}^p(X, \mathfrak{a}, \mu) = \mathscr{L}^p(X, \mathfrak{a}, \mu) \big/ \ker\|\cdot\|_{\mu,p}, \tag{2.2.54}$$

which we still call the space of *p-integrable functions* by some mild abuse of notation. Note that elements in $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ are equivalence classes of functions modulo *zero functions*. Moreover, the seminorm becomes a well-defined norm on this quotient again by Remark 2.2.2, *vi.)*. Working with $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ requires some care: as usual with quotients we have to worry about well-definedness as soon as we work with representatives from $\mathscr{L}^p(X, \mathfrak{a}, \mu)$.

We can now re-interpret our sequence spaces ℓ^p for $p \in [1, \infty]$ as particular cases of the measure-theoretic $\mathscr{L}^p$-spaces and L^p-spaces. Here the measure space is $X = \mathbb{N}$ with $\mathfrak{a} = 2^{\mathbb{N}}$ and the *counting measure*, defined on all subsets of $\mathbb{N}$ by

$$\mu_{\mathrm{count}}(A) = \begin{cases} \#A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite.} \end{cases} \tag{2.2.55}$$

For the counting measure μ_{count} we have $\mu_{\mathrm{count}}(A) = 0$ for $A \subseteq \mathbb{N}$ iff $A = \emptyset$. Hence there are no non-trivial zero sets and thus no non-trivial zero functions in this particular case. Therefore the passage to the quotients as in (2.2.49) and (2.2.54) is not needed.

We leave the measure-theoretic framework for now and go to examples from topology.

Example 2.2.11 (Continuous functions) Let $(M, \mathscr{M})$ be a topological space. In this generality, the space of continuous functions

$$\mathscr{C}(M, \mathbb{K}) = \big\{ f\colon M \longrightarrow \mathbb{K} \;\big|\; f \text{ is continuous} \big\} \tag{2.2.56}$$

can be quite small. In any case, it is a vector space containing at least the constant functions. Thanks to Corollary **??** we know that a continuous function is bounded on each compact set. Thus for a compact subset $K \subseteq M$ one defines

$$\|f\|_K = \sup_{p\in K} |f(p)|. \tag{2.2.57}$$

It turns out that $\|\cdot\|_K$ is a seminorm which, however, is not necessarily a norm. In fact, quite obviously we have

$$\ker\|\cdot\|_K = \big\{ f \in \mathscr{C}(M, \mathbb{K}) \;\big|\; f\big|_K = 0 \big\}. \tag{2.2.58}$$

To arrive at a useful norm for (all) continuous functions we have to impose further conditions. The first option is to consider the *bounded continuous functions*

$$\mathscr{C}_{\mathrm{b}}(M, \mathbb{K}) = \mathscr{C}(M, \mathbb{K}) \cap \mathscr{B}(M), \tag{2.2.59}$$

which clearly form a subspace of both the continuous functions and the bounded functions. For $f \in \mathscr{C}_{\mathrm{b}}(M, \mathbb{K})$ we can use the supremum norm

$$\|f\|_\infty = \sup_{p \in M} |f(p)| \tag{2.2.60}$$

inherited from $\mathscr{B}(M)$. The other option is to consider a compact space $(M, \mathscr{M})$ directly. Then $\mathscr{C}(M, \mathbb{K}) = \mathscr{C}_{\mathrm{b}}(M, \mathbb{K})$ holds by Corollary **??** and we can directly use (2.2.60) as a norm on $\mathscr{C}(M, \mathbb{K})$. In view of later applications, we will typically assume to have a Hausdorff compact space, see also Proposition **??** for some extremely useful consequences. The case of bounded continuous functions will become most interesting for a locally compact Hausdorff space $(M, \mathscr{M})$.

Before we proceed with the examples we need to recall the definition of the *support* of a function:

Definition 2.2.12 (Support) *Let $(M, \mathscr{M})$ be a topological space. Then the support of a function $f\colon M \longrightarrow \mathbb{K}$ is defined by*

$$\operatorname{supp}(f) = \big\{p \in M \bigm| f(p) \neq 0\big\}^{\mathrm{cl}}. \tag{2.2.61}$$

By definition the support of a function is closed. For a closed subset $A \subseteq M$ one then defines

$$\mathscr{C}_A(M, \mathbb{K}) = \big\{f \in \mathscr{C}(M, \mathbb{K}) \bigm| \operatorname{supp}(f) \subseteq A\big\}. \tag{2.2.62}$$

This way we obtain subspaces of $\mathscr{C}(M, \mathbb{K})$ with $\mathscr{C}_A(M, \mathbb{K}) \subseteq \mathscr{C}_B(M, \mathbb{K})$ whenever $A \subseteq B$.

Example 2.2.13 (Compactly supported continuous functions) Let $(M, \mathscr{M})$ be a topological space. Then we consider

$$\mathscr{C}_0(M, \mathbb{K}) = \big\{f \in \mathscr{C}(M, \mathbb{K}) \bigm| \operatorname{supp}(f) \text{ is compact}\big\}, \tag{2.2.63}$$

the space of *compactly supported continuous functions*. Clearly, $\mathscr{C}_0(M, \mathbb{K}) \subseteq \mathscr{C}(M, \mathbb{K})$ is a subspace and we have

$$\mathscr{C}_0(M, \mathbb{K}) = \bigcup_{\substack{K \subseteq M \\ \text{compact}}} \mathscr{C}_K(M, \mathbb{K}). \tag{2.2.64}$$

It is not immediately clear which norm is suitable and characteristic for $\mathscr{C}_0(M, \mathbb{K})$. Of course, $\mathscr{C}_0(M, \mathbb{K}) \subseteq \mathscr{C}_{\mathrm{b}}(M, \mathbb{K})$ and hence we can use the supremum norm $\|\cdot\|_\infty$ for $\mathscr{C}_0(M, \mathbb{K})$. But there are many more interesting norms: let $\rho \in \mathscr{C}(M, \mathbb{K})$ be any continuous function with $\rho > 0$. Then

$$\|f\|_\rho = \sup_{p \in M} \rho(p) |f(p)| \tag{2.2.65}$$

is still a well-defined norm on $\mathscr{C}_0(M, \mathbb{K})$, since for each $f \in \mathscr{C}_0(M, \mathbb{K})$ only the points in the compact support of f contribute to the supremum in (2.2.65). On this compact subset ρ is bounded.

We conclude now this section on normed spaces. There are of course many more function spaces of interest in analysis, like differentiable, smooth, real-analytic or even holomorphic functions on various spaces. However, it turns out that the concept of normed spaces is not wholly adequate when it comes to differentiability.

2.2.2 The topology of Normed Spaces

We continue with the more conceptual and general questions by investigating the topology of normed spaces. In fact, normed spaces are metric spaces in a canonical way:

Proposition 2.2.14 *Let V be a normed space. Then*

$$\mathrm{d}(v, w) = \|v - w\| \tag{2.2.66}$$

defines a metric on V.

Proof: This is an elementary verification based on the properties of a norm. In particular, the triangle inequality (2.2.3) yields the triangle inequality for d. □

Note that for a mere seminorm the definition (2.2.66) yields something rather close to a metric. Only the definiteness $\mathrm{d}(v, w) = 0 \implies v = w$ fails if $\|\cdot\|$ is not a norm. This is one of the main reasons that we prefer norms over seminorms. In particular, the L^p-spaces from Example 2.2.9 and Example 2.2.10 are preferred over the $\mathscr{L}^p$-spaces, even though this involves the quotient procedure.

In the following we will equip a normed space V always with its metric topology induced by the metric (2.2.66) without further notice. With other words, a subset $O \subseteq V$ is open iff for every point $v \in O$ there is a $\varepsilon > 0$ with $\mathrm{B}_\varepsilon(v) \subseteq O$, where we use the balls with respect to the metric (2.2.66).

Remark 2.2.15 Since the topology of a normed space V is metric we get several nice features automatically:

i.) The topology of V is first countable: the metric balls

$$\mathrm{B}_r(v) = \big\{w \bigm| \|v - w\| < r\big\} \tag{2.2.67}$$

form a basis of open neighbourhoods of v, and the countable radii $r = \frac{1}{n}$ for $n \in \mathbb{N}$ already suffice. Similarly, we get a basis of closed neighbourhoods by taking the closed balls $\mathrm{B}_r(p)^{\mathrm{cl}}$ instead.

ii.) The topology of V is Hausdorff and even normal.

iii.) Continuity of maps $\phi\colon V \longrightarrow M$ into some topological space M is equivalent to sequential continuity.

iv.) Closures of subsets of V can be characterized by sequences, i.e. for $A \subseteq V$ we have $v \in A^{\mathrm{cl}}$ iff there is a sequence $(v_n)_{n\in\mathbb{N}}$ of vectors $v_n \in A$ with $v_n \to v$. In particular, general nets are not needed.

v.) A subset $K \subseteq V$ is compact iff it is sequentially compact.

vi.) The metric $\mathrm{d}\colon V \times V \longrightarrow \mathbb{R}_0^+$ is uniformly continuous, hence

$$V \times V \ni (v, w) \mapsto \|v - w\| \in \mathbb{R}_0^+ \tag{2.2.68}$$

is uniformly continuous, see Exercise **??**. In particular, the norm itself is a continuous map

$$\|\cdot\|\colon V \ni v \mapsto \|v\| \in \mathbb{R}_0^+. \tag{2.2.69}$$

vii.) Explicitly, the convergence of a sequence $(v_n)_{n\in\mathbb{N}}$ of vectors $v_n \in V$ to $v \in V$ means that for $\varepsilon > 0$ we find an $N \in \mathbb{N}$ with

$$\|v_n - v\| < \varepsilon, \tag{2.2.70}$$

whenever $n \geq N$. An analogous characterization holds for Cauchy sequences, too. Here we come as close to elementary calculus as possible.

viii.) From the construction of the metric we also see that it is *translation invariant*, i.e. the translations τ_v are isometries. This is clear since

$$\mathrm{d}(\tau_v(w), \tau_v(u)) = \|w + v - (u + v)\| = \|w - u\| = \mathrm{d}(w, u) \tag{2.2.71}$$

for all $v, w, u \in V$.

In general, topological considerations for V can now be reformulated in terms of *estimates* using the norm. This will be a pervasive theme in functional analysis.

Crucial for the following is of course that the vector space operations of V are continuous:

Proposition 2.2.16 *Let V be a normed vector space. Then the canonical topology turns V into a topological vector space.*

PROOF: As we are in the metric situation it suffices to check sequential continuity, see Exercise **??** for a more direct proof. Let $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ be sequences in V converging to v and w, respectively. Moreover, let $(z_n)_{n\in\mathbb{N}}$ be a sequence of scalars $z_n \in \mathbb{K}$ converging to z. Then we have

$$\|v_n + w_n - (v + w)\| \leq \|v_n - v\| + \|w_n - w\| \to 0$$

for $n \to \infty$, showing the continuity of $+$ at (v, w). Similarly, we have

$$\|z_n v_n - zv\| = \|z_n v_n - z_n v + z_n v - zv\| \leq \|z_n(v_n - v)\| + \|(z_n - z)v\| = |z_n|\|v_n - v\| + |z_n - z|\|v\| \to 0$$

for $n \to \infty$, showing the continuity of the scalar multiplication at (z, v). □

With this observation all the results and definitions from Section 2.1 become now available to normed spaces.

The following definition formalizes the dependence of the topology on the norm. It turns out that many seemingly very different norms yield the same topology:

Definition 2.2.17 (Equivalent norms) *Let V be a vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called equivalent if they induce the same topology on V.*

Proposition 2.2.18 *Let V be a vector space with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$.*

i.) The topology $\mathscr{V}_2$ induced by $\|\cdot\|_1$ is finer than the topology $\mathscr{V}_2$ induced by $\|\cdot\|_2$ iff there is a constant $c > 0$ such that

$$\|v\|_2 \leq c\|v\|_1 \tag{2.2.72}$$

holds for all $v \in V$.

ii.) The topologies $\mathscr{V}_1$ and $\mathscr{V}_2$ coincide iff there are constants $c_1, c_2 > 0$ such that

$$\|v\|_1 \leq c_1\|v\|_2 \tag{2.2.73}$$

and

$$\|v\|_2 \leq c_2\|v\|_1 \tag{2.2.74}$$

hold for all $v \in V$.

PROOF: Clearly, *ii.)* follows from *i.)* directly. Since metric topologies are generated by open balls and since the topologies obtained from norms are translation invariant, we only need to show that open balls around one point, say $0 \in V$, of $\mathscr{V}_2$ are also open with respect to $\mathscr{V}_1$ iff (2.2.72) holds. So suppose $\mathscr{V}_1$ is finer and hence the open unit ball $\mathrm{B}_{\|\cdot\|_2,1}(0)$ is open with respect to $\mathscr{V}_1$, too. Thus it is a neighbourhood of 0 with respect to $\mathscr{V}_1$ and hence contains an open ball $\mathrm{B}_{\|\cdot\|_1,\varepsilon}(0)$ for some appropriate $\varepsilon > 0$. This means

$$\|v\|_1 < \varepsilon \implies \|v\|_2 < 1 \tag{$*$}$$

for all $v \in V$. Consider now an arbitrary vector $v \neq 0$. Then for every $\delta > 0$ the vector

$$w = \frac{\varepsilon}{\delta + \|v\|_1} v$$

has norm $\|w\|_1 = \frac{\varepsilon\|v\|_1}{\delta+\|v\|_1} < \varepsilon$ and hence $\|w\|_2 < 1$ follows from $(*)$. This means

$$\varepsilon\|v\|_2 < \delta + \|v\|_1$$

for all $\delta > 0$ and all $v \neq 0$. Setting $c = \frac{1}{\varepsilon}$ then gives (2.2.72). Conversely, assume (2.2.72) holds and let $\mathrm{B}_{\|\cdot\|_2,r}(0)$ be the open ball with respect to $\|\cdot\|_2$ around 0 of radius $r > 0$. We need to show that it is open with respect to $\|\cdot\|_1$ as well. Hence let $v \in \mathrm{B}_{\|\cdot\|_2,r}(0)$ be given, i.e. $\|v\|_2 < r$. Define $\rho = \frac{r-\|v\|_2}{c} > 0$ and consider $u \in \mathrm{B}_{\|\cdot\|_1,\rho}(0)$. Then

$$\|v+u\|_2 \leq \|v\|_2 + \|u\|_2 \leq \|v\|_2 + c\|u\|_1 < \|v\|_2 + c\Big(\frac{r-\|v\|_2}{c}\Big) < r$$

shows $u \in \mathrm{B}_{\|\cdot\|_2,r}(v)$. Hence $\mathrm{B}_{\|\cdot\|_1,\rho}(v) \subseteq \mathrm{B}_{\|\cdot\|_2,r}(0)$ follows for all such $v \in \mathrm{B}_{\|\cdot\|_2,r}(0)$. So $\mathrm{B}_{\|\cdot\|_2,r}(0)$ is open with respect to $\mathscr{V}_1$ which shows that $\mathscr{V}_1$ is finer. □

Hence two norms on V are equivalent iff there are constants $c, C > 0$ with

$$\|v\|_1 \leq c\|v\|_2 \leq C\|v\|_1. \tag{2.2.75}$$

Note that sometimes this estimate is taken as the (very unconceptual) definition of equivalence of norms: what really matters is the coincidence of the induced topologies as this is relevant for all notions of convergence, continuity, etc. Note that also the uniform structures coincide and hence the notions of Cauchy sequences and completeness.

From elementary calculus courses we recall that in finite dimensions all norms are equivalent, see Exercise **??** for a guided proof:

Theorem 2.2.19 (Standard topology of $\mathbb{K}^n$) *Let $n \in \mathbb{N}$. Then the standard topology of $\mathbb{K}^n$ is the unique topology induced by a norm: all norms on $\mathbb{K}^n$ are equivalent.*

2.2.3 Continuous Linear Maps and the Operator Norm

We investigate now the continuity of linear maps between normed spaces more closely. The first step is to check that the two competing definitions of uniform continuity match:

Proposition 2.2.20 *Let $\phi\colon V \longrightarrow W$ be a map between normed spaces. Then ϕ is uniformly continuous in the sense of Definition 2.1.13 iff ϕ is uniformly continuous in the metric sense.*

Proof: This is merely a translation. Suppose that ϕ is continuous in the sense of Definition 2.1.13 and let $\varepsilon > 0$ be given. Since $\mathrm{B}_\varepsilon(0) \subseteq W$ is an open neighbourhood of $0 \in W$ we find an open zero neighbourhood $Z \subseteq V$ with

$$\phi(v) - \phi(v') \in \mathrm{B}_\varepsilon(0), \tag{$*$}$$

whenever $v - v' \in Z$. Since the open balls form a basis of neighbourhoods we find a $\delta > 0$ with $\mathrm{B}_\delta(0) \subseteq Z$. Then $(*)$ means

$$\mathrm{d}(\phi(v), \phi(v')) = \|\phi(v) - \phi(v')\| < \varepsilon,$$

whenever $\mathrm{d}(v, v') = \|v - v'\| < \delta$, which is metric uniform continuity. The converse is shown along the same lines of argument. □

Hence we can speak of uniform continuity in the following without a clash of definitions. As a consequence of Proposition 2.1.14 we get the characterization of continuity for linear maps as follows:

Corollary 2.2.21 *Let $\phi\colon V \longrightarrow W$ be a linear map between normed spaces. Then the following statements are equivalent:*

i.) The map ϕ is uniformly continuous.

ii.) The map ϕ is continuous.

iii.) The map ϕ is continuous at some point $v \in V$.

iv.) The map ϕ is continuous at zero.

This leaves of course the question how we can characterize continuity using the norms more explicitly. Here we have the following fundamental result:

Proposition 2.2.22 *Let $\phi\colon V \longrightarrow W$ be a linear map between normed spaces. Then ϕ is continuous iff there exists a $c \geq 0$ with*

$$\|\phi(v)\|_W \leq c\|v\|_V \tag{2.2.76}$$

for all $v \in V$.

PROOF: There are many equivalent ways to prove this. Assume first that ϕ is continuous and thus continuous at $0 \in V$. Thus $\phi^{-1}(\mathrm{B}_1(0)) \subseteq V$ is an open neighbourhood of zero in V. Hence there exists an $\varepsilon > 0$ with

$$\mathrm{B}_\varepsilon(0) \subseteq \phi^{-1}(\mathrm{B}_1(0)),$$

which means

$$\|v\|_V < \varepsilon \implies \|\phi(v)\|_W < 1$$

for all $v \in V$. As in the proof of Proposition 2.2.18, *i.)*, this implies

$$\|\phi(v)\|_W \leq \frac{1}{\varepsilon}\|v\|_V,$$

which is (2.2.76) for $c = \frac{1}{\varepsilon}$. Conversely, assume that (2.2.76) holds. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in V converging to $0_V \in V$. Then

$$\|\phi(v_n)\|_W \leq c\|v_n\|_V \to 0$$

shows that $(\phi(v_n))_{n\in\mathbb{N}}$ is a zero sequence in W. In other words, ϕ is sequentially continuous at zero. This is all we need to show thanks to Corollary 2.2.21. □

Alternatively, we can argue for the opposite direction as in the proof of Proposition 2.2.18, *i.)*: in fact, in that proof we showed that the identity map $\mathrm{id}\colon (V, \|\cdot\|_1) \longrightarrow (V, \|\cdot\|_2)$ is continuous.

Thanks to Proposition 2.2.22 we can now ask for the best constant $c \geq 0$ such that (2.2.76) holds. This is the *operator norm* of a continuous linear map:

Definition 2.2.23 (Operator norm) *Let $\phi\colon V \longrightarrow W$ be a continuous linear map. Then*

$$\|\phi\| = \sup_{v\in V\setminus\{0\}} \frac{\|\phi(v)\|_W}{\|v\|_V} \tag{2.2.77}$$

is called the operator norm of ϕ.

The operator norm is a norm indeed:

Proposition 2.2.24 *Let V, W, and X be normed spaces.*

i.) For the operator norm of $\phi \in \mathrm{L}(V, W)$ *one has*

$$\|\phi\| = \sup\big\{\|\phi(v)\|_W \bigm| v \in V \textit{ with } \|v\|_V \leq 1\big\} \tag{2.2.78}$$

$$\begin{aligned} \|\phi\| &= \sup\big\{\|\phi(v)\|_W \bigm| v \in V \textit{ with } \|v\|_V = 1\big\} &(2.2.79)\\ &= \inf\big\{c \bigm| \|\phi(v)\|_W \leq c\|v\|_V \textit{ for all } v \in V\big\} &(2.2.80)\end{aligned}$$

and

$$\|\phi(v)\|_W \leq \|\phi\|\|v\|_V \tag{2.2.81}$$

for all $v \in V$*.*

ii.) The operator norm is a norm for $\mathrm{L}(V, W)$ *and one has*

$$\|\mathrm{id}_V\| = 1 \tag{2.2.82}$$

and

$$\|\psi \circ \phi\| \leq \|\psi\|\|\phi\| \tag{2.2.83}$$

for $\phi \in \mathrm{L}(V, W)$ *and* $\psi \in \mathrm{L}(W, X)$*.*

iii.) A continuous linear map $\phi \in \mathrm{L}(V, W)$ *is Lipschitz continuous with Lipschitz constant* $\|\phi\|$*.*

Proof: Since for $v \neq 0$ the vector $\frac{v}{\|v\|_V}$ has norm one we have

$$\|\phi\| = \sup_{v \in V\setminus\{0\}} \frac{\|\phi(v)\|_W}{\|v\|_V} = \sup_{v \in V\setminus\{0\}} \left\|\phi\left(\frac{v}{\|v\|_V}\right)\right\|_W = \sup_{\|\tilde{v}\|_V = 1} \|\phi(\tilde{v})\|_W. \tag{$*$}$$

For a non-zero vector of smaller norm than one we have a $0 < c < 1$ with $v = c\frac{v}{\|v\|_V}$ and hence

$$\|\phi(v)\|_W = \left\|\phi\left(c\frac{v}{\|v\|_V}\right)\right\|_W = c\left\|\phi\left(\frac{v}{\|v\|_V}\right)\right\|_W < \left\|\phi\left(\frac{v}{\|v\|_V}\right)\right\|_W,$$

showing that in the supremum in (2.2.78) the vectors with norm less than one are not relevant. Hence (2.2.78) and (2.2.79) follow by ($*$). From the definition of the operator norm as supremum (2.2.77) we see that $\|\phi(v)\|_W \leq \|\phi\|\|v\|_V$ holds for all $v \in V \setminus \{0\}$. Since this is trivially correct for $v = 0$ as well, we conclude (2.2.81). This also shows "$\geq$" in (2.2.81). The definition of $\|\phi\|$ as (2.2.77) then shows that there is no smaller c than $\|\phi\|$ with (2.2.81), proving equality in (2.2.81). For the second part, recall from Proposition 2.1.10 that $\mathrm{L}(V, W)$ is indeed a vector space and id_V as well as the compositions are again continuous linear maps. Let now $\phi, \phi' \in \mathrm{L}(V, W)$ be given. Then

$$\begin{aligned} \|(\phi + \phi')(v)\|_W &= \|\phi(v) + \phi'(v)\|_W \\ &\leq \|\phi(v)\|_W + \|\phi'(v)\|_W \\ &\leq \|\phi\|\|v\|_V + \|\phi'\|\|v\|_V \\ &= (\|\phi\| + \|\phi'\|)\|v\|_V \end{aligned}$$

according to (2.2.81). Since this is true for all $v \in V \setminus \{0\}$ we can conclude

$$\|\phi + \phi'\| \leq \|\phi\| + \|\phi'\|$$

after dividing by $\|v\|_V$ and taking the supremum over all $v \neq 0$. Moreover, $\|z\phi\| = |z|\|\phi\|$ is clear from the definition. Hence we have a seminorm. If $\phi \neq 0$ is not the zero map then there is a $v \neq 0$ with $\phi(v) \neq 0$. Thus $\frac{\|\phi(v)\|_W}{\|v\|_V} > 0$ for this vector v, showing $\|\phi\| > 0$. Finally, (2.2.82) is clear and for (2.2.83) we consider

$$\|(\psi \circ \phi)(v)\|_X = \|\psi(\phi(v))\|_X \leq \|\psi\|\|\phi(v)\|_W \leq \|\psi\|\|\phi\|\|v\|_V,$$

for $v \neq 0$ by using (2.2.81) twice. Dividing by $\|v\|_V$ and taking the supremum gives (2.2.83). The last part is easy since

$$\mathrm{d}_W(\phi(v), \phi(v')) = \|\phi(v) - \phi(v')\|_W = \|\phi(v - v')\|_W \leq \|\phi\| \|v - v'\|_V = \|\phi\| \, \mathrm{d}_V(v, v')$$

for all $v, v' \in V$ by (2.2.81). Thus ϕ is Lipschitz continuous and $\|\phi\|$ is a Lipschitz constant for ϕ. $\square$

In particular, for a continuous linear functional $\varphi \in \mathrm{L}(V, \mathbb{K}) = V'$ in the topological dual, the operator norm

$$\|\varphi\| = \sup_{v \in V \setminus \{0\}} \frac{|\varphi(v)|}{\|v\|} = \sup_{\|v\|=1} |\varphi(v)| \tag{2.2.84}$$

is also called the *functional norm* of φ. Note that here we always equip $\mathbb{K}$ with the absolute value $|\cdot|$ as canonical norm. In particular, the topological dual V' becomes a normed space itself.

2.2.4 Continuous Multilinear Maps

Linear maps are not sufficient for many applications in mathematics. Hence one has to go beyond linearity to deal with more interesting questions. In general, the theory of arbitrary, say continuous, maps between normed spaces is of course very complicated. Adding differentiability still provides difficulties beyond the finite-dimensional framework. Hence falling back on polynomial functions is an option which allows us to use algebraic properties as well. Since polynomial functions appear in (formal) Taylor expansions, too, their study will also help going beyond a purely algebraic setting.

Now in infinite dimensions polynomials are not automatically continuous as we have seen already for linear polynomials. To get a reasonable theory we have to demand continuity as an extra condition. In this short section we will provide the necessary tools to characterize continuity of polynomial maps.

The good definition of polynomial maps here is to consider multilinear maps of a certain degree and evaluate them on the diagonal. Hence only the symmetric part of the multilinear map plays a role by the usual polarization identities, see Exercise 2.5.10. Then the continuity can be encoded in the continuity of the multilinear map.

With this shift in perspective, it is clear that multilinear maps in general are of interest whether they are symmetric or not. In this section we will formulate their continuity properties.

Definition 2.2.25 (Continuous k-linear maps) *Let $V_1, \ldots, V_k$ and W be topological vector spaces. Then the set of continuous k-linear maps*

$$\Phi \colon V_1 \times \cdots \times V_k \longrightarrow W \tag{2.2.85}$$

is denoted by

$$\mathrm{L}(V_1, \ldots, V_k; W) = \left\{ \Phi \in \mathrm{Hom}(V_1, \ldots, V_k; W) \;\middle|\; \Phi \textit{ is continuous} \right\}. \tag{2.2.86}$$

Here we use the Cartesian product topology on $V_1 \times \cdots \times V_k$ to make sense of continuity. Moreover,

$$\mathrm{Hom}(V_1, \ldots, V_k; W) = \left\{ \Phi \colon V_1 \times \cdots \times V_k \longrightarrow W \;\middle|\; \Phi \text{ is linear in each argument} \right\} \tag{2.2.87}$$

denotes the k-linear maps from $V_1 \times \cdots \times V_k$ to W.

Proposition 2.2.26 *Let $V_1, \ldots, V_k$ and W be topological vector spaces. Then*

$$\mathrm{L}(V_1, \ldots, V_k; W) \subseteq \mathrm{Hom}(V_1, \ldots, V_k; W) \tag{2.2.88}$$

is a subspace.

PROOF: Clearly the zero map is k-linear and continuous. For Φ, Ψ being k-linear and $\lambda, \mu \in \mathbb{k}$ also $\lambda\Phi + \mu\Psi$ is k-linear, i.e. $\mathrm{Hom}(V_1, \ldots, V_k; W) \subseteq \mathrm{Map}(V_1 \times \cdots \times V_k, W)$ is a subspace of the vector space of all maps. This is a result of linear algebra which can be easily verified, see [6, Chapter 3] or [1] for a general introduction to multilinear algebra. The continuity of $\lambda\Phi + \mu\Psi$ follows as in the case for linear maps from the continuity of the vector space operations of W. $\square$

Now we can formulate the extension of Corollary 2.2.21 and Proposition 2.2.22. Note, however, that uniform continuity for k-linear maps will not hold anymore as soon as $k \geq 2$. This is already the case in finite dimensions as e.g. the map $\mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$ is not uniformly continuous.

Proposition 2.2.27 *Let $V_1, \ldots, V_k$ and W be normed spaces. For a k-linear map $\Phi\colon V_1 \times \cdots \times V_k \longrightarrow W$ the following statements are equivalent:*

i.) The map Φ is continuous.

ii.) The map Φ is continuous at $(0, \ldots, 0) \in V_1 \times \cdots \times V_k$.

iii.) There exists a constant $c \geq 0$ with

$$\|\Phi(v_1, \ldots, v_k)\|_W \leq c\|v_1\|_{V_1} \cdots \|v_k\|_{V_k} \tag{2.2.89}$$

for all $v_1 \in V_1, \ \ldots, \ v_k \in V_k$.

PROOF: We have *i.)* $\implies$ *ii.)* in general. Assume *ii.)* holds. Consider $\mathrm{B}_1(0) \subseteq W$ which is a neighbourhood of 0. Hence $\Phi^{-1}(\mathrm{B}_1(0)) \subseteq V_1 \times \cdots \times V_k$ is a neighbourhood of $(0, \ldots, 0) \in V_1 \times \cdots \times V_k$ by continuity at $(0, \ldots, 0)$. It follows that there are $r_1, \ldots, r_k > 0$ with

$$\mathrm{B}_{r_1}(0) \times \cdots \times \mathrm{B}_{r_k}(0) \subseteq \Phi^{-1}(\mathrm{B}_1(0))$$

by the definition of the topology on a finite Cartesian product. This means that

$$\|v_1\|_{V_1} < r_1, \ \ldots, \ \|v_k\|_{V_k} < r_k \implies \|\Phi(v_1, \ldots, v_k)\|_W < 1.$$

Let $\varepsilon > 0$ Then $\frac{r_1}{\|v_1\|_{V_1} + \varepsilon} v_1 \in \mathrm{B}_{r_1}(0), \ldots, \frac{r_k}{\|v_k\|_{V_k} + \varepsilon} v_k \in \mathrm{B}_{r_k}(0)$ and thus

$$\begin{aligned}
1 &> \left\| \Phi\left(\frac{r_1}{\|v_1\|_{V_1} + \varepsilon} v_1, \ldots, \frac{r_k}{\|v_k\|_{V_k} + \varepsilon} v_k \right) \right\|_W \\
&= \left\| \frac{r_1 \cdots r_k}{(\|v_1\|_{V_1} + \varepsilon) \cdots (\|v_k\|_{V_k} + \varepsilon)} \Phi(v_1, \ldots, v_k) \right\|_W \\
&= \frac{r_1 \cdots r_k}{(\|v_1\|_{V_1} + \varepsilon) \cdots (\|v_k\|_{V_k} + \varepsilon)} \|\Phi(v_1, \ldots, v_k)\|_W,
\end{aligned}$$

from which (2.2.89) follows with $c = \frac{1}{r_1 \cdots r_k} > 0$ by taking the limit $\varepsilon \to 0$. Finally, assume *iii.)* holds. We consider sequences $(v_{1,n})_{n \in \mathbb{N}}, \ldots, (v_{k,n})_{n \in \mathbb{N}}$ converging to $v_1 \in V_1, \ldots, v_k \in V_k$, respectively. We want to show continuity at $(v_1, \ldots, v_k) \in V_1 \times \cdots \times V_k$. Since all spaces are metric it suffices to show sequential continuity. To this end, we estimate

$$\begin{aligned}
&\|\Phi(v_{1,n}, \ldots, v_{k,n}) - \Phi(v_1, \ldots, v_k)\|_W \\
&\quad = \big\|\Phi(v_{1,n} - v_1, v_{2,n}, \ldots, v_{k,n}) + \Phi(v_1, v_{2,n}, \ldots, v_{k,n}) - \Phi(v_1, \ldots, v_k)\big\|_W \\
&\quad = \big\|\Phi(v_{1,n} - v_1, v_{2,n}, \ldots, v_{k,n}) + \Phi(v_1, v_{2,n} - v_2, v_{3,n}, \ldots, v_{k,n}) \\
&\qquad\quad + \Phi(v_1, v_2, v_{3,n}, \ldots, v_{k,n}) - \Phi(v_1, v_2, \ldots, v_k)\big\|_W \\
&\quad = \left\| \sum_{\ell=1}^{k} \Phi(v_{1,n} - v_1, \ldots, v_{\ell,n} - v_\ell, v_{\ell+1,n}, \ldots, v_{k,n}) \right\|_W
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\ell=1}^{k} \big\| \Phi(v_{1,n} - v_1, \ldots, v_{\ell,n} - v_\ell, v_{\ell+1,n}, \ldots, v_{k,n}) \big\|_W \\
&\leq \sum_{\ell=1}^{k} c \|v_{1,n} - v_1\|_{V_1} \cdots \|v_{\ell,n} - v_\ell\|_{V_\ell} \|v_{\ell+1,n}\|_{V_{\ell+1}} \cdots \|v_{k,n}\|_{V_k}.
\end{aligned}
$$

In the limits $v_{1,n} \to v_1$, ..., $v_{k,n} \to v_k$ each summand converges to zero showing the continuity at $(v_1, \ldots, v_k)$. As this point was arbitrary, we have continuity everywhere. □

The argument is very close to the one for $k = 1$ in Proposition 2.2.22. Again, we can ask for the best constant in (2.2.89), leading to an operator norm also for multilinear maps:

Definition 2.2.28 (Operator norm II) *Let $V_1, \ldots, V_k$ and W be normed vector spaces. For $\Phi \in \mathrm{L}(V_1, \ldots, V_k; W)$ one defines the operator norm $\|\Phi\|$ by*

$$
\|\Phi\| = \sup_{v_1 \in V_1 \setminus \{0\}, \ldots, v_k \in V_k \setminus \{0\}} \frac{\|\Phi(v_1, \ldots, v_k)\|_W}{\|v_1\|_{V_1} \cdots \|v_k\|_{V_k}}. \tag{2.2.90}
$$

As for the linear case in Proposition 2.2.24, this yields a norm on the vector space $\mathrm{L}(V_1, \ldots, V_k; W)$ of multilinear maps:

Proposition 2.2.29 *Let $V_1, \ldots, V_k$ and W be normed spaces.*

i.) For the operator norm of k-linear maps $\Phi \in \mathrm{L}(V_1, \ldots, V_k; W)$ one has

$$
\begin{aligned}
\|\Phi\| &= \sup\big\{ \|\Phi(v_1, \ldots, v_k)\|_W \bigm| v_i \in V_i \text{ with } \|v_i\|_{V_i} \leq 1 \text{ for } i = 1, \ldots, k \big\} && (2.2.91) \\
&= \sup\big\{ \|\Phi(v_1, \ldots, v_k)\|_W \bigm| v_i \in V_i \text{ with } \|v_i\|_{V_i} \leq 1 \text{ for } i = 1, \ldots, k \big\} && (2.2.92) \\
&= \inf\big\{ c \bigm| \|\Phi(v_1, \ldots, v_k)\|_W \leq c \|v_1\|_{V_1} \cdots \|v_k\|_{V_k} \text{ for all } v_1 \in V_1, \ldots, v_k \in V_k \big\} && (2.2.93)
\end{aligned}
$$

and

$$
\|\Phi(v_1, \ldots, v_k)\|_W \leq \|\Phi\| \|v_1\|_{V_1} \cdots \|v_k\|_{V_k} \tag{2.2.94}
$$

for all $v_1 \in V_1$, ..., $v_k \in V_k$.

ii.) The operator norm is a norm for $\mathrm{L}(V_1, \ldots, V_k; W)$.

iii.) For the insertion $\circ_i$ at the i-th position one obtains

$$
\|\Psi \circ_i \Phi\| \leq \|\Psi\| \|\Phi\|, \tag{2.2.95}
$$

where $\Phi \in \mathrm{L}(V_1, \ldots, V_k; W_i)$ and $\Psi \in \mathrm{L}(W_1, \ldots, W_\ell; U)$ with $i = 1, \ldots, \ell$.

Proof: The arguments from Proposition 2.2.24 can be transferred directly without further difficulties, see also Exercise **??** for details. □

2.2.5 Cauchy Sequences and Completeness

Consider again a normed vector space V. Then we have two competing definitions of Cauchy nets and sequences: the metric definition based on the canonical metric of V and the general definition for topological vector spaces, see again Definition 2.1.15. Fortunately, they yield the same:

Proposition 2.2.30 *Let V be a normed vector space and let $(v_i)_{i \in I}$ be a net in V. Then $(v_i)_{i \in I}$ is a Cauchy net in the sense of Definition 2.1.15 iff it is a Cauchy net in the metric sense for the canonical metric of V.*

Proof: The proof is similar to the one of Proposition 2.2.20. It consists of the same translations of the neighbourhood systems. Suppose $(v_i)_{i\in I}$ is a Cauchy net in the sense of Definition 2.1.15 and let $\varepsilon > 0$ be given. Since $\mathrm{B}_\varepsilon(0)$ is an open neighbourhood of 0 we find an index $i \in I$ with

$$v_j - v_{j'} \in \mathrm{B}_\varepsilon(0)$$

for $j, j' \succcurlyeq i$. This means $\|v_j - v_{j'}\| < \varepsilon$, which is the metric Cauchy condition. Conversely, let $(v_i)_{i\in I}$ be a metric Cauchy net and consider a neighbourhood $U \subseteq V$ of zero. Then there is an $\varepsilon > 0$ with $\mathrm{B}_\varepsilon(0) \subseteq U$. For this ε we find an index $i \in I$ with

$$\|v_j - v_{j'}\| < \varepsilon,$$

whenever $j, j' \succcurlyeq i$ by the metric Cauchy condition. But this implies

$$v_j - v_{j'} \in \mathrm{B}_\varepsilon(0) \subseteq U.$$

Hence $(v_i)_{i\in I}$ is a Cauchy net in the sense of Definition 2.1.15. □

With the theory of uniform spaces at our disposal one would have shown that the uniform structure underlying a topological vector space coincides with the metric uniform structure in case of a normed space. This would have unified the two proofs of Proposition 2.2.20 and Proposition 2.2.30. We refer to [3, Chap 6] for more details on uniform structure.

