Locally Convex Analysis

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Introduction

Chapter 1

Function Spaces

In this first chapter we present the typical examples of spaces in functional analysis which go beyond the easy cases of Banach or Hilbert spaces. While one can start to investigate topological vector spaces directly from a conceptual point of view, it will be useful to have very explicit examples at hand: They will illustrate that functional analysis beyond the Banach space case is not just an abstract game to play but, instead, is required by various applications ranging from the theory of partial differential equations to quantum physics, from differential geometry to representation theory and far beyond.

More precisely, in this chapter we discuss *function spaces* which are all comprised of certain functions

$$f \colon X \longrightarrow \mathbb{K} \tag{1.0.1}$$

defined on a set X with values in either the real or complex numbers, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, as their constituting elements. Of course, one imposes additional conditions such as continuity, integrability, differentiability, etc. on the functions which therefore will refer to additional structures on the set X needed to formulate such properties. Ultimately, we require a function space to have a linear structure, i.e. to be a vector space over $\mathbb{K}$ with respect to the pointwise operations. Thus a function space on X becomes a subspace of $\operatorname{Map}(X, \mathbb{K})$.

In more detail, we first discuss the simplest examples, namely the *sequence spaces*. Here the set X is an index set I, typically just $I = \mathbb{N}$, without any additional structure. Sequence spaces are then subspaces of all sequences $\operatorname{Map}(\mathbb{N}, \mathbb{K})$ subject to, e.g. growth conditions or summability properties.

The next big class of function spaces are functions on topological spaces. Here one requires continuity and perhaps additional boundedness properties. In many interesting situations, X is an open subset of $\mathbb{R}^n$ but also other topological spaces are of interest. A nice feature will be local compactness of X together with the assumption that X is Hausdorff. Beside boundedness also measurability can be required. In the case of an underlying topological space one uses the Borel σ-algebra for this.

After these examples of function spaces the perhaps most important class comes from differentiable functions. Here one considers an open subset $X \subseteq \mathbb{R}^n$ as the domain of definition. Alternatively, most of the interesting function spaces can equally well be defined on a smooth manifold X, reflecting their geometric origin. Now various options are of interest: $\mathscr{C}^k$-functions for $k \in \mathbb{N}$ or $k = \infty$, with or without prescribed behaviour at infinity like boundedness, support properties, etc. Also holomorphic functions on a domain $X \subseteq \mathbb{C}^n$ as well as certain symbol classes fall into this big class of examples. Several first properties and some technical constructions will be discussed on the way.

Finally, integration theory on measurable spaces provides additional classes of functions. Here we will meet the $\mathscr{L}^p$-spaces and their Banach space quotients of $\mathscr{L}^p$-functions modulo zero functions. While we obtain Banach spaces in this way there are also localized versions of interest, again leading to classes of examples beyond the Banach space scenario.

With all these examples at hand we can then formulate the questions we want to address in the subsequent chapters: To start analysis with these functions spaces we need to understand issues of convergence, completeness and continuity of the many operations we can perform on these functions. Here we will quickly see that Banach space techniques are by far not enough.

1.1 Sequence Spaces

A sequence is a map $a\colon \mathbb{N} \longrightarrow \mathbb{K}$ where we always use either real or complex sequences, i.e. $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. The space of all sequences is thus

$$\mathrm{Map}(\mathbb{N}, \mathbb{K}) = \big\{a\colon \mathbb{N} \to \mathbb{K}\big\}. \tag{1.1.1}$$

More generally, it will sometimes be of interest to consider more flexible domains. For a sequence it might be more natural to take other index sets I instead of $\mathbb{N}$. Think for example of the Taylor coefficients of a smooth function f of two variables. These coefficients are labelled by

$$a_{k\ell} = \frac{\partial^{k+\ell} f}{\partial x^k \partial y^\ell}(0), \tag{1.1.2}$$

and hence the more appropriate index set would be $I = \mathbb{N}_0 \times \mathbb{N}_0$ in this case. Hence we will allow any (non-empty) set I as an index set and consider arbitrary maps

$$a\colon I \longrightarrow \mathbb{K} \tag{1.1.3}$$

as generalized sequences. We will always endow $\mathrm{Map}(I, \mathbb{K})$ with the vector space structure originating from the pointwise operations. Then the notion of a sequence refers to the idea that we think of I as a *discrete* topological space leading to

$$\mathscr{C}(I) = \mathrm{Map}(I, \mathbb{K}), \tag{1.1.4}$$

i.e. *every* map $a\colon I \longrightarrow \mathbb{K}$ is continuous. Moreover, we think of the discrete space I to be equipped with the only canonical measure we always have: the counting measure defined on the largest possible σ-algebra 2^I of all subsets of I. This is the Borel σ-algebra when viewing I as a discrete topological space.

Of course, $\mathrm{Map}(I, \mathbb{K})$ is not very interesting per se, since the only interesting quantity is the cardinality of I in this case. Hence we will have to specify subspaces of all sequences in order to obtain more interesting sequence spaces.

1.1.1 Summable Sequences

In order to specify certain summability properties we recall the following basic facts and conventions. For a sequence $a\colon I \longrightarrow \mathbb{K}$ we consider the sum

$$\sum_{i\in K} |a_i| \in \mathbb{R}^+ \tag{1.1.5}$$

for a finite subset $K \subseteq I$ and define

$$\sum_{i\in I} |a_i| = \sup_{K\subseteq I} \sum_{i\in K} |a_i| \in [0, \infty], \tag{1.1.6}$$

where the supremum is taken over all finite subsets $K \subseteq I$.

Proposition 1.1.1 *Let I be a non-empty index set and let $a \in \mathrm{Map}(I, \mathbb{K})$. One has $\sum_{i\in I} |a_i| < \infty$ iff there exists a countable subset $I_0 \subseteq I$ with $a_i = 0$ for $i \in I \setminus I_0$ and either I_0 is finite or the series $\sum_{n=1}^{\infty} a_{i_n}$ converges absolutely for a bijection $\mathbb{N} \ni n \mapsto i_n \in I_0$.*

PROOF: This is of course just an etude in Lebesgue integration theory for the counting measure on I. However, we disguise the measure-theoretic background and give an explicit direct proof. Consider

$$I_n = \left\{ i \in I \;\middle|\; |a_i| \geq \tfrac{1}{n} \right\}$$

for $n \in \mathbb{N}$. Then $I_n \subseteq I_{n+1}$ and the supremum (1.1.6) is clearly $+\infty$ if one of the sets I_n is infinite. Thus we assume that the supremum is finite and hence have $\# I_n < \infty$ for all $n \in \mathbb{N}$. Then their union

$$I_0 = \bigcup_{n \in \mathbb{N}} I_n$$

is at most countably infinite and $a_i = 0$ for $i \in I \setminus I_0$. If I_0 is finite itself then

$$\sup_{K \subseteq I \text{ finite}} \sum_{i \in K} |a_i| = \sum_{i \in I_0} |a_i| < \infty,$$

and we are done. Thus assume that I_0 is countably infinite and choose an enumeration $\mathbb{N} \in n \mapsto i_n \in I_0$. Then for all $N \in \mathbb{N}$ we have

$$\sum_{n=1}^{N} |a_{i_n}| \leq \sup_{K \subseteq I \text{ finite}} \sum_{i \in K} |a_i| < \infty,$$

since $K = \{i_1, \ldots, i_N\} \subseteq I$ is one of the finite subsets the supremum is taken over. But this implies that the series $\sum_{n=1}^{\infty} |a_{i_n}|$ converges and hence $\sum_{n=1}^{\infty} a_{i_n}$ converges absolutely. For the opposite direction, we can assume that the set I_0 of non-zero a_i is infinite, otherwise the supremum is trivially finite. Hence assume that I_0 is infinite and we have an enumeration of I_0 with a convergent series $\sum_{n=1}^{\infty} |a_{i_n}|$. If $K \subseteq I$ is a finite subset then

$$\sum_{i \in K} |a_i| = \sum_{i \in K \cap I_0} |a_i| \leq \sum_{n=1}^{\infty} |a_{i_n}|,$$

since for $i \in K \setminus I_0$ we have that $|a_i| = 0$ does not contribute to the sum. Thus

$$\sup_{K \subseteq I \text{ finite}} \sum_{i \in K} |a_i| \leq \sum_{n=1}^{\infty} |a_{i_n}| < \infty$$

completes the proof. □

In this case we have of course

$$\sum_{i \in I} |a_i| = \sum_{n=1}^{\infty} |a_{i_n}|. \tag{1.1.7}$$

Since we have absolute convergence we also have unconditional convergence of the series $\sum_{n=1}^{\infty} |a_{i_n}|$: this is clear as we can use any bijection $\mathbb{N} \longrightarrow I$ in the above proof. Hence we write

$$\sum_{i \in I} a_i = \sum_{n=1}^{\infty} a_{i_n} \tag{1.1.8}$$

in this situation to stress the independence on the bijection $\mathbb{N} \simeq I_0$. Of course, the same notation is used if I_0 is actually finite. These observations allow us to state the following definitions:

Definition 1.1.2 (Summability) *Let I be a non-empty index set.*

i.) A sequence $a \in \operatorname{Map}(I, \mathbb{K})$ *is called summable if*

$$\sum_{i \in I} |a_i| < \infty. \tag{1.1.9}$$

ii.) The set of summable sequences is denoted by

$$\ell^1(I) = \big\{ a \in \operatorname{Map}(I, \mathbb{K}) \bigm| a \textit{ is summable} \big\}. \tag{1.1.10}$$

iii.) For $a \in \ell^1(I)$ *one defines*

$$\sum_{i \in I} a_i = \begin{cases} \sum_{i \in I_0} a_i & \textit{if } I_0 \textit{ is finite} \\ \sum_{n=1}^{\infty} a_{i_n} & \textit{if } I_0 \textit{ is infinite,} \end{cases} \tag{1.1.11}$$

where $I_0 \subseteq I$ *is the subset of those indices* $i \in I$ *with* $a_i \neq 0$ *and* $\mathbb{N} \in n \mapsto i_n \in I_0$ *is a bijection in the case of* I_0 *being countably infinite.*

iv.) For $I = \mathbb{N}$ *we simply write*

$$\ell^1 = \ell^1(\mathbb{N}). \tag{1.1.12}$$

This will give the first sequence space of interest. Before investigating its properties we need a slight generalization of it: We choose a sequence $\mu \in \operatorname{Map}(I, \mathbb{K})$ with

$$\mu_i > 0 \tag{1.1.13}$$

for all $i \in I$. This allows to consider the μ-weighted counting measure instead of the counting measure on I. In lieu of summing the coefficients of a sequence $a \in \operatorname{Map}(I, \mathbb{K})$ directly, we weight them with the corresponding coefficient of μ. Thus we say that $a \in \operatorname{Map}(I, \mathbb{K})$ is *summable with respect to* μ if

$$\sum_{i \in I} |a_i| \mu_i < \infty, \tag{1.1.14}$$

and set

$$\ell^1(I, \mu) = \big\{ a \in \operatorname{Map}(I, \mathbb{K}) \bigm| a \text{ is summable with respect to } \mu \big\}. \tag{1.1.15}$$

In principle, one can also admit non-negative sequences μ instead of (strictly) positive ones: in this case, some indices might not play a role at all and the coefficients a_i will never enter the discussion as soon as the corresponding weight $\mu_i = 0$ vanishes. Thus we will mainly stick to the strictly positive case (1.1.13) to avoid these trivial generalizations.

For a summable sequence with respect to μ we then can define the sum $\sum_{i \in I} a_i \mu_i$ as before since a is summable with respect to μ iff the sequence $a\mu$ is summable in the sense of Definition 1.1.2. The following basic estimates on series will be used throughout:

Proposition 1.1.3 *Let* I *be a non-empty index set. Moreover, let* $\mu \in \operatorname{Map}(I, \mathbb{K})$ *be a positive sequence.*

i.) For $p, q \in [1, \infty)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *one has Hölder's inequality*

$$\sum_{i \in I} |a_i b_i| \mu_i \leq \left(\sum_{i \in I} |a_i|^p \mu_i \right)^{\frac{1}{p}} \left(\sum_{i \in I} |b_i|^q \mu_i \right)^{\frac{1}{q}} \tag{1.1.16}$$

for all $a, b \in \operatorname{Map}(I, \mathbb{K})$*, viewed as an inequality in* $[0, \infty]$.

ii.) For $p \in [1, \infty)$ one has Minkowski's inequality

$$\left(\sum_{i \in I} |a_i + b_i|^p \mu_i\right)^{\frac{1}{p}} \leq \left(\sum_{i \in I} |a_i|^p \mu_i\right)^{\frac{1}{p}} + \left(\sum_{i \in I} |b_i|^p \mu_i\right)^{\frac{1}{p}} \tag{1.1.17}$$

for all $a, b \in \operatorname{Map}(I, \mathbb{K})$, viewed as an inequality in $[0, \infty]$.

iii.) For $0 < p < \infty$ one has the inequality

$$\sum_{i \in I} |a_i + b_i|^p \mu_i \leq 2^p \left(\sum_{i \in I} |a_i|^p \mu_i + \sum_{i \in I} |b_i|^p \mu_i\right). \tag{1.1.18}$$

PROOF: The proof is entirely parallel to the usual case of the Hölder inequality and the Minkowski inequality for series, see Exercise 1.4.2. □

The last estimate is comparably rough but suffices to obtain the following statement. We can now define not only summable sequences but also p-summable sequences for all $0 < p < \infty$:

Definition 1.1.4 (p-Summable sequences) *Let I be a non-empty index set and let $\mu \in \operatorname{Map}(I, \mathbb{K})$ be a positive sequence. Moreover, let $0 < p < \infty$.*

i.) A sequence $a \in \operatorname{Map}(I, \mathbb{K})$ is called p-summable with respect to μ if

$$\sum_{i \in I} |a_i|^p \mu_i < \infty. \tag{1.1.19}$$

ii.) The set of p-summable sequences with respect to μ is denoted by

$$\ell^p(I, \mu) = \left\{a \in \operatorname{Map}(I, \mathbb{K}) \;\middle|\; a \text{ is } p\text{-summable with respect to } \mu\right\}. \tag{1.1.20}$$

iii.) The case $I = \mathbb{N}$ and $\mu_i = 1$ for all $i \in I$ is abbreviated by

$$\ell^p = \ell^p(\mathbb{N}, \mu = 1). \tag{1.1.21}$$

Proposition 1.1.5 *Let I be a non-empty index set and let $\mu \in \operatorname{Map}(I, \mathbb{K})$ be a positive sequence.*

i.) For all $0 < p < \infty$ the p-summable sequences $\ell^p(I, \mu)$ form a subspace of all sequences.

ii.) Suppose that $\mu_i \geq 1$ for all $i \in I$. Then one has

$$\ell^p(I, \mu) \subseteq \ell^q(I, \mu) \tag{1.1.22}$$

for all $1 \leq p < q < \infty$. Moreover, one has

$$\sqrt[q]{\sum\nolimits_{i \in I} |a_i|^q \mu_i} \leq \sqrt[p]{\sum\nolimits_{i \in I} |a_i|^p \mu_i} \tag{1.1.23}$$

for all sequences $a \in \operatorname{Map}(I, \mathbb{K})$ as inequality in $[0, \infty]$.

iii.) If $\mu' \in \operatorname{Map}(I, \mathbb{K})$ satisfies $\mu_i \leq \mu'_i$ for all $i \in I$ then for all $0 < p < \infty$ one has

$$\ell^p(I, \mu') \subseteq \ell^p(I, \mu). \tag{1.1.24}$$

PROOF: For the first part we note that for $a \in \ell^p(I, \mu)$ and $z \in \mathbb{K}$ we also have $za \in \ell^p(I, \mu)$ since clearly

$$\sum_{i \in I} |za_i|^p \mu_i = |z|^p \sum_{i \in I} |a_i|^p \mu_i < \infty.$$

Moreover, for $a, b \in \ell^p(I, \mu)$ we have $a + b \in \ell^p(I, \mu)$ according to Proposition 1.1.3, *iii.)*. Hence $\ell^p(I, \mu)$ is a subspace since the zero sequence is clearly contained in $\ell^p(I, \mu)$ for all values of p and μ. For the second part, it suffices to check (1.1.23). For abbreviation we set

$$\|a\|_{p,\mu} = \sqrt[p]{\sum\nolimits_{i\in I} |a_i|^p \mu_i} \in [0, \infty],$$

anticipating the Banach norm on $\ell^p(I, \mu)$ already here. We have

$$|a_i|^p \le |a_i|^p \mu_i \le \sum_{i\in I} |a_i|^p \mu_i = \|a\|^p_{p,\mu}.$$

Note that we used $\mu_i \ge 1$ in an essential way. If the quantity $\|a\|_{p,\mu}$ is infinite, the claimed inequality (1.1.23) certainly holds, thus we assume $\|a\|_{p,\mu} < \infty$ in the following. If $\|a\|_{p,\mu} = 0$ then $a = 0$, a case in which we have nothing to show either. For $q > p$ we have $x^q \le x^p$ for all $x \in [0, 1)$. Hence

$$\frac{|a_i|^q}{\|a\|^q_{p,\mu}} \le \frac{|a_i|^p}{\|a\|^p_{p,\mu}}.$$

Taking now the μ-weighted sum preserves this inequality yielding

$$\frac{1}{\|a\|^q_{p,\mu}} \sum_{i\in I} |a_i|^q \mu_i = \sum_{i\in I} \frac{|a_i|^q}{\|a\|^q_{p,\mu}} \mu_i \le \sum_{i\in I} \frac{|a_i|^p}{\|a\|^p_{p,\mu}} \mu_i = \frac{1}{\|a\|^p_{p,\mu}} \sum_{i\in I} |a_i|^p \mu_i = 1.$$

Taking the q-th root gives then the desired inequality (1.1.23). The third part is obvious. □

1.1.2 Bounded Sequences

There are also other types of sequence spaces which we now define as follows:

Definition 1.1.6 (Bounded sequences) *Let I be a non-empty index set. Then*

$$\ell^\infty(I) = \left\{ a \in \operatorname{Map}(I, \mathbb{K}) \;\middle|\; \|a\|_\infty = \sup\nolimits_{i\in I} |a_i| < \infty \right\} \tag{1.1.25}$$

is called the space of bounded sequences. One sets

$$\ell^\infty = \ell^\infty(\mathbb{N}). \tag{1.1.26}$$

Definition 1.1.7 (Finite sequences) *Let I be a non-empty index set. Then the sequences with finitely many coefficients different from zero are denoted by*

$$c_{\circ\circ}(I) = \operatorname{Map}_\circ(I, \mathbb{K}) = \left\{ a \in \operatorname{Map}(I, \mathbb{K}) \;\middle|\; a_i = 0 \textit{ for all but finitely many } i \in I \right\}. \tag{1.1.27}$$

One sets

$$c_{\circ\circ} = c_{\circ\circ}(\mathbb{N}). \tag{1.1.28}$$

For the bounded case one can also consider a version with weights: Let $\mu \in \operatorname{Map}(I, \mathbb{K})$ be a sequence with positive coefficients $\mu_i > 0$ and consider

$$\ell^\infty(I, \mu) = \left\{ a \in \operatorname{Map}(I, \mathbb{K}) \;\middle|\; \sup\nolimits_{i\in I} |a_i| \mu_i < \infty \right\}. \tag{1.1.29}$$

Again, one could relax this to non-negative weights $\mu_i \ge 0$ which mainly is of minor interest. In all these cases one obtains subspaces again:

Proposition 1.1.8 *Let I be a non-empty index set.*

i.) For a positive sequence $\mu \in \operatorname{Map}(I, \mathbb{K})$ *the set* $\ell^\infty(I, \mu)$ *is a subspace of* $\operatorname{Map}(I, \mathbb{K})$ *and*

$$c_{\circ\circ}(I) \subseteq \ell^\infty(I, \mu) \tag{1.1.30}$$

is a subspace therein.

ii.) For all $p \in [1, \infty)$ *and all positive* $\mu \in \operatorname{Map}(I, \mathbb{K})$ *one has*

$$c_{\circ\circ}(I) \subseteq \ell^p(I, \mu) \tag{1.1.31}$$

and

$$\ell^p(I, \mu) \subseteq \ell^\infty(I, \sqrt[p]{\mu}). \tag{1.1.32}$$

More precisely, one has

$$\sup_{i \in I} |a_i| \sqrt[p]{\mu_i} \leq \sqrt[p]{\sum\nolimits_{i \in I} |a_i|^p \mu_i} \tag{1.1.33}$$

for all sequences $a \in \operatorname{Map}(I, \mathbb{K})$ *as an inequality in* $[0, \infty]$.

iii.) For positive sequences $\mu, \mu' \in \operatorname{Map}(I, \mathbb{K})$ *with* $\mu \leq \mu'$ *one has*

$$\ell^\infty(I, \mu') \subseteq \ell^\infty(I, \mu). \tag{1.1.34}$$

PROOF: The first statement is trivial. For the second, we observe that $|a_i|^p \mu_i \leq \sum_{j \in I} |a_j|^p \mu_j$ for all $i \in I$. Hence the supremum over $i \in I$ is still bounded by the right hand side and taking the p-th root preserves this inequality which is (1.1.33). Then (1.1.32) is a consequence. The last part is clear. ☐

1.1.3 Convergent Sequences

The next two examples require to have $I = \mathbb{N}$ even though one can also generalize them to general directed sets instead of just using $\mathbb{N}$, see Exercise 1.4.4. We define the set of convergent sequences and of zero sequences:

Definition 1.1.9 (Convergent Sequences) *The space of convergent sequences is defined by*

$$c = \left\{ a \in \operatorname{Map}(\mathbb{N}, \mathbb{K}) \;\middle|\; \lim\nolimits_{n \to \infty} a_n \text{ exists} \right\}, \tag{1.1.35}$$

and we set

$$c_\circ = \left\{ a \in c \;\middle|\; \lim\nolimits_{n \to \infty} a_n = 0 \right\}. \tag{1.1.36}$$

Once again, we obtain subspaces of sequences which can be related to our previous examples as follows:

Proposition 1.1.10 *For the space of convergent sequences one has the following properties:*

i.) One has the inclusions

$$c_{\circ\circ} \subseteq c_\circ \subseteq c \subseteq \ell^\infty \tag{1.1.37}$$

as subspaces.

ii.) The limit gives a linear functional

$$\lim_{n \to \infty} : c \longrightarrow \mathbb{K}, \tag{1.1.38}$$

such that

$$c_\circ = \ker \lim_{n \to \infty}. \tag{1.1.39}$$

In particular, the codimension of $c_\circ$ *in* c *is one.*

Yet another interpretation of $c_\circ$ and c is obtained as follows. Consider the topological space $X = \mathbb{N}$ with the usual discrete topology. Then we have a one-point compactification $\hat{X} = X \cup \{\infty\}$ where the open neighbourhoods of $\{\infty\}$ are given by the subsets of $\hat{X}$ which contain $\{\infty\}$ and all but finitely many elements of X. Then the continuous functions on $\hat{X}$ can be identified those sequences $a \in \operatorname{Map}(I, \mathbb{K})$ which admit a continuous extension $\hat{a}$ to $\hat{X}$ by a value $\hat{a}(\infty) \in \mathbb{K}$. Clearly, these sequences coincide with the sequences in c. Hence we get the identification

$$c = \mathscr{C}(\mathbb{N} \cup \{\infty\}). \tag{1.1.40}$$

We note that also in this situation we have a weighted version of c. Indeed, let $\mu \in \operatorname{Map}(\mathbb{N}, \mathbb{K})$ be a positive sequence as before. Then we set

$$c(\mathbb{N}, \mu) = \big\{a \in \operatorname{Map}(\mathbb{N}, \mathbb{K}) \bigm| \lim_{n\to\infty} a_n \mu_n \text{ exists}\big\} \tag{1.1.41}$$

and

$$c_\circ(\mathbb{N}, \mu) = \big\{a \in c(\mathbb{N}, \mu) \bigm| \lim_{n\to\infty} a_n \mu_n = 0\big\}. \tag{1.1.42}$$

The analogous results to Proposition 1.1.10 still hold in this more general case, see also Exercise 1.4.5. Finally, note that for all $1 \le p < \infty$ we have

$$\ell^p(\mathbb{N}, \mu) \subseteq c_\circ(\mathbb{N}, \sqrt[p]{\mu}), \tag{1.1.43}$$

see Exercise 1.4.6.

1.1.4 Rapidly Decreasing Sequences

This class of sequences is again formulated for the index set $I = \mathbb{N}$ but also here natural generalizations are possible. As motivation, we consider a fixed weight $\mu \in \operatorname{Map}(\mathbb{N}, \mathbb{K})$ with positive entries $\mu_n \ge 1$. Then we have the canonical inclusions

$$c_{\circ\circ} \subseteq \ell^p(\mathbb{N}, \mu) \subseteq \ell^q(\mathbb{N}, \mu) \subseteq c_\circ(\mathbb{N}, \sqrt[q]{\mu}) \subseteq c(\mathbb{N}, \sqrt[q]{\mu}) \subseteq \ell^\infty(\mathbb{N}, \sqrt[q]{\mu}) \tag{1.1.44}$$

for $1 \le p \le q < \infty$. It requires only some easy examples to see that all the above inclusion are non-trivial, see Exercise 1.4.7.

Now if we also vary the weight μ we can relate the spaces further. To this end we consider weights $\mu^{(k)} \in \operatorname{Map}(\mathbb{N}, \mathbb{K})$ of polynomial growth, i.e.

$$\mu_n^{(k)} = n^k. \tag{1.1.45}$$

Clearly $1 \le \mu^{(k)} \le \mu^{(k+1)}$ and hence we get the inclusions

$$c_{\circ\circ} \subseteq \cdots \subseteq \ell^p(\mathbb{N}, \mu^{(k+1)}) \subseteq \ell^p(\mathbb{N}, \mu^{(k)}) \subseteq \cdots \subseteq \ell^p \tag{1.1.46}$$

for all $1 \le p \le \infty$ as well as

$$c_{\circ\circ} \subseteq \cdots \subseteq c_\circ(\mathbb{N}, \mu^{(k+1)}) \subseteq c_\circ(\mathbb{N}, \mu^{(k)}) \subseteq \cdots \subseteq c_\circ \tag{1.1.47}$$

and

$$c_{\circ\circ} \subseteq \cdots \subseteq c(\mathbb{N}, \mu^{(k+1)}) \subseteq c(\mathbb{N}, \mu^{(k)}) \subseteq \cdots \subseteq c. \tag{1.1.48}$$

This suggests to consider the intersections of the spaces for all $k \in \mathbb{N}$. Surprisingly, the dependence on p disappears in the intersection:

Proposition 1.1.11 *Let $\mu^{(k)} \in \operatorname{Map}(\mathbb{N}, \mathbb{K})$ be given by $\mu^{(k)} = n^k$ for $k \in \mathbb{N}$. Moreover, let $1 \le p < \infty$.*

i.) For all $k \in \mathbb{N}$ one has

$$\ell^\infty(\mathbb{N}, \mu^{(k+2)}) \subseteq \ell^1(\mathbb{N}, \mu^{(k)}). \tag{1.1.49}$$

ii.) The following intersections

$$\bigcap_{k=1}^\infty \ell^1(\mathbb{N}, \mu^{(k)}) = \bigcap_{k=1}^\infty \ell^p(\mathbb{N}, \mu^{(k)}) = \bigcap_{k=1}^\infty c_\circ(\mathbb{N}, \mu^{(k)}) = \bigcap_{k=1}^\infty c(\mathbb{N}, \mu^{(k)}) = \bigcap_{k=1}^\infty \ell^\infty(\mathbb{N}, \mu^{(k)}) \tag{1.1.50}$$

coincide.

PROOF: In view of the above inclusions (1.1.44) as well as the inclusions in (1.1.46), (1.1.47), and (1.1.48) it suffices to check (1.1.49). But this is easy since for $a \in \ell^\infty(\mathbb{N}, \mu^{(k+2)})$ we have

$$c = \sup_{n\in\mathbb{N}} |a_n| n^{k+2} < \infty,$$

and hence $|a_n| \leq \frac{c}{n^{k+2}}$ for all $n \in \mathbb{N}$. This gives

$$\sum_{k=1}^\infty |a_n| n^k \leq \sum_{k=1}^\infty \frac{c}{n^{k+2}} n^k = c \sum_{k=1}^\infty \frac{1}{n^2} < \infty,$$

and thus $a \in \ell^1(\mathbb{N}, \mu^{(k)})$ showing (1.1.49). $\square$

It turns out that this intersection is one of the most important sequence spaces in locally convex analysis:

Definition 1.1.12 (Rapidly decreasing sequences) *The intersection*

$$s = \bigcap_{k=1}^\infty \ell^\infty(\mathbb{N}, \mu^{(k)}) \tag{1.1.51}$$

is called the space of rapidly decreasing sequences or the Schwartz sequence space.

Corollary 1.1.13 *The space of rapidly decreasing sequences is a subspace contained in all the sequence spaces $c_\circ(\mathbb{N}, \mu^{(k)})$, $c(\mathbb{N}, \mu^{(k)})$, $\ell^p(\mathbb{N}, \mu^{(k)})$ and $\ell^\infty(\mathbb{N}, \mu^{(k)})$ for all $1 \leq p < \infty$ and $k \in \mathbb{N}$.*

Corollary 1.1.14 *Let $a \in \operatorname{Map}(\mathbb{N}, \mathbb{K})$ be a sequence. Then the following statements are equivalent.*

i.) The sequence a is rapidly decreasing, i.e. for all $k \in \mathbb{N}$ one has

$$\sup_{n\in\mathbb{N}} |a_n| n^k < \infty. \tag{1.1.52}$$

ii.) For all $p \in [1, \infty)$ and all $k \in \mathbb{N}$ one has

$$\sum_{n=1}^\infty |a_n|^p n^k < \infty. \tag{1.1.53}$$

iii.) For all $k \in \mathbb{N}$ the limit $\lim_{n\to\infty} |a_n| n^k$ exists.

iv.) For all $k \in \mathbb{N}$ one has

$$\lim_{n\to\infty} |a_n| n^k = 0. \tag{1.1.54}$$

Indeed, the intersection of subspaces is again a subspace and the four characterizations in Corollary 1.1.14 correspond precisely to the intersections in Proposition 1.1.11, *ii.)*.

The importance of space s is that many function spaces with entirely different ideas for their definition eventually turn out to be isomorphic to s. We will see examples later on. Thus the space of rapidly decreasing sequences allows us to study various other types of function spaces. Many properties are then easily verified for s while they can be quite mysterious for others.

Also not completely trivial is the fact that the obvious inclusion

$$c_{\circ\circ} \subseteq s \tag{1.1.55}$$

is proper. There are sequences in s which are not contained in $c_{\circ\circ}$, see Exercise 1.4.15 for some examples. In fact, the codimension of $c_{\circ\circ}$ in s is uncountable.

1.1.5 Köthe Spaces

A vast generalization of the space s is provided by a construction of Köthe leading to the notion of a Köthe space. For its definition we need two index sets I and J. The first plays the role of indexing the elements of a sequence and the Köthe space will become a subspace of $\operatorname{Map}(I, \mathbb{K})$. The other index set J is used to index sequences of weights we want to use in the same spirit as in the definition of s. In most cases I and J are countable infinite and can hence be taken as $\mathbb{N}$. However, even if they are countable infinite there might be no natural way to obtain a bijection to $\mathbb{N}$ and thus it can be advantageous to stay with general index sets I and J. Then one defines a Köthe set and the corresponding Köthe space as follows:

Definition 1.1.15 (Köthe space) *Let I and J be non-empty index sets.*

i.) *A Köthe set $\mathscr{P} = \{\mu^{(j)}\}_{j\in J} \subseteq \operatorname{Map}(I, \mathbb{K})$ is a set of non-negative sequences $\mu^{(j)}$ such that for $j, j' \in J$ there exists a $k \in J$ with*

$$\max\{\mu_i^{(j)}, \mu_i^{(j')}\} \leq \mu_i^{(k)} \tag{1.1.56}$$

for all $i \in I$ and such that for all $i \in I$ one finds a $j \in J$ with

$$\mu_i^{(j)} > 0. \tag{1.1.57}$$

ii.) *Let $p \in [1, \infty]$ and let $\mathscr{P} \subseteq \operatorname{Map}(I, \mathbb{K})$ be a Köthe set indexed by J. Then the Köthe space $\Lambda^p(\mathscr{P})$ is defined by*

$$\Lambda^p(I, \mathscr{P}) = \left\{ a \in \operatorname{Map}(I, \mathbb{K}) \;\middle|\; \left(a_i \mu_i^{(j)}\right)_{i\in I} \in \ell^p(I) \text{ for all } j \in J \right\}. \tag{1.1.58}$$

If $I = \mathbb{N}$ we simply write $\Lambda^p(\mathscr{P}) = \Lambda^p(\mathbb{N}, \mathscr{P})$.

Equivalently, we can characterize the Köthe space as intersection

$$\Lambda^p(I, \mathscr{P}) = \bigcap_{j\in J} \ell^p(I, \mu^{(j)}) \tag{1.1.59}$$

of $\mu^{(j)}$-weighted ℓ^p-spaces, where we might want to allow non-negative weights instead of strictly positive ones. This gives immediately the relation to the previous sequence spaces, in particular, all $\ell^p(I, \mu)$ are Köthe spaces themselves by taking just one weight $\mu > 0$ and thus $\mathscr{P} = \{\mu\}$.

Also, the rapidly decreasing sequences are a Köthe space where as Köthe set one takes the weights

$$\mathscr{P}_{\text{rapidly decreasing}} = \left\{ (n^k)_{n\in\mathbb{N}}, k \in \mathbb{N} \right\}. \tag{1.1.60}$$

Note that the conditions for a Köthe set are satisfies indeed. In this case we know that the dependence on $p \in [1, \infty]$ actually disappears. In general, this needs not to be the case.

Köthe spaces are always non-trivial since clearly

$$c_{\circ\circ}(I) \subseteq \Lambda^p(I, \mathscr{P}), \tag{1.1.61}$$

no matter which parameters p and $\mathscr{P}$ we choose.

It is clear that a Köthe space $\Lambda^p(I, \mathscr{P})$ is determined by its *Köthe matrix*

$$K = (K_{ij})_{i \in I, j \in J} \text{ with } K_{ij} = \mu_i^{(j)}. \tag{1.1.62}$$

However, there is a certain redundancy in the description as we can obtain e.g. the same space $\ell^1(I, \mu)$ by replacing μ with 2μ.

Example 1.1.16 (Holomorphic functions) Consider $R > 0$ and let $\mathrm{B}_R(0) \subseteq \mathbb{C}$ be the open disc of radius R in $\mathbb{C}$. Moreover, let $f\colon \mathrm{B}_R(0) \longrightarrow \mathbb{C}$ be a holomorphic function written as convergent Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f}{\partial z^n}(0) z^n. \tag{1.1.63}$$

Since the Taylor series converges absolutely in the interior we get

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left| \frac{\partial^n f}{\partial z^n}(0) \right| \rho^n < \infty \tag{1.1.64}$$

for all $0 < \rho < R$. Conversely, if $a \in \operatorname{Map}(\mathbb{N}, \mathbb{C})$ is a sequence of complex numbers with

$$\sum_{n=0}^{\infty} \frac{1}{n!} |a_n| \rho^n < \infty \tag{1.1.65}$$

for all $0 < \rho < R$ then the a_n are the Taylor coefficients of a holomorphic function, namely

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} a_n z^n. \tag{1.1.66}$$

Clearly, it suffices to consider countably many radii $\rho < R$ converging to R to test this property. Hence the sequences a with (1.1.65) form the Köthe space

$$\Lambda^1\Big(\big\{(\tfrac{1}{n!}\rho^n)_{n \in \mathbb{N}}\big\}_{0 < \rho < R}\Big) = \Lambda^1\Big(\big\{(\tfrac{1}{n!}(R - \tfrac{1}{k})^n)_{n \in \mathbb{N}}\big\}_{k \in \mathbb{N}}\Big), \tag{1.1.67}$$

which is then in bijection to the holomorphic functions on the disc $\mathrm{B}_R(0)$. Clearly, one can also handle the entire functions, i.e. the case $R = \infty$ this way. Here we also need only countably many weight sequences like e.g. $(\frac{1}{n!}k^n)_{n \in \mathbb{N}}$ for all $k \in \mathbb{N}$.

We will see more examples of Köthe spaces later on. In fact, their purpose is at least twofold: first one can generate examples of spaces with certain properties by design. Here we will see that many abstract properties are easily checked for Köthe spaces. Second, one has many other spaces of interest which ultimately turn out to be Köthe spaces. Establishing such isomorphisms is sometimes the only way to prove that some spaces of interest have certain nice properties.

1.2 Bounded and Continuous Functions

We turn now to function spaces where the underlying set X will be endowed with some additional (geometric) structure instead of having a discrete set I as before. Note, however, that many of the sequence spaces can also be viewed as spaces we discuss in this section.

1.2.1 Bounded and Measurable Functions

Let X be a non-empty set then we denote the *bounded functions* on X by

$$\mathscr{B}(X) = \Big\{ f \in \operatorname{Map}(X, \mathbb{K}) \;\Big|\; \sup_{x \in X} |f(x)| < \infty \Big\}. \tag{1.2.1}$$

Clearly, this is a subspace of all functions. The first trivial observation is that uniform approximation preserves boundedness in the following sense:

Proposition 1.2.1 *Let $(f_i)_{i \in I}$ be a net of bounded functions on a non-empty set X such that there exists a function $f\colon X \longrightarrow \mathbb{K}$ with the property that for all $\varepsilon > 0$ there exists an index $i_0 \in I$ with*

$$\sup_{x \in X} |f_i(x) - f(x)| < \varepsilon, \tag{1.2.2}$$

whenever $i \succcurlyeq i_0$. Then $f \in \mathscr{B}(X)$.

PROOF: Note that (1.2.2) implies in particular $f_i(x) \longrightarrow f(x)$ *pointwise*, i.e. for all $x \in X$. Let $\varepsilon > 0$ be fixed and choose i_0 such that (1.2.2) holds. Then for all $x \in X$

$$|f(x)| \le |f(x) - f_{i_0}(x)| + |f_{i_0}(x)| < \varepsilon + \sup_{x \in X} |f_{i_0}(x)| < \infty,$$

since f_{i_0} is bounded. □

We will refer to this situation as *uniform convergence* of the net $(f_i)_{i \in I}$ to the function f. In this sense, uniform convergence preserves boundedness.

If we impose no other structure on X, then $\mathscr{B}(X)$ coincides with $\ell^\infty(X)$. However, in the following we shall specialize further to sets X with extra structure.

The first option is to endow X with the structure of a *measurable space*, i.e. we specify a *σ-algebra* $\mathfrak{a} \subseteq 2^X$ of measurable subsets of X. Then we require both: boundedness and measurability of the functions.

Definition 1.2.2 (Bounded measurable functions) *Let $(X, \mathfrak{a})$ be a measurable space. Then the set of bounded and measurable functions on X is denoted by*

$$\mathscr{BM}(X, \mathfrak{a}) = \big\{ f \in \operatorname{Map}(X, \mathbb{K}) \;\big|\; f \textit{ is bounded and measurable} \big\}. \tag{1.2.3}$$

Recall that $f\colon X \longrightarrow \mathbb{K}$ is *measurable* if $f^{-1}(A) \in \mathfrak{a}$ for all Borel sets $A \subseteq \mathbb{K}$. Note that on $\mathbb{K}$ we take the σ-algebra of Borel sets and *not* the Lebesgue-Borel sets. The *measurable functions* are sometimes denoted by

$$\mathscr{M}(X, \mathfrak{a}) = \big\{ f \in \operatorname{Map}(X, \mathbb{K}) \;\big|\; f \text{ is measurable} \big\}. \tag{1.2.4}$$

Just as $\mathscr{B}(X)$ and $\mathscr{M}(X, \mathfrak{a})$ are subspaces of all functions, so is their intersection

$$\mathscr{BM}(X, \mathfrak{a}) = \mathscr{B}(X) \cap \mathscr{M}(X, \mathfrak{a}) \tag{1.2.5}$$

a subspace of all functions $\operatorname{Map}(X, \mathbb{K})$.

If the σ-algebra $\mathfrak{a}$ is the maximal one, i.e. the whole power set $\mathfrak{a} = 2^X$, then any function is measurable

$$\mathscr{M}(X, 2^X) = \operatorname{Map}(X, \mathbb{K}), \tag{1.2.6}$$

and hence

$$\mathscr{BM}(X, 2^X) = \mathscr{B}(X) \tag{1.2.7}$$

are simply all bounded functions.

The space of measurable functions is actually fairly large and it typically requires complicated arguments to find non-measurable functions at all. One reason can be seen in the following properties which we quote without detailed proofs. Here we refer to textbooks on measure theory like e.g. [14, Theorem 1.14].

Remark 1.2.3 (Measurable functions) Let $(X, \mathfrak{a})$ be a measurable space.

i.) A complex-valued function f is measurable iff $\overline{f}$ is measurable which is equivalent to $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ being measurable.

ii.) If $f_n = \overline{f}_n \in \mathscr{M}(X, \mathfrak{a})$ is a sequence of measurable real-valued functions then the pointwise operations $\sup_{n\in\mathbb{N}} f_n$, $\inf_{n\in\mathbb{N}} f_n$, $\max_{n\in\mathbb{N}} f_n$, $\min_{n\in\mathbb{N}} f_n$, $\limsup_{n\in\mathbb{N}} f_n$, and $\liminf_{n\in\mathbb{N}} f_n$ yield again measurable functions.

iii.) If a sequence of measurable functions $f_n \in \mathscr{M}(X, \mathfrak{a})$ converges pointwise to a function $f \in \operatorname{Map}(X, \mathbb{K})$ then $f \in \mathscr{M}(X, \mathfrak{a})$, too.

iv.) If $A \subseteq X$ is measurable and $f \in \mathscr{M}(X, \mathfrak{a})$ then the restriction $f\big|_A$ is measurable with respect to the σ-algebra

$$\mathfrak{a}\big|_A = \big\{B \subseteq A \;\big|\; B \in \mathfrak{a}\big\} = \big\{B \cap A \;\big|\; B \in \mathfrak{a}\big\} \tag{1.2.8}$$

of A.

v.) Conversely, suppose that $A_n \subseteq X$ is a sequence of measurable subsets of X with $X = \bigcup_{n\in\mathbb{N}} A_n$. If $f_n \in \mathscr{M}(A_n, \mathfrak{a}\big|_{A_n})$ with

$$f_n\Big|_{A_n\cap A_m} = f_m\Big|_{A_n\cap A_m} \tag{1.2.9}$$

are given then there exists a unique function $f \in \operatorname{Map}(X, \mathbb{K})$ with

$$f\Big|_{A_n} = f_n, \tag{1.2.10}$$

which is measurable again. Note that the existence of a necessarily unique function f with (1.2.10) is trivial. It is the measurability which requires some work, see Exercise 1.4.8.

vi.) Again, let $A_n \subseteq X$ be a sequence of measurable subsets of X with $X = \bigcup_{n\in\mathbb{N}} A_n$. Then $f \in \operatorname{Map}(X, \mathbb{K})$ is measurable iff the restrictions $f\big|_{A_n}$ are measurable with respect to the restricted σ-algebras $\mathfrak{a}\big|_{A_n}$. This gives the reverse implication to *iv.)*, see again Exercise 1.4.8. With some more sophisticated language these two features show that measurable functions share a lot of the properties of a sheaf on a topological space. We will come back to this aspect later where we show that in certain situations this can be made more precise.

vii.) Measurable functions can always be extended from a measurable subset to the whole space. If $A \subseteq X$ is measurable and $f \in \mathscr{M}(A, \mathfrak{a}\big|_A)$ then the definition

$$f\Big|_{X\setminus A} = c \tag{1.2.11}$$

with $c \in \mathbb{K}$ gives again a measurable function $f \in \mathscr{M}(X, \mathfrak{a})$. This is of course very different from e.g. continuous functions which cannot always be extended from a subset to the whole space.

In some sense, the measurable functions will typically be the largest class of functions for which one can say something (though not much) interesting.

An important subset of measurable functions are given by the simple functions. They turn out to be of crucial importance for many constructions.

Definition 1.2.4 (Simple functions) *Let $(X, \mathfrak{a})$ be a measurable space. A measurable function $f \in \mathscr{M}(X, \mathfrak{a})$ is called simple if it has only a finite number of values.*

If $z_1, \ldots, z_N \in \mathbb{K}$ are the finitely many pairwise distinct values of a simple function f then $A_n = f^{-1}(\{z_n\}) \subseteq X$ is a measurable subset since $\{z_n\} \subseteq \mathbb{K}$ is measurable. Hence

$$f = \sum_{n=1}^{N} z_n \chi_{A_n} \tag{1.2.12}$$

is a finite linear combination of the characteristic functions χ_{A_n} of measurable subsets. Recall that the *characteristic function* $\chi_B \in \operatorname{Map}(X, \mathbb{K})$ of a subset $B \subseteq X$ is defined by

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{else.} \end{cases} \tag{1.2.13}$$

Since for a measurable $A \subseteq X$ we clearly have $\chi_A \in \mathscr{M}(X, \mathfrak{a})$ this gives an equivalent characterization of simple functions: f is simple iff it is in the $\mathbb{K}$-span of the characteristic functions of the measurable subsets of X. This shows that we have many simple functions.

This richness of simple functions results e.g. in the following approximation property:

Proposition 1.2.5 *Let $(X, \mathfrak{a})$ be a measurable space. Then for every $f \in \mathscr{BM}(X, \mathfrak{a})$ and every $\varepsilon > 0$ there exists a simple function g with*

$$\sup_{x\in X} |f(x) - g(x)| < \varepsilon. \tag{1.2.14}$$

PROOF: Decomposing f into the positive and negative parts of the real and imaginary parts, we see that it suffices to construct a simple function g for a given function $f \in \mathscr{BM}(X, \mathfrak{a})$ with $f \geq 0$. Consider then the subsets

$$A_{n,i} = f^{-1}\big([i2^{-n}, (i+1)2^{-n})\big)$$

for $n \in \mathbb{N}$ and $i = 0, \ldots, n2^n - 1$. Since f is bounded, we can fix n to be large enough with

$$\sup_{x\in X} |f(x)| < n.$$

Then the subsets $\{A_{n,i}\}_{i=0,\ldots,n2^n-1}$ are measurable as pre-images of the measurable intervals and they cover X disjointly, i.e.

$$X = \bigcup_{i=0}^{n2^n-1} A_{n,i}$$

is a disjoint union. Now we define

$$g_n = \sum_{i=0}^{n2^n-1} i2^{-n} \chi_{A_{n,i}}.$$

Then for $x \in A_{n,i}$ we get the estimate

$$g_n(x) = i2^{-n} \leq f(x) < (i+1)2^{-n} = g_n(x) + 2^{-n}$$

and hence $|f(x) - g_n(x)| < 2^{-n}$ for all $x \in X$. Taking n now large enough yields the function g we are looking for. □

1.2.2 Essentially Bounded Functions

Sometimes non-bounded functions behave still very much like bounded ones if the non-boundedness happens only on negligible subsets. To make this more precise we consider again a measurable space $(X, \mathfrak{a})$. Then we need to specify a collection of subsets which we do not want to take into account for questions of boundedness. Here a good choice would be the class of *zero sets* with respect to a given positive measure on $(X, \mathfrak{a})$. Now the measure itself will not be significant since we are only interested in its zero sets. They form a σ-*ideal* in the σ-algebra $\mathfrak{a}$ and hence we require only the specification of a σ-ideal $\mathfrak{n} \subseteq \mathfrak{a}$.

Definition 1.2.6 (Essentially bounded functions) *Let $(X, \mathfrak{a}, \mathfrak{n})$ be a measurable space with a σ-ideal $\mathfrak{n} \subseteq \mathfrak{a}$. Moreover, let $f \in \mathscr{M}(X, \mathfrak{a})$ be a measurable function.*

i.) The essential range of f with respect to $\mathfrak{n}$ are the points

$$\operatorname{ess\,range}(f) = \big\{ z \in \mathbb{K} \bigm| f^{-1}(\mathrm{B}_\varepsilon(z)) \notin \mathfrak{n} \textit{ for all } \varepsilon > 0 \big\}. \tag{1.2.15}$$

ii.) The essential supremum of $|f|$ with respect to $\mathfrak{n}$ is

$$\operatorname*{ess\,sup}_{x \in X}(|f|) = \sup\big\{ |z| \bigm| z \in \operatorname{ess\,range}(f) \big\}. \tag{1.2.16}$$

iii.) The function f is called essentially bounded with respect to $\mathfrak{n}$ if $\operatorname{ess\,sup}_{x\in X}(f) < \infty$. The set of all essentially bounded functions with respect to $\mathfrak{n}$ is denoted by

$$\mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n}) = \big\{ f \in \mathscr{M}(X, \mathfrak{a}) \bigm| f \textit{ is essentially bounded} \big\}. \tag{1.2.17}$$

If we have a positive measure μ on $(X, \mathfrak{a})$, i.e. if $(X, \mathfrak{a}, \mu)$ is a *measure space*, then the μ-zero sets

$$\mathfrak{n}(\mu) = \big\{ A \in \mathfrak{a} \bigm| \mu(A) = 0 \big\} \subseteq \mathfrak{a} \tag{1.2.18}$$

form a σ-ideal. In this case we always use this σ-ideal and write

$$\mathscr{L}^\infty(X, \mathfrak{a}, \mu) = \mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n}(\mu)) \tag{1.2.19}$$

for the essentially bounded functions with respect to μ. Nevertheless, the definition only refers to the σ-ideal and not to a given measure. Some first properties of the essential range and the essential supremum are discussed in Exercise 1.4.19.

In particular, $\mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$ is a subspace of all measurable functions $\mathscr{M}(X, \mathfrak{a})$. It contains the bounded measurable functions

$$\mathscr{BM}(X, \mathfrak{a}) \subseteq \mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n}). \tag{1.2.20}$$

1.2.3 Locally Bounded Functions

Let X be a topological space with topology $\mathscr{X} \subseteq 2^X$. Then we can define several function classes making use of the topology. First we recall that a topology uniquely determines a smallest σ-algebra $\mathfrak{a}(\mathscr{X}) = \mathfrak{a}_{\mathrm{Borel}}(X)$ containing the topology $\mathscr{X}$, the *Borel σ-algebra* of $(X, \mathscr{X})$.

Unless stated otherwise, a topological space will always be equipped with this Borel σ-algebra and all notions involving measurability will refer to $\mathfrak{a}_{\mathrm{Borel}}(X)$.

Definition 1.2.7 (Locally bounded functions) *Let $(X, \mathscr{X})$ be a topological space and let $f\colon X \longrightarrow \mathbb{K}$ be a function. Then f is called locally bounded if for every $x \in X$ one finds a neighbourhood $U \subseteq X$ of x such that f is bounded on U. The set of locally bounded functions is denoted by*

$$\mathscr{B}_{\mathrm{loc}}(X) = \big\{ f \in \operatorname{Map}(X, \mathbb{K}) \bigm| f \textit{ is locally bounded} \big\}. \tag{1.2.21}$$

Again, it is easy to see that $\mathscr{B}_{\mathrm{loc}}(X)$ is a subspace of all functions and in fact also a unital subalgebra. Moreover, one has

$$\mathscr{B}(X) \subseteq \mathscr{B}_{\mathrm{loc}}(X). \tag{1.2.22}$$

Sometimes it will be useful to require in addition that the functions are measurable. This yields the subspace of measurable locally bounded functions

$$\mathscr{BM}_{\mathrm{loc}}(X) \subseteq \mathscr{B}_{\mathrm{loc}}(X) \cap \mathscr{M}(X, \mathfrak{a}_{\mathrm{Borel}}(X)), \tag{1.2.23}$$

containing the bounded measurable functions as a subspace.

For a general topological space one cannot say much about $\mathscr{B}_{\mathrm{loc}}(X)$. This changes whenever X is locally compact and Hausdorff.

Proposition 1.2.8 *Let X be a topological space.*

i.) A locally bounded function $f \in \mathscr{B}_{\mathrm{loc}}(X)$ satisfies

$$\sup_K |f| < \infty \tag{1.2.24}$$

for all compact subsets $K \subseteq X$.

ii.) Suppose that X is in addition locally compact and Hausdorff. Then a function $f \in \mathrm{Map}(X, \mathbb{K})$ is locally bounded iff for all compact subsets $K \subseteq X$ one has

$$\sup_{x \in K} |f(x)| < \infty. \tag{1.2.25}$$

iii.) Suppose that $(f_i)_{i \in I}$ is a net of locally bounded functions such that for all $x \in X$ we have pointwise convergence $f_i(x) \longrightarrow f(x)$ to a function $f \colon X \longrightarrow \mathbb{K}$ and such that for all $\varepsilon > 0$ and all points $x_0 \in X$ one finds an open neighbourhood $U \subseteq X$ of x_0 and an index $i_0 \in I$ such that

$$\sup_{x \in U} |f_i(x) - f(x)| < \varepsilon \tag{1.2.26}$$

for all later indices $i \succcurlyeq i_0$. Then $f \in \mathscr{B}_{\mathrm{loc}}(X)$.

Proof: Assume $f \in \mathscr{B}_{\mathrm{loc}}(X)$ and let $K \subseteq X$ be compact. Then for $x \in K$ we find a neighbourhood $U_x \subseteq X$ of x such that $\sup_{y \in U_x} |f(y)| < \infty$. Without restriction, we can choose U_x to be open. Hence we get an open cover $\{U_x\}_{x \in X}$ of K having a finite subcover, say

$$K \subseteq U_{x_1} \cup \cdots \cup U_{x_N}.$$

Since $\sup_{x \in U_{x_i}} |f(x)| < \infty$ for $i = 1, \ldots, N$ we can conclude that also $\sup_{x \in K} |f(x)|$ is finite, namely the maximum of the previous suprema. This gives the first part. Next, suppose that f satisfies (1.2.25) for all compact subsets K. If X is a locally compact Hausdorff space, every point $x \in X$ has a neighbourhood basis of compact subsets. On these, f is bounded and hence locally bounded, proving the second part. For the third part, we note that the condition (1.2.26) implies the pointwise convergence of $f_i(x_0)$ to $f(x_0)$ since $x_0 \in U$. We included the assumption of pointwise convergence just for clarification. Now let $x_0 \in X$ and choose $\varepsilon > 0$. Let an open neighbourhood $U \subseteq X$ of x_0 and an index $i_0 \in I$ be given such that (1.2.26) holds. Without restriction, we can assume that f_{i_0} is already bounded on U, otherwise we shrink U further. We get for all $x \in U$

$$|f(x)| \leq |f(x) - f_{i_0}(x)| + |f_{i_0}(x)| < \varepsilon + |f_{i_0}(x)|,$$

so that

$$\sup_{x \in U} |f(x)| \leq \varepsilon + \sup_{x \in U} |f_{i_0}(x)| < \infty.$$

This shows $f \in \mathscr{B}_{\mathrm{loc}}(X)$. □

If X is a locally compact Hausdorff space then the convergence condition for the second part can be replaced by the following: for all $\varepsilon > 0$ and all compact subsets $K \subseteq X$ one finds an index $i_0 \in I$ with

$$\sup_{x \in K} |f_i(x) - f(x)| < \varepsilon \tag{1.2.27}$$

if $i \succcurlyeq i_0$ is late enough. The convergence condition (1.2.26) will also be referred to as *locally uniform convergence* of locally bounded functions.

The significance of the first part and (1.2.27) is that we have a common class of subsets on which we have to test for boundedness or convergence, instead of the individual open subsets.

We have now several variations of the locally bounded functions. One important generalization is to consider locally essentially bounded functions where we replace the supremum by the essential supremum. Here we need to specify a σ-ideal of zero sets inside $\mathfrak{a}_{\mathrm{Borel}}(X)$. In many situations this comes either as the actual zero sets of a specific measure or by further features of the underlying topological space.

Definition 1.2.9 (Locally essentially bounded functions) *Let $(X, \mathscr{X})$ be a topological space with a σ-ideal $\mathfrak{n} \subseteq \mathfrak{a}_{Borel}(X)$.*

i.) A measurable function $f\colon X \longrightarrow \mathbb{K}$ is called locally essentially bounded if for every $x \in X$ one finds an open neighbourhood $U \subseteq X$ such that $\operatorname{ess\,sup}_{x \in U} |f(x)| < \infty$ *with respect to $\mathfrak{n}\big|_U$.*

ii.) The space of locally essentially bounded functions is denoted by

$$\mathscr{L}^\infty_{\mathrm{loc}}(X, \mathfrak{n}) = \big\{ f \in \mathscr{M}(X, \mathfrak{a}_{\mathrm{Borel}}(X)) \bigm| f \text{ is locally essentially bounded} \big\}. \tag{1.2.28}$$

Again it is straightforward to see that $\mathscr{L}^\infty_{\mathrm{loc}}(X, \mathfrak{n})$ is a subspace of $\mathscr{M}(X, \mathfrak{a}_{\mathrm{Borel}}(X))$ containing

$$\mathscr{B}\mathscr{M}_{\mathrm{loc}}(X) \subseteq \mathscr{L}^\infty_{\mathrm{loc}}(X, \mathfrak{n}) \tag{1.2.29}$$

as a subspace. For pointwise convergent sequences of functions in $\mathscr{L}^\infty_{\mathrm{loc}}(X, \mathfrak{n})$ we get analogous statements to Proposition 1.2.8, see Exercise 1.4.9.

Remark 1.2.10 Unlike bounded functions, locally bounded functions can be glued together in the same spirit as measurable functions: let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of M and let $f_\alpha \in \mathscr{B}_{\mathrm{loc}}(U_\alpha)$ be given such that

$$f_\alpha\big|_{U_\alpha \cap U_\beta} = f_\beta\big|_{U_\alpha \cap U_\beta}, \tag{1.2.30}$$

whenever the overlap $U_\alpha \cap U_\beta \neq \emptyset$ is nontrivial. Then the unique function $f\colon X \longrightarrow \mathbb{K}$ with $f\big|_{U_\alpha} = f_\alpha$ is again locally bounded. Of course, even if all f_α are bounded, this needs not to be true for f as trivial examples show. Thus local boundedness is indeed a local property while boundedness is not. Note that we can test local boundedness locally in the sense that $f\colon X \longrightarrow \mathbb{K}$ is locally bounded iff for an open cover $\{U_\alpha\}_{\alpha \in I}$ the restrictions $f\big|_{U_\alpha}$ are locally bounded.

1.2.4 Continuous Functions

While the locally bounded functions on a topological space already have some interesting structure, it is the class of continuous functions one is interested most.

In general, a topological space X need not admit any continuous functions beside the constant functions at all. The lack of good separation properties might forbid that. However, if X has reasonable separation properties then one has many continuous functions: We define the set of continuous functions as usual:

Definition 1.2.11 (Continuous functions) *Let $(X, \mathscr{X})$ and $(Y, \mathscr{Y})$ be topological spaces.*

i.) A map $f\colon X \longrightarrow Y$ is called continuous if the pre-image of any open subset of Y is again open in X.

ii.) The set of continuous functions is denoted by

$$\mathscr{C}(X) = \big\{ f \in \operatorname{Map}(X, \mathbb{K}) \bigm| f \text{ is continuous} \big\}. \tag{1.2.31}$$

Of course, there are many equivalent definitions of continuity, the above is the most conceptual but perhaps not too explicit one. Recall that a map $f\colon X \longrightarrow Y$ between topological spaces is called *continuous at* $x \in X$ if for every convergent net $(x_i)_{i \in I}$ with $\lim_{i \in I} x_i = x$ the net $(f(x_i))_{i \in I}$ in Y converges as well and

$$f\Big(\lim_{i \in I} x_i\Big) = \lim_{i \in I} f(x_i). \tag{1.2.32}$$

Then f is continuous iff f is continuous at every point in X. This and many more properties of continuous functions can be found in any textbook on general topology like e.g. [8, 19]. We will frequently use such results without further reference.

Continuous maps are necessarily measurable with respect to the Borel σ-algebras of the underlying topological spaces. In fact, assigning the Borel σ-algebra to a topological space gives a functor from the category of topological spaces to the category of measurable spaces, see Exercise 1.4.20. In particular, we get

$$\mathscr{C}(X) \subseteq \mathscr{M}(X, \mathfrak{a}_{\mathrm{Borel}}(X)). \tag{1.2.33}$$

Since addition and multiplication are continuous operations on $\mathbb{K}$, we infer that $\mathscr{C}(X)$ is a subspace with respect to the pointwise operations, and, in fact, a unital subalgebra. In case of $\mathbb{K} = \mathbb{C}$, also the complex conjugation preserves continuity, turning $\mathscr{C}(X)$ into a unital *-algebra inside all functions.

The next observation is that continuous functions are always locally bounded:

Proposition 1.2.12 *Let* $(X, \mathscr{X})$ *be a topological space.*

i.) *One has*

$$\mathscr{C}(X) \subseteq \mathscr{B}\mathscr{M}_{\mathrm{loc}}(X, \mathfrak{a}_{\mathrm{Borel}}(X)). \tag{1.2.34}$$

ii.) *If* $(f_i)_{i \in I}$ *is a net of continuous functions which converges locally uniformly to* f *then* f *is again continuous.*

Proof: Let $x \in X$ be given and let $\varepsilon > 0$. Then the pre-image $f^{-1}(\mathrm{B}_\varepsilon(f(x))) \subseteq X$ is an open neighbourhood of x on which f takes values in $\mathrm{B}_\varepsilon(f(x))$. Hence f is bounded on this neighbourhood, showing (1.2.34) as we already know (1.2.33). For the second part, let $x_0 \in X$ and assume that $f_i \longrightarrow f$ locally uniformly. We want to show that f is continuous at x_0. We choose $\varepsilon > 0$ and find first an open neighbourhood $U \subseteq X$ of x_0 and an index $i_0 \in I$ such that

$$\sup_{x \in U} |f(x) - f_i(x)| < \frac{\varepsilon}{3} \tag{☺}$$

for all $i \succcurlyeq i_0$ by locally uniform convergence. Now let $(x_j)_{j \in J}$ be a net converging to x_0. Hence we find an index $j_0 \in J$ such that for $j \succcurlyeq j_0$ we have $x_j \in U$. Moreover, we know that f_{i_0} is continuous at x_0 and hence we find an index $j_1 \in J$ with $|f_{i_0}(x_0) - f_{i_0}(x_j)| < \frac{\varepsilon}{3}$ for all $j \succcurlyeq j_1$. Let $k \in J$ be such that $k \succcurlyeq j_0, j_1$. Then we have for all $j \succcurlyeq k$

$$|f(x_0) - f(x_j)| \le |f(x_0) - f_{i_0}(x_0)| + |f_{i_0}(x_0) - f_{i_0}(x_j)| + |f_{i_0}(x_j) - f(x_j)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

since the first and third contribution can be estimated by (☺) while the second can be estimated since $j \succcurlyeq k \succcurlyeq j_1$. Thus f is continuous at x_0, showing the second part. □

Again, if X is even a locally compact Hausdorff space it suffices to test convergence on the compact subsets. The net $(f_i)_{i \in I}$ converges locally uniformly to f iff for all $\varepsilon > 0$ and for all compact subsets $K \subseteq X$ one finds an index $i_0 \in I$ with

$$\sup_{x \in K} |f(x) - f_i(x)| < \varepsilon, \tag{1.2.35}$$

whenever $i \succcurlyeq i_0$, see also (1.2.27). This will provide a substantial simplification. Hence we will mainly focus on this situation later on.

Remark 1.2.13 (Gluing continuous functions) Continuity is again a local property in the following sense. Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of X and let $f\colon X \longrightarrow \mathbb{K}$ be a map. Then f is continuous iff $f\big|_{U_\alpha}$ is continuous. Moreover, if $f_\alpha \in \mathscr{C}(U_\alpha)$ is given with $f_\alpha\big|_{U_\alpha\cap U_\beta} = f_\beta\big|_{U_\alpha\cap U_\beta}$ then the unique function $f\colon X \longrightarrow \mathbb{K}$ with $f\big|_{U_\alpha} = f_\alpha$ is continuous. We will frequently make use of these gluing properties of continuous functions.

While continuous functions are always locally bounded they do not need to be bounded. Hence it is interesting to consider the *bounded continuous functions*. We set

$$\mathscr{C}_{\mathrm{b}}(X) = \mathscr{C}(X) \cap \mathscr{B}(X) = \big\{f \in \mathrm{Map}(X, \mathbb{K}) \;\big|\; f \text{ is bounded and continuous}\big\} \tag{1.2.36}$$

to denote the subspace of continuous bounded functions.

As in Proposition 1.2.1 we can guarantee convergence inside $\mathscr{C}_{\mathrm{b}}(X)$ as follows:

Proposition 1.2.14 *Let X be a topological space and let $(f_i)_{i\in I}$ be a net of bounded continuous functions. Suppose there exists a function $f\colon X \longrightarrow \mathbb{K}$ with the property that for all $\varepsilon > 0$ there exists an index $i_0 \in I$ with*

$$\sup_{x\in X} |f(x) - f_i(x)| < \varepsilon, \tag{1.2.37}$$

whenever $i \succcurlyeq i_0$. Then $f \in \mathscr{C}_{\mathrm{b}}(X)$.

Proof: Indeed, Proposition 1.2.1 gives $f \in \mathscr{B}(X)$ and Proposition 1.2.12, *ii.)*, gives $f \in \mathscr{C}(X)$ since clearly the uniform convergence (1.2.37) implies the locally uniform convergence. □

Of course, the weaker form of locally uniform convergence of bounded continuous functions can very well result in unbounded continuous function in the limit.

We can conclude this discussion by listing some examples of locally compact Hausdorff spaces:

Example 1.2.15 (Locally compact Hausdorff spaces)

i.) If I is an index set as used in Section 1.1 then we can endow it with the discrete topology. This way, I becomes a locally compact Hausdorff space and we get

$$\mathscr{C}(I) = \mathscr{M}(I) = \mathscr{B}_{\mathrm{loc}}(I) = \mathscr{BM}_{\mathrm{loc}}(I) = \mathrm{Map}(I, \mathbb{K}), \tag{1.2.38}$$

as well as

$$\mathscr{C}_{\mathrm{b}}(I) = \mathscr{B}(I) = \mathscr{BM}(I) = \ell^\infty(I). \tag{1.2.39}$$

Moreover, if we take the trivial σ-ideal $\mathfrak{n} = \{\emptyset\}$ then we get in addition

$$\mathscr{L}^\infty(I) = \ell^\infty(I). \tag{1.2.40}$$

ii.) An open subset $X \subseteq \mathbb{R}^n$ is a locally compact Hausdorff space of particular interest, where we take the usual topology induced by $\mathbb{R}^n$. We will mainly be interested in this class of examples.

iii.) Slightly more general, a topological manifold X is a locally compact Hausdorff space. Recall that a topological Hausdorff space is called a topological manifold of dimension $n \in \mathbb{N}_0$ if every point $x \in X$ has an open neighbourhood $U \subseteq X$ such that U is homeomorphic to an open subset of $\mathbb{R}^n$ and if X is second countable. This last condition has many useful consequences. In particular, X admits a sequence of nested compact subsets

$$K_1 \subseteq K_2^\circ \subseteq K_2 \subseteq \cdots \subseteq K_i \subseteq K_{i+1}^\circ \subseteq K_{i+1} \subseteq \cdots \subseteq X, \tag{1.2.41}$$

such that

$$X = \bigcup_{k=0}^\infty K_i. \tag{1.2.42}$$

Note that an open subset $X \subseteq \mathbb{R}^n$ is a topological manifold of dimension n, see also Exercise 1.4.14 for an explicit construction of an exhausting sequence of compact subsets in this case.

iv.) Every compact Hausdorff space X is also locally compact but of course not vice versa. We will see many examples of such spaces which have a very infinite-dimensional flavour. They are not topological manifolds by any means.

1.2.5 Compactly Supported Continuous Functions

Inside all continuous functions one can consider those which have compact support. We recall the definition of the support:

Definition 1.2.16 (Support) *Let $(X, \mathscr{X})$ be a topological space and let $f \in \mathscr{C}(X)$ be a continuous function. Then the support of f is defined by*

$$\operatorname{supp}(f) = \{x \in X \mid f(x) \neq 0\}^{\mathrm{cl}}. \tag{1.2.43}$$

For a closed subset A we define

$$\mathscr{C}_A(X) = \{f \in \mathscr{C}(X) \mid \operatorname{supp}(f) \subseteq A\}, \tag{1.2.44}$$

and set

$$\mathscr{C}_0(X) = \{f \in \mathscr{C}(X) \mid \operatorname{supp}(f) \textit{ is compact}\}. \tag{1.2.45}$$

Of course, one can also define the support for other classes of functions with other regularity than continuity. However, it becomes most useful in the continuous framework. By definition, $\operatorname{supp}(f)$ is a closed subset. Moreover, for $f, g \in \mathscr{C}(X)$ and $z, w \in \mathbb{K}$ we have

$$\operatorname{supp}(zf + wg) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g). \tag{1.2.46}$$

This shows immediately that $\mathscr{C}_A(X)$ is a subspace of $\mathscr{C}(X)$ and $\mathscr{C}_0(X)$ is a subspace as well. Moreover, if $A \subseteq A'$ are closed subsets then

$$\mathscr{C}_A(X) \subseteq \mathscr{C}_{A'}(X). \tag{1.2.47}$$

Finally, we get

$$\mathscr{C}_0(X) = \bigcup_{K \subseteq X \text{ compact}} \mathscr{C}_K(X). \tag{1.2.48}$$

The product of functions also behaves well with respect to supports. In fact, one has

$$\operatorname{supp}(fg) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g) \tag{1.2.49}$$

for $f, g \in \mathscr{C}(X)$. This shows that all conditions on supports lead to ideals in $\mathscr{C}(X)$. In more detail, for closed subsets $A, A' \subseteq X$ we get

$$\mathscr{C}_A(X) \cdot \mathscr{C}_{A'}(X) \subseteq \mathscr{C}_{A \cap A'}(X), \tag{1.2.50}$$

all $\mathscr{C}_A(X) \subseteq \mathscr{C}(X)$ are ideals and also

$$\mathscr{C}_0(X) \subseteq \mathscr{C}(X) \tag{1.2.51}$$

is an ideal in $\mathscr{C}(X)$. Of course, the inclusion (1.2.51) might be trivial, as e.g. for a compact space X we trivially get equality in (1.2.51).

While in general one cannot say much about the amount of compactly supported functions, on a locally compact Hausdorff space we have quite an abundance:

Proposition 1.2.17 *Let X be a locally compact Hausdorff space.*

i.) Let $K \subseteq X$ be compact. Then there exists a function $\chi \in \mathscr{C}_0(X)$ with $\chi\big|_K = 1$. More precisely, if $O \subseteq X$ is open with $K \subseteq O$ then one can arrange $\operatorname{supp}(\chi) \subseteq O$.

ii.) Let $f \in \mathscr{C}(X)$. Then there exists a net $\{g_i\}_{i \in I}$ of continuous functions $g_i \in \mathscr{C}_0(X)$ with compact support which converges locally uniformly to f.

PROOF: Let $K \subseteq X$ be compact. For every point $x \in K$ we find an open neighbourhood $U_x \subseteq X$ with compact closure since every point in a locally compact Hausdorff space has a compact neighbourhood by definition. Now $K \subseteq \bigcup_{x \in K} U_x$ and hence we have finitely many, say $U_{x_1}, \ldots, U_{x_N}$, which already cover $K \subseteq U_{x_1} \cup \cdots \cup U_{x_N}$. We set $U = U_{x_1} \cup \cdots \cup U_{x_N}$ and get $U^{\mathrm{cl}} = U_{x_1}^{\mathrm{cl}} \cup \cdots \cup U_{x_N}^{\mathrm{cl}}$ since we have a finite union. Thus U^{cl} is again compact since each $U_{x_i}^{\mathrm{cl}}$ is compact. Now let O be some open subset with $K \subseteq O$. Then also $U \cap O$ is an open neighbourhood of K, now with compact closure $(U \cap O)^{\mathrm{cl}}$. From the Urysohn Lemma for locally compact Hausdorff spaces, see e.g. [19, Cor. 6.1.3], we infer the existence of a continuous function $\chi \in \mathscr{C}(X)$ with $\operatorname{supp}(\chi) \subseteq U \cap O$ and $\chi\big|_K = 1$. Then $\operatorname{supp}(\chi) \subseteq (U \cap O)^{\mathrm{cl}}$ is compact as claimed. For the second part, we take the set of all compact subsets $I = \{K \subseteq X \mid K \text{ is compact}\}$ with its direction by inclusion: $K \preccurlyeq K'$ if $K \subseteq K'$. This is indeed a directed set. For K we choose now a function $\chi_K \in \mathscr{C}_0(X)$ with $\chi\big|_K = 1$ according to the first part. If $f \in \mathscr{C}(X)$ is an arbitrary continuous function then $(f\chi_K)_{K \in I}$ is a net in $\mathscr{C}_0(X)$ converging locally uniformly to f since for a given compact subset $K \subseteq X$ we simply have the equality $f\big|_{K'} = (f\chi_{K'})\big|_{K'}$, whenever $K' \succcurlyeq K$. □

In particular, $\mathscr{C}_0(X)$ contains many interesting functions. As an example, we note that for a discrete index set we have

$$\mathscr{C}_0(I) = c_{\circ\circ}(I), \tag{1.2.52}$$

since the only compact subsets of I are finite subsets.

1.3 Differentiable Functions

The classes of differentiable functions are perhaps the most important ones in functional analysis whenever one is heading towards applications. But also from a more conceptual point of view these function spaces provide a vast testing ground for various notions in locally convex analysis.

In this section, $X \subseteq \mathbb{R}^n$ will always be an open non-empty subset. Slightly more general, most notions will still make sense if X is replaced by a smooth n-dimensional manifold. Nevertheless, we will mainly be focusing on the local situation in order to avoid the more sophisticated global aspects of function spaces on general smooth manifolds. An appropriate treatment would require some more elaborate technology than we can discuss here.

1.3.1 The Function Spaces $\mathscr{C}^k(X)$

Asking for differentiability of a function $f\colon X \longrightarrow \mathbb{K}$ will not yield interesting function spaces itself since the derivative $Df\colon X \times \mathbb{R}^n \longrightarrow \mathbb{K}$ can still be very complicated. This changes if one requires the derivative to be again continuous. More generally, one iterates this also for higher derivatives leading to the following spaces:

Definition 1.3.1 (Continuously differentiable functions) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.*

i.) A function $f\colon X \longrightarrow \mathbb{K}$ is called of class $\mathscr{C}^k$ for $k \in \mathbb{N}_0$ if f is k-times continuously differentiable.

ii.) A function $f\colon X \longrightarrow \mathbb{K}$ is called smooth or of class $\mathscr{C}^\infty$ if f is of class $\mathscr{C}^k$ for all $k \in \mathbb{N}_0$.

iii.) *One defines*

$$\mathscr{C}^k(X) = \{ f \in \operatorname{Map}(X, \mathbb{K}) \mid f \text{ is } \mathscr{C}^k \}. \tag{1.3.1}$$

The case $k = 0$ simply means to have a continuous function, i.e. we have

$$\mathscr{C}^0(X) = \mathscr{C}(X). \tag{1.3.2}$$

Clearly, all the sets $\mathscr{C}^k(X)$ are subspaces and even subalgebras with

$$\mathscr{C}^k(X) \subseteq \mathscr{C}^\ell(X), \tag{1.3.3}$$

whenever $k \geq \ell$ with $k, \ell \in \mathbb{N}_0 \cup \{\infty\}$.

Having these function spaces, the next step is to transfer the notions of convergence of continuous functions to the present setting. It turns out that the good analog of locally uniform convergence is given by locally uniform convergence of all derivatives. To formulate this efficiently we introduce some standard notation for partial derivatives first.

Let $f \in \mathscr{C}^k(X)$ and let $\ell \in \mathbb{N}_0$ with $\ell \leq k$. Then we consider a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with length

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \leq \ell \tag{1.3.4}$$

to abbreviate the α-th partial derivative of f by

$$\frac{\partial^{|\alpha|} f}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial (x^1)^{\alpha_1} \cdots \partial (x^n)^{\alpha_n}}. \tag{1.3.5}$$

Note that we use the convention that basis vectors $\mathrm{e}_1, \ldots, \mathrm{e}_n \in \mathbb{R}^n$ carry their indices as subscripts while the corresponding linear coordinates $x^1, \ldots, x^n$ have their indices as superscripts. This is the common convention in differential geometry allowing us to use Einstein's summation convention, i.e. we write $x = x^i \mathrm{e}_i$ instead of $x = \sum_{i=1}^n x^i \mathrm{e}_i$ whenever we have to use coordinate expressions.

We also use the abbreviation

$$\alpha! = \alpha_1! \cdots \alpha_n! \tag{1.3.6}$$

for $\alpha \in \mathbb{N}_0^k$ as well as the usual componentwise arithmetic operations like addition of multiindices. Finally, we say $\alpha \leq \beta$ for $\alpha, \beta \in \mathbb{N}_0^k$ if one has $\alpha_i \leq \beta_i$ for all $i = 1, \ldots, n$.

With this notation, the first convergence result which we recall, is the following:

Proposition 1.3.2 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. Suppose $(f_i)_{i \in I}$ is a net of $\mathscr{C}^k$-functions $f_i \in \mathscr{C}^k(X)$ with $k \in \mathbb{N}_0 \cup \{\infty\}$ and $g_\alpha \in \operatorname{Map}(X, \mathbb{K})$ are functions for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$. If for all compact subsets $K \subseteq X$ and all $\varepsilon > 0$ and all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ one finds an index $i_0 \in I$ with*

$$\sup_{x \in K} \left| \frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(x) - g_\alpha(x) \right| < \varepsilon, \tag{1.3.7}$$

whenever $i \succcurlyeq i_0$ then $g = g_0 \in \mathscr{C}^k(X)$ and $g_\alpha = \frac{\partial^{|\alpha|} g}{\partial x^\alpha}$ for all allowed values of α.

PROOF: We know that the statement holds for $k = 0$, this is just a particular case of Proposition 1.2.12 and the observation (1.2.35) for the locally compact Hausdorff space X. Applying this argument to the derivatives $(\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha})_{i \in I}$ and g_α instead, shows that $g_\alpha \in \mathscr{C}^0(X)$ is continuous for all allowed values of α. It remains to show that $g = g_0$ is $\mathscr{C}^k$ and g_α is the α-th partial derivative of g. We proceed inductively on $|\alpha|$. For $|\alpha| = 1$ we consider $j = 1, \ldots, n$ and the difference

$$g(x + t\mathrm{e}_j) - g(x) \overset{(a)}{=} \lim_{i \in I} (f_i(x + t\mathrm{e}_j) - f_i(x))$$

$$
\begin{aligned}
&\overset{(b)}{=} \lim_{i\in I} \int_0^t \frac{\partial f_i}{\partial x^j}(x+\tau \mathrm{e}_j)\,\mathrm{d}\tau \\
&\overset{(c)}{=} \int_0^t \lim_{i\in I} \frac{\partial f_i}{\partial x^j}(x+\tau \mathrm{e}_j)\,\mathrm{d}\tau \\
&\overset{(d)}{=} \int_0^t g_j(x+\tau \mathrm{e}_j)\,\mathrm{d}\tau,
\end{aligned}
$$

for $x \in X$ and $t \in \mathbb{R}$ small enough such that $x + \tau \mathrm{e}_j \in X$ for all $0 \le \tau \le t$. Here we used the convergence of $(f_i)_{i\in I}$ to g in (a), the fundamental theorem of calculus for the $\mathscr{C}^1$-function f_i in (b). Since the limit of the derivatives $(\frac{\partial f_i}{\partial x^j})_{i\in I}$ is still locally uniform on every compact subset, it is uniform on the compact line segment from x to $x + \tau \mathrm{e}_j$. Thus we can exchange the integration with the limit in (c), and use the convergence of $(\frac{\partial f_i}{\partial x^j})_{i\in I}$ to g_j in (d). Since g_j is still continuous, the fundamental theorem of calculus gives the result that g is differentiable with partial derivatives given by g_j. A simple iteration of this argument is now all we need to finish the proof. □

We will see a much more conceptual interpretation of this convergence result later. For the time being it is just one of these many approximation statements one knows from the elementary calculus courses.

As before the continuous functions, also $\mathscr{C}^k$-functions have nice gluing properties:

Remark 1.3.3 (Gluing $\mathscr{C}^k$-functions) Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of X and $f\colon X \longrightarrow \mathbb{K}$ a map. Then f is $\mathscr{C}^k$ iff all the restrictions $f\big|_{U_\alpha}\colon U_\alpha \longrightarrow \mathbb{K}$ are $\mathscr{C}^k$. In this sense, being $\mathscr{C}^k$ is a local property which we can test on small open neighbourhoods of each point. Of course, this is a fairly trivial statement essentially built into the definition of differentiability. Nevertheless, it has tremendous impact at many places. Similarly, we can glue $\mathscr{C}^k$-functions. If $f_\alpha \in \mathscr{C}^k(U_\alpha)$ are given for all $\alpha \in I$ with $f_\alpha\big|_{U_\alpha\cap U_\beta} = f_\beta\big|_{U_\alpha\cap U_\beta}$ whenever the overlap $U_\alpha \cap U_\beta \ne \emptyset$ is non-trivial, then the global function f with $f\big|_{U_\alpha} = f_\alpha$ is $\mathscr{C}^k$.

Before we proceed with more specific subspaces of $\mathscr{C}^k(X)$ we mention also the chain rule for the $\mathscr{C}^k$-functions. We will need a semi-explicit formula for the α-th partial derivative of a composition. The following formulation will be sufficient for our needs later on:

Proposition 1.3.4 *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be open non-empty subsets. Suppose $\phi\colon X \longrightarrow Y$ is $\mathscr{C}^k$ and $f \in \mathscr{C}^k(Y)$. Then the pull-back $\phi^* f = f \circ \phi$ is a $\mathscr{C}^k$-function on X and*

$$
\frac{\partial^{|\alpha|}(\phi^* f)}{\partial x^\alpha}(x) = \sum_{\ell=0}^{|\alpha|} \sum_{|\beta|\le|\alpha|} \sum_{|\gamma_1|+\cdots+|\gamma_\ell|=|\alpha|} c^\alpha_{\beta\gamma_1\cdots\gamma_\ell} \frac{\partial^{|\beta|} f}{\partial y^\beta}(\phi(x)) \frac{\partial^{|\gamma_1|}\phi^{\beta_1}}{\partial x^{\gamma_1}}(x) \cdots \frac{\partial^{|\gamma_\ell|}\phi^{\beta_\ell}}{\partial x^{\gamma_\ell}}(x), \tag{1.3.8}
$$

with some universal coefficients $c^\alpha_{\beta\gamma_1\cdots\gamma_\ell} \in \mathbb{Q}$.

Proof: For $|\alpha| = 0$ the formula is certainly correct with the only non-trivial coefficient $c^0_{00} = 1$. We prove the statement by induction: there are two aspects of (1.3.8). First, we have to show that $\phi^* f$ is actually differentiable at all and, second, we have to show the formula (1.3.8). Thus assume $k \ge 1$ and $|\alpha| = 1$. The chain rule from elementary calculus shows that $\phi^* f$ is again $\mathscr{C}^1$ with

$$
\frac{\partial(\phi^* f)}{\partial x^i}(x) = \sum_{j=1}^n \frac{\partial f}{\partial y^j}(\phi(y)) \frac{\partial \phi^j}{\partial x^i}(x), \tag{☺}
$$

which is (1.3.8) for $|\alpha| = 1$. Now assume the statement is valid for $|\alpha| < k$. Counting the derivatives on the right hand side shows that the right hand side is an algebraic combination of functions having at least a degree of differentiability $k - |\alpha| > 0$. Hence the right hand side is at least $\mathscr{C}^1$ showing that $\frac{\partial^{|\alpha|}(\phi^* f)}{\partial x^\alpha}$ is $\mathscr{C}^1$. Hence we can repeat the computation of one further partial derivative of the right hand side. This gives by the chain rule (☺) as well as the Leibniz rule for products

$$
\begin{aligned}
\frac{\partial}{\partial x^j} \frac{\partial^{|\alpha|}(\phi^* f)}{\partial x^\alpha}(x) &= \frac{\partial}{\partial x^j} \left(\sum_{\ell=0}^{|\alpha|} \sum_{|\beta| \leq |\alpha|} c^\alpha_{\beta\gamma_1\cdots\gamma_\ell} \frac{\partial^{|\beta|} f}{\partial y^\beta}(\phi(x)) \frac{\partial^{|\gamma_1|}\phi^{\beta_1}}{\partial x^{\gamma_1}}(x) \cdots \frac{\partial^{|\gamma_\ell|}\phi^{\beta_\ell}}{\partial x^{\gamma_\ell}}(x) \right) \\
&= \sum_{\ell=0}^{|\alpha|} \sum_{|\beta| \leq |\alpha|} c^\alpha_{\beta\gamma_1\cdots\gamma_\ell} \left(\frac{\partial^{|\beta|+1} f}{\partial y^\beta \partial y^\gamma}(\phi(x)) \frac{\partial \phi^\gamma}{\partial x^j}(x) \frac{\partial^{|\gamma_1|}\phi^{\beta_1}}{\partial x^{\gamma_1}}(x) \cdots \frac{\partial^{|\gamma_\ell|}\phi^{\beta_\ell}}{\partial x^{\gamma_\ell}}(x) \right. \\
&\quad \left. + \frac{\partial^{|\beta|} f}{\partial y^\beta}(\phi(x)) \sum_{r=1}^{\ell} \frac{\partial^{|\gamma_1|}\phi^{\beta_1}}{\partial x^{\gamma_1}}(x) \cdots \frac{\partial^{|\gamma_r|+1}\phi^{\beta_r}}{\partial x^{\gamma_r}\partial x^j}(x) \cdots \frac{\partial^{|\gamma_\ell|}\phi^{\beta_\ell}}{\partial x^{\gamma_\ell}}(x) \right).
\end{aligned}
$$

Rearranging the terms gives now again the form (1.3.8) with $|\alpha|$ replaced by $|\alpha| + 1$ after a suitable definition of the constants: in principle they can be determined recursively from this computation.□

The important point with this technical statement is that the first k partial derivatives of $\phi^* f$ on a compact subset $K \subseteq X$ depend only on the first k partial derivatives of f on the compact subset $\phi(K) \subseteq Y$ together with algebraic combination of the first k partial derivatives of the map ϕ on the compact subset K. This will allow to estimate $\frac{\partial^{|\alpha|}(\phi^* f)}{\partial x^\alpha}$ on K in a very useful and conceptual way later on.

1.3.2 The Test Functions

The next class of interesting differentiable functions are the test functions. As before in the general continuous framework we want to specify the supports.

Let $A \subseteq X$ be a closed subset of a non-empty open subset $X \subseteq \mathbb{R}^n$. Then we set

$$\mathscr{C}^k_A(X) = \mathscr{C}^k(X) \cap \mathscr{C}_A(X) = \left\{ f \in \mathscr{C}^k(X) \;\middle|\; \operatorname{supp} f \subseteq A \right\} \tag{1.3.9}$$

for $k \in \mathbb{N}_0 \cup \{\infty\}$. Again $\mathscr{C}^k_A(X)$ is a subspace of all $\mathscr{C}^k$-functions and, thanks to the locality properties of the support, it turns out to be an ideal in $\mathscr{C}^k(X)$. Moreover, for $A \subseteq A'$ we have

$$\mathscr{C}^k_A(X) \subseteq \mathscr{C}^k_{A'}(X). \tag{1.3.10}$$

Asking for compact supports somewhere will give us the *test functions*:

Definition 1.3.5 (Test functions) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then the $\mathscr{C}^k$-test functions are defined by*

$$\mathscr{C}^k_0(X) = \left\{ f \in \mathscr{C}^k(X) \;\middle|\; \operatorname{supp} f \textit{ is compact} \right\}. \tag{1.3.11}$$

The smooth test functions are called test functions and will alternatively be denoted by

$$\mathscr{D}(X) = \mathscr{C}^\infty_0(X). \tag{1.3.12}$$

Clearly, $\mathscr{C}^k_0(X) \subseteq \mathscr{C}^k(X)$ is an ideal for all $k \in \mathbb{N}_0 \cup \{\infty\}$.

The space $\mathscr{D}(X)$ of test functions is the smallest function space of differentiable functions we have encountered on X so far: it is contained in all others. This raises the question whether $\mathscr{D}(X)$ is too small or maybe even trivial. It is a quite non-trivial fact that this is not the case. The space of test functions is actually still very large and rich as the following statements illustrate. The first construction gives a partition of unity:

Theorem 1.3.6 (Partition of unity) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $\{O_\alpha\}_{\alpha \in I}$ be an open cover of X. Moreover, let $r \in \mathbb{N}$ be fixed. Then there exist countably many functions $\{\chi_m\}_{m \in \mathbb{N}}$ with the following properties:*

i.) For all $m \in \mathbb{N}$ one has $\chi_m \in \mathscr{C}_0^\infty(X)$ with $0 \le \chi_m \le 1$.

ii.) For every $x \in X$ there is an open neighbourhood $U_x \subseteq X$ of x with

$$\chi_m\Big|_{U_x} = 0 \tag{1.3.13}$$

for all but finitely many $m \in \mathbb{N}$.

iii.) For every $m \in \mathbb{N}$ there is a (not necessarily unique) $\alpha_m \in I$ with $\operatorname{supp}(\chi_m) \subseteq O_{\alpha_m}$.

iv.) One has

$$\sum_{m=1}^{\infty} \chi_m^r = 1. \tag{1.3.14}$$

Proof: The condition *ii.)* is also called the local finiteness of the supports of the functions $\{\chi_m\}_{m \in \mathbb{N}}$. This property will ensure that locally around every point $x \in X$ the series (1.3.14) is just a finite sum. We do not have any convergence issues in (1.3.14) once we have guaranteed (1.3.13). In order to construct such functions we first choose an exhausting sequence

$$K_1 \subseteq K_2^\circ \subseteq \cdots \subseteq K_i \subseteq K_{i+1}^\circ \subseteq K_{i+1} \subseteq \cdots \subseteq X$$

of compact subsets $K_i \subseteq X$, i.e. we have $X = \bigcup_{i=1}^\infty K_i$ as before. Note that such compact subsets always exist, see e.g. Exercise 1.4.14 for an elementary construction. Out of these compact subsets we construct a new exhaustion by an onion ring construction. For $i \in \mathbb{N}$ the subset $K_{i+1} \setminus K_i^\circ$ is closed again and, being contained in K_{i+1}, compact. Moreover, $K_{i+2}^\circ \setminus K_{i-1}$ is open and contains $K_{i+1} \setminus K_i^\circ$, see Figure **??**. For $x \in X$ we have an index $i \in \mathbb{N}$ with $x \in K_{i+2}^\circ \setminus K_{i-1}$ since these open rings still provide a cover of X. In a next step, we choose for every $x \in K_{i+2}^\circ \setminus K_{i-1}$ a radius $r_x > 0$ such that there is an $\alpha_x \in I$ with

$$\mathrm{B}_{r_x}(x) \subseteq O_{\alpha_x} \cap K_{i+2}^\circ \setminus K_{i-1}.$$

This is possible since each x is in some O_α and in some $K_{i+2}^\circ \setminus K_{i-1}$. As both are open, their intersection is still open and thus we can find a small open ball contained in the intersection. These $x \in K_{i+2}^\circ \setminus K_{i-1}$ provide now an open cover $\{\mathrm{B}_{\frac{1}{2}r_x}(x)\}_{x \in K_{i+2}^\circ \setminus K_{i-1}}$ of the compact subset $K_{i+1} \setminus K_i^\circ$. Hence finitely many, say $x_{i,1}, \ldots, x_{i,\ell_i}$ with some $\ell_i \in \mathbb{N}$, provide already a cover

$$K_{i+1} \setminus K_i^\circ \subseteq \mathrm{B}_{\frac{1}{2}r_{x_{i,1}}}(x_{i,1}) \cup \cdots \cup \mathrm{B}_{\frac{1}{2}r_{x_{i,\ell_i}}}(x_{i,\ell_i}).$$

Since finally all the $K_{i+1} \setminus K_i^\circ$ together cover X, we get a countable collection of balls $\mathrm{B}_{r_x}(x)$ covering X with each ball contained in some O_α and with the property that the balls are locally finite: indeed, since $\mathrm{B}_{\frac{1}{2}r_{x_{i,s}}}(x_{i,s}) \subseteq K_{i+2}^\circ \setminus K_{i-1}$ the balls for $i, j \in \mathbb{N}$ with $|i - j| > 2$ will no longer intersect: the onion rings in which they are contained are disjoint already. In a last step we construct the necessary functions. First we note that we can find a smooth function $\chi_{i,s} \in \mathscr{C}_0^\infty(X)$ with $0 \le \chi_{i,s} \le 1$,

$$\operatorname{supp} \chi_{i,s} \subseteq \mathrm{B}_{r_{x_{i,s}}}(x_{i,s})$$

and

$$\chi_{i,s}\Big|_{\mathrm{B}_{\frac{1}{2}r_{x_{i,s}}}} = 1$$

for all $i \in \mathbb{N}$ and $s = 1, \ldots, \ell_i$. Such a bump function can be obtained explicitly in Exercise 1.4.12. Their supports are locally finite since the balls $\mathrm{B}_{r_{x_{i,s}}}(x_{i,s})$ are locally finite. Moreover, for every $x \in X$

we have an index $i \in \mathbb{N}, s \in \{1, \ldots, \ell_i\}$ with $x \in \mathrm{B}_{\frac{1}{2}r_{x_{i,s}}}(x_{i,s})$ since these balls still provide a cover of X. Hence $\chi_{i,s}(x) > 0$ for these x and the corresponding (i, s). It follows that

$$\chi = \sum_{i,s} \chi_{i,s}^r \tag{☺}$$

is a locally finite sum of non-negative smooth functions and hence smooth again. Moreover, we conclude that $\chi(x) > 0$ for all $x \in X$ since at least one contribution in (☺) is strictly positive. Thus we put

$$\widetilde{\chi}_{i,s} = \frac{\chi_{i,s}}{\sqrt[r]{\chi}} \in \mathscr{C}_0^\infty(X),$$

and obtain the functions need for the partition of unity. □

Remark 1.3.7 (Partition of unity) The existence of a partition of unity for every open cover is perhaps technical and unfamiliar at first but turns out to be one of the most important tools not only in the theory of function spaces and their locally convex analysis but also far beyond like in differential geometry. We call r also the *order* of the partition of unity. Mainly we will be interested in $r = 1$ but sometimes also quadratic partitions of unity, i.e. $r = 2$, are useful.

The existence of a partition of unity is a characteristic feature of function classes up to $\mathscr{C}^\infty$ but not beyond: real-analytic or holomorphic functions can certainly not be used for a partition of unity. Such a function necessarily vanishes everywhere if X is connected as soon as it has compact support.

One consequence of the partition of unity construction is that we can separate disjoint closed subsets of X by smooth functions: this is a $\mathscr{C}^\infty$-version of the well-known Urysohn Lemma from point-set topology.

Corollary 1.3.8 ($\mathscr{C}^\infty$-Urysohn Lemma) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $A_1, A_2 \subseteq X$ be non-empty disjoint closed subsets. Then there exists a smooth function $f \in \mathscr{C}^\infty(X)$ with*

$$f\big|_{A_1} = 1 \quad \textit{and} \quad f\big|_{A_2} = 0, \tag{1.3.15}$$

and $0 \leq f \leq 1$.

PROOF: The two open subsets $O_1 = X \setminus A_1$ and $O_2 = X \setminus A_2$ provide an open cover of X since $A_1 \cap A_2 = \emptyset$. We have $A_1 \subseteq O_2$ and $A_2 \subseteq O_1$. We choose a partition of unity $\{\chi_n\}_{n\in\mathbb{N}}$ as in Theorem 1.3.6 subordinate to this open cover $\{O_1, O_2\}$ of X. Let $I = \{n \in \mathbb{N} \mid \operatorname{supp}(\chi_n) \subseteq O_2\}$ and set

$$f = \sum_{n\in I} \chi_n.$$

Since the sum is locally finite, $f \in \mathscr{C}^\infty(X)$. For $n \in \mathbb{N} \setminus I$ we then have $\operatorname{supp} \chi_n \subseteq O_1$ and thus

$$f\big|_{A_1} = \sum_{n\in I} \chi_n\big|_{A_1} = \sum_{n\in\mathbb{N}} \chi_n\big|_{A_1} = 1,$$

since we have $\sum_{n\in\mathbb{N}} \chi_n\big|_{A_1} = 1$ everywhere. Analogously,

$$f\big|_{A_2} = \sum_{n\in I} \chi_n\big|_{A_2} = 0.$$

Finally, $0 \leq \chi_n \leq 1$ and $\sum_{n\in\mathbb{N}} \chi_n\big|_{A_1} = 1$ implies that also $0 \leq f \leq 1$ holds everywhere. □

Hence we have many non-trivial smooth functions, and thus also many non-trivial functions of class $\mathscr{C}^k$ for $k \in \mathbb{N}_0$. The compactly supported ones are in fact enough to approximate any given $\mathscr{C}^k$-function in the sense of Proposition 1.3.2. In fact, this is the direct analogue of Proposition 1.2.17 for our more particular space X and the more restrictive class of $\mathscr{C}^k$-function:

Proposition 1.3.9 *Let $X \in \mathbb{R}^n$ be a non-empty open subset and let $f \in \mathscr{C}^k(X)$ for some $k \in \mathbb{N}_0 \cup \{\infty\}$. Then there exists a sequence $(g_i)_{i\in\mathbb{N}}$ of functions $g_n \in \mathscr{C}^k_0(X)$ such that for every $\varepsilon > 0$ and every compact subset $K \subseteq X$ and every multiindex $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ one finds an $N \in \mathbb{N}$ such that*

$$\sup_{x\in K}\left|\frac{\partial^{|\alpha|}f}{\partial x^\alpha}(x) - \frac{\partial^{|\alpha|}g_i}{\partial x^\alpha}(x)\right| < \varepsilon \tag{1.3.16}$$

for all $i \geq N$.

Proof: We choose first an exhausting sequence of compact subsets $\ldots \subseteq K_i \subseteq K^\circ_{i+1} \subseteq K_{i+1} \subseteq \ldots \subseteq X$ of X, i.e. we have $X = \bigcup_{i=1}^\infty K_i$. From Corollary 1.3.8 we find $\mathscr{C}^\infty$-functions $\chi_i \in \mathscr{C}^\infty(X)$ with

$$\chi_i\big|_{K_i} = 1 \quad \text{and} \quad \chi_i\big|_{X\setminus K^\circ_{i+1}} = 0,$$

since clearly the closed subsets K_i and $X \setminus K^\circ_{i+1}$ are disjoint by $K_i \subseteq K^\circ_{i+1}$. We define

$$g_i = f\chi_i.$$

First we notice that $g_i \in \mathscr{C}^k(X)$ since f is $\mathscr{C}^k$. Moreover, since $\operatorname{supp}(\chi_i) \subseteq K_{i+1}$ by construction, $\operatorname{supp}(g_i) \subseteq K_{i+1}$ too, and hence $g_i \in \mathscr{C}^k_0(X)$. Now for $K \subseteq X$ compact we find some index N with $K \subseteq K^\circ_N$ since the open subsets $\{K^\circ_i\}_{i\in I}$ provide a cover of X and we have $K^\circ_i \subseteq K^\circ_{i+1}$. But on K_N we have $\chi_i = 1$ for all $i \geq N$ and thus $f\big|_{K^\circ_N} = g_i\big|_{K^\circ_N}$ for these i. This implies that also all derivatives of f and g_i agree on K°_N since this is an open subset. Hence (1.3.16) is trivially fulfilled. ☐

Again this statement can be seen as supporting the idea that the compactly supported functions of class $\mathscr{C}^k$ are a very large subspace. We will put this result in a more conceptual context later on.

1.3.3 Holomorphic Functions

Very different from smooth functions in many aspects is the class of holomorphic functions. In this section we consider a non-empty open subset $X \subseteq \mathbb{C}^n$ and holomorphic functions on X:

Definition 1.3.10 (Holomorphic functions) *Let $X \subseteq \mathbb{C}^n$ be a non-empty open subset.*

i.) A function $f\colon X \longrightarrow \mathbb{C}$ is called holomorphic on X if it is continuous and holomorphic in each variable while the others are kept fixed, i.e.

$$z^i \mapsto f(z^1, \ldots, z^i, \ldots, z^n) \tag{1.3.17}$$

is a holomorphic function of z^i for all those points such that $z = (z^1, \ldots, z^n) \in X$.

ii.) The set of holomorphic functions on X is denoted by

$$\mathscr{O}(X) = \big\{f\colon X \longrightarrow \mathbb{C} \bigm| f \text{ is holomorphic}\big\}. \tag{1.3.18}$$

The holomorphic functions form a unital subalgebra of the continuous functions

$$\mathscr{O}(X) \subseteq \mathscr{C}(X). \tag{1.3.19}$$

Equivalently, a function f on X is holomorphic if for every $z_0 \in X$ one finds an open *polydisc* $D(z_0, r) \subseteq X$, i.e.

$$D(z_0, r) = \mathrm{B}_{r_1}(z_0^1) \times \cdots \times \mathrm{B}_{r_n}(z_0^n) \tag{1.3.20}$$

with appropriate multiradius $r = (r_1, \ldots, r_n)$ with $r_1, \ldots, r_n > 0$ and center $z_0 \in X$, such that there are coefficients $a_\alpha \in \mathbb{C}$ for all $\alpha \in \mathbb{N}_0^n$ such that

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha (z - z_0)^\alpha \tag{1.3.21}$$

converges absolutely for $z \in U$. Here we write $w^\alpha = (w^1)^{\alpha_1} \cdots (w^n)^{\alpha_n}$. This is in fact not completely trivial and requires the Cauchy integral formula from the one-dimensional theory of complex functions applied now to all n variables. It follows that f is smooth and the coefficients a_α are given by the partial derivatives of f in direction of the holomorphic coordinates, in detail

$$a_\alpha = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z_0). \tag{1.3.22}$$

Complex analysis of holomorphic functions in one and also in several variables is of course a vast and far developed area of mathematics, we can barely scratch the surface of. Instead, we refer to some classical textbooks like [**?**, 10] as well as [3, 4] for the case of several variables. Note that the cases $n = 1$ and $n > 1$ differ substantially in many aspects. In this section we shall just point out some very basic features of the space $\mathscr{O}(X)$ we will need later on.

In Example 1.1.16 we have already established a linear bijection

$$\mathscr{O}(\mathrm{B}_R(0)) \longrightarrow \Lambda^1\Big(\big\{(\tfrac{1}{n!}(R - \tfrac{1}{k})^n)_{n \in \mathbb{N}_0}\big\}_{k \in \mathbb{N}}\Big) \tag{1.3.23}$$

between the holomorphic functions in one dimension on $\mathrm{B}_R(0) \subseteq \mathbb{C}$ and a particular sequence space, the Köthe space in (1.3.23). By the identity principle of holomorphic functions, a holomorphic function $f \in \mathscr{O}(X)$ for a connected X is uniquely determined by its Taylor coefficients a_α in (1.3.22) at some point, thus specifying similar isomorphisms also for more general X instead of $\mathrm{B}_R(0) \subseteq \mathbb{C}$.

Remark 1.3.11 (Gluing holomorphic functions) Holomorphic functions enjoy the same gluing properties as continuous or smooth functions. Being holomorphic is a local feature. Indeed, for an open cover $\{U_\alpha\}_{\alpha \in I}$ of X a function $f\colon X \longrightarrow \mathbb{C}$ is holomorphic iff $f\big|_{U_\alpha}\colon U_\alpha \longrightarrow \mathbb{C}$ is holomorphic for all $\alpha \in I$. Moreover, if one has holomorphic functions $f_\alpha \in \mathscr{O}(U_\alpha)$ with $f_\alpha\big|_{U_\alpha \cap U_\beta} = f_\beta\big|_{U_\alpha \cap U_\beta}$ for non-trivial overlaps $U_\alpha \cap U_\beta \neq \emptyset$ then the global function $f\colon X \longrightarrow \mathbb{C}$ with $f\big|_{U_\alpha} = f_\alpha$ is holomorphic, too.

To obtain a flavour for the rigidity of holomorphic functions we mention the following approximation property. We only need to approximate locally uniformly and obtain an approximation of all derivatives as well:

Proposition 1.3.12 (Weierstraß) *Let $X \subseteq \mathbb{C}^n$ be a non-empty open subset and let $(f_i)_{i \in \mathbb{N}}$ be a sequence of holomorphic functions $f_i \in \mathscr{O}(X)$. Suppose that the functions f_i converge locally uniformly to a function $g\colon X \longrightarrow \mathbb{C}$. Then $g \in \mathscr{O}(X)$ and all derivatives of the f_i converge to the corresponding derivatives of g locally uniformly, i.e.*

$$\sup_{z \in K} \left| \frac{\partial^{|\alpha|} f_i}{\partial z^\alpha}(z) - \frac{\partial^{|\alpha|} g}{\partial z^\alpha}(z) \right| \longrightarrow 0 \tag{1.3.24}$$

for all compact subsets $K \subseteq X$.

We postpone the proof of this classical theorem of Weierstraß in complex analysis since we will find a much clearer formulation later on, including a very conceptual proof to be done in Exercise 2.5.44 as a consequence of Proposition 2.3.28.

We also note that a similar statement in the $\mathscr{C}^k$-setting is completely false for $k \geq 1$. Here quite the opposite is true. We can approximate an arbitrary continuous function by e.g. $\mathscr{C}^\infty$-functions locally uniformly. This and related approximation theorems play of course an important role in analysis in many places. We will formulate and prove them properly once we have developed the necessary machinery of convolution for doing so.

1.3.4 The Schwartz Space

The last function space of differentiable functions we want to introduce in this preliminary chapter is the space of *rapidly decreasing functions*, also called the *Schwartz space*. Here the local property of being smooth is combined with a global property, the growth features at infinity:

Definition 1.3.13 (Schwartz space) *The space of rapidly decreasing functions or the Schwartz space is defined by*

$$\mathscr{S}(\mathbb{R}^n) = \left\{ f \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{C}) \;\middle|\; \sup_{x \in \mathbb{R}^n} \left| (1 + x^2)^{\frac{m}{2}} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| < \infty \textit{ for all } m \in \mathbb{N}_0 \textit{ and all } \alpha \in \mathbb{N}_0^n \right\}. \tag{1.3.25}$$

Here $x^2 = \langle x, x \rangle$ is the Euclidean norm square of $x \in \mathbb{R}^n$.

In other words, all derivatives of the smooth function f have to decrease faster than any polynomial in coordinates. The chosen prefactor $(1 + x^2)^{\frac{m}{2}}$ is a useful convention, equivalently one can consider, e.g.

$$f \in \mathscr{S}(\mathbb{R}^n) \iff \sup_{x \in \mathbb{R}^n} \left| p(x) \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| < \infty \tag{1.3.26}$$

for all $\alpha \in \mathbb{N}_0^n$ and all polynomials $p \in \mathbb{C}[x^1, \ldots, x^n]$, see Exercise 1.4.21 for some alternatives.

It is clear that $\mathscr{S}(\mathbb{R}^n) \subseteq \mathscr{C}^\infty(\mathbb{R}^n)$ is a subspace. Note that for the Schwartz space we admit complex values if not stated otherwise. This will provide several benefits once we discuss the Fourier transform. We have $f \in \mathscr{S}(\mathbb{R}^n)$ iff $\overline{f} \in \mathscr{S}(\mathbb{R}^n)$ iff real and imaginary parts of f are Schwartz functions.

A first example of Schwartz functions is obtained as follows:

Example 1.3.14 (Schwartz functions) Let $n \in \mathbb{N}$.

i.) The test functions $\mathscr{D}(\mathbb{R}^n) = \mathscr{C}_0^\infty(\mathbb{R}^n)$ are clearly contained in the Schwartz functions,

$$\mathscr{C}_0^\infty(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n). \tag{1.3.27}$$

In fact, if $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ satisfies $\operatorname{supp}(f) \subseteq K$ for some compact subset then

$$\begin{aligned} \sup_{x \in \mathbb{R}^n, |\alpha| \leq \ell} (1 + x^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| &= \max_{x \in \mathbb{R}^n, |\alpha| \leq \ell} (1 + x^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| \\ &\leq c_{K,m} \max_{x \in \mathbb{R}^n, |\alpha| \leq \ell} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| \end{aligned}$$

with

$$c_{K,m} = \max_{x \in K} (1 + x^2)^{\frac{m}{2}} < \infty. \tag{1.3.28}$$

ii.) Less obvious is the fact that there are functions in $\mathscr{S}(\mathbb{R}^n)$ which are not in $\mathscr{C}_0^\infty(\mathbb{R}^n)$. Here we point out one particular example, the *Gaussian function*

$$f(x) = \mathrm{e}^{-x^2}. \tag{1.3.29}$$

Indeed, a standard argument shows that there is a polynomial function p_α on $\mathbb{R}^n$ with

$$\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) = p_\alpha(x) \mathrm{e}^{-x^2} \tag{1.3.30}$$

for all $\alpha \in \mathbb{N}_0^n$. Then the exponential factor e^{-x^2} decays fast enough to guarantee $f \in \mathscr{S}(\mathbb{R}^n)$, see also Exercise 1.4.22 for further properties of the Gaussian. We will see more functions in $\mathscr{S}(\mathbb{R}^n)$ soon.

The Schwartz space mixes local with global features. Hence we do not have the nice gluing properties as for smooth or holomorphic functions, see Remark 1.3.3 and Remark 1.3.11, respectively. Moreover, the Schwartz space is not invariant under all diffeomorphisms of $\mathbb{R}^n$, see Exercise 1.4.24 for an explicit example. This makes it impossible to define the Schwartz space in more geometric situations like on smooth manifolds. Nevertheless, one still has some invariance, namely under affine diffeomorphisms: Recall that the affine group

$$\mathrm{Aff}(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R}^n) \ltimes \mathbb{R}^n \tag{1.3.31}$$

acts smoothly on $\mathbb{R}^n$ by translations and matrix multiplication. We denote the action of $(A, a) \in \mathrm{Aff}(\mathbb{R}^n)$ on $x \in \mathbb{R}^n$ by

$$(A, a) \triangleright x = Ax + a. \tag{1.3.32}$$

Alternatively, we write

$$\tau_a(x) = x + a \tag{1.3.33}$$

for the translation part. The pull-backs with affine diffeomorphisms map Schwartz functions to Schwartz functions. Related to this fact is the statement that for $f \in \mathscr{S}(\mathbb{R}^n)$ also the partial derivatives $\partial^\alpha f \in \mathscr{S}(\mathbb{R}^n)$ are Schwartz functions:

Proposition 1.3.15 *Let $A \in \mathrm{GL}_n(\mathbb{R})$ and $a \in \mathbb{R}^n$.*

i.) For all $m, \ell \in \mathbb{N}_0$ one has a continuous function $c_{m,\ell} \colon \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}^n, |\alpha| \le \ell} \left| (1 + x^2)^{\frac{m}{2}} \frac{\partial^{|\alpha|}(A^* f)}{\partial x^\alpha}(x) \right| \le c_{m,\ell}(A) \sup_{x \in \mathbb{R}^n, |\alpha| \le \ell} \left| (1 + x^2)^{\frac{m}{2}} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| \tag{1.3.34}$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$. In particular, $A^* f \in \mathscr{S}(\mathbb{R}^n)$.

ii.) For all $m, \ell \in \mathbb{N}_0$ one has a continuous function $c_{m,\ell} \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ with $c_{m,\ell}(0) = 1$ and $c_{m,\ell} \ge 1$ which is bounded by a polynomial of degree m, such that

$$\sup_{x \in \mathbb{R}^n, |\alpha| \le \ell} \left| (1 + x^2)^{\frac{m}{2}} \frac{\partial^{|\alpha|}(\tau_a^* f)}{\partial x^\alpha}(x) \right| \le c_{m,\ell}(a) \sup_{x \in \mathbb{R}^n, |\alpha| \le \ell} \left| (1 + x^2)^{\frac{m}{2}} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| \tag{1.3.35}$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$. In particular, $\tau_a^* f \in \mathscr{S}(\mathbb{R}^n)$.

iii.) For all $\beta \in \mathbb{N}_0^n$ one has

$$\sup_{x \in \mathbb{R}^n, |\alpha| \le \ell} \left| (1 + x^2)^{\frac{m}{2}} \frac{\partial^{|\alpha|}(\partial^\beta f)}{\partial x^\alpha}(x) \right| \le \sup_{x \in \mathbb{R}^n, |\alpha| \le \ell + |\beta|} \left| (1 + x^2)^{\frac{m}{2}} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| \tag{1.3.36}$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$. In particular, $\partial^\beta f \in \mathscr{S}(\mathbb{R}^n)$.

PROOF: For the first part, denote the operator norm of A by $\|A\|$. Then we have for $f \in \mathscr{S}(\mathbb{R}^n)$

$$\frac{\partial(A^* f)}{\partial x^i}(x) = \frac{\partial}{\partial x^i}(f(Ax)) = \frac{\partial f}{\partial x^j}(Ax) A_i^j$$

by the chain rule. Hence for higher derivatives we get

$$\frac{\partial^{|\alpha|}(A^* f)}{\partial x^\alpha}(x) = \sum_{|\beta| = |\alpha|} \frac{\partial^{|\beta|} f}{\partial x^\beta}(Ax) A_\alpha^\beta,$$

where we set $A_\alpha^\beta = A_{\alpha_1}^{\beta_1} \cdots A_{\alpha_\ell}^{\beta_\ell}$ for abbreviation, if $\ell = |\beta| = |\alpha|$. Since $|A_i^j| \leq \|A\|$ we get the estimate

$$\left|\frac{\partial^{|\alpha|}(A^* f)}{\partial x^\alpha}(x)\right| \leq \sum_{|\beta| = |\alpha|} \left|\frac{\partial^{|\beta|} f}{\partial x^\beta}(Ax)\right| \|A\|^{|\beta|} \leq n^{|\alpha|} \|A\|^{|\alpha|} \sup_{|\beta| = |\alpha|} \left|\frac{\partial^{|\beta|} f}{\partial x^\beta}(Ax)\right|,$$

since the sum has $n^{|\alpha|} = n^\ell$ terms. For the prefactor in (1.3.34) we have the estimate

$$1 + (A^{-1}x)^2 \leq 1 + \|A^{-1}\|^2 \|x\|^2 \leq \max\{1, \|A^{-1}\|^2\}(1 + x^2).$$

Putting things together we get

$$\begin{aligned}
\sup_{x \in \mathbb{R}^n, |\alpha| \leq \ell} (1 + x^2)^{\frac{m}{2}} \left|\frac{\partial^{|\alpha|}(A^* f)}{\partial x^\alpha}(x)\right| &\leq \sup_{x \in \mathbb{R}^n, |\alpha| \leq \ell} (1 + x^2)^{\frac{m}{2}} n^{|\alpha|} \|A\|^{|\alpha|} \left|\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(Ax)\right| \\
&= \sup_{x \in \mathbb{R}^n, |\alpha| \leq \ell} (1 + (A^{-1}x)^2)^{\frac{m}{2}} n^{|\alpha|} \|A\|^{|\alpha|} \left|\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)\right| \\
&\leq \sup_{x \in \mathbb{R}^n, |\alpha| \leq \ell} n^\ell \max\{1, \|A^{-1}\|^m, \|A\|^\ell\}(1 + x^2) \left|\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)\right|.
\end{aligned}$$

Setting

$$c_{m,\ell}(A) = n^\ell \max\{1, \|A^{-1}\|^m, \|A\|^\ell\}$$

will now do the job. Note that $c_{m,\ell}$ is continuous since inversion on $\mathrm{GL}_n(\mathbb{R})$ is continuous and so is the operator norm. For the second part we first note that

$$\frac{\partial^{|\alpha|}(\tau_a^* f)}{\partial x^\alpha}(x) = \frac{\partial^{|\alpha|}}{\partial x^\alpha} f(x + a) = \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x + a)$$

by the chain rule. Next, we have

$$\begin{aligned}
1 + (x - a)^2 &\leq 1 + x^2 + 2\|x\|\|a\| + a^2 \\
&= (1 + x^2)\left(1 + \frac{2\|x\|\|a\| + a^2}{1 + x^2}\right) \\
&\leq (1 + x^2) \max_{\xi \geq 0}\left(1 + \frac{2\xi\|a\| + a^2}{1 + \xi^2}\right).
\end{aligned}$$

Now, the maximum of the right hand side is indeed finite and can be bounded by a quadratic function $q(\|a\|)$ of $\|a\|$ with $q(0) = 1$, see Exercise 1.4.13. Thus

$$1 + (x - a)^2 \leq (1 + x^2) q(\|a\|)$$

follows. This can now be used for the estimate

$$\sup_{x \in \mathbb{R}^n, |\alpha| \leq \ell} (1 + x^2)^{\frac{m}{2}} \left|\frac{\partial^{|\alpha|}(\tau_a^* f)}{\partial x^\alpha}(x)\right| = \sup_{x \in \mathbb{R}^n, |\alpha| \leq \ell} (1 + x^2)^{\frac{m}{2}} \left|\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x + a)\right|$$

$$
\begin{aligned}
&= \sup_{x \in \mathbb{R}^n, |\alpha| \le \ell} (1 + (x-a)^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| \\
&\le q(\|a\|)^{\frac{m}{2}} \sup_{x \in \mathbb{R}^n, |\alpha| \le \ell} (1 + x^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right|,
\end{aligned}
$$

which shows the second part by setting $c_{m,\ell}(a) = q(\|a\|)^{\frac{m}{2}}$. In fact, the constant $c_{m,\ell}(a)$ is independent of ℓ. The last statement is trivial. □

Example 1.3.16 (Gaussian function) Let $A \in \mathrm{M}_n(\mathbb{R})$ be a positive definite (symmetric) matrix and let $a \in \mathbb{R}^n$. Then $A = B^{\mathrm{T}} B$ for some $B \in \mathrm{GL}_n(\mathbb{R})$. Let $G \in \mathscr{S}(\mathbb{R}^n)$ be the Gaussian function from Example 1.3.14, *ii.)*. Then we have

$$
\begin{aligned}
((B,a)^* G)(x) &= G(Bx + a) \\
&= \mathrm{e}^{-\langle Bx+a, Bx+a \rangle} \\
&= \mathrm{e}^{-\langle Bx, Bx \rangle - 2\langle Bx, a \rangle - \langle a, a \rangle} \\
&= \mathrm{e}^{-\langle x, Ax \rangle - 2\langle x, B^{\mathrm{T}} a \rangle - \langle a, a \rangle}.
\end{aligned}
$$

This function is again in $\mathscr{S}(\mathbb{R}^n)$ according to the previous proposition. Hence all functions of the form

$$
G_{A,b}(x) = \mathrm{e}^{-\langle x, Ax \rangle - \langle b, x \rangle} \tag{1.3.37}
$$

with a positive definite A and an arbitrary $b \in \mathbb{R}^n$ are again Schwartz functions. Of course, this can also be seen by a more direct investigation of the needed estimate (1.3.25) for $G_{A,b}$.

One way to produce new examples of Schwartz functions is to multiply a given Schwartz function with a smooth function which grows only in a very moderate way. To understand this mechanism, one introduces the following two additional function spaces:

Definition 1.3.17 (Slowly and very slowly increasing functions) *Let $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ be a smooth function.*

i.) The function f is called slowly increasing if for every $\alpha \in \mathbb{N}_0^n$ one has an $m \in \mathbb{N}_0$ with

$$
\sup_{x \in \mathbb{R}^n} \frac{1}{(1+x^2)^{\frac{m}{2}}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| < \infty. \tag{1.3.38}
$$

The set of slowly increasing functions is then denoted by

$$
\mathscr{O}_M(\mathbb{R}^n) = \left\{ f \in \mathscr{C}^\infty(\mathbb{R}^n) \;\middle|\; f \textit{ is slowly increasing} \right\}. \tag{1.3.39}
$$

ii.) The function f is called very slowly increasing if there exists an $m \in \mathbb{N}_0$ such that for all $\alpha \in \mathbb{N}_0^n$ one has

$$
\sup_{x \in \mathbb{R}^n} \frac{1}{(1+x^2)^{\frac{m}{2}}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| < \infty. \tag{1.3.40}
$$

The set of very slowly increasing functions is then denoted by

$$
\mathscr{O}_C(\mathbb{R}^n) = \left\{ f \in \mathscr{C}^\infty(\mathbb{R}^n) \;\middle|\; f \textit{ is very slowly increasing} \right\}. \tag{1.3.41}
$$

The idea is that a function and its derivatives have a growth at infinity which can be controlled by a polynomial. Clearly, $\mathscr{O}_M(\mathbb{R}^n) \subseteq \mathscr{C}^\infty(\mathbb{R}^n)$ is a subspace and

$$
\mathscr{O}_C(\mathbb{R}^n) \subseteq \mathscr{O}_M(\mathbb{R}^n) \tag{1.3.42}
$$

is a subspace therein. Moreover,

$$\mathscr{S}(\mathbb{R}^n) \subseteq \mathscr{O}_C(\mathbb{R}^n) \tag{1.3.43}$$

is clear and we also have

$$\operatorname{Pol}(\mathbb{R}^n) \subseteq \mathscr{O}_C(\mathbb{R}^n). \tag{1.3.44}$$

Indeed, if $f \in \operatorname{Pol}(\mathbb{R}^n)$ is a polynomial of degree $m \in \mathbb{N}$ then all derivatives $\partial^\alpha f$ are polynomials of degree $m - |\alpha| \leq m$. Hence

$$\sup_{x \in \mathbb{R}^n} \frac{|f(x)|}{(1+x^2)^{\frac{m}{2}}} < \infty \tag{1.3.45}$$

holds and the same is true for all derivatives. Hence $\mathscr{O}_C(\mathbb{R}^n)$ contains already many more interesting functions. Similar estimates as in Proposition 1.3.15 then show that both spaces $\mathscr{O}_C(\mathbb{R}^n)$ and $\mathscr{O}_M(\mathbb{R}^n)$ are invariant under affine diffeomorphisms, see Exercise 1.4.25 for further details.

Example 1.3.18 Let $a \in \mathbb{R}^n$ be a fixed vector. Then the *phase function*

$$e_a(x) = \mathrm{e}^{i\langle a, x\rangle} \tag{1.3.46}$$

is very slowly increasing. In fact, it is bounded with all its derivatives being bounded as well. Thus we have the condition (1.3.40) for $m = 0$. More generally, any function with bounded derivatives belongs to $\mathscr{O}_C(\mathbb{R}^n)$.

The space $\mathscr{O}_M(\mathbb{R}^n)$ is now the largest space of functions such that the multiplications with $f \in \mathscr{O}_M(\mathbb{R}^n)$ map $\mathscr{S}(\mathbb{R}^n)$ into itself:

Proposition 1.3.19 *Let $f \in \mathscr{O}_M(\mathbb{R}^n)$ and let $m_\beta \in \mathbb{N}_0$ for $\beta \in \mathbb{N}_0^n$ be chosen such that*

$$c_\beta(f) = \sup_{x \in \mathbb{R}^n} \frac{1}{(1+x^2)^{\frac{m_\beta}{2}}} \left| \frac{\partial^{|\beta|} f}{\partial x^\beta}(x) \right| < \infty \tag{1.3.47}$$

according to (1.3.38)*. Then for all $g \in \mathscr{S}(\mathbb{R}^n)$ one has*

$$\sup_{x \in \mathbb{R}^n, |\alpha| \leq \ell} (1+x^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|}(fg)}{\partial x^\alpha}(x) \right| \leq c_\ell(f) \sup_{x \in \mathbb{R}^n, |\alpha| \leq \ell} (1+x^2)^{\frac{m'}{2}} \left| \frac{\partial^{|\alpha|} g}{\partial x^\alpha}(x) \right| \tag{1.3.48}$$

with some constant $c_\ell(f) < \infty$ depending on f and some $m' \in \mathbb{N}_0$ depending on f as well. In particular, $fg \in \mathscr{S}(\mathbb{R}^n)$.

Proof: The Leibniz rule gives

$$\frac{\partial^{|\alpha|}(fg)}{\partial x^\alpha}(x) = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \frac{\partial^{|\beta|} f}{\partial x^\beta}(x) \frac{\partial^{|\alpha-\beta|} g}{\partial x^{\alpha-\beta}}(x).$$

We set now $m_\ell = \max_{|\beta| \leq \ell}\{m_\beta\} \in \mathbb{N}_0$ as well as

$$c_\ell(f) = 2^\ell \max_{|\beta| \leq \ell} c_\beta(f) < \infty.$$

Then get

$$\left| \frac{\partial^{|\alpha|}(fg)}{\partial x^\alpha}(x) \right| \leq \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \left| \frac{\partial^{|\beta|} f}{\partial x^\beta}(x) \frac{\partial^{|\alpha-\beta|} g}{\partial x^{\alpha-\beta}}(x) \right|$$

$$
\begin{aligned}
&= \sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} \frac{1}{(1+x^2)^{\frac{m_\beta}{2}}} \left| \frac{\partial^{|\beta|} f}{\partial x^\beta}(x) \right| (1+x^2)^{\frac{m_\beta}{2}} \left| \frac{\partial^{|\alpha-\beta|} g}{\partial x^{\alpha-\beta}}(x) \right| \\
&\le \sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} c_\beta(f) (1+x^2)^{\frac{m_\ell}{2}} \left| \frac{\partial^{|\alpha-\beta|} g}{\partial x^{\alpha-\beta}}(x) \right| \\
&\le c_{|\alpha|}(f)(1+x^2)^{\frac{m_\ell}{2}} \max_{|\gamma| \le |\alpha|} \left| \frac{\partial^{|\gamma|} g}{\partial x^\gamma}(x) \right|,
\end{aligned}
$$

since the sum over the multinomial coefficients alone gives $2^{|\alpha|}$ and every occurring derivative of g is of order $\le |\alpha|$. Hence

$$
\sup_{x \in \mathbb{R}^n, |\alpha| \le \ell} (1+x^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|}(fg)}{\partial x^\alpha}(x) \right| \le c_\ell(f) \sup_{x \in \mathbb{R}^n, |\alpha| \le \ell} (1+x^2)^{\frac{m}{2}+\frac{m_\ell}{2}} \left| \frac{\partial^{|\alpha|} g}{\partial x^\alpha}(x) \right|,
$$

which is (1.3.48) if we set $m' = m + m_\ell$. Thus $fg \in \mathscr{S}(\mathbb{R}^n)$ follows. □

1.4 Exercises

Exercise 1.4.1 (Bijections and algebra morphisms)

Exercise 1.4.2

Exercise 1.4.3 (Weighted Sequences Are Subspaces)

Exercise 1.4.4 (Generalize Convergent Sequences and Zero Sequences to Directed Index Sets)

Exercise 1.4.5 (Weighted Convergent and Zero Sequences for Directed Sets)

Exercise 1.4.6 (The inclusion $\ell^p(\mathbf{N}, \mu) \subseteq c_0(\mathbf{N}, \sqrt[p]{\mu})$)

Exercise 1.4.7 (Non-trivial inclusions of sequence spaces) Let $\mu \in \operatorname{Map}(\mathbb{N}, \mathbb{R})$ be a sequence of positive weights $\mu_n \ge 1$. Show that all the inclusions in (1.1.44) are non-trivial by providing explicit examples.

Exercise 1.4.8 (Glueing of measurable functions) Let $(X, \mathfrak{a})$ be a measurable space with a sequence $A_n \in \mathfrak{a}$ of measurable subsets such that $X = \bigcup_{n \in \mathbb{N}} A_n$.

i.) Show that $f \in \operatorname{Map}(X, \mathbb{K})$ is measurable iff all the restrictions $f\big|_{A_n}$ are measurable with respect to the σ-algebras $\mathfrak{a}\big|_{A_n}$.

ii.) Show that for measurable functions $f_n \in \mathscr{M}(A_n, \mathfrak{a}\big|_{A_n})$ with $f_n\big|_{A_n \cap A_m} = f_m\big|_{A_n \cap A_m}$ there exists a unique and measurable function $f \in \mathscr{M}(X, \mathfrak{a})$ with $f\big|_{A_n} = f_n$ for all $n \in \mathbb{N}$.

Hint: Why is it important that the A_n are measurable and why is it important to have at most countably many A_n?

iii.)

Exercise 1.4.9 (Convergence in $\mathscr{BM}_{\mathrm{loc}}(X)$)

Exercise 1.4.10 (Holomorphic functions)

Exercise 1.4.11 (The chain rule) Let $k \in \mathbb{N} \cup \{\infty\}$. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be non-empty open subsets. Suppose that $\phi \colon X \longrightarrow Y$ is $\mathscr{C}^k$ and let $f \in \mathscr{C}^k(Y)$. Prove that there are universal (independent of ϕ and f) coefficients $c^{\alpha}_{\beta\gamma_1\cdots\gamma_\ell} \in \mathbb{Q}$ for all multiindices $\alpha, \gamma_1, \ldots, \gamma_\ell \in \mathbb{N}_0^n$ as well as $\beta \in \mathbb{N}_0^m$ such that

$$\frac{\partial^{|\alpha|}(\phi^* f)}{\partial x^\alpha}(x) = \sum_{\ell=0}^{|\alpha|} \sum_{|\beta| \le |\alpha|} \sum_{|\gamma_1| + \cdots + |\gamma_\ell| = |\alpha|} c^{\alpha}_{\beta\gamma_1\cdots\gamma_\ell} \frac{\partial^{|\beta|} f}{\partial y^\beta}(\phi(x)) \frac{\partial^{|\gamma_1|} \phi^{\beta_1}}{\partial x^{\gamma_1}}(x) \cdots \frac{\partial^{|\gamma_\ell|} \phi^{\beta_\ell}}{\partial x^{\gamma_\ell}}(x), \tag{1.4.1}$$

where $\phi^i = y^i \circ \phi$ denotes the i-th component of the map ϕ with $i = 1, \ldots, m$ and $\phi^* f = f \circ \phi$ is the pull-back as usual.

Exercise 1.4.12 (Some test functions) Let $\epsilon, r > 0$ and let $n \in \mathbb{N}$.

i.) Compute the derivatives of the function $h \colon \mathbb{R} \longrightarrow \mathbb{R}$ with

$$h(t) = \begin{cases} \mathrm{e}^{-\frac{1}{t}} & \text{for } t > 0 \\ 0 & \text{for } t \le 0, \end{cases} \tag{1.4.2}$$

sufficiently explicit to show that h is everywhere smooth.

Hint: The only interesting point is $t = 0$. Show that $h^{(r)}(t) = p_r(\frac{1}{t})\mathrm{e}^{-\frac{1}{t}}$ with a polynomial p_r.

ii.) Sketch the graph of h.

iii.) Consider the function

$$g_\epsilon(t) = \frac{h(t)}{h(t) + h(\epsilon - t)}. \tag{1.4.3}$$

Show that g_ϵ is smooth and prove

$$g_\epsilon(t) = 0 \text{ for } t \le 0, \quad g_\epsilon(t) = 1 \text{ for } t \ge \epsilon, \quad \text{as well as} \quad g_\epsilon(t) \in [0, 1]. \tag{1.4.4}$$

Sketch the graph of g_ϵ.

iv.) Define the function $\varphi_{r,\epsilon} \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\varphi_{r,\epsilon}(x) = 1 - g_\epsilon(\|x\| - r), \tag{1.4.5}$$

with the Euclidean norm $\|x\| = \sqrt{\langle x, x \rangle}$ for $x \in \mathbb{R}^n$. Show that also $\varphi_{r,\epsilon}$ is smooth and satisfies

$$\varphi_{r,\epsilon}(x) = \begin{cases} 0 & \text{for } \|x\| \ge r + \epsilon \\ 1 & \text{for } \|x\| \le r \end{cases} \quad \text{as well as} \quad \varphi_{r,\epsilon}(x) \in [0, 1]. \tag{1.4.6}$$

Sketch the qualitative form of $\varphi_{r,\epsilon}$ for $n = 2$.

v.) Now let $U \subseteq \mathbb{R}^n$ be an open subset. Prove that for every $x \in U$ one finds an open neighbourhood $V \subseteq U$ of x together with a smooth function $\varphi \in \mathscr{C}^\infty(U)$ such that $\varphi\big|_V = 1$ and $\operatorname{supp} \varphi$ is compact.

vi.) Let $U \subseteq \mathbb{R}^n$ be a non-empty open subset. Prove that the smooth functions with compact support $\mathscr{C}^\infty_0(U) \subseteq \mathscr{C}^\infty(U)$ form an infinite-dimensional subspace of all smooth functions on U. In particular, $\mathscr{C}^\infty(U)$ is infinite-dimensional, too.

Exercise 1.4.13 (An elementary estimate) Let $x, y \in \mathbb{R}^n$ and equip $\mathbb{R}^n$ with the usual Euclidean norm. Show that for $x, y \in \mathbb{R}^n$ one has a quadratic function $q \colon \mathbb{R} \longrightarrow \mathbb{R}$ with $q(0) = 1$ such that

$$\frac{1 + \|x + y\|^2}{1 + \|x\|^2} \le q(\|y\|). \tag{1.4.7}$$

Conclude that the supremum of the left hand side over $x \in \mathbb{R}^n$ can be bounded by a continuous function of y giving the value 1 at $y = 0$.

Exercise 1.4.14 (Exhaustion by compact subsets)

Exercise 1.4.15 (The Schwartz sequence space) Let s denote the Schwartz sequence space of rapidly decreasing sequences and let $c_{\circ\circ}$ be the subspace of sequences with finitely many nonzero entries.

i.) Let $a > 0$. Show that the sequence e_a with $e_a(n) = \mathrm{e}^{-an}$ for $n \in \mathbb{N}$ is an element of s.

ii.) Show that in the quotient space $s/c_{\circ\circ}$ the elements $\{[e_a]\}_{a>0}$ are linearly independent.

Hint: Let $N \in \mathbb{N}$ and $a_1, \ldots, a_N > 0$ as well as $z_1, \ldots, z_N \in \mathbb{C}$ such that $z_1[e_{a_1}] + \cdots + z_N[e_{a_N}] = 0$. Translate this condition of linear (in-)dependence modulo $c_{\circ\circ}$ into a condition for the entries of the representing sequence $z_1 e_{a_1} + \cdots + z_N e_{a_N}$. Then the Vandermonde determinate might be useful.

This shows that s contains "many" more sequences than $c_{\circ\circ}$.

Exercise 1.4.16 (The algebras c and ℓ^∞) Consider $I = \mathbb{N}$ and the sequence spaces $c_\circ \subseteq c \subseteq \ell^\infty$.

i.) Show that ℓ^∞ as well as c are unital subalgebras of $\mathrm{Map}(\mathbb{N}, \mathbb{K})$ and *-subalgebras in the case $\mathbb{K} = \mathbb{C}$.

ii.) Show that the limit yields a character of the algebra $c(\mathbb{N}, \mathbb{K})$, i.e. a multiplicative linear functional normalized to 1 on the unit element of $c(\mathbb{N}, \mathbb{K})$.

iii.) Show that $c_\circ \subseteq \ell^\infty$ is an ideal and even a *-ideal in the case $\mathbb{K} = \mathbb{C}$.

Exercise 1.4.17 (The algebra s)

Exercise 1.4.18 (Nets of measurable functions) Let $(X, \mathfrak{a})$ be a measurable space such that single points $x \in X$ yield measurable subsets $\{x\} \in \mathfrak{a}$. Show that for every function $f \in \mathrm{Map}(X, \mathbb{K})$ one finds a net $(f_i)_{i \in I}$ of simple functions $f_i \in \mathscr{M}(X, \mathfrak{a})$ converging pointwise to f. Conclude that the statements of Remark 1.2.3 typically will fail if sequences are replace by general nets.

Hint: It will be sufficient to consider functions with finite support.

Exercise 1.4.19 (Essential range and essential supremum)

Exercise 1.4.20 (The Borel functor) Consider the Borel σ-algebra $\mathfrak{a}_{\mathrm{Borel}}(X)$ of a topological space X.

i.) Show that a continuous map $f\colon X \longrightarrow Y$ between topological spaces is measurable with respect to the Borel σ-algebras $\mathfrak{a}_{\mathrm{Borel}}(X)$ and $\mathfrak{a}_{\mathrm{Borel}}(Y)$, respectively.

ii.) Show that this gives a functor

$$\mathsf{Borel}\colon \mathsf{top} \longrightarrow \mathsf{Mess} \tag{1.4.8}$$

from the category of topological spaces to the category of measurable spaces with measurable maps as morphisms.

Exercise 1.4.21 (Characterizations of the Schwartz space) Let $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ be a smooth function. Show that the following statements are equivalent:

i.) The function f belongs to $\mathscr{S}(\mathbb{R}^n)$.

ii.) For all $\alpha, \beta \in \mathbb{N}_0^n$ one has

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha \frac{\partial^{|\beta|} f}{\partial x^\beta}(x) \right| < \infty. \tag{1.4.9}$$

iii.) For all polynomials $p \in \mathbb{R}[x^1, \ldots, x^n]$ and all $\beta \in \mathbb{N}_0^n$ one has

$$\sup_{x \in \mathbb{R}^n} \left| p(x) \frac{\partial^{|\beta|} f}{\partial x^\beta}(x) \right| < \infty. \tag{1.4.10}$$

Exercise 1.4.22 (Properties of the Gaussian function)

Exercise 1.4.23 (The algebra structure of $\mathscr{S}(\mathbb{R}^n)$)

Exercise 1.4.24 (The Schwartz space is not diffeomorphism invariant)

Exercise 1.4.25 (Slowly and very slowly increasing functions)

Chapter 2

From Topological to Locally Convex Vector Spaces

As we have seen in the previous chapter, many interesting function spaces carry an analytic nature which, as it turns out, goes beyond the familiar Banach space situation. To treat these examples adequately, one needs new notions: we have a vector space structure over $\mathbb{C}$ or $\mathbb{R}$, but also notions of approximation, convergence, and alike. To start a systematic investigation we need the concept of a topological vector space. Here one brings together the linear structure with a topological structure by asking for some fairly obvious compatibility conditions. We will investigate topological vector spaces to some extend, mainly in order to have clean definitions of continuity, convergence and completeness. It then turns out quickly that a general theory of topological vector spaces is still not specific enough to admit interesting and non-trivial statements.

This changes drastically if one requires the topological vector space to be locally convex. Here we will discuss several equivalent definitions and characterizations. Besides the geometric characterization by means of a neighbourhood basis consisting of convex neighbourhoods the approach using a collection of seminorms is most suited to incorporate our examples from Chapter 1. We will see how to interpret all of our function spaces as locally convex spaces by viewing the defining conditions as seminorms.

The general theory of locally convex spaces allows us then to give various formulations of continuity, convergence and completeness. We will also outline some of the standard constructions like products, direct sums, and quotients. A particular class will then be the Fréchet spaces, where a countability condition enters the arena. This will simplify many situations, but there will be relevant examples of locally convex spaces beyond the case of Fréchet spaces. The most interesting such class will be the LF spaces which provide some interesting new effects.

2.1 Topological Vector Spaces

A topological vector space provides an algebraic structure, the vector space, together with a topological structure in a compatible way. We will discuss several first notions and concepts from topology and investigate how the linear structure helps to understand the topological features. Remarkably, it turns out that one automatically has a uniform structure for free. We always consider real or complex vector spaces and endow $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the standard topology.

2.1.1 The Definition of a Topological Vector Space

The central definition is now easy to state. Since we want a topology on a vector space, it is well-motivated that the structure maps should be continuous:

Definition 2.1.1 (Topological vector space) *A topological vector space V over $\mathbb{K}$ is a vector space over $\mathbb{K}$ endowed with a topology $\mathscr{V}$ such that addition*

$$+\colon V \times V \longrightarrow V \tag{2.1.1}$$

and multiplication with scalars

$$\cdot\colon \mathbb{K} \times V \longrightarrow V \tag{2.1.2}$$

are continuous maps.

If the context is clear, we write just V instead of $(V, \mathscr{V})$. It follows that the additive group $(V, +)$ is a topological group in the sense that all group operations are continuous. Indeed, also the inversion $v \mapsto -v$ is continuous, since we can express it as restriction of the multiplication by scalars to $\{-1\} \times V$. This has already many useful consequences.

Proposition 2.1.2 *Let $(V, \mathscr{V})$ be a topological vector space.*

i.) For every $v \in V$ the translation by v is a homeomorphism

$$\tau_v\colon V \ni w \;\mapsto\; \tau_v(w) = w + v \in V, \tag{2.1.3}$$

which defines a continuous group action of the additive group $(V, +)$ on V.

ii.) For every $z \in \mathbb{K}^\times$ the scaling

$$V \ni v \;\mapsto\; zv \in V \tag{2.1.4}$$

is a homeomorphism which yields a continuous group action of $\mathbb{K}^\times$ on V.

Proof: It is clear that the maps τ_v as well as (2.1.4) are continuous, since they are restrictions of the continuous maps (2.1.1) and (2.1.2). Since τ_v is invertible with inverse τ_{-v}, also its inverse is continuous. This shows the first part and the second is analogous. ☐

This statement can be seen as the invariance of the topology under translations and scalings. It has a trivial, but important corollary:

Corollary 2.1.3 *Let V be a topological vector space and $v \in V$.*

i.) For every (open) neighbourhood $U \subseteq V$ of 0 the subset $\tau_v(U) = v + U$ is an (open) neighbourhood of v.

ii.) If $\mathcal{B}(0)$ is a basis of (open) neighbourhoods of zero, then

$$\mathcal{B}(v) = \bigl\{\tau_v(B) = v + B \bigm| B \in \mathcal{B}(0)\bigr\} \tag{2.1.5}$$

is a basis of (open) neighbourhoods of v. Also

$$\tilde{\mathcal{B}}(v) = \bigl\{\tau_v(-B) = v - B \bigm| B \in \mathcal{B}(0)\bigr\} \tag{2.1.6}$$

is a basis of (open) neighbourhoods of v.

Note that by our conventions, neighbourhoods need not to be open. The corollary is obvious, since the notions of neighbourhood and basis of neighbourhoods is invariant under homeomorphisms in general, in our case the translations τ_v and the reflections $v \mapsto -v$.

Even though this is a simple observation, it has tremendous consequences:

Remark 2.1.4 Let V be a topological vector space.

i.) Since a topology can be reconstructed from the system of all neighbourhoods of all points, the topology of V is already fixed by the collection of neighbourhoods of zero. In fact, a basis of neighbourhoods is already enough.

ii.) The translation invariance also simplifies the notion of convergence considerably. Suppose that $(v_i)_{i \in I}$ is a net in V and $v \in V$. Then we have $v_i \to v$ iff $v_i - v \to 0$ by the homeomorphism property of τ_v. Hence we only have to understand the convergence of nets to zero. If $\mathcal{B}$ is a basis of neighbourhoods of zero, then $v_i \to 0$ iff for every $B \in \mathcal{B}$ one finds an index $i_B \in I$ with $v_i \in B$ for all $i \succcurlyeq i_B$. Hence we have a subnet $(v_{i_B})_{B \in \mathcal{B}}$ indexed by the basis of neighbourhoods $\mathcal{B}$ such that the convergence of $(v_i)_{i \in I}$ to zero is equivalent to the convergence of the subnet $(v_{i_B})_{B \in \mathcal{B}}$ to zero: Of course this is still possible for all topological spaces. But in our case we can use the same index set, say $\mathcal{B}$, for testing convergence of nets also for other limit points, since all neighbourhood systems are pairwise isomorphic.

iii.) If $U \subseteq V$ is an open neighbourhood of zero and $X \subseteq V$ an arbitrary subset, then

$$X + U = \bigcup_{x \in X} \tau_x(U) \tag{2.1.7}$$

is an open neighbourhood of X, since clearly $x \in \tau_x(U)$ for all $x \in X$ and any union of open subsets is open.

iv.) If $U \subseteq V$ is a neighbourhood of zero, then

$$V = \bigcup_{n \in \mathbb{N}} nU. \tag{2.1.8}$$

Indeed, let $v \in V$. Then the continuity of the multiplication with scalars implies that the sequence $(\frac{1}{n}v)_{n \in \mathbb{N}}$ converges to zero. Thus there is an $n_0 \in \mathbb{N}$ with $\frac{1}{n}v \in U$ for all $n \geq n_0$ and hence $v \in nU$.

