Algebra und Dynamik von Quantensystemen Blatt Nr. 1

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Problem 1 (Norm)**.** Which of the maps $\mathcal{M}_2 \to [0, \infty)$

$$M \mapsto |\det M| \tag{1a}$$

$$M \mapsto |\operatorname{tr} M| \tag{1b}$$

$$M \mapsto \sup_{ij} |M_{ij}| \tag{1c}$$

$$M \mapsto \sup_{\substack{v \in \mathbb{C}^2, \\ \|v\|=1}} \|Mv\| \quad (\text{with } \|v\| = \sqrt{|v_1|^2 + |v_2|^2}) \tag{1d}$$

define a norm on the algebra $\mathcal{M}_2$ of complex 2×2 matrices? Which turn $\mathcal{M}_2$ into a C^*-algebra?

Proof. The first is not homogeneous, because multiplying M by a constant multiplies $\det M$ by the squared constant.

The second is not positive definite: $\operatorname{Tr}(\operatorname{diag}(1, -1)) = 0$, but $\operatorname{diag}(1, -1) \neq 0$.

The third is not a norm:

1. It is homogeneous.
2. It is positive definite
3. It satisfies the triangle inequality.
4. It is not submultiplicative: Consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

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thus

$$2 = \|A^2\| \not\leq \|A\|\|A\| = 1.$$

This is the functional norm. □

Problem 2 (Spectrum and Resolvent)**.** Use the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

to parametrize a general complex 2×2 matrix $M \in \mathcal{M}_2$ by four complex numbers $(a_0, \vec{a})$:

$$M(a_0, \vec{a}) = a_0 1 + \vec{a} \cdot \vec{\sigma} = a_0 1 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}. \tag{3}$$

1. For which $(a_0, \vec{a}) \in \mathbb{C}^4$ is $M(a_0, \vec{a})$

   (a) normal,

   (b) isometric,

   (c) unitary,

   (d) self-adjoint,

   (e) positive?

2. Determine the resolvent set $r_{\mathcal{M}_2}(M(a_0, \vec{a}))$ and spectrum $\sigma_{\mathcal{M}_2}(M(a_0, \vec{a}))$ for all $(a_0, \vec{a})$. Handle exceptional cases.

3. Test the general results for the spectrum of normal, isometric, unitary, self-adjoint, and positive matrices.

4. Compute the resolvent

$$R^{a_0, \vec{a}} : r_{\mathcal{M}_2}(M(a_0, \vec{a})) \to \mathcal{M}_2, \quad z \mapsto (z1 - M(a_0, \vec{a}))^{-1} \tag{4}$$

   as a 2×2 matrix (i.e., perform the matrix inversion explicitly!).

5. **NB:** In the lecture on *Tuesday, November 4, 2025*, it will be shown that $P_{\mathcal{C}}^{M(a_0,\vec{a})}$ is indeed a projection, if $\mathcal{C}$ encircles a part of the spectrum $\sigma(M(a_0,\vec{a}))$. You can wait until then to complete the exercise, take a peek at the script, or just do the integral choosing typical examples for $\mathcal{C}$ based on your earlier results for $\sigma(M(a_0,\vec{a}))$.

   Compute the projections

$$P_{\mathcal{C}}^{M(a_0,\vec{a})} = \int_{\mathcal{C}} \frac{dz}{2\pi i}\, R^{a_0,\vec{a}}(z) \tag{5}$$

   for "interesting" $\mathcal{C}$ explicitly. Are there qualitatively different cases to consider?

Proof. 1. (a) We compute $AA^\dagger - A^\dagger A$ explicitly to get

$$\left(\begin{array}{c} 2i\left(a_1\left(a_2\right)^* - a_2\left(a_1\right)^*\right) \\ a_1\left(a_0+a_3\right)^* - \left(a_1+ia_2\right)\left(a_0\right)^* + \left(a_1+ia_2\right)\left(a_3\right)^* + ia_2\left(a_0+a_3\right)^* - 2a_3\left(a_1\right)^* - 2ia_3\left(a_2\right)^* \end{array} \right.$$

□