Thanks to this proposition we can use the metric notions of Cauchy sequences and completions as well as the general results on Cauchy nets and completions of topological vector spaces. In particular, for the question of completeness, Cauchy sequences are sufficient thanks to Proposition 2.1.18.

Corollary 2.2.31 *A normed space V is complete iff it is sequentially complete, i.e. iff every Cauchy sequence converges.*

Theorem 2.2.32 (Completion of a normed space) *Let V be a normed space and let $\widehat{V}$ be its metric completion with inclusion map $\iota\colon V \longrightarrow \widehat{V}$. Then $\widehat{V}$ has a unique structure of a complete normed vector space such that ι becomes a linear and norm-preserving map.*

Proof: First we note that addition is a linear map

$$+\colon V \oplus V \ni (v, w) \;\mapsto\; v + w \in V$$

and multiplication by $z \in \mathbb{K}$ is a linear map

$$z\cdot\colon V \ni v \;\mapsto\; zv \in V$$

for every vector space V. Now V has a metric topology and so has $V \oplus V$ by applying the metric componentwise as in Exercise **??**. We know that $+$ as well as $z\cdot$ are continuous and hence uniformly continuous linear maps by Proposition 2.1.14 or Corollary 2.2.21, since we are in the normed situation. Thanks to Corollary **??** we obtain unique uniformly continuous extensions

$$\widehat{+}\colon \widehat{V} \times \widehat{V} \longrightarrow \widehat{V}$$

and

$$\widehat{z\cdot}\colon \widehat{V} \longrightarrow \widehat{V},$$

which are now the candidates for structure maps for $\widehat{V}$. In addition, the norm itself is a uniformly continuous map

$$\|\cdot\|\colon V \longrightarrow \mathbb{R}_0^+ \subseteq \mathbb{K},$$

extending canonically to

$$\|\cdot\|\colon \widehat{V} \longrightarrow \mathbb{K}$$

by the same argument. We claim that these extensions turn $\widehat{V}$ into a normed vector space. First we note that $\widehat{+}$ and $\widehat{z\cdot}$ yield a vector space structure since the relevant algebraic identities are fulfilled on the dense subspace $V \subseteq \widehat{V}$ and the identities themselves consist of compositions of the continuous maps $\widehat{+}$ and $\widehat{z\cdot}$. Thus these identities are fixed by their values on the dense subspace by Proposition **??** as we are in a metric situation anyway. In conclusion, the identities hold on $\widehat{V}$ as well. Similarly, the extension of the norm fulfills all requirements of a seminorm by the same argument. Since $\mathrm{d}(v, w) = \|v - w\|$ holds for $v, w \in V$ this identity continues to stay valid on $\widehat{V}$. In particular, $\widehat{\mathrm{d}}(v, w) = \|v - w\|$ for the extension of the norm to $\widehat{V}$. Since $\widehat{\mathrm{d}}$ is a metric, it follows that $\|\cdot\|$ is not just a seminorm but stays a norm on $\widehat{V}$. $\square$

The aesthetic drawback of this result is that we have to rely on the fairly complicated construction of the completion of a metric space, i.e. on Corollary **??**, in a crucial way. We will see a completely different approach once we have the Hahn-Banach theorem in our toolbox.

2.3 Examples and Constructions of Banach Spaces

In Theorem 2.2.32 we have seen that every normed space has a completion which is a normed space again. This desirable situation deserves a particular name:

Definition 2.3.1 (Banach space) *A complete normed vector space is called a Banach space.*

In this section we will see the first important examples of Banach spaces together with many useful constructions.

2.3.1 Examples of Banach Spaces

From Theorem 2.2.19 and the componentwise completeness we directly obtain the following example from elementary calculus:

Example 2.3.2 The finite-dimensional space $\mathbb{K}^n$ is a Banach space for every norm we put on it.

More interesting are of course infinite-dimensional examples. We start with the sequence spaces:

Example 2.3.3 (Completeness of c and $c_\circ$) Consider the space of convergent sequences c. To show that c is complete, let $(a^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in c, i.e. each $a^{(n)} = (a_m^{(n)})_{m\in\mathbb{N}}$ is a convergent sequence such that for all $\varepsilon > 0$ we have an $N \in \mathbb{N}$ with

$$\big\|a^{(n)} - a^{(n')}\big\|_\infty = \sup_{m\in\mathbb{N}} \big|a_m^{(n)} - a_m^{(n')}\big| < \varepsilon, \tag{2.3.1}$$

whenever $n, n' \geq N$. In particular, $|a_m^{(n)} - a_m^{(n')}| < \varepsilon$ for all m in this case. This shows that the m-th entries form a Cauchy sequence $(a_m^{(n)})_{n\in\mathbb{N}}$ of scalars in $\mathbb{K}$. Thus we have a limit

$$a_m = \lim_{n\to\infty} a_m^{(n)} \in \mathbb{K} \tag{2.3.2}$$

for each m since $\mathbb{K}$ is complete. This gives us a sequence $a = (a_m)_{m\in\mathbb{N}}$ as a candidate for the limit of the Cauchy sequence $(a^{(n)})_{n\in\mathbb{N}}$. We have

$$\big|a_m - a_m^{(n')}\big| = \lim_{n\to\infty} \big|a_m^{(n)} - a_m^{(n')}\big| \leq \limsup_n \big|a_m^{(n)} - a_m^{(n')}\big| \leq \varepsilon \tag{2.3.3}$$

if $n' \geq N$, for all $m \in \mathbb{N}$. This shows

$$\left\| a - a^{(n')} \right\|_\infty \leq \varepsilon \tag{2.3.4}$$

whenever $n' \geq N$ and hence we have convergence of $(a^{(n)})_{n \in \mathbb{N}}$ to the sequence $a \in \operatorname{Map}(\mathbb{N}, \mathbb{K})$ with respect to the supremum norm. We claim $a \in c$. Indeed, this is an easy $\frac{\varepsilon}{3}$-argument. We have

$$|a_m - a_{m'}| \leq \left|a_m - a_m^{(n)}\right| + \left|a_m^{(n)} - a_{m'}^{(n)}\right| + \left|a_{m'}^{(n)} - a_{m'}\right|. \tag{2.3.5}$$

For $\varepsilon > 0$ we find an $N \in \mathbb{N}$ with $\|a - a^{(n)}\|_\infty < \frac{\varepsilon}{3}$ for $n \geq N$ and hence the first and third contribution are less than $\frac{\varepsilon}{3}$ for all m, m' in this case. Fixing $n \geq N$ we have convergence $a_m^{(n)} \to \lim_{m \to \infty} a_m^{(n)}$ since $a^{(n)} \in c$. Thus we have an $M \in \mathbb{N}$ with $|a_m^{(n)} - a_{m'}^{(n)}| < \frac{\varepsilon}{3}$ for $m, m' \geq M$ since a convergent sequence is a Cauchy sequence. It follows that $|a_m - a_{m'}| < \varepsilon$ for $m, m' \geq M$ and thus $(a_m)_{m \in \mathbb{N}}$ is a convergent sequence, showing $a \in c$ as required. Thus c is complete. Then also

$$c_\circ = \ker \lim \tag{2.3.6}$$

is complete since $\lim\colon c \longrightarrow \mathbb{K}$ is a continuous linear functional and thus $\ker \lim = \lim^{-1}(\{0\})$ is closed. By Proposition 2.1.19 this suffices to conclude that $c_\circ$ is complete indeed. The sequence space $c_{\circ\circ}$ on the other hand is *not* complete. Quite the contrary, $c_{\circ\circ} \subseteq c_\circ$ is dense inside $c_\circ$. To see this, we first note that $c_{\circ\circ} \subseteq c_\circ$ implies

$$c_{\circ\circ}^{\mathrm{cl}} \subseteq c_\circ^{\mathrm{cl}} = c_\circ, \tag{2.3.7}$$

i.e. the limit of finite sequences is a zero sequence. To show equality in (2.3.7), we consider a zero sequence $a \in c_\circ$, which we write as $a = (a_n)_{n \in \mathbb{N}}$. Define the finite sequences

$$a^{(N)} = (a_n^{(N)})_{n \in \mathbb{N}} \quad \text{with} \quad a_n^{(N)} = \begin{cases} a_n & n \leq N \\ 0 & n > N \end{cases} \tag{2.3.8}$$

for all $N \in \mathbb{N}$. By construction we have $a^{(N)} \in c_{\circ\circ}$ since only the first N entries can be non-zero. Since a is a zero sequence, we find for all $\varepsilon > 0$ a $K \in \mathbb{N}$ with $|a_k| < \varepsilon$ for $k \geq K$. Thus

$$\left\| a^{(N)} - a \right\|_\infty = \sup_{k > N} |a_k| \leq \varepsilon \tag{2.3.9}$$

as soon as $N \geq K$, showing convergence $a^{(N)} \to a$ with respect to $\|\cdot\|_\infty$. This shows equality in (2.3.7) and hence $c_\circ$ is the completion of $c_{\circ\circ}$. We can still reinterpret the convergence (2.3.9) in a slightly different way. Recall that for $c_{\circ\circ}$ we had a basis (2.2.19) of sequences $\mathrm{e}_n \in c_{\circ\circ}$ where we have at the m-th position a Kronecker delta δ_{nm}. Matching to this basis we have linear coefficient functionals

$$\varepsilon_n\colon c \longrightarrow \mathbb{K} \tag{2.3.10}$$

defined by

$$\varepsilon_n((a_m)_{m \in \mathbb{N}}) = a_n. \tag{2.3.11}$$

Since

$$|\varepsilon_n((a_m)_{m \in \mathbb{N}})| = |a_n| \leq \|a_n\|_\infty \tag{2.3.12}$$

these functionals are continuous on c with respect to the supremum norm $\|\cdot\|_\infty$. If in addition the sequence $a = (a_m)_{m \in \mathbb{N}}$ is a zero sequence, then the above considerations show that

$$a^{(N)} = \sum_{n=1}^{N} \varepsilon_n(a) \mathrm{e}_n \tag{2.3.13}$$

converges to a, i.e. we get

$$a = \sum_{n=1}^{\infty} \varepsilon_n(a) \mathrm{e}_n. \tag{2.3.14}$$

In fact, a slightly more careful analysis shows that the convergence is *unconditional*, i.e. the order of summation will not play any role. Moreover, the convergence is typically *not absolute*, see also Exercise **??** for these additional statements. This is an effect in infinite dimensions quite in contrast to the finite-dimensional situation, where unconditional convergence is the same as absolute convergence.

We gave a very detailed discussion of this fundamental example of Banach spaces. The next examples are obtained by similar arguments, so we can be more brief. Details can be found in the exercises, see Exercise **??** and Exercise **??**.

Example 2.3.4 (Completeness of ℓ^∞) The sequence space of bounded sequences ℓ^∞ is a Banach space with respect to the supremum norm $\|\cdot\|_\infty$. This is shown along the same lines of thought as for c and $c_\circ$. Now $c \subseteq \ell^\infty$ is a closed subspace since c is complete by Example 2.3.3. Surprisingly enough, ℓ^∞ is larger in a quite drastic way: the quotient vector space ℓ^∞ / c is known to be of uncountable dimension by fairly explicit arguments, see again Exercise 2.5.7. Note, however, that this gives only an estimate on the size of a basis of ℓ^∞, an honest basis can not be obtained in a constructive way but requires the more arcane techniques of the axiom of choice.

Example 2.3.5 (Completeness of ℓ^p for $p \in [1, \infty)$) The sequence spaces ℓ^p of p-summable sequences are Banach spaces for $p \in [1, \infty)$. This can be deduced from general measure-theoretic results, since $\ell^p = \mathrm{L}^p(\mathbb{N}, \mu_{\mathrm{count}})$ as we have seen in Section 2.2.1. However, in this particular case it is quite elementary to show the completeness directly with arguments close to those showing the completeness of c and ℓ^∞. A guided tour can be found in Exercise **??**. In particular, the coefficient functionals ε_n from (2.3.11) yield linear functionals

$$\varepsilon_n \colon \ell^p \longrightarrow \mathbb{K}, \tag{2.3.15}$$

which are continuous with respect to the ℓ^p-norm as well. Indeed, we have

$$|\varepsilon_n(a)| = |a_n| \le \|a\|_p \tag{2.3.16}$$

for all $a \in \ell^p$. As for $c_\circ$ we can now show that

$$a = \sum_{n=1}^{\infty} \varepsilon_n(a) \mathrm{e}_n \tag{2.3.17}$$

is an unconditionally convergent series. In fact, (2.3.17) is even absolutely convergent for $p = 1$ but not absolutely convergent in general otherwise. It follows that

$$c_{\circ\circ} \subseteq \ell^p \tag{2.3.18}$$

is also dense with respect to $\|\cdot\|_p$ for each $p \in [1, \infty)$. This is interesting in so far as we have different completions of $c_{\circ\circ}$ with respect to all the ℓ^p-norms, including the supremum norm as the case $p = \infty$. Hence all the norms $\|\cdot\|_p$ on $c_{\circ\circ}$ are pairwise *inequivalent*, showing that in infinite dimensions the analogue of Theorem 2.2.19 does not hold.

Example 2.3.6 (Completeness of $\mathscr{B}(X)$ and $\mathscr{BM}(X, \mathfrak{a})$) Let X be a non-empty set.

i.) Then the bounded functions $\mathscr{B}(X)$ are complete with respect to the supremum norm. This generalizes Example 2.3.4 which is recovered for $X = \mathbb{N}$. To see this, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of bounded functions $f_n \in \mathscr{B}(X)$ with respect to $\|\cdot\|_\infty$. We consider the evaluation functionals

$$\delta_x \colon \mathscr{B}(X) \longrightarrow \mathbb{K}, \tag{2.3.19}$$

defined by

$$\delta_x(f) = f(x) \tag{2.3.20}$$

for every $x \in X$. Since

$$|\delta_x(f)| = |f(x)| \leq \sup_{y\in X} |f(y)| = \|f\|_\infty, \tag{2.3.21}$$

all δ_x are continuous linear functionals. It follows that $(\delta_x(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$ and hence convergent. But this is just the pointwise convergence of $(f_n)_{n\in\mathbb{N}}$, thus defining a new function $f \in \operatorname{Map}(X, \mathbb{K})$ by

$$f(x) = \lim_{n\to\infty} f_n(x). \tag{2.3.22}$$

The idea is to show $f \in \mathscr{B}(X)$ and $f_n \to f$ uniformly. To this end we consider $x \in X$. Then

$$|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \varepsilon \tag{2.3.23}$$

if $n \geq N$ is chosen such that $\|f_n - f_m\|_\infty < \varepsilon$ by the Cauchy condition. This shows for $n \geq N$

$$\|f - f_n\|_\infty = \sup_{x\in X} \|f(x) - f_n(x)\| \leq \varepsilon, \tag{2.3.24}$$

and hence

$$\|f\|_\infty \leq \|f - f_N\|_\infty + \|f_N\|_\infty \leq \|f_N\|_\infty + \varepsilon < \infty, \tag{2.3.25}$$

which is $f \in \mathscr{B}(X)$. Moreover, (2.3.24) then shows $f_n \to f$ with respect to $\|\cdot\|_\infty$. Hence $\mathscr{B}(X)$ is a Banach space.

ii.) Suppose now that in addition X is equipped with a σ-algebra $\mathfrak{a}$. Then the bounded measurable functions $\mathscr{B}(X, \mathfrak{a})$ form a subspace of $\mathscr{B}(X)$. As convergence in $\mathscr{B}(X)$ with respect to $\|\cdot\|_\infty$ implies pointwise convergence, a Cauchy sequence of measurable bounded functions converges to a limit in $\mathscr{B}(X)$ with respect to $\|\cdot\|_\infty$ and hence pointwise. Since pointwise limits of measurable functions are again measurable in general, see e.g. [4, ???], we conclude that $\mathscr{BM}(X, \mathfrak{a}) \subseteq \mathscr{B}(X)$ is a closed subspace and hence complete itself. How big $\mathscr{BM}(X, \mathfrak{a})$ actually is depends very much on the details of the σ-algebra.

The simple functions inside $\mathscr{BM}(X, \mathfrak{a})$ form a dense subspace of $\mathscr{BM}(X, \mathfrak{a})$. This follows from the following statement from measure theory:

Proposition 2.3.7 *Let $(X, \mathfrak{a})$ be a measurable space.*

i.) For every $f \in \mathscr{BM}(X, \mathfrak{a})$ and every $\varepsilon > 0$ there exists a simple function $g \in \mathscr{BM}(X, \mathfrak{a})$ such that

$$\|f - g\|_\infty < \varepsilon. \tag{2.3.26}$$

ii.) The subspace of simple functions is dense in $\mathscr{BM}(X, \mathfrak{a})$.

The proof can be found in Exercise **??**, the second statement is just the conceptual interpretation of the first.

Still in a measure-theoretic framework is the following example of a Banach space based on Example 2.2.9:

Example 2.3.8 (Completeness of $\mathrm{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$) Let $(X, \mathfrak{a})$ be a measurable space with a σ-ideal $\mathfrak{n} \subseteq \mathfrak{a}$, e.g. coming from the zero sets of a positive measure on $(X, \mathfrak{a})$. Since the essentially bounded functions $\mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$ only carry a seminorm $\|\cdot\|_{\mathfrak{n},\infty}$ in general, we have to pass to the quotient $\mathrm{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$ before asking about completeness. Here results of measure theory show that $\mathrm{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$ is indeed complete, i.e. a Banach space with respect to the essential supremum norm $\|\cdot\|_{\mathfrak{n},\infty}$. Moreover, a bounded measurable function is clearly essentially bounded, no matter which σ-ideal $\mathfrak{n}$ we take. Hence we have a canonical inclusion

$$\mathscr{BM}(X, \mathfrak{a}) \subseteq \mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n}). \tag{2.3.27}$$

Moreover, for $f \in \mathscr{BM}(X, \mathfrak{a})$ we have

$$\|f\|_{\mathfrak{n},\infty} \leq \|f\|_\infty, \tag{2.3.28}$$

following from elementary properties of the essential supremum, see again Exercise **??**. Then (2.3.28) stays valid for the equivalence class $[f] \in \mathrm{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$, showing

$$\|[f]\|_{\mathfrak{n},\infty} \leq \|f\|_\infty, \tag{2.3.29}$$

see again the definition of the norm obtained from a seminorm in Remark 2.2.2, *vi.)*. Hence the map

$$\mathscr{BM}(X, \mathfrak{a}) \ni f \ \mapsto \ [f] \in \mathrm{L}^\infty(X, \mathfrak{a}, \mathfrak{n}) \tag{2.3.30}$$

is continuous and has operator norm at most one by (2.3.29). In fact, considering the constant functions shows that the operator norm is equal to one unless $\mathfrak{n} = \mathfrak{a}$. It is a slightly more involved result that the map (2.3.30) is actually *surjective*.

An example of Banach spaces we will often see and use in the sequel is given by the L^p-spaces:

Example 2.3.9 (Completeness of $\mathrm{L}^p(X, \mathfrak{a}, \mu)$) Consider the p-integrable functions $\mathscr{L}^p(X, \mathfrak{a}, \mu)$ on a measure space $(X, \mathfrak{a}, \mu)$ from Example 2.2.10. To obtain honest norms we first have to pass to the quotient $\mathrm{L}^p(X, \mathfrak{a}, \mu)$. There it is a standard result in measure theory that $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ is complete with respect to the p-norm $\|\cdot\|_p$, i.e. a Banach space, see e.g. [4, ???] for a proof. As for $\mathscr{BM}(X, \mathfrak{a})$ and $\mathrm{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$ the simple functions provide a dense subspace. Now we have to take care that they are elements of $\mathscr{L}^p(X, \mathfrak{a}, \mu)$ at all: a simple function $f \in \mathscr{M}(X, \mathfrak{a})$ is in $\mathscr{L}^p(X, \mathfrak{a}, \mu)$ iff for every value $z \in \mathbb{K} \setminus \{0\}$ one has $\mu(f^{-1}(\{z\})) < \infty$. Then the subspace of those simple functions provides a dense subspace of $\mathrm{L}^p(X, \mathfrak{a}, \mu)$ for every $p \in [1, \infty)$. In fact, by the very definition of the Lebesgue integral, $\mathscr{L}^1(X, \mathfrak{a}, \mu)$ is constructed by approximating with simple functions. Note, however, that now we need to approximate by the norm $\|\cdot\|_p$ instead of the supremum norm $\|\cdot\|_\infty$ or the essential supremum norm $\|\cdot\|_{\mathfrak{n},\infty}$.

In our list of examples in Section 2.2.1, the continuous functions would come next. We will investigate their properties in detail in Section 2.4, including aspects of completeness. Hence we close our list of Banach spaces for the moment and will add further examples later on.

2.3.2 The Banach Spaces $\mathrm{L}(V, W)$ and V'

One crucial observation in the theory of normed spaces is that even if one starts with a mere normed space, one can construct certain Banach spaces automatically.

Proposition 2.3.10 *Let V be a normed space and let W be a Banach space. Then $\mathrm{L}(V, W)$ is complete with respect to the operator norm.*

Proof: Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of continuous linear maps $A_n \in \mathrm{L}(V, W)$ with respect to the operator norm $\|\cdot\|$. Then for $\varepsilon > 0$ we find an $N \in \mathbb{N}$ with

$$\|A_n - A_m\| < \varepsilon,$$

whenever $n, m \geq N$. Let $v \in V$ be given. Then

$$\|A_n v - A_m v\|_W = \|(A_n - A_m)v\|_W \leq \|A_n - A_m\|\|v\|_V.$$

It follows that $(A_n v)_{n\in\mathbb{N}}$ is a Cauchy sequence in W for every v and thus convergent by assumption. We denote the limit by

$$Av = \lim_{n\to\infty} A_n v,$$

thereby defining a map $A\colon V \longrightarrow W$. Since each A_n is linear and since the vector space operations in W are continuous, it follows that A is linear. By convergence $A_n v \to Av$ we find an $M \in \mathbb{N}$ with $\|A_n v - Av\| < \varepsilon\|v\|$ for $n \geq M$. Without restriction, we choose $M \geq N$. Then we get for $n \geq N$

$$\|A_n v - Av\| \leq \|A_n v - A_M v\| + \|A_M v - Av\| < \|A_n - A_M\|\|v\| + \varepsilon\|v\| < 2\varepsilon\|v\|$$

for all $v \in V$. Note that the choice of M depends on v but the resulting estimate holds for all $n \geq N$ uniformly in v. This shows that $A_n - A \in \mathrm{L}(V, W)$ with operator norm $\|A_n - A\| \leq 2\varepsilon$. Then also $A = A - A_n + A_n \in \mathrm{L}(V, W)$ and $A_n \to A$ with respect to the operator norm. This shows the completeness. □

Corollary 2.3.11 *For every normed space V the topological dual $V' = \mathrm{L}(V, \mathbb{K})$ is a Banach space.*

The important point is that V itself needs not to be complete, the topological dual is complete nevertheless. The drawback of this corollary is that at the moment we do not know much about how non-trivial V' actually is. In finite dimensions we have $V' = V^*$ and $\dim V^* = \dim V$. In infinite dimension we need the axiom of choice already to show that the algebraic dual V^* is nontrivial. To find continuous linear functionals beside the zero-functional is even more complicated. We will have to come back to this issue in detail before Corollary 2.3.11 reaches its full potential.

Since V' is a Banach space we can form its topological dual

$$V'' = (V')' = \mathrm{L}(V', \mathbb{K}) = \mathrm{L}(\mathrm{L}(V, \mathbb{K}), \mathbb{K}), \tag{2.3.31}$$

the *topological bidual* of V. In linear algebra one has the canonical linear map

$$\iota\colon V \ni v \;\mapsto\; \iota(v) = \mathrm{ev}_v \in V^{**} \tag{2.3.32}$$

into the algebraic bidual of V, where $\mathrm{ev}_v\colon V^* \ni \alpha \mapsto \mathrm{ev}_v(\alpha) = \alpha(v) \in \mathbb{K}$ is the *evaluation* on $v \in V$. Restricting ev_v to the subspace $V' \subseteq V^*$ yields a continuous linear functional:

Proposition 2.3.12 *Let V be a normed space.*

i.) For every $v \in V$ one has $\mathrm{ev}_v \in V''$.

ii.) The canonical linear map

$$\iota\colon V \ni v \;\mapsto\; \iota(v) = \mathrm{ev}_v \in V'' \tag{2.3.33}$$

is continuous. More precisely, for all $v \in V$ one has

$$\|\iota(v)\| \leq \|v\|_V. \tag{2.3.34}$$

PROOF: Note that V' is equipped with the functional norm and so is V''. Let $\alpha \in V'$, then

$$|\mathrm{ev}_v(\alpha)| = |\alpha(v)| \leq \|\alpha\| \|v\|_V$$

holds for all $v \in V$ by definition of the functional norm. This shows that

$$\iota(v)\colon \alpha \ \mapsto \ \mathrm{ev}_v(\alpha)$$

is continuous with functional norm at most $\|v\|_V$, i.e. we have (2.3.34). This estimate, in turn, shows that ι itself is continuous with operator norm at most one. □

In linear algebra one knows that (2.3.32) is even an injective linear map. At the moment we can not decide this property for the corresponding map (2.3.33), since $V' \subseteq V^*$ could be too small to detect all vectors of V.

With only a little effort one can extend the completeness result for linear maps to the multilinear case. Again, it suffices that the target space is complete, not the domains:

Proposition 2.3.13 *Let $V_1, \ldots, V_k$ be normed spaces and let W be a Banach space. Then the continuous k-linear maps $\mathrm{L}(V_1, \ldots, V_k; W)$ are complete with respect to the operator norm.*

A guided tour through the proof can be found in Exercise **??**.

2.3.3 Quotients of Banach Spaces

The next important construction is also known from linear algebra and needs to be transferred to the functional-analytic framework.

Lemma 2.3.14 *Let V be a vector space with a subspace $U \subseteq V$.*

i.) *For every seminorm p on V*

$$[\mathrm{p}]([v]) = \inf\{\mathrm{p}(v+u) \mid u \in U\} \tag{2.3.35}$$

defines a seminorm on the quotient V/U where $[v] \in V/U$ denotes the equivalence class of $v \in V$.

ii.) *If $\|\cdot\|$ is a norm on V then the corresponding seminorm (2.3.35) is a norm on V/U iff $U = U^{\mathrm{cl}}$ with respect to this norm.*

PROOF: First we note that $[\mathrm{p}]([v]) \geq 0$ for all $[v] \in V/U$. Let $z \in \mathbb{K}$ and $[v] \in V/U$. If $z = 0$ then clearly

$$[\mathrm{p}](z[v]) = [\mathrm{p}]([zv]) = [\mathrm{p}]([0]) = 0.$$

Hence consider $z \neq 0$, then

$$\begin{aligned}
[\mathrm{p}](z[v]) &= \inf\{\mathrm{p}(zv+u) \mid u \in U\} \\
&= \inf\Big\{\mathrm{p}\Big(z\Big(v + \frac{u}{z}\Big)\Big) \;\Big|\; u \in U\Big\} \\
&= \inf\Big\{|z|\mathrm{p}\Big(v + \frac{u}{z}\Big) \;\Big|\; u \in U\Big\} \\
&= |z| \inf\{\mathrm{p}(v+u) \mid u \in U\} \\
&= |z|[\mathrm{p}]([v]),
\end{aligned}$$

since U is a subspace. Finally, let $v, w \in V$ then

$$[\mathrm{p}]([v]+[w]) = [\mathrm{p}]([v+w])$$

$$
\begin{aligned}
&= \inf\bigl\{\mathrm{p}(v + w + u) \bigm| u \in U\bigr\} \\
&= \inf\bigl\{\mathrm{p}(v + u + w + u') \bigm| u, u' \in U\bigr\} \\
&\le \inf\bigl\{\mathrm{p}(v + u) + \mathrm{p}(w + u') \bigm| u, u' \in U\bigr\} \\
&= \inf\bigl\{\mathrm{p}(v + u) \bigm| u \in U\bigr\} + \inf\bigl\{\mathrm{p}(w + u') \bigm| u' \in U\bigr\} \\
&= [\mathrm{p}]([v]) + [\mathrm{p}]([w]),
\end{aligned}
$$

again using that U is a subspace. This shows the first part. Now let $\mathrm{p} = \|\cdot\|$ be the norm of a normed vector space V. Let $[v] \in V/U$ with $[\mathrm{p}]([v]) = 0$ be given. By the definition of the infimum this means for all $\varepsilon > 0$ we find an $u \in U$ with

$$
\|v + u\| < \varepsilon.
$$

Equivalently, we have $v \in U^{\mathrm{cl}}$. Hence $[\mathrm{p}]$ is again a norm iff $U^{\mathrm{cl}} = U$. □

Definition 2.3.15 (Quotient norm) *Let V be a normed space with a closed subspace $U = U^{\mathrm{cl}}$. Then the norm*

$$
\|[v]\| = \inf\bigl\{\|v + u\| \bigm| u \in U\bigr\} \tag{2.3.36}
$$

on V/U is called the quotient norm.

In the following we will equip quotients of a normed space by a closed subspace always with the quotient norm. The following proposition collects some first properties:

Proposition 2.3.16 *Let V be a normed space with a closed subspace $U = U^{\mathrm{cl}}$.*

i.) The quotient map $\mathrm{pr}\colon V \ni v \mapsto \mathrm{pr}(v) = [v] \in V/U$ is continuous and has operator norm

$$
\|\mathrm{pr}\| \le 1. \tag{2.3.37}
$$

ii.) The quotient map pr is an open map.

iii.) If V is complete then V/U is complete, too.

Proof: The first part is easy since

$$
\|\mathrm{pr}(v)\| = \|[v]\| = \inf\bigl\{\|v + u\| \bigm| u \in U\bigr\} \le \|v\|
$$

for all $v \in V$. For the second statement, let $r > 0$ and consider $\mathrm{pr}(\mathrm{B}_r(0)) \subseteq V/U$. We have for $v \in \mathrm{B}_r(0)$

$$
\|\mathrm{pr}(v)\| \le \|v\| < r
$$

by the first part. Hence $\mathrm{pr}(\mathrm{B}_r(0)) \subseteq \mathrm{B}_r(0)$. Conversely, let $[v] \in \mathrm{B}_r(0) \subseteq V/U$. Then $\|[v]\| < r$ shows that there is a representative $v + u \in [v]$ with $\|v + u\| < r$. Hence $v + u \in \mathrm{B}_r(0)$ showing $\mathrm{B}_r(0) \subseteq \mathrm{pr}(\mathrm{B}_r(0))$. Together we get

$$
\mathrm{pr}(\mathrm{B}_r(0)) = \mathrm{B}_r(0).
$$

Next, let $X \subseteq V$ be any subset and $v \in V$. Then

$$
\mathrm{pr}(X + v) = \bigl\{\mathrm{pr}(x + v) \bigm| x \in X\bigr\} = \bigl\{\mathrm{pr}(x) + \mathrm{pr}(v) \bigm| x \in X\bigr\} = \mathrm{pr}(X) + \mathrm{pr}(v)
$$

by the linearity of pr. Hence the translation invariance of the topologies on V and V/U ensures that unions of translated open balls in V, which yield all open subsets of V, are mapped to corresponding unions of open balls in V/U. This shows that pr is an open map. In fact, there is a very general argument from topology behind this, see e.g. [5, Exercise 3.4.8]. It remains to show the completeness of the quotient V/U if V is a Banach space. Let $([v_n])_{n\in\mathbb{N}}$ be a Cauchy sequence in V/U. The difficulty is that arbitrarily chosen pre-images $v_n \in [v_n]$ will not necessarily form a Cauchy sequence since we have the freedom to add any vector from U to the v_n without changing $[v_n]$ but increasing the norm in V. To actually construct a Cauchy sequence of preimages we first choose a subsequence $[v_{n_k}]$ with

$$\big\|[v_{n_k}] - [v_{n_{k+1}}]\big\| < \frac{1}{2^k} \tag{$*$}$$

by applying the Cauchy condition to $\varepsilon = \frac{1}{2^k}$. Let $w_1 \in [v_{n_1}]$ be an arbitrary representative. Then $(*)$ allows us to find a representative $w_2 \in [v_{n_2}]$ with $\|w_1 - w_2\| < \frac{1}{2}$ since

$$\big\|[w_1] - [v_{n_2}]\big\| = \inf\big\{\|w_1 - v_{n_2} + u\| \bigm| u \in U\big\} < \frac{1}{2}.$$

This is the start of an inductive construction: repeating the argument gives representatives $w_k \in [v_{n_k}]$ such that

$$\|w_k - w_{k+1}\| < \frac{1}{2^k}$$

for all $k \in \mathbb{N}$. Hence $(w_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in V since the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ clearly converges. As V is a Banach space we have a limit $w \in V$ with $w_k \to w$. Thus the continuity of pr gives

$$\mathrm{pr}(w_k) = [w_k] = [v_{n_k}] \to \mathrm{pr}(w),$$

showing that $([v_n])_{n\in\mathbb{N}}$ has a convergent subsequence. Thus $[v_n] \to \mathrm{pr}(w)$, too, showing the completeness of $V\big/U$. □

This quotient of normed spaces has nice additional properties from a more categorical point of view:

Proposition 2.3.17 *Let V be a normed space with a closed subspace $U = U^{\mathrm{cl}}$ and equip the quotient $V\big/U$ with the quotient norm.*

i.) The quotient norm on $V\big/U$ equips $V\big/U$ with the finest norm topology such that the quotient map is continuous.

ii.) A linear map $\phi\colon V\big/U \longrightarrow W$ into another normed space is continuous iff the composition $\phi \circ \mathrm{pr}\colon V \longrightarrow W$ is continuous.

PROOF: Let p be a norm on $V\big/U$ such that pr is continuous. This means there is a $c > 0$ with

$$\mathrm{p}(\mathrm{pr}(v)) = \mathrm{p}([v]) \le c\|v\| \tag{$*$}$$

for all $v \in V$. Now $[v] = [v + u]$ for all $u \in U$ and hence $(*)$ gives

$$\mathrm{p}([v]) = \mathrm{p}([v+u]) \le c\|v + u\|$$

for all $u \in U$. Hence taking the infimum over $u \in U$ yields

$$\mathrm{p}([v]) \le c \inf\big\{\|v + u\| \bigm| u \in U\big\} = c\|[v]\|.$$

With Proposition 2.2.18, *i.)*, the first part follows. For the second, assume $\phi \circ \mathrm{pr}$ is continuous. Then we have a constant $c > 0$ with

$$\|(\phi \circ \mathrm{pr})(v)\|_W \le c\|v\|_V$$

for all $v \in V$. As for the first part we infer

$$\big\|\phi([v])\big\|_W = \big\|\phi([v+u])\big\|_W = \|(\phi \circ \mathrm{pr})(v + u)\|_W \le c\|v + u\|_V$$

for all $u \in U$ and hence

$$\big\|\phi([v])\big\|_W \le c \inf\big\{\|v + u\| \bigm| u \in U\big\} = c\big\|[v]\big\|,$$

showing the continuity of ϕ. The converse is clear since pr is continuous by Proposition 2.3.16, *i.)*.□

Remark 2.3.18 The first statement gives an abstract uniqueness of the normed space $V\big/U$ as the finest normed topology with the continuity of pr. The second statement is a universal property analogous to the universal property of quotient vector spaces in linear algebra.

2.3.4 Direct Sum and Tensor Product

While quotients have a unique norm again, the situation for direct sums and tensor products is more complicated. As warming up we consider finite direct sums:

Proposition 2.3.19 *Let $V_1, \ldots, V_N$ be normed vector spaces and let $p \in [1, \infty)$. Moreover, let $V = V_1 \oplus \cdots \oplus V_N$ be the direct sum.*

i.) For $v = v_1 + \cdots + v_N \in V$ with $v_i \in V_i$ for $i = 1, \ldots, N$ one obtains a norm $\|\cdot\|_\infty$ by

$$\|v\|_\infty = \max\{\|v_1\|_{V_1}, \ldots, \|v_n\|_{V_N}\}. \tag{2.3.38}$$

ii.) Analogously,

$$\|v\|_p = \sqrt[p]{\|v_1\|_{V_1}^p + \cdots + \|v_n\|_{V_N}^p} \tag{2.3.39}$$

defines a norm on V.

iii.) All the norms $\|\cdot\|_p$ are equivalent. In fact, one has

$$\|v\|_\infty \leq \|v\|_p \leq \|v\|_q \leq \|v\|_1 \leq N\|v\|_\infty \tag{2.3.40}$$

for all $q, p \in [1, \infty)$ with $q \leq p$.

iv.) The canonical inclusions

$$i_k \colon V_k \longrightarrow V \tag{2.3.41}$$

are norm-preserving continuous linear maps for all $k = 1, \ldots, N$ and all norms $\|\cdot\|_p$ with $p \in [1, \infty]$ on V.

v.) The canonical projections

$$\pi_k \colon V \longrightarrow V_k \tag{2.3.42}$$

are continuous linear maps for all $k = 1, \ldots, N$ and all norms $\|\cdot\|_p$ with $p \in [1, \infty]$ on V.

vi.) The topology induced by all the above norms is the Cartesian product topology on

$$V_1 \oplus \cdots \oplus V_N = V_1 \times \cdots \times V_N. \tag{2.3.43}$$

Proof: The verification of the norm properties for (2.3.38) and (2.3.39) is a simple application of the Minkowski inequality for finite sums. The estimates (2.3.40) are shown as in the case of ℓ^p-spaces based on the Jensen inequality. Note that in (2.3.40) one explicitly needs the number of summands in the last inequality. It follows that the norms are equivalent, i.e. induce the same topology on V. For $v_k \in V_k$ we clearly have

$$\|i_k(v_k)\|_p = \sqrt[p]{\|v_k\|_{V_k}^p} = \|v_k\|_{V_k}$$

and

$$\|i_k(v_k)\|_\infty = \max\{0, \ldots, 0, \|v_k\|_{V_k}, 0, \ldots, 0\} = \|v_k\|_{V_k},$$

showing the fourth part. For $v = v_1 + \cdots + v_N$ one has

$$\|\pi_k(v)\|_{V_k} = \|v_k\|_{V_k} \leq \|v\|_p$$

showing the fifth part, too. The last part is most easily shown using the norm $\|\cdot\|_\infty$ on $V_1 \times \cdots \times V_N$ where the open balls are just the products of the open balls in each component. From this observation the statement follows immediately since open balls generate the norm topology and Cartesian products of open subsets generate the Cartesian product topology. □

Hence the topology of a direct sum is the Cartesian product topology for all the above choices of norms. However, as normed spaces, $(V, \|\cdot\|_p)$ are different for different p. In fact, one can define even more versions of norms on V with similar properties, all leading to the same underlying Cartesian product topology. Hence there is no canonical choice to make a direct sum of normed spaces a normed space again, even though many choices lead to the same topology, the Cartesian product topology.