Corollary 2.1.5 *A topological vector space V is first countable iff it is first countable at 0, i.e. 0 has a countable basis of neighbourhoods.*

Remark 2.1.6 (Subspaces in topological vector spaces) If V is a topological vector space and $W \subseteq V$ is a subspace, we use the induced topology on W. It is then clear that the restrictions of (2.1.1) and (2.1.2) to W stay continuous. Hence any subspace of a topological vector space becomes a topological vector space again. Unless stated otherwise, we will always endow subspaces with the induced topology.

Another consequence of the translation invariance of the topology is the following characterization of closures:

Proposition 2.1.7 *Let V be a topological vector space.*

i.) If $W \subseteq V$ is a subspace, then its closure $W^{\mathrm{cl}} \subseteq V$ is a subspace, too.

ii.) If $X \subseteq V$ is an arbitrary subset, then

$$X^{\mathrm{cl}} = \bigcap_{U \in \mathcal{B}(0)} (X + U), \tag{2.1.9}$$

where $\mathcal{B}(0)$ denotes a basis of neighbourhoods of zero.

Proof: For the first part we choose nets $(v_i)_{i \in I}$ and $(w_i)_{i \in I}$ with $v_i \to v$ and $w_i \to w$, where $v, w \in W^{\mathrm{cl}}$. By Remark 2.1.4, *ii.)*, it is possible to index them by the same index set I. Then the continuity of the addition and multiplication by scalars implies the convergence

$$\lambda v_i + \mu w_i \longrightarrow \lambda v + \mu w$$

for all $\lambda, \mu \in \mathbb{K}$. Since $(\lambda v_i + \mu w_i)_{i \in I}$ is again a net in W, we get $\lambda v + \mu w \in W^{\mathrm{cl}}$, showing the subspace property, since clearly $0 \in W^{\mathrm{cl}}$. For the second part we first recall that $v \in X^{\mathrm{cl}}$ iff for every neighbourhood U of v we have $U \cap X \neq \emptyset$. This is the case iff for every B of a neighbourhood basis of v one has $B \cap X \neq \emptyset$. Now the subsets $\tau_v(U)$ yield such a basis of neighbourhoods if $U \in \mathcal{B}(0)$ ranges over a basis of neighbourhoods of zero. Thus $v \in X^{\mathrm{cl}}$ iff $\big(v + U\big) \cap X \neq \emptyset$, which is the case iff $v \in X - U$. The subsets $-U$ also form a basis of neighbourhoods and hence

$$v \in X^{\mathrm{cl}} \quad \text{iff} \quad v \in \bigcap_{U \in \mathcal{B}(0)} (X + U),$$

which is the second statement. □

As usual, separation properties are of particular importance in topology and hence also for topological vector spaces. Since we have not (yet) required any separation properties explicitly, the following separation of compact and closed subsets in topological vector spaces is remarkable:

Proposition 2.1.8 *Let V be a topological vector space and let $K \subseteq V$ be compact and $A \subseteq V$ be closed with $K \cap A = \emptyset$. Then there exists an open neighbourhood $U \subseteq V$ of zero such that*

$$(K + U) \cap (A + U) = \emptyset. \tag{2.1.10}$$

In particular, V is a T_3-space.

PROOF: First let $O \subseteq V$ be an open neighbourhood of zero. Since taking differences is continuous, we find an open neighbourhood of $(0, 0)$ in $V \times V$, which is mapped into O under the difference map

$$V \times V \ni (v, w) \mapsto v - w \in V.$$

Any open neighbourhood of $(0, 0)$ contains an open neighbourhood of the form $Z \times Z \subseteq V \times V$, where Z is a suitable open neighbourhood of 0 in V. This is a just a basic feature of the product topology. Hence we get an open neighbourhood Z of 0 with $Z - Z \subseteq O$. Now also $-Z$ is an open neighbourhood of 0 and thus $X = Z \cap (-Z)$ is still an open neighbourhood of zero with the two features

$$X + X \subseteq O \quad \text{and} \quad X = -X.$$

Replacing now O by X we find an open neighbourhood $Y \subseteq V$ of zero with $Y = -Y$ and $Y + Y \subseteq X$ and thus

$$Y + Y + Y + Y \subseteq O. \tag{$*$}$$

We apply this construction as follows. For $v \in K$ we consider the open subset $O_v = (V \setminus A) - v \subseteq V$. Indeed, since $A \subseteq V$ is assumed to be closed, O_v is open. Moreover, since $v \in K \subseteq V \setminus A$ we have $0 \in O_v$. Thus we find an open neighbourhood $Y_v \subseteq V$ of zero with ($*$), i.e.

$$Y_v = -Y_v \quad \text{and} \quad Y_v + Y_v + Y_v + Y_v \subseteq O_v = (V \setminus A) - v.$$

Since Y_v contains zero, we conclude

$$Y_v + Y_v + Y_v + v \subseteq O_v + v = V \setminus A \tag{$**$}$$

for all $v \in K$. Next, ($**$) implies

$$v + Y_v + Y_v \subseteq V \setminus (A + Y_v),$$

since again $Y_v = -Y_v$. As $v \in v + Y_v$, we get an open cover of the compact subset K, of which we can select finitely many covering K, too. Thus let $v_1, \ldots, v_n \in K$ be such a choice with

$$K \subseteq (v_1 + Y_{v_1}) \cap \cdots \cap (v_n + Y_{v_n}).$$

We set $U = Y_{v_1} \cap \cdots \cap Y_{v_n}$, which is an open neighbourhood of zero. Now this gives

$$K + U \subseteq \bigcap_{i=1}^{n} (v_i + Y_{v_i} + U) \subseteq \bigcap_{i=1}^{n} (v_i + Y_{v_i} + Y_{v_i}).$$

On the other hand, each $v_i + Y_{v_i} + Y_{v_i}$ has trivial intersection with $A + Y_{v_i}$ and hence also

$$(v_i + Y_{v_i} + Y_{v_i}) \cap (A + (Y_{v_1} \cap \cdots \cap Y_{v_n})) = \emptyset,$$

which results in $(v_i + Y_{v_i} + Y_{v_i}) \cap (A + U) = \emptyset$. Then their finite union has trivial intersection with $A + U$, too, leading to $(K + U) \cap (A + U) = \emptyset$, as wanted. Since a point $K = \{p\}$ is compact, we can separate points from closed subsets A by disjoint open subsets $p + U$ and $A + U$. This is the T_3 property. □

This separation property becomes even more interesting if we add a mild additional one:

Corollary 2.1.9 *Let $(V, \mathscr{V})$ be a topological vector space. Then the following statements are equivalent:*

i.) *The topology $\mathscr{V}$ is T_1, i.e. points are closed.*

ii.) *The topology $\mathscr{V}$ is T_2, i.e. Hausdorff.*

iii.) *The topology $\mathscr{V}$ is regular.*

Proof: From the preceding proposition we know that the topology is T_3. Hence *i.)* and *iii.)* become equivalent as a regular space is, by definition, a T_1- and T_3-space. In general, we have *ii.)* $\implies$ *i.)*, and a regular space is necessarily T_2, i.e. we have *iii.)* $\implies$ *ii.)*, see also [19, Proposition 2.6.8]. □

Thus we will mainly focus on *Hausdorff topological vector spaces*. In fact, some textbooks like [15] take this requirement as part of the definition itself. However, we will see several occasions where, at least as an intermediate step, non-Hausdorff topologies arise naturally. Thus we stick to our more flexible definition.

2.1.2 Continuity of Linear Maps

Having a new class of objects, the topological vector spaces, it is a good custom in mathematics to look for the appropriate categorical framework, i.e. to find a good notion of morphisms. In our case this is fairly obvious:

Definition 2.1.10 (Continuous linear maps and topological dual) *Let V and W be topological vector spaces.*

i.) *A morphism $\phi\colon V \longrightarrow W$ is a continuous linear map.*

ii.) *The set of continuous linear maps from V to W is denoted by*

$$\mathrm{L}(V, W) = \big\{\phi\colon V \longrightarrow W \bigm| \phi \textit{ is linear and continuous}\big\}, \tag{2.1.11}$$

and we set

$$\mathrm{L}(V) = \mathrm{L}(V, V). \tag{2.1.12}$$

iii.) *The set of continuous linear functionals on V, i.e.* $\mathrm{L}(V, \mathbb{K})$, *is called the topological dual of V, denoted by*
$$V' = \mathrm{L}(V, \mathbb{K}). \tag{2.1.13}$$

Here a word of caution is necessary. In the literature one also finds V^* for the topological dual and uses V' for the algebraic dual, i.e. the set of all linear functionals, continuous or not. Similarly, one finds $\mathrm{Hom}(V, W)$ as the space of continuous linear maps. We shall stick to the above version and reserve $\mathrm{Hom}(V, W)$ for the set of *all* linear maps and V^* for the *algebraic dual*, i.e.

$$\mathrm{L}(V, W) = \left\{\phi \in \mathrm{Hom}(V, W) \;\middle|\; \phi \text{ is continuous}\right\} \subseteq \mathrm{Hom}(V, W) \tag{2.1.14}$$

and
$$V' = \left\{\phi \in V^* \;\middle|\; \phi \text{ is continuous}\right\} \subseteq V^*. \tag{2.1.15}$$

The following result shows as expected that the choice of continuous linear maps yields a nice categorical framework:

Proposition 2.1.11 *Let V, W and X be topological vector spaces.*

i.) *For $\phi \in \mathrm{L}(V, W)$ and $\psi \in \mathrm{L}(W, X)$ one has $\psi \circ \phi \in \mathrm{L}(V, X)$. Moreover, $\mathrm{id}_V \in \mathrm{L}(V)$.*

ii.) *For $\phi, \phi' \in \mathrm{L}(V, W)$ and $\lambda, \mu \in \mathbb{K}$ one has $\lambda\phi + \mu\phi' \in \mathrm{L}(V, W)$, i.e. $\mathrm{L}(V, W)$ is a subspace of $\mathrm{Hom}(V, W)$.*

iii.) *The topological vector spaces together with the continuous linear maps form a category.*

PROOF: The composition of linear maps is linear and the composition of continuous maps is continuous. This gives the first part, since $\mathrm{id}_V \in \mathrm{L}(V)$ is obvious. From this the third part is clear. For the second part, let $(v_i)_{i\in I}$ be a convergent net with limit $v \in V$. Then $(\phi(v_i))_{i\in I}$ and $(\phi'(v_i))_{i\in I}$ are convergent by the continuity of ϕ and ϕ', respectively. More precisely, we have $\phi(v_i) \to \phi(v)$ and $\phi'(v_i) \to \phi'(v)$. Thus

$$(\lambda\phi + \mu\phi')(v_i) = \lambda\phi(v_i) + \mu\phi'(v_i) \longrightarrow \lambda\phi(v) + \mu\phi'(v) = (\lambda\phi + \mu\phi')(v)$$

follows from the continuity of the vector space operations in W. Since this holds for all nets, the linear map $\lambda\phi + \mu\phi'$ is continuous. □

Remark 2.1.12 (Topology on $\mathrm{L}(V, W)$) It is tempting to extend the second statement to have a *topological* vector space $\mathrm{L}(V, W)$. This turns out to be not completely trivial. In fact, we will meet several possibilities of introducing a topology on $\mathrm{L}(V, W)$ and, in particular, on V'. In general, they will turn out to be inequivalent.

The category of topological vector spaces will sometimes be denoted by tVS. However, as it turns out, tVS is still too big and too general to admit interesting statements. If we add the Hausdorff condition, we arrive at the full subcategory

$$\mathsf{TVS} \subseteq \mathsf{tVS} \tag{2.1.16}$$

of Hausdorff topological vector spaces. Still, TVS is only slightly more interesting than tVS. We will have to add further features to arrive at interesting subcategories.

The translation invariance of the topology implies that we do not just have a notion of continuity, but even a canonical notion of uniform continuity. The reason is that we can construct a canonical uniform structure on V by translating open neighbourhoods of zero to every other point, see Exercises 2.5.2 and 2.5.3. To keep the discussion simple at this stage, we will not require these exercises and define uniformly continuous maps ad-hoc as follows:

Definition 2.1.13 (Uniformly continuous maps) *Let V and W be topological vector spaces. Then a not necessarily linear map $\phi\colon V \longrightarrow W$ is called uniformly continuous if for all neighbourhoods $U \subseteq W$ of zero there exists a neighbourhood $Z \subseteq V$ of zero such that*

$$\phi(v) - \phi(v') \in U \quad \textit{whenever} \quad v - v' \in Z. \tag{2.1.17}$$

The idea is that one neighbourhood correspondence serves to test the continuity condition everywhere in the same uniform way. Clearly, a uniformly continuous map is continuous, but easy examples from calculus show that the converse needs not to be true. Remarkably, for linear maps the two notions coincide:

Proposition 2.1.14 *Let V and W be topological vector spaces and let $\phi\colon V \longrightarrow W$ be linear. Then the following statements are equivalent:*

i.) The map ϕ is continuous at zero.

ii.) The map ϕ is continuous at some $v \in V$.

iii.) The map ϕ is continuous.

iv.) The map ϕ is uniformly continuous.

PROOF: Let ϕ be continuous at zero and let $U \subseteq W$ be a neighbourhood of zero. Then we find a neighbourhood $Z \subseteq V$ of zero with $\phi(Z) \subseteq U$, i.e. $\phi(v) \in U$ for all $v \in Z$. But then for all $v, v' \in V$ with $v - v' \in Z$ we have

$$\phi(v) - \phi(v') = \phi(v - v') \in U.$$

This shows that ϕ is uniformly continuous. Hence we have *i.)* $\implies$ *iv.)*. The implications *iv.)* $\implies$ *iii.)* and *iii.)* $\implies$ *ii.)* hold in general. Finally, *i.)* and *ii.)* are equivalent, since the translation τ_v by v is a homeomorphism by Proposition 2.1.2, *i.)*. Thus ϕ is continuous at v iff $\phi \circ \tau_v = \tau_{\phi(v)} \circ \phi$ is continuous at zero. □

2.1.3 Completeness

Since we have a uniform structure on a topological vector space according to Exercise 2.5.3, we have a notion of Cauchy nets and completeness from the general theory of uniform spaces. Again, we give a more ad-hoc definition as follows which, of course, reproduces the general definition but is more adapted to our current situation:

Definition 2.1.15 (Cauchy net) *Let V be a topological vector space and let $(v_i)_{i \in I}$ be a net in V. Then $(v_i)_{i \in I}$ is called a Cauchy net if for every neighbourhood $U \subseteq V$ of zero there is an index $i_U \in I$ with*

$$v_j - v_{j'} \in U \tag{2.1.18}$$

for all $j, j' \succcurlyeq i_U$. A Cauchy net with index set $I = \mathbb{N}$ is called a Cauchy sequence.

Some first properties of Cauchy nets are collected in the following proposition:

Proposition 2.1.16 *Let V and W be topological vector spaces.*

i.) If $\phi\colon V \longrightarrow W$ is a uniformly continuous map and if $(v_i)_{i \in I}$ is a Cauchy net in V, then $(\phi(v_i))_{i \in I}$ is a Cauchy net in W.

ii.) A convergent net is a Cauchy net.

iii.) A Cauchy net converges iff it has a convergent subnet.

PROOF: For the first part, let $(v_i)_{i\in I}$ be a Cauchy net in V and let $U \subseteq W$ be a neighbourhood of zero. Then by uniform continuity we find a neighbourhood $Z \subseteq V$ of zero with $\phi(v)-\phi(v') \in U$ for all $v-v' \in Z$. Next, we find an index $i_Z \in I$ with $v_j - v_{j'} \in Z$ for $j, j' \succcurlyeq i_Z$ and hence $\phi(v_j)-\phi(v_{j'}) \in U$ for all $j, j' \succcurlyeq i_Z$. Thus the image net is Cauchy again. For the second statement let $U \subseteq V$ be a neighbourhood of zero and assume that a net $(v_i)_{i\in I}$ converges to $v \in V$. Inside U we find another neighbourhood $Z \subseteq U$ of zero with $Z - Z \subseteq U$, see again the proof of Proposition 2.1.8. Then we find an index $i_Z \in I$ with $v_j \in v + Z$, whenever $j \succcurlyeq i_Z$. Taking differences gives

$$v_j - v_{j'} \in (v+Z) - (v+Z) = Z - Z \subseteq U$$

for all $j, j' \succcurlyeq i_Z$ showing that $(v_i)_{i\in I}$ is Cauchy. Finally, let $(v_i)_{i\in I}$ be a Cauchy net. If $(v_i)_{i\in I}$ converges, then every subnet converges to the same limit again, this holds in general topological spaces, see e.g. [19, Proposition 4.1.12]. Thus assume that $\Phi\colon J \longrightarrow I$ is a cofinal map such that the subnet $(v_{\Phi(j)})_{j\in J}$ of $(v_i)_{i\in I}$ converges to $v \in V$. Let $U \subseteq V$ be a neighbourhood of zero, then we find a neighbourhood $Z \subseteq V$ of zero with $Z + Z \subseteq U$ as before. Now from the convergence of the subnet we get an index j_Z with $v_{\Phi(j)} \in Z + v$, whenever $j \succcurlyeq j_Z$. From the Cauchy property we get an index $i_Z \in I$ with $v_i - v_{i'} \in Z$, whenever $i, i' \succcurlyeq i_Z$. Without restriction, we can assume $\Phi(j_Z) \succcurlyeq i_Z$, since Φ is cofinal. For $i \succcurlyeq \Phi(j_Z)$ we then get

$$v_i - v = v_i - v_{\Phi(j)} + v_{\Phi(j)} - v \in Z + Z \subseteq U$$

by inserting $v_{\Phi(j)}$ for some $j \succcurlyeq j_Z$. This means $v_i \in v + U$ and hence $(v_i)_{i\in I}$ converges to v. ☐

Remark 2.1.17 Since a convergent net always has a subnet indexed by a basis of neighbourhoods of zero according to Remark 2.1.4, *i.)*, we can assume without restriction that Cauchy nets are indexed by such a basis of neighbourhoods already: all questions about convergence can be discussed by the corresponding subnet. In particular, if V is first countable, a Cauchy net converges iff it has a convergent *subsequence*.

Of course, we know already situations from metric spaces in calculus, where Cauchy sequences might not converge. It is a feature of the ambient space, whether the converse of Proposition 2.1.16, *ii.)*, is true as well, called *completeness*:

Definition 2.1.18 (Completeness) *Let V be a topological vector space.*

i.) The space V is called complete if every Cauchy net converges.

ii.) The space V is called sequentially complete if all Cauchy sequences converge.

In some sense, mainly Hausdorff topological vector spaces are interesting here as otherwise we have no unique limit to which a Cauchy net should converge. Thus we will mainly consider Hausdorff topological vector spaces as soon as it comes to questions about completeness.

Clearly, a complete topological space is also sequentially complete. However, we will see examples of sequentially complete spaces, which are *not* complete when discussing dual spaces later. Hence we will need to distinguish the two notions carefully. Nevertheless, in the situation of first countable spaces this simplifies:

Proposition 2.1.19 *Let V be a first countable topological vector space. Then V is complete iff V is sequentially complete.*

PROOF: This follows directly from Proposition 2.1.16, *iii.)*, and Remark 2.1.17. ☐

The topological closure of a subspace is again a subspace by Proposition 2.1.7, *i.)*. If the ambient space is complete, then the closure is again complete:

Proposition 2.1.20 *Let V be a complete Hausdorff topological vector space and let $W \subseteq V$ be a subspace. Then W is complete iff $W = W^{\mathrm{cl}}$.*

PROOF: Assume first that W is complete in the induced topology. Let $v \in W^{\mathrm{cl}}$ be a point in the closure and choose a net $(w_i)_{i \in I}$ in W converging to v. This is always possible, since the topological closure coincides with net-closure. Then $w_i \to v$ implies that $(w_i)_{i \in I}$ is a Cauchy net in V and hence a Cauchy net in W by the definition of the induced topology. By completeness, we have $w_i \to w \in W$. Since in the Hausdorff situation we have at most one limit, $w = v$ follows and hence $v \in W$. This gives $W = W^{\mathrm{cl}}$. Conversely, assume $W = W^{\mathrm{cl}}$ and let $(w_i)_{i \in I}$ be a Cauchy net in W^{cl}. Then it is also a Cauchy net in V and thus convergent to some $v \in V$ by completeness. Since limits of convergent nets belong to the closure, we have $v \in W^{\mathrm{cl}} = W$, showing that W is complete itself. □

Note that for the implication that a complete W is closed, we do net yet need that V is complete itself.

In many constructions in analysis one wants to approximate an ideal element in a topological vector space V by more accessible and easy elements. Often it is possible to show that an approximation is done by a Cauchy net (or even a Cauchy sequence) and hence one needs to know, whether one has an actual limit, i.e. whether V is complete. Since in general this might not be the case in the first place, one wants to add the needed ideal elements to include V into some larger, now complete, topological vector space. This process is then called *completion*. Of course, in a non-Hausdorff situation we could add arbitrarily many limits as limits are not unique. Thus one restricts the notion of a completion to Hausdorff topological vector spaces. The precise definition is then done by means of a universal property.

Definition 2.1.21 (Completion) *Let V be a Hausdorff topological vector space. Then a completion $(\hat{V}, \iota)$ of V is a complete Hausdorff topological vector space $\hat{V}$ together with a continuous linear map $\iota\colon V \longrightarrow \hat{V}$ such that for all other complete Hausdorff topological vector spaces W and continuous linear maps $\phi\colon V \longrightarrow W$ one has a unique linear map $\hat{\phi}\colon \hat{V} \longrightarrow W$ such that*

$$
\begin{tikzcd}
\hat{V} \arrow[r, "\hat{\phi}"] & W \\
V \arrow[u, "\iota"] \arrow[ur, "\phi"'] &
\end{tikzcd}
\tag{2.1.19}
$$

commutes.

Since we defined a completion by means of a universal property, it is clear that any two completions $(\hat{V}, \iota)$ and $(\tilde{V}, j)$ are isomorphic by means of a unique isomorphism I, such that

$$
\begin{tikzcd}
\hat{V} \arrow[rr, "I"] & & \tilde{V} \\
& V \arrow[ul, "\iota"] \arrow[ur, "j"'] &
\end{tikzcd}
\tag{2.1.20}
$$

commutes. This is the usual argument with universal properties, see Exercise 2.5.4.

Thus, as to be expected, it is the existence, which one has to worry about. Here one needs a specific construction of $\hat{V}$ and ι. We will not go into the details here, but only indicate the steps, which are similar to the construction of the completion of a metric space. In fact, one can use the completion of V as a uniform space, see e.g. [8, Chap. 6] for details. More directly, one proceeds as follows: one fixes as index set I a basis of neighbourhoods of zero in V. Then it will suffice to consider

Cauchy nets indexed by I. The set of all Cauchy nets $\mathcal{C}(V)$ with this index set turns out to be a vector space by the pointwise operations in I. One has the subspace $\mathcal{C}_0(V) \subseteq \mathcal{C}(V)$ of zero nets, i.e. nets converging to zero. Moreover, the constant nets provide an injective linear map $i\colon V \to \mathcal{C}(V)$. Taking now the quotient $\hat{V} = \mathcal{C}(V)/\mathcal{C}_0(V)$ gives the candidate for the completion $(\hat{V}, \iota)$, where $\iota\colon V \longrightarrow \hat{V}$ is just the composition of i with the quotient map. One needs to construct now a topology on $\hat{V}$ such that the requirements are fulfilled, a not completely trivial task. A slightly different construction can be found in e.g. [7, Sect. 3.3], see also Exercise 2.5.5. The result is that one has a completion indeed. Moreover, the map $\iota\colon V \longrightarrow \hat{V}$ turns out to be injective, which allows to identify V with the subspace $\iota(V) \subseteq \hat{V}$. Then $\iota(V)^{\mathrm{cl}} = \hat{V}$:

Theorem 2.1.22 (Completion of Topological Hausdorff Vector Space) *Let V be a Hausdorff topological vector space. Then there exists a completion $(\hat{V}, \iota)$ of V, unique up to unique isomorphism. The map ι is injective and $\iota(V)^{\mathrm{cl}} = \hat{V}$.*

For a more specific class of topological vector spaces we will see a much more explicit construction later on in Section 2.4.4.

2.1.4 Subsets in Topological Vector Spaces

The linear structure of a topological vector space allows us to define *balanced* and *convex* subsets. The interplay of these notions with the topological concepts is then of crucial importance. We start with the following definitions from linear algebra:

Definition 2.1.23 (Balanced and convex subsets) *Let V be a vector space over $\mathbb{K}$ and let $X \subseteq V$ be a subset.*

i.) *The subset X is called balanced or circled if for all $z \in \mathbb{K}$ with $|z| \leq 1$ and $v \in X$ one has $zv \in X$.*

ii.) *The subset X is called convex if for all $v, w \in X$ and all $\lambda \in [0, 1]$ one has $\lambda v + (1 - \lambda) w \in X$.*

iii.) *The subset X is called absolutely convex if it is balanced and convex.*

With this definition $\emptyset \subseteq V$ is absolutely convex. A few first properties are now collected in the following remark:

Remark 2.1.24 Let V be a vector space over $\mathbb{K}$.

i.) Arbitrary intersections and unions of balanced subsets are again balanced. A balanced subset necessarily contains 0, as soon as it is non-empty at all.

ii.) Arbitrary intersections of convex subsets are again convex. However, the union of convex subsets needs not to be convex: easy counterexamples for $V = \mathbb{R}^2$ exist.

iii.) It follows that arbitrary intersections of absolutely convex subsets are absolutely convex.

iv.) The vector space V itself absolutely convex. In fact, every subspace is absolutely convex.

Since a given subset needs not to be balanced or convex, one might want to pass to the smallest balanced or convex subset containing it. Since V itself is balanced and convex such a smallest subset always exists:

Proposition 2.1.25 *Let V be a vector space over $\mathbb{K}$ and let $X \subseteq V$ be a subset.*

i.) *The intersection of all balanced subsets containing X is the smallest balanced subset containing X.*

ii.) *The intersection of all convex subsets containing X is the smallest convex subset containing X.*

iii.) *The intersection of all absolutely convex subsets containing X is the smallest absolutely convex subset containing X.*

Proof: Since V itself is absolutely convex, we have such subsets containing X. Then the minimality of their intersection is clear by the above Remark 2.1.24, *i.)*, *ii.)*, and *iii.)*, respectively. $\square$

Definition 2.1.26 (Balanced and convex hull) *Let V be a vector space over $\mathbb{K}$ and let $X \subseteq V$ be a subset.*

i.) *The smallest balanced subset of V containing X is called the balanced hull of X, denoted by* $\operatorname{balanced}(X)$.

ii.) *The smallest convex subset of V containing X is called the convex hull of X, denoted by* $\operatorname{conv}(X)$.

iii.) *The smallest absolutely convex subset of V containing X is called the absolutely convex hull of X, denoted by* $\operatorname{absconv}(X)$.

Since the intersections of Proposition 2.1.25 are hard to describe, we need a more explicit way to obtain the hull operations. This can be accomplished as follows:

Proposition 2.1.27 *Let V be a vector space over $\mathbb{K}$ and let $X \subseteq V$ be a subset.*

i.) *The balanced hull of X is explicitly given by*

$$\operatorname{balanced}(X) = \left\{ zx \;\middle|\; x \in X \textit{ and } z \in \mathbb{K} \textit{ with } |z| \leq 1 \right\}. \tag{2.1.21}$$

ii.) *The convex hull of X is explicitly given by*

$$\operatorname{conv}(X) = \left\{ \sum\nolimits_{i=1}^{n} \lambda_i x_i \;\middle|\; n \in \mathbb{N},\ x_i \in X,\ \textit{and } \lambda_i \geq 0 \textit{ with } \lambda_1 + \cdots + \lambda_n = 1 \right\}. \tag{2.1.22}$$

iii.) *The absolutely convex hull of X is explicitly given by*

$$\operatorname{absconv}(X) = \left\{ \sum\nolimits_{i=1}^{n} z_i x_i \;\middle|\; n \in \mathbb{N},\ x_i \in X,\ \textit{and } z_i \in \mathbb{K} \textit{ with } |z_1| + \cdots + |z_n| \leq 1 \right\}. \tag{2.1.23}$$

iv.) *One has*

$$\operatorname{balanced}(\operatorname{conv}(X)) \subseteq \operatorname{absconv}(X) = \operatorname{conv}(\operatorname{balanced}(X)). \tag{2.1.24}$$

v.) *If X is balanced, then* $\operatorname{conv}(X)$ *is balanced again.*

Proof: First we note that the right hand side of (2.1.21) is balanced and contains X: this is clear by construction. Hence the balanced hull is contained in the right hand side, being the smallest balanced subset of V containing X. Now if $Y \subseteq V$ is any balanced subset with $X \subseteq Y$, then for $z \in \mathbb{K}$ with $|z| \leq 1$ and $x \in X$ one has $zx \in Y$. Thus the right hand side is contained in Y. Together this gives *i.)*. For the second part, we note that the right hand side contains X by considering $n = 1$. Moreover, a simple check shows that it is convex. Conversely, if we have a convex subset $Y \subseteq V$ and $y_1, \ldots, y_n \in Y$ together with $\lambda_1, \ldots, \lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$, then

$$\lambda_1 y_1 + \cdots + \lambda_n y_n \in Y$$

again. Indeed, this follows inductively on n from the convexity of Y: the cases $n = 1$ and $n = 2$ are clear. To pass from n to $n + 1$ we can assume $\lambda = \lambda_{n+1} > 0$, since otherwise the statement is trivial. Since $\lambda_1 + \cdots + \lambda_n = 1 - \lambda_{n+1} = 1 - \lambda$, we have

$$\frac{\lambda_1}{1-\lambda} + \cdots + \frac{\lambda_n}{1-\lambda} = 1,$$

and hence
$$y = \frac{\lambda_1}{1-\lambda} y_1 + \cdots + \frac{\lambda_n}{1-\lambda} y_n \in Y$$
by induction. Thus $\lambda_1 y_1 + \cdots + \lambda_{n+1} y_{n+1} = (1-\lambda) y + \lambda y_{n+1} \in Y$ by convexity, proving the claim. It follows that if $X \subseteq Y$, then the right hand side of (2.1.22) is contained in Y. Together, this gives *ii.)*. The argument for the third part is analogous: the right hand side is both balanced and convex. If Y is balanced and convex, then for $y_1, \ldots, y_n \in Y$ and $z_1, \ldots, z_n \in \mathbb{K}$ with $|z_1| + \cdots + |z_n| \leq 1$ one has
$$z_1 y_1 + \cdots + z_n y_n \in Y$$
again. Indeed, for $n = 1$ this follows from Y being balanced and the case $n = 2$ is obtained as follows: We can assume $z_1 \neq 0 \neq z_2$ without restriction. Then we set
$$\lambda = \frac{|z_1|}{|z_1| + |z_2|} \in (0, 1),$$
and observe that $\frac{z_1}{|z_1|} y_1 \in Y$ and $\frac{z_2}{|z_2|} y_2 \in Y$, since Y is balanced. We then get
$$z_1 y_1 + z_2 y_2 = (|z_1| + |z_2|)\Big(\tfrac{|z_1|}{|z_1|+|z_2|} \tfrac{z_1}{|z_1|} y_1 + \tfrac{|z_2|}{|z_1|+|z_2|} \tfrac{z_2}{|z_2|} y_2\Big) = (|z_1| + |z_2|)(\lambda v_1 + (1-\lambda) v_2) \in Y,$$
since first $v_i = \frac{z_i}{|z_i|} y_i \in Y$, as Y is balanced. Then $y = \lambda v_1 + (1-\lambda) v_2 \in Y$, since Y is convex, and finally, $(|z_1| + |z_2|) y \in Y$, since Y is balanced and $|z_1| + |z_2| \leq 1$. The induction to get from n to $n+1$ is then along the same lines as before. This shows that any absolutely convex subset Y is stable under this operation and hence $X \subseteq Y$ implies that Y contains the right hand side of (2.1.23) as well. As before, this proves that the smallest absolutely convex subset containing X is given by (2.1.23). For the fourth part, we first note that for $X \subseteq Y$ we have for trivial reasons
$$\operatorname{conv}(X) \subseteq \operatorname{conv}(Y),$$
$$\operatorname{balanced}(X) \subseteq \operatorname{balanced}(Y),$$
and
$$\operatorname{absconv}(X) \subseteq \operatorname{absconv}(Y).$$
Since $\operatorname{absconv}(X)$ is convex, we have $\operatorname{absconv}(X) \supseteq \operatorname{conv}(X)$ and thus
$$\operatorname{absconv}(X) = \operatorname{balanced}(\operatorname{absconv}(X)) \supseteq \operatorname{balanced}(\operatorname{conv}(X)),$$
since $\operatorname{absconv}(X)$ is already balanced. Similarly, we get
$$\operatorname{absconv}(X) \supseteq \operatorname{conv}(\operatorname{balanced}(X)).$$
Now suppose X is balanced and hence $X = \operatorname{balanced}(X)$. Then for $y \in \operatorname{conv}(X)$ we have $x_1, \ldots, x_n \in X$ with $\lambda_1, \ldots, \lambda_n \geq 0$ and $\lambda_1 + \cdots + \lambda_n = 1$ such that
$$y = \lambda_1 x_1 + \cdots + \lambda_n x_n$$
by *ii.)*. Let $z \in \mathbb{K}$ with $|z| \leq 1$ be given. Then $z x_i \in X$, since X is balanced. Thus
$$zy = z\lambda_1 x_1 + \cdots + z\lambda_n x_n = \lambda_1(z x_1) + \cdots + \lambda_n(z x_n)$$
is again in $\operatorname{conv}(X)$, showing $\operatorname{balanced}(\operatorname{conv}(X)) = \operatorname{conv}(X)$, whenever X is balanced. In particular,
$$\operatorname{balanced}(\operatorname{conv}(\operatorname{balanced}(X))) = \operatorname{conv}(\operatorname{balanced}(X))$$
holds for every subset $X \subseteq V$. Since $\operatorname{conv}(\operatorname{balanced}(X))$ is convex and balanced, i.e. absolutely convex, we get
$$\operatorname{conv}(\operatorname{balanced}(X)) \supseteq \operatorname{absconv}(X),$$
which is the remaining inclusion required in (2.1.24). □

Note that the inclusion

$$\operatorname{balanced}(\operatorname{conv}(X)) \subseteq \operatorname{absconv}(X) \tag{2.1.25}$$

is typically strict: there are easy examples in $\mathbb{R}^2$ of convex subsets $X = \operatorname{conv}(X)$, where balanced$(X)$ is no longer convex, see also Exercise 2.5.8.

By definition, convex combinations of points in a convex subset end up again in this subset. One can now ask, whether a given point is a convex combination in a non-trivial way or not. This leads to the concept of an *extreme* point:

Definition 2.1.28 (Extreme points) *Let $X \subseteq V$ be a subset of a vector space V over $\mathbb{K}$. Then $v \in X$ is called extreme if for $\lambda \in (0, 1)$ and $v_1, v_2 \in X$ with*

$$v = \lambda v_1 + (1 - \lambda)v_2 \tag{2.1.26}$$

one has $v = v_1 = v_2$. The set of extreme points of X is denoted by extreme(X).

In general, extreme$(X) = \emptyset$. If, e.g. $X \neq \{0\}$ is a subspace itself, then clearly extreme$(X) = \emptyset$. However, under certain assumptions, a subset X has extreme points. For a convex subset X it may happen that X can be reconstructed from its extreme points. However, this will require a non-trivial theorem, the Krein-Milman Theorem in Section 3.5.

We shall now investigate the interplay with the topological concepts. The first statement about extreme points shows that they are necessarily boundary points:

Proposition 2.1.29 *Let V be a topological vector space and let $X \subseteq V$ be a subset. Then*

$$\operatorname{extreme}(X) \subseteq \partial X. \tag{2.1.27}$$

Proof: Suppose $v \in X^\circ$ is an interior point. Then there is an $\epsilon > 0$ with $(1 + \lambda)v \in X^\circ$ for all $|\lambda| < \epsilon$, since the scalings are continuous. Hence

$$v = \tfrac{1}{2}(1 + \lambda)v + \tfrac{1}{2}(1 - \lambda)v$$

is a non-trivial convex combination, whenever $\lambda \neq 0$. Hence v is not extreme. □

Next, we investigate the interior and the closure of balanced and convex subsets:

Proposition 2.1.30 *Let V be a topological vector space.*

i.) If $B \subseteq V$ is a balanced subset, then B^{cl} is balanced. If in addition $0 \in B^\circ$, then B° is balanced, too.

ii.) If $K \subseteq V$ is a convex subset, then K^{cl} and K° are convex, too.

iii.) If $A \subseteq V$ is absolutely convex, then A^{cl} is absolutely convex.

Proof: Let $(v_i)_{i\in I}$ be a net in B converging to $v \in B^{\mathrm{cl}}$. For $z \in \mathbb{K}$ with $|z| \leq 1$ we have $zv_i \in B$ and hence the net $(zv_i)_{i\in I}$ is in B. The continuity of the multiplication with scalars gives $zv = \lim_{i\in I} zv_i$ and thus $zv \in B^{\mathrm{cl}}$. Hence B^{cl} is balanced, too. Now let $v \in B^\circ$ be an interior point and let $U \subseteq V$ be an open neighbourhood of zero such that $v + U \subseteq B^\circ$. Note that being interior implies that we have such an open neighbourhood, see Exercise 2.5.9. Thus for $u \in U$ we have for all $z \in \mathbb{K}$ with $|z| \leq 1$ that $z(v + u) \in B$ since B is balanced. This shows that $zv + zU \subseteq B$. Since for $z \neq 0$ the subset zU is still an open neighbourhood of zero, $zv \in B^\circ$ follows for $z \neq 0$. Since by assumption furthermore $0 \in B^\circ$, we get the same conclusion also for $z = 0$. Note that the assumption $0 \in B^\circ$ is *not superfluous*, as simple examples in $\mathbb{R}^2$ show, see Exercise 2.5.10. The second part is shown essentially in the same manner: the convexity of K^{cl} is clear by the same net-based argument. For K° assume that $\lambda \in (0,1)$. Then λK° and $(1-\lambda)K^\circ$ are again open. Moreover,

$$\lambda K^\circ + (1-\lambda)K^\circ = \bigcup_{x \in \lambda K^\circ} (x + (1-\lambda)K^\circ)$$

is a union of open subsets and hence open again, see Remark 2.1.4, *iii.)*. Thus $\lambda K^\circ + (1-\lambda)K^\circ$ is open and, since K is convex, $\lambda K^\circ + (1-\lambda)K^\circ \subseteq K$. Since $K^\circ \subseteq K$ is by definition the largest open subset contained inside of K, we have

$$\lambda K^\circ + (1-\lambda)K^\circ \subseteq K^\circ$$

for all $\lambda \in (0,1)$. Clearly, this also holds for the extreme cases $\lambda = 0$ and $\lambda = 1$, showing that K° is convex again. The third part is then a consequence of *i.)* and *ii.)*. □

Since the topology is entirely determined by the zero neighbourhoods, we want to investigate now, which sorts of neighbourhood bases one can achieve.

Proposition 2.1.31 *Let V be a topological vector space.*

i.) *Every neighbourhood of zero contains a balanced open neighbourhood of zero.*

ii.) *There exists a basis of closed balanced neighbourhoods of zero.*

iii.) *If $K \subseteq V$ is a convex neighbourhood of zero, then K contains an absolutely convex neighbourhood of zero.*

PROOF: Let $U \subseteq V$ be a zero neighbourhood. Since multiplication by scalars is continuous, there is an $\epsilon > 0$ and an open zero neighbourhood $W \subseteq V$ with $zW \subseteq U$ for all $|z| < \epsilon$. Then the union of all zW with $0 < |z| < \epsilon$ is still open and a zero neighbourhood. By construction, this union is balanced: note that $z = 0$ is not problematic, since $0 \in W$ anyway. This shows the first part. From Proposition 2.1.8 we know that V is a T_3-space. Hence every zero neighbourhood contains a closed zero neighbourhood, see e.g. [19, Proposition 2.6.5]. From the first part, this closed neighbourhood contains a balanced neighbourhood. Since the closure of this balanced neighbourhood is, on the one hand, still contained in the closed neighbourhood, and, on the other hand, again balanced by Proposition 2.1.30, *i.)*, the second part follows as well. Now assume that K is a convex neighbourhood of zero. By the first part we find a balanced zero neighbourhood $U \subseteq V$ with $U \subseteq K$. Then $U \subseteq zK$ for all $z \in \mathbb{K}$ with $|z| = 1$, since $U = zU$, as U is balanced. Thus U is also contained in the intersection

$$U \subseteq W = \bigcap_{|z|=1} zK,$$

which then turns out to be a neighbourhood of zero. Since zK is again convex, also W is convex by Remark 2.1.24, *ii.)*. From Proposition 2.1.30, *ii.)*, we infer that W° is an open convex neighbourhood. Finally, we need to show that W is balanced. Thus let $w = zv \in W$ with some $v \in K$ and $|z| = 1$. Moreover, let $x \in \mathbb{K}$ with $|x| \leq 1$ be given and consider xw. If $x = 0$, then $xw = 0 \in W$. Thus we can assume that $x \neq 0$. Then

$$xw = |x|\tfrac{x}{|x|}w = |x|\tfrac{x}{|x|}w + (1 - |x|) \cdot 0 \in W,$$

since by construction $\frac{x}{|x|}w = \frac{x}{|x|}zv \in W$ again and W is convex with $0 \in W$. This shows that W is balanced. Since $0 \in W^\circ$, also W° is balanced by Proposition 2.1.30, *i.)*. □

There are now topological vector spaces V such that the only convex neighbourhood of zero is V itself, see Exercise 2.5.11. Thus for such topological vector spaces the last statement will not yield anything interesting. It turns out that the existence of a neighbourhood basis of convex neighbourhoods will be the key to a deeper understanding of topological vector spaces. If one does not have such a neighbourhood basis, V behaves rather pathological in many aspects, as we shall see. We will come back to a detailed discussion of the role of convex subsets in Section 2.2.

2.1.5 Bounded Subsets

It is not completely obvious how one can define a *bounded subset* using only the topology of the topological vector space: in a metric space we can replace the metric by another metric such that $d(x, y) \leq 1$ for all points x, y *without* changing the topology. Hence a definition of a subset being bounded with respect to a metric is not a topological concept, but explicitly refers to and depends on the choice of a metric.

For a topological vector space we have no metric available in general, thus this option will drop out anyway. Instead, one bases the definition on the scaling behaviour:

Definition 2.1.32 (Bounded subset) *Let V be a topological vector space. Then a subset $B \subseteq V$ is called bounded if for every neighbourhood $U \subseteq V$ of zero one finds an $r > 0$ with $B \subseteq rU$.*

Thus a bounded subset can be absorbed by any neighbourhood of zero, once we have inflated the neighbourhood sufficiently.

Remark 2.1.33 Let V be a topological vector space.

i.) It suffices to test the boundedness of a subset $B \subseteq V$ for a basis $\mathcal{B}$ of neighbourhoods of zero. Indeed, if U is an arbitrary neighbourhood, we find a $W \in \mathcal{B}$ with $W \subseteq U$. If now $B \subseteq rW$, then also $B \subseteq rU$.

ii.) Since for every open neighbourhood U of zero, we have $V = \bigcup_{n\in\mathbb{N}} nU$ by Remark 2.1.4, *iv.)*, the subsets $\{nU\}_{n\in\mathbb{N}}$ provide an open cover of V. Hence for a compact subset $K \subseteq V$ finitely many $n_1U, \ldots, n_kU$ already cover K. Since $nU \subseteq (n+1)U$, a single nU covers K, showing that K is bounded: every compact subset is bounded.

iii.) Every finite subset is bounded, since finite subsets are compact.

iv.) If a subset $B \subseteq V$ is bounded, then any subset of B is bounded as well.

v.) Finite unions and arbitrary intersections of bounded subsets are again bounded.

vi.) If $B \subseteq V$ is bounded, then B is also bounded for any coarser topological vector space structure on V.

One of the important aspects of bounded subsets is that it is a concept referring to countability in the following sense. We can test for boundedness by considering *sequences*:

Proposition 2.1.34 *Let V be a topological vector space and let $B \subseteq V$ be a subset. Then the following statements are equivalent:*

i.) The subset B is bounded.

ii.) For every sequence $(v_n)_{n\in\mathbb{N}}$ in B and every zero sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$ one has

$$\lim_{n\to\infty} z_n v_n = 0. \tag{2.1.28}$$

iii.) Every countable subset of B is bounded.

PROOF: According to Proposition 2.1.31, *ii.)*, we can find a basis $\mathcal{B}$ of neighbourhoods of zero consisting of balanced open subsets. We use this basis to test boundedness, which suffices according to Remark 2.1.33, *i.)*, and to test convergence. Thus let B be bounded and let $v_n \in B$ for $n \in \mathbb{N}$ and $z_n \in \mathbb{K}$ with $z_n \to 0$ be given. Let $U \in \mathcal{B}$ and fix $r > 0$ with $B \subseteq rU$. Then for all $n \in \mathbb{N}$ with $|z_n| < \frac{1}{r}$ we get $z_n v_n \in z_n rU \subseteq U$, since U is balanced. Hence *ii.)* follows. Conversely, assume that B is unbounded, then we find at least one $U \in \mathcal{B}$, such that for all $n \in \mathbb{N}$ there is a $v_n \in B$ with $v_n \notin nU$ and hence $\frac{1}{n}v_n \notin U$. This implies that $\left(\frac{1}{n}v_n\right)_{n\in\mathbb{N}}$ is not converging to zero. Hence *i.)* and *ii.)* are equivalent. The equivalence of *ii.)* and *iii.)* is clear. $\square$

Note that this uses a feature of convergent *sequences*, namely that they are bounded. This is not necessarily true for convergent nets, see Exercise 2.5.13.

In general, the closure of a subset can increase the subset quite substantially. Thus the following statement is remarkable:

Proposition 2.1.35 *Let V be a topological vector space. Then the closure of a bounded subset is again bounded.*

PROOF: It suffices to test the boundedness of B^{cl} for a bounded subset $B \subseteq V$ with a basis $\mathcal{B}$ of closed neighbourhoods of zero. Since V is a T_3 space by Proposition 2.1.8, we know that such a basis of neighbourhoods exists, see also Proposition 2.1.31, *ii.)*. Now let $U = U^{\mathrm{cl}} \in \mathcal{B}$ and choose $r > 0$ with $B \subseteq rU$. Then $B^{\mathrm{cl}} \subseteq (rU)^{\mathrm{cl}} = rU^{\mathrm{cl}} = rU$, since the scaling with r is a homeomorphism and U is closed. This shows that U^{cl} is bounded. □

An immediate consequence of the definition of a bounded set is the following compatibility of boundedness and continuous linear maps. Note that bounded subsets behave very much like compact subsets: the image of a bounded subset stays bounded, but pre-images of bounded subsets can very well be unbounded. More precisely, we have the following statement:

Proposition 2.1.36 *Let $\phi\colon V \longrightarrow W$ be a sequentially continuous linear map between topological vector spaces. If $B \subseteq V$ is bounded, then $\phi(B) \subseteq W$ is bounded again.*

PROOF: This is clear by Proposition 2.1.34, *ii.)*. □

In elementary calculus one can characterize the compact subsets of $\mathbb{K}^n$ as the bounded and closed subsets. This is either the naive definition or, better, the Theorem of Heine-Borel. In general topological vector spaces their equivalence might fail, even if we add the Hausdorff property to guarantee that compact subsets are closed. Thus the following definition singles out this particularly nice situation:

Definition 2.1.37 (Heine-Borel property) *A Hausdorff topological vector space is said to have the Heine-Borel property if every bounded and closed subset is compact.*

Later on we will see several non-trivial examples beyond the finite-dimensional case.

2.1.6 Finite-Dimensional Topological Vector Spaces

For the standard topology on $\mathbb{K}^n$ this finite-dimensional vector space becomes a Hausdorff topological vector space. In this section we shall show that this is the only possibility for $\mathbb{K}^n$ to become a Hausdorff topological vector space.

Lemma 2.1.38 *Let V be a topological vector space over $\mathbb{K}$ and let $\phi\colon \mathbb{K}^n \longrightarrow V$ be a linear map. Then ϕ is continuous with respect to the standard topology of $\mathbb{K}^n$, i.e.*

$$\mathrm{L}(\mathbb{K}^n, V) = \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, V). \tag{2.1.29}$$

PROOF: Let $\mathrm{e}_1, \ldots, \mathrm{e}_n \in \mathbb{K}^n$ be the standard basis and denote the images by $v_i = \phi(\mathrm{e}_i)$ for $i = 1, \ldots, n$. Since multiplication by scalars is continuous, the linear map $\mathbb{K} \ni z \mapsto zv \in V$ for a fixed $v \in V$ is continuous. This shows the case $n = 1$. The standard topology on $\mathbb{K}^n$ is the product topology and hence the product map

$$\mathbb{K}^n \ni z \;\mapsto\; (z_i v_i)_{i=1,\ldots,n} \in V^n$$

is continuous as well by the universal property of the product topology. Finally, the n-fold addition

$$V^n \ni (u_i)_{i=1,\ldots,n} \;\mapsto\; u_1 + \cdots + u_n \in V$$

is continuous. This follows from the continuity of the addition $V \times V \longrightarrow V$ by induction on n. Then ϕ turns out to be a composition of continuous maps, thus being continuous itself. □

With this observation we can now prove the uniqueness of the Hausdorff topological vector space structure on a finite-dimensional $\mathbb{K}$-vector space:

Theorem 2.1.39 (Finite dimensional topological vector spaces) *Let V be a finite-dimensional Hausdorff topological vector space. Then the choice of a basis yields an isomorphism to $\mathbb{K}^n$ endowed with its standard topology.*

PROOF: Let $v_1, \ldots, v_n \in V$ be a basis. Then the map

$$\phi\colon \mathbb{K}^n \ni z \;\mapsto\; z_1 v_1 + \cdots + z_n v_n \in V$$

is a linear isomorphism, which is continuous by Lemma 2.1.38. For the standard topology the sphere $\mathbb{S}_r^{n-1} \subseteq \mathbb{K}^n$ of radius $r > 0$ is compact. Since ϕ is continuous, $\phi(\mathbb{S}_r^{n-1}) = K_r \subseteq V$ is compact again and closed, since V is Hausdorff. As $0 \notin \mathbb{S}_r^{n-1}$, we also have $0 \notin K_r$, since ϕ is a linear isomorphism. By closedness of K_r, its complement $V \setminus K_r$ is an open neighbourhood of zero. As such, it contains a balanced open neighbourhood $U_r \subseteq V \setminus K_r$ of zero by Proposition 2.1.31, *ii.)*. Let $u \in U_r$. Then $\|\phi^{-1}(u)\| \neq r$. If $\|\phi^{-1}(u)\| > r$, then $\frac{r}{\|\phi^{-1}(u)\|} u \in U_r$, since U_r is balanced. Thus $\phi^{-1}(\frac{r}{\|\phi^{-1}(u)\|} u)$ has now norm

$$\left\| \phi^{-1}\Big(\tfrac{r}{\|\phi^{-1}(u)\|} u\Big) \right\| = r,$$

showing $\phi^{-1}\big(\frac{r}{\|\phi^{-1}(u)\|} u\big) \in \mathbb{S}_r^{n-1}$ and hence $\frac{r}{\|\phi^{-1}(u)\|} u \in K_r$, a contradiction. This leaves $\|\phi^{-1}(u)\| < r$ as the only possibility. Hence $\phi^{-1}(U_r) \subseteq \mathrm{B}_r(0)$. Since the open balls $\mathrm{B}_r(0)$ constitute a basis of neighbourhoods, we conclude that the linear map ϕ^{-1} is continuous at zero and hence everywhere by Proposition 2.1.14. □

This gives now the general result that functional analysis in finite-dimensional real or complex vector spaces is entirely covered by the situation of elementary calculus. We collect a few consequences, which will turn out to be useful at many places:

Corollary 2.1.40 *Let V be a Hausdorff topological vector space and let $W \subseteq V$ be a finite-dimensional subspace.*

i.) *The induced topology on W is the unique standard topology obtained by taking a linear isomorphism to $\mathbb{K}^n$.*

ii.) *The subspace W is closed.*

PROOF: The first part is just a reformulation of Theorem 2.1.39, since the induced topology stays Hausdorff. For the second statement we note that the completeness of $\mathbb{K}^n$ gives the completeness of W. Thus Proposition 2.1.20 applies and yields that $W = W^{\mathrm{cl}}$: note that here we do not need that the ambient space V is complete. □

The first part can be interpreted as follows: in interesting, meaning infinite-dimensional Hausdorff topological vector spaces, it will not be sufficient to restrict the attention to finite-dimensional subspaces, as they will not be able to distinguish different topologies. In fact, we will see that infinite-dimensional vector spaces can carry many different topologies of interest. For instance, we will discuss a multitude of topologies for $\mathrm{L}(V, W)$ in Section 3.3.2.

Corollary 2.1.41 *Let V be a locally compact Hausdorff topological vector space. Then V is finite dimensional.*

PROOF: Being locally compact and Hausdorff means that we have a basis of zero neighbourhoods consisting of compact neighbourhoods of zero. Thus we find an open neighbourhood $U \subseteq V$ of zero with compact closure U^{cl}. According to Remark 2.1.33, *ii.)*, the subset U^{cl} is bounded. Now $\frac{1}{2}U$ is still an open neighbourhood of zero and hence the subsets $\left\{v + \frac{1}{2}U\right\}_{v \in V}$ provide an open cover of V. Thus finitely many cover the compact U^{cl}, say

$$U^{\mathrm{cl}} \subseteq (v_1 + \tfrac{1}{2}U) \cup \cdots \cup (v_N + \tfrac{1}{2}U) \tag{$*$}$$

for some $v_1, \ldots, v_N \in V$. Then $W = \operatorname{span}_{\mathbb{K}}\{v_1, \ldots, v_N\} \subseteq V$ is a finite-dimensional subspace and hence closed by Corollary 2.1.40, *ii.)*. From $(*)$ we see that $U \subseteq W + \frac{1}{2}U$ and hence $\frac{1}{2}U \subseteq \frac{1}{2}W + \frac{1}{4}U \subseteq W + \frac{1}{4}U$, since W is a subspace. Thus $U \subseteq W + \frac{1}{2}U \subseteq W + W + \frac{1}{4}U = W + \frac{1}{4}U$, since again W is a subspace. By induction we see that

$$U \subseteq W + \frac{1}{2^n}U$$

for all $n \in \mathbb{N}$. If $O \subseteq V$ is an arbitrary neighbourhood of zero, the boundedness of U shows that $U \subseteq 2^n O$ for some large enough $n \in \mathbb{N}$, or $\frac{1}{2^n}U \subseteq O$. Hence the subsets $\left\{\frac{1}{2^n}U\right\}_{n \in \mathbb{N}}$ form a basis of neighbourhoods of zero. From Proposition 2.1.7, *ii.)*, we conclude that

$$U \subseteq \bigcap_{n \in \mathbb{N}} (W + \tfrac{1}{2^n}U) = W^{\mathrm{cl}} = W,$$

since W is closed. Hence $U \subseteq W$ follows. But by Remark 2.1.4, *iv.)*, we have $V = \bigcup_{n \in \mathbb{N}} nU$, since U is an open neighbourhood of zero and hence $V \subseteq W$, showing $V = W$. □

Corollary 2.1.42 *Let V be a Hausdorff topological vector space with the Heine-Borel property and a bounded neighbourhood of zero. Then V is finite dimensional.*

PROOF: Indeed, if $U \subseteq V$ is a bounded neighbourhood of zero, then U^{cl} is still bounded by Proposition 2.1.35 and hence compact by the Heine-Borel property. Since U^{cl} is still a neighbourhood of zero, Corollary 2.1.41 applies. □

2.2 Locally Convex Vector Spaces

Topological vector spaces are in many aspects still too general to be interesting and they can show certain pathological behaviours from conceptional points of view. If we add the condition of having a neighbourhood basis of convex subsets, this changes significantly. While this very geometric approach is perhaps the most aesthetic one, we start by investigating seminorms on vector spaces and their induced topologies. Ultimately, both approaches will turn out to be equivalent. We can then formulate the basic concepts of continuity, convergence and boundedness in terms of seminorms as well, which gives more manageable criteria compared to the purely topological characterizations.

2.2.1 Seminorms and Induced Topologies

In elementary calculus one has encountered the notion of a normed vector space, and, as complete version, the notion of a Banach space. Here the choice of the norm is a crucial part of the data. However, we know that the underlying topology is fairly insensitive to the choice of the norm, see Exercise 2.5.12. The main step from normed spaces to general locally convex spaces consists now in allowing not only a single norm, but many seminorms to define the topology. We recall the definition of a *seminorm*:

Definition 2.2.1 (Seminorm) *Let V be a $\mathbb{K}$-vector space. A map $\mathrm{p}\colon V \longrightarrow \mathbb{R}_0^+$ is called seminorm if it is homogeneous, i.e.*

$$\mathrm{p}(zv) = |z|\mathrm{p}(v), \tag{2.2.1}$$

and if it satisfies the triangle inequality

$$\mathrm{p}(v+w) \leq \mathrm{p}(v) + \mathrm{p}(w) \tag{2.2.2}$$

for all $v, w \in V$ and $z \in \mathbb{K}$. A seminorm is called a norm if in addition we have

$$\mathrm{p}(v) > 0 \tag{2.2.3}$$

for $v \neq 0$.

Remark 2.2.2 Let V, W be $\mathbb{K}$-vector spaces.

i.) For every seminorm p on V one has

$$\mathrm{p}(0) = 0. \tag{2.2.4}$$

This is clear from (2.2.1).

ii.) A norm is a seminorm. From Exercise 2.5.14 we know that on V there always exists a norm, in fact many. Thus the set of seminorms is always non-trivial.

iii.) Suppose $\mathrm{p}_1, \ldots, \mathrm{p}_n$ are seminorms on V and $\lambda_1, \ldots, \lambda_n \geq 0$ are non-negative scalars. Then

$$\mathrm{q} = \lambda_1 \mathrm{p}_1 + \cdots + \lambda_n \mathrm{p}_n \quad \text{and} \quad \tilde{\mathrm{q}} = \max_{i=1}^{n} \{\lambda_i \mathrm{p}_i\} \tag{2.2.5}$$

are again seminorms. In particular, the zero map is always a seminorm. Also, the rescaling by a positive number gives again a seminorm. Such a rescaling also respects (2.2.3), i.e. rescaling with positive numbers turns norms into norms. Hence seminorms have the structure of a *convex cone*.

iv.) The set of seminorms is *directed* by the induced direction of $\mathbb{R}_0^+$, i.e. for p, q we define $\mathrm{p} \leq \mathrm{q}$ if for all $v \in V$

$$\mathrm{p}(v) \leq \mathrm{q}(v). \tag{2.2.6}$$

It is easy to see that this gives a partial order $\leq$, which is even a direction: for p and q the seminorm $\mathrm{p} + \mathrm{q}$ satisfies $\mathrm{p} \leq \mathrm{p} + \mathrm{q}$ and $\mathrm{q} \leq \mathrm{p} + \mathrm{q}$.

v.) If $\phi \colon V \longrightarrow W$ is a linear map and q is a seminorm on W, then the *pull-back*

$$\phi^* \mathrm{q} = \mathrm{q} \circ \phi \tag{2.2.7}$$

of q by ϕ is a seminorm on V. The pull-back satisfies the usual properties of a pull-back, i.e. we have

$$\phi^* \circ \psi^* = (\psi \circ \phi)^* \tag{2.2.8}$$

for another linear map $\psi \colon W \longrightarrow X$ and we have that id_V^* is the identity on the set of seminorms. Moreover, pullbacks are compatible with the ordering $\leq$ of seminorms, i.e. we have

$$\mathrm{q} \leq \mathrm{p} \quad \Longrightarrow \quad \phi^* \mathrm{q} \leq \phi^* \mathrm{p} \tag{2.2.9}$$

for seminorms p and q on W. Finally, pull-backs are compatible with the operations of taking weighted sums or weighted maxima as in (2.2.5), see also Exercise 2.5.15. Note that for every linear functional $\varphi \in V^*$ we have the induced seminorm

$$\mathrm{p}_\phi = |\varphi(\,\cdot\,)| = \varphi^* |\cdot|. \tag{2.2.10}$$

Also the *restriction* of a seminorm p to a subspace $W \subseteq V$ can be understood as a pull-back, namely with the inclusion map $\iota_W \colon W \longrightarrow V$. We have

$$\mathrm{p}\Big|_W = \iota_W^* \mathrm{p}. \tag{2.2.11}$$

In particular, the restriction of a seminorm to a subspace is again a seminorm.

vi.) The *kernel* of a seminorm p on V is defined by

$$\ker \mathrm{p} = \big\{v \in V \bigm| \mathrm{p}(v) = 0\big\}. \tag{2.2.12}$$

Even though p is not a linear map, the kernel is easily seen to be a subspace of V. We have

$$\ker(\lambda \mathrm{p}) = \ker(\mathrm{p}) \tag{2.2.13}$$

for all $\lambda > 0$. Moreover, if $\mathrm{p}_1, \ldots, \mathrm{p}_n$ are seminorms, then

$$\ker(\mathrm{p}_1 + \cdots + \mathrm{p}_n) = \ker \max_{i=1}^{n}\{\mathrm{p}_i\} = \ker \mathrm{p}_1 \cap \cdots \cap \ker \mathrm{p}_n, \tag{2.2.14}$$

as an elementary verification shows. For two seminorms p and q on V we have

$$\mathrm{p} \le \mathrm{q} \quad \implies \quad \ker \mathrm{p} \supseteq \ker \mathrm{q}, \tag{2.2.15}$$

but the reverse implication does not hold in general: we have $\ker \mathrm{p} = \{0\}$ iff p is a norm, but there will be different norms on V, as soon as $V \ne \{0\}$. For the pull-back by a linear map $\phi\colon V \longrightarrow W$ we get

$$\ker(\phi^* \mathrm{p}) = \phi^{-1}(\ker \mathrm{p}). \tag{2.2.16}$$

For the examples of functions spaces and sequence spaces it is fairly easy to find seminorms which are naturally attached to the corresponding space. In fact, the basic idea is always to turn the condition on a function to be in the relevant space into a seminorm. We exemplify this in the following list. The actual verification that we indeed have seminorms is a simple exercise, see also Exercise 2.5.16.

Example 2.2.3 (Seminorms for sequence spaces) Let I be a non-empty index set. Then on the sequence space $\mathrm{Map}(I, \mathbb{K})$ from Section 1.1 we define the seminorm

$$|(a_i)_{i\in I}|_j = |a_j| \tag{2.2.17}$$

for $j \in I$. We can interpret $|\cdot|_j$ as the pull-back of the usual absolute value on $\mathbb{K}$ by the j-th coefficient functional $\mathrm{e}^j\colon \mathrm{Map}(I, \mathbb{K}) \longrightarrow \mathbb{K}$. Hence $|\cdot|_j$ is a seminorm by (2.2.10).

i.) For the p-summable sequences $\ell^p(I)$ with $p \in [1, \infty)$ one defines the seminorm

$$\|(a_i)_{i\in I}\|_p = \sqrt[p]{\sum\nolimits_{i\in I} |a_i|^p}. \tag{2.2.18}$$

In fact, this turns out to be a norm on $\ell^p(I)$, see Exercise 2.5.17. More generally, for a collection $\mu = (\mu_i)_{i\in I}$ of positive weights, we define the (semi-) norm

$$\|(a_i)_{i\in I}\|_{p,\mu} = \sqrt[p]{\sum\nolimits_{i\in I} |a_i|^p \mu_i} \tag{2.2.19}$$

on the space of p-summable sequences with respect to μ. Note that for $0 < p < 1$ we do not get seminorms by this construction: the triangle inequality fails.

ii.) For the bounded sequences $\ell^\infty(I)$ one defines

$$\|(a_i)_{i\in I}\|_\infty = \sup_{i\in I} |a_i|, \tag{2.2.20}$$

which is again a norm, called the *sup-norm*. Similarly,

$$\|(a_i)_{i\in I}\|_{\infty,\mu} = \sup_{i\in I} |a_i| \mu_i \tag{2.2.21}$$

is a norm on the bounded sequences with respect to a choice of positive weights $\mu = (\mu_i)_{i\in I}$.

iii.) For the finite sequences $c_{\circ\circ}(I)$ we have the norms $\|\cdot\|_{p,\mu}$ for all $p \in [1, \infty]$ and all weights μ.

iv.) For the convergent sequences c and the zero sequences $c_\circ$ the sup-norm $\|\cdot\|_\infty$ is again a well-defined norm, since $c_\circ \subseteq c \subseteq \ell^\infty$ by Proposition 1.1.10, *i.)*. Also here the weighted versions $c_\circ(\mathbb{N}, \mu)$ and $c(\mathbb{N}, \mu)$ come with the weighted sup-norms $\|\cdot\|_{\infty,\mu}$.

v.) On the space s of rapidly decreasing sequences we have the norms

$$\|(a_i)_{i\in I}\|_{\infty,k} = \sup_{n\in\mathbb{N}} |a_n| n^k \tag{2.2.22}$$

and

$$\|(a_i)_{i\in I}\|_{1,k} = \sum_{n\in\mathbb{N}} |a_n| n^k \tag{2.2.23}$$

for all $k \in \mathbb{N}_0$. Again, we know that for the weights $\mu_n^{(k)} = n^k$ we have norms $\|\cdot\|_{p,\mu^{(k)}}$ on $\ell^p(\mathbb{N}, \mu^{(k)})$ and $s \subseteq \ell^p(\mathbb{N}, \mu^{(k)})$ for all $k \in \mathbb{N}_0$. Hence these norms restrict to s, too.

vi.) Let J be another non-empty index set and consider a Köthe set $\mathcal{P} = \{\mu^{(j)}\}_{j\in J} \subseteq \operatorname{Map}(I, \mathbb{K})$ of weights indexed by J. Then on the Köthe space $\Lambda^p(\mathcal{P}) \subseteq \operatorname{Map}(I, \mathbb{K})$ one has the seminorm $\|\cdot\|_{p,\mu^{(j)}}$ by restriction from $\ell^p(I, \mu^{(j)})$ to $\Lambda^p(\mathcal{P}) \subseteq \ell^p(I, \mu^{(j)})$ for all $j \in J$. This generalizes all seminorms so far by choosing the weights in the Köthe set appropriately.

Example 2.2.4 (Sup-Norm for bounded functions) Let X be a non-empty set and consider $\mathscr{B}(X)$. Then the sup-norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \tag{2.2.24}$$

is a norm on $\mathscr{B}(X)$. It restricts to a norm on the various subspaces of $\mathscr{B}(X)$ discussed in Section 1.2.1.

Example 2.2.5 (Essential sup-norm) Let $(X, \mathfrak{a})$ be a measurable space with a σ-ideal $\mathfrak{n} \subseteq \mathfrak{a}$. Typically, $\mathfrak{n}$ will be the σ-ideal of zero sets with respect to a chosen positive measure μ on $(X, \mathfrak{a})$, but there are other situations of interest as well. On the space $\mathscr{L}^\infty(X, \mathfrak{a}, \mathfrak{n})$ of essentially bounded functions the essential supremum

$$\|f\|_{\infty,\mathfrak{n}} = \operatorname*{ess\,sup}_{x\in X} |f(x)| \tag{2.2.25}$$

with respect to $\mathfrak{n}$ gives a seminorm. Note that here we typically get only a seminorm and *not* a norm. Indeed, if $N \subseteq X$ is a non-empty subset with $N \in \mathfrak{n}$, then the characteristic function χ_N of N is different from zero, but

$$\|\chi_N\|_{\infty,\mathfrak{n}} = 0. \tag{2.2.26}$$

In fact,

$$\ker \|\cdot\|_{\infty,\mathfrak{n}} = \{f \in \mathscr{L}^\infty(X, \mathfrak{a}) \mid f \text{ is a zero function}\}, \tag{2.2.27}$$

where a *zero function* (or *null function*) is a measurable function f such that there is a subset $N \in \mathfrak{n}$ with $f\big|_{X\setminus N} = 0$.

Except for the last example we have seen only norms, but not yet genuine seminorms. This changes now for the following examples:

Example 2.2.6 Let X be a locally compact Hausdorff space. Then on the locally bounded functions $\mathscr{B}_{\mathrm{loc}}(X)$ one defines

$$\|f\|_K = \sup_{x\in K} |f(x)| \tag{2.2.28}$$

for every compact subset $K \subseteq X$. This turns out to be a seminorm. Moreover,

$$\ker \|\cdot\|_K = \{f \in \mathscr{B}_{\mathrm{loc}}(X) \mid f\big|_K = 0\} \tag{2.2.29}$$

is typically non-trivial, unless $K = X$. Thus we have genuine seminorms, but not norms in this case. Note also that for a general topological space X we can still define $\|\cdot\|_K$ on $\mathscr{B}_{\mathrm{loc}}(X)$, but in view of the characterization of locally bounded functions in Proposition 1.2.8, *ii.)*, the locally compact Hausdorff case is the most relevant here. The local essential sup-norms can be defined in an analogous way, see Exercise 2.5.18. Since the continuous functions $\mathscr{C}(X)$ are always locally bounded for a locally compact Hausdorff space, we can use the local sup-norms $\|\cdot\|_K$ also for $\mathscr{C}(X)$ and the various subspaces of $\mathscr{C}(X)$.

The next interesting class are the compactly supported continuous functions on a locally compact Hausdorff space. Since $\mathscr{C}_0(X) \subseteq \mathscr{C}(X) \subseteq \mathscr{B}(X)$ we have the local sup-norms on $\mathscr{C}_0(X)$. But we get many more interesting seminorms as follows:

Example 2.2.7 (Weighted sup-norms) Let X be a locally compact Hausdorff space. For a continuous non-negative function $\mu \in \mathscr{C}(X)$ one defines the *μ-weighted sup-norm*

$$\|f\|_{\infty,\mu} = \sup_{x \in X} |f(x)|\mu(x) \tag{2.2.30}$$

for $f \in \mathscr{C}_0(X)$. This gives again a well-defined seminorm. For $\mu = 1$ this gives the usual sup-norm, but in general we get many more (semi-)norms. Note that the compact support of f yields that the right hand side in (2.2.30) is always finite. In fact, for a compact subset $K \subseteq X$ we have

$$\|f\|_{\infty,\mu} \leq c_K \|f\|_K, \tag{2.2.31}$$

where

$$c_K = \|\mu\|_K < \infty \tag{2.2.32}$$

by the continuity of μ. For further discussion of this example, see Exercise 2.5.38.

The next examples of seminorms are adapted to differentiable functions: Here we want estimates for the function *and* its derivatives leading to the following seminorms:

Example 2.2.8 ($\mathscr{C}^k$-seminorms) Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then for every compact subset $K \subseteq X$ and every $\ell \in \mathbb{N}_0$ with $\ell \leq k$ one defines

$$\mathrm{p}_{K,\ell}(f) = \sup_{\substack{x \in X \\ |\alpha| \leq \ell}} \left| \frac{\partial^{|\alpha|}(f)}{\partial x^\alpha}(x) \right| \tag{2.2.33}$$

for $f \in \mathscr{C}^k(X)$. Since by assumption $\frac{\partial^{|\alpha|}(f)}{\partial x^\alpha}$ is still continuous, the supremum is finite and hence yields a seminorm $\mathrm{p}_{K,\ell}(f)$ on $\mathscr{C}^k(X)$. These seminorms restrict to all the subspaces of $\mathscr{C}^k(X)$.

Example 2.2.9 (Seminorms for holomorphic functions) Let $X \subseteq \mathbb{C}^n$ be a non-empty open subset. For a point $z \in X$ and a poly-radius $R = (r_1, \ldots, r_n)$, where $r_i > 0$ for all $i = 1, \ldots, n$ such that the closed polydisc

$$D(z, R) = \big\{ x \in X \;\big|\; |w_i - z_i| \leq r_i \text{ for } i = 1, \ldots, n \big\} \tag{2.2.34}$$

is contained in X we define

$$\mathrm{p}_{z,R}(f) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \left| \frac{\partial^{|\alpha|}(f)}{\partial x^\alpha}(z) \right| R^\alpha \tag{2.2.35}$$

for $f \in \mathscr{O}(X)$. Here we write $R^\alpha = r_1^{\alpha_1} \cdots r_n^{\alpha_n}$ as usual. From the theory of complex functions one knows that the Taylor expansion of a holomorphic function converges uniformly and absolutely on compact subsets in the domain X, where f is holomorphic. Thus $\mathrm{p}_{z,R}(f)$ defines a convergent power series. It is then easy to see that $\mathrm{p}_{z,R}$ is a seminorm, and, in fact, a norm as soon as X is connected. More properties of these seminorms can be found in Exercise 2.5.20.

The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ combines global behaviour of the functions with local properties. Also here we can turn the defining properties of $\mathscr{S}(\mathbb{R}^n)$ into seminorms:

Example 2.2.10 (Seminorms for $\mathscr{S}(\mathbb{R}^n)$) Let $m, \ell \in \mathbb{N}_0$. Then one defines

$$\mathrm{r}_{m,\ell}(f) = \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq \ell}} \left(1 + x^2\right)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|}(f)}{\partial x^\alpha}(x) \right| \tag{2.2.36}$$

for $f \in \mathscr{S}(\mathbb{R}^n)$. Then $f \in \mathscr{S}(\mathbb{R}^n)$ guarantees that $\mathrm{r}_{m,\ell}(f)$ is finite. In fact, a function $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ is in $\mathscr{S}(\mathbb{R}^n)$ iff $\mathrm{r}_{m,\ell}(f) < \infty$ for all $m, \ell \in \mathbb{N}_0$. It is then easy to see that $\mathrm{r}_{m,\ell}$ is a norm on the Schwartz space for all $m, \ell \in \mathbb{N}_0$, see also Exercise 2.5.21 for some alternative seminorms. Once we have constructed the corresponding locally convex topologies, they will turn out to be equivalent.

From the examples we see that all our function spaces come along with naturally defined seminorms, which reflect the defining properties of the underlying spaces. This turns out to be no coincidence, as the function spaces all are equipped with topologies coming from these seminorms. In order to see this, we consider again the general situation:

Suppose V is a vector space over $\mathbb{K}$ and $\mathrm{p}\colon V \longrightarrow \mathbb{R}_0^+$ is a seminorm on V. In analogy to the case of a norm we define the *open ball* and the *closed ball* of radius $r > 0$ with respect to p around $v \in V$ by

$$\mathrm{B}_{\mathrm{p},r}(v) = \left\{ w \in V \;\middle|\; \mathrm{p}v - w < r \right\} \tag{2.2.37}$$

and

$$\mathrm{B}_{\mathrm{p},r}(v)^{\mathrm{cl}} = \left\{ w \in V \;\middle|\; \mathrm{p}v - w \leq r \right\}, \tag{2.2.38}$$

respectively. Of course, up to now these are just names and we need to find a topology on V first in order to justify the words “open” and “closed”. Note that the balls typically look more like cylinders, since we always have

$$v + \ker \mathrm{p} \subseteq \mathrm{B}_{\mathrm{p},r}(v). \tag{2.2.39}$$

Thus the ball contains a non-trivial affine subspace, as soon as p is a seminorm, but not a norm.

The idea is now very simple: given a collection $\mathcal{P}$ of seminorms on V, we can construct a topology on V such that the open balls $\mathrm{B}_{\mathrm{p},r}$ are indeed open by declaring these open balls to be a subbasis of the topology. Having a subbasis is of course not yet very specific, as every collection of subsets serves as subbasis of a unique topology, see e.g. [19, Proposition 2.2.6]. Thus the interesting question is whether we have actually a basis. Here we want a criterion in terms of the seminorms. Also, we need to understand when the resulting topology will be Hausdorff.

Definition 2.2.11 (Systems of seminorms) *Let V be a $\mathbb{K}$-vector space and let $\mathcal{P}$ be a non-empty set of seminorms on V.*

i.) The collection $\mathcal{P}$ is called filtrating if for every $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ there is a $\mathrm{q} \in \mathcal{P}$ with

$$\mathrm{p}_1, \ldots, \mathrm{p}_n \leq \mathrm{q}, \tag{2.2.40}$$

i.e. it is a directed subset of all seminorms.

ii.) The collection $\mathcal{P}$ is called Hausdorff if for every $v \in V \setminus \{0\}$ there is a $\mathrm{p} \in \mathcal{P}$ with

$$\mathrm{p}(v) > 0. \tag{2.2.41}$$

iii.) The collection $\mathcal{P}$ is called saturated if for every seminorm q on V such that there exist $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ and a constant $c > 0$ with

$$\mathrm{q} \leq c \, \max\{\mathrm{p}_1, \ldots, \mathrm{p}_n\} \tag{2.2.42}$$

one has already $q \in \mathcal{P}$.

iv.) *The saturation* $\mathcal{P}_{\mathrm{sat}}$ *of* $\mathcal{P}$ *is the smallest saturated set of seminorms containing* $\mathcal{P}$.

We collect some first properties of our new definitions:

Lemma 2.2.12 *Let* V *be a* $\mathbb{K}$*-vector space and let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be non-empty sets of seminorms on* V*.*

i.) The set of all seminorms is saturated.

ii.) An arbitrary intersection of saturated sets of seminorms is again saturated.

iii.) A saturated set of seminorms is filtrating.

iv.) A saturated set of seminorms is closed under taking finite linear combinations with non-negative coefficients and under taking finite maxima.

v.) The saturation of $\mathcal{P}$ *is explicitly given by*

$$\mathcal{P}_{\mathrm{sat}} = \left\{ \mathrm{q} \textit{ seminorm on } V \;\middle|\; \textit{there exist } \mathrm{p}_1, \ldots, \mathrm{p}_n \in P \textit{ and } c > 0 \textit{ with } \mathrm{q} \le c \ \max\{\mathrm{p}_1, \ldots, \mathrm{p}_n\} \right\}. \tag{2.2.43}$$

vi.) If $\mathcal{Q} \subseteq \mathcal{P}$*, then* $\mathcal{Q}_{\mathrm{sat}} \subseteq \mathcal{P}_{\mathrm{sat}}$*.*

vii.) One has $(\mathcal{P}_{\mathrm{sat}})_{\mathrm{sat}} = \mathcal{P}_{\mathrm{sat}}$*.*

viii.) The saturation $\mathcal{P}_{\mathrm{sat}}$ *is Hausdorff iff* $\mathcal{P}$ *is Hausdorff.*

PROOF: The first part is clear, as the condition is vacuous. Hence there exists a least one saturated subset of seminorms. If $\{\mathcal{P}_i\}_{i \in I}$ is a set of saturated sets of seminorms, we consider $\mathcal{P} = \bigcap_{i \in I} \mathcal{P}_i$. Suppose q is a seminorm with $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ and $c > 0$ satisfying (2.2.42). Then $\mathcal{P} \subseteq \mathcal{P}_i$ for all $i \in I$ implies that $\mathrm{q} \in \mathcal{P}_i$, as $\mathcal{P}_i$ is saturated. Hence $\mathrm{q} \in \mathcal{P}$, showing the second part. This already implies that the saturation of any collection $\mathcal{P}$ of seminorms exists: we can take the intersection of all saturated systems containing $\mathcal{P}$. For the third part let $\mathcal{P}$ be saturated and let $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$. Then

$$\mathrm{q} = \max\{\mathrm{p}_1, \ldots, \mathrm{p}_n\}$$

is a seminorm which satisfies $\mathrm{q} \le \max\{\mathrm{p}_1, \ldots, \mathrm{p}_n\}$ and hence $\mathrm{q} \in \mathcal{P}$. But clearly $\mathrm{p}_1, \ldots, \mathrm{p}_n \le \mathrm{q}$, showing that $\mathcal{P}$ is filtrating. It also implies that $\mathcal{P}$ is closed under finite maxima. Moreover, let $\mathrm{q} = \mathrm{p}_1 + \cdots + \mathrm{p}_n$ for $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$. Then $\mathrm{q} \le n \max\{\mathrm{p}_1, \ldots, \mathrm{p}_n\}$ and hence $\mathrm{q} \in \mathcal{P}$, showing that $\mathcal{P}$ is closed under finite sums. Finally, for $p \in \mathcal{P}$ and $c > 0$ the seminorm $\mathrm{q} = c\mathrm{p}$ satisfies $\mathrm{q} \le c \max\{p\}$ and thus again $\mathrm{q} \in \mathcal{P}$. This completes the fourth part. Now let $\mathcal{P}$ be arbitrary and consider the collection (2.2.43). If $\mathcal{Q}$ is saturated and $\mathcal{P} \subseteq \mathcal{Q}$, then clearly the right hand side of (2.2.43) is contained in $\mathcal{Q}$. Hence we have "$\supseteq$" in (2.2.43). Conversely, let q be a seminorm with seminorms $\mathrm{q}_1, \ldots, \mathrm{q}_n$ in the right hand side of (2.2.43) such that

$$\mathrm{q} \le c \max\{\mathrm{q}_1, \ldots, \mathrm{q}_n\}.$$

Then for each q_i we find a $c_i > 0$ and $\mathrm{p}_1^{(i)}, \ldots, \mathrm{p}_{m_i}^{(i)} \in \mathcal{P}_i$ with $m_i \in \mathbb{N}$ and

$$\mathrm{q}_i \le c_i \max\left\{\mathrm{p}_1^{(i)}, \ldots, \mathrm{p}_{m_i}^{(i)}\right\}$$

by definition. Thus

$$\mathrm{q} \le c \max\{\mathrm{q}_1, \ldots, \mathrm{q}_n\} \le c \max\{c_1, \ldots, c_n\} \max\{\mathrm{p}_1^{(1)}, \ldots, \mathrm{p}_{m_n}^{(n)}\}.$$

This shows that q belongs to the right hand side of (2.2.43). Hence this collection is saturated and "$\subseteq$" follows. Together, the fifth part is shown. The last three statements are obvious. ☐

In general, it will be complicated to find a more explicit description of the saturation of $\mathcal{P}$, see Exercise 2.5.23 for some mild variations of (2.2.43). It will typically be rather easy to find a filtrating system containing a given collection $\mathcal{P}$, e.g. by taking finite maxima or finite sums. This will not yet be the saturation of $\mathcal{P}$, but is still manageable. The Hausdorff property of $\mathcal{P}$ is typically easy to check. Note that $\mathcal{P}$ is Hausdorff iff $\mathcal{P}_{\mathrm{sat}}$ is Hausdorff. The significance of the saturation is now contained in the following theorem. We consider, for a given system of seminorms, the system of open balls defined by $\mathcal{P}$, i.e.

$$\mathscr{B}(\mathcal{P}) = \left\{ \mathrm{B}_{\mathrm{p},r}(v) \;\middle|\; v \in V, r > 0 \text{ and } p \in \mathcal{P} \right\}, \tag{2.2.44}$$

and take it as a subbasis of a topology on V. We denote this topology by $\mathscr{V}(\mathcal{P})$ and call it the *induced topology* by $\mathcal{P}$.

Theorem 2.2.13 (Topology induced by seminorms) *Let V be a $\mathbb{K}$-vector space and let $\mathcal{P}$ be a non-empty collection of seminorms on V.*

i.) If $\mathcal{P}$ is filtrating, then $\mathcal{B}(\mathcal{P})$ is a basis of $\mathscr{V}(\mathcal{P})$.

ii.) The topologies induced by $\mathcal{P}$ and $\mathcal{P}_{\mathrm{sat}}$ coincide.

iii.) The subbasis $\mathcal{B}(\mathcal{P}_{\mathrm{sat}})$ is a basis of $\mathscr{V}(\mathcal{P})$ and

$$\mathcal{B}(\mathcal{P}_{\mathrm{sat}}) = \left\{ \mathrm{B}_{\mathrm{p},1}(v) \;\middle|\; v \in V \text{ and } \mathrm{p} \in \mathcal{P}_{\mathrm{sat}} \right\}. \tag{2.2.45}$$

iv.) A subset $U \subseteq V$ is open iff for all $v \in U$ there exists an $r > 0$ and a $\mathrm{p} \in \mathcal{P}_{\mathrm{sat}}$ with $\mathrm{B}_{\mathrm{p},r}(v) \subseteq U$. If $\mathcal{P}$ is filtrating, it suffices to take $\mathrm{p} \in \mathcal{P}$.

v.) The topology $\mathscr{V}(\mathcal{P})$ induced by $\mathcal{P}$ makes V a topological vector space.

vi.) The topology $\mathscr{V}(\mathcal{P})$ induced by $\mathcal{P}$ is Hausdorff iff $\mathcal{P}$ is Hausdorff.

vii.) A seminorm p on V is continuous with respect to $\mathscr{V}(\mathcal{P})$ iff $\mathrm{p} \in \mathcal{P}_{\mathrm{sat}}$.

PROOF: For the first part, let $\mathcal{P}$ be filtrating and let $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$, $v_1, \ldots, v_n \in V$ and $r_1, \ldots, r_n > 0$. Suppose that the intersection

$$U = \mathrm{B}_{\mathrm{p}_1,r_1}(v_1) \cap \cdots \cap \mathrm{B}_{\mathrm{p}_n,r_n}(v_n)$$

is non-empty. Let $u \in U$, then $\delta_i = r_i - \mathrm{p}_i(v_i - u) > 0$. Consider now a seminorm $\mathrm{q} \in \mathcal{P}$ with $\mathrm{p}_1, \ldots, \mathrm{p}_n \le \mathrm{q}$ and set $\delta = \min\{\delta_1, \ldots, \delta_n\} > 0$. We claim that $\mathrm{B}_{\mathrm{q},\delta}(u) \subseteq U$. Indeed, let $u' \in \mathrm{B}_{\mathrm{q},\delta}(u)$ be given, then

$$\mathrm{p}_i(u' - u) \le \mathrm{q}(u' - u) < \delta \le \delta_i.$$

This gives by the triangle inequality for the seminorm p_i

$$\mathrm{p}_i(u' - v_i) \le \mathrm{p}_i(u' - u) + \mathrm{p}(u - v_i) < \delta_i + \mathrm{p}_i(u - v_i) < r_i$$

and thus $u' \in \mathrm{B}_{\mathrm{p}_i,r_i}(v_i)$ for all $i = 1, \ldots, n$. Hence $u' \in U$, showing $\mathrm{B}_{\mathrm{q},\delta}(u) \subseteq U$. It follows that the intersection U of the open balls $\mathrm{B}_{\mathrm{p}_1,r_1}(v_1), \ldots, \mathrm{B}_{\mathrm{p}_n,r_n}(v_n)$ can be written as the union over balls $\mathrm{B}_{\mathrm{q},\delta}(u)$ for $u \in U$ with appropriate q and $\delta > 0$. Hence it can be written as a union of elements in $\mathcal{B}(\mathcal{P})$. This shows that $\mathcal{B}(\mathcal{P})$ is actually a basis of the topology. For the second statement, $\mathcal{P} \subseteq \mathcal{P}_{\mathrm{sat}}$ implies immediately $\mathcal{B}(\mathcal{P}) \subseteq \mathcal{B}(\mathcal{P}_{\mathrm{sat}})$ and hence $\mathscr{V}(\mathcal{P}) \subseteq \mathscr{V}(\mathcal{P}_{\mathrm{sat}})$. Thus the topology $\mathscr{V}(\mathcal{P})$ is coarser than the topology $\mathscr{V}(\mathcal{P}_{\mathrm{sat}})$. We need to show that it is also finer, i.e. that we have "$\supseteq$". Hence let $\mathrm{B}_{\mathrm{q},r}(v)$ be an open ball with radius $r > 0$ around $v \in V$ for $\mathrm{q} \in \mathcal{P}_{\mathrm{sat}}$. We find $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ and $c > 0$ with $\mathrm{q} \le c \max\{\mathrm{p}_1, \ldots, \mathrm{p}_n\}$ by Lemma 2.2.12, *v.)*. We set $\delta = \frac{r - \mathrm{q}(v-u)}{c}$ for $u \in \mathrm{B}_{\mathrm{q},r}(v)$ and conclude $\delta > 0$. Now suppose $u' \in \mathrm{B}_{\mathrm{p}_1,\delta}(u) \cap \cdots \cap \mathrm{B}_{\mathrm{p}_n,\delta}(u)$, then

$$\mathrm{q}(v - u') \le \mathrm{q}v - u + \mathrm{q}(u - u') \le \mathrm{q}(v - u) + c \max_{i=1}^{n}\{p_i(u - u')\} < \mathrm{q}(v - u) + c\delta = r.$$

Hence $u' \in \mathrm{B}_{\mathrm{q},r}(v)$ and thus we conclude that

$$\mathrm{B}_{\mathrm{p}_1,\delta}(u) \cap \cdots \cap \mathrm{B}_{\mathrm{p}_n,\delta}(u) \subseteq \mathrm{B}_{\mathrm{q},r}(v)$$

for every $u \in \mathrm{B}_{\mathrm{q},r}(v)$. This shows that $\mathrm{B}_{\mathrm{q},r}(v)$ is a union of open subsets of the topology $\mathscr{V}(\mathcal{P})$ and hence open in the topology $\mathscr{V}(\mathcal{P})$. Thus $\mathscr{V}(\mathcal{P}_{\mathrm{sat}}) \subseteq \mathscr{V}(\mathcal{P})$ follows, concluding the second part. Since $\mathcal{P}_{\mathrm{sat}}$ is filtrating by Lemma 2.2.12, *iii.)*, the subbasis $\mathcal{B}(\mathcal{P}_{\mathrm{sat}})$ is a basis. Moreover, we have

$$\mathrm{B}_{c\mathrm{p},cr}(v) = \mathrm{B}_{\mathrm{p},r}(v)$$

for all seminorms p. Hence

$$\big\{\mathrm{B}_{\mathrm{p},1}(v) \bigm| \mathrm{p} \in \mathcal{P}_{\mathrm{sat}}, v \in V\big\} = \big\{\mathrm{B}_{r\mathrm{p},r}(v) \bigm| \mathrm{p} \in \mathcal{P}_{\mathrm{sat}}, r > 0 \text{ and } v \in V\big\} = \mathcal{B}(\mathcal{P})$$

follows, since with $\mathrm{p} \in \mathcal{P}_{\mathrm{sat}}$ also $r\mathrm{p} \in \mathcal{P}_{\mathrm{sat}}$ for all $r > 0$. Since $\mathcal{B}(\mathcal{P})$ is a basis of $\mathscr{V}(\mathcal{P})$, whenever $\mathcal{P}$ is filtrating, a subset $U \subseteq V$ is open, whenever it is a union of elements in $\mathcal{B}(\mathcal{P})$. This shows the fourth part, since $\mathcal{P}_{\mathrm{sat}}$ is always filtrating. The fifth part is now the interesting one. To check continuity of $+\colon V \times V \longrightarrow V$ we consider an element $\mathrm{B}_{\mathrm{q},r}$ of the basis $\mathcal{B}(\mathcal{P}_{\mathrm{sat}})$. Then its pre-image under $+$ is the subset of those $(u, u') \in V \times V$ with $\delta = \mathrm{q}(u + u' - v) < r$. For such a point (u, u') we claim that $\mathrm{B}_{\mathrm{q},\frac{r-\delta}{2}}(u) \times \mathrm{B}_{\mathrm{q},\frac{r-\delta}{2}}(u')$ is mapped into $\mathrm{B}_{\mathrm{q},r}(v)$. Indeed, let $w \in \mathrm{B}_{\mathrm{q},\frac{r-\delta}{2}}(u)$ and $w' \in \mathrm{B}_{\mathrm{q},\frac{r-\delta}{2}}(u')$ be given, then the triangle inequality gives

$$\mathrm{q}(w + w' - v) \le \mathrm{q}(w - u) + \mathrm{q}(w' - u') + \mathrm{q}(u + u' - v) < \frac{r - \delta}{2} + \frac{r - \delta}{2} + \delta = r.$$

This shows that the pre-image of $\mathrm{B}_{\mathrm{q},r}(v)$ under $+$ is open. Since the subsets $\mathrm{B}_{\mathrm{q},r}(v)$ form a basis, this already implies that $+$ is continuous. For the multiplication with scalars we argue similarly. Let $(z, u) \in \mathbb{C} \times V$ with $\delta = \mathrm{q}(zu - v) < r$, i.e. a pair (z, u) with zu in the open ball $\mathrm{B}_{\mathrm{q},r}(v)$. We consider $u' \in \mathrm{B}_{\mathrm{q},\tilde{r}}(u)$ and $z' \in \mathrm{B}_{r'}(z)$ and estimate

$$\begin{aligned}
\mathrm{q}(z'u' - v) &\le \mathrm{q}((z' - z)u') + \mathrm{q}(z(u' - u)) + \mathrm{q}(zu - v) \\
&\le |z' - z|\mathrm{q}(u') + |z|\mathrm{q}(u' - u) + \delta \\
&\le r'\mathrm{q}(u' - u) + r'\mathrm{q}(u) + |z|\tilde{r} + \delta \\
&\le r'\tilde{r} + r'\mathrm{q}(u) + |z|\tilde{r} + \delta.
\end{aligned}$$

Now we fix $r' = \min\{\frac{r-\delta}{3(|z|+1)}, 1\}$ and $\tilde{r} = \min\{\frac{r-\delta}{3(\mathrm{q}(u)+q)}, 1\}$. Then $\mathrm{q}(z'z' - v) < r$ will follow. Thus this open neighbourhood $\mathrm{B}_{\tilde{r}}(z) \times \mathrm{B}_{\mathrm{q},r'}(u)$ of (z, u) is contained in the preimage of $\mathrm{B}_{\mathrm{q},r}(v)$ again, showing the continuity of the multiplication with scalars. The next part is easy: If the topology is Hausdorff, we can separate $v \neq 0$ from 0 by open subsets and hence by balls from $\mathcal{B}(\mathcal{P})$. This means there is a $\mathrm{p} \in \mathcal{P}$ and an $r > 0$ with $v \notin \mathrm{B}_{\mathrm{p},r}(0)$. Thus $\mathrm{p}(v) \ge r > 0$. Conversely, if $\mathcal{P}$ is Hausdorff and $u \neq v$, then we have a seminorm $\mathrm{p} \in \mathcal{P}$ with $r = \mathrm{p}(u - v) > 0$. But then the open balls $\mathrm{B}_{\mathrm{p},\frac{r}{2}}(u)$ and $\mathrm{B}_{\mathrm{p},\frac{r}{2}}(v)$ are disjoint by the triangle inequality. Hence $\mathscr{V}(\mathcal{P})$ is Hausdorff. For the last part, let $\mathrm{q} \in \mathcal{P}_{\mathrm{sat}}$. For $a < b$ we have $v \in \mathrm{q}^{-1}((a, b))$ iff $a < \mathrm{q}(v) < b$. Let $r > 0$ be such that $a < \mathrm{q}(v) - r$ and $\mathrm{q}(v) + r < b$. Then for $u \in \mathrm{B}_{\mathrm{q},r}(v)$ we have by the triangle inequality

$$\mathrm{q}(u) \le \mathrm{q}(u - v) + \mathrm{q}(v) < r + \mathrm{q}(v) < b$$

and

$$a < \mathrm{q}(v) - r \le \mathrm{q}(u - v) + \mathrm{q}(u) - r < \mathrm{q}(u),$$

showing $\mathrm{B}_{\mathrm{q},r}(v) \subseteq \mathrm{q}^{-1}\big((a, b)\big)$. From this we conclude that $\mathrm{q}^{-1}\big((a, b)\big)$ is open and since the open intervals of $\mathbb{R}$ form a basis of the standard topology of $\mathbb{R}$, the seminorm q is continuous. Conversely, let q be a continuous seminorm on V with respect to $\mathscr{V}(\mathcal{P}) = \mathscr{V}(\mathcal{P}_{\mathrm{sat}})$. Then $\mathrm{B}_{\mathrm{q},1}(0) = \mathrm{q}^{-1}\big([0, 1)\big)$ is open. Since $\mathcal{B}(\mathcal{P}_{\mathrm{sat}})$ forms a basis, we find a $\mathrm{q} \in \mathcal{P}_{\mathrm{sat}}$ such that $\mathrm{B}_{\mathrm{p},1}(0) \subseteq \mathrm{B}_{\mathrm{q},1}(0)$. Hence $\mathrm{p}(v) < 1$ implies $\mathrm{q}(v) < 1$ for $v \in V$. We claim that this implies $\mathrm{q} \le \mathrm{p}$. Indeed, we first consider $v \in V$ with $\mathrm{p}(v) = 0$. Then $0 = |z|\mathrm{p}(v) = \mathrm{p}(zv)$ for all $z \in \mathbb{K}$. Hence $|z|\mathrm{q}(v) = \mathrm{q}(zv) < 1$ for all $z \in \mathbb{K}$ giving $\mathrm{q}(v) = 0$, too. Thus $\mathrm{q}(v) \le \mathrm{p}(v)$ holds in this case. If $\mathrm{p}(v) > 0$, then for all $c > 1$ we have $\mathrm{p}(\frac{v}{c\mathrm{p}(v)}) = \frac{1}{c} < 1$ and hence $\mathrm{q}(\frac{v}{c\mathrm{p}(v)}) = \frac{\mathrm{q}(v)}{c\mathrm{p}(v)} < 1$. Since $c > 1$ is arbitrary, this implies $\mathrm{q}(v) \le \mathrm{p}(v)$ also in this case and hence $\mathrm{q} \le \mathrm{p}$. Since $\mathrm{p} \in \mathcal{P}_{\mathrm{sat}}$, this implies $\mathrm{q} \in \mathcal{P}_{\mathrm{sat}}$ by Lemma 2.2.12, *v.)*. ☐

This construction of a topological vector space out of a collection of seminorms has a tremendous amount of useful consequences. Before entering a detailed discussion, we list a first few consequences:

Corollary 2.2.14 *Let V be a $\mathbb{K}$-vector space and let $\mathcal{P}$ be a collection of seminorms on V.*

i.) *For every $\mathrm{p} \in \mathcal{P}_{\mathrm{sat}}$ the closed balls $\mathrm{B}_{\mathrm{p},r}(v)^{\mathrm{cl}}$ with $v \in V$ and $r > 0$ are closed and the open ball*

$$\mathrm{B}_{\mathrm{p},r}(v) \subseteq \mathrm{B}_{\mathrm{p},r}(v)^{\mathrm{cl}} \tag{2.2.46}$$

is sequentially dense in the closed ball.

ii.) *If $\mathcal{P}$ is filtrating, then the closed balls $\{\mathrm{B}_{\mathrm{p},r}(0)^{\mathrm{cl}}\}_{r>0,\mathrm{p}\in\mathcal{P}}$ as well as the open balls $\{\mathrm{B}_{\mathrm{p},r}(0)\}_{r>0,\mathrm{p}\in\mathcal{P}}$ form bases of neighbourhoods of zero, consistent of closed or open absolutely convex subsets, respectively.*

Proof: First we recall that $\mathrm{B}_{\mathrm{p},r}(v)$ is indeed open, since $\mathcal{V}(\mathcal{P}) = \mathcal{V}(\mathcal{P}_{\mathrm{sat}})$ and the topology is defined by means of the balls as basis. Since p is continuous, $\mathrm{p}^{-1}([0,r]) = \mathrm{B}_{\mathrm{p},r}(0)^{\mathrm{cl}}$ is a closed subset. If $u \in \mathrm{B}_{\mathrm{p},r}(0)^{\mathrm{cl}}$, then $\mathrm{p}(u) \le r$ and hence $\mathrm{p}((1-\frac{1}{n})u) = (1-\frac{1}{n})\mathrm{p}(u) \le (1-\frac{1}{n})r < r$. Thus $(1-\frac{1}{n})u \in \mathrm{B}_{\mathrm{p},r}(0)$ for all $n \in \mathbb{N}$. Since $(1-\frac{1}{n}u) \longrightarrow u$ in any topological vector space, $\mathrm{B}_{\mathrm{p},r}(0) \subseteq \mathrm{B}_{\mathrm{p},r}(0)^{\mathrm{cl}}$ is therefore sequentially dense. This shows the first part for $v = 0$. Since in a topological vector space, the topology is translationally invariant, as shown all the way back in Proposition 2.1.2, *i.)*, the result holds for general $v \in V$. For the second part we know that the open balls $\{\mathrm{B}_{\mathrm{p},r}(0)\}_{r>0,\mathrm{p}\in\mathcal{P}}$ form a basis of neighbourhoods. Clearly, open as well as closed balls are absolutely convex by the triangle inequality. Taking closures gives then the second statement, since by

$$\mathrm{B}_{\mathrm{p},\frac{r}{2}}(0)^{\mathrm{cl}} \subseteq \mathrm{B}_{\mathrm{p},r}(0) \subseteq \mathrm{B}_{\mathrm{p},r}(0)^{\mathrm{cl}}$$

we still have a basis of neighbourhoods of zero. ☐

Thus we have achieved one of our original goals to justify the notions of a closed and open ball $\mathrm{B}_{\mathrm{p},r}(v)$, as introduced in (2.2.38) and (2.2.37), respectively.

In a next step we want to compare the induced topologies of two different sets of seminorms. Here we get the following characterization:

Proposition 2.2.15 *Let V be a $\mathbb{K}$-vector space and let $\mathcal{P}$ and $\mathcal{Q}$ be systems of seminorms on V. Then the following statements are equivalent:*

i.) *For every $\mathrm{q} \in \mathcal{Q}$ there is a $c > 0$ and seminorms $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ with*

$$\mathrm{q} \le c \max\{\mathrm{p}_1, \ldots, \mathrm{p}_n\}. \tag{2.2.47}$$

ii.) *One has $\mathcal{Q}_{\mathrm{sat}} \subseteq \mathcal{P}_{\mathrm{sat}}$.*

iii.) *The topology $\mathcal{V}(\mathcal{Q})$ is coarser than the topology $\mathcal{V}(\mathcal{P})$.*

iv.) *Every seminorm $\mathrm{q} \in \mathcal{Q}$ is continuous with respect to $\mathcal{V}(\mathcal{P})$.*

Proof: In view of Lemma 2.2.12, *v.)*, we see that *i.)* implies $\mathcal{Q} \subseteq \mathcal{P}_{\mathrm{sat}}$ and thus $\mathcal{Q}_{\mathrm{sat}} \subseteq \mathcal{P}_{\mathrm{sat}}$ by *vi.)* and *vii.)* of the same lemma. Hence *i.)* $\implies$ *ii.)* follows. Assume *ii.)*. Since the topologies

$\mathscr{V}(\mathcal{Q}_{\mathrm{sat}})$ and $\mathscr{V}(\mathcal{P})_{\mathrm{sat}}$ are built from open balls, it is clear that $\mathscr{V}(\mathcal{Q}_{\mathrm{sat}}) \subseteq \mathscr{V}(\mathcal{P}_{\mathrm{sat}})$, since we simply have more open balls, when using more seminorms. Then Theorem 2.2.13, *ii.)*, gives the implication *i.)* $\implies$ *iii.)*, since $\mathscr{V}(\mathcal{Q}) = \mathscr{V}(\mathcal{Q}_{\mathrm{sat}})$ and $\mathscr{V}(\mathcal{P}) = \mathscr{V}(\mathcal{P}_{\mathrm{sat}})$. Assume *iii.)*. Then $\mathrm{q} \in \mathcal{Q}$ is continuous with respect to $\mathscr{V}(\mathcal{Q})$ according to Theorem 2.2.13, *vii.)*. Hence it is continuous for the finer topology $\mathscr{V}(\mathcal{P})$, as well. Finally, assume *iv.)*. Then $\mathrm{q} \in \mathcal{Q}$ implies $\mathrm{q} \in \mathcal{P}_{\mathrm{sat}}$ by Theorem 2.2.13, *vii.)*, which is *i.)* according to Lemma 2.2.12, *v.)*. ☐

We take this characterization now as motivation for the following definition:

Definition 2.2.16 (Equivalent systems of seminorms) *Let $\mathcal{P}$ and $\mathcal{Q}$ be systems of seminorms on a $\mathbb{K}$-vector space V.*

i.) The system $\mathcal{P}$ dominates $\mathcal{Q}$ if $\mathcal{Q}_{\mathrm{sat}} \subseteq \mathcal{P}_{\mathrm{sat}}$.

ii.) The systems $\mathcal{P}$ and $\mathcal{Q}$ are called equivalent if $\mathcal{Q}_{\mathrm{sat}} = \mathcal{P}_{\mathrm{sat}}$.

For a discussion of some more properties of our new definition, see Exercise 2.5.24.

2.2.2 Characterization of Locally Convex Spaces

When constructing a topological vector space from a system of seminorms according to Theorem 2.2.13, we have seen in Corollary 2.2.14, *ii.)*, that the resulting topology admits a basis of neighbourhoods of zero consisting of (absolutely) *convex* subsets. In view of the general situation discussed in Proposition 2.1.31 this is a particular feature not present in general, see again Exercise 2.5.11. It turns out that having a neighbourhood basis of convex neighbourhoods is such a pleasant property that it justifies to concentrate on these topological vector spaces.

Definition 2.2.17 (Locally convex space) *Let $(V, \mathscr{V})$ be a topological vector space. If V admits a neighbourhood basis of zero consisting of convex subsets, then V is called locally convex.*

For simplicity, a locally convex topological vector space will just be called a *locally convex space*. From Proposition 2.1.30 and Proposition 2.1.31 we conclude that for a locally convex space we have a neighbourhood basis of zero, consisting of absolutely convex and open neighbourhoods.

The subcategory of locally convex spaces among all topological vector spaces will be denoted by

$$\mathsf{lcs} \subseteq \mathsf{tVS}, \tag{2.2.48}$$

the subcategory of Hausdorff locally convex spaces is then denoted by

$$\mathsf{LCS} \subseteq \mathsf{TVS}. \tag{2.2.49}$$

In both cases we consider lcs and LCS as full subcategories, i.e. we take all continuous linear maps as morphisms.

One consequence of Theorem 2.2.13, namely Corollary 2.2.14, *ii.)*, is that if the topology is induced by a collection of seminorms, then we obtain a locally convex space. Remarkably, the converse is true as well: all locally convex spaces arise by this construction. To see this we need to construct seminorms out of convex neighbourhoods of zero. This is accomplished by the *Minkowski functionals*:

Definition 2.2.18 (Minkowski functional) *Let $U \subseteq V$ be an absorbing subset of a $\mathbb{K}$-vector space V. Then the Minkowski functional $\mathrm{p}_U \colon V \longrightarrow \mathbb{R}$ of U is defined by*

$$\mathrm{p}_U(v) = \inf\big\{\lambda > 0 \bigm| v \in \lambda U\big\}. \tag{2.2.50}$$

Here an *absorbing subset* U means that for every $v \in V$ there exists a $\lambda > 0$ with $v \in \lambda U$. This makes the definition well-defined, as the set of λ's on the right hand side is non-empty, thus having a finite infimum. Note that in a topological vector space, every neighbourhood of zero is absorbing, see Remark 2.1.4, *iv.)*. This is just the continuity of the multiplication by scalars. For a general absorbing subset U, the Minkowski functional is not very interesting. This changes if U is in addition absolutely convex:

Lemma 2.2.19 *Let $U \subseteq V$ be an absorbing subset of a $\mathbb{K}$-vector space V.*

i.) *One has $\mathrm{p}_U \geq 0$ and $\mathrm{p}_U(0) = 0$.*

ii.) *If U is balanced, then*

$$\mathrm{p}_U(zv) = |z|\mathrm{p}_U(v) \tag{2.2.51}$$

for all $z \in \mathbb{K}$ and $v \in V$.

iii.) *If U is convex, then p_U is subadditive, i.e.*

$$\mathrm{p}_U(v + w) \leq \mathrm{p}_U(v) + \mathrm{p}_U(w) \tag{2.2.52}$$

for all $v, w \in V$.

PROOF: The first statement is clear from the definition. For the second, let $z \neq 0$ and $v \in V$ be given. Then $zv = |z|\frac{z}{|z|}v \in |z|\lambda U$ iff $v \in \lambda U$, since U is balanced, i.e. $\frac{z}{|z|}U \subseteq U$. Hence (2.2.51) follows as the case $z = 0$ is covered by the first part. The interesting one is the third: assume that U is convex and let $v, w \in V$. Let $\epsilon > 0$, then by definition we have $v \in \lambda U$ and $w \in \mu U$, where $\lambda = \mathrm{p}_U(v) + \epsilon > 0$ and $\mu = \mathrm{p}_U(w) + \epsilon > 0$. Hence $\frac{1}{\lambda}v, \frac{1}{\mu}w \in U$. Since U is convex,

$$\frac{\lambda}{\lambda + \mu}\frac{1}{\lambda}v + \frac{\mu}{\lambda + \mu}\frac{1}{\mu}w \in U,$$

which means $v + w \in (\lambda + \mu)U = (\mathrm{p}_U(v) + \mathrm{p}_U(w) + 2\epsilon)U$. Since $\epsilon > 0$ is arbitrary, $\mathrm{p}_U(v + w) \leq \mathrm{p}_U(v) + \mathrm{p}_U(w)$ follows. □

Thus every absorbing and absolutely convex subset $U \subseteq V$ defines a seminorm p_U. It turns out that the open and closed balls with respect to p_U allow to determine U as follows:

Lemma 2.2.20 *Let $U \subseteq V$ be absolutely convex and absorbing. Then one has*

$$\mathrm{B}_{\mathrm{p}_U,1}(0) \subseteq U \subseteq \mathrm{B}_{\mathrm{p}_U,1}(0)^{\mathrm{cl}}. \tag{2.2.53}$$

PROOF: Suppose $v \in \mathrm{B}_{\mathrm{p}_U,1}(0)$, i.e. $\mathrm{p}_U(v) < 1$. Then for every $\epsilon > 0$ we have $v \in (\mathrm{p}_U(v) + \epsilon)U$ and taking ϵ small enough shows $v \in U$. Next, let $v \in U$, then $\mathrm{p}_U(v) = \inf\{\lambda > 0 \mid v \in \lambda U\} \leq 1$. □

Even though p_U does not allow to reconstruct U completely, it is almost fixed by (2.2.53). Conversely, if we have a seminorm p, we know that its open and closed unit ball are absolutely convex and absorbing. Hence they define a seminorm by the corresponding Minkowski functionals. This reproduces the seminorm we started with:

Lemma 2.2.21 *Let p be a seminorm on V. Then the Minkowski functionals of $\mathrm{B}_{\mathrm{p},1}(0)$ and $\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ coincide with p.*

PROOF: Let $U = \mathrm{B}_{\mathrm{p},1}(0)$. We know that U is absolutely convex and absorbing. Hence it has a seminorm p_U as Minkowski functional. Now

$$\mathrm{p}_U(v) = \inf\big\{\lambda > 0 \;\big|\; v \in \lambda U\big\}$$

$$\begin{aligned}
&= \inf\bigl\{\lambda > 0 \bigm| v \in \lambda \mathrm{B}_{\mathrm{p},1}(0)\bigr\} \\
&= \inf\bigl\{\lambda > 0 \bigm| v \in \mathrm{B}_{\mathrm{p},\lambda}(0)\bigr\} \\
&= \inf\bigl\{\lambda > 0 \bigm| \mathrm{p}(v) < \lambda\bigr\} \\
&= \mathrm{p}(v).
\end{aligned}$$

Analogously, we have for $U = \mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ the same answer, since also

$$\inf\bigl\{\lambda > 0 \bigm| \mathrm{p}(v) \le \lambda\bigr\} = \mathrm{p}(v)$$

in the last step. □

With this circle of constructions being closed, we can now find the reformulation of the condition for a locally convex vector space:

Theorem 2.2.22 (Characterization of locally convex spaces) *Let V be a topological vector space over $\mathbb{K}$. Then the following statements are equivalent:*

i.) The topological vector space V is locally convex.

ii.) There exists a system $\mathcal{P}$ of seminorms inducing the topology.

iii.) The continuous seminorms on V induce the topology of V.

PROOF: The equivalence of the statements *ii.)* and *iii.)* is the content of Theorem 2.2.13, *ii.)* and *vii.)*. The implication *ii.)* $\implies$ *i.)* is the statement of Corollary 2.2.14, *ii.)*. Thus assume that V admits a basis $\mathcal{B}$ of zero neighbourhoods, consisting of convex subsets. Without restriction we can assume that they are already absolutely convex according to Proposition 2.1.30 and Proposition 2.1.31. Finally, being neighbourhoods of zero, they are absorbing by Remark 2.1.4, *iv.)*. Hence for every $U \in \mathcal{B}$ we get a Minkowski functional p_U, which is a seminorm. We claim that p_U is continuous. Indeed, from Lemma 2.2.19 we get for every $\epsilon > 0$

$$\mathrm{p}_U^{-1}\bigl([0,\epsilon)\bigr) = \mathrm{B}_{\mathrm{p},\epsilon}(0) = \epsilon \mathrm{B}_{\mathrm{p},1}(0) \subseteq \epsilon U$$

and for $0 < \epsilon' < \epsilon$ we have

$$\epsilon' U \subseteq \mathrm{B}_{\mathrm{p},\epsilon'}(0)^{\mathrm{cl}} \subseteq \mathrm{B}_{\mathrm{p},\epsilon}(0).$$

This shows that $\mathrm{p}_U^{-1}([0,\epsilon))$ contains the open neighbourhood $\epsilon' U$ and thus is a neighbourhood itself. This is the continuity of p_U at zero. Since p_U is a seminorm, it follows that p_U is continuous everywhere, see Exercise 2.5.25. Since p_U is continuous, $\mathrm{B}_{\mathrm{p}_U,\epsilon}(0) = \mathrm{p}^{-1}([0,\epsilon))$ is an open neighbourhood of zero. We have the mutual inclusion

$$\epsilon' U \subseteq \mathrm{B}_{\mathrm{p}_U,\epsilon}(0) \subseteq \epsilon U \tag{$*$}$$

for all $0 < \epsilon' < \epsilon$, which shows that the collection of all the balls $\mathrm{B}_{\mathrm{p}_U,\epsilon}(0)$ forms a basis of neighbourhoods as well. Thus the induced topology by all the seminorms $\{\mathrm{p}_U\}_{U\in\mathcal{B}}$ is finer than the original one. But $(*)$ also shows that the basis $\mathcal{B}$ yields a finer neighbourhood basis than the induced one. Since both topologies are determined by the neighbourhoods of zero, they coincide. This gives the implication *i.)* $\implies$ *ii.)*. □

In fact, the above proof even shows a slightly stronger statement: if $\mathcal{B}$ is a basis of neighbourhoods of zero consisting of absolutely convex subsets, then $\mathcal{P} = \bigl\{\mathrm{p}_U\bigr\}_{U\in\mathcal{B}}$ is a system of seminorms inducing the original topology of V.

Remark 2.2.23 (Locally convex spaces)

i.) The definition of a locally convex space is the *geometric* version: we characterize the topology by geometric properties of the zero neighbourhoods. In contrast to this, the construction by seminorms can be seen as the *analytic* side of the story. Seminorms typically arise naturally in various estimates reflecting the analytic nature of many locally convex spaces, in particular for the case of function spaces. The theorem thus states that both concepts are equivalent.

ii.) For practical purposes it is often very simple to find seminorms adapted to the question one wants to discuss. Thus one typically starts with a small and easy to handle system $\mathcal{P}$ of seminorms and uses $\mathcal{P}$ to induce a locally convex topology $\mathscr{V}(\mathcal{P})$ by Theorem 2.2.13. However, it then might be rather complicated to investigate conceptual properties of $\mathscr{V}(\mathcal{P})$. In some sense the usage of seminorms can be compared to the usage of coordinates, while the direct usage of neighbourhood systems is more intrinsic. In any case, only both aspects together unfold the true nature of locally convex spaces.

iii.) Having a locally convex space turns out not only to be very useful, as we can argue with seminorms, but also fairly common. In fact, it is quite tricky to find examples of topological vector spaces, which are *not* locally convex. Such examples turn out to have pathological behaviour in many aspects. We will discuss a selected few of the nicer ones down the line.

iv.) A subspace $W \subseteq V$ of a locally convex space V is again locally convex with respect to the subspace topology. Indeed, if $U \subseteq V$ is a convex neighbourhood of zero, then $U \cap W \subseteq W$ is still convex and hence a convex neighbourhood of zero in W with respect to the subspace topology. Also, for a continuous seminorm $\mathrm{p}\colon V \longrightarrow [0, \infty)$ the restriction $\mathrm{p}\big|_W \colon W \longrightarrow [0, \infty)$ is still a seminorm and hence a continuous seminorm in the subspace topology. It requires a little argument to see that the passage to subspaces is compatible with taking the Minkowski functionals, i.e. for an absorbing convex subset $U \subseteq V$ one has

$$\mathrm{p}_{U \cap W} = \mathrm{p}_U\Big|_W, \tag{2.2.54}$$

see Exercise 2.5.26.

Corollary 2.2.24 *Let V be a finite-dimensional $\mathbb{K}$-vector space. Then the unique Hausdorff topology on V turning it into a topological vector space is locally convex.*

Proof: We have a system of seminorms inducing this topology, e.g. the absolute values of the coordinates with respect to a chosen basis $\mathrm{e}_1, \ldots, \mathrm{e}_n \in V$ defined by

$$\mathrm{p}_k(v) = |v^k|,$$

where $v = \sum_{k=1}^n v^k e_k$, as usual. Alternatively, we can use any norm on V, compare again with Theorem 2.1.39. ☐

Corollary 2.2.25 *Let V be a finite-dimensional $\mathbb{K}$-vector space. For any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V one finds $c_1, c_2 > 0$ with*

$$c_1 \|v\|_2 \le \|v\|_1 \le c_2 \|v\|_2 \tag{2.2.55}$$

for all $v \in V$.

Proof: This is of course well-known from elementary calculus, but follows now more conceptually from the uniqueness in Theorem 2.1.39 and the characterization of equivalent systems of (semi-)norms in Proposition 2.2.15. ☐

These two corollaries show quite drastically that the whole interest in locally convex analysis and in topological vector spaces becomes trivial in finite dimensions. In particular, non-locally convex topological vector spaces are necessarily infinite-dimensional.

2.2.3 Continuity of Linear Maps

In this small section we translate the general characterization of continuity for linear maps to the specific locally convex case. The presence of seminorms simplifies the test for continuity considerably.

Proposition 2.2.26 *Let $\phi\colon V \longrightarrow W$ be a linear map between locally convex spaces. Then the following statements are equivalent:*

i.) The map ϕ is continuous.

ii.) For every continuous seminorm q on W the pull-back $\phi^*\mathrm{q} = \mathrm{q} \circ \phi$ is a continuous seminorm on V.

iii.) For every continuous seminorm q on W and for every defining system $\mathcal{P}$ of seminorms on V there exist seminorms $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ and a $c > 0$ with

$$\mathrm{q}(\phi(v)) \leq c \max\{\mathrm{p}_1(v), \ldots, \mathrm{p}_n(v)\} \tag{2.2.56}$$

for all $v \in V$.

iv.) There exist defining systems $\mathcal{Q}$ and $\mathcal{P}$ of continuous seminorms on W and V, respectively, such that for all $\mathrm{q} \in \mathcal{Q}$ there exist seminorms $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ and a $c > 0$ with (2.2.56).

v.) There exists a defining system $\mathcal{Q}$ of continuous seminorms on W and a filtrating defining system $\mathcal{P}$ of continuous seminorms on V such that for all $\mathrm{q} \in \mathcal{Q}$ there exist $c > 0$ and $\mathrm{p} \in \mathcal{P}$ with

$$\mathrm{q}(\phi(v)) \leq c\mathrm{p}(v) \tag{2.2.57}$$

for all $v \in V$.

PROOF: We show *i.)* $\implies$ *ii.)* $\implies$ *iii.)* $\implies$ *iv.)* $\implies$ *v.)* $\implies$ *i.)*. Thus let ϕ be continuous. Then first $\phi^*\mathrm{q} = \mathrm{q} \circ \phi$ is a seminorm by Remark 2.2.2, *v.)*. It is continuous, since the compositions of continuous maps are continuous, which gives *ii.)*. Assume *ii.)*. Then $\phi^*\mathrm{q}$ is a continuous seminorm, which can be estimated by (2.2.56) with respect to any defining system of continuous seminorms $\mathcal{P}$ on V according to Theorem 2.2.13, *vii.)*, and Lemma 2.2.12, *v.)*. The implication *iii.)* $\implies$ *iv.)* is trivial. Since we can always pass from a defining system of seminorms $\mathcal{P}$ to a filtrating such system by taking finite maxima, the implication *iv.)* $\implies$ *v.)* is clear as well. The interesting implication is thus the last one, *v.)* $\implies$ *i.)*. Suppose we have found $\mathcal{Q}$ and $\mathcal{P}$ as specified. Then the open balls $\{\mathrm{B}_{\mathrm{p},r}(0)\}_{\mathrm{p}\in\mathcal{P}, r>0}$ form a basis of neighbourhoods of zero in V. Then (2.2.57) simply means that for $\mathrm{q} \in \mathcal{Q}$ and the corresponding $c > 0$ and $\mathrm{p} \in \mathcal{P}$ we have

$$\mathrm{B}_{\mathrm{p},\frac{c}{r}}(0) \subseteq \phi^{-1}(\mathrm{B}_{\mathrm{q},r}(0))$$

for all $r > 0$. Since we can test continuity at a point by means of a neighbourhood (sub-)basis of that point, we conclude that ϕ is continuous at zero. By Proposition 2.1.14 this proves *i.)*. $\square$

There are some more variations of the characterization of continuity by seminorms: in particular if *iv.)* or *v.)* hold for some pair of system $\mathcal{Q}$ and $\mathcal{P}$, then it also holds for every pair. If $\mathcal{P}$ is even closed under taking positive multiples, then we can arrange $c = 1$ in (2.2.57). Of course, the idea will always be to use *small* systems $\mathcal{Q}$ to test the required conditions. Occasionally, we will refer to estimates like these as *continuity estimates*.

Corollary 2.2.27 *Let V be a locally convex space. Then a linear functional $\varphi\colon V \longrightarrow \mathbb{K}$ is continuous iff there exists a continuous seminorm p on V with*

$$(|\cdot|^*\varphi)(v) = |\phi(v)| \leq \mathrm{p}(v) \tag{2.2.58}$$

for all $v \in V$.

This simple observation can now be used to characterize the topological dual V' of V. In particular, the more continuous seminorms we have on V, the more continuous functionals we can have. This will ultimately lead to large topological duals V' of locally convex spaces, a statement which becomes more precise, once we discuss the Hahn-Banach Theorem in Section 3.1.

Example 2.2.28 Let V be a finite-dimensional vector space over $\mathbb{K}$ with its standard topology. Then every linear map $\phi\colon V \longrightarrow W$ into a locally convex space is continuous. Indeed, since on V *every* seminorm is continuous by the uniqueness of the (locally convex) standard topology, this is clear. Note that a slightly more involved argument is needed to show the same statement for an arbitrary topological vector space, see Exercise **??**.

2.2.4 Convergence and Completeness

Also the notions of convergence of sequences and nets, as well as completeness can be made more explicit in the locally convex situation. To avoid trivialities we will assume that all involved spaces are Hausdorff in this subsection. The first statement clarifies the notion of convergence:

Proposition 2.2.29 *Let V be a Hausdorff locally convex space and let $(v_i)_{i\in I}$ be a net in V indexed by a directed set I. Then the following statements are equivalent:*

i.) The net $(v_i)_{i\in I}$ converges to $v \in V$.

ii.) For all defining systems $\mathcal{P}$ of continuous seminorms on V, all $\mathrm{p} \in \mathcal{P}$, and all $\epsilon > 0$ one finds an index $i_0 \in I$ such that for all $i \succcurlyeq i_0$ one has

$$\mathrm{p}(v_i - v) < \epsilon. \tag{2.2.59}$$

iii.) There exists a defining system $\mathcal{P}$ of continuous seminorms on V with (2.2.59).

Proof: We prove *i.)* $\implies$ *ii.)* $\implies$ *iii.)* $\implies$ *i.)*. Since a continuous seminorm is, in particular, continuous and the topology is translationally invariant, $v_i \to v$ is equivalent to $v_i - v \to 0$. Hence *ii.)* is clear, as $\mathrm{p}(0) = 0$ for all seminorms p. The implication *ii.)* $\implies$ *iii.)* is obvious. Finally assume *iii.)*. This means

$$v_i - v \in \mathrm{B}_{\mathrm{p},\epsilon}(0)$$

for $i \succcurlyeq i_0$. Now the open balls $\{\mathrm{B}_{\mathrm{p},r}(0)\}_{r>0,\mathrm{p}\in\mathcal{P}}$ form a subbasis of open neighbourhoods of zero, which suffices to test convergence. Hence *iii.)* implies $v_i - v \to 0$ and thus $v_i \to v$ by the translation invariance of the topology. □

Again, the advantage will be to use a very small and easy system of defining seminorms to test, whether a net converges or not. This makes the condition *iii.)* the most effective one in practice.

For Cauchy nets we obtain the following, completely analogous statement:

Proposition 2.2.30 *Let V be a Hausdorff locally convex space and let $(v_i)_{i\in I}$ be a net in V indexed by a directed set I. Then the following statements are equivalent:*

i.) The net $(v_i)_{i\in I}$ is a Cauchy net.

ii.) For all defining systems $\mathcal{P}$ of continuous seminorms on V, all $\mathrm{p} \in \mathcal{P}$, and all $\epsilon > 0$ one finds an index $i_0 \in I$ such that for all $i, j \succcurlyeq i_0$ one has

$$\mathrm{p}(v_i - v_j) < \epsilon. \tag{2.2.60}$$

iii.) There exists a defining system $\mathcal{P}$ of continuous seminorms on V with (2.2.60).

Proof: The proof is based on the very same argument as the previous one, namely the observation that (2.2.60) means $v_i - v_j \in \mathrm{B}_{\mathrm{p},\epsilon}(0)$. □

In Theorem 2.1.22 we mentioned (without giving a detailed proof) that to every Hausdorff topological vector space V, one can construct a completion $\widehat{V}$, containing V as a dense subspace. If V is locally convex, then $\widehat{V}$ stays locally convex. This fact is most easily seen using the seminorms:

Theorem 2.2.31 (Completion of a locally convex space) *Let V be a Hausdorff locally convex space with completion $\iota\colon V \longrightarrow \widehat{V}$. Then $\widehat{V}$ is again locally convex. More precisely, a seminorm $\widehat{\mathrm{p}}$ on $\widehat{V}$ is continuous iff its restriction $\widehat{\mathrm{p}}\big|_{\iota(V)}$ is continuous. Any continuous seminorm on V has a unique extension to a continuous seminorm on $\widehat{V}$.*

PROOF: Let p be a continuous seminorm on V. Then the triangle inequality for p shows

$$|\mathrm{p}(v) - \mathrm{p}(v')| \leq \mathrm{p}(v - v')$$

for every $v, v' \in V$. Thus if $v - v' \in \mathrm{B}_{\mathrm{p},\epsilon}(0)$, then $\mathrm{p}(v) - \mathrm{p}(v') \in (-\epsilon, \epsilon)$, which is the uniform continuity of p, since it suffices to test this for the basis of neighbourhoods $\{(-\epsilon, \epsilon)\}_{\epsilon>0}$ of $0 \in \mathbb{K}$. Now every uniformly continuous map defined on a subspace with a complete target extends to the completion of the subspace in a unique and continuous way. This is true for general uniform spaces and their completions, see e.g. [8, Chap. 6]. In detail, one can show that this extension is well-defined by setting

$$\widehat{\mathrm{p}}\Big(\lim_{i\in I} v_i\Big) = \lim_{i\in I} \mathrm{p}(v)$$

for a Cauchy net $(v_i)_{i\in I} \subseteq V$ approximating the point $\lim_{i\in I} v_i \in \widehat{V}$. Note that this incorporates the continuity of the extension. It is then easy to see that for a seminorm p also $\widehat{\mathrm{p}}$ is a seminorm on $\widehat{V}$. Thus we can extend every continuous seminorm on V to a continuous seminorm on $\widehat{V}$. Conversely, if $\widehat{\mathrm{p}}$ is a continuous seminorm on V, then the continuity of $\iota\colon V \longrightarrow \widehat{V}$ shows that $\mathrm{p} = \widehat{\mathrm{p}} \circ \iota = \iota^*\widehat{\mathrm{p}}$ is a continuous seminorm on $\widehat{V}$. Extending this uniquely to $\widehat{V}$ then reproduces $\widehat{\mathrm{p}}$, as $\iota(V) \subseteq \widehat{V}$ is dense and hence the continuous map $\widehat{\mathrm{p}}$ is uniquely determined by its restriction to V. Together this shows that the two operations are inverse to each other and hence yield a bijection between the continuous seminorms on V and on $\widehat{V}$. So far, this holds for a general topological vector space. Thus assume that V is locally convex with topology $\mathscr{V}$ induced by the continuous seminorms. Denote by $\widehat{\mathscr{V}}$ the topology of the completion $\widehat{V}$ and let $\widetilde{\mathscr{V}}$ be the locally convex topology induced by all continuous seminorms on $\widehat{V}$. We then have $\widetilde{\mathscr{V}} \subseteq \widehat{\mathscr{V}}$ and want to show equality. First we note that the map $i\colon V \longrightarrow \big(\widehat{V}, \widetilde{\mathscr{V}}\big)$ is still continuous, as $\widetilde{\mathscr{V}}$ is coarser than $\widehat{\mathscr{V}}$. We claim that the image $i(V) \subseteq \big(\widehat{V}, \widetilde{\mathscr{V}}\big)$ is still dense. But this is also clear, as for a coarser topology closures can only become larger. Thus we have a continuous inclusion with a dense image of V in $\big(\widehat{V}, \widetilde{\mathscr{V}}\big)$, which is equivalent to the statement that $\big(\widehat{V}, \widetilde{\mathscr{V}}\big)$ is also a completion of V. By Exercise 2.5.4 every two completions are isomorphic by a unique linear homeomorphism $I\colon \big(\widehat{V}, \widetilde{\mathscr{V}}\big) \longrightarrow \big(\widehat{V}, \widehat{\mathscr{V}}\big)$ such that

$$\begin{tikzcd}
\big(\widehat{V}, \widetilde{\mathscr{V}}\big) \arrow[rr, "I"] & & \big(\widehat{V}, \widehat{\mathscr{V}}\big) \\
& V \arrow[ul, "i"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

commutes. But clearly $I = \mathrm{id}_{\widehat{V}}$ is the unique extension of $\iota\colon V \longrightarrow \big(\widehat{V}, \widehat{\mathscr{V}}\big)$ and hence the two topologies actually coincide. ☐

Thus we stay within the subcategory of locally convex Hausdorff spaces LCS, when it comes to completions. Of course, it still remains to be shown in detail that a completion actually *exists*. We still postpone this for a bit and show the existence in the locally convex setting in Subsection 2.4.4.

We discuss now a first application of our considerations to convergence. The following definition is motivated by the familiar situation of convergence of series in $\mathbb{K}$:

Definition 2.2.32 (Absolute and unconditional convergence) *Let V be a locally convex space, let I be a non-empty index set, and let $(v_i)_{i\in I}$ be a net in V.*

i.) The sum $\sum_{i\in I} v_i$ is called absolutely convergent if for every continuous seminorm p on V the series $\sum_{i\in I} \mathrm{p}(v_i)$ converges in $\mathbb{R}$.

ii.) The sum $\sum_{i\in I} v_i$ converges unconditionally to $v \in V$ if

$$\lim_{K\in J} \left(\sum_{i\in K} v_i \right)_{K\subseteq I \textit{ finite}} = v, \tag{2.2.61}$$

where $J \subseteq 2^I$ is the directed set of finite subsets of I. In this case we write

$$\sum_{i\in I} v_i = v. \tag{2.2.62}$$

Recall that for any set I, the set J of finite subsets is a directed subset of 2^I with respect to the direction coming from set-theoretic inclusion. If $I = \mathbb{N}$ and $V = \mathbb{K}$, then this indeed reproduces the usual unconditional convergence of the series. The second definition also makes sense for a topological vector space, which is not necessarily locally convex. Clearly, to test absolute convergence, seminorms from a defining system of continuous seminorms suffice.

Proposition 2.2.33 *Let V be a complete locally convex space and let I be a non-empty index set. Then an absolutely convergent sum $\sum_{i\in I} v_i$ is unconditionally convergent to a unique limit v and*

$$\mathrm{p}(v) \le \sum_{i\in I} \mathrm{p}(v_i) \tag{2.2.63}$$

holds for all continuous seminorms p on V.

Proof: Let $(v_i)_{i\in I}$ be a net such that $\sum_{i\in I} v_i$ is absolutely convergent. Then for every continuous seminorm p on V the series $\sum_{i\in I} \mathrm{p}(v_i)$ is absolutely convergent. In particular, at most countably many $\mathrm{p}(v_i)$ are different from zero since $\mathrm{p}(v_i) \ge 0$. Let $\epsilon > 0$, then there is a finite subset $K_\epsilon \subseteq I$ such that $\sum_{i\in I\setminus K_\epsilon} \mathrm{p}(v_i) < \epsilon$. Hence for any two finite subsets $K, K' \subseteq I$ with $K_\epsilon \subseteq K, K'$, we get

$$\mathrm{p}\left(\sum_{i\in K} v_i - \sum_{i\in K'} v_i \right) = \mathrm{p}\left(\sum_{i\in K\setminus K'} v_i - \sum_{i\in K'\setminus K} v_i \right) \le \sum_{i\in (K\cup K')\setminus(K\cap K')} \mathrm{p}(v_i) \le \sum_{i\in I\setminus K_\epsilon} \mathrm{p}(v_i) < \epsilon,$$

since the terms for equal indices cancel and $K_\epsilon \subseteq K \cap K'$. Thus the net $(\sum_{i\in K} v_i)_{K\subseteq I \text{ finite}}$ is a Cauchy net in V. As V is complete, it converges to some limit $v \in V$. The continuity of the seminorm p and the triangle inequality for finite sums gives (2.2.63). □

Remark 2.2.34 In finite dimensions, $V = \mathbb{K}$ or $V = \mathbb{K}^n$, absolute convergence is, in fact, equivalent to unconditional convergence. Remarkably, this fails in infinite dimensions, as simple examples show, see Exercise 2.5.27.

If the index set I is $\mathbb{N}$, we also write

$$\sum_{n\in\mathbb{N}} v_n = \sum_{n=1}^{\infty} v_n \tag{2.2.64}$$

for the limit in the case of unconditional convergence, even though the right hand side suggests to have only convergence for the specified order of summation. Conditionally convergent series will not play a very important role in the sequel.

The last statement about convergence is the following fairly general result on the exchanging of limits. The formulation is slightly technical, but it will turn out to be a very useful result. Suppose we have two directed index sets I and J. Then on $I \times J$ we have the canonical direction defined by $(i, j) \preccurlyeq (i', j')$ if $i \preccurlyeq i'$ and $j \preccurlyeq j'$. Given then a net $(v_{ij})_{(i,j)\in I\times J}$ in V, one calls this net *uniformly convergent* for all $i \in I$ to some $v_i \in V$ if for every zero neighbourhood $U \subseteq V$ there exists a $j_0 \in J$ such that

$$v_{ij} - v_i \in U \tag{2.2.65}$$

for all $j \succcurlyeq j_0$ and $i \in I$. The crucial point is that j_0 depends on U, but is universal for all $i \in I$. As usual, it suffices to take neighbourhoods from a subbasis of neighbourhoods. Moreover, if V is locally convex, then this translates into the condition that for every continuous seminorm p (from a defining system of continuous seminorms) and every $\epsilon > 0$ one finds a $j_0 \in J$ such that

$$\mathrm{p}(v_{ij} - v_i) < \epsilon, \tag{2.2.66}$$

whenever $j \succcurlyeq j_0$. With these considerations in mind, we can now prove the following exchange of limits statement:

Proposition 2.2.35 *Let V be a complete Hausdorff locally convex vector space. Moreover, let I, J be non-empty directed sets and $(v_{ij})_{(i,j)\in I\times J}$ be a net in V such that $(v_{ij})_{j\in J}$ converges uniformly to some $v_i \in V$ for all $i \in I$ and such that $v_j = \lim_{i\in I} v_{ij}$ exists for all $j \in J$, not necessarily uniformly. Then also the nets $(v_{ij})_{(i,j)\in I\times J}$ and $(v_{ij})_{j\in J}$ converge and one has*

$$\lim_{(i,j)\in I\times J} v_{ij} = \lim_{i\in I} \lim_{j\in J} v_{ij} = \lim_{j\in J} \lim_{i\in I} v_{ij}. \tag{2.2.67}$$

Proof: Let $\epsilon > 0$ and let p be a continuous seminorm on V. Moreover, let $j_0 \in J$ be such that $\mathrm{p}(v_{ij} - v_i) < \epsilon$ for all $j \succcurlyeq j_0$ and $i \in I$. Then we have

$$\mathrm{p}(v_{ij} - v_{ij_0}) \le \mathrm{p}(v_{ij} - v_i) + \mathrm{p}(v_i - v_{ij_0}) < 2\epsilon$$

for all $j \succcurlyeq j_0$ and $i \in I$. Since $(v_{ij})_{i\in I}$ also converges for all $j \in J$, we fix $i_0 \in I$ such that for $i \succcurlyeq i_0$ one has

$$\mathrm{p}(v_{ij_0} - v_{j_0}) < \epsilon.$$

Note that now i_0 depends on j_0, as we did not ask for uniform convergence for this order of indices. Then

$$\mathrm{p}(v_{ij_0} - v_{i_0j_0}) \le \mathrm{p}(v_{ij_0} - v_{j_0}) + \mathrm{p}(v_{j_0} - v_{i_0j_0}) < 2\epsilon$$

for all $i \succcurlyeq i_0$, using again uniform convergence in $i \in I$ to estimate the first term by ϵ. Now let $i, i' \succcurlyeq i_0$ and $j, j' \succcurlyeq j_0$ be given. Then

$$\mathrm{p}(v_{ij} - v_{i'j'}) \le \mathrm{p}(v_{ij} - v_{ij_0}) + \mathrm{p}(v_{ij_0} - v_{i_0j_0}) + \mathrm{p}(v_{i_0j_0} - v_{i'j_0}) + \mathrm{p}(v_{i'j_0} - v_{i'j'}) < 8\epsilon.$$

This shows that the net $(v_{ij})_{(i,j)\in I\times J}$ is a Cauchy net and hence convergent due to completeness of V. Hence we have established the existence of the first limit in (2.2.67). Since the individual limits $\lim_{i\in I} v_{ij}$ and $\lim_{j\in J} v_{ij}$ all exist by assumption (the second even uniformly in i), one can use a standard result on convergence of nets in T_3-spaces from point-set topology: the conclusion of [18, Exercise 4.4.5] is that also the two iterated limits exist and coincide with the limit of the net $(v_{ij})_{(i,j)\in I\times J}$. Note that we indeed need a T_3-space, which we know to be the case for topological vector spaces in general by Proposition 2.1.8. □

We will see several applications of this statement in the sequel.

Remark 2.2.36 We conclude our considerations on convergence by noting that all statements of this section stay valid if completeness is replaced by sequential completeness and if all index sets are required to be countable.

2.2.5 First Countability and Fréchet Spaces

In a general locally convex space the topology can be very large. This amounts to two effects: on the one hand, second countability might fail, which is a global feature. On the other hand, already first countability can fail, i.e. we can have neighbourhood systems of zero, which do not allow for a countable basis of neighbourhoods. In this situation, the usage of general nets instead of sequences becomes unavoidable if we want to check continuity, compute closures, or discuss completeness. Thus a particularly nice class of locally convex spaces will be those, where we have a countable basis of zero neighbourhoods. As we shall see in various examples, the second countability can still fail.

The first important observation is that for a first countable Hausdorff locally convex space, we can find a *metric* inducing the topology:

Theorem 2.2.37 (Metrizable locally convex spaces) *Let V be a Hausdorff locally convex space. Then the following statements are equivalent:*

i.) The space V is first countable.

ii.) There exists a sequence of continuous seminorms

$$\mathrm{p}_1 \leq \mathrm{p}_2 \leq \cdots , \tag{2.2.68}$$

which defines the topology of V.

iii.) There exists a countable system $\mathcal{P}$ of continuous seminorms inducing the topology of $\mathcal{P}$.

iv.) There exists a metric d on V, which determines the topology and is invariant under translations, i.e. fulfills

$$d(v+u, w+u) = d(v, w) \tag{2.2.69}$$

for all $u, v, w \in V$.

v.) The topology is metrizable, i.e. there exists a metric on V inducing the topology.

Proof: Suppose *i.)* and let $\{U_n\}_{n\in\mathbb{N}}$ be a countable basis of absolutely convex neighbourhoods of zero, the existence of which is guaranteed by Proposition 2.1.31, *iii.)*. According to Theorem 2.2.13, the Minkowski functionals p_{U_n} of these U_n are continuous seminorms inducing the topology. Then we define

$$\mathrm{p}_n = \max\{\mathrm{p}_{U_1}, \ldots, \mathrm{p}_{U_n}\}$$

for $n \in \mathbb{N}$ and obtain an equivalent system of seminorms, now satisfying (2.2.68). This shows *i.)* $\implies$ *iii.)* $\implies$ *ii.)* and *ii.)* $\implies$ *iii.)* is trivial. Now assume *ii.)*. Then we define

$$d(v, w) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mathrm{p}_n(v-w)}{1+\mathrm{p}_n(v-w)} \tag{$*$}$$

for $v, w \in V$. The series clearly converges and takes values in $[0, 1]$. We have $d(v, w) = d(w, v)$ and $d(v, v) = 0$ iff $v = 0$, since the collection of seminorms $\{\mathrm{p}_n\}_{n\in\mathbb{N}}$ is Hausdorff, see Theorem 2.2.13, *vi.)*. Thus d is a metric, once it fulfils the triangle inequality. To this end we note that the function $f(x) = \frac{x}{1+x}$ for $x \in \mathbb{R}_0^+$ is strictly monotonous with inverse $f^{-1}(y) = \frac{y}{1-y}$ for $y \in [0, 1)$. A standard estimate shows that for every (semi-)metric $\tilde{d}$ also $f \circ \tilde{d}$ is a (semi-) metric, see Exercise 2.5.28 or e.g. [18, Exercise 2.7.1, *v.)*]. Taking the convergent series $(*)$ preserves the validity of the triangle inequality. Finally, we note that d is translationally invariant: indeed, (2.2.69) is fulfilled by construction. Thus we have found a translationally invariant metric on V. Next we show that the metric open balls around zero are neighbourhood with respect to the original topology. Thus let $\epsilon > 0$ be given. Then we choose $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2},$$

which is clearly possible. We also can choose a $\delta > 0$ such that

$$\frac{\delta}{1+\delta} < \frac{\epsilon}{2} \sum_{n=1}^{N} \frac{1}{2^n},$$

using the bijectivity of the function f. Since $\sum_{n=1}^{N} \frac{1}{2^n} < 1$, we can always arrange δ with this property. Now consider $v \in \mathrm{B}_{\mathrm{p}_N,\delta}(0)$, then by (2.2.68) we have

$$\mathrm{p}_1(v) \leq \cdots \leq \mathrm{p}_N(v) < \delta,$$

and thus

$$d(v, 0) = \sum_{n=1}^{N} \frac{1}{2^n} \frac{\mathrm{p}_n(v)}{1+\mathrm{p}_n(v)} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \frac{\mathrm{p}_n(v)}{1+\mathrm{p}_n(v)} < \sum_{n=1}^{N} \frac{1}{2^n} \frac{\delta}{1+\delta} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \epsilon,$$

using again that f is monotonous. This shows

$$\mathrm{B}_{\mathrm{p}_N,\delta}(0) \subseteq \mathrm{B}_{d,\epsilon}(0),$$

and hence every metric ball around zero is a neighbourhood with respect to the original topology. Conversely, let $N \in \mathbb{N}$ and $\epsilon > 0$ be given. Then for $v \in \mathrm{B}_{d,\epsilon}(v)$, we have for all $n \in \mathbb{N}$

$$\frac{\mathrm{p}_n(v)}{1+\mathrm{p}_n(v)} < 2^n \epsilon.$$

Now we choose $\delta > 0$ small enough, such that $2^N \delta < 1$ and

$$\frac{2^N \delta}{1 - 2^N \delta} < \epsilon,$$

using once more the monotonous function f and its inverse. Then we obtain $\mathrm{p}_N(v) < \epsilon$, showing

$$\mathrm{B}_{d,\delta}(0), \subseteq \mathrm{B}_{\mathrm{p}_N,\epsilon}(0).$$

This shows that for every seminorm p_N the open balls around zero are neighbourhoods of zero in the metric topology of d. Since both topologies are translationally invariant, they coincide. This completes *ii.)* $\implies$ *iv.)*. The remaining implication *iv.)* $\implies$ *v.)* and *v.)* $\implies$ *i.)* are trivial. ☐

In the preceding theorem we assumed the space V to be Hausdorff. Some of the equivalences still hold if we drop that assumption, see Exercise 2.5.29.

