Homework for the Lecture

Functional Analysis

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Homework Sheet No 5

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(24 Points. Discussion 18. 11. 2024)

Homework 5-1: Completeness of ℓ^p

(6 Points) Prove that $(\ell^p, \|\cdot\|_p)$ is complete for every $p \in [1, \infty]$.
Hint: Start with the completeness of ℓ^∞. Then, try to proceed similarily for $p < \infty$.

Homework 5-2: A Dense Subspace of ℓ^p

(2 Points) Show that the space $c_{00} \subseteq \ell^p$ is dense for every $p \in [1, \infty)$. Is it dense as a subspace of ℓ^∞?

Homework 5-3: The Dual Space of ℓ^p

i.) **(4 Points)** Let $p, q \in (1, \infty)$ such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \tag{5.1}$$

with $r \in [1, \infty)$. Prove that the product of sequences

$$\ell^p \times \ell^q \ni ((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) \mapsto (x_n y_n)_{n\in\mathbb{N}} \tag{5.2}$$

yields a continuous bilinear map $m : \ell^p \times \ell^q \to \ell^r$. Compute its operator norm.
Hint: It might be helpful to generalize Young's inequality.

ii.) **(1 Point)** Let now $p \in [1, \infty)$ and $q \in (1, \infty]$ be conjugated to p. Here, we set $q := \infty$ in the case $p = 1$. Show that the multiplication m from part *i.)* induces a continuous linear map $\phi : \ell^q \to (\ell^p)'$.

iii.) **(6 Points)** Show that ϕ is invertible with ϕ^{-1} being an isometry.
Hint: It could be helpful to use a Schauder basis of ℓ^p.

Homework 5-4: The Stone-Weierstraß Theorem: Part I

Let X be a compact Hausdorff space. Consider the continuous functions $\mathscr{C}(X) = \mathscr{C}(X, \mathbb{C})$ with the usual supremum norm.

i.) **(1 Point)** For two functions $f, g \in \mathscr{C}(X, \mathbb{R})$ write $\max(f, g)$ and $\min(f, g)$ as a linear combination of $f \pm g$ and $|f \pm g|$ to show $\max(f, g), \min(f, g) \in \mathscr{C}(X, \mathbb{R})$ again.

Let now $\mathscr{A} \subseteq \mathscr{C}(X)$ be a *-subalgebra, that is $\mathscr{A}$ is a subspace of $\mathscr{C}(X)$ which is closed under multiplication and complex conjugation of functions. Assume $\mathscr{A}$ to be point-separating, i.e. for different $x, y \in X$ there is a function $g \in \mathscr{A}$ with $g(x) \neq g(y)$. Moreover, assume the constant one-function to be contained in $\mathscr{A}$. We consider a fixed $f \in \mathscr{C}(X)$ in the sequel.

ii.) **(2 Points)** Use Homework 4-4 to conclude that for $f = \overline{f}$ and $g = \overline{g}$ both in $\mathscr{A}$ one has $\max(f, g), \min(f, g) \in \mathscr{A}^{\mathrm{cl}}$.

iii.) **(2 Points)** Let $y, z \in X$. Show that there is a function $g \in \mathscr{A}$ with $g(y) = f(y)$ as well as $g(z) = f(z)$.
Hint: Consider the function $\tilde{g}(x) = f(y)h(x) - f(z)h(x) - f(y)h(z) + f(z)h(y)$ for a suitable $h \in \mathscr{A}$.