Proposition 2.3.20 *Let $V_1, \ldots, V_N$ be normed vector spaces and let $p \in [1, \infty]$. Equip $V = V_1 \oplus \cdots \oplus V_N$ with the norm $\|\cdot\|_p$ as before.*

i.) A sequence $(v^{(n)})_{n\in\mathbb{N}}$ of vectors $v^{(n)} \in V$ converges to $v \in V$ iff for all $k = 1, \ldots, N$ the component sequences $(v_k^{(n)})_{n\in\mathbb{N}}$ converge to the component v_k.

ii.) A sequence $(v^{(n)})_{n\in\mathbb{N}}$ of vectors $v^{(n)} \in V$ is a Cauchy sequence in V iff for all $k = 1, \ldots, N$ the component sequences $(v_k^{(n)})_{n\in\mathbb{N}}$ are Cauchy sequences in V_k.

iii.) The direct sum is complete iff all summands $V_1, \ldots, V_N$ are complete.

PROOF: The first statement holds for Cartesian products in general. Since the topology on V is the Cartesian product topology, we can apply this to prove the first part. The second statement is shown analogously: we can consider e.g. $p = \infty$ since all p yield the same underlying topological vector space and Cauchy sequences only use this information. With $\|\cdot\|_\infty$ the necessary estimates for *ii.)* are easy. Then *iii.)* is an immediate consequence. □

Note that both propositions generalize the passage from $\mathbb{K}$ to $\mathbb{K}^N = \mathbb{K} \oplus \cdots \oplus \mathbb{K}$. If we have infinite direct sums the situation becomes much more complicated: this is already to be expected from the examples of sequence spaces. Indeed, since

$$c_{\circ\circ} = \operatorname{span}_{\mathbb{K}}\{\mathrm{e}_n\}_{n\in\mathbb{N}}, \tag{2.3.44}$$

we can view $c_{\circ\circ}$ as the $\mathbb{N}$-fold direct sum of copies of $\mathbb{K}$. Then we have various p-norms $\|\cdot\|_p$ on $c_{\circ\circ}$ leading to different completions $c_\circ$ and ℓ^p, respectively. In particular, the norms $\|\cdot\|_p$ for different $p \in [1, \infty]$ are pairwise inequivalent since their completions differ, see again Exercise **??**. Moreover, we see that a direct sum $c_{\circ\circ}$ of complete spaces $\mathbb{K}$ does not need to be complete anymore. Hence from the perspective of normed spaces there is no reasonable way to put a norm on a direct sum in general. Nevertheless, there are norms inspired by the ℓ^p-norms for arbitrary direct sums as well. In Exercise **??** some examples and their features are discussed.

The last canonical construction from linear algebra, which we want to transfer to the realm of normed and Banach spaces, is the tensor product. The following construction yields a specific norm on the tensor product of normed spaces which characterizes continuity of the tensor product map. To construct this norm, let $V_1, \ldots, V_k$ be normed spaces. Recall from linear algebra that the tensor product is a pair $(V_1 \otimes \cdots \otimes V_k, \otimes)$ of a vector space $V_1 \otimes \cdots \otimes V_k$ together with a k-linear map

$$\otimes\colon V_1 \times \cdots \times V_k \longrightarrow V_1 \otimes \cdots \otimes V_k \tag{2.3.45}$$

with the universal property that for every other k-linear map

$$\Phi\colon V_1 \times \cdots \times V_k \longrightarrow W \tag{2.3.46}$$

into another vector space W there is a unique linear map $\phi\colon V_1 \otimes \cdots \otimes V_k \longrightarrow W$ such that the diagram

$$\begin{tikzcd} V_1 \otimes \cdots \otimes V_k \arrow[r, "\phi"] & W \\ V_1 \times \cdots \times V_k \arrow[u, "\otimes"] \arrow[ur, "\Phi"'] & \end{tikzcd} \tag{2.3.47}$$

commutes. Further details on the linear-algebraic aspects of tensor products can be found e.g. in [6, Chapter 3] or [1].

The idea is now that we want to establish a norm on $V_1 \otimes \cdots \otimes V_k$ in such a way that ϕ is continuous whenever Φ was continuous. To this end we want the tensor product map in (2.3.45) to be continuous. According to the characterization from Proposition 2.2.27, this means that there is a $c > 0$ with

$$\|v_1 \otimes \cdots \otimes v_k\|_{V_1 \otimes \cdots \otimes V_k} \leq c \|v_1\|_{V_1} \cdots \|v_k\|_{V_k} \tag{2.3.48}$$

for all $v_1 \in V_1, \ldots, v_k \in V_k$. Here we write $v_1 \otimes \cdots \otimes v_k$ instead of $\otimes(v_1, \ldots, v_k)$ as usual. Without restriction we can assume even $c = 1$ after rescaling of the still-to be found norm $\|\cdot\|_{V_1 \otimes \cdots \otimes V_k}$. The problem is that the requirement

$$\|v_1 \otimes \cdots \otimes v_k\|_{V_1 \otimes \cdots \otimes V_k} \leq \|v_1\|_{V_1} \cdots \|v_k\|_{V_k} \tag{2.3.49}$$

does not yet fix the norm we are looking for, since a general element in $V_1 \otimes \cdots \otimes V_k$ is a linear combination of factorizing tensors like $v_1 \otimes \cdots \otimes v_k$. To implement (2.3.49) in a most efficient way one defines

$$\|v\|_{V_1 \otimes \cdots \otimes V_k} = \inf \Big\{ \sum\nolimits_\ell \|v_1^\ell\|_{V_1} \cdots \|v_k^\ell\|_{V_k} \;\Big|\; \sum\nolimits_\ell v_1^\ell \otimes \cdots \otimes v_k^\ell = v \Big\}, \tag{2.3.50}$$

i.e. the infimum taken over all possibilities to express $v \in V_1 \otimes \cdots \otimes V_k$ as a linear combination of factorizing tensors.

Lemma 2.3.21 *Let $V_1, \ldots, V_k$ be normed spaces. Then $\|\cdot\|_{V_1 \otimes \cdots \otimes V_k}$ as in (2.3.50) defines a seminorm on $V = V_1 \otimes \cdots \otimes V_k$ with*

$$\|v_1 \otimes \cdots \otimes v_k\|_V \leq \|v_1\|_{V_1} \cdots \|v_k\|_{V_k} \tag{2.3.51}$$

for all $v_1 \in V_1, \ldots, v_k \in V_k$.

Proof: Clearly, $\|v\|_V \geq 0$ for all $v \in V_1 \otimes \cdots \otimes V_k$. Let $z \in \mathbb{K}$, then

$$zv = \sum\nolimits_\ell z v_1^\ell \otimes \cdots \otimes v_k^\ell \quad \text{iff} \quad v = \sum\nolimits_\ell v_1^\ell \otimes \cdots \otimes v_k^\ell.$$

Thus the infimum needed for $\|zv\|_V$ is taken over the same linear combinations as the infimum needed for v alone, including the additional prefactor z in the first tensor factor. This shows

$$\begin{aligned}
\|zv\|_{V_1 \otimes \cdots \otimes V_k} &= \inf \Big\{ \sum\nolimits_\ell \|z v_1^\ell\|_{V_1} \cdots \|v_k^\ell\|_{V_k} \;\Big|\; \sum\nolimits_\ell z v_1^\ell \otimes \cdots \otimes v_k^\ell = zv \Big\} \\
&= |z| \inf \Big\{ \sum\nolimits_\ell \|v_1^\ell\|_{V_1} \cdots \|v_k^\ell\|_{V_k} \;\Big|\; \sum\nolimits_\ell v_1^\ell \otimes \cdots \otimes v_k^\ell = v \Big\} \\
&= |z| \|v\|_{V_1 \otimes \cdots \otimes V_k}.
\end{aligned}$$

For $v = \sum_\ell v_1^\ell \otimes \cdots \otimes v_k^\ell$ and $w = \sum_\ell w_1^\ell \otimes \cdots \otimes w_k^\ell$ the linear combination $\sum_\ell v_1^\ell \otimes \cdots \otimes v_k^\ell + \sum_\ell w_1^\ell \otimes \cdots \otimes w_k^\ell$ gives one way to decompose $v + w$. Hence taking the infimum yields

$$\begin{aligned}
\|v + w\|_{V_1 \otimes \cdots \otimes V_k} &= \inf \Big\{ \sum\nolimits_\ell \|x_1^\ell\|_{V_1} \cdots \|x_k^\ell\|_{V_k} \;\Big|\; \sum\nolimits_\ell x_1^\ell \otimes \cdots \otimes x_k^\ell = v + w \Big\} \\
&\leq \inf \Big\{ \sum\nolimits_\ell \|v_1^\ell\|_{V_1} \cdots \|v_k^\ell\|_{V_k} + \sum\nolimits_\ell \|w_1^\ell\|_{V_1} \cdots \|w_k^\ell\|_{V_k} \\
&\qquad\qquad \Big|\; \sum\nolimits_\ell v_1^\ell \otimes \cdots \otimes v_k^\ell = v \text{ and } \sum\nolimits_\ell w_1^\ell \otimes \cdots \otimes w_k^\ell = w \Big\} \\
&= \inf \Big\{ \sum\nolimits_\ell \|v_1^\ell\|_{V_1} \cdots \|v_k^\ell\|_{V_k} \;\Big|\; \sum\nolimits_\ell v_1^\ell \otimes \cdots \otimes v_k^\ell = v \Big\} \\
&\quad + \inf \Big\{ \sum\nolimits_\ell \|w_1^\ell\|_{V_1} \cdots \|w_k^\ell\|_{V_k} \;\Big|\; \sum\nolimits_\ell w_1^\ell \otimes \cdots \otimes w_k^\ell = w \Big\} \\
&= \|v\|_{V_1 \otimes \cdots \otimes V_k} + \|w\|_{V_1 \otimes \cdots \otimes V_k},
\end{aligned}$$

showing the triangle inequality. $\square$

Thus it is tempting to take $\|\cdot\|_{V_1\otimes\cdots\otimes V_k}$ for a norm on $V_1 \otimes \cdots \otimes V_k$. We will do so later. However, at the moment we are still lacking the appropriate tools to actually show that (2.3.50) is a *norm* and not just a seminorm. Hence we have to postpone a further analysis of the tensor product of normed spaces until Section 3.1.4.

2.4 The Banach Space of Continuous Bounded Functions

For a topological space M the continuous real- or complex-valued functions $\mathscr{C}(M) = \mathscr{C}(M, \mathbb{K})$ form a $\mathbb{K}$-vector space which we can endow with nice topological structures itself. In the simplest scenario we want to discuss here, either M is compact or we consider the bounded continuous functions $\mathscr{C}_{\mathrm{b}}(M)$ from Example 2.2.11, which coincide with $\mathscr{C}(M)$ if M is compact.

In this section we will investigate the properties of continuous bounded functions in quite some detail, proving thereby some of the most fundamental theorems about continuous functions.

2.4.1 Completeness of $\mathscr{C}_{\mathrm{b}}(M)$

In order to have the supremum norm available, we need to restrict ourselves to bounded continuous functions. They turn out to form a Banach space:

Proposition 2.4.1 *Let M be a topological space. Then $\mathscr{C}_{\mathrm{b}}(M, \mathbb{K})$ becomes a Banach space over $\mathbb{K}$ when equipped with the supremum norm $\|\cdot\|_\infty$.*

PROOF: Recall that $\mathscr{C}_{\mathrm{b}}(M, \mathbb{K})$ indeed forms a vector space over $\mathbb{K}$, see again Exercise 2.5.3, on which the supremum norm $\|\cdot\|_\infty$ is defined. We need to show the completeness. Thus let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of continuous bounded functions $f_n \in \mathscr{C}_{\mathrm{b}}(M, \mathbb{K})$ with respect to $\|\cdot\|_\infty$. From Example 2.3.6 we know that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a bounded function $f \in \mathscr{B}(M, \mathbb{K})$. It remains to check that this limit f is continuous. To this end, let $p \in M$ and let $\varepsilon > 0$ be given. From the uniform convergence $f_n \to f$ we find an index $N \in \mathbb{N}$ with

$$\|f - f_n\|_\infty < \frac{\varepsilon}{3}$$

for all $n \geq N$. In particular, for all $q \in M$ we have $|f(q) - f_n(q)| < \frac{\varepsilon}{3}$ as well. We fix such an $n \geq N$. Since f_n is continuous, we find a neighbourhood $U \subseteq M$ of p such that

$$|f_n(p) - f_n(q)| < \frac{\varepsilon}{3}$$

for all $q \in U$. For this U we then get

$$\begin{aligned}
|f(p) - f(q)| &\leq |f(p) - f_n(p)| + |f_n(p) - f_n(q)| + |f_n(q) - f(q)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon,
\end{aligned}$$

as soon as $q \in U$. This shows $U \subseteq f^{-1}(\mathrm{B}_\varepsilon(f(p)))$ and hence $f^{-1}(\mathrm{B}_\varepsilon(f(p)))$ is a neighbourhood of p, proving the continuity of f at p since the ε-balls $\mathrm{B}_\varepsilon(f(p)) \subseteq \mathbb{K}$ form a neighbourhood basis of $f(p)$. As p was arbitrary, the continuity of f follows. □

Since on a compact space M any continuous function is automatically bounded by Corollary **??**, we get the following result:

Corollary 2.4.2 *Let M be a compact topological space. Then $\mathscr{C}(M, \mathbb{K})$ is a Banach space over $\mathbb{K}$.*

As a concluding remark we should note that this observation still poses several difficulties:

Remark 2.4.3 Let M be a topological space.

i.) Up to now it is not clear how rich $\mathscr{C}_{\mathrm{b}}(M, \mathbb{K})$ really is. The constant functions are always part of $\mathscr{C}_{\mathrm{b}}(M, \mathbb{K})$ but it is very non-trivial in general to show that $\mathscr{C}_{\mathrm{b}}(M, \mathbb{K})$ is actually larger than just $\mathbb{K}$. We will see some answers in the next section. The reason for the difficulty is that for too coarse topologies on M it becomes too hard for non-constant functions to be continuous.

ii.) The nice Banach space structure refers to the supremum norm and, hence, relies on bounded continuous functions in a crucial way. There are of course situations where also unbounded continuous functions are of interest. Then $\|\cdot\|_\infty$ is not available anymore. As an alternative, one can take local supremum seminorms

$$\|f\|_K = \sup_{p \in K} |f(p)| \tag{2.4.1}$$

for compact subsets $K \subseteq M$. Thanks to Corollary **??** this supremum exists and is actually a maximum. The non-trivial question is whether and how one can then use all these seminorms to formulate a statement analogous to Proposition 2.4.1. This will require to go beyond normed spaces and Banach spaces and hence beyond the scope of this chapter. In the theory of more general locally convex spaces one can come back to this again.

2.4.2 Urysohn's Lemma and Tietze's Theorem

We are now interested in proving the existence of non-constant continuous functions. Their existence is tied to the T_4-property of the topology. In fact, we have the following characterization:

Theorem 2.4.4 (Urysohn's Lemma) *Let $(M, \mathscr{M})$ be a topological space. Then the following statements are equivalent:*

i.) The space $(M, \mathscr{M})$ has the T_4-property.

ii.) For disjoint closed subsets $A, B \subseteq M$ there exists a continuous function

$$f \colon M \longrightarrow [0,1] \tag{2.4.2}$$

with the property

$$f\Big|_A = 0 \quad \textit{and} \quad f\Big|_B = 1. \tag{2.4.3}$$

PROOF: Suppose $A, B \subseteq M$ are closed and $A \cap B = \emptyset$. If f is a continuous function satisfying (2.4.3) then

$$O_1 = f^{-1}\big([0, \tfrac{1}{2})\big) \quad \text{and} \quad O_2 = f^{-1}\big((\tfrac{1}{2}, 1]\big)$$

are open subsets of M by the continuity of f which are disjoint and satisfy

$$A \subseteq O_1 \quad \text{and} \quad B \subseteq O_2.$$

Hence M is T_4. It is the other implication *i.)* $\implies$ *ii.)* which is non-trivial and requires some clever idea. Let A, B be given as before and assume that M satisfies T_4. Then consider the dyadic numbers

$$\mathscr{D} = \left\{ \frac{p}{2^q} \;\middle|\; p, q \in \mathbb{N} \right\}.$$

For $t \in \mathscr{D}$ with $t > 1$ we set $G_t = M$. Moreover, for $t = 1$ we set $G_1 = M \setminus B$ which is an open subset with $A \subseteq G_1$. We want to squeeze-in additional open subsets G_t increasing with t for $t \in \mathscr{D} \cap [0,1]$. The T_4-property allows us to construct such G_t inductively: We choose an open subset $G_0 \subseteq M$ with

$$A \subseteq G_0 \subseteq G_0^{\mathrm{cl}} \subseteq G_1 = M \setminus B,$$

Figure 2.1: The inductive construction of the subsets G_t for $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

which is possible thanks to the T_4-property, see again Proposition ??, ??. Since G_0^{cl} is again closed and G_1 is open, applying the T_4-property once more yields a new open subset $G_{\frac{1}{2}}$ in between, i.e.

$$G_0^{\mathrm{cl}} \subseteq G_{\frac{1}{2}} \subseteq G_{\frac{1}{2}}^{\mathrm{cl}} \subseteq G_1,$$

see also Figure 2.1. Inductively, we get open subsets $G_{\frac{p}{2^q}}$ for every $q \in \mathbb{N}$ and $p = 1, \ldots, 2^q - 1$ such that

$$G_{\frac{p-1}{2^q}}^{\mathrm{cl}} \subseteq G_{\frac{p}{2^q}} \subseteq G_{\frac{p}{2^q}}^{\mathrm{cl}} \subseteq G_{\frac{p+1}{2^q}},$$

by applying the T_4-property to the previous chain of open subsets in form of Proposition ??, ??, filling the gaps in the middle. The function we are looking for is then defined pointwise by

$$f(p) = \inf\{t \mid p \in G_t\} \in [0, 1].$$

Since $A \subseteq G_0$ we have $f\big|_A = 0$. Moreover, since $B \subseteq G_t = M$ for all $t > 1$ but $B \cap G_1 = \emptyset$ we see that $f\big|_B = 1$. The values of f are clearly in $[0, 1]$. It remains to check that f is actually continuous. Hence consider $p \in M$ and let $\varepsilon > 0$. We find $0 < \delta, \delta' < \varepsilon$ such that

$$f(p) + \delta, f(p) - \delta' \in \mathscr{D}.$$

For a point $q \in G_{f(p)+\delta}$ we have $f(q) \leq f(p) + \delta$ and thus $f(q) < f(p) + \varepsilon$. Similarly, $q \notin G_{f(p)-\delta'}^{\mathrm{cl}}$ implies $f(q) \geq f(p) - \delta'$ and thus $f(q) > f(p) - \varepsilon$. In conclusion we have

$$f(p) - \varepsilon < f(q) < f(p) + \varepsilon,$$

whenever $q \in G_{f(p)+\delta} \setminus G_{f(p)-\delta'}^{\mathrm{cl}}$. Since this is an open neighbourhood containing p, and since $\varepsilon > 0$ was arbitrary, we conclude that f is continuous at p. $\square$

The T_4-property thus ensures the existence of interesting continuous functions. Since we have already many examples of T_4-spaces this theorem applies to quite many situations. In particular, a compact Hausdorff space is a T_4-space by Proposition ??. Also metric spaces are T_4 automatically by Proposition ??. However, sometimes T_4 is not present or at least difficult to see. In some cases of interest we nevertheless can construct interesting continuous functions, too. Important are locally compact Hausdorff spaces as generalization of compact Hausdorff spaces:

Proposition 2.4.5 *Let $(M, \mathscr{M})$ be a locally compact Hausdorff space.*

i.) For every compact subset $K \subseteq M$ and every open neighbourhood $O \subseteq M$ of K, i.e. $K \subseteq O$, there exists another open subset $U \subseteq M$ with

$$K \subseteq U \subseteq U^{\mathrm{cl}} \subseteq O, \tag{2.4.4}$$

such that U^{cl} is still compact.

ii.) For every compact subset $K \subseteq M$ and every closed subset $B \subseteq M$ with $K \cap B = \emptyset$ there exists a continuous function

$$f\colon M \longrightarrow [0, 1] \tag{2.4.5}$$

with

$$f\Big|_K = 0 \quad \textit{and} \quad f\Big|_B = 1. \tag{2.4.6}$$

PROOF: Since every point $p \in M$ has a neighbourhood basis consisting of compact neighbourhoods, we find for $p \in M$ an open subset $U_p \subseteq M$ with compact closure U_p^{cl} such that $U_p^{\mathrm{cl}} \subseteq O$, see again Proposition **??**, **??**. The collection $\{U_p\}_{p \in K}$ of all such chosen open subsets provides an open cover of K for which we find a finite subcover, say $U_{p_1}, \ldots, U_{p_n}$, since K is compact. Then

$$U = U_{p_1} \cup \cdots \cup U_{p_n}$$

contains K and $U^{\mathrm{cl}} = U_{p_1}^{\mathrm{cl}} \cup \cdots \cup U_{p_n}^{\mathrm{cl}}$ is still compact as the finite union of compact subsets is again compact. Since $U_p^{\mathrm{cl}} \subseteq O$ for all $p \in K$ we have $U^{\mathrm{cl}} \subseteq O$, too, proving *i.)*. The second part can then be done as Theorem 2.4.4 with A being replaced by K. Note that only the repetitive usage of (2.4.4) as replacement for the T_4-property in form of Proposition**??**, **??**, is needed. □

Needless to say that in both situations of either Theorem 2.4.4 or Proposition 2.4.5 we can achieve other numerical values $a, b \in \mathbb{K}$ with

$$f\Big|_A = a \quad \text{and} \quad f\Big|_B = b \tag{2.4.7}$$

beside $a = 0$ and $b = 1$. Moreover, there are analogous statements for finitely many disjoint closed subsets instead of two, where we can achieve different numerical values on each of the subsets.

For Tietze's theorem we need a little lemma as preparation:

Lemma 2.4.6 *Let $(M, \mathscr{M})$ be a T_4-space and let $A \subseteq M$ be a non-empty closed subset. Let $f \in \mathscr{C}_{\mathrm{b}}(A, \mathbb{R})$ be a continuous bounded function on A. Then there exists a continuous bounded function $g \in \mathscr{C}_{\mathrm{b}}(M, \mathbb{R})$ on M such that*

$$|g(p) - f(p)| \le \frac{2}{3}\|f\|_\infty \tag{2.4.8}$$

for all $p \in A$ and

$$\|g\|_\infty \le \frac{1}{3}\|f\|_\infty. \tag{2.4.9}$$

PROOF: Of course $\|g\|_\infty$ refers to the supremum norm of g over M while $\|f\|_\infty$ only uses the supremum over A where f is defined. We first decompose A into three pieces

$$B = f^{-1}\big([-\|f\|_\infty, -\tfrac{1}{3}\|f\|_\infty]\big) \quad \text{and} \quad C = f^{-1}\big([\tfrac{1}{3}\|f\|_\infty, \|f\|_\infty]\big),$$

as well as the open rest where $-\frac{1}{3}\|f\|_\infty < f < \frac{1}{3}\|f\|_\infty$. Clearly, B and C are closed in A and disjoint since f is continuous. Since A is closed itself, B and C are also closed when viewed as subsets of M. On the subset B we define the function g to be constant

$$g\Big|_B = -\frac{1}{3}\|f\|_\infty.$$

Then (2.4.9) will be satisfied on B and also (2.4.8) is satisfied on B by the defining property of B. Analogously, on C we define g to be constant

$$g\Big|_C = \frac{1}{3}\|f\|_\infty,$$

too and get the same conclusions. Thus on the union $B \cup C$ of the two closed subsets B and C the function g takes values between $-\frac{1}{3}\|f\|_\infty$ and $\frac{1}{3}\|f\|_\infty$, namely precisely these two. By Urysohn's lemma, see Theorem 2.4.4, we find an extension of g to a continuous function $g \in \mathscr{C}_{\mathrm{b}}(M, \mathbb{R})$ with

$$-\frac{1}{3}\|g\|_\infty \le g(p) \le \frac{1}{3}\|g\|_\infty$$

for all points $p \in M$. Since this in particular applies to the points $p \in A \setminus (B \cup C)$, we see that here (2.4.8) is satisfied, too, since f satisfies

$$-\frac{1}{3}\|f\|_\infty = -\frac{1}{3}\|g\|_\infty \leq f(p) \leq \frac{1}{3}\|g\|_\infty = \frac{1}{3}\|f\|_\infty$$

for those $p \in A \setminus (B \cup C)$. In total, this shows the lemma. □

The lemma is now the key to prove Tietze's extension theorem, where we will make use of the completeness of $\mathscr{C}_\mathrm{b}(M, \mathbb{R})$ in a crucial way:

Theorem 2.4.7 (Tietze) *Let $(M, \mathscr{M})$ be a topological space. Then the following two statements are equivalent:*

i.) The space $(M, \mathscr{M})$ has the T_4-property.

ii.) For every closed subset $A \subseteq M$ and every continuous map $f \colon A \longrightarrow [a, b]$ with $a, b \in \mathbb{R}$ there is a continuous extension $F \colon M \longrightarrow [a, b]$ of f, i.e.

$$F\Big|_A = f. \tag{2.4.10}$$

Proof: First we assume *ii.)*. Consider two disjoint closed subsets $B, C \subseteq M$. Then $A = B \cup C$ is closed again and the function

$$f\big|_B = -1 \quad \text{and} \quad f\big|_C = 1$$

is a continuous function $f \colon A \longrightarrow [-1, 1]$. By assumption, it has a continuous extension $F \colon M \longrightarrow [-1, 1]$. Setting now

$$O_1 = F^{-1}\big((-\infty, -\tfrac{1}{2})\big) \quad \text{and} \quad O_2 = F^{-1}\big((\tfrac{1}{2}, \infty)\big)$$

yields open subsets by the continuity of F such that

$$B \subseteq O_1, \quad C \subseteq O_2, \quad \text{and} \quad O_1 \cap O_2 = \emptyset.$$

Hence we separated B and C showing the T_4-property. The converse is the actual theorem of Tietze and requires some more sophisticated idea. Assume that M is a T_4-space and let $f \colon A \longrightarrow [a, b]$ be given. Without restriction we can add a suitable constant to f in order to achieve $\|f\|_\infty = b$ and $a = -\|f\|_\infty$. From Lemma 2.4.6 we obtain a continuous function $g_1 \in \mathscr{C}_\mathrm{b}(M, \mathbb{R})$ with values in $[-\frac{1}{3}\|f\|_\infty, \frac{1}{3}\|f\|_\infty]$ and

$$|f(p) - g_1(p)| \leq \frac{2}{3}\|f\|_\infty.$$

We set $f_1 = f - g_1\big|_A$ which is a continuous function on A with $\|f_1\|_\infty \leq \frac{2}{3}\|f\|_\infty$. We repeat the construction and obtain $g_2, g_3, \ldots \in \mathscr{C}_\mathrm{b}(M, \mathbb{R})$ together with

$$f_k = f - g_1\big|_A - g_2\big|_A - \cdots - g_k\big|_A,$$

such that

$$\|g_k\|_\infty \leq \frac{1}{3}\Big(\frac{2}{3}\Big)^{k-1}\|f\|_\infty \tag{$*$}$$

and

$$\big\|f - g_1\big|_A - \cdots - g_k\big|_A\big\|_\infty \leq \Big(\frac{2}{3}\Big)^k\|f\|_\infty. \tag{$**$}$$

Since the functions $g_k \in \mathscr{C}_\mathrm{b}(M, \mathbb{R})$ are globally defined on M the estimate $(*)$ shows that

$$g = \sum_{k=1}^{\infty} g_k$$

converges absolutely with respect to the supremum norm, yielding a continuous function $g \in \mathscr{C}_{\mathrm{b}}(M, \mathbb{R})$ by Proposition 2.4.1 with

$$\|g\|_\infty \leq \sum_{k=1}^{\infty} \|g_k\|_\infty \overset{(*)}{\leq} \sum_{k=1}^{\infty} \frac{1}{3}\Big(\frac{2}{3}\Big)^{k-1} \|f\|_\infty = \|f\|_\infty.$$

Moreover, taking the limit $k \to \infty$ in $(**)$ shows that on A we have $f = g\big|_A$. Hence we have found the extension $F = g$ as wanted. □

Again, there are variants of Tietze's extension theorem for more general spaces subject to further conditions similar to the Urysohn lemma as discussed e.g. in Proposition 2.4.5, see also Exercise **??**.

2.4.3 The Stone-Weierstraß Theorem

A classical theorem of Weierstraß asserts that the polynomial functions on the interval $[0, 1]$ are dense in all continuous functions with respect to the supremum norm. In this section we will formulate and prove a drastic generalization of this statement, the Stone-Weierstraß theorem.

To appreciate this generalization we need some algebraic features of $\mathscr{C}(M, \mathbb{K})$ beyond the purely linear ones. The product of $\mathscr{C}(M, \mathbb{K})$ is a continuous bilinear map. In case of $\mathbb{K} = \mathbb{C}$ the pointwise complex conjugation is also continuous. More precisely, we have the following properties:

Lemma 2.4.8 *Let $(M, \mathscr{M})$ be a topological space.*

i.) The pointwise product is a continuous bilinear map

$$\mathscr{C}_{\mathrm{b}}(M, \mathbb{K}) \times \mathscr{C}_{\mathrm{b}}(M, \mathbb{K}) \longrightarrow \mathscr{C}_{\mathrm{b}}(M, \mathbb{K}) \tag{2.4.11}$$

with

$$\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty \tag{2.4.12}$$

for all $f, g \in \mathscr{C}_{\mathrm{b}}(M, \mathbb{K})$.

ii.) The pointwise complex conjugation is a continuous anti-linear map

$$\overline{}\colon \mathscr{C}_{\mathrm{b}}(M, \mathbb{C}) \longrightarrow \mathscr{C}_{\mathrm{b}}(M, \mathbb{C}) \tag{2.4.13}$$

with

$$\|\overline{f}\|_\infty = \|f\|_\infty \tag{2.4.14}$$

and

$$\|\overline{f} f\|_\infty = \|f\|_\infty^2 \tag{2.4.15}$$

for all $f \in \mathscr{C}_{\mathrm{b}}(M, \mathbb{C})$.

Proof: In view of our general results on the continuity of multilinear maps from Proposition 2.2.27, *iii.)*, it suffices to check (2.4.12) to get the continuity of the product. This is a simple property of the supremum. The second part is even easier. Note that complex conjugation is $\mathbb{R}$-linear and hence its continuity can be encoded by the estimate (equality) in (2.4.14). The second statement is also a simple but decisive property of the supremum. □

It is a good occasion to recall the relevant vocabulary implicitly used in this lemma:

Definition 2.4.9 (C^*-Algebra) *Let $\mathscr{A}$ be a vector space over $\mathbb{K}$ endowed with a bilinear map $\mu\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$.*

i.) The pair $(\mathscr{A}, \mu)$ *is called an associative algebra if* μ *is associative, i.e. we have*

$$\mu(a, \mu(b, c)) = \mu(\mu(a, b), c) \tag{2.4.16}$$

for all $a, b, c \in \mathscr{A}$*. In this case we write* $ab = \mu(a, b)$ *to simplify notation. In this case we call* μ *the product or multiplication of the algebra* $\mathscr{A}$*.*

ii.) An associative algebra $(\mathscr{A}, \mu)$ *over* $\mathbb{K} = \mathbb{C}$ *together with an anti-linear map* ${}^*\colon \mathscr{A} \ni a \mapsto a^* \in \mathscr{A}$ *is called* **-algebra if*

$$(ab)^* = b^* a^* \quad \textit{and} \quad (a^*)^* = a \tag{2.4.17}$$

for all $a, b \in \mathscr{A}$*. In this case the map* * *is called the* **-involution of the* **-algebra* $(\mathscr{A}, \mu, {}^*)$*.*

iii.) An associative algebra $(\mathscr{A}, \mu)$ *is called a normed algebra if it is equipped with a norm* $\|\cdot\|$ *such that*

$$\|ab\| \le \|a\| \|b\| \tag{2.4.18}$$

for all $a, b \in \mathscr{A}$*. If in addition* $\|\cdot\|$ *is complete then* $(\mathscr{A}, \mu, \|\cdot\|)$ *is called a Banach algebra.*

iv.) A **-algebra* $(\mathscr{A}, \mu, {}^*)$ *is called a normed* **-algebra if it is equipped with a norm* $\|\cdot\|$ *turning it into a normed algebra such that in addition*

$$\|a^*\| = \|a\| \tag{2.4.19}$$

holds for all $a \in \mathscr{A}$*. If in addition* $\|\cdot\|$ *is complete then* $(\mathscr{A}, \mu, {}^*, \|\cdot\|)$ *is called a Banach* **-algebra.*

v.) A Banach **-algebra is called* C^**-algebra if in addition one has*

$$\|a^* a\| = \|a\|^2 \tag{2.4.20}$$

for all $a \in \mathscr{A}$*.*

There is of course a huge story behind these notions. The first part makes sense for algebras over other fields $\Bbbk$ instead of the real or complex numbers. In the context of algebras, the complex algebras are easier to handle so that our focus will mainly be put on this case. The *-involution makes explicit reference to the case $\mathbb{K} = \mathbb{C}$. The estimate (2.4.18) just means that the product is continuous with operator norm

$$\|\mu\| \le 1, \tag{2.4.21}$$

according to our results on multilinear maps from Proposition 2.2.27, see also Definition 2.2.28. The *-involution of a normed *-algebra is a continuous anti-linear map as well with operator norm

$$\|{}^*\| = 1. \tag{2.4.22}$$

This is precisely the condition (2.4.19), except for the trivial case $\mathscr{A} = \{0\}$.

If we have an element $\mathbb{1} \in \mathscr{A}$ with

$$\mathbb{1} a = a = a \mathbb{1} \tag{2.4.23}$$

for all $a \in \mathscr{A}$ then $\mathbb{1}$ is necessarily unique. One calls $\mathbb{1}$ the *unit* and $\mathscr{A}$ *unital* in this case. In the normed case we require in addition

$$\|\mathbb{1}\| = 1. \tag{2.4.24}$$

In the unital case, the product μ has operator norm $\|\mu\| = 1$. If we have

$$ab = ba \tag{2.4.25}$$

for all $a, b \in \mathscr{A}$ one calls the algebra *commutative*. The conclusion of Lemma 2.4.8 is now the following result:

Proposition 2.4.10 *Let $(M, \mathscr{M})$ be a topological space. Then $\mathscr{C}_{\mathrm{b}}(M, \mathbb{C})$ is a commutative unital C^*-algebra. In particular, $\mathscr{C}(M, \mathbb{C})$ is a commutative unital C^*-algebra for every compact space M.*

In view of Urysohn's lemma and Tietze's theorem, it will be most interesting to require a T_4-property for the spaces we want to investigate. Since a Hausdorff compact space is automatically a T_4-space by Proposition **??**, we will focus on such spaces from now on. By Proposition 2.4.5, also for locally compact Hausdorff spaces we can expect many continuous functions making $\mathscr{C}_{\mathrm{b}}(M, \mathbb{C})$ an interesting C^*-algebra. This justifies to consider the Hausdorff case also in the locally compact framework.

We are now interested in *-subalgebras of $\mathscr{C}_{\mathrm{b}}(M, \mathbb{C})$, i.e. subspaces $\mathscr{A} \subseteq \mathscr{C}_{\mathrm{b}}(M, \mathbb{C})$ which are preserved under taking products and under complex conjugation. The topological closure with respect to the supremum norm is then again such a *-subalgebra. In fact, this holds in general:

Proposition 2.4.11 *Let $\mathscr{A}$ be a Banach *-algebra with a *-subalgebra $\mathscr{B} \subseteq \mathscr{A}$. Then $\mathscr{B}^{\mathrm{cl}} \subseteq \mathscr{A}$ is still a *-subalgebra and a Banach *-algebra by itself.*

PROOF: We need to show that the closure contains ab and a^* whenever it contains a, b. We choose sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ with $a_n, b_n \in \mathscr{B}$ such that $a_n \to a$ and $b_n \to b$. Note that in the case of a Banach algebra we have a metric topology and hence sequences are sufficient to approximate elements of the closure. Since the product and the *-involution are continuous for a normed *-algebra as we have seen, we get $a_n b_n \to ab$ and $a_n^* \to a^*$ showing $ab \in \mathscr{B}^{\mathrm{cl}}$ and $a^* \in \mathscr{B}^{\mathrm{cl}}$, respectively. ☐

Corollary 2.4.12 *Let $\mathscr{A} \subseteq \mathscr{C}_{\mathrm{b}}(M, \mathbb{C})$ be a *-subalgebra of the continuous bounded functions on a topological space $(M, \mathscr{M})$. Then $\mathscr{A}^{\mathrm{cl}} \subseteq \mathscr{C}_{\mathrm{b}}(M, \mathbb{C})$ is a C^*-subalgebra.*

The question we want to address now is how large the closure of a *-subalgebra will actually become. To this end we mention the following technical lemma from elementary calculus:

Lemma 2.4.13 *Let $p_0 \in \mathbb{R}[x]$ be $p_0(x) = 0$ and define $p_n \in \mathbb{R}[x]$ recursively by*

$$p_{n+1}(x) = p_n(x) + \frac{1}{2}(x - p_n^2(x)). \tag{2.4.26}$$

Then one has the following properties:

i.) The polynomials p_n have rational coefficients, $p_n \in \mathbb{Q}[x]$ and satisfy $p_n(0) = 0$ for all $n \in \mathbb{N}$.

ii.) For $x \in [0, 1]$ and for all $n \in \mathbb{N}$ one has

$$0 \leq \sqrt{x} - p_n(x) \leq \frac{2\sqrt{x}}{2 + n\sqrt{x}} < \frac{2}{n}. \tag{2.4.27}$$

iii.) For all $x \in [0, 1]$ one has

$$\lim_{n\to\infty} p_n(x) = \sqrt{x}. \tag{2.4.28}$$

In fact, the convergence $p_n \to \sqrt{\cdot}$ is uniform on $[0, 1]$.