Having a metrizable topology has of course many useful and far-reaching consequences. The metric itself is of course not unique at all. We have chosen a sequence (2.2.68) of increasing seminorms defining the topology. There we have already and abundance of possibilities. Also the construction of the metric out of the seminorms is by far not the only possible one, see e.g. Exercise 2.5.31 for some alternatives. Moreover, note that the balls for the metric d we have constructed are *not* convex in general, see Exercise 2.5.30. Using a different construction involving maxima, this can, however, be fixed, which we discuss in Exercise 2.5.32. In conclusion, we should not focus on the metric itself, but on the underlying topology instead. The metric is a useful tool, but not the structure we are primarily interested in.

Since our notion of uniformity is based on differences in V and the usual zero neighbourhoods, the metric concepts coincide with this thanks to the translation invariance. We summarize this in the following proposition:

Proposition 2.2.38 *Let V be a metrizable locally convex vector space and let $(v_i)_{i\in I}$ be a net in V.*

i.) The net $(v_i)_{i\in I}$ converges to $v \in V$ iff it converges with respect to any translation invariant metric d which is defining the same topology.

ii.) The net $(v_i)_{i\in I}$ is a Cauchy net iff it is a Cauchy net with respect to any translationally invariant metric d which is defining the same topology.

PROOF: Since convergence is a purely topological notion, the first part is clear and does not yet use the translation invariance. For the second, we recall that $(v_i)_{i\in I}$ is Cauchy iff for every zero neighbourhood $U \subseteq V$ one finds an index $i_0 \in I$ with $v_i - v_j \in U$, whenever $i, j \succcurlyeq i_0$. Since the metric ϵ-balls $\mathrm{B}_{d,\epsilon}(0)$ form a basis of zero neighbourhoods, we get an $\epsilon > 0$ with $\mathrm{B}_{dd,\epsilon}(0) \subseteq U$. Thus this is equivalent to $d(v_i - v_j, 0) < \epsilon$ for $i, j \succcurlyeq i_0$. Since d is translationally invariant, this is equivalent to

$$d(v_i, v_j) = d(v_i - v_j, 0) < \epsilon,$$

i.e. $(v_i)_{i\in I}$ is Cauchy with respect to d. □

This has the convenient consequence that we can test completeness of metrizable locally convex spaces by using *sequences* only:

Corollary 2.2.39 *A metrizable locally convex space is complete iff it is sequentially complete.*

PROOF: Indeed, this is true for any metric space and hence Proposition 2.2.38, *ii.)*, tells us that we use the correct notion of Cauchy nets and Cauchy sequences. □

Many of the functional spaces in analysis are in fact metrizable and complete. This situation deserves special attention and a specific name:

Definition 2.2.40 (Fréchet Space) *A metrizable locally convex space, which is complete, is called a Fréchet space.*

Theorem 2.2.37 gives now an easy to check criterion for metrizability: typically it is easy to find countably many seminorms defining the topology.

Remark 2.2.41 (Fréchet vs. Banach spaces) From elementary calculus one knows that a *Banach space* is a complete normed space $(V, \|\cdot\|)$. Here we have even one single norm $\|\cdot\|$ inducing the topology. Hence every Banach space is a Fréchet space. Note, however, that the choice of norm $\|\cdot\|$ is part of the data of a Banach space: $(V, \|\cdot\|)$ and $(V, 2\|\cdot\|)$ are *different* Banach spaces, but they *coincide* as Fréchet spaces, as the induced topologies are the same.

We conclude the chapter with a somewhat technical corollary about particularly nice defining systems of seminorms for metrizable locally convex spaces, which will be convenient in the sequel:

Corollary 2.2.42 *Let V be a metrizable locally convex space.*

i.) There exists a sequence $\{\mathrm{p}_n\}_{n\in\mathbb{N}}$ of continuous seminorms defining the topology such that

$$\mathrm{p}_n \le \frac{1}{2}\mathrm{p}_{n+1} \tag{2.2.70}$$

for all $n \in \mathbb{N}$.

ii.) There exists a basis $\{U_n\}_{n\in\mathbb{N}}$ of closed neighbourhoods of zero with

$$U_n = -U_n \tag{2.2.71}$$

and

$$U_{n+1} + U_{n+1} \subseteq U_n \tag{2.2.72}$$

for $n \in \mathbb{N}$.

PROOF: Let $\{\tilde{\mathrm{p}}_n\}_{n\in\mathbb{N}}$ be a defining system of continuous seminorms for V with $\tilde{\mathrm{p}}_n \leq \tilde{\mathrm{p}}_{n+1}$ for all $n \in \mathbb{N}$ according to Theorem 2.2.37, *ii.)*. Define then

$$\mathrm{p}_n = \frac{1}{2^n}\tilde{\mathrm{p}}_n,$$

which does the job for (2.2.70). Given such seminorms, the closed unit balls

$$U_n = \mathrm{B}_{\mathrm{p}_n,1}(0)^{\mathrm{cl}}$$

satisfy both (2.2.71) and (2.2.72). □

2.3 The Main Examples

Having established now the basic notions of locally convex spaces, we are now in the position to investigate the main examples from functional analysis more closely.

2.3.1 The Sequence Spaces

Let I be an arbitrary non-empty index set and consider again the space of all sequences indexed by I, i.e. the space $\operatorname{Map}(I,\mathbb{K})$. Since we have no other structure to use than the components, we endow $\operatorname{Map}(I,\mathbb{K})$ with the following topology:

Definition 2.3.1 (The locally convex space $\operatorname{Map}(I,\mathbb{K})$**)** *Let I be a non-empty index set. Then* $\operatorname{Map}(I,\mathbb{K})$ *is equipped with the locally convex topology induced by the seminorms* $\{|\cdot|_i\}_{i\in I}$ *from* (2.2.17).

This topology is still fairly uninteresting, as it does not yet reflect any features of I besides its cardinality. We simply have no properties of I to talk about. Nevertheless, one has the following statement:

Proposition 2.3.2 *Let I be a non-empty index set. Then* $\operatorname{Map}(I,\mathbb{K})$ *is complete. If I is countable,* $\operatorname{Map}(I,\mathbb{K})$ *is a Fréchet space.*

PROOF: Let $(a_j)_{j\in J}$ be a net in $\operatorname{Map}(I,\mathbb{K})$. Then each $a_j \in \operatorname{Map}(I,\mathbb{K})$ is in fact a collection $a_j = (a_{ji})_{i\in I}$ of numbers in $\mathbb{K}$. The Cauchy condition means that for every $i \in I$ and $\epsilon > 0$ we find an index j_0, such that

$$|a_j - a_{j'}|_i = |a_{ji} - a_{j'i}| < \epsilon,$$

whenever $j \succcurlyeq j_0$. This shows that convergence in $\operatorname{Map}(I,\mathbb{K})$ is simply the componentwise convergence in $\mathbb{K}$ and the Cauchy condition is the componentwise, too. Since $\mathbb{K}$ is complete, $\operatorname{Map}(I,\mathbb{K})$ turns out to be complete as well. If I is countable, we have a countable system of defining seminorms $\{|\cdot|_i\}_{i\in I}$. Hence $\operatorname{Map}(I,\mathbb{K})$ is a Fréchet space by Theorem 2.2.37, *iii.)*. □

The topology of $\operatorname{Map}(I,\mathbb{K})$ is thus also called the topology of componentwise or *pointwise convergence*. Note that if I is uncountable, then $\operatorname{Map}(I,\mathbb{K})$ is not metrizable, which we show in Exercise 2.5.33. For most purposes, this topology is way too coarse to be of any interest, see Exercise 2.5.34. Nevertheless, it appears in several applications and can serve as a useful example or counterexample, see also Exercise 2.5.49.

The following sequence spaces are familiar from elementary calculus and provide first standard examples of Banach spaces thereby justifying our previous choices of the topology:

Proposition 2.3.3 *Let I be a non-empty index set and $p \in [1,\infty]$. Moreover, let $\mu = (\mu_i)_{i\in I}$ with $\mu_i > 0$ for all $i \in I$.*

i.) *The p-summable sequences $\ell^p(I, \mu)$ form a Banach space with respect to the norm $\|\cdot\|_{p,\mu}$ from Example 2.2.3, i.) and ii.).*

ii.) *Consider $I = \mathbb{N}$. Then the μ-convergent sequences $c(\mathbb{N}, \mu)$ and the μ-zero sequences $c_\circ(\mathbb{N}, \mu)$ are closed subspaces of $\ell^\infty(I, \mu)$ and hence Banach spaces themselves. Taking the limit is a continuous linear functional on $c_\circ(\mathbb{N}, \mu)$.*

iii.) *The finite sequences $c_{\circ\circ}(I)$ are dense in each $\ell^p(I, \mu)$ for $p \in [1, \infty)$ and in $c(\mathbb{N}, \mu)$ if $I = \mathbb{N}$.*

PROOF: We know already that $\|\cdot\|_{p,\mu}$ is a norm for all $p \in [1, \infty]$ and μ, where

$$\|\cdot\|_{p,\mu}$$

is the μ-weighted sup-norm. Thus we have a single norm and hence a metrizable locally convex topology. The completeness reduces to sequential completeness, which is then a standard result on convergence properties of series, see Exercise 2.5.35. The second statement is similar and uses e.g. Proposition 2.2.33 as a conceptual explanation or an elementary argument, see again Exercise 2.5.35. Finally, the density of $c_{\circ\circ}(I)$ is again an elementary estimate. □

While the sequence spaces $\ell^p(I, \mu)$, $c(I, \mu)$ and $c_\circ(I, \mu)$ are just Banach spaces, the Schwartz sequence space is a genuine locally convex space:

Definition 2.3.4 (The Fréchet space s) *On the space s of rapidly decreasing sequences one defines the locally convex topology induced by the seminorms*

$$\|(a_n)_{n\in\mathbb{N}}\|_{\infty,k} = \sup_{n\in\mathbb{N}} |a_n| n^k \tag{2.3.1}$$

for all $k \in \mathbb{N}_0$.

We immediately warrant the words we used in the definition. In particular, the completeness result justifies the choice of the topology of the Schwartz space:

Proposition 2.3.5 *The rapidly decreasing sequences s are a Fréchet space and $c_{\circ\circ} \subseteq s$ is dense. For every $p \in [1, \infty)$ the systems $\{\|\cdot\|_{p,k}\}_{k\in\mathbb{N}}$ of seminorms provide equivalent systems of continuous seminorms.*

PROOF: As we have countably many seminorms (2.3.1) the locally convex topology will automatically be first countable. We need to check sequential completeness. Thus let $(a_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in s, i.e. $a_i = (a_{in})_{n\in\mathbb{N}}$ is a rapidly decreasing sequence for all $i \in \mathbb{N}$. Moreover, for all k and $\epsilon > 0$ we find an index $i_0 \in \mathbb{N}$ with

$$\|a_j - a_{j'}\|_{\infty,k} < \epsilon$$

for all $j, j' \geq i_0$. This means

$$\sup_{n\in\mathbb{N}} |a_{jn} - a_{j'n}| n^k < \epsilon,$$

and hence

$$|a_{jn} - a_{j'n}| < \frac{\epsilon}{n^k} \tag{$*$}$$

for all $n \in \mathbb{N}$ and $j, j' \geq i_0$. In particular, $(a_{jn})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for all $n \in \mathbb{N}$, thus convergent to some $a_n \in \mathbb{C}$. We claim that $a = (a_n)_{n\in\mathbb{N}}$ is the limit of $(a_j)_{j\in\mathbb{N}}$ in s, which we are searching for. Let $j' \geq i_0$ as before, then

$$|a_n - a_{j'n}| = \lim_{j\to\infty} |a_{jn} - a_{jn'}| \leq \frac{\epsilon}{n^k} \tag{$**$}$$

by $(*)$. This shows

$$|a_n| \leq |a_{j'n}| + |a_n - a_{j'n}| < \frac{\|a_{j'}\|_{\infty,k}}{n^k} + \frac{\epsilon}{n^k},$$

since $a_j \in s$ and $j' \succcurlyeq i_0$. Hence

$$\|a\|_{\infty,k} = \sup_{n\in\mathbb{N}} n^k |a_n| \leq \sup_{n\in\mathbb{N}} n^k |a_{j'n}| + \epsilon = \|a_{j'}\|_{\infty,k} + \epsilon.$$

Thus the convergence of the sequence $(a_i)_{i\in\mathbb{N}}$ to a with respect to the seminorm $\|\cdot\|_{\infty,k}$ follows. Since k was arbitrary, the convergence $\lim_{i\to\infty} a_i = a$ in the topology of s follows, too, proving the completeness. Next, let $a \in s$ be given. Then we define

$$a_i = (a_{in})_{n\in\mathbb{N}} \quad \text{by} \quad a_{in} = \begin{cases} a_n & \text{for } i \leq n \\ 0 & \text{else.} \end{cases}$$

By definition we have $a_i \in c_{\circ\circ}$. We claim that $a_i \longrightarrow a$ in the topology of s. Hence let $\epsilon > 0$ and $k \in \mathbb{N}$ be given. From $a \in s$ we know

$$|a_n| \leq \frac{\|a\|_{\infty,k+1}}{n^{k+1}}$$

for all $n \in \mathbb{N}$. Hence

$$\|a - a_i\|_{\infty,k} = \sup_{n\in\mathbb{N}} |a_n - a_{in}| n^k = \sup_{n>i} |a_n| n^k \leq \frac{\|a\|_{\infty,k+1}}{i},$$

since the difference $a - a_i$ has zeros as the first n entries. This right hand side converges to zero for $i \longrightarrow \infty$ and hence $a_i \longrightarrow a$ follows. The equivalence of the different systems of seminorms was implicitly shown in Proposition 1.1.11 and is discussed in Exercise 2.5.36. ☐

As a slight generalization of the above construction, we get locally convex topologies for all Köthe spaces:

Definition 2.3.6 (Locally convex topology of Köthe spaces) *Let I and J be non-empty index sets and consider a Köthe set $\mathcal{P} = \{\mu^{(j)}\}_{j\in J} \subseteq \operatorname{Map}(I,\mathbb{K})$ of weights on I indexed by J. Let $p \in [1,\infty]$. Then the Köthe space $\Lambda^p(I,\mathcal{P})$ is endowed with the locally convex topology induced by the seminorms $\{\|\cdot\|_{p,\mu^{(j)}}\}_{j\in J}$ where*

$$\|a\|_{p,\mu^{(j)}} = \begin{cases} \sqrt[p]{\sum_{i\in I} |a_i|^p \mu_i^{(j)}} & \text{if } p \in [1,\infty) \\ \sup_{i\in I} |a_i| \mu_i^{(j)} & \text{if } p = \infty. \end{cases} \tag{2.3.2}$$

Proposition 2.3.7 *Let I and J be non-empty index sets and consider a Köthe set $\mathcal{P} = \{\mu^{(j)}\}_{j\in J} \subseteq \operatorname{Map}(I,\mathbb{K})$ of weights on I indexed by J and let $p \in [1,\infty]$.*

i.) The Köthe space $\Lambda^p(I,\mathcal{P})$ is a complete locally convex space.

ii.) If J is countable, the Köthe space $\Lambda^p(I,\mathcal{P})$ is a Fréchet space.

iii.) For $p \in [1,\infty)$ the finite sequences $c_{\circ\circ}(I) \subseteq \Lambda^p(I,\mathcal{P})$ are dense.

PROOF: The proof of this proposition together with some additional statements is discussed in Exercise 2.5.37. ☐

Remark 2.3.8 (The role of $c_{\circ\circ}$) While the systems of seminorms for the sequence spaces $\ell^p(I,\mu)$, s and $\Lambda^p(I,\mathcal{P})$ are simple to find and well motivated by the definitions of the sequence spaces, we do not yet have a reasonable candidate for a locally convex topology on $c_{\circ\circ}$. In fact, all of the above seminorms restrict to $c_{\circ\circ}$ and could be used. However, the finite sequences are *dense* in all the above sequence spaces. Thus $c_{\circ\circ}$ will *not* be a complete locally convex space with respect to any of the above systems of seminorms. The completion would be one of the above sequence spaces and hence strictly larger. It remains to be clarified, which the locally convex topology on $c_{\circ\circ}$ could be. We postpone the definite answer to Section 2.4.3.

2.3.2 Bounded and Continuous Functions

We continue now with the main examples of functions spaces. The following definition is just $\ell^\infty(I)$, interpreted slightly differently:

Definition 2.3.9 (The Banach spaces $\mathscr{B}(X)$ and $\mathscr{BM}(X, \mathfrak{a})$) *Let X be a non-empty set.*

i.) The bounded functions on X is the space $\mathscr{B}(X)$ with the supremum norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|. \tag{2.3.3}$$

ii.) If in addition $\mathfrak{a}$ is a σ-algebra on X, then the bounded measurable functions $\mathscr{BM}(X, \mathfrak{a}) \subset \mathscr{B}(X)$ inherit the supremum norm.

The topology is also called the topology of uniform convergence.

As a consequence of the convergence statement in Proposition 1.2.1, we immediately obtain the completeness of $\mathscr{B}(X)$ and $\mathscr{BM}(X, \mathfrak{a})$ which we take as a sign that the chosen topology was reasonable.

Proposition 2.3.10 *Let X be a non-empty set.*

i.) The normed space $(\mathscr{B}(X), \|\cdot\|_\infty)$ is complete, i.e. a Banach space.

ii.) If $\mathfrak{a}$ is a σ-algebra on X, then the subspace $\mathscr{BM}(X, \mathfrak{a}) \subset \mathscr{B}(X)$ is closed and hence a Banach space itself.

PROOF: Since $\mathscr{B}(X) = \ell^\infty(X)$, the first part is just Proposition 2.3.3, *i.)*, for $p = \infty$ and trivial weight μ. More elementary, it follows directly from Proposition 1.2.1. The second part follows, since limits with respect to $\|\cdot\|_\infty$ are in particular pointwise limits and measurability is preserved under pointwise limits in general, see Remark 1.2.3, *iii.)*. Note that in the normed case, sequences are all we need by Corollary 2.2.39. □

Proposition 2.3.11 *Let $(X, \mathfrak{a})$ be a measurable space. Then the simple functions on X form a dense subspace of $\mathscr{BM}(X, \mathfrak{a})$.*

PROOF: This is just Proposition 1.2.5. □

The density of the simple functions can then be taken as the starting point to construct interesting linear maps defined on $\mathscr{BM}(X, \mathfrak{a})$. The integration can be seen as an example.

More interesting are now the local versions of $\mathscr{B}(X)$ and $\mathscr{BM}(X)$. Here we need not only a topological space X, but we require to have a locally compact Hausdorff space. The reason is that in this case we have neighbourhood bases consisting of compact subsets. We can use them to define seminorms as follows:

Definition 2.3.12 (The locally convex space $\mathscr{B}_{\mathrm{loc}}(X)$) *Let X be a locally compact Hausdorff space. Then $\mathscr{B}_{\mathrm{loc}}(X)$ is equipped with the locally convex topology induced by the seminorms*

$$\|f\|_K = \sup_{x \in K} |f(x)|, \tag{2.3.4}$$

where $f \in \mathscr{B}_{\mathrm{loc}}(X)$ and $K \subseteq X$ runs through all compact subsets of X. This topology is also referred to as the topology of locally uniform convergence.

Thanks to Proposition 1.2.8, *ii.)*, we know that $\|f\|_K < \infty$ for all compact subsets $K \subseteq X$ is equivalent to $f \in \mathscr{B}_{\mathrm{loc}}(X)$. It is clear that $\|\cdot\|_K$ is a seminorm. Since we can have $f \in \mathscr{B}_{\mathrm{loc}}(X)$ with $f\big|_K = 0$, but $f \neq 0$ for a general compact subset $K \subseteq X$, we obtain only a seminorm $\|\cdot\|_K$, but not a norm in general. The next result shows again that this locally convex topology was a natural choice:

Proposition 2.3.13 *Let X be a locally compact Hausdorff space.*

i.) The locally bounded functions $\mathscr{B}_{\mathrm{loc}}(X)$ are a complete Hausdorff locally convex space. The topology is finer than the locally convex topology of pointwise convergence induced by the inclusion $\mathscr{B}_{\mathrm{loc}}(X) \subseteq \mathrm{Map}(X, \mathbb{K})$.

ii.) If X is second countable, then $\mathscr{B}_{\mathrm{loc}}(X)$ is a Fréchet space. In this case, $\mathscr{B}\mathscr{M}_{\mathrm{loc}}(X)$ is a closed subspace and hence Fréchet itself.

iii.) If X is compact, then $\mathscr{B}_{\mathrm{loc}}(X)$ is normable and choosing the norm $\|\cdot\|_X$ gives the equality of locally convex spaces $\mathscr{B}_{\mathrm{loc}}(X) = \mathscr{B}(X)$.

Proof: Since $K = \{x\}$ is compact for all $x \in X$ and thus gives a seminorm $\|\cdot\|_{\{x\}} = |\cdot|_x$, the topology of $\mathscr{B}_{\mathrm{loc}}(X)$ is finer than the one inherited from $\mathrm{Map}(X, \mathbb{K})$ as in Definition 2.3.1 and hence Hausdorff. In particular, a Cauchy net $\{f_i\}_{i \in I}$ in $\mathscr{B}_{\mathrm{loc}}(X)$ is a Cauchy net for pointwise convergence and hence it converges pointwise. Let $f(x) = \lim_{i \in I} f_i(x)$ be its pointwise limit. Then for every compact subset $K \subseteq X$ and $\epsilon > 0$ we find an index $i_0 \in I$ with

$$|f_i(x) - f_j(x)| < \epsilon$$

for all $x \in K$ and $i, j \succcurlyeq i_0$. Hence for those x we have

$$|f(x) - f_j(x)| = \lim_{i \in I} |f_i(x) - f_j(x)| \leq \epsilon \tag{$*$}$$

and thus $f \in \mathscr{B}_{\mathrm{loc}}(X)$ by Proposition 1.2.8, *iii.)*, follows, as X has a basis of compact neighbourhoods around every point. Moreover, $(*)$ means $\|f - f_j\|_K \leq \epsilon$ and hence we have not only pointwise convergence, but convergence in the topology of $\mathscr{B}_{\mathrm{loc}}(X)$. This shows the completeness of $\mathscr{B}_{\mathrm{loc}}(X)$ and hence *i.)*. If X is second countable, we find an exhausting sequence of compact subsets $K_1 \subseteq K_1^\circ \subseteq K_2 \subseteq \cdots \subseteq X$ with $X = \bigcup_{n \in \mathbb{N}} K_n$, see Exercise 1.4.14. Since clearly

$$\|\cdot\|_K \leq \|\cdot\|_{K'},$$

whenever $K \subseteq K'$, the seminorms $\{\|\cdot\|_{K_n}\}_{n \in \mathbb{N}}$ are already enough to define the topology of $\mathscr{B}_{\mathrm{loc}}(X)$. This shows the second statement thanks to Theorem 2.2.37, *ii.)*. Since measurable functions are already closed under taking pointwise limits of sequences, $\mathscr{B}\mathscr{M}_{\mathrm{loc}}(X) \subseteq \mathscr{B}_{\mathrm{loc}}(X)$ is a closed subspace and hence a Fréchet space itself. The last part is clear, since $\|\cdot\|_X = \|\cdot\|_\infty$ and hence we are back at Proposition 2.3.10. □

In view of our discussion in Remark 2.2.41 we emphasize here that, even for a compact Hausdorff space X, the spaces $\mathscr{B}(X)$ and $\mathscr{B}_{\mathrm{loc}}(X)$ are considered to be *different*: the first is viewed as Banach space with the supremum norm, the second is a Fréchet space without a preferred choice of a norm inducing its topology. The underlying locally convex spaces are of course the same.

We can now move to continuous functions. Since by Proposition 1.2.12, *ii.)*, we know $\mathscr{C}(X) \subseteq \mathscr{B}\mathscr{M}_{\mathrm{loc}}(X)$, we can use the same seminorms for $\mathscr{C}(X)$ again:

Definition 2.3.14 (The locally convex space $\mathscr{C}(X)$) *Let X be a locally compact Hausdorff space. Then $\mathscr{C}(X)$ is equipped with the locally convex topology induced by the local supremum seminorms $\{\|\cdot\|_K\}_{K \subseteq X \text{compact}}$.*

Note that for continuous functions we even have

$$\|f\|_K = \max_{x \in K} |f(x)| \tag{2.3.5}$$

instead of a mere supremum, since continuous functions attain a maximum on each compact subset.

According to Proposition 1.2.12, *ii.)*, locally uniform limits of continuous functions are again continuous. This gives immediately the following statement:

Corollary 2.3.15 *Let X be a locally compact Hausdorff space. Then $\mathscr{C}(X)$ is a complete locally convex space. If X is second countable, $\mathscr{C}(X)$ is a Fréchet space. If X is compact, $\mathscr{C}(X)$ becomes a Banach space with respect to the supremum norm $\|\cdot\|_\infty$.*

Again, for a compact X one should be careful, whether one considers $\mathscr{C}(X)$ as Banach space by specifying $\|\cdot\|_\infty$ as part of the data or by considering $\mathscr{C}(X)$ as a Fréchet space, which can be normed. Of course,

$$\mathscr{C}(X) \subseteq \mathscr{BM}_{\mathrm{loc}}(X) \subseteq \mathscr{B}_{\mathrm{loc}}(X) \tag{2.3.6}$$

is now a sequence of closed subspaces.

In Section 1.2.5 we investigated several subspaces of $\mathscr{C}(X)$, characterized by the support properties of the functions. Here we have the following result:

Proposition 2.3.16 *Let X be a locally compact Hausdorff space and let $A \subseteq X$ be a closed subset.*

i.) The subspace $\mathscr{C}_A(X) \subseteq \mathscr{C}(X)$ is closed.

ii.) The subspace of compactly supported functions $\mathscr{C}_0(X)$ is dense in $\mathscr{C}(X)$.

iii.) If X is second countable, $\mathscr{C}_0(X) \subseteq \mathscr{C}(X)$ is sequentially dense.

PROOF: Suppose $\{f_i\}_{i\in I}$ is a convergent net in $\mathscr{C}(X)$ with $f_i \in \mathscr{C}_A(X)$ for all $i \in I$ and limit $f \in \mathscr{C}(X)$. Then $f_i(x) = 0$ for all $i \in I$ and $x \in X \setminus A$ by definition of the support. Since the locally uniform convergence implies pointwise convergence, we get $f(x) = 0$ for $x \in X \setminus A$ and thus $\operatorname{supp} f \subseteq A$, showing *i.)*. The second part is Proposition 1.2.17, *ii.)*. The third part is clear, since $\mathscr{C}(X)$ is a Fréchet space in this situation according to Corollary 2.3.15. □

Remark 2.3.17 (The role of $\mathscr{C}_0(X)$) For a non-compact locally compact Hausdorff space X we have $\mathscr{C}_0(X) \neq \mathscr{C}(X)$ as e.g. the constant functions are not in $\mathscr{C}_0(X)$. Hence $\mathscr{C}_0(X)$ is *not* complete with respect to the seminorms $\{\|\cdot\|_K\}_{K\subseteq X \text{ compact}}$. This reflects the difficulty that locally uniform convergence can not capture the global feature of having compact support. At the moment we have no good candidate for a better locally convex topology as the one inherited from $\mathscr{C}(X)$, such that $\mathscr{C}_0(X)$ becomes a complete subspace. Note that the analogy to the role of the sequence space $c_{\circ\circ}(I)$ of finite sequences is in fact the same situation: if we topologize the index set I discretely, we have $\operatorname{Map}(I, \mathbb{K}) = \mathscr{C}(I)$ and $c_{\circ\circ}(I) = \mathscr{C}_0(I)$, since in a discrete space I the compact subsets are exactly the finite subsets. We will have to return to this difficulty later.

A last statement about continuous functions is the continuity of pull-backs:

Proposition 2.3.18 *Let $\phi\colon X \longrightarrow Y$ be a continuous map between locally compact Hausdorff spaces. Then the pull-back*

$$\phi^*\colon \mathscr{C}(Y) \longrightarrow \mathscr{C}(X) \tag{2.3.7}$$

is continuous.

PROOF: According to Proposition 2.2.26, *v.)*, we have to show that for every seminorm $\|\cdot\|_K$ with $K \subseteq X$ compact, the pull-back seminorm $\|\cdot\|_K \circ \phi^*$ is continuous on $\mathscr{C}(Y)$, since the seminorms $\{\|\cdot\|_K\}_{K\subseteq X \text{ compact}}$ provide a defining system of continuous seminorms for $\mathscr{C}(X)$. Now for $f \in \mathscr{C}(Y)$ we have

$$(\|\cdot\|_K \circ \phi^*)(f) = \|\phi^* f\|_K = \sup_{x\in K} |f(\phi(x))| = \sup_{y\in\phi(K)} |f(y)| = \|f\|_{\phi(K)}.$$

Since $\phi(K) \subseteq Y$ is again compact as the image of a compact set under a continuous map, we indeed have a continuous seminorm on $\mathscr{C}(X)$. □

This induces a functor $\mathscr{C}\colon \mathsf{LCHTop} \longrightarrow \mathsf{LCS}$ from locally compact Hausdorff topological spaces to locally convex spaces, see Exercise 2.5.40. In fact, we end up in locally convex algebras which we have not yet studied properly.

2.3.3 Integrable Functions

2.3.4 Differentiable Functions

For an open subset $X \subseteq \mathbb{R}^n$ we have already discussed the space of k-times continuously differentiable functions $\mathscr{C}^k(X)$ in Section 2.3.4, where $k \in \mathbb{N}_0 \cup \{\infty\}$. The naive convergence results are now put on a solid locally convex basis.

Definition 2.3.19 (The Fréchet space $\mathscr{C}^k(X)$) *Let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then $\mathscr{C}^k(X)$ is endowed with the locally convex topology induced by the seminorms $\{\mathrm{p}_{K,\ell}\}_{K \subseteq X \text{compact}, \ell \in \mathbb{N}_0,\, \ell \leq k}$ from* (2.2.33)*.The topology is called the topology of locally uniform convergence of all derivatives up to order k or the $\mathscr{C}^k$-topology.*

We indeed obtain a Fréchet space $\mathscr{C}^k(X)$ with this definition, stressing as usual that the topology was chosen appropriately:

Proposition 2.3.20 *Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and let $X \subseteq \mathbb{R}^n$ be a non-empty open set.*

i.) The $\mathscr{C}^k$-topology turns $\mathscr{C}^k(X)$ into a Fréchet space.

ii.) If $A \subseteq X$ is closed, then $\mathscr{C}^k_A(X) \subseteq \mathscr{C}^k(X)$ is a closed subspace.

iii.) The subspace $\mathscr{C}^k_0(X) \subseteq \mathscr{C}^k(X)$ is (sequentially) dense.

PROOF: The $\mathscr{C}^k$-topology is clearly finer than the $\mathscr{C}^0$-topology and hence finer than the topology of pointwise convergence. Thus a Cauchy net $(f_i)_{i \in I}$ in $\mathscr{C}^k(X)$ has a pointwise limit f. Repeating now the argument of Proposition 2.3.13, *i.)*, shows that f_i converges to f locally uniformly and $\partial^\alpha f_i$ converges to some function g_α also locally uniformly for all multiindices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$. Indeed, we just apply that proposition to every derivative $\partial^\alpha f_i$ individually. From Proposition 1.3.2 we infer that $g_\alpha = \partial^\alpha f$. Then the $\mathscr{C}^k(X)$-convergence of the functions f_i to f is clear. Thus $\mathscr{C}^k(X)$ is complete. Since X admits a compact exhaustion $K_1 \subseteq K_1^\circ \subseteq K_2 \subseteq \cdots \subseteq X$ with $X = \bigcup_{n \in \mathbb{N}} K_n$, see again Exercise 1.4.14, we get countably many seminorms $\{\mathrm{p}_{K_n,\ell}\}_{n,\ell \in \mathbb{N}_0, \ell \leq k}$. Now for a general compact subset $K \subseteq X$ we clearly have

$$\mathrm{p}_{K,\ell} \leq \mathrm{p}_{K_n,\ell},$$

as soon as $K \subseteq K_n$. But the open cover $X = \bigcup_{n=1}^\infty K_n$ shows that there is such a compact K_n for every compact subset K. Hence the countably many seminorms $\{\mathrm{p}_{K_n,\ell}\}_{n,\, \ell \in \mathbb{N}_0,\, \ell \leq k}$ already define the $\mathscr{C}^k(X)$-topology. By Theorem 2.2.37, *iii.)*, we obtain a Fréchet space. The argument for the second statement is the same as for the special case $k = 0$ in Proposition 2.3.16, *i.)*. Finally, the last statement is the approximation obtained in Proposition 1.2.17. □

As for $c_{\circ\circ}(I)$ and $\mathscr{C}_0(X)$ before, also $\mathscr{C}^k_0(X)$ seems to carry the wrong topology if we just take the subspace topology inherited from $\mathscr{C}^k(X)$. With respect to the $\mathscr{C}^k$-topology $\mathscr{C}^k_0(X)$ is *not* complete. We will need a finer topology for $\mathscr{C}^k_0(X)$ to achieve its completeness.

For $\mathscr{C}^k(X)$-functions we get a new class of continuous maps, the differential operators. We recall the following definition, which is not the most conceptual, but completely sufficient for our purposes:

Definition 2.3.21 (Differential operator) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $k \in \mathbb{N}_0 \cup \{\infty\}$. A smooth differential operator D of order $r \in \mathbb{N}_0$ on X is a linear map of the form*

$$D = \sum_{|\alpha| \leq r} \frac{1}{\alpha!} D^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} \tag{2.3.8}$$

with smooth functions $D^\alpha \in \mathscr{C}^\infty(X)$. The set of such operators is denoted by $\mathrm{DiffOp}^r(X)$*. We set*

$$\mathrm{DiffOp}^\bullet(X) = \bigcup_{r \geq 0} \mathrm{DiffOp}^r(X). \tag{2.3.9}$$

With this definition, D becomes a linear map

$$D\colon \mathscr{C}^k(X) \longrightarrow \mathscr{C}^{k-r}(X) \tag{2.3.10}$$

as long as $k \geq r$. In particular, D is an endomorphism of $\mathscr{C}^\infty(X)$. If the coefficients D^α of D have a lower regularity, then the result of D acting on $\mathscr{C}^k(X)$ might end up in less regular functions. This is of course also an interesting and important case, but we focus on the smooth one with $D^\alpha \in \mathscr{C}^\infty(X)$ here.

Proposition 2.3.22 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then a smooth differential operator D of order $r \in \mathbb{N}_0$ is a continuous map*

$$D\colon \mathscr{C}^k(X) \longrightarrow \mathscr{C}^{k-r}(X), \tag{2.3.11}$$

as long as $k \geq r$.

PROOF: We consider first the case $r = 0$, i.e. D is just a multiplication operator with a smooth function $D^0 \in \mathscr{C}^\infty(X)$. Let $f \in \mathscr{C}^k(X)$, then $D^0 f \in \mathscr{C}^k(X)$ again. For every $\ell \in \mathbb{N}_0$ with $\ell \leq k$ and every compact subset $K \subseteq X$ we have

$$\begin{aligned}
\mathrm{p}_{K,\ell}(D(f)) &= \sup_{\substack{x\in K\\ |\alpha|\leq\ell}} \left| \frac{\partial^{|\alpha|}(D^0 f)}{\partial x^\alpha}(x) \right| \\
&= \sup_{\substack{x\in K\\ |\alpha|\leq\ell}} \left| \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \frac{\partial^{|\beta|} D^0}{\partial x^\beta}(x) \frac{\partial^{|\alpha-\beta|} f}{\partial x^\beta}(x) \right| \\
&\leq \sup_{\substack{x\in K\\ |\alpha|\leq\ell}} \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \left| \frac{\partial^{|\beta|} D^0}{\partial x^\beta}(x) \right| \left| \frac{\partial^{|\alpha-\beta|} f}{\partial x^\beta}(x) \right| \\
&\leq 2^\ell \sup_{\substack{x\in K\\ |\beta|\leq\ell}} \left| \frac{\partial^{|\beta|} D^0}{\partial x^\beta}(x) \right| \sup_{\substack{x\in K\\ |\alpha|\leq\ell}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| \\
&= 2^\ell \mathrm{p}_{K,\ell}(D^0) \mathrm{p}_{K,\ell}(f).
\end{aligned}$$

Using the criterion from Proposition 2.2.26, *v.)*, we see that D is continuous in the $\mathscr{C}^k$-topology. The next case is a differential operator consisting of a partial derivative $D = \partial^\beta$ for some $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq r$. Then we get for $\ell \leq k - r$

$$\mathrm{p}_{K,\ell}(D(f)) = \sup_{\substack{x\in K\\ |\alpha|\leq\ell}} \left| \frac{\partial^{|\beta|}}{\partial x^\beta} \left(\frac{\partial^{|\alpha|} f}{\partial x^\alpha} \right)(x) \right| \leq \mathrm{p}_{K,\ell+r}(f),$$

and hence the continuity. Since compositions of continuous linear maps are continuous again and since continuous linear maps form a subspace of all linear maps according to Proposition 2.1.11, *ii.)*, the general case follows. ☐

Of course, one can also get a direct estimate for the general case. To this end we define the seminorm

$$\mathrm{p}_{K,\ell}(D) = \sup_{\substack{x\in K\\ |\alpha|\leq\ell\\ 0\leq|\beta|\leq r}} \left| \frac{\partial^{|\alpha|} D^\beta}{\partial x^\alpha}(x) \right| \tag{2.3.12}$$

for the differential operator D itself. This simply means to take the usual $\mathrm{p}_{K,\ell}$-seminorm for each component D^β and then the maximum over β. Clearly, this defines a seminorm on $\operatorname{DiffOp}^r(X)$ and $\operatorname{DiffOp}^r(X)$ becomes a Fréchet space by taking $\{\mathrm{p}_{K,\ell}\}_{K \subseteq X \text{ compact}, \ell \in \mathbb{N}_0}$, see Exercise 2.5.43. Then the combined continuity estimate for a general differential operator $D \in \operatorname{DiffOp}^r(X)$ reads

$$\mathrm{p}_{K,\ell}(D(f)) \le \mathit{const}\, \mathrm{p}_{K,\ell}(D)\mathrm{p}_{K,\ell}(f), \tag{2.3.13}$$

where $f \in \mathscr{C}^k(X)$ with $k - r \ge 0$ and $\ell \le k$, as before. When we come to continuity of multilinear maps, we will find a re-interpretation of this estimate, see Section **??**.

Corollary 2.3.23 *The $\mathscr{C}^k$-topology on $\mathscr{C}^k(X)$ is the coarsest locally convex topology such that every smooth differential operator D of order at most k gives a continuous linear map*

$$D\colon \mathscr{C}^k(X) \longrightarrow \mathscr{C}^0(X). \tag{2.3.14}$$

Proof: We have seen that every such D is actually continuous in Proposition 2.3.22. Conversely, if every D is continuous, then in particular $D = \partial^\beta$ for $|\beta| \le k$ is continuous. Hence the seminorm $\|\cdot\|_K \circ \partial^\beta$ has to be continuous and also the finite maximum over all $|\beta| \le \ell$ is a continuous seminorm for $\ell \le k$. But this gives

$$\sup_{|\beta|\le\ell} \|\cdot\|_K \circ \partial^\beta = \sup_{\substack{x\in X\\|\beta|\le\ell}} \left| \frac{\partial^{|\beta|}}{\partial x^\beta}(\,\cdot\,)(x) \right| = \mathrm{p}_{K,\ell}(\,\cdot\,).$$

Thus any $\mathrm{p}_{K,\ell}$ is necessarily continuous. □

Hence the definition of the $\mathscr{C}^k$-topology is not a coincidence, but the minimal possibility to make the process of differentiation a continuous operation itself. Since differentiable functions are made for being differentiated, this seems to be a strong motivation for the $\mathscr{C}^k$-topology.

Differential operators are not the only operators on differentiable functions one could be interested. As already for the continuous functions, pull-backs are non-local operations of interest. This is different from differential operators, which are *local* in the sense that

$$\operatorname{supp}(D(f)) \subseteq \operatorname{supp}(f) \tag{2.3.15}$$

for all differential operators D and all sufficiently differentiable functions f. The pull-back of course will change the support in general. To ensure continuity, we have to use differentiable maps to pull back:

Proposition 2.3.24 *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be non-empty open subsets and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then the pull-back*

$$\phi^*\colon \mathscr{C}^k(Y) \longrightarrow \mathscr{C}^k(X) \tag{2.3.16}$$

with a $\mathscr{C}^k$-map $\phi\colon X \longrightarrow Y$ is continuous in the $\mathscr{C}^k$-topology.

Proof: First we note that for $f \in \mathscr{C}^k(Y)$ also $\phi^* f = f \circ \phi$ is $\mathscr{C}^k$ by the chain rule. More precisely, we consider a compact subset $K \subseteq X$ and $\ell \in \mathbb{N}_0$ with $\ell \le k$. Then we know that $\phi(K) \subseteq Y$ is compact again. Using the chain rule from Proposition 1.3.4 we estimate

$$\begin{aligned}
\mathrm{p}_{K,\ell}(\phi^*(f)) &= \sup_{\substack{x\in K\\|\alpha|\le\ell}} \left| \frac{\partial^{|\alpha|}(f\circ\phi)}{\partial x^\alpha}(x) \right| \\
&= \sup_{\substack{x\in K\\|\alpha|\le\ell}} \left| \sum_{r=0}^{|\alpha|} \sum_{|\beta|\le|\alpha|} \sum_{|\gamma_1|+\cdots+|\gamma_r|=|\alpha|} c^\alpha_{\beta\gamma_1\cdots\gamma_r} \frac{\partial^{|\beta|} f}{\partial y^\beta}(\phi(x)) \frac{\partial^{|\gamma_1|}\phi^{\beta_1}}{\partial x^{\gamma_1}}(x) \cdots \frac{\partial^{|\gamma_r|}\phi^{\beta_r}}{\partial x^{\gamma_r}}(x) \right|
\end{aligned}$$

$$\leq \sup_{\substack{x\in K\\|\alpha|\leq\ell}} \sum_{|\beta|\leq|\alpha|} \sum_{|\gamma_1|+\cdots+|\gamma_r|=|\alpha|} c^{\alpha}_{\beta\gamma_1\cdots\gamma_r} \left|\frac{\partial^{|\beta|}f}{\partial y^\beta}(\phi(x))\right| \left|\frac{\partial^{|\gamma_1|}\phi^{\beta_1}}{\partial x^{\gamma_1}}(x)\right| \cdots \left|\frac{\partial^{|\gamma_r|}\phi^{\beta_r}}{\partial x^{\gamma_r}}(x)\right|$$
$$\leq \mathit{const} \sup_{\substack{y\in\phi(K)\\|\beta|\leq\ell}} \left|\frac{\partial^{|\beta|}f}{\partial y^\beta}\right|(y),$$

since all the derivatives of ϕ are bounded on the compact subset $K \subseteq X$ and all sums are finite. Hence all these contributions can be estimated by a single, large enough constant depending on ϕ. Thus we arrive at an estimate of the form

$$\mathrm{p}_{K,\ell}(\phi^*(f)) \leq \mathit{const}\, \mathrm{p}_{\phi(K),\ell}(f),$$

which gives the desired continuity, again by Proposition 2.2.26, *v.)*. ☐

The next question we want to address is how the different $\mathscr{C}^k$-topologies relate. Of course we have the continuous inclusions

$$\mathscr{C}^\infty(X) \subseteq \cdots \subseteq \mathscr{C}^{k+1}(X) \subseteq \mathscr{C}^k(X) \subseteq \cdots \subseteq \mathscr{C}^1(X) \subseteq \mathscr{C}^0(X) = \mathscr{C}(X). \tag{2.3.17}$$

But unlike for the inclusions $\mathscr{C}^k_A(X) \subseteq \mathscr{C}^k(X)$, the above subspaces will not be closed. Quite contrary, the subspaces are dense in all following ones. This can be shown with some convolution tricks, which we will investigate later in more detail.

Proposition 2.3.25 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then the inclusions in* (2.3.17) *are continuous and*

$$\mathscr{C}^\infty_0(X) \subseteq \mathscr{C}^k(X) \tag{2.3.18}$$

is sequentially dense for the $\mathscr{C}^k$-topology.

Proof: We simply increase the number of seminorms in the defining sets, when we move from the right to the left in (2.3.17). This shows that the $\mathscr{C}^\infty$-topology is the finest and the $\mathscr{C}^k$-topology is finer than the $\mathscr{C}^\ell$-topology, whenever $k \geq \ell$. It is the second statement, which is more interesting. First we consider a compactly supported function $f \in \mathscr{C}^k_0(X)$, which we want to approximate by $\mathscr{C}^\infty_0(X)$-functions in the $\mathscr{C}^k$-topology. We choose a bump function $\chi \in \mathscr{C}^\infty_0(\mathbb{R}^n)$ with

$$\operatorname{supp}(\chi) \subseteq \mathrm{B}_1(0) \quad \text{and} \quad \int_{\mathbb{R}^n} \chi(x)\, \mathrm{d}^n x = 1.$$

From Exercise 1.4.12 we know that such a bump function exists. Then we consider the functions $\chi_m(x) = m^n\chi(mx)$, which are still smooth, have support now in $\mathrm{B}_{\frac{1}{m}}(0)$, and are still normalized to integral equal to 1. We consider the convolution integral

$$(\chi_m * f)(x) = \int_{\mathbb{R}^n} \chi_m(y) f(x-y)\, \mathrm{d}^n y.$$

Some simple estimates show that $\chi_m * f$ is actually smooth: convolution adds regularities. Moreover,

$$\operatorname{supp}(\chi_m * f) \subseteq \operatorname{supp}(\chi_m) + \operatorname{supp}(f) \subseteq \operatorname{supp}(f) + \mathrm{B}_{\frac{1}{m}}(0).$$

Since $\operatorname{supp}(f) \subseteq X$ is compact, finitely many of the balls $\mathrm{B}_{\frac{1}{m}}(x)$ for $x \in \operatorname{supp}(f)$ cover $\operatorname{supp}(f)$ already. Taking now m large enough shows that $\operatorname{supp}(f) + \mathrm{B}_{\frac{1}{m}}(0) \subseteq X$ for all $m \geq m_0$. Hence we see that for these $m \geq m_0$ we have

$$\chi_m * f \in \mathscr{C}^\infty_0(X).$$

A last estimate using the definition of χ_m gives then the $\mathscr{C}^k$-convergence $\chi_m * f \longrightarrow f$. We will come back to convolution in much generality later in Section 5.4.2, where we will supply the missing details. Since we already know that $\mathscr{C}_0^k(X) \subseteq \mathscr{C}^k(X)$ is dense, we conclude that $\mathscr{C}_0^\infty(X) \subseteq \mathscr{C}^k(X)$ is dense as well. $\square$

In particular, the restriction of the $\mathscr{C}^\ell$-topology to $\mathscr{C}^k(X)$ for $\ell < k$ gives a strictly coarser topology on $\mathscr{C}^k(X)$ than the $\mathscr{C}^k$-topology.

2.3.5 Holomorphic Functions

As expected from complex function theory, the space of holomorphic functions $\mathscr{O}(X)$ for some open $X \subseteq \mathbb{C}^n$ behaves very well and, in many aspects, better than the smooth analogues from real analysis.

In Example 2.2.9 we have seen a system of seminorms for $\mathscr{O}(X)$ based on the Taylor expansion of holomorphic functions. Since $\mathscr{O}(X) \subseteq \mathscr{C}^\infty(X, \mathbb{C}) \subseteq \mathscr{C}^k(X, \mathbb{C})$, we can also use all the $\mathscr{C}^k$-topologies for $\mathscr{O}(X)$ with $k \in \mathbb{N}_0 \cup \{\infty\}$. Surprisingly, it turns out that all these topologies coincide.

Definition 2.3.26 ($\mathscr{O}$-Topology) *Let $X \subseteq \mathbb{C}^n$ be a non-empty open subset. Then the $\mathscr{O}$-topology for $\mathscr{O}(X)$ is the locally convex topology induced by the seminorms $\{\mathrm{p}_{z,R}\}_{z \in X,\ R \text{ small enough}}$ from (2.2.35), where $z \in X$ and $R = (r_1, \ldots, r_n)$ is a small enough polyradius such that the closed polydisc $D(z, R) \subseteq X$ is still contained in X.*

First we note that we have the trivial estimate

$$\mathrm{p}_{z,R} \leq \mathrm{p}_{z,R'}, \tag{2.3.19}$$

whenever $R \leq R'$. Moreover, in general the seminorms $\{\mathrm{p}_{z,R}\}_{z \in X,\ R \text{ small enough}}$ are not yet filtrating: we will need to take finite sums or finite maxima to have a filtrating system of seminorms.

Lemma 2.3.27 *Let $X \subseteq \mathbb{C}^n$ be a non-empty open subset.*

i.) Let $K \subseteq X$ be compact. Then there exist finitely many $z_1, \ldots, z_N \in X$ with corresponding polyradii $R_1, \ldots, R_N$ such that

$$K \subseteq D(z_1, R_1) \cup \cdots \cup D(z_N, R_N) \subseteq X. \tag{2.3.20}$$

In this case, one has

$$\|f\|_K \leq \mathrm{p}_{z_1,R_1}(f) + \cdots + \mathrm{p}_{z_N,R_N}(f) \tag{2.3.21}$$

for all $f \in \mathscr{O}(X)$.

ii.) For all $z \in X$ with $R' < R$ small enough such that $D(z, R) \subseteq X$ one has

$$\mathrm{p}_{z,R'}(f) \leq c\|f\|_{\partial D(z,R)} \tag{2.3.22}$$

for all $f \in \mathscr{O}(X)$ with a constant $c > 0$.

Proof: Let $z \in X$ and R a polyradius such that $D(z, R) \subseteq X$. Then for all $w \in D(z, R)^\circ$ one has

$$f(w) = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z)(w - z)^\alpha$$

by the convergent Taylor expansion for any holomorphic function $f \in \mathscr{O}(X)$. Hence the very rough estimate

$$|f(w)| \leq \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \right| \left|(w - z)^\alpha\right| \leq \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \right| R^\alpha = \mathrm{p}_{z,R}(f)$$

shows

$$\|f\|_{D(z,R)} \leq \mathrm{p}_{z,R}(f).$$

Since every compact subset $K \subseteq X$ is covered by finitely many open polydisks $D(z,R)^\circ$ with $D(z,R) \subseteq X$, the first statement follows. The second is slightly more subtle, as we use the Cauchy formula

$$f(z) = \frac{1}{(2\pi \mathrm{i})^n} \oint_{\partial \mathrm{B}_{r_1}(z_1)} \cdots \oint_{\partial \mathrm{B}_{r_n}(z_n)} \frac{f(w)}{(w_1 - z_1)\cdots(w_n - z_n)} \mathrm{d}w_1 \cdots \mathrm{d}w_n$$

for holomorphic functions. Here $z \in X$ and $R = (r_1, \ldots r_N)$ is a polyradius such that $D(z,R) \subseteq X$. One can now differentiate into the integral to obtain

$$\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) = \frac{\alpha!}{(2\pi \mathrm{i})^n} \oint_{\partial \mathrm{B}_{r_1}(z_1)} \cdots \oint_{\partial \mathrm{B}_{r_n}(z_n)} \frac{f(w)}{(w_1 - z_1)^{\alpha_1+1}\cdots(w_n - z_n)^{\alpha_n+1}} \mathrm{d}w_1 \cdots \mathrm{d}w_n.$$

This gives the rough estimate

$$\left|\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z)\right| \leq \frac{\alpha!}{(2\pi)^n}(2\pi r_1)\cdots(2\pi r_n)\frac{\max_{x\in\partial D(z,R)}|f(w)|}{r_1^{\alpha_1+1}\cdots r_n^{\alpha_n+1}} \leq \frac{\alpha!}{R^\alpha}\|f\|_{\partial D(z,R)}.$$

Inserting this into the definition of $\mathrm{p}_{r,R'}$ we get

$$\mathrm{p}_{r,R'}(f) = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!}\left|\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z)\right|(R')^\alpha \leq \sum_{\alpha=0}^{\infty}\left(\frac{R'}{R}\right)^\alpha \|f\|_{\partial D(z,R)},$$

where $(\frac{R'}{R})^\alpha = (\frac{r_1'}{r_1})^{\alpha_1}\cdots(\frac{r_n'}{r_n})^{\alpha_n}$. Since by assumption $r_i' < r_i$ for all $i = 1, \ldots, n$, this series converges and gives a finite constant $c > 0$. In fact, we have

$$c = \sum_{\alpha=0}^{\infty}\left(\frac{R'}{R}\right)^\alpha = \sum_{\alpha_1=0}^{\infty}\cdots\sum_{\alpha_n=0}^{\infty}\left(\frac{r_1'}{r_1}\right)^{\alpha_1}\cdots\left(\frac{r_n'}{r_n}\right)^{\alpha_1} = \frac{1}{1-\frac{r_1'}{r_1}}\cdots\frac{1}{1-\frac{r_n'}{r_n}}.$$

□

The consequence of this simple observation is now the following proposition, which is clarifying the structure of $\mathscr{O}(X)$ and the $\mathscr{O}$-topology:

Proposition 2.3.28 *Let $X \subseteq \mathbb{C}^n$ be a non-empty open subset.*

i.) The space of holomorphic functions $\mathscr{O}(X)$ is complete with respect to the $\mathscr{O}$-topology.

ii.) The $\mathscr{O}$-topology coincides with the $\mathscr{C}^0$-topology.

iii.) The differentiation

$$\frac{\partial}{\partial z^i}\colon \mathscr{O}(X) \longrightarrow \mathscr{O}(X) \tag{2.3.23}$$

is a continuous map with respect to the $\mathscr{O}$-topology.

iv.) The space $\mathscr{O}(X) \subseteq \mathscr{C}^0(X)$ is a closed subspace.

v.) The $\mathscr{C}^k$-topologies coincide on $\mathscr{O}(X)$ for all $k \in \mathbb{N}_0 \cup \{\infty\}$.

PROOF: The mutual estimates of the seminorms of the $\mathscr{O}$-topology and the $\mathscr{C}^0$-topology obtained in Lemma 2.3.27 show the second statement according to the general results of Proposition 2.2.15. We prove the first statement: Starting off we note that we can restrict ourselves to Cauchy sequences, since the $\mathscr{C}^0$-topology is first countable for $X \subseteq \mathbb{C}^n$. Thus let $f_i \in \mathscr{O}(X)$ for $i \in \mathbb{N}$ be given such that $(f_i)_{i\in\mathbb{N}}$ is a Cauchy sequence. Moreover, let $z \in X$ and a polyradius R be given with $D(z,R) \subseteq X$.

Denote by $a_{i,\alpha} = \frac{\partial^{|\alpha|} f_i}{\partial z^\alpha}(z)$ the sequence of the α-th Taylor coefficients of the functions f_i at z. Then for $\epsilon > 0$ we have an $N \in \mathbb{N}$ with

$$\mathrm{p}_{z,R}(f_i - f_j) = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} |a_{i,\alpha} - a_{j,\alpha}| R^\alpha < \epsilon$$

for $i, j \geq N$ by the Cauchy property. But this implies first that $a_{i,\alpha} \longrightarrow a_\alpha$ is convergent for $i \to \infty$. Moreover, since the seminorm $\mathrm{p}_{z,R}$ can be viewed as a weighted ℓ^1-norm on the sequence of Taylor coefficients, the sequence $(a_\alpha)_{\alpha \in \mathbb{N}_0^n}$ is again ℓ^1-summable, i.e.

$$\sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} |a_\alpha| R^\alpha < \infty. \tag{$*$}$$

In addition, we have ℓ^1-convergence of $(a_{i,\alpha})_{i \in \mathbb{N}, \alpha \in \mathbb{N}_0^n}$ to $(a_\alpha)_{\alpha \in \mathbb{N}_0^n}$: This is the completeness result of Proposition 2.3.3, *i.)*. Now $(*)$ simply means that the series

$$f(w) = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z)(w - z)^\alpha \tag{$**$}$$

converges for $w \in D(z, R)^\circ$ and defines a holomorphic function $f \in \mathscr{O}(D(z, R)^\circ)$. Moreover, on $D(z, R)^\circ$ we have pointwise convergence of the functions f_i to this f. But $z \in X$ was arbitrary. This implies that the pointwise limit f of the f_i exists on X and is locally given by a holomorphic function. Since being holomorphic is a local property, we conclude $f \in \mathscr{O}(X)$. Finally, the ℓ^1-convergence of the Taylor coefficients of the f_i to the sequence $(a_\alpha)_{\alpha \in \mathbb{N}_0^n}$ as above gives directly the $\mathrm{p}_{z,R}$-convergence of the f_i to f as in $(**)$ such that the a_α are the Taylor coefficients of f. This shows the first statement. The third is now very easy. We have for $z \in X$ with $R' < R$ such that $D(z, R) \subseteq X$ the estimate

$$\begin{aligned}
\mathrm{p}_{z,R'}\left(\frac{\partial f}{\partial z^i}\right) &= \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \left| \frac{\partial^{|\alpha+1|} f}{\partial z^\alpha \partial z^i}(z) \right| (R')^\alpha \\
&= \sum_{\alpha=0}^{\infty} \frac{\alpha_i + 1}{(\alpha + \mathrm{e}_i)!} \left| \frac{\partial^{|\alpha+1|} f}{\partial z^{\alpha + \mathrm{e}_i}}(z) \right| (R')^\alpha \\
&\leq c \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \right| R^\alpha
\end{aligned}$$

with a constant $c > 0$ such that for all n one has

$$(n + 1)(r_i')^n \leq c r_i^{n+1}.$$

Note that is possible to find such a $c > 0$, since $r_i' < r_i$ by assumption. But then we conclude

$$\mathrm{p}_{z,R'}\left(\frac{\partial f}{\partial z^i}\right) \leq c\,\mathrm{p}_{z,R}(f),$$

from which the continuity of $\frac{\partial}{\partial z^i}$ follows. The fourth part is now a simple consequence of the first and second part: $\mathscr{O}(X)$ is complete in the $\mathscr{C}^0$-topology. The last statement is clear by the third. For a holomorphic function we can express the partial derivatives in direction of real and imaginary parts by z-derivatives only. Hence the continuity of (2.3.23) gives the continuity of $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^i}$, as well. Then we conclude that the $\mathrm{p}_{K,\ell}$-seminorms are continuous in the $\mathscr{O}$-topology. Hence they define a coarser locally convex topology. But the $\mathscr{C}^k$-topology is also finer than the $\mathscr{C}^0$-topology, which coincides with the $\mathscr{O}$-topology. Thus they all coincide. □

As an amusing consequence the previously mentioned theorem of Weierstraß, see Proposition 1.3.12, becomes a trivial consequence, see also Exercise 2.5.44. In Exercises 2.5.45, 2.5.46, 2.5.47, 2.5.48 we discuss further properties of the locally convex space $\mathscr{O}(X)$.

2.3.6 The Schwartz Space

For the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ we have seen seminorms $\mathrm{r}_{m,\ell}$ in Example 2.2.10, capturing the defining properties of Schwartz functions. We take them to introduce a locally convex topology for $\mathscr{S}(\mathbb{R}^n)$:

Definition 2.3.29 (Topology of the Schwartz space) *The $\mathscr{S}$-topology on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is the locally convex topology induced by the seminorms $\{\mathrm{r}_{m,\ell}\}_{m,\ell\in\mathbb{N}_0}$.*

Since we have countably many seminorms, the $\mathscr{S}$-topology is first countable for this trivial reason. Moreover, we have estimates

$$\mathrm{r}_{m',\ell'} \leq \mathrm{r}_{m,\ell}, \tag{2.3.24}$$

whenever $m' \leq m$ and $\ell' \leq \ell$. Hence the above system of seminorms is already filtrating. There are several other, equivalent systems of seminorms for the $\mathscr{S}$-topology, which are sometimes useful, see again Exercise 2.5.21.

Proposition 2.3.30 *The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is a Fréchet space with respect to the $\mathscr{S}$-topology, which is finer than the $\mathscr{C}^\infty$-topology inherited from the inclusion $\mathscr{S}(\mathbb{R}^n) \subset \mathscr{C}^\infty(\mathbb{R}^n)$.*

PROOF: The $\mathscr{S}$-topology is first countably by its very definition, as already mentioned. If $K \subseteq \mathbb{R}^n$ is compact and $\ell \in \mathbb{N}_0$, then

$$\mathrm{p}_{K,\ell}(f) = \sup_{\substack{x\in K \\ |\alpha|\leq\ell}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| \leq \sup_{\substack{x\in \mathbb{R}^n \\ |\alpha|\leq\ell}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| = \mathrm{r}_{0,\ell}(f)$$

shows that the seminorms $\{\mathrm{p}_{K,\ell}\}_{K\subseteq\mathbb{R}^n \text{compact}, \ell\in\mathbb{N}_0, \ell\leq k}$ are dominated by the seminorms $\{\mathrm{r}_{m,\ell}\}_{m,\ell\in\mathbb{N}_0}$. Hence the $\mathscr{S}$-topology is finer than the $\mathscr{C}^\infty$-topology. In particular, it is Hausdorff as well. It remains to check the completeness, where it suffices to consider Cauchy sequences. Thus let $f_i \in \mathscr{S}(\mathbb{R}^n)$ with $i \in \mathbb{N}$ form a Cauchy sequence with respect to the $\mathscr{S}$-topology. Then $(f_i)_{i\in I}$ is a Cauchy sequence with respect to the coarser $\mathscr{C}^\infty$-topology, too, thus $\mathscr{C}^\infty$-convergent to some function $f \in \mathscr{C}^\infty(\mathbb{R}^n)$. As $\mathscr{C}^\infty$-convergence implies pointwise convergence of all derivatives, we get for all $x \in \mathbb{R}^n$, $m \in \mathbb{N}_0$, and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq \ell$ the estimate

$$(1+x^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) - \frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(x) \right| = \lim_{j\to\infty} (1+x^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|} f_j}{\partial x^\alpha}(x) - \frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(x) \right| \leq \limsup_{j\to\infty} \mathrm{r}_{m,\ell}(f_j - f_i) < \epsilon,$$

as soon as i is large enough. Taking the supremum over $x \in \mathbb{R}^n$ we conclude

$$\mathrm{r}_{m,\ell}(f - f_i) \leq \epsilon \tag{$*$}$$

for large enough i. But $\mathrm{r}_{m,\ell}(f) \leq \mathrm{r}_{m,\ell}(f - f_i) + \mathrm{r}_{m,\ell}(f_i)$ then implies $\mathrm{r}_{m,\ell}(f) < \infty$ and thus $f \in \mathscr{S}(\mathbb{R}^n)$. Then $(*)$ gives the $\mathscr{S}$-convergence to f, showing completeness. □

From now on, the Schwartz space will always be equipped with this $\mathscr{S}$-topology, turning it into a Fréchet space. As it turns out, it will be one of the most important function spaces. We collect a few further properties of this Fréchet space, some of which we have already obtained without having the corresponding conceptual interpretations:

Proposition 2.3.31 *Let $n \in \mathbb{N}$.*

i.) The subspace $\mathscr{C}_0^\infty(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n)$ is dense in the $\mathscr{S}$-topology.

ii.) The $\mathscr{S}$-topology is $\mathrm{Aff}(\mathbb{R}^n)$-invariant. More precisely, for $A \in \mathrm{GL}_n(\mathbb{R})$ and $a \in \mathbb{R}^n$ the pull-backs with A and τ_a are homeomorphisms of $\mathscr{S}(\mathbb{R}^n)$.

iii.) *For all* $i = 1, \ldots, n$ *the derivative* $\frac{\partial}{\partial x^i}$ *gives a continuous linear map*

$$\frac{\partial}{\partial x^i} \colon \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}(\mathbb{R}^n). \tag{2.3.25}$$

Moreover, for every $f \in \mathscr{S}(\mathbb{R}^n)$ *one has*

$$\lim_{a \to 0} \tau_a^* f = f \tag{2.3.26}$$

and

$$\frac{\partial f}{\partial x^i} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(\tau_{\epsilon e_i}^* f - f\big) \tag{2.3.27}$$

in the $\mathscr{S}$*-topology.*

iv.) *For all* $f \in \mathscr{O}_M(\mathbb{R}^n)$ *the multiplication by* f *is a continuous map*

$$f \colon \mathscr{S}(\mathbb{R}^n) \ni g \ \mapsto \ fg \in \mathscr{S}(\mathbb{R}^n). \tag{2.3.28}$$

Proof: For the first part, let $f \in \mathscr{S}(\mathbb{R}^n)$ be given. Moreover, let $\epsilon > 0$ and fix $m, \ell \in \mathbb{N}_0$. Furthermore we fix a bump function $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with

$$0 \le \chi \le 1, \quad \operatorname{supp} \chi \subseteq \mathrm{B}_2(0), \quad \text{and} \quad \chi\big|_{\mathrm{B}_1(0)} = 1.$$

For $R \in \mathbb{N}$ we then consider the function $\chi_R(x) = \chi(\frac{x}{R})$, which is still smooth, satisfies $\operatorname{supp} \chi_R \subseteq \mathrm{B}_{2R}(0)$ and $\chi_R\big|_{\mathrm{B}_R(0)} = 1$. Moreover, for every derivative we get

$$\frac{\partial^{|\alpha|} \chi_R}{\partial x^\alpha}(x) = \frac{1}{R^{|\alpha|}} \frac{\partial^{|\alpha|} \chi}{\partial x^\alpha}(\tfrac{x}{R}),$$

and thus

$$\sup_{x \in \mathbb{R}^n} \left| \frac{\partial^{|\alpha|}(1 - \chi_R)}{\partial x^\alpha}(x) \right| = \frac{1}{R^{|\alpha|}} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial^{|\alpha|}(1 - \chi)}{\partial x^\alpha}(\tfrac{x}{R}) \right| = \frac{c_\alpha}{R^{|\alpha|}}, \tag{$*$}$$

where $c_\alpha > 0$ is the supremum over the α-th derivative of the original function $1 - \chi$. After this preparation we can show that the sequence $(\chi_R f)_{R \in \mathbb{N}}$ converges to f in the $\mathscr{S}$-topology. Indeed, let $R \in \mathbb{N}$ be large enough such that

$$\sup_{x \in \mathbb{R}^n \setminus \mathrm{B}_R(0)^{\mathrm{cl}}} (1 + x^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| < \epsilon$$

for all $\alpha \in \mathbb{N}_0$ with $|\alpha| \le \ell$. This is possible, since we know $\mathrm{r}_{m+1,\ell}(f) < \infty$ and $(1 + x^2)^{\frac{m}{2}}$ grows to ∞ for $\|x\| \longrightarrow \infty$. Then

$$\begin{aligned}
&\sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \le \ell}} (1 + x^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) - \frac{\partial^{|\alpha|}(\chi_R f)}{\partial x^\alpha}(x) \right| \\
&\stackrel{(a)}{=} \sup_{\substack{x \in \mathbb{R}^n \setminus \mathrm{B}_R(0)^{\mathrm{cl}} \\ |\alpha| \le \ell}} (1 + x^2)^{\frac{m}{2}} \left| \frac{\partial^{|\alpha|}((1 - \chi_R) f)}{\partial x^\alpha}(x) \right| \\
&\stackrel{(b)}{=} \sup_{\substack{x \in \mathbb{R}^n \setminus \mathrm{B}_R(0)^{\mathrm{cl}} \\ |\alpha| \le \ell}} (1 + x^2)^{\frac{m}{2}} \left| \sum_{\beta = 0}^{\alpha} \binom{\alpha}{\beta} \frac{\partial^{|\beta|} f}{\partial x^\beta}(x) \frac{\partial^{|\alpha - \beta|}(1 - \chi_R)}{\partial x^{\alpha - \beta}}(x) \right|
\end{aligned}$$

$$\overset{(c)}{\leq} \sup_{\substack{x\in\mathbb{R}^n\setminus \mathrm{B}_R(0)^{\mathrm{cl}}\\ |\alpha|\leq\ell}} (1+x^2)^{\frac{m}{2}} 2^\ell \left|\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)\right| \max_{|\gamma|\leq\ell} \frac{c_\gamma}{R^{|\gamma|}}$$

$$\overset{(d)}{<} \mathit{const}\, \epsilon.$$

Here we used in (a) that χ_R is equal to 1 on $\mathrm{B}_R(0)$. In (b) we employ the Leibniz rule for higher derivatives, while (c) estimates the derivatives of $1-\chi_R$ according to $(*)$. Finally, in (d) we collect the numerical constants from before and obtain a constant depending on ℓ, but *not* on R any more. This shows $\mathrm{r}_{m,\ell}(f-\chi_R f) < \mathit{const}\,\epsilon$ for large enough n and hence $\chi_R f \to f$ in the $\mathscr{S}$-topology. As $\chi_R f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$, the first part follows. The second statement is a consequence of the technical Proposition 1.3.15, *i.)* and *ii.)*, where we estimated

$$\mathrm{r}_{m,\ell}(A^* f) \leq c_{m\ell}(A)\mathrm{r}_{m,\ell}(f)$$

and

$$\mathrm{r}_{m,\ell}(\tau_a^* f) \leq c_{m,\ell}(a)\mathrm{r}_{m,\ell}(f)$$

for all $A\in \mathrm{GL}_n(\mathbb{R})$ and $a\in\mathbb{R}^n$. Note that we have even a very good control over the two constants $c_{m,\ell}(A)$ and $c_{m,\ell}(a)$, leading to further properties of this group action. The third part is again contained in Proposition 1.3.15, now in *iii.)*. This gives directly the continuity statement (2.3.25). To arrive at (2.3.26) and (2.3.27) we have to work a bit more carefully. We have

$$\begin{aligned}
\mathrm{r}_{m,\ell}(\tau_a^* f - f) &= \sup_{\substack{x\in\mathbb{R}^n\\ |\alpha|\leq\ell}} (1+x^2)^{\frac{m}{2}} \left|\frac{\partial^{|\alpha|}\tau_a^* f}{\partial x^\alpha}(x) - \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)\right| \\
&= \sup_{\substack{x\in\mathbb{R}^n\\ |\alpha|\leq\ell}} (1+x^2)^{\frac{m}{2}} \left|\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x+a) - \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)\right| \\
&= \sup_{\substack{x\in\mathbb{R}^n\\ |\alpha|\leq\ell}} (1+x^2)^{\frac{m}{2}} \left|\sum\nolimits_{i=1}^n a^i \int_0^1 \left(\frac{\partial}{\partial x^i}\frac{\partial^{|\alpha|} f}{\partial x^\alpha}\right)(x+ta)\,\mathrm{d}t\right| \\
&\leq n\|a\| \sup_{\substack{x\in\mathbb{R}^n\\ |\alpha|\leq\ell}} \max_{\substack{t\in[0,1]\\ i=1,\ldots,n}} (1+x^2)^{\frac{m}{2}} \left|\frac{\partial^{|\alpha|+1} f}{\partial x^{\alpha+\mathrm{e}_i}}(x+ta)\right| \\
&\leq n\|a\| \sup_{\substack{x\in\mathbb{R}^n\\ |\alpha|\leq\ell}} \max_{\substack{t\in[0,1]\\ i=1,\ldots,n}} (1+x^2)^{\frac{m}{2}} \frac{\mathrm{r}_{m,\ell+1}(f)}{(1+(x+ta)^2)^{\frac{m}{2}}} \\
&\leq n\|a\|\mathrm{r}_{m,\ell+1}(f) \sup_{\substack{x\in\mathbb{R}^n\\ t\in[0,1]}} \left(\frac{1+x^2}{1+(x+ta)^2}\right)^{\frac{m}{2}} \\
&\leq q(\|a\|) n\|a\|\mathrm{r}_{m,\ell+1}(f),
\end{aligned}$$

where we have used the same estimate in the last step, as in the proof of Proposition 1.3.15, *ii.)*, with a quadratic function $q(\|a\|)$ of the norm of a, satisfying $q(0)=1$. In particular, $q(\|a\|)$ stays bounded for $a \longrightarrow 0$ and hence the prefactor $\|a\|$ brings this seminorm estimate to zero in the limit $a\to 0$. Thus (2.3.26) follows. The second assertion is shown similarly. We have

$$\mathrm{r}_{m,\ell}\left(\frac{1}{\epsilon}(\tau_{\epsilon\mathrm{e}_i}^* f - f) - \frac{\partial f}{\partial x^i}\right) = \sup_{\substack{x\in\mathbb{R}^n\\ |\alpha|\leq\ell}} (1+x^2)^{\frac{m}{2}} \left|\frac{1}{\epsilon}\left(\frac{\partial^{|\alpha|}\tau_{\epsilon\mathrm{e}_i}}{\partial x^\alpha}(x) - \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)\right) - \frac{\partial^{|\alpha|+1} f}{\partial x^{\alpha+\mathrm{e}_i}}(x)\right|$$

$$
\begin{aligned}
&= \sup_{\substack{x\in\mathbb{R}^n\\|\alpha|\le\ell}} (1+x^2)^{\frac{m}{2}} \left| \frac{1}{\epsilon}\left(\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x+\epsilon \mathrm{e}_i) - \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)\right) - \frac{\partial^{|\alpha|+1} f}{\partial x^{\alpha+\mathrm{e}_i}}(x)\right| \\
&= \sup_{\substack{x\in\mathbb{R}^n\\|\alpha|\le\ell}} (1+x^2)^{\frac{m}{2}} \left| \int_0^1 \frac{\partial^{|\alpha|+1} f}{\partial x^{\alpha+\mathrm{e}_i}}(x+t\epsilon \mathrm{e}_i)\,\mathrm{d}t - \frac{\partial^{|\alpha|+1} f}{\partial x^{\alpha+\mathrm{e}_i}}(x)\right| \\
&= \sup_{\substack{x\in\mathbb{R}^n\\|\alpha|\le\ell}} (1+x^2)^{\frac{m}{2}} \left| \int_0^1 \left(\frac{\partial^{|\alpha|+1} f}{\partial x^{\alpha+\mathrm{e}_i}}(x+t\epsilon \mathrm{e}_i) - \frac{\partial^{|\alpha|+1} f}{\partial x^{\alpha+\mathrm{e}_i}}(x)\right)\mathrm{d}t\right| \\
&= \epsilon \sup_{\substack{x\in\mathbb{R}^n\\|\alpha|\le\ell}} (1+x^2)^{\frac{m}{2}} \left| \int_0^1 t\epsilon \int_0^1 \frac{\partial^{|\alpha|+2} f}{\partial x^{\alpha+2\mathrm{e}_i}}(x+ts\epsilon \mathrm{e}_i)\,\mathrm{d}s\,\mathrm{d}t\right| \\
&\le \epsilon \sup_{\substack{x\in\mathbb{R}^n\\|\alpha|\le\ell}} (1+x^2)^{\frac{m}{2}} \sup_{t,s\in[0,1]} \left| \frac{\partial^{|\alpha|+2} f}{\partial x^{\alpha+2\mathrm{e}_i}}(x+ts\epsilon \mathrm{e}_i)\right| \\
&\le \epsilon \sup_{\substack{x\in\mathbb{R}^n\\|\alpha|\le\ell}} (1+x^2)^{\frac{m}{2}} \sup_{t\in[0,1]} \frac{\mathrm{r}_{m,\ell+2}(f)}{(1+(x+t\epsilon \mathrm{e}_i)^2)^{\frac{m}{2}}}.
\end{aligned}
$$

From here we can argue as before that the rational function of x and t stays bounded for $\epsilon \longrightarrow \infty$ and thus the supremum is finite. The prefactor ϵ gives then the desired convergence for (2.3.27). The last assertion is the statement of Proposition 1.3.19, interpreted now in terms of the seminorms of $\mathscr{S}(\mathbb{R}^n)$. ☐

Remark 2.3.32 It is less evident how the multipliers $\mathscr{O}_M(\mathbb{R}^n)$ and the convoluters $\mathscr{O}_C(\mathbb{R}^n)$ can be endowed with a locally convex topology. In fact, this is possible, but needs some more preparation and concepts we still need to develop. We also note that the $\mathscr{S}$-topology on $\mathscr{C}_0^\infty(\mathbb{R}^n)$ will not make this a complete space, even though it is finer than the $\mathscr{C}^\infty$-topology. We will still need a yet finer topology for the test functions $\mathscr{C}_0^\infty(\mathbb{R}^n)$.

2.4 Basic Constructions

Having the category of locally convex spaces, one may wonder which of the standard constructions from a categorical point of view are possible here. We will not give a completely systematic treatment, but collect a few important ones. Moreover, it will be crucial to understand the relations of these constructions with the concepts we have already introduced.

2.4.1 Subspace of Locally Convex Spaces

Implicitly, we have already made use of the induced topology on a subspace. For convenience we collect the most important properties in the following proposition:

Proposition 2.4.1 *Let V be a locally convex space and let $W \subseteq V$ be a subspace equipped with the subspace topology.*

i.) The subspace topology on W is locally convex. Moreover, a system of defining seminorms of it is given by the restriction of the continuous seminorms of V.

ii.) The subspace W is closed iff it is complete in the subspace topology.

iii.) The subspace topology of W is first countable if the topology of V was first countable.

iv.) A subset $B \subseteq W$ is bounded in the subspace topology iff $B \subseteq V$ is bounded.

v.) If V has the Heine-Borel property and W is closed, then W has the Heine-Borel property, too.

Proof: The first part was mentioned in Remark 2.2.23, *iv.)*, and discussed further in Exercise 2.5.26. The second statement holds for general topological vector spaces, see Proposition 2.1.20. The third part is also true for general topological spaces and their subspaces. For the fourth part one can e.g. use the characterization of bounded subsets from Proposition 2.1.34, *ii.)*, and the fact that a sequence $(w_n)_{n\in\mathbb{N}}$ with $w_n \in W$ converges to 0 in the subspace topology iff it converges to 0 in the ambient topology. Then the last statement is a combination of *iv.)* and the fact that for a closed subspace W a subset $A \subseteq W$ is closed in V iff it is closed in the subspace topology. □

Note that one can have a first countable subspace $W \subseteq V$ without V being first countable: e.g. every finite-dimensional subspace W is first countable according to Theorem 2.1.39, no matter what the topology of V is. Also note that a continuous seminorm p on V restricts to a continuous seminorm $\mathrm{p}\big|_W$, but the converse is not clear yet: can every continuous seminorm q on W be continuously extended to a continuous seminorm on V? We will see an answer to this question in Corollary 2.4.3. Finally, we note that subspaces can have the Heine-Borel property, even though the ambient V may fail to have it: again, the finite-dimensional subspaces provide quick examples.

The following technical lemma shows how one can obtain the open subsets in a closed subspace: first it is clear by definition that any open $U \subseteq W$ in a subspace $W \subseteq V$ is the intersection of an open $O \subseteq V$ with W, i.e. $U = W \cap O$. If U is convex in addition, we can construct a convex O with the following additional properties:

Lemma 2.4.2 *Let V be a locally convex space with a subspace $W \subseteq V$ equipped with the subspace topology.*

i.) If $U \subseteq W$ is an absolutely convex neighbourhood of zero, then there exists an absolutely convex neighbourhood $O \subseteq V$ of zero such that

$$U = W \cap O. \tag{2.4.1}$$

ii.) If in addition W is closed and $v \in V \setminus W$, then the neighbourhood O in i.) can be chosen such that $v \notin O$.

Proof: Since U is a zero neighbourhood, we find a zero neighbourhood $\tilde{U} \subseteq V$ with $\tilde{U} \cap W \subseteq U$. As V is locally convex, we can assume $\tilde{U}$ to be absolutely convex, see Proposition 2.1.31, *iii.)*. Now set

$$O = \operatorname{absconv}(U \cup \tilde{U}) \subseteq V.$$

This subset is absolutely convex by construction and a zero neighbourhood in V, since $\tilde{U} \subseteq O$. Since U and $\tilde{U}$ are both absolutely convex, every vector in O can be written as $\alpha x + \beta y$ with $|\alpha| + |\beta| \leq 1$ as well as $x \in U$ and $y \in \tilde{U}$. Now consider $z \in O \cap W$, then $\alpha x = z - \beta y \in W$, since $\beta y \in U \subseteq W$ and W is a subspace. Then $\alpha x \in W \cap \tilde{U} \subseteq U$, since $\tilde{U}$ and U are absolutely convex, leading to $z \in U$, again by absolute convexity of U. Hence $O \cap W \subseteq U$. The inclusion $U \subseteq O \cap W$ being obvious, we have proved the first statement. An alternative argument can be found in Exercise **??**. For the second, assume that W is closed. If $v \in V \setminus W$, then we can separate v from W, since V is T_3, see Proposition 2.1.8. Hence let $\tilde{U} \subseteq V$ be an open absolutely convex neighbourhood of zero with $\tilde{U} \cap W \subseteq U$, as before. Thanks to the T_3-property, we can assume that $v + \tilde{U} \cap W = \emptyset$ after a possible shrinking of $\tilde{U}$. As before we have for $O = \operatorname{absconv}(U \cup \tilde{U})$ the property $O \cap W = U$. Suppose $v \in O$, then $v = \alpha x + \beta y$ with $|\alpha| + |\beta| \leq 1$as well as $x \in U$ and $y \in \tilde{U}$. Since $\tilde{U}$ is absolutely convex, $\beta y = v - \alpha x \in v + \tilde{U}$, a contradiction since $\beta y \in W$. □

Corollary 2.4.3 *Let V be a locally convex space with a subspace $W \subseteq V$ equipped with the subspace topology. If p is a continuous seminorm on W, then there exists a continuous seminorm q on V with*

$$\mathrm{p} = \mathrm{q}\big|_W. \tag{2.4.2}$$

PROOF: Indeed, consider $U = \mathrm{B}_{\mathrm{p},1}(0) \subseteq W$, then $\mathrm{p}_U = \mathrm{p}$ by Lemma 2.2.20, since U is open. Moreover, U is absolutely convex and hence we find an absolutely convex neighbourhood $O \subseteq V$ of zero with $U = W \cap O$. Consider now the Minkowski functional of this subset, $\mathrm{q} = \mathrm{p}_O$. Then we have $\mathrm{B}_{\mathrm{q},1}(0) \subseteq O \subseteq \mathrm{B}_{\mathrm{q},1}(0)^{\mathrm{cl}}$, again by Lemma 2.2.20. From Remark 2.2.23, *iv.)*, we get

$$\mathrm{q}\Big|_W = \mathrm{p}_O\Big|_W = \mathrm{p}_{O\cap W} = \mathrm{p}_U = \mathrm{p}. \qquad \square$$

2.4.2 Initial Topologies and the Cartesian Product

The next construction is the initial topology determined by a collection of maps: here one wants to find a coarsest topology on the source, such that all these maps are continuous. In case of topological vector spaces and locally convex spaces, we arrive at the following situation: let V be a vector space with linear maps $\phi_i \colon V \longrightarrow V_i$ into topological vector spaces V_i for $i \in I$ with some index set I. Then the first question is whether the initial topology turns V into a topological vector space at all. If the maps ϕ_i are not linear, there is indeed no reason that this is the case. However, staying in the right category yields the following result:

Proposition 2.4.4 *Let V be a $\mathbb{K}$-vector space and let $\phi_i \colon V \longrightarrow V_i$ be linear maps to topological vector spaces V_i for $i \in I$ with some index set I.*

i.) The initial topology on V turns V into a topological vector space.

ii.) If $\mathcal{B}_i$ is a subbasis of neighbourhoods of zero in V_i for each $i \in I$, then

$$\mathcal{B} = \left\{ \phi_i^{-1}(U_i) \;\middle|\; U_i \in \mathcal{B}_i,\ i \in I \right\} \tag{2.4.3}$$

forms a subbasis of neighbourhoods of zero for the initial topology of V.

iii.) If all the V_i are locally convex, then the initial topology turns V into a locally convex space as well. More precisely, if $\mathcal{P}_i$ is a system of defining continuous seminorms for V_i for each $i \in I$, then

$$\mathcal{P} = \left\{ \mathrm{p}_i \circ \phi_i \;\middle|\; \mathrm{p}_i \in \mathcal{P}_i,\ i \in I \right\} \tag{2.4.4}$$

is a defining system of continuous seminorms for the initial topology of V.

PROOF: First we need to show that the vector space operations of V are continuous with respect to the initial topology. Since the ϕ_i are linear, we have

$$\phi_i \circ + = +_i \circ (\phi_i \times \phi_i) \quad \text{and} \quad \phi_i \circ \cdot = \cdot_i \circ (\mathrm{id}_{\mathbb{K}} \times \phi_i)$$

for all $i \in I$, where we write $+_i \colon V_i \times V_i \longrightarrow V_i$ and $\cdot_i \colon \mathbb{K} \times V_i \longrightarrow V_i$ for the vector space operations of V_i and we write $+$ and $\cdot$ for the corresponding ones of V. Since for the initial topology the maps ϕ_i are continuous and since $+_i$ and $\cdot_i$ are continuous for all $i \in I$, too, the universal property of the initial topology gives the continuity of $+$ and $\cdot$ for V as well, see Exercise 2.5.50. This shows the first part. For the second part we recall that $\phi_i^{-1}(U_i) \subseteq V$ is a neighbourhood of zero for each $U_i \in \mathcal{B}_i$, since ϕ_i is continuous and linear. Then the general properties of neighbourhoods and the initial topology show that we obtain a subbasis by taking all of these. The initial topology is the coarsest topology, such that all ϕ_i are continuous. For the third part, we consider first

$$\mathcal{B}_i = \left\{ \mathrm{B}_{\mathrm{p},\epsilon}(0) \;\middle|\; \mathrm{p} \in \mathcal{P}_i,\ \epsilon > 0 \right\},$$

which gives a subbasis of neighbourhoods of zero in V_i. Then $\mathcal{B}$ as in (2.4.3) is given by

$$\mathcal{B} = \left\{ \mathrm{B}_{\phi_i^*\mathrm{p},\epsilon}(0) \;\middle|\; \mathrm{p} \in \mathcal{P}_i,\ \epsilon > 0,\ i \in I \right\},$$

since $\phi_i^{-1}(\mathrm{B}_{\mathrm{p},\epsilon}(0)) = \mathrm{B}_{\phi_i^*\mathrm{p},\epsilon}(0)$. Hence we have a subbasis of absolutely convex neighbourhoods of zero for V. This shows that V is locally convex. Clearly, $\mathcal{P}$ as in (2.4.4) is the system of seminorms corresponding to this $\mathcal{B}$. $\square$

In general, the Hausdorff property of the V_i needs not to be inherited by V if the maps ϕ_i are not injective enough, see Exercise 2.5.51 for some further details. As we will see in the discussion of the weak* topology, the completeness of each V_i does *not* yet imply that the initial topology is complete again. Here one needs additional properties to guarantee the completeness. Finally, also the behaviour of bounded subsets is complicated in general, see Exercise 2.5.52.

One of the most important initial topologies is the product topology. Here one considers a collection $\{V_i\}_{i\in I}$ of topological vector spaces for some index set I. Then one endows their Cartesian product

$$V = \prod_{i\in I} V_i \tag{2.4.5}$$

with the initial topology induced by all the projection maps

$$\mathrm{pr}_i \colon V \longrightarrow V_i. \tag{2.4.6}$$

The Cartesian product preserves now many nice features of its factors. Some of the following results are completely topological and hold for topological spaces in general.

Proposition 2.4.5 *Let $\{V_i\}_{i\in I}$ be a collection of topological vector spaces for some index set I and endow $V = \prod_{i\in I} V_i$ with the product topology.*

i.) The Cartesian product V is Hausdorff iff all V_i are Hausdorff.

ii.) A net $(v^\alpha)_{\alpha\in J}$ in V is a Cauchy net iff the component nets $(v_i^\alpha = \mathrm{pr}_i(v^\alpha))_{\alpha\in J}$ are Cauchy nets in V_i for all $i \in I$.

iii.) A net $(v^\alpha)_{\alpha\in J}$ in V converges to some $v \in V$ iff the component nets $(v_i^\alpha = \mathrm{pr}_i(v^\alpha))_{\alpha\in J}$ converge to $v_i = \mathrm{pr}_i(v)$ in V_i for all $i \in I$.

iv.) The Cartesian product V is complete iff all the V_i are complete.

v.) The Cartesian product V is sequentially complete iff all the V_i are sequentially complete.

vi.) For all $i \in I$ the canonical inclusion map

$$\iota_i \colon V_i \longrightarrow V, \tag{2.4.7}$$

defined by its components

$$\iota_i(v)_j = \begin{cases} v & \textit{for } j = i \\ 0 & \textit{else}, \end{cases} \tag{2.4.8}$$

is a homeomorphism onto its image with inverse pr_i. The image $\iota_I(V_i) \subseteq V$ is closed.

Proof: The first part as well as the third hold for general topological spaces and their Cartesian products, see e.g. [19, Exercise 3.4.1 and Exercise 4.4.4]. Thus consider a net $(v^\alpha)_{\alpha\in J}$ in V, which is completely determined by its component nets $(v_i^\alpha)_{\alpha\in J}$ with $i \in I$. Assume first that $(v^\alpha)_{\alpha\in J}$ is a Cauchy net. Then for every neighbourhood $U_i \subseteq V_i$ of zero, the pre-image $\mathrm{pr}_i^{-1}(U_i) \subseteq V$ is a neighbourhood of zero. Thus we find an index $\alpha_0 \in J$ with

$$v^\alpha - v^\beta \in \mathrm{pr}_i^{-1}(U_i) \tag{$*$}$$

for all $\alpha, \beta \succcurlyeq \alpha_0$ by the Cauchy condition. But $(*)$ means

$$v_i^\alpha - v_i^\beta = \mathrm{pr}_i(v^\alpha - v^\beta) \in U_i.$$

Thus $(v_i^\alpha)_{\alpha\in J}$ is a Cauchy net in V_i, too. As $i \in I$ was arbitrary, we obtain " $\Longrightarrow$ ". The converse implication is shown along the same lines, since it always suffices to test the Cauchy condition for a subbasis of neighbourhoods of zero. Then the fourth and fifth parts are clear. For the sixth part we first note that the inclusion map is linear and a section of pr_i, i.e. we have

$$\mathrm{pr}_i \circ \iota_i = \mathrm{id}_{V_i}. \tag{$**$}$$

Now restricting pr_i to the image $\iota_i(V_i) \subseteq V$ gives a surjective continuous linear map onto V_i, which is injective thanks to $(**)$ and thus bijective. From the characterization of the Cartesian product topology it is clear that this restriction is still an open map. This implies that pr_i is a homeomorphism and its set-theoretic inverse ι_i is continuous. The closedness of the image can be seen e.g. using convergent nets and part *iii.)*. □

In other words, also in the context of topological vector spaces, a Cartesian product allows us to do everything componentwise. Note that the last part has an analogue for general Cartesian products, but requires a non-canonical choice to obtain a section. In our case we can set all other components to *zero* in (2.4.8). We will always identify V_i with its image in V.

First countability is not preserved in Cartesian products in general, as a general index set can be too big. However, for at most *countable* Cartesian products we obtain the following result:

Proposition 2.4.6 *Let I be an at most countable index set and let $\{V_i\}_{i \in I}$ be a collection of topological vector spaces.*

i.) The Cartesian product $V = \prod_{i \in I} V_i$ is first countable iff V_i is first countable for all $i \in I$.

ii.) The Cartesian product $V = \prod_{i \in I} V_i$ is a Fréchet space iff V_i is a Fréchet space for all $i \in I$.

PROOF: From Proposition 2.4.4, *ii.)*, it is clear that we get a countable neighbourhood basis for V out of at most countably many countable neighbourhood bases of the V_i. Hence V is first countable if all V_i are. Conversely, if V is first countable, then necessarily all its subspaces are first countable as well. From Proposition 2.4.5, *vi.)*, we infer that all V_i are first countable, completing the proof of the first part. Then the second one follows from the first and the result of Proposition 2.4.5, *iv.)*, and Proposition 2.4.4, *i.)*. □

Example 2.4.7 A first example is given by the sequence space $\mathrm{Map}(I, \mathbb{K})$ of all sequence, indexed by some set I. From the definition of its locally convex topology in Definition 2.3.1 we see that it is precisely the Cartesian product topology on

$$\mathrm{Map}(I, \mathbb{K}) = \mathbb{K}^I \tag{2.4.9}$$

of the I-fold Cartesian product of copies of $\mathbb{K}$. Hence the results of Proposition 2.3.3 can be now understood more conceptually from the point of view of the product topology. Note that, in particular, completeness is now obvious by the completeness of $\mathbb{K}$.

2.4.3 Final Topologies, Quotients, and the Direct Sum

While the initial topology for any system of linear maps $\{\phi_i \colon V \longrightarrow V_i\}_{i \in I}$ into topological vector spaces yields already a topological vector space V, this needs not to be true for the final topology. Here we need to restrict ourselves to the universal property within the class of topological vector spaces or locally convex spaces and linear maps. Since we are interested mainly in the locally convex setting anyway, we consider the final topology only in this category:

Definition 2.4.8 (Final locally convex topology) *Let $\phi_i \colon V_i \longrightarrow V$ be linear maps from locally convex spaces V_i into a vector space V for $i \in I$ for some index set I. Then the finest locally convex topology on V such that the maps ϕ_i are continuous is called the final locally convex topology.*

From the definition it is not completely clear, whether the final locally convex topology actually exists, let alone what properties it inherits from the locally convex spaces V_i. We collect a few first results on the final locally convex topology:

Proposition 2.4.9 *Let I be an index set and let $\phi_i\colon V_i \longrightarrow V$ be linear maps from locally convex spaces V_i to a vector space V for each $i \in I$.*

i.) The final locally convex topology on V exists.

ii.) A linear map $\psi\colon V \longrightarrow W$ into another locally convex space W is continuous with respect to the final locally convex topology on V iff $\psi \circ \phi_i\colon V_i \longrightarrow W$ is continuous for all $i \in I$.

iii.) A seminorm p on V is continuous with respect to the final locally convex topology iff the pullback-seminorm $\phi_i^*\mathrm{p} = \mathrm{p} \circ \phi_i$ is continuous on V_i for all $i \in I$.

Proof: Consider the coarsest locally convex topology on V, i.e. the only continuous seminorm on V is the zero seminorm. This is a (non-Hausdorff for $V \neq \{0\}$) locally convex topology, such that the maps ϕ_i are certainly continuous: any map into V is continuous for this topology. Hence the set of those locally convex topologies on V, for which all ϕ_i are continuous, is non-empty. Since the union of locally convex topologies, corresponding to the union of their sets of continuous seminorms, is still locally convex, we can simply take the maximal such locally convex topology. This shows the uniqueness and existence, thereby allowing us to speak of *the* final locally convex topology in the following. From the construction it is also clear that the final locally convex topology corresponds to the set of all those seminorms p on V, for which $\phi_i^*\mathrm{p}$ is a continuous seminorm on V_i for all $i \in I$. Indeed, if p is continuous with respect to the final locally convex topology, then the composition $\phi_i^*\mathrm{p} = \mathrm{p} \circ \phi_i$ is of course still continuous. On the other hand, if $\phi_i^*\mathrm{p}$ is continuous for all $i \in I$, we can include p into the set of continuous seminorms without spoiling the continuity of the ϕ_i, see Proposition 2.2.26, *ii.)*. Thus p already belonged to the continuous seminorms, showing the third part. Since again by the same proposition $\psi\colon V \longrightarrow W$ is continuous iff for all continuous seminorms q on W, the seminorm $\mathrm{q} \circ \psi$ is continuous on V, the last part follows as well. □

While the characterization of the continuous seminorms on V is fairly clear and explicit, it does not yet guarantee good properties of the final locally convex topology that easily. In fact, it can happen that V becomes non-Hausdorff, even though all V_i are Hausdorff, see e.g. Exercise 2.5.54.

The final locally convex topology can be quite complicated to understand in general. Instead of entering this discussion here, we focus on several important, but particular situations. The first is a quotient construction:

Proposition 2.4.10 *Let V be a locally convex spaces with a subspace $U \subseteq V$. Then the quotient V/U equipped with the final locally convex topology with respect to the projection $\mathrm{pr}\colon V \longrightarrow V/U$ enjoys the following properties:*

i.) For every seminorm p on V the definition

$$[p]([v]) = \inf\{\mathrm{p}(v+u) \mid u \in U\} \tag{2.4.10}$$

yields a seminorm on the quotient V/U. Every seminorm on V/U is of this form. In fact, for a seminorm q on V/U one has

$$\mathrm{q} = [\mathrm{pr}^*\,\mathrm{q}] = [\mathrm{q} \circ \mathrm{pr}]. \tag{2.4.11}$$

ii.) A seminorm $[\mathrm{p}]$ on V/U is continuous for the locally convex quotient topology iff it corresponds to a continuous seminorm p on V.

iii.) The quotient V/U is Hausdorff iff the subspace U is closed.

iv.) The quotient map $\mathrm{pr}\colon V \longrightarrow V/U$ is open.

PROOF: The check that [p] gives actually a well-defined seminorm is contained in Exercise 2.5.55. From the general statement in Proposition 2.4.9, *iii.)*, we see that a seminorm q on V/U is continuous iff $\mathrm{pr}^* \mathrm{q}$ is continuous on V. With the first part this shows *ii.)*. For the third one suppose first that V/U is Hausdorff. Then $\{0\} \subseteq V/U$ is closed and hence $\mathrm{pr}^{-1}(\{0\}) = U$ is closed by the continuity of pr, too. Conversely, let $U = U^{\mathrm{cl}}$ be closed. Assume that $[\mathrm{p}]([v]) = 0$ for all continuous seminorms [p] on V/U, which are of the form (2.4.10). Thus

$$\inf\{\mathrm{p}(v+u) \mid u \in U\} = 0$$

for a representative $v \in V$ of $[v]$ for all continuous seminorms p on V. Thus for a given continuous seminorm p on V and $\epsilon > 0$ we find $u_{\mathrm{p},\epsilon}$ with $\mathrm{p}(v - u_{\mathrm{p},\epsilon}) < \epsilon$. But this implies $v \in U^{\mathrm{cl}} = U$ and hence $[v] = 0$. This completes the third part. For the last part let p be a continuous seminorm on V with corresponding continuous seminorm [p] on V/U. For $v \in \mathrm{B}_{\mathrm{p},1}(0)$ we have

$$[\mathrm{p}]([v]) = \inf\{\mathrm{p}(v+u) \mid u \in U\} \leq \mathrm{p}(v) < 1,$$

and thus $[v] \in \mathrm{B}_{[\mathrm{p}],1}(0)$. Conversely, let $v \in V$ be such that $[v] \in \mathrm{B}_{[\mathrm{p}],1}(0)$. Then there is a representative $v+u \in [v]$ for which $\mathrm{p}(v+u) < 1$. In particular, $v+u \in \mathrm{B}_{\mathrm{p},1}(0)$. This shows $\mathrm{pr}(\mathrm{B}_{\mathrm{p},1}(0)) = \mathrm{B}_{[\mathrm{p}],1}(0)$. As the open unit balls with respect to all continuous seminorms constitute a basis of zero neighbourhoods in both cases and since the topology is translation invariant, this already implies that pr is open. □

Remarkably, we do *not* need to assume that V is Hausdorff in the third part. This allows to pass from a non-Hausdorff locally convex space to its Hausdorffization:

Definition 2.4.11 (Hausdorffization) *Let V be a not necessarily Hausdorff locally convex space. Then $V/\{0\}^{\mathrm{cl}}$ is called the Hausdorffization of V.*

Note that the closedness of the singleton $\{0\}$ and by translation invariance the closedness of all singletons is equivalent to the Hausdorff property. This does not hold for general topological spaces, but is a consequence of the T_3-property, which we have seen for topological *vector* spaces in Proposition 2.1.8.

Hausdorffization behaves well categorically:

Proposition 2.4.12 *Hausdorffization is a functor*

$$\mathsf{lcs} \longrightarrow \mathsf{LCS} \tag{2.4.12}$$

PROOF: This is Exercise 2.5.56. □

A particular example of the Hausdorffization is now given for the case of a single seminorm:

Example 2.4.13 (Hausdorffization) Let V be a vector space with a seminorm p. Then the locally convex topology on V defined by this single seminorm gives

$$\{0\}^{\mathrm{cl}} = \ker \mathrm{p}. \tag{2.4.13}$$

Thus on $V/\{0\}^{\mathrm{cl}} = V/\ker \mathrm{p}$ we have a Hausdorff locally convex topology, which is determined by a single seminorm [p]. This can only happen if [p] is a *norm* and not just a seminorm. In fact, this is exactly the passage from $\mathscr{L}^p(X)$ to $\mathrm{L}^p(X)$, as done in calculus, see Exercise 2.5.57.

A useful application of the locally convex quotient is now the following construction of seminorms:

Corollary 2.4.14 *Let V be a locally convex space with a closed subspace $U \subseteq V$ and $v \in V \setminus U$. Then there exists a continuous seminorm p on V with*

$$\mathrm{p}\Big|_U = 0 \quad \textit{and} \quad \mathrm{p}(v) > 0. \tag{2.4.14}$$

PROOF: Whether V is Hausdorff or not, the quotient V/U is Hausdorff by Proposition 2.4.10, *iii.)*. Since $v \notin U$, we know $[v] \neq 0$ and hence there is a continuous seminorm q on V/U with $\mathrm{q}([v]) > 0$. But then the pullback $\mathrm{p} = \mathrm{pr}^* \mathrm{q}$ is a continuous seminorm on V by Proposition 2.4.10, *i.)*, which will do the job. □

Again, we do not need to assume that V is Hausdorff itself. The closedness of U suffices here. In some sense, the corollary can be seen as a seminorm version of the statement that topological vector spaces are T_3, see again Proposition 2.1.8.

While the quotient construction is very useful to obtain Hausdorffness, completeness can be easily destroyed by passing to quotients. Here one even has the following general result that every topological (whether locally convex or not) vector space is actually a quotient, see [?]. We will not enter the discussion of such counterexamples, but present the following positive result on completeness. In the first countable case quotients behave nicely:

Proposition 2.4.15 *Let $U \subseteq V$ be a closed subspace in a Fréchet space V. Then V/U is a Fréchet space again. If V is even a Banach space, then V/U is a Banach space with respect to the quotient norm.*

PROOF: We fix an increasing system

$$\mathrm{p}_1 \leq \mathrm{p}_2 \leq \cdots \leq \mathrm{p}_n \leq \mathrm{p}_{n+1} \leq \cdots \tag{$*$}$$

of continuous seminorms on V, which defines the topology according to Theorem 2.2.37, *ii.)*. From the very definition of the quotient seminorms in (2.4.10) it is clear that this gives

$$[\mathrm{p}_1] \leq [\mathrm{p}_2] \leq \cdots \leq [\mathrm{p}_n] \leq [\mathrm{p}_{n+1}] \leq \cdots \tag{$**$}$$

as a defining system of seminorms for the quotient V/U. In particular, it follows that the locally convex quotient will be Hausdorff by the assumption $U = U^{\mathrm{cl}}$ and first countable. For the question of completeness, it thus suffices to consider a Cauchy sequence $([v_n])_{n\in\mathbb{N}}$ in V/U, where we have some representatives $v_n \in V$. Of course, the sequence $(v_n)_{n\in\mathbb{N}}$ may not be a Cauchy sequence, since we can replace v_n by $v_n + u_n$ with an arbitrary sequence $(u_n)_{n\in\mathbb{N}}$ in U. The idea is thus to find appropriate representatives, which actually do form a Cauchy sequence in V. Since $([v_n])_{n\in\mathbb{N}}$ is a Cauchy sequence, it is a Cauchy sequence for every continuous seminorm $[\mathrm{p}_n]$ on V/U by Proposition 2.2.30. Hence we can find an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of indices, such that for all $k \in \mathbb{N}$ we have

$$[\mathrm{p}_k]([v_{n_k}] - [v_{n_{k+1}}]) < \frac{1}{2^k},$$

by applying the Cauchy condition to the seminorm $[\mathrm{p}_k]$ with $\epsilon = \frac{1}{2^k}$. Now we choose a starting representative $w_1 \in [v_{k_1}]$. Then $[\mathrm{p}_1]([v_{n_1}] - [v_{n_2}]) = [\mathrm{p}_1]([w_1 - v_{n_2}]) < \frac{1}{2}$ allows to find a vector $u \in U$ such that $\mathrm{p}(w_1 - v_{n_2} - u) < \frac{1}{2}$ by the very definition of the quotient seminorm as an infimum in (2.4.10). Setting now $w_2 = v_{n_2} - u \in [v_{n_2}]$ gives $\mathrm{p}_1(w_1 - w_2) < \frac{1}{2}$ and thus the start for our inductive construction: repeating this argument gives a sequence $(w_k)_{k\in\mathbb{N}}$ with

$$w_k \in [v_{n_k}] \quad \text{and} \quad \mathrm{p}_k(w_k - w_{k+1}) < \frac{1}{2^k}$$

for $k \in \mathbb{N}$. From $(*)$ this implies that for $k_0 \leq k$ we have

$$\mathrm{p}_{k_0}(w_k - w_{k+1}) < \frac{1}{2^k},$$

too, and hence $(w_k)_{k\in\mathbb{N}}$ is a Cauchy sequence for the seminorm p_{k_0} by the usual telescope argument. Since k_0 is arbitrary, $(w_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in V and thus converging to some $w \in V$ by completeness. Hence the continuity of the canonical projection to V/U gives

$$\lim_{k\to\infty} [w_k] = [w].$$

But $[w_k] = [v_{n_k}]$ and thus a subsequence $([v_{n_k}])_{k\in\mathbb{N}}$ of the Cauchy sequence $([v_n])_{n\in\mathbb{N}}$ is converging to $[w]$. This implies the convergence

$$\lim_{n\to\infty} [v_n] = [w],$$

and hence the completeness of the quotient V/U. If V was even a Banach space with Banach norm $\|\cdot\|$, we know that the quotient seminorm of $\|\cdot\|$ is a norm already, see Example 2.4.13. Hence the quotient is a Banach space with respect to this quotient norm. □

The first countability is really crucial for this statement, as the above mentioned counterexamples demonstrate. Nevertheless, many of the relevant function spaces fall under this assumption and thus the result of Proposition 2.4.15 becomes very useful.

The next particular case of a final locally convex topology is the direct sum:

Definition 2.4.16 (Locally convex direct sum) *Let I be a non-empty index set and let V_i be a locally convex space for all $i \in I$. Then the direct sum*

$$V = \bigoplus_{i\in I} V_i \tag{2.4.15}$$

equipped with the final locally convex topology of all the canonical inclusion maps

$$\iota_i \colon V_i \longrightarrow V \tag{2.4.16}$$

is called the locally convex direct sum.

Here the inclusion maps ι_i in (2.4.16) are of course the obvious ones, placing a vector $v_i \in V_i$ at the i-th position in the (algebraic) direct sum of all the involved vector spaces. In particular, every map ι_i is injective. This allows to identify V_i with its image in V, at least algebraically. Some of the first properties of the locally convex direct sum are now collected in the following proposition:

Proposition 2.4.17 *Let I be a non-empty index set and let $\{V_i\}_{i\in I}$ be a collection of locally convex spaces with locally convex direct sum $V = \bigoplus_{i\in I} V_i$.*

i.) A seminorm p on V is continuous iff the restrictions $\mathrm{p}\big|_{V_i}$ are continuous for all $i \in I$. Conversely, given continuous seminorms $\{\mathrm{p}_i \colon V_i \longrightarrow \mathbb{R}_0^+\}_{i\in I}$ for all V_i, then the seminorm

$$\mathrm{p} = \sum_{i\in I} \mathrm{p}_i \tag{2.4.17}$$

is well-defined on V and continuous. The set of all such seminorms yields a defining system of continuous seminorms for the locally convex direct sum.

ii.) The inclusion maps $\iota_i \colon V_i \longrightarrow V$ are homeomorphisms onto their images for all $i \in I$. Moreover, $\iota_i(V_i) \subseteq V$ is closed if all V_i are Hausdorff.

iii.) *The locally convex direct sum V is Hausdorff iff all V_i are Hausdorff.*

iv.) *The topology of the locally convex direct sum is finer than the inherited product topology from the canonical inclusion $V \subseteq \prod_{i\in I} V_i$.*

v.) *The locally convex direct sum is complete Hausdorff iff V_i is complete Hausdorff for all $i \in I$.*

vi.) *If I is finite, then the locally convex direct sum coincides with the Cartesian product $\prod_{i\in V_i}$.*

PROOF: From the general characterization of continuous seminorms as in Proposition 2.4.9, *iii.)*, we see that p is continuous iff $\mathrm{p}_i = \iota_i^*\mathrm{p} = \mathrm{p}\big|_{V_i}$ is continuous. Since in a direct sum, every vector $v \in V$ has only finitely many components v_i different from zero, the definition (2.4.17) yields a well-defined seminorm on V. Clearly, we have $\mathrm{p}\big|_{V_i} = \mathrm{p}_i$ for this p and hence p is continuous. Finally, let q be an arbitrary continuous seminorm on V, then

$$\mathrm{q}(v) = \mathrm{q}\Big(\sum\nolimits_{i\in I} v_i\Big) \le \sum_{i\in I} \mathrm{q}\big|_{V_i}(v_i) = \mathrm{p}(v)$$

for $v = \sum_{i\in I} v_i$ with p defined according to (2.4.17) from the components $\mathrm{p}_i = \mathrm{q}\big|_{V_i}$. Thus p dominates q, proving that the seminorms obtained from (2.4.17) yield already a defining system. For the second part we see that the induced topology on $\iota_i(V_i) \subseteq V$ coincides with the original topology, since we simply recover precisely all continuous seminorms of V_i from the process discussed in *i.)*. Next, since each V_i is Hausdorff, we can characterize the vectors v in $\iota_i(V_i) \subseteq V$ as those, for which $\mathrm{p}_j(v) = 0$ for all $j \ne i$ and all continuous seminorms p_j of V_j extended to V. Since $\ker \mathrm{p}_j \subseteq V$ is closed by the continuity according to *i.)*, also the intersection of all these subspaces is closed, showing that $\iota_i(V_i) \subseteq V$ is closed, which is *ii.)*. From the first part, the third follows immediately. Now consider the projection $\mathrm{pr}_i\colon V \longrightarrow V_i$ for $i \in I$. Then we have from linear algebra

$$\mathrm{pr}_i \circ \iota_j = \delta_{ij}\, \mathrm{id}_{V_i}$$

for all $j \in I$, which is certainly a continuous linear map $V_j \longrightarrow V_i$. By the universal property of the final locally convex topology, $\mathrm{pr}_i\colon V \longrightarrow V_i$ is therefore continuous, see the general result in Proposition 2.4.9, *ii.)*. But then the identity map including V into the Cartesian product $\prod_{i\in I} V_i$ is continuous by the universal property of the product topology. Thus the product topology is coarser than the topology of the locally convex direct sum, showing *iv.)*. Alternatively, every continuous seminorm on $\prod_{i\in I} V_i$ from the defining system (2.4.4) is clearly contained in the system described in *i.)*: they are the single p_i extended to V by zero. This also shows the last part, since here already $V = \prod_{i\in I} V_i$ as vector spaces and the defining systems of seminorms (2.4.4) and (2.4.17) yield the same topology, as the sums in (2.4.17) are finite in this case. It remains to check the fifth part. Suppose first that V is complete and Hausdorff. Then every subspace is Hausdorff and closed subspaces are complete again. Hence V_i is complete by the second part. Conversely, assume V_i is complete and Hausdorff for all $i \in I$. Let $(v_\alpha)_{\alpha\in J}$ be a Cauchy net in V. Since the projections $\mathrm{pr}_i\colon V \longrightarrow V_i$ are continuous for all $i \in I$, also the component nets $(v_{\alpha,i} = \mathrm{pr}_i(v_\alpha))_{\alpha\in J}$ form a Cauchy net in V_i for all $i \in I$. By completeness we have convergence to some $v_i = \lim_{\alpha\in J} v_{\alpha,i} \in V_i$. Denote by $I_0 = \{i \in I \mid v_i \ne 0\}$ the set of indices, for which these components are non-trivial. We can choose continuous seminorms p_i on V_i such that

$$\mathrm{p}_i(v_i) = 2$$

for all $i \in I_0$, according to the Hausdorff property of V_i. Now $\mathrm{p} = \sum_{i\in I_0} \mathrm{p}_i$ is a continuous seminorm on V according to the first part. As $(v_\alpha)_{\alpha\in J}$ is a Cauchy sequence, there is an index $\alpha_0 \in J$ with

$$1 > \mathrm{p}\big(v_\alpha - v_\beta\big) = \sum_{j\in I_0} \mathrm{p}_j\big(v_{\alpha,j} - v_{\beta,j}\big) \ge \mathrm{p}_i\big(v_{\alpha,i} - v_{\beta,i}\big)$$

for $\alpha, \beta \succcurlyeq \alpha_0$ and $i \in I_0$. By convergence of $(v_{\alpha,i})_{\alpha\in J}$ in V_i we can take the limit over $\beta \in J$ to obtain $\mathrm{p}_i\big(v_{\alpha,i} - v_i\big) < 1$ for all $\alpha \succcurlyeq \alpha_0$ and $i \in I_0$. The reverse triangle inequality implies now

$$\mathrm{p}_i(v_{\alpha_0,i}) \geq \mathrm{p}_i(v_i) - \mathrm{p}_i\big(v_{\alpha_0,i} - v_i\big) \geq 2 - 1 = 1$$

for all $i \in I_0$. Finally, adding over $i \in I_0$ results in

$$\infty > \mathrm{p}(v_{\alpha_0}) = \sum_{i\in I_0} \mathrm{p}_i(v_{\alpha_0,i}) \geq \sum_{i\in I_0} 1 = \big|I_0\big|,$$

wherefore the set I_0 is finite. This means that the limit of the Cauchy net $(v_\alpha)_{\alpha\in J}$ is an element in the direct sum $\bigoplus_{i\in I} V_i$, showing the completeness. □

It follows that for infinite I and non-zero vector spaces V_i the direct sum has a *strictly* finer topology than the one inherited from the Cartesian product. In fact, for the Cartesian product topology,

$$\bigoplus_{i\in I} V_i \subseteq \prod_{i\in I} V_i \tag{2.4.18}$$

is always a dense subspace, see Exercise 2.5.58.

Example 2.4.18 (The space $c_{\circ\circ}(I)$) Let I be a non-empty index set and consider once again $c_{\circ\circ}(I)$. We can view this as the direct sum of I copies of the scalars $\mathbb{K}$. Hence we can endow $c_{\circ\circ}(I)$ with the topology of this locally convex direct sum. Since $\mathbb{K}$ is complete, also $c_{\circ\circ}(I)$ is complete: we have found the appropriate locally convex topology for $c_{\circ\circ}(I)$. Moreover, since all seminorms on $\mathbb{K}$ are continuous for trivial reasons, simply *all* seminorms on $c_{\circ\circ}(I)$ are continuous by Proposition 2.4.17, *i.)*. Hence $c_{\circ\circ}(I)$ carries the *finest* possible locally convex topology.

Of course, on every vector space we have the finest locally convex topology, simply by declaring all seminorms to be continuous. Here we have the following characterization:

Corollary 2.4.19 *Let V be a locally convex space over $\mathbb{K}$. Then the following statements are equivalent:*

i.) The topology is the finest locally convex topology.

ii.) The topology is the final locally convex topology with respect to the inclusions of the finite-dimensional subspaces of V.

iii.) For every basis $B \subseteq V$ one has the isomorphism

$$V \cong \bigoplus_{b\in B} \mathbb{K}b \cong c_{\circ\circ}(B) \tag{2.4.19}$$

of locally convex spaces.

PROOF: We have already discussed the equivalence *i.)* $\iff$ *iii.)*. The topology in *ii.)* is clearly coarser than the final locally convex topology obtained from including only the one-dimensional subspaces originating from a choice of a basis, which is the finest. If p is a seminorm on V, then the restriction to a finite-dimensional subspace is continuous, as on a finite-dimensional subspace any seminorm is continuous by Theorem 2.1.39. Thus p is continuous with respect to the final locally convex topology defined in *ii.)*, see again Proposition 2.4.9, *iii.)*. This proves that every seminorm is continuous. Thus also *ii.)* yields the finest locally convex topology. □

2.4.4 The Completion

While the more obvious way to construct the completion of a Hausdorff topological vector space V consists in considering the space of all Cauchy nets indexed by a basis of neighbourhoods of zero modulo zero nets, see the discussion around Theorem 2.1.22, in the case of locally convex spaces there is a simpler construction. We assume that the completion of a normed space to a Banach space is known: this can be done along the above lines of arguments, but relying on Cauchy sequences instead of Cauchy nets, see also Exercise 2.5.59. A reinterpretation of Example 2.4.13 gives now the following result:

Lemma 2.4.20 *Let V be a locally convex space and let p be a continuous seminorm on V.*

i.) On the quotient $V/\ker \mathrm{p}$ the quotient seminorm

$$\|\cdot\|_{\mathrm{p}} = [\mathrm{p}] \tag{2.4.20}$$

is a norm.

ii.) The quotient map $V \longrightarrow V/\ker \mathrm{p}$ is continuous.

iii.) For all $v \in V$ one has

$$\|[v]\|_{\mathrm{p}} = \mathrm{p}(v). \tag{2.4.21}$$

PROOF: We consider V equipped with the locally convex topology defined by p alone: clearly this is coarser than the original topology of V and hence the identity map

$$\mathrm{id}\colon V_{\mathrm{orig}} \longrightarrow (V, \{\mathrm{p}\}) \tag{$*$}$$

is continuous. In this coarser topology we have $\{0\}^{\mathrm{cl}} = \ker \mathrm{p}$ by Example 2.4.13, leading to a norm $\|\cdot\|_{\mathrm{p}}$. Thus also the second part follows by $(*)$, since the quotient map is always continuous. Finally, $\big\|[v]\big\|_{\mathrm{p}} \le \mathrm{p}(v)$ is clear by definition of the quotient norm (2.4.10). Conversely, assume the infimum $\big\|[v]\big\|_{\mathrm{p}}$ is strictly smaller than $\mathrm{p}(v)$. Then there exists a $w \in \ker \mathrm{p}$ with $\mathrm{p}(v + w) < \mathrm{p}(v)$ and hence

$$\mathrm{p}(v) \le \mathrm{p}(v + w) + \mathrm{p}(w) = \mathrm{p}(v + w) < \mathrm{p}(v),$$

a contradiction. □

This motivates the following definition:

Definition 2.4.21 (Local Banach space) *Let V be a locally convex space with a continuous seminorm p. Then the Banach space completion*

$$V_{\mathrm{p}} = \widehat{V/\ker \mathrm{p}} \tag{2.4.22}$$

of the normed space $(V/\ker \mathrm{p}, \|\cdot\|_{\mathrm{p}})$ is called the local Banach space of V at p. Its norm will still be denoted by $\|\cdot\|_{\mathrm{p}}$.

Since a Banach space completion is fairly easy to obtain, see also Exercise 2.5.60 for yet another alternative construction, we use the local Banach spaces to construct the completion of V. We consider the set of all continuous seminorms $\mathcal{P}_V$ of V and from the Cartesian product

$$\mathcal{V} = \prod_{\mathrm{p} \in \mathcal{P}_V} V_{\mathrm{p}} \tag{2.4.23}$$

of all local Banach spaces of V. The product of all the corresponding quotient maps $V \ni v \mapsto [v]_{\mathrm{p}} \in V_{\mathrm{p}}$ gives a linear map

$$\iota\colon V \ni v \ \mapsto \ ([v]_{\mathrm{p}})_{\mathrm{p} \in \mathcal{P}_V} \in \prod_{\mathrm{p} \in \mathcal{P}_V} V_{\mathrm{p}}. \tag{2.4.24}$$

If V is Hausdorff this results in an embedding:

Proposition 2.4.22 *Let V be a Hausdorff locally convex space.*

i.) *The canonical map $\iota\colon V \longrightarrow \prod_{\mathrm{p}\in\mathcal{P}_V} V_{\mathrm{p}}$ is an embedding with respect to the Cartesian product topology on the right hand side.*

ii.) *Taking the closure of the image gives a completion $(\iota(V)^{\mathrm{cl}}, \iota)$ of V.*

PROOF: Since V is Hausdorff, we have $v = 0$ iff $\mathrm{p}(v) = 0$ for all $\mathrm{p} \in \mathcal{P}_V$. Hence the map ι is injective. Moreover, from Lemma 2.4.20, *ii.)*, we see that the collection

$$\tilde{\mathcal{P}}_V = \left\{ \tilde{\mathrm{p}} = \|\cdot\|_{\mathrm{p}} \circ \mathrm{pr}_{\mathrm{p}} \;\middle|\; \mathrm{p} \in \mathcal{P}_V \right\}$$

of seminorms on the Cartesian product reduces to $\mathcal{P}_V$, when restricted to the image $\iota(V) \subseteq \prod_{\mathrm{p}\in\mathcal{P}_V} V_{\mathrm{p}}$. Here we denote the quotient maps by $\mathrm{pr}_{\mathrm{p}}\colon V \longrightarrow V_{\mathrm{p}}$. But $\tilde{\mathcal{P}}_V$ is a generating set of seminorms for the product topology according to the general properties of initial topologies, as discussed in Proposition 2.4.4, *iii.)*. Thus ι is an embedding. For the second part we note that the product of Banach spaces might no longer a Banach space, but stays complete thanks to Proposition 2.4.5, *iv.)*. Then the closure $\iota(V)^{\mathrm{cl}}$ inside $\prod_{\mathrm{p}\in\mathcal{P}_V} V_{\mathrm{p}}$ is complete as well. Hence we have found a complete space containing V as dense subspace, which is equivalent to a completion by Exercise 2.5.6. □

Note that this way we have obtained the proof of Theorem 2.2.31: completions of Hausdorff locally convex spaces exist and are again locally convex. Moreover, in the above construction we can replace the set of all continuous seminorms by any defining subset. This allows us to shrink $\mathcal{P}_V$ considerably. As a consequence, we get the following corollary:

Corollary 2.4.23 *Let V be a Hausdorff locally convex space.*

i.) *If V is a normed space, its completion $\hat{V}$ is a Banach space.*

ii.) *If V is first countable, its completion $\hat{V}$ is a Fréchet space.*

Of course, this first statement is somewhat redundant, as we have already used the completion of normed spaces to Banach spaces in the construction. Nevertheless, the construction reduces to the completion of normed spaces, once we can take as defining system of seminorms $\mathcal{P}_V = \{\|\cdot\|\}$. Finally, we note that the construction itself is functorial in every step. This gives yet another proof of the functoriality of the completion, see also Exercise 2.5.7.

2.4.5 Projective and Inductive Limits

Let $(I, \preccurlyeq)$ be a directed set. In this section we are interested in two categorical constructions for functors

$$\Phi\colon I \longrightarrow \mathsf{lcs}, \tag{2.4.25}$$

where we view I as a category in the usual way: the objects are elements of I and we have a single morphism from i to j iff $i \preccurlyeq j$. This determines the composition of morphisms uniquely and clearly leads to a category. In fact, a pre-ordered set is already sufficient. We discuss this in Exercise 2.5.61.

Now a functor (2.4.25) consists of the following data: for every index $i \in I$ we are given a locally convex space $V_i = \Phi(i)$. Second, whenever $i \preccurlyeq j$, we are given a continuous linear map

$$\Phi(i \preccurlyeq j) = \phi_{i\preccurlyeq j}\colon V_i \longrightarrow V_j \tag{2.4.26}$$

if the functor Φ is covariant. In the contravariant case, i.e. for a covariant functor $I^{\mathrm{opp}} \longrightarrow V$, we have

$$\Phi(i \preccurlyeq j) = \phi_{i\preccurlyeq j}\colon V_j \longrightarrow V_i \tag{2.4.27}$$

instead. The functoriality then means

$$\phi_{j \preccurlyeq k} \circ \phi_{i \preccurlyeq j} = \phi_{i \preccurlyeq k} \quad \text{and} \quad \phi_{i \preccurlyeq i} = \mathrm{id}_{V_i} \tag{2.4.28}$$

in the covariant case, while

$$\phi_{i \preccurlyeq j} \circ \phi_{j \preccurlyeq k} = \phi_{i \preccurlyeq k} \quad \text{and} \quad \phi_{i \preccurlyeq i} = \mathrm{id}_{V_i} \tag{2.4.29}$$

in the contravariant case. These two situations can symbolically be depicted as diagrams

$$\begin{tikzcd} \cdots \arrow[r] & V_i \arrow[r, "\phi_{i \preccurlyeq j}"] \arrow[rr, bend right, "\phi_{i \preccurlyeq k}"'] & V_j \arrow[r, "\phi_{j \preccurlyeq k}"] & V_k \arrow[r] & \cdots \end{tikzcd} \tag{2.4.30}$$

and

$$\begin{tikzcd} \cdots \arrow[r] & V_k \arrow[r, "\phi_{j \preccurlyeq k}"] \arrow[rr, bend right, "\phi_{i \preccurlyeq k}"'] & V_j \arrow[r, "\phi_{i \preccurlyeq j}"] & V_i \arrow[r] & \cdots \end{tikzcd} . \tag{2.4.31}$$

We call (2.4.30) also an *inductive system*, while (2.4.31) is referred to as a *projective system.* When asking now for a categorical limit of (2.4.30) or (2.4.31), one is interested in the object standing at the "end" or the "beginning", respectively. This idea is made precise in the following definition, which we spell out directly for locally convex spaces. Note, however, that the target category lcs in (2.4.25) can be replaced by any other category.

Definition 2.4.24 (Inductive limit) *Let I be a directed set and let $\Phi\colon I \longrightarrow \mathsf{lcs}$ be a covariant functor. Then a locally convex space V together with continuous linear maps $\phi_i\colon \Phi(i) = V_i \longrightarrow V$ is called inductive (or direct) limit of Φ if the following holds:*

i.) For all $i \preccurlyeq j$ on has

$$\phi_i = \phi_j \circ \phi_{i \preccurlyeq j}, \tag{2.4.32}$$

where $\phi_{i \preccurlyeq j} = \Phi(i \preccurlyeq j)$ is the image of the unique morphism $i \longrightarrow j$.

ii.) The collection $(V, \{\phi_i\}_{i \in I})$ is universal with respect to (2.4.32)*, i.e. if $(\tilde{V}, \{\tilde{\phi}_i\}_{i \in I})$ is another choice satisfying* (2.4.32)*, then there exists a unique continuous linear map*

$$\phi\colon V \longrightarrow \tilde{V} \tag{2.4.33}$$

with $\tilde{\phi}_i = \phi \circ \phi_i$ for all $i \in I$.

In category theory such a limit is referred to as a *colimit* in the recent literature. Nevertheless, we will stick to the more traditional notion of an inductive limit. Dually, the contravariant version reads as follows:

Definition 2.4.25 (Projective limit) *Let I be a directed set and let $\Phi\colon I \longrightarrow \mathsf{lcs}$ be a contravariant functor. Then a locally convex space V together with continuous linear maps $\phi_i\colon V \longrightarrow V_i = \Phi(i)$ is called projective (or inverse) limit of Φ if the following holds:*

i.) For all $i \preccurlyeq j$ on has

$$\phi_i = \phi_{i \preccurlyeq j} \circ \phi_j, \tag{2.4.34}$$

where $\phi_{i \preccurlyeq j} = \Phi(i \preccurlyeq j)$ is the image of the unique morphism $i \longrightarrow j$.

ii.) The collection $(V, \{\phi_i\}_{i\in I})$ *is universal with respect to* (2.4.34)*, i.e. if* $(\tilde{V}, \{\tilde{\phi}_i\}_{i\in I})$ *is another choice satisfying* (2.4.34)*, then there exists a unique continuous linear map*

$$\phi\colon \tilde{V} \longrightarrow V \tag{2.4.35}$$

with $\tilde{\phi}_i = \phi_i \circ \phi$ *for all* $i \in I$*.*

Again, the modern notation is to call such a universal object simply a *limit* of Φ. That being said, we stick to the more traditional notion of a projective limit in the following.

If an inductive system admits an inductive limit V, we also write

$$V = \varinjlim_{i\in I} V_i = \varinjlim V_i = \operatorname{ind}\lim_{i\in I} V_i \tag{2.4.36}$$

and omit the linear maps $\phi_i\colon V_i \longrightarrow V$ (and possibly the index set I in the notation). Similarly, we write

$$V = \varprojlim_{i\in I} V_i = \varprojlim V_i = \operatorname{proj}\lim_{i\in I} V_i \tag{2.4.37}$$

for a projective limit V provided the projective system has such a limit.

As to be expected, the universal property immediately gives the uniqueness of the projective or inductive limit up to a unique isomorphism:

Lemma 2.4.26 *Let* I *be a directed set.*

i.) For a covariant functor $\Phi\colon I \longrightarrow \mathsf{lcs}$ *any two inductive limits* $(V, \{\phi_i\}_{i\in I})$ *and* $(\tilde{V}, \{\tilde{\phi}_i\}_{i\in I})$ *are isomorphic with a unique isomorphism* $\Xi\colon V \longrightarrow \tilde{V}$ *such that*

$$\tilde{\phi}_i = \Xi \circ \phi_i \tag{2.4.38}$$

for all $i \in I$*.*

ii.) For a contravariant functor $\Phi\colon I \longrightarrow \mathsf{lcs}$ *any two projective limits* $(V, \{\phi_i\}_{i\in I})$ *and* $(\tilde{V}, \{\tilde{\phi}_i\}_{i\in I})$ *are isomorphic with a unique isomorphism* $\Xi\colon V \longrightarrow \tilde{V}$ *such that*

$$\phi_i = \tilde{\phi}_i \circ \Xi \tag{2.4.39}$$

for all $i \in I$*.*

PROOF: The proof is entirely diagrammatic: suppose we have two inductive limits as indicated. Since V is an inductive limit, we get a unique continuous linear map $\Xi\colon V \longrightarrow \tilde{V}$ with (2.4.38). Since $\tilde{V}$ is also an inductive limit, we get a unique continuous linear map $\tilde{\Xi}\colon \tilde{V} \longrightarrow V$ with

$$\phi_i = \tilde{\Xi} \circ \tilde{\phi}_i \tag{2.4.40}$$

for all $i \in I$. But then $\phi_i = \tilde{\Xi} \circ \Xi \circ \phi_i$ holds for all $i \in I$. Applying the universal property to V itself, we see that $\mathrm{id}_V = \tilde{\Xi} \circ \Xi$ is the only possible continuous linear map with this feature. Analogously, we get $\mathrm{id}_{\tilde{V}} = \Xi \circ \tilde{\Xi}$, proving the first statement. The second is analogous. $\square$

Thus we can speak of *the* inductive or projective limit by some slight abuse of language. As usual for universal objects, it is the existence we have to worry about. This can not be shown by purely categorical means. We start with the projective limit, which is somewhat easier:

Theorem 2.4.27 (Projective limit) *Let* I *be a directed set and let* $(\{V_i\}_{i\in I}, \{\phi_{i\preccurlyeq j}\colon V_j \longrightarrow V_i\}_{i\preccurlyeq j})$ *be a projective system of locally convex spaces.*

i.) *The projective limit* $\varprojlim V_i$ *exists. More precisely, it is given by*

$$\varprojlim_{i\in I} V_i = \left\{ (v_i)_{i\in I} \in \prod\nolimits_{i\in I} V_i \;\middle|\; \phi_{i\preccurlyeq j}(v_j) = v_i \textit{ for all } i \preccurlyeq j \right\} \tag{2.4.41}$$

with the subspace topology inherited from the Cartesian product together with the maps

$$\phi_i = \mathrm{pr}_i \big|_{\varprojlim V_i} \colon \varprojlim_{i\in I} V_i \longrightarrow V_i. \tag{2.4.42}$$

For every consistent family $(v_i)_{i\in I}$ *with* $\phi_{i\preccurlyeq j}(v_j) = v_i$ *whenever* $i \preccurlyeq j$, *there exists a unique element* $v \in \varprojlim V_i$ *with* $\phi_i(v) = v_i$ *for all* $i \in I$.

ii.) *Suppose* $\mathcal{P}_i$ *is a defining system of continuous seminorms on* V_i *for each* $i \in I$. *Then*

$$\mathcal{P} = \left\{ \mathrm{p}\colon \varprojlim\nolimits_{i\in I} V_i \longrightarrow \mathbb{R}_0^+ \;\middle|\; \mathrm{p} = \mathrm{p}_i \circ \phi_i,\ i \in I,\ \mathrm{p}_i \in \mathcal{P}_i \right\} \tag{2.4.43}$$

yields a defining system of continuous seminorms on the projective limit. In particular, the locally convex topology on $\varprojlim V_i$ *is the initial locally convex topology with respect to the maps* ϕ_i.

iii.) *The projective limit of Hausdorff locally convex spaces is Hausdorff again.*

iv.) *The projective limit of complete locally convex spaces is complete again.*

v.) *If* I *is countable, then the projective limit of Fréchet spaces is a Fréchet space again.*

Proof: First recall that the projective limit in the category of vector spaces always exists and is given by the subspace (2.4.41) and the linear maps (2.4.42), see also Exercise 2.5.62. Since for the product topology the projections pr_i are continuous, their restrictions to a subspace stay continuous. Now let $(\tilde{V}, \{\tilde{\phi}_i\}_{i\in I})$ be another locally convex space $\tilde{V}$ with continuous linear maps $\tilde{\phi}_i\colon \tilde{V} \longrightarrow V_i$ such that $\tilde{\phi}_i = \phi_{i\preccurlyeq j} \circ \tilde{\phi}_j$ for all $i \preccurlyeq j$. Then the unique linear map $\phi\colon \tilde{V} \longrightarrow \varprojlim V_i$ with $\tilde{\phi}_i = \phi_i \circ \phi$ is given by its components

$$\phi\colon \tilde{V} \ni \tilde{v} \mapsto (\tilde{\phi}_i(\tilde{v}))_{i\in I} \in \varprojlim_{i\in I} V_i \subseteq \prod_{i\in I} V_i,$$

since $\phi_i = \mathrm{pr}_i \big|_{\varprojlim V_i}$. Note that the image is indeed in the subspace $\varprojlim V_i \subseteq \prod_{i\in I} V_i$. Now, since all the $\tilde{\phi}_i$ are continuous, the map ϕ is continuous, too, by the universal property of the Cartesian product topology. This shows that we have found a projective limit. The remaining statement is obvious by the concrete description of $\varprojlim V_i$, since such a $v = (v_i)_{i\in I}$ is an element of the subspace $\varprojlim V_i \subseteq \prod_{i\in I} V_i$, showing the first part. For the second we note that the seminorms $\{\mathrm{p}_i \circ \mathrm{pr}_i\}_{i\in I,\, \mathrm{p}_i\in\mathcal{P}_i}$ form a defining system for the product topology by (2.4.4). Restricting to the subspace $\varprojlim V_i$ gives the defining system $\mathcal{P}$ from (2.4.43). By Proposition 2.4.4, *iii.)*, this characterizes the initial topology with respect to the maps ϕ_i. The parts *iii.)* and *iv.)* follow directly from the corresponding statements on the Cartesian product in Proposition 2.4.5, *i.)* and *iv.)*, together with the trivial observation that $\varprojlim V_i$ is a closed subspace. Indeed, the condition in (2.4.41) is closed, since the maps $\phi_{i\preccurlyeq j}$ are continuous. The last part is then a consequence of *ii.)*, *iii.)* and *iv.)*. ☐

Example 2.4.28 (The $\mathscr{C}^\infty$-topology) Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. Then the $\mathscr{C}^\infty$-functions $\mathscr{C}^\infty(X)$ can be identified with the projective limit of all $\mathscr{C}^k$-functions, i.e.

$$\mathscr{C}^\infty(X) \dashrightarrow \cdots \longrightarrow \mathscr{C}^k(X) \longrightarrow \mathscr{C}^{k-1}(X) \longrightarrow \cdots \longrightarrow \mathscr{C}^1(X) \longrightarrow \mathscr{C}^0(X). \tag{2.4.44}$$

First, the maps are canonical inclusions, from which

$$\mathscr{C}^\infty(X) = \varprojlim \mathscr{C}^k(X) \tag{2.4.45}$$

as vector spaces follows at once. But (2.4.45) also holds as locally convex spaces by Theorem 2.4.27, *ii.)*, and the explicit form of the defining seminorms $\mathrm{p}_{K,\ell}$ of the $\mathscr{C}^k$-topologies: According to that statement, we have to take *all* the seminorms $\{\mathrm{p}_{K,\ell} \circ \iota_k\}_{k \in \mathbb{N}_0}$ with the inclusion

$$\iota_k \colon \mathscr{C}^\infty(X) \longrightarrow \mathscr{C}^k(X), \tag{2.4.46}$$

and all compact $K \subseteq X$ and $\ell \leq k$. But this just yields the defining seminorms of the $\mathscr{C}^\infty$-topology.

The case of the inductive limit is slightly more complicated, as the Hausdorff property can fail in general. Nevertheless, we get the following construction: First we recall that in linear algebra any inductive system of vector spaces has an inductive limit:

Lemma 2.4.29 *Let $\{V_i\}_{i \in I}$ be an inductive system of vector spaces over a field $\Bbbk$ with corresponding linear maps $\phi_{i \preccurlyeq j} \colon V_i \longrightarrow V_j$ for $i \preccurlyeq j$. Then there exists a vector space V with linear maps $\phi_i \colon V_i \longrightarrow V$ for every $i \in I$, satisfying $\phi_i = \phi_j \circ \phi_{i \preccurlyeq j}$, which is universal with respect to this property. The vector space V is spanned by the images of the maps ϕ_i.*

Proof: One way to construct this inductive limit is to consider the direct sum $\tilde{V} = \bigoplus_{i \in I} V_i$ of all the vector spaces V_i. Then we have obvious linear maps

$$\tilde{\phi}_i \colon V_i \longrightarrow \tilde{V}, \tag{$*$}$$

namely the inclusions of elements of V_i at the i-th position. However, in general we do *not* yet have the property $\tilde{\phi}_i = \tilde{\phi}_j \circ \phi_{i \preccurlyeq j}$. Thus we enforce this by specifying a subspace

$$\tilde{V}_0 = \operatorname{span}_{\Bbbk}\Big\{ (\tilde{\phi}_i - \tilde{\phi}_j \circ \phi_{i \preccurlyeq j})(v_i) \;\Big|\; i \preccurlyeq j,\ v_i \in V_i \Big\},$$

and taking the quotient $V = \tilde{V}/\tilde{V}_0$. The maps $(*)$ descend to linear maps $\phi_i \colon V_i \longrightarrow V$ by taking the composition with the quotient map $\tilde{V} \longrightarrow V$. If now $v_i \in V_i$ and $i \preccurlyeq j$, then $(\tilde{\phi}_i - \tilde{\phi}_j \circ \phi_{i \preccurlyeq j})(v_i) \in \tilde{V}_0$ by definition. Hence $\phi_i = \phi_j \circ \phi_{i \preccurlyeq j}$ follows. Finally, we have to show the universal property. Thus let U be another vector space with linear maps $\psi_i \colon V_i \longrightarrow U$ such that $\psi_i = \psi_j \circ \phi_{i \preccurlyeq j}$ whenever $i \preccurlyeq j$. Since $\tilde{V}$ is the direct sum, we get a unique linear map

$$\tilde{\Psi} \colon \tilde{V} = \bigoplus_{i \in I} V_i \longrightarrow U,$$

such that $\tilde{\Psi} \circ \tilde{\phi}_i = \psi_i$, namely $\tilde{\Psi} = \bigoplus_{i \in I} \psi_i$. Note that there it is crucial to use the direct sum. Clearly, for $v_i \in V_i$ and $i \preccurlyeq j$ we have

$$\tilde{\Psi}\big(\tilde{\phi}_i(v_i) - \tilde{\phi}_j \circ \phi_{i \preccurlyeq j}(v_i)\big) = \psi_i(v_i) - \psi_j \circ \phi_{i \preccurlyeq j}(v_i) = 0,$$

and hence $\tilde{V}_0 \subseteq \ker \Psi$. This shows that $\tilde{\Psi}$ descends to the quotient and yields a well-defined linear map

$$\Psi \colon V = \tilde{V}/\tilde{V}_0 \longrightarrow U,$$

still satisfying $\psi \circ \phi_i = \psi_i$, as the latter can be checked on representatives. Finally, suppose that $\Psi' \colon V \longrightarrow U$ is another linear map with $\Psi' \circ \phi_i = \psi_i$. Since the images of the $\tilde{\phi}_i$ span the direct sum $\tilde{V}$, the images of the maps $\phi_i \colon V_i \longrightarrow V$ span V. This shows that $\Psi' = \Psi$ is necessarily unique. □

The idea is now to endow the linear-algebraic inductive limit with a locally convex topology in such a way, that all the topological requirements of Definition 2.4.24 are fulfilled as well. This can indeed be done:

Theorem 2.4.30 (Inductive limit) *Let I be a directed set and let $(\{V_i\}_{i\in I}, \{\phi_{i\preccurlyeq j}\colon V_i \longrightarrow V_j\}_{i\preccurlyeq j})$ be an inductive system of locally convex spaces.*

i.) The final locally convex topology on the linear algebraic inductive limit $V = \varinjlim V_i$ turns V into the inductive limit of locally convex spaces.

ii.) A seminorm p on the locally convex inductive limit $V = \varinjlim V_i$ is continuous iff for all $i \in I$ the pull-back seminorm $\phi_i^*\mathrm{p} = \mathrm{p} \circ \phi_i$ is continuous on V_i.

iii.) A linear map $\Psi\colon V = \varinjlim V_i \longrightarrow W$ from the locally convex inductive limit V into another locally convex space W is continuous iff for all $i \in I$ the linear maps $\psi \circ \phi_i\colon V_i \longrightarrow W$ are continuous.

PROOF: Let $V = \varinjlim V_i$ be the linear-algebraic inductive limit according to the previous lemma. We equip V with the final locally convex topology with respect to all the maps $\phi_i\colon V_i \longrightarrow V$. Then every ϕ_i is continuous by construction. Now let $\tilde{V}$ be another locally convex space with continuous linear maps $\tilde{\phi}_i\colon V_i \longrightarrow \tilde{V}$, such that $\tilde{\phi}_i = \tilde{\phi}_j \circ \phi_{i\preccurlyeq j}$ for $i \preccurlyeq j$. By Lemma 2.4.29 we obtain a unique linear map $\psi\colon V \longrightarrow \tilde{V}$ such that

$$\psi \circ \phi_i = \tilde{\phi}_i$$

for all $i \in I$. The universal property of the final locally convex topology from Proposition 2.4.9, *ii.)*, then shows that ψ is continuous. This is all we need for the locally convex inductive limit, thus proving the first part. The second and the third part are then just properties of the final locally convex topology from Proposition 2.4.9, *ii.)* and *iii.)*. ☐

Unlike for the projective limit, it is typically rather difficult to decide whether $\varinjlim V_i$ is Hausdorff or (sequentially) complete: there are examples of Hausdorff or (sequentially) complete V_i, such that their locally convex inductive limit is not Hausdorff or not (sequentially) complete, see also Exercise 2.5.63.

