Homework for the Lecture

Functional Analysis

Stefan Waldmann
Christopher Rudolph

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Homework Sheet No 13

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(24 Points. Discussion 27. 01. 2025)

Homework 13-1: Compact Operators

Let V, W be Banach spaces and $K : V \to W$ be a linear map. We call K compact if the image of the closed unit Ball lies within a compact subset of W, i.e. $K\left(\mathrm{B}_1(0)^{\mathrm{cl}}\right)^{\mathrm{cl}} \subset W$ is compact.

i.) **(1 Point)** Show that every compact operator is continuous.

ii.) **(5 Points)** Characterize a compact operator in terms of sequences.

iii.) **(2 Points)** Show that every finite rank operator is compact.

iv.) **(5 Points)** Show that the set $\mathfrak{K}(V, W)$ of compact operators is a closed subspace of $L(V, W)$ endowed with the operator norm topology.

v.) **(1 Point)** Prove the following: The identity id_V is compact iff V is finite dimensional.

vi.) **(1 Point)** Conclude that id_V cannot be a limit of finite rank operators (with respect to the operator norm topology) if V has infinite dimension.

vii.) **(2 Points)** Now, assume $W = V$. Show that for every $K \in \mathfrak{K}(V) := \mathfrak{K}(V, V)$ and $A \in L(V)$ one has $A \circ K \in \mathfrak{K}(V)$ and $K \circ A \in \mathfrak{K}(V)$.

Homework 13-2: Completeness Relation in Some Operator Topologies

(3 Points) Consider a Hilbert space $(\mathfrak{H}, \langle \cdot, \cdot \rangle)$ with a countable Hilbert basis $(e_n)_{n \in \mathbb{N}} \subset \mathfrak{H}$. Let $P_n \in \mathscr{B}(\mathfrak{H})$ denote the projection onto the subspace spanned by e_n, i.e. $P_n \phi := \langle e_n, \phi \rangle e_n$. Check if the sequence $(\sum_{n=1}^{N} P_n)_{N \in \mathbb{N}}$ converges to $\mathbb{1}$ with respect to the

- weak topology
- strong topology
- operator norm topology.

Also study convergence of the sequence $(P_n)_{n\in\mathbb{N}}$.

Homework 13-3: The Operator Product

(4 Points) Let $(\mathfrak{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. For two bounded linear operators $A, B \in \mathscr{B}(\mathfrak{H})$, we define their operartor product by

$$m(A, B) := AB := A \circ B. \tag{13.1}$$

Study (separate) continuity of m with respect to the

- weak topology
- strong topology
- operator norm topology.