PROOF: The proof is entirely elementary but requires a tedious induction for the second part. With the estimate (2.4.27) the third part is clear, some more hints can be found in Exercise 2.5.13. ☐

This construction has several interesting applications throughout analysis. For us the following result is important:

Proposition 2.4.14 *Let $\mathscr{A} \subseteq \mathscr{C}_{\mathrm{b}}(M, \mathbb{C})$ be a *-subalgebra of the continuous bounded functions on a topological space $(M, \mathscr{M})$.*

i.) *If* $f \in \mathscr{A}$ *is real-valued with* $f \geq 0$ *then* $\sqrt{f} \in \mathscr{A}^{\mathrm{cl}}$.

ii.) *If* $f \in \mathscr{A}$ *then* $|f| \in \mathscr{A}^{\mathrm{cl}}$.

iii.) *If* $f, g \in \mathscr{A}$ *are real-valued then* $\min(f,g), \max(f,g) \in \mathscr{A}^{\mathrm{cl}}$.

PROOF: Let $f \in \mathscr{A}$ with $f \geq 0$ be given. If $f = 0$ then the statement is trivially true. Otherwise we replace f by $\frac{f}{\|f\|_\infty}$ and obtain a function with non-negative values and supremum equal to one. Hence we can assume $\|f\|_\infty = 1$ from the beginning. We use the polynomials p_n from Lemma 2.4.13 to obtain new continuous functions $p_n \circ f \in \mathscr{C}(M, \mathbb{C})$. In fact, we get for all $p \in M$

$$0 \leq \sqrt{f(p)} - (p_n \circ f)(p) \leq \frac{2\sqrt{f(p)}}{2 + n\sqrt{f(p)}} < \frac{2}{n}. \tag{$*$}$$

Since p_n is a polynomial, we have $p_n \circ f \in \mathscr{A}$. Then $(*)$ shows

$$\left\|\sqrt{f} - p_n \circ f\right\|_\infty < \frac{2}{n},$$

and hence $p_n \circ f \to \sqrt{f}$ uniformly. Thus $\sqrt{f} \in \mathscr{A}^{\mathrm{cl}}$ follows. The second part follows at once since $|f| = \sqrt{\overline{f}f}$ and $\overline{f}f \in \mathscr{A}$ satisfies the assumptions of *i.)* since $\mathscr{A}$ is a *-subalgebra. Finally, also the third part follows since

$$\min(f,g) = \frac{1}{2}(f + g - |f - g|) \in \mathscr{A}^{\mathrm{cl}}$$

as well as

$$\max(f,g) = \frac{1}{2}(f + g + |f - g|) \in \mathscr{A}^{\mathrm{cl}}$$

since all terms are in $\mathscr{A}^{\mathrm{cl}}$ by *ii.)*. □

This observation has many generalizations to the theory of C^*-algebras. We will use it now to prove the Stone-Weierstraß theorem:

Theorem 2.4.15 (Stone-Weierstraß) *Let* $(M, \mathscr{M})$ *be a compact Hausdorff space. If* $\mathscr{A} \subseteq \mathscr{C}(M, \mathbb{C})$ *is a point-separating unital* **-subalgebra, then*

$$\mathscr{A}^{\mathrm{cl}} = \mathscr{C}(M, \mathbb{C}). \tag{2.4.29}$$

Here *point-separating* means that for $p \neq q$ we find a function $f \in \mathscr{A}$ with

$$f(p) \neq f(q). \tag{2.4.30}$$

Note that $\mathscr{C}(M, \mathbb{C})$ is point-separating by Urysohn's lemma and the fact that a compact Hausdorff space is T_4, see again Proposition **??**. If there were two points $p \neq q$ in M which can not be separated by any function $f \in \mathscr{A}$, then also in the closure of $\mathscr{A}$ there would be no such function. This explains why the assumption of $\mathscr{A}$ being point-separating is needed. The role of being a *-subalgebra with $\mathbb{1} \in \mathscr{A}$ is explored further in Exercise **??**.

PROOF (OF THEOREM 2.4.15): Let $f \in \mathscr{C}(M, \mathbb{C})$ be given with $\varepsilon > 0$. We need to construct a function $g_\varepsilon \in \mathscr{A}$ with $\|f - g_\varepsilon\|_\infty < \varepsilon$. Since $\mathscr{A} \subseteq \mathscr{A}^{\mathrm{cl}}$ is dense in its closure by the very definition of a closure, we construct a $g_\varepsilon \in \mathscr{A}^{\mathrm{cl}}$ which suffices to show the density of $\mathscr{A}$ in $\mathscr{C}(M, \mathbb{C})$, too. Since $f = \mathrm{Re}(f) + \mathrm{i}\,\mathrm{Im}(f)$ and since $\mathscr{A}$ is assumed to be a *-subalgebra, we can approximate $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ separately by real-valued functions from $\mathscr{A}$. Thus it suffices to consider $f = \overline{f}$ from the beginning. Suppose $p \neq q$ then we find a real-valued $g = \overline{g} \in \mathscr{A}$ with $g(p) \neq g(q)$. We define a new function h by

$$h = \frac{f(p) - f(q)}{g(p) - g(q)} g - \frac{f(p)g(q) - f(q)g(p)}{g(p) - g(q)} \mathbb{1} \in \mathscr{A},$$

since $\mathscr{A}$ is unital. Clearly, $h = \overline{h}$ and

$$h(p) = f(p) \quad \text{and} \quad h(q) = f(q). \tag{$*$}$$

For $p = q$ we can take the constant function $h = f(p)\mathbb{1} \in \mathscr{A}$ to achieve $(*)$ as well. Thus we get a function $h \in \mathscr{A}$ with $(*)$ for any two given points $p, q \in M$. Next we fix $q \in M$ and use a function $h_p \in \mathscr{A}$ with $(*)$ for $p \in M$. This allows us to define

$$U_p = \{p' \in M \mid h_p(p') < f(p') + \varepsilon\},$$

which is an open subset of M by the continuity of h_p and f. Moreover, $p \in U_p$. Hence we have an open cover $\{U_p\}_{p\in M}$ of the compact space M of which we can select finitely many, say $U_{p_1}, \ldots, U_{p_n}$, still covering M. We define

$$g_q = \min\{h_{p_1}, \ldots, h_{p_n}\} \in \mathscr{A}^{\mathrm{cl}},$$

where we use Proposition 2.4.14, *iii.)*. Being a pointwise minimum we get

$$g_q(p) < f(p) + \varepsilon$$

for all $p \in M$. Since all functions h_p satisfy $h_p(q) = f(q)$, we also get

$$g_q(q) = f(q).$$

In a second step we define

$$V_q = \{p \in M \mid g_q(p) > f(p) + \varepsilon\},$$

which yields again an open cover of M by the continuity of each g_q and $q \in V_q$. Again we find finitely many $V_{q_1}, \ldots, V_{q_m}$ covering M. Then

$$g = \max\{g_{q_1}, \ldots, g_{q_m}\} \in \mathscr{A}^{\mathrm{cl}}$$

is a function in the closure of $\mathscr{A}$ according to Proposition 2.4.14, *iii.)*, which satisfies $\|f - g\|_\infty < \varepsilon$ by construction. □

Corollary 2.4.16 (Weierstraß) *The complex polynomials $\mathbb{C}[x]$ are uniformly dense in the continuous functions $\mathscr{C}([a, b], \mathbb{C})$ on a compact interval.*

Proof: Indeed $\mathbb{C}[x]$ is a *-subalgebra containing the constants. It is point-separating since already $x \in \mathbb{C}[x]$ separates points. □

This was Weierstraß' original formulation. In fact, the *Bernstein polynomials* provide a rather explicit way to approximate a continuous function $f \in \mathscr{C}([0, 1], \mathbb{C})$ by polynomials, see also Exercise 2.5.14. However, the Stone-Weierstraß theorem in its full generality goes far beyond this situation.

2.4.4 The Arzelà-Ascoli Theorem

We consider again a compact Hausdorff space $(M, \mathscr{M})$. Since $\mathscr{C}(M, \mathbb{K})$ is an infinite-dimensional Banach space except for trivial situations, it is not possible to characterize compact subsets by the Heine-Borel criterion: the compact subsets of $\mathscr{C}(M, \mathbb{K})$ are bounded and closed but in general a bounded and closed subset does not need to be compact at all. We will see explicit examples later on, see e.g. Theorem 3.2.11.

Hence we need to find a better criterion to determine the compact subsets of $\mathscr{C}(M, \mathbb{K})$, which will be achieved by the Arzelà-Ascoli theorem.

First we recall the notion of a totally bounded subset of a metric space:

Definition 2.4.17 (Totally bounded subset) *Let (M, d) be a metric space. Then a subset $K \subseteq M$ is called totally bounded if for every $\varepsilon > 0$ there are finitely many points $p_1, \ldots, p_n \in M$ with*

$$K \subseteq \mathrm{B}_\varepsilon(p_1) \cup \cdots \cup \mathrm{B}_\varepsilon(p_n). \tag{2.4.31}$$

It is clear that a compact subset is totally bounded since $\{\mathrm{B}_\varepsilon(p)\}_{p \in K}$ provides an open cover of K having a finite subcover by compactness. Moreover, we have the following properties of totally bounded subsets:

Proposition 2.4.18 *Let (M, d) be a metric space with $K \subseteq M$.*

i.) The subset K is totally bounded iff the metric space $(K, \mathrm{d}\big|_{K \times K})$ is totally bounded, i.e. we can find the points $p_1, \ldots, p_n$ with (2.4.31) already inside K.

ii.) If K is totally bounded and $K' \subseteq K$ then K' is totally bounded as well.

iii.) The subset K is totally bounded iff K^{cl} is totally bounded.

iv.) Suppose (M, d) is complete. Then K is totally bounded iff K^{cl} is compact.

Proof: For the first part, let $\varepsilon > 0$ be given and let $p_1, \ldots, p_n \in M$ be points with $K \subseteq \mathrm{B}_{\frac{\varepsilon}{2}}(p_1) \cup \cdots \cup \mathrm{B}_{\frac{\varepsilon}{2}}(p_n)$, i.e. we apply (2.4.31) to $\frac{\varepsilon}{2} > 0$. Either $K \cap \mathrm{B}_{\frac{\varepsilon}{2}}(p_i) = \emptyset$, then we do not need $\mathrm{B}_{\frac{\varepsilon}{2}}(p_i)$ for covering K. Otherwise, choose $q_i \in K \cap \mathrm{B}_{\frac{\varepsilon}{2}}(p_i)$. Then the triangle inequality shows $\mathrm{B}_{\frac{\varepsilon}{2}}(p_i) \subseteq \mathrm{B}_\varepsilon(q_i)$ and thus

$$K \subseteq \mathrm{B}_\varepsilon(q_1) \cup \cdots \cup \mathrm{B}_\varepsilon(q_n),$$

showing the first part. The second is trivial. For the third, assume K is totally bounded and let $\varepsilon > 0$ be given. Then we find $p_1, \ldots, p_n \in K$ with

$$K \subseteq \mathrm{B}_{\frac{\varepsilon}{2}}(p_1) \cup \cdots \cup \mathrm{B}_{\frac{\varepsilon}{2}}(p_n),$$

and thus we have

$$K^{\mathrm{cl}} \subseteq \mathrm{B}_{\frac{\varepsilon}{2}}(p_1)^{\mathrm{cl}} \cup \cdots \cup \mathrm{B}_{\frac{\varepsilon}{2}}(p_n)^{\mathrm{cl}} \subseteq \mathrm{B}_\varepsilon(p_1) \cup \cdots \cup \mathrm{B}_\varepsilon(p_n).$$

This shows that K^{cl} is totally bounded, too. The opposite is clear by *ii.)* since $K \subseteq K^{\mathrm{cl}}$. Next, assume that (M, d) is complete. Assume first that K^{cl} is compact. As already mentioned, then it is totally bounded. Hence also $K \subseteq K^{\mathrm{cl}}$ is totally bounded by *ii.)*. Conversely, assume K is totally bounded. Then also K^{cl} is totally bounded by *iii.)* and complete by Proposition **??**, **??**. Hence it suffices to consider $M = K = K^{\mathrm{cl}}$ from the beginning. Assume $\{O_i\}_{i \in I}$ is an open cover of M without a finite subcover. We set $K_0 = M$ and $\varepsilon_0 = \frac{1}{2}$. Then we have points $p_1, \ldots, p_n \in M$ with

$$K_0 \subseteq \mathrm{B}_{\varepsilon_0}(p_1) \cup \cdots \cup \mathrm{B}_{\varepsilon_0}(p_n) \subseteq \mathrm{B}_{\varepsilon_0}(p_1)^{\mathrm{cl}} \cup \cdots \cup \mathrm{B}_{\varepsilon_0}(p_n)^{\mathrm{cl}}.$$

At least one of these closed balls can not have a finite subcover of the cover $\{O_i\}_{i \in I}$, otherwise we would have a finite subcover for K_0, too. Let K_1 be such a closed ball. Since K_1 is still totally bounded we can repeat the construction, now for $\varepsilon_1 = \frac{1}{4}$. This leads to a sequence of closed subsets

$$K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$$

of diameter at most $\operatorname{diam} K_n \leq 2\varepsilon_n = \frac{1}{2^n}$, such that none of the closed subsets has a finite subcover. In particular, none is empty. Choosing $p_n \in K_n$ gives then a sequence with the property that for every $N \in \mathbb{N}$ all but finitely many members of the sequence are in K_N having diameter at most $\frac{1}{2^N}$. Thus this sequence is a Cauchy sequence, convergent to some $p \in M$ by completeness. Since $p \in M$ is in at least one O_{i_0}, we conclude that $K_n \subseteq O_{i_0}$ for large enough n, a contradiction. $\square$

The last part can also be interpreted as follows: since every metric space (M, d) has a completion $(\widehat{M}, \widehat{\mathrm{d}})$, a subset $K \subseteq M$ is totally bounded iff its closure inside $\widehat{M}$ is compact. Thus one calls totally bounded subsets also *pre-compact*. From that point of view we will be interested in finding and characterizing totally bounded subsets.

The next notion we need to formulate the Arzelà-Ascoli theorem is the following:

Definition 2.4.19 (Equicontinuity) *Let $(M, \mathscr{M})$ be a topological space and let $\mathscr{F} \subseteq \mathscr{C}(M, \mathbb{K})$ be a subset of continuous functions on M. Then $\mathscr{F}$ is called equicontinuous at $p \in M$ if for all $\varepsilon > 0$ there is a neighbourhood $U \subseteq M$ of p such that for all $q \in U$ and all $f \in \mathscr{F}$ one has*

$$|f(p) - f(q)| < \varepsilon. \tag{2.4.32}$$

The set $\mathscr{F}$ is called equicontinuous if it is equicontinuous at all points $p \in M$.

The important point is that we can use the same neighbourhood U for all elements of $\mathscr{F}$ in the continuity estimate. For a single function, i.e. $\mathscr{F} = \{f\}$ equicontinuity is just continuity. If $\mathscr{F} = \{f_1, \ldots, f_n\}$ consists of finitely many functions, we can take intersections of finitely many neighbourhoods of p and still have a neighbourhood. With this argument one sees that also in this case $\mathscr{F}$ is equicontinuous. The definition becomes interesting for infinite $\mathscr{F}$.

If $\mathscr{F} \subseteq \mathscr{C}(M, \mathbb{K})$ is a pre-compact subset then $\mathscr{F}^{\mathrm{cl}} \subseteq \mathscr{C}(M, \mathbb{K})$ is compact and hence the evaluation functional

$$\delta_p \colon \mathscr{C}(M, \mathbb{K}) \ni f \ \mapsto \ f(p) \in \mathbb{K} \tag{2.4.33}$$

takes values in a bounded subset of $\mathbb{K}$ when restricted to $\mathscr{F}^{\mathrm{cl}}$, since δ_p is clearly continuous with respect to the supremum norm. In fact, recall from Example 2.3.6, *i.)*, that

$$|\delta_p(f)| = |f(p)| \leq \|f\|_\infty \tag{2.4.34}$$

holds for all $p \in M$ showing $\|\delta_p\| \leq 1$. Testing on $f = 1$ gives then

$$\|\delta_p\| = 1. \tag{2.4.35}$$

Thus a necessary condition for a pre-compact, i.e. totally bounded subset $\mathscr{F} \subseteq \mathscr{C}(M, \mathbb{K})$ in the Banach space $\mathscr{C}(M, \mathbb{K})$ is to be *pointwise bounded*, i.e.

$$\sup_{f \in \mathscr{F}} |\delta_p(f)| < \infty \tag{2.4.36}$$

for all points $p \in M$.

Together with equicontinuity the pointwise boundedness already characterizes the pre-compact subsets of the Banach space $\mathscr{C}(M, \mathbb{K})$. This is the theorem of Arzelà and Ascoli:

Theorem 2.4.20 (Arzelà-Ascoli) *Let $(M, \mathscr{M})$ be a compact Hausdorff space. Then a subset $\mathscr{F} \subseteq \mathscr{C}(M, \mathbb{K})$ is totally bounded iff $\mathscr{F}$ is equicontinuous and pointwise bounded.*

PROOF: Assume first that $\mathscr{F}$ is equicontinuous and pointwise bounded. To show total boundedness we choose $\varepsilon > 0$. For every $p \in M$ we choose a neighbourhood $U_p \subseteq M$ of p with

$$|f(p) - f(q)| < \varepsilon \tag{$*$}$$

for all $q \in U_p$ and all $f \in \mathscr{F}$ by equicontinuity. Since M is compact, finitely many $U_{p_1}, \ldots, U_{p_n}$ cover M. Since every $p \in M$ is in at least one U_{p_i} we get from the pointwise boundedness of $\mathscr{F}$ at that matching point p_i

$$\sup_{f \in \mathscr{F}} |f(p)| \leq \sup_{f \in \mathscr{F}} \{|f(p_i)| + |f(p_i) - f(p)|\} \overset{(a)}{\leq} \sup_{f \in \mathscr{F}} |f(p_i)| + \varepsilon \leq \max_{i=1}^{n} \sup_{f \in \mathscr{F}} |f(p_i)| + \varepsilon, \tag{$**$}$$

which is now a bound independent of i and hence valid for all $p \in M$. Here we used $(*)$ for p_i in (a). This means that $\mathscr{F}$ is a bounded subset for the supremum norm, i.e.

$$\sup_{f \in \mathscr{F}} \|f\|_\infty \le \sup_{p \in M} \sup_{f \in \mathscr{F}} |f(p)| \le R < \infty,$$

where we can take R from the above estimate $(**)$. We use the above selected points to define the map

$$\Phi \colon \mathscr{F} \ni f \ \mapsto\ (f(p_1), \ldots, f(p_n)) \in \mathbb{K}^n,$$

which then takes values in the compact subset $(\mathrm{B}_R(0)^{\mathrm{cl}})^n \subseteq \mathbb{K}^n$, i.e.

$$\Phi(\mathscr{F}) \subseteq (\mathrm{B}_R(0)^{\mathrm{cl}})^n.$$

Hence also the closure of the image

$$K = \Phi(\mathscr{F})^{\mathrm{cl}} \subseteq (\mathrm{B}_R(0)^{\mathrm{cl}})^n$$

is compact. If we use the maximum norm $\|\cdot\|_{\max}$ for $\mathbb{K}^n$ instead of the usual Euclidean norm then the Cartesian product $(\mathrm{B}_R(0)^{\mathrm{cl}})^n$ of the closed balls $\mathrm{B}_R(0)^{\mathrm{cl}} \subseteq \mathbb{K}$ is the n-dimensional closed "ball" with respect to $\|\cdot\|_{\max}$. Since K is compact and since $\Phi(f) \in \mathrm{B}_\varepsilon(\Phi(f))$ we find finitely many $f_1, \ldots, f_N \in \mathscr{F}$ such that

$$K \subseteq \mathrm{B}_\varepsilon(\Phi(f_1)) \cup \cdots \cup \mathrm{B}_\varepsilon(\Phi(f_N)).$$

Hence for $f \in \mathscr{F}$ we have at least one $k \in \{1, \ldots, N\}$ with

$$\|\Phi(f) - \Phi(f_k)\|_{\max} < \varepsilon,$$

which means

$$|f(p_i) - f_k(p_i)| < \varepsilon \tag{$\star$}$$

for $i = 1, \ldots, n$ since we use the maximum norm. Moreover, for $p \in U_{p_i}$ we have

$$|f(p) - f(p_i)| < \varepsilon \quad \text{and} \quad |f_k(p) - f_k(p_i)| < \varepsilon \tag{$\star\star$}$$

by equicontinuity at p_i. Putting both estimates $(\star)$ and $(\star\star)$ together yields

$$|f(p) - f_k(p)| \le |f(p) - f(p_i)| + |f(p_i) - f_k(p_i)| + |f_k(p_i) - f_k(p)| < 3\varepsilon,$$

independently of the auxiliary point p_i used to derive $(\star)$ and $(\star\star)$. Hence $\|f - f_k\|_\infty \le 3\varepsilon$ follows. This shows that for every $f \in \mathscr{F}$ there is a 3ε-ball around some f_k to which f belongs. As $\varepsilon > 0$ was arbitrary, $\mathscr{F}$ is totally bounded as claimed.

Conversely, let $\mathscr{F}$ be totally bounded. As we have already argued, $\mathscr{F}$ is then necessarily pointwise bounded. Moreover, for $\varepsilon > 0$ we find $f_1, \ldots, f_N \in \mathscr{F}$ with $\mathscr{F} \subseteq \mathrm{B}_\varepsilon(f_1) \cup \cdots \cup \mathrm{B}_\varepsilon(f_N) \subseteq \mathscr{C}(M, \mathbb{K})$, where now $\mathrm{B}_\varepsilon(f_i)$ denotes the open ball around f_i with respect to $\|\cdot\|_\infty$. This means that for $f \in \mathscr{F}$ we find at least one $k \in \{1, \ldots, N\}$ with $\|f - f_k\|_\infty < \varepsilon$. Let $p \in M$ and choose a neighbourhood $U_{k,p} \subseteq M$ of p such that

$$|f_k(p) - f_k(q)| < \varepsilon \tag{☺}$$

for all $q \in U_{k,p}$ using the continuity of f_k. Then $U_p = U_{1,p} \cap \cdots \cap U_{N,p}$ is still a neighbourhood of p such that $q \in U_p$ satisfies $(*)$ now for all $k = 1, \ldots, N$. For all $f \in \mathscr{F}$ and $q \in U_p$ we thus get

$$|f(p) - f(q)| \le |f(p) - f_k(p)| + |f_k(p) - f_k(q)| + |f_k(q) - f(q)| < 3\varepsilon$$

by (☺) and $\|f - f_k\|_\infty < \varepsilon$. This shows equicontinuity of $\mathscr{F}$ at p since $\varepsilon > 0$ was arbitrary. Since $p \in M$ is arbitrary, $\mathscr{F}$ is equicontinuous. $\square$

Since $\mathscr{C}(M, \mathbb{K})$ is already a complete metric space, we get the following conclusion from Proposition 2.4.18, *iv.)*:

Corollary 2.4.21 *Let $(M, \mathscr{M})$ be a compact Hausdorff space and let $\mathscr{F} \subseteq \mathscr{C}(M, \mathbb{K})$. Then the following statements are equivalent:*

i.) The subset $\mathscr{F}$ is totally bounded.

ii.) The subset $\mathscr{F}^{\mathrm{cl}}$ is compact.

iii.) The subset $\mathscr{F}$ is equicontinuous and pointwise bounded.

iv.) Every sequence in $\mathscr{F}$ has a convergent subsequence.

Of course, the limit of such a convergent subsequence will be in $\mathscr{F}^{\mathrm{cl}}$ in general and not necessarily in $\mathscr{F}$. The equivalence of *ii.)* and *iv.)* holds since $\mathscr{C}(M, \mathbb{K})$ is metric and hence compactness and sequential compactness coincide by Proposition **??**. It will be the equivalence of *iii.)* and *iv.)* which is used very often in (functional) analysis. Note that *ii.)* is the conceptual explanation.

There are various generalizations of the Arzelà-Ascoli theorem. Most notably, we can replace $\mathbb{K}$-valued continuous functions by continuous functions with values in a metric space. Also concerning the domain one can find several generalizations.

The applications of this theorem are manifold. Some we will see later.

2.5 Exercises

Exercise 2.5.1 (Separation properties of a topological vector space) Provide a proof for Proposition 2.1.7.

Hint:

Exercise 2.5.2 (The vector space $\mathrm{L}(V, W)$) Let V and W be topological vector spaces. Show that $\mathrm{L}(V, W)$ is a subspace of $\mathrm{Hom}_{\mathbb{K}}(V, W)$ by characterizing continuity with nets. This gives an alternative proof of Proposition 2.1.10, *iii.)*.

Exercise 2.5.3 (Continuous vector-valued maps) Let $(M, \mathscr{M})$ be a topological space and let W be a topological vector space. Show that the set of continuous maps $\mathscr{C}(M, W) \subseteq \mathrm{Map}(M, W)$ is a subspace of the space of all W-valued maps with respect to the pointwise vector space operations. Use this result to conclude that $\mathrm{L}(V, W)$ is a subspace of $\mathrm{Map}(V, W)$ for another topological vector space V.

Hint: The most conceptual proof is obtained along the lines of the proof of Proposition 2.1.10, *iii.)*.

Exercise 2.5.4 (Comparing topologies on topological vector spaces) Let V be a vector space with two topologies $\mathscr{V}_1$ and $\mathscr{V}_2$ for both of which it becomes a topological vector space.

Exercise 2.5.5 (Linear functionals are open maps) Let V be a topological vector space. Show that a (not necessarily continuous) linear functional $\varphi \in V^*$ is an open map $\varphi\colon V \longrightarrow \mathbb{K}$ iff $\varphi \neq 0$.

Exercise 2.5.6 (Some inequalities) In this exercise we collect some useful and well-known inequalities for series.

Exercise 2.5.7 (The quotient $\ell^\infty / c_\circ$) To compare the "size" of ℓ^∞ with its subspace $c_\circ$ we consider the quotient $\ell^\infty / c_\circ$. Find a family of elements $\{v_i\}_{i \in I}$ of uncountably many sequences $v_i \in \ell^\infty$ such that their classes $\{[v_\lambda]\}_{i \in I}$ are linearly independent in $\ell^\infty / c_\circ$.

Hint:

Exercise 2.5.8 (Weighted ℓ^p-spaces)

Exercise 2.5.9 (L^p-Spaces for different measures) Recall that two positive measures μ and ν on a measurable space $(X, \mathfrak{a})$ are called equivalent if they are mutually absolutely continuous, i.e. they have the same zero sets.

i.) Show that $\mathrm{L}^\infty(X, \mathfrak{a}, \mu) = \mathrm{L}^\infty(X, \mathfrak{a}, \nu)$ if μ and ν are equivalent.

ii.) Let $p \in [1, \infty)$. Give an example of a measure μ on $\mathbb{R}$ equivalent to the Lebesgue measure such that there are two functions $f \in \mathrm{L}^p(\mathbb{R}, \mu)$ with $f \notin \mathrm{L}^p(\mathbb{R}, \mathrm{d}x)$ and $g \in \mathrm{L}^p(\mathbb{R}, \mathrm{d}x)$ but $g \notin \mathrm{L}^p(\mathbb{R}, \mu)$.

Hint: One can find the measure μ and the functions f and g even in such a way, that the properties hold for all p at once.

Thus the L^p-spaces are very sensitive to the measure for $p \in [1, \infty)$ but rather robust for $p = \infty$. This is the reason why we define $\mathrm{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$ directly for a σ-ideal $\mathfrak{n}$ instead for a specific measure.

Exercise 2.5.10 (Polarization)

Exercise 2.5.11 (Duals of sequence spaces)

Exercise 2.5.12 (The topological annihilator I)

Exercise 2.5.13 (Approximating the square root) Consider again the recursively defined polynomials p_n from Lemma 2.4.13.

i.) Compute the first polynomials p_1, p_2, p_3, p_4 explicitly and sketch their graphs on the unit interval $[0, 1]$.

ii.) Show inductively that $p_n \in x\mathbb{Q}[x]$.

iii.) Show with a first inductive proof that p_n maps $[0, 1]$ back into $[0, 1]$.

iv.) Use *iii.)* to show the improved estimate (2.4.27) also by induction.

v.) Conclude that the polynomials approximate the square root uniformly on $[0, 1]$.

vi.) A slightly easier but less explicit argument is to us *Dini's theorem* to show that the polynomials p_n approximate the square root uniformly. To this end, show that pointwise $p_n(x)$ is an monotonously increasing sequence converging pointwise to the square root without using the above explicit estimate.

Exercise 2.5.14 (Bernstein polynomials and Weierstraß' theorem)

Exercise 2.5.15 (Fourier modes) Consider the continuous functions $\mathscr{C}(\mathbb{S}^1, \mathbb{C})$ on the unit circle $\mathbb{S}^1 \subseteq \mathbb{C}$. We use the quasi-global angle coordinate $\varphi \in (0, 2pi)$ to parameterize $\mathbb{S}^1$. Then the n-th Fourier mode is defined by

$$\mathrm{e}_n(\varphi) = \mathrm{e}^{\mathrm{i}n\varphi} \tag{2.5.1}$$

for $n \in \mathbb{Z}$.

Exercise 2.5.16 (Arzelà-Ascoli theorem for vector-valued functions)

Chapter 3

Main Theorems for Banach Spaces

In this chapter we will formulate and prove the main structural results on general Banach spaces. Of course, particular classes of Banach spaces will have more specific features. However, here we focus on general aspects applicable to all Banach spaces. As it turns out, many of them have generalizations even beyond the Banach space setting.

Most important and fundamental is the Hahn-Banach theorem. We will see several formulations and a wealth of consequences. The overall result will be that for normed spaces the topological dual space is large. We get existence of many continuous linear functions fulfilling additional hand-designed properties. Since already the existence of linear functionals without continuity requires some form of choice in general, also the Hahn-Banach theorem will rely on the axiom of choice in a mild form, making it a highly non-constructive statement. Nevertheless, without this theorem functional analysis quickly becomes extremely boring and restrictive.

The next important statement is the Banach-Alaoglu theorem which introduces a new topology on the topological dual of a Banach space, the weak* topology. This, together with its close relative of the weak topology, gives a first example of a topological vector space beyond the normed case. In fact, the topology will be encoded by many seminorms instead of a single norm. The result will then be a compactness statement, the closed unit ball in the topological dual of a Banach space is compact for the weak* topology. This is in quite some contrast to the norm topology, where closed unit balls are compact iff the Banach space is finite-dimensional. One major application of the Banach-Alaoglu theorem will be the fact that the norm topology and the weak topology have the same bounded subsets.

In Section **??** we have already seen a first formulation of the principle of uniform boundedness. With the Banach-Steinhaus theorem we will now pick up this question again in the context of Banach spaces leading to many extremely useful consequences. In particular, it will allow us to study pointwise limits of sequences of continuous linear maps. This will turn out to be very helpful when constructing continuous linear maps.

The open mapping theorem and the closed graph theorem yield important tools for determining the continuity of linear maps, in particular for inverses of linear maps. One way to apply them is to prove uniqueness statements about Banach space topologies.

Finally, the theorems of Krein and Milman show the existence of extreme points of convex subsets in Banach spaces. On the conceptual level they are needed to show the existence of certain linear functionals in various fields in functional analysis, like e.g. the existence of pure states of C^*-algebras. On a more practical level they guarantee the existence of solutions to various variational or optimization problems.

All of the above fundamental results have generalizations to situations beyond Banach spaces. With the weak and weak* topology we have seen first examples of locally convex topologies. Many more function spaces in analysis can be endowed with such locally convex topologies. Hence generalizations of the above main theorems are needed, a development beyond our current aims. Here one needs to consult the literature [] on locally convex analysis.

3.1 The Hahn-Banach Theorem

The first main theorem in Banach space theory we will investigate is the Hahn-Banach theorem with some of its most important consequences.