Example 2.4.31 (Direct locally convex sum) Let Λ be an arbitrary index set and let $\{V_\lambda\}_{\lambda\in\Lambda}$ be a collection of locally convex spaces. We can turn this now into an inductive system by taking as directed set I the set of finite subsets of Λ, ordered by inclusion. For $i = \{\lambda_1, \ldots, \lambda_r\}$ we define

$$V_i = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_r}, \tag{2.4.47}$$

and have the canonical inclusion maps

$$\phi_{i\preccurlyeq j}\colon V_i \longrightarrow V_j, \tag{2.4.48}$$

whenever $i \preccurlyeq j$. It is then easy to see that we get

$$\varinjlim_{i\in I} V_i = \bigoplus_{\lambda\in\Lambda} V_\lambda \tag{2.4.49}$$

for the locally convex inductive limit. Indeed, the linear-algebraic inductive limit is the direct sum. Moreover, the final locally convex topology with respect to all inclusions

$$V_i = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_r} \longrightarrow \bigoplus_{\lambda\in\Lambda} V_\lambda \tag{2.4.50}$$

for $i \in I$ coincides with the final locally convex topology obtained from the inclusions

$$V_\lambda \longrightarrow \bigoplus_{\lambda\in\Lambda} V_\lambda, \tag{2.4.51}$$

since in (2.4.50) we take *finite* direct sums on the left hand side. Thus in this case, $\varinjlim V_i$ inherits the nice properties like Hausdorffness or completeness from the individual $\{V_\lambda\}_{\lambda\in\Lambda}$, since the direct sum has this feature according to Proposition 2.4.17, *iii.)* and *v.)*.

Proposition 2.4.32 *Let $V = \varinjlim V_i$ be the locally convex inductive limit of an inductive system of locally convex spaces $\{V_i\}_{i \in I}$ with corresponding linear maps $\phi_i \colon V_i \longrightarrow V$.*

i.) The linear maps $\phi_i \colon V_i \longrightarrow V$ are injective for all $i \in I$ iff the maps $\phi_{i \preccurlyeq j} \colon V_i \longrightarrow V_j$ are injective for all $i \preccurlyeq j$.

ii.) Suppose V is Hausdorff. If $\phi_i \colon V_i \longrightarrow V$ is injective for some $i \in I$, then the corresponding V_i is Hausdorff, too.

PROOF: The first statement holds in much larger generality and can be shown e.g. for the linear-algebraic situation, see Exercise 2.5.64. The second statement is clear, since for $v_i \in V_i$ with $v_i \neq 0$ we have $\phi_i(v_i) \neq 0$ and hence there is a continuous seminorm p on V with $\mathrm{p}(\phi_i(v_i)) > 0$. But $\mathrm{p} \circ \phi_i$ is a continuous seminorm on V_i, showing that V_i is Hausdorff, too. $\square$

In a next step we want to characterize the neighbourhoods of zero in the inductive limit. As usual, the translation invariance of the topology allows us to focus on the zero neighbourhoods in order to understand the entire inductive limit. The following result gives now a simple criterion for an absolutely convex set to be a zero neighbourhood:

Proposition 2.4.33 *Let $V = \varinjlim V_i$ be a locally convex inductive limit of an inductive system of locally convex spaces $\{V_i\}_{i \in I}$ with corresponding canonical continuous linear maps $\phi_i \colon V_i \longrightarrow V$. Let $U \subseteq V$ be an absolutely convex subset. Then U is a zero neighbourhood iff $\phi_i^{-1}(U) \subseteq V_i$ is a zero neighbourhood for all $i \in I$.*

PROOF: Suppose $\phi_i^{-1}(U)$ is a zero neighbourhood for all $i \in I$. We first show that U is absorbing. Since V is a quotient of the direct sum, for $v \in V$ we find $v_{i_1} \in V_{i_1}, \ldots, v_{i_N} \in V_{i_N}$ such that

$$v = \sum_{r=1}^{N} \phi_{i_r}(v_{i_r}).$$

Since $U_i = \phi_i^{-1}(U)$ is absorbing for all $i \in I$ as a zero neighbourhood, we find positive $\lambda_{i_1}, \ldots, \lambda_{i_N} \in (0, \infty)$ with

$$\frac{1}{\lambda_{i_r}} v_{i_r} \in U_{i_r}$$

for all $r = 1, \ldots, N$. Taking $\lambda = \max\{\lambda_{i_1}, \ldots, \lambda_{i_N}\}$ then gives for all $r = 1, \ldots, N$

$$\frac{1}{\lambda} v_{i_r} \in U_{i_r},$$

since U_{i_r} is balanced. Hence

$$\frac{1}{\lambda} \phi_{i_r}(v_{i_r}) = \phi_{i_r}\big(\tfrac{1}{\lambda} v_{i_r}\big) \in \phi_{i_r}(U_{i_r}) \subseteq U$$

gives an element in U. But then

$$\frac{1}{\lambda N} v = \frac{1}{N} \sum_{r=1}^{N} \phi_{i_r}\big(\tfrac{1}{\lambda} v_{i_r}\big) \in U,$$

since we have a convex combination of elements of U. This shows $\frac{1}{\lambda N} v \in U$ and thus U is absorbing, as claimed. Hence U defines a Minkowski functional p_U, which is a seminorm on V, such that

$$\mathrm{B}_{\mathrm{p}_U, 1}(0) \subseteq U \subseteq \mathrm{B}_{\mathrm{p}_U, 1}(0)^{\mathrm{cl}}, \tag{$*$}$$

according to Lemma 2.2.20 and Lemma 2.2.21. We want to show that p_U is actually continuous: up to now the closed ball in $(*)$ is just an abbreviation as in (2.2.38). Thus let $i \in I$ and consider the seminorm $\mathrm{p}_U \circ \phi_i$ on V_i. We have for $v_i \in V_i$

$$\begin{aligned}(\mathrm{p}_U \circ \phi_i)(v_i) &= \inf\big\{\lambda > 0 \bigm| \phi_i(v_i) \in \lambda U\big\} \\ &= \inf\big\{\lambda > 0 \bigm| \phi_i\big(\tfrac{1}{\lambda} v_i\big) \in U\big\} \\ &= \inf\big\{\lambda > 0 \bigm| \tfrac{1}{\lambda} v_i \in \phi_i^{-1}(U)\big\} \\ &= \mathrm{p}_{U_i}(v_i),\end{aligned}$$

where $U_i = \phi_i^{-1}(U)$ as before. Since U_i is a zero neighbourhood by assumption, p_{U_i} is continuous, showing that p_U is continuous thanks to Theorem 2.4.30, *iii.)*. But then U is a zero neighbourhood according to $(*)$. The opposite implication is trivial, since all the maps ϕ_i are continuous. □

2.4.6 Strict Inductive Limits and LF Spaces

In general, a locally convex inductive limit $V = \varinjlim V_i$ can be quite complicated. Many nice properties of the V_i will not be inherited by V without extra assumptions. From Proposition 2.4.32 we infer that it is of particular interest to consider injective maps $\phi_{i \preccurlyeq j}$ for all indices $i \preccurlyeq j$ of the directed set I, since in this case also the canonical maps $\phi_i \colon V_i \longrightarrow V$ will be injective. Elements in the kernel of the ϕ_i will not contribute to V anyway. Having injective linear maps

$$\phi_{i \preccurlyeq j} \colon V_i \longrightarrow V_j \tag{2.4.52}$$

for $i \preccurlyeq j$ it is natural to compare the original topology of V_i to the one induced by the inclusion $V_i \cong \phi_{i \preccurlyeq j}(V_i) \subseteq V_j$. Since (2.4.52) is continuous, the induced topology is coarser than the original one. The interesting case is hence, when they actually coincide:

Definition 2.4.34 (Strict inductive limit) *A locally convex inductive limit $V = \varinjlim V_i$ of an inductive system of locally convex space is called strict if the corresponding continuous linear maps $\phi_{i \preccurlyeq j} \colon V_i \longrightarrow V_j$ for $i \preccurlyeq j$ are always embeddings.*

While arbitrary strict inductive limits are still rather complicated, for a *countable* strict inductive limit the situation starts to become accessible: First we note that an inductive system over the directed set $I = \mathbb{N}$ is just a sequence of locally convex spaces V_n for $n \in \mathbb{N}$ together with continuous linear maps

$$\phi_{n \le n+1} \colon V_n \longrightarrow V_{n+1}. \tag{2.4.53}$$

Indeed, the remaining maps $\phi_{n \le m} \colon V_n \longrightarrow V_m$ for $n \le m$ are determined to be

$$\phi_{n \le m} = \phi_{m-1 \le m} \circ \cdots \circ \phi_{n \le n+1} \tag{2.4.54}$$

by functoriality, since I is not just directed, but totally ordered. Thus, for $I = \mathbb{N}$ it suffices to specify linear maps (2.4.53) to obtain an inductive system. Moreover, the system will be strict iff the maps (2.4.53) are embeddings: this holds, since the composition (2.4.54) of embeddings is automatically an embedding again in this case. Finally, for a strict system, the final maps $V_m \longrightarrow V = \varinjlim V_n$ are injective. Putting these results together, we see that a countable strict inductive limit $V = \varinjlim V_n$ is given by a sequence of locally convex spaces $\{V_n\}_{n \in \mathbb{N}}$ being nested subspaces

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq V_{n+1} \subseteq \cdots \subseteq V = \bigcup_{n=1}^{\infty} V_n, \tag{2.4.55}$$

such that the subspace topology on $V_n \subseteq V_{n+1}$ coincides with the original topology of V_n and the locally convex topology on V is the finest locally convex topology, such that all the inclusions

$$V_n \subseteq V \tag{2.4.56}$$

are continuous, too. However, it is not clear a priori, whether the induced subspace topology from (2.4.56) reproduces the original topology on V_n. It turns out that this is necessarily the case for a countable strict inductive limit.

The following technical construction gives us zero neighbourhoods of the inductive limit built out of ascending sequences of zero neighbourhoods of each individual V_n in the case of a countable strict inductive limit:

Lemma 2.4.35 *Let $\{V_n\}_{n\in\mathbb{N}}$ be a sequence of locally convex spaces forming a countable strict inductive system*

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq V_{n+1} \subseteq \cdots \subseteq V \tag{2.4.57}$$

with inductive limit $V = \varinjlim V_n$ identified with $V = \bigcup_{n\in\mathbb{N}} V_n$ as vector space.

i.) Suppose $U_n \subseteq V_n$ is an absolutely convex (open) zero neighbourhood for all $n \in \mathbb{N}$ such that

$$U_n = V_n \cap U_{n+1}. \tag{2.4.58}$$

Then $U = \bigcup_{n\in\mathbb{N}} U_n \subseteq V$ is an absolutely convex (open) neighbourhood of zero with $U_n = V_n \cap U$.

ii.) Conversely, if $U \subseteq V$ is an absolutely convex (open) zero neighbourhood, then $U_n = V_n \cap U$ gives absolutely convex (open) zero neighbourhoods in V_n with (2.4.58) and $U = \bigcup_{n\in\mathbb{N}} U_n$.

PROOF: Assume we are given the $U_n \subseteq V_n$. Then (2.4.58) implies $U_n \subseteq U_{n+1}$, since $V_n \subseteq V_{n+1}$ and $V_{n+1} \cap U_{n+1} = U_{n+1}$. Thus the union $U = \bigcup_{n\in\mathbb{N}} U_n$ stays absolutely convex. Next, we have

$$U \cap V_n = \Big(\bigcup_{m\in\mathbb{N}} U_m\Big) \cap V_n = \bigcup_{m\in\mathbb{N}} \big(U_m \cap V_n\big) = U_n,$$

since for $m \geq n$ we get inductively $U_m \cap V_n = U_n$ from the assumption (2.4.58), since $V_n \subseteq V_{n+1}$ anyway. For $m < n$ we have $U_m \cap V_n \subseteq U_m \subseteq U_n$ not enlarging the union. Finally, U is a zero neighbourhood since the pre-images in each V_n are zero neighbourhoods, namely the U_n. This follows from the defining property of the final locally convex topology. Note that U and the U_n need to be absolutely convex for this. Moreover, if in addition the U_n are open, then also U is open by the defining property of the final locally convex topology, see also Exercise 2.5.53. Conversely, let $U \subseteq V$ be an absolutely convex neighbourhood of zero. Then the pre-images under the inclusions $V_n \subseteq V$ are absolutely convex and still neighbourhoods of zero, since V carries the final locally convex topology. Forming again their union reproduces U, since $V = \bigcup_{n\in\mathbb{N}} V_n$. Again, if U is open, the U_n are open, being pre-images of U under the continuous inclusions. □

Using this lemma, we can now show that also the final maps $V_n \longrightarrow V$ into the inductive limit are embeddings. Moreover, we can prove that the Hausdorff property is inherited:

Proposition 2.4.36 *Let $\{V_n\}_{n\in\mathbb{N}}$ be a sequence of locally convex spaces forming a countable strict inductive system with inductive limit $V = \varinjlim V_n$.*

i.) For all $n \in \mathbb{N}$ the inclusions $V_n \subseteq V$ are embeddings.

ii.) The inductive limit V is Hausdorff iff for all $n \in \mathbb{N}$ the spaces V_n are Hausdorff.

Proof: Let $n_0 \in \mathbb{N}$ be fixed and let $U_{n_0} \subseteq V_{n_0}$ be an absolutely convex zero neighbourhood. Since $V_{n_0} \subseteq V_{n_0+1}$ is an embedding by assumption, Lemma 2.4.35 gives an absolutely convex zero neighbourhood $U_{n_0+1} \subseteq V_{n_0+1}$ with

$$U_{n_0} = U_{n_0+1} \cap V_{n_0}.$$

Continuing with U_{n_0+1} we get by the same argument absolutely convex zero neighbourhoods $U_{n_0+k} \subseteq V_{n_0+k}$ for all $k \in \mathbb{N}$ with the property

$$U_{n_0+k} = V_{n_0+k} \cap U_{n_0+k+1}.$$

For $n < n_0$ we know that $V_n \subseteq V_{n_0}$ is an embedding, too, and hence defining $U_n = V_n \cap U_{n_0}$ is an absolutely convex zero neighbourhood for all $1 \leq n \leq n_0$. Clearly, $U_n = V_n \cap U_{n+1}$ holds for all such n, since $V_n \subseteq V_{n+1}$. Together we arrive at a sequence $\{U_n\}_{n\in\mathbb{N}}$ of absolutely convex zero neighbourhoods with U_{n_0} being the one we started with and

$$U_n = V_n \cap U_{n+1} \tag{$*$}$$

for all $n \in \mathbb{N}$. From Lemma 2.4.35, *i.)*, we conclude that their union

$$U = \bigcup_{n\in\mathbb{N}} U_n \subseteq V$$

is an absolutely convex zero neighbourhood in the inductive limit. From $(*)$ we see that

$$U \cap V_n = U_n \tag{$**$}$$

for all $n \in \mathbb{N}$. Thus the subspace topology inherited from $V_{n_0} \subseteq V$ is finer than the original topology of V_{n_0}, as by $(**)$ we obtain every U_{n_0} as intersection of an absolutely convex zero neighbourhood in V with V_{n_0}. Conversely, the original topology is clearly finer than the induced subspace topology, since by the very definition of the locally convex inductive limit topology, the inclusion $V_{n_0} \subseteq V$ is continuous. As n_0 was arbitrary, the first part is shown. For the second part, let $v \in V$ be non-zero. Then there exists an $n \in \mathbb{N}$ with $v \in V_n$, since $V = \bigcup_{n\in\mathbb{N}} V_n$. If V_n is Hausdorff, we can separate v from zero by an absolutely convex neighbourhood $U_n \subseteq V_n$ of zero. Since $V_n \subseteq V$ is an embedding, this U_n is the intersection of a zero neighbourhood $U \subseteq V$ with V_n. Thus $v \notin U$ allows to separate v from 0 in V, showing that $v \notin \{0\}^{\mathrm{cl}}$. This gives $\{0\}^{\mathrm{cl}} = \{0\}$ and hence V is T_1 and thus T_2 by Corollary 2.1.9. The other direction is clear, since every subspace $V_n \subseteq V$ of a Hausdorff space is again Hausdorff. Note that we use again that $V_n \subseteq V$ is an embedding. ☐

Before we proceed, it is illustrative to consider the following pretty bad example. This will show that in general the subspaces V_n need not to be closed:

Example 2.4.37 Let V be an incomplete Hausdorff locally convex space with completion $\iota\colon V \longrightarrow \hat{V}$. This can be viewed as countable strict inductive system

$$V \xrightarrow{\iota} \hat{V} = \hat{V} = \hat{V} = \cdots = \hat{V}. \tag{2.4.59}$$

Since it stabilizes, the inductive limit topology on $\hat{V}$ is just the original locally convex topology of the completion. Note that $V \subseteq \hat{V}$ is dense and thus not closed, as soon as $V \neq \hat{V}$.

The same effect can be observed in the following, somewhat more sophisticated example:

Example 2.4.38 Let V be an infinite-dimensional Banach space and let $V_0 \subseteq V$ be a dense subspace, such that there exist countably many linearly independent unit vectors $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ in V with

$$V_0 \oplus \operatorname{span}_{\mathbb{K}}\{\mathrm{e}_n\}_{n\in\mathbb{N}} = V. \tag{2.4.60}$$

It is easy to find examples for this situation. Start e.g. with ℓ^1, containing the dense subspace $c_{\circ\circ} \subseteq \ell^1$. Since $c_{\circ\circ}$ has countable dimension, but ℓ^1 has necessarily uncountable dimension, see Exercise 2.5.65, there exists a subspace $V_0 \subseteq \ell^1$ with $c_{\circ\circ} \subseteq V_0$ and such that V_0 has a linear algebraic complement of countable dimension: this is a simple application of Zorn's lemma. Then we consider

$$V_n = V_0 \oplus \operatorname{span}_{\mathbb{K}}\{\mathrm{e}_1, \ldots, \mathrm{e}_n\} \tag{2.4.61}$$

with the obvious inclusions $V_n \subseteq V_{n+1}$, leading to $V = \bigcup_{n\in\mathbb{N}} V_n$. On each V_n we use the subspace topology inherited from V, making each $V_n \subseteq V_{n+1}$ an embedding. We claim that the inductive limit topology on V is the original Banach space topology. We denote by $\iota_n \colon V_n \longrightarrow \hat{V}_n = V$ the inclusion map of V_n into its completion, which is V for all $n \in \mathbb{N}$. Since these maps are consistent with the inductive system $V_n \subseteq V_{n+1}$ we get an induced continuous linear map $\varinjlim V_n \longrightarrow V$, such that

$$\begin{tikzcd} V_n \arrow[r] \arrow[dr, "\iota_n"'] & \varinjlim V_n \arrow[d] \\ & \hat{V}_n = V \end{tikzcd} \tag{2.4.62}$$

commutes for all $n \in \mathbb{N}$. Since $\varinjlim V_n = \bigcup_{n\in\mathbb{N}} V_n$, we can conclude that this map is the identity, thereby showing that the inductive limit topology is finer than the Banach space topology of V. The difficulty is now that the inductive limit needs not to be complete a priori. Thus we need to argue as follows. Let $\jmath_n \colon V_n \longrightarrow \varinjlim V_n$ be the canonical continuous maps from the defining property of the inductive limit. As linear maps we of course have $\iota_n = \jmath_n$, but we use another symbol to distinguish the notion of continuity. Since $\iota_n \colon V_n \longrightarrow \hat{V}_n = V$ is the completion of V_n, we get a unique continuous linear map $\hat{\jmath}_n \colon \hat{V}_n = V \longrightarrow \widehat{\varinjlim V_n}$, such that

$$\begin{tikzcd} \hat{V}_n = V \arrow[r, "\hat{\jmath}_n"] & \widehat{\varinjlim V_n} \\ V_n \arrow[u, "\iota_n"] \arrow[r, "\jmath_n"'] & \varinjlim V_n \arrow[u] \end{tikzcd} \tag{2.4.63}$$

commutes for all $n \in \mathbb{N}$. This is the unique extension by continuity. Since the identity $\mathrm{id} \colon \varinjlim V_n \longrightarrow V$ is continuous, we get a unique extension

$$\begin{tikzcd} \widehat{\varinjlim V_n} \arrow[r, "\hat{\mathrm{id}}"] & V \\ \varinjlim V_n \arrow[u] \arrow[ur, "\mathrm{id}"'] \end{tikzcd}, \tag{2.4.64}$$

as V is already complete. Of course, $\hat{\mathrm{id}}$ stays surjective. In a last step, we note that the inclusion $V_{n-1} \subseteq V_n$ leads to the identity map under completion, since V_{n-1} and V_n are both dense in V. This leads to the commuting diagram

$$
\begin{tikzcd}
V = \hat{V}_{n-1} \arrow[r, "\mathrm{id}"] \arrow[rr, bend left, "\hat{\jmath}_{n-1}"] & V = \hat{V}_n \arrow[r, "\hat{\jmath}_n"] & \widehat{\varinjlim V_n} \arrow[r, "\hat{\mathrm{id}}"] & V \\
V_{n-1} \arrow[u, "\iota_{n-1}"] \arrow[r, "\subseteq"] \arrow[rr, bend right, "\jmath_{n-1}"'] & V_n \arrow[u, "\iota_n"] \arrow[r, "\jmath_n"] & \varinjlim V_n \arrow[u, "\iota"] \arrow[ur, "\mathrm{id}"']
\end{tikzcd}
\tag{2.4.65}
$$

of continuous linear maps. In particular, the completed maps $\hat{\jmath}_{n-1} = \hat{\jmath}_n$ coincide. Therefore this map $\hat{\jmath} = \hat{\jmath}_n$ is in fact independent of $n \in \mathbb{N}$. The composition $\mathrm{id}_V \circ \jmath_n$ coincides with id_V on $V_n \subseteq V$. By continuity and density, we can conclude that $\hat{\mathrm{id}}_V \circ \hat{\jmath} = \mathrm{id}_V$. This gives a continuous projection $\mathrm{pr} = \hat{\jmath} \circ \hat{\mathrm{id}} \colon \widehat{\varinjlim V_n} \longrightarrow \widehat{\varinjlim V_n}$ and hence a decomposition

$$
\widehat{\varinjlim V_n} = \ker \mathrm{pr} \oplus \operatorname{im} \mathrm{pr} = \ker \mathrm{pr} \oplus \ker(1 - \mathrm{pr}) \tag{2.4.66}
$$

into two complementary subspaces. Since $\hat{\mathrm{id}}$ is surjective, we conclude that $\operatorname{im} \mathrm{pr} = \operatorname{im} \hat{\jmath}$. From the commuting diagram we note that for all $n \in \mathbb{N}$ we have

$$
\operatorname{im}(\iota \circ \jmath_n) \subseteq \operatorname{im} \hat{\jmath}_n = \operatorname{im} \hat{\jmath}, \tag{2.4.67}
$$

since $\iota \circ \jmath_n = \hat{\jmath}_n \circ \iota_n$. But the union of the images $\operatorname{im} \jmath_n$ is $\varinjlim V_n$. Hence we have

$$
\operatorname{im} \iota \subseteq \operatorname{im} \hat{\jmath}. \tag{2.4.68}
$$

Since $\operatorname{im} \hat{\jmath}$ is a closed subspace in $\widehat{\varinjlim V_n}$ and since $\operatorname{im} \iota = \iota(\varinjlim V_n)$ is dense in $\widehat{\varinjlim V_n}$, the inclusion (2.4.68) shows

$$
\operatorname{im} \hat{\jmath} = \widehat{\varinjlim V_n}. \tag{2.4.69}
$$

Thus $\ker \mathrm{pr} = \{0\}$ follows and hence $\mathrm{pr} = \mathrm{id}_{\widehat{\varinjlim V_n}}$ is the identity on $\widehat{\varinjlim V_n}$. Hence

$$
V \xrightarrow{\hat{\jmath}} \widehat{\varinjlim V_n} \xrightarrow{\hat{\mathrm{id}}} V \tag{2.4.70}
$$

are mutually inverse continuous linear maps. Finally, let $(v_\alpha)_{\alpha \in I}$ be a Cauchy net in $\varinjlim V_n$, which converges in $\widehat{\varinjlim V_n}$, i.e. we have a $v \in \widehat{\varinjlim V_n}$ with $\iota(v_\alpha) \to v$. Since $\hat{\jmath}$ is surjective, there is a unique $v' \in V$ with $\hat{\jmath}(v') = v$. Since there exists an $n \in \mathbb{N}$ with $v' \in V_n$, the diagram (2.4.65) gives $\iota(v') = v$ and thus ι is surjective. We conclude that $\varinjlim V_n = \widehat{\varinjlim V_n}$ is already complete and hence $\hat{\mathrm{id}} = \mathrm{id}$. But then $\hat{\jmath} = \mathrm{id}$, as well, showing that the topology of V is finer then the topology of $\widehat{\varinjlim V_n}$.

Thus the two coincide. For later use we mention already here two features of this strict countable inductive limit of normed spaces:

i.) None of the subspaces $V_n \subseteq V$ is closed. Instead, all of them are dense in the (complete) limit V.

ii.) The sequence $(\frac{1}{n}\mathrm{e}_n)_{n \in \mathbb{N}}$ in V converges for trivial reasons to zero. Nevertheless, in each V_n there are only finitely many elements of this sequence, namely $\mathrm{e}_1, \frac{1}{2}\mathrm{e}_2, \ldots, \frac{1}{n}\mathrm{e}_n$.

It turns out that less pathological behaviour can only be expected if the individual subspaces V_n become *closed* subspaces of their inductive limit V. The previous examples shows that this needs not to be the case, as soon as the V_n are non-complete. If the V_n are complete Hausdorff, the situation becomes nicer:

Proposition 2.4.39 *Let $\{V_n\}_{n\in\mathbb{N}}$ be a sequence of complete Hausdorff locally convex spaces forming a strict countable inductive system with inductive limit $V = \varinjlim V_n$. Then for all $n \in \mathbb{N}$ the subspace $V_n \subseteq V$ is closed.*

PROOF: Let $(v_\alpha)_{\alpha\in I}$ be a net in V with $v_\alpha \in V_n$ for all $\alpha \in I$, converging to $v \in V_n^{\mathrm{cl}} \subseteq V$. Since the subspace topology of $V_n \subseteq V$ is the original topology by Proposition 2.4.36, *i.)*, the Cauchy net $(v_\alpha)_{\alpha\in I}$ in V is in fact a Cauchy net in V_n in the original topology. By completeness, it converges to a limit in V_n, which, by the continuity of the embedding $V_n \subseteq V$, coincides with v. Thus $V_n = V_n^{\mathrm{cl}} \subseteq V$. □

In a strict countable inductive limit we can have continuous seminorms growing arbitrarily fast as n grows. This is the content of the following construction:

Proposition 2.4.40 *Let $\{V_n\}_{n\in\mathbb{N}}$ be a sequence of locally convex spaces forming a strict countable inductive system with inductive limit $V = \varinjlim V_n$.*

i.) Suppose p_n are continuous seminorms on V such that $\mathrm{p}_n\big|_{V_{n-1}} = 0$ for all $n \in \mathbb{N}$. Then

$$\mathrm{p} = \max_{n\in\mathbb{N}}\{\mathrm{p}_n\} \tag{2.4.71}$$

is a continuous seminorm on V, too.

ii.) Assume in addition that $V_n \subseteq V$ is closed for all $n \in \mathbb{N}$. Suppose $v_n \in V_n \setminus V_{n-1}$ is given for all $n \in \mathbb{N}$ together with an increasing sequence $0 < r_1 \le r_2 \le \cdots$ of real numbers. Then there exists a continuous seminorm p on V with

$$\mathrm{p}(v_n) = r_n \tag{2.4.72}$$

for all $n \in \mathbb{N}$.

PROOF: Let $v \in V$, then $v \in V_{n_0}$ for some $n_0 \in \mathbb{N}$. It follows that $\mathrm{p}_k(v) = 0$ for $k > n_0$ and hence $\mathrm{p}(v) = \max_{n\in\mathbb{N}}\{\mathrm{p}_n(v)\} = \max_{1\le n\le n_0}\{\mathrm{p}_n(v)\}$ is well-defined. Hence

$$\mathrm{p}\big|_{V_n} = \max\left\{\mathrm{p}_1\big|_{V_n}, \ldots, \mathrm{p}_n\big|_{V_n}\right\}$$

is a well-defined seminorm on V_n. As all p_n are continuous seminorms on V, their restrictions to V_n are continuous. Thus $\mathrm{p}\big|_{V_n}$ is a continuous seminorm for all $n \in \mathbb{N}$. From Theorem 2.4.30, *ii.)*, we conclude that p is a continuous seminorm on V. This first part holds also in greater generality, see Exercise 2.5.66. It is thus the second statement, which is nontrivial: here we have to construct the seminorm. Since $v_n \in V_n \setminus V_{n-1}$ and $V_{n-1} \subseteq V$ is closed, we have a continuous seminorm p_n on V with

$$\mathrm{p}_n\big|_{V_{n-1}} = 0 \quad\text{and}\quad \mathrm{p}_n(v_n) = r_n$$

by Corollary 2.4.14. Note that any value $r_n > 0$ can be obtained by rescaling the seminorm appropriately. We apply the first part to obtain $\mathrm{p} = \max_{n\in\mathbb{N}}\{\mathrm{p}_n\}$, which then solves (2.4.72), since the sequence $(r_n)_{n\in\mathbb{N}}$ is increasing. □

This fairly simple observation has two important consequences: we can characterize bounded subsets and Cauchy sequences. Note that the extra assumption on the closedness of $V_n \subseteq V$ can be guaranteed easily by e.g. Proposition 2.4.39.

Proposition 2.4.41 *Let $\{V_n\}_{n\in\mathbb{N}}$ be a sequence of locally convex spaces forming a strict countable inductive system with inductive limit $V = \varinjlim V_n$. Assume that $V_n \subseteq V$ is closed for all $n \in \mathbb{N}$.*

i.) A subset $B \subseteq V$ is bounded iff there exists an $n_0 \in \mathbb{N}$ with $B \subseteq V_{n_0}$ and B is bounded in V_{n_0}.

ii.) *A sequence $(v_k)_{k\in\mathbb{N}}$ in V is a Cauchy sequence iff there exists an $n_0 \in \mathbb{N}$ with $v_k \in V_{n_0}$ for all $k \in \mathbb{N}$ and $(v_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in V_{n_0}.*

iii.) *The inductive limit V is sequentially complete iff V_n is sequentially complete for all $n \in \mathbb{N}$.*

PROOF: Suppose that $B \subseteq V$ is bounded, but $B \cap (V_k \setminus V_{k-1})$ is non-empty for more than finitely many k. This allows to find a sequence $v_n \in B$ with a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ such that $v_n \in V_{k_n} \setminus V_{k_{n-1}}$. By Proposition 2.4.40, *ii.)*, we find a continuous seminorm p on V with e.g.

$$\mathrm{p}(v_n) = n. \tag{$*$}$$

But $(*)$ clearly contradicts the boundedness of B. The converse if obvious by Proposition 2.1.36 and the continuity of the inclusion $V_{n_0} \subseteq V$. Next, suppose $(v_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in V. Since Cauchy sequences are bounded subsets, we can apply the first part to conclude $v_k \in V_{n_0}$ for some suitable n_0. Since $V_{n_0} \subseteq V$ is an embedding, the sequence $(v_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in V_{n_0}, too. The converse is again obvious, since the continuous inclusion $V_{n_0} \subseteq V$ maps Cauchy sequences to Cauchy sequences. The last part is then clear by the second. □

Up to now we can not extend this proposition to Cauchy nets. The reason is that Cauchy nets are not necessarily bounded. Note also that the additional closedness assumption follows automatically if the V_n are complete:

Corollary 2.4.42 *Let $\{V_n\}_{n\in\mathbb{N}}$ be a sequence of complete locally convex spaces forming a strict countable inductive system with inductive limit $V = \varinjlim V_n$. Then V is sequentially complete.*

However, this corollary is still a bit disappointing, since we invest completeness and gain only sequential completeness. This is even worse in so far, as the inductive limit tends to be non-metrizable, even if we start with metrizable V_n, i.e. Fréchet spaces. Fortunately, much more is true: completeness is inherited directly. Nevertheless, this requires a bit more preparation and effort. We start with the following refinement of the statement of Lemma 2.4.35:

Lemma 2.4.43 *Let $\{V_n\}_{n\in\mathbb{N}}$ be a sequence of locally convex spaces forming a strict countable inductive system with inductive limit $V = \varinjlim V_n$. Suppose that $U_n \subseteq V_n$ is an absolutely convex open zero neighbourhood for all $n \in \mathbb{N}$ such that*

$$U_{n+1} \cap V_n \subseteq U_n \tag{2.4.73}$$

for all $n \in \mathbb{N}$.

i.) *Defining recursively $O_1 = U_1$ and*

$$O_n = \operatorname{absconv}(O_{n-1} \cup U_n) \tag{2.4.74}$$

yields open absolutely convex zero neighbourhoods $O_n \subseteq V_n$ with $O_{n+1} \cap V_n = O_n$ and $U_n \subseteq O_n$ for all $n \in \mathbb{N}$.

ii.) *The union $O = \bigcup_{n\in\mathbb{N}} O_n \subseteq V$ of these is an open absolutely convex zero neighbourhood with $O \cap V_n = O_n$.*

iii.) *For a fixed $k \in \mathbb{N}$ define*

$$\tilde{U}_n = \begin{cases} V_n & \text{for } n \le k \\ U_n & \text{for } n > k. \end{cases} \tag{2.4.75}$$

Then the corresponding $\tilde{O}_n$ according to i.) yield $\tilde{O} = \bigcup_{n\in\mathbb{N}} \tilde{O}_n$, given explicitly by

$$\tilde{O} = V_k + O. \tag{2.4.76}$$

Proof: First we note that the recursive definition of O_n can be made more explicit by taking

$$O_n = \operatorname{absconv}(U_1 \cup \cdots \cup U_n) \subseteq V_n.$$

Indeed, since each U_n is already absolutely convex, this follows directly from Proposition 2.1.27, *iii.)*, after a simple induction on n. Another induction shows that O_n is again open. Indeed, $O_1 = U_1$ is open by assumption. Suppose O_{n-1} is open. Then for every $v \in O_{n-1}$ there exists another $u \in O_{n-1}$ and a $\lambda \in (0, 1)$ with $v = \lambda u$. Thus we see that the convex combinations needed in the absolutely convex hull $\operatorname{absconv}(O_{n-1} \cup U_n)$ can be obtained by taking $\lambda u + (1-\lambda)u$ with $\lambda \in [0, 1)$ and $u \in O_{n-1}$ and $v \in U_n$ only. Hence

$$O_n = \operatorname{absconv}(O_{n-1} \cup U_n) = \bigcup_{\lambda \in [0,1)} \bigcup_{u \in O_{n-1}} (\lambda u + (1-\lambda)U_n)$$

is a union of translates of the open subsets $(1-\lambda)U_n$, thus open itself. Note that $O_{n-1} \subseteq V_n$ is of course not open. Now $U_1, \ldots, U_n \subseteq V_n$ and hence

$$(U_1 \cup \cdots \cup U_{n+1}) \cap V_n = U_1 \cup \cdots \cup U_n \cup (U_{n+1} \cap V_n) \subseteq U_1 \cup \cdots \cup U_n$$

by (2.4.73). Since V_n is already absolutely convex, taking the absolutely convex hull gives $O_{n+1} \cap V_n = O_n$ and clearly $U_n \subseteq O_n$ by construction. This completes the proof of the first part. The second is just Lemma 2.4.35, *i.)*. For the third part, let $k \in \mathbb{N}$ be given and define $\tilde{U}_n$ as in (2.4.75). This gives another sequence of absolutely convex open zero neighbourhoods, to which we can apply the construction of the first part, yielding $\tilde{O}_n$ and $\tilde{O}$. Note that we indeed have $\tilde{U}_{n+1} \cap V_n \subseteq \tilde{U}_n$ for all $n \in \mathbb{N}$ and that $\tilde{U}_n$ is an absolutely convex open zero neighbourhood. Now V_k is a subspace of V and hence $V_k + O$ is an absolutely convex open zero neighbourhood in V. By Lemma 2.4.35, *ii.)*, we have

$$V_k + O = \bigcup_{n \in \mathbb{N}} W_n \quad \text{with} \quad W_n = (V_k + O) \cap V_n,$$

with absolutely convex open zero neighbourhoods W_n. For $n \leq k$ we have the inclusion $\tilde{U}_n = V_n \subseteq (V_k + O) \cap V_n$, since O is a zero neighbourhood and $V_k \cap V_n = V_n$. For $n > k$ we have

$$\tilde{U}_n = U_n \subseteq O_n = O \cap V_n \subseteq (V_k + O) \cap V_n,$$

since V_k is a vector space containing O. It follows that $\tilde{U}_n \subseteq W_n$ for all $n \in \mathbb{N}$ and hence $\tilde{O}_n \subseteq W_n$, since W_n is already absolutely convex. Taking the union over n then gives $\tilde{O} \subseteq V_k + O$. For the opposite inclusion we note that trivially $U_n \subseteq \tilde{U}_n$ and hence $O_n \subseteq \tilde{O}_n$ for all $n \in \mathbb{N}$, leading to $O \subseteq \tilde{O}$. But also $V_k = \tilde{U}_k \subseteq \tilde{O}_k \subseteq \tilde{O}$. Thus consider $v \in V_k$ and $u \in O$. Since O is open, we can write $u = \lambda \tilde{u}$ with $\lambda \in (0, 1)$ and $\tilde{u} \in O$. Thus

$$v + u = (1-\lambda)\left(\tfrac{1}{1-\lambda} v\right) + \lambda \tilde{u}$$

is a convex combination of $\frac{1}{1-\lambda} v \in V_k \subseteq \tilde{O}$ and $\tilde{u} \in O \subseteq \tilde{O}$, hence again contained in the convex subset $\tilde{O}$. This shows $V_k + O \subseteq \tilde{O}$ and thus proves the last part. ☐

The next technical lemma is the key to understand the completeness of strict countable inductive limits:

Lemma 2.4.44 *Let $\{V_n\}_{n \in \mathbb{N}}$ be a sequence of locally convex spaces forming a strict countable inductive system with inductive limit $V = \varinjlim V_n$. Let $(v_\alpha)_{\alpha \in I}$ be a Cauchy net in V. Then there exists an $n_0 \in \mathbb{N}$ and a Cauchy net $(w_\beta)_{\beta \in J}$ in V_{n_0} with a cofinal map $J \ni \beta \mapsto \alpha(\beta) \in I$ such that*

$$\lim_{\beta \in J} (w_\beta - v_{\alpha(\beta)}) = 0. \tag{2.4.77}$$

Proof: Let $\mathcal{B}$ be a basis of open absolutely convex zero neighbourhoods in V, which we view as directed set as usual, i.e. $U \preccurlyeq \tilde{U}$ for $U, \tilde{U} \in \mathcal{B}$ if $\tilde{U} \subseteq U$. For $\alpha \in I$ and $U \in \mathcal{B}$ there is a minimal $n(\alpha, U)$ such that $V_{n(\alpha,U)} \cap (v_\alpha + U) \neq \emptyset$. Indeed, the set of $n \in \mathbb{N}$ with $V_n \cap (v_\alpha + U) \neq \emptyset$ is non-empty, as e.g. $n \in \mathbb{N}$ with $v_\alpha \in V_n$ belongs to this set. For every $\alpha \in I$ and $U \in \mathcal{B}$ we can thus choose a

$$w_{(\alpha,U)} \in V_{n(\alpha,U)} \cap (v_\alpha + U).$$

This gives a net $(w_{(\alpha,U)})_{(\alpha,U) \in I \times \mathcal{B}}$, where we use the canonical direction on the product $I \times \mathcal{B}$, i.e. $(\alpha, U) \preccurlyeq (\tilde{\alpha}, \tilde{U})$ if $\alpha \preccurlyeq \tilde{\alpha}$ and $U \preccurlyeq \tilde{U}$. The projection $I \times \mathcal{B} \ni (\alpha, U) \mapsto \alpha \in I$ is clearly a cofinal map. Since $w_{(\alpha,U)} - v_\alpha \in U$ by the choice of $w_{(\alpha,U)}$, we see that the net $(w_{(\alpha,U)} - v_\alpha)_{(\alpha,U)\in I\times\mathcal{B}}$ is convergent to zero, i.e. we have

$$\lim_{(\alpha,U)\in I\times\mathcal{B}} (w_{(\alpha,U)} - v_\alpha) = 0$$

in V. Moreover, the net $(w_{(\alpha,U)})_{(\alpha,U)\in I\times\mathcal{B}}$ is a Cauchy net in V. Indeed, let $O \subseteq V$ be a zero neighbourhood. Then we find a $U_0 \in \mathcal{B}$ such that $3U_0 \subseteq O$. Moreover, let $\alpha_0 \in I$ be such that $v_\alpha - v_{\alpha'} \in U_0$ for $\alpha, \alpha' \succcurlyeq \alpha_0$, which is possible be the Cauchy property. Then for $(\alpha, U), (\alpha', U') \succcurlyeq (\alpha_0, U_0)$ we have

$$w_{(\alpha,U)} - w_{(\alpha',U')} \in v_\alpha + U - (v_{\alpha'} + U') \subseteq v_\alpha - v_{\alpha'} + U - U \subseteq U_0 + U_0 + U_0 \subseteq O,$$

where we used the fact that U_0 is absolutely convex. This shows that $(w_{(\alpha,U)})_{(\alpha,U)\in I\times\mathcal{B}}$ is a Cauchy net in V. However, it has no reason to be contained in a single V_{n_0} yet. The idea is to pass to a suitable subnet of $(w_{(\alpha,U)})_{(\alpha,U)\in I\times\mathcal{B}}$. We define

$$J_m = \{(\alpha, U) \in I \times \mathcal{B} \mid n(\alpha, U) \leq m\} \subseteq I \times \mathcal{B}$$

for $m \in \mathbb{N}$. It follows that for $(\alpha, U) \in J_m$ we have $w_{(\alpha,U)} \in V_m$. Thus we need to show that there is an $n_0 \in \mathbb{N}$ with $J_{n_0} \subseteq I \times \mathcal{B}$ being a cofinal subset: in this case $(w_{(\alpha,U)})_{(\alpha,U)\in J_{n_0}}$ is a subnet meeting all requirements. Hence we assume that none of the subsets J_{n_0} is cofinal to eventually reach a contradiction. Since J_1 is not cofinal, we find $(\alpha_1, W_1) \in I \times \mathcal{B}$ such that $(\beta, W) \succcurlyeq (\alpha_1, W_1)$ implies $(\beta, W) \notin J_1$. Now we inductively construct an increasing sequence $(\alpha_1, W_1) \preccurlyeq (\alpha_2, W_2) \preccurlyeq \cdots$ with the property that $(\beta, W) \succcurlyeq (\alpha_n, W_n)$ implies $(\beta, W) \notin J_n$ for $n \in \mathbb{N}$. Indeed, suppose we have found $(\alpha_1, W_1) \preccurlyeq \cdots \preccurlyeq (\alpha_n, W_n)$ with this property. Since J_{n+1} is not cofinal, we find a $(\alpha, U) \in I \times \mathcal{B}$ with $(\beta, W) \succcurlyeq (\alpha, U)$ implies $(\beta, W) \notin J_{n+1}$. Now we can choose $(\alpha_{n+1}, W_{n+1}) \succcurlyeq (\alpha, U), (\alpha_n, W_n)$, since $I \times \mathcal{B}$ is directed, which allows to proceed inductively. Since each $W_n \subseteq V$ is an absolutely convex open neighbourhood of zero, the intersections $U_n = V_n \cap W_n$ give absolutely convex open neighbourhoods of zero in V_n. Moreover, by construction we have $W_n \supseteq W_{n+1}$ and hence

$$U_{n+1} \cap V_n = (W_{n+1} \cap V_{n+1}) \cap V_n \subseteq W_n \cap V_n = U_n.$$

Thus we can perform the construction of Lemma 2.4.43, *i.)*, to obtain absolutely convex open zero neighbourhoods $O_n \subseteq V_n$ with $O_{n+1} \cap V_n = O_n$ and $U_n \subseteq O_n$ leading to an absolutely convex open zero neighbourhood $O = \bigcup_{n\in\mathbb{N}} O_n \subseteq V$. To this O we apply the Cauchy condition: we find an index $\alpha_0 \in I$ with $v_\alpha - v_{\alpha'} \in O$ for all $\alpha, \alpha' \succcurlyeq \alpha_0$. Since $V = \bigcup_{n\in\mathbb{N}} V_n$, we have a $k \in \mathbb{N}$ such that $v_{\alpha_0} \in V_k$. For this k we consider the index α_k of the above constructed sequence. For these two indices α_0 and α_k we find a later $\beta \succcurlyeq \alpha_0, \alpha_k$, since I is ordered. We consider the corresponding v_β to arrive at the desired contradiction. First we have $\beta \succcurlyeq \alpha_0$ and hence $v_\beta - v_{\alpha_0} \in O$ by the Cauchy condition. Thus

$$v_\beta \in v_{\alpha_0} + O \subseteq V_k + O. \tag{$*$}$$

On the other hand, $\beta \succcurlyeq \alpha_k$ and hence $(\beta, W_k) \succcurlyeq (\alpha_k, W_k)$. By construction of (α_k, W_k) this means $(\beta, W_k) \notin J_k$, which means $n(\beta, W_k) > k$. Thus $V_k \cap (v_\beta + W_k) = \emptyset$ and hence

$$v_\beta \notin V_k - W_k = V_k + W_k, \tag{$**$}$$

since W_k is balanced. Finally, we make use of Lemma 2.4.43, *iii.)*. The subset O is obtained from the U_n and

$$V_k + O = \bigcup_{n\in\mathbb{N}} \tilde{U}_n \quad \text{with} \quad \tilde{U}_n = \begin{cases} V_n & \text{for } n \le k \\ U_n & \text{for } n > k. \end{cases}$$

Analogously, the subset W_k is obtained from $Y_n = W_k \cap V_n$ as $W_k = \bigcap_{n\in\mathbb{N}} Y_n$ by Lemma 2.4.35, *ii.)*, since here we already have $Y_{n+1} \cap V_n = Y_n$. The corresponding $\tilde{Y}_n$ then yields the subset $V_k + W_k$ according to the construction in Lemma 2.4.43, *iii.)*. We have

$$\tilde{U}_n = \left.\begin{cases} V_n & \text{for } n \le k \\ W_n \cap V_n & \text{for } n > k \end{cases}\right\} = \left.\begin{cases} V_n & \text{for } n \le k \\ W_k \cap V_n & \text{for } n > k \end{cases}\right\} = \tilde{Y}_n,$$

since $W_n \subseteq W_k$ for $n > k$ by the construction of the sequence $(\alpha_1, W_1) \preccurlyeq (\alpha_2, W_2) \preccurlyeq \cdots$. But this implies

$$V_k + O \subseteq V_k + W_k.$$

With this inclusion, the two properties $(*)$ and $(**)$ of v_β become a contradiction. □

Remark 2.4.45 It is perhaps remarkable that in general the new net $(w_\beta)_{\beta\in J}$ can not be chosen to be a subnet of the Cauchy net $(v_\alpha)_{\alpha\in I}$ directly. In general a Cauchy net has no subnet contained in a fixed V_{n_0}. This already fails for sequences, as in Example 2.4.38. The reason is that the V_n might not be closed in V.

In any case, this technical statement is now enough to obtain the following completeness result:

Theorem 2.4.46 (Completeness of inductive limits) *Let $\{V_n\}_{n\in\mathbb{N}}$ be a sequence of complete locally convex spaces forming a strict countable inductive system. Then their inductive limit $V = \varinjlim V_n$ is again complete.*

PROOF: This is now a trivial consequence of the previous Lemma 2.4.44. Let $(v_\alpha)_{\alpha\in I}$ be a Cauchy net in V. Then we have an index $n_0 \in \mathbb{N}$ and a Cauchy net $(w_\beta)_{\beta\in J}$ with a cofinal map $J \ni \beta \mapsto \alpha(\beta) \in I$ such that

$$\lim_{\beta\in J}(w_\beta - v_{\alpha(\beta)}) = 0. \tag{$*$}$$

As V_{n_0} is complete, we have a limit $w = \lim_{\beta\in J} w_\beta \in V_{n_0}$. This gives a convergent subnet $(v_{\alpha(\beta)})_{\beta\in J}$ of the Cauchy net with limit $\lim_{\beta\in J} v_{\alpha(\beta)} = w$ thanks to $(*)$. But then also $(v_\alpha)_{\alpha\in I}$ converges to $w \in V$, showing the completeness. □

Remark 2.4.47 The inductive limit can very well be complete, even if the participating spaces V_n are not, see again Example 2.4.37 and Example 2.4.38.

There are now trivial cases of strict countable inductive limits, namely those where the sequence stabilizes at some point. If we exclude these cases, one arrives at the definition of a LF space:

Definition 2.4.48 (LF and LB spaces) *Let V be a locally convex space.*

i.) The space V is called an LF (limit of Fréchet) space if its topology can be obtained as countable strict inductive limit of a sequence $\{V_n\}_{n\in\mathbb{N}}$ of Fréchet spaces with

$$V_{n+1} \setminus V_n \neq \emptyset \tag{2.4.78}$$

for all $n \in \mathbb{N}$.

ii.) An LF space is called LB (limit of Banach) space if the V_n can be chosen to be Banach spaces.

The non-triviality assumption (2.4.78) forbids LF spaces to be just Fréchet spaces. Instead, they turn out to be fairly complicated, as we obtain our first relevant examples of locally convex spaces, which are not Baire spaces. Here a Baire space is a topological space in which complements of meager subsets are necessarily dense, see e.g. [19, sect. 7.1]. Moreover, it gives the perhaps most important class of examples of locally convex spaces which are not first countable:

Proposition 2.4.49 *Let V be an LF space.*

i.) The space V is meager (or of first category).

ii.) The space V is a complete locally convex space.

iii.) The space V is not a Baire space.

iv.) The space V is not first countable.

PROOF: Let $V \cong \varinjlim V_n$ with Fréchet spaces V_n forming a countable strict inductive system. Then each V_n can be viewed as closed subspace of V by Proposition 2.4.39, since V_n is complete. The assumption (2.4.78) then shows $V_n \neq V$ for all $n \in \mathbb{N}$ but $V = \bigcup_{n\in\mathbb{N}} V_n$ as a vector space. For a proper subspace V_n in a topological space V we necessarily have $V_n^\circ = \emptyset$ by the translation invariance of the topology and Remark 2.1.4, *iv.)*, see also Exercise 2.5.1. Thus V is a countable union of nowhere dense subsets, which is the first statement. From this it follows that V can not be a Baire space. The completeness is ensured by Theorem 2.4.46. But then V can not be first countable, since otherwise it would be a Fréchet space and hence a complete metric space by Theorem 2.2.37, contradicting Baire's Theorem for complete metric spaces, see e.g. [18, Theorem 7.2.1]. □

The following standard example of an LF space is the main motivation for the definition of LF spaces. It also serves as the starting point for Schwartz's theory of distributions which we will discuss in detail in Chapter 5. At last, we found the "correct" locally convex topology for the space of test functions:

Proposition 2.4.50 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $k \in \mathbb{N}_0 \cup \{\infty\}$. The locally convex inductive limit topology on*

$$\mathscr{C}_0^k(X) = \varinjlim_{K\subseteq X} \mathscr{C}_K^k(X), \tag{2.4.79}$$

where $K \subseteq X$ ranges over all compact subsets of X and where we use the canonical inclusions $\mathscr{C}_K^k(X) \longrightarrow \mathscr{C}_{K'}^k(X)$ for $K \subseteq K'$, turns $\mathscr{C}_0^k(X)$ into an LF space.

PROOF: First we note that the inclusion maps

$$\mathscr{C}_K^k(X) \longrightarrow \mathscr{C}_{K'}^k(X)$$

for $K \subseteq K'$ are closed embeddings of Fréchet spaces with respect to the $\mathscr{C}^k$-topology on these subspaces of $\mathscr{C}^k(X)$. Thus the inductive limit is strict. To see that there exists a countable sequence of Fréchet spaces resulting in the same topology we choose a sequence

$$K_1 \subseteq K_2^\circ \subseteq K_2 \subseteq \cdots K_n \subseteq K_{m+1}^\circ \subseteq K_{m+1} \subseteq \cdots \subseteq X$$

of exhausting compact subsets $X = \bigcup_{m\in\mathbb{N}} K_m$ according to Exercise 1.4.14. This allows the endow $\mathscr{C}_0^k(X)$ with the structure of an LF space by taking just $\varinjlim \mathscr{C}_{K_m}^k(X)$. Of course, a priori, the topology depends on the choice of the K_m. Denote this LF space by V, then we have for every compact $K \subseteq X$ an index $m \in \mathbb{N}$ with $K \subseteq K_m$. Hence we get a system of continuous linear map leading to the commutative diagram

$$
\begin{tikzcd}
\mathscr{C}^k_K(X) \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[rrd] & \mathscr{C}^k_{K'}(X) \arrow[r] \arrow[d] \arrow[rd] & \mathscr{C}^k_0(X) \arrow[d, "\mathrm{id}"] \\
\mathscr{C}^k_{K_m}(X) \arrow[r] \arrow[rr, bend right] & \mathscr{C}^k_{K_{m'}}(X) \arrow[r] & V
\end{tikzcd}
\tag{$*$}
$$

of inclusions. Now the composition $\mathscr{C}^k_K(X) \longrightarrow \mathscr{C}^k_{K_m}(X) \longrightarrow V$ is continuous and satisfies the compatibility needed to apply the defining property of inductive limits to $\mathscr{C}^k_0(X)$: we get a unique continuous linear map $\mathscr{C}^k_0(X) \longrightarrow V$, such that the resulting diagram $(*)$ commutes. Clearly, this map is the identity, showing that the topology of $\mathscr{C}^k_0(X)$ is finer than the one of V. But the converse also holds, as $\mathscr{C}^k_{K_m}(X)$ are part of the defining inductive system for $\mathscr{C}^k_0(X)$. This results in a unique continuous linear map $\mathrm{id}\colon V \longrightarrow \mathscr{C}^k_0(X)$ by applying the defining property of inductive limits now to V. Hence the topologies coincide, which gives the result. We clearly have proper inclusions $\mathscr{C}^k_{K_m}(X) \subsetneq \mathscr{C}^k_{K_{m+1}}(X)$, whenever $K \subsetneq K'$ by the Urysohn Lemma, see Corollary 1.3.8. ☐

Of course, on could have defined the topology on $\mathscr{C}^k_0(X)$ directly by choosing the sequence $\{K_n\}_{n\in\mathbb{N}}$ of exhausting compact subsets of X. This would directly give an LF topology for $\mathscr{C}^k_0(X)$. However, one would need to show that the topology is actually independent of the choice of the compact subsets to have a reasonable definition. This is circumvented with the above definition. We will come back this fundamental example of an LF space several times. Related constructions are discussed in the Exercises 2.5.67, 2.5.68, 2.5.70, 2.5.71, 2.5.72, 2.5.73 and 2.5.74. Note also that this is perhaps the first really relevant example of a non-first countable topological space.

2.5 Exercises

Exercise 2.5.1 (Open interior of subspaces) Let $W \subseteq V$ be a subspace of a topological vector space V. Show that $W^\circ = \emptyset$ iff $W \neq V$.

Exercise 2.5.2 (Uniformities)

Exercise 2.5.3 (Canonical uniformity on topological vector spaces)

Exercise 2.5.4 (Uniqueness up to unique isomorphism)

Exercise 2.5.5 (Completion of topological vector space)

Exercise 2.5.6 (Characterization of the completion)

Exercise 2.5.7 (Completion functor)

Exercise 2.5.8 (Composition of balanced and convex hulls)

Exercise 2.5.9 (Interior and closure in topological vector spaces)

Exercise 2.5.10 (Non balanced open interior of balanced set) Find an easy example of a balanced set $B \subseteq \mathbb{R}^2$ with B° being non balanced. Why does your counterexample fail in one dimension?

Hint: The 11th Doctor is inspiration enough.

Exercise 2.5.11 (The space $\mathrm{L}^p([0,1], \mathrm{d}x)$ for $0 < p < 1$) Consider a set X with σ-algebra $\mathfrak{a} \subseteq 2^X$ and a positive measure μ defined on $\mathfrak{a}$. Let $0 < p < 1$ and consider the set of p-integrable functions

$$\mathscr{L}^p(X, \mathfrak{a}, \mu) = \left\{ f \in \mathscr{M}(X, \mathfrak{a}) \;\middle|\; \int_X |f(x)|^p \mu < \infty \right\}. \tag{2.5.1}$$

i.) Show that $\mathscr{L}^p(X, \mathfrak{a}, \mu)$ is a vector space and prove that

$$d(f, g) = \int_X |f(x) - g(x)|^p \,\mathrm{d}\mu \tag{2.5.2}$$

defines a translation invariant *pseudo-metric* turning $\mathscr{L}^p(X, \mathfrak{a}, \mu)$ into a (possibly non-Hausdorff) topological vector space

Hint: The relevant estimates can be obtained from Exercise **??**.

ii.) Show that $\{0\}^{\mathrm{cl}} \subseteq \mathscr{L}^p(X, \mathfrak{a}, \mu)$ consists of the functions vanishing outside of a zero set. Conclude that the Hausdorffization

$$\mathrm{L}^p(X, \mathfrak{a}, \mu) = \mathscr{L}^p(X, \mathfrak{a}, \mu) \big/ \{0\}^{\mathrm{cl}} \tag{2.5.3}$$

becomes a metric topological vector space.

iii.) Suppose $U \subseteq \mathrm{L}^p([0,1], \mathrm{d}x)$ is a convex subset with non-empty open interior $U^\circ \neq \emptyset$. Show that $U = \mathrm{L}^p([0,1], \mathrm{d}x)$. In particular, $\mathrm{L}^p([0,1], \mathrm{d}x)$ is not locally convex.

Hint: Argue that without restriction $0 \in U^\circ$. Let $\epsilon > 0$ with $\mathrm{B}_{d,\epsilon}(0) \subseteq U^\circ$. For $f \in \mathrm{L}^p([0,1], \mathrm{d}x)$ consider the function

$$F(x) = \int_0^x |f(t)|^p \,\mathrm{d}t$$

for $x \in [0,1]$. Show that F is continuous and monotonously increasing. Find for some $N \in \mathbb{N}$ large enough with $\epsilon N^{1-p} > d(f, 0)$ points $x_0 = 0 < x_1 < x_2 < \cdots < x_N = 1$ such that $F(x_k) = d(f, 0)\frac{k}{N}$ for all $k = 0, \ldots, N$. Consider then the sum of the functions $f_k = N f \chi_{(x_{k-1}, x_k]}$ for $k = 1, \ldots, N$ to show $f \in U$.

iv.) Show that the topological dual of $\mathrm{L}^p([0,1], \mathrm{d}x)$ is the zero vector space.

Exercise 2.5.12 (Equivalent norms)

Exercise 2.5.13 (Boundedness of convergent sequences)

Exercise 2.5.14 (Existence of norms)

Exercise 2.5.15 (Pull-back of seminorms)

Exercise 2.5.16 (Title needed)

Exercise 2.5.17 (The p-summable sequences)

Exercise 2.5.18 (Local essential supremum norm)

Exercise 2.5.19 (Local supremum norm with weights)

Exercise 2.5.20 (Seminorms for holomorphic functions)

Exercise 2.5.21 (Seminorms for the Schwartz space)

Exercise 2.5.22

Exercise 2.5.23 (Saturation)

Exercise 2.5.24 (Domination)

Exercise 2.5.25 (Continuity of seminorms)

Exercise 2.5.26 (Subspaces of locally convex spaces)

Exercise 2.5.27 (Unconditional and absolute convergence) Let $V = \ell^2$. Show that the series

$$\sum_{n=0}^{\infty} \frac{1}{n} \mathrm{e}_n \tag{2.5.4}$$

is unconditionally convergent, but not absolutely.

Exercise 2.5.28 (A bounded metric) A *pseudo-metric* on a set M is a map $d\colon M \times M \longrightarrow [0, \infty)$ with $d(x, y) = d(y, x)$, which satisfies the triangle inequality and $d(x, x) = 0$. Consider the function $f\colon [0, \infty) \ni x \mapsto \frac{x}{1+x} \in [0, 1)$.

i.) Show that f is a strictly increasing homeomorphism and compute its the inverse.

ii.) Assume that d is a pseudo-metric on M. Show that $f \circ d$ is again a pseudo-metric. Show that if in addition d is a metric, then $f \circ d$ is again a metric.

iii.) Show that for pseudo-metrics $\{d_n\}_{n \in \mathbb{N}}$ on M and $\alpha_n \geq 0$ the sum

$$d^{(N)}(x, y) = \sum_{n=1}^{N} \alpha_n d_n(x, y) \tag{2.5.5}$$

is again a pseudo-metric. Show that $d^{(\infty)}$ is a pseudo-metric if the series converges at all.

Exercise 2.5.29

Exercise 2.5.30 (Metric balls can be non-convex)

Exercise 2.5.31

Exercise 2.5.32 (Convex metric balls)

Exercise 2.5.33 (A non-metrizable locally convex space) Let I be an uncountable set. Show that $\operatorname{Map}(I, \mathbb{K})$ is not a Fréchet space.

Hint: Use Theorem 2.2.37, *i.)*.

Exercise 2.5.34 (Pointwise convergence is pointless)

Exercise 2.5.35 (The Banach space $\ell^p(I, \mu)$) Let $\mu = (\mu_i)_{i \in I}$ with $\mu_i > 0$ for all $i \in i$.

i.) Show that for elements in $\ell^p(I, \mu)$ and $c_\circ(I, \mu)$ at most countably many components are different from zero.

ii.) Show completeness of $\ell^p(I, \mu)$, $c(I, \mu)$ and $c_\circ(I, \mu)$.

iii.) Conclude taking limits is continuous by looking at its kernel.

iv.) Show density of $c_{\circ\circ}(I)$ in all of the other examples.

v.) Show that the canonical map $\ell^p \longrightarrow \ell^q$ is continuous with dense image.

Exercise 2.5.36 (The Schwartz sequence space)

Exercise 2.5.37 (Köthe space)

Exercise 2.5.38 (Weighted supremum norms)

Exercise 2.5.39 (Continuous linear maps)

Exercise 2.5.40 (The functor $\mathscr{C}$)

Exercise 2.5.41 (Vanishing ideals)

Exercise 2.5.42 (Algebraic properties of differential operators)

Exercise 2.5.43 (Differential operators as Fréchet space) Let $X \subseteq \mathbb{R}^n$ be open and non-empty. Show that the differential operators $\mathrm{DiffOp}^r(X)$ of order r are a Fréchet space with respect to the seminorms (2.3.12).

Exercise 2.5.44 (A theorem of Weierstraß) Prove Proposition **??** using Proposition 2.3.28.

Exercise 2.5.45 (Runge domains)

Exercise 2.5.46 (Pullback by holomorphic functions)

Exercise 2.5.47 (A non-filtrating system of seminorms)

Exercise 2.5.48 (Holomorphic Differential Operators)

Exercise 2.5.49 (Absolute convergence)

Exercise 2.5.50

Exercise 2.5.51 (Hausdorffness of the initial topology)

Exercise 2.5.52 (Boundedness for the initial topology) Let V be a $\mathbb{K}$-vector space and let $\phi_i \colon V \longrightarrow V_i$ be linear maps to topological vector spaces V_i for $i \in I$ with some index set I. Show that, if

$$B = \bigcap_{i \in I} \ker \phi_i \subseteq V \tag{2.5.6}$$

is non-empty, then it is not bounded. Note that $\phi_i(B) = \{0\}$ is always bounded.

Exercise 2.5.53 (Open subsets in the final locally convex topology)

Exercise 2.5.54

Exercise 2.5.55 (Locally convex quotient seminorms) Check properties of [p] defined in (2.4.10).

Exercise 2.5.56 (Hausdorffization)

Exercise 2.5.57 (Hausdorffization of $\mathscr{L}^p$-spaces)

Exercise 2.5.58 (Direct sum as a subspace of the Cartesian product) Let I be a non-empty index set and let $\{V_i\}_{i \in I}$ be a collection of locally convex spaces with locally convex direct sum $V = \bigoplus_{i \in I} V_i$.

i.) Show that the direct sum with the subspace topology inherited from $\prod_{i \in I} V_i$ is a dense subspace.

ii.) Show that the completion $\widehat{\bigoplus_{i \in I} V_i}$ is the direct sum of the completions $\hat{V}_i$.

iii.) Characterize the completion of the Cartesian product.

Exercise 2.5.59 (Completion of a normed space I)

Exercise 2.5.60 (Completion of a normed space II)

Exercise 2.5.61 (Pre-ordered sets as categories)

Exercise 2.5.62 (Projective limit of vector spaces)

Exercise 2.5.63

Exercise 2.5.64

Exercise 2.5.65 (Infinite-dimensional Banach spaces)

Exercise 2.5.66

Exercise 2.5.67 (The LF space $\mathscr{C}_0(X)$)

Exercise 2.5.68 (Functions with space-like compact support on Minkowski space)

Exercise 2.5.69 (Continuity for LF spaces)

Exercise 2.5.70

Exercise 2.5.71

Exercise 2.5.72

Exercise 2.5.73 (The LB space $\mathscr{C}_0^k(X)$)

Exercise 2.5.74 (Norms on LB spaces)

Chapter 3

The Main Theorems

In this chapter we will discuss some of the main results on locally convex spaces, including their applications. The perhaps most important statement is the Hahn-Banach Theorem, since it relies heavily on the locally convex nature: the topological dual space V' of a locally convex space is big in a very precise way. We can separate vectors from one another and also vectors from disjoint closed subspaces, we can extend continuous linear functionals and much more. All these statements will be consequences or incarnations of the Hahn-Banach Theorem. The other important theorems are the Principle of Uniform Boundedness in form of the Banach-Steinhaus Theorem, the Open Mapping Theorem and the Closed Graph Theorem. All three have a multitude of application. The Banach-Alaoglu Theorem relates then again the space V to its dual and will be one of the cornerstones of duality theory. With the Bipolar Theorem we can then characterize the bounded subsets of a locally convex space in a very efficient way. Finally, we discuss extreme points of convex subsets and their existence in the Krein-Milman Theorem. The main theme of this chapter is that many of the well-known fundamental theorems in the theory of Banach spaces have far-reaching generalizations in the context of general locally convex spaces.

3.1 The Hahn-Banach Theorem and Separation

The Hahn-Banach Theorem comes in several different flavours. The first formulation allows us to extend linear functionals from subspaces to the ambient space. While in linear algebra this can be done by e.g. choosing a basis of the subspace and extending it to a basis of the total space, in the context of locally convex spaces we want to have a compatibility with the topology. If the original functional was continuous, also the extension is required to be continuous. For a locally convex space this will immediately give the existence of many continuous linear functionals. But the Hahn-Banach Theorem can also be interpreted more geometrically as a separation property. Indeed, an interesting question is whether one can separate two disjoint subsets by a hyperplane defined by a continuous linear functional. In general, this is of course not possible, but under certain convexity assumptions one can show the existence of such separating hyperplanes.

3.1.1 The Hahn-Banach Theorem

We start with the algebraic and analytic formulation of the Hahn-Banach Theorem. The following definition combines linear functionals and seminorms:

Definition 3.1.1 (Sublinear functionals) *Let V be a vector space over $\mathbb{K}$. Then a map $\mathrm{p}\colon V \longrightarrow \mathbb{R}$ is called sublinear if*

$$\mathrm{p}(v + w) \leq \mathrm{p}(v) + \mathrm{p}(w) \tag{3.1.1}$$

and

$$\mathrm{p}(\lambda v) = \lambda \mathrm{p}(v) \tag{3.1.2}$$

for all $v, w \in V$ and $\lambda \geq 0$. The set of sublinear functionals on V is denoted by $V^\sharp$.

Clearly, every $\mathbb{R}$-linear functional is sublinear, but also every seminorm is a sublinear functional. Note that the definition refers only to the underlying structure as a vector space over $\mathbb{R}$, no matter whether $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

Remark 3.1.2 Let V be a $\mathbb{K}$-vector space.

i.) We can use the ordering of $\mathbb{R}$ to define a partial order on the set of sublinear functionals. For $\mathrm{p}, \mathrm{q} \in V^\sharp$ one defines $\mathrm{p} \preccurlyeq \mathrm{q}$ if $\mathrm{p}(v) \leq \mathrm{q}(v)$ for all $v \in V$. This clearly gives a partial order.

ii.) If $\mathrm{p}, \mathrm{q} \in V^\sharp$ and $\alpha, \beta \geq 0$, then also $\alpha\mathrm{p} + \beta\mathrm{q} \in V^\sharp$.

iii.) The set $V^\sharp$ is not only partially ordered, but also directed, since for $\mathrm{p}, \mathrm{q} \in V^\sharp$ we have $\max\{\mathrm{p}, \mathrm{q}\} \in V^\sharp$, dominating both p and q.

The partial order of $V^\sharp$ has minimal elements, which can be characterized explicitly. The minimal elements are precisely the $\mathbb{R}$-linear functionals:

Lemma 3.1.3 *Let V be a real vector space. Then the set of minimal elements of $V^\sharp$ coincides with the algebraic dual $V^* \subseteq V^\sharp$.*

Proof: First we prove that a linear functional $\varphi \in V^*$ is minimal. Thus let $\mathrm{p} \in V^\sharp$ with $\mathrm{p} \preccurlyeq \varphi$ be given. Then $0 = \mathrm{p}(0) \leq \mathrm{p}(v) + \mathrm{p}(-v)$ for all $v \in V$. Now $\mathrm{p}(v) \leq \varphi(v)$, as well as $\mathrm{p}(-v) \leq \varphi(-v) = -\varphi(v)$, showing $\mathrm{p}(v) + \mathrm{p}(-v) \leq \varphi(v) - \varphi(v) = 0$ and hence $\mathrm{p}(v) = -\mathrm{p}(-v)$. Together, this gives $\mathrm{p}(v) \leq \varphi(v)$ and $-\mathrm{p}(v) = \mathrm{p}(-v) \leq \varphi(-v) = -\varphi(v)$. Hence $\mathrm{p} = \varphi$. Thus φ is indeed minimal. Conversely, assume $\mathrm{p} \in V^\sharp$ is minimal. We want to show that p is actually linear. Fix $w \in V$ and define the map

$$\mathrm{q}\colon V \ni v \;\mapsto\; \inf_{\lambda \geq 0} \big\{ \mathrm{p}(v + \lambda w) - \lambda \mathrm{p}(w) \big\} \in [-\infty, \infty). \tag{$*$}$$

We claim that this infimum is actually not $-\infty$ and hence we have a map $\mathrm{q}\colon V \longrightarrow \mathbb{R}$. Indeed, from $\lambda\mathrm{p}(w) = \mathrm{p}(\lambda w) \leq \mathrm{p}(v + \lambda w) + \mathrm{p}(-v)$ we see that

$$-\mathrm{p}(-v) \leq \mathrm{p}(v + \lambda w) - \lambda \mathrm{p}(w)$$

for all $\lambda \geq 0$ and $v \in V$. Thus $\mathrm{q}(v) \geq -\mathrm{p}(-v) > -\infty$ for all $v \in V$. Taking $\lambda = 0$ in $(*)$ shows $\mathrm{q}(v) \leq \mathrm{p}(v)$ for all $v \in V$. We claim that $\mathrm{q} \in V^\sharp$. For $\mu > 0$ we get

$$\begin{aligned}
\mathrm{q}(\mu v) &= \inf_{\lambda \geq 0} \big\{ \mathrm{p}(\mu v + \lambda w) - \lambda \mathrm{p}(w) \big\} \\
&= \inf_{\lambda \geq 0} \big\{ \mu \mathrm{p}(v + \tfrac{\lambda}{\mu} w) - \mu \tfrac{\lambda}{\mu} \mathrm{p}(w) \big\} \\
&= \mu \inf_{\lambda \geq 0} \big\{ \mathrm{p}(v + \lambda w) - \lambda \mathrm{p}(w) \big\} \\
&= \mu \mathrm{q}(v).
\end{aligned}$$

Since $\mathrm{q}(0) = 0$, we have $\mathrm{q}(\mu v) = \mu \mathrm{q}(v)$ for all $\mu \geq 0$, showing (3.1.2) for q. To show (3.1.1) we fix $\epsilon > 0$ and $v_1, v_2 \in V$. Then we find $\lambda_1, \lambda_2 \geq 0$ such that

$$\mathrm{q}(v_i) + \frac{\epsilon}{2} \geq \mathrm{p}(v_i + \lambda_i w) - \lambda_i \mathrm{p}(w)$$

for $i = 1, 2$. Taking the sum of these two inequalities gives

$$\mathrm{q}(v_1) + \mathrm{q}(v_2) \geq \mathrm{p}(v_1 + \lambda_1 w) - \lambda_1 \mathrm{p}(w) + \mathrm{p}(v_2 + \lambda_2 w) - \lambda_2 \mathrm{p}(w) - \epsilon$$

$$
\begin{aligned}
&\geq \mathrm{p}(v_1 + v_2 + (\lambda_1 + \lambda_2)) - (\lambda_1 + \lambda_2)\mathrm{p}(w) - \epsilon \\
&\geq \mathrm{q}(v_1 + v_2) - \epsilon,
\end{aligned}
$$

since p is sublinear. As $\epsilon > 0$ is arbitrary, we get $\mathrm{q}(v_1 + v_2) \leq \mathrm{q}(v_1) + \mathrm{q}(v_2)$ for all $v_1, v_2 \in V$ and hence $\mathrm{q} \in V^\sharp$, as claimed. Since p is minimal by assumption and clearly $\mathrm{q} \preccurlyeq \mathrm{p}$, we have $\mathrm{q} = \mathrm{p}$. Hence we get

$$
\mathrm{p}(v) = \mathrm{q}(v) \leq \mathrm{p}(v + w) - \mathrm{p}(w) \leq \mathrm{p}(v),
$$

showing $\mathrm{p}(v + w) = \mathrm{p}(v) + \mathrm{p}(w)$. But this already implies $\mathbb{R}$-linearity, since $\mathrm{p}(\lambda v) = \lambda \mathrm{p}(v)$ for $\lambda \geq 0$ and $\mathrm{p}(-v) + \mathrm{p}(v) = 0$ according to $\mathrm{p}(0) = 0$ and the additivity. □

The next lemma shows that for every element in $V^\sharp$ we indeed have a smaller minimal, i.e. linear, one. As this already implies that V^* is non-vacuous, the need to use the Axiom of Choice in form of Zorn's Lemma is not surprising. Again, it suffices to consider V as a real vector space.

Lemma 3.1.4 *Let V be a real vector space. Then for every $\mathrm{p} \in V^\sharp$ there exists a $\varphi \in V^*$ with $\varphi \preccurlyeq \mathrm{p}$.*

PROOF: Let $\mathrm{p} \in V^\sharp$ be given and consider those elements of $V^\sharp$, which are smaller than p, i.e.

$$
\mathcal{P} = \left\{ \mathrm{q} \in V^\sharp \;\middle|\; \mathrm{q} \preccurlyeq \mathrm{p} \right\},
$$

and endow $\mathcal{P}$ with the partial order $\preccurlyeq$ inherited from $V^\sharp$. Since $\mathrm{p} \in \mathcal{P}$, we have $\mathcal{P} \neq \emptyset$. Let $\mathcal{Q} = \{\mathrm{q}_i\}_{i \in I} \subseteq \mathcal{P}$ be a totally ordered subset. We claim that for a given $v \in V$ the set $\{\mathrm{q}_i(v)\}_{i \in I} \subseteq \mathbb{R}$ is bounded from below. Assume the converse and let $\mathrm{q}_{i_1}, \mathrm{q}_{i_2}, \ldots$ be a sequence in $\mathcal{Q}$ with $\mathrm{q}_{i_n}(v) \leq -n$. Since $\mathcal{Q}$ is totally ordered, the finite subset $\mathrm{q}_{i_1}, \ldots, \mathrm{q}_{i_n}$ has a minimum $\mathrm{p}_n = \min\{\mathrm{q}_{i_1}, \ldots, \mathrm{q}_{i_n}\} \in V^\sharp$. We have $\mathrm{p}_n(v) \leq -n$ and $\mathrm{p}_{n+1} \preccurlyeq \mathrm{p}_n$ for all $n \in \mathbb{N}$. Moreover, the sublinearity gives

$$
0 \leq \mathrm{p}_n(v) + \mathrm{p}_n(-v) \leq -n + \mathrm{p}_1(-v)
$$

for all $n \in \mathbb{N}$, which is nonsense. Thus the pointwise infimum

$$
\mathrm{p}_{\mathrm{inf}}(v) = \inf_{i \in I} \left\{ \mathrm{q}_i(v) \right\} \in \mathbb{R}
$$

is finite and yields a map $\mathrm{p}_{\mathrm{inf}} \colon V \longrightarrow \mathbb{R}$. It is now easy to see that the pointwise infimum of sublinear maps is again a sublinear map, provided it is pointwise finite at all. Since we have shown the latter, we have $\mathrm{p}_{\mathrm{inf}} \in V^\sharp$ and $\mathrm{p}_{\mathrm{inf}} \preccurlyeq \mathrm{q}_i \preccurlyeq \mathrm{q}$ for all $i \in I$. This shows that every totally ordered subset $\mathcal{Q} \subseteq \mathcal{P}$ has an infimum $\mathrm{p}_{\mathrm{inf}} \in \mathcal{P}$. Now we are in the position to use Zorn's Lemma to infer the existence of a minimal element φ in $\mathcal{P}$. If φ is such a minimal element, it is also minimal in $V^\sharp$, for if there would be a $\tilde{\mathrm{p}} \preccurlyeq \varphi$, then also $\tilde{\mathrm{p}} \preccurlyeq \mathrm{p}$ and thus $\tilde{\mathrm{p}} \in \mathcal{P}$. Hence $\varphi \preccurlyeq \tilde{\mathrm{p}}$ and $\tilde{\mathrm{p}} = \varphi$ follows. From the previous lemma we conclude $\varphi \in V^*$. □

The first version of the Hahn-Banach Theorem is now the following extension of a linear functional bounded by a sublinear functional:

Theorem 3.1.5 (Hahn-Banach) *Let V be a vector space over $\mathbb{K}$ and let $U \subseteq V$ be a subspace. Let $\mathrm{p} \in V^\sharp$ and $\varphi \in U^*$ be given such that*

$$
\operatorname{Re}(\varphi(u)) \leq \mathrm{p}(u) \tag{3.1.3}
$$

holds for all $u \in U$. Then there exists an extension $\Phi \in V^*$ of φ to a $\mathbb{K}$-linear functional on V, such that for all $v \in V$ one has

$$
\operatorname{Re}(\Phi(v)) \leq \mathrm{p}(v). \tag{3.1.4}
$$

Proof: First we consider the real case $\mathbb{K} = \mathbb{R}$, where we have $\operatorname{Re}(\varphi(u)) = \varphi(u)$. By assumption, we have $\varphi(u) = -\varphi(-u) \geq -\mathrm{p}(-u)$ for all $u \in U$, as well as $\mathrm{p}(-u) \leq \mathrm{p}(v-u) + \mathrm{p}(-v)$ for all $u \in U$ and $v \in V$ by sublinearity. Together, this gives

$$-\mathrm{p}(v) = \mathrm{p}(-u) - \mathrm{p}(-v) - \mathrm{p}(-u) \leq \mathrm{p}(v-u) + \mathrm{p}(-v) - \mathrm{p}(-v) + \varphi(u) = \mathrm{p}(v-u) + \varphi(u).$$

It follows that

$$\tilde{\mathrm{p}}(v) = \inf_{u \in U} \left\{ \mathrm{p}(v-u) + \varphi(u) \right\} > -\infty$$

is finite for all $v \in V$. With a similar argument as in the proof of Lemma 3.1.3, we see that $\tilde{\mathrm{p}} \in V^\sharp$ is actually sublinear, see also Exercise 3.6.1. Moreover, taking $u = 0$ gives immediately $\tilde{\mathrm{p}} \preccurlyeq \mathrm{p}$. For $v \in U \subseteq V$ we even get $\tilde{\mathrm{p}}(v) \leq \varphi(v)$, again by considering $u = 0$ in the above infimum. From Lemma 3.1.4 we get a $\Phi \in V^*$ with $\Phi \preccurlyeq \tilde{\mathrm{p}} \preccurlyeq \mathrm{p}$. Restricting Φ to U gives then $\Phi\big|_U \preccurlyeq \varphi$. But $\varphi \in U^*$ is already minimal, hence $\Phi\big|_U = \varphi$ follows. This shows the real case. For the complex case we can use the real case to obtain an extension $\Psi \in \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$ of $\operatorname{Re}(\varphi)$ with $\Psi \preccurlyeq \mathrm{p}$. Then we define

$$\Phi(v) = \Psi(v) - \mathrm{i}\Psi(\mathrm{i}v),$$

which is easily seen to be a $\mathbb{C}$-linear functional on V. Moreover, $\operatorname{Re}(\Phi) = \Psi$ and for $u \in U$ we have

$$\Phi(u) = \Psi(u) - \mathrm{i}\Psi(\mathrm{i}u) = \operatorname{Re}(\varphi(u)) - \mathrm{i}\operatorname{Re}(\varphi(\mathrm{i}u)) = \varphi(u),$$

since φ is already $\mathbb{C}$-linear. Hence $\Phi\big|_U = \varphi$. Finally, $\operatorname{Re}(\Phi(v)) = \Psi(v) \leq \mathrm{p}(v)$ for all $v \in V$ by the choice of Ψ. ☐

The above formulation of the Hahn-Banach Theorem is still quite technical. However, as the following corollaries show, we will obtain many useful consequences at once. We start with the following application:

Corollary 3.1.6 *Let V be a vector space over $\mathbb{K}$ and let $U \subseteq V$ be a subspace. If p is a seminorm on V and $\varphi \in U^*$ is a linear functional on U with $|\varphi(u)| \leq \mathrm{p}(u)$ for all $u \in U$, then there exists an extension $\Phi \in V^*$ of φ with*

$$|\Phi(v)| \leq \mathrm{p}(v) \tag{3.1.5}$$

for all $v \in V$.

Proof: We consider first the complex case, the real case is even simpler. By the Hahn-Banach Theorem 3.1.5 we find an extension $\Phi \in V^*$ of φ with $\operatorname{Re}(\Phi(v)) \leq \mathrm{p}(v)$ for all $v \in V$. Write $\Phi(v) = \mathrm{e}^{\mathrm{i}\alpha} r$ for $r \geq 0$ and some phase $\mathrm{e}^{\mathrm{i}\alpha} \in \mathbb{S}^1$. Then

$$|\Phi(v)| = r = \mathrm{e}^{-\mathrm{i}\alpha}\Phi(v) = \Phi(\mathrm{e}^{-\mathrm{i}\alpha}v) = \operatorname{Re}(\Phi(\mathrm{e}^{-\mathrm{i}\alpha}v)) \leq \mathrm{p}(\mathrm{e}^{-\mathrm{i}\alpha}v) = \mathrm{p}(v)$$

shows (3.1.5), since Φ is linear and p is a seminorm. For the real case we have $\mathrm{e}^{-\mathrm{i}\alpha} \in \{-1, 1\}$. ☐

This observation can be used to show the existence of continuous linear functionals on locally convex spaces:

Corollary 3.1.7 *Let V be a locally convex space with a subspace $U \subseteq V$. If $\varphi \in U'$ is a continuous linear functional on U, then there is a continuous linear extension $\Phi \in V'$ of φ.*

Proof: Indeed, the continuity of φ with respect to the subspace topology means that there is a continuous seminorm p on V with $|\varphi(u)| \leq \mathrm{p}(u)$ for all $u \in U$, see Corollary 2.2.27. Then we can apply Corollary 3.1.6. ☐

Note that we even can arrange the extension Φ in such a way that we stay with the same continuity estimate, i.e. if $|\varphi(u)| \leq \mathrm{p}(u)$ for all $u \in U$, we can use the same seminorm p to get

$$|\Phi(v)| \leq \mathrm{p}(v). \tag{3.1.6}$$

While in the general locally convex case this might not be very significant, it gives a strong result in those cases, where a specific choice of a seminorm is important, e.g. in the case of Banach spaces.

In the case of a Hausdorff locally convex space, we have many continuous seminorms by Theorem 2.2.13, *ii.)*. This allows to characterize Hausdorff locally convex spaces as follows:

Corollary 3.1.8 *Let V be a locally convex space. Then V is Hausdorff iff for every $0 \neq v \in V$ there exists a continuous linear functional $\varphi \in V'$ with*

$$\varphi(v) \neq 0. \tag{3.1.7}$$

PROOF: Assume V is Hausdorff and let $v \neq 0$. Then we have a continuous seminorm p on V with $\mathrm{p}(v) > 0$ according to Theorem 2.2.13, *vi.)*. Consider the one-dimensional subspace $U = \operatorname{span}_{\mathbb{K}}\{v\} \subseteq V$ and define $\varphi \in U^*$ by $\varphi(v) = \mathrm{p}(v)$. It follows that $|\varphi(u)| = \mathrm{p}(u)$ for all $u \in U$ and hence $\varphi \in U'$. Then we can apply Corollary 3.1.7. The converse is clear, for if $\varphi \in V'$ satisfies $\varphi(v) \neq 0$, then $|\varphi(v)| > 0$ and $\mathrm{p} = |\cdot| \circ \varphi$ is a continuous seminorm on V. □

A generalization of this corollary can be obtained by replacing $0 \in V$ with a closed subspace: then this becomes a first instance of a separation statement:

Corollary 3.1.9 *Let V be a locally convex space. Suppose $U = U^{\mathrm{cl}} \subseteq V$ is a closed subspace and $v \in V \setminus U$. Then there exists a continuous linear functional $\varphi \in V'$ with*

$$\varphi(v) \neq 0 \quad \textit{and} \quad \varphi\Big|_U = 0. \tag{3.1.8}$$

PROOF: Since U is closed, the quotient V/U becomes a Hausdorff locally convex space with $[v] \neq 0$, since $v \notin U$, see Proposition 2.4.10, *iii.)*. Then Corollary 3.1.8 provides a continuous linear $\psi \in (V/U)'$ with $\psi([v]) \neq 0$. Now $\varphi = \mathrm{pr}^* \psi = \psi \circ \mathrm{pr}$ will do the job, where $\mathrm{pr}\colon V \longrightarrow V/U$ is the continuous canonical projection as usual. □

3.1.2 Separation Properties

While the Hahn-Banach Theorem guarantees first of all a wealth of non-trivial continuous linear functionals on locally convex spaces, another interpretation comes from the following separation problems: given two disjoint subsets $A, B \subseteq V$ in a real vector space V, can one separate them by a *hyperplane*? Here we mean a codimension one affine subspace such that A lies on one side and B on the other. Since such a subspace can be encoded by a linear functional φ as a shifted version of the kernel $\ker \varphi$, the question actually makes sense in this context and means that for all $a \in A$ and $b \in B$ one wants

$$\varphi(a) < \varphi(b). \tag{3.1.9}$$

Of course, for general A and B the condition $A \cap B = \emptyset$ does not yet imply the existence of $\varphi \in V^*$ with (3.1.9). Trivial counterexamples are easily found. This changes, when A and B are assumed to be convex. In our context, we also want the separating functional φ to be continuous in addition. This leads to the first formulation of the separation theorem, still working for a general topological vector space:

Theorem 3.1.10 (Separation I) *Let V be a topological vector space over $\mathbb{K}$ with two non-empty convex disjoint subsets $A, B \subseteq V$.*

i.) If A is open, then there exists a continuous linear functional $\Phi \in V'$ and a real number $\alpha \in \mathbb{R}$ such that

$$\operatorname{Re}(\Phi(v)) < \alpha \leq \operatorname{Re}(\Phi(u)) \tag{3.1.10}$$

for all $v \in A$ and $u \in B$.

ii.) If A and B are open, there exists a continuous linear functional $\Phi \in V'$ and a real number $\alpha \in \mathbb{R}$ such that

$$\operatorname{Re}(\Phi(v)) < \alpha < \operatorname{Re}(\Phi(u)) \tag{3.1.11}$$

for all $v \in A$ and $u \in B$.

Proof: If V is a complex vector space and we have found an $\mathbb{R}$-linear functional $\Psi\colon V \longrightarrow \mathbb{R}$ with (3.1.10), then $\Phi(v) = \Psi(v) - \mathrm{i}\Psi(\mathrm{i}v)$ defines a $\mathbb{C}$-linear functional satisfying (3.1.10), as $\operatorname{Re}\Phi = \Psi$. Hence it suffices to consider the case $\mathbb{K} = \mathbb{R}$ from the beginning. We consider first the case, where A is open and B is arbitrary. Then for $v_0 \in A$ and $w_0 \in B$ the subset

$$C = A - B + u_0 = \bigcup_{w \in B} (A - w) + u_0$$

with $u_0 = w_0 - v_0$ is a union of open subsets, since the topology is translation invariant. Hence C is open. Moreover, taking $w = w_0$ shows that $v_0 - w_0 + w_0 - v_0 = 0 \in C$. Thus C is a neighbourhood of zero and hence absorbing. Finally, C turns out to be convex, as a simple verification shows. Let p_C be the Minkowski functional of C, which is a well-defined sublinear functional on V by Lemma 2.2.19. Now, $A \cap B = \emptyset$ implies $u_0 \notin C$ and hence $\mathrm{p}_C(u_0) \geq 1$ follows. We define an $\mathbb{R}$-linear functional ψ on the subspace $\operatorname{span}_{\mathbb{R}}\{u_0\}$ by $\psi(\lambda u_0) = \lambda$ for $\lambda \in \mathbb{R}$. For $\lambda \geq 0$ we have

$$\psi(\lambda u_0) = \lambda \leq \lambda \mathrm{p}_C(u_0) = \mathrm{p}_C(\lambda u_0),$$

while trivially $\psi(\lambda u_0) = \lambda < 0 \leq \mathrm{p}_C(\lambda u_0)$ for $\lambda < 0$. Thus the linear functional ψ is dominated by the sublinear functional p_C. According to the Hahn-Banach Theorem 3.1.5 we get an extension Ψ of ψ to V as an $\mathbb{R}$-linear functional still satisfying

$$\Psi(v) \leq \mathrm{p}_C(v)$$

for all $v \in V$. For $v \in C$ we know $\mathrm{p}_C(v) \leq 1$ and hence $\Psi(v) \leq 1$ as well. For $v \in -C$ we get analogously $\Psi(v) \geq -1$, resulting in

$$|\Psi(v)| \leq 1$$

for $v \in C \cap (-C)$. This intersection is still an open neighbourhood of zero and hence Ψ is bounded on this zero neighbourhood. By Exercise 3.6.2 the linear functional Ψ is continuous. We claim that this Ψ does the job: for $v \in A$ and $u \in B$ we get

$$\Psi(v) - \Psi(u) + 1 = \Psi(v - u + u_0) \leq \mathrm{p}_C(v - u + u_0) < 1,$$

since by the openness of C we get from $v - u + u_0 \in C$ also $v - u + u_0 \in (1 - \epsilon)C$ for some small enough $\epsilon > 0$. Hence the Minkowski functional yields a value strictly less than one. It follows that

$$\Psi(v) < \Psi(u) \tag{$*$}$$

for all $v \in A$ and $u \in B$. The two cases (3.1.10) and (3.1.11) can now be explained as follows. We know that both subsets $\Psi(A)$ and $\Psi(B)$ are convex, since Ψ is linear. Moreover, Ψ is an open map and maps A to an open subset of $\mathbb{R}$. Finally, $\Psi(A)$ is bounded to the right by $(*)$. This leaves the only possibility

$$\Psi(A) = (\omega, \alpha)$$

with $\omega \in [-\infty, \alpha)$ and some $\alpha \in \mathbb{R}$. It follows from $(*)$ that α will do the job for (3.1.10). If in addition B is open, too, then $\Psi(B)$ is open, convex and bounded to the left by $(*)$. Hence $\Psi(B) = (\beta, \omega')$ with some $\omega' \in (\beta, +\infty]$ and $\beta \geq \alpha$. Any real number in $[\alpha, \beta]$, e.g. α itself, will then satisfy (3.1.11). $\square$

Remark 3.1.11 Already simple examples for the case $V = \mathbb{R}$ show that the estimates (3.1.10) and (3.1.11) can not be improved in general, see also Exercise 3.6.3 for some further illustration.

The usual difficulty with this formulation of the separation theorem is the assumption to have at least one of the convex subsets to be *open*. As we have seen in Exercise 2.5.11 there are topological vector spaces V, where the only open non-empty convex subset is the entire space V itself. Thus the theorem holds, but for very trivial reasons: the assumptions are never satisfied. This changes in the locally convex situation, where we can expect substantially stronger statements:

Theorem 3.1.12 (Separation II) *Let V be a locally convex space and let $K, C \subseteq V$ be disjoint non-empty convex subsets, such that K is compact and C is closed. Then there exists a continuous linear functional Φ on V and real numbers $\alpha < \beta$ such that*

$$\operatorname{Re}(\Phi(v)) < \alpha < \beta < \operatorname{Re}(\Phi(u)) \tag{3.1.12}$$

for all $v \in K$ and $u \in C$.

PROOF: By Proposition 2.1.8 we can separate K and C by means of an open neighbourhood $U \subseteq V$ of zero in such a way that

$$(K + U) \cap (C + U) = \emptyset.$$

Since V is locally convex, we can choose U to be convex in addition. Thus we can apply the previous theorem to the open convex subset

$$A = K + U = \bigcup_{x \in K} (x + U)$$

and the convex subset $B = C$. Note that A is indeed convex, since K *and* U are convex, see also Exercise 3.6.4. Thus let $\Psi \colon V \longrightarrow \mathbb{R}$ be the continuous $\mathbb{R}$-linear functional separating A and B according to Theorem 3.1.10 with a point $\beta \in \mathbb{R}$ between $\Psi(A)$ and $\Psi(B)$, i.e.

$$\Psi(v) < \beta \leq \Psi(u)$$

for all $v \in A$ and $u \in B$. Since A is open, $\Psi(A)$ is open as well and since $K \subseteq A$ is compact and Ψ is continuous, $\Psi(K) \subseteq \Psi(A)$ is compact. Thus $\Psi(K) \subseteq [\omega, \alpha]$ with some $\omega \leq \alpha < \beta$. This shows the claim in the real case. The complex case is then obtained by setting $\Phi(v) = \Psi(v) - \mathrm{i}\Psi(\mathrm{i}v)$ for $v \in V$ as usual. □

Remark 3.1.13 Exchanging Φ by $\Phi' = -\Phi$ and setting $\alpha' = \beta$ and $\beta' = -\alpha$ gives

$$\operatorname{Re}(\Phi'(u)) < \alpha' < \beta' < \operatorname{Re}(\Phi'(v)) \tag{3.1.13}$$

for $u \in C$ and $v \in K$. Thus the two sets K and C enter symmetrically in the statement of the theorem.

We have now several corollaries to Theorem 3.1.12, which specify the separation even further.

Corollary 3.1.14 *Let V be a locally convex space and let K and C be disjoint non-empty convex subsets of V, such that K is compact and C is closed and absolutely convex. Then there exists a continuous linear functional $\Phi \in V'$ such that*

$$\sup_{u \in C} |\Phi(u)| < \inf_{v \in K} |\Phi(v)|. \tag{3.1.14}$$

PROOF: Let $\Phi \in V'$ and $\alpha < \beta$ satisfy

$$\operatorname{Re}(\Phi(u)) < \alpha < \beta < \operatorname{Re}(\Phi(v)) \tag{$*$}$$

for $u \in C$ and $v \in K$ according to Theorem 3.1.12 and Remark 3.1.13. Since C is absolutely convex, we have $\mathrm{e}^{\mathrm{i}\varphi} u \in C$ for all $u \in C$ and all phases $\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{K}$. Let $\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{K}$ be the phase, such that for $u \in C$ we have $\mathrm{e}^{\mathrm{i}\varphi}\Phi(u) = |\Phi(u)|$. Then

$$\operatorname{Re}(\Phi(\mathrm{e}^{\mathrm{i}\varphi} u)) = \operatorname{Re}(\mathrm{e}^{\mathrm{i}\varphi}\Phi(u)) = \operatorname{Re}(|\Phi(u)|) = |\Phi(u)|.$$

Since $\mathrm{e}^{\mathrm{i}\varphi} u \in C$, we have $\big|\Phi(u)\big| < \alpha$ by $(*)$. Thus we infer $\sup_{u\in C}|\Phi(u)| \leq \alpha$. On the other side we have $\operatorname{Re}(\Phi(v)) > \beta$ for $v \in K$ and hence $|\Phi(v)| \geq |\operatorname{Re}(\Phi(v))| \geq \operatorname{Re}(\Phi(v)) > \beta$ holds as well. Thus $\inf_{v\in K}|\Phi(v)| \geq \beta$ follows, showing (3.1.14) since $\alpha < \beta$. □

Since $K = \{v\}$ is always convex and compact, we get the following particular case:

Corollary 3.1.15 *Let V be a locally convex space and let $C \subseteq V$ be an absolutely convex closed subset. If $v \in V$ is not in C, then there exists a continuous linear functional $\Phi \in V'$ with $\big|\Phi(u)\big| \leq 1$ for all $u \in C$ and $\Phi(v) > 1$.*

PROOF: Let $\tilde{\Phi} \in V'$ be a continuous linear functional with

$$\gamma = \sup_{u\in C}|\tilde{\Phi}(u)| < |\tilde{\Phi}(v)|$$

as guaranteed by the previous corollary. Rescaling $\tilde{\Phi}$ by γ and by the phase of $\tilde{\Phi}(v)$ gives the functional Φ we are looking for. □

We will see many further applications of the separation theorems for locally convex spaces in the following.

3.1.3 The Weak Topology

As a first application of the Hahn-Banach Theorem we can define and investigate the *weak topology*. Later on, we will put this in a much more general context, but for the time being we take the following ad-hoc definition:

Definition 3.1.16 (Weak topology) *Let V be a topological vector space. Then the weak topology on V is the locally convex topology determined by the seminorms*

$$\mathcal{P}_{\mathrm{weak}} = \big\{\mathrm{p}_\varphi \;\big|\; \varphi \in V'\big\}, \tag{3.1.15}$$

where $\mathrm{p}_\varphi(v) = |\varphi(v)|$ for $v \in V$.

Of course, in general it might happen that we have only very few continuous linear functionals on V and hence $\mathcal{P}_{\mathrm{weak}}$ might be very small or even trivial, as in the case of the space $\mathscr{L}^p([0,1], \mathrm{d}x)$ for $0 < p < 1$, see once again Exercise 2.5.11. However, if V is a Hausdorff locally convex space, the weak topology is Hausdorff, too:

Proposition 3.1.17 *Let V be a topological vector space.*

i.) The weak topology on V is the initial topology with respect to the continuous linear functionals V' in the original topology.

ii.) The weak topology on V is the coarsest topology on V, such that all $\varphi \in V'$ are continuous. In particular, it is coarser than the original topology of V.

iii.) *A linear functional $\varphi \in V^*$ is continuous iff it is weakly continuous.*

iv.) *Suppose V is a locally convex space. Then the weak topology is Hausdorff iff the original topology of V is Hausdorff.*

PROOF: The first part is clear by Proposition 2.4.4, *iii.)*. Thus the second follows by the very properties of an initial topology. For the third part we note that $\varphi \in V'$ is weakly continuous by *ii.)*. Conversely, if φ is weakly continuous, it is also continuous with respect to any finer topology. It is the fourth statement, which uses the Hahn-Banach Theorem in form of Corollary 3.1.8: for $v \neq 0$ we have a $\varphi \in V'$ with $\varphi(v) \neq 0$ and hence $\mathrm{p}_\varphi(v) > 0$. Thus $\mathcal{P}_{\mathrm{weak}}$ is Hausdorff, which means that the weak topology is Hausdorff, too, by Theorem 2.2.13, *vi.)*. The converse is trivial since for every continuous $\varphi \in V'$ the absolute value $|\varphi(\,\cdot\,)|$ provides a continuous seminorm which then can be used to separate also with respect to the original topology. $\square$

The weak topology can also be understood as the topology of *componentwise convergence*. Indeed, we have the following characterization of weakly convergent nets:

Proposition 3.1.18 *Let V be a topological vector space.*

i.) *A net $(v_i)_{i\in I}$ in V is weakly converging to $v \in V$ iff for all $\varphi \in V'$ one has*

$$\lim_{i\in I} \varphi(v_i) = \varphi(v). \tag{3.1.16}$$

ii.) *A net $(v_i)_{i\in I}$ in V is a weak Cauchy net iff for all $\varphi \in V'$ the net $\big(\varphi(v_i)\big)_{i\in I}$ is a Cauchy net in $\mathbb{K}$.*

PROOF: From the characterization of convergent or Cauchy nets in locally convex spaces as in Proposition 2.2.29 and Proposition 2.2.30 this follows directly from the definition of $\mathcal{P}_{\mathrm{weak}}$. $\square$

In finite dimensions the uniqueness of the Hausdorff locally convex topology shows that it coincides with the weak topology: convergence in $\mathbb{K}^n$ means componentwise convergence. In general, however, the weak topology is typically strictly coarser than the original topology, even if the latter was already locally convex and Hausdorff.

For a general topological vector space V we say that V' *separates points* if for $v, w \in V$ with $v \neq w$ we find a $\varphi \in V'$ with $\varphi(v) \neq \varphi(w)$. Hence this is equivalent to the statement that the weak topology is Hausdorff. Thus Hausdorff locally convex spaces form a large class of examples with point separating topological duals spaces. As a simple consequence, the original topology is Hausdorff as well.

As yet another corollary to Theorem 3.1.12 we get now the following version of separation:

Corollary 3.1.19 *Let V be a topological vector space such that V' separates points. If $K_1, K_2 \subseteq V$ are non-empty disjoint convex compact subsets, then there exists a $\Phi \in V'$ with*

$$\sup_{v\in K_1} |\Phi(v)| < \inf_{u\in K_2} |\Phi(u)|. \tag{3.1.17}$$

PROOF: The compact subsets K_1 and K_2 are also weakly compact, as the weak topology is coarser and hence has less open covers. Since it is Hausdorff by the assumption that V' separates points, compact subsets are closed in the weak topology. Thus K_2 is closed and we can apply Theorem 3.1.12 to find a weakly continuous Φ with (3.1.17), since the weak topology is locally convex. But weakly continuous linear functionals are also continuous by Proposition 3.1.17, *iii.)*. $\square$

Finally, weakly dense subsets do not need to be dense in the original topology as the weak topology is coarser. Thus the following proposition for convex subsets is surprising and, in fact, very useful:

Proposition 3.1.20 *Let V be a Hausdorff locally convex space and let $A \subseteq V$ be a convex subset. Then the weak closure of A coincides with the closure of A in the original topology.*

PROOF: We denote the weak closure of A by A^{wcl}. Then we know $A^{\mathrm{cl}} \subseteq A^{\mathrm{wcl}}$. Suppose $v_0 \in A^{\mathrm{wcl}} \setminus A^{\mathrm{cl}}$. Now A^{cl} is still convex and the singleton $\{v_0\}$ is compact and convex. Thus we can apply Theorem 3.1.12 to find a $\Phi \in V'$ and some $\alpha \in \mathbb{R}$ with

$$\mathrm{Re}(\Phi(v_0)) < \alpha < \mathrm{Re}(\Phi(v)) \tag{$*$}$$

for all $v \in A^{\mathrm{cl}}$. Since Φ is weakly continuous, too, the subset

$$U = \{v \in V \mid \mathrm{Re}(\Phi(v)) < \alpha\}$$

is an open subset in the weak topology and hence a weak neighbourhood of $v_0 \in U$. Thus $(*)$ gives $U \cap A \subseteq U \cap A^{\mathrm{cl}} = \emptyset$. But this shows that there is a weak neighbourhood U of v_0 with trivial intersection with A. This implies $v_0 \notin A^{\mathrm{wcl}}$, a contradiction. □

Thus for *convex* subsets the notions of closedness, closure and hence also density with respect to the weak and the original topology coincide. Since subspaces are always convex, we get the following important particular case:

Corollary 3.1.21 *Let V be a Hausdorff locally convex space with a subspace $U \subseteq V$. Then*

$$U^{\mathrm{wcl}} = U^{\mathrm{cl}}. \tag{3.1.18}$$

3.2 Polars and Further Topologies

The next important theorems we want to discuss are the Bipolar Theorem and the Banach-Alaoglu Theorem. To this end we need to find topologies on the topological dual V' of the locally convex space V. Here one has several possibilities adapted to different purposes.

3.2.1 Polars and the Weak* Topology

We consider a topological vector space V with its topological dual V'. In order to have an interesting framework, we want V' to be non-trivial, which is most easily guaranteed by requiring V to be locally convex as usual. Nevertheless, for generic topological vector spaces we can define the polars of subsets in V and in V' as follows:

Definition 3.2.1 (Polar) *Let V be a topological vector space with topological dual V'.*

i.) The polar $A^{\circledast}$ of a subset $A \subseteq V$ is defined by

$$A^{\circledast} = \{\varphi \in V' \mid |\varphi(v)| \leq 1 \textit{ for all } v \in A\}. \tag{3.2.1}$$

ii.) The polar $B_{\circledast}$ of a subset $B \subseteq V'$ is defined by

$$B_{\circledast} = \{v \in V \mid |\varphi(v)| \leq 1 \textit{ for all } \varphi \in B\}. \tag{3.2.2}$$

Remark 3.2.2 The notions of polars of subsets in V and in V' are not completely dual: for $B \subseteq V'$ the polar in the sense of the first part of the definition would be a subset $B^{\circledast} \subseteq (V')'$, the topological dual of the topological dual. This of course would require to specify a topology on V' first, a challenge we have not yet met. Thus one should perhaps call $B_{\circledast} \subseteq V$ better the *pre-polar* of B. In any case, we will distinguish the two in our notation, where we follow [13, Section 3.5]. Also note that in the literature, the polar $A^{\circledast}$ is often denoted by A° causing a clash of notations with the open interior of A.

The definition becomes somewhat boring once the topological dual V' is too trivial. Thus we will mainly focus on (Hausdorff) locally convex space, just to make sure that we have enough continuous linear functionals to make polars interesting, see also Exercise 3.6.5. We collect several elementary properties of polars:

Proposition 3.2.3 *Let V be a locally convex space.*

i.) For $A \subseteq V$ one has $A \subseteq (A^{\circledast})_{\circledast}$.

ii.) For $A \subseteq B \subseteq V$ one has $B^{\circledast} \subseteq A^{\circledast}$.

iii.) For any set I and subsets $A_i \subseteq V$ for $i \in I$ one has

$$\left(\bigcup\nolimits_{i\in I} A_i\right)^{\circledast} = \bigcap_{i\in I} A_i^{\circledast}. \tag{3.2.3}$$

iv.) One has $(\emptyset)^{\circledast} = \{0\}^{\circledast} = V'$.

v.) For $z \in \mathbb{K}^{\times}$ and $A \subseteq V$ one has $(zA)^{\circledast} = \frac{1}{z}A^{\circledast}$.

vi.) For a subspace $W \subseteq V$ one has

$$W^{\circledast} = \left\{\varphi \in V' \mid \varphi\big|_W = 0\right\} = W^{\mathrm{ann}}. \tag{3.2.4}$$

vii.) If $0 \in A \cap B$ for $A, B \subseteq V$ one has

$$A^{\circledast} \cap B^{\circledast} \subseteq 2(A+B)^{\circledast} \subseteq 2(A^{\circledast} \cap B^{\circledast}). \tag{3.2.5}$$

viii.) For $A \subseteq V$ and $B \subseteq V'$ one has

$$A \subseteq B_{\circledast} \iff A^{\circledast} \supseteq B. \tag{3.2.6}$$

Proof: The first part is clear from the definition. The second statement is also clear, as for $B^{\circledast}$ we have to pose more conditions. For the third part, let $\varphi \in (\bigcap_{i\in I} A_i)^{\circledast}$. This means

$$|\varphi(v_i)| \le 1 \qquad \text{for all } v_i \in A_i, \text{ and all } i \in I,$$

and hence $\varphi \in A_i^{\circledast}$ for all $i \in I$, showing "$\subseteq$". The converse is analogously shown. Part *iv.)* is again clear from the definition. The fifth claim is also clear, since $\varphi \in (zA)^{\circledast}$ means $|\varphi(zv)| = |(z\varphi)(v)| \le 1$ for all $v \in A^{\circledast}$. But this implies $z\varphi \in A^{\circledast}$ or $\varphi \in \frac{1}{z}A^{\circledast}$, since $z \ne 0$. For a subspace $W \subseteq V$ we have $|\varphi(w)| \le 1$ for all $w \in W$ iff $|\lambda w| \le 1$ for all $w \in W$ and $\lambda \in \mathbb{K}$, since $\lambda w \in W$. Thus $|\varphi(w)| \le \frac{1}{|\lambda|}$ for all $\lambda \ne 0$, implying $\varphi(w) = 0$. This shows $\varphi \in W^{\mathrm{ann}}$. The converse is trivial. Here we take of course the *topological* annihilator of W, i.e. $W^{\mathrm{ann}} \subseteq V'$ and not the (typically bigger) algebraic annihilator in V^*. Next let $\varphi \in A^{\circledast} \cap B^{\circledast}$, then $|\varphi(v+w)| \le |\varphi(v)| + |\varphi(w)| \le 2$ for all $v \in A$ and $w \in B$. Thus $\frac{1}{2}\varphi \in (A+B)^{\circledast}$. Since $0 \in A \cap B$, we get from $|\varphi(v+w)| \le 1$ for all $v \in A$ and $w \in B$ the separate estimates $|\varphi(v)| \le 1$ and $|\varphi(w)| \le 1$. Hence $\varphi \in A^{\circledast} \cap B^{\circledast}$. Rescaling by 2 gives then (3.2.5). Finally, assume $A \subseteq B_{\circledast}$, which means for all $v \in A$ and all $\varphi \in B$ we have $|\varphi(v)| \le 1$. But this shows that equivalently $\varphi \in A^{\circledast}$. ☐

We have again the dual statements for the polars $B_{\circledast}$ of subsets $B \subseteq V'$, see Exercise 3.6.6, which are shown analogously. The next properties give some additional topological insight:

Proposition 3.2.4 *Let V be a locally convex space. For $B \subseteq V'$ the polar $B_{\circledast} \subseteq V$ is absolutely convex and weakly closed.*

PROOF: Let $z \in \mathbb{K}$ with $|z| \leq 1$ and $v \in B_{\circledast}$. Then for all $\varphi \in B$ we have $|\varphi(zv)| = |z||\varphi(v)| \leq |z| \leq 1$, showing $zv \in B_{\circledast}$. Moreover, for $v, w, \in B_{\circledast}$ and $\lambda \in [0,1]$ we have for all $\varphi \in B$

$$|\varphi(\lambda v + (1-\lambda)w)| \leq \lambda|\varphi(v)| + (1-\lambda)|\varphi(w)| \leq \lambda + (1-\lambda) = 1$$

and hence $\lambda v + (1-\lambda)w \in B_{\circledast}$. Together, this shows that $B_{\circledast}$ is absolutely convex. Since the weak topology of V is determined by the seminorms $\mathrm{p}_\varphi(v) = |\varphi(v)|$ with $\varphi \in V'$ we see that

$$B_{\circledast} = \bigcap_{\varphi \in B} \mathrm{p}_\varphi^{-1}([0,1]),$$

which is closed as the intersection of closed subsets. □

Corollary 3.2.5 *Let V be a locally convex space and $B \subseteq V'$. Then the polar $B_{\circledast} \subseteq V$ is closed.*

PROOF: Indeed, it is closed in a coarser topology than the original topology of V, namely the weak one. □

Before we can formulate the analogous statement for a polar $A^{\circledast} \subseteq V'$ of a subset $A \subseteq V$ we need to to specify a topology on V'. We will meet many different such topologies later on, but the following weak* topology is adapted to this question:

Definition 3.2.6 (Weak* topology) *Let V be a locally convex space. Then the weak* topology for V' is the subspace topology*

$$V' \subseteq \mathrm{Map}(V, \mathbb{K}), \tag{3.2.7}$$

where $\mathrm{Map}(V, \mathbb{K})$ is endowed with the locally convex topology of pointwise convergence, as in Definition 2.3.1.

With other words, we inherit the seminorms from $\mathrm{Map}(V, \mathbb{K})$, explicitly given by

$$\mathrm{p}_v(\varphi) = |\varphi(v)| \tag{3.2.8}$$

with $v \in V$. This topology is rather boring in many aspects and completely ignores any structure we might have on V. Only the *set* V matters to define the topology on $\mathrm{Map}(V, \mathbb{K})$. However, restricting it to the specific V' gives interesting effects later on. A first impression can be found in Exercise 3.6.8.

The weak* topology on V' is very coarse and we will meet finer topologies in the following. Nevertheless, polars behave nicely with respect to the weak* topology:

Proposition 3.2.7 *Let V be a locally convex space and $A \subseteq V$. Then the polar $A^{\circledast} \subseteq V'$ is absolutely convex and closed in the weak* topology of V'. It is also closed in any finer topology in V'.*

PROOF: Identical arguments to the ones used in Proposition 3.2.4 yield the claim. □

The central result of this section is now the following Bipolar Theorem, which determines the bipolar explicitly:

Theorem 3.2.8 (Bipolar Theorem) *Let V be a Hausdorff locally convex space and let $A \subseteq V$ be a non-empty subset. Then the bipolar $(A^{\circledast})_{\circledast}$ of A coincides with the (weak) closure of the absolutely convex hull of A, i.e.*

$$\left(A^{\circledast}\right)_{\circledast} = \mathrm{absconv}(A)^{\mathrm{wcl}} = \mathrm{absconv}(A)^{\mathrm{cl}}. \tag{3.2.9}$$

PROOF: Since $\operatorname{absconv}(A)$ is convex, Proposition 3.2.4 guarantees that the closure and the weak closure of $\operatorname{absconv}(A)$ coincide. By Proposition 3.2.3, *i.)*, we have $A \subseteq (A^{\circledast})_{\circledast}$ and hence $\operatorname{absconv}(A) \subseteq \operatorname{absconv}((A^{\circledast})_{\circledast}) = (A^{\circledast})_{\circledast}$, since a polar is absolutely convex by Proposition 3.2.4. Since by the same proposition, $(A^{\circledast})_{\circledast}$ is also weakly closed, we conclude that the weak closure of $\operatorname{absconv}(A)$ is contained in $(A^{\circledast})_{\circledast}$, showing "$\supseteq$" in (3.2.9). Thus assume $v \in V$ is not in the closure of $\operatorname{absconv}(A)$. From Corollary 3.1.15 we obtain a functional $\varphi \in V'$, separating $\operatorname{absconv}(A)^{\mathrm{cl}}$ from v, i.e. $\varphi(v) > 1$, but $|\varphi(u)| \leq 1$ for all $u \in \operatorname{absconv}(A)^{\mathrm{cl}}$. Hence

$$\varphi \in (\operatorname{absconv}(A)^{\mathrm{cl}})^{\circledast} \subseteq A^{\circledast}$$

by the very definition of the polar and Proposition 3.2.3, *ii.)*. But then $\varphi(v) > 1$ shows $v \notin (A^{\circledast})_{\circledast}$, which gives "$\subseteq$" in (3.2.9). □

We collect a few easy corollaries to this fundamental result. More serious implications will be discussed later. The first is a reformulation for a particular subset $A \subseteq V$.

Corollary 3.2.9 *Let V be a Hausdorff locally convex space and let p be a continuous seminorm on V. For all $r > 0$ one has*

$$(\mathrm{B}_{\mathrm{p},r}(0)^{\circledast})_{\circledast} = \mathrm{B}_{\mathrm{p},r}(0)^{\mathrm{cl}}. \tag{3.2.10}$$

PROOF: Indeed, $\mathrm{B}_{\mathrm{p},r}(0)$ is already absolutely convex, thus we only need to take its closure to get the bipolar. □

Corollary 3.2.10 *Let V be a Hausdorff locally convex space. For $A \subseteq V$ one has $((A^{\circledast})_{\circledast})^{\circledast} = A^{\circledast}$ and*

$$A^{\circledast} = (\operatorname{absconv}(A)^{\mathrm{cl}})^{\circledast}. \tag{3.2.11}$$

PROOF: Taking polars implements a Galois correspondence by Proposition 3.2.3, *i.)* and *ii.)*. Thus the polar coincides with the triple polar. Then

$$(\operatorname{absconv}(A)^{\mathrm{cl}})^{\circledast} = ((A^{\circledast})_{\circledast})^{\circledast} = A^{\circledast}$$

by the Bipolar Theorem 3.2.8. □

3.2.2 The Banach-Alaoglu Theorem and Bounded Subsets

The weak* topology on V' is fairly coarse. While for many applications this can be seen as a serious drawback, see e.g. once more Exercise 3.6.8, the question about compact subsets becomes easier. Since there are not so many open coverings, it is easy for a subset to be compact. In fact, polars of neighbourhoods of zero turn out to be compact:

Theorem 3.2.11 (Banach-Alaoglu) *Let V be a Hausdorff locally convex space and let $U \subseteq V$ be a neighbourhood of zero. Then the polar $U^{\circledast} \subseteq V'$ of U is compact with respect to the weak* topology.*

PROOF: Let p be a continuous seminorm on V, such that U contains the closed unit ball $\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}} \subseteq U$. We know $U^{\circledast} \subseteq (\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}})^{\circledast}$ and $U^{\circledast}$ is a weak* closed subset of $(\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}})^{\circledast}$ by Proposition 3.2.4. Thus is suffices to show that $(\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}})^{\circledast}$ is compact, as then every closed subset of this compact set would be compact as well. The idea is to embed this into a large compact Hausdorff space as a closed subset. We consider $\operatorname{Map}(V, \mathbb{K})$ with the Cartesian product topology from Example 2.4.7. Then the elements $\varphi \in (\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}})^{\circledast}$ can are particular elements in $\operatorname{Map}(V, \mathbb{K})$ leading to

$$\left(\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}\right)^{\circledast} \subseteq \operatorname{Map}(V, \mathbb{K}) \cong \mathbb{K}^{V}.$$

Note that the weak* topology is nothing else than the Cartesian product topology. By our choice of the seminorm p we have

$$|\varphi(v)| \leq \mathrm{p}(v) \tag{$*$}$$

for all $\varphi \in U^{❄}$ and all $v \in V$. Indeed, if $\mathrm{p}(v) = 0$, then $\lambda v \in \mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ for all $\lambda \in \mathbb{K}$ and hence $1 \geq |\varphi(\lambda v)| = |\lambda||\varphi(v)|$, which means $\varphi(v) = 0$. For $\mathrm{p}(v) \neq 0$ we have $\frac{v}{\mathrm{p}(v)} \in \mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ and hence $|\varphi(\frac{v}{\mathrm{p}(v)})| \leq 1$, which gives $(*)$, too. Now the set of *all* maps $\varphi \in \mathrm{Map}(V, \mathbb{K})$ satisfying $(*)$ is clearly closed in the weak* topology, i.e. the product topology. Since also the *linear* maps $V^* \subseteq \mathrm{Map}(V, \mathbb{K})$ are closed in the weak* topology, their intersection is still closed. But this is precisely $(\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}})^{❄}$ and hence

$$\left(\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}\right)^{❄} = V^* \cap K \quad \text{with} \quad K = \left\{\varphi \in \mathrm{Map}(V, \mathbb{K}) \;\middle|\; |\varphi(v)| \leq \mathrm{p}(v) \text{ for all } v \in V\right\}.$$

Note that the intersection of V^* and K automatically ends up in V' since the linear maps in K are continuous by the definition of K. Now $K \subseteq \mathrm{Map}(V, \mathbb{K})$ is not just closed but also compact. Indeed, we have

$$K = \prod_{v \in V} \left\{z \in \mathbb{K} \;\middle|\; |z| \leq \mathrm{p}(v)\right\} = \prod_{v \in V} \mathrm{B}_{\mathrm{p}(v)}(0)^{\mathrm{cl}}$$

under the identification $\mathrm{Map}(V, \mathbb{K}) = \mathbb{K}^V$. Since the discs $\mathrm{B}_{\mathrm{p}(v)}(0)^{\mathrm{cl}} \subseteq \mathbb{K}$ are compact, K is compact itself by Tikhonov's Theorem, see e.g. [19, Theorem 5.3.1]. But then the intersection $V^* \cap K$ is compact again as it is a closed subset of K. □

One of the first applications is that we can characterize bounded subsets in locally convex spaces now very easily. First we note that boundedness has a simple formulation using seminorms:

Proposition 3.2.12 *Let $B \subseteq V$ be a subset in a locally convex space. Then B is bounded iff every continuous seminorm of V is bounded on B.*

PROOF: Let B be bounded. For a continuous seminorm p on V the open unit ball $\mathrm{B}_{\mathrm{p},1}(0) \subseteq V$ is open. Thus there exists an $r > 0$ with $B \subseteq r\mathrm{B}_{\mathrm{p},1}(0) = \mathrm{B}_{\mathrm{p},r}(0)$, which means $\mathrm{p}(v) \leq r$ for all $v \in B$. Conversely, assume every continuous seminorm is bounded on B and let U be an open zero neighbourhood. Then there exists a continuous seminorm p on V with $\mathrm{B}_{\mathrm{p},1}(0) \subseteq U$. By assumption one finds $r > 0$ with $\mathrm{p}(v) < r$ for all $v \in B$, which gives $B \subseteq \mathrm{B}_{\mathrm{p},r}(0) = r\mathrm{B}_{\mathrm{p},1}(0) \subseteq rU$. Hence B is bounded. □

Since we have to consider all rescalings anyway, one can test the boundedness of $B \subseteq V$ also with any defining system of continuous seminorms. Thus the locally convex situation is rather easy and reproduces the basic idea of boundedness. Nevertheless, testing the boundedness requires a good understanding of all the continuous seminorms of V. Thus the following statement is very welcome: already very few and particularly simple seminorms suffice.

Theorem 3.2.13 (Boundedness and weak boundedness) *Let V be a Hausdorff locally convex space. Then a subset $B \subseteq V$ is bounded iff it is weakly bounded.*

PROOF: Since the weak topology is coarser than the original one, every bounded subset is weakly bounded, see Remark 2.1.33, *vi.)*. Thus assume that B is weakly bounded. This means that for every $\varphi \in V'$ we have a constant $c_\varphi > 0$ with

$$\mathrm{p}_\varphi(v) = |\varphi(v)| \leq c_\varphi \tag{$*$}$$

for all $v \in B$. Now let p be a continuous seminorm on V. By the Banach-Alaoglu Theorem 3.2.11 we know that the polar $K = \mathrm{B}_{\mathrm{p},1}(0)^{❄} \subseteq V'$ is compact with respect to the weak* topology. The main step is to show that the constants c_φ in $(*)$ stay bounded, when we vary φ in the compact subset K, i.e. we want to show

$$\sup_{\varphi \in K} \sup_{v \in B} |\varphi(v)| < \infty. \tag{$**$}$$

To this end, we consider the intersection of all the weakly* closed unit balls

$$A = \bigcap_{v \in B} \mathrm{B}_{\mathrm{p}_v, 1}(0)^{\mathrm{cl}} = \bigcap_{v \in B} \left\{ \varphi \in V \;\middle|\; \mathrm{p}_v(\varphi) = |\varphi(v)| \leq 1 \right\}$$

of the seminorms p_v with $v \in B$. This is again a weakly* closed subset of V'. For $\varphi \in K$ we have by $(*)$

$$\varphi \in \bigcap_{v \in B} \mathrm{B}_{\mathrm{p}_v, c_\varphi}(0)^{\mathrm{cl}} = c_\varphi \bigcap_{v \in B} \mathrm{B}_{\mathrm{p}_v, 1}(0)^{\mathrm{cl}} = c_\varphi A.$$

Taking $n \in \mathbb{N}$ large enough, we have for every $\varphi \in K$ an $n \in \mathbb{N}$ with $\varphi \in nA$. Hence

$$K = \bigcup_{n \in \mathbb{N}} (K \cap nA) \tag{$\star$}$$

is a countable union of weakly* closed subsets $K \cap nA$. Since K is compact and thus locally compact in the Hausdorff weak* topology, we can apply Baire's Theorem to the union $(\star)$: there is an $n_0 \in \mathbb{N}$ such that $K \cap n_0 A$ has a non-trivial open interior with respect to the relative topology of K. This follows, since locally convex Hausdorff spaces are Baire spaces, see e.g. [18, Theorem 7.2.2]. Thus let $\varphi_0 \in K \cap n_0 A$ be an interior point. Then we get a weakly* open neighbourhood of φ_0, whose intersection with K is contained in $K \cap n_0 A$. Since the open balls with respect to the seminorms $\{\mathrm{p}_v\}_{v \in V}$ constitute a subbasis of the weak* topology, we can assume that this open neighbourhood is of the form $\varphi_0 + U$ with

$$U = \mathrm{B}_{\mathrm{p}_{w_1}, 1}(0) \cap \cdots \cap \mathrm{B}_{\mathrm{p}_{w_N}, 1}(0)$$

for some appropriate $w_1, \ldots, w_N \in V$. Since $\mathbb{K}$ is compact, also $K - \varphi_0$ is compact and thus bounded itself. Thus there is an $r > 0$ with $K - \varphi_0 \subseteq rU$. Without restriction, we can take $r > 1$. Then for every $\varphi \in K$ the convex combination

$$\psi = (1 - \tfrac{1}{r})\varphi_0 + \tfrac{1}{r}\varphi$$

is again in K, as a polar is absolutely convex. Thus $\psi - \varphi_0 = \frac{1}{r}(\varphi - \varphi_0) \in \frac{1}{r}(K - \varphi_0) \subseteq U$ by our choice of r. This means

$$\psi = (1 - \tfrac{1}{r})\varphi_0 + \tfrac{1}{r}\varphi \in \varphi_0 + U \subseteq K \cap n_0 A,$$

and hence $|\psi(v)| \leq n_0$ for all $v \in B$ by definition of A. Together with $\varphi \in K \cap n_0 A$ and hence $|\varphi_0(v)| \leq n_0$ for all $v \in B$ this gives

$$|\varphi(v)| = |r\psi(v) - r\varphi_0(v) + \varphi_0(v)| \leq r n_0 + (r-1) n_0 = c$$

for all $v \in B$, independently of $\varphi \in K$. This finally shows $(**)$ with the supremum being estimated by c. The next step is very easy: for all $v \in B$ we have $|\varphi(\frac{v}{c})| \leq 1$ for all $\varphi \in K = \mathrm{B}_{\mathrm{p},1}(0)^{\circledast}$. Thus $\frac{v}{c} \in (\mathrm{B}_{\mathrm{p},1}(0)^{\circledast})_{\circledast} = \mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ by the Bipolar Theorem 3.2.8. This gives $B \subseteq c\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}} \subseteq (c+1)\mathrm{B}_{\mathrm{p},1}(0)$ and hence B is bounded in the original topology. □

This characterization of bounded subsets is very convenient in many places. Note also the first appearance of Baire's Theorem. There will be more results in locally convex analysis depending on Baire's Theorem in a crucial way. One way to interpret this result is that boundedness can be checked *componentwise*. This makes it perhaps most transparent why this theorem is so useful.

3.2.3 From the Weak* to the Strong Topology

Using the notion of polars the weak* topology on the dual V' can also be characterized by the following neighbourhood basis of zero: if $v \in V$, then the polar of the one-pointed set $\{v\}$ is

$$\{v\}^{\circledast} = \{\varphi \in V' \mid |\varphi(v)| \leq 1\} = \mathrm{B}_{\mathrm{p}_v,1}(0)^{\mathrm{cl}}. \tag{3.2.12}$$

Hence for finitely many $v_1, \ldots, v_n \in V$ we have

$$\{v_1, \ldots, v_n\}^{\circledast} = (\{v_1\} \cup \cdots \cup \{v_n\})^{\circledast} = \{v_1\}^{\circledast} \cap \cdots \cap \{v_n\}^{\circledast} = \mathrm{B}_{\mathrm{p}_{v_1},1}(0)^{\mathrm{cl}} \cap \cdots \cap \mathrm{B}_{\mathrm{p}_{v_n},1}(0)^{\mathrm{cl}} \tag{3.2.13}$$

by Proposition 3.2.3, *iii.)*, or a direct elementary argument. Thus the polars of the finite subsets of V constitute a basis of neighbourhoods of zero for the weak* topology. The idea is now to generalize this to more subsets than just finite ones, thus yielding new interesting topologies.

Lemma 3.2.14 *Let V be a Hausdorff locally convex space. Let $B \subseteq V$ be a non-empty subset and define $\mathrm{p}_B \colon V' \longrightarrow [0, \infty]$ by*

$$\mathrm{p}_B(\varphi) = \sup_{v \in B} |\varphi(v)|. \tag{3.2.14}$$

i.) The subset B is bounded iff $\mathrm{p}_B(\varphi) < \infty$ for all $\varphi \in V'$.

ii.) For a bounded subset B the map p_B is a seminorm on V' and

$$B^{\circledast} = \mathrm{B}_{\mathrm{p}_B,1}(0)^{\mathrm{cl}}. \tag{3.2.15}$$

PROOF: The first part is just Theorem 3.2.13. Indeed, $\mathrm{p}_B(\varphi) < \infty$ simply means that the seminorm p_φ is bounded on B for all $\varphi \in V'$. Hence B is weakly bounded, thus bounded. The converse is clear by the characterization of bounded sets using seminorms and the fact that all p_φ are continuous. For the second part, it is clear that $\mathrm{p}_B(\varphi)$ satisfies the properties of a seminorm. Then $\mathrm{p}_B(\varphi) \leq 1$ is equivalent to $|\varphi(v)| \leq 1$ for all $v \in B$ and hence (3.2.15) follows. □

This simple observation opens the door to define many further topologies on V' by taking collections of seminorms p_B for certain collections of bounded subsets of V.

Lemma 3.2.15 *Let V be a Hausdorff locally convex space and let $\mathcal{B} \subseteq 2^V$ be a collection of bounded subsets of V.*

i.) The system of seminorms $\{\mathrm{p}_B\}_{B \in \mathcal{B}}$ is filtrating if for $B_1, B_2 \in \mathcal{B}$ one finds a $B \in \mathcal{B}$ such that $B_1 \cup B_2 \subseteq B$.

ii.) The system of seminorms $\{\mathrm{p}_B\}_{B \in \mathcal{B}}$ is Hausdorff if the system $\{\lambda B\}_{\lambda > 0,\, B \in \mathcal{B}}$ is a cover of V.

PROOF: Clearly, for bounded subsets $A \subseteq B$ we have

$$\mathrm{p}_A(\varphi) = \sup_{v \in A} |\varphi(v)| \leq \sup_{v \in B} |\varphi(v)| = \mathrm{p}_B(\varphi),$$

and thus $\mathrm{p}_A \leq \mathrm{p}_B$. Hence if for $B_1, B_2 \in B$ we find a $B \in \mathcal{B}$ with $B_1 \cup B_2 \subseteq B$, then

$$\mathrm{p}_{B_1} \leq \mathrm{p}_{B_1 \cup B_2} \leq \mathrm{p}_B \quad \text{and} \quad \mathrm{p}_{B_2} \leq \mathrm{p}_{B_1 \cup B_2} \leq \mathrm{p}_B.$$

By induction we see that the system $\{\mathrm{p}_B\}_{B \in \mathcal{B}}$ is filtrating. Now assume that we get a cover of V by taking all rescaled copies of elements of $\mathcal{B}$. Suppose $\varphi \in V'$ is different from zero. Then we have some $v_0 \in V$ with $\varphi(v_0) \neq 0$ and hence we find a $\lambda > 0$ and some $B \in \mathcal{B}$ with $v_0 \in \lambda B$. But then $\frac{1}{\lambda} v_0 \in B$ and thus

$$\mathrm{p}_B(\varphi) = \sup_{v \in B} |\varphi(v)| \geq |\varphi(\tfrac{1}{\lambda} v_0)| = \tfrac{1}{\lambda} |\varphi(v_0)| > 0,$$

showing that $\{\mathrm{p}_B\}_{B \in \mathcal{B}}$ is Hausdorff. □

Note that the conditions are only sufficient, e.g. obtaining a cover of an absorbing subset of V would be sufficient, as well. For some slight generalization, see Exercise 3.6.9. Nevertheless, the above conditions are fairly easy to check. We can state now the following definition:

Definition 3.2.16 ($\mathcal{B}$-Topology on V') *Let V be a Hausdorff locally convex space with a collection of bounded subsets $\mathcal{B}$ such that*

i.) *for $B_1, B_2 \in \mathcal{B}$ there exists a $B \in \mathcal{B}$ with $B_1 \cup B_2 \subseteq B$,*

ii.) *the subsets $\{\lambda B\}_{\lambda>0,\, B\in\mathcal{B}}$ constitute a cover of V.*

We call such a system admissible. Then the locally convex topology induced by the seminorms $\{\mathrm{p}_B\}_{B\in\mathcal{B}}$ is called the $\mathcal{B}$-topology on V'.

Some first consequences are collected in the following proposition:

Proposition 3.2.17 *Let V be a Hausdorff locally convex space and let $\mathcal{B}$ be an admissible collection of bounded subsets and equip V' with the corresponding $\mathcal{B}$-topology.*

i.) *The $\mathcal{B}$-topology is locally convex and Hausdorff.*

ii.) *The polars of the subsets λB with $B \in \mathcal{B}$ and $\lambda > 0$ form a basis of closed absolutely convex neighbourhoods of zero for the $\mathcal{B}$-topology.*

iii.) *The $\mathcal{B}$-topology is finer than the weak* topology.*

iv.) *A net $(\varphi_i)_{i\in I}$ in V' converges to $\varphi \in V'$ with respect to the $\mathcal{B}$-topology iff it converges uniformly on all subsets $B \in \mathcal{B}$.*

PROOF: The first statement is clear by construction and Lemma 3.2.15, *ii.)*. Also the second statement is clear by Lemma 3.2.14, *ii.)*, and Lemma 3.2.15, *i.)*. Since the subsets λB cover V, for every point $v \in V$ we find a $B \in \mathcal{B}$ with $v \in \lambda B$. Thus $\mathrm{p}_v = \mathrm{p}_{\{v\}} \leq \mathrm{p}_{\lambda B} = \frac{1}{\lambda}\mathrm{p}_B$ shows that the seminorms $\{\mathrm{p}_v\}$ of the weak* topology can be estimated by the seminorms of the $\mathcal{B}$-topology. Finally, let $(\varphi_i)_{i\in I}$ be a net in V'. This converges to $\varphi \in V'$ in the $\mathcal{B}$-topology iff for the filtrating system $\{\mathrm{p}_B\}_{B\in\mathcal{B}}$ of seminorms we have for every $\epsilon > 0$ and $B \in \mathcal{B}$ an index $j \in I$ with

$$\mathrm{p}_B(\varphi_i - \varphi) < \epsilon$$

for $i \succcurlyeq j$. But $\mathrm{p}_B(\varphi_i - \varphi) = \sup_{v\in B}|\varphi_i(v) - \varphi(v)|$, showing also the last claim. □

We list now some important examples:

Example 3.2.18 ($\mathcal{B}$-topologies on V') Let V be a Hausdorff locally convex space.

i.) The *weak* topology* is obtained by taking the system of all finite subsets of V as bounded sets: this is clearly admissible and reproduces by (3.2.12) the weak* topology. This yields the coarsest possible $\mathcal{B}$-topology. We denote this also by $\sigma(V', V)$.

ii.) The other extreme is the *strong topology.* Here we simply take all bounded subsets of V and thus all the seminorms p_B with $B \subseteq V$ bounded. Clearly, this is admissible as well, and yields the finest topology among the $\mathcal{B}$-topologies. The usual notation for this topology is $\beta(V', V)$ and it also called the *topology of bounded convergence.*

iii.) Another system is to take the system of compact subsets of V, which are always bounded. Since singletons are compact and since unions of finitely many compact subsets are compact again, this provides an admissible system of bounded subsets. The topology corresponding to it will be called the *topology of compact convergence.* It is denoted by $c(V', V)$.

iv.) Slightly coarser than $c(V', V)$ is the *topology of convex compact convergence*: here we take the system of convex compact subsets. While it is easy to see that convex compact subsets cover all of V, the first property in Definition 3.2.16 requires a little more thought: for two convex compact subsets K_1 and K_2 their union $K_1 \cup K_2$ is contained in a convex compact subset again, see Exercise 3.6.10. Thus we have met all requirements. The resulting topology of convex compact convergence will be denoted by $\gamma(V', V)$.

Clearly, we can compare the above four topologies and arrange them by being finer as follows:

$$\sigma(V', V) \leq \gamma(V', V) \leq c(V', V) \leq \beta(V', V). \tag{3.2.16}$$

The dual, equipped with one of these topologies, will also be denoted by $V'_\sigma, V'_\gamma, V'_c, V'_\beta$, respectively.

Since we have equipped V' now with various topologies, we can ask for its topological dual. Here we get the following characterization:

Proposition 3.2.19 *Let V be a Hausdorff locally convex space. Then the topological dual of the dual V' with respect to the weak* topology is again V by the linear isomorphism*

$$\mathrm{ev}\colon V \ni v \ \mapsto \ \mathrm{ev}_v \in (V'_\sigma)', \tag{3.2.17}$$

where the evaluation $\mathrm{ev}_v\colon V' \longrightarrow \mathbb{K}$ at v is defined as usual by

$$\mathrm{ev}_v(\varphi) = \varphi(v) \tag{3.2.18}$$

for $\varphi \in V'$.

Proof: From linear algebra and the Hahn-Banach Theorem one knows that $\mathrm{ev}\colon V \longrightarrow (V')^*$ is injective. Furthermore, it clearly maps into $(V'_\sigma)'$, i.e. ev_v is weakly* continuous for all $v \in V$: indeed, $|\cdot| \circ \mathrm{ev}_v = \mathrm{p}_v$. It is the surjectivity we have to prove. Thus let $\Phi\colon V'_\sigma \longrightarrow \mathbb{K}$ be a weakly* continuous linear map. This means that we have finitely many $v_1, \ldots, v_n \in V$ with

$$|\Phi(\varphi)| \leq \max_{i=1}^{n} \mathrm{p}_{v_i}(\varphi).$$

Note that we do not need to scale the right hand side, as this amounts to a scaling of the vectors $v_1, \ldots, v_n$. It follows that $\varphi(v_1) = \cdots = \varphi(v_n) = 0$ implies $\Phi(\varphi) = 0$ or

$$\bigcap_{i=1}^{n} \ker \mathrm{ev}_{v_i} \subseteq \ker \Phi.$$

A simple linear-algebraic lemma, see Exercise 3.6.11, then shows $\Phi \in \operatorname{span}_{\mathbb{K}}\{\mathrm{ev}_{v_1}, \ldots, \mathrm{ev}_{v_n}\}$ and thus

$$\Phi = \lambda_1 \mathrm{ev}_{v_1} + \cdots + \lambda_n \mathrm{ev}_{v_n} = \mathrm{ev}_{\lambda_1 v_1 + \cdots + \lambda_n v_n}$$

for appropriate $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$. Thus (3.2.17) is surjective. □

This result leads to a useful reformulation of the Bipolar Theorem, which is discussed in Exercise 3.6.12.

In general, the topological dual of V' with respect to the other topologies might be larger, as the topologies get finer. Since the strong topology is the finest, we take this as landmark, motivating the following definition:

Definition 3.2.20 (Reflexive space) *A Hausdorff locally convex space V is called semi-reflexive if the evaluation map yields a linear isomorphism*

$$\mathrm{ev}\colon V \longrightarrow (V'_\beta)'. \tag{3.2.19}$$

It is called reflexive if (3.2.19) *is an isomorphism of locally convex spaces, when $(V'_\beta)'$ is equipped with the strong topology again, i.e.*

$$\mathrm{ev}\colon V \xrightarrow{\cong} (V'_\beta)'_\beta. \tag{3.2.20}$$

We will have to come back to the question, which locally convex spaces are reflexive. It turns out to be a rather subtle question, whose answer ultimately may depend on detailed information about V.

We continue this section with a discussion of the continuity properties of transposed maps. Let

$$\Phi\colon V \longrightarrow W \tag{3.2.21}$$

be a continuous linear map between Hausdorff locally convex spaces. Then the pull-back with Φ yields a linear map

$$\Phi^*\colon W^* \longrightarrow V^* \tag{3.2.22}$$

as usual, which restricts to a linear map

$$\Phi'\colon W' \longrightarrow V', \tag{3.2.23}$$

since the composition $\Phi^*\varphi = \varphi \circ \Phi$ of a continuous linear functional φ with the continuous linear map Φ is again continuous. One calls Φ' also the *transpose* or *dual* of Φ. We have the usual rules for transposing, see Exercise 3.6.13. The interesting question is now, whether (3.2.23) is again a *continuous* linear map, when W' and V' are equipped with some $\mathcal{B}$-topologies. Of course, this will depend on the *pair* of $\mathcal{B}$-topologies we use, but there is a simple sufficient criterion:

Proposition 3.2.21 *Let $\Phi\colon V \longrightarrow W$ be a continuous linear map between Hausdorff locally convex spaces. Moreover, let $\mathcal{B}_V$ and $\mathcal{B}_W$ be admissible systems of bounded subsets of V and W, respectively, such that for all $A \in \mathcal{B}_V$ there exists a $B \in \mathcal{B}_W$ with $\Phi(A) \subseteq B$. Then the transpose*

$$\Phi'\colon W'_{\mathcal{B}_W} \longrightarrow V'_{\mathcal{B}_V} \tag{3.2.24}$$

is continuous.

Proof: To check continuity we use the defining systems of seminorms. Thus let p_A on $V'_{\mathcal{B}_V}$ with $A \in \mathcal{B}_V$ be given and choose $B \in \mathcal{B}_W$ with $\Phi(A) \subseteq B$. Then

$$\mathrm{p}_A(\Phi'\varphi) = \sup_{v\in A}|(\varphi \circ \Phi)(v)| \le \sup_{w\in B}|\varphi(w)| = \mathrm{p}_B(\varphi),$$

since $\Phi(v) \in B$. This shows the continuity of Φ'. □

In the previous examples this gives the following good functorial behaviour:

Corollary 3.2.22 *Let V and W be Hausdorff locally convex spaces and let $\Phi\colon V \longrightarrow W$ be a continuous linear map. Then the transpose $\Phi'\colon W' \longrightarrow V'$ gives continuous linear maps*

$$\Phi'\colon W'_\beta \longrightarrow V'_\beta, \tag{3.2.25}$$

$$\Phi'\colon W'_c \longrightarrow V'_c, \tag{3.2.26}$$

$$\Phi'\colon W'_\gamma \longrightarrow V'_\gamma, \tag{3.2.27}$$

$$\Phi'\colon W'_\sigma \longrightarrow V'_\sigma. \tag{3.2.28}$$

Proof: Indeed, Φ maps bounded subsets to bounded ones by Proposition 2.1.36. It maps compact subsets to compact ones by continuity and convex subsets to convex ones by linearity. Finally, it maps finite subsets to finite subsets. □

Of course, taking a finer topology on W' and a coarser one of V' retains continuity for trivial reasons. Hence also e.g.

$$\Phi' \colon W'_\gamma \longrightarrow V'_\sigma \tag{3.2.29}$$

is continuous, etc.

