Intro to Diffgeo

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Definition 1. An isometric function between two metric spaces (X, d_1), (Y, d_2) is a function that preserves norms.

Corollary 2. *Isometric functions are injective.*

Definition 3. If it is surjective (and hence bijective), it is called an isometry.

Theorem 4. *Isometries on euclidean space ($\mathbb{R}^n$) are linear. Additionally, they are the composition of a translation and a rotation.*

$$F = t_v \circ L_A.$$

Where $L_A : \mathbb{R}^n \to \mathbb{R}^n$, with $L_A(x) = Ax$ and $A \in O(n)$.

Definition 5. An isometry is called orientation preserving if $\det(A) = 1$ and orientation reversenig if $\det(A) = -1$.

Definition 6. The cross product is defined by $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

Theorem 7. *The cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the unique product sending $x, y \in \mathbb{R}^3$ to the vector z such that*

$$det(x, y, v) = \langle v, z \rangle \qquad \forall v \in \mathbb{R}^3.$$

Theorem 8. ***(Properties of the Cross Product)***

1. *$\times$ is $\mathbb{R}$-bilinear*

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2. $y \times x = -x \times y$

3. Suppose $A \in SO(3, \mathbb{R})$. *Then*

$$Ax \times Ay = A(x \times y).$$

Proof.

$$\begin{aligned}
&\langle Ax \times Ay, Av \rangle \\
=&\det(Ax, Ay, Av) \\
=&\det(A) \cdot \det(x, y, v) \\
=&\det(A) \cdot \langle x \times y, v \rangle \\
=&\det(A) \cdot \langle A(x \times y), v \rangle
\end{aligned}$$

So if $\det(A) = 1$,

$$\langle Ax \times Ay, w \rangle = \langle A(x \times y), w \rangle \ \forall w \in \mathbb{R}^3.$$

If $\det(A) = -1$, then

$$\langle Ax \times Ay, w \rangle = -\langle A(x \times y), w \rangle \ \forall w \in \mathbb{R}^3.$$

Hence,

$$Ax \times Ay = -A(x \times y) \forall x, y \in \mathbb{R}^3.$$

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