3.1.1 Sublinear Functionals and the Hahn-Banach Theorem

For the formulation of the Hahn-Banach theorem we need the notion of *sublinear functionals.* The vector space can be either real or complex but the values of the sublinear functionals will be real:

Definition 3.1.1 (Sublinear functionals) *Let V be a vector space over $\mathbb{K}$.*

i.) *A map $\mathrm{p}\colon V \longrightarrow \mathbb{R}$ is called sublinear if*

$$\mathrm{p}(v+w) \le \mathrm{p}(v) + \mathrm{p}(w) \tag{3.1.1}$$

and

$$\mathrm{p}(\lambda v) = \lambda \mathrm{p}(v) \tag{3.1.2}$$

for all $v, w \in V$ and $\lambda \ge 0$.

ii.) *The set of sublinear functionals on V is denoted by $V^{\#}$.*

iii.) *For $\mathrm{p}, \mathrm{q} \in V^{\#}$ one defines $\mathrm{p} \preccurlyeq \mathrm{q}$ if for all $v \in V$ one has*

$$\mathrm{p}(v) \le \mathrm{q}(v). \tag{3.1.3}$$

Proposition 3.1.2 *Let V be a vector space over $\mathbb{K}$.*

i.) *The set of sublinear functionals is a convex cone in $\operatorname{Map}(V, \mathbb{R})$, i.e. for $\alpha, \beta \ge 0$ and $\mathrm{p}, \mathrm{q} \in V^{\#}$ one has $\alpha\mathrm{p} + \beta\mathrm{q} \in V^{\#}$.*

ii.) *The relation $\preccurlyeq$ turns $V^{\#}$ into a partially ordered and directed set.*

PROOF: The first assertion is trivial. For the second we note that $\mathrm{p} \preccurlyeq \mathrm{p}$ holds trivially. Then $\mathrm{p} \preccurlyeq \mathrm{q}$ and $\mathrm{q} \preccurlyeq \mathrm{p}$ implies $\mathrm{p} = \mathrm{q}$. Moreover, $\mathrm{p} \preccurlyeq \mathrm{q}$ and $\mathrm{q} \preccurlyeq \mathrm{r}$ imply $\mathrm{p} \preccurlyeq \mathrm{r}$. Finally, for $\mathrm{p}, \mathrm{q} \in V^{\#}$ the pointwise maximum $\max\{\mathrm{p}, \mathrm{q}\} \in V^{\#}$ is again sublinear and dominates both p and q. $\square$

For the Hahn-Banach theorem we need two preparatory lemmas: The first determines the minimal elements of the partially ordered set $(V^{\#}, \preccurlyeq)$:

Lemma 3.1.3 *Let V be a real vector space. Then the set of minimal elements in $V^{\#}$ is the algebraic dual $V^* \subseteq V^{\#}$.*

PROOF: Assume $\varphi \in V^*$. Then clearly $\varphi \in V^{\#}$ as well. We need to show that φ is minimal. Hence consider a sublinear functional $\mathrm{p} \in V^{\#}$ with $\mathrm{p} \preccurlyeq \varphi$. First we note that in general $\mathrm{p}(0) = 0$ by (3.1.2) and hence $0 \le \mathrm{p}(v) + \mathrm{p}(-v)$ for all $v \in V$. Since $\mathrm{p}(v) \le \varphi(v)$ and $\mathrm{p}(-v) \le \varphi(-v) = -\varphi(v)$, we can conclude

$$0 \le \mathrm{p}(v) + \mathrm{p}(-v) \le \varphi(v) + \varphi(-v) = 0,$$

and thus $\mathrm{p}(v) = -\mathrm{p}(-v)$. Then

$$-\mathrm{p}(v) = \mathrm{p}(-v) \le \varphi(-v) = -\varphi(v)$$

shows $\mathrm{p}(v) \geq \varphi(v)$ and thus $\mathrm{p}(v) = \varphi(v)$ follows. This shows $\mathrm{p} = \varphi$ and hence φ is minimal as claimed. The converse requires a bit more work. Assume that $\mathrm{p} \in V^\#$ is minimal. For $w \in V$ fixed we define the map

$$\mathrm{q}\colon V \ni v \;\mapsto\; \inf_{\lambda \geq 0} \big\{ \mathrm{p}(v + \lambda w) - \lambda \mathrm{p}(w) \big\} \in [-\infty, \infty). \tag{$*$}$$

We claim that $\mathrm{q}(v) > -\infty$ for all $v \in V$. To see this we use first

$$\lambda \mathrm{p}(w) = \mathrm{p}(\lambda w) \leq \mathrm{p}(v + \lambda w) + \mathrm{p}(-v),$$

and conclude

$$-\mathrm{p}(-v) \leq \mathrm{p}(v + \lambda w) - \lambda \mathrm{p}(w)$$

for all $\lambda \geq 0$. Thus $\mathrm{q}(v) \geq -\mathrm{p}(-v) > -\infty$ for all $v \in V$. This shows $\mathrm{q}\colon V \longrightarrow \mathbb{R}$ as claimed. Taking $\lambda = 0$ in $(*)$ gives $\mathrm{q}(v) \leq \mathrm{p}(v)$ for all $v \in V$. We claim that $\mathrm{q} \in V^\#$. First let $\mu > 0$ then

$$\begin{aligned}
\mathrm{q}(\mu v) &= \inf_{\lambda \geq 0} \big\{ \mathrm{p}(\mu v + \lambda w) - \lambda \mathrm{p}(w) \big\} \\
&= \inf_{\lambda \geq 0} \big\{ \mu \mathrm{p}(v + \tfrac{\lambda}{\mu} w) - \mu \tfrac{\lambda}{\mu} \mathrm{p}(w) \big\} \\
&= \mu \inf_{\lambda \geq 0} \big\{ \mathrm{p}(v + \tfrac{\lambda}{\mu} w) - \tfrac{\lambda}{\mu} \mathrm{p}(w) \big\} \\
&= \mu \mathrm{q}(v).
\end{aligned}$$

Since also $\mathrm{q}(0) = 0$ we have $\mathrm{q}(\mu v) = \mu \mathrm{q}(v)$ for all $\mu \geq 0$ and all $v \in V$. To check (3.1.1) we consider $v_1, v_2 \in V$ and fix $\varepsilon > 0$. By definition $(*)$ we find $\lambda_1, \lambda_2 \geq 0$ with

$$\mathrm{q}(v_i) + \frac{\varepsilon}{2} \geq \mathrm{p}(v_i + \lambda_i w) - \lambda_i \mathrm{p}(w)$$

for $i = 1, 2$. Hence

$$\begin{aligned}
\mathrm{q}(v_1 + v_2) &\leq \mathrm{p}(v_1 + v_2 + \lambda_1 w + \lambda_2 w) - (\lambda_1 + \lambda_2)\mathrm{p}(w) \\
&\leq \mathrm{p}(v_1 + \lambda_1 w) - \lambda_1 \mathrm{p}(w) + \mathrm{p}(v_2 + \lambda_2 w) - \lambda_2 \mathrm{p}(w) \\
&\leq \mathrm{q}(v_1) + \mathrm{q}(v_2) + \varepsilon,
\end{aligned}$$

since p is sublinear. Since $\varepsilon > 0$ was arbitrary, we conclude $\mathrm{q}(v_1 + v_2) \leq \mathrm{q}(v_1) + \mathrm{q}(v_2)$, finally showing $\mathrm{q} \in V^\#$. Since we already know that $\mathrm{q} \leq \mathrm{p}$ pointwise, we conclude $\mathrm{p} = \mathrm{q}$ since p is minimal by assumption. Hence from $(*)$ we get

$$\mathrm{p}(v) = \mathrm{q}(v) \leq \mathrm{p}(v + w) - \mathrm{p}(w) \leq \mathrm{p}(v),$$

using once more that p is sublinear. Thus

$$\mathrm{p}(v + w) - \mathrm{p}(w) = \mathrm{p}(v)$$

follows, proving that p is additive. Since we already know $\mathrm{p}(\lambda v) = \lambda \mathrm{p}(v)$ for $\lambda \in \mathbb{R}_0^+$, this is sufficient to conclude $\mathrm{p} \in V^*$. ☐

The second lemma shows that for every element $\mathrm{p} \in V^\#$ we find a smaller minimal one. Here we use the axiom of choice in form of Zorn's lemma:

Lemma 3.1.4 *Let V be a real vector space and $\mathrm{p} \in V^\#$. Then there is a $\varphi \in V^*$ with*

$$\varphi \preccurlyeq \mathrm{p}. \tag{3.1.4}$$

Proof: Consider the subset

$$\mathcal{P} = \left\{ \mathrm{q} \in V^{\#} \;\middle|\; \mathrm{q} \preccurlyeq \mathrm{p} \right\} \subseteq V^{\#}$$

of those sublinear functionals dominated by p. We know $\mathrm{p} \in \mathcal{P}$ and want to show that $\mathcal{P}$ contains a linear functional. Suppose $\mathcal{Q} = \{\mathrm{q}_i\}_{i \in I} \subseteq \mathcal{P}$ is a totally ordered subset. We claim that in this case for a given vector $v \in V$ the set $\{\mathrm{q}_i(v)\}_{i \in I} \subseteq \mathbb{R}$ is bounded from below. Assuming this is false, we find a sequence $\mathrm{q}_{i_1}, \mathrm{q}_{i_2}, \ldots$ with $\mathrm{q}_{i_n}(v) \leq -n$ for all $n \in \mathbb{N}$. Since $\mathcal{Q}$ is totally ordered, a finite subset $\mathrm{q}_{i_1}, \ldots, \mathrm{q}_{i_n}$ has a minimum

$$\mathrm{p}_n = \min \left\{ \mathrm{q}_{i_1}, \ldots, \mathrm{q}_{i_n} \right\} \in V^{\#}$$

satisfying $\mathrm{p}_n(v) \leq -n$ and $\mathrm{p}_{n+1} \preccurlyeq \mathrm{p}_n$ for all $n \in \mathbb{N}$. Note that p_n is just one of the $\mathrm{q}_{i_1}, \ldots, \mathrm{q}_{i_n}$. By sublinearity we have

$$0 = \mathrm{p}_n(0) \leq \mathrm{p}_n(v) + \mathrm{p}_n(-v) \leq -n + \mathrm{p}_1(-v),$$

a contradiction. Thus

$$\mathrm{p}_{\mathrm{inf}}(v) = \inf_{i \in I} \left\{ \mathrm{q}_i(v) \right\} = \lim_{i \in I} \mathrm{q}_i(v) \in \mathbb{R}$$

is finite and yields a map $\mathrm{p}_{\mathrm{inf}} \colon V \longrightarrow \mathbb{R}$. Note that the infimum is in fact a limit for the totally ordered set of the q_i. Here we use the fact that the total order yields a direction and hence a net. Thus the pointwise infimum of these sublinear functionals is again sublinear once it takes values in $\mathbb{R}$ at all. It follows that $\mathrm{p}_{\mathrm{inf}} \in V^{\#}$ and $\mathrm{p}_{\mathrm{inf}} \preccurlyeq \mathrm{q}_i \preccurlyeq \mathrm{p}$ showing $\mathrm{p}_{\mathrm{inf}} \in \mathcal{P}$. This finally shows that every totally ordered subset $\mathcal{Q} \subseteq \mathcal{P}$ has an infimum. By Zorn's lemma we infer the existence of a minimal element $\varphi \in \mathcal{P}$. This is also minimal in $V^{\#}$ since if $\mathrm{q} \in V^{\#}$ satisfies $\mathrm{q} \preccurlyeq \varphi$ then $\mathrm{q} \preccurlyeq \mathrm{p}$ as well, hence $\mathrm{q} \in \mathcal{P}$ and thus $\mathrm{q} = \varphi$. From Lemma 3.1.3 we get that $\varphi \in V^*$ is actually linear. □

The first version of the Hahn-Banach theorem can now be stated as follows:

Theorem 3.1.5 (Hahn-Banach) *Let V be a vector space over $\mathbb{K}$ and let $U \subseteq V$ be a subspace. Moreover, let $\mathrm{p} \in V^{\#}$ and $\varphi \in U^*$ be given such that*

$$\mathrm{Re}(\varphi(u)) \leq \mathrm{p}(u) \tag{3.1.5}$$

for all $u \in U$. Then there exists an extension $\Phi \in V^*$ of φ to a $\mathbb{K}$-linear functional on V such that

$$\mathrm{Re}(\Phi(v)) \leq \mathrm{p}(v) \tag{3.1.6}$$

for all $v \in V$.

Proof: We consider $\mathbb{K} = \mathbb{R}$ first. For $u \in U$ we have by (3.1.5)

$$\varphi(u) = -\varphi(-v) \geq -\mathrm{p}(-u)$$

as well as $\mathrm{p}(-u) \leq \mathrm{p}(v - u) + \mathrm{p}(-v)$ for $v \in V$. Together we have

$$\begin{aligned} -\mathrm{p}(-v) &= \mathrm{p}(-u) - \mathrm{p}(-v) - \mathrm{p}(-u) \\ &\leq \mathrm{p}(v - u) + \mathrm{p}(-v) - \mathrm{p}(-v) + \varphi(u) \\ &= \mathrm{p}(v - u) + \varphi(u). \end{aligned}$$

Hence

$$\tilde{\mathrm{p}}(v) = \inf_{u \in U} \left\{ \mathrm{p}(v - u) + \varphi(u) \right\} > -\infty$$

follows for all $v \in V$. Similarly to the proof of Lemma 3.1.3 one then verifies that the resulting map $\tilde{\mathrm{p}} \colon V \longrightarrow \mathbb{R}$ is sublinear, i.e. $\tilde{\mathrm{p}} \in V^{\#}$. Moreover, as a subspace U contains $u = 0$, we see $\tilde{\mathrm{p}} \preccurlyeq \mathrm{p}$. Taking a vector $v \in U$ we get

$$\tilde{\mathrm{p}}(v) = \inf_{u \in U} \left\{ \mathrm{p}(v - u) + \varphi(u) \right\} \leq \mathrm{p}(v - u) + \varphi(v) = \varphi(v)$$

and hence $\tilde{p}\big|_U \preccurlyeq \varphi$. From Lemma 3.1.4 we get a linear functional $\Phi \in V^*$ with $\Phi \preccurlyeq \tilde{p}$. Restricting this linear functional to U gives

$$\Phi\big|_U \preccurlyeq \tilde{p}\big|_U \preccurlyeq \varphi.$$

Since φ was linear it is minimal in $U^{\#}$. Hence $\Phi\big|_U = \tilde{p}\big|_U = \varphi$ follows from Lemma 3.1.3, showing the case $\mathbb{K} = \mathbb{R}$.

For $\mathbb{K} = \mathbb{C}$ we first obtain an extension $\Psi \in \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$ of $\psi = \operatorname{Re}(\varphi)$ by considering the complex vector space as a real one. To obtain a complex-linear functional out of Ψ still obeying the estimate (3.1.6) we define $\Phi \colon V \longrightarrow \mathbb{C}$ by

$$\Phi(v) = \Psi(v) - \mathrm{i}\Psi(\mathrm{i}v)$$

for all $v \in V$. Then $\Phi(\mathrm{i}v) = \mathrm{i}\Phi(v)$ from which the complex linearity of Φ follows at once. We have $\operatorname{Re}(\Phi(v)) = \Psi(v)$ by construction and for $u \in U$ we obtain

$$\Phi(u) = \Psi(u) - \mathrm{i}\Psi(\mathrm{i}u) = \psi(u) - \mathrm{i}\psi(\mathrm{i}u) = \varphi(u),$$

since Ψ extends $\psi = \operatorname{Re}(\varphi)$. Finally, $\operatorname{Re}(\Phi(v)) = \Psi(v) \le \mathrm{p}(v)$ holds as well. □

With this still rather general and abstract version of the Hahn-Banach theorem we get more convenient formulations of extensions as follows:

Corollary 3.1.6 *Let V be a normed space with a subspace $U \subseteq V$, endowed with the induced norm. Suppose that $\varphi \in U'$ is a continuous linear functional on U. Then there exists an extension $\Phi \in V'$ of φ with*

$$\|\Phi\| = \|\varphi\|. \tag{3.1.7}$$

Proof: Here $\|\varphi\|$ is the functional norm with respect to the restriction of the norm on V to the subspace U, i.e. we have

$$|\varphi(u)| \le \|\varphi\| \|u\|$$

for all $u \in U$, and $\|\varphi\|$ is the smallest such constant. Then we have

$$\operatorname{Re}(\varphi(u)) \le |\operatorname{Re}\varphi(u)| \le |\varphi(u)| \le \|\varphi\| \|u\|$$

for all $u \in U$. With $\mathrm{p}(v) = \|\varphi\| \|v\|$ we obtain a sublinear functional $\mathrm{p} \in V^{\#}$ to which we can then apply the Hahn-Banach theorem. We get a $\mathbb{K}$-linear extension Φ of φ to V with

$$\operatorname{Re}(\Phi(v)) \le \|\varphi\| \|v\|.$$

First we consider the complex case $\mathbb{K} = \mathbb{C}$. For every $v \in V$ we have a unique phase $\mathrm{e}^{\mathrm{i}\alpha} \in \mathrm{U}(1) \subseteq \mathbb{C}$ with $\Phi(v) = \mathrm{e}^{\mathrm{i}\alpha}|\Phi(v)|$. Hence

$$|\Phi(v)| = \mathrm{e}^{-\mathrm{i}\alpha}\Phi(v) = \operatorname{Re}(\mathrm{e}^{-\mathrm{i}\alpha}\Phi(v)) = \operatorname{Re}(\Phi(\mathrm{e}^{-\mathrm{i}\alpha}v)) \le \|\varphi\| \|\mathrm{e}^{-\mathrm{i}\alpha}v\| = \|\varphi\| \|v\|$$

follows since $\mathrm{e}^{-\mathrm{i}\alpha}\Phi(v)$ is real and Φ is linear. This shows $|\Phi(v)| \le \|\varphi\| \|v\|$ and hence $\Phi \in V'$ with $\|\Phi\| \le \|\varphi\|$. Because Φ is an extension of φ we have $\|\Phi\| \ge \|\varphi\|$ and thus (3.1.7) in total. The real case is even easier since we only have to take care of a sign instead of a phase. □

To make use of this corollary we need a continuous linear functional φ on a subspace to start with. Fortunately, here we have many candidates:

Corollary 3.1.7 *Let V be a normed vector space with a finite dimensional subspace $U \subseteq V$. Then every linear functional $\varphi \in U^* = U'$ has an extension $\Phi \in V'$ with*

$$\|\Phi\| = \|\varphi\|.$$

PROOF: Indeed, for a finite-dimensional subspace $U \subseteq V$ all norms are equivalent and hence $U^* = U'$. □

Since in finite dimensions we have many linear functionals, we get the following consequence:

Corollary 3.1.8 *Let V be a normed space. For every vector $v \in V \setminus \{0\}$ we find a continuous linear functional $\varphi \in V'$ with $\|\varphi\| = 1$ such that*

$$\varphi(v) = \|v\|. \tag{3.1.8}$$

PROOF: On the one-dimensional subspace $U = \operatorname{span}_{\mathbb{K}}\{v\} \subseteq V$ we take the unique linear functional defined by (3.1.8). This is well-defined since $\{v\}$ is a basis of U. Clearly, this functional has functional norm equal to one. □

In particular, we can detect whether a vector $v \in V$ is zero or not by looking at all its "coordinates" $\{\varphi(v)\}_{\varphi \in V'}$ obtained from continuous linear functionals.

Remark 3.1.9 It should be noted that the Hahn-Banach theorem gives the existence of various continuous linear functionals subject to the above conditions. However, uniqueness is by no means guaranteed. Quite contrary, already in the finite-dimensional situations extensions like the ones above are typically far from being unique, though always continuous. This is of course then one of the big drawbacks of the Hahn-Banach theorem as it only gives a (non-constructive) existence.

3.1.2 Separation Properties

The Hahn-Banach theorem can also be used to improve the separation properties in a normed space from the purely topological to a more geometric context. To see this we need to consider seminorms more carefully: Recall that a subset $A \subseteq V$ in a vector space over $\mathbb{K}$ is called *absorbing* if for every $v \in V$ there is a $\lambda > 0$ with $v \in \lambda A$. Moreover, A is called *balanced* (or *circled*) if for $v \in A$ and $z \in \mathbb{K}$ with $|z| \leq 1$ one has $zv \in A$, too. In Exercise 3.5.1 one can find some first properties of balanced and convex subsets.

Proposition 3.1.10 *Let V be a vector space over $\mathbb{K}$.*

i.) If $\mathrm{p}\colon V \longrightarrow \mathbb{R}_0^+$ is a seminorm, the two subsets

$$\mathrm{B}_{\mathrm{p},1}(0) = \{v \in V \mid \mathrm{p}(v) < 1\} \tag{3.1.9}$$

and

$$\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}} = \{v \in V \mid \mathrm{p}(v) \leq 1\} \tag{3.1.10}$$

are convex, absorbing, and balanced.

ii.) If $C \subseteq V$ is a convex, absorbing, and balanced subset then

$$\mathrm{p}_C(v) = \inf\{\lambda \in \mathbb{R} \mid v \in \lambda C, \lambda > 0\} \tag{3.1.11}$$

is a seminorm on V.

iii.) One has for every convex, absorbing, and balanced subset C the inclusions

$$\mathrm{B}_{\mathrm{p}_C,1}(0) \subseteq C \subseteq \mathrm{B}_{\mathrm{p}_C,1}(0)^{\mathrm{cl}}. \tag{3.1.12}$$

iv.) For every seminorm p one has

$$\mathrm{p}_{\mathrm{B}_{\mathrm{p},1}(0)} = \mathrm{p} = \mathrm{p}_{\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}}. \tag{3.1.13}$$

Proof: For the first part, the convexity of each of the two subsets follows from the homogeneity and the triangle inequality of p at once. Moreover, if $v \in V$ then either $\mathrm{p}(v) = 0$ and hence $v \in \mathrm{B}_{\mathrm{p},1}(0) \subseteq \mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ or we have $\mathrm{p}(v) > 0$. In that case $v \in 2\mathrm{p}(v)\mathrm{B}_{\mathrm{p},1}(0)$. This shows that $\mathrm{B}_{\mathrm{p},1}(0)$ and thus also $\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ is absorbing. Clearly $\mathrm{B}_{\mathrm{p},1}(0)$ and $\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ are balanced. Conversely, let C be convex, absorbing, and balanced. For p_C defined by (3.1.14) we first note $\mathrm{p}_C(v) \geq 0$ for all $v \in V$ since there is always a $\lambda > 0$ with $v \in \lambda C$ as C is absorbing. Moreover, $\mathrm{p}_C(0) = 0$. Let $z \neq 0$ and $v \in V$ then $zv = \frac{z}{|z|}|z|v \in |z|C$ whenever $v \in C$ since C is balanced. This yields $\mathrm{p}_C(zv) = |z|\mathrm{p}_C(v)$ at once. With $\mathrm{p}_C(0) = 0$ we also get this for $z = 0$. Finally, we need to check the triangle inequality for p_C. Let $v, w \in V$ and fix $\varepsilon > 0$. Consider $\lambda = \mathrm{p}_C(v) + \varepsilon$ and $\mu = \mathrm{p}_C(w) + \varepsilon$. Since C is convex, also any rescaled version λC is convex. If $v \in \lambda C$ then $v \in \lambda' C$ for any $\lambda' \geq \lambda$, too, since for a convex λC the whole connecting segment from $0 \in \lambda C$ to v belongs to λC. It follows that $\frac{1}{\lambda}v$ and $\frac{1}{\mu}w$ are in C by definition of p_C. Since C is convex, we get

$$\frac{\lambda}{\lambda+\mu}\frac{1}{\lambda}v + \frac{\mu}{\lambda+\mu}\frac{1}{\mu}w \in C,$$

which means $v + w \in (\lambda+\mu)C = (\mathrm{p}_C(v) + \mathrm{p}_C(w) + 2\varepsilon)C$. Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\mathrm{p}_C(v+w) \leq \mathrm{p}_C(v) + \mathrm{p}_C(w).$$

Hence the triangle inequality holds, too. Note that for the triangle inequality we only needed the convexity of C. Together, this finally shows that p_C is a seminorm. For the third part assume $v \in \mathrm{B}_{\mathrm{p}_C,1}(0)$, i.e. we have $\mathrm{p}_C(v) < 1$. Then for every $\varepsilon > 0$ we have $v \in (\mathrm{p}_C(v) + \varepsilon)C$. Taking ε small enough gives $v \in C$. If $v \in C$ then $\mathrm{p}_C(v) \leq 1$ by definition of p_C. Finally, for the last part let p be a seminorm and consider $C = \mathrm{B}_{\mathrm{p},1}(0)$. Then

$$\mathrm{p}_C(v) = \inf\{\lambda > 0 \mid v \in \lambda\mathrm{B}_{\mathrm{p},1}(0)\} = \inf\{\lambda > 0 \mid \mathrm{p}(v) < \lambda\} = \mathrm{p}(v).$$

Analogously, one has for $C = \mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$

$$\mathrm{p}_C(v) = \inf\left\{\lambda > 0 \;\middle|\; v \in \lambda\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}\right\} = \inf\{\lambda > 0 \mid \mathrm{p}(v) \leq \lambda\} = \mathrm{p}(v),$$

completing the proof. ☐

Definition 3.1.11 (Minkowski functional) *Let $C \subseteq V$ be an absorbing subset in a vector space over $\mathbb{K}$. Then $\mathrm{p}_C \colon V \longrightarrow \mathbb{R}_0^+$ with*

$$\mathrm{p}_C(v) = \inf\{\lambda > 0 \mid v \in \lambda C\} \tag{3.1.14}$$

is called the Minkowski functional of C.

It follows that the Minkowski functionals of absorbing, balanced and convex subsets C are seminorms which in turn allow us to reconstruct the subset C almost, i.e. up to (3.1.13) in Proposition 3.1.10, *iv.)*. Note also that an absorbing and convex subset C has a Minkowski functional p_C which is sublinear, see also Exercise 3.5.2. Conversely, and this is the easier part, every seminorm p determines its *open* and *closed unit ball* $\mathrm{B}_{\mathrm{p},1}(0)$ and $\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ which are absorbing, balanced, and convex subsets reproducing the seminorm as their Minkowski functionals. Note that we do not have a topology yet which would justify the usage of notions like "open" and "closed". At this stage this is a mere analogy to the case of a norm and its unit ball. We will come back to Minkowski functionals several times. In the theory of locally convex spaces they play a crucial role when defining locally convex topologies.

With the notion of the Minkowski functional we are now able to discuss the following version of the separation properties following from the Hahn-Banach theorem.

Theorem 3.1.12 (Separation I) *Let V be a normed space over $\mathbb{K}$ with two non-empty disjoint convex subsets $A, B \subseteq V$.*

i.) Suppose A is open. Then there exists a continuous linear functional $\Phi \in V'$ and a real number $\alpha \in \mathbb{R}$ such that

$$\mathrm{Re}(\Phi(v)) < \alpha \leq \mathrm{Re}(\Phi(u)) \tag{3.1.15}$$

for all $v \in A$ and all $u \in B$.

ii.) Suppose that both subsets A and B are open. Then there exists a continuous linear functional $\Phi \in V'$ and a real number $\alpha \in \mathbb{R}$ such that

$$\mathrm{Re}(\Phi(v)) < \alpha < \mathrm{Re}(\Phi(u)) \tag{3.1.16}$$

for all $v \in A$ and all $u \in B$.

PROOF: Assume the case $\mathbb{K} = \mathbb{R}$ is solved. Then the complex case can be easily obtained as follows. Let $\Psi \colon V \longrightarrow \mathbb{R}$ be a continuous $\mathbb{R}$-linear functional with either (3.1.15) or (3.1.16) for an appropriate $\alpha \in \mathbb{R}$. Then we define $\Phi(v) = \Psi(v) - \mathrm{i}\Psi(\mathrm{i}v)$ for $v \in V$, yielding a $\mathbb{C}$-linear continuous linear functional such that $\mathrm{Re}(\Phi(v)) = \Psi(v)$ for all $v \in V$. Hence Φ solves (3.1.15) or (3.1.16), respectively.

It remains to show the real case $\mathbb{K} = \mathbb{R}$. Thus assume first that A is open. We fix vectors $v_0 \in A$ and $w_0 \in B$ and set $u_0 = w_0 - v_0$, which is non-zero by assumption. Then

$$C = A - B + u_0 = \bigcup_{w \in B} (A - w + u_0)$$

is an open subset since translations of open subsets are open and so are unions. We have $0 \in C$. Since any neighbourhood of zero is absorbing, see (2.1.7), the subset C is absorbing. A simple check shows that for convex A and B also $A - B$ is convex. The translation by u_0 preserves convexity, too, and hence C is convex. We consider now the Minkowski functional p_C of C. Since C might not be balanced, p_C might not be a seminorm. However, C being convex is already enough to conclude that p_C is sublinear, i.e. $\mathrm{p}_C \in V^{\#}$, see again the proof of Proposition 3.1.10, *ii.)*, as well as Exercise 3.5.2. Since $A \cap B = \emptyset$ we have $u_0 \notin C$ and thus $\mathrm{p}_C(u_0) \geq 1$ follows from the definition of p_C. We consider now the one-dimensional subspace $U = \mathrm{span}_{\mathbb{K}}\{u_0\}$ and define a linear functional $\psi \in U^* = U'$ by specifying ψ on the basis vector u_0 to be

$$\psi(u_0) = 1 \leq \mathrm{p}_C(u_0).$$

For positive scalings $\lambda > 0$ we have $\lambda \mathrm{p}_C(u_0) = \mathrm{p}_C(\lambda u_0)$, and for negative $\lambda < 0$ we trivially have $\psi(\lambda u_0) = \lambda < 0 \leq \mathrm{p}_C(\lambda u_0)$. Together this shows

$$\psi(u) \leq \mathrm{p}_C(u)$$

for all $u \in U$. Hence we are in the position to apply the Hahn-Banach theorem to get an extension $\Psi \in V^*$ with

$$\Psi(v) \leq \mathrm{p}_C(v)$$

for all $v \in V$. If $v \in C$ we have $\mathrm{p}_C(v) \leq 1$ by definition and hence $\Psi(v) \leq 1$ follows. Now consider $v \in -C$. Then $-v \in C$ shows

$$\Psi(v) \geq -1.$$

This implies that for $v \in C \cap (-C)$ we get the estimate

$$|\Psi(v)| \leq 1.$$

Since C was an open neighbourhood of zero, also $C \cap (-C)$ is an open neighbourhood of zero. As such it contains an open ball $\mathrm{B}_\varepsilon(0) \subseteq C \cap (-C)$ for a suitably chosen $\varepsilon > 0$. It follows that

$$\|v\| < \varepsilon \implies |\Psi(v)| \leq 1.$$

As we have seen in the proof of Proposition 2.2.22 this implies that Ψ is continuous, i.e. $\Psi \in V'$. We claim that this functional Ψ will do the job. For $v \in A$ and $u \in B$ we have

$$\Psi(v) - \Psi(u) + 1 = \Psi(v - u + u_0) \leq \mathrm{p}_C(v + u + u_0) < 1,$$

since by the openness of C we first get $v - u + u_0 \in C$ and hence $v - u + u_0 \in (1 - \varepsilon)C$, too, for some small $\varepsilon > 0$. It follows that

$$\Psi(v) < \Psi(u) \tag{$*$}$$

for all $v \in A$ and $u \in B$. Since Ψ is linear, the subsets $\Psi(A)$ and $\Psi(B)$ are still convex subsets of $\mathbb{R}$. As a non-constant linear functional, Ψ is an open map, see also Exercise 2.5.5, from which we deduce that $\Psi(A) \subseteq \mathbb{R}$ is open. Finally, $(*)$ shows that $\Psi(A) \subseteq \mathbb{R}$ is bounded from above. Thus the only remaining convex open subsets bounded from above are of the form

$$\Psi(A) = (\omega, \alpha)$$

with $\omega \in [-\infty, \alpha)$ and $\alpha \in \mathbb{R}$. Then this α will do the job for (3.1.15).

Suppose finally that also B is open. Then the analogous argument shows that $\Psi(B) \subseteq \mathbb{R}$ is open, convex, and bounded from below, hence of the form

$$\Psi(B) = (\beta, \Omega)$$

with $\beta \in \mathbb{R}$ and $\Omega \in (\beta, \infty]$. From $(*)$ we get $\alpha \leq \beta$. Then every number in $[\alpha, \beta]$, like e.g. α itself, will satisfy (3.1.16). $\square$

It is an easy exercise to check that the estimates (3.1.15) and (3.1.16) can not be improved in general: here one finds counterexamples in finite dimensions already, see also Exercise 3.5.3.

For normed spaces we can give now a simple proof of Proposition 2.1.7.

Proposition 3.1.13 *Let V be a normed space and let $K \subseteq V$ be a compact subset and $A \subseteq V$ be a closed subset, such that $K \cap A = \emptyset$. Then there exists an $\varepsilon > 0$ with*

$$(K + \mathrm{B}_\varepsilon(0)) \cap (A + \mathrm{B}_\varepsilon(0)) = \emptyset. \tag{3.1.17}$$

PROOF: Since a normed space is a metric space, we know that for a point $v \in V$ and a closed subset $A \subseteq V$ we have $\mathrm{d}(v, A) = 0$ iff $v \in A$, where

$$\mathrm{d}(v, A) = \inf\big\{\|v - w\| \bigm| w \in A\big\},$$

see also Exercise **??**. Define now $\varepsilon_v = \mathrm{d}(v, A) > 0$ for $v \in K$. Then

$$\mathrm{B}_{\frac{\varepsilon_v}{4}}(v) \cap (A + \mathrm{B}_{\frac{\varepsilon_v}{4}}(0)) = \emptyset,$$

since the distance from v to A is ε_v. The open balls $\{\mathrm{B}_{\frac{\varepsilon_v}{4}}(v)\}_{v \in K}$ cover K and by compactness finitely many will cover K already. We take $\varepsilon = \frac{1}{4}\min\{\varepsilon_{v_1}, \ldots, \varepsilon_{v_n}\} > 0$ for these finitely many balls and obtain (3.1.17). $\square$

We use this result to show the following version of the separation theorem:

Theorem 3.1.14 (Separation II) *Let V be a normed space and let $K \subseteq V$ be a compact, non-empty convex subset and let $C \subseteq V$ be a closed, non-empty convex subset with $K \cap C = \emptyset$. Then there exists a continuous linear functional $\Phi \in V'$ together with $\alpha, \beta \in \mathbb{R}$ such that*

$$\operatorname{Re}(\Phi(v)) < \alpha < \beta < \operatorname{Re}(\Phi(u)) \tag{3.1.18}$$

for all $v \in K$ and $u \in C$.

Proof: We consider the real case $\mathbb{K} = \mathbb{R}$ first. Let $\varepsilon > 0$ be such that the ε-ball separates K and C according to Proposition 3.1.13, i.e. we have

$$(K + \mathrm{B}_\varepsilon(0)) \cap (C + \mathrm{B}_\varepsilon(0)) = \emptyset.$$

Since $\mathrm{B}_\varepsilon(0)$ is convex, the left hand side $A = K + \mathrm{B}_\varepsilon(0)$ is a convex and open subset. Moreover $B = C$ is convex and disjoint from A. According to Theorem 3.1.12, *i.)*, we find $\tilde{\beta} \in \mathbb{R}$ and a continuous linear functional $\Psi \colon V \longrightarrow \mathbb{R}$ with

$$\Psi(v) < \tilde{\beta} \le \Psi(u)$$

for all $v \in A$ and $u \in B$. Again $\Psi(A) \subseteq \mathbb{R}$ is open and $\Psi(K) \subseteq \Psi(A)$ is still compact. Since $\Psi(A)$ is bounded by $\tilde{\beta}$ from above, we conclude that

$$\Psi(K) \subseteq [\omega, \alpha]$$

for some $\omega < \alpha < \tilde{\beta}$. Hence the claim follows by taking some $\beta \in (\alpha, \tilde{\beta})$. The complex case is again obtained from the real one by the now familiar trick of considering $\Phi(v) = \Psi(v) - \mathrm{i}\Psi(\mathrm{i}v)$ for $v \in V$ as before. $\square$

We can exchange Φ by $\Phi' = -\Phi$ and set $\alpha' = -\beta$ and $\beta' = -\alpha$ to obtain

$$\operatorname{Re}(\Phi'(u)) < \alpha' < \beta' < \operatorname{Re}(\Phi(v)) \tag{3.1.19}$$

for all $u \in C$ and $v \in K$. Thus K and C enter completely symmetric in Theorem 3.1.14.

The following two corollaries will illustrate the usage of these separation theorems further:

Corollary 3.1.15 *Let V be a normed space with a non-empty convex compact subset $K \subseteq V$ and a non-empty balanced convex closed subset $C \subseteq V$ such that $K \cap C = \emptyset$. Then there exists a continuous linear functional $\Phi \in V'$ such that*

$$\sup_{u \in C} |\Phi(u)| < \inf_{v \in K} |\Phi(v)|. \tag{3.1.20}$$

Proof: Consider $\mathbb{K} = \mathbb{C}$ first. Let $\Phi \in V'$ and $\alpha < \beta$ satisfy

$$\operatorname{Re}(\Phi(u)) < \alpha < \beta < \operatorname{Re}(\Phi(v)) \tag{$*$}$$

for all $u \in C$ and $v \in K$ according to Theorem 3.1.14 in the version (3.1.19). Since C is balanced, we have $\mathrm{e}^{\mathrm{i}\varphi} u \in C$ whenever $u \in C$ and $\mathrm{e}^{\mathrm{i}\varphi} \in \mathrm{U}(1) \subseteq \mathbb{C}$. For $u \in C$ we write $\Phi(u) = \mathrm{e}^{-\mathrm{i}\varphi} |\Phi(u)|$ with the appropriate phase $\mathrm{e}^{-\mathrm{i}\varphi}$. Then

$$\operatorname{Re}(\Phi(\mathrm{e}^{\mathrm{i}\varphi} u)) = \operatorname{Re}(\mathrm{e}^{\mathrm{i}\varphi} \Phi(u)) = \operatorname{Re}(|\Phi(u)|) = |\Phi(u)|.$$

It follows that $|\Phi(u)| < \alpha$ by $(*)$ and hence

$$\sup_{u \in C} |\Phi(u)| \le \alpha.$$

For vectors $v \in K$ we have $\operatorname{Re}(\Phi(v)) > \beta$ showing $|\Phi(v)| > \beta$. Hence

$$\inf_{v \in K} |\Phi(v)| \ge \beta$$

follows. Together with $(*)$ this gives (3.1.20). The real case $\mathbb{K} = \mathbb{R}$ is easier with the same idea. $\square$

Corollary 3.1.16 *Let V be a normed space and let $C \subseteq V$ be a closed balanced convex subset. For $v \in V \setminus C$ there exists a continuous linear functional $\Phi \in V'$ with*

$$\Phi(v) > 1 \quad \text{and} \quad |\Phi(u)| \leq 1 \tag{3.1.21}$$

for all $u \in C$.

PROOF: We take the compact subset $K = \{v\}$ and apply the previous Corollary 3.1.15. Hence we find a $\widetilde{\Phi} \in V'$ with

$$\gamma = \sup_{u \in C} |\widetilde{\Phi}(u)| < |\widetilde{\Phi}(v)|.$$

Rescaling $\widetilde{\Phi}$ by γ and the phase of $\widetilde{\Phi}(v)$ then gives the continuous linear functional Φ we are looking for. □

Surprisingly enough, all the separation statements in this section will generalize almost verbatim to locally convex spaces instead of normed spaces. Here we refer to [] for further discussions on locally convex spaces.

3.1.3 The Weak Topology

Every normed space comes automatically with another topology, the *weak topology*. This is the first example of a so-called locally convex topology as we construct it by means of (many) seminorms instead of a single norm.

Definition 3.1.17 (Weak topology) *Let V be a topological vector space.*

i.) For $\varphi \in V'$ one defines the seminorm $\mathrm{p}_\varphi \colon V \longrightarrow \mathbb{R}_0^+$ by

$$\mathrm{p}_\varphi(v) = |\varphi(v)| \tag{3.1.22}$$

for $v \in V$ and calls

$$\mathrm{B}_{\mathrm{p}_\varphi, r}(v) = \big\{ w \in V \bigm| \mathrm{p}_\varphi(v - w) < r \big\} \tag{3.1.23}$$

the open ball around v of radius $r > 0$ with respect to p_φ.

ii.) The weak topology on V is the unique topology generated by all open balls (3.1.23) *for $\varphi \in V'$, $v \in V$ and $r > 0$.*

We have seen these open balls $\mathrm{B}_{\mathrm{p}_\varphi, r}(v)$ already in Proposition 3.1.10 for an arbitrary seminorm p instead of p_φ. Note that p_φ is indeed a seminorm, see again Remark 2.2.2, *v.)*. Some first properties of the weak topology are summarized in the following proposition where we focus on the case of a normed space V for simplicity.

Proposition 3.1.18 *Let V be a normed space.*

i.) The weak topology turns V into a topological vector space.

ii.) The weak topology is coarser than the original norm topology.

iii.) A linear functional $\varphi \in V^*$ is continuous iff it is weakly continuous.

iv.) The weak topology is Hausdorff.

v.) The seminorm p_φ for $\varphi \in V'$ is continuous in the weak topology.

PROOF: First we need to show that the addition of vectors and the multiplication with scalars are continuous with respect to the weak topology. This is true in a much larger context: every topology generated from the open balls with respect to an arbitrary collection of seminorms turns a vector space into a topological vector space, see also Exercise 3.5.5. To check the continuity of $+$ we make use of the defining subbasis of open balls $\mathrm{B}_{\mathrm{p}_\varphi, r}(v)$ as follows. Let $(v, v') \in V \times V$ with $v + v' \in V$ and let $\varphi \in V'$ as well as $r > 0$ be given. Since the open balls form a subbasis of the topology, it suffices to show that the preimage of $\mathrm{B}_{\mathrm{p}_\varphi, r}(v + v')$ under $+$ is a neighbourhood of the point $(v, v') \in V \times V$, see again **??**. Consider the (open) neighbourhood $U = \mathrm{B}_{\mathrm{p}_\varphi, \frac{r}{2}}(v) \times \mathrm{B}_{\mathrm{p}_\varphi, \frac{r}{2}}(v')$ of $(v, v') \in V \times V$. Then for $(u, u') \in U$ we have

$$\mathrm{p}_\varphi(u + u' - (v + v')) \leq \mathrm{p}_\varphi(u + v) + \mathrm{p}_\varphi(u' + v') < \frac{r}{2} + \frac{r}{2} = r,$$

and thus $u + u' \in \mathrm{B}_{\mathrm{p}_\varphi, r}(v + v')$ follows. This shows that U belongs to the preimage of $\mathrm{B}_{\mathrm{p}_\varphi, r}(v + v')$ under addition. Hence we have shown that $+$ is continuous at (v, v') since U is a neighbourhood. As (v, v') was arbitrary, $+$ is continuous. Similarly, let $(z, v) \in \mathbb{K} \times V$ and consider $\mathrm{B}_{\mathrm{p}_\varphi, r}(zv)$ for some $\varphi \in V'$ and $r > 0$. Again we need to show that the preimage of this neighbourhood is a neighbourhood of $(z, v) \in \mathbb{K} \times V$ to conclude continuity of the multiplication by scalars. Consider $(z', v') \in \mathbb{K} \times V$ with

$$z' \in \mathrm{B}_\delta(z) \quad \text{with} \quad \delta = \frac{r}{2(\mathrm{p}_\varphi(v) + 1)} > 0$$

and

$$v' \in \mathrm{B}_{\mathrm{p}_\varphi, \tilde{\delta}}(v) \quad \text{with} \quad \tilde{\delta} = \frac{r}{2(|z| + \delta)} > 0.$$

Then

$$\begin{aligned} \mathrm{p}_\varphi(z'v' - zv) &\leq \mathrm{p}_\varphi(z'v' - z'v) + \mathrm{p}_\varphi(z'v - zv) \\ &= |z'|\mathrm{p}_\varphi(v - v') + |z' - z|\mathrm{p}_\varphi(v) \\ &< (|z| + \delta)\frac{r}{2(|z| + \delta)} + \frac{r}{2(\mathrm{p}_\varphi(v) + 1)}\mathrm{p}_\varphi(v) \\ &< r \end{aligned}$$

shows $z'v' \in \mathrm{B}_{\mathrm{p}_\varphi, r}(zv)$. This completes the proof of the first part. Note that we have not yet used any specific features of the seminorms p_φ beside being seminorms. Hence we can and will use this argument also in other situations.