We conclude this section with a discussion of the topological dual of a normed space V. If $\|\cdot\|$ denotes the norm on V and $\varphi \in V'$ is a continuous linear functional, the *functional* (or *operator*) *norm* $\|\varphi\|$ of φ is defined as

$$\|\varphi\| = \sup_{v \in V \setminus \{0\}} \frac{|\varphi(v)|}{\|v\|}, \tag{3.2.30}$$

i.e. the optimal constant, such that the continuity estimate

$$|\varphi(v)| \leq \|\varphi\| \cdot \|v\| \tag{3.2.31}$$

holds. Note that here we make explicit use of the norm $\|\cdot\|$ on V.

Proposition 3.2.23 *Let V be a normed space.*

i.) The functional norm turns V' into a Banach space.

ii.) The Banach space topology on V' coincides with the strong topology.

Proof: It is a standard exercise to check that (3.2.30) defines a norm on V'. Note that the finiteness of the supremum on the right hand side of (3.2.30) is equivalent to the continuity of φ. The completeness can also be checked by hand. Alternatively, we will see another argument later, see also Exercise 3.6.14. Here, we want to focus on the second statement. Since V is normed, a subset $B \subseteq V$ is bounded iff there exists a $R \in \mathbb{R}$ with

$$\sup_{v \in B} \|v\| \leq R. \tag{$*$}$$

This follows directly from Proposition 3.2.12 and the fact that every continuous seminorm on V is dominated by a suitable multiple of the norm. Then $(*)$ simply means $B \subseteq \mathrm{B}_R(0)^{\mathrm{cl}}$. Hence the seminorm p_B of the strong topology on V' can be estimated by $\mathrm{p}_{\mathrm{B}_R(0)^{\mathrm{cl}}}$. From the linearity of φ we see that

$$\mathrm{p}_{\mathrm{B}_R(0)^{\mathrm{cl}}}(\varphi) = \sup_{v \in \mathrm{B}_R(0)^{\mathrm{cl}}} |\varphi(v)| = \sup_{\|v\| = R} |\varphi(v)| = R\|\varphi\|,$$

since the vectors in $\mathrm{B}_R(0)^{\mathrm{cl}}$ with norm less than R do not contribute to the supremum. Thus every continuous seminorms of the strong topology can be estimated by the functional norm. Conversely, the above computation shows $\|\cdot\| = \mathrm{p}_{\mathrm{B}_1(0)^{\mathrm{cl}}}$ and thus the functional norm is also continuous in the strong topology. This completes the proof. $\square$

Remark 3.2.24 Thus the strong topology is the relevant generalization of the norm topology on the dual of a normed space. In particular, for normed spaces the notion of reflexivity as in Definition 3.2.20 reproduces the usual notion of reflexivity familiar from elementary functional analysis. We can anticipate already here that the strong topology will typically have nice features. In particular, the completeness of strong duals will follow with a small extra assumption, satisfied automatically for normed spaces. We will discuss this in Section 3.3.2.

3.3 Barrels and Uniform Boundedness

The next important concept we want to investigate is uniform boundedness. Then the famous Banach-Steinhaus Theorem characterizes equicontinuous families of linear maps using boundedness properties. The crucial assumption is that the underlying space is barrelled, a notion which we explain first. Many spaces are barrelled, which makes the Banach-Steinhaus Theorem applicable in various situation.

3.3.1 Barreled and Bornological Spaces

In a topological vector space we defined bounded sets by their property that they can be absorbed by every neighbourhood of zero. Rescaling the neighbourhood large enough gives a big enough subset to contain the bounded subset. However, given the bounded subsets, other subsets besides the neighbourhoods of zero might have this property as well. They are called *bornivorous*:

Definition 3.3.1 (Bornivorous subsets) *Let V be a topological vector space and let $A \subseteq V$ be a subset. Then A is called bornivorous if for all bounded subsets $B \subseteq V$ there exists an $r > 0$ with $B \subseteq rA$.*

Then by definition, zero neighbourhoods are bornivorous. It is thus interesting to understand which other bornivorous subsets one has to expect. This leads to the following definition:

Definition 3.3.2 (Barrelled and infrabarrelled spaces) *Let V be a topological vector space.*

i.) A closed absolutely convex absorbing subset of V is called a barrel.

ii.) The topological vector space V is called infrabarrelled if every bornivorous barrel is a zero neighbourhood.

iii.) The topological vector space V is called barrelled if every barrel is a zero neighbourhood.

For some first properties of barrels, see Exercise 3.6.15. As yet another variation one considers only absolutely convex subsets which are bornivorous, and wants them to be zero neighbourhoods.

Definition 3.3.3 (Bornological space) *Let V be a topological vector space. Then V is called bornological if all absolutely convex bornivorous subsets are zero neighbourhoods.*

Clearly, a barrelled space is infrabarrelled and a bornological space is infrabarrelled, as well.

In the following we will mainly be interested in Hausdorff locally convex spaces instead of general topological vector spaces. Nevertheless, the above definitions make sense as soon as we have a topological vector space.

First we show that all three types of spaces behave well under quotients and finite direct sums:

Proposition 3.3.4 *Let $W \subseteq V$ be a closed subspace of a Hausdorff locally convex space and equip the quotient V/W with its quotient locally convex topology.*

i.) If V is barrelled, then V/W is barrelled, too.

ii.) If V is infrabarrelled, then V/W is infrabarrelled, too.

iii.) If V is bornological, then V/W is bornological, too.

Proof: Let $\mathrm{pr}\colon V \longrightarrow V/W$ denote the quotient map, which is continuous, linear, and open, see Proposition 2.4.10, *iv.)*. The preimage of an absolutely convex subset in V/W is absolutely convex, since pr is linear. Now assume $A \subseteq V/W$ is a barrel. Then $\mathrm{pr}^{-1}(A)$ is closed and absolutely convex, too. Moreover, if $v \in V$, then $\mathrm{pr}(v) \in \lambda A$ for some $\lambda > 0$, as A is absorbing. This means $v \in \lambda\,\mathrm{pr}^{-1}(A)$ and hence also $\mathrm{pr}^{-1}(A)$ is absorbing. Together this shows that $\mathrm{pr}^{-1}(A)$ is a barrel. If in addition $B \subseteq V$ is bounded and A is bornivorous, then $\mathrm{pr}(B) \subseteq \lambda A$ for some $\lambda > 0$, since the image of a bounded subset under a continuous linear map is again bounded by Proposition 2.1.36. But then $B \subseteq \mathrm{pr}^{-1}(\mathrm{pr}(B)) \subseteq \mathrm{pr}^{-1}(\lambda A) = \lambda\,\mathrm{pr}^{-1}(A)$ shows that $\mathrm{pr}^{-1}(A)$ is bornivorous again. Combining this shows that for a bornivorous barrel A also $\mathrm{pr}^{-1}(A)$ is a bornivorous barrel. Finally, for a bornivorous absolutely convex A also $\mathrm{pr}^{-1}(A)$ is bornivorous and absolutely convex again. Using now the assumption that V is barrelled (infrabarrelled or bornivorous) we conclude that $\mathrm{pr}^{-1}(A)$ is a neighbourhood of zero. But then $A = \mathrm{pr}(\mathrm{pr}^{-1}(A))$ is a neighbourhood of zero again, since pr is open. $\square$

Proposition 3.3.5 *Let V and W be Hausdorff locally convex spaces and equip $V \times W$ with the product topology.*

i.) If V and W are barrelled, then $V \times W$ is barrelled, too.

ii.) If V and W are infrabarrelled, then $V \times W$ is infrabarrelled, too.

iii.) If V and W are bornological, then $V \times W$ is bornological, too.

PROOF: We have the continuous inclusions

$$\iota_V \colon V \longrightarrow V \times W \quad \text{and} \quad \iota_W \colon W \longrightarrow V \times W.$$

Thus for a bounded subset $B \subseteq V$ also $\iota_V(B) = B \times \{0\}$ is bounded in $V \times W$, similarly for $B' \subseteq W$. Next, let $A \subseteq V \times W$ be absolutely convex, then $\iota_V^{-1}(A) \cong A \cap \big(V \times \{0\}\big)$ and $\iota_W^{-1}(A) \cong A \cap \big(\{0\} \times W\big)$ are absolutely convex, too. Since ι_V and ι_W are continuous, $\iota_V^{-1}(A)$ and $\iota_W^{-1}(A)$ are closed for all closed A. Finally, assume A is bornivorous, then for any bounded $B \subseteq V$ we have a $\lambda > 0$ with $\iota_V(B) \subseteq \lambda A$ and thus $B = \iota_V^{-1}(\iota_V(B)) \subseteq \iota_V^{-1}(\lambda A) = \lambda \iota_V^{-1}(A)$. Thus $\iota_V^{-1}(A)$ and analogously $\iota_W^{-1}(A)$ are bornivorous, too. In conclusion, if A is a barrel, a bornivorous barrel, or a bornivorous absolutely convex subsets, respectively, then in either of the three cases the assumptions on V and W imply that $\iota_V^{-1}(A)$ and $\iota_W^{-1}(A)$ are zero neighbourhoods. But then A itself is zero neighbourhood by convexity, since for $v \in \iota_V^{-1}(A)$ and $w \in \iota_W^{-1}(A)$ we have

$$\tfrac{1}{2}(v, 0) + \tfrac{1}{2}(0, w) = \tfrac{1}{2}(v, w) \in A.$$

This shows $\frac{1}{2}\iota_V^{-1}(A) \times \frac{1}{2}\iota_W^{-1}(A) \subseteq A$ and thus A is a zero neighbourhood, too, as claimed. □

It follows that finite-dimensional spaces are barrelled and bornological, since $\mathbb{K}$ is. More generally, it is once more Baire's Theorem, which gives interesting examples:

Proposition 3.3.6 *Let V be a topological vector space, which is a Baire space. Then V is barreled.*

PROOF: Recall that V is a Baire space if any countable union of closed subsets of V without inner points has no inner points, see e.g. [19, Definition 7.1.6]. Thus let $A \subseteq V$ be a barrel. Since A is absolutely convex and absorbing, we have

$$V = \bigcup_{n=1}^{\infty} nA$$

and each nA is again a barrel. In particular, nA is closed. Since V has inner points, at least one of these closed subsets nA has an interior point, say $v_0 \in n_0 A$. But then already $\frac{1}{n_0} v_0 \in A$ is an inner point, as the rescaling is a homeomorphism. Suppose $\frac{1}{n_0} v_0 \neq 0$, then also $-\frac{1}{n_0} v_0$ is an inner point, since A is absolutely convex and $v \mapsto -v$ is a homeomorphism. Now the interior of A is still convex by Proposition 2.1.30, *ii.)*. In particular, $0 = \frac{1}{2}(\frac{1}{n_0} v_0) + \frac{1}{2}(-\frac{1}{n_0} v_0)$ is a convex combination of two inner points, thus inner itself. This shows that A is a neighbourhood of zero. □

Corollary 3.3.7 *Every Fréchet space is barrelled.*

PROOF: Being complete metric spaces, Baire's Theorem guarantees that a Fréchet space is a Baire space, see [19, Theorem 7.2.1]. □

In particular, Banach spaces as particular cases of Fréchet spaces are barrelled. The corollary is somehow trivial, as Fréchet and Banach spaces are easily seen to be Baire spaces. However, there are barrelled spaces beyond the Fréchet situation, which are not at all Baire spaces:

Proposition 3.3.8 *Let $V = \varinjlim V_i$ be a locally convex inductive limit of barrelled spaces $\{V_i\}_{i \in I}$, forming an inductive system. Then V is barrelled again.*

PROOF: Let $B \subseteq V$ be a barrel and denote the canonical maps by $\phi_i \colon V_i \longrightarrow V$. Then $\phi_i^{-1}(B)$ is closed, as each ϕ_i is continuous. Moreover, $\phi_i^{-1}(B)$ stays absolutely convex and absorbing. Hence $\phi_i^{-1}(B) = B_i \subseteq V_i$ is a barrel for all $i \in I$ and thus a zero neighbourhood. With Proposition 2.4.33 we see that $B \subseteq V$ is a zero neighbourhood, as well. □

Corollary 3.3.9 *Any LF space is barrelled.*

On the other hand there are examples of normed spaces, which are not barrelled, even though completeness is not necessary in general. One can also find examples of non-complete, but barrelled spaces. For more examples and some non-examples, see Exercise 3.6.16.

We turn now to examples of bornological spaces. Here completeness is not essential, but first countability is sufficient:

Proposition 3.3.10 *A first countable Hausdorff locally convex space V is bornological.*

PROOF: Let $\{U_n\}_{n \in \mathbb{N}}$ be a countable basis of zero neighbourhoods, which we can arrange to satisfy $U_1 \supseteq U_2 \supseteq \ldots$. Now consider an absolutely convex subset $A \subseteq V$, which is not a zero neighbourhood. Then also nA is not a zero neighbourhood. Thus U_n can not be contained entirely in nA for all $n \in \mathbb{N}$. This allows to choose vectors $v_n \in U_n \setminus nA \neq \emptyset$. Since $v_n \in U_n$, the sequence $(v_n)_{n \in \mathbb{N}}$ converges to zero. Thus the set of all these points is bounded: in fact $B = \{v_n\}_{n \in \mathbb{N}} \cup \{0\}$ is a compact subset and hence bounded. However, since $v_n \notin nA$ the set A can not absorb this bounded set B and thus A is not bornivorous. □

Corollary 3.3.11 *Normed spaces and Fréchet spaces are bornological.*

Also beyond the first countable situation we have bornological spaces:

Proposition 3.3.12 *Let $V = \varinjlim V_i$ be a locally convex inductive limit of bornological locally convex spaces $\{V_i\}_{i \in I}$ forming an inductive system. Then V is bornological, too.*

PROOF: The argument is similar to the proof of Proposition 3.3.8. Let $\phi_i \colon V_i \longrightarrow V$ denote the canonical continuous linear maps. Suppose that $A \subseteq V$ is absolutely convex and bornivorous. Let $B_i \subseteq V_i$ be bounded, then $\phi_i(B_i) \subseteq V$ is still bounded, hence absorbed by A with some scaling λ, i.e. $\phi_i(B_i) \subseteq \lambda A$. This means that $B_i \subseteq \phi_i^{-1}(\lambda A) = \lambda \phi_i^{-1}(A)$. Hence the subset $A_i = \phi_i^{-1}(A) \subseteq V_i$ is bornivorous for all $i \in I$. As preimage of an absolutely convex set under a linear map, A_i is absolutely convex again. By assumption, $A_i \subseteq V_i$ is a zero neighbourhood, as V_i is bornological by assumption. Then again Proposition 2.4.33 shows that $A \subseteq V$ is a zero neighbourhood, too, proving that V is bornological. □

Corollary 3.3.13 *Any LF space is bornological.*

A first reason to consider bornological spaces is, among many others to come, the following characterization of continuous linear maps. Note that in general a continuous linear map is sequentially continuous, but not necessarily vice versa if the topology fails to be first countable. Also, a sequentially continuous linear map is bounded, but not necessarily vice versa. In the bornological situation all three concepts coincide:

Proposition 3.3.14 *Let V and W be Hausdorff locally convex spaces with V being bornological. Then for a linear map $\phi \colon V \longrightarrow W$ the following statements are equivalent:*

i.) *The map ϕ is continuous.*

ii.) *The map ϕ is sequentially continuous.*

iii.) *The map ϕ is bounded, i.e. maps bounded subsets of V to bounded subsets of W.*

PROOF: We have *i.)* $\implies$ *ii.)* in general. The implication *ii.)* $\implies$ *iii.)* is Proposition 2.1.36 and also holds in general. Thus we need to show *iii.)* $\implies$ *i.)* using the assumption that V is bornological. Let $U \subseteq W$ be an absolutely convex neighbourhood of zero in W. Then $\phi^{-1}(U) \subseteq V$ is still absolutely convex and bornivorous: indeed, the preimage of bornivorous subsets under bounded linear maps is bornivorous, since for a bounded subset $B \subseteq V$ the subset $\phi(B) \subseteq W$ is bounded. Hence we find an $r > 0$ with $\phi(B) \subseteq rU$ and thus $B \subseteq \phi^{-1}(rU) = r\phi^{-1}(U)$ shows that $\phi^{-1}(U)$ is bornivorous. By assumption that V is bornological, we see that $\phi^{-1}(U)$ is a zero neighbourhood, which implies the continuity of ϕ, as we have a basis of absolutely convex zero neighbourhoods in W by the very definition of a locally convex space. □

In particular, the equivalence of *i.)* and *ii.)* is already very helpful. Note that many of our important examples of locally convex spaces are bornological, in particular the LF spaces. Here the equivalence of continuity and sequential continuity is non-trivial, as we know that LF spaces are *not* first countable by Proposition 2.4.49.

3.3.2 Topologies for Continuous Linear Maps

To appreciate the significance of barrelled spaces we first need to establish topologies on the space of continuous linear maps. Motivated by the case of the topological dual V' with the topologies arising from systems of bounded subsets of V as discussed in Definition 3.2.16, we extend this to general linear maps as follows:

Lemma 3.3.15 *Let V and W be locally convex spaces. For a bounded subset $B \subseteq V$ and a continuous seminorm q on W*

$$\mathrm{p}_{B,\mathrm{q}}(\Phi) = \sup_{v \in B} \mathrm{q}(\Phi(v)) \tag{3.3.1}$$

defines a seminorm on $\mathrm{L}(V, W)$.

PROOF: Since Φ is continuous, $\mathrm{q} \circ \Phi$ is a continuous seminorm on V. Hence it is bounded on the bounded subset B, which shows $\mathrm{p}_{B,\mathrm{q}}(\Phi) < \infty$. The properties of a seminorm are then straightforward. □

We take now these seminorms as starting point to define topologies on $\mathrm{L}(V, W)$. Note that this is entirely parallel to the case of $V' = \mathrm{L}(V, \mathbb{K})$, since here we only need one norm of $\mathbb{K}$, namely the absolute value. In the general case, we make use of all continuous seminorms. For some first properties of our new definition, see Exercise 3.6.17.

Definition 3.3.16 ($\mathcal{B}$-Topologies on $\mathrm{L}(V, W)$) *Let V and W be locally convex spaces.*

i.) *A system $\mathcal{B}$ of bounded subsets of V, satisfying the two properties that for $B_1, B_2 \in \mathcal{B}$ one has a $B \in \mathcal{B}$ with $B_1 \cup B_2 \subseteq B$ and that $\{\lambda B\}_{\lambda > 0,\, B \in \mathcal{B}}$ provides a cover of V, is called admissible.*

ii.) *For an admissible system $\mathcal{B}$ of bounded subsets of V the $\mathcal{B}$-topology on $\mathrm{L}(V, W)$ is the locally convex topology determined by the seminorms $\{\mathrm{p}_{B,\mathrm{q}}\}_{B \in \mathcal{B},\, \mathrm{q}\ continuous}$, where q runs through all continuous seminorms on W. We denote the space of continuous linear maps endowed with this topology by $\mathrm{L}_{\mathcal{B}}(V, W)$.*

iii.) *The $\mathcal{B}$-topology for the system $\mathcal{B}$ of finite subsets of V is called the topology of pointwise convergence and is denoted by $\mathrm{L}_\sigma(V, W)$.*

iv.) *The $\mathcal{B}$-topology for the system $\mathcal{B}$ of convex compact subsets of V is called the topology of pointwise convergence and is denoted by $\mathrm{L}_\gamma(V, W)$.*

v.) *The $\mathcal{B}$-topology for the system $\mathcal{B}$ of compact subsets of V is called the topology of compact convergence and is denoted by $\mathrm{L}_c(V, W)$.*

vi.) *The $\mathcal{B}$-topology for the system $\mathcal{B}$ of bounded subsets of V is called the topology of bounded convergence and is denoted by $\mathrm{L}_\beta(V, W)$.*

Remark 3.3.17 Let V and W be locally convex spaces.

i.) For $W = \mathbb{K}$ the above constructions reduce to the various $\mathcal{B}$-topologies on V', leading to

$$\mathrm{L}_\sigma(V, \mathbb{K}) = V'_\sigma, \tag{3.3.2}$$

$$\mathrm{L}_\gamma(V, \mathbb{K}) = V'_\gamma, \tag{3.3.3}$$

$$\mathrm{L}_c(V, \mathbb{K}) = V'_c, \tag{3.3.4}$$

$$\mathrm{L}_\beta(V, \mathbb{K}) = V'_\beta. \tag{3.3.5}$$

ii.) If W is Hausdorff, then the two conditions on an admissible systems $\mathcal{B}$ of bounded subsets guarantee that every $\mathcal{B}$-topology is also Hausdorff.

iii.) The topology of pointwise convergence is the coarsest of all the $\mathcal{B}$-topologies, the topology of bounded convergence is the finest $\mathcal{B}$-topology. We have of course again

$$\mathrm{L}_\sigma(V, W) \leq \mathrm{L}_\gamma(V, W) \leq \mathrm{L}_c(V, W) \leq \mathrm{L}_\beta(V, W), \tag{3.3.6}$$

as well as

$$\mathrm{L}_\sigma(V, W) \leq \mathrm{L}_\mathcal{B}(V, W) \leq \mathrm{L}_\beta(V, W) \tag{3.3.7}$$

for all admissible systems $\mathcal{B}$ of bounded subsets.

iv.) It is clear that for a defining system $\mathcal{Q}$ of continuous seminorms on W the $\mathcal{B}$-topology is already determined by the seminorms $\{\mathrm{p}_{B,\mathrm{q}}\}_{B\in\mathcal{B},\, \mathrm{q}\in\mathcal{Q}}$, see again Exercise 3.6.17.

v.) The names for the special cases of $\mathcal{B}$-topologies are established in the literature, which is why we adapted them here. However, it would be more precise to speak of them as the topology of "uniform convergence on X subsets" instead of the "topology of X convergence".

vi.) We denote the closures of a subset A with respect to the above topologies also $A^{\sigma\text{-cl}}$ or $A^{\beta\text{-cl}}$ etc. for abbreviation.

The $\mathcal{B}$-topology will typically depend on the choice of $\mathcal{B}$ in a rather sensible way. In particular, the four topologies in (3.3.6) will typically be all distinct. We do not provide explicit examples, but we will see the difference between the extreme cases $\mathrm{L}_\sigma(V, W)$ and $\mathrm{L}_\beta(V, W)$ in the following.

As already for the topological duals, the topology of pointwise convergence is very coarse, resulting in the general statement that $\mathrm{L}_\sigma(V, W)$ is not complete besides in trivial situations. Instead, $\mathrm{L}_\sigma(V, W)$ tends to be dense in the space of all linear maps, continuous or not. This changes for the finer strong topology: here we have a fair chance that $\mathrm{L}_\beta(V, W)$ is complete itself. Of course, $\mathrm{L}_\beta(V, W)$ is typically *not* first countable, so we have to deal with Cauchy nets instead of mere Cauchy sequences. We formulate the desired completeness result in two steps, first for a general $\mathcal{B}$-topology:

Theorem 3.3.18 (Completeness of $\mathrm{L}_\mathcal{B}(V, W)$) *Let V and W be two Hausdorff locally convex spaces and suppose that W is complete. Let $\mathcal{B}$ be a collection of bounded subsets of V, satisfying the properties needed in Definition 3.3.16, ii.), such that the following holds: A linear map $\phi\colon V \longrightarrow W$ is continuous iff $\phi\big|_B\colon B \longrightarrow W$ is continuous for all $B \in \mathcal{B}$. In this case $\mathrm{L}_\mathcal{B}(V, W)$ is complete.*

PROOF: Of course, for a continuous linear map ϕ any restriction $\phi\big|_B$ is continuous for trivial reasons. It is the opposite direction we are interested in. The collection $\mathcal{B}$ is rich enough to test continuity. Thus consider a Cauchy net $(\phi_\alpha)_{\alpha\in I}$ in $\mathrm{L}_{\mathcal{B}}(V, W)$. Since the $\mathcal{B}$-topology is finer than the topology of pointwise convergence, $(\phi_\alpha)_{\alpha\in I}$ is also a Cauchy net in $\mathrm{L}_\sigma(V, W)$. It follows that for all $v \in V$ the net $(\phi_\alpha(v))_{\alpha\in I}$ in W is a Cauchy net, hence convergent in W by assumption. We denote the limit by $\phi(v) = \lim_{\alpha\in I} \phi_\alpha(v)$, thereby specifying a map $\phi\colon V \longrightarrow W$. Clearly, ϕ is linear again, see also Exercise 3.6.19. In addition, we know that $(\phi_\alpha)_{\alpha\in I}$ is a Cauchy net with respect to the $\mathcal{B}$-topology. Thus for $B \in \mathcal{B}$, a continuous seminorm q on W, and for $\epsilon > 0$ we find an index $\alpha_0 \in I$ with

$$\mathrm{p}_{B,\mathrm{q}}(\phi_\alpha - \phi_\beta) = \sup_{v\in B} \mathrm{q}(\phi_\alpha(v) - \phi_\beta(v)) < \epsilon, \tag{$*$}$$

whenever $\alpha, \beta \succcurlyeq \alpha_0$. Hence we conclude that on $B \in \mathcal{B}$ the pointwise convergence $\phi_\alpha(v) \to \phi(v)$ is actually *uniform*. For $v, v' \in B$ this implies

$$\mathrm{q}(\phi(v) - \phi(v')) \le \mathrm{q}(\phi(v) - \phi_\alpha(v)) + \mathrm{q}(\phi_\alpha(v) - \phi_\alpha(v')) + \mathrm{q}(\phi_\alpha(v') - \phi(v'))$$

for all $\alpha \in I$. Now the pointwise convergence $\phi_\alpha(v) \to \phi(v)$ and $\phi_\alpha(v') \to \phi(v')$ gives indices $\beta_0, \beta_0' \in I$ with $\mathrm{q}(\phi(v) - \phi_\alpha(v)) < \epsilon$ and $\mathrm{q}(\phi(v') - \phi_\alpha(v')) < \epsilon$, whenever $\alpha \succcurlyeq \beta_0$ and $\alpha \succcurlyeq, \beta_0'$, respectively. The continuity of ϕ_α gives then a continuous seminorm p on V, such that $\mathrm{p}(v - v') \le 1$ implies $\mathrm{p}(\phi_\alpha(v) - \phi_\alpha(v')) < \epsilon$. We conclude that on $\mathcal{B}$ the limit ϕ is continuous. The (non-trivial) assumption about the system $\mathcal{B}$ then implies $\phi \in \mathrm{L}(V, W)$. Finally, taking the limit over α' in $(*)$ gives for all $\alpha \succcurlyeq \alpha_0$

$$\mathrm{p}_{B,\mathrm{q}}(\phi - \phi_\alpha) = \sup_{v\in B} \mathrm{p}(\phi(v) - \phi_\alpha(v)) \le \epsilon,$$

which is the convergence $\phi_\alpha \to \phi$ in the $\mathcal{B}$-topology. □

Of course, the theorem is still pretty useless, as the assumption about the collection $\mathcal{B}$ seems to be highly technical and not easy to check at all. There is one exception: the system of all bounded subsets for a bornological space.

Corollary 3.3.19 *Let V be a bornological Hausdorff locally convex space and let W be a complete locally convex space. Then $\mathrm{L}_\beta(V, W)$ is complete.*

PROOF: Indeed, suppose V is bornological. Assume $\phi\colon V \longrightarrow W$ is a linear map, such that $\phi\big|_B\colon B \longrightarrow W$ is continuous for all bounded subsets $B \subseteq V$. We claim that then ϕ maps bounded subsets to bounded ones. Indeed, let $B \subseteq V$ be bounded and let $(w_n)_{n\in\mathbb{N}}$ be a sequence in $\phi(B)$. We choose corresponding pre-images $v_n \in B$, i.e. $\phi(v_n) = w_n$. Since B is bounded, we have for every zero sequence $(z_n)_{n\in\mathbb{N}}$ of scalars the convergence $z_n v_n \to 0$. Since the set $\{z_n v_n\}_{n\in\mathbb{N}} \cup \{0\}$ is still bounded in V, the map ϕ is continuous there, implying $\phi(z_n v_n) = z_n\phi(v_n) = z_n w_n \to 0$. This implies that $\phi(B)$ is still bounded. Now V is assumed to be bornological and hence a bounded linear map ϕ is continuous by Proposition 3.3.14. This shows that V satisfies the assumption of Theorem 3.3.18 for the collection of all bounded subsets. □

Luckily, many interesting locally convex spaces are known to be bornological: normed spaces, Fréchet spaces and LF spaces to name a few. Applied to the duals, we can take advantage of the completeness of the scalars $\mathbb{K}$:

Corollary 3.3.20 *Let V be a normed space, a Fréchet space, or an LF space. Then V_β' is complete.*

We conclude our discussion of $\mathrm{L}_\beta(V, W)$ for the particular case of normed spaces V and W, generalizing the situation of Proposition 3.2.23. First we recall that for $\phi \in \mathrm{L}(V, W)$ the *operator norm* is defined by

$$\|\phi\| = \sup_{\|v\|_V = 1} \|\phi(v)\|_W, \tag{3.3.8}$$

where $\|\cdot\|_V$ and $\|\cdot\|_W$ are the norms of V and W, respectively. This is easily verified to be a norm on $\mathrm{L}(V, W)$. We then get the following result, analogous to the particular case of the topological dual space:

Proposition 3.3.21 *Let V and W be normed spaces. Then the operator norm defines the topology of bounded convergence on $\mathrm{L}(V, W)$. In particular, $\mathrm{L}_\beta(V, W)$ becomes a Banach space with respect to* (3.3.8) *if W is a Banach space.*

PROOF: The proof is analogous to the one of Proposition 3.2.23. □

3.3.3 Equicontinuity and the Banach-Steinhaus Theorem

In general, the weak* topology on V' is not complete in a quite drastic way, see Exercise **??**. However, the density of V'_σ in the algebraic dual V^*_σ requires general nets and not just sequences. Since on the other hand, the convergence criterion of the weak* topology is particularly simple, it would be desirable to find situations where V'_σ is at least *sequentially* complete. In fact, many constructions in functional analysis produce linear functionals as weak* convergent sequences of continuous ones. Slightly more general, the same type of questions arises for linear maps with a general locally convex target W instead of $\mathbb{K}$.

Remarkably, one only needs to pose a condition on the domain V and not on the target W to find solutions to these kinds of questions. To this end, we state the definition of *equicontinuity*. In principle one can formulate this for a general uniform space as target, but we restrict ourselves to a topological vector space:

Definition 3.3.22 (Equicontinuity) *Let X be a topological space and let W be a topological vector space. A set $\mathcal{F} \subseteq \mathrm{Map}(X, W)$ of W-valued functions on X is called equicontinuous at $x \in X$ if for every zero neighbourhood $O \subseteq W$ there exits a neighbourhood $U \subseteq X$ of x such that*

$$f(x) - f(y) \in O \tag{3.3.9}$$

for all $f \in \mathcal{F}$ and all $y \in U$. The set $\mathcal{F}$ is called equicontinuous if it is equicontinuous at all points $x \in X$.

Remark 3.3.23 It is clear that for a set $\mathcal{F} = \{f\}$ consisting of a single function, equicontinuity of $\mathcal{F}$ at $x \in X$ is just the continuity of f at x. Also for a finite set of functions $\mathcal{F} = \{f_1, \ldots, f_n\}$ equicontinuity at x means just that all functions are continuous at x. This follows directly by taking finite intersections of neighbourhoods at x. The situation becomes interesting for infinite sets $\mathcal{F}$. Then equicontinuity of $\mathcal{F}$ still implies continuity of each element of $\mathcal{F}$. However, we have the stronger property that the same continuity data "for all O exists a U" applies now to *all* elements of $\mathcal{F}$.

If the domain is now even a topological vector space V itself and all maps are linear, the translation invariance of the topology immediately gives the following equivalent formulations:

Proposition 3.3.24 *Let V and W be topological vector spaces and let $\Phi \subseteq \mathrm{Hom}(V, W)$. Then the following statements are equivalent:*

i.) The set Φ is equicontinuous.

ii.) The set Φ is equicontinuous at some point in V.

iii.) The set Φ is equicontinuous at zero.

iv.) For every zero neighbourhood $O \subseteq W$ there exists a zero neighbourhood $U \subseteq V$ such that for all $v \in U$ one has

$$\phi(v) \in O \tag{3.3.10}$$

for all $\phi \in \Phi$.

v.) For all zero neighbourhoods $O \subseteq W$ *the set*

$$U = \bigcap_{\phi \in \Phi} \phi^{-1}(O) \tag{3.3.11}$$

is a zero neighbourhood $U \subseteq V$.

In particular, $\Phi \subseteq \mathrm{L}(V, W)$ *follows from equicontinuity.*

PROOF: The translation invariance of the topology of V gives the equivalence of the first three statements. The third is just equivalent to *iv.)* by using the fact that $\phi(0) = 0$ for a linear map. This is also seen to be equivalent to *v.)*: starting with U from *iv.)* we have $U \subseteq \bigcap_{\phi \in \Phi} \phi^{-1}(O)$ and hence also $\bigcap_{\phi \in \Phi} \phi^{-1}(O)$ is a neighbourhood of zero. The opposite direction is clear, as *v.)* gives us a candidate U for the zero neighbourhood needed in *iv.)*. □

If both topological vector spaces V and W are locally convex, we also have a seminorm based version of equicontinuity:

Proposition 3.3.25 *Let V and W be locally convex spaces. A subset $\Phi \subseteq \mathrm{L}(V, W)$ is equicontinuous iff for all continuous seminorms q on W there exists a continuous seminorm p on V, such that*

$$(\phi^* \mathrm{q})(v) = \mathrm{q}(\phi(v)) \leq \mathrm{p}(v) \tag{3.3.12}$$

for all $\phi \in \Phi$ and all $v \in V$. It suffices to check (3.3.12) *for a defining system of continuous seminorms on W.*

PROOF: This follows e.g. from Proposition 3.3.24, *iv.)*, and the fact that we can test (3.3.10) for open balls with respect to continuous seminorms q on W instead of general (open) zero neighbourhoods. □

This formulation of equicontinuity makes it perhaps most transparent, what "equi" is all about: we have the *same continuity estimates* for all elements in an equicontinuous set $\Phi \subseteq \mathrm{L}(V, W)$.

Equicontinuous subsets of $\mathrm{L}(V, W)$ behave much nicer than general subsets in many aspects. As a first example one has the following statement on the weak closures, i.e. the closure with respect to pointwise convergence. Note that the statement is non-trivial insofar, as $\mathrm{L}_\sigma(V, W) \subseteq \mathrm{Hom}_\sigma(V, W)$ is not closed in general, see again Exercise 3.6.8.

Proposition 3.3.26 *Let V and W be locally convex spaces and let $\Phi \subseteq \mathrm{L}(V, W)$ be an equicontinuous subset. Then the closure $\Phi^{\sigma\text{-cl}}$ with respect to the σ-topology in $\mathrm{Hom}(V, W)$ is actually contained in $\mathrm{L}_\sigma(V, W)$. Moreover, $\Phi^{\sigma\text{-cl}} \subseteq \mathrm{L}_\sigma(V, W)$ is again equicontinuous.*

PROOF: Let q be a continuous seminorm on W with a corresponding continuous seminorm on V such that $\phi^* \mathrm{q} = \mathrm{q} \circ \phi \leq \mathrm{p}$ for all $\phi \in \Phi$. Thanks to Proposition 3.3.25 we can characterize equicontinuity like that. If now $(\phi_\alpha)_{\alpha \in I}$ is a net in Φ converging to $\phi \in \Phi^{\sigma\text{-cl}} \subseteq \mathrm{Hom}_\sigma(V, W)$ in the topology of pointwise convergence, then $\mathrm{q}\big(\phi_\alpha(v)\big) \leq \mathrm{p}(v)$ gives

$$\mathrm{q}(\phi(v)) \leq \sup_{\alpha \in I} \mathrm{q}(\phi_\alpha(v)) \leq \mathrm{p}(v),$$

and thus $\phi \in \mathrm{L}(V, W)$. Moreover, we have the same continuity estimate for ϕ as for the elements in Φ. It follows by Proposition 3.3.25 that $\Phi^{\sigma\text{-cl}}$ is still equicontinuous. □

The statement is still correct for general topological vector spaces, but in the locally convex case it almost becomes a triviality. The next statement is of similar quality. On equicontinuous subsets the various topologies defined on $\mathrm{L}(V, W)$ in Definition 3.3.16 coincide:

Proposition 3.3.27 *Let V and W be locally convex spaces and let $\Phi \subseteq \mathrm{L}(V, W)$ be equicontinuous. Then all $\mathcal{B}$-topologies coarser than the topology of compact convergence (i.e. convergence on compact subsets) coincide on Φ.*

PROOF: If $\mathcal{B}$ is any admissible set of bounded subsets of V to define a $\mathcal{B}$-topology, we know that

$$\sigma \le \mathcal{B} \le c$$

for the topologies, where we also use the assumption. Thus for the restrictions to Φ we still have

$$\sigma\big|_\Phi \le \mathcal{B}\big|_\Phi \le c\big|_\Phi.$$

It thus suffices to show $\sigma\big|_\Phi = c\big|_\Phi$. To this end let $K \subseteq V$ be a compact subset and q be a continuous seminorm on W with corresponding continuous p on V, such that $\phi^*\mathrm{q} = \mathrm{q} \circ \phi \le \mathrm{p}$ for all $\phi \in \Phi$. Furthermore let $\epsilon > 0$ be arbitrary and $\phi_0 \in \Phi$ be given. A neighbourhood basis of ϕ_0 in the c-topology is then given by the intersections $\Phi \cap \mathrm{B}_{\mathrm{p}_{K,\mathrm{q}},r}(\phi_0)$ for $r > 0$ and all such K, where $\mathrm{p}_{K,\mathrm{q}} = \sup_{v \in K} \mathrm{q}(\phi(v))$ is the corresponding seminorm of the c-topology. We need to show that such a subset is also a σ-neighbourhood. Thus let $v_1, \ldots, v_n \in V$ be finitely many points with

$$K \subseteq (v_1 + \mathrm{B}_{\mathrm{p},\epsilon}(0)) \cup \cdots \cup (v_n + \mathrm{B}_{\mathrm{p},\epsilon}(0))$$

by compactness of K. For $\phi \in \Phi$ we then get for $v \in K$ and $i \in \{1, \ldots, n\}$ such that $v \in v_i + \mathrm{B}_{\mathrm{p},\epsilon}(0)$ the estimate

$$\begin{aligned}
\mathrm{q}((\phi - \phi_0)(v)) &= \mathrm{q}((\phi - \phi_0)(v_i)) + \mathrm{q}((\phi - \phi_0)(v - v_i)) \\
&\le \mathrm{q}((\phi - \phi_0)(v_i)) + \mathrm{q}(\phi(v - v_i)) + \mathrm{q}(\phi_0(v - v_i)) \\
&\le \mathrm{q}((\phi - \phi_0)(v_i)) + \mathrm{p}(v - v_i) + \mathrm{p}(v - v_i) \\
&< \max_{i=1}^{n} \mathrm{q}((\phi - \phi_0)(v_i)) + 2\epsilon.
\end{aligned}$$

Hence we get

$$\mathrm{p}_{K,\mathrm{q}}(\phi - \phi_0) = \sup_{v \in K} \mathrm{q}((\phi - \phi_0)(v)) \le \max_{i=1}^{n} \mathrm{q}((\phi - \phi_0)(v_i)) + 2\epsilon.$$

Fix, say, $\epsilon < \frac{1}{4}$. Then this estimate shows that those $\phi \in \Phi$ with $\phi - \phi_0$ in the σ-open zero neighbourhood

$$U = \mathrm{B}_{\mathrm{p}_{v_1,\mathrm{q}},\frac{1}{2}}(0) \cap \cdots \cap \mathrm{B}_{\mathrm{p}_{v_n,\mathrm{q}},\frac{1}{2}}(0),$$

where we use the σ-continuous seminorms $\mathrm{p}_{v_1,\mathrm{q}} = \mathrm{q}(\phi(v_i))$, are contained in the c-open neighbourhood $\mathrm{B}_{\mathrm{p}_{K,\mathrm{q}},1} + \phi_0$ of ϕ_0, since

$$\mathrm{p}_{K,\mathrm{q}}(\phi - \phi_0) < \max_{i=1}^{n} \mathrm{q}((\phi - \phi_0)(v_i)) + \tfrac{1}{2} < 1.$$

This shows $U \cap \Phi \subseteq (\mathrm{B}_{\mathrm{p}_{K,\mathrm{q}},1} + \phi_0) \cap \Phi$. Hence the σ-neighbourhoods provide a finer neighbourhood basis than the c-neighbourhoods. □

Concerning the topology of bounded convergence we get the following result:

Proposition 3.3.28 *Let V and W be locally convex spaces and let $\Phi \subseteq \mathrm{L}(V, W)$ be equicontinuous. Then $\Phi \subseteq \mathrm{L}_\beta(V, W)$ is bounded and thus Φ is bounded in all coarser $\mathcal{B}$-topologies, as well.*

Proof: Since the β-topology of bounded convergence is the finest of the $\mathcal{B}$-topologies, any bounded subset in this topology will also be bounded in any coarser $\mathcal{B}$-topology. Thus let q be a continuous seminorm on W with corresponding continuous seminorm p on V, such that $\phi^*\mathrm{q} = \mathrm{q} \circ \phi \leq \mathrm{p}$ for all $\phi \in \Phi$. Moreover, choose a bounded subset $B \subseteq V$. Then for all $\phi \in \Phi$ we have

$$\mathrm{p}_{B,\mathrm{q}}(\phi) = \sup_{v \in B} \mathrm{q}(\phi(v)) \leq \sup_{v \in B} \mathrm{p}(v) = c < \infty,$$

with a constant $c > 0$, which is independent of ϕ, since B is bounded. Thus the seminorm $\mathrm{p}_{B,\mathrm{q}}$ is bounded by c on the subset Φ. As these seminorms define the β-topology, Φ is bounded. □

If $W = \mathbb{K}$, i.e. for equicontinuous sets of linear functionals on V we have the following nice consequence:

Proposition 3.3.29 *Let V be a Hausdorff locally convex space and let $\Phi \subseteq V'$ be a subset.*

i.) The subset Φ is equicontinuous iff

$$\Phi \subseteq U^{\circledast} \tag{3.3.13}$$

for a zero neighbourhood $U \subseteq V$.

ii.) In this case, $\Phi^{\sigma\text{-cl}} \subseteq V'_\sigma$ is compact.

Proof: According to Proposition 3.3.25 the subset $\Phi \subseteq V'$ is equicontinuous iff there is a continuous seminorm p on V with

$$|\varphi(v)| \leq \mathrm{p}(v)$$

for all $\varphi \in \Phi$ and $v \in V$. This is equivalent to $|\varphi(v)| \leq 1$ for all $v \in \mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}}$ and hence to $\varphi \in (\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}})^{\circledast}$. Since for every zero neighbourhood $U \subseteq V$ we have a continuous seminorm p with $\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}} \subseteq U$ and thus $U^{\circledast} \subseteq (\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}})^{\circledast}$, the first part follows. Since a polar $U^{\circledast} \subseteq V'_\sigma$ is weak* compact by the Banach-Alaoglu Theorem 3.2.11, the second statement is clear. □

We have now convinced ourselves that equicontinuous sets of linear maps enjoy nice properties. However, it might be fairly complicated to actually verify the equicontinuity in any of its forms discussed in Proposition 3.3.24 or Proposition 3.3.25. In general, there is not much one can do about this. If, however, the domain is barrelled, the fundamental theorem of Banach-Steinhaus gives the following characterization:

Theorem 3.3.30 (Banach-Steinhaus) *Let V and W be locally convex spaces with V being barrelled. For a subset $\Phi \subseteq \mathrm{L}(V, W)$ the following statements are equivalent:*

i.) The subset Φ is bounded in $\mathrm{L}_\sigma(V, W)$.

ii.) The subset Φ is bounded in $\mathrm{L}_\beta(V, W)$.

iii.) The subset Φ is equicontinuous.

Proof: Thanks to Proposition 3.3.26 we have *iii.)* $\implies$ *ii.)* for general V. Also the implication *ii.)* $\implies$ *i.)* is trivial, since the σ-topology is coarser than the β-topology. It is the implication *i.)* $\implies$ *iii.)* that requires V to be barrelled. Thus suppose that Φ is bounded with respect to the topology of pointwise convergence and let q be a continuous seminorm on W. Since every $\phi \in \Phi$ is continuous itself, the preimages $\phi^{-1}(\mathrm{B}_{\mathrm{q},1}(0)^{\mathrm{cl}})$ are closed. Moreover, as $\mathrm{B}_{\mathrm{q},1}(0)^{\mathrm{cl}}$ is absolutely convex, also $\phi^{-1}(\mathrm{B}_{\mathrm{q},1}(0)^{\mathrm{cl}})$ is absolutely convex. Finally, being pointwise bounded means that we have a constant $c_v > 0$ for all $v \in V$, such that

$$\sup_{\phi \in \Phi} \mathrm{q}(\phi(v)) \leq c_v.$$

Given $v \in V$ we thus have $\mathrm{q}(\frac{1}{c_v}\phi(v)) \leq 1$ and thus

$$\frac{1}{c_v} v \in \bigcap_{\phi \in \Phi} \phi^{-1}(\mathrm{B}_{\mathrm{q},1}(0)^{\mathrm{cl}}) = A$$

or $v \in c_v A$. Hence A is absorbing. Being an intersection of closed absolutely convex subsets, A is closed and absolutely convex, too. Thus A is a barrel and hence a neighbourhood of zero by assumption. From Proposition 3.3.24, *iv.)*, we see that Φ is equicontinuous. □

Remark 3.3.31 (Banach-Steinhaus Theorem) Under slightly milder assumptions one can still get some of the implications.

i.) If V is infra-barrelled, then we still have *ii.)* $\iff$ *iii.)* in Theorem 3.3.30.

ii.) If V is sequentially complete, then we still have *i.)* $\iff$ *ii.)* in Theorem 3.3.30.

The proof of these equivalences are discussed in Exercise **??**.

The importance of the Banach-Steinhaus Theorem, also referred to as the *Principle of Uniform Boundedness*, can hardly be overestimated. Its usefulness originates of course in the amount of barrelled spaces we have. Fréchet spaces and in particular Banach spaces are barrelled and so are LF spaces. Baire's Theorem is then hidden in these examples, the actual proof of Theorem 3.3.30 is not very spectacular in the end.

To formulate some consequences, we recall some practical ways to test for boundedness:

Corollary 3.3.32 *Let V and W be locally convex spaces with V being barrelled. Then $\Phi \subseteq \mathrm{L}(V, W)$ is equicontinuous iff for all continuous seminorms* q *on W one has*

$$\sup_{\phi \in \Phi} \mathrm{q}(\phi(v)) < \infty \tag{3.3.14}$$

for all $v \in V$.

Corollary 3.3.33 *Let V be a Banach space and let W be a normed space. Then $\Phi \subseteq \mathrm{L}(V, W)$ is equicontinuous iff*

$$\sup_{\phi \in \Phi} \|\phi(v)\| < \infty \tag{3.3.15}$$

for all $v \in V$.

Corollary 3.3.34 *Let V be a barreled space. Then $\Phi \subseteq V'$ is equicontinuous iff*

$$\sup_{\varphi \in \Phi} |\varphi(v)| < \infty \tag{3.3.16}$$

for all $v \in V$.

Indeed, these are always the statements that Φ is bounded in the topology of pointwise convergence.

A first not so obvious consequence is the following statement, which turns out to be very useful in many places:

Proposition 3.3.35 *Let V and W be Hausdorff locally convex spaces with V being barrelled. Suppose that $(\phi_i)_{i \in I}$ is a net in $\mathrm{L}(V, W)$, such that for all $v \in V$ the net $(\phi_i(v))_{i \in I}$ converges in W. Suppose that there is a subnet $(\phi_{i(j)})_{j \in J}$ of $(\phi_i)_{i \in I}$, which is pointwise bounded. Then*

$$\phi(v) = \lim_{i \in I} \phi_i(v) \tag{3.3.17}$$

defines a continuous linear map $\phi \in \mathrm{L}(V, W)$ and

$$\phi = \lim_{j \in J} \phi_{i(j)} \tag{3.3.18}$$

in the topology of convergence on compact subsets.

PROOF: Let $\Phi = (\phi_{i(j)})_{j \in J}$ be the subnet of $(\phi_i)_{i \in I}$, with some cofinal map $J \ni j \mapsto i(j) \in I$ as usual, such that Φ is pointwise bounded. Then $\Phi \subseteq \mathrm{L}(V, W)$ is equicontinuous by the Banach Steinhaus Theorem 3.3.30. The pointwise convergence (3.3.17) defines a linear map $\phi \in \mathrm{Hom}(V, W)$, a priori not necessarily continuous. However, we also have the pointwise convergence

$$\phi(v) = \lim_{j \in J} \phi_{i(j)}(v),$$

and thus $\phi \in \Phi^{\sigma\text{-cl}}$. Proposition 3.3.26 shows that the subnet converges uniformly on compact subsets and $\phi \in \mathrm{L}(V, W)$. □

The original net will at least converge pointwise to ϕ. It is mainly the continuity statement one is interested in. A typical situation, where we can make use of this proposition is the following:

Corollary 3.3.36 *Let V be a barrelled space and let W be a complete locally convex space. Suppose $(\phi_i)_{i \in I}$ is a pointwise Cauchy net in $\mathrm{L}_\sigma(V, W)$, which is pointwise bounded. Then $(\phi_i)_{i \in I}$ converges to some $\phi \in \mathrm{L}_\sigma(V, W)$ pointwise.*

PROOF: Being a Cauchy net in the topology of pointwise convergence means that for all $v \in V$ the net $(\phi_i(v))_{i \in I}$ is a Cauchy net. Since W is assumed to be complete,

$$\phi(v) = \lim_{i \in I} \phi_i(v)$$

converges for all $v \in V$ and yields a linear map $\phi \in \mathrm{Hom}(V, W)$. Since in addition, $(\phi_i)_{i \in I}$ is assumed to be pointwise bounded, we are in the situation of Proposition 3.3.35. □

The boundedness of convergent *sequences* then gives the following consequence:

Corollary 3.3.37 *Let V be a barrelled space and let W be a sequentially complete space. Then $\mathrm{L}_\sigma(V, W)$ is sequentially complete.*

PROOF: Indeed, let $(\phi_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathrm{L}_\sigma(V, W)$. This means that $(\phi_n(v))_{n \in \mathbb{N}}$ is a Cauchy sequence in W for all $v \in V$, thus convergent by assumption. Being a convergent sequence, $(\phi_n(v))_{n \in \mathbb{N}}$ is bounded in W, which means that $(\phi_n)_{n \in \mathbb{N}}$ is pointwise bounded. Thus by Proposition 3.3.35 we have convergence of $(\phi_n)_{n \in \mathbb{N}}$ to a continuous limit $\phi \in \mathrm{L}_\sigma(V, W)$. □

The statement of this last corollary is somewhat complementary to the one of Corollary 3.3.19. Note that besides trivial cases, $\mathrm{L}_\sigma(V, W)$ will not be complete, but dense in $\mathrm{Hom}_\sigma(V, W)$. Thus the above corollary is in some sense the best result we can expect.

Corollary 3.3.38 *Let V be a barrelled space. Then V'_σ is sequentially complete. In particular, the dual V'_σ is sequentially complete for a Fréchet space or an LF space V.*

This innocent looking corollary is at the very heart of distribution theory, since there, many distributions are defined by pointwise convergent sequences. Hence the result will be again a distribution. We will come back to this in greater detail in Chapter 5.

A last version is for Banach spaces, which are barrelled by Corollary 3.3.7. Here the *uniform boundedness* becomes most transparent:

Corollary 3.3.39 *Let V be a Banach space and W be a normed space. Let $\Phi \in \mathrm{L}(V, W)$ be a family of continuous linear maps with*

$$\sup_{\phi \in \Phi} \|\phi(v)\| < \infty \tag{3.3.19}$$

for all $v \in V$. Then

$$\sup_{\phi \in \Phi} \|\phi\| < \infty. \tag{3.3.20}$$

PROOF: Since W is normed, (3.3.19) simply means that Φ is pointwise bounded. Using the operator norm in (3.3.20) we see that (3.3.20) is the statement that Φ is bounded in $\mathrm{L}_\beta(V, W)$, see Proposition 3.3.21. Thus the Banach-Steinhaus Theorem 3.3.30 gives the desired implication. $\square$

3.4 Open Mappings and Closed Graphs

The next two classical theorems in functional analysis are the Open Mapping Theorem and the Closed Graph Theorem. For the first one we need again barrelledness together with a countability argument. It will have many applications, in particular concerning the uniqueness of Fréchet topologies. In the literature one finds various generalizations of the formulation we present, see e.g. [16, Section IV.8]. Nevertheless, our more modest approach covers essentially all situations in daily life. Closely related to the Open Mapping Theorem is the Closed Graph Theorem, which will often be used to infer the continuity of certain linear maps.

3.4.1 The Open Mapping Theorem

Recall that a map between topological spaces is called open if it maps open subsets to open subsets. The following definition weakens this property slightly:

Definition 3.4.1 (Nearly open map) *Let $\phi\colon X \longrightarrow Y$ be a map between topological spaces. Then ϕ is called nearly open if for all open subsets $O \subseteq X$ one has*

$$\phi(O) \subseteq (\phi(O)^{\mathrm{cl}})^\circ. \tag{3.4.1}$$

With other words, $\phi(O)$ might not yet be open, but up to a closure. Note that we do not require ϕ to be continuous, even though this will be the most interesting case.

Example 3.4.2 Let X, Y be topological spaces.

i.) Every open map $\phi\colon X \longrightarrow Y$ is nearly open. Indeed, $\phi(O)^\circ \subseteq (\phi(O)^{\mathrm{cl}})^\circ$ holds in general. Thus (3.4.1) trivially follows from $\phi(O) = \phi(O)^\circ$ for all open $O \subseteq X$ in case of an open map ϕ.

ii.) Suppose that $X \subseteq Y$ is a dense subset. Then the canonical inclusion map $\iota\colon X \longrightarrow Y$ is nearly open. Indeed, if $O \subseteq X$ is open, then we have an open subset $U \subseteq Y$ with $O = U \cap X$. Since $X^{\mathrm{cl}} = Y$, we get

$$O^{\mathrm{cl}} = (X \cap U)^{\mathrm{cl}} = U^{\mathrm{cl}}, \tag{3.4.2}$$

which we prove in Exercise 3.6.21. Since $U = U^\circ$ is open, we have $U \subseteq (U^{\mathrm{cl}})^\circ$. This gives (3.4.1). Clearly, this leads to many examples of nearly open, but non-open maps, like e.g. the inclusion

$$\iota\colon \mathbb{Q} \longrightarrow \mathbb{R}. \tag{3.4.3}$$

Since for topological vector spaces and, in particular, for locally convex spaces we typically encode the topology in a system of zero neighbourhoods, we shall now reformulate Definition 3.4.1 using neighbourhoods:

Proposition 3.4.3 *Let $\phi\colon X \longrightarrow Y$ be a map between topological spaces. Then the following statements are equivalent:*

i.) The map ϕ is nearly open.

ii.) There exists a basis $\mathcal{B}$ of the topology of X, such that for all $O \in \mathcal{B}$ one has

$$\phi(O) \subseteq (\phi(O)^{\mathrm{cl}})^\circ. \tag{3.4.4}$$

iii.) For every $x \in X$ one has a basis of neighbourhoods $\mathcal{B}_x$, such that for all $U \in \mathcal{B}_x$

$$\phi(x) \in (\phi(U)^{\mathrm{cl}})^{\circ}. \tag{3.4.5}$$

PROOF: The implication *i.)* $\implies$ *ii.)* is trivial, as (3.4.1) holds for all open subsets and hence for all open subsets in any given basis of the topology. Thus assume *ii.)* and let $x \in X$. Then $\mathcal{B}_x = \{U \in \mathcal{B} \mid x \in U\}$ is a basis of (open) neighbourhoods for x, satisfying (3.4.5), since $\phi(x) \in \phi(U) \subseteq (\phi(U)^{\mathrm{cl}})^{\circ}$. Finally, assume *iii.)* and let $\mathcal{B}_x$ be given for all $x \in X$. Let $O \subseteq X$ be open and $x \in O$. Then we find a $B \in \mathcal{B}_x$ with $B \subseteq O$, as O is a neighbourhood of x and $\mathcal{B}_x$ a basis of neighbourhoods of x. We conclude $\phi(B)^{\mathrm{cl}} \subseteq \phi(O)^{\mathrm{cl}}$, leading to $(\phi(B)^{\mathrm{cl}})^{\circ} \subseteq (\phi(O)^{\mathrm{cl}})^{\circ}$. But $\phi(x) \in (\phi(B)^{\mathrm{cl}})^{\circ}$ by (3.4.5) and hence $\phi(x) \in (\phi(O)^{\mathrm{cl}})^{\circ}$ follows. As $x \in O$ was arbitrary, we have $\phi(O) \subseteq (\phi(O)^{\mathrm{cl}})^{\circ}$, which gives *i.)*.□

One could call a map $\phi\colon X \longrightarrow Y$ with the property *iii.)* at $x \in X$ also *nearly open at x*. Then the equivalence *i.)* $\iff$ *iii.)* reads that ϕ is nearly open iff it is nearly open at all points. Note also that the neighbourhood basis in *iii.)* needs not to consist of open neighbourhoods. An analogous characterization holds for open maps, see Exercise 3.6.22.

For topological vector spaces we get the following version, making use of the translation invariance of the topology as usual:

Proposition 3.4.4 *A linear map $\phi\colon V \longrightarrow W$ between topological vector spaces is nearly open iff it is nearly open at one point iff it is nearly open at $0 \in V$.*

PROOF: In view of the prior proposition it suffices to show the second equivalence. Thus let $v \in V$. Since for every neighbourhood $U \subseteq V$ of zero, $v + U$ is a neighbourhood of v and vice versa, being nearly open at zero is equivalent to being nearly open at v, because

$$(\phi(v + U)^{\mathrm{cl}})^{\circ} = ((\phi(v) + \phi(U))^{\mathrm{cl}})^{\circ} = \phi(v) + (\phi(U)^{\mathrm{cl}})^{\circ}$$

by the linearity of ϕ and the translation invariance of the topology on W. □

In particular, for $\phi\colon V \longrightarrow W$ nearly open, $\phi(U)^{\mathrm{cl}} \subseteq W$ is a zero neighbourhood for all zero neighbourhoods $U \subseteq V$, see also Exercise 3.6.23.

It is now the target, which provides an easy condition for a (continuous) linear map to be nearly open:

Proposition 3.4.5 *Let $\phi\colon V \longrightarrow W$ be a surjective linear map between Hausdorff locally convex spaces, such that W is in addition barrelled. Then ϕ is nearly open.*

PROOF: We use Proposition 3.4.4 to test for nearly openness at zero and we use Proposition 3.4.3 to use a particular neighbourhood system: let $\mathcal{B}_0$ be the absolutely convex zero neighbourhoods in V. Let furthermore $U \in \mathcal{B}_0$ be given. Then $\phi(U)$ is absolutely convex, since ϕ is linear. Being a zero neighbourhood, U is absorbing. Thus for $v \in V$ we have a $\lambda \in \mathbb{K}$ with $v \in \lambda U$. Now ϕ is surjective and hence for every $w \in W$ we have a preimage $v \in V$ with $\phi(v) = w$. Taking the appropriate scaling λ shows $w \in \phi(\lambda U) = \lambda\phi(U)$. This proves that $\phi(U)$ is absorbing, too. The closure $\phi(U)^{\mathrm{cl}}$ is then absolutely convex, absorbing, and closed, i.e. a barrel. By assumption, $\phi(U)^{\mathrm{cl}} \subseteq W$ is a zero neighbourhood. Thus $\phi(0) = 0 \in (\phi(U)^{\mathrm{cl}})^{\circ}$ follows. This is all we need to show to conclude that ϕ is nearly open. □

Remarkably, we do not even need the linear map to be continuous. Before we investigate whether a nearly open linear map is open, we note that an open linear map is necessarily surjective: indeed, if $\phi(V) = \operatorname{im}\phi \subseteq W$ is an open subset, it follows that $\operatorname{im}\phi = W$, since a subspace is open iff it is the whole space, see also Exercise 3.6.24.

The Open Mapping Theorem can now be seen as a converse: a nearly open map is open under the following assumptions:

Theorem 3.4.6 (Open Mapping Theorem) *Let $\phi\colon V \longrightarrow W$ be a continuous linear map from a Fréchet space V to a Hausdorff locally convex space W. Then ϕ is nearly open iff ϕ is open.*

Proof: Assume ϕ is nearly open. Let $U \subseteq V$ be a zero neighbourhood: we need to show that $\phi(U) \subseteq W$ is a zero neighbourhood as well. This will prove openness of ϕ, see Exercise 3.6.22 for the characterization of open maps using neighbourhoods. By Corollary 2.2.42 we find seminorms $\mathrm{p}_1, \mathrm{p}_2, \ldots$ on V defining the topology such that $\mathrm{p}_n \leq \frac{1}{2}\mathrm{p}_{n+1}$ for all $n \in \mathbb{N}$. Without restriction, we can assume $\mathrm{B}_{\mathrm{p}_1,1}(0)^{\mathrm{cl}} \subseteq U$, since the closed unit balls $\mathrm{B}_{\mathrm{p}_n,1}(0)^{\mathrm{cl}}$ constitute a basis of zero neighbourhoods. We abbreviate these by $B_n = \mathrm{B}_{\mathrm{p}_n,1}(0)^{\mathrm{cl}}$. Since ϕ is nearly open, we know that $\phi(B_n)^{\mathrm{cl}} \subseteq W$ is a zero neighbourhood, see also Proposition 3.4.3. It follows that for all $n \in \mathbb{N}$ one has

$$\phi(B_n)^{\mathrm{cl}} \subseteq \phi(B_n) + \phi(B_{n+1})^{\mathrm{cl}}, \tag{$*$}$$

see also Proposition 2.1.7, *ii.)*. Now consider a point $w_2 \in \phi(B_2)^{\mathrm{cl}}$. The aim is to show that for such a point we actually have $w_2q \in \phi(B_1) \subseteq \phi(U)$, establishing $\phi(B_2)^{\mathrm{cl}} \subseteq \phi(U)$. Since $\phi(B_2)^{\mathrm{cl}}$ is a zero neighbourhood, $\phi(U)$ is a zero neighbourhood, which will imply that ϕ is open. From $(*)$ we see $w_2 \in \phi(B_2) + \phi(B_3)^{\mathrm{cl}}$ and thus we find $v_2 \in B_2$ and $w_3 \in \phi(B_3)^{\mathrm{cl}}$ with

$$w_2 = \phi(v_2) + w_3.$$

Repeating the argument for $w_3 \in \phi(B_3)^{\mathrm{cl}} \subseteq \phi(B_3) + \phi(B_4)^{\mathrm{cl}}$ gives $w_4 \in \phi(B_4)^{\mathrm{cl}}$, as well as $v_3 \in B_3$ with $w_3 = \phi(v_3) + w_4$. Inductively, we obtain $w_n \in \phi(B_n)^{\mathrm{cl}}$ and $v_n \in B_n$ with

$$w_n = \phi(v_n) + w_{n+1} \tag{☺}$$

for all $n \geq 2$. Since $\mathrm{p}_n(v_n) \leq 1$ we conclude that

$$\mathrm{p}_n(v_{n+k}) \leq \frac{1}{2^k}\mathrm{p}_{n+k}(v_{n+k}) \leq \frac{1}{2^k}$$

for all $k \geq 0$. This shows that the series

$$v = \sum_{k=2}^{\infty} v_k \tag{$**$}$$

converges absolutely with respect to the seminorm p_n for all $n \in \mathbb{N}$. As these seminorms define the topology of V, we have absolutely convergence of $(**)$, leading to a limit $v \in V$, since V is complete. Moreover, we have

$$\mathrm{p}_1(v) \leq \sum_{k=2}^{\infty} \mathrm{p}_1(v_k) \leq \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} \leq 1,$$

showing $v \in B_1$. Recursively inserting the defining relation (☺) gives now

$$\begin{aligned} w_2 &= \phi(v_2) + w_3 \\ &= \phi(v_2) + \phi(v_3) + w_4 \\ &= \sum_{k=2}^{n} \phi(v_k) + w_{n+1} \\ &= \phi\Big(\textstyle\sum_{k=2}^{n} v_k\Big) + w_{n+1}, \end{aligned}$$

and hence

$$w_{n+1} = w_2 - \phi\Big(\textstyle\sum_{k=2}^{n} v_k\Big).$$

The limit $n \to \infty$ of the right hand side exists thanks to the continuity of ϕ and yields

$$\lim_{n\to\infty} w_n = w_2 - \phi(v).$$

Since the closed balls B_n form a basis of zero neighbourhoods in V, the continuity of ϕ implies that for every closed zero neighbourhood $A \subseteq W$ we have an $n \in \mathbb{N}$ with $B_n \subseteq \phi^{-1}(A)$ and hence $\phi(B_n) \subseteq \phi(B_n)^{\mathrm{cl}} \subseteq A^{\mathrm{cl}} = A$. This implies that the sequence $w_n \in \phi(B_n)^{\mathrm{cl}}$ converges to zero in W. We conclude $w_2 = \phi(v) \in \phi(B_1) \subseteq \phi(U)$. This finally shows $\phi(B_2)^{\mathrm{cl}} \subseteq \phi(U)$, proving that $\phi(U)$ is a zero neighbourhood, as required. □

This formulation of the Open Mapping Theorem might still look quite technical and unfamiliar. Thus we first note a few direct corollaries, which gives useful consequences:

Corollary 3.4.7 *A continuous linear map $\phi\colon V \longrightarrow W$ between Fréchet spaces is surjective iff it is open.*

Proof: A Fréchet space W is barrelled by Corollary 3.3.7 and thus a surjective map $\phi\colon V \longrightarrow W$ is nearly open by Proposition 3.4.5. It is open by the Open Mapping Theorem 3.4.6. The converse always holds, see again Exercise 3.6.24. □

Corollary 3.4.8 *A continuous linear bijection between Fréchet spaces is a topological isomorphism.*

Proof: Indeed, a continuous open bijection has a continuous inverse in general. □

Corollary 3.4.9 *Let V be a vector space with two Hausdorff locally convex first countable topologies τ_1 and τ_2, such that $\tau_1 \subseteq \tau_2$. If both topologies are complete, then $\tau_1 = \tau_2$.*

This is sometimes a very convenient criterion to compare Fréchet topologies, see e.g. Exercise 3.6.25 for an explicit application.

Corollary 3.4.10 *Let $\phi\colon V \longrightarrow W$ be a surjective continuous linear map from a Fréchet space V to a barrelled Hausdorff locally convex space. Then the induced map*

$$\phi\colon V/\ker\phi \longrightarrow W \tag{3.4.6}$$

is a topological isomorphism. In particular, W is a Fréchet space.

Proof: From liner algebra we know that the induced map (3.4.6) is a linear bijection. Since the locally convex quotient topology is final for the quotient map $\mathrm{pr}\colon V \longrightarrow V/\ker\phi$, the induced map ϕ stays continuous. According to Proposition 2.4.15 the quotient is again complete, i.e. a Fréchet space. Hence the surjective map ϕ is nearly open and open by the Open Mapping Theorem 3.4.6. Thus it is a topological isomorphism. □

Note that W turns out to be necessarily complete itself. Of course, in most cases we prove that W is barrelled by first showing that it is a Fréchet space.

Nevertheless, there are more general scenarios, where a surjective continuous linear map is open. Variants of this generalization of the Open Mapping Theorem can be found e.g. in [6, Chap. 3, §17] or [16, Chap. 8, §8].

Finally, we note that all the above statements hold for Banach spaces as particular cases of Fréchet spaces, thereby reproducing the well-known statements from the functional analysis of Banach spaces.

3.4.2 The Closed Graph Theorem

Of equal importance and in fact closely related to the Open Mapping Theorem is the Closed Graph Theorem. As motivation we recall the following trivial observation from general point set topology. Suppose $\phi\colon X \longrightarrow Y$ is a continuous map between topological spaces, such that Y is Hausdorff. Then the *graph*

$$\operatorname{graph}(\phi) = \big\{(x, \phi(x)) \bigm| x \in X\big\} \subseteq X \times Y \tag{3.4.7}$$

of ϕ is a closed subset of the Cartesian product $X \times Y$. However, it is fairly easy to construct examples, where the graph is closed, but the map ϕ is not continuous, see e.g. Exercise 3.6.26. In case of a linear map between vector spaces the graph of ϕ is always a subspace. We are now interested in the question, whether a closed graph implies continuity. As for the Open Mapping Theorem, one has many formulations in different generalities. The following version of the Closed Graph Theorem is sufficient for many purposes:

Theorem 3.4.11 (Closed Graph Theorem) *Let $\phi\colon V \longrightarrow W$ be a linear map between Hausdorff locally convex spaces, such that V is barrelled and W is a Fréchet space. Then ϕ is continuous iff* $\operatorname{graph}(\phi) \subseteq V \times W$ *is closed.*

Proof: As already discussed, any continuous map has a closed graph if the target is Hausdorff. It is the opposite implication we are interested in and where we need the additional assumptions for. Consider the canonical continuous linear projections

$$\mathrm{pr}_V\colon V \times W \longrightarrow V \quad \text{and} \quad \mathrm{pr}_W\colon V \times W \longrightarrow W.$$

The restriction of pr_V to the graph $\operatorname{graph}(\phi)$ is still continuous and bijective

$$\mathrm{pr}_V\Big|_{\operatorname{graph}(\phi)}\colon \operatorname{graph}(\phi) \longrightarrow V,$$

with inverse given by the map $v \mapsto (v, \phi(v))$. Hence we have

$$\phi = \mathrm{pr}_W \circ \big(\mathrm{pr}_V\big|_{\operatorname{graph}(\phi)}\big)^{-1},$$

a fact, which holds in general. The idea is now to prove that the inverse of $\mathrm{pr}_V\big|_{\operatorname{graph}(\phi)}$ is continuous, then the continuity of ϕ will follow. With the *additional* assumption that also V is a Fréchet space, we can directly argue with the Open Mapping Theorem: the product $V \times W$ is again a Fréchet space and the closed subspace $\operatorname{graph}(\phi)$ is still a Fréchet space. Thus the surjective continuous bijection $\mathrm{pr}_V\big|_{\operatorname{graph}(\phi)}\colon \operatorname{graph}\phi \longrightarrow V$ is open and thus its inverse is continuous by Corollary 3.4.8. However, with the given assumption of a barrelled space V we have to be a bit more careful. Let $U \subseteq W$ be an absolutely convex zero neighbourhood, then we need to show that $\phi^{-1}(U) \subseteq V$ is a zero neighbourhood as well. As in the proof of the Open Mapping Theorem, we choose a sequence of continuous seminorms $\{\mathrm{p}_n\}_{n\in\mathbb{N}}$ on W defining the topology, such that $\mathrm{p}_n \leq \frac{1}{2}\mathrm{p}_{n+1}$ for all $n \in \mathbb{N}$. This is again possible by Corollary 2.2.42. Without restriction, $\mathrm{B}_{\mathrm{p}_1,1}(0)^{\mathrm{cl}} \subseteq U$, where we set again

$$B_n = \mathrm{B}_{\mathrm{p}_n,1}(0)^{\mathrm{cl}}.$$

Since ϕ is linear, the preimage of the barrels B_n are absorbing and absolutely convex. Thus $\phi^{-1}(B_n)^{\mathrm{cl}}$ is a barrel and hence a zero neighbourhood in V for all $n \in \mathbb{N}$ by assumption. Now we want to prove that $\phi^{-1}(B_2)^{\mathrm{cl}} \subseteq \phi^{-1}(U)$, which will identify $\phi^{-1}(U)$ as a zero neighbourhood. Hence let $v \in \phi^{-1}(B_2)$ be given. Then

$$\phi^{-1}(B_n)^{\mathrm{cl}} \subseteq \phi^{-1}(B_n) + \phi^{-1}(B_{n+1})^{\mathrm{cl}}$$

as before shows that $v = v_2 + \tilde{v}_3$ with $v_2 \in \phi^{-1}(B_2)$ and $\tilde{v}_3 \in \phi^{-1}(B_3)^{\mathrm{cl}}$. Inductively, we construct $v_n \in \phi^{-1}(B_n)$ and $\tilde{v}_n \in \phi^{-1}(B_n)^{\mathrm{cl}}$ such that

$$\tilde{v}_n = v_n + \tilde{v}_{n+1}.$$

Thus $\phi(v_n) \in B_n$ and hence

$$v = v_2 + \tilde{v}_3 = v_2 + v_3 + \tilde{v}_4 = \cdots = \sum_{k=2}^{n} v_k + \tilde{v}_{k+1}$$

for all $n \in \mathbb{N}$. Therefore $v - \sum_{k=2}^{n} v_k = \tilde{v}_{k+1} \in \phi^{-1}(B_{k+1})^{\mathrm{cl}}$. The same seminorm estimate as in the proof of the Open Mapping Theorem shows the absolute convergence of the series

$$w = \sum_{k=2}^{\infty} \phi(v_k), \tag{$*$}$$

since $\phi(v_k) \in B_k$ with $w \in B_1$. Here we use that W is complete. However, since we do not yet know the convergence of the series $\sum_{k=2}^{\infty} v_k$, we can not conclude $\phi(v) = w$ just yet. This is the point, where we utilize the closedness of $\operatorname{graph}(\phi)$. If we knew $\phi(v) = w$, then $v \in \phi^{-1}(B_1) \subseteq \phi^{-1}(U)$, showing that $\phi^{-1}(U)$ is a zero neighbourhood and hence showing the continuity of ϕ. The idea is now to check $(v, w) \in \operatorname{graph}(\phi)$, which means $\phi(v) = w$ by checking $(v, w) \in \operatorname{graph}(\phi)^{\mathrm{cl}} = \operatorname{graph}(\phi)$. Thus let $O \subseteq V$ be an absolutely convex zero neighbourhood. Then

$$v - \sum_{k=2}^{n} v_k = \tilde{v}_{n+1} \in \phi^{-1}(B_{n+1})^{\mathrm{cl}} \subseteq \phi^{-1}(B_{n+1}) + O,$$

since O is a zero neighbourhood. Thus we find $\hat{v}_{n+1} \in \phi^{-1}(B_{n+1})$ and $u_n \in O$ with

$$v - \sum_{k=2}^{n} v_k = \hat{v}_{n+1} + u_n.$$

It follows that $\hat{v}_{n+1} + \sum_{k=2}^{n} v_k \in v + O$ for $n \in \mathbb{N}$. In the absolutely convergent series $(*)$ for w, we split the series at n into

$$w = \sum_{k=2}^{n} \phi(v_k) + w_n \quad \text{with} \quad w_n = \sum_{k=n+1}^{\infty} \phi(v_k).$$

Since $\phi(v_k) \in B_k$, we get

$$\mathrm{p}_n(w_n) \leq \sum_{k=n+1}^{\infty} \mathrm{p}_n(v_k) = \sum_{k=2}^{\infty} \mathrm{p}_n(v_{n+k}) \leq \sum_{k=2}^{\infty} \frac{1}{2^k} \mathrm{p}_{n+k}(v_{n+k}) \leq \sum_{k=2}^{\infty} \frac{1}{2^k} \leq 1,$$

which means $w_n \in B_n$. Together, this gives

$$\mathrm{p}_{n-1}(\phi(\hat{v}_{n+1}) - w_n) \leq \mathrm{p}(\phi(\hat{v}_{n+1})) + \mathrm{p}_{n-1}(w_n) \leq \frac{1}{4} \mathrm{p}_{n+1}(\phi(\hat{v}_{n+1})) + \frac{1}{2} \mathrm{p}_n(w_n) \leq 1,$$

leading to $\phi(\hat{v}_{n+1}) + w_n \in B_{n-1}$. This shows

$$\phi\Big(\hat{v}_{n+1} + \sum\nolimits_{k=2}^{n} v_k\Big) = \phi(\hat{v}_{n+1}) + w - w_n \in w + B_{n+1}.$$

Now in the Cartesian product $V \times W$ we get

$$(v, w) + (-u_n, \phi(\hat{v}_{n+1}) - w_n) = \left(\hat{v}_{n+1} + \sum\nolimits_{k=2}^{n} v_k, \phi\left(\hat{v}_{n+1} + \sum\nolimits_{k=2}^{n} v_k\right)\right) \in \operatorname{graph}(\phi).$$

On the other hand, $(-u_n, \phi(\hat{v}_{n+1}) - w_n) \in O \times B_{n-1}$. This shows that

$$((v, w) + O \times B_{n-1}) \cap \operatorname{graph}(\phi) \neq \emptyset,$$

as we have constructed the above point in this intersection. Since $O \subseteq V$ was an arbitrary absolutely convex zero neighbourhood and since the closed balls $\{B_n\}_{n \in \mathbb{N}}$ provide a basis of neighbourhoods, the sets $O \times B_{n-1}$ yield a basis of zero neighbourhoods in $V \times W$. It follows that $(v, w) \in \operatorname{graph}(\phi)^{\mathrm{cl}}$ and hence $(v, w) \in \operatorname{graph}(\phi)$ by closedness of $\operatorname{graph}(\phi)$. Thus $\phi(v) = w$ and we have finally shown that $\phi^{-1}(U)$ is a zero neighbourhood. □

As mentioned in the proof, the case of a Fréchet space V follows quite directly from the Open Mapping Theorem. We note this as a special case:

Corollary 3.4.12 *Let $\phi \colon V \longrightarrow W$ be a linear map between Fréchet spaces. Then ϕ is continuous iff $\operatorname{graph}(\phi) \subseteq V \times W$ is closed.*

While this is perhaps the most important formulation of the Closed Graph Theorem, the following situation is also important.

Corollary 3.4.13 *Let $\phi \colon V \longrightarrow W$ be a linear map from an LF space to a Fréchet space. Then ϕ is continuous iff $\operatorname{graph}(\phi) \subseteq V \times W$ is closed.*

3.5 Extreme Points and the Krein-Milman Theorem

The last topic in this chapter are the theorems of Krein and Milman on the existence and characterization of extreme points of (convex) subsets. The concept of extreme points, see Definition 2.1.28, refers only to the vector space structure, but not to a topology. Nevertheless, extreme points turn out to be related to topological concepts, as they are necessarily boundary points, as already mentioned in Proposition 2.1.29. We shall now investigate the existence of extreme points. Proposition 2.1.29 gives only a necessary condition, but no sufficient one. Indeed, it is easy to find subsets, say in $\mathbb{R}^2$, with a non-trivial boundary, but still no extreme points. Note that such subsets can even be taken to be convex, see also Exercise 3.6.27 for some first examples and non-examples for extreme points.

As a slight generalization of extreme points of a subset $X \subseteq V$ in a vector space V one defines a non-empty subset $A \subseteq X$ to be *extreme* if $\lambda v + (1 - \lambda) w \in A$ for $\lambda \in (0, 1)$ and $v, w \in X$ implies already $v, w \in A$. This is consistent with Definition 2.1.28, in the sense that $p \in X$ is extreme iff $\{p\}$ is extreme as a subset. Thus the extreme points of a subset can be seen as the extreme subsets consisting of a single point: they are the minimal extreme subsets. This already hints at a possible application of Zorn's Lemma, which we employ in our first formulation of the Krein-Milman Theorem:

Theorem 3.5.1 (Krein-Milman I) *Let V be a topological vector space such that V' separates points. Then for a non-empty compact convex subset $K \subseteq V$ one has*

$$K = (\operatorname{conv}(\operatorname{extreme}(K)))^{\mathrm{cl}}. \tag{3.5.1}$$

PROOF: We consider the set of all extreme compact subsets of K, denoted by

$$\mathcal{P} = \big\{ A \subseteq K \bigm| A \text{ compact and extreme} \big\}.$$

Clearly, $K \in \mathcal{P}$ and thus $\mathcal{P}$ is non-empty. Now let $\{A_i\}_{i \in I}$ be a set of elements of $\mathcal{P}$, such that their intersection is non-empty. We claim that in this case $A = \bigcap_{i \in I} A_i \in \mathcal{P}$ again. Indeed, V is Hausdorff, as V' separates points. Hence the intersection of compact subsets gives again a compact subset. For $v, w \in K$ and $\lambda \in (0, 1)$ with $\lambda v + (1-\lambda)w \in A$ we have $\lambda v + (1-\lambda)w \in A_i$ for all $i \in I$. As the A_i are extreme for all $i \in I$, we conclude $v, w \in A_i$ for all $i \in I$. Thus $v, w \in A$ follows, showing $A \in \mathcal{P}$, as claimed. Next, let $A \in \mathcal{P}$ be fixed and consider $\varphi \in V'$. Since $\operatorname{Re}\varphi$ is continuous and $A \neq \emptyset$,

$$m_{A,\varphi} = \sup_{w \in A} \operatorname{Re}(\varphi(w)) = \max_{w \in A} \operatorname{Re}(\varphi(w)) < \infty.$$

Thus we can consider the subset

$$A_\varphi = \big\{ v \in A \bigm| \operatorname{Re}(\varphi(v)) = m_{A,\varphi} \big\},$$

which is non-empty. We claim that $A_\varphi \in \mathcal{P}$. Thus let $v, w \in K$ and $\lambda \in (0, 1)$ with $\lambda v + (1-\lambda)w \in A_\varphi$ be given. Since $A_\varphi \subseteq A$ and A is extreme, $v, w \in A$ follows. Thus $\operatorname{Re}(\varphi(v)), \operatorname{Re}(\varphi(w)) \leq m_{A,\varphi}$ by definition of $m_{A,\varphi}$. Hence

$$m_{A,\varphi} = \operatorname{Re}(\varphi(\lambda v + (1-\lambda)w)) = \lambda \operatorname{Re}(\varphi(v)) + (1-\lambda) \operatorname{Re}(\varphi(w))$$

shows that $\operatorname{Re}(\varphi(v)) = \operatorname{Re}(\varphi(w)) = m_{A,\varphi}$ is the only possibility. This gives $v, w \in A_\varphi$, proving the second claim $A_\varphi \in \mathcal{P}$. The third step of the proof consists now in an application of Zorn's Lemma. We consider $A \in \mathcal{P}$ and denote by $\mathcal{P}_A \subseteq \mathcal{P}$ those elements in $\mathcal{P}$, which are subsets of A. Clearly, $A \in \mathcal{P}_A$ and hence $\mathcal{P}_A$ is non-empty. The usual set-inclusion gives a partial ordering of $\mathcal{P}_A$. Suppose $\mathcal{B} \subseteq \mathcal{P}_A$ is a totally ordered subset. Then we consider the intersection

$$C = \bigcap_{B \in \mathcal{B}} B.$$

Since $\mathcal{B}$ is totally ordered, we see that a finite intersection $B_1 \cap \cdots \cap B_n$ of elements in $\mathcal{B}$ is just the smallest of the subsets. In particular, a finite intersection is non-empty. Next, $A \setminus B$ for $B \in \mathcal{B}$ is an open subset of A. We want to show that $C \neq \emptyset$. Suppose the converse, $C = \emptyset$. Then all the open subsets $\{A \setminus B\}_{B \in \mathcal{B}}$ yield an open cover of A, thus finitely many, say $A \setminus B_1, \ldots, A \setminus B_n$ already cover the compact subset A. Since $\mathcal{B}$ is totally ordered, $B_1 \supseteq \cdots \supseteq B_n$ without restriction. Then

$$A \setminus B_n = A \setminus (B_1 \cap \cdots \cap B_n) = (A \setminus B_1) \cup \cdots \cup (A \setminus B_n) = A,$$

a contradiction to $B_n \neq \emptyset$. Hence $C \neq \emptyset$ follows. We have $C \subseteq A$ and from our first claim we know that $C \in \mathcal{P}$, thus $C \in \mathcal{P}_A$. Clearly, C is the minimum of the totally ordered decreasing subset $\mathcal{B} \subseteq \mathcal{P}_A$. This allows now to apply Zorn's Lemma: the partially ordered set $\mathcal{P}_A$ has minimal elements. Let $B \in \mathcal{P}_A$ be such a minimal element. Then B can not have proper subsets also belonging to $\mathcal{P}_A$ by minimality. In particular, for all $\varphi \in V'$ the subsets B_φ have to coincide with B, i.e. for all $v \in B$ we have $\operatorname{Re}(\varphi(v)) = \max_{w \in B} \operatorname{Re}(\varphi(w))$. It follows that $\operatorname{Re}\varphi$ is constant on B. Replacing φ by $\mathrm{i}\varphi$ in the complex case shows that also $\operatorname{Im}\varphi$ is constant. In total, φ is constant on B. Since by assumption V' is point-separating, B consists of a single element. This is then an extreme point of the original subset K. In particular, $\operatorname{extreme}(K) \neq \emptyset$. Actually, we have shown more: for every extreme closed subset $A \subseteq K$ we have extreme points of K in A, i.e.

$$\operatorname{extreme}(K) \cap A \neq \emptyset. \tag{$*$}$$

Since K is convex, we get

$$\operatorname{conv}(\operatorname{extreme}(K)) \subseteq \operatorname{conv}(K) = K,$$

and since K is closed, we have

$$\tilde{K} = \operatorname{conv}(\operatorname{extreme}(K))^{\mathrm{cl}} \subseteq K^{\mathrm{cl}} = K.$$

It follows that the subset $\tilde{K} \subseteq K$ is compact itself. Suppose now that $\tilde{K} \neq K$ and let $v_0 \in K \setminus \tilde{K}$ be a point in the complement of $\tilde{K}$ in K. Since then $\{v_0\}$ and $\tilde{K}$ are both compact and convex, we find a $\varphi \in V'$ with

$$\sup_{v \in \tilde{K}} \operatorname{Re}(\varphi(v)) < \inf_{v \in \{v_0\}} \operatorname{Re}(\varphi(v)) = \operatorname{Re}(\varphi(v_0))$$

according to the separation statements in Corollary 3.1.19. It follows that the maximum $m_{K,\varphi}$ of $\operatorname{Re}\varphi$ on K is strictly larger than the values of $\operatorname{Re}\varphi$ on $\tilde{K}$. Hence we conclude that

$$K_\varphi \cap \tilde{K} = \emptyset$$

for the corresponding subset K_φ. But $K_\varphi \in \mathcal{P}$ and hence we have a contradiction to $(*)$, finally showing $\tilde{K} = K$. □

Remark 3.5.2 (Krein-Milman Theorem) The importance of the Krein-Milman Theorem lies, first of all, in the existence of extreme points. As simple examples in the plane $\mathbb{R}^2$ already show, none of the assumptions can be dropped in general. Only the compact convex subsets K admit extreme points in general, i.e. $\operatorname{extreme}(K) \neq \emptyset$. In addition, the statement goes far beyond mere existence. The amount of extreme points is sufficient to reconstruct the original compact subset by taking convex combinations of the extreme points. However, in general one needs to take a topological closure afterwards, as simple examples show, see e.g. Exercise 3.6.28.

If the ambient space V is even a Hausdorff locally convex space, we obtain a version of the Krein-Milman Theorem for arbitrary compact subsets. Note that in the previous proof the separation properties of compact subsets were needed to apply Corollary 3.1.19. In the locally convex situation, we can separate better:

Theorem 3.5.3 (Krein Milman II) *Let V be a Hausdorff locally convex space. For a non-empty compact subset $K \subseteq V$ one has*

$$K \subseteq (\operatorname{conv}(\operatorname{extreme}(K)))^{\mathrm{cl}}. \tag{3.5.2}$$

PROOF: Note that we do not assume K to be convex here. Thus we only can hope for an inclusion (3.5.2) instead of an equality. By construction $C = (\operatorname{conv}(\operatorname{extreme}(K)))^{\mathrm{cl}}$ is closed and convex. We proceed as in the proof of Theorem 3.5.1 by defining $\mathcal{P}$ and showing the existence of minimal elements of $\mathcal{P}$ using Zorn's Lemma. Note that here we never used that K itself was convex. Assuming now that (3.5.2) does not hold gives us a point $v_0 \in K$ not being contained in the closed convex set C. Thus we can separate $\{v_0\}$ from C with a $\varphi \in V'$ according to Theorem 3.1.12 to arrive at

$$\sup_{v \in \tilde{K}} \operatorname{Re}(\varphi(v)) < \operatorname{Re}(\varphi(v_0)).$$

From there we can argue again as in the previous proof of Theorem 3.5.1. □

The difference to the previous version of the Krein-Milman Theorem is that the closed convex subset generated by the extreme points of K needs no longer to be compact itself. Thus in a general topological vector space we can not proceed with the proof, since here Corollary 3.1.19 only allows to separate disjoint compact subsets instead of a compact and a closed one.

In the situation of Theorem 3.5.3 we know that closures of convex subsets are convex again, leading to the inclusion $\operatorname{conv}(K)^{\mathrm{cl}} \subseteq (\operatorname{conv}(\operatorname{extreme}(K)))^{\mathrm{cl}}$. Since trivially $\operatorname{extreme}(K) \subseteq K$, we have the opposite inclusion, as well. Hence (3.5.2) implies

$$\operatorname{conv}(K)^{\mathrm{cl}} = (\operatorname{conv}(\operatorname{extreme}(K)))^{\mathrm{cl}} \tag{3.5.3}$$

for any compact subset $K \subseteq V$ in a Hausdorff locally convex space V. Note again that the subset in (3.5.3) needs not to be compact any more. This equality of course raises the question, what the extreme points of $\operatorname{conv}(K)^{\mathrm{cl}}$ are. In principle, one can expect to have more extreme points for the larger subset $\operatorname{conv}(K)^{\mathrm{cl}}$ than for K itself. Before answering this question, we need a better understanding of the convex hull of a union of compact convex subsets, which is also of independent interest:

Proposition 3.5.4 *Let V be a topological vector space with compact convex subsets $K_1, \ldots, K_n \subseteq V$. Then the convex hull $\operatorname{conv}(K_1 \cup \cdots \cup K_n)$ is again compact.*

Proof: Since each of the subsets $K_1, \ldots, K_n$ is assumed to be convex, the convex hull of their union is obtained from convex combinations as

$$\operatorname{conv}(K_1 \cup \cdots \cup K_n) = \left\{ \sum_{i=1}^{n} \lambda_i v_i \;\middle|\; v_i \in K_i \text{ and } \lambda_i \geq 0 \text{ for all } i = 1, \ldots, n \text{ with } \lambda_1 + \cdots + \lambda_n = 1 \right\}.$$

This is clear by Proposition 2.1.27, *ii.)*. Consider now the map

$$\psi \colon V^n \times \Delta_n \longrightarrow V,$$

defined by

$$\psi(v_1, \ldots, v_n, \lambda) = \sum_{i=1}^{n} \lambda_i v_i,$$

where

$$\Delta_n = \big\{ \lambda \in \mathbb{R}^n \;\big|\; \lambda_i \geq 0 \text{ for all } i = 1, \ldots, n \text{ with } \lambda_1 + \cdots + \lambda_n = 1 \big\}$$

is the standard simplex in $\mathbb{R}^n$. Clearly, $\Delta_n \subseteq \mathbb{R}^n$ is compact and

$$\operatorname{conv}(K_1 \cup \cdots \cup K_n) = \psi\big(K_1 \times \cdots \times K_n \times \Delta_n\big).$$

Hence the convex hull is the image of a compact subset under the obviously continuous map ψ, thus compact itself. □

This allows for a description of all convex compact subsets of $\mathbb{R}^n$, which we do in Exercise 3.6.29. In the Fréchet situation, we even get that convex hulls of arbitrary convex subsets remain compact, see Exercise 3.6.30.

The next theorem clarifies now the amount of extreme points of $\operatorname{conv}(K)^{\mathrm{cl}}$ under the additional assumption that $\operatorname{conv}(K)^{\mathrm{cl}}$ stays compact. We do not get new points outside of K:

Theorem 3.5.5 (Milman) *Let V be a Hausdorff locally convex space. Suppose $K \subseteq V$ is compact, such that $\operatorname{conv}(K)^{\mathrm{cl}}$ is still compact. Then*

$$\operatorname{extreme}(\operatorname{conv}(K)^{\mathrm{cl}}) \subseteq K. \tag{3.5.4}$$

Proof: Assume $v_0 \in \operatorname{extreme}(\operatorname{conv}(K)^{\mathrm{cl}})$ is an extreme point not contained in K, i.e. $v_0 \notin K$. Since V is Hausdorff we can separate v_0 from K by an absolutely convex zero neighbourhood $U \subseteq V$ via

$$(v_0 + U^{\mathrm{cl}}) \cap K = \emptyset,$$

according to Proposition 2.1.8 and the fact that in a locally convex space we have a basis of closed absolutely convex zero neighbourhoods. The compactness of K gives finitely many points $v_1, \ldots, v_n \in K$ such that $K \subseteq (v_1 + U) \cup \cdots \cup (v_n + U)$. Consider now the subsets

$$A_i = \operatorname{conv}(K \cap (v_i + U))^{\mathrm{cl}} \subseteq \operatorname{conv}(K)^{\mathrm{cl}}.$$

By construction, they are closed and stay convex. Since $A_i \subseteq \operatorname{conv}(K)^{\mathrm{cl}}$, these subsets are compact. Finally, we have

$$K = (K \cap (v_1 + U)) \cup \cdots \cup (K \cap (v_n + U)) \subseteq A_1 \cup \cdots \cup A_n.$$

Thus taking the convex hull and then the closure gives

$$\operatorname{conv}(K)^{\mathrm{cl}} \subseteq \operatorname{conv}(A_1 \cup \cdots \cup A_n)^{\mathrm{cl}} = \operatorname{conv}(A_1 \cup \cdots \cup A_n),$$

where the last equality holds by the compactness of the subsets $A_1, \ldots, A_n$ and Proposition 3.5.4: in the Hausdorff case, the compact subset $\operatorname{conv}(A_1 \cup \cdots \cup A_n)$ is necessarily closed. Conversely, $A_i \subseteq \operatorname{conv}(K)^{\mathrm{cl}}$ shows that we have the equality

$$\operatorname{conv}(K)^{\mathrm{cl}} = \operatorname{conv}(A_1 \cup \cdots \cup A_n).$$

Thus for v_0 we find a convex combination

$$v_0 = \lambda_1 w_1 + \cdots + \lambda_n w_n$$

for some $w_i \in A_i$, $\lambda_i \geq 0$ and $\lambda_1 + \cdots + \lambda_n = 1$. Without restriction, we can assume $\lambda_2 + \cdots + \lambda_n > 0$ and hence

$$v_0 = \lambda_1 w_1 + (1 - \lambda_1) \frac{\lambda_2 w_2 + \cdots + \lambda_n w_n}{\lambda_2 + \cdots + \lambda_n} = \lambda_1 w_1 + (1 - \lambda_1) w,$$

where w is a convex combination of $w_2, \ldots, w_n$ and hence $w \in \operatorname{conv}(K)^{\mathrm{cl}}$. By assumption, v_0 is an extreme point of $\operatorname{conv}(K)^{\mathrm{cl}}$. Thus either $v_0 = w_1$ or $v_0 = w$. If $v_0 = w$, we can repeat the argument with $w_2, \ldots, w_n$ instead of $w_1, \ldots, w_n$ and ultimately find an index $i = 1, \ldots, n$ with $v_0 = w_i \in A_i$. Since U^{cl} is (absolutely) convex, we note that $A_i \subseteq v_i + U^{\mathrm{cl}}$ and thus $v_0 \in v_i + U^{\mathrm{cl}} \subseteq K + U^{\mathrm{cl}}$. Since $U^{\mathrm{cl}} = -U^{\mathrm{cl}}$ by assumption, we see that $v_0 + U^{\mathrm{cl}} \cap K \neq \emptyset$, a contradiction. □

Remark 3.5.6 (States in C^*-algebras) The above statements on the existence of extreme points will play a central role at various places in functional analysis. One important consequence is e.g. the existence of *pure*, i.e. extreme, states of C^*-algebras. Recall that a state on a unital C^*-algebra $\mathscr{A}$ is a linear functional $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$, satisfying

$$\omega(a^* a) \geq 0 \quad \text{and} \quad \omega(\mathbb{1}) = 1. \tag{3.5.5}$$

One notes that such functionals are necessarily continuous and form a weak* closed subset of the closed unit ball $\mathrm{B}_1(0)^{\mathrm{cl}} \subseteq \mathscr{A}'$ in the topological dual of $\mathscr{A}$. Hence by the Banach-Alaoglu Theorem 3.2.11, the states form a convex and weak*-compact subset. By Theorem 3.5.1 every state can be approximated by convex combinations of pure states in the weak* topology, a fact which has major impact for operator algebra theory, see e.g. the classical monographs [1, 2] for further reading. We will see such positive functionals again in Section **??**.

3.6 Exercises

Exercise 3.6.1 (Hahn-Banach Theorem) Show that the map $\tilde{\mathrm{p}}$ constructed in the proof of Theorem 3.1.5 is sublinear.

Exercise 3.6.2 (Continuity of linear functionals) Let $\varphi\colon V \longrightarrow \mathbb{K}$ be a linear functional on a topological vector space V. Show that φ is continuous iff there exists an open neighbourhood of $0 \in V$ on which it is bounded.

Exercise 3.6.3 (Separation estimates are sharp) Find convex subsets $A, B \subseteq \mathbb{R}$ such that the separation in the sense of (3.1.10) or (3.1.11) is optimal.

Exercise 3.6.4

Exercise 3.6.5 (Polars for topological vector spaces) Check which of the propositions at the beginning of Section 3.2.1 stay valid to topological vector spaces.

Exercise 3.6.6

Exercise 3.6.7 (Galois correspondences)

Exercise 3.6.8

Exercise 3.6.9 Let V be a Hausdorff locally convex space and let $\mathcal{B} \subseteq 2^V$ be a collection of bounded subsets of V. Show that if

$$\left(\bigcup_{\lambda>0,\ B\in\mathcal{B}} \lambda B \right)^{\mathrm{cl}} \tag{3.6.1}$$

is absorbing, then the associated system of seminorms $\{\mathrm{p}_B\}_{B\in\mathcal{B}}$ is Hausdorff.

Exercise 3.6.10

Exercise 3.6.11

Exercise 3.6.12

Exercise 3.6.13

Exercise 3.6.14 Let V be a normed space.

i.) Show that V' is complete by considering Cauchy sequences.

ii.) Show that V' is complete by invoking the Banach-Steinhaus Theorem **??**.

Exercise 3.6.15 (Barrels)

Exercise 3.6.16

Exercise 3.6.17

Exercise 3.6.18 (Normed spaces are bornological) Give a direct and elementary argument that normed spaces are bornological.

Exercise 3.6.19

Exercise 3.6.20 (Banach-Steinhaus Theorem with seminorms)

Exercise 3.6.21 (Inclusion of dense subset is nearly open)

Exercise 3.6.22

Exercise 3.6.23

Exercise 3.6.24

Exercise 3.6.25

Exercise 3.6.26

Exercise 3.6.27

Exercise 3.6.28

Exercise 3.6.29

Exercise 3.6.30

Chapter 4

Vector-Valued Functions

In Chapter 1 we provided many examples of topological and locally convex vector spaces consisting of functions with particular properties. In fact, the relevant topologies were built in a way reflecting the defining properties of the functions under consideration. While very different concerning their properties they all had one feature in common: the values of the functions were in $\mathbb{K}$. Such scalar functions are of course not the only option. Both from a conceptual point of view and from a practical perspective one needs to go beyond the scalar-valued case. As we still want to have vector spaces of functions we require the values to be inside an a priori fixed vector space V.

In the general case, endowing the target vector space V now with additional topological structure will allow us to define function spaces referring to this structure like continuous or differentiable functions. In particular, this will lead to many important new classes of examples of locally convex spaces. In addition, we will see many new techniques of handling problems in functional analysis by taking the point of view of vector-valued functions.

Even if we start with scalar-valued functions the canonical *currying isomorphism*

$$\mathrm{Map}(X \times Y, \mathbb{K}) \cong \mathrm{Map}(X, \mathrm{Map}(Y, \mathbb{K})) \tag{4.1}$$

for two sets X and Y suggests to investigate vector-valued functions on X, namely with values in the vector space $V = \mathrm{Map}(Y, \mathbb{K})$, might be an appropriate tool to understand scalar-valued functions on X depending parametrically on Y (or vice versa). When adding additional structure like continuity this becomes quickly an important and non-trivial issue. The change of perspective in (4.1) then becomes helpful.

In this chapter we will discuss the basic constructions like Riemann integration of vector-valued functions and differentiation. As it turns out, Riemann integration is a quite powerful tool requiring only a very mild completeness assumption for the target. To have a clear formulation of the relevant concept of convergence we start this chapter with a short digression on bornologies and their relations to locally convex spaces. After Riemann integration and differentiation we then move to further concepts like asymptotic expansions and the notion of vector-valued holomorphic functions. For all these topics we will meet important applications on the way.

4.1 Bornological Vector Spaces

4.1.1 Bornologies

In this short section we give a brief and by no means complete overview on the concepts of bornologies and bounded maps including the resulting notions of convergence and completeness.

The behaviour of bounded subsets in $\mathbb{R}^n$ or in general topological vector spaces can be axiomatized as follows. This leads to the general definition of a *bornology* on a set in the very same spirit as one defines a topology as an axiomatization of the features of open subsets in $\mathbb{R}^n$:

Definition 4.1.1 (Bornology) *Let X be a set. A bornology $\mathscr{B}$ on X is a collection $\mathscr{B} \subseteq 2^X$ of subsets of X, called the bounded subsets of X, satisfying the following properties:*

i.) *The bounded subsets cover X, i.e. $X = \bigcup_{B \in \mathscr{B}} B$.*

ii.) *For $B \in \mathscr{B}$ and $A \subseteq B$ one has $A \in \mathscr{B}$.*

iii.) *For $B_1, \ldots, B_n \in \mathscr{B}$ one has $B_1 \cup \cdots \cup B_n \in \mathscr{B}$.*

As for topologies one has a few immediate properties and constructions. We list some important ones here and refer to Exercise 4.5.1 for further details.

Clearly the usual bounded subsets of a topological vector space satisfy the above requirements: this gives us an immediate wealth of examples.

Remark 4.1.2 Let $\mathscr{B}$ be a bornology on a set X.

i.) Every finite subset $\{x_1, \cdots, x_n\}$ of X belongs to X. Indeed, for $x \in X$ one finds a bounded subset $B \subseteq X$ with $x \in B$ by the cover property. Hence $\{x\}$ is bounded by the second property and the third property shows that all finite subsets are bounded.

ii.) Conversely, taking $\mathscr{B}_{\mathrm{finite}}$ to be the collection of finite subsets one sees immediately that $\mathscr{B}_{\mathrm{finite}}$ is a bornology. Clearly, we have $\mathscr{B}_{\mathrm{finite}} \subseteq \mathscr{B}$ by *i.)* and hence $\mathscr{B}_{\mathrm{finite}}$ is the smallest bornology on X.

iii.) Analogously, taking the full power set $\mathscr{B}_{\max} = 2^X$ we also get a bornology, the largest bornology on X. We have $X \in \mathscr{B}$ iff $\mathscr{B} = \mathscr{B}_{\max}$.

iv.) If $\{\mathscr{B}_\alpha\}_{\alpha \in I}$ is any collection of bornologies on X then the intersection $\mathscr{B} = \bigcap_{\alpha \in I} \mathscr{B}_\alpha$ is again a bornology. Indeed, the second and third part are verified directly. The covering property follows from $\mathscr{B}_{\mathrm{finite}} \subseteq \mathscr{B}_\alpha$ by *ii.)* for all α and hence $\mathscr{B}_{\mathrm{finite}} \subseteq \mathscr{B}$.

v.) Suppose $\mathcal{C} \subseteq 2^X$ is a collection of subsets of X. Then there exists a unique smallest bornology of X containing all the subsets from $\mathcal{C}$. Indeed, there exist bornologies containing $\mathcal{C}$ like $\mathscr{B}_{\max}$. Thus taking the intersection of all bornologies containing $\mathcal{C}$ is still a bornology by *iv.)*, now obviously being the smallest with this property. This way, we can *generate* a bornology with certain subsets of X being bounded in an optimal way, see also Exercise 4.5.2.

vi.) If $\mathscr{B}_i$ are bornologies on X_i for $i = 1, 2$ then the product subsets $B_1 \times B_2 \subseteq X_1 \times X_2$ for $B_i \in \mathscr{B}_i$ can be used to generate a bornology on the Cartesian product, called the *product bornology*. Note that we still need to enlarge the class of subsets from product subsets to a true bornology. Not all bounded subsets of the product bornology are products of bounded subsets.

Having the notion of a bornology we can proceed as in general topology to establish a category of bornological spaces:

Definition 4.1.3 (Bornological spaces) *Let X be a set.*

i.) *The set X together with a bornology $\mathscr{B}_X$ is called a bornological space $(X, \mathscr{B}_X)$.*

ii.) *A map $f\colon (X, \mathscr{B}_X) \longrightarrow (Y, \mathscr{B}_Y)$ between bornological spaces is called bounded (or bornological) if f maps bounded subsets of X to bounded subsets of Y.*

iii.) *The category of bornological spaces with bounded maps between them as morphisms is denoted by* Born.

Remark 4.1.4 (The category Born) It is a straightforward check to verify that we indeed end up with a category: the composition of bounded maps is bounded again. One should note that "bounded" is perhaps a misnomer since it typically will not imply that the image $f(X) \subseteq Y$ is a bounded subset: if X itself is not a bounded set, i.e. if $\mathscr{B}_X$ is not $\mathscr{B}_{\max}$, then this need not be the case. Hence "locally bounded" is a more precise, though non-common attribute instead.

We also note that there is a certain clash of notations concerning a bornological locally convex space in the sense of Definition 3.3.3 which we will need to resolve later.

4.1.2 Vector Bornologies

While general bornological spaces already capture many interesting features of bounded subsets and boundedness, the concept of a bornology becomes most fruitful when combined with the linear structure of a vector space. Here we have two versions which can be seen as analogs of topological vector spaces and locally convex vector spaces:

Definition 4.1.5 (Vector bornology) *Let V be a vector space over $\mathbb{K}$. A bornology $\mathscr{B}$ on V is called vector (or linear) bornology if*

i.) *for $B_1, B_2 \in \mathscr{B}$ one has $B_1 + B_2 \in \mathscr{B}$,*

ii.) *for $B \in \mathscr{B}$ and $\lambda \in \mathbb{K}$ one has $\lambda B \in \mathscr{B}$,*

iii.) *for $B \in \mathscr{B}$ one has* $\operatorname{balanced}(B) \in \mathscr{B}$.

If in addition one has

iv.) *for $B \in \mathscr{B}$ also* $\operatorname{conv}(B) \in \mathscr{B}$

then $\mathscr{B}$ is called a convex vector bornology.

The definition of a bornivorous subset in a vector space V with vector bornology $\mathscr{B}$ can be stated as before in Definition 3.3.1: the subset A is *bornivorous* if for all $B \in \mathscr{B}$ there exists a $\lambda > 0$ with $B \subseteq \lambda A$. This matches the previous definition on view of the following example:

Example 4.1.6 (Canonical vector bornology) Let V be a topological vector space. Then the collection of bounded subsets of V forms a vector bornology, called the *von Neumann* or *canonical* bornology of V. If V is locally convex the canonical bornology of V is a convex vector bornology, see also Exercise 4.5.3 for some more details.

In general, a bornivorous subset of a vector space with vector bornology is absorbing:

Proposition 4.1.7 *Let V be a vector space over $\mathbb{K}$ with vector bornology.*

i.) *A bornivorous subset of V is absorbing.*

ii.) *A union of bornivorous subsets is again bornivorous.*

iii.) *If $A \subseteq V$ is bornivorous and $\lambda \neq 0$ then λA is bornivorous, too.*

iv.) *If $A \subseteq V$ is bornivorous and $A \subseteq C$ then C is bornivorous, too.*

v.) *If $A \subseteq V$ is bornivorous then also* $\operatorname{balanced}(A)$, $\operatorname{conv}(A)$ *and* $\operatorname{absconv}(A)$ *are bornivorous.*

PROOF: Let $v \in V$ then $\{v\}$ is a bounded subset. If $A \subseteq V$ is bornivorous we find a $\lambda > 0$ with $\{v\} \subseteq \lambda A$ which means $v \in \lambda A$. Hence A is absorbing. The fourth part is clear from the definition and gives the second part at once. Since for a bounded subset B also $\frac{1}{\lambda}B$ is bounded if $\lambda \neq 0$, the third part follows. Then the fifth part is a combination of the previous statements. □

Having a topological vector space, Example 4.1.6 shows that we can construct a vector bornology out of it. Conversely, a vector bornology allows us to build a matching topology as well. We focus on the locally convex case directly:

Proposition 4.1.8 *Let V be a vector space over $\mathbb{K}$ with a convex vector bornology $\mathscr{B}$.*

i.) *The seminorms*

$$\mathcal{P}(\mathscr{B}) = \left\{ \mathrm{p} \textit{ seminorm on } V \;\middle|\; \mathrm{p}\big|_B \textit{ is bounded for all } B \in \mathscr{B} \right\} \tag{4.1.1}$$

constitute a saturated system of seminorms on V with a corresponding locally convex topology $\mathscr{V}(\mathscr{B})$

ii.) *The locally convex topology $\mathcal{V}(\mathcal{B})$ is the finest locally convex topology such that all $B \in \mathcal{B}$ are bounded with respect to $\mathcal{V}(\mathcal{B})$.*

iii.) *The locally convex topology $\mathcal{V}(\mathcal{B})$ is Hausdorff iff there exists no $B \in \mathcal{B}$ containing a one-dimensional subspace.*

Proof: First we note that $\mathcal{P}(\mathcal{B})$ is actually saturated. Let q be a seminorm such that we find a $c > 0$ and $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}(\mathcal{B})$ with $\mathrm{q} \leq c \max\{\mathrm{p}_1, \ldots, \mathrm{p}_n\}$. Then for $B \in \mathcal{B}$ we have for all $v \in B$

$$\mathrm{q}(v) \leq c \max\{\mathrm{p}_1(v), \ldots, \mathrm{p}_n(v)\} \leq \max\{c_1, \ldots, c_n\}$$

with $c_i = \sup_{v \in B} \mathrm{p}_i(v) < \infty$. Hence q is bounded on B showing $\mathrm{q} \in \mathcal{P}(\mathcal{B})$. For the corresponding locally convex topology $\mathcal{V}(\mathcal{B})$ we have by Proposition 3.2.12 that a subset $B \subseteq V$ is bounded iff $\mathrm{p}\big|_B$ is bounded for all continuous seminorms p on V, i.e. for all $\mathrm{p} \in \mathcal{P}(\mathcal{B})$. Thus every $B \in \mathcal{B}$ is bounded with respect to $\mathcal{V}(\mathcal{B})$ by the very definition of $\mathcal{P}(\mathcal{B})$. Moreover, the system of seminorms $\mathcal{P}(\mathcal{B})$ is clearly chosen in a maximal fashion with respect to this property, hence the second part follows. Finally, assume that $\mathcal{P}(\mathcal{B})$ is not Hausdorff. Then there is a $v \neq 0$ with $\mathrm{p}(v) = 0$ for all $\mathrm{p} \in \mathcal{P}(\mathcal{B})$. In this case, $\mathrm{p}(\lambda v) = 0$ for all $\lambda \in \mathbb{K}$, too, showing that $\operatorname{span}_{\mathbb{K}}\{v\}$ is a bounded subset. Conversely, assume that we have a bounded subset containing a one-dimensional subspace, given by $\operatorname{span}_{\mathbb{K}}\{v\}$ for some $v \neq 0$. For all $\mathrm{p} \in \mathcal{P}(\mathcal{B})$ we then find a $c > 0$ with $\mathrm{p}\big|_{\operatorname{span}_{\mathbb{K}}\{v\}} < c$. This means $\mathrm{p}(\lambda v) = |\lambda| \mathrm{p}(v) < c$ for all $\lambda \in \mathbb{K}$ which is only possible for $\mathrm{p}(v) = 0$. Thus $\mathrm{p} \in \mathcal{P}(\mathcal{B})$ is not Hausdorff. ☐

One also calls $\mathcal{V}(\mathcal{B})$ the *induced* locally convex topology of $\mathcal{B}$. A bornology satisfying the property in *iii.)* of the proposition will also be called *Hausdorff*. This matches of course very well with the intuition of bounded subsets: a set containing a proper subspace should not be considered to be bounded.

The above proposition leaves the question whether the construction of the topology $\mathcal{V}(\mathcal{B})$ out of the bornology $\mathcal{B}$ produces new bounded subsets of V. Note that there will be many locally convex topologies having the same set of bounded subsets, see e.g. Theorem 3.2.13. Thus we cannot expect the bornology $\mathcal{B}$ to determine a unique locally convex topology. It is only the *finest* one with the property that bounded subsets stay bounded:

Proposition 4.1.9 *Let V be a vector space over $\mathbb{K}$.*

i.) *If $\mathcal{B} \subseteq \widetilde{\mathcal{B}}$ are two convex vector bornologies on V then for the corresponding induced topologies one has $\mathcal{V}(\widetilde{\mathcal{B}}) \subseteq \mathcal{V}(\mathcal{B})$.*

ii.) *If $\mathcal{V} \subseteq \widetilde{\mathcal{V}}$ are two locally convex topologies on V then for the von Neumann bornologies one has $\mathcal{B}(\widetilde{\mathcal{V}}) \subseteq \mathcal{B}(\mathcal{V})$.*

iii.) *The locally convex topology $\mathcal{V}(\mathcal{B})$ induced by a convex vector bornology is bornological.*

iv.) *A locally convex topology $\mathcal{V}$ on V is bornological iff $\mathcal{V} = \mathcal{V}(\mathcal{B}(\mathcal{V}))$.*

v.) *For a locally convex topology $\mathcal{V}$ on V one has $\mathcal{B}(\mathcal{V}) = \mathcal{B}(\mathcal{V}(\mathcal{B}(\mathcal{V})))$.*

Proof: More bounded subsets give more conditions on a seminorm to belong to the system (4.1.1). Hence the first part follows. Conversely, more open subsets put more conditions on a subset to be bounded, implying the second statement. Let $\mathcal{B}$ be a convex vector bornology and $\mathcal{V}(\mathcal{B})$ the induced locally convex topology. Suppose $A \subseteq V$ is absolutely convex and bornivorous, i.e. for all $B \in \mathcal{B}$ one has $\lambda > 0$ with $B \subseteq \lambda A$. This implies that the Minkowski functional p_A of A satisfies $\mathrm{p}_A\big|_B \leq \lambda$, showing $\mathrm{p}_A \in \mathcal{P}(\mathcal{B})$. Now

$$\mathrm{B}_{\mathrm{p}_A,1}(0) \subseteq A \subseteq \mathrm{B}_{\mathrm{p}_A,1}(0)^{\mathrm{cl}}$$

according to Lemma 2.2.20 and hence A is a zero neighbourhood since it contains the open unit ball $\mathrm{B}_{\mathrm{p}_A,1}(0)$ of the continuous seminorm p_A. Thus $\mathcal{V}(\mathcal{B})$ is bornological, showing the third part.

Suppose now that $\mathcal{V}$ is a bornological locally convex topology on V and consider its von Neumann bornology $\mathscr{B}(\mathcal{V})$. Being bornological implies that for every absolutely convex bornivorous subset $A \subseteq V$ the seminorm p_A is continuous, since $\mathrm{B}_{\mathrm{p}_A,1}(0)^{\mathrm{cl}}$ contains the zero neighbourhood A. But then p_A is bounded on every bounded subset $B \in \mathscr{B}(\mathcal{V})$, thereby belonging to $\mathcal{P}(\mathscr{B}(\mathcal{V}))$. This shows that $\mathcal{V}(\mathscr{B}(\mathcal{V})) \supseteq \mathcal{V}$. Conversely, let $\mathrm{p} \in \mathcal{P}(\mathscr{B}(\mathcal{V}))$ be a seminorm which is bounded on all bounded subsets of the von Neumann bornology of $\mathcal{V}$. This means that for $B \in \mathscr{B}(\mathcal{V})$ one finds a $\lambda > 0$ with $B \subseteq \mathrm{B}_{\mathrm{p}_A,\lambda}(0)^{\mathrm{cl}} \subseteq \lambda \mathrm{B}_{\mathrm{p}_A,1}(0)^{\mathrm{cl}}$. Hence the absolutely convex subset $\mathrm{B}_{\mathrm{p}_A,1}(0)^{\mathrm{cl}}$ is bornivorous. Being bornological this implies that $\mathrm{B}_{\mathrm{p}_A,1}(0)^{\mathrm{cl}}$ is a zero neighbourhood with respect to $\mathcal{V}$ and hence p is continuous. Thus all seminorms in $\mathcal{P}(\mathscr{B}(\mathcal{V}))$ are continuous showing $\mathcal{V}(\mathscr{B}(\mathcal{V})) \subseteq \mathcal{V}$ which completes the proof of the fourth part as by the third we already know that $\mathcal{V}(\mathscr{B}(\mathcal{V}))$ is always bornological. Finally, let $\mathcal{V}$ be a locally convex topology on V with bounded subsets $\mathscr{B}(\mathcal{V})$. Then we know $\mathcal{V}(\mathscr{B}(\mathcal{V})) \supseteq \mathcal{V}$ according to Proposition 4.1.8, *ii.)*. Hence $\mathscr{B}(\mathcal{V}) \supseteq \mathscr{B}(\mathcal{V}(\mathscr{B}(\mathcal{V})))$ by *ii.)*. But $\mathscr{B}(\mathcal{V}) \subseteq \mathscr{B}(\mathcal{V}(\mathscr{B}(\mathcal{V})))$ by Proposition 4.1.8, *ii.)*, again. Hence the last part follows. □

This proposition has a remarkable consequence: for every locally convex topology there exists a unique finer bornological one having the same bounded subsets:

Corollary 4.1.10 *Let $(V, \mathcal{V})$ be a locally convex space. Then there exists a unique bornological locally convex topology on V having the same bounded subsets as the original topology. It is finer that the original topology and explicitly given by $\mathcal{V}(\mathscr{B}(\mathcal{V}))$.*

Definition 4.1.11 (Bornologification) *Let $(V, \mathcal{V})$ be a locally convex space. Then the locally convex space $(V, \mathcal{V}(\mathscr{B}(\mathcal{V})))$ is called the bornologification of $(V, \mathcal{V})$.*

Yet another way to put this is to say that V is bornological iff it coincides with its bornologification. We can use this statement to obtain immediate examples of non-bornological spaces:

Example 4.1.12 Let V be a locally convex Hausdorff space such that the weak topology of V is strictly weaker that the original one. By Theorem 3.2.13 we know that the bounded subsets of V coincide with the weakly bounded subsets. Hence the bornologification of the weak (locally convex) topology is finer that the original topology and, by assumption, strictly finer than the weak topology. This is a rather typical situation showing that weak topologies tend to be non-bornological.

Focusing on the bornology will have certain advantages as it gives a better robustness. Many very different locally convex topologies can have the same underlying bornology. This gives the following consequence: if one is able to show that a certain, a priori topological, question ultimately depends only on the von Neumann bornology then it will not matter too much which details about the topology are known. In this case we speak of a *bornological concept*.

4.1.3 Mackey Convergence and Completeness

A vector bornology on a vector space V allows for an intrinsic concept of convergence based on bounded subsets instead of open ones. The basic idea is that we scale differences to become small. In detail, one states the following definition:

Definition 4.1.13 (Mackey convergence) *Let V be a vector space over $\mathbb{K}$ and $\mathscr{B}$ a vector bornology on V. Moreover, let $(v_i)_{i \in I}$ be a net in $\mathcal{V}$.*

i.) The net $(v_i)_{i \in I}$ Mackey converges to $v \in V$ if there exists a bounded subset $B \in \mathscr{B}$ and a net $(\lambda_i)_{i \in I}$ of real numbers $\lambda_i \in \mathbb{R}$ with $\lim_{i \in I} \lambda_i = 0$ such that

$$v_i - v \in \lambda_i B \tag{4.1.2}$$

for all $i \in I$.

ii.) The net $(v_i)_{i \in I}$ *is called a Mackey-Cauchy net if there exists a bounded subset* $B \in \mathscr{B}$ *and a net* $(\lambda_{i,j})_{(i,j) \in I \times I}$ *of real numbers* $\lambda_{i,j} \in \mathbb{R}$ *with* $\lim_{(i,j) \in I \times I} \lambda_{i,j} = 0$ *such that*

$$v_i - v_j \in \lambda_{i,j} B \tag{4.1.3}$$

for all $i, j \in I$.

Remark 4.1.14 As usual, we are mainly concerned about the case of a convex vector bornology $\mathscr{B}$. In this case, we can require the subset B in the definition to be absolutely convex without restriction: indeed, suppose we found a $\widetilde{B} \in \mathscr{B}$ satisfying either (4.1.2) or (4.1.3) then $B = \operatorname{absconv}(\widetilde{B})$ is still bounded and contains $\widetilde{B}$. Then this B will also satisfy (4.1.2) or (4.1.3), respectively.

The idea of Mackey convergence is that the differences become small in a way which is controlled by a small scaling, the zero net $(\lambda_i)_{i \in I}$, and a fixed reference in V, the bounded subset B. The next lemma shows that one can adjust the net of scaling factors slightly:

Lemma 4.1.15 *Let* V *be a vector space over* $\mathbb{K}$ *with a vector bornology* $\mathscr{B}$. *If a net* $(v_i)_{i \in I}$ *is Mackey convergent to* v *then the scaling factors* $(\lambda_i)_{i \in I}$ *can be chosen to be non-negative for late enough indices. If* I *admits a cofinal sequence, then we can choose* $\lambda_i > 0$.

PROOF: From $\lim_{i \in I} \lambda_i = 0$ we get an index $i_0 \in I$ with $|\lambda_i| \le 1$ for $i \succcurlyeq i_0$. Define then for $i \succcurlyeq i_0$

$$\mu_i = \sup\{|\lambda_j| \mid j \succcurlyeq i\},$$

satisfying $0 \le \mu_i \le 1$. We have $v_i - v \in \lambda_i B \subseteq \mu_i B$ where we assume B to be circled from the beginning which is no restriction by the properties of a vector bornology: we can pass from B to $-B \cup B$ if needed. If in addition I admits a cofinal sequence, say $(i_n)_{n \in \mathbb{N}}$, we define

$$\mu_i = \max\{|\lambda_i|, \ |\lambda_{i_n}| \text{ with } i_n \succcurlyeq i, \ \tfrac{1}{n} \text{ with } i_n \succcurlyeq i\},$$

which will then be positive and still satisfy $v_i - v \in \mu_i B$. □

The last feature is sometimes convenient as now we can equivalently write $\frac{1}{\mu_i}(v_i - v) \in B$ instead of $v_i - v \in \mu_i B$.

To get another interpretation of Mackey convergence we use the following *localization*. Let $B \in \mathscr{B}$ be a bounded subset which we can assume to be absolutely convex from the beginning as we shall consider a *convex* vector bornology from now on. The linear span of B is then denoted by

$$V_B = \operatorname{span}_{\mathbb{K}} B \subseteq V. \tag{4.1.4}$$

In general, $B \subseteq V$ has no reason to be absorbing. However, as subset $B \subseteq V_B$ of this subspace it is absorbing by the very definition of a span and the convexity of B. Thus we have a seminorm, the Minkowski functional p_B on this subspace. In general, p_B is just a seminorm on V_B. If, however, the bornology $\mathscr{B}$ is Hausdorff, we have a norm:

Lemma 4.1.16 *Let* V *be a vector space over* $\mathbb{K}$ *and let* $\mathscr{B}$ *be a convex vector bornology on* V. *Then* $\mathscr{B}$ *is Hausdorff iff for all* $B \in \mathscr{B}$ *the seminorm* p_B *on* V_B *is a norm.*

PROOF: Assume first that $\mathscr{B}$ is Hausdorff, i.e. no $B \in \mathscr{B}$ contains a one-dimensional subspace. The Minkowski functional p_B gives then the usual relation

$$\mathrm{B}_{\mathrm{p}_B,1}(0) \subseteq B \subseteq \mathrm{B}_{\mathrm{p}_B,1}(0)^{\mathrm{cl}}$$

for the defining absolutely convex and absorbing subset $B \subseteq V_B$. We have in general

$$\ker \mathrm{p}_B \subseteq \mathrm{B}_{\mathrm{p}_B,\varepsilon}(0)$$

for all $\varepsilon > 0$, see (2.2.39). Thus $\ker \mathrm{p}_B \subseteq B$ follows. Hence $\ker \mathrm{p}_B = \{0\}$ which means that p_B is a norm. Conversely, assume that p_B is a norm and B contains a one-dimensional subspace, say $\operatorname{span}_{\mathbb{K}}\{v\} \subseteq B$ with $v \in V_B$. Then the definition of p_B gives $\mathrm{p}_B(v) = 0$ at once, a contradiction. □

Thus we expect to have a reasonable behaviour of Mackey convergence once we restrict ourselves to Hausdorff bornologies. In this case, the norm p_B on V_B will also be denoted by $\|\cdot\|_B$ to emphasize that we actually have a norm. The next lemma relates the norms for varying $B \in \mathscr{B}$:

Lemma 4.1.17 *Let V be a vector space over $\mathbb{K}$ with Hausdorff convex vector bornology. For absolutely convex $B, \widetilde{B} \in \mathscr{B}$ with $B \subseteq \widetilde{B}$ we then have*

$$\|v\|_{\widetilde{B}} \leq \|v\|_B \tag{4.1.5}$$

for all $v \in V_B$. Hence the inclusion $V_B \subseteq V_{\widetilde{B}}$ is continuous.

PROOF: First we note that $V_B \subseteq V_{\widetilde{B}}$. Then (4.1.5) is clear from the definition of $\|\cdot\|_B$ as the Minkowski functional of B. Indeed, we have

$$\begin{aligned}\|v\|_B &= \inf\big\{\lambda > 0 \bigm| v \in \lambda B\big\}\\ &\geq \inf\Big\{\lambda > 0 \Bigm| v \in \lambda \widetilde{B}\Big\}\\ &= \|v\|_{\widetilde{B}},\end{aligned}$$

since the condition $v \in \lambda\widetilde{B}$ is easier to fulfill for a larger $\widetilde{B} \supseteq B$. □

The next proposition relates now Mackey convergence to usual topological convergence in the normed spaces:

Proposition 4.1.18 *Let V be a vector space over $\mathbb{K}$ with a Hausdorff convex vector bornology $\mathscr{B}$. Moreover, let $(v_i)_{i\in I}$ be a net in V.*

i.) The net $(v_i)_{i\in I}$ Mackey converges to $v \in V$ iff there exists an absolutely convex $B \in \mathscr{B}$ with $v_i \in V_B$ for all $i \in I$ such that $(v_i)_{i\in I}$ converges to $v \in V_B$ in the normed space V_B.

ii.) The net $(v_i)_{i\in I}$ is a Mackey-Cauchy net iff there exists an absolutely convex $B \in \mathscr{B}$ with $v_i \in V_B$ for all $i \in I$ such that $(v_i)_{i\in I}$ is a Cauchy net in the normed space V_B.

PROOF: Suppose that $(v_i)_{i\in I}$ Mackey converges to $v \in V$ with an absolutely convex $\widetilde{B} \in \mathscr{B}$ and a zero net $(\lambda_i)_{i\in I}$ of real numbers such that $v_i - v \in \lambda_i\widetilde{B}$. Consider then

$$B = \operatorname{absconv}(\{v\} \cup \widetilde{B}), \tag{☺}$$

which is still an element of $\mathscr{B}$ as $\mathscr{B}$ is a convex vector bornology. Clearly $\tilde{B} \in \mathscr{B}$ and hence $v_i - v \in \lambda_i B$ still holds. But $v \in B$ and hence $v \in V_B$, too. Now $v_i \in \lambda_i B + v \subseteq V_B$ yields a bounded subset B with $v_i \in V_B$ for all $i \in I$. The definition of $\|\cdot\|_B$ as Minkowski functional of $B \in V_B$ the yields

$$\|v_i - v\|_B \leq \lambda_i,$$

showing that $\lim_{i\in I} v_i = v$ in the normed space V_B. Conversely, assume we have found a $B \in \mathscr{B}$ with the above requirements, i.e. $\|v_i - v\|_B \longrightarrow 0$. Then

$$\lambda_i = 2\|v_i - v\|_B$$

defines a net of real numbers converging to zero, and, by definition of the infimum in the Minkowski functional, we have for every $\varepsilon > 0$

$$v_i - v \in (\|v_i - v\|_B + \varepsilon)B. \tag{4.1.6}$$

Taking $\varepsilon = \|v_i - v\|_B$ then gives $v_i - v \in \lambda_i B$ —note that the case $\|v_i - v\|_B = 0$ is trivially covered— which is the Mackey convergence we want. For the second statement one can proceed analogously with the only important modification being that instead of (☺) one takes $B = \operatorname{absconv}(\{v_{i_0}\} \cup \widetilde{B})$ for some $i_0 \in I$ to guarantee $v_i \in V_B$ for all $i \in I$. □

This observation has a very important consequence. Since in a normed space completeness is equivalent to sequential completeness, we get the same behaviour for Mackey convergence. The only little subtlety is in the fact that we might have to adapt the corresponding bounded subset accordingly.

Proposition 4.1.19 *Let V be a vector space over $\mathbb{K}$ with a Hausdorff convex vector bornology $\mathscr{B}$. Then the following statements are equivalent:*

i.) Every Mackey-Cauchy net in V Mackey converges.

ii.) Every Mackey-Cauchy sequence in V Mackey converges.

PROOF: We need to show *ii.)* $\implies$ *i.)*. Thus let $(v_i)_{i\in I}$ be a Mackey-Cauchy net and choose the appropriate absolutely convex $B \in \mathscr{B}$ with $v_i \in V_B$ for all $i \in I$ and $(v_i)_{i\in I}$ being a Cauchy net with respect to $\|\cdot\|_B$ according to Proposition 4.1.18, *ii.)*. Denote the completion of V_B by $\widehat{V}_B$ as usual. Then $(v_i)_{i\in I}$ converges to some $v \in \widehat{V}_B$ with respect to $\|\cdot\|_B$. Since $\widehat{V}_B$ is a Banach space, we get indices $i_n \in I$ with $\|v_i - v\|_B < \frac{1}{n}$ for all $i \succcurlyeq i_n$ and $n \in \mathbb{N}$. Without restriction, we can inductively arrange the indices i_n such that

$$i_n \preccurlyeq i_m \text{ for } n \le m.$$

Then the sequence $(v_{i_n})_{i\in\mathbb{N}}$ is a Cauchy sequence in the normed space V_B since it converges to $v \in \widehat{V}_B$. According to Proposition 4.1.18, *ii.)*, it is a Mackey-Cauchy sequence in V. By assumption, $(v_{i_n})_{n\in\mathbb{N}}$ is Mackey convergent to some $w \in V$. By Proposition 4.1.18, *i.)*, we find an absolutely convex $\widetilde{B} \in \mathscr{B}$ such that $v_{i_n} \in V_{\widetilde{B}}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} v_{i_n} = w \in V_{\widetilde{B}}$ in the norm sense of $V_{\widetilde{B}}$. Finally, consider $C = \operatorname{absconv}(B \cup \widetilde{B})$ which is still an element of $\mathscr{B}$. Thus we get the inclusions

$$V_B \subseteq V_C \quad \text{and} \quad V_{\widetilde{B}} \subseteq V_C, \tag{☺}$$

which are continuous according to Lemma 4.1.17. In $V_{\widetilde{B}}$ the net $(v_i)_{i\in I}$ is still a Cauchy net and $(v_{i_n})_{n\in\mathbb{N}}$ is still a Cauchy sequence, in fact convergent to $w \in V_{\widetilde{B}} \subseteq V_C$. Since (☺) is continuous, we can extend (☺) to an inclusion $\widehat{V}_B \subseteq \widehat{V}_C$ of the completions. Now $(v_i)_{i\in I}$ converges to $v \in \widehat{V}_B$ which then becomes an element of $v \in \widehat{V}_C$ under the inclusion. Since $V_{\widetilde{B}} \subseteq \widehat{V}_C$ and $(v_{i_n})_{n\in\mathbb{N}}$ converges to v in $\widehat{V}_B$, we eventually see that $v = w$ as in the Banach space $\widehat{V}_C$ the limits are unique. This ultimately gives a limit $w = v$ of the Cauchy net $(v_i)_{n\in I}$ which is already in $V_C \subseteq \widehat{V}_C$. Hence Proposition 4.1.18, *i.)*, shows that $(v_i)_{i\in I}$ Mackey converges to $w \in V_C \subseteq V$. □

Definition 4.1.20 (Mackey completeness) *Let V be a vector space over $\mathbb{K}$ with a Hausdorff convex vector bornology $\mathscr{B}$. Then $\mathscr{B}$ is called Mackey complete if every Mackey-Cauchy net converges in V.*

Equivalently, $\mathscr{B}$ is Mackey complete if every Mackey-Cauchy *sequence* Mackey converges in V. This brings an unexpected countability feature into the picture which makes the theory of Mackey convergence fairly easy and attractive for many purposes.

The interesting situation is of course the case of the canonical bornology of a Hausdorff locally convex space. In this case, we can compare Mackey convergence to the usual, topological notion of convergence in V.

Proposition 4.1.21 *Let V be a Hausdorff locally convex space with its canonical bornology $\mathscr{B}$.*

i.) For every absolutely convex bounded subset $B \in V$ the inclusion $V_B \subseteq V$ is continuous.

ii.) Every Mackey-Cauchy net in V is a Cauchy net.

iii.) Every Mackey convergent net in V is convergent.

Proof: Let p be a continuous seminorm on V. Then we find a constant $c_{\mathrm{p}} > 0$ such that $\mathrm{p}(v) \le c_{\mathrm{p}}$ for all $v \in B$ since B is bounded. If $\lambda > 0$ with $v \in \lambda B$ then $\mathrm{p}(v) = \lambda \mathrm{p}(\frac{1}{\lambda} v) \le \lambda c_{\mathrm{p}}$ holds. Thus

$$\mathrm{p}(v) \le \inf\{\lambda > 0 \mid v \in \lambda B\} c_{\mathrm{p}} = c_{\mathrm{p}} \|v\|_B \tag{4.1.7}$$

for $v \in V_B$. This is the continuity estimate needed in the first part. From Proposition 4.1.18 we get the other two statements at once. □

Moreover, one has explicit examples showing that convergence in V does *not* imply Mackey convergence and similarly for the Cauchy condition, see e.g. Exercise 4.5.4. Since Mackey completeness is a sequential concept by Proposition 4.1.19 and since Mackey-Cauchy sequences are Cauchy sequences, Mackey completeness turns out to be a fairly weak property:

Corollary 4.1.22 *Let V be a sequentially complete locally convex space. Then the canonical bornology is Mackey complete.*

In the case where the canonical bornology of a Hausdorff locally convex space V is Mackey complete, we also say that V is *Mackey complete*. A slightly more explicit characterization of Mackey completeness of a locally convex space V is obtained using the subspaces V_B of V:

Proposition 4.1.23 *Let V be a Hausdorff locally convex space. Then the following statements are equivalent:*

i.) The canonical bornology of V is Mackey complete.

ii.) For every absolutely convex closed bounded subset $B \subseteq V$ the normed space V_B is complete, i.e. a Banach space.

iii.) For every bounded subset $B \subseteq V$ there exists an absolutely convex bounded subsets $\widetilde{B} \supseteq B$ such that $V_{\widetilde{B}}$ is complete.

Proof: We show *i.)* $\implies$ *ii.)* $\implies$ *iii.)* $\implies$ *i.)*. Suppose V is Mackey complete and let $B \subseteq V$ be an absolutely convex closed bounded subset. Let $(v_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in the normed space V_B. Note, that sequences are enough to test for completeness. By Proposition 4.1.18 the sequence $(v_n)_{n\in\mathbb{N}}$ is a Mackey-Cauchy sequence in V, hence Mackey-convergent to some $v \in V$. Now let $\varepsilon > 0$, then we have a $N \in \mathbb{N}$ with $\|v_n - v_m\|_B < \varepsilon$, whenever $n, m > N$. This means $v_n - v_m \in \varepsilon B$ by definition of $\|\cdot\|_B$ as Minkowski functional of B. Since B and thus εB are in addition closed, taking the limit $m \to \infty$ in the topology of V gives $v_n - v \in \varepsilon B$. Thus $v \in V_B$ follows. But $v_n - v \in \varepsilon B$ for all large n then shows $v_n \to v$ with respect to $\|\cdot\|_B$ as well. This establishes the completeness of V_B, hence *ii.)*. The implication *ii.)* $\implies$ *iii.)* is trivial since with B also $\operatorname{absconv}(B)$ is bounded. Finally, assume *iii.)* and let $(v_n)_{n\in\mathbb{N}}$ be a Mackey-Cauchy sequence. As usual we get a bounded $\widetilde{B}$ with $v_n - v_m \in \varepsilon_{n,m}\widetilde{B}$ for a net $(\varepsilon_{n,m})_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ converging to zero. By *iii.)* we can enlarge $\widetilde{B}$ to an absolutely convex bounded B such that V_B is now a Banach space. Without restriction, we can assume $v_1 \in B$ and hence $v_n \in V_B$, with respect to $\|\cdot\|_B$. Thus, by *iii.)*, we have a limit $v \in V_B$ which by Proposition 4.1.18 *ii.)*, shows that $(v_n)_{n\in\mathbb{N}}$ Mackey converges to v, which gives *i.)*. □

4.2 Bounded and Continuous Functions

The first function spaces of vector-valued functions we discuss are spaces of (locally) bounded and of continuous functions. Thus we can proceed very much parallel to the scalar case in Section 1.2. For notions of boundedness we have to rely on a bornology on the target space while continuity requires a topology as usual. In addition, we will also discuss spaces of locally Lipschitz functions.

4.2.1 Bounded and Locally Bounded Functions

Let X be a topological space which later on will be taken to be more specific. Moreover, let V be a vector space over $\mathbb{K}$ endowed with a vector bornology $\mathscr{B}$. This will allow us to speak of bounded subsets of V with respect to $\mathscr{B}$. Hence we can state the following definition:

Definition 4.2.1 (Locally bounded function) *Let X be a topological space and let V be a vector space endowed with a vector bornology $\mathscr{B}$.*

i.) *A function $f\colon X \longrightarrow V$ is called locally bounded if for every $x \in X$ there exists a neighbourhood $U \subseteq X$ of x such that $f(U) \subseteq V$ is a bounded subset.*

ii.) *A function $f\colon X \longrightarrow V$ is called bounded if $f(X) \in \mathscr{B}$.*

iii.) *The set of locally bounded functions is denoted by*

$$\mathscr{B}_{\mathrm{loc}}(X, V) = \big\{ f\colon X \longrightarrow V \bigm| f \textit{ is locally bounded} \big\}. \tag{4.2.1}$$

iv.) *The set of bounded functions is denoted by*

$$\mathscr{B}(X, V) = \big\{ f\colon X \longrightarrow V \bigm| f \textit{ is bounded} \big\}. \tag{4.2.2}$$

In this generality, not much can be said beside the fact that both sets $\mathscr{B}(X, V) \subseteq \mathscr{B}_{\mathrm{loc}}(X, V) \subseteq \operatorname{Map}(X, V)$ are vector spaces for the pointwise operations:

Proposition 4.2.2 *Let X be a topological space and let V be a vector space with vector bornology $\mathscr{B}$. Then*

$$\mathscr{B}(X, V) \subseteq \mathscr{B}_{\mathrm{loc}}(X, V) \subseteq \operatorname{Map}(X, V) \tag{4.2.3}$$

are subspaces of the space of all maps with respect to the usual pointwise vector space operations.

PROOF: Indeed, this is fairly obvious. For $f \in \mathscr{B}_{\mathrm{loc}}(X, V)$ and $z \in \mathbb{K}$ it is easy to see that zf is locally bounded again since for $f(U) \in \mathscr{B}$ we have $(zf)(U) = zf(U) \in \mathscr{B}$ as $\mathscr{B}$ is a vector bornology. Similarly, if $f, g \in \mathscr{B}_{\mathrm{loc}}(X, V)$ and $x \in X$ we find neighbourhoods $U, \widetilde{U} \subseteq X$ of x with $f(U), g(\widetilde{U}) \in \mathscr{B}$. But then $U \cap \widetilde{U}$ is still a neighbourhood of x and $(f+g)(U \cap \widetilde{U}) \subseteq f(U \cap \widetilde{U}) + g(U \cap \widetilde{U}) \subseteq f(U) + g(\widetilde{U}) \in \mathscr{B}$, since $\mathscr{B}$ is a vector bornology. The case of bounded functions is analogous. □

The construction of $\mathscr{B}_{\mathrm{loc}}(X, V)$ and $\mathscr{B}(X, V)$ is functorial for continuous maps between the domain and bounded linear maps between the targets:

Proposition 4.2.3 *Let X, Y be topological spaces and let V, W be vector spaces with vector bornologies.*

i.) *For every continuous map $\Phi\colon X \longrightarrow Y$ the pull-back yields a linear map*

$$\Phi^*\colon \mathscr{B}_{\mathrm{loc}}(Y, V) \longrightarrow \mathscr{B}_{\mathrm{loc}}(X, V), \tag{4.2.4}$$

restricting to a linear map

$$\Phi^*\colon \mathscr{B}(Y, V) \longrightarrow \mathscr{B}(X, V). \tag{4.2.5}$$

ii.) *For every bounded linear map $A\colon V \longrightarrow W$ the composition with A yields a linear map*

$$A\colon \mathscr{B}_{\mathrm{loc}}(X, V) \longrightarrow \mathscr{B}_{\mathrm{loc}}(X, W), \tag{4.2.6}$$

restricting to a linear map

$$A\colon \mathscr{B}(X, V) \longrightarrow \mathscr{B}(X, W). \tag{4.2.7}$$

PROOF: Let $x \in X$ then we have for $f \in \mathscr{B}_{\mathrm{loc}}(Y, V)$ an open neighbourhood $U \subseteq Y$ such that $f(U) \subseteq V$ is bounded. Since $\Phi^{-1}(U)$ is an open neighbourhood of x with $\Phi(\Phi^{-1}(U)) \subseteq U$ we get that $(\Phi^* f)(\Phi^{-1}(U)) \subseteq V$ is bounded. This shows (4.2.4) and (4.2.5) is even simpler. Next, let $x \in X$ and $f \in \mathscr{B}_{\mathrm{loc}}(X, V)$ be given. Let $U \subseteq X$ be an open neighbourhood of x such that $f(U) \subseteq V$ is bounded. If A is a bounded map then $A(f(U)) \subseteq W$ is bounded again, proving (4.2.6). Again, (4.2.7) is clear. □

Thus we obtain contravariant functors

$$\mathscr{B}_{\mathrm{loc}}(\,\cdot\,, V)\colon \mathsf{top} \longrightarrow \mathsf{Vect}, \tag{4.2.8}$$

and

$$\mathscr{B}(\,\cdot\,, V)\colon \mathsf{top} \longrightarrow \mathsf{Vect}. \tag{4.2.9}$$

for every vector space V with vector bornology. Similarly, we have covariant functors

$$\mathscr{B}_{\mathrm{loc}}(X, \,\cdot\,)\colon \mathsf{Vectborn} \longrightarrow \mathsf{Vect}, \tag{4.2.10}$$

and

$$\mathscr{B}(X, \,\cdot\,)\colon \mathsf{Vectborn} \longrightarrow \mathsf{Vect}. \tag{4.2.11}$$

Of course, the to be checked functoriality is obvious in all cases.

We specialize now in two directions: first we assume X to be Hausdorff and locally compact in order to have a basis of compact neighbourhoods for all points. Second, we focus on the case where V is a (Hausdorff) locally convex space with the bornology being the canonical one. The first observation is that we can test for (local) boundedness in a scalar way as follows:

Proposition 4.2.4 *Let X be a topological space and let V be a Hausdorff locally convex space. Then a function $f\colon X \longrightarrow V$ is (locally) bounded iff for every $\phi \in V'$ the scalar function $\phi \circ f\colon X \longrightarrow \mathbb{K}$ is (locally) bounded.*

PROOF: Indeed, the canonical bornology of V is the same for the weak topology by Theorem 3.2.13.□

We want to endow $\mathscr{B}_{\mathrm{loc}}(X, V)$ and $\mathscr{B}(X, V)$ now with locally convex topologies. Analogously to the local supremum norms form the scalar case, see Definition 2.3.12, we define seminorms $\mathrm{p}_{K,q}$ on $\mathscr{B}_{\mathrm{loc}}(X, V)$ by

$$\mathrm{p}_{K,q}(f) = \sup_{x \in K} \mathrm{q}(f(x)), \tag{4.2.12}$$

where $K \subseteq X$ is compact and q is a continuous seminorm on V. The first observation is that (4.2.12) is actually a well-defined seminorm:

Lemma 4.2.5 *Let X be a locally compact Hausdorff space and let V be a locally convex space. For a function $f\colon X \longrightarrow V$ one has $f \in \mathscr{B}_{\mathrm{loc}}(X, V)$ iff $\mathrm{p}_{K,q} < \infty$ for all compact subsets $K \subseteq X$ and all continuous seminorms* q. *Moreover, this case, $\mathrm{p}_{K,q}$ is a seminorm on $\mathscr{B}_{\mathrm{loc}}(X, V)$.*

PROOF: Suppose $f \in \mathscr{B}_{\mathrm{loc}}(X, V)$ and let $K \subseteq X$ be compact. For $x \in K$ we have an open neighbourhood $U_x \subseteq X$ of x with $f(U_x) \subseteq V$ being bounded. By compactness, finitely many $x_1, \ldots, x_n \in K$ suffice to have a cover $U_{x_1} \cup \cdots \cup U_{x_n} \supseteq K$. Moreover, $f(U_k)$ is bounded iff for all continuous seminorms q on V one has a constant $c_x > 0$ with

$$\mathrm{q}(f(y)) \leq c_k$$

for all $y \in U_k$. Together, this gives

$$\mathrm{q}(f(y)) \leq c_{x_1} + \cdots + c_{x_n} < \infty$$

for all $y \in K$. Conversely, if $\mathrm{p}_{K,\mathrm{q}}(f) < \infty$ for all continuous seminorms q then $f(K) \subseteq V$ is bounded. Since X is locally compact and Hausdorff every $x \in X$ has a compact neighbourhood $K \subseteq X$. On this neighbourhood, f is bounded, showing $f \in \mathscr{B}_{\mathrm{loc}}(X, V)$. □

Note that for a Hausdorff locally convex space V the previous Proposition 4.2.4 shows that $f\colon X \longrightarrow V$ is locally bounded iff for all $\varphi \in V'$ and all compact $K \subseteq X$ one has

$$\mathrm{p}_{K,\mathrm{p}_\varphi}(f) = \sup_{x\in K} |\varphi(f(x))| < \infty \tag{4.2.13}$$

This is sometimes very convenient for testing. Note that in general there are much more continuous seminorms q on V than the ones obtained from the p_φ with $\varphi \in V'$: The latter specify the weak topology.

We can use now either the seminorms $\mathrm{p}_{K,\mathrm{p}_\varphi}$ from (4.2.13) or all seminorms $\mathrm{p}_{K,\mathrm{q}}$ from (4.2.12) to specify a locally convex topology on $\mathscr{B}_{\mathrm{loc}}(X, V)$. Since V is typically not complete with respect to the weak topology we can not expect much completeness for $\mathscr{B}_{\mathrm{loc}}(X, V)$ for this choice. Hence one opts for the finer topology:

Definition 4.2.6 ($\mathscr{B}_{\mathrm{loc}}$-Topology and $\mathscr{B}$-topology) *Let X be a locally compact Hausdorff space and let V be a locally convex space.*

i.) The $\mathscr{B}_{\mathrm{loc}}$-topology on $\mathscr{B}_{\mathrm{loc}}(X, V)$ is the locally convex topology induced by all the seminorms $\mathrm{p}_{K,\mathrm{q}}$ where $K \subseteq X$ is compact and q is a continuous seminorm on V.

ii.) The $\mathscr{B}$-topology on $\mathscr{B}(X, V)$ is the locally convex topology induced by all the seminorms

$$\mathrm{q}_\infty(f) = \sup_{x\in X} \mathrm{q}(f(x)) \tag{4.2.14}$$

where q is a continuous seminorm on V.

It is clear that q_∞ is a well-defined seminorm on the bounded functions and, in fact, $f\colon X \longrightarrow V$ is bounded iff $\mathrm{q}_\infty(f) < \infty$ for all continuous seminorms q on V, see also Exercise 4.5.5.

One also calls the $\mathscr{B}_{\mathrm{loc}}$-topology the locally uniform topology since convergence with respect to the $\mathscr{B}_{\mathrm{loc}}$-topology simply means uniform convergence with respect to the continuous seminorms of V on all compact subsets. Similarly, the $\mathscr{B}$-topology is referred to as the topology of uniform convergence. Both topologies are finer than the topology of pointwise convergence, see also Exercise 4.5.6.

The following statement shows that the choice for the locally convex topologies on $\mathscr{B}(X, V)$ and $\mathscr{B}_{\mathrm{loc}}(X, V)$ was reasonable in so far as completeness properties of V are inherited by $\mathscr{B}(X, V)$ and $\mathscr{B}_{\mathrm{loc}}(X, V)$ respectively:

Proposition 4.2.7 *Let X be a locally compact Hausdorff space and let V be a (sequentially) complete locally convex space. The $\mathscr{B}(X, V)$ and $\mathscr{B}_{\mathrm{loc}}(X, V)$ are (sequentially) complete, too.*

Proof: First we note that $\mathscr{B}_{\mathrm{loc}}(X, V)$ is Hausdorff whenever V is Hausdorff. Indeed, if $f \neq 0$ then there is a point $x \in X$ with $f(x) \neq 0$ and thus $\mathrm{p}_{\{x\},\mathrm{q}}(f) = \mathrm{q}(f(x)) \neq 0$ for a suitable continuous seminorm q on V. Now assume that $(f_i)_{i\in I}$ is a Cauchy net in V. Thus completeness of V guarantees that

$$f(x) = \lim_{i\in I} f_i(x)$$

gives a pointwise limit $f\colon X \longrightarrow V$. From here we can proceed literally as in the proof of Proposition 2.3.13 by replacing the absolute value with a continuous seminorm q of V. The case of a sequentially complete V is analogous: We can use the same arguments for a sequence $(f_n)_{n\in\mathbb{N}}$ instead of a general net. Finally, the statements for $\mathscr{B}(X, V)$ are obtained the same way, see also Proposition 2.3.11 and Proposition 2.3.3. We only have to repeat those arguments for every continuous seminorm q on V instead of the absolute value. □

The spaces $\mathscr{B}(X, V)$ and $\mathscr{B}_{\mathrm{loc}}(X, V)$ inherit also other properties from V, see in particular Exercise 4.5.7 and Exercise 4.5.8.

Finally, the functors from Proposition 4.2.3, *i.)*, are now compatible with the $\mathscr{B}_{\mathrm{loc}}$-topology or $\mathscr{B}$-topology. One has the following result:

Proposition 4.2.8 *Let X, Y be locally compact Hausdorff spaces and let V, W be Hausdorff locally convex spaces.*

i.) Every continuous map $\Phi\colon X \longrightarrow Y$ induces a continuous linear map

$$\Phi^*\colon \mathscr{C}(Y,V) \longrightarrow \mathscr{C}(X,V), \tag{4.2.15}$$

restricting to a continuous linear map

$$\Phi^*\colon \mathscr{C}_{\mathrm{b}}(Y,V) \longrightarrow \mathscr{C}_{\mathrm{b}}(X,V). \tag{4.2.16}$$

ii.) The composition with a continuous linear map $A\colon V \longrightarrow W$ yields a continuous linear map

$$A\colon \mathscr{C}(X,V) \longrightarrow \mathscr{C}(X,W), \tag{4.2.17}$$

restricting to a continuous linear map

$$A\colon \mathscr{C}_{\mathrm{b}}(X,V) \longrightarrow \mathscr{C}_{\mathrm{b}}(X,W). \tag{4.2.18}$$

PROOF: First it is clear that for $f \in \mathscr{C}(Y,V)$ also $\Phi^*(f) = f \circ \Phi$ is continuous. Moreover, a pull-back is necessarily linear. We need to check continuity. Thus let $K \subseteq X$ be compact and let q be a continuous seminorm on V. Then for $f \in \mathscr{C}(Y,V)$ we have

$$\mathrm{q}_K(\Phi^* f) = \sup_{x\in K} \mathrm{q}(f(\Phi(x))) = \sup_{y\in\Phi(K)} \mathrm{q}(f(x)) = \mathrm{q}_{\Phi(K)}(f).$$

Since $\Phi(K) \subseteq Y$ is indeed compact, the continuity of (4.2.15) follows. For (4.2.16) we can take $K = X$ and get the estimate

$$\mathrm{q}_\infty(\Phi^* f) = \mathrm{q}_X(\Phi^* f) = \mathrm{q}_{\Phi(X)}(f) \le \mathrm{q}_Y(f) = \mathrm{q}_\infty(f),$$

which is the continuity of (4.2.16). For the second part we note that for a continuous seminorm q on W we find a continuous seminorm p on V with $\mathrm{q}(A(v)) \le \mathrm{p}(v)$ for all $v \in V$. Hence for $f \in \mathscr{C}(X,V)$ we get

$$\mathrm{q}(A(f(x))) \le \mathrm{p}(f(x))$$

for all $x \in X$ which gives $\mathrm{q}_K(A \circ f) \le \mathrm{p}_K(f)$ for all subsets $K \subseteq X$ at once. Taking K compact yields the continuity of (4.2.17), taking $K = X$ yields the continuity of (4.2.18). ☐

This results in a good bivariant functorial behaviour of $\mathscr{C}(\,\cdot\,,\,\cdot\,)$ and $\mathscr{C}_{\mathrm{b}}(\,\cdot\,,\,\cdot\,)$ analogously to the ones of $\mathscr{B}_{\mathrm{loc}}(\,\cdot\,,\,\cdot\,)$ and $\mathscr{B}(\,\cdot\,,\,\cdot\,)$ in Proposition 4.2.3, *i.)*.

4.2.2 Continuous Functions

Next we consider continuous functions. While for general topological spaces X we have a meaningful definition of continuous functions $f\colon X \longrightarrow V$ with values in a topological vector space, for locally compact Hausdorff spaces we get additional important features analogously to the scalar case.

Proposition 4.2.9 *Let X be a topological space and let V be a topological vector space. Then the continuous functions*

$$\mathscr{C}(X,V) \subseteq \mathrm{Map}(X,V) \tag{4.2.19}$$

form a subspace with respect to the pointwise operations.

PROOF: This is entirely standard. We know that constant functions are continuous and hence $0 \in \mathscr{C}(X, C)$. Moreover, the diagonal map $\Delta\colon X \longrightarrow X \times X$ is continuous in general. Thus for $f, g \in \mathscr{C}(X, V)$ we have for $z, w \in \mathbb{K}$

$$\begin{aligned}(zf + wg)(x) &= zf(x) + wg(x) \\ &= (+ \circ (z\,\cdot\, \times w\,\cdot\,) \circ (f \times g) \circ \Delta)(x),\end{aligned}$$

which is a composition of continuous maps $f \times g\colon X \times X \longrightarrow V \times V$ since the vector space operations $+$ and $z\,\cdot$ for $z \in \mathbb{K}$ are continuous. □

In general, a continuous functions needs not to be locally bounded. However, if X is a locally compact Hausdorff space this changes:

Proposition 4.2.10 *Let X be a locally compact Hausdorff space and let V be a Hausdorff locally convex space over $\mathbb{K}$.*

i.) The continuous functions $f\colon X \longrightarrow V$ are locally bounded.

ii.) The continuous functions form a closed subspace

$$\mathscr{C}(X, V) \subseteq \mathscr{B}_{\mathrm{loc}}(X, V) \tag{4.2.20}$$

iii.) If V is (sequentially) complete the continuous functions $\mathscr{C}(X, V)$ are (sequentially) complete, too, with respect to the inherited topology from (4.2.20).

PROOF: Again, we can proceed essentially as in the scalar case. If $x \in X$ is a point we find a compact neighbourhood $K \subseteq X$ of x. Then a continuous $f \in \mathscr{C}(X, V)$ yields a compact subset $f(K) \subseteq V$ which is necessarily bounded by Remark 2.1.33 *ii.)*. This shows $f \in \mathscr{B}_{\mathrm{loc}}(X, V)$. The second part follows analogously to the scalar case discussed in Proposition 1.2.12 *ii.)*, with the only modification that we have to replace the absolute value of scalar values by a continuous seminorm q on V of the vector values. Then the usual $\frac{\varepsilon}{2}$- argument still applies. Finally, the third part is then an immediate consequence of the corresponding completeness results of the ambient space $\mathscr{B}_{\mathrm{loc}}(X, V)$ from Proposition 4.2.7. □

Similarly, the bounded continuous functions

$$\mathscr{C}_{\mathrm{b}}(X, V) = \mathscr{B}(X, V) \cap \mathscr{C}(X, V) \tag{4.2.21}$$

form a closed subspace of the bounded functions $\mathscr{B}(X, V)$ which is (sequentially) complete if V is (sequentially) complete.

Definition 4.2.11 ($\mathscr{C}$-topology) *Let X be a locally compact Hausdorff space and let V be a Hausdorff locally convex space over $\mathbb{K}$. Then the $\mathscr{B}_{\mathrm{loc}}$-topology restricted to $\mathscr{C}(X, V)$ is called the $\mathscr{C}$-topology or the topology of locally uniform convergence. The $\mathscr{B}$-topology restricted to $\mathscr{C}_{\mathrm{b}}(X, V)$ is called the $\mathscr{C}_{\mathrm{b}}$-topology or the topology of uniform convergence.*

More explicitly, the relevant seminorms are given by those from 4.2.12, i.e. the $\mathscr{C}$-topology is the locally convex topology induced by the seminorms

$$\mathrm{q}_K(f) = \mathrm{p}_{K,q}(f) = \sup_{x \in K} \mathrm{q}(f(x)) \tag{4.2.22}$$

for compact subsets $K \subseteq X$ and continuous seminorms q on V. Again, an exhausting set of compact seminorms on V are already sufficient to specify the $\mathscr{C}$-topology, see also Exercise 4.5.9 for some simple consequences.

Similarly, for the $\mathscr{C}_{\mathrm{b}}$-topology we use the seminorms

$$\mathrm{q}_{\infty}(f) = \sup_{x \in X} \mathrm{q}(f(x)) \tag{4.2.23}$$

with continuous seminorms q on V. Again, a defining subset of continuous seminorms suffices to determine the $\mathscr{C}_{\mathrm{b}}$-topology. Note also that for the case $V = \mathbb{K}$ we reproduce the formerly defined scalar versions of the $\mathscr{C}$-topology and the $\mathscr{C}_{\mathrm{b}}$-topology, respectively.

We conclude this section with a remark on linear maps. If we take for X a topological vector space itself, then Corollary 2.1.40 shows that the only locally compact Hausdorff topological vector spaces X are given by $\mathbb{K}^n$ for $n \in \mathbb{N}$, up to isomorphism. In this case, Lemma 2.1.38 shows that linear maps $\mathbb{K}^n \to V$ are always continuous, no matter which topological vector space we take. This allows now to consider

$$\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, V) \subseteq \mathscr{C}(\mathbb{K}^n, V) \tag{4.2.24}$$

with the induced $\mathscr{C}$-topology. This turns $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, V)$ into a closed subspace and the $\mathscr{C}$-topology can be described in many different equivalent ways: As a first application of the $\mathscr{C}$-topology making use of the point of view of vector-valued functions we discuss a particular case of the exponential law for continuous maps between topological spaces, also referred to as currying. We restrict ourselves here to a situation which is technically easier than more general scenarios but still sufficient for our purposes:

Theorem 4.2.12 (Currying) *Let X, Y be locally compact Hausdorff spaces and let V be a Hausdorff locally convex space.*

i.) Let $f \in \mathscr{C}(X \times Y, V)$. Then the function $c(f)\colon X \to \mathscr{C}(Y, V)$, defined by

$$c(f)(x) = (y \mapsto f(x, y)) \tag{4.2.25}$$

is continuous.

ii.) The resulting map

$$c\colon \mathscr{C}(X \times Y, V) \longrightarrow \mathscr{C}(X, \mathscr{C}(Y, V)) \tag{4.2.26}$$

is an isomorphism of locally convex spaces.

Proof: First we note that the map $y \mapsto f(x, y)$ is continuous for every fixed $x \in X$. Hence $c(f) \in \mathrm{Map}(X, \mathscr{C}(Y, V))$ follows. First we need to check that $c(f)$ is actually continuous. Thus fix a compact subset $K_2 \subseteq Y$ and a continuous seminorm q on V with resulting seminorm q_{K_2} for $\mathscr{C}(Y, V)$. Moreover, let $x_0 \in X$ be given with a net $(x_i)_{i \in I}$ in X converging to x_0. Then we have to show

$$\mathrm{q}_{K_2}(c(f)(x_i) - c(f)(x_0)) \to 0.$$

Thus consider $y \in K_2$ and $\varepsilon > 0$. Since f is continuous at $(x_0, y) \in X \times Y$ we find open subsets $O_y \subseteq X$ and $U_y \subseteq Y$ such that $x_0 \in O_y$ and $y \in U_y$ with

$$\mathrm{q}\big(f(x', y') - f(x_0, y)\big) < \varepsilon \tag{$*$}$$

whenever $(x', y') \in O_y \times U_y$. Finitely many of the U_y cover the compact subset K_2, say

$$K_2 \subseteq U_{y_1} \cup \cdots \cup U_{y_n}$$

Define now the open neighbourhood

$$O = O_{y_1} \cap \cdots \cap O_{y_n}$$

of x_0. Since this is an open neighbourhood of x_0 we find an index $i_0 \in I$ with $x_i \in O$ for $i \succeq i_0$ by convergence of $(x_i)_{i \in I}$ to x_0. For such i and all $y \in K_2$ we have $(x_i, y) \in O \times U_{y_\ell} \subseteq O_{y_\ell} \times U_{y_\ell}$ for at least one index $\ell = 1, \ldots, n$. Thus $(*)$ applies and gives

$$q(f(x_i, y) - f(x_0, y)) < \varepsilon.$$

Taking the supremum over y yields

$$q_{K_2}(f(x_i, \cdot\,) - f(x_0, \cdot\,)) \leq \varepsilon$$

and hence the convergence of $f(x_i, \cdot\,) \to f(x_0, \cdot\,)$ in the $\mathscr{C}$-topology of $\mathscr{C}(Y, V)$ follows. This is the continuity of $c(f)$ required for the first part. Now consider a function $F \in \mathscr{C}(X, \mathscr{C}(Y, V))$. The obvious candidate for the pre-image of F is

$$c^{-1}(F)(x, y) = (F(x))(y).$$

We first need to show that $c^{-1}(F)$ is continuous. Thus let $(x_i, y_i)_{i \in I}$ be a net in $X \times Y$ converging to (x_0, y_0). Let q be a continuous seminorm on V. Then we fix a compact neighbourhood of y_0 in Y, which is possible as Y is locally compact. Then we have an $i_0 \in I$ with $i \succeq i_0$ implying $y_i \in K_2$. For such indices i we get

$$\begin{aligned}
\mathrm{q}\big(c^{-1}(F)(x_i, y_i) - c^{-1}(F)(x_0, y_0)\big) &= \mathrm{q}(F(x_i)(y_i) - F(x_0)(y_0)) \\
&\leq \mathrm{q}(F(x_i)(y_i) - F(x_0)(y_0)) + \mathrm{q}(F(x_0)(y_i) - F(x_0)(y_0)) \\
&\leq \mathrm{q}_{K_2}(F(x_i) - F(x_0)) + \mathrm{q}(F(x_0)(y_i) - F(x_0)(y_0)).
\end{aligned}$$

Now the first contribution becomes small for late i since F is continuous and $x_i \to x_0$. The second contribution becomes small for late i since $F(x_0) \in \mathscr{C}(Y, V)$ is continuous and $y_i \to y_0$. In total, this shows that $c^{-1}(F)$ is a continuous function. This establishes a linear isomorphism c in (4.2.26).

It remains to show that under this isomorphism the two locally convex topologies coincide. The compact subsets of $X \times Y$ are contained in factorising compact subsets $K_1 \times K_2$ with $K_1 \subseteq X$ and $K_2 \subseteq Y$ both compact. Thus a defining system of seminorms for $\mathscr{C}(X \times Y, V)$ is given by

$$\mathrm{q}_{K_1 \times K_2}(f) = \sup_{(x,y) \in K_1 \times K_2} \mathrm{q}(f(x, y))$$

where q runs through the continuous seminorms of V. The $\mathscr{C}$-topology of $\mathscr{C}(Y, V)$ is determined by the seminorms q_{K_2} and hence the $\mathscr{C}$-topology of $\mathscr{C}(X, \mathscr{C}(Y, V))$ is determined by the seminorms q_{K_1}. Now

$$\begin{aligned}
(\mathrm{q}_{K_2})_{K_1}(c(f)) &= \sup_{x \in K_1} \mathrm{q}_{K_2}(c(f)(x)) \\
&= \sup_{x \in K_1} \sup_{y \in K_2} \mathrm{q}(c(f)(x)(y)) \\
&= \sup_{(x,y) \in K_1 \times K_2} (q(f(x, y))) \\
&= \mathrm{q}_{K_1 \times K_2}(f).
\end{aligned}$$

Thus the two systems of seminorms simply coincide under the identification c. $\square$

The map c is called the currying map. If one forgets about continuity but considers just maps between sets, one has

$$\operatorname{Map}(X, \operatorname{Map}(Y, Z)) \cong \operatorname{Map}(X \times Y, Z). \tag{4.2.27}$$

Writing $Z^Y = \operatorname{Map}(Y, Z)$ as common in set-theory, 4.2.27 becomes

$$\left(Z^Y\right)^X \cong Z^{X \times Y}. \tag{4.2.28}$$

This explains the name "exponential law" for the isomorphism (4.2.27). One has many other scenarios where an exponential law holds. Note that in the previous proof it was crucial that Y is locally compact to get the continuity of $c^{-1}(F)$. In fact, for general topological spaces X, Y and Z this is the step which can fail, see e.q. the discussion in $[\cdots]$

Consider the seminorms for $f \in \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, V)$ defined by

$$\|f\|_{\mathrm{q},\infty} = \max_{i=1,\ldots,n} \mathrm{q}(f(e_i)), \tag{4.2.29}$$

$$\|f\|_{\mathrm{q},p} = \sqrt[p]{\sum_{i=1}^{n} q(f(e_i))^p}, \tag{4.2.30}$$

$$\|f\|_q = \sup_{u \neq 0} \frac{q(f(u))}{\|u\|}, \tag{4.2.31}$$

for a continuous seminorm q on V, where $e_1, \ldots, e_n \in \mathbb{K}^n$ is a basis, $p \in [1, \infty)$, and $\|\cdot\|$ is an arbitrary auxilliary norm on $\mathbb{K}^n$. Clearly, these define seminorms as a simple verification shows.

Proposition 4.2.13 *Let $n \in \mathbb{N}$ and let V be a Hausdorff locally convex space. Moreover, let $e_1, \ldots, e_n \in \mathbb{K}^n$ be a basis, let $p \in [1, \infty]$, and let $\|\cdot\|$ be an auxilliary norm on $\mathbb{K}^n$. Then the systems of seminorms $\{\|\cdot\|_{\mathrm{q},\infty}\}_{\mathrm{q}\,\text{continuous}}$, $\{\|\cdot\|_{\mathrm{q},p}\}_{\mathrm{q}\,\text{continuous}}$, and $\{\|\cdot\|_{\mathrm{q}}\}_{\mathrm{q}\,\text{continuous}}$, where q runs through all continuous seminorms on V, induce the $\mathscr{C}$-topology on $\operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^n, V) \subseteq \mathscr{C}(\mathbb{K}^n, V)$. Moreover, $\operatorname{Hom}(\mathbb{K}^n, V)$ is a closed subspace of $\mathscr{C}(\mathbb{K}^n, V)$.*

PROOF: The proof consists in elementary mutual estimates of the three kinds of seminorms and the comparison with the seminorms $\mathrm{p}_{K,\mathrm{q}}$ of the $\mathscr{C}$-topology, see Exercise 4.5.10. ☐

4.2.3 Lipschitz Continuous Functions

Recall that a map $f \colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called a Lipschitz map if there exists a constant $c > 0$ sucht that

$$d_Y(f(x), f(x')) \leq c\, d_X(x, x'). \tag{4.2.32}$$

In this case c is called a Lipschitz constant for f. Clearly, such a map is necessarily continuous but not all continuous maps are Lipschitz. Thus being Lipschitz is not just a topological concept. Perhaps surprising is the fact that it is not entirely metric either. Instead, we only need the metric on X but on Y a bornology would suffice to define a Lipschitz map. The idea is that (4.2.32) is equivalent to

$$\sup_{x \neq x'} \frac{d_Y(f(x), f(x'))}{d_X(x, x')} < \infty, \tag{4.2.33}$$

which can be re-interpreted that the distances of $f(x)$ and $f(x')$ are bounded even after dividing by the distance of x and x'. This allows to formulate the following definition:

Definition 4.2.14 (Lipschitz functions) *Let (X, d_X) be a metric space and let V be a vector space with vector bornology $\mathscr{B}$.*

i.) *A function $f\colon X \to V$ is called locally Lipschitz if for every $x \in X$ there is a neighbourhood $U \subseteq X$ of x such that the set*

$$\left\{ \frac{f(x) - f(y)}{d_X(x, y)} \big| x, y \in U, x \neq y \right\} \subseteq V \tag{4.2.34}$$

is bounded.

ii.) *A function $f\colon X \to V$ is called Lipschitz if it satisfies* (4.2.34) *with $U = X$.*

iii.) *The set of locally Lipschitz functions from X to V is denoted by*

$$\mathscr{Lip}(X, V) = \left\{ f\colon X \to V \big| f \text{ is locally Lipschitz} \right\} \tag{4.2.35}$$

As in the (locally) bounded case one can prove that $\mathscr{Lip}(X, V)$ is a subspace of all maps $\operatorname{Map}(X, V)$.

Proposition 4.2.15 *Let (X, d) be a metric space and let V be a vector space over $\mathbb{K}$ equipped with a vector bornology $\mathscr{B}$. Then $\mathscr{Lip}(X, V) \subseteq \operatorname{Map}(X, V)$ is a subspace containing all constant maps.*

Proof: Clearly, the constant maps are locally Lipschitz and, in fact, even globally Lipschitz. In particular, $0 \in \mathscr{Lip}(X, V)$. Let $f, g \in \mathscr{Lip}(X, V)$ be given and let $x_0 \in X$. Then we find neighbourhoods $U, \widetilde{U} \subseteq X$ of x_0 with bounded subsets $B_1, B_2 \in \mathscr{B}$ such that

$$\frac{f(x) - f(y)}{d(x, y)} \in B_1$$

and

$$\frac{g(x) - g(y)}{d(x, y)} \in B_2$$

for $x_0 \neq y$ and for $x, y \in U$ and$x, y \in \widetilde{U}$, respectively. But then we get for $x, y \in U \cap \widetilde{U}$ with $x \neq y$ the property

$$\frac{(zf + wg)(x) - (zf + wg)(y)}{d(x, y)} = \frac{zf(x) - zf(y)}{d(x, y)} + \frac{wg(x) - wg(y)}{d(x, y)} \in zB_1 + wB_2$$

for $z, w \in \mathbb{K}$.

Since $U \cap \widetilde{U}$ is still a neighbourhood of x_0 and since the vector bornology $zB_1 + wB_2$ is still bounded, this shows $zf + wg \in \mathscr{Lip}(X, V)$. □

Again, without any further assumption one can not say much interesting about $\mathscr{Lip}(X, V)$. This changes when the bornology on V is the canonical bornology of a Hausdorff locally convex topology on V. One reason can be seen that then the Lipschitz condition can be tested with continuous linear functionals, see also Proposition 4.2.3 for the analogous statement on local boundedness:

Proposition 4.2.16 *Let X be a metric space and let V be a Hausdorff locally convex space. Then one has*

$$\mathscr{Lip}(X, V) = \left\{ f\colon X \longrightarrow V \big| \phi \circ f \in \mathscr{Lip}(X, \mathbb{K}) \text{ for all } \varphi \in V' \right\} \tag{4.2.36}$$

Proof: A subset $B \subseteq V$ is bounded iff it is weakly bounded by Theorem 3.2.13. For any subset $U \subseteq X$ and

$$B = \left\{ \frac{1}{d(x, y)} (f(x) - f(y)) \big| x, y \in U, x \neq y \right\}$$

we have for φ in V'

$$\phi(B) = \left\{ \frac{1}{d(x, y)} ((\phi \circ f)(x) - (\varphi \circ f)(y)) \big| x, y \in U, x \neq y \right\} \subseteq \mathbb{K}.$$

Thus the local Lipschitz conditions on f transfer directly to the local Lipschitz conditions for $\varphi \circ f$ and vice versa. □

Corollary 4.2.17 *Let X be a metric space and let V be a Hausdorff locally convex space.*

i.) For every compact subset $K \subseteq X$ one has for $f \in \mathscr{L}ip(X, V)$ the property that

$$B = \left\{ \frac{1}{d(x,y)} (f(x) - f(y)) \big| x, y \in K, x \neq y \right\} \tag{4.2.37}$$

is bounded.

ii.) If in addition X is locally compact then $f \in \mathscr{L}ip(X, V)$ iff (4.2.37) *is bounded for all compact $K \subseteq V$.*

PROOF: Suppose $f \in \mathscr{L}ip(X, V)$ the for every $\phi in V'$ the scalar function $\phi \circ f \colon X \longrightarrow \mathbb{K}$ is locally Lipschitz. For scalar Lipschitz functions it is well-known that they satisfy a Lipschitz condition on every compact subset, see also Exercise 4.5.12. But this means that the subset $\phi(B)$ is a bounded subset of $\mathbb{K}$ for all $\phi \in V'$, hence bounded by Theorem 3.2.13 again. This shows the first statement. For the second, assume that B as in (4.2.37) is bounded for every compact K. Since every point $x \in X$ has a compact neighbourhood, we get the local Lipschitz property at once. The converse is covered by part *i.)*. □

Since we can test the boundedness of subsets on a Hausdorff locally convex space V by seminorms, the corollary suggests to look at the following seminorms

$$\widetilde{\mathrm{p}}_{K,q}(f) = \sup \left\{ \frac{1}{d(x,y)} \mathrm{q}(f(x) - f(y)) \big| x, y \in K, x \neq y \right\} \tag{4.2.38}$$

to test for the local Lipschitz condition. the difficulty with the collection of these seminorms alone is that they can not yet induce a Hausdorff topology: We clearly have

$$\widetilde{\mathrm{p}}_{K,q}(v) = 0 \tag{4.2.39}$$

for every constant function $v \in V$ and any compact $K \subseteq X$ and continuous seminorm q on V for trivial reasons. Thus we have to add artificially seminorms to get a Hausdorff system of seminorms for $\mathscr{L}ip(X, V)$. The easiest way to achieve this is to use the pointwise evaluations.

Definition 4.2.18 (Lip-topology) *Let X be a locally compact metric space and let V be a Hausdorff locally convex space. Then the $\mathscr{L}ip$-topology is the locally convex topology induced by the seminorms $\widetilde{\mathrm{p}}_{K,q}$ for all compact $K \subseteq X$ and all continuous seminorms q together with the pointwise seminorms q_x for all continuous seminorms q on V and $x \in X$, where*

$$\mathrm{q}_x(f) = \mathrm{q}(f(x)). \tag{4.2.40}$$

Proposition 4.2.19 *Let X be a locally compact metric space and let V be a Hausdorff locally convex space.*

i.) Every locally Lipschitz function $f \colon X \longrightarrow V$ is continuous.

ii.) The inclusion

$$\mathscr{L}ip(X, V) \subseteq \mathscr{C}(X, V) \tag{4.2.41}$$

is continuous.

iii.) If V is (sequentially) complete then $\mathscr{L}ip(X, V)$ is (sequentially) complete.

PROOF: Let $x_0 \in X$ be given and assume $f \in \mathscr{L}ip(X, V)$ is locally Lipschitz. Moreover, let q be a continuous seminorm on V. If $(x_n)_{n \in \mathbb{N}}$ is a sequence converging to x_0 then

$$\mathrm{q}(f(x_n) - f(x_0)) \leq \widetilde{\mathrm{p}}_{K,\mathrm{q}}(f) d(x_n, x_0) \tag{$*$}$$

for every compact subset $K \subseteq X$ with $x_n \in K$. Since X is locally compact, there is a $\varepsilon > 0$ such that $K = B_\varepsilon(x_0)^{\mathrm{cl}}$ is compact. Moreover, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in B_\varepsilon(x_0)$ by convergence. Thus $(*)$ holds for this compact subset and all $n \geq N$. As then $d(x_n, x_0) \longrightarrow 0$ for $n \longrightarrow \infty$ we get convergence $\mathrm{q}(f(x_n) - f(x_0)) \longrightarrow 0$. Since this holds for all seminorms of V, we have convergence $f(x_n) \longrightarrow f(x_0)$ showing sequential continuity of f. As a metric space is first countable, sequential continuity is the same as continuity, finally proving $f \in \mathscr{C}(X, V)$. For the second part, let $K \subseteq X$ be compact and let q be a continuous seminorm on V. Let $y, x \in K$. Then we know

$$\mathrm{q}(f(x) - f(y)) \leq \widetilde{\mathrm{p}}_{K,\mathrm{q}}(f) d(x, y) \tag{$**$}$$

from the definition of the seminorm $\widetilde{\mathrm{p}}(f)$ as in (4.2.38). Now

$$\begin{aligned}
\mathrm{q}(f(y)) &\leq \mathrm{q}(f(x) - f(x) + f(y)) \\
&\leq \mathrm{q}(f(x)) + \mathrm{q}(f(x) - f(x)) \\
&\leq \mathrm{q}(f(x)) + \widetilde{\mathrm{p}}_{K,\mathrm{q}}(f) d(x, y) \\
&\leq \mathrm{q}_x(f) + \operatorname{diam}(K) \widetilde{\mathrm{p}}_{K,\mathrm{q}}(f),
\end{aligned}$$

where $\operatorname{diam}(K)$ is the diameter of the compact subset

$$\operatorname{diam}(K) = \sup_{x,y \in K} d(x, y) < \infty.$$

Taking now the supremum over $y \in K$ gives the estimate

$$\mathrm{q}_K(f) \leq \mathrm{q}_x(f) + \operatorname{diam}(K) \widetilde{\mathrm{p}}_{K,\mathrm{q}}(f),$$

thus proving the continuity of the inclusion. In particular, the $\mathscr{L}ip$-topology is finer as the induced $\mathscr{C}$- topology and hence Hausdorff, too. For the third part, assume that $(f_i)_{i \in I}$ is a Cauchy net in $\mathscr{L}ip(X, V)$. Then it is also a Cauchy net in $\mathscr{C}(X, V)$, hence convergent to some $f \in \mathscr{C}(X, V)$ if V is complete. As the $\mathscr{C}$-topology is finer as the topology of pointwise convergence, $(f_i)_{i \in I}$ converges also with respect to the seminorms q_x of the pointwise topology. Now let $K \subseteq X$ be compact and let q be a continuous seminorm. For $\varepsilon > 0$ we find an index $i_0 \in I$ with

$$\widetilde{\mathrm{p}}_{K,\mathrm{q}}(f_i - f_j) < \varepsilon \quad \text{for } i, j \succeq i_0,$$

and thus for $x, y \in K$ with $x \neq y$

$$\frac{1}{d(x, y)} \mathrm{q}(f_i(x) - f_j(x) - f_i(y) + f_j(y) < \varepsilon.)$$

Since $f_i(x) \longrightarrow f(x)$ and $f_i(y) \longrightarrow f(y)$ by pointwise convergence, we get the inequality

$$\frac{1}{d(x, y)} \mathrm{q}(f(x) - f_j(x) - f(y) + f_j(y) \leq \varepsilon.)$$

in the limit. But this means $\widetilde{\mathrm{p}}_{K,\mathrm{q}}(f - f_j) \leq \varepsilon$ and hence

$$\widetilde{\mathrm{p}}_{K,\mathrm{q}}(f) \leq \widetilde{\mathrm{p}}_{K,\mathrm{q}}(f - f_j) + \widetilde{\mathrm{p}}_{K,\mathrm{q}}(f_j) \leq \varepsilon + \widetilde{\mathrm{p}}_{K,\mathrm{q}}(f_j)$$

as soon as $j \succeq i_0$. Now this implies then on one hand $\widetilde{\mathrm{p}}_{K,\mathrm{q}}(f) < \infty$ and hence $f \in \mathscr{L}ip(X, V)$. On the other hand we conclude from $\widetilde{\mathrm{p}}_{K,\mathrm{q}}(f - f_j) < \varepsilon$ that the net $(f_i)_{i \in I}$ converges to F in the $\mathscr{L}ip$-topology. The sequence case is analogous. □

Beside for very trivial situations the inclusion (4.2.41) is strict and the $\mathscr{L}ip$-topology is strictly finer than the $\mathscr{C}$-topology. Further properties of locally Lipschitz functions are discussed in Exercise 4.5.13.

There is a fundamental difference between locally Lipschitz functions and continuous functions which results from Proposition 4.2.16. The local Lipschitz condition can be tested by continuous linear functionals while this is not true for continuity: In general one has weakly continuous maps $f\colon X \longrightarrow V$ which are **not** continuous for the original and thus typically **finer** topology of V. This has several important implications. Some of them are fundamental for the development of the **convenient calculus**, a version of calculus in infinite dimensions based on convenient locally convex vector spaces, see e.q. [9].

Related to Lipschitz continuity is uniform continuity. While there is a general definition of uniform continuity in the context of uniform spaces, see [...] , we confine ourselves to the metric situation, see also Definition 2.1.13:

Definition 4.2.20 (Uniform continuity) *Let X be a metric space and let V be a topological vector space. A function $f\colon X \longrightarrow V$ is called uniformly continuous if for ever zero neighbourhood $U \subseteq V$ there exists a $\delta > 0$ such that*

$$f(x) - f(y) \in U \quad \text{whenever } d(x, y) < \delta. \tag{4.2.42}$$

If X is a metrizable topological vector space it can be easily checked that this definition is consistent with our previous version of uniform continuity in Definition 2.1.13, see also Exercise 4.5.14. We collect some properties of uniformly continuous functions which are well-known in the scalar case $V = \mathbb{K}$ from elementary calculus. As before, we focus on a Hausdorff locally convex space V as target:

Proposition 4.2.21 *Let X be a metric space and let V be a Hausdorff locally convex space.*

i.) A function $f\colon X \longrightarrow V$ is uniformly continuous if for every continuous seminorm q on V one finds a $\delta > 0$ such that

$$\mathrm{q}(f(x) - f(y)) < 1 \quad \text{whenever } d(x, y) < \delta. \tag{4.2.43}$$

ii.) A uniformly continuous function $f\colon X \longrightarrow V$ is continuous.

iii.) If X is compact, a continuous function $f\colon X \longrightarrow V$ is uniformly continuous.

iv.) A globally Lipschitz continuous function $f\colon X \longrightarrow V$ is uniformly continuous.

Proof: The first part is clear as the open unit Balls $B_{\mathrm{q},1}(0)$ with respect to continuous seminorms form a basis of open neighbourhoods of 0. The uniform continuity directly gives the continuity. The third part is more interesting. Suppose a continuous function $f\colon X \longrightarrow V$ is not uniformly continuous. Thus we find a continuous seminorm q on V and points $x_n, x'_n \in X$ with $d(x_n, x'_n) < \frac{1}{n}$ but $\mathrm{q}(f(x_n) - f(x'_n)) \geq 1$ for all $n \in \mathbb{N}$. Since X is sequentially compact we find convergent subsequences $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{n}}$ and $(x'_{n_k})_{k\in\mathbb{N}}$ of $(x'_n)_{n\in\mathbb{n}}$ converging to x and x', respectively. Since f is continuous, we have $f(x_{n_k}) \longrightarrow f(x)$ and $f(x'_{n_k}) \longrightarrow f(x')$, respectively. However, $d(x_n, x'_n) < \frac{1}{n}$ shows that $x = x'$ and hence we get a contradiction to $\mathrm{q}(f(x_{n_k}) - f(x'_{n_k})) \geq 1$ for all $k \in \mathbb{N}$. Finally, assume f is globally Lipschitz, i.e. we have $\mathrm{p}_{X,\mathrm{q}} < \infty$ for a continuous seminorm q on V. This means that

$$\mathrm{q}(f(x) - f(y)) \leq \widetilde{\mathrm{p}}(f) d(x, y)$$

for all $x, y \in X$. Taking e.g. $\delta = \frac{1}{1+\widetilde{\mathrm{p}}}(f)$ will then do the job for (4.2.43). □

4.2.4 Compactly Supported Continuous Functions

We consider again a locally compact Hausdorff space X and a Hausdorff locally convex space V yielding the Hausdorff locally convex space $\mathscr{C}(X, V)$ as before. Then we can still define the support as in the scalar case using the distinguished element of V.

Definition 4.2.22 (Support) *Let X be a topological space and V a vector space. For a function $f\colon X \longrightarrow V$ one defines the support of f by*

$$\operatorname{supp}(f) = \{x \in X | f(x) \neq 0\}. \tag{4.2.44}$$

As usual, the definition becomes interesting as soon as we consider more particular functions with values in more particular vector spaces: For a locally compact Hausdorff space and a Hausdorff locally convex space X we define for a subset $A \subseteq X$ the subspaces

$$\mathscr{C}_A(X, v) = \{f \in \mathscr{C}(X, V) | \operatorname{supp}(f) \subseteq A\}. \tag{4.2.45}$$

Moreover, we define

$$\mathscr{C}_0(X, V) = \{f \in \mathscr{C}(X, V) | \operatorname{supp}(f) \text{ is compact}\}. \tag{4.2.46}$$

As usual, the most interesting situation is when A is closed as well since $\operatorname{supp}(f)$ is closed by definition. As for the scalar situation we obtain the following properties:

Proposition 4.2.23 *Let X be a locally compact Hausdorff space and let V be a Hausdorff locally convex space.*

i.) For every closed subset $A \subseteq X$ the subspace $\mathscr{C}_A(X, V) \subseteq \mathscr{C}(X, V)$ is closed.

ii.) For two closed subsets $A \subseteq B \subseteq X$ the inclusion

$$\mathscr{C}_A(X, V) \subseteq \mathscr{C}_B(X, V) \tag{4.2.47}$$

is an embedding with closed image.

iii.) The compactly supported functions $\mathscr{C}_0(X, V) \subseteq \mathscr{C}(X, V)$ are dense. If X is in addition second countable then $\mathscr{C}_0(X, V)$ is even sequentially dense in $\mathscr{C}(X, V)$.

PROOF: The arguments are identical to the scalar version $V = \mathbb{K}$ in Proposition 2.3.16. For the second part we note that the continuous seminorms of $\mathscr{C}_A(X, V)$ are restrictions of corresponding seminorms of $\mathscr{C}_B(X, V)$ since any compact $K \subseteq A$ is also compact in B. Thus we have an embedding. Moreover, the image is closed by *i.)* ☐

We can now copy the construction of the LF space $\mathscr{C}_0(X)$ from the scalar case to this vector-valued situation. For a general X and a general V we can not hope to obtain an LF space. Nevertheless, the inductive limit is fairly well-behaved: Since as a vector space we have

$$\mathscr{C}_0(X, V) = \bigcup_{\substack{K \subseteq X \\ K \text{ compact}}} \mathscr{C}_K(X, V), \tag{4.2.48}$$

we endow $\mathscr{C}_0(X, V)$ with the locally convex inductive limit topology arising from this inductive system

$$\mathscr{C}_K(X, V) \longrightarrow \mathscr{C}_{K'}(X, V) \quad \text{for } K \subseteq K' \tag{4.2.49}$$

i.e.

$$\mathscr{C}_0(X, V) = \varinjlim_{K \subset X \text{ compact}} \mathscr{C}_K(X, V). \tag{4.2.50}$$

Proposition 4.2.24 *Let X be a locally compact Hausdorff space and let V be a Hausdorff locally convex space.*

i.) The inductive system (4.2.49) *is strict with closed images.*

ii.) The locally convex inductive limit topology on $\mathscr{C}_0(X, V)$ is finer than the $\mathscr{C}$-topology inherited by $\mathscr{C}_0(X, V) \subseteq \mathscr{C}(X, V)$. In particular, $\mathscr{C}_0(X, V)$ is Hausdorff.

iii.) Suppose X is second countable and V is a Fréchet space. Then $\mathscr{C}_0(X, V)$ is an LF space.

PROOF: We already know that for all $K \subseteq K'$ the inclusion (4.2.50) is an embedding with closed image. This gives the first part at once. Now let $\widetilde{K} \subseteq X$ be a compact subset and let q be a continuous seminorm on V. Then the seminorm $\mathrm{q}_{\widetilde{K}}$ on $\mathscr{C}(X, V)$ yields for every $K \subseteq X$ the restriction

$$\mathrm{q}_{\widetilde{K}}\big|_{\mathscr{C}_K(X,V)} = \mathrm{q}_{\widetilde{K}\cap K}\big|_{\mathscr{C}_K(X,V)}.$$

In particular, $\widetilde{K} \cap K \subseteq K$ is compact and hence $\mathrm{q}_{\widetilde{K}\cap K}$ is a continuous seminorm for $\mathscr{C}_K(X, V)$. By Theorem 2.4.27, *ii.)*, this gives a continuous seminorm $\mathrm{q}_{\widetilde{K}}$ on the locally convex inductive limit $\mathscr{C}_0(X, V)$. This shows the second claim. Note however, that the locally convex inductive limit topology of $\mathscr{C}_0(X, V)$ will typically be **strictly** finer than the subspace topology inherited from $\mathscr{C}(X, V)$. For the last part, we first note that for every compact $K \subseteq X$ the subspace $\mathscr{C}_K(X, V) \cong \mathscr{C}(K, V)$ is a Fréchet space whenever V is a Fréchet space, see also Exercise 4.5.15. From here we can proceed as in Proposition 2.4.50 to show that any exhaustion $K_1 \subseteq K_1^\circ \subseteq K_2 \subseteq \ldots \subseteq X$ of X by a sequence of compact subsets yields the same locally convex inductive limit

$$\mathscr{C}_0(X, V) \cong \varinjlim_n (X, V).$$

Since in the second countable case, such a sequence always exists, we obtain indeed an LF space. □

As in the scalar case, we will refer to the locally convex inductive limit topology of $\mathscr{C}_0(X, V)$ as the **$\mathscr{C}_0$-topology**. Note that the third part of the proposition applies, in particular, to open subsets $X \subseteq \mathbb{R}^n$ or to topological manifolds: This gives yet another large class of LF spaces as we can now use all Fréchet spaces V as targets.

4.3 The Riemann Integral

One of the main techniques and challenges is the integration of vector-valued functions. Here one can take at least two points of view: One can try to incorporate fairly general functions, defined on fairly general domains X. This will ultimately require a measure-theoretic approach to integration. The principle obstacle one has to overcome is then that the decomposition of scalar functions in positive and negative parts, which is at the heart of the measure-theoretic definition of an integral, is no longer meaningful if the functions take values in an infinite-dimensional space. The best one can ask for is to decompose components into positive and negative parts, where components are understood as composition with elements from V'. Ultimately this leads to a "weak" definition of an integral suitable for many purposes but not suited for others. A more direct definition then requires substantially more effort. The second point of view is to consider fairly simple domains X and fairly nice functions. Surprisingly, this is powerful enough for many situations and, in fact, has several advantages compared to a measure-theoretic integral. Beside its simplicity it is the very mild requirement concerning completeness properties of the target space V which makes the Riemann integral attractive. In addition, many derived concepts like oscillatory integrals require such a Riemann integral instead of a measure-theoretic one.

4.3.1 Riemann Integrable Functions

As before, we consider a Hausdorff locally convex space V as target for the vector-valued functions we want to integrate. As domain we start with something very simple and somewhat extremely special, an n-dimensional compact interval with non-empty open interior

$$I = [a_1, b_1] \times \ldots \times [a_n, b_n] \subseteq \mathbb{R}^n, \tag{4.3.1}$$

where $a_1 < b_1, \ldots, a_n < b_n$. For such an interval the Euclidian volume is denoted by

$$\operatorname{vol}(I) = (b_1 - a_1) \ldots (b_n - a_n). \tag{4.3.2}$$

Of course, in the case $n = 1$ we speak of the length of the interval while in the case $n = 2$ we refer to $\operatorname{vol}(I)$ as its surface.

Since we consider **closed** intervals I in the following, the locally bounded functions are just the bounded ones, i.e. we have

$$\mathscr{B}_{\mathrm{loc}}(I, V) = \mathscr{B}(I, V). \tag{4.3.3}$$

This is also a topological equality: The $\mathscr{B}_{\mathrm{loc}}$-topology is induced by the seminorms

$$\mathrm{q}_\infty(f) = \mathrm{q}_I(f) = \sup_{x \in I} \mathrm{q}(f(x)) \tag{4.3.4}$$

for continuous seminorms q on V. Thus the $\mathscr{B}_{\mathrm{loc}}$-topology simply coincides with the $\mathscr{B}$-topology, see also Exercise 4.5.17.

To construct the Riemann integral we recall the following notions from elementary calculus: A **partition** of I consists of finitely many compact subintervals $\mathcal{I} = \{I_1, \ldots, I_N\}$ of I with the property that

$$I_j^\circ \cap I_{j'}^\circ = \emptyset \tag{4.3.5}$$

for $j \neq j'$ and

$$I = \bigcup_{j=1}^{N} I_j. \tag{4.3.6}$$

The maximum length of the edges of the subintervals of a partition will be denoted by

$$\Delta \mathcal{I} = \max_{j=1}^{N} \max_{k=1}^{n} (b_k^j - a_k^j), \tag{4.3.7}$$

where we write $I_j = [a_1^j, b_1^j] \times \ldots \times [a_n^j, b_n^j]$ for the j-th interval of $\mathcal{I}$.

In a next step one chooses points $\xi_j \in I_j$ in the subintervals of a partition $\mathcal{I}$ for all $j = 1, \ldots, N$. The collections of these points will be abbreviated by $\Xi = \{\xi_1, \ldots, \xi_N\}$. The partition $\mathcal{I}$ together with a choice of points Ξ in the subintervals of the partition then allows us to define the **Riemann sum**

$$\Sigma_{\mathcal{I}\Xi}(f) = \sum_{j=1}^{N} f(\xi_j) \operatorname{vol}(I_j) \tag{4.3.8}$$

of a function $f: I \longrightarrow V$ with respect to $\mathcal{I}$ and Ξ.

In order to define the Riemann integral we want to take the limit of the Riemann sums for finer and finer subintervals. To make this precise, we denote the set of all pairs $(\mathcal{I}, \Xi)$ of partitions $\mathcal{I}$ of I and choices of points Ξ in $\mathcal{I}$ by

$$\mathscr{J}.2.14 = \{(\mathcal{I}, \Xi) | \mathcal{I} \text{ is a partition of } I \text{ and } \Xi \text{ a collection of points in } \mathcal{I}\}. \tag{4.3.9}$$

Then $\mathscr{J}$ becomes a directed set by defining $(\mathcal{I}, \Xi)$ to be earlier as $(\mathcal{I}', \Xi')$ if every subinterval $I'_{j'}$ of $\mathcal{I}'$ is contained in some (necessarily unique) subinterval I_j of $\mathcal{I}$. We make no restriction on the relations of the points Ξ and Ξ' and write $(\mathcal{I}, \Xi) \preceq (\mathcal{I}', \Xi')$ in this case. It is clear that this defines a direction $\preceq$ on $\mathscr{J}$: Note that two partitions have a common refinement which is then later than both of them. The remaining proposition of a directed set are trivial.

Definition 4.3.1 (Riemann integral) *Let $I \subseteq \mathbb{R}^n$ be a compact interval with non-empty open interior and let V be a Hausdorff topological vector space. A function $f\colon I \longrightarrow V$ is called Riemann integrable over I if the limit $\lim_{(\mathfrak{I},\Xi)\in\mathcal{J}} \Sigma_{\mathfrak{I},\Xi}(f)$ of its Riemann sums exists. In this case we call the limit*

$$\int_I f(x)\,\mathrm{d}^n x = \lim_{(\mathfrak{I},\Xi)\in\mathcal{J}} \Sigma_{\mathfrak{I},\Xi}(f) \tag{4.3.10}$$

the Riemann integral of f over I. The set of Riemann integrable functions is denoted by

$$\mathscr{R}(I, V) = \{f\colon V \longrightarrow I \mid f \text{ is Riemann integrable over } I\}. \tag{4.3.11}$$

As usual, the restriction to **Hausdorff** topological vector spaces is reasonable as otherwise we can not expect a unique limit and thus a well-defined value of te integral. Some first properties of the Riemann integral are listed in the next proposition:

Proposition 4.3.2 *Let $I \subseteq \mathbb{R}^n$ be a compact interval with non-empty open interior and let V be a Hausdorff topological vector space.*

i.) The constant functions are Riemann integrable and for all $v \in V$ we have

$$\int_I v\,\mathrm{d}^n x = v \cdot \operatorname{vol}(I). \tag{4.3.12}$$

ii.) If $v \in V$ and $\chi \in \mathscr{R}(I, \mathbb{K})$ is a scalar Riemann integrable function then $f\colon X \to V$ defined by $f(x) = \chi(x)v$ for $x \in I$ is Riemann integrable and

$$\int_I f(x)\,\mathrm{d}^n x = v \cdot \int_I \chi(x)\,\mathrm{d}^n x. \tag{4.3.13}$$

iii.) The Riemann integrable functions form a subspace of $\operatorname{Map}(I, V)$.

iv.) The Riemann integral is a linear map

$$\int_I \cdot\,\mathrm{d}^n x\colon \mathscr{R}(I, V) \longrightarrow V. \tag{4.3.14}$$

v.) Let $A\colon V \longrightarrow W$ be a continuous linear map into another Hausdorff topological vector space W. Then for every $f \in \mathscr{R}(I, V)$ one has $A \circ f \in \mathscr{R}(I, W)$ and

$$A\left(\int_I f(x)\,\mathrm{d}^n x\right) = \int_I (A \circ f)(x)\,\mathrm{d}^n x. \tag{4.3.15}$$

PROOF: For the first part we note that for every $(\mathfrak{I}, \Xi) \in \mathcal{J}$ we have

$$\Sigma_{(\mathfrak{I},\Xi)} v = \sum_{j=1}^{N} v\operatorname{vol}(I_j) = v\operatorname{vol} I.$$

Thus the net of Riemann sums is constant and thus convergent with limit given by (4.3.12). Next, consider a scalar function $\chi \in \mathscr{R}(I, \mathbb{K})$ and a fixed vector $v \in V$. For a $(\mathfrak{I}, \Xi) \in \mathcal{J}$ we then have

$$\Sigma_{(\mathfrak{I},\Xi)}(\chi v) = \sum_{j=1}^{n} \chi(\xi_j) v \operatorname{vol}(I_j) = \Sigma_{(\mathfrak{I},\Xi)}(\chi) \cdot v$$

from which the result follows since multiplication by scalars is continuous. The third part is clear since the Riemann sums are linear in f and since the vector space operations are continuous. This also gives the linearity statement (4.3.14) of the fourth part. Finally, for the fifth statement we have

$$A \int_I f(x)\,\mathrm{d}^n x = A \lim_{(\mathfrak{I},\Xi)\in\mathcal{J}} \Sigma_{(\mathfrak{I},\Xi)}(f)$$

$$
\begin{aligned}
&= \lim_{(\mathcal{I},\Xi)\in\mathscr{I}} A\Sigma_{(\mathcal{I},\Xi)}(f) \\
&= \lim_{(\mathcal{I},\Xi)\in\mathscr{I}} \Sigma_{(\mathcal{I},\Xi)}(A \circ f) \\
&= \int_I (A \circ f)(x)\, \mathrm{d}^n x,
\end{aligned}
$$

since A is continuous and linear. □

We specialize now to the locally convex case again. Here we get the following remarkable result based on the properties of the scalar Riemann integral:

Proposition 4.3.3 *Let $I \subseteq \mathbb{R}^n$ be a compact interval with non-empty open interior and let V be a Hausdorff locally convex space.*

i.) *One has $\mathscr{R}(I, V) \subseteq \mathscr{B}(I, V)$.*

ii.) *Suppose $f : I \longrightarrow V$ has image contained in a finite-dimensional subspace $W \subseteq V$. Then $f \in \mathscr{R}(I, V)$ iff each component $\phi \circ f \in \mathscr{R}(I, V)$ for $\phi \in V'$ iff each component of f with respect to a basis of W is Riemann integrable.*

PROOF: Let $\phi \in V'$ then we have $\phi \circ f \in \mathscr{R}(I, \mathbb{K})$ for every $f \in \mathscr{R}(I, V)$. For the scalar Riemann integral it is well-known that $\mathscr{R}(I, \mathbb{K}) \subseteq \mathscr{B}(I, \mathbb{K})$, see e.g. [...] Hence f takes values in a weakly bounded subset of V. But weakly bounded subsets are bounded by Theorem 3.2.13. Hence $f \in \mathscr{B}(I, V)$ follows. Next, assume $f(I) \subseteq W$ for a finite-dimensional subspace W and let $e_1, \ldots, e_k \in W$ be a basis. Thus we have scalar functions $I \longrightarrow \mathbb{K}$ with $f(x) = \sum_{\alpha=1}^k f^\alpha(x) e_\alpha$. In fact, $f^\alpha = e^{\alpha_0} f$ where $e^1, \ldots, e^k \in W^*$ denotes the dual basis. Since V is Hausdorff and locally convex we can extend the $e^1, \ldots, e^k$ to continuous linear functionals $\phi^1, \ldots, \phi^k \in V'$ by the Hahn-Banach Theorem, see Corollary 3.1.7. Now suppose that f is Riemann integrable, then $\phi \circ f$ is Riemann integrable by Proposition 4.3.2 *v.)*, for all $\phi \in V'$. In particular, $f^\alpha = \phi^\alpha \circ f \in \mathscr{R}(I, \mathbb{K})$ for $\alpha = 1, \ldots, k$. Conversely, assume $f^\alpha = \phi^\alpha \circ f \in \mathscr{R}(I, \mathbb{K})$. Then also $f^\alpha e_\alpha \in \mathscr{R}(I, V)$ by Proposition 4.2.21 *ii.)*. By linearity we get $f = \sum_{\alpha=1}^k f^\alpha e_\alpha \in \mathscr{R}(I, V)$. □

Remark 4.3.4 The last statement shows that vector-valued Riemann integration becomes more involved once the image of the function is not contained in a finite-dimensional subspace. Otherwise one can rely on the scalar situation by considering components.

Remark 4.3.5 (Improper Riemann integrals) Occasionally, we will also need domains in $\mathbb{R}^n$ which are note compact intervals. This requires then the case of improper Riemann integrals. The general idea is that the domain $X \subseteq \mathbb{R}^n$ will be exhausted by compact intervals in a specific way. Then the integrations over each compact interval are well-defined and one has to take a further limit in order to define the improper Riemann integral over X. We will not go into details at this time but specify the relevant exhaustion and limit procedures in the situations where we actually need them. Off course, on important question will be how much the result depends on the way the exhaustion is performed.

4.3.2 The Riemann Integral of Continuous Functions

Already in the scalar situation it is fairly complicated to characterize the space $\mathscr{R}(I, \mathbb{K})$ of Riemann integrable functions as this is done e.g. in Lebesgués famous theorem, see [...] . In the vector-valued situation things get more complicated due to the nature of V: The new feature is that we need some completeness assumption on V which for $\mathbb{K}$ are automatic.

Even though the definition of the Riemann integral requires a net limit, the following theorem shows that **sequential** completeness will be sufficient for continuous functions to be Riemann integrable:

Theorem 4.3.6 (Riemann integral of continuous functions) *Let $I \subseteq \mathbb{R}^n$ be a compact interval with non-empty open interior and let V be a sequentially complete locally convex space. Then*

$$\mathscr{C}(I, V) \subseteq \mathscr{R}(I, V). \tag{4.3.16}$$

PROOF: Let $f \in \mathscr{C}(I, V)$ be given. First we show that the net of Riemann sums of f is a Cauchy net. To this end, let q be a continuous seminorm on V and let $\varepsilon > 0$. Since the continuous function f is uniformly continuous on the compact interval I, we find a $\delta > 0$ such that

$$\mathrm{q}(f(x) - f(y)) < \varepsilon \quad \text{whenever } \|x - y\|_\infty < \delta. \tag{$*$}$$

Here we use the max-norm of $\mathbb{R}^n$ as metric on I. Now let $\mathcal{I} = \{I_1, \ldots, I_N\}$ be a partition of I such that $\Delta\mathcal{I} < \delta$. If $\mathcal{I}'$ is any refinement of $\mathcal{I}$, i.e. $\mathcal{I} \preceq \mathcal{I}'$, then we have for a subinterval I_j from $\mathcal{I}$ unique subintervals $I'_{k_1}, \ldots, I'_{k_r} \in \mathcal{I}'$ such that $I_j = I'_{k_1} \cup \cdots \cup I'_{k_r}$. Next, let Ξ and Ξ' be choices of points in I matching the partitions $\mathcal{I}$ and $\mathcal{I}'$, respectively. Then

$$\begin{aligned} \mathrm{q}\left(f(\xi_j)\mathrm{vol}(I_j) - \sum_{i=1}^r f(\xi'_{k_i})\mathrm{vol}(I'_{k_i}) \right) &= \mathrm{q}\left(\sum_{i=1}^r \big(f(\xi_j) - f(\xi'_{k_i})\big)\mathrm{vol}(I'_{k_i}) \right) \\ &\le \sum_{i=1}^r \mathrm{q}\big(f(\xi_j) - f(\xi'_{k_i})\big)\mathrm{vol}(I'_{k_i}) \\ &\overset{(a)}{<} \varepsilon \mathrm{vol}(I_j), \end{aligned}$$

since $\mathrm{vol}(I_j) = \mathrm{vol}(I'_{k_1}) + \cdots + \mathrm{vol}(I'_{k_r})$ and since the points $\xi_j, \xi_{k_1}, \ldots, \xi_{k_r} \in I_j$ have distances less than δ as $\Delta\mathcal{I} < \delta$. This means that we can make use of $(*)$ in (a). Taking now the sum over $j = 1, \ldots, N$ first gives the Riemann sums with respect to $(\mathcal{I}, \Xi)$ and $(\mathcal{I}', \Xi')$ and thus

$$\mathrm{q}\big(\Sigma_{(\mathcal{I},\Xi)}(f) - \Sigma_{(\mathcal{I}',\Xi')}(f)\big) < \varepsilon \mathrm{vol}(I).$$

Here it is crucial that in the sum over j also all the subintervals of $\mathcal{I}'$ occur **once**. Now we fix a partition $\mathcal{I}_0$ with points Ξ_0 such that $\Delta I_0 < \delta$. If $\mathcal{I}$, $\mathcal{I}'$ are refinements of $\mathcal{I}_0$ with corresponding points Ξ and Ξ_0 then

$$\begin{aligned} \mathrm{q}\big(\Sigma_{(\mathcal{I},\Xi)}(f) - \Sigma_{(\mathcal{I}',\Xi')}(f)\big) &\le \mathrm{q}\big(\Sigma_{(\mathcal{I},\Xi)}(f) - \Sigma_{(\mathcal{I}_0,\Xi_0)}(f)\big) + \mathrm{q}\big(\Sigma_{(\mathcal{I}',\Xi')}(f) - \Sigma_{(\mathcal{I}_0,\Xi_0)}(f)\big) \\ &< 2\varepsilon \mathrm{vol}(I). \end{aligned} \tag{$**$}$$

Note that here the partitions $\mathcal{I}$, $\mathcal{I}'$ do not have a relation with respect to $\preceq$. Thus the Riemann sums of a continuous function form a Cauchy net, without any further assumptions on V. If V is complete the claim (4.3.16) follows directly. However, we only want to assume that V is sequentially complete. In this situation we need to show that we can define a suitable subsequence of the above net: We divide each interval $[a_k, b_k]$ in $I = [a_1, b_1] \times \cdots \times [a_n, b_b]$ into 2^m equal pieces for $m \in \mathbb{N}$. Taking Cartesian products yields a partition $\mathcal{I}_m$ of I consisting of 2^{mn} subintervals, all having equal k-th length and hence equal volume given by $\frac{1}{2^{mn}}\mathrm{vol}(I)$. We choose matching points Ξ_m for each $\mathcal{I}_m$. By construction, $\mathcal{I}_{m+1} \succeq \mathcal{I}_m$ for all $m \in \mathbb{N}$ and $\Delta\mathcal{I}_m = \frac{1}{2^{mn}}\mathrm{vol}(I) \longrightarrow 0$. Thus there exists an $m_0 \in \mathbb{N}$ with $\Delta\mathcal{I}_{m_0} < \delta$, allowing us to apply $(**)$. This shows that for $m, m' \ge m_0$ we get

$$\mathrm{q}\big(\Sigma_{(\mathcal{I}_m,\Xi_m)}(f) - \Sigma_{(\mathcal{I}_{m'},\Xi_{m'})}(f)\big) < \varepsilon \mathrm{vol}(I).$$

This shows that the Riemann sums for these partitions form a Cauchy **sequence**, therefore converging to some

$$v = \lim_{m \to \infty} \Sigma_{(\mathcal{I}_m,\Xi_m)}(f) \in V.$$

Fix now $m_1 \in \mathbb{N}$ such that $\mathrm{q}\Big(\Sigma_{(\mathcal{I}_{m_1}, \Xi_{m_1})}(f) - v\Big) < \varepsilon$ and such that $m_1 \geq m_0$. If now $\mathcal{I}$ is an arbitrary partition of I with corresponding choice Ξ of points such that $\mathcal{I} \succeq \mathcal{I}_{m_1}$ we get

$$\begin{aligned}\mathrm{q}\big(\Sigma_{(\mathcal{I},\Xi)}(f) - v\big) &\leq \mathrm{q}\Big(\Sigma_{(\mathcal{I},\Xi)}(f) - \Sigma_{(\mathcal{I}_{m_1}, \Xi_{m_1})}(f)\Big) + \mathrm{q}\Big(\Sigma_{(\mathcal{I}_{m_1}, \Xi_{m_1})}(f) - v\Big) \\ &\overset{(**)}{<} \varepsilon \mathrm{vol}(I) + \varepsilon.\end{aligned}$$

From this we conclude that the whole net of Riemann sums converges to v. This concludes the proof.□

Remarkably, the statement requires only a fairly mild completeness assumption. While we have many examples of complete locally convex spaces among the interesting function spaces, there are also important examples of non-complete ones: The duals of locally convex spaces in the weak-* topology are essentially never complete. However, we have seen already several situations, see e.g. Corollary 3.3.38, where weak-* duals are sequentially complete. Hence our integration theory applies to these situations as well. Note that the very coarse weak-* topology will make it fairly easy to test the continuity.

We continue now with some additional features of the Riemann integral of continuous functions.

Proposition 4.3.7 *Let $I \subseteq \mathbb{R}^n$ be a compact interval with non-empty open interior and let V be a sequentially complete locally convex space.*

i.) For every $f \in \mathscr{C}(I, V)$ and every continuous seminorm q on V one has

$$\mathrm{q}\bigg(\int_I f(x)\, \mathrm{d}^n x\bigg) \leq \int_I (\mathrm{q} \circ f)(x)\, \mathrm{d}^n x \leq \mathrm{vol}(I) \mathrm{q}_I(f). \tag{4.3.17}$$

ii.) The Riemann integral is a continuous linear functional

$$\int_I \cdot\, \mathrm{d}^n x \colon \mathscr{C}(I, V) \longrightarrow V \tag{4.3.18}$$

with respect to the $\mathscr{C}$-topology.

Proof: Clearly, for every Riemann sum of f with respect to a partition $\mathcal{I}$ of I and points Ξ we have

$$\begin{aligned}\mathrm{q}\big(\Sigma_{(\mathcal{I},\Xi)}(f)\big) &= \mathrm{q}\left(\sum_{j=1}^N f(\xi_j) \mathrm{vol}(I_j)\right) \\ &\leq \sum_{j=1}^N \mathrm{q}(f(\xi_j)) \mathrm{vol}(I_j) \\ &= \Sigma_{(\mathcal{I},\Xi)}.\end{aligned}$$

Since the function $\mathrm{q} \circ f$ is continuous on I the scalar Riemann integral as net limit of the right hand side exists and gives

$$\lim_{(\mathcal{I},\Xi) \in \mathcal{J}} \Sigma_{(\mathcal{I},\Xi)}(\mathrm{q} \circ f) = \int_I (\mathrm{q} \circ f)(x)\, \mathrm{d}^n x.$$

The seminorm q is continuous and hence the limit of the left hand side gives thus

$$\lim_{(\mathcal{I},\Xi) \in \mathcal{J}} \mathrm{q}\big(\Sigma_{(\mathcal{I},\Xi)}(f)\big) = \mathrm{q}\bigg(\lim_{(\mathcal{I},\Xi) \in \mathcal{J}} \Sigma_{(\mathcal{I},\Xi)}(f)\bigg) = \mathrm{q}\bigg(\int_I f(x)\, \mathrm{d}^n x\bigg). \tag{4.3.19}$$

Taking limits preserves inequalities in $\mathbb{R}$ and thus the first inequality in (4.3.17) follows. The second is clear from the scalar theory. Then the second statement follows from the combined estimate in (4.3.17) since the $\mathscr{C}$-topology of $\mathscr{C}(I, V)$ for a compact I is induced by the seminorm q_I. □

This continuity statement will be very useful when it comes to exchanging of limits and integration.

Proposition 4.3.8 *Let $\mathcal{I} = \{I_1, \ldots, I_N\}$ be a partition of the compact interval $I \subseteq \mathbb{R}^n$ with non-empty open interior. Moreover, let $f \in \mathscr{C}(I, V)$, with a sequentially complete locally convex space V. Then one has*

$$\int_I f(x)\,\mathrm{d}^n x = \sum_{j=1}^{N} \int_{I_j} f\big|_{I_j}(x)\,\mathrm{d}^n x. \tag{4.3.20}$$

PROOF: First we note that $f\big|_{I_j} \in \mathscr{C}(I_j, V)$ are still continuous and hence Riemann integrable. Thus the right hand side makes sense at all. Since we know that all involved limit exist, we can evaluate them on suitable cofinal subsets of the indexing set $\mathcal{J}$. Thus consider partition partitions of I which are refinements of the given partition $\mathcal{I}$. This gives clearly a cofinal subset of all partitions. For such a partition, the Riemann sum of f can simply be written as the sum of corresponding Riemann sums of the restrictions $f\big|_{I_j}$. □

We conclude this section with some remarks on the special case $n = 1$. For $I = [a, b] \subseteq \mathbb{R}$ one traditionally writes

$$\int_a^b f(x)\,\mathrm{d}x = \int_I f(x)\,\mathrm{d}x. \tag{4.3.21}$$

Then the last proposition says that for every $c \in (a, b)$ we have

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x \tag{4.3.22}$$

for all $f \in \mathscr{C}([a, b], V)$. It is consistent with this property to define

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x. \tag{4.3.23}$$

Indeed, with this convention, (4.3.22) holds for all three points a, b, c in the domain I of a continuous function, whether $a < c < b$ holds or not. This convention will be very convenient at many places and avoids to make case-by-case studies. In higher dimensions we can also make a corresponding definition leading then to a discussion of **orientation**. Ultimately, it is completely parallel to the scalar version.

4.3.3 Fubini Theorem for Continuous Functions

As in the scalar case it will be advantageous to compute the n-dimensional integrals as iterated one-dimensional integrals by means of Fubini's Theorem. One can now either directly prove this by evaluating the limits on particular cofinal partitions. Here we take a slightly different road involving the vector-valued functions point of view more consequently.

We start with the following simple observation. Suppose $I_1 \subseteq \mathbb{R}^{n_1}$ and $I_2 \subseteq \mathbb{R}^{n_2}$ are compact intervals with non-empty open interior. Then $I = I_1 \times I_2 \subseteq \mathbb{R}^n$ with $n = n_1 + n_2$ is again such an interval. Then we can integrate a function $f \in \mathscr{C}(I_1 \times I_2, V)$ only over the I_2-variables and get a function of the I_1-variables. This turns out to be a continuous operation:

Proposition 4.3.9 *Let $I = I_1 \times I_2 \subseteq \mathbb{R}^n$ with compact intervals $I_1 \subseteq \mathbb{R}^{n_1}$ and $I_2 \subseteq \mathbb{R}^{n_2}$ with non-empty open interior and let V be a sequentially complete locally convex space.*

i.) For $f \in \mathscr{C}(I, V)$ the functions

$$I_1 \ni x_1 \mapsto \int_{I_2} f(x_1, x_2)\,\mathrm{d}^{n_2} x_2 \in V \quad \text{and} \quad I_2 \ni x_2 \mapsto \int_{I_1} f(x_1, x_2)\,\mathrm{d}^{n_1} x_1 \in V \tag{4.3.24}$$

are continuous.

ii.) The integral yields continuous linear maps

$$\int_{I_2} \cdot \, \mathrm{d}^{n_2}x \colon \mathscr{C}(I,V) \longrightarrow \mathscr{C}(I_1,V) \quad \text{and} \quad \int_{I_1} \cdot \, \mathrm{d}^{n_1}x \colon \mathscr{C}(I,V) \longrightarrow \mathscr{C}(I_2,V). \tag{4.3.25}$$

PROOF: The traditional way to check this proposition is to establish some straight forward estimates. However, having the currying from Theorem 4.2.12 this becomes a triviality. We have $\mathscr{C}(I,V) \cong \mathscr{C}(I_1, \mathscr{C}(I_2,V))$ from Theorem 4.2.12 via the currying map. Then the integral over I_2 is a continuous linear functional

$$\int_{I_2} \cdot \, \mathrm{d}^{n_2}x \colon \mathscr{C}(I_2,V) \longrightarrow V \tag{4.3.26}$$

by Proposition 4.3.7 *ii.)*. This results in a continuous linear map

$$\int_{I_2} \cdot \, \mathrm{d}^{n_2}x \colon \mathscr{C}(I_1, \mathscr{C}(I_2,V)) \longrightarrow \mathscr{C}(I_1,V) \tag{4.3.27}$$

from the functorial behavior of $\mathscr{C}(I_1, \cdot\,)$ as discussed in Proposition 4.2.8 *ii.)*. Unwinding the definition of the currying isomorphism this is precisely the integration needed in (4.3.24), thereby proving both statements at once. ☐

The classical Fubini theorem has two aspects: We can exchange limits, i.e. integration or, as second feature, a net limit is computed as iterated net limit. Both questions refer to the general situation of net limits in locally convex spaces:

Theorem 4.3.10 (Fubini) *Let $I = I_1 \times I_2 \subseteq \mathbb{R}^n$ with compact intervals $I_1 \subseteq \mathbb{R}^{n_1}$ and $I_2 \subseteq \mathbb{R}^{n_2}$ with non-empty open interior and $n = n_1 + n_2$. Moreover, let V be a sequentially complete locally convex space. For $f \in \mathscr{C}(I,V)$ one has:*

$$\int_I f(x)\, \mathrm{d}^n x = \int_{I_1} \left(\int_{I_2} f(x_1,x_2)\, \mathrm{d}^{n_2}x_2 \right) \mathrm{d}^{n_1}x_1 = \int_{I_2} \left(\int_{I_1} f(x_1,x_2)\, \mathrm{d}^{n_1}x_1 \right) \mathrm{d}^{n_2}x_2. \tag{4.3.28}$$

PROOF: First we note that all (iterated) Riemann integrals exist thanks to the continuity of the involved functions, see also Proposition 4.3.9. Now we consider Riemann sums for the left hand side of a particular limit: A partition $\mathcal{J}$ of I is called factorizing if its subintervals arise from cartesian products of subintervals of partitions $\mathcal{J}_1$ of I_1 and $\mathcal{J}_2$ of I_2. Clearly, not every partition of I is factorizing but every partition of I has a factorizing refinement. Hence it will be sufficient to compute the net limits using such factorizing partitions $\mathcal{J} = \mathcal{J}_1 \mathcal{J}_2$. This way, we obtain a net indexed by the product of the directed sets of all pairs $(\mathcal{J}_1, \Xi_1)$ of partitions of I_1 with points in there and of all pairs $(\mathcal{J}_2, \Xi_2)$ of partitions of I_2 with points in there. For this net we thus get

$$\lim_{(\mathcal{J}_1 \times \mathcal{J}_2, \Xi_1 \times \Xi_2)} \big(\Sigma_{(\mathcal{J}_1 \times \mathcal{J}_2, \Xi_1 \times \Xi_2)}(f)\big)_{(\mathcal{J}_1 \times \mathcal{J}_2, \Xi_1 \times \Xi_2)} = \int_I f(x)\, \mathrm{d}^n x. \tag{$*$}$$

Now denote by $g(x_1) = \int_{I_2} f(x_1,x_2)\, \mathrm{d}^{n_2}x$ the function obtained by first integrating over the I_2-variables. Then the limit of the double-indexed net in $(*)$ for fixed $(\mathcal{J}_1, \Xi_1)$ is just the corresponding Riemann sum

$$\Sigma_{(\mathcal{J}_1, \Xi_1)}(g) = \lim_{(\mathcal{J}_2, \Xi_2)} \Sigma_{(\mathcal{J}_1 \times \mathcal{J}_2, \Xi_1 \times \Xi_2)}(f)$$

of g. Thus this limit exists and the subsequent limit over $(\mathcal{J}_1, \Xi_1)$ again exists yielding the integral of g. Now we are in the position to use the general result of exchanging net limits from e.g. [18][Exercise 4.4.5], since all involved spaces certainly satisfy the required T_3-axiom according to Proposition 2.1.8. This gives directly the first equality in (4.3.28). The other is analogous. ☐

From this point of view, Fubini's Theorem arises from two facts: The Currying isomorphism guaranteeing the continuity of the function after integrating over some variables and hence the existence of the iterated integrals, and, second, the general exchanging of limits for nets in T_3-spaces.

Iterating the procedure in (4.3.28) we then arrive at one-dimensional integrals:

Corollary 4.3.11 *Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ and let V be a sequentially complete locally convex space. Then*

$$\int_I f(x)\, \mathrm{d}^n x = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{4.3.29}$$

holds for all $f \in \mathscr{C}(I, V)$.

As a last remark we note that already in the scalar case $V = \mathbb{C}$ there are counterexamples that the mere Riemann integrability of the function f is **not** sufficient for Fubini's Theorem to hold.

4.3.4 The Riemann Integral of Lipschitz Functions

If the locally convex space V is not (sequentially) complete, the Riemann integral of a continuous function still exists as an element of its completion $\hat{V}$. In fact, it takes a value in the smallest sequentially closed subset of V inside $\hat{V}$: This could be viewed as the sequential completion. If we consider now even a Lipschitz function $f \in \mathscr{L}ip(I, V)$ then the Riemann integral takes its values in an even smaller subspace: Mackey completeness is sufficient. We formulate this result directly for a Mackey complete space V:

Proposition 4.3.12 *Let $I \subseteq \mathbb{R}^n$ be a compact interval with non-empty open interior and let V be a Mackey complete locally convex space. Then we have*

$$\mathscr{L}ip(I, V) \subseteq \mathscr{R}(I, V). \tag{4.3.30}$$

PROOF: We need to show that the net of Riemann sums of $f \in \mathscr{L}ip(I, V)$ is in fact a Mackey-Cauchy net. The interval I is compact itself, so a locally Lipschitz function is globally Lipschitz. Hence the subset

$$B = \operatorname{absconv}\left\{ \frac{f(x) - f(y)}{\|x - y\|} \,\middle|\, x \neq y, x, y \in I \right\}$$

is bounded: Indeed, from Corollary **??** we know that before taking the absolute convex hull the subset is bounded since f is Lipschitz. Taking the absolute convex hull preserves boundedness in a locally convex space, see (...). As in the proof of Theorem 4.3.6 we fix a partition $\mathcal{I} = \{I_1, \ldots, I_N\}$ of I with $\delta = \Delta\mathcal{I}$. For a refinement $\mathcal{I}' = \{I'_1, \ldots, I'_N\}$ we then have for every $j = 1, \ldots, N$

$$I_j = I'_{k_1} \cup \cdots \cup I'_{k_{r_j}}$$

for some unique subintervals $I'_{k_1}, \ldots, I'_{k_{r_j}} \in \mathcal{I}'$ depending on j. Next, we fix collections of points Ξ and Ξ' corresponding to $\mathcal{I}$ and $\mathcal{I}'$. Then

$$\begin{aligned} f(\xi_j)\mathrm{vol}(I_j) - \sum_{i=1}^{r_j} f(\xi'_{k_i})\mathrm{vol}(I'_{k_i}) &= \sum_{i=1}^{r_j} \big(f(\xi_j) - f(\xi'_{k_i})\big)\mathrm{vol}(I'_{k_i}) \\ &= \sum_{i=1}^{r_j} \frac{f(\xi_j) - f(\xi'_{k_i})}{\delta} \delta \mathrm{vol}(I'_{k_i}). \end{aligned}$$

Now the difference $v_j = \frac{f(\xi_j) - f(\xi'_{k_i})}{\delta}$ is contained in B since $\|\xi_j - \xi'_{k_i}\| \leq \delta$ by the refinement condition. Hence for the difference of the Riemann sums we obtain

$$\begin{aligned}\Sigma_{(\mathcal{I},\Xi)}(f) - \Sigma_{(\mathcal{I}',\Xi')}(f) &= \sum_{j=1}^{N} \sum_{i=1}^{r_j} \big(f(\xi_j)\mathrm{vol}(I_j) - f(\xi'_{k_i})\mathrm{vol}(I'_{k_i}) \big) \\ &= \sum_{j=1}^{N} v_j \delta \mathrm{vol}(I_j).3.12\end{aligned}$$

with $\sum_{j=1}^{N} \mathrm{vol}(I_j)\delta = \delta \mathrm{vol}(I)$ and $v_j \in I$. Hence the difference of the Riemann sums is in

$$\Sigma_{(\mathcal{I},\Xi)}(f) - \Sigma_{(\mathcal{I}',\Xi')}(f) \in \delta \mathrm{vol}(I) B$$

since B is absolute convex. If (I'', Ξ'') is now another refinement of $\mathcal{I}$ then

$$\Sigma_{(\mathcal{I}',\Xi')}(f) - \Sigma_{(\mathcal{I}'',\Xi'')}(f) \in 2\delta' \mathrm{vol}(I) B.$$

As δ was arbitrary, taking e.g.

$$\lambda_{(I',\Xi'),(I'',\Xi'')} = 2\mathrm{vol}(I) \max\big\{\Delta\mathcal{I}', \Delta\mathcal{I}''\big\}$$

gives a net converging to zero such that

$$\Sigma_{(\mathcal{I}',\Xi')}(f) - \Sigma_{(\mathcal{I}'',\Xi'')}(f) \in \lambda_{(I',\Xi'),(I'',\Xi'')} B.$$

But this shows that the net of Riemann sums is in fact a Mackey-Cauchy net. ☐

Thus the slightly more restrictive assumption of Lipschitz continuity gives a nice relaxation of the completeness assumption on V for the Riemann integrability of functions. Recall that Mackey completeness is a fairly weak completeness property, even weaker than sequential completeness.

Remark 4.3.13 (Mackey completeness) The statement of 4.3.12 has a remarkable converse if for every $f \in \mathscr{L}ip(I, V)$, with $I = [a, b] \subseteq \mathbb{R}$ a compact interval, the Riemann integral of f over I exists then the target space V is necessarily Mackey complete. Thus the assumption in the proposition is optimal, see [9][Thm 2.14].

Since Lipschitz continuous functions are continuous, we get similar seminorm estimates for the Riemann integral under the assumption of Proposition 4.3.12:

Corollary 4.3.14 *Let $I \subseteq \mathbb{R}^n$ be a compact interval with non-empty open interior and let V be a Mackey complete locally convex space.*

i.) For every $f \in \mathscr{L}ip(I, V)$ one has for every continuous seminorm q on V

$$\mathrm{q}\bigg(\int_I f(x)\,\mathrm{d}^n x\bigg) \leq \int_I (\mathrm{q} \circ f)(x)\,\mathrm{d}^n x \leq \mathrm{vol}(I)\mathrm{q}_I(f). \tag{4.3.31}$$

ii.) The Riemann integral is a continuous linear functional

$$\int_I \cdot\,\mathrm{d}^n x \colon \mathscr{L}ip(I, V) \longrightarrow V \tag{4.3.32}$$

with respect to the $\mathscr{L}ip$-topology.

PROOF: We know that $\mathrm{q} \circ f$ is continuous and hence Riemann integrable. Since f is Riemann integrable by Proposition 4.3.12, we can prove the first inequality the same way as we did this for continuous functions in the sequentially complete case, see Proposition 4.3.7 *i.)*. The second inequality is then clear from the scalar case. For the continuity we note that (4.3.32) is continuous in the $\mathscr{C}$-topology by (4.3.31). Since the $\mathscr{L}ip$-topology is finer we have $\mathscr{L}ip$-continuity as well. □

The main point in the second statement is that the integral takes values in V only, and not in the completion $\hat{V}$.

Also the possibility to chop an integration into smaller pieces is valid in the $\mathscr{L}ip$ case with only Mackey complete target:The restrictions of a $\mathscr{L}ip$ function to subintervals stays a $\mathscr{L}ip$ function, see also Exercise 4.5.19 for more details.

4.4 Differentiable Functions

4.4.1 Differentiability

4.4.2 The Fundamental Theorem of Calculus and Applications

4.4.3 Smooth Functions

4.4.4 The $\mathscr{C}^k$-, $\mathscr{L}ip^k$-, and $\mathscr{C}^\infty$-Topology

4.4.5 Compactly Supported $\mathscr{C}^k$-Functions

4.4.6 Smooth Curves

4.5 Exercises

Exercise 4.5.1 (Bornologies)

Exercise 4.5.2 (Generating bornologies)

Exercise 4.5.3 (The von Neumann bornology)

Exercise 4.5.4 (Mackey convergence)

Exercise 4.5.5 (Properties of $\mathscr{B}$-topology)

Exercise 4.5.6 (l_c for $\operatorname{Map}(X, V)$**)**

Exercise 4.5.7 $\mathscr{B}_{\mathrm{loc}}$ is Fréchet if
$\mathscr{B}$ is Banach if
$\mathscr{B} = \mathscr{B}_{\mathrm{loc}}$ if X compact

Exercise 4.5.8 Module structure of $\mathscr{B}(X, V)$ and $\mathscr{B}_{\mathrm{loc}}(X, V)$ for $\mathscr{B}(X)$ and $\mathscr{B}_{\mathrm{loc}}(X)$.

Exercise 4.5.9 ($\mathscr{C}$-topology) *i.)* $\mathscr{C}$-topology features: Banach, Fréchet, ...
ii.) Special case X compact.

Exercise 4.5.10 (Generating seminorms for $\mathscr{C}$-topology) Proof Proposition 4.2.13.

Exercise 4.5.11 Directly $\mathscr{L}ip(X, V) \subseteq \mathscr{C}(X, V)$ via zero neigbourhood iso.

Exercise 4.5.12 Scalar case of *i.)* with hints. Subtle point: Easier with sequences, needs to show that Lipschitz is continuous first hence bounded.

Exercise 4.5.13 (Properties of locally Lipschitz functions) *i.)* $\mathscr{L}ip(X) \circlearrowleft \mathscr{L}ip(X, V)$

ii.) $\mathscr{L}ip(X)$ dense in $\mathscr{C}(X)$ via Stone Weierstraß

iii.) $\mathscr{L}ip$ and bounded linear maps $V \to W$

iv.) Fréchet, Banach?

Exercise 4.5.14 (Uniform continuity) Check the consistence of the new definition in 4.2.20 with the previous one in 2.1.13.

Exercise 4.5.15 Show that $\mathscr{C}_K(X, V) \cong \mathscr{C}(K, V)$ are a Fréchet space if V is a Fréchet space.

Exercise 4.5.16 Properties of $\mathscr{C}_0(X, V)$ for particular X, V
V: Banach gives LB
explicit seminorms in LB case

Exercise 4.5.17 Show that on a closed interval in $\mathbb{R}^n$ the $\mathscr{B}_{\mathrm{loc}}$-topology coincides with the $\mathscr{B}$-topology.

Exercise 4.5.18 Discuss why the assumption $\mathring{I} \neq \emptyset$ for the Riemann integration in $\mathbb{R}^n$ is reasonable.

Exercise 4.5.19 Show that the restriction of a $\mathscr{L}ip$ function to a subinterval stays a $\mathscr{L}ip$ function.

Chapter 5

Distributions

In this chapter we start investigating the dual space of the test function space $\mathscr{C}_0^\infty(X)$ more closely: these are the *distributions* or *generalized functions*. Much before the advent of a mathematically sound theory of distributions, heuristic methods using the "δ-function" of Dirac proved to be extremely successful in solving various problems in physics involving ordinary or partial differential equations. It is fair to say that leaving out ordinary and partial differential equations from physics, not much is left. This explains well the tremendous interest in concepts like the "δ-function".

However, it was very clear from the beginning that a function δ with the property that $\delta(x) = 0$ for all $x \neq 0$, but

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1 \tag{5.1}$$

is simply not possible in any sense. No matter which theory of integration one uses, such a function is not available: statements like "$\delta(0)$ has to be infinite in such a way that (5.1) holds" are not justifiable within the usual bounds of mathematics.

The way out is to interpret δ not as a function, but as a *linear functional* on a space of suitable test functions. Then (5.1) becomes

$$\delta \colon f \;\mapsto\; f(0), \tag{5.2}$$

which is of course entirely harmless. The δ-functional is just the evaluation of a test function at zero. But now new questions and difficulties arise. One needs to establish algebraic and analytic features of such linear functionals, which allow one to perform the heuristic manipulations needed in physical applications, now on a rigorous basis. Important examples are operations like:

i.) Linear combinations.

ii.) Differentiation.

iii.) Multiplication and convolutions.

iv.) Convergence of sequences and series.

v.) Automatic exchanging of limits.

vi.) Interpretation as functions.

To achieve this goal one needs to specify the precise class of linear functionals one wants to consider. In principle, several options are possible. The seminal work *Theorie des Distributions* of Laurent Schwartz answered this question in a very satisfactory way. One needs to specify a very small class of test functions of a very high regularity, such that the topological dual of it can contain extraordinarily singular functionals. This leads to the choice of the smooth functions with compact support as test functions and their topological dual as space of distributions. Here one needs of course a reasonable choice of a topology on the space of test functions. The natural one is the LF topology we have already seen in Proposition 2.4.50. Nevertheless, other types of test functions and therefore distributions can be considered as well. Some we will see as being included in the above choice, some others are still more general and require yet other tools and techniques.

Some of the difficulties when working with distributions can be handled within the locally convex analytic world, once sufficient care is taken. Some others, however, are much more severe and, for some, can not be solved at all. It turns out that the algebra structure we have for all types of reasonable function classes does not carry over to distributions. From an algebraic point of view this is not very surprising, as for an algebra $\mathscr{A}$ the dual $\mathscr{A}^*$ does not inherit any algebra multiplication. Unfortunately, this identifies distribution theory as an essentially *linear* concept. In view of the many applications in partial differential equations, but also beyond, say, in quantum field theory, this is a serious drawback. It takes a non-trivial amount of new and sophisticated ideas to surpass this bound to linear problems. Nevertheless, one can go beyond and many very recent developments show that interesting problems can be formulated and solved this way. The other difficulty we will have to discuss is in which sense distributions can still be seen as functions. Clearly, any attempt to assign a value to $\delta(0)$ is doomed to fail and hence distributions are, at best, *generalized functions*. To make this more precise requires a choice to be made. The transformation law of test functions under diffeomorphisms is *different* from the induced transformation law of functionals, as transposition reverses the order. This will lead to two interpretations of distributions: either as linear functionals or as generalized functions.

With this motivation in mind we shall now enter the vast landscape of distribution theory. In a first step we recall the basic features of the relevant test function test function spaces. From the general theory of locally convex spaces we then can infer some first fundamental properties of distributions before we move on to some more specific features like the support properties and convolution products of convolutions. Having established the basic calculus of distributions, we then can give some first applications to ordinary and partial differential equations.

5.1 Test Functions and Distributions

In this preliminary section we recall some of the basic features of the test function spaces obtained so far. This will allow us to conclude some general properties of their topological duals, the distributions.

5.1.1 The Test Function Space $\mathscr{C}_0^\infty(X)$

The most common space of test functions used to define distributions is given by $\mathscr{C}_0^\infty(X)$, where $X \subseteq \mathbb{R}^n$ is a non-empty open subset. We will meet functions with lesser regularity and other support properties as well, but $\mathscr{C}_0^\infty(X)$ is the most important choice. Even though we will not be concerned with the geometric versions here, we just mention that essentially everything carries over to the case where X is a (Hausdorff, second-countable) smooth manifold. Then the (non-)compactness of X will be an important question, as it decides whether $\mathscr{C}_0^\infty(X) \subseteq \mathscr{C}^\infty(X)$ is a proper subspace or not. However, we shall stick to $X \subseteq \mathbb{R}^n$ in the following.

Inside $\mathscr{C}^\infty(X)$ we have the closed subspaces

$$\mathscr{C}_A^\infty(X) = \left\{ f \in \mathscr{C}^\infty(X) \;\middle|\; \operatorname{supp}(f) \subseteq A \right\} \subseteq \mathscr{C}^\infty(X) \tag{5.1.1}$$

for every closed subset $A \subseteq X$, where we always use the $\mathscr{C}^\infty$-topology from Definition 2.3.19 for $\mathscr{C}^\infty(X)$. Recall from Proposition 2.4.50 as well as Section **??** that the natural topology on $\mathscr{C}_0^\infty(X)$ is then given by the locally convex inductive limit topology

$$\mathscr{C}_0^\infty(X) = \varinjlim_{\substack{K \subseteq X \\ \text{compact}}} \mathscr{C}_K^\infty(X), \tag{5.1.2}$$

where we use all compact subsets of X in (5.1.2). We collect the basic properties of this $\mathscr{C}_0^\infty$-topology in the following theorem:

Theorem 5.1.1 (Test functions) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.*

i.) *The $\mathscr{C}_0^\infty$-topology on $\mathscr{C}_0^\infty(X)$ is finer than the subspace topology inherited from $\mathscr{C}_0^\infty(X) \subseteq \mathscr{C}^\infty(X)$. In particular, it is Hausdorff.*

ii.) *The $\mathscr{C}_0^\infty$-topology is the finest locally convex topology on $\mathscr{C}_0^\infty(X)$ such that the inclusions*

$$\mathscr{C}_K^\infty(X) \subseteq \mathscr{C}_0^\infty(X) \tag{5.1.3}$$

are continuous for all compact $K \subseteq X$. These inclusions are embeddings with closed images and the subspace topology on $\mathscr{C}_K^\infty(X)$ from (5.1.3) is the $\mathscr{C}^\infty$-topology inherited from (5.1.1).

iii.) *If $K_1 \subseteq K_2^\circ \subseteq K_2 \subseteq \cdots \subseteq X$ is an exhausting sequence by compact subsets, i.e. $X = \bigcup_{n=1}^\infty K_n$, then the locally convex inductive limit topology of*

$$\mathscr{C}_0^\infty(X) = \varinjlim_{n \in \mathbb{N}} \mathscr{C}_{K_n}^\infty(X) \tag{5.1.4}$$

coincides with the $\mathscr{C}_0^\infty$-topology. In particular, this gives $\mathscr{C}_0^\infty(X)$ the structure of an LF space.

iv.) *The $\mathscr{C}_0^\infty$-topology is complete.*

v.) *A sequence $f_n \in \mathscr{C}_0^\infty(X)$ converges to $f \in \mathscr{C}_0^\infty(X)$ in the $\mathscr{C}_0^\infty$-topology iff there exists a compact subset $K \subseteq X$ with $f_n \in \mathscr{C}_K^\infty(X)$ and $f_n \to f \in \mathscr{C}_K^\infty(X)$ in the $\mathscr{C}^\infty$-topology of $\mathscr{C}_K^\infty(X)$.*

vi.) *A subset $B \subseteq \mathscr{C}_0^\infty(X)$ is bounded iff there exists a compact subset $K \subseteq X$ such that $B \subseteq \mathscr{C}_K^\infty(X)$ and B is bounded in the $\mathscr{C}^\infty$-topology of $\mathscr{C}_K^\infty(X)$.*

vii.) *Suppose $\Phi\colon \mathscr{C}_0^\infty(X) \longrightarrow V$ is a linear map into a locally convex space V. Then Φ is continuous iff for all compact subsets $K \subseteq X$ the restriction*

$$\Phi\Big|_{\mathscr{C}_K^\infty(X)} \colon \mathscr{C}_K^\infty(X) \longrightarrow V \tag{5.1.5}$$

is continuous with respect to the $\mathscr{C}^\infty$-topology of $\mathscr{C}_K^\infty(X)$. If $K_1 \subseteq K_2^\circ \subseteq K_2 \subseteq \cdots \subseteq X$ is an exhausting sequence of compact subsets, then Φ is continuous iff

$$\Phi\Big|_{\mathscr{C}_{K_n}^\infty(X)} \colon \mathscr{C}_{K_n}^\infty(X) \longrightarrow V \tag{5.1.6}$$

is continuous for all $n \in \mathbb{N}$.

viii.) *The $\mathscr{C}_0^\infty$-topology makes $\mathscr{C}_0^\infty(X)$ a meager space. It is not a Baire space and it is not first countable.*

ix.) *Let $\rho = \{\rho_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ be a set of continuous functions $\rho_\alpha \in \mathscr{C}(X)$ with $\rho_\alpha(x) \geq 0$ such that for every compact $K \subseteq X$ only finitely many ρ_α are non-zero on K. Then*

$$\mathrm{p}_\rho(f) = \sup_{\substack{x \in X \\ \alpha \in \mathbb{N}_0^n}} \rho_\alpha(x) \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| \tag{5.1.7}$$

defines a continuous seminorm on $\mathscr{C}_0^\infty(X)$ and the $\mathscr{C}_0^\infty$-topology is induced by all those p_ρ.

x.) *The $\mathscr{C}_0^\infty$-topology makes $\mathscr{C}_0^\infty(X)$ a barrelled and bornological space.*

Proof: All the statements have already been shown elsewhere, we just collected all the results in one theorem for convenience. We start with the first part of *vii.)*, which is the the universal property of a locally convex inductive limit according to Theorem 2.4.30, *iii.)*. Since the subspaces $\mathscr{C}_K^\infty(X) \subseteq \mathscr{C}^\infty(X)$ carry the subspace topology, the first statement becomes a consequence of part *vii.)*. The third part is the content of Proposition 2.4.50. The second part is a general feature of strict inductive limits. In fact, every compact subset $K \subseteq X$ can be taken as starting point $K_1 = K$ of an exhausting sequence of compact subsets as needed in part *iii.)*. Then Proposition 2.4.36 shows that the inclusions (5.1.3) are embeddings and Proposition 2.4.39 shows that the subspaces are closed. The fourth part is again a general consequence of strict countable inductive limits, see Theorem 2.4.46, and holds in general for LF spaces by Proposition 2.4.49. The convergence of sequences in strict countable inductive limits is described in general in Proposition 2.4.41, *ii.)*, and yields the fifth part at once. Proposition 2.4.41, *i.)*, gives the sixth part directly. Since the $\mathscr{C}_0^\infty$-topology is already determined by an exhausting sequence of compact subsets, the remaining statement in *vii.)* is simply a consequence of the universal property of the locally convex inductive limit (5.1.4). Then part *viii.)* is the content of Proposition 2.4.49, *i.)*, *iii.)* and *iv.)*. The explicit description of a defining system of seminorms in *ix.)* will not be used much and is proved in Exercise 5.5.1. Finally, *x.)* is a general property of LF spaces, see Corollary 3.3.9 and Corollary 3.3.13. □

In the following, it will be convenient to rely just on this theorem, when dealing with $\mathscr{C}_0^\infty(X)$. In particular, a clumsy but more elementary approach to the $\mathscr{C}_0^\infty$-topology without developing the theory of LF spaces consists in basing the definition of the $\mathscr{C}_0^\infty$-topology directly on the defining system of seminorms in (5.1.7) and derive all required properties from here. This point of view is taken e.g. in [5].

Remark 5.1.2 (Test functions $\mathscr{C}_0^k(X)$) For $k \in \mathbb{N}_0$ we get an analogous test function space $\mathscr{C}_0^k(X)$. Here all the statements of the theorem stay valid with an obvious modification of part *ix.)*: the seminorms defining the LF topology of $\mathscr{C}_0^k(X)$ are now given by

$$\mathrm{p}_\rho(f) = \sup_{\substack{x\in X\\ |\alpha|\le k}} \rho_\alpha(x) \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right| \tag{5.1.8}$$

parametrized by continuous functions $\rho_\alpha \in \mathscr{C}(X)$ with $\rho_\alpha(x) \ge 0$ for all $x \in X$. Here we only need to control finitely many derivatives and hence the local finiteness of the supports of the ρ_α as needed in Theorem 5.1.1, *ix.)*, is automatically fulfilled without restrictions, as there are only finitely many ρ_α at all. We will sometimes use these test functions as well.

Let us also recall the functorial properties of the $\mathscr{C}_0^\infty$-topology and some standard continuous linear maps. We start with the following observation:

Proposition 5.1.3 *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be non-empty open subsets and let $\phi\colon X \longrightarrow Y$ be a proper smooth map. Then the pull-back*

$$\phi^*\colon \mathscr{C}_0^\infty(Y) \longrightarrow \mathscr{C}_0^\infty(X) \tag{5.1.9}$$

is a continuous linear map.

Proof: Let $K \subseteq Y$ be compact. Then properness means that $\phi^{-1}(K) \subseteq X$ is compact as well. This shows that ϕ^* gives a linear map

$$\phi^*\colon \mathscr{C}_K^\infty(Y) \longrightarrow \mathscr{C}_{\phi^{-1}(K)}^\infty(X) \tag{$*$}$$

for all compact $K \subseteq Y$. Hence the pull-back of a function in $\mathscr{C}_0^\infty(Y)$ is again compactly supported. From Proposition 2.3.24 we know that the pull-back

$$\phi^*\colon \mathscr{C}^\infty(Y) \longrightarrow \mathscr{C}^\infty(X)$$

is continuous with respect to the $\mathscr{C}^\infty$-topologies. Hence it restricts to a continuous linear map $(*)$ for all K, since we use the subspace topologies. From the continuous inclusion (5.1.3) in Theorem 5.1.1, *ii.)*, we see that also

$$\phi^*\colon \mathscr{C}^\infty_K(Y) \longrightarrow \mathscr{C}^\infty_0(X)$$

is continuous for all compact $K \subseteq Y$. Hence the universal property of the locally convex inductive limit topology according to Theorem 5.1.1, *vii.)*, gives the continuity of (5.1.9). ☐

Note that the additional requirement of properness is crucial, since otherwise $\phi^* f$ might have a non-compact support. Of course, Proposition 5.1.3 can now be interpreted as obtaining a functor $\mathscr{C}^\infty_0$ from open subsets of $\mathbb{R}^n$ to LF spaces, see also Exercise 5.5.2. A similar argument shows that (smooth) differential operators are continuous, too:

Proposition 5.1.4 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let D be a smooth differential operator of order $r \in \mathbb{N}_0$. Then*

$$D\colon \mathscr{C}^\infty_0(X) \longrightarrow \mathscr{C}^\infty_0(X) \tag{5.1.10}$$

is a continuous linear map.

Proof: Since for a differential operator we have

$$\operatorname{supp}(D(f)) \subseteq \operatorname{supp}(f)$$

for all functions $f \in \mathscr{C}^\infty(X)$, we get for every compact subset $K \subseteq X$ a restriction satisfying

$$D\colon \mathscr{C}^\infty_K(X) \longrightarrow \mathscr{C}^\infty_K(X). \tag{$*$}$$

According to Proposition 2.3.22 the differential operator $D\colon \mathscr{C}^\infty(X) \longrightarrow \mathscr{C}^\infty(X)$ is continuous and hence yields a continuous restriction $(*)$ to the subspace $\mathscr{C}^\infty_K(X) \subseteq \mathscr{C}^\infty(X)$. Again, $\mathscr{C}^\infty_K(X) \subseteq \mathscr{C}^\infty_0(X)$ is a continuous inclusion and thus

$$D\colon \mathscr{C}^\infty_K(X) \longrightarrow \mathscr{C}^\infty_0(X)$$

is continuous for all compact $K \subseteq X$. Then again Theorem 5.1.1, *vii.)*, gives the continuity of (5.1.10). ☐

Remark 5.1.5 Again, both propositions have immediate analogues for compactly supported functions of finite differentiability class $\mathscr{C}^k$, see also Exercise 5.5.3 for details.

5.1.2 The Definition of Distributions

Any reasonable sort of function on X can be viewed as a linear functional on the test function space. This idea can be made precise for the large class of $\mathscr{L}^1_{\mathrm{loc}}$-functions. Recall from Exercise **??** that for $f \in \mathscr{L}^1_{\mathrm{loc}}(X)$ and $\varphi \in \mathscr{C}^\infty_0(X)$ the (Lebesgue) integral

$$I_f(\varphi) = \int_X f(x)\varphi(x)\,\mathrm{d}^n x \tag{5.1.11}$$

is well-defined: since $\operatorname{supp}(\varphi)$ is compact, $f\big|_{\operatorname{supp}(\varphi)}$ is integrable by definition of $\mathscr{L}^1_{\mathrm{loc}}$ and hence $f\varphi$ is integrable over $\operatorname{supp}(\varphi)$, too, yielding already the full integral (5.1.11). More precisely, one even has the following continuity estimate:

Proposition 5.1.6 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.*

i.) *For $f \in \mathscr{L}^1_{\mathrm{loc}}(X)$ and $\varphi \in \mathscr{C}^\infty_K(X)$, with $K \subseteq X$ compact, one has*

$$|I_f(\varphi)| \le \|f\|_{1,K} \mathrm{p}_{K,0}(\varphi). \tag{5.1.12}$$

ii.) *One has $I_f(\varphi) = 0$ for all $\varphi \in \mathscr{C}^\infty_0(X)$ iff $[f] = 0$ in $\mathrm{L}^1_{\mathrm{loc}}(X)$.*

Proof: Let $K \subseteq X$ be compact. Then we estimate

$$|I_f(\varphi)| \le \left| \int_X f(x)\varphi(x) \,\mathrm{d}^n x \right| \le \int_K |f(x)| \,\mathrm{d}^n x \sup_{x \in K} |\varphi(x)| = \|f\|_{1,K} \mathrm{p}_{K,0}(\varphi),$$

since outside of K the function φ is already identically zero. This is (5.1.12). The second part is slightly more involved: either one uses convolution to show that $\mathscr{C}^\infty_0(X) \subseteq \mathrm{L}^1_{\mathrm{loc}}(X)$ is dense or uses the fact that almost all points of X are Lebesgue points of f, see Exercise 5.5.4 for this approach. We avoid using convolution at this stage as we want to have an independent and more direct proof of this second statement. This will allow us to use it when we will discuss convolution. ☐

Corollary 5.1.7 *The assignment*

$$I \colon \mathrm{L}^1_{\mathrm{loc}}(X) \ni f \;\mapsto\; I_f \in \mathscr{C}^\infty_0(X)' \tag{5.1.13}$$

is a well-defined, injective linear map, which is continuous if we equip $\mathscr{C}^\infty_0(X)'$ with the strong topology.

Proof: Clearly, for a zero function $f \in \mathscr{L}^1_{\mathrm{loc}}(X)$ we have $I_f = 0$ by (5.1.12), since $\|f\|_{1,K} = 0$ for all compact subsets. Moreover, the estimate (5.1.12) shows that $I_f \in \mathscr{C}^\infty_0(X)'$ is a continuous linear functional. Proposition 5.1.6, *ii.)*, is the injectivity of (5.1.13). Finally, recall that the seminorms of the strong topology on the dual $\mathscr{C}^\infty_0(X)'$ are determined by bounded subsets $B \subseteq \mathscr{C}^\infty_0(X)$ and

$$\mathrm{p}_B(u) = \sup_{\varphi \in B} |u(\varphi)|$$

for $u \in \mathscr{C}^\infty_0(X)'$, see Example 3.2.18, *ii.)*. If B is bounded, we have a compact subset $K \subseteq X$ with $B \subseteq \mathscr{C}^\infty_K(X) \subseteq \mathscr{C}^\infty_0(X)$ by the general characterisation of bounded subsets in LF spaces applied to our case in Theorem 5.1.1, *vi.)*. Hence the boundedness of B implies

$$c_B = \sup_{\varphi \in B} \mathrm{p}_{K,0}(\varphi) < \infty.$$

Then (5.1.12) gives

$$\mathrm{p}_B(I_f) = \sup_{\varphi \in B} |I_f(\varphi)| \le \sup_{\varphi \in B} \|f\|_{1,K} \mathrm{p}_{K,0}(\varphi) = c_B \|f\|_{1,K},$$

which is the continuity of (5.1.13) in the strong topology. ☐

Since now all of our interesting function spaces can be (injectively) included into $\mathrm{L}^1_{\mathrm{loc}}(X)$, we get inclusions into $\mathscr{C}^\infty_0(X)'$ as well. This motivates now to call the linear functionals $\mathscr{C}^\infty_0(X)'$ *generalized functions* or *distributions*:

Definition 5.1.8 (Distribution) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. A continuous linear functional*

$$u \colon \mathscr{C}^\infty_0(X) \longrightarrow \mathbb{K} \tag{5.1.14}$$

is called distribution or generalized function on X. The space $\mathscr{C}^\infty_0(X)'$ of all distributions is equipped with the strong topology unless stated otherwise.

Remark 5.1.9 Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.

i.) Being a dual space we also have other important locally convex topologies at our disposal. Besides the strong topology based on the system of bounded subsets of $\mathscr{C}_0^\infty(X)$ the weak* topology will be used frequently. Here a defining system of seminorms is given by

$$\mathrm{p}_\varphi(\varphi) = |u(\varphi)|, \tag{5.1.15}$$

where $\varphi \in \mathscr{C}_0^\infty(X)$ and $u \in \mathscr{C}_0^\infty(X)'$.

ii.) The continuity condition for $u \in \mathscr{C}_0^\infty(X)'$ can be stated as follows: for every compact $K \subseteq X$ we have an index $\ell \in \mathbb{N}_0$ and a constant $c_{K,\ell}$ such that

$$|u(\varphi)| \le c_{K,\ell}\mathrm{p}_{K,\ell}(\varphi) \tag{5.1.16}$$

for all $\varphi \in \mathscr{C}_K^\infty(X)$. Indeed, this follows at once from the universal property of the locally convex inductive limit topology, see Theorem 5.1.1, *vii.)*, and the fact that the seminorms $\mathrm{p}_{K,\ell}$ determine the $\mathscr{C}^\infty$-topology on $\mathscr{C}_K^\infty(X)$. Alternatively, one can use the seminorms (5.1.7). However, this is typically more clumsy than the above criterion (5.1.16).

iii.) A linear functional $u\colon \mathscr{C}_0^\infty(X) \longrightarrow \mathbb{K}$ is continuous iff it is sequentially continuous. Indeed, if $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of test functions converging to $\varphi \in \mathscr{C}_0^\infty(X)$, then we find a compact subset $K \subseteq X$ with $\varphi_n, \varphi \in \mathscr{C}_K^\infty(X)$ for all $n \in \mathbb{N}$ and $\varphi_n \to \varphi$ in the $\mathscr{C}^\infty$-topology of $\mathscr{C}_K^\infty(X)$. So if u is sequentially continuous, then $u(\varphi_n) \to u(\varphi)$ and hence $u\big|_{\mathscr{C}_K^\infty(X)}$ is sequentially continuous. Since $\mathscr{C}_K^\infty(X)$ is a Fréchet space, $u\big|_{\mathscr{C}_K^\infty(X)}$ is continuous and thus u is continuous by Theorem 5.1.1, *vii.)*. In fact, this is a general feature of LF spaces, see Exercise 2.5.69. Note that this equivalence also holds for locally convex target spaces V different from the scalars $\mathbb{K}$.

We collect now several types of examples:

Example 5.1.10 (Distributions) Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.

i.) Every $f \in \mathrm{L}^1_{\mathrm{loc}}(X)$ can be considered as a distribution via the previous continuous inclusion $I\colon \mathrm{L}^1_{\mathrm{loc}}(X) \longrightarrow \mathscr{C}_0^\infty(X)'$ from Corollary 5.1.7. This gives already very many interesting examples, as $\mathrm{L}^1_{\mathrm{loc}}(X)$ contains most of our interesting function spaces. This is one of the motivations to call distributions also generalized functions. In particular, most of our function spaces on X are injectively included in $\mathrm{L}^1_{\mathrm{loc}}(X)$ and hence such functions are immediately identified with distributions.

ii.) More generally, reasonable Borel measures are distributions. Suppose μ is a positive Borel measure, i.e. a measure defined on a σ-algebra $\mathfrak{a}$ containing the Borel sigma algebra $\mathfrak{a}(X)$ given by all Borel measure subsets of X. We require that $\mu(K) < \infty$ for every compact subset $K \subseteq X$. In this case, every continuous function with compact support is μ-integrable and we have

$$\left| \int_X f \,\mathrm{d}\mu \right| \le \mu(K)\mathrm{p}_{K,0}(f), \tag{5.1.17}$$

whenever $f \in \mathscr{C}_0(X)$ with $\operatorname{supp}(f) \subseteq K$. As $\mathscr{C}_0^\infty(X) \subseteq \mathscr{C}_0(X)$, we get a distribution by integration

$$\mathscr{C}_0^\infty(X) \ni \varphi \mapsto \int_X \varphi \,\mathrm{d}\mu \in \mathbb{K} \tag{5.1.18}$$

by the explicit continuity estimate from (5.1.16).

iii.) Consider complex measures $\mathrm{Meas}(X)$ on X with respect to the Borel σ-algebra. Recall that $\mathrm{Meas}(X)$ is a complex Banach space with respect to the Banach norm being

$$\|\mu\| = |\mu|(X), \tag{5.1.19}$$

where $|\mu|$ is the total variation of μ and $\|\mu\|$ is the variational norm of μ, see e.g. [14, Chapter 6] for details on complex measures. For every $f \in \mathscr{BM}(X)$ we have the estimate

$$\left|\int_X f \,\mathrm{d}\mu\right| \leq \|f\|_\infty \|\mu\|, \tag{5.1.20}$$

resulting in the estimate

$$\left|\int_X f \,\mathrm{d}\mu\right| \leq \mathrm{p}_{K,0}(\varphi)\|\mu\| \tag{5.1.21}$$

for all test functions $\varphi \in \mathscr{C}^\infty_K(X)$ with $K \subseteq X$ being compact. Note that the constant $\|\mu\|$ in (5.1.21) is even independent of the compact subset K. It follows that the integration

$$\mathscr{C}^\infty_0(X) \ni \varphi \mapsto \int_X \varphi \,\mathrm{d}\mu \in \mathbb{C} \tag{5.1.22}$$

is a distribution. Moreover, a complex Borel measure μ is zero iff the integral with every continuous compactly supported function is zero. Since $\mathscr{C}^\infty_0(X) \subseteq \mathscr{C}(X)$ is dense and the integration is even continuous in the sup-norm by (5.1.20), we conclude that the distribution (5.1.18) is zero iff $\mu = 0$. This results in a linear map

$$\mathrm{Meas}(X) \ni \mu \mapsto \left(\varphi \mapsto \int_X \varphi \,\mathrm{d}\mu\right) \in \mathscr{C}^\infty_0(X)', \tag{5.1.23}$$

which is injective. As for $\mathrm{L}^1_{\mathrm{loc}}(X)$ this inclusion is continuous with respect to the strong topology of $\mathscr{C}^\infty_0(X)'$. The argument of the proof of Corollary 5.1.7 can be re-used and based on the estimate (5.1.21) instead of (5.1.12). Finally, note that our definition is consistent with (5.1.13) on the overlap, which we discuss in Exercise 5.5.5.

iv.) A particular case of a Borel measure is the point measure δ_{x_0} at a point $x_0 \in X$. This gives finally the Dirac δ-function, now of course as a functional

$$\delta_{x_0} \colon \mathscr{C}^\infty_0(X) \ni \varphi \mapsto \varphi(x_0) \in \mathbb{K}. \tag{5.1.24}$$

The continuity estimate to show $\delta_{x_0} \in \mathscr{C}^\infty_0(X)'$ is of course trivial: we have

$$|\delta_{x_0}(\varphi)| = \left.\begin{cases} 0 & \text{if } x_0 \notin K \\ |\varphi(x)| & \text{if } x_0 \in K \end{cases}\right\} \leq \mathrm{p}_{K,0}(\varphi) \tag{5.1.25}$$

for all compact subsets $K \subseteq X$ and $\varphi \in \mathscr{C}^\infty_K(X)$. This gives now, at last, a mathematically satisfying definition of what δ really should be. In the end, a perhaps disappointingly easy solution, which, however, will turn out to be not that innocent as we shall see.

We consider now some more specific examples in one dimension. Here some standard constructions are needed:

Example 5.1.11 (Heaviside function) Consider $X = \mathbb{R}$ and the function $\alpha \colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\alpha(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases} \tag{5.1.26}$$

This is clearly a continuous function and hence locally integrable. Thus we can view α as a distribution I_α. Moreover, α is differentiable for all $x \in \mathbb{R} \setminus \{0\}$ with derivative given by the *Heaviside function*

$$\theta(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases} \tag{5.1.27}$$

Indeed, we have $\alpha'(x) = \theta(x)$ for all $x \neq 0$. Also θ is locally integrable and hence a distribution: note that for I_θ it will not matter how we define $\theta(0)$, i.e. the value on the only point where α is not differentiable, since for the usage as distribution, only $\theta \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ matters.

Example 5.1.12 (Principal value I) The function $f\colon \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ given by

$$f(x) = \frac{1}{x} \tag{5.1.28}$$

is not locally integrable as a function on $\mathbb{R}$, no matter how we define $f(0)$. Nevertheless, we can construct a distribution on $\mathbb{R}$ in the spirit of using I_f. First we note that for $\varphi \in \mathscr{C}^\infty_0(\mathbb{R})$ the function $x \mapsto \varphi(x)f(x)$ is *not* integrable in general. Hence a naive definition of I_f as in (5.1.11) is not available. However, for $\epsilon > 0$ the integral

$$\int_{|x|\geq\epsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x = \int_{-\infty}^{-\epsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x + \int_{\epsilon}^{\infty} \frac{\varphi(x)}{x}\,\mathrm{d}x \tag{5.1.29}$$

is non-problematic. The idea is now that a *simultaneous* limit $\epsilon \to 0$ of both parts exists, since φ is $\mathscr{C}^\infty$ around 0. Indeed, we rewrite (5.1.29) as

$$\begin{aligned}
\int_{-\infty}^{-\epsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x + \int_{\epsilon}^{\infty} \frac{\varphi(x)}{x}\,\mathrm{d}x &= \int_{\epsilon}^{\infty} \frac{-\varphi(-x)}{x}\,\mathrm{d}x + \int_{\epsilon}^{\infty} \frac{\varphi(x)}{x}\,\mathrm{d}x \\
&= \int_{\epsilon}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,\mathrm{d}x \\
&= \int_{\epsilon}^{\infty} \int_{-1}^{1} \varphi'(tx)\,\mathrm{d}t\,\mathrm{d}x,
\end{aligned}$$

since φ' is continuous. We use this to show that the limit $\epsilon \to 0$ exists. Indeed, we get an estimate

$$\left| \int_{\epsilon}^{\infty} \int_{-1}^{1} \varphi'(tx)\,\mathrm{d}t\,\mathrm{d}x \right| \leq \int_{\epsilon}^{a} 2 \sup_{y\in[-a,a]} |\varphi'(y)| \leq 2a \sup_{y\in[-a,a]} |\varphi'(y)| \leq 2a\mathrm{p}_{[-a,a],1}(\varphi) \tag{5.1.30}$$

for all $\varphi \in \mathscr{C}^\infty_0(\mathbb{R})$ with $\operatorname{supp}(\varphi) \in [-a, a]$. Thus on the one hand, the limit $\epsilon \to 0$ exists, which allows us to define a linear functional

$$\left(\mathrm{vp}\,\frac{1}{x}\right)(\varphi) = \lim_{\epsilon\to 0} \int_{|x|\geq\epsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x \tag{5.1.31}$$

for $\varphi \in \mathscr{C}^\infty_0(\mathbb{R})$. On the other hand, (5.1.30) shows the continuity estimate

$$\left| \left(\mathrm{vp}\,\frac{1}{x}\right)(\varphi) \right| \leq 2a\mathrm{p}_{[-a,a],1}(\varphi) \tag{5.1.32}$$

for those $\varphi \in \mathscr{C}^\infty_0(\mathbb{R})$ with $\operatorname{supp}(\varphi) \subseteq [-a, a]$. Hence $\mathrm{vp}\,\frac{1}{x}$ is indeed a distribution, called the *principal value* of $\frac{1}{x}$. Remarkably, we needed the seminorm $\mathrm{p}_{[-a,a],1}$, controlling the *first derivative* of φ and not just the values like we did before. One can show that this is indeed necessary, see also Exercise 5.5.6.

Example 5.1.13 Let $\rho \in \mathscr{C}_0(\mathbb{R})$ be a continuous function with $\operatorname{supp}(\rho) \subseteq [0, 1]$ and $\rho(\frac{1}{2}) = 1$. Then $\rho \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R})$ and thus we have a distribution. Now we translate ρ by integers to get functions $\rho_n(x) = \rho(x - n)$ having support in $[n, n + 1]$. We define now a distribution by

$$u(\varphi) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \rho_n(x)\varphi^{(n)}(x)\,\mathrm{d}x. \tag{5.1.33}$$

To check that u is indeed a distribution, we fix a compact subset $K \subseteq \mathbb{R}$, say $K \subseteq [-N, N]$. Then for $\varphi \in \mathscr{C}^\infty_K(\mathbb{R})$ the integrals $\sum_{n=0}^\infty \int \rho_n(x)\varphi^{(n)}(x)\,\mathrm{d}x$ are zero, once $n > N$. Hence the series is a finite sum for all $\varphi \in \mathscr{C}^\infty_K(\mathbb{R})$, but the number of terms grows with increasing N. Thus we get an estimate of the form

$$
\begin{aligned}
|u(\varphi)| &\le \sum_{n=0}^{N} \int_{-\infty}^{\infty} |\rho_n(x)\varphi^{(n)}(x)|\,\mathrm{d}x \\
&\le \sum_{n=0}^{N} \int_{-\infty}^{\infty} |\rho_n(x)|\mathrm{p}_{[0,N],N}(\varphi)\,\mathrm{d}x \\
&= c_N \mathrm{p}_{[0,N],N}(\varphi)
\end{aligned}
\tag{5.1.34}
$$

with a constant c_N built out of the data in front of $\mathrm{p}_{[0,N],N}(\varphi)$. This shows again two things: the map u is a distribution and we need to control higher and higher order derivatives of φ with N increasing. There is no uniform bound on the order of differentiation needed.

5.1.3 Convergence and Completeness

For the distributions we have the strong topology as well as the weak* topology arising from the fact that $\mathscr{C}^\infty_0(X)'$ is a dual space, namely of $\mathscr{C}^\infty_0(X)$. We already know that the weak* topology, being algebraic in nature, has very poor completeness properties. The topological dual V' of a Hausdorff locally convex space V is always weak* dense in the algebraic dual V^*. Since V^* is weak* complete, V' is weak* complete iff $V' = V^*$, a situation being rarely of interest. Using the axiom of choice it is now fairly easy to see that for $\mathscr{C}^\infty_0(X)$ we do have discontinuous linear functionals and thus $\mathscr{C}^\infty_0(X)'$ is *not* complete in the weak* topology, see Exercise 5.5.7. However, completeness might not be the interesting question compared to *sequential* completeness. Here we have the following positive result for general reasons:

Proposition 5.1.14 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. Then $\mathscr{C}^\infty_0(X)'_\sigma$ is sequentially complete with respect to the weak* topology.*

PROOF: Indeed, this is one of the many useful applications of the Banach-Steinhaus Theorem 3.3.30. According to Corollary 3.3.38 duals of LF spaces are weak* sequentially complete. ☐

The weak* topology is the topology of pointwise convergence on $\mathscr{C}^\infty_0(X)$. In particular, a sequence $(u_m)_{m\in\mathbb{N}}$ of distributions $u_n \in \mathscr{C}^\infty_0(X)'$ converges to a distribution u in the weak* topology iff for all $\varphi \in \mathscr{C}^\infty_0(X)$ one has

$$
u(\varphi) = \lim_{m\to\infty} u_m(\varphi).
\tag{5.1.35}
$$

This gives, among other things, a new and in fact very weak notion of convergence also for all kind of other function spaces like e.g. $\mathrm{L}^1_{\mathrm{loc}}(X)$, which we can view as subspace of $\mathscr{C}^\infty_0(X)'$ as usual. Hence a sequence $(f_m)_{m\in\mathbb{N}}$ of functions $f_m \in \mathrm{L}^1_{\mathrm{loc}}(X)$ converges in the weak* topology inherited from $\mathrm{L}^1_{\mathrm{loc}}(X) \subseteq \mathscr{C}^\infty_0(X)'$ to a function f or a general distribution u iff

$$
u(\varphi) = \lim_{m\to\infty} I_{f_m}(\varphi), \quad \text{i.e.} \quad u(\varphi) = \lim_{m\to\infty} \int_X f_m(x)\varphi(x)\,\mathrm{d}^n x
\tag{5.1.36}
$$

holds for all test functions $\varphi \in \mathscr{C}^\infty_0(X)$. In this case one also says that $(f_m)_{m\in\mathbb{N}}$ *converges in the sense of distributions.* As $\mathrm{L}^1_{\mathrm{loc}}(X)$ and hence all other kind of interesting function spaces include continuously into $\mathscr{C}^\infty_0(X)'_\beta$ and hence also into $\mathscr{C}^\infty_0(X)'_\sigma$, original convergence implies convergence in the sense of distributions:

Remark 5.1.15 Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.

i.) Let $u_m \in \mathscr{C}_0^\infty(X)'$ be distributions such that for all test functions $\varphi \in \mathscr{C}_0^\infty(X)$ the limit

$$u(\varphi) = \lim_{m \to \infty} u_m(\varphi) \tag{5.1.37}$$

of numbers exists. Then this defines a functional u, which is easily seen to be linear and, by sequential weak* completeness of $\mathscr{C}_0^\infty(X)'$, it is a distribution $u \in \mathscr{C}_0^\infty(X)'$. This illustrates once more the easy usage of convergence in the sense of distributions. In fact, it is very hard to actually construct non-examples of distributions at all, see again Exercise 5.5.7.

ii.) Let $f_m\colon X \longrightarrow \mathbb{K}$ and $f\colon X \longrightarrow \mathbb{K}$ be functions for $m \in \mathbb{N}$.

- If $f_m \in \mathscr{C}^k(X)$ with $f_m \to f$ in the $\mathscr{C}^k$-topology for $k \in \mathbb{N}_0 \cup \{\infty\}$, then $f_m \to f$ in the sense of distributions.
- If $f_m \in \mathscr{L}^p(X)$ with $f_m \to f$ in the $\mathscr{L}^p$-topology for some $p \in [1, \infty]$, then $f_m \to f$ in the sense of distributions.
- If $f_m \in \mathscr{C}_0^k(X)$ with $f_m \to f$ in the $\mathscr{C}_0^k$-topology for some $k \in \mathbb{N}_0 \cup \{\infty\}$, then $f_m \to f$ in the sense of distributions.

In fact, in all these cases, the continuous inclusions of all the above function spaces into $\mathrm{L}^1_{\mathrm{loc}}(X)$, see **??**, and the subsequent continuous inclusion into $\mathscr{C}_0^\infty(X)'_\beta$ gives not only weak* convergence, but also *strong convergence* of distributions, i.e. uniform convergence on bounded subsets of test functions.

iii.) Let $\mu_m \in \operatorname{Meas}(X)$ be complex Borel measures for $m \in \mathbb{N}$ converging to $\mu \in \operatorname{Meas}(X)$ in the norm sense. Then $\mu_m \to \mu$ in the sense of distributions again in the strong topology by Example 5.1.10, *iii.)*. This implies that

$$\lim_{m \to \infty} \int_X \varphi \, \mathrm{d}\mu_m = \int_X \varphi \, \mathrm{d}\mu \tag{5.1.38}$$

holds for all test functions $\varphi \in \mathscr{C}_0^\infty(X)$. Sometimes (5.1.38) is also referred to as *weak convergence of measures.*

We come now to some first examples of convergent sequences of distributions. Most interesting examples arise, when we approach a distribution, which is *not* a function by functions. Here the approximations of the δ-functional are of archetypical importance:

Proposition 5.1.16 *Let $f \in \mathscr{L}^1(\mathbb{R}^n, \mathrm{d}^n x)$ be an integrable function with $\int f(x) \, \mathrm{d}^n x = 1$. Then*

$$f_\epsilon(x) = \tfrac{1}{\epsilon^n} f(\tfrac{1}{\epsilon} x) \tag{5.1.39}$$

for $\epsilon > 0$ is still integrable and one has

$$\lim_{\epsilon \searrow 0} f_\epsilon = \delta \tag{5.1.40}$$

in the sense of distributions.

Proof: First we note that $f_\epsilon \in \mathscr{L}^1(\mathbb{R}^n, \mathrm{d}^n x)$. In fact,

$$\int |f_\epsilon(x)| \, \mathrm{d}^n x = \int \tfrac{1}{\epsilon^n} |f(\tfrac{1}{\epsilon})| \, \mathrm{d}^n x = \int |f(y)| \, \mathrm{d}^n y = 1$$

after the substitution $y = \frac{1}{\epsilon} x$. To show the weak* convergence, let $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ be given. Then

$$I_{f_\epsilon}(\varphi) = \int f_\epsilon(x) \varphi(x) \, \mathrm{d}^n x = \int f(y) \varphi(\epsilon y) \, \mathrm{d}^n y$$

by the same substitution $y = \frac{1}{\epsilon}x$ as before. Now $|f(y)\varphi(\epsilon y)| \le c|f(y)|$, where $c < \infty$ is the maximum of $|\varphi|$, existing since $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ has compact support and is continuous. Thus the functions $y \mapsto f(y)\varphi(\epsilon y)$ are dominated by the integrable function $c|f|$ for all $\epsilon > 0$. Thus we can use Lebesgue's dominated convergence to get

$$\begin{aligned}
\lim_{\epsilon\to 0} I_{f_\epsilon}(\varphi) &= \lim_{\epsilon\to 0} \int f(y)\varphi(\epsilon y)\,\mathrm{d}^n y \\
&= \int f(y) \lim_{\epsilon\to 0} \varphi(\epsilon y)\,\mathrm{d}^n y \\
&= \varphi(0) \int f(y)\,\mathrm{d}^n y \\
&= \varphi(0).
\end{aligned}$$

This shows (5.1.40). ☐

Corollary 5.1.17 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset with $0 \in X$.*

i.) For every $\chi \in \mathscr{C}_0^\infty(X)$ and $\chi_\epsilon = \frac{1}{\epsilon}\chi(\frac{1}{\epsilon}x)$ one has

$$\lim_{\epsilon\to 0} \chi_\epsilon = c\delta \quad \textit{with} \quad c = \int_X \chi(x)\,\mathrm{d}^n. \tag{5.1.41}$$

ii.) The δ-functional is in the sequential closure of $\mathscr{C}_0^\infty(X)$ in the weak topology.*

PROOF: Indeed, $\chi \in \mathscr{L}^1(X, \mathrm{d}^n x)$ and a trivial trivial extension by zero gives $\chi \in \mathscr{L}^1(\mathbb{R}^n, \mathrm{d}^n x)$. Note that χ is still smooth, i.e. $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$. With the previous Proposition 5.1.16 we get (5.1.41) after rescaling χ with c if $c \ne 0$. If $c = 0$, then the proof of the proposition shows that (5.1.41) holds in this case as well. The second part is then clear. ☐

In fact, we will substantially extend the second statement once we investigate convolutions in Section 5.4.3.

Example 5.1.18 (Approximations of δ) The following explicit cases are used frequently to approximate the δ-functional:

i.) The perhaps simplest approximation of the δ-functional is given by the piecewise continuous function $f \in \mathscr{L}^1(\mathbb{R}, \mathrm{d}x)$ with

$$f(x) = \begin{cases} 0 & \text{for } |x| > \frac{1}{2} \\ 1 & \text{for } |x| \le \frac{1}{2}, \end{cases} \tag{5.1.42}$$

for which we can apply Proposition 5.1.16. Here we get

$$f_\epsilon(x) = \begin{cases} 0 & \text{for } |x| > \frac{\epsilon}{2} \\ \frac{1}{\epsilon} & \text{for } |x| \le \frac{\epsilon}{2}, \end{cases} \tag{5.1.43}$$

which comes close to the heuristic idea of the "δ-function" as $\epsilon \searrow 0$. In higher dimensions one gets similar, piecewise continuous versions.

ii.) Another, now real-analytic version is given by $f \in \mathscr{C}^\omega(\mathbb{R})$ with

$$f_\epsilon(x) = \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + x^2} \tag{5.1.44}$$

in this case.

iii.) In arbitrary dimensions we can use the normalized Gaussian function $f \in \mathscr{C}^\omega(\mathbb{R}^n)$ given by

$$f(x) = \frac{1}{(4\pi)^{n/2}} \mathrm{e}^{-\frac{1}{4}x^2}, \tag{5.1.45}$$

where $x^2 = \langle x, x \rangle$ is an abbreviation for the Euclidean norm square. By Proposition 5.1.16 the functions

$$f_\epsilon(x) = \frac{1}{(4\pi\epsilon^2)^{n/2}} \mathrm{e}^{-\frac{1}{4\epsilon^2}x^2} \tag{5.1.46}$$

approximate the δ-functional for $\epsilon \searrow 0$. Note that setting $t = \epsilon^2$ this rescaled Gaussian function satisfies the *heat equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t = \Delta f_t \tag{5.1.47}$$

on $\mathbb{R}^+ \times \mathbb{R}^n$ as an easy verification shows, see also Exercise 5.5.8. The function

$$f_t(x) = \frac{1}{(4\pi t)^{n/2}} \mathrm{e}^{-\frac{x^2}{4t}} \tag{5.1.48}$$

is also called the (Euclidean) *heat kernel*. We have

$$\lim_{t \searrow 0} f_t = \delta. \tag{5.1.49}$$

Though not related to the general result of Corollary 5.1.17, the following gives another approximation of a genuine distribution by locally integrable functions:

Example 5.1.19 (Principal value II) The way we defined the principal value of $\frac{1}{x}$ is exactly a limit in the sense of distributions. Indeed, consider the function $f_\epsilon \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{d}x)$ defined by

$$f_\epsilon(x) = \left.\begin{cases} 0 & \text{for } |x| \le \epsilon \\ \frac{1}{x} & \text{for } |x| \ge \epsilon, \end{cases}\right\} = \frac{\chi_{\mathbb{R}\setminus[-\epsilon,\epsilon]}(x)}{x}, \tag{5.1.50}$$

with the usual characteristic function $\chi_{\mathbb{R}\setminus[-\epsilon,\epsilon]}$ of $\mathbb{R} \setminus [-\epsilon, \epsilon]$. Then the distribution I_{f_ϵ} associated to it satisfies

$$\lim_{\epsilon \searrow 0} I_{f_\epsilon}(\varphi) = \int_{-\infty}^{\infty} f_\epsilon(x)\varphi(x)\,\mathrm{d}x = \lim_{\epsilon \searrow 0} \int_{|x| \ge \epsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x = \Big(\operatorname{vp}\frac{1}{x}\Big)(\varphi) \tag{5.1.51}$$

by the very definition of the distribution $\operatorname{vp}\frac{1}{x}$ in Example 5.1.12, see (5.1.31). Thus we have in the sense of distributions

$$\lim_{\epsilon \searrow 0} f_\epsilon = \operatorname{vp}\frac{1}{x}. \tag{5.1.52}$$

Hence also the principal value $\operatorname{vp}\frac{1}{x}$ is in the sequential closure of $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ inside $\mathscr{C}^\infty_0(\mathbb{R})'$.

Some more examples of distributions arising as the limit of functions are discussed in Exercise 5.5.9. We conclude this section with a remark on the completeness properties of $\mathscr{C}^\infty_0(X)'$ in other topologies. Quite remarkably, we get completeness and not just sequential completeness with respect to the strong topology:

Proposition 5.1.20 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. Then $\mathscr{C}^\infty_0(X)'_\beta$ is complete with respect to the strong topology.*

Proof: Indeed, here we can rely on Corollary 3.3.20 for general LF spaces which is a consequence of Theorem 3.3.18 and the fact that LF spaces are bornological. $\square$

5.1.4 Gluing and Restricting

This short technical section establishes the basic gluing features of distributions. In a more sophisticated language, distributions form a sheaf. Without using this notion we will nevertheless establish the relevant properties.

Consider again an open subset $X \subseteq \mathbb{R}^n$, non-empty to avoid trivialities. Moreover, if $U \subseteq X$ is open, then we can extend every $\varphi \in \mathscr{C}_0^\infty(U)$ to a test function $\varphi \in \mathscr{C}_0^\infty(X)$ by zero. This gives a continuous linear map

$$\iota_U \colon \mathscr{C}_0^\infty(U) \longrightarrow \mathscr{C}_0^\infty(X), \tag{5.1.53}$$

see Exercise 5.5.10. Note that we use the same symbol $\iota_U \colon U \longrightarrow X$ for the set-inclusion of U into X. Note also that $\iota_U \colon U \longrightarrow X$ is, in general, not a proper map at all, so we have no pull-back of test functions to open subsets.

Nevertheless, the continuity of (5.1.53) allows to use the *transpose map*

$$\iota_U^* \colon \mathscr{C}_0^\infty(X)' \longrightarrow \mathscr{C}_0^\infty(U)', \tag{5.1.54}$$

which we denote by ι_U^* instead of ι_U' to stress its role as a *pull-back*. Note that from the general theory, the continuity of ι_U implies the continuity of the restriction map ι_U^* with respect to either the weak* topology of distribution or the strong topology, see Corollary 3.2.22. Being a restriction, we will also use the notation

$$u\big|_U = \iota_U^* u \tag{5.1.55}$$

in the following.

Restrictions to yet smaller open subsets are compatible in the sense that we have for open subsets $U \subseteq V \subseteq X$

$$\iota_U^* = \iota_U^* \circ \iota_V^*, \tag{5.1.56}$$

meaning that first restricting to V and then to U is the same as restricting to U right from the beginning. Together with the trivial observation that $\iota_X^* = \mathrm{id}_{\mathscr{C}_0^\infty(X)'}$ we arrive at the statement that distributions form a presheaf. The following theorem then states that we actually have a sheaf instead of a presheaf only. However, we will not use the language of sheaves anywhere, but formulate the theorem more directly:

Theorem 5.1.21 (Gluing of Distributions) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. Moreover, let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X.*

i.) If $u \in \mathscr{C}_0^\infty(X)'$ is a distribution with $u\big|_{U_\alpha} = 0$ for all $\alpha \in I$, then $u = 0$.

ii.) If $u_\alpha \in \mathscr{C}_0^\infty(U_\alpha)'$ are distributions for all $\alpha \in I$ such that

$$u_\alpha\Big|_{U_\alpha \cap U_\beta} = u_\beta\Big|_{U_\alpha \cap U_\beta} \tag{5.1.57}$$

for all $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a (unique) $u \in \mathscr{C}_0^\infty(X)'$ with

$$u\Big|_{U_\alpha} = u_\alpha \tag{5.1.58}$$

for all $\alpha \in I$.