The second part is even easier. Let $\varphi \in V'$ and $r > 0$ be given. Then we have the estimate

$$\mathrm{p}_\varphi(v - v') \leq |\varphi(v - v')| \leq \|\varphi\| \|v - v'\| < r$$

for all $v' \in V$ with $\|v - v'\| < \frac{r}{\|\varphi\| + 1} = \delta$, i.e. for all $v' \in \mathrm{B}_\delta(v)$. This shows that the (open) neighbourhood $\mathrm{B}_{\mathrm{p}_\varphi, r}(v)$ of v in the weak topology is also a neighbourhood in the original norm topology, as it contains the open norm-ball $\mathrm{B}_\delta(v)$. From this we deduce that every weakly open subset is norm-open, showing the second statement, see also Exercise 1.5.2.

Now let $\varphi \in V^*$. Suppose φ is weakly continuous. Since the norm topology is finer than the weak topology, φ is also norm-continuous, i.e. $\varphi \in V'$. Conversely, let $\varphi \in V'$ be norm-continuous. Then

$$\varphi^{-1}(\mathrm{B}_r(0)) = \big\{v \in V \bigm| |\varphi(v)| < r\big\} = \big\{v \in V \bigm| \mathrm{p}_\varphi(v) < r\big\} = \mathrm{B}_{\mathrm{p}_\varphi, r}(0)$$

shows that the preimage of the zero neighbourhood $\mathrm{B}_r(0) \subseteq \mathbb{K}$ is the r-ball of the weak topology with respect to the seminorm p_φ and hence a weak zero neighbourhood. This proves continuity of φ at zero in the weak topology which implies continuity everywhere by our general considerations from Proposition 2.1.14.

The fourth part is now the Hahn-Banach theorem: for $v \neq 0$ we can find a $\varphi \in V'$ with $r = \varphi(v) > 0$. Hence $v \notin \mathrm{B}_{\mathrm{p}_\varphi, \frac{r}{2}}(0)$ and $0 \notin \mathrm{B}_{\mathrm{p}_\varphi, \frac{r}{2}}(v)$ gives the separation of v from 0 by weakly open balls. By the usual translation invariance of the topology in a topological vector space it suffices to separate $v \neq 0$ from 0 to conclude the Hausdorff property everywhere.

Finally, for *v.)* we note that $\mathrm{p}_\varphi = |\cdot| \circ \varphi$ is a composition of the weakly continuous functional φ, see *iii.)*, and the absolute value, hence again weakly continuous. □

Since the weak topology is coarser, every norm-convergent net or sequence is also weakly convergent. The opposite is not necessarily true, opening the door to new and interesting convergence results which are not available for the stronger norm convergence. The next proposition characterizes now weak convergence as a sort of *componentwise* convergence:

Proposition 3.1.19 *Let V be a normed space.*

i.) A net $(v_i)_{i \in I}$ in V is weakly convergent to $v \in V$ iff for every $\varphi \in V'$ one has

$$\lim_{i \in I} \varphi(v_i) = \varphi(v). \tag{3.1.24}$$

ii.) A net $(v_i)_{i \in I}$ in V is a weak Cauchy net iff for every $\varphi \in V'$ the net $(\varphi(v_i))_{i \in I}$ is a Cauchy net in $\mathbb{K}$.

PROOF: We check the convergence using the subbasis of open balls $\mathrm{B}_{\mathrm{p}_\varphi, r}(v)$ for $r > 0$ and $\varphi \in V'$. Convergence is equivalent to the existence of an index $i \in I$ with $v_j \in \mathrm{B}_{\mathrm{p}_\varphi, r}(v)$ for all $j \succcurlyeq i$. This means

$$\mathrm{p}_\varphi(v_j - v) = |\varphi(v_j - v)| = |\varphi(v_j) - \varphi(v)| < r,$$

which is equivalent to $\lim_{i \in I} \varphi(v_i) = \varphi(v)$ since r was arbitrary. As $\varphi \in V'$ was arbitrary, too, the first part follows. For the second, the Cauchy condition is equivalent to the existence of an index $i \in I$ such that $v_j - v_{j'} \in \mathrm{B}_{\mathrm{p}_\varphi, r}(0)$ for all $j, j' \succcurlyeq i$, again for all neighbourhoods of zero from the subbasis $\{\mathrm{B}_{\mathrm{p}_\varphi, r}(0)\}_{\varphi \in V', r > 0}$. But this means

$$\mathrm{p}_\varphi(v_j - v_{j'}) = |\varphi(v_j - v_{j'})| = |\varphi(v_j) - \varphi(v_{j'})| < r,$$

which shows the equivalence in the second part, see also Exercise 3.5.5 for the general result. □

Since the weak topology is coarser, a norm dense subset is also weakly dense. The opposite needs not to be true: there might be weakly dense subsets which are not dense in the original norm topology. This will be one of the interesting applications of the weak topology since we might be able to approximate vectors weakly but not in the norm sense. From this point of view the following proposition is valuable since it shows that on convex subsets these differences disappear:

Proposition 3.1.20 *Let V be a normed space with a convex subset C. Then the weak closure and the norm closure of C coincide.*

PROOF: We denote the weak closure of C by C^{wcl}. Since the weak topology is coarser, we have $C^{\mathrm{cl}} \subseteq C^{\mathrm{wcl}}$ in general. We need to show the opposite inclusion. Suppose $v_0 \in C^{\mathrm{wcl}} \setminus C^{\mathrm{cl}}$. The closure of a convex subset C is still convex by the continuity of the vector space operations, see also Exercise 3.5.1. Since the set $\{v_0\}$ is compact and convex, we can apply the separation theorem in the version of Theorem 3.1.14 to $K = \{v_0\}$ and C^{cl}. We obtain a continuous linear functional $\Phi \in V'$ with $\alpha \in \mathbb{R}$ such that

$$\mathrm{Re}(\Phi(v_0)) < \alpha < \mathrm{Re}(\Phi(v)) \tag{$*$}$$

for all $v \in C^{\mathrm{cl}}$. Since Φ is weakly continuous by Proposition 3.1.18, *iii.)*, the subset

$$U = \big\{ v \in V \bigm| \mathrm{Re}(\Phi(v)) < \alpha \big\}$$

is a weakly open subset and thus a weak neighbourhood of v_0. Thus by $(*)$ we see $U \cap C \subseteq U \cap C^{\mathrm{cl}} = \emptyset$. This is a contradiction to $v_0 \in C^{\mathrm{wcl}}$ since every weak neighbourhood of v_0 has to intersect C nontrivially. □

Corollary 3.1.21 *Let $U \subseteq V$ be a subspace in a normed space. Then the closures of U in the original norm topology and in the weak topology coincide.*

PROOF: Indeed, a subspace is convex. $\square$

3.1.4 Applications

The Hahn-Banach theorem has many applications. We will illustrate this now with some examples. The first gives us an almost trivial construction of the completion of a normed space.

Proposition 3.1.22 *Let V be a normed space. Then the canonical map*

$$\iota \colon V \longrightarrow V'' \tag{3.1.25}$$

is norm-preserving and hence injective.

PROOF: Recall that for $v \in V$ we have defined $\iota(v) \colon V' \longrightarrow \mathbb{K}$ by

$$\iota(v)\varphi = \varphi(v).$$

From Proposition 2.3.12 we know $\iota(v) \in V''$ and $\iota \colon V \longrightarrow V''$ is continuous with operator norm at most one. Without the Hahn-Banach theorem the topological dual could be trivial and hence the map ι could be the zero map. This does not happen since for every $v \in V$ we have a continuous linear functional $\varphi \in V'$ with $\|\varphi\| = 1$ and $\varphi(v) = \|v\|$ according to Corollary 3.1.8. Hence for this functional

$$|\iota(v)\varphi| = |\varphi(v)| = \|v\| \|\varphi\|$$

follows, showing that

$$\|\iota(v)\| = \sup_{\psi \in V' \setminus \{0\}} \frac{|\iota(v)\psi|}{\|\psi\|} \geq \|v\|.$$

Since we already know $\|\iota(v)\| \leq \|v\|$ we have $\|\iota(v)\| = \|v\|$ for all $v \in V$. $\square$

Theorem 3.1.23 (Completion of normed space) *Every normed space can be completed to a Banach space. More precisely, for a normed space V,*

$$\iota(V)^{\mathrm{cl}} \subseteq V'' \tag{3.1.26}$$

provides a completion.

PROOF: We only need to check the more specific statement. The map $\iota \colon V \longrightarrow \iota(V) \subseteq V''$ is a norm-preserving bijection onto a subspace of the bidual V'' by Proposition 3.1.22. Since $V'' = (V')'$ is complete by Corollary 2.3.11, the closure $\iota(V)^{\mathrm{cl}}$ yields a complete space thanks to Proposition 2.1.19, in which $\iota(V)$ is dense by construction. $\square$

The next application of the Hahn-Banach theorem also makes use of the non-triviality of V' in a crucial way. First we note the following general fact about dualizing continuous linear maps:

Proposition 3.1.24 *Let V, W, X be topological vector spaces and let $A, \tilde{A} \colon V \longrightarrow W$ and $B \colon W \longrightarrow X$ be continuous linear maps.*

i.) For every $\varphi \in W'$ one has $\varphi \circ A \in V'$, thus defining a map $A' \colon W' \longrightarrow V'$ by

$$A' \colon W' \ni \varphi \mapsto A'\varphi = \varphi \circ A \in V'. \tag{3.1.27}$$

ii.) One has

$$(\mathrm{id}_V)' = \mathrm{id}_{V'} \tag{3.1.28}$$

and

$$(B \circ A)' = A' \circ B'. \tag{3.1.29}$$

iii.) One has

$$(\alpha A + \tilde{\alpha}\tilde{A})' = \alpha A' + \tilde{\alpha}\tilde{A}' \tag{3.1.30}$$

for all $\alpha, \tilde{\alpha} \in \mathbb{K}$.

PROOF: It is clear that the composition $\varphi \circ A \colon V \longrightarrow \mathbb{K}$ is again continuous and linear, showing $A'\varphi \in V'$ as claimed. The second and third part are clear, since $A' = A^*\big|_{V'}$ is just the usual linear-algebraic dual map restricted to the topological dual. $\square$

Definition 3.1.25 (Dual maps) *Let $A \colon V \longrightarrow W$ be a continuous linear map. Then $A' \colon W' \longrightarrow V'$ is called the dual map of A.*

For general topological vector spaces one can not say much more than that. The difficulty is simply the lack of reasonable topologies on the topological duals and on the space $\mathrm{L}(V, W)$ of maps itself. Focusing on normed spaces we have a norm topology on the topological duals and on $\mathrm{L}(V, W)$. Hence we can speak of and prove continuity of A'. The Hahn-Banach theorem allows us to compute the norm of the adjoint map as follows:

Proposition 3.1.26 *Let V, W be normed spaces with a continuous linear map $A \colon V \longrightarrow W$. Then*

$$\|A\| = \sup_{\substack{\varphi \in W' \setminus \{0\} \\ v \in V \setminus \{0\}}} \frac{|\varphi(Av)|}{\|\varphi\|_{W'} \|v\|_V} = \sup_{\substack{\varphi \in W', \|\varphi\|_{W'} = 1 \\ v \in V, \|v\|_V = 1}} |\varphi(Av)| = \|A'\|. \tag{3.1.31}$$

In particular, $A' \colon W' \longrightarrow V'$ is continuous with respect to the norm topologies of the topological duals.

PROOF: The second equality is clear since the norms are homogeneous and hence multiples of φ and v drop out of the quotients. To check the first equality, for $v \in V$ we find a $\psi \in W'$ with $\psi(Av) = \|Av\|_W$ and $\|\psi\|_{W'} = 1$ by the Hahn-Banach theorem in form of Corollary 3.1.8. Then $|\varphi(Av)| \leq \|\varphi\|_{W'} \|Av\|_W$ for all $\varphi \in W'$, and together with the specific ψ we found, this gives the equality

$$\sup_{\varphi \in W', \|\varphi\| = 1} |\varphi(Av)| = \|Av\|_W.$$

Taking the supremum over $v \in V$ with $\|v\| = 1$ then gives the equality on the left in (3.1.31). For the last we use the definitions of the operator norm and the functional norm to get

$$\|A'\| = \sup_{\varphi \in W', \|\varphi\| = 1} \|A'\varphi\|_{V'} = \sup_{\varphi \in W', \|\varphi\| = 1} \sup_{v \in V, \|v\| = 1} |(A'\varphi)(v)| = \sup_{\substack{\varphi \in W', \|\varphi\| = 1 \\ v \in V, \|v\| = 1}} |\varphi(Av)|.$$

This shows also the continuity of A'. Note that the existence of sufficiently many non-trivial continuous linear functionals $\varphi \in W'$ is crucial for the validity of (3.1.31). $\square$

The last application we want to mention is the non-triviality of the tensor product of norms: Recall from Lemma 2.3.21 that on the (algebraic) tensor product $V_1 \otimes \cdots \otimes V_k$ of normed spaces $V_1, \ldots, V_k$ we have a seminorm $\|\cdot\|_{V_1 \otimes \cdots \otimes V_k}$ defined by

$$\|v\|_{V_1 \otimes \cdots \otimes V_k} = \inf \Big\{ \sum\nolimits_i \|v_1^i\|_{V_1} \cdots \|v_k^i\|_{V_k} \;\Big|\; \sum\nolimits_i v_1^i \otimes \cdots \otimes v_k^i = v \Big\}, \tag{3.1.32}$$

where the infimum is taken over all possibilities to express $v \in V_1 \otimes \cdots \otimes V_k$ as a sum of factorizing tensors. Without the Hahn-Banach theorem one can not say much more than that (3.1.32) is a seminorm. Using the Hahn-Banach theorem we get the following:

Proposition 3.1.27 *Let $V_1, \ldots, V_k$ be normed spaces.*

i.) Let $\varphi_1 \in V_1', \ldots, \varphi_k \in V_k'$ be continuous linear functionals. Then for all $v \in V_1 \otimes \cdots \otimes V_k$ one has
$$|(\varphi_1 \otimes \cdots \otimes \varphi_k)(v)| \leq \|\varphi_1\|_{V_1'} \cdots \|\varphi_k\|_{V_k'} \|v\|_{V_1 \otimes \cdots \otimes V_k}. \tag{3.1.33}$$

ii.) The seminorm (3.1.32) on $V_1 \otimes \cdots \otimes V_k$ is actually a norm.

iii.) For a factorizing tensor $v_1 \otimes \cdots \otimes v_k \in V_1 \otimes \cdots \otimes V_k$ one has
$$\|v_1 \otimes \cdots \otimes v_k\|_{V_1 \otimes \cdots \otimes V_k} = \|v_1\|_{V_1} \cdots \|v_k\|_{V_k}. \tag{3.1.34}$$

iv.) A k-linear map $\Phi \colon V_1 \times \cdots \times V_k \longrightarrow W$ into some other Banach space W is continuous iff the corresponding linear map $\phi \colon V_1 \otimes \cdots \otimes V_k \longrightarrow W$ is continuous.

Proof: Let $v \in V_1 \otimes \cdots \otimes V_k$ be given with a decomposition
$$v = \sum_{i=1}^{N} v_1^i \otimes \cdots \otimes v_k^i \tag{$*$}$$
into a sum of factorizing tensors. Then
$$\begin{aligned}|(\varphi_1 \otimes \cdots \otimes \varphi_k)(v)| &= \left|\sum_{i=1}^{N} \varphi_1(v_1^i) \cdots \varphi_k(v_k^i)\right| \\ &\leq \sum_{i=1}^{N} |\varphi_1(v_1^i)| \cdots |\varphi_k(v_k^i)| \\ &\leq \sum_{i=1}^{N} \|\varphi_1\|_{V_1'} \|v_1^i\|_{V_1} \cdots \|\varphi_k\|_{V_k'} \|v_k^i\|_{V_k} \\ &= \|\varphi_1\|_{V_1'} \cdots \|\varphi_k\|_{V_k'} \sum_{i=1}^{N} \|v_1^i\|_{V_1} \cdots \|v_k^i\|_{V_k}.\end{aligned}$$
Since this holds for all decompositions $(*)$ of v, we can take the infimum over all these decompositions yielding the estimate in (3.1.33). Next, let $v \in V_1 \otimes \cdots \otimes V_k$ with $v \neq 0$ be given. Again we can write v as in $(*)$. Then the spans
$$\begin{aligned}U_1 &= \operatorname{span}_{\mathbb{K}} \left\{v_1^1, \ldots, v_1^N\right\} \subseteq V_1 \\ &\vdots \\ U_k &= \operatorname{span}_{\mathbb{K}} \left\{v_k^1, \ldots, v_k^N\right\} \subseteq V_k\end{aligned}$$
are finite-dimensional by definition. Hence we find bases $\{\mathrm{e}_\alpha^{(1)}\}_{\alpha=1,\ldots,\dim U_1}, \ldots, \{\mathrm{e}_\alpha^{(k)}\}_{\alpha=1,\ldots,\dim U_k}$ of these subspaces with corresponding dual bases $\{\mathrm{e}^\alpha_{(1)}\}_{\alpha=1,\ldots,\dim U_1}, \ldots, \{\mathrm{e}^\alpha_{(k)}\}_{\alpha=1,\ldots,\dim U_k}$, respectively. Thus we can rewrite $(*)$ as
$$v = \sum_{\alpha_1=1}^{\dim U_1} \cdots \sum_{\alpha_k=1}^{\dim U_k} v^{\alpha_1 \cdots \alpha_k} \mathrm{e}_{\alpha_1}^{(1)} \otimes \cdots \otimes \mathrm{e}_{\alpha_k}^{(k)},$$
now with unique coefficients $v^{\alpha_1 \cdots \alpha_k} \in \mathbb{K}$, explicitly given by
$$v^{\alpha_1 \cdots \alpha_k} = \left(\mathrm{e}^{\alpha_1}_{(1)} \otimes \cdots \otimes \mathrm{e}^{\alpha_k}_{(k)}\right)(v). \tag{$**$}$$

Since we have finite-dimensional subspaces U_i, each $\mathrm{e}^{\alpha}_{(i)}$ is continuous by the uniqueness of the (norm) topology on U_i for $i = 1, \ldots, k$. Hence by the Hahn-Banach theorem we get a continuous extension $\varphi^{\alpha}_{(i)} \in V_i'$ for all $i = 1, \ldots, k$ and $\alpha = 1, \ldots, \dim U_i$, see again Corollary 3.1.6. Then $\varphi^{\alpha_1}_{(1)} \otimes \cdots \otimes \varphi^{\alpha_k}_{(k)}$ satisfies (3.1.33) and we have (∗∗). Together this shows

$$|v^{\alpha_1 \cdots \alpha_k}| \le \left\|\varphi^{\alpha_1}_{(1)}\right\|_{V_1'} \cdots \left\|\varphi^{\alpha_k}_{(k)}\right\|_{V_k'} \|v\|_{V_1 \otimes \cdots \otimes V_k}.$$

Hence from $\|v\|_{V_1 \otimes \cdots \otimes V_k} = 0$ we get $v^{\alpha_1 \cdots \alpha_k} = 0$ for all possible index combinations which means $v = 0$. This proves the second assertion. For the third part, let $v_1 \in V_1, \ldots, v_k \in V_k$ be given. Then we find continuous linear functionals with functional norm one such that

$$\varphi_1(v_1) = \|v_1\|_{V_1}, \quad \ldots, \quad \varphi_k(v_k) = \|v_k\|_{V_k},$$

again by Corollary 3.1.8. Thus for any decomposition $v_1 \otimes \cdots \otimes v_k = \sum_i v_1^i \otimes \cdots \otimes v_k^i$ we get

$$\begin{aligned}
\|v_1\|_{V_1} \cdots \|v_k\|_{V_k} &= \varphi_1(v_1) \cdots \varphi_k(v_k) \\
&= |\varphi_1(v_1) \cdots \varphi_k(v_k)| \\
&= |(\varphi_1 \otimes \cdots \otimes \varphi_k)(v_1 \otimes \cdots \otimes v_k)| \\
&= \left|(\varphi_1 \otimes \cdots \otimes \varphi_k)\Big(\sum\nolimits_i v_1^i \otimes \cdots \otimes v_k^i\Big)\right| \\
&= \left|\sum\nolimits_i \varphi_1(v_1^i) \cdots \varphi_k(v_k^i)\right| \\
&\le \sum\nolimits_i |\varphi_1(v_1^i)| \cdots |\varphi_k(v_k^i)| \\
&\le \sum\nolimits_i \|v_1^i\|_{V_1} \cdots \|v_k^i\|_{V_k},
\end{aligned}$$

since the functionals $\varphi_1, \ldots, \varphi_k$ have functional norm one. Taking the infimum over all possible decompositions gives the estimate

$$\|v_1\|_{V_1} \cdots \|v_k\|_{V_k} \le \|v_1 \otimes \cdots \otimes v_k\|_{V_1 \otimes \cdots \otimes V_k}.$$

As $v_1 \otimes \cdots \otimes v_k = v_1 \otimes \cdots \otimes v_k$ is one of the possible decompositions, the opposite estimate is trivially true as well, showing the third part. The last part is now easy. Suppose $\Phi\colon V_1 \times \cdots \times V_k \longrightarrow W$ is a k-linear map with corresponding linear map $\phi\colon V_1 \otimes \cdots \otimes V_k \longrightarrow W$, uniquely determined by

$$\phi(v_1 \otimes \cdots \otimes v_k) = \Phi(v_1, \ldots, v_k)$$

as usual. Suppose first that ϕ is continuous, then

$$\|\Phi(v_1, \ldots, v_k)\|_W = \|\phi(v_1 \otimes \cdots \otimes v_k)\|_W \le \|\phi\| \|v_1 \otimes \cdots \otimes v_k\|_{V_1 \otimes \cdots \otimes V_k} \le \|\phi\| \|v_1\|_{V_1} \cdots \|v_k\|_{V_k},$$

which shows the continuity of Φ according to Proposition 2.2.27 with operator norm

$$\|\Phi\| \le \|\phi\|,$$

see Proposition 2.2.29. Conversely, let Φ be continuous and $v = \sum_i v_1^i \otimes \cdots \otimes v_k^i \in V_1 \otimes \cdots \otimes V_k$. Then

$$\begin{aligned}
\|\phi(v)\|_W &= \left\|\phi\Big(\sum\nolimits_i v_1^i \otimes \cdots \otimes v_k^i\Big)\right\|_W \\
&= \left\|\sum\nolimits_i \phi(v_1^i \otimes \cdots \otimes v_k^i)\right\|_W \\
&= \left\|\sum\nolimits_i \Phi(v_1^i, \ldots, v_k^i)\right\|_W
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum\nolimits_i \|\Phi(v_1^i, \ldots, v_k^i)\|_W \\
&\leq \sum\nolimits_i \|\Phi\| \|v_1^i\|_{V_1} \cdots \|v_k^i\|_{V_k}.
\end{aligned}
$$

Since this holds for all possible decompositions of v, we can take the infimum over these decompositions. This yields

$$
\|\phi(v)\|_W \leq \|\Phi\| \|v\|_{V_1 \otimes \cdots \otimes V_k},
$$

which is the continuity of ϕ together with the operator norm estimate $\|\phi\| \leq \|\Phi\|$. □

Corollary 3.1.28 *Let $V_1, \ldots, V_k$ and W be normed spaces. For a continuous k-linear map $\Phi \in \mathrm{L}(V_1, \ldots, V_k; W)$ with corresponding continuous linear map $\phi \in \mathrm{L}(V_1 \otimes \cdots \otimes V_k, W)$ one has*

$$
\|\Phi\| = \|\phi\|. \tag{3.1.35}
$$

Hence the canonical isomorphism

$$
\mathrm{L}(V_1, \ldots, V_k; W) \cong \mathrm{L}(V_1 \otimes \cdots \otimes V_k, W) \tag{3.1.36}
$$

is an isomorphism of normed spaces.

Proof: Proposition 3.1.27, *iv.)*, shows that we have a canonical identification (3.1.36) from the linear-algebraic identification

$$
\mathrm{Hom}(V_1, \ldots, V_k; W) \cong \mathrm{Hom}(V_1 \otimes \cdots \otimes V_k, W).
$$

The estimate (3.1.35), which was obtained in the proof of Proposition 3.1.27, *iv.)*, then gives the second claim. □

Definition 3.1.29 (Projective tensor product) *The tensor product $V_1 \otimes \cdots \otimes V_k$ of normed spaces $V_1, \ldots, V_k$ with the norm* (3.1.32) *is called the projective tensor product of $V_1, \ldots, V_k$. We write $V_1 \otimes_\pi \cdots \otimes_\pi V_k$ if we equip the tensor product with this norm.*

Remark 3.1.30 (Cross norms) While this proposition gives now a decent norm on the tensor product of normed spaces, it turns out that there are many other norms possible such that the factorizing property

$$
\|v_1 \otimes \cdots \otimes v_k\|_{V_1 \otimes \cdots \otimes V_k} = \|v_1\|_{V_1} \cdots \|v_k\|_{V_k} \tag{3.1.37}
$$

holds. In fact, we will see important examples when we discuss Hilbert spaces, where another norm with (3.1.37) shows up equally naturally, though yielding a different topology. Norms on the tensor product of normed spaces with the property (3.1.37) are sometimes also called *cross norms*.

3.2 Polars and the Banach-Alaoglu Theorem

The next important theorem of general character is the Banach-Alaoglu theorem which states that the closed unit ball in the topological dual space of a normed space is compact in a topology which is coarser than the norm topology but still has useful features. This weak* topology will be important in many other situations, too. In the course of these investigations we will meet a particular type of Banach spaces, the reflexive Banach spaces, where information about the weak* topology can be turned into information about the weak topology we already know.

3.2.1 Polars

The notion of the *polar* of a subset uses only the topology of a topological vector space, so that we can state the following definition in the general form:

Definition 3.2.1 (Polar) *Let V be a topological vector space with topological dual V'.*

i.) The polar $A^{\circledast}$ of a subset $A \subseteq V$ is defined by

$$A^{\circledast} = \left\{\varphi \in V' \;\middle|\; |\varphi(v)| \leq 1 \textit{ for all } v \in A\right\}. \tag{3.2.1}$$

ii.) The polar $B_{\circledast}$ of a subset $B \subseteq V'$ is defined by

$$B_{\circledast} = \left\{v \in V \;\middle|\; |\varphi(v)| \leq 1 \textit{ for all } \varphi \in B\right\}. \tag{3.2.2}$$

Note that this definition is *not* completely symmetric: the polar of a subset of V' is a subset of V and not of V''. In principle, for a normed space V, we can treat V' as a Banach space itself and consider $B^{\circledast} \subseteq (V')' = V''$ in the sense of the first part of the definition as a subset of the bidual of V. We try to indicate this difference by placing $\circledast$ at the bottom. A more careful notion would be to call $B_{\circledast}$ e.g. the *pre-polar* of B, which unfortunately is not common at all. For a general topological vector space we do not even have a topology on V' yet for which we could speak about V''. Note also that sometimes the notation A° is used for the polar. This would conflict with our notation for the open interior of A, and hence we use a more matching symbol for the polar. Finally, we also note that one can define polars in a bigger context of dual pairings, where the above to versions become more symmetric, see Exercise **??**.

We collect a few first formal and simple properties of polars of subsets in a topological vector space.

Proposition 3.2.2 *Let V be a topological vector space.*

i.) For $A \subseteq V$ one has $A \subseteq (A^{\circledast})_{\circledast}$.

ii.) For $A \subseteq B \subseteq V$ one has $B^{\circledast} \subseteq A^{\circledast}$.

iii.) For any set I and subsets $A_i \subseteq V$ for $i \in I$ one has

$$\left(\bigcup_{i \in I} A_i\right)^{\circledast} = \bigcap_{i \in I} A_i^{\circledast}. \tag{3.2.3}$$

iv.) One has $\emptyset^{\circledast} = \{0\}^{\circledast} = V'$.

v.) For $z \in \mathbb{K}^{\times}$ and $A \subseteq V$ one has $(zA)^{\circledast} = \frac{1}{z} A^{\circledast}$.

vi.) For a subspace $W \subseteq V$ one has

$$W^{\circledast} = \left\{\varphi \in V' \;\middle|\; \varphi\big|_W = 0\right\} = W^{\mathrm{ann}}, \tag{3.2.4}$$

i.e. the topological annihilator of W from Exercise 2.5.12.

vii.) If $0 \in A \cap B$ for $A, B \subseteq V$ one has

$$A^{\circledast} \cap B^{\circledast} \subseteq 2(A+B)^{\circledast} \subseteq 2(A^{\circledast} \cap B^{\circledast}). \tag{3.2.5}$$

viii.) For $A \subseteq V$ and $B \subseteq V'$ one has

$$A \subseteq B_{\circledast} \quad \textit{iff} \quad A^{\circledast} \supseteq B. \tag{3.2.6}$$

PROOF: The first two statements follow directly from the definition. For the next statement, let $\varphi \in \left(\bigcup_{i\in I} A_i\right)^{\circledast}$, i.e. we have $|\varphi(v_i)| \leq 1$ for all $v_i \in A_i$ and all $i \in I$. It follows that $\varphi \in A_i^{\circledast}$ for all $i \in I$ which is "$\subseteq$" in (3.2.3). Conversely, suppose $\varphi \in \bigcap_{i\in I} A_i^{\circledast}$ then $|\varphi(v_i)| \leq 1$ for all $v_i \in A_i$ and $i \in I$ which means $\varphi \in \left(\bigcup_{i\in I} A_i\right)^{\circledast}$. For *iv.)* we just need the definition again. Since the pairing of $\varphi \in V'$ and $v \in V$ is bilinear we have $|\varphi(zv)| = |(z\varphi)(v)|$ from which *v.)* follows at once. Next let $W \subseteq V$ be a subspace and $\varphi \in W^{\circledast}$. Then $|\varphi(w)| \leq 1$ for all $w \in W$. Since with $w \in W$ also $\lambda w \in W$ for all $\lambda > 0$ we conclude $\lambda|\varphi(w)| = |\varphi(\lambda w)| \leq 1$ which implies $\varphi(w) = 0$. The converse inclusion in (3.2.4) is trivial. Now let $\varphi \in A^{\circledast} \cap B^{\circledast}$ then $|\varphi(v+w)| \leq |\varphi(v)| + |\varphi(w)| \leq 2$ for all $v \in A$ and $w \in B$. This shows $\frac{1}{2}\varphi \in (A+B)^{\circledast}$. Next, let $\varphi \in (A+B)^{\circledast}$ then $|\varphi(v+w)| \leq 1$ for all $v \in A$ and $w \in B$. Since $0 \in A \cap B$ by assumption, we can set either $v = 0$ or $w = 0$ to obtain $|\varphi(v)| \leq 1$ and $|\varphi(w)| \leq 1$, respectively. This shows $\varphi \in A^{\circledast} \cap B^{\circledast}$ so that a rescaling by 2 gives the remaining inclusion in (3.2.5). Finally, $A \subseteq B_{\circledast}$ means that for all $v \in A$ and all $\varphi \in B$ one has $|\varphi(v)| \leq 1$ which is $B \subseteq A^{\circledast}$. $\square$

It is now a straightforward verification that analogous statements also hold for the polars of subsets in V', see Exercise 3.5.12 for details.

Polars of subsets in V' turn out to be weakly closed and absolutely convex: recall that a subset of a vector space is called *absolutely convex* if it is balanced and convex, see again Exercise 3.5.1.

Proposition 3.2.3 *Let V be a topological vector space and let $B \subseteq V'$. Then $B_{\circledast} \subseteq V$ is absolutely convex and weakly closed.*

PROOF: Recall that the definition of the weak topology on a vector space V only requires the knowledge of the topological dual V'. In particular, we have the weak topology for every topological vector space, even though it might fail to be Hausdorff in general, see also Exercise 3.5.6. Let $z \in \mathbb{K}$ with $|z| \leq 1$ be given, then for $v \in B_{\circledast}$ we have for all $\varphi \in B$

$$|\varphi(zv)| = |z||\varphi(v)| \leq |z| \leq 1.$$

Hence $zv \in B_{\circledast}$, showing that the polar $B_{\circledast}$ is balanced. Analogously, for $\lambda \in [0,1]$ and $v, w \in B_{\circledast}$ we have for all $\varphi \in B$

$$|\varphi(\lambda v + (1-\lambda)w)| = |\lambda\varphi(v) + (1-\lambda)\varphi(w)| \leq \lambda|\varphi(v)| + (1-\lambda)|\varphi(w)| \leq 1.$$

Thus $\lambda v + (1-\lambda)w \in B_{\circledast}$ and $B_{\circledast}$ is convex. Finally, we observe that by definition of the polar

$$B_{\circledast} = \left\{v \in V \;\middle|\; |\varphi(v)| \leq 1 \text{ for all } \varphi \in B\right\} = \bigcap_{\varphi \in B} \mathrm{p}_{\varphi}^{-1}([0,1]),$$

where $\mathrm{p}_{\varphi}(v) = |\varphi(v)|$ as usual. Since the seminorms p_{φ} of the weak topology are weakly continuous, $B_{\circledast}$ is weakly closed. $\square$

In particular, the polar is also closed in any finer topology than the weak topology, like e.g. the original norm topology in case of a normed space V.

Before we can formulate the appropriate closedness statements for polars $A^{\circledast} \subseteq V'$ of subsets $A \subseteq V$, we need to equip V' with the correct topology. If V is a normed space we have (at least) two options: we can take the weak topology of V', viewed as Banach space with respect to its functional norm. This is the topology induced by seminorms

$$\mathrm{p}_X(\varphi) = |X(\varphi)| \quad \text{for } X \in V'' \tag{3.2.7}$$

as in Definition 3.1.17 with V being replaced by V' throughout. It turns out that this is not quite adequate here. In the normed case we know that canonically $V \subseteq V''$ via the evaluation maps. Hence we can use the seminorms (3.2.7) only for those $X \in V''$ coming from V, i.e. for $X = \mathrm{ev}_v$ with $v \in V$. This results in the seminorms

$$\mathrm{p}_v(\varphi) = |\mathrm{ev}_v(\varphi)| = |\varphi(v)|, \tag{3.2.8}$$

now viewed as seminorms on V' labeled by $v \in V$. This can now be done whether V was normed or not:

Definition 3.2.4 (Weak* topology) *Let V be a vector space.*

i.) For $v \in V$ one defines the seminorm p_v on V^* by

$$\mathrm{p}_v(\varphi) = |\varphi(v)| \tag{3.2.9}$$

for $\varphi \in V^*$ and calls

$$\mathrm{B}_{\mathrm{p}_v,r}(\varphi) = \left\{\psi \in V^* \;\middle|\; \mathrm{p}_v(\varphi - \psi) < r\right\} \subseteq V^* \tag{3.2.10}$$

the open ball with respect to p_v around φ of radius $r > 0$.

ii.) The weak topology on V^* is the unique topology generated by all the open balls* (3.2.10) *for $\varphi \in V^*$ and $r > 0$ as well as $v \in V$.*

iii.) If V is in addition a topological vector space we use the same construction for the topological dual $V' \subseteq V^*$ by restricting the seminorms p_v to V' with corresponding open balls

$$\mathrm{B}_{\mathrm{p}_v,r}(\varphi) = \left\{\psi \in V' \;\middle|\; \mathrm{p}_v(\varphi - \psi) < r\right\} \subseteq V' \tag{3.2.11}$$

for $\varphi \in V'$ and $r > 0$.

With other words, the weak* topology of V' is the subspace topology inherited from the weak* topology of V^*, see also Exercise 1.5.3. Note that the definition of the weak* topology on the algebraic dual V^* requires no topological information on V but is completely intrinsic. The following properties of the weak* topology are shown exactly the same way as the corresponding ones for the weak topology in Proposition 3.1.18 and Proposition 3.1.19:

Proposition 3.2.5 *Let V be a vector space.*

i.) The weak topology turns V^* into a topological vector space.*

ii.) The seminorms p_v for $v \in V$ are continuous with respect to the weak topology.*

iii.) The weak topology is Hausdorff.*

iv.) A net $(\varphi_i)_{i\in I}$ in V^* is weak convergent to $\varphi \in V^*$ iff for every $v \in V$ one has*

$$\lim_{i\in I} \varphi_i(v) = \varphi(v), \tag{3.2.12}$$

i.e. $(\varphi_i)_{i\in I}$ converges pointwise to φ on V.

v.) A net $(\varphi_i)_{i\in I}$ in V^* is a weak Cauchy net iff for all $v \in V$ the net $(\varphi_i(v))_{i\in I}$ is a Cauchy net in $\mathbb{K}$.*

Note that the Hausdorff property is now entirely trivial and not relying on the Hahn-Banach theorem since for $\varphi \neq 0$ we have a vector $v \in V$ with $\varphi(v) \neq 0$ by the definition of linear functionals. Then p_v allows to separate this φ from 0. In case of a topological vector space these properties are inherited by the topological dual $V' \subseteq V^*$.