Proof: As a tool we choose a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha \in I}$. Recall that we find functions $\chi_\alpha \in \mathscr{C}_0^\infty(U_\alpha)$ with locally finite supports and $\sum_{\alpha \in I} \chi_\alpha = 1$ by Theorem 1.3.6. Note that we can achieve that the index set of the test functions is I by adding sufficiently many zero functions. Locally finite means that for every point $x \in X$ there is an open neighbourhood $U \subseteq X$ of x with $\mathrm{supp}(\chi_\alpha) \cap U = \emptyset$ for all but finitely many $\alpha \in I$. The usage of such a partition of unity will be crucial for both parts of the theorem. As an immediate consequence of the local finiteness we get the result that for every compact $K \subseteq X$ there are finitely many indices $\alpha_1, \ldots, \alpha_r \in I$ depending on K such that

$$(\chi_{\alpha_1} + \cdots + \chi_{\alpha_r})\big|_K = 1$$

and $\chi_\alpha\big|_K = 0$ for $\alpha \notin \{\alpha_1, \ldots, \alpha_r\}$. Hence we have

$$\varphi = \chi_{\alpha_1}\varphi + \cdots + \chi_{\alpha_r}\varphi \tag{$*$}$$

for all $\varphi \in \mathscr{C}^\infty_K(X)$. Now suppose $\varphi \in \mathscr{C}^\infty_0(X)$ is given and $u \in \mathscr{C}^\infty_0(X)'$ satisfies the assumptions of *i.)*. Then $\varphi \in \mathscr{C}^\infty_{\operatorname{supp}(\varphi)}(X)$ and hence $(*)$ holds for the appropriate indices. Since $\operatorname{supp}(\chi_\alpha\varphi) \subseteq U$ for all $\alpha \in I$ we get

$$\begin{aligned}
u(\varphi) &= u(\chi_{\alpha_1}\varphi + \cdots + \chi_{\alpha_r}\varphi) \\
&= u(\chi_{\alpha_1}\varphi) + \cdots + u(\chi_{\alpha_r}\varphi) \\
&= u_{\alpha_1}(\chi_{\alpha_1}\varphi) + \cdots + u_{\alpha_r}(\chi_{\alpha_r}\varphi) \\
&= 0
\end{aligned}$$

by the linearity of u and the assumption $u\big|_{U_\alpha} = 0$ for all $\alpha \in I$. Hence $u = 0$ follows proving *i.)*. For the existence of u in the second statement we define

$$u(\varphi) = \sum_{\alpha \in I} u_\alpha(\chi_\alpha\varphi) \tag{$**$}$$

for $\varphi \in \mathscr{C}^\infty_0(X)$. If $\varphi \in \mathscr{C}^\infty_K(X)$, then it is clear from $(*)$ that only finitely man indices $\alpha_1, \ldots, \alpha_r \in I$ depending on K but independent of φ contribute to this sum. Thus $(**)$ is a well-defined map. Clearly, u is linear as testing linearity involves only finitely many test functions, which have supports in a common large enough compact subset $K \subseteq X$. Hence the same $\alpha_1, \ldots, \alpha_r \in I$ can be used for all the involved functions and hence the linearity of $u_{\alpha_1}, \ldots, u_{\alpha_r}$ gives the linearity of u. Finally, it is easy to see that u is a distribution. Indeed, let $(\varphi_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathscr{C}^\infty_0(X)$ with limit $\varphi \in \mathscr{C}^\infty_0(X)$. By Theorem 5.1.1, *v.)*, we have a compact subset $K \subseteq X$ with $\varphi_n, \varphi \in \mathscr{C}^\infty_K(X)$ for all $n \in \mathbb{N}$. Hence

$$u(\varphi_n) = u_{\alpha_1}(\chi_{\alpha_1}\varphi_n) + \cdots + u_{\alpha_r}(\chi_{\alpha_r}\varphi_n)$$

with the same $\alpha_1, \ldots, \alpha_r \in I$ for all $n \in \mathbb{N}$, as we can use the same compact subset K for all. Now

$$\chi_{\alpha_i}\varphi_n \longrightarrow \chi_{\alpha_i}\varphi$$

by the continuity of the multiplication with a fixed function χ_{α_i} according to e.g. Proposition 5.1.4: this is a very particular case of a differential operator. Since $u_{\alpha_1}, \ldots, u_{\alpha_r}$ are continuous, we conclude $u(\varphi_n) \to u(\varphi)$. Thus u is sequentially continuous when restricted to $\mathscr{C}^\infty_K(X)$. As this is a Fréchet space, $u\big|_{\mathscr{C}^\infty_K(X)}$ is continuous. From the universal property of the locally convex inductive limit we can conclude that u is continuous, i.e. $u \in \mathscr{C}^\infty_0(X)'$ as wanted, see Theorem 5.1.1, *vii.)*, and Exercise 2.5.69. It remains to check that $u\big|_{U_\alpha} = u_\alpha$. Here the compatibility (5.1.57) enters. Let $\varphi \in \mathscr{C}^\infty_0(X)$ with $\operatorname{supp}(\varphi) \subseteq U_\alpha$ for some $\alpha \in I$. Let $\alpha_1, \ldots, \alpha_r \in I$ be the finitely many indices with $\chi_{\alpha_1} + \cdots + \chi_{\alpha_r}\big|_{\operatorname{supp}(\varphi)} = 1$ and thus

$$\varphi = \chi_{\alpha_1}\varphi + \cdots + \chi_{\alpha_r}\varphi$$

with

$$\operatorname{supp}(\chi_{\alpha_i}\varphi) \subseteq \operatorname{supp}(\chi_{\alpha_i}) \cap \operatorname{supp}(\varphi) \subseteq U_{\alpha_i} \cap U_\alpha.$$

Since $u_\alpha\big|_{U_{\alpha_i}\cap U_\alpha} = u_{\alpha_i}\big|_{U_{\alpha_i}\cap U_\alpha}$ and $\chi_{\alpha_i}\varphi \in \mathscr{C}_0^\infty(U_{\alpha_i}\cap U_\alpha)$ we thus get

$$\begin{aligned} u(\varphi) &= u(\chi_{\alpha_1}\varphi + \cdots + \chi_{\alpha_r}\varphi) \\ &= u_{\alpha_1}(\chi_{\alpha_1}\varphi) + \cdots + u_{\alpha_r}(\chi_{\alpha_r}\varphi) \\ &= u_\alpha(\chi_{\alpha_1}\varphi) + \cdots + u_\alpha(\chi_{\alpha_r}\varphi) \\ &= u_\alpha(\chi_{\alpha_1}\varphi) + \cdots + \chi_{\alpha_r}\varphi) \\ &= u_\alpha(\varphi) \end{aligned}$$

by the linearity of u_α. This shows $u\big|_{U_\alpha} = u_\alpha$ as wanted. The remaining uniqueness statement is just *i.)* applied to the difference of two hypothetical gluings of the u_α, which completes the proof. □

Remark 5.1.22 The two above properties distinguish a presheaf from a sheaf, as already mentioned. As a conclusion, distributions indeed behave like functions, as they can be restricted and are determined by their restrictions. They also can be glued from consistent local data. However, we emphasize that we can only restrict distributions to *open* subsets, but not to smaller ones. In particular, and unlike for functions, distributions can not be restricted to single points $\{x\} \subseteq X$. This was a feature already the "function" spaces $\mathrm{L}^p(X)$ or $\mathrm{L}^p_{\mathrm{loc}}(X)$ had: here also a restriction to open subsets is possible, but not to points. This is a crucial difference to the function spaces $\mathscr{L}^p(X)$ and $\mathscr{L}^p_{\mathrm{loc}}(X)$.

The possibility of restriction allows us now to define the *support* of a distribution as follows:

Definition 5.1.23 (Support of distributions) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $u \in \mathscr{C}_0^\infty(X)'$ be a distribution. Then the support of u is defined by*

$$\operatorname{supp}(u) = \big\{x \in X \;\big|\; \textit{for all open } U \subseteq X \textit{ with } x \in U \textit{ one has } u\big|_U \neq 0\big\}. \tag{5.1.59}$$

Proposition 5.1.24 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $u \in \mathscr{C}_0^\infty(X)'$.*

i.) The support $\operatorname{supp}(u) \subseteq X$ *is the smallest closed subset of X such that $u\big|_{X\setminus\operatorname{supp}(u)} = 0$. In particular,*

$$\operatorname{supp}(u) = (\operatorname{supp}(u))^{\mathrm{cl}}. \tag{5.1.60}$$

ii.) For $\varphi \in \mathscr{C}_0^\infty(X)$ with $\operatorname{supp}(\varphi) \cap \operatorname{supp} u = \emptyset$ *one has*

$$u(\varphi) = 0. \tag{5.1.61}$$

iii.) For $g \in \mathscr{C}(X)$ one has

$$\operatorname{supp}(I_g) = \operatorname{supp}(g). \tag{5.1.62}$$

PROOF: Let $x \in X \setminus \operatorname{supp}(u)$. Then we find an open neighbourhood $U \subseteq X$ of x with $u\big|_U = 0$. If $y \in U$, then U is also a neighbourhood of y and thus $y \notin \operatorname{supp}(u)$ follows. In particular, $U \subseteq X \setminus \operatorname{supp}(u)$ and thus $X \setminus \operatorname{supp}(u)$ is open, i.e. $\operatorname{supp}(u)$ is closed. Next, we show that $u\big|_{X\setminus\operatorname{supp}(u)} = 0$. Indeed, let $x \in X \setminus \operatorname{supp}(u)$, then we find an open neighbourhood $O_x \subseteq X$ with $u\big|_{O_x} = 0$ according to the definition of $\operatorname{supp}(u)$. Since $\operatorname{supp}(u)$ is closed, we can assume that $O_x \subseteq X \setminus \operatorname{supp}(u)$ already, taking the intersection with $X \setminus \operatorname{supp}(u)$ if needed. Here we use that also for all smaller open subsets $U \subseteq O_x$ we have $u\big|_U = 0$ by (5.1.54). Thus the open subsets $\{O_x\}_{x\in X\setminus\operatorname{supp}(u)}$ provide an open cover of $X \setminus \operatorname{supp}(u)$ with $u\big|_{O_x} = 0$ for all $x \in X \setminus \operatorname{supp}(u)$. By Theorem 5.1.21, *i.)*, we get $u\big|_{X\setminus\operatorname{supp}(u)} = 0$. Finally, we need to show that $\operatorname{supp}(u)$ is the smallest such subset. Assume $A \subseteq \operatorname{supp}(u)$ is a closed subset with $u\big|_{X\setminus A} = 0$ and let $x \in \operatorname{supp}(u) \setminus A$. Then $x \in X \setminus A$ and hence we have an open neighbourhood, namely $X \setminus A$, of x on which u vanishes. Thus $x \notin \operatorname{supp}(u)$, which is a contradiction:

we have $A = \operatorname{supp}(u)$, proving the first part. The second is clear since $\operatorname{supp}(\varphi) \cap \operatorname{supp}(u) = \emptyset$ means $\varphi \in \mathscr{C}_0^\infty(X \setminus \operatorname{supp}(u))$ and hence

$$u(\varphi) = u\big|_{X \setminus \operatorname{supp}(u)}(\varphi) = 0$$

by *ii.)*. Finally, let $g \in \mathscr{C}(X)$ be continuous. Consider first a point $x \in X$ with $g(x) \neq 0$. Then either $\operatorname{Re}(g)$ or $\operatorname{Im}(g)$ are different from zero in an open neighbourhood U of x by continuity. Without restriction, we assume $\operatorname{Re}(g) \neq 0$ on U. Shrinking U if necessary, we can assume $\operatorname{Re}(g) > 0$ without restriction, passing from g to $-g$ otherwise, without changing the support properties. If O is now an open neighbourhood of x, then we find a test function $\chi \in \mathscr{C}_0^\infty(O \cap U)$ with $\chi(x) > 0$ and hence

$$I_g(\varphi) = \int_X g\chi \, \mathrm{d}^n x = \int_{X \setminus \operatorname{supp}(g)} g\chi \, \mathrm{d}^n x = 0,$$

showing $I_g\big|_{X \setminus \operatorname{supp}(g)} = 0$. But then $x \notin \operatorname{supp}(I_g)$. Hence $X \setminus \operatorname{supp}(g) \subseteq X \setminus \operatorname{supp}(I_g)$, showing $\operatorname{supp}(I_g) \subseteq \operatorname{supp}(g)$, proving the third statement. □

Remark 5.1.25 (Support of distributions) Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.

i.) The explicit definition of the support given in (5.1.59) coincides with the sheaf-theoretic definition of support.

ii.) The consistency of the definition in (5.1.59) with the usual definition of supports of functions (5.1.62) uses the continuity of g in an essential way. One can also determine the support of I_g for $g \in \mathscr{L}^p_{\mathrm{loc}}(X)$ or for distributions arising from (complex) Borel measures as in Example 5.1.10. Then we will need an essential support instead of the (topological) support, see Exercise 5.5.11.

iii.) We have certain basic features of the support, which are immediate form the definition. For $u, v \in \mathscr{C}_0^\infty(X)'$ and $z \neq 0$ one has

$$\operatorname{supp}(zu) = \operatorname{supp}(u) \tag{5.1.63}$$

and

$$\operatorname{supp}(u + v) \subseteq \operatorname{supp}(u) \cup \operatorname{supp}(v), \tag{5.1.64}$$

where strict inclusions can occur in (5.1.64), e.g. for $v = -u$: it is clear that

$$\operatorname{supp}(0) = \emptyset, \tag{5.1.65}$$

e.g. by (5.1.62). Note that, conversely, $\operatorname{supp}(u) = \emptyset$ implies $u = 0$.

Example 5.1.26 (Support of δ-functional) Let $x_0 \in X$, then the δ-functional δ_{x_0} at x_0 has the support

$$\operatorname{supp}(\delta_{x_0}) = \{x_0\}. \tag{5.1.66}$$

This is obvious as for $\varphi \in \mathscr{C}_0^\infty(X)$ with $\operatorname{supp}(\varphi) \subseteq X \setminus \{x_0\}$ we have $\varphi(x_0) = 0$ and hence $\delta_{x_0}(\varphi) = 0$, proving $\operatorname{supp}(\delta_{x_0}) \subseteq \{x_0\}$. Since $\delta_{x_0} \neq 0$, we get (5.1.66).

Note that the δ-functional behaves very different from a continuous function with respect to the support. If $g \in \mathscr{C}(X)$, then $\operatorname{supp}(g)$ is determined by its open interior, i.e.

$$\operatorname{supp}(g) = (\operatorname{supp}(g)^\circ)^{\mathrm{cl}}. \tag{5.1.67}$$

Indeed, the points $x \in X$ with $g(x) \neq 0$ are inside $\operatorname{supp}(g)^\circ$ thanks to the continuity of g and their closure is the support by definition, showing (5.1.67).

To quantify the amount of singularity we will need to develop additional quantities beyond the support. A very rough but already interesting possibility is the singular support:

Definition 5.1.27 (Singular support) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $u \in \mathscr{C}_0^\infty(X)'$ be a distribution.*

i.) A point $x \in X$ is called regular for u if there exists an open neighbourhood $U \subseteq X$ of x and a smooth function $f \in \mathscr{C}^\infty(U)$ with

$$u\Big|_U = I_f. \tag{5.1.68}$$

Otherwise x is called singular for u.

ii.) The set of singular points of u is called the singular support of u, denoted by

$$\operatorname{sing\,supp}(u) = \big\{x \in X \bigm| x \textit{ is singular for } u\big\}. \tag{5.1.69}$$

The following proposition will now describe some first features of the singular support in analogy to the usual support:

Proposition 5.1.28 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $u \in \mathscr{C}_0^\infty(X)'$ be a distribution.*

i.) The singular support $\operatorname{sing\,supp}(u)$ *of u is the smallest closed subset of X, such that u is represented by a smooth function on the complement $X \setminus \operatorname{sing\,supp}(u)$. In particular,*

$$\operatorname{sing\,supp}(u) = \operatorname{sing\,supp}(u)^{\mathrm{cl}}. \tag{5.1.70}$$

ii.) One has

$$\operatorname{sing\,supp}(u) \subseteq \operatorname{supp}(u). \tag{5.1.71}$$

iii.) One has $\operatorname{sing\,supp}(u) = \emptyset$ *iff $u = I_f$ with $f \in \mathscr{C}^\infty(X)$.*

PROOF: If $x \in X$ is regular, then we have an open neighbourhood $U \subseteq X$ with $x \in U$ and $u\big|_U = I_f$ for some $f \in \mathscr{C}^\infty(U)$. But then all other points of U are regular as well with the same U and the same f to satisfy the condition required in (5.1.68). Hence the set of regular points is open, turning $\operatorname{sing\,supp}(u)$ into a closed subset. Now consider $x \in X \setminus \operatorname{sing\,supp}(u)$. Then we have an open neighbourhood $U_x \subseteq X$ for every such x and a smooth function $f_x \in \mathscr{C}^\infty(U_x)$ with $u\big|_{U_x} = I_{f_x}$. For $x, y \in X \setminus \operatorname{sing\,supp}(u)$ this implies

$$I_{f_x}\big|_{U_x \cap U_y} = u\big|_{U_x \cap U_y} = I_{f_y}\big|_{U_x \cap U_y},$$

and hence

$$f_x\big|_{U_x \cap U_y} = f_y\big|_{U_x \cap U_y}$$

as the assignment $f \mapsto I_f$ is injective on smooth functions by Corollary 5.1.7 and the fact that $\mathscr{C}^\infty(X)$ injects into $\mathrm{L}^1_{\mathrm{loc}}(X)$, see again Exercise 2.5.57. By the gluing properties of smooth functions according to Remark 1.3.3 we obtain a smooth function $f \in \mathscr{C}^\infty(X \setminus \operatorname{sing\,supp}(u))$. Hence

$$u\big|_{U_x} = I_{f_x} = I_f\big|_{U_x}$$

for all $x \in \operatorname{sing\,supp}(u)$ and thus $u\big|_{X \setminus \operatorname{sing\,supp}(u)} = I_f$ by Theorem 5.1.21, *i.)*. Thus u is represented by f on $X \setminus \operatorname{sing\,supp}(u)$. Finally, we need to show that $\operatorname{sing\,supp}(u)$ is the smallest closed subset with this property. Thus let $A \subseteq \operatorname{sing\,supp}(u)$ be a closed subset and assume that $u\big|_{X \setminus A}$ is represented by $f \in \mathscr{C}^\infty(X \setminus A)$. Then $x \in X \setminus A$ has an open neighbourhood, namely $X \setminus A$, with $u\big|_{X \setminus A} = I_f$, showing that x is regular. This means $x \in X \setminus \operatorname{sing\,supp}(u)$ and hence $\operatorname{sing\,supp}(u) \subseteq A$, proving $A = \operatorname{sing\,supp}(u)$. Since the zero function is smooth and $u\big|_{X \setminus \operatorname{sing\,supp}(u)} = 0 = I_0$, the second part is clear. So is the last part. □

The definition of the singular support indeed captures some aspects of singular behaviour. However, with this definition also e.g. continuous functions can have singularities, wherever they are not smooth:

Example 5.1.29 We consider $X = \mathbb{R}$ with the continuous function $\alpha\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\alpha(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 & \text{for } x > 0, \end{cases} \tag{5.1.72}$$

as in Example 5.1.11. Then all points $x \neq 0$ are regular, since the zero function is smooth on $\mathbb{R}^-$ and so is the identity function $x \mapsto x$ on $\mathbb{R}^+$. The point $x = 0$ has no open neighbourhood, on which α is differentiable, let alone smooth. Hence

$$\operatorname{sing\,supp}(\alpha) = \{0\}, \tag{5.1.73}$$

while $\operatorname{supp}(\alpha) = [0, \infty)$. An analogous example is the Heaviside function θ from Example 5.1.11. Again, we find

$$\operatorname{sing\,supp}(\theta) = \{0\} \tag{5.1.74}$$

and $\operatorname{supp}(\theta) = [0, \infty)$. Finally, for the δ-distribution we get

$$\operatorname{sing\,supp}(\delta) = \{0\} = \operatorname{supp}(\delta). \tag{5.1.75}$$

We will see that the three results are in fact related.

More examples for possible singular supports are discussed in Exercise 5.5.13.

5.1.5 Order of Distributions

In this very brief section we discuss the order of distributions. We have seen that continuity estimates of distributions involve the seminorms $\mathrm{p}_{K,\ell}$ for various $\ell \in \mathbb{N}_0$. The regular distributions and all distributions of the form I_f with $f \in \mathrm{L}^1_{\mathrm{loc}}(X)$ require only $\ell = 0$, but we have seen examples, where higher ℓ are needed like e.g. for the principal value $\operatorname{vp}\frac{1}{x}$. This motivates the following definition:

Definition 5.1.30 (Order of distributions) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $u \in \mathscr{C}^\infty_0(X)'$ be a distribution.*

i.) The order of u at $x \in X$ is defined by

$$\operatorname{ord}_x(u) = \inf_{x \in U \subseteq X} \big\{ \ell \in \mathbb{N}_0 \bigm| \textit{there exists } c > 0 \textit{ with } |u(\varphi)| \leq c\mathrm{p}_{U^{\mathrm{cl}},\ell}(\varphi) \textit{ for all } \varphi \in \mathscr{C}^\infty_{U^{\mathrm{cl}}}(X) \big\}, \tag{5.1.76}$$

where U ranges over the open neighbourhoods of x with compact closures.

ii.) The order of u on a subset $A \subseteq X$ is defined by

$$\operatorname{ord}_A(u) = \sup_{x \in A} \operatorname{ord}_x(u) \in \mathbb{N}_0 \cup \{\infty\}. \tag{5.1.77}$$

iii.) The (global) order of u is defined by

$$\operatorname{ord}(u) = \operatorname{ord}_X(u) = \sup_{x \in X} \operatorname{ord}_x(u) \in \mathbb{N}_0 \cup \{\infty\}. \tag{5.1.78}$$

Remark 5.1.31 If we would allow all compact subsets containing $x \in X$ in (5.1.76) instead of the neighbourhoods, we would always have $\operatorname{ord}_x(u) = 0$. Indeed, since $\mathscr{C}^\infty_{\{x\}}(X) = \{0\}$, the condition $|u(\varphi)| \leq 0$ is trivially fulfilled for all $\varphi \in \mathscr{C}^\infty_{\{x\}}(X)$. Also note that for every neighbourhood U of x with compact closure U^{cl} we always can find a $c > 0$ with $|u(\varphi)| \leq c \mathrm{p}_{U^{\mathrm{cl}},\ell}(\varphi)$ for all $\varphi \in \mathscr{C}^\infty_{U^{\mathrm{cl}}}(X)$ for *some* $\ell \in \mathbb{N}_0$. This is just the continuity condition of u. This shows that the order at $x \in X$ is well-defined and hence

$$\operatorname{ord}_x(u) \in \mathbb{N}_0 \tag{5.1.79}$$

holds for all $x \in X$ and all $u \in \mathscr{C}^\infty_0(X)'$.

Some first properties of the order of a distribution are now collected in the following proposition:

Proposition 5.1.32 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.*

i.) The order $\operatorname{ord}_x(u)$ of a distribution $u \in \mathscr{C}^\infty_0(X)'$ at $x \in X$ is a well-defined number $\operatorname{ord}_x(u) \in \mathbb{N}_0$.

ii.) The order of a distribution $u \in \mathscr{C}^\infty_0(X)'$ is locally bounded. More precisely, $x \mapsto \operatorname{ord}_x(u)$ is locally non-increasing in the sense that for $x \in X$ there exists an open neighbourhood $U \subseteq X$ of x such that

$$\operatorname{ord}_y(u) \leq \operatorname{ord}_x(u) \tag{5.1.80}$$

for all $y \in U$.

iii.) For $u, v \in \mathscr{C}^\infty_0(X)'$ and $z, w \in \mathbb{K}$ one has for all $x \in X$

$$\operatorname{ord}_x(zu + wv) \leq \max\{\operatorname{ord}_x(u), \operatorname{ord}_x(v)\}. \tag{5.1.81}$$

iv.) The order of a distribution $u \in \mathscr{C}^\infty_0(X)'$ at $x \in X$ is a local concept, i.e. for all open neighbourhoods $U \subseteq X$ of x one has

$$\operatorname{ord}_x(u) = \operatorname{ord}_x\left(u\big|_U\right). \tag{5.1.82}$$

v.) For $A \subseteq B \subseteq X$ one has

$$\operatorname{ord}_A(u) \leq \operatorname{ord}_B(u) \tag{5.1.83}$$

for all $u \in \mathscr{C}^\infty_0(X)'$.

PROOF: The first statement was already discussed in the previous Remark 5.1.31. For the second statement we observe that the infimum in the definition of the order at x is actually a minimum, since the possible values of ℓ are discrete. Moreover, if $U \subseteq X$ is an open neighbourhood of x with compact closure such that

$$|u(\varphi)| \leq c\mathrm{p}_{U^{\mathrm{cl}},\ell}(\varphi)$$

holds for all $\varphi \in \mathscr{C}^\infty_{U^{\mathrm{cl}}}(X)$, then $\operatorname{ord}_x(u) \leq \ell$. But that is true for all $y \in U$ as well and hence $\operatorname{ord}_y(u) \leq \ell$, too. This shows the second statement. In particular, $\operatorname{ord}_U(u) \leq \ell$ holds. The third part is clear, since if c, c' are constants such that

$$|u(\varphi)| \leq c\mathrm{p}_{U^{\mathrm{cl}},\ell}(\varphi) \quad \text{and} \quad |v(\varphi)| \leq c'\mathrm{p}_{U^{\mathrm{cl}},\ell'}(\varphi)$$

for all $\varphi \in \mathscr{C}^\infty_{U^{\mathrm{cl}}}(X)$, then

$$|(zu + wv)(\varphi)| \leq c''\mathrm{p}_{U^{\mathrm{cl}},\ell''}(\varphi)$$

with $c'' = |z|c + |w|c'$ and $\ell'' = \max\{\ell, \ell'\}$. In particular, this gives (5.1.81). The fourth part now follows, since if $V \subseteq U$ are open neighbourhoods of x with U^{cl} compact, then

$$\mathrm{p}_{V^{\mathrm{cl}},\ell}(\varphi) \leq \mathrm{p}_{U^{\mathrm{cl}},\ell}(\varphi).$$

Thus the definition of $\operatorname{ord}_x(u)$ using the infimum (5.1.76) gives (5.1.82) at once. The last statement is clear. $\square$

Note that a strict inequality in (5.1.81) is possible. We discuss this and some more elementary property of the order of distributions in Exercise 5.5.14. We return to our examples and discuss their orders:

Example 5.1.33 (Order of distributions)

i.) The global order of a distribution needs not to be finite. Here the distribution from Example 5.1.13 provides a simple example.

ii.) While the order is locally non-increasing, it can definitely drop strictly. Consider the distribution

$$u \colon \mathscr{C}_0^\infty(\mathbb{R}) \ni \varphi \;\mapsto\; \varphi(0)' \in \mathbb{K}. \tag{5.1.84}$$

We clearly have $\operatorname{supp}(u) = \operatorname{sing\,supp}(u) = \{0\}$ and the order of u is one at $x = 0$. However, at every other point, the order is zero, i.e.

$$\operatorname{ord}_x(u) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x \neq 0. \end{cases} \tag{5.1.85}$$

Taking higher derivatives instead we obtain distributions, where the other drops an arbitrary (finite) step.

iii.) The order of a distribution I_f with $f \in \mathrm{L}^1_{\mathrm{loc}}(X)$ or distributions originating from positive or complex Borel measures are all of the global order zero. On the other hand, the principal value $\operatorname{vp}\frac{1}{x}$ is of order zero at all points $x \neq 0$, since by Proposition 5.1.32, *iv.)*, we have for $x \neq 0$

$$\operatorname{ord}_x\left(\operatorname{vp}\tfrac{1}{x}\right) = \operatorname{ord}_x\left(\operatorname{vp}\tfrac{1}{x}\big|_{\mathbb{R}\setminus\{0\}}\right) = \operatorname{ord}_x\left(I_{1/x}\big|_{\mathbb{R}\setminus\{0\}}\right) = 0, \tag{5.1.86}$$

since for $\varphi \in \mathscr{C}_0^\infty(\mathbb{R} \setminus \{0\})$ we have $(\operatorname{vp}\frac{1}{x})(\varphi) = I_{1/x}(\varphi)$. For $x = 0$, however, we get only the estimate $\operatorname{ord}_0(\operatorname{vp}\frac{1}{x}) \leq 1$ from (5.1.32). It requires a little extra argument to show that we actually have

$$\operatorname{ord}_0(\operatorname{vp}\tfrac{1}{x}) = 1, \tag{5.1.87}$$

see again Exercise 5.5.6.

Corollary 5.1.34 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $u \in \mathscr{C}_0^\infty(X)'$ be a distribution. If $x \in X$ satisfies $\operatorname{ord}_x(u) \geq 1$, then $x \in \operatorname{sing\,supp}(u)$.*

Proof: Indeed, if $x \notin \operatorname{sing\,supp}(u)$, then there exists an open neighbourhood $U \subseteq X$ of x and a smooth function $f \in \mathscr{C}^\infty(U)$ with $u\big|_U = I_f$. But then Proposition 5.1.32, *iv.)*, shows

$$\operatorname{ord}_x(u) = \operatorname{ord}_x(u\big|_U) = \operatorname{ord}_x(I_f) = 0$$

by Example 5.1.33, *iii.)*. □

Note, however, that points x with $\operatorname{ord}_x(u) = 0$ can very well belong to the singular support: the order of the δ-functional is globally zero, but $0 \in \mathbb{R}$ is in the singular support. To compute the order of a compact subset the next proposition gives an alternative possibility. In the case of non-compact subsets the possibly infinite order would spoil the following construction:

Proposition 5.1.35 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $K \subseteq X$ be compact. For $u \in \mathscr{C}_0^\infty(X)'$ one has*

$$\operatorname{ord}_K(u) = \inf_{K \subseteq U} \left\{ \ell \in \mathbb{N}_0 \;\middle|\; \textit{there exists } c > 0 \textit{ with } |u(\varphi)| \leq c\mathrm{p}_{U^{\mathrm{cl}},\ell}(\varphi) \textit{ for all } \varphi \in \mathscr{C}^\infty_{U^{\mathrm{cl}}}(X) \right\}, \tag{5.1.88}$$

where $U \subseteq X$ ranges over those open subsets such that $K \subseteq U$ and U^{cl} is compact.

PROOF: First we note that for such U we always find some $\ell \in \mathbb{N}_0$ and some $c > 0$ such that $|u(\varphi)| \leq c\mathrm{p}_{U^{\mathrm{cl}},\ell}(\varphi)$, since U^{cl} is compact: this is the continuity of u. Hence the infimum $\ell_0 \in \mathbb{N}_0$ on the right hand side is well-defined and, in fact, a minimum. Suppose now $x \in K$. Since $x \in K \subseteq U$, the open subset U is a neighbourhood of x with compact closure. Thus the definition of $\mathrm{ord}_x(u)$ gives immediately $\mathrm{ord}_x(u) \leq \ell_0$, as u is one of the neighbourhoods the infimum (5.1.77) is taken over. This shows

$$\mathrm{ord}_K(u) = \sup_{x \in K} \mathrm{ord}_x(u) \leq \ell_0.$$

We need to show equality. Thus we assume that $\mathrm{ord}_K(u) < \ell_0$ and hence $\mathrm{ord}_x(u) < \ell_0$ for all $x \in K$. This means that for all $x \in K$ we find an open ball $\mathrm{B}_{r_x}(x)$ with $\mathrm{B}_{r_x}(x)^{\mathrm{cl}} \subseteq X$ and

$$|u(\varphi)| \leq c_x \mathrm{p}_{\mathrm{B}_{r_x}(x)^{\mathrm{cl}},\ell_x}(\varphi)$$

for some $c_x > 0$ and some $\ell_x < \ell_0$. By compactness of K finitely many of the open balls $\mathrm{B}_{r_x/3}(x)$ cover K, say

$$K \subseteq \mathrm{B}_{r_{x_1}/3}(x_1) \cup \cdots \cup \mathrm{B}_{r_{x_N}/3}(x_N) = U.$$

For the open cover $\tilde{U} = \mathrm{B}_{r_{x_1}/2}(x) \cup \cdots \cup \mathrm{B}_{r_{x_N}/(2)}(x)$ of K, we choose partition of unity, i.e. we find $\chi_1, \ldots, \chi_N \in \mathscr{C}^\infty(X)$ with $\mathrm{supp}(\chi_i) \subseteq \mathrm{B}_{r_{x_i}/2}(x_i)$ for $i = 1, \ldots, N$ and

$$\chi_1 + \cdots + \chi_N = 1_{\tilde{U}}$$

on $\tilde{U}$. If now $\varphi \in \mathscr{C}^\infty_{U^{\mathrm{cl}}}(X)$, then

$$\varphi = \chi_1\varphi + \cdots + \chi_N\varphi$$

follows with $\mathrm{supp}(\chi_i\varphi) \subseteq \mathrm{B}_{r_{x_i}/2}(x_i)$ for all $i = 1, \ldots, N$. Hence

$$\begin{aligned}
|u(\varphi)| &= |u(\chi_1\varphi + \cdots + \chi_N\varphi)| \\
&\leq c_{x_1}\mathrm{p}_{\mathrm{B}_{r_{x_1}}(x_1)^{\mathrm{cl}},\ell_{x_1}}(\chi_1\varphi) + \cdots + c_{x_1}\mathrm{p}_{\mathrm{B}_{r_{x_N}}(x_N)^{\mathrm{cl}},\ell_{x_N}}(\chi_N\varphi) \\
&\leq c\mathrm{p}_{U^{\mathrm{cl}},\ell}(\varphi)
\end{aligned}$$

by the usual continuity estimate for products and the fact that $\mathrm{supp}(\varphi) \subseteq U^{\mathrm{cl}}$. Here

$$\ell = \max\{\ell_{x_1}, \ldots, \ell_{x_N}\} < \ell_0$$

will do the job. But this shows that the infimum (5.1.88) is at most ℓ, since U is a valid choice to compute (5.1.88), contradicting our assumption. $\square$

Note that a similar construction for a non-compact subset $A \subseteq X$ instead of a compact K is not feasible, since A would not have neighbourhoods U with compact closure: thus an estimate like

$$|u(\varphi)| \leq c\mathrm{p}_{U^{\mathrm{cl}},\ell}(\varphi) \tag{5.1.89}$$

for $A \subseteq U$ might now hold at all, turning the right hand side into an undefined infimum: of course, assigning $\inf \emptyset = \infty$ would match here, as well.

5.2 Calculus with Distributions

One of the motivations to consider distributions was to extend the usual rules of calculus to functions beyond differentiable functions. Of course, one can not dream about differentiating e.g. continuous but non-differentiable functions in the usual pointwise sense: the relevant limits simply do not exist. However, in a distributional sense such manipulations will become available. All of the operations we want to introduce for distributions originate from *dualizing* the corresponding operations on test functions.

5.2.1 The Module Structure of Distributions

The first operation is very simple and comes from the following module structure of test functions:

Proposition 5.2.1 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $k \in \mathbb{N}_0 \cup \{\infty\}$.*

i.) For every compact subset $K \subseteq X$ the test functions $\mathscr{C}^k_K(X)$ are a module over $\mathscr{C}^k(X)$ with respect to the pointwise operations.

ii.) For every compact subset $K \subseteq X$ and all $\ell \in \mathbb{N}_0$ with $\ell \leq k$ one has the continuity estimate

$$\mathrm{p}_{K,\ell}(f\varphi) \leq \mathit{const}\, \mathrm{p}_{K,\ell}(f)\mathrm{p}_{K,\ell}(\varphi) \tag{5.2.1}$$

for $f \in \mathscr{C}^k(X)$ and $\varphi \in \mathscr{C}^k_K(X)$ with a constant depending on ℓ only.

iii.) The test functions $\mathscr{C}^k_0(X)$ become a module over $\mathscr{C}^k(X)$ via the pointwise operations. For each $f \in \mathscr{C}^k(X)$ the multiplication with f is a continuous endomorphism of the LF space $\mathscr{C}^k_0(X)$.

Proof: The first statement is clear. Note that

$$\operatorname{supp}(f\varphi) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(\varphi)$$

for all $f \in \mathscr{C}^k(X)$ and $\varphi \in \mathscr{C}^k_0(X)$. Thus the subspaces $\mathscr{C}^k_K(X) \subseteq \mathscr{C}^k_0(X)$ are invariant under the module multiplication. The estimate (5.2.1) is a simple exercise based on the Leibniz rule, see also Exercise 5.5.15. In fact, we used this estimate in the proof of Proposition 2.3.22 already in a slightly different context. We will see later that (5.2.1) means that the *bilinear* map

$$\mathscr{C}^k(X) \times \mathscr{C}^k_K(X) \ni (f, \varphi) \;\mapsto\; f\varphi \in \mathscr{C}^k_K(X)$$

is continuous. However, for the time being, we will not need this result, but only the continuity of the linear map

$$\mathscr{C}^k_K(X) \ni \varphi \;\mapsto\; f\varphi \in \mathscr{C}^k_K(X)$$

for each fixed f. It follows that also

$$\mathscr{C}^k_K(X) \ni \varphi \;\mapsto\; f\varphi \in \mathscr{C}^k_0(X)$$

is continuous for all compact $K \subseteq X$. Hence the linear map

$$\mathscr{C}^k_0(X) \ni \varphi \;\mapsto\; f\varphi \in \mathscr{C}^k_0(X)$$

is continuous by the usual universal property of an inductive limit, see again Theorem 5.1.1, *vii.)*. □

The continuity of the multiplication operator gives us immediately a continuous transpose map:

Definition 5.2.2 (Module structure of distributions) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. For a distribution $u \in \mathscr{C}^\infty_0(X)'$ and a smooth function $f \in \mathscr{C}^\infty(X)$ one defines the new distribution fu by*

$$(fu)(\varphi) = u(f\varphi). \tag{5.2.2}$$

Proposition 5.2.3 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.*

i.) The multiplication (5.2.2) of distributions with smooth functions endows $\mathscr{C}^\infty_0(X)'$ with a module structure over $\mathscr{C}^\infty(X)$.

ii.) For every $f \in \mathscr{C}^\infty(X)$ the multiplication by f is a continuous linear map

$$f\colon \mathscr{C}^\infty_0(X)'_\sigma \longrightarrow \mathscr{C}^\infty_0(X)'_\sigma \tag{5.2.3}$$

as well as a continuous linear map

$$f\colon \mathscr{C}^\infty_0(X)'_\beta \longrightarrow \mathscr{C}^\infty_0(X)'_\beta. \tag{5.2.4}$$

iii.) *The canonical inclusion* $I\colon \mathrm{L}^1_{\mathrm{loc}}(X) \longrightarrow \mathscr{C}^\infty_0(X)'$ *is a module morphism, i.e. for all* $f \in \mathscr{C}^\infty(X)$ *and* $g \in \mathrm{L}^1_{\mathrm{loc}}(X)$ *one has*

$$I_{fg} = f I_g. \tag{5.2.5}$$

iv.) *For* $U \subseteq X$ *open and* $f \in \mathscr{C}^\infty(X)$ *one has for all* $u \in \mathscr{C}^\infty_0(X)'$

$$(fu)\Big|_U = f\Big|_U \cdot u\Big|_U, \tag{5.2.6}$$

i.e. the module structure is local.

v.) *For* $f \in \mathscr{C}^\infty(X)$ *and* $u \in \mathscr{C}^\infty_0(X)'$ *one has*

$$\operatorname{supp}(fu) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(u) \tag{5.2.7}$$

and

$$\operatorname{sing\,supp}(fu) \subseteq \operatorname{supp}(f) \cap \operatorname{sing\,supp}(u). \tag{5.2.8}$$

vi.) *For* $f \in \mathscr{C}^\infty(X)$ *and* $u \in \mathscr{C}^\infty_0(X)'$ *one has*

$$\operatorname{ord}_x(fu) \leq \operatorname{ord}_x(u). \tag{5.2.9}$$

PROOF: First we note that by definition the map $u \mapsto fu$ is the transpose of the module multiplication by f on the $\mathscr{C}^\infty(X)$-module $\mathscr{C}^\infty_0(X)$. Hence $fu \in \mathscr{C}^\infty_0(X)'$ for all $u \in \mathscr{C}^\infty_0(X)'$ as needed. The algebraic properties of a $\mathscr{C}^\infty(X)$-module are then quickly verified. Note that we do not have to worry about left/right module structures, since $\mathscr{C}^\infty(X)$ is commutative. This shows the first part, the second is a consequence of general continuity properties of dual maps, as discussed in Corollary 3.2.22. For the third part we first recall that $\mathrm{L}^1_{\mathrm{loc}}(X)$ is indeed a $\mathscr{C}^\infty(X)$-module for the pointwise operations, see also Exercise 5.5.16. To check (5.2.5), let $f \in \mathscr{C}^\infty(X)$, $g \in \mathrm{L}^1_{\mathrm{loc}}(X)$, and $\varphi \in \mathscr{C}^\infty_0(X)$ be given. Then

$$I_{fg}(\varphi) = \int_X f(x) g(x) \varphi(x) \,\mathrm{d}^n x = I_g(f\varphi).$$

Next, consider an open subset $U \subseteq X$ and $\varphi \in \mathscr{C}^\infty_0(U)$, which we view as a test function $\varphi \in \mathscr{C}^\infty_0(X)$, as usual. Then

$$(fu)\big|_U(\varphi) = (fu)(\varphi) = u(f\varphi) = u\big|_U\big(f\big|_U\varphi\big),$$

since $\operatorname{supp}(f\varphi) \subseteq \operatorname{supp}(\varphi) \subseteq U$ again and we do not need other information than $f\big|_U$ to compute $f\varphi$. This proves (5.2.6). Then (5.2.7) is obtained as follows. For $\varphi \in \mathscr{C}^\infty_0(X)$ with $\operatorname{supp}(\varphi) \subseteq X \setminus \operatorname{supp}(u)$ we also have $\operatorname{supp}(f\varphi) \subseteq \operatorname{supp}(\varphi) \subseteq X \setminus \operatorname{supp}(u)$. Hence $(fu)(\varphi) = u(f\varphi) = 0$ for those φ, showing $\operatorname{supp}(fu) \subseteq \operatorname{supp}(u)$. If $\operatorname{supp}(\varphi) \subseteq X \setminus \operatorname{supp}(f)$, then $f\varphi = 0$ and thus $(fu)(\varphi) = u(f\varphi) = 0$, too, showing $\operatorname{supp}(fu) \subseteq \operatorname{supp}(f)$, which gives (5.2.7). Now let $g \in \mathscr{C}^\infty(X \setminus \operatorname{sing\,supp}(u))$ be the (unique) smooth function such that $u\big|_{X\setminus\operatorname{sing\,supp}(u)} = I_g$. Then

$$\begin{aligned}
(fu)\Big|_{X\setminus\operatorname{sing\,supp}(u)} &\overset{(5.2.6)}{=} f\Big|_{X\setminus\operatorname{sing\,supp}(u)} u\Big|_{X\setminus\operatorname{sing\,supp}(u)} \\
&= f\Big|_{X\setminus\operatorname{sing\,supp}(u)} I_g \\
&\overset{(5.2.5)}{=} I_{f|_{X\setminus\operatorname{sing\,supp}(u)} g},
\end{aligned}$$

with $f\big|_{X\setminus\operatorname{sing\,supp}(u)} g \in \mathscr{C}^\infty(X \setminus \operatorname{sing\,supp}(u))$. Thus fu is represented by this smooth function on the open subset $X \setminus \operatorname{sing\,supp}(u)$. This shows

$$\operatorname{sing\,supp}(fu) \subseteq \operatorname{sing\,supp}(u).$$

Since we already know $\operatorname{sing\,supp}(fu) \subseteq \operatorname{supp}(fu) \subseteq \operatorname{supp}(f)$ by Proposition 5.1.28, *ii.)*, and (5.2.7), we conclude (5.2.8). Next, let $K \subseteq X$ be compact and let $\ell \in \mathbb{N}_0$ and $c > 0$ be such that

$$|u(\varphi)| \leq c \cdot \mathrm{p}_{K,\ell}(\varphi)$$

for all $\varphi \in \mathscr{C}^\infty_K(X)$. Then we get the continuity estimate

$$|(fu)(\varphi)| = |u(f\varphi)| \leq c \cdot \mathrm{p}_{K,\ell}(f\varphi) \leq \tilde{c} \cdot \mathrm{p}_{K,\ell}(f) \cdot \mathrm{p}_{K,\ell}(\varphi)$$

with some numerical constant $\tilde{c}$, according to the continuity of the module structure in (5.2.1). Hence we get the same ℓ for K also for the new distribution fu. This gives (5.2.9) at once. ☐

The proposition can be seen as a collection of compatibility properties. The module structure is compatible with the sheaf-structures of $\mathscr{C}^\infty(X)$ and $\mathscr{C}^\infty_0(X)'$, the inclusion of functions is compatible with the module structure, the notions of supports, and singular supports as well. Finally, the last statement can be seen as the dual statement that the module structure of $\mathscr{C}^\infty_0(X)$ as $\mathscr{C}^\infty(X)$-module is of order zero: in (5.2.1) the number of derivatives stays the same. We consider now some examples:

Example 5.2.4 Consider the δ-distribution at $0 \in \mathbb{R}$ and the smooth function $x \in \mathbb{R}$. Then we have

$$x\delta = 0, \tag{5.2.10}$$

since for every $\varphi \in \mathscr{C}^\infty_0(\mathbb{R})$ we have $x\varphi(x)\big|_{x=0} = 0$. Since $\operatorname{supp}(x) = \mathbb{R}$ and $\operatorname{supp}(\delta) = \operatorname{sing\,supp}(\delta) = \{0\}$, we see that the inclusions in (5.2.7) and (5.2.8) can be strict. Also the order can drop strictly. Here we once again consider the distribution

$$u(\varphi) = \varphi'(0). \tag{5.2.11}$$

Then $\operatorname{sing\,supp}(u) = \{0\}$ and $\operatorname{ord}_0(u) = 1$, as we have seen in Example 5.1.33, *ii.)*. However, $x^2 u = 0$, since $(x^2\varphi(x))'\big|_{x=0} = 0$ for all $\varphi \in \mathscr{C}^\infty_0(\mathbb{R})$. In Exercise 5.5.17 we discuss the multiplication of $\operatorname{vp}\frac{1}{x}$ with the function x.

Example 5.2.5 (Partition of unity) Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X and let $\{\chi_\beta\}_{\beta \in J}$ be a partition of unity subordinate to this cover, i.e. $x_\beta \in \mathscr{C}^\infty_0(X)$ with a refinement map $J \ni \beta \mapsto \alpha(\beta) \in I$ such that $\operatorname{supp}(\chi_\beta) \subseteq U_{\alpha(\beta)}$ and $\sum_{\beta \in J} \chi_\beta = 1$. Here the sum is locally finite, as usual. For a distribution $u \in \mathscr{C}^\infty_0(X)'$ we define then the distributions

$$u_\beta = \chi_\beta u. \tag{5.2.12}$$

We have $\operatorname{supp}(u_\beta) \subseteq U_{\alpha(\beta)}$ and the support of each u_β is even compact, since $\operatorname{supp}(\chi_\beta)$ is. We claim that we can add all u_β to recover u. Here the argument with locally finite supports is slightly more involved, since we can not add distributions in a pointwise sense unlike we do for functions. Nevertheless, we have for every $\varphi \in\in \mathscr{C}^\infty_0(X)$ only finitely many $\beta \in J$ with $\operatorname{supp}(\chi_\beta) \cap \operatorname{supp}(\varphi) \neq \emptyset$. Hence only finitely many $\beta_1, \ldots, \beta_k$ gives $u_{\beta_i}(\varphi) \neq 0$, turning the sum of all these values into a well-defined quantity. In fact,

$$\sum_{\beta \in J} u_\beta(\varphi) = \sum_{i=1}^k u_{\beta_i}(\varphi) = \sum_{i=1}^k u(\chi_{\beta_i}\varphi) = u\Bigg(\sum_{i=1}^k \chi_{\beta_i}\varphi\Bigg) = u(\varphi). \tag{5.2.13}$$

Thus for all $\varphi \in \mathscr{C}^\infty_0(X)$ we get (5.2.13) and hence

$$\sum_{\beta \in J} u_\beta = u \tag{5.2.14}$$

holds in the weak* sense. Indeed, the series converges in the weak* topology, since for each $\varphi \in \mathscr{C}^\infty_0(X)$ the evaluation on φ gives even a finite sum only. This is a very useful way to construct localized versions of distributions with supports in prescribed open subsets.

The somewhat trivial convergence (5.2.14) suggests to investigate the continuity properties of the module structure in both argument. Note that the next statement does not follow directly from the general properties of transposition unlike as the continuity result in Proposition 5.2.3, *ii.)*.

Proposition 5.2.6 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. For every $u \in \mathscr{C}_0^\infty(X)'$ the module multiplication yields a continuous linear map*

$$\mathscr{C}^\infty(X) \ni f \mapsto fu \in \mathscr{C}_0^\infty(X)'_\beta, \tag{5.2.15}$$

and thus a continuous linear map

$$\mathscr{C}^\infty(X) \ni f \mapsto fu \in \mathscr{C}_0^\infty(X)'_\sigma. \tag{5.2.16}$$

Proof: The continuity of (5.2.16) follows from (5.2.15), since the strong topology is finer than the weak* topology in general. Thus let $B \subseteq \mathscr{C}_0^\infty(X)$ be a bounded subset. From Theorem 5.1.1, *vi.)*, we get a compact subset $K \subseteq X$ with $B \subseteq \mathscr{C}_K^\infty(X)$ and B is bounded in this Fréchet space. For this K we find a $c > 0$ and an $\ell \in \mathbb{N}_0$ such that

$$|u(\varphi)| \leq c \cdot \mathrm{p}_{K,\ell}(\varphi)$$

for all $\varphi \in \mathscr{C}_K^\infty(X)$ by continuity of u. This gives

$$\begin{aligned}
\mathrm{p}_B(fu) &= \sup_{\varphi\in B} |(fu)(\varphi)| \\
&= \sup_{\varphi\in B} |u(f\varphi)| \\
&\overset{(a)}{\leq} \sup_{\varphi\in B} c\mathrm{p}_{K,\ell}(f\varphi) \\
&\overset{(b)}{\leq} \sup_{\varphi\in B} \tilde{c}\mathrm{p}_{K,\ell}(f)\mathrm{p}_{K,\ell}(\varphi) \\
&\overset{(c)}{\leq} c'\mathrm{p}_{K,\ell}(f),
\end{aligned}$$

where we used $f\varphi \in \mathscr{C}_K^\infty(X)$ in (a), the continuity of the module structure of $\mathscr{C}_K^\infty(X)$ in (b), and the boundedness of B in (c), yielding some numerical constants $\tilde{c}$ and c'. This is the required continuity estimate for (5.2.15). ☐

5.2.2 Differentiation of Distributions

From Proposition 5.1.4 we know that arbitrary smooth differential operators are continuous linear maps on test functions. Thus we can transpose them to obtain weak* continuous or strongly continuous linear maps on distributions. However, the compatibility with the inclusion of various function spaces into the distributions leads to a slightly modified definition. Note, however, that this is just a convention, which turns out to be very useful.

Lemma 5.2.7 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. Let $\varphi \in \mathscr{C}_0^\infty(X)$ and let $f \in \mathscr{C}^1(X)$. Then for all $i = 1, \ldots, n$ one has*

$$I_f\Big(\frac{\partial\varphi}{\partial x^i}\Big) = -I_{\frac{\partial f}{\partial x^i}}(\varphi). \tag{5.2.17}$$

Proof: This is just integration by parts: we have

$$I_f\Big(\frac{\partial\varphi}{\partial x^i}\Big) = \int_X f(x)\frac{\partial\varphi}{\partial x^i}(x)\,\mathrm{d}^n x = -\int_X \frac{\partial f}{\partial x^i}(x)\varphi(x)\,\mathrm{d}^n x = -I_{\frac{\partial f}{\partial x^i}}(\varphi),$$

since the support of φ and hence of $f\frac{\partial\varphi}{\partial x^i}$ is compact. We do not get any boundary contributions. ☐

Thus we define the differentiation of distributions using the transpose of the differentiations of test functions *with* the additional sign (5.2.17) such that the inclusion

$$I \colon \mathscr{C}^1(X) \longrightarrow \mathscr{C}_0^\infty(X)' \tag{5.2.18}$$

becomes compatible with differentiation:

Definition 5.2.8 (Differentiation of distributions) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. For $u \in \mathscr{C}_0^\infty(X)'$ we define*

$$\frac{\partial u}{\partial x^i} \colon \mathscr{C}_0^\infty(X) \longrightarrow \mathbb{K} \tag{5.2.19}$$

for $i = 1, \ldots, n$ by

$$\frac{\partial u}{\partial x^i}(\varphi) = -u\Big(\frac{\partial \varphi}{\partial x^i}\Big). \tag{5.2.20}$$

Thus we have the transpose of the derivative of test functions up to a sign. The continuity properties are of course not affected by this, leading to the following basic properties:

Proposition 5.2.9 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.*

i.) For every $u \in \mathscr{C}_0^\infty(X)'$ and $i = 1, \ldots, n$ one has $\frac{\partial u}{\partial x^i} \in \mathscr{C}_0^\infty(X)'$.

ii.) Partial derivatives of distributions commute, i.e.

$$\frac{\partial}{\partial x^j}\frac{\partial u}{\partial x^i} = \frac{\partial}{\partial x^i}\frac{\partial u}{\partial x^j} \tag{5.2.21}$$

for all $i, j = 1, \ldots, n$ and $u \in \mathscr{C}_0^\infty(X)'$.

iii.) Differentiation of distributions is a continuous linear map, i.e. one has the continuity in the weak topology*

$$\frac{\partial}{\partial x^i} \colon \mathscr{C}_0^\infty(X)'_\sigma \longrightarrow \mathscr{C}_0^\infty(X)'_\sigma \tag{5.2.22}$$

as well as in the strong topology

$$\frac{\partial}{\partial x^i} \colon \mathscr{C}_0^\infty(X)'_\beta \longrightarrow \mathscr{C}_0^\infty(X)'_\beta \tag{5.2.23}$$

for all $i = 1, \ldots, n$.

iv.) Differentiation of distributions extends the usual differentiation, i.e. for $f \in \mathscr{C}^1(X)$ one has

$$\frac{\partial I_f}{\partial x^i} = I_{\frac{\partial I_f}{\partial x^i}} \tag{5.2.24}$$

for all $i = 1, \ldots, n$.

v.) Differentiation of distributions is local, i.e. for $u \in \mathscr{C}_0^\infty(X)'$ and an open subset $U \subseteq X$ one has

$$\frac{\partial u}{\partial x^i}\Big|_U = \frac{\partial}{\partial x^i}\big(u\big|_U\big) \tag{5.2.25}$$

for all $i = 1, \ldots, n$.

vi.) Differentiation of distributions does not increase the support or the singular support, i.e. one has

$$\operatorname{supp}\Big(\frac{\partial u}{\partial x^i}\Big) \subseteq \operatorname{supp}(u) \tag{5.2.26}$$

and

$$\operatorname{sing\,supp}\Big(\frac{\partial u}{\partial x^i}\Big) \subseteq \operatorname{sing\,supp}(u) \tag{5.2.27}$$

for all $u \in \mathscr{C}_0^\infty(X)'$ and $i = 1, \ldots, n$.

vii.) *Differentiation of distributions increases the order at most by one, i.e. for all $x \in X$ one has*

$$\operatorname{ord}_x\left(\frac{\partial u}{\partial x^i}\right) \leq \operatorname{ord}_x(u) + 1 \tag{5.2.28}$$

for all $u \in \mathscr{C}_0^\infty(X)'$ and $i = 1, \ldots, n$.

viii.) *One has the Leibniz rule*

$$\frac{\partial}{\partial x^i}(fu) = \frac{\partial f}{\partial x^i} u + f \frac{\partial u}{\partial x^i} \tag{5.2.29}$$

with respect to the module structure, where $f \in \mathscr{C}^\infty(X)$, $u \in \mathscr{C}_0^\infty(X)'$ and $i = 1, \ldots, n$.

PROOF: Since $\frac{\partial}{\partial x^i} \colon \mathscr{C}_0^\infty(X) \longrightarrow \mathscr{C}_0^\infty(X)$ is a continuous linear map, its transpose is a well-defined linear map from the topological dual to the topological dual. This shows *i.)*. Moreover, the usual continuity properties of transposed maps according to Corollary 3.2.22 give the third part, as the additional minus sign does not change the continuity estimates. For the second part we know that partial derivatives commute for test functions. Since transposition is linear and functorial, the same holds for the partial derivatives of distributions. The fourth part is the motivation for the definition (5.2.20) and was shown in Lemma 5.2.7. Now let $U \subseteq X$ be open and $\varphi \in \mathscr{C}_0^\infty(U)$. Then

$$\frac{\partial u}{\partial x^i}\Big|_U(\varphi) = \frac{\partial u}{\partial x^i}(\varphi) = -u\Big(\frac{\partial \varphi}{\partial x^i}\Big) = -u\Big|_U\Big(\frac{\partial \varphi}{\partial x^i}\Big),$$

since $\frac{\partial \varphi}{\partial x^i} \in \mathscr{C}_0^\infty(U)$ again: the locality of the usual differentiation implies the one for distributions. The sixth part is a trivial consequence of this, since

$$\frac{\partial u}{\partial x^i}\Big|_{X \setminus \operatorname{supp}(u)} \overset{(5.2.25)}{=} \frac{\partial}{\partial x^i}\big(u\big|_{X \setminus \operatorname{supp}(u)}\big) = \frac{\partial}{\partial x^i}(0) = 0,$$

and if $u\big|_{X \setminus \operatorname{sing\,supp}(u)} = I_g$ with $g \in \mathscr{C}^\infty(X \setminus \operatorname{sing\,supp}(u))$, then

$$\frac{\partial u}{\partial x^i}\Big|_{X \setminus \operatorname{sing\,supp}(u)} \overset{(5.2.25)}{=} \frac{\partial}{\partial x^i}\big(u\big|_{X \setminus \operatorname{sing\,supp}(u)}\big) = \frac{\partial}{\partial x^i} I_g \overset{(5.2.24)}{=} I_{\frac{\partial g}{\partial x^i}}$$

with $\frac{\partial g}{\partial x^i} \in \mathscr{C}^\infty(X)$. Hence the two claims (5.2.26) and (5.2.27) have been shown. The next statement follows as $\frac{\partial}{\partial x^i}$ is of order one on test functions, i.e. we have for trivial reasons

$$\mathrm{p}_{K,\ell}\Big(\frac{\partial \varphi}{\partial x^i}\Big) \leq \mathrm{p}_{K,\ell+1}(\varphi).$$

With the definition of the order at x in (5.1.76) we note that

$$\Big|\frac{\partial u}{\partial x^i}(\varphi)\Big| = \Big|-u\Big(\frac{\partial \varphi}{\partial x^i}\Big)\Big| \leq c \cdot \mathrm{p}_{K,\ell}\Big(\frac{\partial \varphi}{\partial x^i}\Big) \leq c \cdot \mathrm{p}_{K,\ell+1}(\varphi),$$

whenever $|u(\varphi)| \leq c\mathrm{p}_{K,\ell}(\varphi)$ with $\varphi \in \mathscr{C}_K^\infty(X)$. This gives (5.2.28) at once. Finally, (5.2.29) follows again from the general rules of transposition, since the module structure of $\mathscr{C}_0^\infty(X)$ satisfies this Leibniz rule. In detail, we have

$$\begin{aligned}
\Big(\frac{\partial}{\partial x^i}(fu)\Big)(\varphi) &= -(fu)\Big(\frac{\partial \varphi}{\partial x^i}\Big) \\
&= -u\Big(f\frac{\partial \varphi}{\partial x^i}\Big) \\
&= -u\Big(\frac{\partial}{\partial x^i}(f\varphi)\Big) + u\Big(\frac{\partial f}{\partial x^i}\varphi\Big)
\end{aligned}$$

$$= \frac{\partial u}{\partial x^i}(f\varphi) + \Big(\frac{\partial f}{\partial x^i} u\Big)(\varphi)$$
$$= \Big(f \frac{\partial u}{\partial x^i} + \frac{\partial f}{\partial x^i} u\Big)(\varphi),$$

which proves (5.2.29). □

Remark 5.2.10 (Differentiation of Distributions)

i.) The innocent looking first part of the proposition is perhaps the most remarkable. It shows that many functions, which are not differentiable in the usual sense, becomes infinitely often differentiable in the sense of distributions: every function $f \in \mathrm{L}^1_{\mathrm{loc}}(X)$ is now differentiable! However, the price to pay is that the derivative $\frac{\partial}{\partial x^i} I_f$ might not be a function in the traditional sense anymore. It is typically *not* in the image of the map $I\colon \mathrm{L}^1_{\mathrm{loc}}(X) \longrightarrow \mathscr{C}^\infty_0(X)'$. Instead, it becomes a more general distribution.

ii.) The second statement allows to use multiindex notation for higher derivatives of distributions as well. We set
$$\frac{\partial^{|\alpha|} u}{\partial x^\alpha} = \frac{\partial^{|\alpha|} u}{\partial (x^1)^{\alpha_1} \cdots \partial (x^n)^{\alpha_n}} \tag{5.2.30}$$
for $\alpha \in \mathbb{N}_0^n$ and $u \in \mathscr{C}^\infty_0(X)'$. Explicitly, we have
$$\frac{\partial^{|\alpha|} u}{\partial x^\alpha}(\varphi) = (-1)^{|\alpha|} u\Big(\frac{\partial^{|\alpha|} \varphi}{\partial x^\alpha}\Big) \tag{5.2.31}$$
for all $\varphi \in \mathscr{C}^\infty_0(X)$.

iii.) With the more sophisticated language of sheaves, the fifth part of the proposition says that differentiation is a sheaf morphism. For practical purposes this is a very handy observation.

iv.) The statement about the singular support gives finally the justification for the definition of the singular support. We based it on *smooth* functions in Definition 5.1.27, because now differentiation does not produce singularities. Either, we had singularities already before differentiation or not, but $\frac{\partial u}{\partial x^i}$ will not have more singular points than u.

We come now to some first examples. The first relates the Heaviside function to the δ-distribution:

Example 5.2.11 With the notation of Example 5.1.29 we see that
$$\alpha' = \theta \tag{5.2.32}$$
in the sense of distributions. Since this is of fundamental importance, we will illustrate the argument in some detail. Let $\varphi \in \mathscr{C}^\infty_0(\mathbb{R})$ be a test function. Then
$$\begin{aligned}
\alpha'(\varphi) &= -\alpha(\varphi') \\
&= -\int_0^\infty x\varphi'(x)\,\mathrm{d}x \\
&= -\int_0^\infty ((x\varphi)'(x) - \varphi(x))\,\mathrm{d}x \\
&= -x\varphi(x)\Big|_0^\infty + \int_0^\infty \varphi(x)\,\mathrm{d}x \\
&= 0 + \theta(\varphi)
\end{aligned}$$
by an elementary integration by parts and the fact that $x\varphi(x)$ vanishes at infinity thanks to the support requirement. Note that the most naive knowledge of Riemann integration is completely sufficient to prove this relation (5.2.32). Equally simple is the next: we have
$$\theta' = \delta. \tag{5.2.33}$$

Again, we check this relation by an elementary computation. For $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$ we have

$$\theta'(\varphi) = -\theta(\varphi') = -\int_0^\infty \varphi'(x)\,\mathrm{d}x = -\varphi(x)\Big|_0^\infty = \varphi(0),$$

since φ vanishes at infinity. Hence we get (5.2.33). Remarkably, we can view the combination of these two computations as a differential equation

$$\Delta\alpha = \delta, \tag{5.2.34}$$

where $\Delta = \frac{\partial^2}{\partial x^2}$ denotes the one-dimensional Euclidean Laplace operator. Note that this way, δ becomes the second derivative of the *continuous* (non-negative) function α.

The next example was implicitly used already at several points:

Example 5.2.12 (Derivatives of δ) Consider again the δ-distribution, now back in n dimensions. For every multiindex $\alpha \in \mathbb{N}_0^n$ we then have

$$(\partial^\alpha\delta)(\varphi) = (-1)^{|\alpha|}\frac{\partial^{|\alpha|}\varphi}{\partial x^\alpha}(0) \tag{5.2.35}$$

for $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$. This way we have computed the derivatives of the δ-distribution in arbitrary orders. We see that in this case

$$\operatorname{supp}(\partial^\alpha\delta) = \operatorname{sing\,supp}(\partial^\alpha\delta) = \{0\} \tag{5.2.36}$$

and

$$\operatorname{ord}_x(\partial^\alpha\delta) = \begin{cases} |\alpha| & \text{for } x = 0 \\ 0 & \text{for } x \neq 0. \end{cases} \tag{5.2.37}$$

This also shows that the estimates on the support, the singular support, and the order in Proposition 5.2.9, *vi.)* and *vii.)*, can not be improved in general.

Example 5.2.13 (Charge densities) One of the main motivations for introducing distributions in physics are charge densities of various kind. Out of the given charge density ρ one then can compute potentials and forces. In the context of electric charges, the electric potential ϕ is given by

$$\phi(\vec{x}) = \int_{\mathbb{R}^3} \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}\,\mathrm{d}^3y, \tag{5.2.38}$$

where we have incorporated several physical constants in the definition of ρ. To fill this definition with mathematical life, we have to specify the type of density ρ one can use. In view of the integral we face two possible difficulties here. We have to expect singular behaviour for $\vec{y} = \vec{x}$ and we have to take care of the non-compact domain of integration. Both difficulties enforce us to have certain regularity requirements on ρ. The following two cases are of particular interest:

1. A point charge q at $\vec{x}_0 \in \mathbb{R}^3$. In this case the Coulomb potential is

   $$\phi(\vec{x}) = \frac{q}{|\vec{x}-\vec{x}_0|}. \tag{5.2.39}$$

   In view of the general formula (5.2.38) we can interpret this as having the charge density

   $$\rho = q\delta_{\vec{x}_0}, \tag{5.2.40}$$

   i.e. a δ-distribution located at $\vec{x}_0$. Note that we can not directly interpret (5.2.38) for $\rho = q\delta$ as a pairing of a distribution with a test function, since the integral kernel $\frac{1}{|\vec{x}-\vec{y}|}$ is of course not at all a test function. This already indicates that our definitions and tools are not yet developed far enough to treat physically relevant examples. Nevertheless, beside for $\vec{x} = \vec{x}_0$ an evaluation of the δ-distribution on the "test function" $\frac{1}{|\vec{x}-\vec{y}|}$ is possible in a naive way, reproducing (5.2.39).

2. A physical dipole consists of two opposite charges $\pm q$ sitting at e.g. $\pm\epsilon\vec{e}_3 \in \mathbb{R}^3$. The potential is
$$\phi_{\mathrm{dipole}}(\vec{x}) = \frac{q}{|x - \epsilon\vec{e}_3|} - \frac{q}{|x + \epsilon\vec{e}_3|}, \tag{5.2.41}$$
while the charge density is
$$\rho_{\mathrm{dipole}} = q\delta_{\epsilon\vec{e}_3} - q\delta_{-\epsilon\vec{e}_3}. \tag{5.2.42}$$
The mathematical dipole is now an idealization of the field generated by ϕ_{dipole} at large distances $|\vec{x}| \gg \epsilon$. One obtains this in either of the following two ways: one can perform a Taylor expansion of ϕ_{dipole} around $\epsilon = 0$ and observes that for a fixed dipole moment
$$p = 2q\epsilon \tag{5.2.43}$$
the higher order terms in the Taylor expansion of ϕ_{dipole} behave like higher orders in ϵ, while the remaining limit gives the potential of the mathematical dipole
$$\phi_{\mathrm{dipole,\ math}}(\vec{x}) = \frac{p\vec{e}_3 \cdot \vec{x}}{|\vec{x}|^3}. \tag{5.2.44}$$
Alternatively, one can directly compute the limit $\epsilon \to 0$ of the density (5.2.42) for $q = \frac{p}{2\epsilon}$ in
$$\rho_{\mathrm{dipole,\ math}} = \lim_{\epsilon\searrow 0} \frac{p}{2\epsilon}(\delta_{\epsilon\vec{e}_3} - \delta_{-\epsilon\vec{e}_3}) \tag{5.2.45}$$
in the sense of distributions. This gives for $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^3)$
$$\rho_{\mathrm{dipole,\ math}}(\varphi) = \lim_{\epsilon\searrow 0} \frac{p}{2\epsilon}(\varphi(\epsilon\vec{e}_3) - \varphi(-\epsilon\vec{e}_3)) = p\frac{\partial\varphi}{\partial x^3}(0). \tag{5.2.46}$$
This way, the charge density of a mathematical dipole is given by a *derivative* of the δ-distribution. This matches the computation of the Taylor expansion of ϕ_{dipole} and hence (5.2.45), see also Exercise 5.5.18 for more details. Again, we have to take this with a grain of salt, as the above manipulations leave the solid ground of distributions as continuous linear functionals on the space $\mathscr{C}_0^\infty(\mathbb{R}^3)$ of test functions. Here we need to upgrade our technology still further.

For a general charge density ρ one calls the Taylor coefficients of the potential the monopole, the dipole, the quadrupole moment of ρ, and all higher multipoles. A similar limiting process gives then the mathematical quadrupole etc. as higher derivatives of δ, see Exercise 5.5.19.

Combining the differentiation of distributions with the module structure allows us to act on distributions with arbitrary (smooth) differential operators. Indeed, to achieve compatibility with the inclusion
$$I\colon \mathscr{C}^k(X) \longrightarrow \mathscr{C}_0^\infty(X)' \tag{5.2.47}$$
we define the action of differential operators as follows:

Definition 5.2.14 (Differential operators for distributions) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let*
$$D = \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} \tag{5.2.48}$$
be a differential operator of order $k \in \mathbb{N}_0$ with smooth coefficients $D^\alpha \in \mathscr{C}^\infty(X)$. Then one defines for a distribution $u \in \mathscr{C}_0^\infty(X)'$ the map
$$Du\colon \mathscr{C}_0^\infty(X) \longrightarrow \mathbb{K} \tag{5.2.49}$$

by

$$(Du)(\varphi) = \sum_{|\alpha|\leq k} \frac{(-1)^{|\alpha|}}{\alpha!} u\Big(\frac{\partial^{|\alpha|}}{\partial x^\alpha}(D^\alpha \varphi)\Big), \tag{5.2.50}$$

where $\varphi \in \mathscr{C}_0^\infty(X)$ *is a test function.*

Corollary 5.2.15 *Let* $X \subseteq \mathbb{R}^n$ *be a non-empty open subset and let* D *be a differential operator of order* $k \in \mathbb{N}_0$ *with smooth coefficients.*

i.) The definition (5.2.50) *yields a continuous linear map*

$$D \colon \mathscr{C}_0^\infty(X)' \longrightarrow \mathscr{C}_0^\infty(X)' \tag{5.2.51}$$

in the weak as well as in the strong topologies.*

ii.) If $f \in \mathscr{C}^k(X)$*, then one has*

$$I_{Df} = DI_f. \tag{5.2.52}$$

iii.) For any open $U \subseteq X$ *and* $u \in \mathscr{C}_0^\infty(X)'$ *one has*

$$Du\Big|_U = D_U\big(u\big|_U\big), \tag{5.2.53}$$

where $D_U = \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha\big|_U \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ *is the usual restriction of* D *to* $\mathscr{C}^\infty(U)$.

iv.) For all $u \in \mathscr{C}_0^\infty(X)'$ *one has*

$$\operatorname{supp}(Du) \subseteq \operatorname{supp}(u) \tag{5.2.54}$$

and

$$\operatorname{sing\,supp}(Du) \subseteq \operatorname{sing\,supp}(u), \tag{5.2.55}$$

as well as

$$\operatorname{ord}_x(Du) \leq \operatorname{ord}_x(u) + k \tag{5.2.56}$$

for all $x \in X$.

PROOF: The first statement is a combination of the continuity properties of the module structure as in Proposition 5.2.3, *ii.)*, and the continuity of differentiation as in Proposition 5.2.9, *iii.)*. The second statement is obtained by k-fold integration by parts. Note that the test function φ is first multiplied by the coefficient functions and afterwards the result gets differentiated. The remaining statements are combinations of the corresponding statements on the module structure and the differentiation, see again Proposition 5.2.3 and Proposition 5.2.9. □

Having defined the action of differential operators on distributions, we can consider linear partial differential equations of the form

$$Du = f \tag{5.2.57}$$

for a differential operator D with smooth coefficients and a given inhomogeneity $f \in \mathscr{C}_0^\infty(X)'$, looking for a *distribution* u solving (5.2.57). Depending on the data f we may or may not need a distribution with interesting singularities. Since D will not add new points to the singular support, we have

$$\operatorname{sing\,supp}(f) \subseteq \operatorname{sing\,supp}(u). \tag{5.2.58}$$

This means that u has to be at least as singular as f is, concerning the singular support. However, even if we have $f \in \mathscr{C}^\infty(X)$ and thus (5.2.58) becomes a vacuous condition on u, we can have genuinely distributional solutions to (5.2.57). Thus it will be of great importance to understand how many distributional solutions we have to expect compared to the classical solutions of (5.2.57). The non-classical solutions are sometimes called *weak* solutions, even though this notion is often used for more specific types of distributional solutions.

In general, it will depend very much on the details of the differential operator D, whether weak solutions are automatically classical ones or not. In the rest of this chapter, we will start investigating some very simple cases. The first key observation is the distributional version of the fundamental theorem of calculus in one dimension:

Theorem 5.2.16 (Primitive of distribution) *Let $I \subseteq \mathbb{R}$ be an open interval.*

i.) For every $f \in \mathscr{C}_0^\infty(I)'$ there exists a distribution $u \in \mathscr{C}_0^\infty(I)'$ with

$$u' = f. \tag{5.2.59}$$

ii.) Any two solutions $u, v \in \mathscr{C}_0^\infty(I)'$ ofq (5.2.59) differ by a constant, i.e. $u - v = const$.

iii.) We have $u' = 0$ for $u \in \mathscr{C}_0^\infty(I)'$ iff $u = const$.

Proof: We first claim that the kernel of the constant distribution 1, i.e. the integration over I, is given as follows. Suppose $1(\varphi) = \int_I \varphi(x)\,\mathrm{d}x = 0$, then

$$\psi(x) = \int_{-\infty}^x \varphi(t)\,\mathrm{d}t \quad \text{for } x \in I$$

is a smooth function on I, where we extend φ as usual to $\mathbb{R}$ by zero. Moreover, if $\operatorname{supp}(\varphi) \subseteq [a, b] \subseteq I$, then $\psi(x) = 0$, whenever $x \leq a$. But also $\psi(x) = 0$ for $x \geq b$, since in this case $1(\varphi) = 0$ gives

$$\int_{-\infty}^x \varphi(t)\,\mathrm{d}t = \int_{-\infty}^b \varphi(t)\,\mathrm{d}t = 0$$

already. Thus $\psi \in \mathscr{C}_0^\infty(I)$. If φ has a non-vanishing integral, i.e. $1(\varphi) \neq 0$, then the function ψ can not have compact support, now matter which constant we add to ψ. Instead, we have

$$\psi(+\infty) - \psi(-\infty) = \int_{-\infty}^\infty \psi(t)\,\mathrm{d}t \neq 0.$$

However, we can use a fixed test function $\chi \in \mathscr{C}_0^\infty(I)$ to cure this. Take an always existing test function with

$$\int_I \chi(t)\,\mathrm{d}t = 1,$$

and put

$$P(\varphi)(x) = \int_{-\infty}^x (\varphi(t) - 1(\varphi)\chi(t))\,\mathrm{d}t.$$

First we note that the function $x \mapsto \int_{-\infty}^x (\varphi(t) - 1(\varphi)\chi(t))\,\mathrm{d}t$ has compact support in I. Indeed, assume $\operatorname{supp}(\varphi) \subseteq [a, b]$ and $\operatorname{supp}(\chi) \subseteq [c, d]$ with $[a, b], [c, d] \subseteq I$. Then for $t \leq \min\{a, c\}$ we have $\varphi(t) - 1(\varphi)\chi(t) = 0$ and hence $P(\varphi)(x) = 0$ for $x \leq \min\{a, c\}$. Also, for $t \geq \max\{b, d\}$ we have $\varphi(t) = 0$, as well as $\chi(t) = 0$. In addition, the integral over φ gives $1(\varphi)$, while the integral over χ gives 1 by assumption. Thus $P(\varphi)(x) = 0$ for all $x \geq \max\{b, d\}$, showing

$$\operatorname{supp}(P(\varphi)) \subseteq [\min\{a, c\}, \max\{b, d\}]. \tag{$*$}$$

Next, standard exchange of integration and differentiation gives $P(\varphi) \in \mathscr{C}^\infty(I)$ and thus $P(\varphi) \in \mathscr{C}_0^\infty(I)$ again. We claim that the corresponding linear map

$$P\colon \mathscr{C}_0^\infty(I) \longrightarrow \mathscr{C}_0^\infty(I) \tag{$\star$}$$

is continuous. Indeed, let $[a, b] \subseteq I$ be a fixed compact subinterval and define $a' = \min\{a, c\}$ and $b' = \max\{b, d\}$ as above. Then

$$P \colon \mathscr{C}^\infty_{[a,b]}(I) \longrightarrow \mathscr{C}^\infty_{[a',b']}(I) \tag{☺}$$

as we saw in $(*)$. Now let $\ell \in \mathbb{N}_0$, then we estimate $\mathrm{p}_{[a',b'],\ell}(P(\varphi))$ for $\varphi \in \mathscr{C}^\infty_{[a,b]}(I)$ as follows. For $\alpha \geq 1$ we get

$$\begin{aligned}
\sup_{x \in [a',b']} \left| \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} P(\varphi)(x) \right| &= \sup_{x \in [a',b']} \left| \frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}x^{\alpha-1}} (\varphi(x) - 1(\varphi)\chi(x)) \right| \\
&\leq \mathrm{p}_{[a',b'],\alpha-1}(\varphi) + |1(\varphi)| \mathrm{p}_{[a',b'],\alpha-1}(\chi) \\
&= \mathrm{p}_{[a,b],\alpha-1}(\varphi) + (b-a) \mathrm{p}_{[a,b],0}(\varphi) \mathrm{p}_{[c,d],\alpha-1}(\chi) \\
&\leq \mathit{const}\, \mathrm{p}_{[a,b],\alpha-1}(\varphi).
\end{aligned}$$

For $\alpha = 0$ we have directly

$$\sup_{x \in [a',b']} |P(\varphi)(x)| \leq (b-a) \mathrm{p}_{[a,b],0}(\varphi) + |1(\varphi)| \mathrm{p}_{[c,d],0}(\chi)(d-c) \leq \mathit{const}\, \mathrm{p}_{[a,b],0}(\varphi).$$

Together, this gives the estimate

$$\mathrm{p}_{[a',b'],\alpha}(P(\varphi)) \leq \mathit{const}\, \mathrm{p}_{[a,b],\alpha'}(\varphi)$$

with $\alpha' = \alpha$ for $\alpha = 0$ and $\alpha' = \alpha - 1$ otherwise. Thus P in (☺) is continuous. From the characterization of the continuity of linear maps on the LF space $\mathscr{C}^\infty_0(I)$ as in Theorem 5.1.1, *vii.)*, we conclude that P is a continuous linear map, as claimed in $(\star)$. The map P can now be seen as a left inverse of differentiation. Indeed, we have $1(\varphi') = 0$ by the fundamental theorem of calculus and the compact support of φ. Thus

$$P(\varphi')(x) = \int_{-\infty}^x \varphi(t) \,\mathrm{d}t - 1(\varphi') \int_{-\infty}^x \chi(t) \,\mathrm{d}t = \varphi(x).$$

Now we can solve (5.2.59) as follows: given $f \in \mathscr{C}^\infty_0(I)'$ we define

$$u(\varphi) = -f(P(\varphi)).$$

Clearly, this is a distribution. Indeed, the explicit continuity uses the continuity of f, given by

$$|f(\varphi)| \leq \mathit{const}\, \mathrm{p}_{[a,b],\ell}(\varphi)$$

for some constant and some $\ell \in \mathbb{N}_0$, where $\varphi \in \mathscr{C}^\infty_{[a,b]}(I)$. Then

$$\begin{aligned}
|u(\varphi)| &= |f(P(\varphi))| \\
&\leq \mathit{const}\, \mathrm{p}_{[a,b],\ell}(P(\varphi)) \\
&\leq \mathit{const}\, \mathrm{p}_{[a',b'],\ell}(P(\varphi)) \\
&\leq \mathit{const}\, \mathrm{p}_{[a,b],\ell}(\varphi).
\end{aligned}$$

In particular, we see that u has the same local orders as f. Finally, we have

$$u'(\varphi) = -u(\varphi') = f(P(\varphi')) = f(\varphi),$$

and hence (5.2.59) follows. For the second statement, let $u, v \in \mathscr{C}^\infty_0(I)'$ be solutions to (5.2.59) for a given $f \in \mathscr{C}^\infty_0(I)'$. Then $w = u - v$ solves $w' = 0$. Hence $0 = w'(\varphi) = -w(\varphi')$ and thus

$$0 = w(P(\varphi)') = w(\varphi - 1(\varphi)\chi) = w(\varphi) - 1(\varphi) w(\chi).$$

This shows $w(\varphi) = 1(\varphi) w(\chi)$, which proves that w is the constant distribution $w(\chi)1$. The third part is included in this statement. □

Corollary 5.2.17 *Let $I \subseteq \mathbb{R}$ be an open interval and let $f \in \mathscr{C}_0^\infty(I)'$ be a distribution. Then $u \in \mathscr{C}_0^\infty(I)'$ with $u' = f$ satisfies*

$$\operatorname{ord}_x(u) \leq \operatorname{ord}_x(f) \tag{5.2.60}$$

for all $x \in I$.

In fact, this was obtained in the proof. Note that this estimate on the order is non-trivial insofar, as $u' = f$ only gives $\operatorname{ord}_x(f) \leq \operatorname{ord}_x(u) + 1$ for general reasons.

Corollary 5.2.18 *Let $I \subseteq \mathbb{R}$ be an open interval. Then $u \in \mathscr{C}_0^\infty(I)'$ satisfies $u^{(\ell)} = 0$ iff u is a polynomial of degree $\ell - 1$.*

This trivial corollary, obtained by induction on ℓ, can be seen as a first regularity result that weak solutions of $u^{(\ell)} = 0$ are in fact strong solutions.

We conclude this section now with a last characterization of constant distributions. In fact, the result is not so much about distributions but the test functions, which are linear combinations of derivatives. It generalizes Theorem 5.2.16, *iii.)*, to arbitrary dimensions:

Theorem 5.2.19 (Constant distributions) *Let $u \in \mathscr{C}_0^\infty(\mathbb{R}^n)'$ be a distribution. Then we have*

$$\frac{\partial u}{\partial x^i} = 0 \quad \textit{for all}\, i = 1, \ldots, n \tag{5.2.61}$$

iff u is constant.

Proof: In fact, we do not have to assume that u is a *continuous* linear functional at all. Any linear functional in $\mathscr{C}_0^\infty(\mathbb{R}^n)^*$ vanishing on partial derivatives is a multiple of the integral. To see this, we first note that for $n = 1$ we have the following statement. Denote the integral again as the constant distribution $1 \in \mathscr{C}_0^\infty(\mathbb{R})'$, then

$$\ker(1) = \operatorname{im}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{$*$}$$

by the construction in the proof of Theorem 5.2.16. We claim that this stays correct in all dimensions, i.e. we have

$$\ker(1) = \operatorname{span}_{\mathbb{R}}\left\{\frac{\partial \varphi}{\partial x^i} \;\middle|\; i = 1, \ldots, n,\ \varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)\right\}. \tag{$**$}$$

The inclusion "$\supseteq$" is trivial by the fundamental theorem of calculus, it is the inclusion "$\subseteq$" we have to worry about. Thus let $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^{n+1})$ with $1(\varphi) = 0$ be given. We prove that statement by induction on n, where the case $n = 0$ is $(*)$. We define $\psi\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\psi(x^1, \ldots, x^n) = \int_{-\infty}^{\infty} \varphi(x^1, \ldots, x^n, x^{n+1})\, \mathrm{d}x^{n+1}.$$

Since the support of φ is compact in $\mathbb{R}^{n+1}$, the support of ψ is compact in $\mathbb{R}^n$ and ψ is smooth. By our induction assumption we have functions $\psi_1, \ldots, \psi_n \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with

$$\psi = \frac{\partial \psi_1}{\partial x^1} + \cdots + \frac{\partial \psi_n}{\partial x^n}.$$

Choose again a function $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$ with $1(\chi) = 1$ and define

$$\varphi_i(x^1, \ldots, x^{n+1}) = \psi_i(x^1, \ldots, x^n)\chi(x^{n+1})$$

for $i = 1, \ldots, n$. This gives $\varphi_i \in \mathscr{C}_0^\infty(\mathbb{R}^{n+1})$. Moreover, the function

$$\tilde{\varphi} = \varphi - \sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial x^i}$$

is a test function $\tilde{\varphi} \in \mathscr{C}_0^\infty(\mathbb{R}^{n+1})$, satisfying

$$\begin{aligned}\int_{-\infty}^{\infty} \tilde{\varphi}(x^1, \ldots, x^{n+1}) \, \mathrm{d}x^{n+1} &= \int_{-\infty}^{\infty} \varphi(x^1, \ldots, x^{n+1}) \, \mathrm{d}x^{n+1} - \sum_{i=1}^{n} \int_{-\infty}^{\infty} \frac{\partial \psi_i}{\partial x^i}(x^1, \ldots, x^n) \chi(x^{n+1}) \, \mathrm{d}x^{n+1} \\ &= \psi(x^1, \ldots, x^n) - \sum_{i=1}^{n} \frac{\partial \psi_i}{\partial x^i}(x^1, \ldots, x^n) \\ &= 0\end{aligned}$$

for all $x^1, \ldots, x^n \in \mathbb{R}$. Viewing $\tilde{\varphi}$ as a function of x^{n+1} only, we get from the one-dimensional case a function φ_{n+1} depending parametrically on $x^1, \ldots, x^n$ and on x^{n+1} such that

$$\tilde{\varphi}(x^1, \ldots, x^n, x^{n+1}) = \frac{\partial \varphi_{n+1}}{\partial x^{n+1}}(x^1, \ldots, x^n, x^{n+1}).$$

Note that φ_{n+1} can be chosen to be smooth in $x^1, \ldots, x^n$, since the construction in the proof of Theorem 5.2.16 leading to $(*)$ was continuous in the $\mathscr{C}_0^\infty$-topology. The operator P clearly preserves the smooth dependence on parameters. Also, φ_{n+1} has still compact support in $x^1, \ldots, x^n$, leading to $\varphi_{n+1} \in \mathscr{C}_0^\infty(\mathbb{R}^{n+1})$ in total. But then $(**)$ follows. From here the conclusion of the theorem is trivial linear algebra since the kernel $\ker(1) \subseteq \mathscr{C}_0^\infty(\mathbb{R}^n)$ has codimension one, as $1 \neq 0$. Now let $u \colon \mathscr{C}_0^\infty(\mathbb{R}^n) \longrightarrow \mathbb{K}$ be a linear functional with $\ker(u) \supseteq \ker 1$, i.e. (5.2.61). Then with the χ fulfilling $1(\chi) = 1$ from above we have

$$u(\varphi) = u(\varphi - 1(\varphi)\chi + 1(\varphi)\chi) = u(\varphi - 1(\varphi)\chi) + 1(\varphi)u(\chi) = 1(\varphi)u(\chi),$$

no matter whether u is continuous or not. □

Remark 5.2.20 (Constant distributions) The result of the above discussion has a remarkable and quite remote interpretation. What we actually computed in Theorem 5.2.19 was the top-degree de Rham cohomology with compact support for $\mathbb{R}^n$. Indeed, we identify a test function $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with the differential form

$$\omega_\varphi = \varphi \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{5.2.62}$$

of degree n, which clearly has compact support. For dimensional reasons, $\mathrm{d}\omega_\varphi = 0$, i.e. ω_φ is closed. Now ω_φ is exact if we find a $(n-1)$-form α such that $\mathrm{d}\alpha = \omega_\varphi$. This α can be written as

$$\alpha = \sum_{i=1}^{n} (-1)^{i+1} \varphi_i \, \mathrm{d}x^1 \wedge \cdots \overset{i}{\wedge} \cdots \wedge \mathrm{d}x^n \tag{5.2.63}$$

with uniquely determined functions $\varphi_1, \ldots, \varphi_n$, since the $(n-1)$-forms $\mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$, $\mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \cdots \wedge \mathrm{d}x^n$, $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1}$ form a (global) frame. We have a compactly supported α iff all the coefficient functions $\varphi_1, \ldots, \varphi_n$ have compact support. Finally, the usual calculus of the exterior derivative shows

$$\mathrm{d}\alpha = \sum_{i=1}^{n} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}\varphi_i \wedge \cdots \wedge \mathrm{d}x^n = \sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial x^i} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n,$$

thereby explaining our sign convention. Thus the statement shows that the quotient of all compactly supported n-forms modulo the exterior derivatives of compactly supported $(n-1)$-forms is *one-dimensional.* Hence its linear algebraic dual is one-dimensional as well, and the integration functional 1 is a non-zero vector in this dual, hence a basis. We will not make use of this differential-geometric interpretation of Theorem 5.2.19 any more, but stay comforted with the reassuring result that constant distributions can be characterized by the usual infinitesimal criterion, i.e. by their vanishing derivatives.

5.2.3 Push-Forward and Pull-Back of Distributions

While the convention with the additional minus sign in the definition of derivatives of distributions was fairly easy to motivate and not to disturbing, the definition of pull-backs and push-forwards is less clear. In fact, we will meet two principle versions, both being useful and being used. This is a bit unfortunate, as one has to state clearly to which convention one wants to stick. Even worse, there are scenarios, where both competing definitions occur simultaneously. Caution is needed in these cases.

We start with the simplest case of the push-forward. Recall that a smooth map

$$\Phi\colon X \longrightarrow Y \tag{5.2.64}$$

between open subsets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ allows us to pull-back smooth functions on Y to smooth functions on X by pre-composition. This pull-back

$$\Phi^*\colon \mathscr{C}^\infty(Y) \longrightarrow \mathscr{C}^\infty(X) \tag{5.2.65}$$

is continuous with respect to the $\mathscr{C}^\infty$-topologies. However, in general, the pull-back of a test function is no longer a test function. The support of $\Phi^*\varphi$ can become non-compact, even though $\operatorname{supp}(\varphi)$ is compact. As soon as can guarantee this for all $\varphi \in \mathscr{C}_0^\infty(Y)$, we have a proper smooth map Φ and hence a continuous linear map

$$\Phi^*\colon \mathscr{C}_0^\infty(Y) \longrightarrow \mathscr{C}_0^\infty(X) \tag{5.2.66}$$

according to Proposition 5.1.3, see also Exercise 5.5.22. This allows to transpose (5.2.66) to a map between distributions:

Definition 5.2.21 (Push-Forward of distributions) *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be non-empty open subsets with a proper smooth map $\Phi\colon X \longrightarrow Y$. Then the push-forward of $u \in \mathscr{C}_0^\infty(X)'$ by Φ is the map*

$$\Phi_* u\colon \mathscr{C}_0^\infty(Y) \longrightarrow \mathbb{K} \tag{5.2.67}$$

defined by

$$(\Phi_* u)(\varphi) = u(\Phi^*\varphi) \tag{5.2.68}$$

for $\varphi \in \mathscr{C}_0^\infty(Y)$.

The following proposition is then immediate from the definition and general properties of transposed maps:

Proposition 5.2.22 *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be non-empty open subsets with a proper smooth map $\Phi\colon X \longrightarrow Y$.*

i.) For all $u \in \mathscr{C}_0^\infty(X)'$ one has $\Phi_* u \in \mathscr{C}_0^\infty(Y)'$.

ii.) The push-forward of distributions with Φ is a continuous linear map

$$\Phi_*\colon \mathscr{C}_0^\infty(X)'_\sigma \longrightarrow \mathscr{C}_0^\infty(Y)'_\sigma \tag{5.2.69}$$

as well as

$$\Phi_*\colon \mathscr{C}_0^\infty(X)'_\beta \longrightarrow \mathscr{C}_0^\infty(Y)'_\beta. \tag{5.2.70}$$

iii.) If $\Psi\colon Y \longrightarrow Z \subseteq \mathbb{R}^k$ is yet another proper smooth map, then $\Psi \circ \Phi\colon X \longrightarrow Z$ is proper, too. We have

$$(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_* \tag{5.2.71}$$

as well as

$$(\mathrm{id}_X)_* = \mathrm{id}_{\mathscr{C}_0^\infty(X)'}. \tag{5.2.72}$$

iv.) *For every* $u \in \mathscr{C}_0^\infty(X)'$ *we have*

$$\operatorname{supp}(\Phi_* u) \subseteq \Phi(\operatorname{supp}(u)). \tag{5.2.73}$$

v.) *For every* $u \in \mathscr{C}_0^\infty(X)'$ *we have*

$$\operatorname{ord}_x(\Phi_* u) \leq \operatorname{ord}_{\Phi^{-1}(\{y\})}(u) \tag{5.2.74}$$

for all $y \in Y$.

PROOF: Since by definition the push-forward of a distribution is obtained as dual of the continuous pull-back Φ^* of test functions, the first part is clear: $\Phi_* u$ is again continuous and linear. Then the second is a general feature of dualizing, which we have seen now already many times, see Corollary 3.2.22. The third part is obvious. Compositions of proper maps are proper in general and thus (5.2.71) follows, while (5.2.72) is trivial. The fourth part is slightly less easy. First we note that $\Phi(\operatorname{supp}(u)) \subseteq Y$ is a closed subset, since proper maps into a locally compact Hausdorff space like Y are closed maps and compact subsets in X are closed, see e.g. [19, Exercise 5.5.10]. Now consider $\varphi \in \mathscr{C}_0^\infty(Y)$ with $\operatorname{supp}(\varphi) \subseteq Y \setminus \Phi(\operatorname{supp}(u))$. Then $\Phi^*\varphi \in \mathscr{C}_0^\infty(X)$ has support in

$$\operatorname{supp}(\Phi^*\varphi) \subseteq \Phi^{-1}(Y \setminus \Phi(\operatorname{supp}(u))) \subseteq X \setminus \operatorname{supp}(u).$$

Thus $(\Phi_* u)(\varphi) = u(\Phi^*\varphi) = 0$ for all such test functions, proving (5.2.73). The last statement is in some sense a consequence of the fact that the pull-back map Φ^* is of order zero. The continuity estimate for Φ^* in Proposition 2.3.24 gives

$$\mathrm{p}_{K,\ell}(\Phi^*\varphi) \leq \mathit{const}\, \mathrm{p}_{\Phi(K),\ell}(\varphi)$$

for every compact subset $K \subseteq X$ and $\varphi \in \mathscr{C}^\infty(Y)$ with the same $\ell \in \mathbb{N}_0$ on both sides. Now if $y \in Y$ and $U \subseteq Y$ is an open neighbourhood of y with compact closure $U^{\mathrm{cl}} \subseteq Y$, then we note that $\Phi^{-1}(U^{\mathrm{cl}}) \subseteq X$ is still compact, since Φ is proper. Suppose for this open subset $\Phi^{-1}(U) \subseteq \Phi^{-1}(U^{\mathrm{cl}})$ we have an estimate like

$$|u(\varphi)| \leq c\mathrm{p}_{\Phi^{-1}(U^{\mathrm{cl}}),\ell}(\varphi)$$

for all $\varphi \in \mathscr{C}_0^\infty(X)$ with $\operatorname{supp}(\varphi) \subseteq \Phi^{-1}(U^{\mathrm{cl}})$ for an appropriate $c > 0$ and some $\ell \in \mathbb{N}_0$. Then for $\psi \in \mathscr{C}_0^\infty(Y)$ with $\operatorname{supp}(\psi) \subseteq U^{\mathrm{cl}}$ we get

$$|(\Phi_* u)(\psi)| = |u(\Phi^*\psi)| \leq \mathit{const}\, \mathrm{p}_{\Phi^{-1}(U^{\mathrm{cl}}),\ell}(\Phi^*\psi) \leq \mathit{const}'\mathrm{p}_{U^{\mathrm{cl}},\ell}(\psi),$$

still with the same $\ell \in \mathbb{N}_0$. Taking now the infimum over all such U gives a minimal ℓ, which can still be estimated by the minimal ℓ for all those points in $\Phi^{-1}(\{y\})$, as the above $\Phi^{-1}(U)$ are neighbourhoods of $x \in \Phi^{-1}(\{y\})$, but maybe not all such neighbourhoods. This gives the last statement. □

The push-forward is very similar to the push-forward of measures. In fact, for complex Borel measures the push-forward in the sense of measures and in the sense of distributions coincide:

Proposition 5.2.23 *Let* $X \subseteq \mathbb{R}^n$ *and* $Y \subseteq \mathbb{R}^m$ *be non-empty open subsets with a proper smooth map* $\Phi\colon X \longrightarrow Y$. *Then the push-forward of a complex Borel measure* $\mu \in \operatorname{Meas}(X)$ *in the sense of measures coincides with the push-forward of* μ *in the sense of distributions.*

PROOF: First we recall that as a continuous map Φ is measurable. The push-forward $\Phi_*\colon \operatorname{Meas}(X) \longrightarrow \operatorname{Meas}(Y)$ is then defined by

$$(\Phi_*\mu)(B) = \mu(\Phi^{-1}(B))$$

for all Borel subsets $B \subseteq Y$. For the integration with respect to μ and $\Phi_*\mu$ one has the general result that for all bounded measurable functions $f \in \mathscr{B}\mathscr{M}(Y)$ one has

$$\int_X \Phi^* f \, \mathrm{d}\mu = \int_Y f \, \mathrm{d}\Phi_*\mu, \tag{$*$}$$

see e.g. [?]. Note that for complex measures we do not need the assumption that Φ is proper. Since $\mathscr{C}_0^\infty(Y) \subseteq \mathscr{B}\mathscr{M}(Y)$, we can apply $(*)$ to test functions $\varphi \in \mathscr{C}_0^\infty(Y)$. The right hand side is then given by the distribution determined by the push-forward measure $\Phi_*\mu$. The left hand side is by definition the push-forward of the distribution given by the measure μ. □

The same statement holds for positive Borel measures such that compact subsets have finite measure, see Exercise 5.5.23. Here we need a proper map to guarantee that compact subsets still have finite measure with respect to the push-forward. We come now to some further illustrating examples of push-forwards. Note that the first task always consists in showing the properness of the map involved:

Example 5.2.24 Consider the smooth map

$$\Phi \colon \mathbb{R} \ni x \ \mapsto \ x^2 \in \mathbb{R}, \tag{5.2.75}$$

which is clearly a proper map.

i.) Let $u = \delta_{\pm 1} \in \mathscr{C}_0^\infty(\mathbb{R})'$ be the δ-distribution at $x = \pm 1$. Then

$$\Phi_* \delta_{\pm 1} = \delta_1. \tag{5.2.76}$$

Indeed, let $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$ be given, then $(\Phi^*\varphi)(x) = \varphi(x^2)$. Thus

$$(\Phi_*\delta_{\pm 1})(\varphi) = \delta_{\pm 1}(\Phi^*\varphi) = (\Phi^*\varphi)(\pm 1) = \varphi((\pm 1)^2) = \varphi(1). \tag{5.2.77}$$

ii.) Consider next $u = \delta_1 - \delta_{-1} \in \mathscr{C}_0^\infty(\mathbb{R})'$. Then

$$\Phi_* u = 0 \tag{5.2.78}$$

according to the linearity of Φ_* and *i.)*. This shows that the inclusion (5.2.73) can be strict, since $\operatorname{supp}(u) = \{1\} \cup \{-1\}$ and thus $\Phi(\operatorname{supp}(u)) = \{1\}$ is strictly larger than $\operatorname{supp}(\Phi_* u) = \emptyset$.

iii.) Now consider $\delta'_{\pm 1}$, i.e. the negative of the evaluation of the first derivative at ± 1. We have

$$\begin{aligned} (\Phi_*(\delta'_{\pm 1}))(\varphi) &= \delta'_{\pm 1}(\varphi \circ \Phi) \\ &= -(\varphi \circ \Phi)'(\pm 1) \\ &= -(2x\varphi'(x^2))\Big|_{x=\pm 1} \\ &= \mp 2\varphi'(1) \\ &= \pm \delta'_1(\varphi). \end{aligned}$$

It follows that

$$\Phi_*(\delta'_1 + \delta'_{-1}) = 0. \tag{5.2.79}$$

Thus also the inequality (5.2.74) can be proper as $\operatorname{ord}_{\pm 1}(\delta'_1 + \delta'_{-1}) = 1$, while the push-forward with Φ is zero and hence has order 0.

Example 5.2.25 There is, in general, no compatibility of push-forward with the canonical inclusion $I\colon \mathrm{L}^1_{\mathrm{loc}}(X) \longrightarrow \mathscr{C}^\infty_0(X)'$. In fact, the push-forward can produce singularities. To see this, we consider a smooth curve $\gamma\colon \mathbb{R} \longrightarrow \mathbb{R}^n$. In general, γ needs not to be proper. However, it is easy to find proper examples like e.g. the embedding

$$\gamma(t) = (t, 0, \ldots, 0). \tag{5.2.80}$$

In fact, if $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$ has at least one component γ^i which is proper, then γ is proper, too, see again Exercise 5.5.22. Another way to say that γ is proper is that for every $R > 0$ there is a time t_R such that $\|\gamma(t)\| > R$ for $|t| > t_R$, i.e. γ leaves every compact subset after a finite time.

i.) Consider first the δ-distribution δ at zero on $\mathbb{R}$, then

$$\gamma_*\delta = \delta_{\gamma(0)}. \tag{5.2.81}$$

ii.) If we consider δ' at 0 on $\mathbb{R}$, then the chain rule gives

$$\gamma_*\delta' = -\dot{\gamma}(0)\delta_{\gamma(0)}. \tag{5.2.82}$$

Indeed, for a test function $\varphi \in \mathscr{C}^\infty_0(\mathbb{R})$ we have

$$(\gamma_*\delta')(\varphi) = \delta'(\varphi \circ \gamma) = -(\varphi \circ \gamma)'(0) = -\dot{\gamma}(0)\varphi(\gamma(0)),$$

showing (5.2.82).

iii.) Consider now $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$, viewed as distribution $I_f \in \mathscr{C}^\infty_0(\mathbb{R})'$. Then

$$(\gamma_* I_f)(\varphi) = \int_{-\infty}^{\infty} f(t)\varphi(\gamma(t))\,\mathrm{d}t \tag{5.2.83}$$

is a distribution with quite complicated singularities. In fact, even if $f \in \mathscr{C}^\infty(\mathbb{R})$, the push-forward (5.2.83) can not be written as a distribution of the form $I_g \in \mathscr{C}^\infty_0(\mathbb{R}^n)'$ with some $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$, as soon as $n > 1$. In particular, $\gamma_* I_f$ has a non-trivial singular support along the trace of γ, depending of course on f, while I_f has none. For $f = 1$ one has

$$(\gamma_* I_1)(\varphi) = \int_{-\infty}^{\infty} \varphi(\gamma(t))\,\mathrm{d}t \tag{5.2.84}$$

and hence

$$\operatorname{supp}(\gamma_* I_1) = \operatorname{sing\,supp}(\gamma_* I_1) = \gamma(\mathbb{R}), \tag{5.2.85}$$

as soon as $n > 1$.

iv.) More generally, we consider as a higher dimensional analogue of γ a proper smooth map $\sigma\colon X \longrightarrow Y$ with open subsets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ with $n < m$. Suppose in addition σ is an embedding with non-degenerate tangent map. In this case, the image $\sigma(X) \subseteq Y$ is an n-dimensional closed submanifold of Y. A distribution on X, viewed as distribution on $\sigma(X)$ can be used as a distribution on Y with singular support being the support of the distribution in X. We discuss this construction in Exercise 5.5.24.

We consider now the pull-back of distributions. Here we restrict ourselves to the case of a diffeomorphism

$$\Phi\colon X \longrightarrow Y \tag{5.2.86}$$

between open subsets $X, Y \subseteq \mathbb{R}^n$. To motivate the definition of a pull-back with Φ we first consider the pull-back of functions, viewed as distributions. Here the following lemma is of crucial importance:

Lemma 5.2.26 *Let $\Phi\colon X \longrightarrow Y$ be a diffeomorphism between open subsets $X, Y \subseteq \mathbb{R}^n$.*

i.) The pull-back with Φ *yields a continuous linear map*

$$\Phi^* \colon \mathscr{C}_0^\infty(Y) \longrightarrow \mathscr{C}_0^\infty(X), \tag{5.2.87}$$

extending to a continuous linear map

$$\Phi^* \colon \mathrm{L}^1_{\mathrm{loc}}(Y) \longrightarrow \mathrm{L}^1_{\mathrm{loc}}(X). \tag{5.2.88}$$

ii.) For $f \in \mathrm{L}^1_{\mathrm{loc}}(Y)$ *one has*

$$\left(\left|\det \frac{\partial \Phi}{\partial x}\right| I_{\Phi^* f}\right)(\varphi) = I_f(\Phi_* \varphi) \tag{5.2.89}$$

for all test functions $\varphi \in \mathscr{C}_0^\infty(X)$*, where* $\frac{\partial \Phi}{\partial x}$ *denotes the Jacobi matrix of* Φ *and* $\Phi_* \varphi = (\Phi^{-1})^* \varphi$*, as usual.*

PROOF: A diffeomorphism is, in particular, a homeomorphism and as such a proper map. Thus (5.2.87) follows again from Proposition **??**. For $f \in \mathrm{L}^1_{\mathrm{loc}}(Y)$ and a compact subset $K \subseteq X$ we consider the local L^1-seminorm of the measurable function $f \circ \Phi$, i.e.

$$\|\Phi^* f\|_{\mathrm{L}^1, K} = \int_K |f(\Phi(x))| \, \mathrm{d}^n x.$$

From the transformation formula for integrals we get

$$\int_K |(\Phi^* f)(x)| \, \mathrm{d}^n x = \int_{\Phi(K)} |f(y)| \left|\det \frac{\partial \Phi^{-1}}{\partial y}(y)\right| \mathrm{d}^n y \le \mathit{const} \int_{\Phi(K)} |f(y)| \, \mathrm{d}^n y,$$

where $\frac{\partial \Phi^{-1}}{\partial y}$ is the Jacobi matrix of the inverse of the diffeomorphism Φ and the constant can be taken to be the maximum of the Jacobi determinant over the compact subset $\Phi(K) \subseteq Y$. This shows the continuity estimate

$$\|\Phi^* f\|_{\mathrm{L}^1, K} \le \mathit{const} \, \|f\|_{\mathrm{L}^1, \Phi(K)},$$

needed for the continuity (5.2.88). In particular, $\Phi^* f \in \mathrm{L}^1_{\mathrm{loc}}(X)$, which we need to make sense out of the statement (5.2.89). Now let $\varphi \in \mathscr{C}_0^\infty(X)$ be a test function. Then we compute

$$\begin{aligned}
\left(\left|\det \frac{\partial \Phi}{\partial x}\right| I_{\Phi^* f}\right)(\varphi) &= \int_X (\Phi^* f)(x) \left|\det \frac{\partial \Phi}{\partial x}\right| \varphi(x) \, \mathrm{d}^n x \\
&= \int_X (\Phi^* f)(x) (\varphi \circ \Phi^{-1} \circ \Phi)(x) \left|\det \frac{\partial \Phi}{\partial x}\right| \mathrm{d}^n x \\
&= \int_Y f(y) (\varphi \circ \Phi^{-1})(y) \, \mathrm{d}^n y \\
&= I_f(\Phi_* \varphi),
\end{aligned}$$

again using the transformation formula. Note that $\Phi_* \varphi = \varphi \circ \Phi^{-1}$ is indeed a test function on Y. $\square$

Later on we will prove that $\mathscr{C}_0^\infty(Y) \subseteq \mathrm{L}^1_{\mathrm{loc}}(Y)$ is actually *dense* and hence the extension in (5.2.88) is uniquely fixed by continuity already. Here we only use the fact, that we have a continuous map between the locally integrable functions, a fact which is the above elementary estimate.

It is this transformation property involving the additional Jacobi determinant, which we incorporate now into the definition of the pull-back. This way, the pull-back of a distribution is *not* just the dualized version of the push-forward of test functions, but gets an additional prefactor. Note that this is a *choice* one makes in order to emphasize the role of distributions as *generalized functions*:

Definition 5.2.27 (Pull-back of distributions) *Let $\Phi\colon X \longrightarrow Y$ be a diffeomorphism between open subsets $X, Y \subseteq \mathbb{R}^n$. Then the pull-back of $u \in \mathscr{C}_0^\infty(Y)'$ by Φ is the map*

$$\Phi^* u\colon \mathscr{C}_0^\infty(X) \longrightarrow \mathbb{K} \tag{5.2.90}$$

defined by

$$(\Phi^* u)(\varphi) = \left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}(y)\right| u\right)(\Phi_* \varphi) \tag{5.2.91}$$

for $\varphi \in \mathscr{C}_0^\infty(X)$, where $\Phi_* \varphi = \varphi \circ \Phi^{-1}$ is the usual push-forward of test functions.

Note that for a diffeomorphism Φ we can indeed push forward test functions and get a test function on Y again. In fact, we have

$$\operatorname{supp}(\Phi_* \varphi) = \Phi(\operatorname{supp}(\varphi)) \tag{5.2.92}$$

for all (test) functions $\varphi \in \mathscr{C}^0(X)$ if $\Phi\colon X \longrightarrow Y$ is a diffeomorphism. Here a $\mathscr{C}^0$-homeomorphism would be sufficient already. We collect now some first properties of the pull-back of distributions, explaining in particular the appearance of the Jacobi determinant in (5.2.91):

Proposition 5.2.28 *Let $\Phi\colon X \longrightarrow Y$ be a diffeomorphism between open subsets $X, Y \subseteq \mathbb{R}^n$.*

i.) For all $u \in \mathscr{C}_0^\infty(Y)'$ one has $\Phi^* u \in \mathscr{C}_0^\infty(X)'$.

ii.) The pull-back of distributions with Φ yields continuous linear maps

$$\Phi^*\colon \mathscr{C}_0^\infty(Y)'_\sigma \longrightarrow \mathscr{C}_0^\infty(X)'_\sigma \tag{5.2.93}$$

and

$$\Phi^*\colon \mathscr{C}_0^\infty(Y)'_\beta \longrightarrow \mathscr{C}_0^\infty(X)'_\beta, \tag{5.2.94}$$

respectively.

iii.) If $\Psi\colon Y \longrightarrow Z$ is another diffeomorphism to another open subset $Z \subseteq \mathbb{R}^n$, then

$$(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^* \tag{5.2.95}$$

and

$$\mathrm{id}_X^* = \mathrm{id}_{\mathscr{C}_0^\infty(X)'}. \tag{5.2.96}$$

In particular, Φ^* is invertible with inverse given by

$$(\Phi^*)^{-1} = (\Phi^{-1})^*. \tag{5.2.97}$$

iv.) For $f \in \mathrm{L}^1_{\mathrm{loc}}(Y)$ one has

$$\Phi^* I_f = I_{\Phi^* f}. \tag{5.2.98}$$

v.) For $f \in \mathscr{C}^\infty(Y)$ and $u \in \mathscr{C}_0^\infty(Y)'$ one has

$$\Phi^*(fu) = \Phi^*(f)\Phi^*(u). \tag{5.2.99}$$

vi.) The pull-back is compatible with locality in the sense that for an open $U \subseteq X$ one has

$$(\Phi^* u)\Big|_U = \left(\Phi\big|_U\right)^* \left(u\big|_{\Phi(U)}\right) \tag{5.2.100}$$

for all $u \in \mathscr{C}_0^\infty(Y)'$.

vii.) *For $u \in \mathscr{C}_0^\infty(Y)'$ one has*

$$\operatorname{supp}(\Phi^* u) = \Phi^{-1}(\operatorname{supp}(u)) \tag{5.2.101}$$

as well as

$$\operatorname{sing\,supp}(\Phi^* u) = \Phi^{-1}(\operatorname{sing\,supp}(u)). \tag{5.2.102}$$

viii.) *For $u \in \mathscr{C}_0^\infty(Y)'$ and $x \in X$ one has*

$$\operatorname{ord}_x(\Phi^* u) = \operatorname{ord}_{\Phi(x)}(u). \tag{5.2.103}$$

ix.) *The chain rule holds in the sense that for $u \in \mathscr{C}_0^\infty(Y)'$ and $i = 1, \ldots, n$ one has*

$$\frac{\partial}{\partial x^i}(\Phi^* u) = \sum_{j=0}^{n} \frac{\partial \Phi^j}{\partial x^i} \Phi^*\Big(\frac{\partial u}{\partial y^j}\Big). \tag{5.2.104}$$

Proof: First we note that for a diffeomorphism Φ^{-1} the Jacobi determinant is nowhere vanishing. In particular, on connected components of Y it has constant sign. Thus $\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right|$ is a again a smooth function on Y. Moreover, we already noted that $\Phi_* \varphi = \varphi \circ \Phi^{-1} = (\Phi^{-1})^* \varphi$ is again a test function, since Φ^{-1} is clearly a proper smooth map. Moreover, $\varphi \mapsto (\Phi^{-1})^* \varphi$ is continuous by Proposition 5.1.3. Since the module structure of $\mathscr{C}_0^\infty(Y)$ is continuous as well by Proposition 5.2.1, *ii.)*, the overall composition needed in $\Phi^* u$ is a continuous linear functional, showing the first part. More precisely, the operations needed in $\Phi^* u$ are compositions of duals of the module multiplication by $\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right|$ and the pull-back $(\Phi^{-1})^*$. Hence the second statement follows from general continuity properties of dual maps, as used already often, see again Corollary 3.2.22. The third part is a simple consequence of the multiplicativity of Jacobi determinants. Note that (5.2.97) follows trivially from (5.2.95) and (5.2.96). The fourth statement is actually the motivation for the somewhat non-intuitive definition with the presence of the Jacobi determinant. It was shown in Lemma 5.2.26, *ii.)*. For the fifth part, we compute

$$\begin{aligned}
\Phi^*(fu)(\varphi) &= \left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right| fu\right)(\varphi \circ \Phi^{-1}) \\
&= u\left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right| f(\varphi \circ \Phi^{-1})\right) \\
&= \left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right| u\right)\big((f \circ \Phi \circ \Phi^{-1})(\varphi \circ \Phi^{-1})\big) \\
&= \left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right| u\right)((\Phi^*(f)\varphi)\Phi^{-1}) \\
&= (\Phi^* u)(\Phi^*(f)\varphi) \\
&= \big(\Phi^*(f)\Phi^*(u)\big)(\varphi)
\end{aligned}$$

for $\varphi \in \mathscr{C}_0^\infty(X)$, which is (5.2.99). Next let $U \subseteq X$ be open. Then $\Phi(U) \subseteq Y$ is open as well for a diffeomorphism, explaining why the right hand side of (5.2.100) makes sense at all. Also $\Phi\big|_U \colon U \longrightarrow \Phi(U)$ is still a diffeomorphism. Thus let $\varphi \in \mathscr{C}_0^\infty(U)$. Then $\operatorname{supp}(\varphi \circ \Phi^{-1}) = \Phi(\operatorname{supp}(\varphi))$ by the observation in (5.2.92). Hence we get

$$\begin{aligned}
(\Phi^* u)\big|_U(\varphi) &= (\Phi^* u)(\varphi) \\
&= \left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right| u\right)(\varphi \circ \Phi^{-1}) \\
&= \left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right| u\big|_{\Phi(U)}\right)(\varphi \circ \Phi^{-1})
\end{aligned}$$

$$= \big(\Phi\big|_U\big)^*\big(u\big|_{\Phi(U)}\big)(\varphi),$$

where we use the locality of the module structure of $\mathscr{C}_0^\infty(Y)$ and the fact that the Jacobi determinant depends on Φ^{-1} in a local way. This gives (5.2.100). From here, (5.2.101) and (5.2.102) are easy. Indeed,

$$\Phi^* u\Big|_{X\setminus\Phi^{-1}(\operatorname{supp}(u))} = \big(\Phi\big|_{X\setminus\Phi^{-1}(\operatorname{supp}(u))}\big)^*\big(u\big|_{Y\setminus\operatorname{supp}(u)}\big) = 0,$$

showing $\operatorname{supp}(\Phi^* u) \subseteq \Phi^{-1}(\operatorname{supp}(u))$. Using the fact that Φ^* is invertible with inverse $(\Phi^{-1})^*$ we get the inclusion

$$\operatorname{supp}(u) = \operatorname{supp}((\Phi^{-1})^*\Phi^* u) \subseteq (\Phi^{-1})^{-1}(\operatorname{supp}(\Phi^* u)) = \Phi(\operatorname{supp}(\Phi^* u)),$$

which gives equality in (5.2.101). Analogously, we know $u\big|_{Y\setminus\operatorname{sing\,supp}(u)} = I_f$ for some smooth function $f \in \mathscr{C}^\infty(Y \setminus \operatorname{sing\,supp}(u))$. Thus

$$\begin{aligned}
\Phi^* u\Big|_{X\setminus\Phi^{-1}(\operatorname{sing\,supp}(u))} &= \big(\Phi\big|_{X\setminus\Phi^{-1}(\operatorname{sing\,supp}(u))}\big)^*\big(u\big|_{Y\setminus\operatorname{sing\,supp}(u)}\big) \\
&= \big(\Phi\big|_{X\setminus\Phi^{-1}(\operatorname{sing\,supp}(u))}\big)^*(I_f) \\
&= I_{\left(\Phi|_{X\setminus\Phi^{-1}(\operatorname{sing\,supp}(u))}\right)^* f}
\end{aligned}$$

with $\Phi^* f \in \mathscr{C}^\infty(X \setminus \Phi^{-1}(\operatorname{sing\,supp}(u)))$, since the pull-back of smooth functions is smooth. This shows

$$\operatorname{sing\,supp}(\Phi^* u) \subseteq \Phi^{-1}(\operatorname{sing\,supp}(u)).$$

With the same argument as above we get equality. Let $x \in X$ and let $U \subseteq X$ be an open neighbourhood of x with compact closure $U^{\mathrm{cl}} \subseteq X$. Then $\Phi(U) \subseteq Y$ is an open neighbourhood of $y = \Phi(x)$ with compact closure $\Phi(U)^{\mathrm{cl}} = \Phi(U^{\mathrm{cl}})$ and any such set is of this form, since the diffeomorphism is, in particular, a homeomorphism. The distributions u and $\left|\det \frac{\partial\Phi^{-1}}{\partial y}\right| u$ have the same order at every $y \in Y$, since $\left|\det \frac{\partial\Phi^{-1}}{\partial y}\right|$ is a non-vanishing smooth function and the module structure of $\mathscr{C}_0^\infty(Y)'$ is of order zero, see Proposition 5.2.3, *vi.)*, and Exercise **??**. Thus let $\ell \in \mathbb{N}_0$ be such that

$$\left|\left(\left|\det \frac{\partial\Phi^{-1}}{\partial y}\right| u\right)(\psi)\right| \le \mathit{const}\, \mathrm{p}_{\Phi(U)^{\mathrm{cl}},\ell}(\psi)$$

for all test functions $\psi \in \mathscr{C}_0^\infty(Y)$ with $\operatorname{supp}(\psi) \subseteq \Phi(U)^{\mathrm{cl}}$. Since the continuity estimate for the pull-back of test functions does not increase the needed order, see Proposition 2.3.24,

$$|(\Phi^* u)(\varphi)| \le \mathit{const}\, \mathrm{p}_{\Phi(U)^{\mathrm{cl}},\ell}(\varphi)$$

for all $\varphi \in \mathscr{C}_0^\infty(X)$ with $\operatorname{supp}(\varphi) \subseteq U^{\mathrm{cl}}$. Note that the pull-back with Φ^{-1} gives a bijection between the test functions ψ and φ. Since this holds for all U, we conclude $\operatorname{ord}_x(\Phi^* u) \le \operatorname{ord}_y(u)$. Using again the inverse $(\Phi^{-1})^*$ of Φ^* we get the opposite estimate, showing the equality in (5.2.103). Finally, the last part is a computational etude in differentiating determinants, see Exercise 5.5.25. Note that we will find a more conceptual way to prove this relation later in **??**. □

We can view all these little results as a compatibility of our previous concepts for distributions with the action of diffeomorphisms. The whole calculus behaves well under diffeomorphisms. Note that the last statement (5.2.104) becomes the usual chain rule for distributions, which are $\mathscr{C}^1$-functions, i.e. we have directly

$$\frac{\partial}{\partial x^i}(\Phi^* I_f) = \frac{\partial}{\partial x^i}(I_{\Phi^* f})$$

$$
\begin{aligned}
&= I_{\frac{\partial}{\partial x^i}(\Phi^* f)} \\
&= I_{\frac{\partial}{\partial x^i}(f \circ \Phi)} \\
&= \sum_{j=1}^{n} I_{\frac{\partial f}{\partial y^j} \circ \Phi \frac{\partial \Phi^j}{\partial x^i}} \\
&= \sum_{j=1}^{n} \frac{\partial \Phi^j}{\partial x^i} I_{\Phi^* \frac{\partial f}{\partial y^j}} \\
&= \sum_{j=1}^{n} \frac{\partial \Phi^j}{\partial x^i} \Phi^* I_{\frac{\partial f}{\partial y^j}},
\end{aligned}
\tag{5.2.105}
$$

using (5.2.98) as well as the chain rule for $f \in \mathscr{C}^1(Y)$. This gives the more elementary proof of (5.2.104) for $u = I_f$.

Remark 5.2.29 (Generalized functions) As we already indicated, the definition of $\Phi^* u$ for $u \in \mathscr{C}_0^\infty(Y)'$ is essentially a *choice*: we could have done equally well without the Jacobi determinant. However, in this case the identification of functions, say from $\mathscr{L}^1_{\mathrm{loc}}(Y)$, with distributions would *not* be diffeomorphism invariant. The relation (5.2.98) would be less simple in the sense that it would contain a Jacobi determinant here. Thus the point of view one usually takes is that distributions are *generalized functions*, as they transform like functions under pull-backs with diffeomorphisms. The other choice would match better to the idea that distributions are dual to test functions and should thus be treated as *generalized densities*. Indeed, if we view complex measures as distributions, then our definition of pull-back will *not* match the usual definition of a pull-back of measures, but involves an additional Jacobi determinant, see also Exercise 5.5.26. The full impact of this distinction will be seen if one treats distributions on manifolds. The good compatibilities under diffeomorphisms allow us to transplant everything we have done so far to a smooth manifold by means of a smooth atlas. Then we arrive at the picture that test functions pair with generalized densities, while distributions in our sense, i.e. generalized functions, pair with test densities. For manifolds this difference is clearly implemented. But for open subsets of $\mathbb{R}^n$ we have the Lebesgue measure, being a canonical trivialization of the density bundle, admitting this confusing identification.

To emphasize the point of view that distributions are generalized functions, i.e. transform according to Definition 5.2.27 under diffeomorphisms, we adopt the following notation:

Notation 5.2.30 (Generalized functions) Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. When viewing distributions as generalized functions, we write

$$
\mathscr{C}^{-\infty}(X) = \mathscr{C}_0^\infty(X)'. \tag{5.2.106}
$$

Moreover, we write

$$
\mathscr{C}_0^{-\infty}(X) = \left\{ u \in \mathscr{C}^{-\infty}(X) \;\middle|\; \operatorname{supp}(u) \text{ is compact} \right\} \tag{5.2.107}
$$

for the compactly supported distributions.

As we shall see soon, this notation turns out to be very useful. In particular, it suggests that there will be other spaces like $\mathscr{C}^{-k}(X)$ for all $k \in \mathbb{N}_0$. We will come back to this soon in Section 5.3.1. Before, we mention some standard examples of pull-backs appearing often in the applications:

Example 5.2.31 (Pull-back of δ-functional) Let $X, Y \subseteq \mathbb{R}^n$ be non-empty open subsets and consider a diffeomorphism $\Phi\colon X \longrightarrow Y$. For $y_0 \in Y$ we get

$$
\Phi^* \delta_{y_0} = \frac{1}{\left|\det \frac{\partial \Phi}{\partial x}(x_0)\right|} \delta_{x_0}, \tag{5.2.108}
$$

where $x_0 = \Phi^{-1}(y_0) \in X$ is the pre-image of y_0. Indeed, for $\varphi \in \mathscr{C}_0^\infty(X)$ we compute

$$\begin{aligned}
(\Phi^*\delta_{y_0})(\varphi) &= \left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right|\delta_{y_0}\right)(\varphi \circ \Phi^{-1}) \\
&= \left|\det \frac{\partial \Phi^{-1}}{\partial y}(y_0)\right|\delta_{y_0}(\varphi \circ \Phi^{-1}) \\
&= \frac{1}{\left|\det \frac{\partial \Phi}{\partial x}(x_0)\right|}\varphi(\Phi^{-1}(y_0)),
\end{aligned}$$

which gives (5.2.108).

Example 5.2.32 (Pull-back with affine transformations) Let $A \in \mathrm{GL}_n(\mathbb{R})$ and $a \in \mathbb{R}^n$. Then the translation $\tau_a\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by a is $\tau_a(x) = x+a$, as usual. Moreover, the linear map $A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism, as well. We have for the Jacobi matrices

$$\frac{\partial \tau_a}{\partial x} = \mathbb{1} \tag{5.2.109}$$

and

$$\frac{\partial A}{\partial y} = A, \tag{5.2.110}$$

i.e. *constant* Jacobi matrices. This gives for a distribution $u \in \mathscr{C}^{-\infty}(\mathbb{R}^n)$ and a test function $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$

$$(\tau_a^* u)(\varphi) = u(\varphi \circ \tau_{-a}) \tag{5.2.111}$$

as well as

$$(A^*u)(\varphi) = \frac{1}{|\det A|}u(\varphi \circ A^{-1}). \tag{5.2.112}$$

As a last word of caution we note that the push-forward

$$\Phi_*\colon \mathscr{C}_0^\infty(X)' \longrightarrow \mathscr{C}_0^\infty(Y)' \tag{5.2.113}$$

with a diffeomorphism is *not* the inverse of the pull-back Φ^*. Instead, when defining the push-forward of distributions with proper smooth maps we took the point of view of simply dualizing, see (5.2.68), *without* a Jacobi determinant. In fact, as we are interested in proper smooth maps also between open subsets X and Y with different dimensions, there is simply no Jacobi determinant available in general.

Proposition 5.2.33 *Let $X, Y \subseteq \mathbb{R}^n$ be non-empty open subsets and let $\Phi\colon X \longrightarrow Y$ be a diffeomorphism. Moreover, let $u \in \mathscr{C}_0^\infty(X)'$ and $v \in \mathscr{C}_0^\infty(Y)'$ be distributions. Then*

$$\Phi^*(\Phi_* u) = \frac{1}{\left|\det \frac{\partial \Phi}{\partial x}\right|}u \tag{5.2.114}$$

and

$$\Phi_*(\Phi^* v) = \left|\det \frac{\partial \Phi^{-1}}{\partial y}\right|v. \tag{5.2.115}$$

PROOF: Let $\varphi \in \mathscr{C}_0^\infty(X)$, then

$$\begin{aligned}
\Phi^*(\Phi_* u)(\varphi) &= \left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right|\Phi_* u\right)(\varphi \circ \Phi^{-1}) \\
&= \Phi_* u\left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right|(\varphi \circ \Phi^{-1})\right)
\end{aligned}$$

$$
\begin{aligned}
&= u\left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right| \circ \Phi(\varphi \circ \Phi^{-1} \circ \Phi)\right) \\
&= u\left(\frac{1}{\left|\det \frac{\partial \Phi}{\partial x}\right|}\varphi\right) \\
&= \left(\frac{1}{\left|\det \frac{\partial \Phi}{\partial x}\right|}u\right)(\varphi).
\end{aligned}
$$

Similarly, for $\psi \in \mathscr{C}_0^\infty(Y)$ we get

$$
\Phi_*(\Phi^* v)(\psi) = (\Phi^* v)(\psi \circ \Phi) = \left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right| v\right)(\psi \circ \Phi \circ \Phi^{-1}) = \left(\left|\det \frac{\partial \Phi^{-1}}{\partial y}\right| v\right)(\psi),
$$

completing the proof. □

The additional Jacobi determinant can thus been seen as a consequence of treating distributions inconsistently: once we view them as generalized generalized densities, once as generalized functions.

5.2.4 Invariant Distributions

As everywhere in mathematics, symmetries play a crucial role also in distribution theory. Having definitions for the action of diffeomorphisms on generalized functions, we can ask for their invariance properties. Besides the global formulation of invariance, we are also interested in an infinitesimal version on vector fields instead of diffeomorphisms.

Remark 5.2.34 (Brief reminder on vector fields and flows) Let $X \subseteq \mathbb{R}^n$ be a non-empty open subsets. Then a smooth map

$$
V\colon X \longrightarrow \mathbb{R}^n \tag{5.2.116}
$$

is called a *vector field* on X. We use vector fields as directional derivatives depending on a base point. In detail, if $f \in \mathscr{C}^\infty(X)$, then we define a new function $V(f)$ by

$$
V(f)\Big|_x = \sum_{i=1}^n V^i(x) \frac{\partial f}{\partial x^i}(x), \tag{5.2.117}
$$

where $V^i \in \mathscr{C}^\infty(X)$ are the components of V. In addition, we can use a vector field to define a first order autonomous ordinary differential equation on X: we look for a curve

$$
\gamma_{x_0}\colon I_{x_0} \longrightarrow X, \tag{5.2.118}
$$

where $I_{x_0} \subseteq \mathbb{R}$ is an open interval containing $0 \in I_{x_0}$ and $x_0 \in X$, such that

$$
\dot{\gamma}_{x_0}(t) = V(\gamma_{x_0}(t)) \tag{5.2.119}
$$

holds for all $t \in I_{x_0}$ and we have the initial condition

$$
\gamma_{x_0}(0) = x_0. \tag{5.2.120}
$$

The theory of ordinary differential equations gives us a unique solution γ_{x_0} on a maximal such interval I_{x_0} for each initial condition. Moreover, the subset

$$
\mathcal{U} = \bigcup_{x_0 \in X} I_{x_0} \times \{x_0\} \subseteq \mathbb{R} \times X \tag{5.2.121}
$$

turns out to be open. The map

$$\Phi^V : \mathcal{U} \longrightarrow X \tag{5.2.122}$$

defined by

$$\Phi^V(t, x) = \gamma_x(t) \tag{5.2.123}$$

is called the *flow* of V. One can then show that Φ^V is in fact a smooth map on $\mathcal{U}$. If $\mathcal{U} = \mathbb{R} \times X$, we call the vector field V *complete*. If the reference to the vector field V is clear from the context, we simply write Φ instead of Φ^V. In this case we also write

$$\Phi_t = \Phi(t, \cdot)\colon X \longrightarrow X \tag{5.2.124}$$

for every fixed time $t \in \mathbb{R}$. Finally, we note that

$$\Phi(t, \Phi(s, x)) = \Phi(t + s, x) \tag{5.2.125}$$

and

$$\Phi(0, x) = x \tag{5.2.126}$$

hold, whenever the points are in the domain, where the flow is defined. This follows from the uniqueness of the solution for a given initial condition. In the case of a complete vector field, these two flow properties translate into the *one-parameter group* properties

$$\Phi_t \circ \Phi_s = \Phi_{t+s} \quad \text{and} \quad \Phi_0 = \mathrm{id}_X\,. \tag{5.2.127}$$

In particular, every $\Phi_t\colon X \longrightarrow X$ is a diffeomorphism with inverse $\Phi_t^{-1} = \Phi_{-t}$. Conversely, every smooth map

$$\Phi\colon X \times \mathbb{R} \longrightarrow X \tag{5.2.128}$$

satisfying (5.2.125) and (5.2.126) is actually the flow of a complete vector field V, given by

$$V(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, x)\Big|_{t=0}. \tag{5.2.129}$$

Even if the vector field is not complete, we can find for every $x \in X$ an open neighbourhood $U \subseteq X$ and an $\epsilon > 0$ such that

$$U \times (-\epsilon, \epsilon) \subseteq \mathcal{U} \tag{5.2.130}$$

is contained in the domain of Φ. In this case, the maps

$$\Phi_t = \Phi(t, \cdot)\colon U \longrightarrow X \tag{5.2.131}$$

are defined for all $t \in (-\epsilon, \epsilon)$. These maps are diffeomorphisms $\Phi_t\colon U \longrightarrow \Phi_t(U) \subseteq X$ onto their images, which now may depend on t. These facts and much more can be found in any reasonable textbook on ordinary differential equations, see e.g. [**?**], as well as Exercise 5.5.27 for some further details. A more geometric treatment can be found in textbooks on differential geometry like [11, Chapter 9] or [12, Sect. 1.3].

In the following we shall use the interplay of flows and vector fields to pass from global to infinitesimal characterizations of invariance and back. We start with the global invariance:

Definition 5.2.35 (Invariance of distributions) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $\Phi\colon X \longrightarrow X$ be a diffeomorphism. Then $u \in \mathscr{C}^{-\infty}(X)$ is called* invariant *under Φ if*

$$\Phi^* u = u. \tag{5.2.132}$$

If G is a group and $\Phi\colon g \mapsto \Phi_g \in \mathrm{Diffeo}(X)$ is a group morphism, we say that $u \in \mathscr{C}^{-\infty}(X)$ is G-invariant if

$$\Phi_g^* u = u \tag{5.2.133}$$

holds for all $g \in G$.

This is of course a bit of an abuse of notation, since not only the group G is relevant in (5.2.133), but also the way it acts on X, i.e. the map Φ. However, in many cases this is clear from the context, justifying the above definition. Note that invariance under a single diffeomorphism Φ always can be seen as invariance under a group action $\Phi\colon \mathbb{Z} \longrightarrow \operatorname{Diffeo}(X)$ of the group $G = \mathbb{Z}$ on X via

$$\Phi(n) = \Phi^n, \tag{5.2.134}$$

since $\Phi^* u = u$ iff $(\Phi^n)^* u = u$ for all $n \in \mathbb{Z}$. Thus we can focus on the case of a group action from the beginning. Here we get the following basic features:

Proposition 5.2.36 *Let $\Phi\colon G \longrightarrow \operatorname{Diffeo}(X)$ be an action of a group G on a non-empty open subset $X \subseteq \mathbb{R}^n$ by diffeomorphisms.*

i.) *The G-invariant smooth functions*

$$\mathscr{C}^\infty(X)^G = \left\{ f \in \mathscr{C}^\infty(X) \;\middle|\; \Phi_g^* f = f \textit{ for all } g \in G \right\} \tag{5.2.135}$$

form a unital subalgebra of $\mathscr{C}^\infty(X)$.

ii.) *The G-invariant distributions*

$$\mathscr{C}^{-\infty}(X)^G = \left\{ f \in \mathscr{C}^{-\infty}(X) \;\middle|\; \Phi_g^* u = u \textit{ for all } g \in G \right\} \tag{5.2.136}$$

form a subspace of $\mathscr{C}^{-\infty}(X)$ and a module over the subalgebra $\mathscr{C}^\infty(X)^G$.

iii.) *Let $f \in \mathrm{L}^1_{\mathrm{loc}}(X)$. Then $I_f \in \mathscr{C}^{-\infty}(X)^G$ iff f is G-invariant, i.e. iff for all $g \in G$ one has*

$$\Phi_g^* f = f. \tag{5.2.137}$$

PROOF: The first part is clear, since $\Phi_g^*\colon \mathscr{C}^\infty(X) \longrightarrow \mathscr{C}^\infty(X)$ is an algebra automorphism. For the second we note that each Φ_g^* is linear, hence the condition $\Phi_g^* u = u$ for all $g \in G$ gives a vector space. Moreover, since by Proposition 5.2.28, *v.)*, we have

$$\Phi_g^*(fu) = \Phi_g^*(f)\Phi_g^*(u)$$

for all $f \in \mathscr{C}^\infty(X)$, the invariant distributions form a module over $\mathscr{C}^\infty(X)^G$. Finally, from Proposition 5.2.28, *iv.)*, we have

$$\Phi_g^* I_f = I_{\Phi^* f}$$

for all $f \in \mathrm{L}^1_{\mathrm{loc}}(X)$. Since the map $I\colon \mathrm{L}^1_{\mathrm{loc}}(X) \longrightarrow \mathscr{C}^{-\infty}(X)$ is injective, the last statement follows at once. $\square$

Example 5.2.37 (Invariant distributions) We consider $X = \mathbb{R}$.

i.) Denote the reflection at the origin by

$$\operatorname{inv}\colon \mathbb{R} \ni x \mapsto -x \in \mathbb{R}. \tag{5.2.138}$$

Then the δ-functional at 0 is invariant under the action of inv. This is obvious.

ii.) Consider the $\mathbb{Z}$ action on $\mathbb{R}$ by translations, i.e. $\tau\colon \mathbb{Z} \longrightarrow \operatorname{Diffeo}(\mathbb{R})$ with

$$\tau_n(x) = x + n. \tag{5.2.139}$$

Then the *Dirac comb*

$$u = \sum_{n \in \mathbb{Z}} \delta_n \tag{5.2.140}$$

is a $\mathbb{Z}$-invariant distribution. Note that (5.2.140) is clearly convergent in the weak* sense.

While these examples are of a rather discrete nature, we are now interested in the situation with a continuous symmetry. In fact, we consider a one-parameter group $\{\Phi_t\}_{t\in\mathbb{R}}$ of diffeomorphisms depending also smoothly on the parameter t: according to Remark 5.2.34 this is a flow of a vector field $V \in \mathscr{C}^\infty(X, \mathbb{R}^n)$.

Theorem 5.2.38 (Derivative of distribution) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. Moreover, let $V\colon X \longrightarrow \mathbb{R}^n$ be a complete smooth vector field with flow Φ. Finally, let $k \in \mathbb{N} \cup \{\infty\}$.*

i.) For all $\alpha \in \mathbb{C}$ and all $t \in \mathbb{R}$ the function

$$(t, x) \;\mapsto\; \det\Big(\frac{\partial \Phi_t}{\partial x}\Big)^\alpha (x) \tag{5.2.141}$$

is smooth.

ii.) For all $\alpha \in \mathbb{C}$ and for all functions $\varphi \in \mathscr{C}^k(X)$ one has

$$\lim_{t\to 0} \frac{1}{t}\bigg(\det\Big(\frac{\partial \Phi_t}{\partial x}\Big)^\alpha \Phi_t^* \varphi - \varphi\bigg) = V\varphi + \alpha \operatorname{div}(V)\varphi \tag{5.2.142}$$

in the $\mathscr{C}^{k-1}$-topology.

iii.) For all $\alpha \in \mathbb{C}$ and for all $\varphi \in \mathscr{C}_0^k(X)$ one has

$$\lim_{t\to 0} \frac{1}{t}\bigg(\det\Big(\frac{\partial \Phi_t}{\partial x}\Big)^\alpha \Phi_t^* \varphi - \varphi\bigg) = V\varphi + \alpha \operatorname{div}(V)\varphi \tag{5.2.143}$$

in the $\mathscr{C}_0^{k-1}$-topology of the LF space $\mathscr{C}_0^{k-1}(X)$.

iv.) For all distributions $u \in \mathscr{C}^{-\infty}(X)$ one has

$$\lim_{t\to 0} \frac{1}{t}(\Phi_t^* u - u) = Vu \tag{5.2.144}$$

in the weak topology of $\mathscr{C}^{-\infty}(X)$. Here V acts on u as a first order differential operator, i.e.*

$$Vu = \sum_{i=1}^n V^i \frac{\partial u}{\partial x^i}. \tag{5.2.145}$$

v.) For all times $t \in \mathbb{R}$ one has

$$\Phi_t^*(Vu) = V\Phi_t^* u \tag{5.2.146}$$

for all $u \in \mathscr{C}^{-\infty}(X)$.

PROOF: Since Φ_t is a one-parameter group of diffeomorphism we have $\det\big(\frac{\partial \Phi_0}{\partial x}\big) = 1$ at $t = 0$. By continuity, $\big(\frac{\partial \Phi_t}{\partial x}\big) > 0$ for all $t \in \mathbb{R}$ and thus the α-th power is well-defined and smooth. For the second part, we consider a compact subset $K \subseteq X$ and $\ell \in \mathbb{N}_0$ with $\ell \leq k$. The compactness implies that for all $\epsilon > 0$

$$\Phi([-\epsilon, \epsilon] \times K) = \tilde{K} \subseteq X$$

is still compact. Note that if V has an incomplete flow, we could still find some small $\epsilon > 0$ such that the flow Φ is defined on $[-\epsilon, \epsilon] \times K$. This follows from the fact that the domain $\mathcal{U} \subseteq \mathbb{R} \times X$ of Φ is open and contains $\{0\} \times X$, see Remark 5.2.34. One can use this observation to extend the statement to vector fields with incomplete flows, see Exercise 5.5.28. We denote the Jacobi determinant of Φ by

$$J_t(x) = J(t, x) = \det\Big(\frac{\partial \Phi_t}{\partial x}(x)\Big)$$

for abbreviation. For $x \in K$ and $t \neq 0$ we then get

$$
\begin{aligned}
\frac{1}{t}&(J_t^\alpha \Phi_t^* \varphi - \varphi)(x) \\
&= \frac{1}{t}(J_t(x)^\alpha \varphi(\Phi_t(x)) - \varphi(x)) \\
&\stackrel{(a)}{=} \frac{1}{t}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}(J_{ts}(x)^\alpha \varphi(\Phi_{ts}(x)))\,\mathrm{d}s \\
&\stackrel{(b)}{=} \frac{1}{t}\int_0^1 \Big(\alpha J_{ts}(x)^{\alpha-1}\dot{J}_{ts}(x) t \varphi(\Phi_{ts}(x)) + J_{ts}(x)^\alpha \sum_{i=1}^n \frac{\partial \varphi}{\partial x^i}(\Phi_{ts}(x)) \frac{\partial \Phi^i}{\partial t}(ts, x) t\Big)\,\mathrm{d}s \\
&= \int_0^1 \Big(\alpha J_{ts}(x)^{\alpha-1}\dot{J}_{ts}(x) \varphi(\Phi_{ts}(x)) + J_{ts}(x)^\alpha \sum_{i=1}^n \frac{\partial \varphi}{\partial x^i}(\Phi_{ts}(x)) \frac{\partial \Phi^i}{\partial t}(ts, x)\Big)\,\mathrm{d}s,
\end{aligned}
\tag{$*$}
$$

where in (a) we used $J(0, x) = 1$ for all $x \in X$, as well as the fundamental theorem of calculus. In (b) we just computed the derivative by means of the chain rule. Now on the compact subset K and for $t \in [-\epsilon, \epsilon]$ all functions under the integral are at least $\mathscr{C}^0$ and thus bounded on the compact subset $[-\epsilon, \epsilon] \times K$. This allows to exchange limits with the integral, using e.g. dominated convergence. Moreover, we can estimate the limits even uniformly for $x \in K$. In particular, if $\ell \leq k-1$, we can first compute additional ℓ derivatives in x-direction of the left hand side by considering the integral over the corresponding derivatives. This shows that we have a $\mathscr{C}^{k-1}$-limit, once we are able to show the existence of the pointwise limit of $(*)$ for $t \to 0$. But this is now easy, as we simply can set $t = 0$ on the right hand side of $(*)$. The only interesting terms are then the following: First we note once again

$$
J(0, x) = 1 \quad \text{for all} \quad x \in X.
$$

Second,

$$
\dot{J}_0(x) = \frac{\mathrm{d}}{\mathrm{d}t} \det\left(\frac{\partial \Phi_t}{\partial x}\right)\Big|_{t=0} = \det\left(\frac{\partial \Phi_t}{\partial x}\right)\Big|_{t=0} \operatorname{tr}\left(\frac{\partial^2 \Phi_t}{\partial x \partial t}\right)\Big|_{t=0} = \operatorname{tr}\left(\frac{\partial}{\partial x^j} V^i\right) = \operatorname{div}(V),
$$

by the usual rules to differentiate a determinant, see Exercise 5.5.30, and the defining differential equation for the flow. Here the divergence of V is explicitly given by

$$
\operatorname{div}(V) = \sum_{i=1}^n \frac{\partial V^i}{\partial x^i}.
$$

Together, this gives (5.2.142), since $\frac{\partial \Phi^i}{\partial t}\big|_{t=0} = V^i$. The second statement is then also clear. For $\varphi \in \mathscr{C}_0^k(X)$ we have a compact subset $K = \operatorname{supp}(\varphi)$ and for $t \in [-\epsilon, \epsilon]$ we have

$$
\operatorname{supp}(\Phi_t^* \varphi) \subseteq \tilde{K} = \Phi([-\epsilon, \epsilon] \times K).
$$

Since we can use then the compact subset $\tilde{K}$ to estimate the first $\ell \leq k-1$ derivatives uniformly, we get convergence of (5.2.143) in the subspace topology of $\mathscr{C}_{\tilde{K}}^{k-1}(X) \subseteq \mathscr{C}^{k-1}(X)$. By the usual properties of convergence in LF spaces, see again Theorem 5.1.1, *v.)*, we can conclude that (5.2.143) is even a limit in the $\mathscr{C}_0^{k-1}$-topology. Now let $u \in \mathscr{C}^{-\infty}$ be a generalized function. To test (5.2.144) we fix a test function $\varphi \in \mathscr{C}_0^\infty(X)$. Then for $t \neq 0$ we have

$$
\frac{1}{t}(\Phi_t^* u - u)(\varphi) = \frac{1}{t}\bigg(\Big(\det\left(\frac{\partial \Phi_t^{-1}}{\partial y}\right) u\Big)(\Phi_* \varphi) - u(\varphi)\bigg) = u\bigg(\frac{1}{t}\Big(\det\left(\frac{\partial \Phi_t^{-1}}{\partial y}\right)\Phi_{-t}^* \varphi - \varphi\Big)\bigg).
$$

Since u is continuous and we have the convergence of (5.2.143) in the $\mathscr{C}_0^\infty$-topology, we get

$$
\begin{aligned}
\lim_{t\to 0}\frac{1}{t}(\Phi_t^* u - u)(\varphi) &= u\Big(\lim_{t\to 0}\frac{1}{t}\Big(\det\Big(\frac{\partial\Phi_t^{-1}}{\partial y}\Big)\Phi_{-t}^*\varphi - \varphi\Big)\Big)\\
&= -u(V(\varphi) + \operatorname{div}(V)\varphi)\\
&= -\sum_{i=1}^n u\Big(V^i\frac{\partial\varphi}{\partial x^i} + \frac{\partial V^i}{\partial x^i}\varphi\Big)\\
&= \Big(\sum_{i=1}^n V^i\frac{\partial u}{\partial x^i}\Big)(\varphi),
\end{aligned}
$$

which is (5.2.144). Finally, we have the one-parameter group property $\Phi_t \circ \Phi_s = \Phi_{t+s}$. Hence

$$
\begin{aligned}
\Phi_t^* V u &= \Phi_t^* \lim_{s\to 0}\frac{1}{s}(\Phi_s^* u - u)(\varphi)\\
&\overset{(a)}{=} \lim_{s\to 0}\frac{1}{s}(\Phi_t^*\Phi_s^* u - \Phi_t^* u)\\
&= \lim_{s\to 0}\frac{1}{s}(\Phi_s^*(\Phi_t^* u) - \Phi_t^* u)\\
&= V\Phi_t^* u,
\end{aligned}
$$

where we have used (5.2.144) twice and the fact that the pull-back of distributions Φ_t^* is weak* continuous in (a). □

Remark 5.2.39 Again, a more geometric point of view would clarify the statement considerably. The limit in Theorem 5.2.38, *iii.)*, is better to be interpreted as the *Lie derivative* of the α-density given by $\varphi|\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n|^\alpha$, see e.g. [**?**, Section 3.2.3 and 3.2.4]. Note, also, that one has suitable adaptions for the case of non-complete vector field, see again Exercise 5.5.28.

Corollary 5.2.40 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $V\colon X \longrightarrow \mathbb{R}^n$ be a vector field with complete flow Φ. Then a distribution $u \in \mathscr{C}^{-\infty}(X)$ is invariant under the one-parameter group $\{\Phi_t\}_{t\in\mathbb{R}}$ iff*

$$
Vu = 0. \tag{5.2.147}
$$

PROOF: Indeed, suppose $\Phi_t^* u = u$ for all $t \in \mathbb{R}$. Then differentiating at $t = 0$ gives (5.2.147) immediately. Conversely, (5.2.147) implies directly $\Phi_t^* u = u$ by (5.2.146). □

This gives now a simple criterion for invariance: the infinitesimal and easy to check condition (5.2.147) is equivalent to the global invariance property

$$
\Phi_t^* u = u. \tag{5.2.148}
$$

Note, however, that the actual solutions of (5.2.147) might still be difficult to determine, as we have a (first order, linear) partial differential equation to solve. We illustrate this with the following example:

Example 5.2.41 (Translation invariance) Consider $X = \mathbb{R}^n$ and $v \in \mathbb{R}^n$. Then the constant vector field $V(x) = v$ has the flow

$$
\Phi_t(x) = x + tv, \tag{5.2.149}
$$

i.e. the translations in direction of v. Thus $u \in \mathscr{C}^{-\infty}(\mathbb{R}^n)$ is translationally invariant in direction of v iff

$$
v^i\frac{\partial u}{\partial x^i} = 0. \tag{5.2.150}
$$

Moreover, u is translationally invariant in *all* directions iff (5.2.150) holds for all $v \in \mathbb{R}^n$. This is the case iff $\frac{\partial u}{\partial x^i} = 0$ for all $i = 1, \ldots, n$. Note that it still needs a non-trivial statement, namely Theorem 5.2.19, to determine all solutions of (5.2.150) for all $v \in \mathbb{R}^n$. Only the multiples of the Lebesgue integral are translationally invariant.

Example 5.2.42 Consider again $X = \mathbb{R}^n$ and $A \in \mathrm{M}_n(\mathbb{R})$. This determines a linear vector field V_A by

$$V_A(x) = Ax. \tag{5.2.151}$$

It has complete flow given by the linear matrix exponential

$$\Phi_t^{V_A}(x) = \mathrm{e}^{tA}x, \tag{5.2.152}$$

where $t \in \mathbb{R}$. Now the invariance of $u \in \mathscr{C}^{-\infty}(\mathbb{R}^n)$ under V_A reads $V_A u = 0$ and hence

$$A_i^j x^i \frac{\partial u}{\partial x^i} = 0. \tag{5.2.153}$$

Specializing A appropriately, we can encode e.g. rotational invariance etc., see Exercise 5.5.31.

5.3 Subspaces of Distributions

In this section we characterize several distinguished subspaces of all distributions $\mathscr{C}^{-\infty}(X)$ by their orders and support properties. This will allow us to identify them with other dual spaces leading to a quite detailed hierarchy inside $\mathscr{C}^{-\infty}(X)$. We will conclude this section with a discussion of positive functionals which automatically will be continuous and of order zero. This admits a characterization as integrations with respect to positive Borel measures.

5.3.1 Distributions of Finite Order

We have seen examples of distributions, which have infinite global order, see Example 5.1.13. On the other hand, for any distribution, the order is locally finite. We are now interested in those distributions having a finite order globally:

Definition 5.3.1 (Distributions of finite global order) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. Then the distributions of global order $k \in \mathbb{N}_0 \cup \{\infty\}$ are denoted by*

$$\mathscr{C}^{-k}(X) = \left\{ u \in \mathscr{C}_0^\infty(X)' \;\middle|\; \operatorname{ord}_x(u) \leq k \textit{ for all } x \in X \right\}. \tag{5.3.1}$$

This recovers our notation for $k = \infty$. Moreover, one should be careful, as $\mathscr{C}^{-0}(X)$ is *not* the same as the continuous functions $\mathscr{C}(X) = \mathscr{C}^0(X)$. Thus in (5.3.1) we have $-0 \neq 0$. We have already seen several standard examples:

Example 5.3.2 (Distributions of finite global order) Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.

i.) For every $f \in \mathrm{L}^1_{\mathrm{loc}}(X)$ one has $I_f \in \mathscr{C}^{-0}(X)$. Hence

$$\mathrm{L}^1_{\mathrm{loc}}(X) \subseteq \mathscr{C}^{-0}(X). \tag{5.3.2}$$

ii.) If $\mu \in \mathrm{Meas}(X)$ is a complex measure, then $\mu \in \mathscr{C}^{-0}(X)$. This way we get

$$\mathrm{Meas}(X) \subseteq \mathscr{C}^{-0}(X). \tag{5.3.3}$$

iii.) The δ-distributions are elements of $\mathscr{C}^{-0}(X)$.

iv.) The principal value of $\frac{1}{x}$ is a distribution of order one. Thus

$$\mathrm{vp}\,\frac{1}{x} \in \mathscr{C}^{-1}(X). \tag{5.3.4}$$

Some first properties of distribution of order at most k are collected in the following proposition:

Proposition 5.3.3 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and $k \in \mathbb{N}_0$.*

i.) The distributions $\mathscr{C}^{-k}(X) \subseteq \mathscr{C}^{-\infty}(X)$ of order at most k form a subspace of all distributions.

ii.) The subspace $\mathscr{C}^{-k}(X) \subseteq \mathscr{C}^{-\infty}(X)$ is a submodule with respect to the $\mathscr{C}^{\infty}(X)$-module structure.

iii.) If $\alpha \in \mathbb{N}_0^n$, then differentiation yields a linear map

$$\partial^\alpha \colon \mathscr{C}^{-k}(X) \longrightarrow \mathscr{C}^{-(k+|\alpha|)}(X). \tag{5.3.5}$$

iv.) If $u \in \mathscr{C}^{-k}(X)$ and $U \subseteq X$ is an open subset, then

$$u\Big|_U \in \mathscr{C}^{-k}(U). \tag{5.3.6}$$

v.) If $\Phi \colon X \longrightarrow Y \subseteq \mathbb{R}^m$ is a proper smooth map to some other open subset, then the push-forward yields a linear map

$$\Phi_* \colon \mathscr{C}^{-k}(X) \longrightarrow \mathscr{C}^{-k}(Y). \tag{5.3.7}$$

vi.) If $\Phi \colon X \longrightarrow Y \subseteq \mathbb{R}^n$ is a diffeomorphism, then the pull-back yields a linear map

$$\Phi^* \colon \mathscr{C}^{-k}(Y) \longrightarrow \mathscr{C}^{-k}(X). \tag{5.3.8}$$

PROOF: The first part follows from Proposition 5.1.32, *iv.)*. The second statement is a consequence of Proposition 5.2.3, *vi.)*. The next part is the statement of Proposition 5.2.9, *vii.)*. The locality property (5.3.6) is then clear, since the order is, by its very definition, a pointwise concept, see Proposition 5.1.32, *iv.)*. Finally, both push-forwards and pull-backs do not increase the order by Proposition 5.2.22, *v.)*, and Proposition 5.2.28, *viii.)*, respectively. ☐

The interesting question is now how one can give a more intrinsic characterization of $\mathscr{C}^{-k}(X)$. This is clarified in the following proposition:

Proposition 5.3.4 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and $k \in \mathbb{N}_0$.*

i.) Every distribution $u \in \mathscr{C}^{-k}(X)$ extends uniquely to a continuous linear functional

$$u \colon \mathscr{C}_0^k(X) \longrightarrow \mathbb{K}. \tag{5.3.9}$$

ii.) Every continuous linear functional $u \in \mathscr{C}_0^k(X)'$ restricts to a distribution

$$u\Big|_{\mathscr{C}_0^\infty(X)} \in \mathscr{C}^{-k}(X) \tag{5.3.10}$$

of order at most k.

PROOF: Here we equip $\mathscr{C}_0^k(X)$ with its natural LF topology, as usual. Since u is of order k globally, we find continuity estimates for compact subsets $K \subseteq X$ of the form

$$|u(\varphi)| \leq \mathit{const}\, \mathrm{p}_{K,k}(\varphi) \tag{$*$}$$

for all $\varphi \in \mathscr{C}_0^\infty(X)$ with $\operatorname{supp}(\varphi) \subseteq K$. But this means that u is continuous with respect to the subspace topology inherited from

$$\mathscr{C}_0^\infty(X) \subseteq \mathscr{C}_0^k(X).$$

From the construction in Proposition 2.3.25 we infer that for every $\varphi \in \mathscr{C}_0^k(X)$ we have a sequence $(\varphi_m)_{m\in\mathbb{N}}$ of test functions $\varphi_m \in \mathscr{C}_0^\infty(X)$, such that there is a common compact subset $K \subseteq X$ with

$$\operatorname{supp}(\varphi_m),\ \operatorname{supp}(\varphi) \subseteq K$$

and $\varphi_m \to \varphi$ in the $\mathscr{C}_K^k$-topology. Indeed, K can be taken to be the closure of any open neighbourhood $\operatorname{supp}(\varphi) + \mathrm{B}_\epsilon(0)$ of $\operatorname{supp}(\varphi)$, where $\epsilon > 0$ is sufficiently small such that the closure is still a subset of X. This shows that $\mathscr{C}_0^\infty(X)$ is sequentially dense in $\mathscr{C}_0^k(X)$ for all $k \in \mathbb{N}_0$. By the usual extension of linear continuous maps to completions, we get a unique extension u to $\mathscr{C}_0^k(X)$: we can view the complete LF space $\mathscr{C}_0^k(X)$ as the completion of its dense subspace $\mathscr{C}_0^\infty(X)$ with respect to the subspace topology. This proves the first part. Conversely, if $u \in \mathscr{C}_0^k(X)'$ is a continuous linear functional, its continuity gives estimates of the above form for all compact $K \subseteq X$ and $\varphi \in \mathscr{C}_0^k(X)$. But this means that the restriction $u\big|_{\mathscr{C}_0^\infty(X)}$ is a distribution of order at most k. □

Remark 5.3.5 In the sequel, we will use this result to identify $\mathscr{C}^{-k}(X)$ with the dual $\mathscr{C}_0^k(X)'$, i.e. we have

$$\mathscr{C}^{-k}(X) \cong \mathscr{C}_0^k(X)', \tag{5.3.11}$$

by extending by continuity and restriction. Indeed, the two constructions in Proposition 5.3.4 are clearly inverse to each other. Having now an isomorphism to a dual space, $\mathscr{C}^{-k}(X)$ inherits a weak* topology, as well. We will equip $\mathscr{C}^{-k}(X)$ with this weak* topology instead of the subspace topology inherited from the weak* topology of $\mathscr{C}^{-\infty}(X)$. Similarly, we get a strong topology from (5.3.11), which we can compare to the strong topology of $\mathscr{C}^{-\infty}(X)$ restricted to $\mathscr{C}^{-k}(X)$.

The relations between the topologies arising from the identification (5.3.11) for various $k \in \mathbb{N}_0 \cup \{\infty\}$ are summarized in the next proposition:

Proposition 5.3.6 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.*

i.) For all $k \in \mathbb{N}_0$ the distributions $\mathscr{C}^{-k}(X)$ of order at most k are sequentially complete with respect to the weak topology coming from $\mathscr{C}^{-k}(X)_\sigma \cong \mathscr{C}_0^k(X)'_\sigma$.*

ii.) For all $k \in \mathbb{N}_0$ the distributions $\mathscr{C}^{-k}(X)$ of order at most k are complete with respect to the strong topology coming from $\mathscr{C}^{-k}(X)_\beta \cong \mathscr{C}_0^k(X)'_\beta$.

iii.) For all $k \in \mathbb{N}_0$ we have the continuous inclusions

$$\mathscr{C}^{-0}(X)_\sigma \subseteq \mathscr{C}^{-k}(X)_\sigma \subseteq \mathscr{C}^{-(k+1)}(X)_\sigma \subseteq \mathscr{C}^{-\infty}(X)_\sigma \tag{5.3.12}$$

as well as

$$\mathscr{C}^{-0}(X)_\beta \subseteq \mathscr{C}^{-k}(X)_\beta \subseteq \mathscr{C}^{-(k+1)}(X)_\beta \subseteq \mathscr{C}^{-\infty}(X)_\beta. \tag{5.3.13}$$

iv.) We have the continuous inclusions

$$\mathrm{L}^1_{\mathrm{loc}}(X) \subseteq \mathscr{C}^{-0}(X)_\beta \tag{5.3.14}$$

as well as

$$\mathrm{Meas}(X) \subseteq \mathscr{C}^{-0}(X)_\beta. \tag{5.3.15}$$

Proof: The first part is a general consequence for duals of LF spaces like $\mathscr{C}_0^k(X)$, see Corollary 3.3.38. Also the completeness in the strong topology holds in general by Corollary 3.3.20. The seminorms of the weak* topology are built from elements in the corresponding space of test functions $\varphi \in \mathscr{C}_0^k(X)$ by $\mathrm{p}_\varphi(u) = |u(\varphi)|$. Having less functions gives less seminorms and thus a coarser topology, explaining (5.3.12). For (5.3.13) we need seminorms of the form $\mathrm{p}_B(u) = \sup_{\varphi\in B}|u(\varphi)|$ with $B \subseteq \mathscr{C}_0^k(X)$ being bounded. Again, the same argument applies, since every bounded subset $B \subseteq \mathscr{C}_0^{k+1}(X)$ stays bounded in $\mathscr{C}_0^k(X)$: here we use the continuity of the inclusion $\mathscr{C}_0^{k+1}(X) \subseteq \mathscr{C}_0^k(X)$. More directly, the continuity of this inclusion dualizes to a strongly continuous linear map by general arguments on duals, see Corollary 3.2.22. Finally, the estimate (5.1.12) uses only seminorms of $\mathscr{C}_0^0(X)$ and hence the argument from Corollary 5.1.7 can be used to sharpen the statement to (5.3.14). Similarly, the estimate (5.1.21) gives the continuity of (5.3.15) by the same reasoning. □

Having now these finer topologies on $\mathscr{C}^{-k}(X)$ we can re-evaluate the continuity statements of our calculus operations. We will not go into the details here, but discuss some examples in Exercise 5.5.33.

5.3.2 Distributions with Compact Support

As for functions, we have a definition of supports also for generalized functions. In fact, this comes ultimately from the sheaf-theoretic definition of supports. Using this, we can define distributions with compact support. We formulate this directly for distributions of finite global order as well:

Definition 5.3.7 (Compactly supported distributions) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and $k \in \mathbb{N}_0 \cup \{\infty\}$. Then we define*

$$\mathscr{C}_0^{-k}(X) = \Big\{ u \in \mathscr{C}^{-k}(X) \;\Big|\; \operatorname{supp}(u) \textit{ is compact} \Big\}. \tag{5.3.16}$$

As for distributions in general, we are mainly interested in $\mathscr{C}_0^{-\infty}(X)$, but also the combination of compact support and finite global order is helpful in many places. Clearly, we have

$$\mathscr{C}_0^{-k}(X) = \mathscr{C}_0^{-\infty}(X) \cap \mathscr{C}^{-k}(X). \tag{5.3.17}$$

We have seen already numerous examples of compactly supported distributions:

Example 5.3.8 Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.

i.) If $f \in \mathscr{L}^1_{\mathrm{loc}}(X)$ vanishes outside a compact subset $K \subseteq X$ almost everywhere, then $\operatorname{supp} I_f \subseteq K$. In particular, if $f \in \mathscr{L}^1_{\mathrm{loc}}(X)$ satisfies $f\big|_{X \setminus K} = 0$, we have

$$I_f \in \mathscr{C}_0^{-0}(X). \tag{5.3.18}$$

ii.) The δ-functional δ_{x_0} at $x_0 \in X$ is a distribution with a single point as support. Hence

$$\delta_{x_0} \in \mathscr{C}_0^{-0}(X) \tag{5.3.19}$$

follows. Moreover, if $\alpha \in \mathbb{N}_0^n$ is some multiindex, we have

$$\partial^\alpha \delta_{x_0} \in \mathscr{C}_0^{-|\alpha|}(X). \tag{5.3.20}$$

While we can have compactly supported distributions of arbitrarily high, but finite order, they can not have infinite order. This and some additional first properties are collected in the next proposition:

Proposition 5.3.9 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and $k \in \mathbb{N}_0 \cup \{\infty\}$.*

i.) The compactly supported distributions $\mathscr{C}_0^{-k}(X)$ of order at most k form a subspace of all distributions $\mathscr{C}^{-\infty}(X)$.

ii.) The subspace $\mathscr{C}_0^{-k}(X)$ is a submodule of $\mathscr{C}^{-\infty}(X)$ with respect to the $\mathscr{C}^\infty(X)$-module structure.

iii.) If $\alpha \in \mathbb{N}_0^n$, then differentiation yields a linear map

$$\partial^\alpha \colon \mathscr{C}_0^{-k}(X) \longrightarrow \mathscr{C}_0^{-(k+|\alpha|)}(X). \tag{5.3.21}$$

iv.) *If* $u \in \mathscr{C}_0^{-\infty}(X)$ *is a distribution with compact support, then there is a* $k \in \mathbb{N}_0$ *with* $u \in \mathscr{C}_0^{-k}(X)$*, i.e. u has finite global order.*

PROOF: The first part is clear by Remark 5.1.25, *iii.)*, for $k = \infty$. Then (5.3.17) together with the fact that $\mathscr{C}^{-k}/(X)$ is a subspace, i.e. Proposition 5.3.3, *i.)*, gives the general case $k \in \mathbb{N}_0 \cup \{\infty\}$. The second statement is a consequence of the locality properties of the module structure according to Proposition 5.2.3, *v.)*, for $k = \infty$, as well as the fact that $\mathscr{C}^{-k}(X)$ is a submodule for all k by Proposition 5.3.3, *ii.)*. The third statement is the locality of differentiation as in Proposition 5.2.9, *v.)*. The last part is at the heart of the definition of the order: it is locally bounded by Proposition 5.1.32, *ii.)*. Hence the order is bounded on a compact subset like the support of $u \in \mathscr{C}_0^{-\infty}(X)$ and thus globally bounded, since $\operatorname{ord}_x(u) = 0$ for $x \in X \setminus \operatorname{supp}(u)$. This completes the proof. □

Clearly, the compactly supported distributions of order at most $k \in \mathbb{N}_0$ give raise to a nested system

$$\mathscr{C}_0^{-0}(X) \subseteq \mathscr{C}_0^{-1}(X) \subseteq \cdots \mathscr{C}_0^{-k}(X) \subseteq \mathscr{C}_0^{-(k+1)}(X) \subseteq \cdots \subseteq \mathscr{C}_0^{-\infty}(X) \tag{5.3.22}$$

of subspaces of $\mathscr{C}_0^{-\infty}(X)$. The last part of the proposition then states that (5.3.22) is exhaustive, i.e. we have

$$\mathscr{C}_0^{-\infty}(X) = \bigcup_{k \in \mathbb{N}_0} \mathscr{C}_0^{-k}(X). \tag{5.3.23}$$

Recall that we also have a nested system

$$\mathscr{C}^{-0}(X) \subseteq \mathscr{C}^{-1}(X) \subseteq \cdots \mathscr{C}^{-k}(X) \subseteq \mathscr{C}^{-(k+1)}(X) \subseteq \cdots \subseteq \mathscr{C}^{-\infty}(X) \tag{5.3.24}$$

of subspaces of $\mathscr{C}^{-\infty}(X)$. However, contrary to (5.3.22) this system is *not* exhaustive, as one always has distributions of infinite order according to Example 5.1.13 and its variants, see also Exercise 5.5.34. As for finite order distributions, we want to obtain a more intrinsic interpretation of $\mathscr{C}_0^{-k}(X)$ as a dual space of some suitable space of (test) functions. Here we get the following characterization:

Proposition 5.3.10 *Let* $X \subseteq \mathbb{R}^n$ *be a non-empty open subset.*

i.) *Let* $k \in \mathbb{N}_0$*. A compactly supported distribution* $u \in \mathscr{C}_0^{-k}(X)$ *of order at most* k *extends uniquely to a continuous linear functional*

$$u \colon \mathscr{C}^k(X) \longrightarrow \mathbb{K}. \tag{5.3.25}$$

ii.) *Let* $k \in \mathbb{N}_0 \cup \{\infty\}$*. The restriction*

$$u\Big|_{\mathscr{C}_0^\infty(X)} \colon \mathscr{C}_0^\infty(X) \longrightarrow \mathbb{K} \tag{5.3.26}$$

of a continuous linear functional $u \in \mathscr{C}^k(X)$ *to the test functions is a compactly supported distribution*

$$u\Big|_{\mathscr{C}_0^\infty(X)} \in \mathscr{C}_0^{-k}(X) \tag{5.3.27}$$

of order at most k*. If* $k = \infty$*, then* $u\big|_{\mathscr{C}_0^\infty(X)}$ *is actually of finite order.*

PROOF: For the first part note that $\mathscr{C}^k(X)$ is equipped with its usual Fréchet topology. From Proposition 2.3.25 an extension of u by continuity is necessarily unique if it exists at all since $\mathscr{C}_0^k(X) \subseteq \mathscr{C}^k(X)$ is dense. The continuity of u gives for all $x \in X$ an open neighbourhood $U \subseteq X$ of x with compact closure U^{cl} such that

$$|u(\varphi)| \leq \mathit{const}\, \mathrm{p}_{U^{\mathrm{cl}},k}(\varphi)$$

holds for all $\varphi \in \mathscr{C}^\infty_{U^{\mathrm{cl}}}(X)$. Here we use the fact that the global order of u is $k \in \mathbb{N}_0$. As a compact set, $\operatorname{supp}(u)$ can be covered by finitely many such neighbourhoods. Their union has still compact closure and hence we get a compact subset K with $\operatorname{supp}(u) \subseteq K^\circ$ and

$$|u(\varphi)| \leq \mathit{const}\, \mathrm{p}_{K,k}(\varphi) \tag{$*$}$$

for all $\varphi \in \mathscr{C}^\infty_K(X)$. Next, let $\chi \in \mathscr{C}^\infty_0(X)$ be a smooth function with compact support in K° such that there exists another open subset $U \subseteq X$ with

$$\operatorname{supp}(u) \subseteq U \subseteq U^{\mathrm{cl}} \subseteq K^\circ$$

and $\chi\big|_{U^{\mathrm{cl}}} = 1$. This implies $(1-\chi)u = 0$ by the locality of the module structure. Note that such a function always exists by the smooth Urysohn Lemma, see Corollary 1.3.8. It follows that $\chi u = u$ and hence

$$u(\varphi) = (\chi u)(\varphi) = u(\chi\varphi) \tag{$**$}$$

holds for all $\varphi \in \mathscr{C}^\infty_0(X)$, no matter where their support is located. Hence we get

$$|u(\varphi)| \leq \mathit{const}'\mathrm{p}_{K,k}(\varphi) \tag{☺}$$

according to $(*)$, as $\chi\varphi$ has support in K, and the continuity properties of the $\mathscr{C}^\infty(X)$-module structure, see again Proposition 5.2.1, *ii.)*. Now (☺) holds for all $\varphi \in \mathscr{C}^\infty_0(X)$. Indeed, this is the continuity statement in the Fréchet topology of $\mathscr{C}^k(X)$ we are looking for. By the density of $\mathscr{C}^\infty_0(X) \subseteq \mathscr{C}^k(X)$ as in Proposition 2.3.25 we get a unique extension of u by continuity. In fact, we even have an explicit construction for the case $k = \infty$. For $f \in \mathscr{C}^\infty(X)$ we have $\chi f \in \mathscr{C}^\infty_0(X)$ and hence we can apply u directly to χf. Therefore we define

$$\hat{u}\colon \mathscr{C}^\infty(X) \longrightarrow \mathbb{K}$$

by

$$\hat{u}(f) = u(\chi f).$$

From (☺) we see that

$$|\hat{u}(f)| = |u(\chi f)| \leq \mathit{const}'\mathrm{p}_{K,k}(\chi f) \leq \mathit{const}''\mathrm{p}_{K,k}(f)$$

by the continuity of the product in $\mathscr{C}^\infty(X)$. Hence $\hat{u}$ is continuous in the $\mathscr{C}^\infty$-topology and $(**)$ shows that $\hat{u}$ is an extension of u and hence the unique such extension. Note that we can first extend u to $\mathscr{C}^\infty(X)$ this way and extend it to $\mathscr{C}^k(X)$ by the continuity and density argument afterwards. For the second statement, let $u \in \mathscr{C}^k(X)'$ be a continuous linear functional. Then we have a compact $K \subseteq X$ and an $\ell \in \mathbb{N}_0$ with $\ell \leq k$ such that

$$|u(f)| \leq \mathit{const}\, \mathrm{p}_{K,\ell}(f) \tag{$\star$}$$

holds for all $f \in \mathscr{C}^k(X)$. In particular, $(\star)$ holds for all $\varphi \in \mathscr{C}^\infty_0(X)$ as well, showing that $u \in \mathscr{C}^{-\infty}(X)$. This is also clear by abstract arguments, as $\mathscr{C}^\infty_0(X) \subseteq \mathscr{C}^k(X)$ is continuously included. Nevertheless, we need the explicit estimate $(\star)$. Now let $\varphi \in \mathscr{C}^\infty_0(X)$ with $\operatorname{supp}(\varphi) \subseteq X \setminus K$ be given. Then $\mathrm{p}_{K,\ell}(\varphi) = 0$ and hence

$$|u(\varphi)| \leq \mathit{const}\, \mathrm{p}_{K,\ell}(\varphi) = 0 \tag{5.3.28}$$

shows $u(\varphi) = 0$. This gives $\operatorname{supp}(u) \subseteq K$ and thus $u \in \mathscr{C}^{-\infty}_0(X)$. From $(\star)$ we see that the order of u is at most $\ell \leq k$. □

Remark 5.3.11 Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.

i.) The abstract extension from the dense subspace $\mathscr{C}_0^\infty(X)$ to its completion $\mathscr{C}^k(X)$ in the $\mathscr{C}^k$-topology is obtained for $k = \infty$ by the explicit formula

$$\hat{u}(f) = u(\chi f), \tag{5.3.29}$$

where $\chi \in \mathscr{C}_0^\infty(X)$ is constant 1 on an open neighbourhood of $\operatorname{supp}(u)$. In particular, (5.3.29) does not depend on the choice of such a function χ. For $k \in \mathbb{N}_0$ we can use (5.3.29) to extend u to $\mathscr{C}^\infty(X)$. Afterwards, we still need to extend to $\mathscr{C}^k(X)$ by an approximation thanks to the density $\mathscr{C}^\infty(X) \subseteq \mathscr{C}^k(X)$.

ii.) Being isomorphic to a dual space, we can and will equip $\mathscr{C}_0^{-k}(X) \cong \mathscr{C}^k(X)'$ with various locally convex topologies like the weak* or strong topology. As usual, we indicate the chosen topology by a subscript like $\mathscr{C}_0^{-k}(X)_\sigma$ or $\mathscr{C}_0^{-k}(X)_\beta$.

iii.) As a consequence we see that the continuous linear functionals of $\mathscr{C}^\infty(X)$ automatically have compact support and finite order.

As for the finite order distributions, the additional topologies of $\mathscr{C}_0^{-k}(X)$ have some interesting properties. Here we list some of them:

Proposition 5.3.12 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and $k \in \mathbb{N}_0 \cup \{\infty\}$.*

i.) The compactly supported distributions $\mathscr{C}_0^{-k}(X)$ of order at most k are sequentially complete with respect to the weak* topology inherited from $\mathscr{C}_0^{-k}(X)_\sigma \cong \mathscr{C}^k(X)'_\sigma$.

ii.) The compactly supported distributions $\mathscr{C}_0^{-k}(X)$ of order at most k are complete with respect to the strong topology inherited from $\mathscr{C}_0^{-k}(X)_\beta \cong \mathscr{C}^k(X)'_\beta$.

iii.) One has the continuous inclusions

$$\mathscr{C}_0^{-0}(X)_\sigma \subseteq \mathscr{C}_0^{-1}(X)_\sigma \subseteq \cdots \subseteq \mathscr{C}_0^{-k}(X)_\sigma \subseteq \mathscr{C}_0^{-(k+1)}(X)_\sigma \subseteq \cdots \subseteq \mathscr{C}_0^{-\infty}(X)_\sigma \tag{5.3.30}$$

as well as

$$\mathscr{C}_0^{-0}(X)_\beta \subseteq \mathscr{C}_0^{-1}(X)_\beta \subseteq \cdots \subseteq \mathscr{C}_0^{-k}(X)_\beta \subseteq \mathscr{C}_0^{-(k+1)}(X)_\beta \subseteq \cdots \subseteq \mathscr{C}_0^{-\infty}(X)_\beta. \tag{5.3.31}$$

Proof: The first two statements are the usual properties of duals of Fréchet spaces with respect to the weak* and the strong topologies, see again Corollary 3.3.38 and Corollary 3.3.20. The continuity of the inclusions

$$\mathscr{C}^k(X) \subseteq \mathscr{C}^{k-1}(X)$$

for all $k \in \mathbb{N}$ as well as the continuity of $\mathscr{C}^\infty(X) \subseteq \mathscr{C}^k(X)$ dualizes to either (5.3.30) or (5.3.31). Finally, the map $I \colon \mathscr{C}_0^0(X) \longrightarrow \mathscr{C}^{-\infty}(X)$ gives the last missing inclusion, since $\operatorname{supp}(I_f) = \operatorname{supp}(f)$ for $f \in \mathscr{C}^0(X)$ shows that $\mathscr{C}_0^0(X)$ is mapped into $\mathscr{C}_0^{-0}(X)$. The continuity estimate for I is now similar to the continuity of $\mathrm{L}^1_{\mathrm{loc}}(X) \subseteq \mathscr{C}^{-\infty}(X)_\beta$, which we obtained in Corollary 5.1.7. For $f \in \mathscr{C}_K^0(X)$ with $K \subseteq X$ compact and $\varphi \in \mathscr{C}^0(X)$ we have

$$|I_f(\varphi)| \le \int_X |f(x)||\varphi(x)| \,\mathrm{d}^n x = \int_K |f(x)||\varphi(x)| \,\mathrm{d}^n x \le \operatorname{vol}(K) \mathrm{p}_{K,0}(f) \mathrm{p}_{K,0}(\varphi).$$

Hence for a bounded subset $B \subseteq \mathscr{C}^0(X)$ we get

$$\mathrm{p}_B(I_f) = \sup_{\varphi \in B} |I_f(\varphi)| \le \operatorname{vol}(K) \mathrm{p}_{K,0}(f) \sup_{\varphi \in B} \mathrm{p}_{K,0}(\varphi).$$

Since B is bounded the last supremum is finite, resulting in an estimate of the form $\mathrm{p}_B(I_f) \le$ *const* $\mathrm{p}_{K,0}(f)$. This is the continuity of $\mathscr{C}_K^0(X) \subseteq \mathscr{C}_0^{-0}(X)_\beta$ for every compact $K \subseteq X$. From the universal property of inductive limits, this gives the continuity $\mathscr{C}_0^0(X) \subseteq \mathscr{C}_0^{-0}(X)_\beta$ we are looking for. ☐

Remark 5.3.13 Having the topologies inherited form the dual $\mathscr{C}^k(X)'$, we can also investigate the continuity of differentiation

$$\partial^\alpha \colon \mathscr{C}_0^{-k}(X) \longrightarrow \mathscr{C}_0^{-(k+|\alpha|)}(X) \tag{5.3.32}$$

for multiindices $\alpha \in \mathbb{N}_0^n$, as well as the module structure, i.e. the multiplication by a function $f \in \mathscr{C}^\infty(X)$. It turns out that ∂^α as well as the module structure are continuous with respect to either the weak* or the strong topologies by the usual dualization argument. See Exercise 5.5.35 for these statements as well as slight generalization thereof.

The situation becomes now more interesting, when we discuss smooth maps. For general distributions we had a push-forward only for proper smooth maps. This changes for the compactly supported distributions: now we can use *any* smooth map to push forward:

Definition 5.3.14 (Push-forward of compactly supported distributions) *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be non-empty open subsets and let $\Phi \colon X \longrightarrow Y$ be a smooth map. Then the push-forward $\Phi_* u$ of a compactly supported distributions $u \in \mathscr{C}_0^{-\infty}(X)$ is defined to be the map*

$$\Phi_* u \colon \mathscr{C}^\infty(Y) \longrightarrow \mathbb{K} \tag{5.3.33}$$

with

$$(\Phi_* u)(\varphi) = u(\Phi^* u) \tag{5.3.34}$$

for $\varphi \in \mathscr{C}^\infty(Y)$.

The crucial point is that the pull-back of a smooth function is always a smooth function. This leads immediately to the following properties:

Proposition 5.3.15 *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be non-empty open subsets and let $\Phi \colon X \longrightarrow Y$ be a smooth map.*

i.) For all $u \in \mathscr{C}_0^{-\infty}(X)$ one has $\Phi_* u \in \mathscr{C}_0^{-\infty}(Y)$.

ii.) The push-forward of $u \in \mathscr{C}_0^{-\infty}(X)$ coincides with the push-forward of u in the sense of distributions, as soon as Φ is proper.

iii.) The push-forward is a continuous linear map with respect to the weak topology*

$$\Phi_* \colon \mathscr{C}_0^{-\infty}(X)_\sigma \longrightarrow \mathscr{C}_0^{-\infty}(Y)_\sigma \tag{5.3.35}$$

as well as for the strong topology

$$\Phi_* \colon \mathscr{C}_0^{-\infty}(X)_\beta \longrightarrow \mathscr{C}_0^{-\infty}(Y)_\beta. \tag{5.3.36}$$

iv.) If $\Psi \colon Y \longrightarrow Z \subseteq \mathbb{R}^k$ is yet another smooth map, then we have

$$(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_* \tag{5.3.37}$$

and

$$(\mathrm{id}_X)_* = \mathrm{id}_{\mathscr{C}_0^{-\infty}(X)}. \tag{5.3.38}$$

v.) For every $u \in \mathscr{C}_0^{-\infty}(X)$ one has

$$\operatorname{supp}(\Phi_* u) \subseteq \Phi(\operatorname{supp}(u)). \tag{5.3.39}$$

vi.) For every $u \in \mathscr{C}_0^{-k}(X)$ one has

$$\operatorname{ord}(\Phi_* u) \leq k. \tag{5.3.40}$$

Proof: The main point is that the pull-back of smooth functions is a continuous linear map

$$\Phi^*: \mathscr{C}^\infty(Y) \longrightarrow \mathscr{C}^\infty(X) \tag{$*$}$$

according to Proposition 2.3.24. Then $\Phi_* u = u \circ \Phi^*$ is again continuous, showing *i.)*. Parts *ii.)* and *iii.)* are the usual statements about the dualization of continuous linear maps from Corollary 3.2.22. The fourth part is clear from the corresponding properties of the pull-back of smooth functions. Next, let $\varphi \in \mathscr{C}^\infty(Y)$ with $\operatorname{supp}(\varphi) \subseteq Y \setminus \Phi(\operatorname{supp}(u))$ be given. Then $\Phi^*\varphi \in \mathscr{C}^\infty(X)$ has support

$$\operatorname{supp}(\Phi^*\varphi) \subseteq X \setminus \Phi^{-1}(\Phi(\operatorname{supp}(u))) \subseteq X \setminus \operatorname{supp}(u),$$

showing $(\Phi_* u)(\varphi) = u(\Phi^*\varphi) = 0$. Since this holds for all such functions φ, the fifth part is shown. Finally, any distribution $u \in \mathscr{C}_0^{-\infty}(X)$ with compact support has finite order, say $k \in \mathbb{N}_0$ by Proposition 5.3.9, *iv.)*. Hence it is a continuous linear functional $u: \mathscr{C}^k(X) \longrightarrow \mathbb{K}$ by Proposition 5.3.10, *i.)*. Since the pull-back $(*)$ is $\mathscr{C}^k$-continuous as well, we get a continuous linear map $\Phi_* u = u \circ \Phi^*: \mathscr{C}^k(Y) \longrightarrow \mathbb{K}$ which corresponds uniquely to $\Phi_* u \in \mathscr{C}_0^{-\infty}(Y)$, again by Proposition 5.3.10, *ii.)*. Hence $\Phi_* u \in \mathscr{C}_0^{-k}(Y)$. ☐

Example 5.3.16 Consider $X = \{0\}$ as a zero dimensional vector space $\mathbb{R}^0$. Then any map $\varphi: X \longrightarrow \mathbb{K}$ is smooth and has compact support: the conditions are trivially fulfilled. We have

$$\mathscr{C}_0^\infty(\{0\}, \mathbb{K}) \cong \mathbb{K} \tag{5.3.41}$$

by identifying a function with its value at 0. The evaluation $\mathrm{ev}: \varphi \mapsto \varphi(0)$ can be seen as the regular distribution 1, i.e. as Lebesgue integral in zero dimensions. Now any map $\Phi: X \longrightarrow Y \subseteq \mathbb{R}^m$ into an open subset is smooth for trivial reasons. It is also proper. Hence we can push-forward ev to a distribution on Y either using Proposition 5.2.22 or Proposition 5.3.15, since $\operatorname{supp}(\mathrm{ev})$ is compact. Either way we get

$$\Phi_* \mathrm{ev} = \delta_{\Phi(0)}, \tag{5.3.42}$$

turning the most regular distribution ev into the δ-distribution at the image point $\Phi(0) \in Y$.

While this example gives yet another interpretation of the δ-distribution, it is certainly not that interesting. However, the push-forward of compactly supported distributions is a very valuable extension of the push-forward of general distributions by proper smooth maps. Note that we have to take care of compact supports in one way or the other.

5.3.3 Distributions with Discrete or Finite Support

The next class of distributions we are interested in are those, where the support is a discrete subset of X or even a finite subset:

Remark 5.3.17 (Discrete subsets) Let $X \subseteq \mathbb{R}^n$ be open and let $D = D^{\mathrm{cl}} \subseteq X$ be a closed discrete subset. Here *discrete* means that for each $x \in D$ we have an open neighbourhood $U_x \subseteq X$ of x with $D \cap U_x = \{x\}$, i.e. the induced topology is the discrete one. Now let $K \subseteq X$ be compact. We claim that $D \cap K$ is finite. Indeed, assume that $D \cap K$ is infinite, then we would get an accumulation point $x_\infty \in D^{\mathrm{cl}} = D$ as well, which gives immediately a contradiction to $D \cap U_{x_\infty} = \{x_\infty\}$. Since X can be exhausted by countably many compact subsets, we conclude that D is at most countably infinite with no accumulation points in X. Conversely, any sequence $(x_k)_{k\in\mathbb{N}}$ in X without accumulation points (in X) is indeed a closed and discrete subset. Hence D is either finite or a sequence $(x_k)_{k\in\mathbb{N}}$ without accumulation points.

We will use this characterization of discrete subsets now as follows. The idea is that a distribution u with discrete support can be written as (convergent) series of distributions with supports being single points:

Proposition 5.3.18 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $u \in \mathscr{C}^{-\infty}(X)$ have discrete support* $\operatorname{supp}(u) = \{x_k \mid k \in I \subseteq \mathbb{N}\}$*, where I is either finite or $I = \mathbb{N}$.*

i.) There exist $r_k > 0$ such that

$$\operatorname{supp}(u) \cap \mathrm{B}_{r_k}(x_k) = \{x_k\} \tag{5.3.43}$$

for all $k \in I$.

ii.) The restrictions $u\big|_{\mathrm{B}_{r_k}(x_k)} \in \mathscr{C}_0^{-\infty}(\mathrm{B}_{r_k}(x_k))$ are distributions with compact support

$$\operatorname{supp}\left(u\big|_{\mathrm{B}_{r_k}(x_k)}\right) = \{x_k\} \tag{5.3.44}$$

for all $k \in I$.

iii.) The push-forwards

$$u_k = \left(\iota_{\mathrm{B}_{r_k}(x_k)}\right)_*\left(u\big|_{\mathrm{B}_{r_k}(x_k)}\right) \tag{5.3.45}$$

are distributions $u_k \in \mathscr{C}_0^{-\infty}(X)$ with support

$$\operatorname{supp}(u_k) = \{x_k\} \tag{5.3.46}$$

for all $k \in I$.

iv.) One has

$$u = \sum_{k \in I} u_k \tag{5.3.47}$$

as absolutely convergent series in the strong topology of $\mathscr{C}^{-\infty}(X)$.

Proof: The first statement is just the fact that $\operatorname{supp}(u)$ is discrete and, of course, closed as any support is. First we note that the restrictions $u\big|_{\mathrm{B}_{r_k}(x_k)}$ are well-defined distributions, since $\mathrm{B}_{r_k}(x_k) \subseteq X$ is open. For a test function $\varphi \in \mathscr{C}_0^\infty(\mathrm{B}_{r_k}(x_k))$ with $\operatorname{supp}(\varphi) \cap \{x_k\} = \emptyset$ we have $\operatorname{supp}(\varphi) \cap \operatorname{supp}(u) = \emptyset$, when viewing φ as a test function on X, since the other points of $\operatorname{supp}(u)$ are outside of the open ball $\mathrm{B}_{r_k}(x_k)$ by (5.3.43). In particular, the restriction has now compact support, a single point. Hence we can indeed push forward $u\big|_{\mathrm{B}_{r_k}(x_k)}$ to a distribution u_k on X with support contained in $\{x_k\}$. Explicitly, for $\varphi \in \mathscr{C}^\infty(X)$ one has

$$u_k = u\Big|_{\mathrm{B}_{r_k}(x_k)}\left(\varphi\big|_{\mathrm{B}_{r_k}(x_k)}\right),$$

which is well-defined as the restriction has compact support. We need a more explicit description of u_k to handle estimates as well. Thus let $\chi_k \in \mathscr{C}_0^\infty(X)$ satisfy $\operatorname{supp}(\chi_k) \subseteq \mathrm{B}_{r_k}(x_k)$ and

$$\chi_k\Big|_{\mathrm{B}_{r_k/2}(x_k)} = 1.$$

Moreover, we set

$$\chi = \sum_{k \in I} \chi_k,$$

which is a smooth function on X, as the sum is locally finite. Note that χ is constant one on an open neighbourhood of $\operatorname{supp}(u)$ and hence

$$\operatorname{supp}(1 - \chi) \cap \operatorname{supp}(u) = \emptyset$$

shows $\chi u = u$. Now the same argument applies to each u_k and hence

$$\chi_k u_k = u_k$$

follows for all $k \in I$. This gives for $\varphi \in \mathscr{C}_0^\infty(X)$

$$u_k(\varphi) = u_k(\chi_k \varphi) = \big(u\big|_{\mathrm{B}_{r_k}(x_k)}\big)(\chi_k \varphi) = u(\chi_k \varphi),$$

since $\operatorname{supp}(\chi_k \varphi) \subseteq \mathrm{B}_{r_k}(x_k)$ and hence the restriction of this (test) function to the ball $\mathrm{B}_{r_k}(x_k)$ has compact support, allowing us to apply u directly. From this and $x_k \in \operatorname{supp}(u)$ we see that u_k is not zero and hence (5.3.46) follows. Now let $B \subseteq \mathscr{C}_0^\infty(X)$ be a bounded subset. Then we have a compact subset $K \subseteq X$ with $B \subseteq \mathscr{C}_K^\infty(X)$ being bounded in the Fréchet topology of $\mathscr{C}_K^\infty(X)$, see again Theorem 5.1.1, *vi.)*. From the first part we infer that at most finitely many $k_1, \ldots, k_N \in I$ lead to points $x_{k_1}, \ldots, x_{k_N} \in K$. For all other indices $\ell \in I \setminus \{k_1, \ldots, k_N\}$ we get

$$u_\ell(\varphi) = 0$$

by $\operatorname{supp}(u_\ell) \cap \operatorname{supp}(\varphi) = \emptyset$ for all $\varphi \in \mathscr{C}_K^\infty(X)$ and hence for all $\varphi \in B$. Hence either

$$\mathrm{p}_B(u_\ell) = \sup_{\varphi \in B} |u_\ell(\varphi)| = 0$$

or for $k \in \{k_1, \ldots, k_N\}$

$$\mathrm{p}_B(u_k) = \sup_{\varphi \in B} |u_k(\varphi)| = c_k$$

with some constant $c_k > 0$. Hence for the absolute convergence of the series (5.3.47) we have

$$\sum_{k \in I} \mathrm{p}_B(u_k) = \sum_{k \in \{k_1, \ldots, k_N\}} \mathrm{p}_B(u_k) = c_{k_1} + \cdots + c_{k_N} < \infty.$$

It remains to check the actual identity. This follows from

$$\sum_{k \in I} u_k(\varphi) = \sum_{k \in I} u(\chi_k \varphi) = u\Big(\sum_{k \in I} \chi \varphi\Big) = u(\varphi)$$

for all $\varphi \in \mathscr{C}_0^\infty(X)$. Note that the sum is always finite. □

The importance of this proposition is not so much the somewhat trivial convergence statement, but the fact that we can focus our attention to distributions with a single point as support. Distributions with finite or discrete support can be assembled from their restrictions as linear combinations of distributions with single points as supports. For the latter we have the following explicit description:

Proposition 5.3.19 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $x_0 \in X$. If $u \in \mathscr{C}^{-\infty}(X)$ has support*

$$\operatorname{supp}(u) = \{x_0\}, \tag{5.3.48}$$

then there are constants $c_\alpha \in \mathbb{K}$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ such that

$$u = \sum_{|\alpha| \leq k} c_\alpha \partial^\alpha \delta_{x_0}. \tag{5.3.49}$$

Proof: Without restriction we can assume $0 \in X$ and $x_0 = 0$, since translations of δ-distributions are δ-distributions at the translated points, see also Exercise 5.5.29. Since u has compact support, we have a finite global order $k \in \mathbb{N}_0$. Consider now a test function $\varphi \in \mathscr{C}_0^\infty(X)$ with

$$\frac{\partial^{|\alpha|}\varphi}{\partial x^\alpha}(0) = 0 \tag{$*$}$$

for all α with $|\alpha| \leq k$. We claim that $u(\varphi) = 0$, which will be the main step of the proof. Let $\epsilon > 0$ be given. Then $(*)$ and the continuity of the derivatives of φ show that there is a radius $0 < R < 1$ such that $\mathrm{B}_R(0) \subseteq X$ and

$$|(\partial^\alpha \varphi)(x)| \leq \epsilon \quad \text{for } x \in \mathrm{B}_R(0)^{\mathrm{cl}}. \tag{$**$}$$

We claim that then we have

$$|(\partial^\alpha \varphi)(x)| \leq \epsilon n^{k-|\alpha|}|x|^{k-|\alpha|} \tag{$\star$}$$

for all $x \in \mathrm{B}_R(0)^{\mathrm{cl}}$. Indeed, if $k - |\alpha| = 0$ this reduces to $(**)$, serving as starting point for an induction. Thus assume $(\star)$ holds for α with $k - |\alpha| = \ell$. With the mean value theorem we find a ξ on the connecting line from 0 to x such that for $k - |\alpha| = \ell + 1$

$$(\partial^\alpha\varphi)(x) = (\partial^\alpha\varphi)(x) - (\partial^\alpha\varphi))(0) = \langle(\nabla\partial^\alpha\varphi)(\xi), x\rangle,$$

where ∇ denotes the gradient as usual. Hence

$$\begin{aligned}
|(\partial^\alpha\varphi)(x)| &= |\langle(\nabla\partial^\alpha\varphi)(\xi), x\rangle| \\
&= \left|\sum_{i=1}^n \left(\frac{\partial}{\partial x^i}\partial^\alpha\varphi\right)(\xi)x^i\right| \\
&\overset{(a)}{\leq} \epsilon n n^{k-|\alpha|-1}|x|^{k-|\alpha|-1}|x| \\
&= \epsilon n^{k-|\alpha|}|x|^{k-|\alpha|},
\end{aligned}$$

where in (a) we used $(\star)$ for the case $k - |\alpha| = \ell$, i.e. our induction hypothesis. This proves $(\star)$ by induction. Next, we choose a bump function $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with $\chi\big|_{\mathrm{B}_{1/2}(0)^{\mathrm{cl}}} = 1$ and $\operatorname{supp}(\chi) \subseteq \mathrm{B}_1(0)$. Rescaling with R from above gives a bump function $\chi_R(x) = \chi(\frac{x}{R})$ with $\operatorname{supp}(\chi_R) \subseteq \mathrm{B}_R(0)$ and $\chi_R\big|_{\mathrm{B}_{R/2}(0)^{\mathrm{cl}}} = 1$. We estimate now the function $\chi_R\varphi$ with respect to the seminorm $\mathrm{p}_{\mathrm{B}_R(0)^{\mathrm{cl}},k}$. Here we have with the Leibniz rule

$$\begin{aligned}
\mathrm{p}_{\mathrm{B}_R(0)^{\mathrm{cl}},k}(\chi_R\varphi) &\leq \sup_{\substack{x\in\mathrm{B}_R(0)^{\mathrm{cl}} \\ |\alpha|\leq k}} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \left|(\partial^{\alpha-\beta}\chi_R)(x)\right|\left|\partial^\beta\varphi(x)\right| \\
&\overset{(\star)}{\leq} \sup_{\substack{x\in\mathrm{B}_R(0)^{\mathrm{cl}} \\ |\alpha|\leq k}} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \frac{1}{R^{|\alpha|-|\beta|}}\left|(\partial^{\alpha-\beta}\chi)\left(\frac{x}{R}\right)\right| \cdot \epsilon n^{k-|\beta|}|x|^{k-|\alpha|} \\
&\leq \epsilon c\,\mathrm{p}_{\mathrm{B}_1(0)^{\mathrm{cl}}}(\chi),
\end{aligned}$$

with a constant $c > 0$ incorporating the binomial coefficients and the remaining constants, since $|x| < R < 1$. Note that c is independent of R, this is the important aspect. Since χ_R is constant to one in an open neighbourhood of $\operatorname{supp}(u)$, we have $u = \chi_R u$, as usual. Finally, the continuity of u and $\operatorname{ord}(u) = k$ give a general estimate

$$|u(\psi)| \leq c'\mathrm{p}_{\mathrm{B}_R(0)^{\mathrm{cl}},k}(\psi)$$

for all $\psi \in \mathscr{C}_0^\infty(X)$ with $\operatorname{supp}(\psi) \subseteq \mathrm{B}_R(0)^{\mathrm{cl}}$. Together, we have for φ as above

$$|u(\varphi)| = |(\chi_R u)(\varphi)| \leq c'\mathrm{p}_{\mathrm{B}_R(0)^{\mathrm{cl}},k}(\chi_R\varphi) \leq \epsilon c c'\mathrm{p}_{\mathrm{B}_R(0)^{\mathrm{cl}},k}(\chi),$$

now with constants depending on R, but still for all $\epsilon > 0$. This shows $u(\varphi) = 0$, as we have claimed. In other words, we get

$$\ker(u) \supseteq \bigcap_{|\alpha| \leq k} \ker(\partial^\alpha \delta).$$

From here we can use a simple lemma from linear algebra that in this situation

$$u \in \operatorname{span}_{\mathbb{K}} \left\{ \partial^\alpha \;\middle|\; |\alpha| \leq k \right\}$$

follows, see Exercise 3.6.11. □

Thus this observation determines the structure of distributions with discrete support completely:

Corollary 5.3.20 *Let $X \subseteq \mathbb{R}^n$ be an non-empty open subset and let $u \in \mathscr{C}^{-\infty}(X)$ be a distribution with discrete support. Then there exist unique $k_x \in \mathbb{N}_0$ and $c_{x,\alpha} \in \mathbb{K}$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k_x$ for all $x \in \operatorname{supp}(u)$ such that*

$$u = \sum_{x \in \operatorname{supp}(u)} \sum_{|\alpha| \leq k_x} c_{x,\alpha} \partial^\alpha \delta_x. \tag{5.3.50}$$

The series converges absolutely in the strong topology if $\operatorname{supp}(u)$ *is infinite.*

Proof: The remaining uniqueness of the coefficients $c_{x,\alpha}$ is clear, as the derivatives of δ are all linearly independent, see also Exercise 5.5.36. □

5.3.4 Positive Functionals

The last type of distributions we consider has its motivation in the theory of *-algebras: we are interested in positive functionals. Here one has several possibilities to define positivity. Naively, a linear functional is positive if it maps positive functions to positive numbers. In this context, it is good custom to use "positive" in a non-strict sense, i.e. x is positive if $x \geq 0$. In particular, the zero functional will be a positive functional.

The slight difficulty arises now in the question, which functions should be considered positive. While a naive "f is positive if for all $x \in X$ one has $f(x) \geq 0$" is tempting, a more algebraic notion turns out to be advantageous. In view of more general *-algebras one calls f *algebraically positive* if it is a *sum of squares*, i.e.

$$f = \sum_{i=1}^{N} \overline{g}_i g_i \tag{5.3.51}$$

for some other functions in the same function algebra under consideration. Clearly, (5.3.51) implies $f(x) \geq 0$ for all $x \in X$ in the domain of definition, but the converse is not necessarily true. In fact, it is one of the famous Hilbert problems to answer this question for polynomials in several variables. Here it is known that for one variable one has the equivalence, but for several it fails. We base our definition of positive linear functionals now on (5.3.51) in accordance with the general framework of *-algebras, see also [17] for an extensive discussion of *-algebras and their representation theory.

Definition 5.3.21 (Positive functional) *Let $\mathscr{A}$ be a *-algebra over $\mathbb{C}$, i.e. an associative algebra over $\mathbb{C}$ with a *-involution. Then a linear functional*

$$\omega\colon \mathscr{A} \longrightarrow \mathbb{C} \tag{5.3.52}$$

is called positive if for all $a \in \mathscr{A}$ one has

$$\omega(a^* a) \geq 0. \tag{5.3.53}$$

We are mainly interested in the case, where $X \subseteq \mathbb{R}^n$ is a non-empty open subset and $\mathscr{A} = \mathscr{C}_0^\infty(X)$. However, the above definition is entirely algebraic and makes sense for any *-algebra $\mathscr{A}$, whether it is an algebra of functions or not. In particular, many non-commutative *-algebras appear naturally at various places in mathematics.

As a first result, we note that positive functionals satisfy a fundamental inequality, the *Cauchy-Schwarz inequality*:

Proposition 5.3.22 *Let $\mathscr{A}$ be a *-algebra over $\mathbb{C}$ and $\omega\colon \mathscr{A} \longrightarrow \mathbb{C}$ be a positive linear functional. Then for all $a, b \in \mathscr{A}$ one has*

$$\omega(a^*b) = \overline{\omega(b^*a)} \tag{5.3.54}$$

as well as

$$|\omega(a^*b)| \leq \omega(a^*a)\omega(b^*b). \tag{5.3.55}$$

Proof: Consider $\lambda, \mu \in \mathbb{C}$ and

$$p(\lambda, \mu) = \omega((\lambda a + \mu b)^*(\lambda a + \mu b)).$$

This is a quadratic polynomial, which satisfies $p(\lambda, \mu) \geq 0$ for all $\lambda, \mu \in \mathbb{C}$. Using the linearity of ω we get the explicit form of p. Inserting now particular values for λ and μ gives the two statements (5.3.54) and (5.3.55), see Exercise 5.5.37. $\square$

It is this simple inequality, which is responsible for numerous consequences about positive functionals in the theory of *-algebras. We will only need the very first and simplest applications to conclude nice continuity properties of positive functionals.

If the *-algebra $\mathscr{A}$ is *unital*, then (5.3.54) implies

$$\omega(a^*) = \overline{\omega(a)} \tag{5.3.56}$$

for all $a \in \mathscr{A}$. In addition, setting $b = \mathbb{1}$ in (5.3.55), one has

$$|\omega(a)|^2 \leq \omega(a^*a)\omega(\mathbb{1}). \tag{5.3.57}$$

In the case of smooth functions, we have to following important statement:

Proposition 5.3.23 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.*

i.) If $\omega\colon \mathscr{C}_0^\infty(X, \mathbb{C}) \longrightarrow \mathbb{C}$ is a positive linear functional, then ω is a distribution of order zero, i.e. $\omega \in \mathscr{C}^{-0}(X, \mathbb{C})$.

ii.) If $\omega\colon \mathscr{C}^\infty(X, \mathbb{C}) \longrightarrow \mathbb{C}$ is a positive linear functional, then ω is a distribution of order zero with compact support, i.e. $\omega \in \mathscr{C}_0^{-0}(X, \mathbb{C})$.

Proof: In both cases we need to show the continuity of ω. Note that we do not assume any sort of continuity, it follows solely from positivity. For the first case, let $K \subseteq X$ be compact and choose a cut-off function $\chi \in \mathscr{C}_0^\infty(X)$ with $\chi\big|_K = 1$. Then the functional

$$\omega_\chi\colon \mathscr{C}^\infty(X, \mathbb{C}) \ni f \;\mapsto\; \omega_\chi(f) = \omega(\overline{\chi} f \chi) \in \mathbb{C}$$

is a well-defined linear functional on all smooth functions, since $\bar{\chi} f \chi \in \mathscr{C}_0^\infty(X)$ indeed has compact support. Moreover, $\omega_\chi(\overline{f} f) = \omega(\overline{\chi f} f \chi) \geq 0$, since $f\chi \in \mathscr{C}_0^\infty(X, \mathbb{C})$ and ω is positive. Next, consider a function $f \in \mathscr{C}^\infty(X, \mathbb{C})$ with $f(x) \geq 0$ for all $x \in X$. Given $\epsilon > 0$, we have a strictly positive function $f + \epsilon 1 > 0$, which has a smooth square root $\sqrt{f + \epsilon 1} \in \mathscr{C}^\infty(X, \mathbb{C})$. If now $\Omega\colon \mathscr{C}^\infty(X, \mathbb{C}) \longrightarrow \mathbb{C}$ is a positive linear functional, then

$$0 \leq \Omega(\overline{\sqrt{f + \epsilon 1}} \sqrt{f + \epsilon 1}) = \Omega(f + \epsilon 1) = \Omega(f) + \epsilon \Omega(1)$$

holds for all $\epsilon > 0$. Hence $\Omega(f) \geq 0$ follows. We apply this now to ω_χ: for $\varphi \in \mathscr{C}_0^\infty(X, \mathbb{C})$ the functions $\|\varphi\|_\infty 1 \pm \operatorname{Re}(\varphi)$ and $\|\varphi\|_\infty 1 \pm \operatorname{Im}(\varphi)$ are smooth and non-negative. Hence we conclude

$$0 \leq \omega_\chi(\|\varphi\|_\infty 1 \pm \operatorname{Re}(\varphi)) = \|\varphi\|_\infty \omega(\overline{\chi}\chi) \pm \omega_\chi(\operatorname{Re}(\varphi))$$

and

$$0 \leq \omega_\chi(\|\varphi\|_\infty 1 \pm \operatorname{Im}(\varphi)) = \|\varphi\|_\infty \omega(\overline{\chi}\chi) \pm \omega_\chi(\operatorname{Im}(\varphi)).$$

If now $\varphi \in \mathscr{C}_0^\infty(X, \mathbb{C})$ has support in K, then $\chi\varphi = \varphi$ and hence $\chi \operatorname{Re}(\varphi) = \operatorname{Re}(\varphi)$ as well as $\chi \operatorname{Im}(\varphi) = \operatorname{Im}(\varphi)$. This gives the estimates

$$\pm\omega(\operatorname{Re}(\varphi)) \leq \omega(\overline{\chi}\chi)\|\varphi\|_\infty$$

and

$$\pm\omega(\operatorname{Im}(\varphi)) \leq \omega(\overline{\chi}\chi)\|\varphi\|_\infty.$$

Together, we get

$$|\omega(\varphi)| \leq 2\omega(\overline{\chi}\chi)\|\varphi\|_\infty$$

for $\varphi \in \mathscr{C}_K^\infty(X, \mathbb{C})$. This is the $\mathscr{C}_0^0$-continuity of ω, proving the first part. For the second statement, let $\omega\colon \mathscr{C}^\infty(X) \longrightarrow \mathbb{C}$ be positive. We choose an exhausting sequence

$$K_1^\circ \subseteq K_1 \subseteq K_2^\circ \subseteq \cdots \subseteq K_k^\circ \subseteq K_k \subseteq K_{k+1}^\circ \subseteq \cdots \subseteq X$$

of compact subsets $K_k \subseteq X$, i.e. $\bigcup_{k\in\mathbb{N}} K_k = X$. We claim that ω has compact support, i.e. there exists a $k_0 \in \mathbb{N}$ with

$$\omega(f) = 0$$

for all $f \in \mathscr{C}^\infty(X)$, such that $\operatorname{supp}(f) \cap K_{k_0} = \emptyset$. Assume this is not the case. Then we find $f_k \in \mathscr{C}^\infty(X)$ with $\operatorname{supp}(f_k) \subseteq X \setminus K_k$ for all $k \in \mathbb{N}$ and $\omega(f_k) \neq 0$. This implies $\omega(f_k^2) > 0$ according to the Cauchy-Schwarz inequality (5.3.57). Since ω is linear, we can even normalize the functions such that we end up with $f_k \in \mathscr{C}^\infty(X)$ satisfying

$$\operatorname{supp}(f_k) \subseteq X \setminus K_k \quad \text{and} \quad \omega(f_k) = 1.$$

Since we have an exhausting sequence of compact subsets, the supports of the functions f_k are locally finite. Hence for all $N \in \mathbb{N}$

$$F_N = 1 + \sum_{k=N}^\infty f_k \in \mathscr{C}^\infty(X)$$

is a well-defined smooth function with $F_N(x) > 0$ for all $x \in X$. As noted before, $\omega(F_N) \geq 0$ for all $N \in \mathbb{N}$. We have by linearity of ω for all $N \in \mathbb{N}$

$$\omega(F_1) = \omega(F_{N+1}) + \sum_{k=1}^N \omega(f_k) \geq N,$$

which is absurd. Hence ω has compact support in some $K_{k_0}^\circ$. This allows us to choose a $\chi \in \mathscr{C}_0^\infty(X)$ with $\chi\big|_{K_{k_0}} = 1$. Then for all $f \in \mathscr{C}^\infty(X)$ we have $f = \overline{\chi} f \chi + (1 - \overline{\chi}\chi) f$, where now $\operatorname{supp}((1 - \bar{\chi}\chi)f) \cap K_{k_0} = \emptyset$. Thus we get

$$\omega(f) = \omega(\overline{\chi} f \chi). \tag{$*$}$$

Since $\mathscr{C}_0^\infty(X) \subseteq \mathscr{C}^\infty(X)$ we know from the first part that $\omega\big|_{\mathscr{C}_0^\infty(X)}$ is a distribution of order zero. With $(*)$ we see that ω can be extended to $\mathscr{C}^\infty(X)$ by means of the multiplication with the cut-off function $\overline{\chi}\chi$. This shows that ω is the usual extension of $\omega\big|_{\mathscr{C}_0^\infty(X)}$ to $\mathscr{C}^\infty(X)$ for a compactly supported distribution: Clearly, $(*)$ is still $\mathscr{C}^\infty$-continuous. As we had order zero already before, $(*)$ shows that we stay with order zero, i.e. $\omega \in \mathscr{C}_0^{-0}(X)$ as claimed. ☐

It is now a classical theorem, which we will not prove here, that the positive functionals on the continuous functions are given by positive measures. More precisely, we have the Riesz Representation Theorem:

Theorem 5.3.24 (Riesz Representation Theorem) *Let X be a locally compact Hausdorff space. Let*

$$\omega\colon \mathscr{C}_0^0(X, \mathbb{C}) \longrightarrow \mathbb{C} \tag{5.3.58}$$

be a positive linear functional. Then there exists a σ-algebra $\mathfrak{a}_\omega$ containing the Borel σ-algebra $\mathfrak{a}_{Borel}(X)$ together with a unique positive complete measure μ_ω on $\mathfrak{a}_\omega$ such that

$$\omega(\varphi) = \int_X \varphi \,\mathrm{d}\mu_\omega \tag{5.3.59}$$

for all $\varphi \in \mathscr{C}_0^0(X, \mathbb{C})$. The measure μ is finite on compact subsets. If in addition, X is σ-compact, then μ_ω is regular.

The proof of this fundamental theorem can be found in e.g. [14, Theorem 2.14]. Note that in our situation $X \subseteq \mathbb{R}^n$ meets the requirements and is σ-compact: we get a regular measure μ_ω for free.

Corollary 5.3.25 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset and let $\omega\colon \mathscr{C}_0^\infty(X, \mathbb{C}) \longrightarrow \mathbb{C}$ be a positive linear functional. Then*

$$\omega(\varphi) = \int_X \varphi \,\mathrm{d}\mu_\omega \tag{5.3.60}$$

for a unique positive regular complete Borel measure μ_ω on X.

PROOF: Indeed, by Proposition 5.3.23, *i.)*, the functional ω is a distribution of order zero and hence extends to $\mathscr{C}_0^0(X, \mathbb{C})$ by continuity according to Proposition 5.3.10, *i.)*. The positivity $\omega(\overline{\varphi}\varphi) \geq 0$ also holds for the extension by continuity. Thus Theorem 5.3.24 applies. □

In the case of a positive functional on $\mathscr{C}^\infty(X, \mathbb{C})$ we have a compactly supported measure instead.

Example 5.3.26 The δ-functional at $x_0 \in X$ is clearly positive, since

$$\delta_{x_0}(\overline{\varphi}\varphi) = |\varphi(x_0)|^2. \tag{5.3.61}$$

The corresponding measure is the point measure concentrated at x_0. Sets containing x_0 have measure 1, all others have measure zero. Here we can take the maximal σ-algebra 2^X.

The positivity of the δ-functionals has now a nice consequence. Even if not all non-negative functions can be written as sums of squares, we can characterize them by positive functionals:

Corollary 5.3.27 *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. Then $\varphi \in \mathscr{C}_0^\infty(X, \mathbb{C})$ satisfies*

$$\varphi(x) \geq 0 \tag{5.3.62}$$

for all $x \in X$ iff for all positive linear functionals $\omega\colon \mathscr{C}_0^\infty(X, \mathbb{C})$ we have

$$\omega(f) \geq 0. \tag{5.3.63}$$

The same holds for $\mathscr{C}^\infty(X, \mathbb{C})$ instead of $\mathscr{C}_0^\infty(X, \mathbb{C})$.

PROOF: In view of the positivity of the δ-functional, (5.3.63) clearly implies (5.3.62). That (5.3.62) implies (5.3.63) was shown in the proof of Proposition 5.3.23 or follows from the positivity properties of integrations with respect to positive measures and the Riesz Representation Theorem in form of Corollary 5.3.25. □

Remark 5.3.28 (States) There is a physical interpretation of positive functionals one should mention here. The function algebra $\mathscr{C}^\infty(X, \mathbb{C})$ or any interesting subalgebra thereof can be viewed as observable algebra of a classical mechanical system, once X has the interpretation of a phase space. Then the positive functionals are identified with the *states* of the physical system, the evaluations

$$\mathrm{E}_\omega(f) = \omega(f) \tag{5.3.64}$$

are the (classical) expectation values of the observable f in the state ω. Here one imposes a normalization $\omega(1) = 1$ on ω to have a sound interpretation as expectation functional. The δ-functionals are then the *pure states* corresponding to specific points in the phase space. But also *mixed states*, i.e. the non-trivial measures on X, are of interest.

5.4 Convolution

In this section we consider various convolution integrals. The main idea is that the translation group $(\mathbb{R}^n, +)$ acts in a sufficiently nice way on many function spaces on $\mathbb{R}^n$ by pull-backs with the usual translations. This allows us to average the translated function by means of another function as weight leading to convolution integrals of the form

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x - y) \, \mathrm{d}^n y. \tag{5.4.1}$$

Of course, we need to specify f and g to make sense out of such integrals. It will turn out that a convolution product $f * g$ typically inherits the better properties of both functions. This observation makes convolutions the ideal tool to find approximations of rough functions by more smooth ones. In fact, we will see that one can even define a convolution with distributions leading ultimately to a way of approximating very singular distributions by very regular test functions.

Since convolutions can easily be differentiated, one important application using convolution products are fundamental solutions to linear partial differential equations with constant coefficients. They can be employed to construct solutions to the general inhomogeneous equation, once a fundamental solution is known.

5.4.1 The Action of Translations

The key to understand convolution also from a more conceptual point of view is to analyze the representation of $(\mathbb{R}^n, +)$ on the function spaces of interest. As before, we denote the translations by

$$\tau_a \colon \mathbb{R}^n \ni x \;\mapsto\; x + a \in \mathbb{R}^n, \tag{5.4.2}$$

where $a \in \mathbb{R}^n$. This gives an action by diffeomorphisms, i.e. each τ_a is a diffeomorphism and

$$\tau_a \circ \tau_b = \tau_{a+b} \tag{5.4.3}$$

as well as

$$\tau_0 = \mathrm{id}_{\mathbb{R}^n}. \tag{5.4.4}$$

We have now induced actions on our function spaces by pull-backs. We start with the space of *all* maps, which is slightly boring from the locally convex point of view, but contains all interesting function spaces. We define the representation of $(\mathbb{R}^n, +)$ on functions by pull-backs, i.e. we consider

$$\tau \colon \mathbb{R}^n \times \operatorname{Map}(\mathbb{R}^n, \mathbb{K}) \longrightarrow \operatorname{Map}(\mathbb{R}^n, \mathbb{K}) \tag{5.4.5}$$

with

$$\tau_a(f) = \tau_{-a}^* f, \tag{5.4.6}$$

i.e.

$$\tau_a(f)\big|_x = f(x-a) \tag{5.4.7}$$

for $a \in \mathbb{R}^n$ and $f \in \operatorname{Map}(\mathbb{R}^n, \mathbb{K})$. The appearance of the additional minus sign is of mere aesthetic reasons. The translations are a left action, hence the pull-backs yield a right representation, which we turn into a left representation by (5.4.6). Of course, $(\mathbb{R}^n, +)$ is abelian and hence the difference between left and right actions is absent. If, however, one wants to generalize convolutions to non-abelian groups, it becomes important.

Proposition 5.4.1 *Let $a \in \mathbb{R}^n$ and $f \in \operatorname{Map}(\mathbb{R}^n, \mathbb{K})$.*

i.) The representation τ consists of continuous linear maps

$$\tau_a \colon \operatorname{Map}(\mathbb{R}^n, \mathbb{K}) \longrightarrow \operatorname{Map}(\mathbb{R}^n, \mathbb{K}) \tag{5.4.8}$$

with respect to the locally convex topology of pointwise convergence.

ii.) Let $j \in \operatorname{Map}(\mathbb{R}^n, \mathbb{K})$ be given. The map

$$\mathbb{R}^n \ni a \;\mapsto\; \tau_a(f) \in \operatorname{Map}(\mathbb{R}^n, \mathbb{K}) \tag{5.4.9}$$

is continuous iff it is continuous at $a = 0$ iff f is continuous.

PROOF: To check the first part we use the seminorms

$$|f|_x = |f(x)|$$

for $x \in \mathbb{R}^n$, which define the topology of pointwise convergence. Then we have

$$|\tau_a(f)|_x = |\tau_a(f)(x)| = |f(x-a)| = |f|_{x-a},$$

which is the continuity estimate needed for the first part. For the second, let again $x \in \mathbb{R}^n$. We first consider continuity at $a = 0$. Since $\tau_0 = \mathrm{id}_{\mathbb{R}^n}$ by (5.4.4), we need to show $\tau_a(f) \to f$ pointwise for $a \to 0$. We have

$$|\tau_a(f) - f|_x = |f(x-a) - f(x)|,$$

which converges to zero for $a \to 0$ iff f is continuous at x. Since x is arbitrary, we get the equivalence of the second and third statement. Thus assume we have continuity at $a = 0$. Then we need to check continuity at $a_0 \in \mathbb{R}^n$. By the representation property we have

$$\tau_a(f) - \tau_{a_0}(f) = \tau_{a_0}(\tau_{a-a_0}(f) - \tau_0(f)) = \tau_{a_0}(\tau_{a-a_0}(f) - f).$$

Since each τ_{a_0} is continuous in the topology of pointwise convergence, the continuity at zero implies continuity everywhere. □

It is a general feature of group representations that continuity in the group parameter needs to be checked only at one point in the group, say at the group unit, as soon as each group element acts in a continuous way. We will make use of this principle frequently.

We are now back to the continuous functions, as soon as we want continuity of the map (5.4.8) in the weakest of our locally convex topologies: the topology of pointwise convergence. This is quite surprising and gives yet another interpretation of what continuity of functions on $\mathbb{R}^n$ is really all about.

If we consider continuous functions anyway, we also have other topologies at our disposal. We can upgrade the continuity statement in this situation:

Proposition 5.4.2 *For $f \in \mathscr{C}(\mathbb{R}^n)$ the map*

$$\mathbb{R}^n \ni a \ \mapsto \ \tau_a(f) \in \mathscr{C}(\mathbb{R}^n) \tag{5.4.10}$$

is continuous in the $\mathscr{C}$-topology and each $\tau_a\colon \mathscr{C}(\mathbb{R}^n) \longrightarrow \mathscr{C}(\mathbb{R}^n)$ is continuous, too.

Proof: In some sense this is the statement that continuous functions are locally uniformly continuous. Let $K \subseteq \mathbb{R}^n$ be compact. We show the continuity at $a = 0$, since we can then transport the continuity to any other point using the continuous map

$$\tau_a\colon \mathscr{C}(\mathbb{R}^n) \longrightarrow \mathscr{C}(\mathbb{R}^n)$$

and the representation property. Note that τ_a is continuous, as it is a pull-back with a continuous map, namely the homeomorphism $\tau_{-a}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Now let $\|a\| \leq 1$, then on the compact subset $K + \mathrm{B}_1(0)^{\mathrm{cl}}$ we have a $\delta > 0$ with $|f(x) - f(y)| < \epsilon$ for all $x, y \in K + \mathrm{B}_1(0)^{\mathrm{cl}}$ with $\|x - y\| < \delta$. This is the uniform continuity of f on this compact subset. Hence we can apply this to $x - a, x \in K + \mathrm{B}_1(0)^{\mathrm{cl}}$ with $x \in K$, leading to

$$\|\tau_a(f) - f\|_K = \sup_{x \in K} |f(x-a) - f(x)| \leq \epsilon$$

for $\|x - a - x\| < \delta$, i.e. for $\|a\| < \delta$. This is the continuity of (5.4.10) in the $\mathscr{C}$-topology at $a = 0$. The second statement is clear and was shown earlier in a larger context. □

Hence we obtain a continuous representation of the translation group on $\mathscr{C}(\mathbb{R}^n)$. We have the following variant of importance:

Corollary 5.4.3 *For $f \in \mathscr{C}_0(\mathbb{R}^n)$ the map*

$$\mathbb{R}^n \ni a \ \mapsto \ \tau_a(f) \in \mathscr{C}_0(\mathbb{R}^n) \tag{5.4.11}$$

is continuous in the $\mathscr{C}_0$-topology and each

$$\tau_a\colon \mathscr{C}_0(\mathbb{R}^n) \longrightarrow \mathscr{C}_0(\mathbb{R}^n) \tag{5.4.12}$$

is continuous for $a \in \mathbb{R}^n$.

Proof: First recall that (5.4.12) is already known in a much larger context of pull-backs with proper maps and $\mathscr{C}_0(X)$ for arbitrary locally compact Hausdorff spaces X, see Exercise 2.5.67. We need to check the first statement for continuity only at $a = 0$. Thus let $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$. For $\|a\| \leq 1$ we have $\tau_a(x) \in \operatorname{supp}(f) + \mathrm{B}_1(0)^{\mathrm{cl}} \subseteq \mathbb{R}^n$, whenever $x \in \operatorname{supp}(f)$. Since the former subset is also compact, we get for $a \to 0$ the uniform convergence

$$\|\tau_a(f) - f\|_{\operatorname{supp}(f) + \mathrm{B}_1(0)^{\mathrm{cl}}} \longrightarrow 0$$

for $a \to 0$ as before. This shows that for every given $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ we have convergence $\tau_a(f) \to f$ in some $\mathscr{C}_K(\mathbb{R}^n)$ for a suitable K. From the universal property of the LF topology of $\mathscr{C}_0(\mathbb{R}^n)$ we infer that $\tau_a(f) \to f$ in $\mathscr{C}_0(\mathbb{R}^n)$, as well. This shows (5.4.12). □

Remark 5.4.4 Quite remarkably, the action of the translations on the Banach space of bounded continuous functions $\mathscr{C}_b(\mathbb{R}^n)$ is more involved. On the one hand, each map τ_a satisfies

$$\|\tau_a(f)\|_\infty = \|f\|_\infty \tag{5.4.13}$$

for all $f \in \mathscr{C}_b(\mathbb{R}^n)$ or even for $f \in \mathscr{B}(\mathbb{R}^n)$. Thus, each individual map $\tau_a\colon \mathscr{C}_b(\mathbb{R}^n) \longrightarrow \mathscr{C}(\mathbb{R}^n)$ is continuous. Moreover, it turns out that the map

$$\mathbb{R}^n \ni a \ \mapsto \ \tau_a(f) \in \mathscr{C}_b(\mathbb{R}^n) \tag{5.4.14}$$

is continuous iff the function f is actually uniformly continuous, see also Exercise 5.5.38.

While this clarifies the continuity of the map sending $a \in \mathbb{R}^n$ to the translated function $\tau_a(f)$ we also want to understand differentiability properties of it. Again, the question can be set as two-fold: Is $\tau_a(f)$ depending on a in a differentiable way and is the difference quotient approximating the derivative also in an interesting topology? Here we get the following result, specializing the results of Proposition **??** to our situation:

Proposition 5.4.5 *Let $f \in \mathscr{C}(\mathbb{R}^n)$. Then the following statements are equivalent:*

i.) The map $\mathbb{R}^n \ni a \mapsto \tau_a(f) \in \mathscr{C}(\mathbb{R}^n)$ is differentiable at $a = 0$ with respect to the $\mathscr{C}$-topology.

ii.) The map $\mathbb{R}^n \ni a \mapsto \tau_a(f) \in \mathscr{C}(\mathbb{R}^n)$ is differentiable everywhere with respect to the $\mathscr{C}$-topology.

iii.) One has $f \in \mathscr{C}^1(\mathbb{R}^n)$.

In this case, for every $a \in \mathbb{R}^n$ one has

$$(Df)a = \lim_{t \to 0} \frac{1}{t}(\tau_{-ta}(f) - f) \tag{5.4.15}$$

in the $\mathscr{C}$-topology.

PROOF: Suppose *i.)* holds, i.e. there exists a derivative $F \in \operatorname{Hom}(\mathbb{R}^n, \mathscr{C}(\mathbb{R}^n))$ of the function $a \mapsto \tau_a(f)$ at $a = 0$. Then by definition we have for the remainder

$$r(a) = \tau_a(f) - f - Fa \in \mathscr{C}(\mathbb{R}^n) \tag{$*$}$$

the property that

$$\lim_{a \to 0} \frac{1}{\|a\|} r(a) = 0 \tag{$**$}$$

in the $\mathscr{C}$-topology, see Definition **??**. We first consider a very simple seminorm of $\mathscr{C}(\mathbb{R}^n)$, namely the one from the pointwise convergence. We get from $(*)$

$$\begin{aligned} |r(a)|_x &= |\tau_a(f) - f - Fa|_x \\ &= |\tau_a(f)(x) - f(x) - F(x)a| \\ &= |f(x - a) - f(x) - F(x)a|, \end{aligned}$$

where we write $F(x) \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{K})$ for the evaluation of F at $x \in \mathbb{R}^n$. Then $(**)$ for this seminorm gives

$$\lim_{a \to 0} \frac{1}{|a|} |f(x - a) - f(x) - F(x)a| = 0,$$

which simply means that $f \in \mathscr{C}(\mathbb{R}^n)$ is differentiable at $x \in \mathbb{R}^n$ with derivative given by $-F(x) \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{K})$. The minus sign is due to our convention for the representation τ as a *left* representation. Hence f is everywhere differentiable with derivative $Df = -F \in \operatorname{Hom}(\mathbb{R}^n, \mathscr{C}(\mathbb{R}^n))$. In particular, Df is continuous and thus $f \in \mathscr{C}^1(\mathbb{R}^n)$. This shows *i.)* $\implies$ *iii.)*. The reverse implication *iii.)* $\implies$ *i.)* was obtained already in a more general context in Proposition **??**. Finally, since each τ_a is $\mathscr{C}$-continuous, we get from the representation property and the assumption *i.)* the limit

$$\begin{aligned} &\lim_{a \to 0} \frac{1}{\|a\|} (\tau_{a_0 + a}(f) - \tau_{a_0}(f) - \tau_{a_0}(F)a) \\ &\quad = \lim_{a \to 0} \frac{1}{\|a\|} \tau_{a_0}(\tau_a(f) - f - Fa) \\ &\quad = \tau_{a_0} \lim_{a \to 0} \frac{1}{\|a\|} (\tau_a(f) - f - Fa) \\ &\quad = 0, \end{aligned}$$

which proves that $\tau_a(f)$ is differentiable at $a_0 \in \mathbb{R}^n$ with derivative given by $\tau_{a_0}(F)$. This shows *i.)* $\implies$ *ii.)*, the converse is trivial. $\square$

As before, we have a version for compactly supported functions based on the LF topology of $\mathscr{C}_0(\mathbb{R}^n)$ instead:

Corollary 5.4.6 *Let $f \in \mathscr{C}_0(\mathbb{R}^n)$ be given. Then the following statements are equivalent:*

i.) The map $\mathbb{R}^n \ni a \mapsto \tau_a(f) \in \mathscr{C}_0(\mathbb{R}^n)$ is differentiable at $a = 0$ with respect to the $\mathscr{C}_0$-topology.

ii.) The map $\mathbb{R}^n \ni a \mapsto \tau_a(f) \in \mathscr{C}_0(\mathbb{R}^n)$ is differentiable everywhere with respect to the $\mathscr{C}_0$-topology.

iii.) One has $f \in \mathscr{C}_0^1(\mathbb{R}^n)$.

In this case, for every $a \in \mathbb{R}^n$ one has

$$(Df)a = \lim_{t \to 0} \frac{1}{t}(\tau_{-ta}(f) - f) \tag{5.4.16}$$

in the $\mathscr{C}_0$-topology.

Proof: For a given $f \in \mathscr{C}_0(\mathbb{R}^n)$ and $\|a\| \le 1$ everything we need to do takes place in *one* big enough compact subset. Hence we can use the above statements from Proposition 5.4.5 and the fact that convergence of sequences in a single $\mathscr{C}_K(\mathbb{R}^n)$ is convergence in $\mathscr{C}_0(\mathbb{R}^n)$. □

In a next step we want to extend these results to higher derivatives. This is now fairly easy, since differentiability is defined inductively for higher derivatives. Hence we can inductively use the above results. This results in two types of questions: is the map $a \mapsto \tau_a(f)$ of higher class $\mathscr{C}^k$ and can the limit of the difference quotient be taken also in the stronger $\mathscr{C}^k$-topologies instead of the $\mathscr{C}$-topology? Fortunately, we have the following answer to both questions:

Proposition 5.4.7 *Let $k \in \mathbb{N}$ and $f \in \mathscr{C}(\mathbb{R}^n)$ be given. Then the following statements are equivalent:*

i.) The map $\mathbb{R}^n \ni a \mapsto \tau_a(f) \in \mathscr{C}(\mathbb{R}^n)$ is k-times differentiable with respect to the $\mathscr{C}$-topology.

ii.) One has $f \in \mathscr{C}^k(\mathbb{R}^n)$.

In this case, for every $a \in \mathbb{R}^n$ one has

$$(Df)a = \lim_{t \to 0} \frac{1}{t}(\tau_{-ta}(f) - f) \tag{5.4.17}$$

in the $\mathscr{C}^{k-1}$-topology.

Proof: The equivalence of the first two statements follows by induction and by considering the components of the iterated derivatives form Proposition 5.4.5. Then (5.4.17) was already shown in a slightly more general context in Proposition **??**, based in a simple estimate using the mean value theorem. □

Corollary 5.4.8 *Let $f \in \mathscr{C}^\infty(\mathbb{R}^n)$. Then for every $a \in \mathbb{R}^n$ one has*

$$(Df)a = \lim_{t \to 0} \frac{1}{t}(\tau_{-ta}(f) - f) \tag{5.4.18}$$

in the $\mathscr{C}^\infty$-topology.

Proof: This is either the above Proposition 5.4.7 for all $k \in \mathbb{N}$ or a consequence of the more general statement on flows acting on functions in Theorem 5.2.38, *ii.)*, applied to the flow τ_{-ta} of the constant vector field a, see again Example 5.2.41. □

Corollary 5.4.9 *Let $k \in \mathbb{N} \cup \{\infty\}$ and $f \in \mathscr{C}_0(\mathbb{R}^n)$. Then the following statements are equivalent:*

i.) The map $\mathbb{R}^n \ni a \mapsto \tau_a(f) \in \mathscr{C}_0(\mathbb{R}^n)$ is k-times differentiable with respect to the $\mathscr{C}_0$-topology.

ii.) *One has* $f \in \mathscr{C}_0^k(\mathbb{R}^n)$.
In this case, for every $a \in \mathbb{R}^n$ *one has*

$$(Df)a = \lim_{t \to 0} \frac{1}{t}(\tau_{-ta}(f) - f) \tag{5.4.19}$$

in the $\mathscr{C}_0^{k-1}$*-topology.*

Corollary 5.4.10 *Let* $k \in \mathbb{N}_0 \cup \{\infty\}$.

i.) *For* $f \in \mathscr{C}_0^k(\mathbb{R}^n)$ *the map*

$$\mathbb{R}^n \ni a \;\mapsto\; \tau_a(f) \in \mathscr{C}_0^k(\mathbb{R}^n) \tag{5.4.20}$$

is continuous in the $\mathscr{C}_0^k$*-topology.*

ii.) *For* $f \in \mathscr{C}^k(\mathbb{R}^n)$ *the map*

$$\mathbb{R}^n \ni a \;\mapsto\; \tau_a(f) \in \mathscr{C}^k(\mathbb{R}^n) \tag{5.4.21}$$

is continuous in the $\mathscr{C}^k$*-topology.*

PROOF: As before, it suffices to check this at $a = 0$ and to use the representation properties of τ together with the continuity of the maps τ_a afterwards. First, we note that

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}\tau_a(f) = \tau_a\Big(\frac{\partial^{|\alpha|} f}{\partial x^\alpha}\Big)$$

for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$, since translations commute with partial derivatives. Hence the continuity in the $\mathscr{C}$-topology or $\mathscr{C}_0$-topology according to Proposition 5.4.2 and Corollary 5.4.3, respectively, shows that

$$a \;\mapsto\; \tau_a\Big(\frac{\partial^{|\alpha|} f}{\partial x^\alpha}\Big) = \frac{\partial^{|\alpha|}}{\partial x^\alpha}\tau_a(f)$$

is continuous in the $\mathscr{C}$-topology or $\mathscr{C}_0$-topology, respectively. The estimates for this continuity give then estimates for

$$\mathrm{p}_{K,\ell}(\tau_a(f) - f) \to 0 \tag{$*$}$$

for $a \to 0$ and $K \subseteq \mathbb{R}^n$ compact and $\ell \in \mathbb{N}_0$ with $\ell \leq k$. This gives the result for *ii.)*. The case of the LF-topology of $\mathscr{C}_0^k(\mathbb{R}^n)$ is then obtained analogously to Corollary 5.4.3 based on $(*)$ instead of $\mathrm{p}_{K,0}$ only. $\square$

These statements clarify the dependence on a translation parameter for most of our interesting function spaces in the relevant topologies. Surprisingly enough, adding a global boundedness condition makes the statements much more complicated. Some first impressions can be found in Exercise 5.5.39. Note also that each map τ_a is a continuous linear endomorphism of all the above function spaces. This has been obtained earlier, since the underlying translations on $\mathbb{R}^n$ are diffeomorphisms. While with the $\mathscr{C}^k$-functions we have found the "correct" function classes to discuss continuity and differentiability of the map $a \mapsto \tau_a(f)$ for topologies finer than the topology of pointwise convergence, there are also other locally convex topologies of interest beyond this scenario. The normable topologies of L^p-functions are not comparable to the topology of pointwise convergence. In fact, a comparison does not even make any sense as in the L^p-spaces we need to take equivalence classes of functions instead of individual functions. Hence the conclusion of Proposition 5.4.1, *ii.)* is not available and, in fact, not true at all. Surprisingly enough, we get continuous dependence on the translation parameter. The first statements in this direction are the following global versions:

Proposition 5.4.11 *Let* $p \in [1, \infty)$.

i.) *The seminorm* $\|\cdot\|_p$ *on* $\mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ *is translation invariant, i.e. for all measurable functions* $f \in \mathscr{M}(\mathbb{R}^n)$ *we have* $\tau_a(f) \in \mathscr{M}(\mathbb{R}^n)$ *and*

$$\|f\|_p = \|\tau_a(f)\|_p \tag{5.4.22}$$

for all $a \in \mathbb{R}^n$ *as equality in* $[0, \infty]$.

ii.) *The representation* τ *restricts to a representation on* $\mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ *with isometric maps* τ_a *for all* $a \in \mathbb{R}^n$.

iii.) *The subspace of zero functions in* $\mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ *is* τ*-invariant.*

iv.) *The representation* τ *induces a representation on the Banach space* $\mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ *with isometric maps* τ_a *for all* $a \in \mathbb{R}^n$.

v.) *For every function* $f \in \mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ *the map*

$$\mathbb{R}^n \ni a \ \mapsto\ \tau_a(f) \in \mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x) \tag{5.4.23}$$

is continuous.

vi.) *For every function* $f \in \mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ *the map*

$$\mathbb{R}^n \ni a \ \mapsto\ \tau_a(f) \in \mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x) \tag{5.4.24}$$

is continuous.

Proof: The first statement is just the fact that the Lebesgue measure is translation invariant. Note that the homeomorphisms τ_a preserve the notion of measurability, as we use the Borel σ-algebra to define the measure $\mathrm{d}^n x$ and its invariance shows that also its completion to the Lebesgue-Borel σ-algebra is compatible with homeomorphisms. Then the second and third statement are immediate consequences of (5.4.22). Since $\mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x) = \mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)/\ker(\|\cdot\|_p)$ the representation τ descends to a well-defined representation on the quotient. Here we have an isometric representation, as these properties can be checked on representatives. The fifth part is now the non-trivial statement and based on a general result on positive Borel measures on σ-compact locally compact Hausdorff spaces. In fact, by a variant of Lusin's Theorem we infer that the functions $\mathscr{C}_0(\mathbb{R}^n)$ are dense in $\mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$, see e.g. [14, Theorem 3.14]. Now let $g \in \mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ and $\epsilon > 0$ be arbitrary. The density of $\mathscr{C}_0(\mathbb{R}^n) \subseteq \mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ allows us to find a $f_\epsilon \in \mathscr{C}_0(\mathbb{R}^n)$ with $\|f_\epsilon - g\|_p < \frac{\epsilon}{3}$. Moreover, we find a $\delta > 0$ with $\|\tau_a(f_\epsilon) - f_\epsilon\|_p < \frac{\epsilon}{3}$, whenever $\|a\| < \delta$ by the just proven continuity at $a = 0$ for $f_\epsilon \in \mathscr{C}_0(\mathbb{R}^n)$. Hence

$$\begin{aligned}
\|\tau_a(g) - g\|_p &\leq \|\tau_a(g) - \tau_a(f_\epsilon)\|_p + \|\tau_a(f_\epsilon) - f_\epsilon\|_p + \|f_\epsilon - g\|_p \\
&< \|f_\epsilon - g\|_p + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&< \epsilon,
\end{aligned}$$

using the isometry property (5.4.22). This shows the continuity of (5.4.23) at $a = 0$ for an arbitrary function in $\mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$. As before, this is all we need to show *v.)*. Passing to the quotient preserves this continuity, completing the proof. □

Remark 5.4.12 As already for $\mathscr{C}_b(\mathbb{R}^n)$ the statement becomes wrong for $p = \infty$. We do not have continuity of the map $a \mapsto \tau_a(f)$ for all (essentially) bounded functions $f \in \mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$. The above argument relying on general facts from measure theory, i.e. on Lusin's Theorem, can be used to extend the statement considerably. We will also meet the second part of the proof, the $\frac{\epsilon}{3}$ argument, again: having continuity on a dense subspace allows us to infer continuity in the whole space.

Without the global integrability we still get a continuity statement for the locally p-integrable functions:

Proposition 5.4.13 *Let $p \in [1, \infty)$.*

i.) For every $a \in \mathbb{R}^n$ the translation by a yields a continuous linear map

$$\tau_a \colon \mathscr{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}^n x) \longrightarrow \mathscr{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}^n x). \tag{5.4.25}$$

In fact, for every compact subset $K \subseteq \mathbb{R}^n$ one has

$$\|\tau_a(f)\|_{p,K} = \|f\|_{p,\tau_{-a}(K)} \tag{5.4.26}$$

for $f \in \mathscr{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}^n x)$.

ii.) The representation τ induces a representation

$$\tau \colon \mathbb{R}^n \times \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}^n x) \longrightarrow \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}^n x) \tag{5.4.27}$$

by continuous linear maps.

iii.) For every $f \in \mathscr{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}^n x)$ the map

$$\mathbb{R}^n \ni a \ \mapsto \ \tau_a(f) \in \mathscr{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}^n x) \tag{5.4.28}$$

is continuous. Also the map

$$\mathbb{R}^n \ni a \ \mapsto \ \tau_a(f) \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}^n x) \tag{5.4.29}$$

is continuous for every class $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}^n x)$.

PROOF: For *i.)* it suffices to check (5.4.22), which is again just the translation invariance of the Lebesgue measure. Then the second part follows as well, since the Hausdorffization is obtained by the quotient by the zero functions, which are a translation invariant subspace by Proposition 5.4.11, *iii.)*. For the continuity we argue as in Proposition 5.4.11, *v.)*. First, $\mathscr{C}_0(\mathbb{R}^n) \subseteq \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}^n x)$ is still dense. Then we have a continuous map $a \mapsto \tau_a(f)$ if $f \in \mathscr{C}_0(\mathbb{R}^n)$. The continuity of the inclusion into $\mathscr{L}^p_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}^n x)$ and the $\frac{\epsilon}{3}$-argument then complete the proof of (5.4.28). Finally, (5.4.29) is a consequence of (5.4.28). □

Again, this is a quite remarkable result that for these fairly irregular functions we can generate a continuity property, once the correct topologies are chosen. Note again that this shows drastically the difference to the topology of pointwise convergence.

The translations do not only act on spaces of functions, but also on spaces of generalized functions. Since each translation is a diffeomorphism, we can dualize as usual, see Example 5.2.32. While the action on functions is not always smooth in the translation parameter, this turns out to be the case for all distributions with an almost ridiculously simple proof:

Proposition 5.4.14 *Let $k \in \mathbb{N}_0$.*

i.) For a distribution $u \in \mathscr{C}^{-\infty}(\mathbb{R}^n)$ the map

$$\mathbb{R}^n \ni a \ \mapsto \ \tau_a(u) \in \mathscr{C}^{-\infty}(\mathbb{R}^n) \tag{5.4.30}$$

is smooth in the weak topology.*

ii.) For $u \in \mathscr{C}^{-\infty}(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$ one has

$$\lim_{t \to 0} \frac{1}{t}(\tau_{-ta}(u) - u) = \sum_{i=1}^n a^i \frac{\partial u}{\partial x^i} \tag{5.4.31}$$

in the weak topology.*

iii.) *For* $u \in \mathscr{C}^{-k}(\mathbb{R}^n)$ *the map* (5.4.30) *is continuous in the weak* topology of* $\mathscr{C}^{-k}(\mathbb{R}^n)$.

iv.) *If* $u \in \mathscr{C}^{-k}(\mathbb{R}^n)$, *then the limit* (5.4.31) *holds in the weak* topology of* $\mathscr{C}^{-(k+1)}(\mathbb{R}^n)$.

v.) *If* $u \in \mathscr{C}_0^{-\infty}(\mathbb{R}^n)$, *then the limit* (5.4.31) *holds and the map* (5.4.30) *is smooth in the weak* topology of* $\mathscr{C}_0^{-\infty}(\mathbb{R}^n)$.

vi.) *If* $u \in \mathscr{C}_0^{-k}(\mathbb{R}^n)$, *then the limit* (5.4.31) *holds in the weak* topology of* $\mathscr{C}_0^{-(k+1)}(\mathbb{R}^n)$.

Proof: Let $u \in \mathscr{C}^{-\infty}(\mathbb{R}^n)$ be a distribution. To test the continuity of (5.4.30) in the weak* topology, we have to evaluate on test functions $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$. We first check continuity at $a = 0$. Here we have

$$\begin{aligned} \mathrm{p}_\varphi(\tau_a(u) - u) &= |(\tau_a(u) - u)(\varphi)| \\ &= |u(\varphi \circ \tau_a) - u(\varphi)| \\ &= |u(\tau_{-a}(\varphi) - \varphi)| \\ &\to 0 \end{aligned}$$

for $a \to 0$, since $\tau_{-a}(\varphi)$ is continuous at $a = 0$ in the $\mathscr{C}_0^\infty$-topology according to Corollary 5.4.9. In fact, the map is even smooth. Note that the dualization of the pull-back $\tau_a(\varphi) = \tau_{-a}^*(\varphi)$ to get $\tau_a(u)$ is using the conventions of generalized functions. Nevertheless, the Jacobi determinant of the translations is constant and equal to one anyway. With the same Corollary 5.4.9 we get the weak* continuity for $u \in \mathscr{C}^{-k}(\mathbb{R}^n)$, since then we use $\varphi \in \mathscr{C}_0^k(\mathbb{R}^n)$. This shows the continuity of (5.4.30) at $a = 0$ in both cases. Since each map $\tau_a \colon \mathscr{C}^{-k}(\mathbb{R}^n) \longrightarrow \mathscr{C}^{-k}(\mathbb{R}^n)$ is weak* continuous by Proposition 5.2.28, *ii.)*, the usual group representation argument applies and gives continuity everywhere. Next, we consider the limit (5.4.31). To test this convergence we need to evaluate on test functions $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ as usual. Now

$$\begin{aligned} &\mathrm{p}_\varphi\Big(\frac{1}{t}(\tau_{-ta}(u) - u) - \sum_{i=1}^n a^i \frac{\partial u}{\partial x^i}\Big) \\ &= \left|\frac{1}{t}(\tau_{-ta}(u)(\varphi) - u(\varphi)) - \sum_{i=1}^n a^i \frac{\partial u}{\partial x^i}(\varphi)\right| \\ &= \left|\frac{1}{t}(u(\tau_a(\varphi)) - u(\varphi)) + \sum_{i=1}^n u\Big(a^i \frac{\partial \varphi}{\partial x^i}\Big)\right| \\ &= \left|u\Big(\frac{1}{t}(\tau_a(\varphi)) - \varphi) + \sum_{i=1}^n a^i \frac{\partial \varphi}{\partial x^i}\Big)\right|. \end{aligned} \tag{$*$}$$

From Corollary 5.4.9 we get that the limit

$$\lim_{t \to 0} \frac{1}{t}(\tau_{-ta}(\varphi) - \varphi) = \sum_{i=1}^n a^i \frac{\partial \varphi}{\partial x^i}$$

exists in the $\mathscr{C}_0^\infty$-topology. Since u is continuous, we can exchange this $\mathscr{C}_0^\infty$-limit with the application of u and see that $(*)$ converges to zero for $t \to 0$. This is the weak* convergence (5.4.31). If $u \in \mathscr{C}^{-k}(\mathbb{R}^n)$ or $u \in \mathscr{C}_0^{-k}(\mathbb{R}^n)$, we can use test functions φ from $\mathscr{C}_0^{k+1}(\mathbb{R}^n)$ or $\mathscr{C}^{k+1}(\mathbb{R}^n)$, wherefore Corollary 5.4.9 and Proposition 5.4.7 give the same convergence result for these test functions. This proves *iv.)* and *vi.)*. Note that the loss of one degree of differentiability means that (5.4.31) holds only in the weaker $\mathscr{C}^{-(k+1)}$-topology and not in the $\mathscr{C}^{-k}$-topology. We need to check smoothness of (5.4.31) in the two cases $u \in \mathscr{C}^{-\infty}(\mathbb{R}^n)$ and $u \in \mathscr{C}_0^{-\infty}(\mathbb{R}^n)$. The second part shows that (5.4.30) is differentiable at $a = 0$ with partial derivatives given by $\frac{\partial u}{\partial x^i}$. Since the τ_a form a group representation, we get differentiability at all other points with

$$\lim_{t \to 0} \frac{1}{t}(\tau_{a_0 - ta}(u) - \tau_{a_0}(u)) = \sum_{i=1}^n a^i \frac{\partial \tau_{a_0}(u)}{\partial x^i} = \tau_{a_0}\Big(\sum_{i=1}^n a^i \frac{\partial u}{\partial x^i}\Big),$$

since the weak* continuous τ_{a_0} commutes with the weak* limit. This shows that (5.4.30) is differentiable everywhere with partial derivatives given by $\tau_a \frac{\partial u}{\partial x^i}$. But these are again distributions of the same sort, so we have continuous partial derivatives. This gives a $\mathscr{C}^1$-function (5.4.30). Iterating this argument shows smoothness in both cases $u \in \mathscr{C}^{-\infty}(\mathbb{R}^n)$ and $u \in \mathscr{C}_0^{-\infty}(\mathbb{R}^n)$. □

The reason for this seemingly strong result, that translated distributions depend smoothly on the translation parameter, is to be seen in the very weak topology of distributions: the weak* topology of $\mathscr{C}^{-\infty}(\mathbb{R}^n)$ or $\mathscr{C}_0^{-\infty}(\mathbb{R}^n)$ is simply weak enough to make it very easy for maps *into* the distributions to become continuous or smooth.

As a conclusion of this section we arrive at the result that the translations act on all kind of function spaces including the generalized functions in such a way that the dependence on the translation parameter is at least continuous if not smooth. This allows to use the integration techniques developed for vector-valued Riemann integrals in Section 4.3 in this context. Indeed, all the target spaces in question are complete enough: we have either completeness or at least sequential completeness of all of them, including the various spaces of distributions. This will allow us to integrate over the translation parameter without further difficulties.

5.4.2 Convolution of Functions

We shall now start with our investigation of the convolution integral (5.4.1) for various classes of functions. Here the main objectives are to identify combinations of function such that (5.4.1) is defined at all and to determine properties of the convolution.

We will make use of the pull-back with the group inversion

$$\mathbb{R}^n \ni x \mapsto -x \in \mathbb{R}^n \tag{5.4.32}$$

of the additive group $(\mathbb{R}^n, +)$. Traditionally, on the level of functions this will be denoted by

$$\check{\cdot} : \mathrm{Map}(\mathbb{R}^n, \mathbb{K}) \ni f \mapsto \check{f} = (x \mapsto f(-x)) \in \mathrm{Map}(\mathbb{R}^n, \mathbb{K}), \tag{5.4.33}$$

yielding an involutive map. Since (5.4.32) is a diffeomorphism with Jacobi determinant having absolute value one, we have simple extensions of $\check{\cdot}$ also to distributions. Moreover, since $\check{\cdot}$ is involutive, we do not need to distinguish pull-back from push-forward.

Remark 5.4.15 (Interpretations of convolution) Let $f, g \in \mathrm{Map}(\mathbb{R}^n, \mathbb{K})$ be two functions subject to further conditions for which we want to define the convolution $f * g$. Assuming nice enough properties on f and g we still have (at least) two principle interpretations for the convolution product:

i.) First, we can interpret (5.4.1) as a pointwise definition of a new function of $x \in \mathbb{R}^n$. This naive definition can then be re-written as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x - y)\, \mathrm{d}^n y = \int_{\mathbb{R}^n} f(y) \tau_x(\check{g})(y)\, \mathrm{d}^n y = I_f(\tau_x(\check{g})). \tag{5.4.34}$$

This suggests to pose conditions on f and g in such a way that I_f becomes a well-defined linear functional on the space containing $\tau_x(\check{g})$. Of course, we expect some locally convex spaces, where g belongs and which carry a good enough representation of the translation group, such that I_f is even a continuous linear functional. Since the Lebesgue measure is translation invariant and invariant under the reflection (5.4.33), we also expect that

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x - y)\, \mathrm{d}^n y = \int_{\mathbb{R}^n} f(x - z) g(z)\, \mathrm{d}^n z = (g * f)(x) \tag{5.4.35}$$

by a change of variables $z = x - y$. Hence we can hope for

$$(f * g)(x) = I_g(\tau_x(\check{f})) \tag{5.4.36}$$

as well, by the above commutativity of $*$. If f and g are from different function spaces, this heuristic suggests to look for a mutual duality of the two spaces in question.

ii.) The second interpretation of the convolution is to view the map

$$\mathbb{R}^n \ni y \;\mapsto\; \tau_y(g) \in \operatorname{Map}(\mathbb{R}^n, \mathbb{K}) \tag{5.4.37}$$

as a function-valued map, which we want to integrate over $\mathbb{R}^n$. This brings us into the realm of vector-valued (Riemann) integrals as discussed in Section 4.3. Of course, g should again belong to a sufficiently nice subspace of $\operatorname{Map}(\mathbb{R}^n, \mathbb{K})$ such that (5.4.37) fulfils the needed properties to make use of the integration techniques from Section 4.3. Then the function $\tau_y(g)\colon x \mapsto g(x-y)$ is the *value* of the map (5.4.37) at the point $y \in \mathbb{R}^n$. The convolution becomes

$$g * f = \int_{\mathbb{R}^n} g(y)\tau_y(f)\,\mathrm{d}^n y, \tag{5.4.38}$$

once the conditions for (Riemann) integrals are met.

The remaining question is then, of course, which of the interpretations is possible and whether they yield the same result: ideally, we have a coincidence between the two principle interpretations together with the commutativity of the convolution product.

A first step consists in discussing very large function spaces with minimal requirements on the regularity of the functions. An interesting scenario will be to consider $\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ for one factor. This space is in some sense the largest reasonable space of functions, which we can use in distribution theory. However, to define integrals, we then need the other factor to take care of compact supports without spoiling the (local) integrability of $\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$. Here we consider the following new space:

Definition 5.4.16 (The space $\mathscr{BM}_0(X)$) *Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset.*

i.) *For a compact subset $K \subseteq X$ one defines*

$$\mathscr{BM}_K(X) = \big\{ f \in \mathscr{BM}(X) \;\big|\; \operatorname{supp}(f) \subseteq K \big\}, \tag{5.4.39}$$

endowed with the supremum norm $\|\cdot\|$ inherited from $\mathscr{BM}(X)$.

ii.) *One defines the space of compactly supported bounded measurable functions by*

$$\mathscr{BM}_0(X) = \big\{ f \in \mathscr{BM}(X) \;\big|\; \operatorname{supp}(f) \textit{ is compact} \big\}, \tag{5.4.40}$$

endowed with the locally convex inductive limit topology from

$$\mathscr{BM}_0(X) = \varinjlim_{\substack{K \subseteq X \\ \textit{compact}}} \mathscr{BM}_K(X). \tag{5.4.41}$$

We collect a few first properties of this space:

Proposition 5.4.17 *Let $X \subseteq \mathbb{R}^n$ be an non-empty open subset.*

i.) *For every compact $K \subseteq X$ the subspace $\mathscr{BM}_K(X) \subseteq \mathscr{BM}(X)$ is closed and hence a Banach space.*

ii.) *For compact subsets $K \subseteq K' \subseteq X$ the inclusion*

$$\mathscr{BM}_K(X) \subseteq \mathscr{BM}_{K'}(X) \tag{5.4.42}$$

is an embedding with closed image.

iii.) *For all compact $K \subseteq X$ the inclusion*

$$\mathscr{C}_K(X) \subseteq \mathscr{B}\mathscr{M}_K(X) \tag{5.4.43}$$

is an embedding with closed image.

iv.) *The locally convex inductive limit topology turns $\mathscr{B}\mathscr{M}_0(X)$ into an LB space.*

v.) *For all compact $K \subseteq X$ we have*

$$\Big| \int_{\mathbb{R}^n} f(y) g(y) \, \mathrm{d}^n y \Big| \le \|f\|_{\infty,K} \cdot \|g\|_{1,K} \tag{5.4.44}$$

for all $f \in \mathscr{B}\mathscr{M}_K(X)$ and $g \in \mathscr{L}^1_{\mathrm{loc}}(X)$.

vi.) *For $f \in \mathscr{B}\mathscr{M}_0(X)$ the map*

$$I_f \colon \mathscr{L}^1_{\mathrm{loc}}(X) \ni g \ \mapsto\ I_f(g) = \int_{\mathbb{R}^n} f(y) g(y) \, \mathrm{d}^n y \in \mathbb{K} \tag{5.4.45}$$

is a continuous linear functional.

vii.) *For $g \in \mathscr{L}^1_{\mathrm{loc}}(X)$ the map*

$$I_g \colon \mathscr{B}\mathscr{M}_0(X) \ni f \ \mapsto\ I_g(f) = \int_{\mathbb{R}^n} f(y) g(y) \, \mathrm{d}^n y \in \mathbb{K} \tag{5.4.46}$$

is a continuous linear functional.

Proof: We know that $\mathscr{B}\mathscr{M}(X)$ is a Banach space with respect to the supremum norm according to Proposition 2.3.13, *ii.)*. The support condition for $\mathscr{B}\mathscr{M}_K(X) \subseteq \mathscr{B}\mathscr{M}(X)$ is then clearly preserved under uniform convergence, showing the first part. The second part is clear, as both spaces inherit the supremum norm, and so is the third. The fourth part follows closely the line of argument for $\mathscr{C}^k_0(X)$ as in Proposition 2.4.50: by *ii.)*, the locally convex inductive limit is strict. Choosing an exhausting sequence of compact subsets in X then gives a countable strict inductive limit of Banach spaces, i.e. the structure of an LB space. A last verification shows that the topologies coincide, completing the proof of *iv.)*. The estimate (5.4.44) is clear by the support properties of f. But then *vi.)* as well as *vii.)* follow at once. □

Moreover, the continuous functions with compact support are continuously included into $\mathscr{B}\mathscr{M}_0(X)$, i.e.

$$\mathscr{C}_0(X) \subseteq \mathscr{B}\mathscr{M}_0(X), \tag{5.4.47}$$

which follows form the usual properties of inductive limits. This inclusion is the main reason, why we are interested in $\mathscr{B}\mathscr{M}_0(X)$. Nevertheless, the space $\mathscr{B}\mathscr{M}_0(X)$ might also have some independent interest, as it demonstrates how continuity of nowhere comes from convolution, see also Exercise 5.5.40.

Theorem 5.4.18 (Convolution I) *Let $f \in \mathscr{B}\mathscr{M}_0(\mathbb{R}^n)$ and $g \in \mathscr{L}^1_{\mathrm{loc}}(X)$ be given.*

i.) *For all $x \in \mathbb{R}^n$ the functions $y \mapsto f(y)g(x-y)$ and $z \mapsto f(x-z)g(z)$ are integrable and one has*

$$\int_{\mathbb{R}^n} f(y) g(x-y) \, \mathrm{d}^n y = \int_{\mathbb{R}^n} f(x-z) g(z) \, \mathrm{d}^n z. \tag{5.4.48}$$

The value of the integral only depends on the class of g in $\mathrm{L}^1_{\mathrm{loc}}(X)$.

ii.) *For the pointwise convolution integrals of f and g one has*

$$f * g = g * f. \tag{5.4.49}$$

iii.) *The reflection as well as the translations map* $\mathscr{BM}_0(\mathbb{R}^n)$ *continuously onto itself.*

iv.) *The pointwise convolution* f *and* g *satisfies*

$$(f * g)(x) = I_f(\tau_x(\check{g})) \tag{5.4.50}$$

as well as

$$(f * g)(x) = I_g(\tau_x(\check{f})). \tag{5.4.51}$$

v.) *One has* $f * g \in \mathscr{C}(\mathbb{R}^n)$.

vi.) *For the support of the convolution* $f * g$ *one has*

$$\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f) + \operatorname{supp}(g), \tag{5.4.52}$$

where $\operatorname{supp}(g)$ *denotes the essential support of* g, *i.e. the smallest closed subset, outside of which* g *is a zero function.*

vii.) *For every compact subset* $K \subseteq \mathbb{R}^n$ *one has*

$$\mathrm{p}_{K,0}(f * g) \leq \|f\|_\infty \|g\|_{1,\operatorname{supp}(f)+K}. \tag{5.4.53}$$

viii.) *The convolution is a separately continuous map bilinear map*

$$\mathscr{BM}_0(\mathbb{R}^n) \times \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^n) \longrightarrow \mathscr{C}(X), \tag{5.4.54}$$

i.e. after fixing one argument one has a continuous linear map for the other argument.

Proof: The function $y \mapsto f(y)g(x-y)$ has compact support inside of $\operatorname{supp}(f)$. Since g is locally integrable, so is $y \mapsto g(x-y)$. Since f is bounded and measurable, we have an integrable function as integrand on the left hand side. Analogously, we argue for the right hand side, showing that both integrals are well-defined. They are equal, since we can perform the change of variables $z = x - y$. Finally, for the right hand side it is clear that a zero function g yields a zero integral, no matter what f is. This shows the first part and the second is just a reformulation. The third part is obvious, as both the reflection and the translations are diffeomorphisms, not changing the compactness of the supports. The continuity is also clear, as for every compact subset $K \subseteq \mathbb{R}^n$ we get continuous linear maps

$$\check{\ } \colon \mathscr{BM}_K(\mathbb{R}^n) \longrightarrow \mathscr{BM}_{\check{K}}(\mathbb{R}^n) \longrightarrow \mathscr{BM}_0(\mathbb{R}^n)$$

and

$$\tau_x \colon \mathscr{BM}_K(\mathbb{R}^n) \longrightarrow \mathscr{BM}_{\tau_x(K)}(\mathbb{R}^n) \longrightarrow \mathscr{BM}_0(\mathbb{R}^n).$$

Then the universal property of locally convex inductive limits shows that $\check{\ }$ and τ_x are also continuous linear maps

$$\check{\ }, \tau_x \colon \mathscr{BM}_0(\mathbb{R}^n) \longrightarrow \mathscr{BM}_0(\mathbb{R}^n).$$

The next part is just a reinterpretation of (5.4.48). However, we know that I_f as well as I_g are now continuous linear functionals. From Proposition 5.4.13, *iii.)*, we infer that $x \mapsto \tau_x(\check{g})$ is a continuous map from $\mathbb{R}^n$ to $\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ for every fixed g. Hence the composition with the continuous linear functional I_f, see Proposition 5.4.17, *vi.)*, yields a continuous map. This shows *iii.)*. For the next part, we first recall that for a closed subset $A \subseteq \mathbb{R}^n$ and a compact subset $K \subseteq \mathbb{R}^n$ also $A + K$ is closed, see Exercise 5.5.41. Hence the subset $\operatorname{supp}(f) + \operatorname{supp}(g) \subseteq \mathbb{R}^n$ is closed. Now consider $x \in X \setminus (\operatorname{supp}(f) + \operatorname{supp}(g))$. If now $y \in \operatorname{supp}(f)$, then $x \notin \operatorname{supp}(f)$ or $g(x-y) = 0$, except for a zero set of y's. In both situations, the convolution integral for this x will be zero. This proves (5.4.52). Now let $K \subseteq \mathbb{R}^n$ be compact. Then for $x \in K$ we have

$$|(f * g)(x)| = \left| \int_{\mathbb{R}^n} f(y) g(x-y) \, \mathrm{d}^n y \right|$$

$$\begin{aligned}
&= \left| \int_{\operatorname{supp}(f)} f(y) g(x-y) \, \mathrm{d}^n y \right| \\
&\leq \|f\|_\infty \int_{\operatorname{supp}(f)} |g(x-y)| \, \mathrm{d}^n y \\
&= \|f\|_\infty \|g\|_{1, \operatorname{supp}(f)+K}.
\end{aligned}$$

Since g is integrable over compact subsets, this gives (5.4.53). We use this to check the separate continuity of (5.4.54). First fix $f \in \mathscr{BM}_0(\mathbb{R}^n)$. For all $K \subseteq \mathbb{R}^n$ we then get

$$\mathrm{p}_{K,0}(f * g) \leq \mathit{const} \, \|g\|_{1, \operatorname{supp}(f)+K},$$

which is the continuity in $g \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$, since $\operatorname{supp}(f) + K$ is compact. Conversely, let $g \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ be fixed and let $K, \tilde{K} \subseteq \mathbb{R}^n$ be compact. Then for all $f \in \mathscr{BM}_{\tilde{K}}(\mathbb{R}^n)$ we have

$$\mathrm{p}_{K,0}(f * g) \leq \|f\|_\infty \|g\|_{1, K+\tilde{K}} = \mathit{const} \, \|f\|_\infty$$

with a constant depending only on $\tilde{K}$, but not on f. This shows that

$$\mathscr{BM}_{\tilde{K}} \ni f \mapsto f * g \in \mathscr{C}(\mathbb{R}^n)$$

is continuous for all $\tilde{K}$. From the universal property of the locally convex inductive limit this implies the continuity of

$$\mathscr{BM}_0 \ni f \mapsto f * g \in \mathscr{C}(\mathbb{R}^n),$$

thereby completing the proof. □

Remark 5.4.19 (Regularization by convolution) This is a first instance of the regularizing powers of convolution. Even though both factors f and g are allowed to be fairly general, the *continuity of the representation* of $\mathbb{R}^n$ on $\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ yields the continuity of the convolution product $f * g$. Moreover, the non-locality is controlled in a reasonable way by (5.4.52) and can be made very mild by choosing the support of f to be a very small neighbourhood of 0. We will see many applications of these regularizing effects in the sequel. Also note that the convolution descends to the quotient

$$*\colon \mathscr{BM}_0(\mathbb{R}^n) \times \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n) \longrightarrow \mathscr{C}(\mathbb{R}^n) \tag{5.4.55}$$

with the same continuity properties.

In a next step we want to increase the regularity from continuity to differentiability. Here we invest a bit more than just functions from $\mathscr{BM}_0(\mathbb{R}^n)$ to arrive at the following statement:

Proposition 5.4.20 *Let $k \in \mathbb{N}_0 \cup \{\infty\}$.*

i.) For $f \in \mathscr{C}^k_0(\mathbb{R}^n)$ and $g \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ we have $f * g \in \mathscr{C}^k(\mathbb{R}^n)$ and

$$\partial^\alpha (f * g) = (\partial^\alpha f) * g \tag{5.4.56}$$

holds for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq k$.

ii.) For every compact subset $K \subseteq \mathbb{R}^n$ and every $\ell \in \mathbb{N}_0$ with $\ell \leq k$ one has

$$\mathrm{p}_{K,\ell}(f * g) \leq \mathrm{p}_{\tilde{K},\ell}(f) \|g\|_{1, \tilde{K}+K} \tag{5.4.57}$$

for all $f \in \mathscr{C}^k_{\tilde{K}}(\mathbb{R}^n)$ with $\tilde{K} \subseteq \mathbb{R}^n$ being compact and $g \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$.

iii.) The convolution yields a separately continuous bilinear map

$$*\colon \mathscr{C}_0^k(\mathbb{R}^n) \times \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^n) \longrightarrow \mathscr{C}^k(\mathbb{R}^n), \tag{5.4.58}$$

inducing a separately continuous bilinear map

$$*\colon \mathscr{C}_0^k(\mathbb{R}^n) \times \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n) \longrightarrow \mathscr{C}^k(\mathbb{R}^n), \tag{5.4.59}$$

PROOF: We know from Corollary 5.4.10 that for $f \in \mathscr{C}_0^k(\mathbb{R}^n)$ the partial derivatives of f can be computed as limit in the $\mathscr{C}_0$-topology of the difference quotients, i.e.

$$\frac{\partial f}{\partial x^i} = \lim_{t \to 0} \frac{1}{t}(\tau_{-t\mathrm{e}_i}(f) - f),$$

where we assume $k \geq 1$ to avoid trivialities. Thus

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{t}(\tau_{-t\mathrm{e}_i}(f * g) - f * g) \\
&\quad = \lim_{t \to 0} \frac{1}{t}(\tau_{-t\mathrm{e}_i} I_g(\tau_\bullet(\check{f})) - I_g(\tau_\bullet(\check{f}))) \\
&\quad \stackrel{(a)}{=} \lim_{t\to 0} I_g\Big(\frac{1}{t}(\tau_{\bullet + t\mathrm{e}_i}(\check{f}) - \tau_\bullet(\check{f}))\Big) \\
&\quad \stackrel{(b)}{=} I_g\Big(\lim_{t\to 0} \frac{1}{t}(\tau_{\bullet + t\mathrm{e}_i}(\check{f}) - \tau_\bullet(\check{f}))\Big) \\
&\quad \stackrel{(c)}{=} I_g\Big(\tau_\bullet\Big(\lim_{t\to 0}\frac{1}{t}(\tau_{t\mathrm{e}_i}(\check{f}) - \check{f})\Big)\Big) \\
&\quad = I_g\Big(\tau_\bullet\Big(-\frac{\partial \check{f}}{\partial x^i}\Big)\Big) \\
&\quad \stackrel{(d)}{=} I_g\Big(\tau_\bullet\Big(\frac{\partial f}{\partial x^i}\Big)^{\check{}}\Big) \\
&\quad = \frac{\partial f}{\partial x^i} * g,
\end{aligned}$$

where in (a) we use the fact that $(\tau_{-a}h)(x) = h(x + a)$ for any function h, in our case the function $x \mapsto \tau_x(\check{f})$. In (b) we use the fact that the limit exists in the $\mathscr{C}_0$-topology and I_g is $\mathscr{C}_0$-continuous. Next, $\tau_{x+t\mathrm{e}_i} = \tau_x \circ \tau_{t\mathrm{e}_i}$ by the representation property and thus (c) follows from the continuity of each τ_x in the $\mathscr{C}_0$-topology. Finally, in (d) we use the chain rule for the reflection map. This shows the first part for $k = 1$, since every $\frac{\partial f}{\partial x^i} * g$ is again continuous by Theorem 5.4.18, *v.)*. A simple induction on k gives the general case. Now suppose $f \in \mathscr{C}^k_{\tilde{K}}(\mathbb{R}^n)$. Then we get for every multiindex $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ the estimate

$$\mathrm{p}_{K,0}(\partial^\alpha(f * g)) \stackrel{i.)}{=} \mathrm{p}_{K,0}((\partial^\alpha f) * g) = \mathrm{p}_{\tilde{K},0}((\partial^\alpha f)) \|g\|_{1,\tilde{K}+K}$$

by the estimate (5.4.53) from Theorem 5.4.18, *viii.)*. Taking now the maximum over $|\alpha| \leq \ell$ gives (5.4.57). Then the third part follows, too. □

Again, we see that the better regularity properties of each factor are inherited by their convolution. The next scenario also supports this general idea:

Proposition 5.4.21 *Let $k, \ell \in \mathbb{N}_0 \cup \{\infty\}$.*

i.) For $f \in \mathscr{C}_0^k(\mathbb{R}^n)$ and $g \in \mathscr{C}^\ell(\mathbb{R}^n)$ we have $f * g \in \mathscr{C}^{k+\ell}(\mathbb{R}^n)$ and

$$\partial^{\alpha+\beta}(f * g) = (\partial^\alpha f) * (\partial^\beta g) \tag{5.4.60}$$

for $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ and $|\beta| \leq \ell$.

ii.) *For all compact subsets* $K, \tilde{K} \subseteq \mathbb{R}^n$ *one has*

$$\mathrm{p}_{K,k+\ell}(f * g) \le \operatorname{vol}(K + \tilde{K}) \mathrm{p}_{\tilde{K},k}(f) \mathrm{p}_{\tilde{K}+K,\ell}(g) \tag{5.4.61}$$

for $f \in \mathscr{C}^k_{\tilde{K}}(\mathbb{R}^n)$ *and* $g \in \mathscr{C}^\ell(\mathbb{R}^n)$.

iii.) *The convolution yields a separately continuous bilinear map*

$$* \colon \mathscr{C}^k_0(\mathbb{R}^n) \times \mathscr{C}^\ell(\mathbb{R}^n) \longrightarrow \mathscr{C}^{k+\ell}(\mathbb{R}^n). \tag{5.4.62}$$

PROOF: With the statement of Proposition 5.4.20, *i.)*, we get

$$\partial^\alpha (f * g) = (\partial^\alpha f) * g$$

already, where $|\alpha| \le k$. Then $\partial^\alpha f \in \mathscr{C}_0(\mathbb{R}^n)$ is still continuous with compact support. Hence we only need to show (5.4.60) for $k = 0$ and thus $\alpha = 0$, but for general $\beta \in \mathbb{N}_0^n$ with $|\beta| \le \ell$. Here we argue as in the proof of Proposition 5.4.20, *i.)*, but with the roles of f and g exchanged. Now we use Corollary 5.4.8 to show that

$$\lim_{t \to 0} \frac{1}{t} (\tau_{-te_i}(f * g) - f * g) = I_f\Big(\tau_\bullet \Big(\frac{\partial g}{\partial x^i}\Big)\check{}\Big) = f * \Big(\frac{\partial g}{\partial x^i}\Big)$$

along the same lines of computation as in the proof of Proposition 5.4.20, *i.)*, with the only difference that I_f is $\mathscr{C}_0$-continuous and the limit is in the $\mathscr{C}_0$-topology. By induction on ℓ this settles the first assertion. For the second statement, we use Proposition 5.4.20, *ii.)*, to get

$$\mathrm{p}_{K,0}\big((\partial^\alpha f) * (\partial^\beta g)\big) \le \mathrm{p}_{\tilde{K},0}(\partial^\alpha f) \|\partial^\beta g\|_{1,\tilde{K}+K} \le \operatorname{vol}(\tilde{K} + K) \mathrm{p}_{\tilde{K},0}(\partial^\alpha f) \mathrm{p}_{K,0}(\partial^\beta g)$$

by the usual estimate between a local L^1-seminorm and the corresponding local supnorm. Since $K + \tilde{K}$ is compact, the volume is indeed finite. Taking then the maximum over $|\alpha| \le k$ and $|\beta| \le \ell$ and using the first part shows (5.4.61). But then the last statement is clear. □

Proposition 5.4.22 *Let* $k, \ell \in \mathbb{N}_0 \cup \{\infty\}$. *For* $f \in \mathscr{C}^k_0(\mathbb{R}^n)$ *and* $g \in \mathscr{C}^\ell(\mathbb{R}^n)$ *the convolution* $f * g \in \mathscr{C}^{k+\ell}(\mathbb{R}^n)$ *coincides with the Riemann integral*

$$f * g = \int_{\mathbb{R}^n} f(y) \tau_y(g) \, \mathrm{d}^n y. \tag{5.4.63}$$

PROOF: First we note that $y \mapsto \tau_y(g)$ is a continuous function on $\mathbb{R}^n$ with values in $\mathscr{C}^\ell(\mathbb{R}^n)$ by Proposition 5.4.7 with respect to the $\mathscr{C}^\ell$-topology. In fact, it is even $\mathscr{C}^\ell$ itself. Since f has compact support, the product function $y \mapsto f(y)\tau_y(g)$ has compact support, too. Since the target space $\mathscr{C}^\ell(\mathbb{R}^n)$ is complete, the function is Riemann integrable by Theorem **??**. Note that we do not need an improper Riemann integral, since the compact support allows us to integrate over a large enough compact interval in $\mathbb{R}^n$. This shows that the right hand side is a well-defined element of $\mathscr{C}^\ell(\mathbb{R}^n)$. Since for each $x \in \mathbb{R}^n$ the δ-functional δ_x is a continuous linear functional on $\mathscr{C}^\ell$ we get

$$\begin{aligned} \delta_x\bigg(\int_{\mathbb{R}^n} f(y)\tau_y(g) \, \mathrm{d}^n y\bigg) &= \int_{\mathbb{R}^n} \delta_x(f(y)\tau_y(g)) \, \mathrm{d}^n y \\ &= \int_{\mathbb{R}^n} f(y)\tau_y(g)\Big|_x \, \mathrm{d}^n y \\ &= \int_{\mathbb{R}^n} f(y) g(x - y) \, \mathrm{d}^n y \\ &= (f * g)(x), \end{aligned}$$

according to Proposition **??**. □

If we replace the $\mathscr{L}^1_{\mathrm{loc}}(X)(\mathbb{R}^n)$ factor by some globally integrable function, we can improve the continuity estimates of the convolution once more. Here the following situation is of interest:

Proposition 5.4.23 *Let $p \in [1, \infty)$ with conjugate index $q \in (1, \infty]$.*

i.) The pointwise convolution integral

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}^n y \tag{5.4.64}$$

is well-defined for $f \in \mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ and $g \in \mathscr{L}^q(\mathbb{R}^n, \mathrm{d}^n x)$ and all $x \in \mathbb{R}^n$ and coincides with

$$(g * f)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}^n y. \tag{5.4.65}$$

ii.) For $f \in \mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ and $g \in \mathscr{L}^q(\mathbb{R}^n, \mathrm{d}^n x)$ one has

$$(f * g)(x) = I_f(\tau_x(\check{g})) = I_g(\tau_x(\check{f})). \tag{5.4.66}$$

In particular, $f * g = 0$ if one function is a zero function.

iii.) The convolution $f * g$ of $f \in \mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ and $g \in \mathscr{L}^q(\mathbb{R}^n, \mathrm{d}^n x)$ is a continuous bounded function

$$f * g \in \mathscr{C}_b(\mathbb{R}^n) \tag{5.4.67}$$

with

$$\|f * g\|_\infty \leq \|f\|_p \|g\|_q. \tag{5.4.68}$$

iv.) The convolution yields a continuous bilinear map

$$*\colon \mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x) \times \mathscr{L}^q(\mathbb{R}^n, \mathrm{d}^n x) \longrightarrow \mathscr{C}_b(\mathbb{R}^n), \tag{5.4.69}$$

inducing a continuous bilinear map

$$*\colon \mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x) \times \mathrm{L}^q(\mathbb{R}^n, \mathrm{d}^n x) \longrightarrow \mathscr{C}_b(\mathbb{R}^n). \tag{5.4.70}$$

PROOF: We know that for $g \in \mathscr{L}^q(\mathbb{R}^n, \mathrm{d}^n x)$ the function $\tau_x(\check{g})$ is still in $\mathscr{L}^q(\mathbb{R}^n, \mathrm{d}^n x)$ and has the same $\mathscr{L}^q$-seminorm, see Proposition 5.4.11, *i.)*. Since we have a pairing of $\mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ with $\mathscr{L}^q(\mathbb{R}^n, \mathrm{d}^n x)$, the integral (5.4.64) is well-defined. Analogously, one gets (5.4.65) and the two coincide by the usual change of variables $z = x - y$. The second part is then just a re-interpretation. Since for a zero function f or g we have $I_f = 0$ or $I_g = 0$, respectively, the other assertion in *ii.)* follows as well. The third part is interesting: we know from Proposition 5.4.11 that the translations act continuously on $\mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ by isometric maps τ_x such that $x \mapsto \tau_x(\check{f})$ is continuous for all $f \in \mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$. Since I_g is a *continuous* linear functional on $\mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ with functional norm $\|g\|_q$ we get

$$|(f * g)(x)| = |I_g(\tau_x(\check{f}))| \leq \|g\|_q \|\tau_x(\check{f})\|_p = \|g\|_q \|f\|_p$$

for all $x \in \mathbb{R}^n$, proving (5.4.67). Then the fourth part is clear by the estimate (5.4.68). Indeed, we will see later in Section **??** that such an estimate suffices to check continuity for bilinear maps, a fact which can easily be checked by hand in this particular situation. □

If we have compact support and continuity for one of the functions in addition, then we can preserve the integrability features. This can be seen most easily using the vector-valued Riemann integral:

Proposition 5.4.24 *Let $p \in [1, \infty)$ and $k \in \mathbb{N}_0 \cup \{\infty\}$.*

i.) *For $f \in \mathscr{C}^k_0(\mathbb{R}^n)$ and $g \in \mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ one has $f * g \in \mathscr{C}^k(\mathbb{R}^n)$ and the convolution coincides with the vector valued Riemann integral*

$$f * g = \int_{\mathbb{R}^n} f(y)\tau_y(g)\, \mathrm{d}^n y \tag{5.4.71}$$

ii.) *For all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le k$ one has*

$$\partial^\alpha (f * g) = (\partial^\alpha f) * g \in \mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x). \tag{5.4.72}$$

iii.) *For all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le k$ and for all compact subsets $K \subseteq \mathbb{R}^n$ there exists a constant $c_K > 0$ such that*

$$\|\partial^\alpha (f * g)\|_p \le c_K \|\partial^\alpha f\|_\infty \|g\|_p \tag{5.4.73}$$

for all $f \in \mathscr{C}^k_K(\mathbb{R}^n)$ and $g \in \mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$.

Proof: Since $\mathscr{L}^p(\mathbb{R}^n, \mathrm{d}^n x) \subseteq \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}^n x)$, we have $f * g \in \mathscr{C}^k(\mathbb{R}^n)$ by Proposition 5.4.20, *i.)*, we have to focus on (5.4.71). First we recall that $y \mapsto \tau_y(g)$ is a continuous map into $\mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ by Proposition 5.4.11, *v.)*. Hence also $y \mapsto f(y)\tau_y(g)$ is a continuous map from $\mathbb{R}^n$ to $\mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$, now with compact support thanks to f. This makes the Riemann integral in (5.4.71) well-defined, since $\mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$, being a Banach space, is complete. This is also the reason why we pass to equivalence classes of functions, since otherwise we would have an ambiguity, which limit one should take. Now fix a function $\varphi \in \mathrm{L}^q(\mathbb{R}^n, \mathrm{d}^n x)$ with $q \in (1, \infty]$ dual to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then I_φ is a continuous linear functional on $\mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ and, in fact, all are of this type. Hence the continuity of the $\mathscr{L}^p$-valued Riemann integral gives

$$I_\varphi\biggl(\int_{\mathbb{R}^n} f(y)\tau_y(g)\, \mathrm{d}^n y\biggr) = \int_{\mathbb{R}^n} f(y) I_\varphi(\tau_y(g))\, \mathrm{d}^n y$$

by Proposition **??**. Note that we can not directly use a δ-functional as in the proof of Proposition 5.4.23, since they are *not* continuous on $\mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$. Nevertheless, we have

$$I_\varphi(\tau_y(g)) = \int_{\mathbb{R}^n} (\varphi\tau_y(g))\Big|_z\, \mathrm{d}^n y = \int_{\mathbb{R}^n} \varphi(z) g(z - y)\, \mathrm{d}^n z,$$

leading to

$$\begin{aligned} I_\varphi\biggl(\int_{\mathbb{R}^n} f(y)\tau_y(g)\, \mathrm{d}^n y\biggr) &= \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} \varphi(z) g(z-y)\, \mathrm{d}^n z\, \mathrm{d}^n y \\ &= \int_{\mathbb{R}^n} \varphi(z) \int_{\mathbb{R}^n} f(y) g(z-y)\, \mathrm{d}^n y\, \mathrm{d}^n z \\ &= I_\varphi(f * g), \end{aligned} \tag{$*$}$$

by the usual Fubini Theorem. Indeed, first note that f has compact support and hence the absolute values $|f(y)\varphi(z)g(z-y)|$ are integrable over $\mathbb{R}^n \times \mathbb{R}^n$. Next, we only need $y \in \operatorname{supp}(f)$, which is compact. Thus

$$\begin{aligned} &\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(y)\varphi(z)g(z-y)|\, \mathrm{d}^n y\, \mathrm{d}^n z \\ &\le \|f\|_\infty \int_{\operatorname{supp}(f) \times \mathbb{R}^n} |\varphi(z) g(z-y)| \\ &\le \|f\|_\infty \sqrt[q]{\int_{\operatorname{supp}(f) \times \mathbb{R}^n} |\varphi(z)|^q\, \mathrm{d}^n y\, \mathrm{d}^n z}\, \sqrt[p]{\int_{\operatorname{supp}(f) \times \mathbb{R}^n} |g(z-y)|^p\, \mathrm{d}^n y\, \mathrm{d}^n z} \end{aligned}$$

$$\leq \|f\|_\infty \sqrt[q]{\mathrm{vol}(\mathrm{supp}(f))}\|\varphi\|_q \sqrt[p]{\mathrm{vol}(\mathrm{supp}(f))}\|g\|_p$$

by Hölder's inequality and the fact that the translations act isometrically on $\mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n z)$. Hence we indeed can apply Fubini's theorem in $(*)$. Since the continuous linear functional I_φ exhaust the topological dual of $\mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$ and we are in a Hausdorff situation, this proves the equality in (5.4.71). In particular, $f * g \in \mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$. Since for $f \in \mathscr{C}^k_0(\mathbb{R}^n)$ we already know $f * g \in \mathscr{C}^k(\mathbb{R}^n)$ with $\partial^\alpha(f * g) = (\partial^\alpha f) * g$ according to Proposition 5.4.20, *i.)*, we can successively apply the first statement to all partial derivatives $\partial^\alpha f$ of f as long as $|\alpha| \leq k$. This gives (5.4.72). The continuity of the Riemann integral then leads to

$$\begin{aligned}
\|\partial^\alpha(f * g)\|_p &\stackrel{i.)}{=} \left\| \int_{\mathbb{R}^n} (\partial^\alpha f)(y) \tau_y(g) \,\mathrm{d}^n y \right\|_p \\
&\leq \int_{\mathbb{R}^n} |(\partial^\alpha f)(y)| \|\tau_y(g)\|_p \,\mathrm{d}^n y \\
&\leq \|\partial^\alpha f\|_\infty \int_K \|\tau_y(g)\|_p \,\mathrm{d}^n y \\
&= \|\partial^\alpha f\|_\infty \int_K \|g\|_p \,\mathrm{d}^n y \\
&= \|\partial^\alpha f\|_\infty \mathrm{vol}(K) \|g\|_p,
\end{aligned}$$

since the representation τ is isometric on $\mathrm{L}^p(\mathbb{R}^n, \mathrm{d}^n x)$. □

Since $\mathscr{C}^k_0(\mathbb{R}^n)$ is contained in $\mathscr{L}^q(\mathbb{R}^n, \mathrm{d}^n x)$ for all $q \in [1, \infty]$, we can combine the estimates (5.4.73) and (5.4.68) to get additional estimates for these convolutions, see also Exercise 5.5.42.

5.4.3 Convolution with Distributions

5.4.4 Convolution Products of Distributions

5.4.5 Algebras and Modules

5.5 Exercises

Exercise 5.5.1 (Seminorms for $\mathscr{C}^\infty_0(X)$)

Exercise 5.5.2 (Test function functor) Reinterpret Proposition 5.1.3 as a functoriality statement.

Exercise 5.5.3 (Less regular test functions) Let $X \subseteq \mathbb{R}^n$ be open and $k \in \mathbb{N}_0$.

i.) Let $Y \subseteq \mathbb{R}^m$ be another open set and $\phi\colon X \longrightarrow Y$ be a proper $\mathscr{C}^k$-map. Show that the pull-back

$$\phi^*\colon \mathscr{C}^k_0(Y) \longrightarrow \mathscr{C}^k_0(X) \tag{5.5.1}$$

is a continuous map.

ii.) Let $D \in \mathrm{DiffOp}(X)$ of regularity $\ell \geq k$ and with order $r \leq k$. Show that it is a continuous map

$$D\colon \mathscr{C}^k_0(X) \longrightarrow \mathscr{C}^k_0[k-r](X). \tag{5.5.2}$$

Exercise 5.5.4

Exercise 5.5.5

Exercise 5.5.6 (Principal value I)

Exercise 5.5.7 (Discontinuous linear functionals on $\mathscr{C}_0^\infty(X)$)

Exercise 5.5.8 (Heat kernel)

Exercise 5.5.9

Exercise 5.5.10

Exercise 5.5.11 (The essential support)

Exercise 5.5.12

Exercise 5.5.13 (Singular supports)

Exercise 5.5.14 (Order of distributions)

Exercise 5.5.15

Exercise 5.5.16

Exercise 5.5.17 (Principal value II)

Exercise 5.5.18 (Dipoles)

Exercise 5.5.19 (Quadrupoles)

Exercise 5.5.20 (Products $f\delta^k$)

Exercise 5.5.21 (Partition of unity in strong sense)

Exercise 5.5.22 (Proper smooth maps)

Exercise 5.5.23 (Push-forward of positive Borel measure) Formulate and prove a version of Proposition 5.2.23 for positive Borel measures μ, such that $\mu(K) < \infty$ for all compact K.

Exercise 5.5.24 (Distributions on submanifolds)

Exercise 5.5.25 Prove (5.2.104).

Exercise 5.5.26

Exercise 5.5.27 (Ordinary Differential Equations)

Exercise 5.5.28 Formulate and prove Theorem 5.2.38 for incomplete vector fields V.

Exercise 5.5.29 (Translations commute with differentiation)

Exercise 5.5.30 (Differentiating determinants)

Exercise 5.5.31

Exercise 5.5.32 (Dirac comb)

Exercise 5.5.33

i.) $\mathscr{C}^k(X)$-module structure of $\mathscr{C}^{-k}(X)$.

ii.) Continuity of differential operators of low enough degree.

iii.) push-forwards and pull-backs.

Exercise 5.5.34 (Distribution of infinite order) Let $X \subseteq \mathbb{R}^n$ be a non-empty open subset. Construct a distribution of infinite order on X.

Exercise 5.5.35

Exercise 5.5.36 Let $x \in X \subseteq \mathbb{R}^n$ for some open subset X and $k \in \mathbb{N}$. Show that the set

$$\left\{\partial^\alpha \delta_x \;\middle|\; |\alpha| \leq k\right\} \subseteq \mathscr{C}^{-\infty}(X) \tag{5.5.3}$$

is linearly independent.

Exercise 5.5.37 (Cauchy-Schwarz inequality)

Exercise 5.5.38 (Bounded uniformly continuous functions)

Exercise 5.5.39

Exercise 5.5.40

Exercise 5.5.41 Let V be a topological vector space. Show that for a closed subset $C \subseteq V$ and a compact subset $K \subseteq V$ their sum $C + K$ is again closed in V.

Exercise 5.5.42

Exercise 5.5.43 (No surjection $\mathscr{C}^\infty(X) \longrightarrow \mathscr{C}^\infty_0(Y)$) Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be non-empty open subsets. Show that there is no continuous surjection $\mathscr{C}^\infty(X) \longrightarrow \mathscr{C}^\infty_0(Y)$.

Hint: Corollary 3.4.10.

Exercise 5.5.44 (Boundary values of holomorphic functions)

Exercise 5.5.45 (Order of distributions)

Exercise 5.5.46 (The LB space $\mathrm{L}^1_0(X)$)

Exercise 5.5.47 (Invariant distributions)

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