With the help of the weak* topology we can now formulate the appropriate closedness of polars in V' as follows with an identical argumentation as in Proposition 3.2.3.

Proposition 3.2.6 *Let V be a topological vector space with $A \subseteq V$. Then the polar $A^{\circledast} \subseteq V'$ is absolutely convex and weak* closed.*

The importance is now that for a normed space V the weak* topology is coarser than the norm topology of V':

Proposition 3.2.7 *Let V be a normed space. Then the weak* topology is coarser than the norm topology of V'.*

PROOF: For every $v \in V$ and $\varphi \in V'$ one has

$$\mathrm{p}_v(\varphi) = |(\varphi)(v)| \leq \|v\|\|\varphi\|.$$

Hence we can estimate p_v by means of the functional norm of V'. With this we can continue to argue as in Proposition 3.1.18, *ii.)*. $\square$

As a consequence of this and Proposition 3.2.6, polars $A^{\circledast} \subseteq V'$ of subsets $A \subseteq V$ will be norm-closed, too.

We can now formulate the main theorem of this section for a normed space:

Theorem 3.2.8 (Bipolar theorem) *Let V be a normed space and let $A \subseteq V$ be a non-empty subset. Then the bipolar $(A^{\circledast})_{\circledast} \subseteq V$ coincides with the (weak) closure of the absolutely convex hull of A, i.e. we have*

$$(A^{\circledast})_{\circledast} = (\mathrm{absconv}(A))^{\mathrm{wcl}} = (\mathrm{absconv}(A))^{\mathrm{cl}}. \tag{3.2.13}$$

PROOF: By definition, $\mathrm{absconv}(A)$ is convex. Thus the weak closure coincides with the norm closure according to Proposition 3.1.20, showing the second equality. The polar $(A^{\circledast})_{\circledast}$ is absolutely convex and weakly closed according to Proposition 3.2.3. Thus $A \subseteq (A^{\circledast})_{\circledast}$ from Proposition 3.2.2, *i.)*, shows $\mathrm{absconv}(A) \subseteq \mathrm{absconv}((A^{\circledast})_{\circledast}) = (A^{\circledast})_{\circledast}$ and hence

$$\mathrm{absconv}(A)^{\mathrm{wcl}} \subseteq ((A^{\circledast})_{\circledast})^{\mathrm{wcl}} = (A^{\circledast})_{\circledast},$$

which is "$\supseteq$" in (3.2.13). For the opposite inclusion assume $v \in V$ is not in $\mathrm{absconv}(A)^{\mathrm{wcl}} = \mathrm{absconv}(A)^{\mathrm{cl}}$. From the Hahn-Banach theorem in form of Corollary 3.1.16 we obtain a continuous linear functional $\phi \in V'$ separating v from $\mathrm{absconv}(A)^{\mathrm{wcl}}$, i.e. we have

$$|\phi(u)| \leq 1 \quad \text{for } u \in \mathrm{absconv}(A)^{\mathrm{wcl}}$$

but $\phi(v) > 1$. Hence $\phi \in (\mathrm{absconv}(A)^{\mathrm{cl}})^{\circledast} \subseteq A^{\circledast}$ by the definition of the polar and Proposition 3.2.2, *ii.)*. Since $\phi(v) > 1$ we conclude $v \notin (A^{\circledast})_{\circledast}$, which shows "$\subseteq$" in (3.2.13), completing the proof. $\square$

Corollary 3.2.9 *Let V be a normed space. Then for all $r > 0$ one has*

$$(\mathrm{B}_r(0)^{\circledast})_{\circledast} = \mathrm{B}_r(0)^{\mathrm{cl}}. \tag{3.2.14}$$

PROOF: Indeed, the ball $\mathrm{B}_r(0)$ is clearly absolutely convex already. $\square$

One may wonder if one can enlarge bipolars further by continuity, taking polars over and over again. It turns out that this will not lead to new subsets:

Corollary 3.2.10 *Let V be a normed space with $A \subseteq V$. Then*

$$((A^{\circledast})_{\circledast})^{\circledast} = A^{\circledast} \tag{3.2.15}$$

and

$$A^{\circledast} = (\mathrm{absconv}(A)^{\mathrm{cl}})^{\circledast}. \tag{3.2.16}$$

PROOF: For the first part one only needs the properties of Proposition 3.2.2, *i.)* and *ii.)*. Indeed, we have $A \subseteq (A^{\circledast})_{\circledast}$ and hence $((A^{\circledast})_{\circledast})^{\circledast} \subseteq A^{\circledast}$. Conversely, we replace A by $A^{\circledast}$ and use the analogous inclusion $A^{\circledast} \subseteq ((A^{\circledast})_{\circledast})^{\circledast}$ to the first part for subsets of V', see also Exercise 3.5.12 and Exercise 3.5.15. This gives the first equality (3.2.15). For the second we use the bipolar theorem to obtain

$$(\operatorname{absconv}(A)^{\mathrm{cl}})^{\circledast} = ((A^{\circledast})_{\circledast})^{\circledast} = A^{\circledast}. \qquad \square$$

3.2.2 The Banach-Alaoglu Theorem

One of the main reasons to consider the very coarse weak* topology is that it admits interesting compact subsets. To appreciate this we first note the following non-compactness statement:

Theorem 3.2.11 (Riesz) *Let V be a normed space. Then the closed unit ball $\mathrm{B}_1(0)^{\mathrm{cl}}$ is compact iff V is finite-dimensional.*

PROOF: Consider the open cover $\{\mathrm{B}_{\frac{1}{2}}(v)\}_{v \in \mathrm{B}_1(0)^{\mathrm{cl}}}$ of $\mathrm{B}_1(0)^{\mathrm{cl}}$ which has a finite subcover under the assumption of compactness. Thus

$$\begin{aligned} \mathrm{B}_1(0)^{\mathrm{cl}} &\subseteq \mathrm{B}_{\frac{1}{2}}(v_1) \cup \cdots \cup \mathrm{B}_{\frac{1}{2}}(v_n) \\ &= (v_1 + \mathrm{B}_{\frac{1}{2}}(0)) \cup \cdots \cup (v_n + \mathrm{B}_{\frac{1}{2}}(0)) \\ &\subseteq \operatorname{span}_{\mathbb{K}}\{v_1, \ldots, v_n\} + \mathrm{B}_{\frac{1}{2}}(0) \\ &\subseteq \operatorname{span}_{\mathbb{K}}\{v_1, \ldots, v_n\} + \mathrm{B}_{\frac{1}{2}}(0)^{\mathrm{cl}} \end{aligned} \tag{$*$}$$

for some suitable choice $v_1, \ldots, v_n \in \mathrm{B}_1(0)^{\mathrm{cl}}$. We denote their span by

$$W = \operatorname{span}_{\mathbb{K}}\{v_1, \ldots, v_n\} \subseteq V.$$

From $(*)$ we get

$$\mathrm{B}_{\frac{1}{2}}(0)^{\mathrm{cl}} \subseteq \frac{1}{2}W + \frac{1}{2}\mathrm{B}_{\frac{1}{2}}(0)^{\mathrm{cl}} = W + \mathrm{B}_{\frac{1}{4}}(0)^{\mathrm{cl}},$$

since W is a vector space and hence

$$\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq W + \mathrm{B}_{\frac{1}{2}}(0)^{\mathrm{cl}} \subseteq W + W + \mathrm{B}_{\frac{1}{4}}(0)^{\mathrm{cl}} = W + \mathrm{B}_{\frac{1}{4}}(0)^{\mathrm{cl}},$$

again using that W is a vector space. By induction this gives

$$\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq W + \mathrm{B}_{2^{-n}}(0)^{\mathrm{cl}}$$

for all $n \in \mathbb{N}$ and thus

$$\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq \bigcap_{n=1}^{\infty} (W + \mathrm{B}_{2^{-n}}(0)^{\mathrm{cl}}).$$

We claim that

$$\bigcap_{n=1}^{\infty} (W + \mathrm{B}_{2^{-n}}(0)^{\mathrm{cl}}) = W^{\mathrm{cl}} \tag{$**$}$$

holds for all subspaces $W \subseteq V$. Indeed, let w be a vector in the subset on the left. Then for all $n \in \mathbb{N}$ one finds $w_n \in W$ and $u_n \in \mathrm{B}_{2^{-n}}(0)^{\mathrm{cl}}$ with $w = w_n + u_n$. Hence $w_n = w - u_n \longrightarrow w$ for $n \to \infty$, showing $w \in W^{\mathrm{cl}}$. Conversely, if $w \in W^{\mathrm{cl}}$ we find $w_n \in W$ with $\|w - w_n\| < 2^{-n}$ and hence $w = w_n + (w - w_n) \in W + \mathrm{B}_{2^{-n}}(0)^{\mathrm{cl}}$, showing $(**)$ in general. Back to our situation, we have $W = W^{\mathrm{cl}}$ since $\dim W \leq n$ is finite-dimensional. It follows that $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq W$ and hence $V = W$. $\square$

Thus the familiar Heine-Borel property, namely that bounded and closed subsets of $\mathbb{K}^n$ coincide with the compact subsets, fails immediately in infinite dimensional normed spaces. From this point of view, the next statement is quite surprising:

Theorem 3.2.12 (Banach-Alaoglu) *Let $U \subseteq V$ be a neighbourhood of zero in a normed space. Then the polar $U^{\circledast} \subseteq V'$ is compact with respect to the weak* topology.*

Proof: Since U is a neighbourhood of zero, it contains a closed ball $\mathrm{B}_r(0)^{\mathrm{cl}} \subseteq U$ for some $r > 0$. Then $U^{\circledast} \subseteq (\mathrm{B}_r(0)^{\mathrm{cl}})^{\circledast}$ by Proposition 3.2.2, *ii.)*, and both polars are closed with respect to the weak* topology by Proposition 3.2.6. To show their compactness it is sufficient to show the compactness of the larger one $(\mathrm{B}_r(0)^{\mathrm{cl}})^{\circledast}$. We consider the topological dual now as a subset $V' \subseteq \mathrm{Map}(V, \mathbb{K}) = \mathbb{K}^V$ and claim that the weak* topology is just the Cartesian product topology. To this end, we recall that

$$\mathrm{B}_{\mathrm{p}_v, r}(\varphi) = \big\{\psi \in V' \bigm| |\varphi(v) - \psi(v)| < r\big\} = V' \cap \mathrm{pr}_v^{-1}(\mathrm{B}_r(\varphi(v)))$$

where $\mathrm{pr}_v \colon \mathbb{K}^V \ni (\varphi(u))_{u \in V} \mapsto \varphi(v)$ is the v-th component and where we identify a map $\varphi \colon V \longrightarrow \mathbb{K}$ with the indexed set $(\varphi(u))_{u \in V}$ of its values to obtain $\varphi \in \mathbb{K}^V$ as usual. Thus the open balls in the weak* topology are exactly the preimages of open balls in $\mathbb{K}$ under one of the projections of the Cartesian product. Since the latter form a subbasis of the product topology and the former a subbasis of the weak* topology on V', the claim is shown. Now the proof can be finished easily: the polar $(\mathrm{B}_r(0)^{\mathrm{cl}})^{\circledast}$ becomes

$$\begin{aligned}
(\mathrm{B}_r(0)^{\mathrm{cl}})^{\circledast} &= \Big\{\varphi \in V' \Bigm| |\varphi(v)| \le 1 \text{ for all } v \in \mathrm{B}_r(0)^{\mathrm{cl}}\Big\} \\
&\overset{(a)}{=} \Big\{\varphi \in V' \Bigm| |\varphi(v)| \le r \text{ for all } v \in \mathrm{B}_1(0)^{\mathrm{cl}}\Big\} \\
&\overset{(b)}{=} \big\{\varphi \in V' \bigm| |\varphi(r)| \le r\|v\| \text{ for all } v \in V\big\},
\end{aligned}$$

where in (a) we just rescale the condition by r. For (b), we note that for $v \neq 0$ we have $\frac{v}{\|v\|} \in \mathrm{B}_1(0)^{\mathrm{cl}}$ and hence $\big|\varphi(\frac{v}{\|v\|})\big| \le 1$, too, which gives $|\varphi(v)| \le r\|v\|$ for all $v \in V \setminus \{0\}$ and for $v = 0$ we have $|\varphi(0)| = 0 \le r\|0\|$ anyway. Hence

$$\begin{aligned}
(\mathrm{B}_r(0)^{\mathrm{cl}})^{\circledast} &= \big\{\varphi \in V' \bigm| |\varphi(v)| \le r\|v\| \text{ for all } v \in V\big\} \\
&= V' \cap \Big\{\varphi \in \mathrm{Map}(V, \mathbb{K}) \Bigm| \varphi(v) \in \mathrm{B}_{r\|v\|}(0)^{\mathrm{cl}} \text{ for all } v \in V\Big\} \\
&= V^* \cap \underbrace{\prod_{v \in V} \mathrm{B}_{r\|v\|}(0)^{\mathrm{cl}}}_{K},
\end{aligned}$$

where we interpret the condition for all $v \in V$ now as being in the Cartesian product of closed balls $\mathrm{B}_{r\|v\|}(0)^{\mathrm{cl}} \subseteq \mathbb{K}$. Since these closed balls are compact, this Cartesian product is compact, too, by Tikhonov's theorem, see Theorem **??**. Note that the intersection of K with linear maps V^* automatically yields continuous linear maps in V', i.e.

$$V^* \cap K = V' \cap K.$$

Finally, it is easy to see that the algebraic dual $V^* \subseteq \mathrm{Map}(V, \mathbb{K})$ is closed with respect to the product topology: indeed, let $(\varphi_i)_{i \in I}$ be a net of linear functionals converging to some $\varphi \in \mathrm{Map}(V, \mathbb{K})$. Then

$$\begin{aligned}
\varphi(\alpha v + \beta w) &= \lim_{i \in I} \varphi_i(\alpha v + \beta w) \\
&= \lim_{i \in I} \alpha\varphi_i(v) + \beta\varphi_i(w)
\end{aligned}$$

$$= \alpha \lim_{i \in I} \varphi_i(v) + \beta \lim_{i \in I} \varphi_i(w)$$
$$= \alpha\varphi(v) + \beta\varphi(w),$$

since convergence in the product topology is componentwise convergence. It follows that $V^* \cap K$ is still compact. ☐

As particular case we obtain the following statement of compactness.

Corollary 3.2.13 *Let V be a normed space. Then the closed unit ball $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V'$ in the topological dual of V is compact in the weak* topology.*

Proof: From the definition of a polar we get

$$\mathrm{B}_1(0)^{\mathrm{cl}} = (\mathrm{B}_1(0)^{\mathrm{cl}})^{\circledast},$$

where one ball is in V and the other in V'. This follows at once from

$$\|\varphi\| = \sup_{v \in \mathrm{B}_1(0)^{\mathrm{cl}}} |\varphi(v)|. \qquad \square$$

Remark 3.2.14 Here we see the big difference between the norm topology on V' and the weak* topology in infinite dimensions, as $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V'$ is of course *not* compact in the norm topology according to Theorem 3.2.11.

Remark 3.2.15 Note that the usage of Tikhonov's theorem gives (covering) compactness of $\mathrm{B}_1(0)^{\mathrm{cl}}$ in general, but not necessarily sequential compactness. In fact, one can not improve the Banach-Alaoglu theorem in this direction without further assumptions. This will sometimes limit the effectiveness of Theorem 3.2.12.

3.3 The Principle of Uniform Boundedness

The next big complex of questions is about boundedness in various situations. After clarifying what bounded subsets in normed and, more generally, topological vector spaces are, we formulate the fundamental theorem of Banach-Steinhaus on uniform boundedness. This will have many important applications not only to the norm topology but also to the weak and weak* topology.

3.3.1 Bounded Subsets of Normed Spaces

A subset $B \subseteq V$ in a normed space is *bounded* in the metric sense if there is a $R > 0$ with

$$B \subseteq \mathrm{B}_R(0). \tag{3.3.1}$$

We can reformulate this now as follows, only making use of zero neighbourhoods:

Proposition 3.3.1 *Let V be a normed space and let $B \subseteq V$. Then B is bounded iff for every zero neighbourhood $U \subseteq V$ one finds $r > 0$ with*

$$B \subseteq rU. \tag{3.3.2}$$

Proof: Any neighbourhood of zero contains a small ball $\mathrm{B}_\varepsilon(0) \subseteq U$. Hence scaling this ball sufficiently gives the equivalence of (3.3.2) with (3.3.1). ☐

This simple observation is now the motivation for the following general definition which is consistent with the previous definition of boundedness in the normed case thanks to Proposition 3.3.1:

Definition 3.3.2 (Bounded subset) *Let V be a topological vector space. A subset $B \subseteq V$ is called bounded if for every zero neighbourhood $U \subseteq V$ there is an $r > 0$ with $B \subseteq rU$.*

While being applicable in general, we are interested mainly in the cases of the weak topology on V and the weak* topology on V':

Proposition 3.3.3 *Let V be a normed space.*

i.) A subset $B \subseteq V$ is weakly bounded iff for every $\varphi \in V'$ the subset

$$\varphi(B) \subseteq \mathbb{K} \tag{3.3.3}$$

is bounded.

ii.) A subset $B \subseteq V'$ is bounded in the weak topology iff for every $v \in V$ the subset*

$$\big\{\varphi(v) \bigm| \varphi \in B\big\} \subseteq \mathbb{K} \tag{3.3.4}$$

is bounded.

PROOF: Let $U \subseteq V$ be a weak zero neighbourhood. By the definition of the weak topology it contains an intersection

$$O = \mathrm{B}_{\mathrm{p}_{\varphi_1},1}(0) \cap \cdots \cap \mathrm{B}_{\mathrm{p}_{\varphi_n},1}(0) \tag{$*$}$$

of open unit balls for the seminorms $\mathrm{p}_{\varphi_1}, \ldots, \mathrm{p}_{\varphi_n}$ with $\varphi_1, \ldots, \varphi_n \in V'$ since such open balls provide a subbasis. Note that we can assume to have unit balls since we can rescale the functionals φ_i accordingly if needed since $\mathrm{B}_{\mathrm{p},r}(0) = \mathrm{B}_{\frac{1}{r}\mathrm{p},1}(0)$. First suppose that B is weakly bounded. Then for every $\mathrm{B}_{\mathrm{p}_\varphi,1}(0)$ we find $r_\varphi > 0$ with $B \subseteq r_\varphi \mathrm{B}_{\mathrm{p}_\varphi,1}(0) = \mathrm{B}_{\mathrm{p}_\varphi,r_\varphi}(0)$ and thus

$$\mathrm{p}_\varphi(v) < r_\varphi$$

for all $v \in B$, which shows (3.3.3). Conversely, suppose (3.3.3) holds. Then for U with O as in ($*$) we have an $r > 0$ with

$$B \subseteq r\mathrm{B}_{\mathrm{p}_{\varphi_i},1}(0) = \mathrm{B}_{\mathrm{p}_{\varphi_i},r}(0)$$

by taking the maximum of the $r_{\varphi_1}, \ldots, r_{\varphi_n}$ with

$$\varphi_i(B) \subseteq \mathrm{B}_{r_{\varphi_i}}(0) \subseteq \mathbb{K}.$$

It follows that $B \subseteq rO \subseteq rU$. The second statement is shown analogously. ☐

Remark 3.3.4 Equivalently we can say that a subset is bounded iff all the seminorms defining the weak and weak* topology, respectively, are bounded on B. This characterization is still valid in much larger generality whenever the topology is coming from seminorms.

3.3.2 The Banach-Steinhaus Theorem

The core of this section is the following theorem of Banach-Steinhaus which is a simple application of the principle of uniform boundedness as in Proposition **??**. Hence it relies crucially on Baire's theorem:

Theorem 3.3.5 (Banach-Steinhaus) *Let V be a Banach space and let W be a normed space. Suppose $\mathcal{A} \subseteq \mathrm{L}(V, W)$ is a set of continuous linear maps which is pointwise bounded, i.e. for every $v \in V$ one has*

$$\sup_{A \in \mathcal{A}} \|Av\| < \infty. \tag{3.3.5}$$

Then $\mathcal{A}$ is bounded in the operator norm

$$\sup_{A \in \mathcal{A}} \|A\| < \infty. \tag{3.3.6}$$

Proof: For every $A \in \mathcal{A}$ the map

$$V \ni v \;\mapsto\; \|Av\| \in \mathbb{R} \tag{$*$}$$

is continuous. The assumption (3.3.5) then shows that $(*)$ is pointwise bounded. Since V is a complete metric space it is a Baire space by Theorem **??**, so that Proposition **??** applies. There exists a non-empty open subset $O \subseteq V$ with

$$c = \sup_{A \in \mathcal{A}} \sup_{v \in O} \|Av\| < \infty.$$

Without restriction, $O = \mathrm{B}_r(v_0)$ for some $r > 0$ and $v_0 \in V$. Then we have

$$\|Av\| \leq \|Av_0\| + \|A(v - v_0)\| \leq \|Av_0\| + c \leq c_0 + c$$

for all $v \in \mathrm{B}_r(0)$, where

$$c_0 = \sup_{A \in \mathcal{A}} \|Av_0\|.$$

It follows that

$$\|v\| < r \implies \|Av\| \leq c_0 + c.$$

This implies

$$\|A\| \leq \frac{c_0 + c}{r},$$

independently of A. Hence (3.3.6) follows. □

This theorem has many very useful consequences. The first is the following:

Theorem 3.3.6 (Boundedness and weak boundedness) *Let V be a normed space. Then $B \subseteq V$ is bounded iff it is weakly bounded.*

Proof: A bounded subset is clearly bounded in the weak topology, too, since every weak neighbourhood of zero is also a neighbourhood of zero in the norm topology. We need to show the opposite implication. Thus let $B \subseteq V$ be weakly bounded. Thus for all $\varphi \in V'$ we have

$$\sup_{v \in B} |\varphi(v)| = \sup_{v \in B} |\mathrm{ev}_v(\varphi)| < \infty.$$

Applying the Banach-Steinhaus theorem to the continuous linear maps $\{\mathrm{ev}_v\}_{v \in B} \subseteq V''$ and the Banach space V' gives the norm boundedness

$$\sup_{v \in B} \|\mathrm{ev}_v\| < \infty.$$

Since $\|\mathrm{ev}_v\| = \|v\|$ according to Proposition 3.1.22, the norm-boundedness of B follows. □

Note that it is typically much easier to check weak boundedness as this corresponds to "componentwise" boundedness in finite dimensions. This makes Theorem 3.3.6 a valuable tool in many situations. A first application is the following:

Corollary 3.3.7 *Let V be a normed space. If $(v_n)_{n\in\mathbb{N}}$ is a sequence in V converging weakly to $v \in V$ then $(v_n)_{n\in\mathbb{N}}$ is norm bounded.*

PROOF: Weak convergence means that for every $\varphi \in V'$ we have $\varphi(v_n) \longrightarrow \varphi(v)$ according to Proposition 3.1.19, *i.)*. Hence the sequence $(\varphi(v_n))_{n\in\mathbb{N}}$ of scalars converges and is therefore bounded. This means that the sequence $(v_n)_{n\in\mathbb{N}}$ is weakly bounded, hence bounded by Theorem 3.3.6. $\square$

This corollary will then give yet another reformulation of the Banach-Steinhaus theorem:

Corollary 3.3.8 *Let V be a Banach space and let W be a normed space. Suppose $\mathcal{A} \subseteq \mathrm{L}(V, W)$ is a subset of continuous linear maps such that for every $\varphi \in W'$ and for every $v \in V$ one has*

$$\sup_{A\in\mathcal{A}} |\varphi(Av)| < \infty. \tag{3.3.7}$$

Then one has

$$\sup_{A\in\mathcal{A}} \|A\| < \infty. \tag{3.3.8}$$

PROOF: Let $v \in V$ be fixed, then (3.3.7) implies that the set of vectors $\{Av\}_{A\in\mathcal{A}} \subseteq W$ is weakly bounded. Hence by Theorem 3.3.6 this subset is bounded, i.e.

$$\sup_{A\in\mathcal{A}} \|Av\| < \infty.$$

Now the Banach-Steinhaus theorem applies and gives (3.3.8). $\square$

3.3.3 Topologies for $\mathrm{L}(V, W)$

We know already the operator norm topology for $\mathrm{L}(V, W)$ where V and W are normed spaces. This topology is complete whenever W is complete. However, norm convergence is quite restrictive in many situations since the norm topology of $\mathrm{L}(V, W)$ is simply too fine. As for V and V' we want to establish coarser topologies for $\mathrm{L}(V, W)$ as well. For normed spaces we have at least the following two options:

Definition 3.3.9 (Strong and weak operator topology) *Let V and W be normed spaces.*

i.) For $v \in V$ one defines the seminorm $\|\cdot\|_v$ by

$$\|A\|_v = \|Av\|_W \tag{3.3.9}$$

for $A \in \mathrm{L}(V, W)$.

ii.) For $v \in V$ and $\varphi \in W'$ one defines the seminorm $\|\cdot\|_{\varphi,v}$ by

$$\|A\|_{\varphi,v} = |\varphi(Av)| \tag{3.3.10}$$

for $A \in \mathrm{L}(V, W)$.

iii.) The strong operator topology of $\mathrm{L}(V, W)$ is the topology generated by all the open balls

$$\mathrm{B}_{\|\cdot\|_v, r}(A) = \big\{B \in \mathrm{L}(V, W) \bigm| \|A - B\|_v < r\big\}, \tag{3.3.11}$$

where $r > 0$ and $v \in V$ as well as $A \in \mathrm{L}(V, W)$.

iv.) The weak operator topology of $\mathrm{L}(V, W)$ *is the topology generated by all the open balls*

$$\mathrm{B}_{\|\cdot\|_{\varphi,v},r}(A) = \big\{ B \in \mathrm{L}(V, W) \bigm| \|A - B\|_{\varphi,v} < r \big\}, \tag{3.3.12}$$

where $\varphi \in W'$, $v \in V$, *and* $r > 0$ *as well as* $A \in \mathrm{L}(V, W)$.

Proposition 3.3.10 *Let* V *and* W *be normed spaces.*

i.) The strong operator topology turns $\mathrm{L}(V, W)$ *into a Hausdorff topological vector space.*

ii.) The strong operator topology is coarser than the norm topology of $\mathrm{L}(V, W)$. *One has*

$$\|A\|_v \le \|v\| \|A\| \tag{3.3.13}$$

for all $v \in V$.

iii.) A net $(A_i)_{i \in I}$ *of operators* $A_i \in \mathrm{L}(V, W)$ *is convergent to* $A \in \mathrm{L}(V, W)$ *in the strong operator topology iff for every* $v \in V$ *one has*

$$\lim_{i \in I} A_i v = Av, \tag{3.3.14}$$

i.e. A_i *converges pointwise on* V *to* A.

iv.) A net $(A_i)_{i \in I}$ *of operators* $A_i \in \mathrm{L}(V, W)$ *is a Cauchy net in the strong operator topology, if for every* $v \in V$ *the net* $(A_i v)_{i \in I}$ *is a Cauchy net in* V.

Proof: Again, the essence of the proof is contained in the analogous Proposition 3.1.18 and Proposition 3.1.19. For *i.)* one argues literally the same, noting that $\|A\|_v = 0$ for all $v \in V$ iff $A = 0$ for the Hausdorff property. The second part follows from the estimate (3.3.13) as in the proof of Proposition 3.1.18, *ii.)*. This estimate is of course just the usual estimate $\|Av\| \le \|A\| \|v\|$. Then *iii.)* and *iv.)* are shown analogously to Proposition 3.1.19. □

We will see examples in infinite dimensions where the strong operator topology is different from the norm topology. If V and W are both finite dimensional, the strong operator topology coincides with the norm topology by Theorem 2.2.19 applied to the finite-dimensional space $\mathrm{L}(V, W)$.

Corollary 3.3.11 *Let* V *be a normed space. Then the strong operator topology on* $V' = \mathrm{L}(V, \mathbb{K})$ *coincides with the weak* topology.*

Proof: In $\mathbb{K}$ we have the absolute value as norm. Hence the defining seminorms (3.3.9) become

$$\|\varphi\|_v = \|\varphi(v)\|_{\mathbb{K}} = |\varphi(v)| = \mathrm{p}_v(\varphi),$$

which are the defining seminorms of the weak* topology. □

For the weak operator topology we get similar statements:

Proposition 3.3.12 *Let* V *and* W *be normed spaces.*

i.) The weak operator topology turns $\mathrm{L}(V, W)$ *into a Hausdorff topological vector space.*

ii.) The weak operator topology is coarser than the strong operator topology of $\mathrm{L}(V, W)$. *One has*

$$\|A\|_{\varphi,v} \le \|\varphi\| \|A\|_v \tag{3.3.15}$$

for all $\varphi \in W'$ *and* $v \in V$. *In particular, the weak operator topology is also coarser than the norm topology.*

iii.) *A net* $(A_i)_{i \in I}$ *of operators* $A_i \in \mathrm{L}(V, W)$ *converges to* $A \in \mathrm{L}(V, W)$ *in the weak operator topology iff for every* $\varphi \in W'$ *and every* $v \in V$ *one has*

$$\lim_{i \in I} \varphi(A_i v) = \varphi(Av). \tag{3.3.16}$$

In other words, the net $(A_i v)_{i \in I}$ *converges weakly to* Av *for all* $v \in V$*.*

iv.) *A net* $(A_i)_{i \in I}$ *of operators* $A_i \in \mathrm{L}(V, W)$ *is a Cauchy net in the weak operator topology if for every* $\varphi \in W'$ *and every* $v \in V$ *the net* $(\varphi(A_i v))_{i \in I}$ *is a Cauchy net in* $\mathbb{K}$*.*

Proof: The only interesting part is perhaps *ii.)*. Here the estimate (3.3.15) can be used to show that every zero neighbourhood in the weak operator topology is also a zero neighbourhood in the strong operator topology by showing that

$$\mathrm{B}_{\|\,\cdot\,\|_v, r}(0) \subseteq \mathrm{B}_{\|\,\cdot\,\|_{\varphi, v}, r'}(0)$$

for all $r' > 0$ and $r = \frac{r'}{\|\varphi\|}$ if $\varphi \neq 0$. For $\varphi = 0$ the inclusion trivially holds for all r. The remaining statements follow the general pattern of topologies defined by systems of seminorms. □

Again, since on $\mathbb{K}$ we have only a one-dimensional space of linear functionals, the weak operator topology on $V' = \mathrm{L}(V, \mathbb{K})$ is the same as the weak* topology. However, we will see examples where the strong operator topology is strictly finer than the weak operator topology. Hence we end up with three typically different topologies for $\mathrm{L}(V, W)$. One should also note that, being a normed space, $\mathrm{L}(V, W)$ carries a weak topology inherited from its topological dual $\mathrm{L}(V, W)'$. Since the functionals

$$\Psi_{\varphi, v} \colon \mathrm{L}(V, W) \ni A \;\mapsto\; \Psi_{\varphi, v}(A) = \varphi(Av) \in \mathbb{K} \tag{3.3.17}$$

are continuous, i.e.

$$|\varphi(Av)| \leq \|\varphi\| \|v\| \|A\| \tag{3.3.18}$$

for all $A \in \mathrm{L}(V, W)$, they belong to $\mathrm{L}(V, W)'$. Hence the seminorms (3.3.10) are particular seminorms of the weak topology of $\mathrm{L}(V, W)$, i.e.

$$\|\,\cdot\,\|_{\varphi, v} = \mathrm{p}_{\Psi_{\varphi, v}}, \tag{3.3.19}$$

see again Definition 3.1.17 now applied to $\mathrm{L}(V, W)$. However, the topological dual $\mathrm{L}(V, W)'$ is typically strictly larger than the span of the linear functionals from (3.3.17). Hence the weak topology has more seminorms p_Ψ with $\Psi \in \mathrm{L}(V, W)'$ than the ones of the form (3.3.19). It follows that the weak operator topology has less open balls and therefore the weak operator topology is coarser than the weak topology. It turns out that the weak topology is not of the same importance, though. In any case, one should be very careful what to call "weak topology".

In a next step we want to investigate completeness properties of $\mathrm{L}(V, W)$ with respect to the strong and weak operator topologies. The first observation is that completeness is never fulfilled as soon as the domain V is infinite dimensional. Here the following example is quite drastic:

Example 3.3.13 (Non-completeness of V') Let V be an infinite-dimensional normed space. Then the topological dual is not complete with respect to the weak* topology. In fact, the algebraic dual V^* is complete and $V' \subseteq V^*$ is dense in the weak* topology. The proof can be found in Exercise **??** and relies heavily on the usage of Cauchy nets and not just Cauchy sequences.

The situation changes if we ask for sequential completeness:

Theorem 3.3.14 (Sequential completeness of $\mathrm{L}(V, W)$**)** *Let* V *and* W *be Banach spaces. Then* $\mathrm{L}(V, W)$ *is sequentially complete with respect to the strong operator topology.*

PROOF: Let $(A_n)_{n\in\mathbb{N}}$ with $A_n \in \mathrm{L}(V,W)$ be a Cauchy sequence in the strong operator topology. This means that for all $v \in V$ the sequence $(A_n v)_{n\in\mathbb{N}}$ is a Cauchy sequence in W. As W is a Banach space, we have a limit denoted by

$$Av = \lim_{n\to\infty} A_n v.$$

It is clear that this defines a linear map $A \in \mathrm{Hom}(V,W)$ since the vector space operations of W are continuous and each A_n is linear. The question is whether $A \in \mathrm{L}(V,W)$ is a continuous linear map. Since we have a convergent sequence $(A_n v)_{n\in\mathbb{N}}$ this sequence is bounded in W. Hence we have a constant $C_v > 0$ with

$$\sup_{n\in\mathbb{N}} \|A_n v\| \leq C_v$$

for all $v \in V$. Note that this is the step which is not possible for a Cauchy net: a convergent net needs not to be bounded at all. Nevertheless, see also Exercise 3.5.16 for a slight generalization. The Banach-Steinhaus theorem then shows that there is a constant $C > 0$ with

$$\sup_{n\in\mathbb{N}} \|A_n\| \leq C.$$

Hence

$$\|Av\| = \lim_{n\to\infty} \|A_n v\| \leq \lim \sup_{n\in\mathbb{N}} \|A_n v\| \leq \lim \sup_{n\in\mathbb{N}} \|A_n\| \|v\| \leq C\|v\|$$

shows that $\|Av\| \leq C\|v\|$ for all $v \in V$. Hence $A \in \mathrm{L}(V,W)$ follows. Then the strong convergence $A_n \longrightarrow A$ is clear by construction since $\|A_n - A\|_v = \|A_n v - Av\|_W \longrightarrow 0$ for $n \to \infty$. Hence we have the sequential completeness. ☐

Corollary 3.3.15 *Let V be a Banach space. Then V' is sequentially complete with respect to the weak* topology.*

PROOF: This is now clear with Corollary 3.3.11. ☐

In view of Example 3.3.13 this shows the crucial difference of completeness and sequential completeness for the weak* topology. Fortunately, in many constructions, Corollary 3.3.15 is all we need.

We will come back to the investigation of the weak* topology of V' and its relation to the weak topology of V' inherited from V'' when discussing reflexive Banach spaces.

3.3.4 Reflexivity and Separability

With the weak* compactness of $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V'$ we have a valuable tool at our disposal. However, as noted in Remark 3.2.15, we will not have sequential compactness in general. Since sequential compactness is easier to handle in many aspects, we want to understand now sufficient conditions guaranteeing this additional feature. The second purpose of this section is to understand the weak topology better: since the weak* topology is sequentially complete it raises the question whether the weak topology of V is sequentially complete, too. We also want to understand the weak compactness of $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V$.

First we need to recall the following definition from topology. A topological space $(M, \mathscr{M})$ is called *separable* if there is a countable subset $X \subseteq M$ with

$$X^{\mathrm{cl}} = M. \tag{3.3.20}$$

The idea is that the existence of such a countable dense subset implies that M is not too large.

Example 3.3.16 Finite-dimensional Hausdorff topological vector spaces are separable. In the real case we have

$$(\mathbb{Q}^n)^{\mathrm{cl}} = \mathbb{R}^n, \tag{3.3.21}$$

while in the complex case we have

$$((\mathbb{Q} + \mathrm{i}\mathbb{Q})^n)^{\mathrm{cl}} = \mathbb{C}^n. \tag{3.3.22}$$

Both subsets $\mathbb{Q}^n$ and $(\mathbb{Q} + \mathrm{i}\mathbb{Q})^n$ are of course countable. Since the topology of finite-dimensional Hausdorff topological vector spaces is unique, this is all we have to show.

Slightly more interesting are the following examples of separable sequence spaces:

Example 3.3.17 (Separable sequence spaces) Let $p \in [1, \infty)$. Then the sequence space ℓ^p is separable. Indeed,

$$X = \operatorname{span}_{\mathbb{Q}}\{\mathrm{e}_n \mid n \in \mathbb{N}\} \tag{3.3.23}$$

with the Schauder basis $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ as in **??** is a dense and countable subset. The trick is to use that $c_{\circ\circ} = \operatorname{span}_{\mathbb{K}}\{\mathrm{e}_n \mid n \in \mathbb{N}\} \subseteq \ell^p$ is dense according to Example 2.3.5. With the same subset X we see that also $c_\circ$ is separable. Since $c_\circ \subseteq c$ has codimension one, we can take the $\mathbb{Q}$-multiples of the constant sequence $1 \in c$ together with X to conclude that also c is separable. It requires a slightly more involved argument to show that ℓ^∞ is *not* separable, see also Exercise **??**.

One first consequence of being separable is now the following:

Proposition 3.3.18 *Let V be a separable Banach space. Then $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V'$ is first countable in the weak* topology and hence sequentially compact.*

PROOF: Let $X = \{v_1, v_2, \ldots\} \subseteq V$ be a countable dense subset and let $\varphi \in \mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V'$ be given. From the definition of the weak* topology we obtain a basis of neighbourhoods of $\varphi \in V'$ by taking finite intersections of weak*-open balls of the form $\mathrm{B}_{\mathrm{p}_v,\varepsilon}(\varphi)$ where $\varepsilon > 0$ and $v \in V$. Since $\varepsilon\mathrm{p}_v = \mathrm{p}_{\varepsilon v}$ it suffices to consider balls $\mathrm{B}_{\mathrm{p}_v,1}(\varphi)$ of radius one. Thus we denote the corresponding intersections with unit ball $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V'$ by

$$U_{w_1,\ldots,w_n}(\varphi) = \mathrm{B}_{\mathrm{p}_{w_1},1}(\varphi) \cap \cdots \cap \mathrm{B}_{\mathrm{p}_{w_n},1}(\varphi) \cap \mathrm{B}_1(0)^{\mathrm{cl}}$$

for $w_1, \ldots, w_n \in V$. These sets provide now a basis of neighbourhoods of φ in the subspace topology of $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V'$. Let such a neighbourhood be fixed, then we find a vector $v_{m_i} \in X$ with

$$\|w_i - v_{m_i}\| < \frac{1}{3}$$

for every $i = 1, \ldots, n$ since $X^{\mathrm{cl}} = V$ by assumption. Let now $\psi \in U_{3v_{m_1},\ldots,3v_{m_n}}(\varphi)$. Then on the one hand we have $\|\psi\| \leq 1$ and on the other hand we have $|\psi(v_{m_i}) - \varphi(v_{m_i})| < \frac{1}{3}$ for all $i = 1, \ldots, n$. Together, this gives

$$\begin{aligned}
|\psi(w_i) - \varphi(w_i)| &\leq |\psi(w_i) - \psi(v_{m_i})| + |\psi(v_{m_i}) - \varphi(v_{m_i})| + |\varphi(v_{m_i}) - \varphi(w_i)| \\
&< \|\psi\|\|w_i - v_{m_i}\| + \frac{1}{3} + \|\varphi\|\|v_{m_i} - w_i\| \\
&< 1
\end{aligned}$$

for all $i = 1, \ldots, n$. Hence $\psi \in U_{w_1,\ldots,w_n}(\varphi)$ follows, thereby showing

$$U_{3v_{m_1},\ldots,3v_{m_n}}(\varphi) \subseteq U_{w_1,\ldots,w_n}(\varphi).$$

Thus the collection $\{U_{3v_{m_1},\ldots,3v_{m_n}}(\varphi)\}_{\substack{n\in\mathbb{N} \\ m_1,\ldots,m_n\in\mathbb{N}}}$ provides a basis of neighbourhoods as well. Clearly, this is now a countable basis. Note that it is important to use $\|\psi\|, \|\varphi\| \leq 1$ in the above estimate, i.e. we need to use the induced weak* topology of $\mathrm{B}_1(0)^{\mathrm{cl}}$. The second statement is then clear from general results, since a compact and first countable space is always sequentially compact, see **??**. □

The following proposition explains why many of our sequence spaces are separable. It will also be helpful to use this criterion in other situations:

Proposition 3.3.19 *Let V be a normed space with separable topological dual V'. Then V is separable, too.*

PROOF: Let $Y = \{\varphi_1, \varphi_2, \ldots\} \subseteq V'$ be a countable norm-dense subset of V'. Then we can assume $\varphi_n \neq 0$ as for Y also $Y \setminus \{0\}$ is still dense if $0 \in Y$. Then $\psi_n = \frac{\varphi_n}{\|\varphi_n\|} \in \partial \mathrm{B}_1(0)^{\mathrm{cl}}$ are in the unit sphere in V'. From $Y^{\mathrm{cl}} = V'$ one then deduces that the set of the ψ_n is dense in $\partial \mathrm{B}_1(0)^{\mathrm{cl}}$. Now we choose $v_n \in V$ with $|\psi_n(v_n)| \geq \frac{1}{2}$ and $\|v_n\| = 1$, which is possible thanks to $\|\psi_n\| = 1$. We claim that the subspace

$$U = \operatorname{span}_{\mathbb{K}}\{v_1, v_2, \ldots\} \subseteq V \tag{$*$}$$

of these vectors is dense in V. Assume the converse and consider a vector $v \in V \setminus U^{\mathrm{cl}}$ in the set-theoretic complement. Without restriction we can assume that $\|v\| = 1$ since U^{cl} is a subspace, too. In the quotient space V/U^{cl} we have $\mathrm{pr}(v) = [v] \neq 0$ since $v \notin U^{\mathrm{cl}}$. Hence we find by the Hahn-Banach theorem applied to V/U^{cl} a continuous linear functional $\Phi \in (V/U^{\mathrm{cl}})'$ with norm $\|\Phi\| = 1$ and $\Phi([v]) \neq 0$. Define now

$$\varphi = \Phi \circ \mathrm{pr} \in V',$$

which is a continuous linear functional with $\varphi(v) \neq 0$ and $\varphi\big|_{U^{\mathrm{cl}}} = 0$. Rescaling φ if needed allows us to achieve $\|\varphi\| = 1$ in addition. Then we find an $n \in \mathbb{N}$ such that φ is approximated well by some ψ_n, say $\|\psi_n - \varphi\| < \frac{1}{2}$. Together, this gives

$$\frac{1}{2} \leq |\psi_n(v_n)| = |\psi_n(v_n) - \varphi(v_n)| \leq \|\psi_n - \varphi\| < \frac{1}{2},$$

since $\varphi(v_n) = 0$ and $\|v_n\| = 1$. Thus we have reached a contradiction. In conclusion, $U^{\mathrm{cl}} = V$ and U was already a dense subspace. From $(*)$ we see that in the real case

$$X = \operatorname{span}_{\mathbb{Q}}\{v_1, v_2, \ldots\}$$

or

$$X = \operatorname{span}_{\mathbb{Q}+\mathrm{i}\mathbb{Q}}\{v_1, v_2, \ldots\}$$

in the complex case will be a dense countable subset of V. □

Example 3.3.20 The converse does not always hold. Consider the separable space $V = \ell^1$ then its dual space $V' \cong \ell^\infty$ is not separable, see also Exercise 2.5.11.

Separability will have its appearance also at other places beyond the sequential compactness of $\mathrm{B}_1(0)^{\mathrm{cl}}$ in the weak* topology. To make this and other nice properties of the weak* topology available to the original space V itself, we want a more symmetric relation between V and its topological dual V'. Both are normed and both have a coarser topology, the weak topology of V and the weak* topology of V'. To relate these two coarser topologies, we first note the following result:

Remark 3.3.21 Let V be a normed space. Then we have the canonical inclusion

$$\iota \colon V \longrightarrow V'' \tag{3.3.24}$$

into its topological bidual V''. Since $V'' = (V')'$ is a topological dual, namely of V', it carries the weak* topology. If we pull-back this topology to V we reproduce the weak topology, see Exercise 3.5.14. Conversely, we can equip V' with the weak topology, i.e. using seminorms p_Ψ with $\Psi \in (V')' = V''$. For $\Psi = \iota(v)$ with $v \in V$ this yields

$$\mathrm{p}_\Psi = \mathrm{p}_v \tag{3.3.25}$$

and hence the weak topology of V' is finer than the weak* topology, see again Exercise 3.5.14. In general, this is strictly finer.

The definition of a reflexive space is now one conceptually very important way to guarantee that the weak and the weak* topology of V' coincide:

Definition 3.3.22 (Reflexive space) *A normed space V is called reflexive if the canonical linear map $\iota\colon V \longrightarrow V''$ is surjective.*

Since we already know that ι is a norm-preserving and hence injective linear map, for a reflexive space V the map ι is a norm-preserving continuous linear bijection. Hence also its inverse ι^{-1} is norm-preserving and thus continuous. In conclusion, ι and ι^{-1} are mutually inverse isometric isomorphisms.

Proposition 3.3.23 *Let V be a normed space.*

i.) If V is reflexive then V is a Banach space.

ii.) If V is reflexive then the weak and the weak topology on V' coincide.*

iii.) If $U \subseteq V$ is a closed subspace of a reflexive space V then U is reflexive, too.

iv.) Suppose V is complete. Then V is reflexive iff V' is reflexive.

PROOF: Since for a reflexive space V the canonical map $\iota\colon V \longrightarrow V''$ is an isomorphism of normed spaces and since V'' is complete, also V is complete, i.e. a Banach space. The second statement is clear by our motivating discussion in Remark 3.3.21 since in (3.3.25) all seminorms of the weak topology of V' are the seminorms p_v of the weak* topology. Now let $U = U^{\mathrm{cl}} \subseteq V$ be a closed subspace of a reflexive space V, where we denote the inclusion map by $j\colon U \longrightarrow V$. Dualizing gives the restriction map

$$j'\colon V' \longrightarrow U',$$

which is surjective according to the Hahn-Banach theorem, see Corollary 3.1.6. Now we dualize once more to obtain

$$j''\colon U'' \longrightarrow V'',$$

which is again injective since j' was surjective. Now let $X \in U''$ be given. Then we know by the reflexivity of V that there is a necessarily unique vector $v \in V$ with

$$j''(X) = \iota(v).$$

We evaluate this on $\varphi \in V'$ and get

$$\varphi(v) = \iota(v)\varphi = j''(X)\varphi = X(j'(\varphi)) = X(\varphi\big|_U) \tag{$**$}$$

since $j'(\varphi) = \varphi\big|_U$ is just the restriction to U. This can only happen if $v \in U$. Indeed, assume $v \notin U$. Then we consider again the quotient Banach space $\mathrm{pr}\colon V \longrightarrow V\big/U$ and have $\mathrm{pr}(v) = [v] \neq 0$. By the Hahn-Banach theorem we find a $\Phi \in (V\big/U)'$ with $\Phi([v]) \neq 0$. Then $\varphi = \Phi \circ \mathrm{pr} \in V'$ satisfies $\varphi(v) \neq 0$ but $\varphi\big|_U = 0$ as $U = \ker \mathrm{pr}$. Thus $(**)$ gives the contradiction. It follows that $v \in U$ and hence $X = \iota(v)$ shows that U is reflexive, too. The last part follows now quickly. Suppose V is reflexive and hence $V \cong V''$ via ι_V. Let $\Psi \in V'''$ be given. By assumption, any $X \in V''$ is of the form $X = \iota_V(v)$ for some unique $v \in V$. Hence $\varphi = \Psi \circ \iota_V \in V'$ satisfies

$$(\iota_{V'}(\varphi))(X) = X(\varphi) = \iota_V(v)(\varphi) = \varphi(v) = \Psi(\iota_V(v)) = \Psi(X).$$

This shows $\Psi = \iota_{V'}(\varphi)$ proving that also $\iota_{V'}\colon V' \longrightarrow V'''$ is surjective, i.e. V' is reflexive. Applying this result now to V' itself shows that if V' is reflexive then V'' is reflexive, too. Hence the closed subspace $\iota_V(V) \subseteq V''$ is reflexive by *iii.)*, showing that V is reflexive. Note that $\iota_V(V)$ is indeed closed as ι_V is isometric and V is complete. □

With the next result we make the Banach-Alaoglu theorem available for the weak topology as well. A first idea would be to focus on a reflexive space which is separable. Then the combination of Proposition 3.3.18 and Proposition 3.3.23, *ii.)*, would yield that the closed unit ball $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V \cong V''$ is sequentially compact. Surprisingly, we do not need the separability at all:

Proposition 3.3.24 *Let V be a reflexive Banach space. Then the closed unit ball $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V$ is sequentially compact in the weak topology.*

PROOF: The idea is that the separability comes from the sequence itself: consider a sequence $(v_n)_{n\in\mathbb{N}}$ of vectors $v_n \in \mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V$ for which we want to find a convergent subsequence. We consider

$$U = \operatorname{span}_{\mathbb{K}}\{v_1, v_2, \ldots\} \subseteq V$$

and obtain a closed subspace $U^{\mathrm{cl}} \subseteq V$ which is now separable as we have seen that now already several times. Moreover, U^{cl} is reflexive as closed subspace of a reflexive space according to Proposition 3.3.23, *iii.)*. Then $(U^{\mathrm{cl}})'$ is reflexive and, since $(U^{\mathrm{cl}})'' \cong U^{\mathrm{cl}}$ is separable, also separable itself by Proposition 3.3.19. According to Proposition 3.3.18, the closed unit ball $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq (U^{\mathrm{cl}})'' \cong U^{\mathrm{cl}}$ is sequentially compact in the weak* topology of $(U^{\mathrm{cl}})''$. Applying Proposition 3.3.23, *ii.)*, to the reflexive space $(U^{\mathrm{cl}})'$, we see that the weak* topology on $(U^{\mathrm{cl}})''$ coincides with the weak topology. Since $(U^{\mathrm{cl}})''' \cong (U^{\mathrm{cl}})'$ the weak topology of $(U^{\mathrm{cl}})''$ coincides with the weak topology of U^{cl} under the canonical isomorphism $U^{\mathrm{cl}} \cong (U^{\mathrm{cl}})''$. This shows that ultimately $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq U^{\mathrm{cl}}$ is sequentially compact in the weak topology of U^{cl}. However, we want to prove the sequential compactness in the weak topology inherited by V. Luckily, these two coincide in general as by the Hahn-Banach theorem every $\psi \in (U^{\mathrm{cl}})'$ is the restriction of a $\varphi \in V'$ to U^{cl}. Hence the seminorm p_ψ is the restriction of the seminorm p_φ to U^{cl} which shows that the open balls with respect to p_ψ are the open balls with respect to p_φ intersected with U^{cl}. Thus the sequential compactness of $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq U^{\mathrm{cl}}$ in the weak topology shows that $(v_n)_{n\in\mathbb{N}}$ has a convergent subsequence with respect to the weak topology of V.□

A first application of reflexivity is now the following:

Corollary 3.3.25 *Let V be a reflexive Banach space.*

i.) *A norm-bounded sequence in V has a weakly convergent subsequence.*

ii.) *Every weak Cauchy sequence converges weakly.*

PROOF: The first statement follows from the sequential compactness of $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq V$ in the weak topology: we can rescale to get the same result for every $\mathrm{B}_R(0)^{\mathrm{cl}}$. The second statement follows from Corollary 3.3.15 applied to V'' and the fact that the weak* topology of V'' coincides with the weak topology of V under the canonical isomorphism $\iota\colon V \longrightarrow V''$ for a reflexive space V. □

Again, as a concluding warning: unless for finite dimensions, V is not weakly complete as the weak topology is the weak* topology of V'' and thus Example 3.3.13 applies.

3.4 Open Mappings and Closed Graphs

3.4.1 The Open Mapping Theorem

Recall that a map between topological spaces is called open if images of open subsets are open again, see Definition ??, ??. In general, this is quite unrelated to continuity. For topological vector spaces we are interested in linear maps as usual, where we get the following characterization, again relying on the translation invariance of the topology:

Proposition 3.4.1 *Let $\phi\colon V \longrightarrow W$ be a linear map between topological vector spaces. Then ϕ is an open map iff for every zero neighbourhood $Z \subseteq V$ the image $\phi(Z) \subseteq W$ is a zero neighbourhood as well. It suffices to test this on a basis of zero neighbourhoods of V.*

PROOF: Since we have $\phi(A) \subseteq \phi(B)$ for all $A \subseteq B \subseteq V$ in general and since $\phi(B)$ is a zero neighbourhood whenever $\phi(A) \subseteq \phi(B)$ is a zero neighbourhood, it is clear that it suffices to test the property for zero neighbourhoods Z from a basis of zero neighbourhoods. Hence consider a non-empty open subset $O \subseteq V$. For $v \in O$ the subset $O - v$ is a zero neighbourhood. By assumption, we find a zero neighbourhood $Z \subseteq V$ with $Z \subseteq O - v$ from the basis such that $\phi(Z) \subseteq W$ is a zero neighbourhood. The linearity of ϕ gives

$$\phi(Z) \subseteq \phi(O) - \phi(v),$$

and hence $\phi(O)-\phi(v) \subseteq W$ is a zero neighbourhood as well. This implies that $\phi(O)$ is a neighbourhood of $\phi(v)$. As this holds for all $v \in O$, and hence for all points $\phi(v) \in \phi(O)$, we conclude that $\phi(O)$ is a neighbourhood of all its points, i.e. an open subset by Proposition **??**, **??**. This shows that ϕ is an open map. For the converse, assume ϕ is an open map and let $Z \subseteq V$ be a zero neighbourhood. Then it contains an open neighbourhood of zero $O \subseteq Z$ which is mapped to an open subset $\phi(O)$, still containing $0 \in \phi(O)$. Thus $\phi(Z) \supseteq \phi(O)$ is again a zero neighbourhood. □

Corollary 3.4.2 *An open linear map $\phi\colon V \longrightarrow W$ between topological vector spaces is surjective.*

PROOF: If ϕ is an open map then $\phi(V) \subseteq W$ is open and contains 0, i.e. a zero neighbourhood. Then

$$W = \bigcup_{n=1}^{\infty} n\phi(V) = \bigcup_{n=1}^{\infty} \phi(nV) = \phi(V)$$

follows from (2.1.7) and the fact that $\phi(V)$ is a subspace. □

The open mapping theorem can now be understood as a converse of this corollary, answering the question when a surjective linear map is open. In general this is not true, but with some extra assumptions one obtains the following result:

Theorem 3.4.3 (Open mapping theorem) *Let $\phi\colon V \longrightarrow W$ be a continuous linear map between Banach spaces. Then ϕ is open iff ϕ is surjective.*

PROOF: The trivial direction is covered by Corollary 3.4.2. Thus assume that ϕ is surjective. For every $R > 0$ we know that

$$V = \bigcup_{n=1}^{\infty} n\mathrm{B}_R(0) = \bigcup_{n=1}^{\infty} \mathrm{B}_{nR}(0),$$

and hence

$$W = \phi(V) = \bigcup_{n=1}^{\infty} \phi(n\mathrm{B}_R(0)) = \bigcup_{n=1}^{\infty} n\phi(\mathrm{B}_R(0))$$

holds by surjectivity of ϕ. Taking closures can only enlarge subsets, implying that we have

$$W = \bigcup_{n=1}^{\infty} (n\phi(\mathrm{B}_R(0)))^{\mathrm{cl}} \tag{$*$}$$

as well. Since scaling by $n \in \mathbb{N}$ is a homeomorphism, we have $(n\phi(\mathrm{B}_R(0)))^{\mathrm{cl}} = n(\phi(\mathrm{B}_R(0)))^{\mathrm{cl}}$. From $(*)$ we see that W is a countable union of closed subsets. Since W is a Baire space according to Theorem **??**, we find an $n_0 \in \mathbb{N}$ such that $(n_0\phi(\mathrm{B}_R(0)))^{\mathrm{cl}} = n_0(\phi(\mathrm{B}_R(0)))^{\mathrm{cl}}$ has an interior point.

Since scaling with n_0 is a homeomorphism, already $(\phi(\mathrm{B}_R(0)))^{\mathrm{cl}}$ has an interior point. We conclude that for all $R > 0$ the closure of the image of $\mathrm{B}_R(0)$ has an interior point. Note that in this step the completeness of W enters in a crucial way.

In a second step we want to show that $0 \in (\phi(\mathrm{B}_R(0)))^{\mathrm{cl}}$ is such an interior point. To this end, suppose that $w_0 \in (\phi(\mathrm{B}_R(0)))^{\mathrm{cl}}$ is an interior point. Then we have a $\delta > 0$ with $\mathrm{B}_\delta(w_0) \subseteq (\phi(\mathrm{B}_R(0)))^{\mathrm{cl}}$. For $w \in \mathrm{B}_\delta(0)$ we get

$$w = w_0 + w - w_0 \in \mathrm{B}_\delta(w_0) - \mathrm{B}_\delta(w_0),$$

and hence

$$\begin{aligned} w &\in \mathrm{B}_\delta(w_0) - \mathrm{B}_\delta(w_0) \\ &\subseteq (\phi(\mathrm{B}_R(0)))^{\mathrm{cl}} - (\phi(\mathrm{B}_R(0)))^{\mathrm{cl}} \\ &\subseteq \big(\phi(\mathrm{B}_R(0))) - \phi(\mathrm{B}_R(0))\big)^{\mathrm{cl}} \\ &\subseteq \big(\phi(\mathrm{B}_R(0) - \mathrm{B}_R(0))\big)^{\mathrm{cl}} \\ &\subseteq \big(\phi(\mathrm{B}_{2R}(0))\big)^{\mathrm{cl}}, \end{aligned}$$

since ϕ is linear. This shows that $\mathrm{B}_\delta(0) \subseteq (\phi(\mathrm{B}_{2R}(0)))^{\mathrm{cl}}$ and since $R > 0$ was arbitrary, we conclude that 0 is always an interior point of $(\phi(\mathrm{B}_R(0)))^{\mathrm{cl}}$ for all $R > 0$.

For the third step we will need the completeness of V as well. Let $R > 0$ and fix a $\delta > 0$ such that

$$\mathrm{B}_\delta(0) \subseteq (\phi(\mathrm{B}_R(0)))^{\mathrm{cl}}, \tag{$**$}$$

according to the second step. We want to show that $(**)$ holds already without taking the closure of the right hand side, i.e.

$$\mathrm{B}_\delta(0) \subseteq \phi(\mathrm{B}_R(0)). \tag{☺}$$

First we note that the homeomorphism property of scalings, the linearity of ϕ, and $(**)$ show that for all $\lambda > 0$ one has

$$\mathrm{B}_{\lambda\delta}(0) = \lambda \mathrm{B}_\delta(0) \subseteq \lambda(\phi(\mathrm{B}_R(0)))^{\mathrm{cl}} = (\lambda\phi(\mathrm{B}_R(0)))^{\mathrm{cl}} = (\phi(\mathrm{B}_{\lambda R}(0)))^{\mathrm{cl}}.$$

Moreover, as the right hand side is closed we infer

$$\mathrm{B}_{\lambda\delta}(0)^{\mathrm{cl}} \subseteq (\phi(\mathrm{B}_{\lambda R}(0)))^{\mathrm{cl}}$$

for all $\lambda > 0$. We fix now $\varepsilon > 0$ and consider a vector $w_0 \in \mathrm{B}_\delta(0)$ different from zero. Then

$$w_0 \in \mathrm{B}_{\|w_0\|}(0)^{\mathrm{cl}} \subseteq \mathrm{B}_\delta(0) \subseteq (\phi(\mathrm{B}_R(0)))^{\mathrm{cl}}$$

shows that we can find a vector $v_0 \in \mathrm{B}_R(0)$ with

$$\|w_0 - \phi(v_0)\| < \frac{\varepsilon\delta}{2},$$

since w_0 is in the closure of the image of the open R-ball and hence approximated by such images up to $\frac{\varepsilon\delta}{2}$. Denote

$$w_1 = w_0 - \phi(v_0) \in \mathrm{B}_{\frac{\varepsilon\delta}{2}}(0).$$

Applying the argument again, we find a $v_1 \in \mathrm{B}_{\frac{\varepsilon R}{2}}(0)$ with

$$\|w_1 - \phi(v_1)\| < \frac{\varepsilon\delta}{4}.$$

Improving the approximation in every step by a factor 2 gives inductively vectors $v_1, v_2, \ldots \in V$ with

$$v_n \in \mathrm{B}_{\frac{\varepsilon R}{2^n}}(0) \quad \text{and} \quad \|w_0 - \phi(v_0) - \cdots - \phi(v_n)\| < \frac{\varepsilon R}{2^{n+1}}. \tag{$\star$}$$

Now the series $v = \sum_{n=0}^{\infty} v_n$ converges absolutely with

$$\|v\| \leq \sum_{n=0}^{\infty} \|v_n\| < R + \sum_{n=1}^{\infty} R \frac{\varepsilon}{2^n} = R(1+\varepsilon),$$

i.e. $v \in \mathrm{B}_{(1+\varepsilon)R}(0)$. Moreover, since ϕ is continuous we infer from ($\star$) that

$$w_0 = \lim_{N \to \infty} \sum_{n=0}^{N} \phi(v_n) = \phi(v).$$

This shows that $w_0 \in \phi(\mathrm{B}_{(1+\varepsilon)R}(0))$ without the closure. Rescaling again shows

$$\mathrm{B}_{\frac{\delta}{1+\varepsilon}}(0) \subseteq \phi(\mathrm{B}_R(0)).$$

Since

$$\mathrm{B}_\delta(0) = \bigcup_{\varepsilon > 0} \mathrm{B}_{\frac{\delta}{1+\varepsilon}}(0),$$

we finally managed to show (☺).

In a last step we conclude that ϕ is an open map. This is now a consequence of the characterization of open linear maps as in Proposition 3.4.1. ☐

Remark 3.4.4 One finds examples which show that the completeness of neither the domain V nor the target W can be relaxed. We indeed need both spaces to be Banach spaces and not just normed spaces, see Exercise **??**.

Some first consequences of the open mapping theorem are obtained from the following corollaries.

Corollary 3.4.5 *Let $\phi\colon V \longrightarrow W$ be a continuous bijective linear map between two Banach spaces. Then $\phi^{-1}\colon W \longrightarrow V$ is continuous, too.*

PROOF: Since ϕ is surjective, it is an open map by the open mapping theorem. A continuous open bijection between topological spaces is always a homeomorphism, see **??**, i.e. its inverse map ϕ^{-1} is continuous. ☐

In fact, we get the following property for the operator norm of the inverse:

Corollary 3.4.6 *Let $\phi\colon V \longrightarrow W$ be a linear bijection between Banach spaces. Then the following statements are equivalent:*

i.) There exists a $c \geq 0$ with

$$\|\phi(v)\|_W \leq c\|v\|_V \tag{3.4.1}$$

for all $v \in V$.

ii.) There exists a $C \geq 0$ with

$$\|\phi(v)\|_W \geq C\|v\|_V \tag{3.4.2}$$

for all $v \in V$.

In this case, the smallest such c is the operator norm of ϕ while the largest C is the inverse of the operator norm of ϕ^{-1}.

PROOF: From the previous corollary we see that ϕ is continuous iff ϕ^{-1} is continuous. The continuity of ϕ is equivalent to *i.)* with the operator norm $\|\phi\|$ being the infimum of all the c with (3.4.1), see again Proposition 2.2.24, *i.)*. For ϕ^{-1} the continuity is equivalent to the existence of a $\tilde{c} \geq 0$ with

$$\|\phi^{-1}(w)\|_V \leq \tilde{c}\|w\|_W$$

for all $w \in W$. Since ϕ is bijective, this becomes

$$\|v\|_V = \|\phi^{-1}(\phi(v))\|_V \leq \tilde{c}\|\phi(v)\|_W$$

for all $v \in V$ and hence (3.4.2) with $C = \frac{1}{\tilde{c}}$. Note that the largest C corresponds to the smallest $\tilde{c}$. □

Note that for the operator norms one has the inequality

$$1 = \|\phi \circ \phi^{-1}\| \leq \|\phi\|\|\phi^{-1}\|. \tag{3.4.3}$$

However, in typical situations this inequality is strict, i.e. we do not have $\|\phi^{-1}\| = \|\phi\|^{-1}$ in general. Here one finds already counterexamples in 2-dimensional normed spaces.

Corollary 3.4.7 *Let V be a vector space with two Banach norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If one has*

$$\|v\|_1 \leq c\|v\|_2 \tag{3.4.4}$$

for some $c \geq 0$ and all v, then the two norms are equivalent.

PROOF: The estimate (3.4.4) means that the topology $\mathscr{V}_2$ is finer than the topology $\mathscr{V}_1$, see Proposition 2.2.18, *i.)*. Hence $\mathrm{id}\colon (V, \mathscr{V}_2) \longrightarrow (V, \mathscr{V}_1)$ is a continuous linear bijection, having a continuous inverse by Corollary 3.4.6. Note that it is important that both norms are complete norms. □

3.4.2 The Closed Graph Theorem

The closed graph theorem can now be obtained easily from the open mapping theorem:

Theorem 3.4.8 (Closed graph theorem) *Let $\phi\colon V \longrightarrow W$ be a linear map between Banach spaces. Then ϕ is continuous iff its graph*

$$\operatorname{graph}(\phi) = \big\{(v, \phi(v)) \bigm| v \in V\big\} \subseteq V \times W \tag{3.4.5}$$

is closed in the product topology.

PROOF: It is a general fact from topology that the graph of a continuous map between Hausdorff spaces is closed, see Exercise **??**. Thus assume that $\operatorname{graph}(\phi)$ is closed. We know that the Cartesian product $V \times W$ is again a Banach space, e.g. by using the norm $\|(v,w)\|_\infty = \max\{\|v\|_V, \|w\|_W\}$ from Proposition 2.3.19 and Proposition 2.3.20, *iii.)*. The closed subspace $\operatorname{graph}(\phi) \subseteq V \times W = V \oplus W$ is then complete and hence a Banach subspace of $V \oplus W$. The two projections

$$V \xleftarrow{\mathrm{pr}_V} V \oplus W \xrightarrow{\mathrm{pr}_W} W$$

are continuous surjective linear maps. The restrictions to the subspace are still continuous and linear. Now

$$\mathrm{pr}_V\big|_{\operatorname{graph}(\phi)}\colon \operatorname{graph}(\phi) \longrightarrow V$$

is bijective and has, by Corollary 3.4.5, a continuous inverse

$$\big(\mathrm{pr}_V\big|_{\operatorname{graph}(\phi)}\big)^{-1}\colon V \longrightarrow \operatorname{graph}(\phi),$$

explicitly given by $V \ni v \mapsto (v, \phi(v)) \in \operatorname{graph}(\phi)$. Hence

$$\phi = \mathrm{pr}_W \circ \big(\mathrm{pr}_V\big|_{\operatorname{graph}(\phi)}\big)^{-1}\colon V \longrightarrow W$$

is continuous. □

The following criterion for the closedness of the graph is often useful:

Corollary 3.4.9 *Let $\phi\colon V \longrightarrow W$ be a linear map between Banach spaces. Then the following statements are equivalent:*

i.) *The map ϕ is continuous.*

ii.) *If $(v_n)_{n\in\mathbb{N}}$ is a sequence of vectors $v_n \in V$ such that the limits $v = \lim_{n\to\infty} v_n$ and $w = \lim_{n\to\infty} \phi(v_n)$ both exist, then $\phi(v) = w$ holds.*

PROOF: Indeed, the second statement just means that the graph of ϕ is sequentially closed. Since we are in a first countable situation, $\operatorname{graph}(\phi)$ is closed and Theorem 3.4.8 applies. ☐

3.5 Exercises

Exercise 3.5.1 (Balanced, convex and absolutely convex subsets)

Exercise 3.5.2 (Minkowski functionals) Let V be a vector space with an absorbing and convex subset $C \subseteq V$. Show that the Minkowski functional p_C is sublinear, i.e. $\mathrm{p}_C \in V^\sharp$.

Exercise 3.5.3 (Separation in the plane) Consider $V = \mathbb{R}^2$. Draw appropriate pictures to illustrate that the assumptions of the separation theorems, i.e. Theorem 3.1.12 and Theorem 3.1.14, can not be relaxed in general.

Exercise 3.5.4 (The topological annihilator II)

Exercise 3.5.5 (Topological vector spaces from seminorms) Let V be a vector space and let $\mathcal{P} \subseteq \operatorname{Map}(V, \mathbb{R})$ be a collection of seminorms on V. The aim of this exercise is to show in general how one can construct a topology out of such a set $\mathcal{P}$, abstracting the construction of the weak topology from Proposition 3.1.18.

i.)

Exercise 3.5.6 (Weak topology of a topological vector space) Let V be a topological vector space with topological dual V'. Consider the weak topology of V and discuss which properties shown in the normed case in Proposition 3.1.18 and Proposition 3.1.19 can be transferred to this more general scenario, possibly after slightly adapting them.

Exercise 3.5.7 (Dualizing is functorial)

Exercise 3.5.8 (Weak* continuity of the dual map) Let $A\colon V \longrightarrow W$ be a continuous linear map between Banach spaces. Show that the dual map $A'\colon W' \longrightarrow V'$ is also continuous with respect to the weak* topologies of W' and V', respectively.

Exercise 3.5.9 (Tensor products and dualizing)

Exercise 3.5.10 (Associativity of projective tensor product)

Exercise 3.5.11 (Completed projective tensor product)

Exercise 3.5.12 (Pre-polars)

Exercise 3.5.13 (Dual pairings and polars)

Exercise 3.5.14 (The weak and the weak*-topology for V, V', and V'')

Exercise 3.5.15 (Galois connection)

Exercise 3.5.16 (Convergence of bounded Cauchy nets)

Exercise 3.5.17 (Completions stay separable) Let V be a normed space with completion $\widehat{V}$. Show that $\widehat{V}$ is separable for a separable V.

Exercise 3.5.18 (Non-separability of ℓ^∞) Show that the sequence space ℓ^∞ is non-separable.

Hint: Define $\mathrm{e}_I \in \ell^\infty$ for $I \subseteq \mathbb{N}$ by $\mathrm{e}_I(i) = 1$ for $i \in I$ and $\mathrm{e}_I(i) = 0$ for $i \notin I$. Compute $\|\mathrm{e}_I - \mathrm{e}_J\|_\infty$.

Exercise 3.5.19 (Operator norm of the inverse) Let $R > 1$. Give an explicit examples of a Banach space V and a continuous bijective linear map $\phi \in \mathrm{L}(V)$ such that $\|\phi\| = 1$ but $\|\phi^{-1}\| \geq R$. Hence the inequality (3.4.3) can be violated arbitrarily bad.

Chapter 4

Hilbert Spaces

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List of Symbols

(M, d)	Metric space
$\mathrm{B}_r(p)$	Open metric ball
$\mathrm{B}_r(p)^{\mathrm{cl}}$	Closed metric ball
A^{cl}	Closure of subset $A \subseteq M$
$(\widehat{M}, \widehat{\mathrm{d}})$	Completion of a metric space (M, d)
$(V, \mathscr{V})$	Topological vector space
$\tau_v \colon V \longrightarrow V$	Translation by $v \in V$
p	Seminorm
$\lVert \cdot \rVert$	Norm
$\phi^*\mathrm{p} = \mathrm{p} \circ \phi$	Pull-back of seminorm p by linear map ϕ
$\ker \mathrm{p}$	Kernel of seminorm p
c	Space of convergent sequences
$c_\circ$	Space of zero sequences
$c_{\circ\circ}$	Space of finite sequences
$\lim \colon c \longrightarrow \mathbb{K}$	Limit as linear functional
$\{\mathrm{e}_n\}_{n \in \mathbb{N}}$	Canonical basis of $c_{\circ\circ}$
ℓ^∞	Bounded sequences
$\lVert \cdot \rVert_\infty$	Supremum norm for bounded sequences
ℓ^p	p-Summable sequences
$\lVert \cdot \rVert_p$	p-Norm for p-summable sequences
$\mathscr{B}(X)$	Bounded functions
$\lVert \cdot \rVert_\infty$	Supremum norm on bounded functions
$\mathfrak{a} \subseteq 2^X$	σ-Algebra on set X
$(X, \mathfrak{a})$	Measurable space
$\mathfrak{a}_{\mathrm{Borel}}(M)$	Borel σ-algebra of topological space M
$\mathscr{M}(X, \mathfrak{a})$	Measurable functions on measurable space $(X, \mathfrak{a})$
$\mathscr{B}\mathscr{M}(X, \mathfrak{a})$	Bounded measurable functions
χ_A	Characteristic function of a subset A

$\mathfrak{n} \subseteq \mathfrak{a}$	σ-Ideal in σ-algebra
$\operatorname{ess\,range}(f)$	Essential range of function
$\\|f\\|_{\mathfrak{n},\infty} = \operatorname{ess\,sup}(\|f\|)$	Essential supremum norm
$\mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$	Essentially bounded functions
$\mathrm{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$	Equivalence classes of essentially bounded functions
$(X, \mathfrak{a}, \mu)$	Measure space
$\mathscr{L}^p(X, \mathfrak{a}, \mu)$	p-Integrable functions
$\\|\cdot\\|_{\mu,p}$	p-Norm with respect to measure μ
$\mathrm{L}^p(X, \mathfrak{a}, \mu)$	Equivalence classes of p-integrable functions
μ_{count}	Counting measure
$\\|\cdot\\|_K$	Supremum norm over compact subset K
$\mathscr{C}_{\mathrm{b}}(M, \mathbb{K})$	Continuous bounded functions
$\operatorname{supp}(f)$	Support of a function
$\mathscr{C}_A(M, \mathbb{K})$	Continuous functions with support in A
$\mathscr{C}_0(M, \mathbb{K})$	Continuous functions with compact support
$\\|\cdot\\|$	Weighted supremum norm with weight ρ
$\mathrm{d}(v, w) = \\|v - w\\|$	Metric of a normed space
$\mathrm{B}_r(v)$	Metric ball in normed space
$\\|\phi\\|$	Operator norm of $\phi\colon V \longrightarrow W$
$\mathrm{Hom}_{\mathbb{K}}(V_1, \ldots, V_k; W)$	Multilinear maps $V_1 \times \cdots \times V_k \longrightarrow W$
$\mathrm{L}(V_1, \ldots, V_k; W)$	Continuous multilinear maps $V_1 \times \cdots \times V_k \longrightarrow W$
$\\|\Phi\\|$	Operator norm of multilinear map Φ
$\widehat{V}$	Completion of a normed space V
$\delta_x\colon \mathscr{B}(X, \mathbb{K}) \longrightarrow \mathbb{K}$	Evaluation functional at $x \in X$
$\iota\colon V \longrightarrow V''$	Canonical inclusion into bidual
$\mathrm{ev}_v\colon V' \longrightarrow \mathbb{K}$	Evaluation on $v \in V$
$[\mathrm{p}]$	Quotient seminorm
$\mathrm{pr}\colon V \longrightarrow V\big/U$	Quotient space with quotient map
$V_1 \oplus \cdots \oplus V_N$	Finite direct sum of normed spaces
$V_1 \otimes \cdots \otimes V_N$	Tensor product (of normed spaces)
$\mathscr{A}$	Algebra (with additional properties)
$^*\colon \mathscr{A} \longrightarrow \mathscr{A}$	*-Involution of *-algebra
$\mathbb{1}$	Unit element

$\|\cdot\|_{V_1 \otimes \cdots \otimes V_k}$	Projective norm on tensor product
$V_1 \otimes_\pi \cdots \otimes_\pi V_k$	Projective tensor product (of normed spaces)

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