Doubt Clearing Session

Course on Mathematics & Puzzles for Interview Preparation

Arjun Arul • Lesson 4 • Feb 14, 2021

Maths and Puzzles

Lecture 2

CONCEPT

Binary Exponentiation

Recurrence Relation

$$a^n = \begin{cases} 1 & n == 0 \\ (a^{n/2})^2 & n \, is \, even \, and \, n > 0 \\ a \cdot (a^{\frac{n-1}{2}})^2 & n \, is \, odd \, and \, n > 0 \end{cases}$$

Recursive Implementation

```
function binpow(a, b) {
    if (b == 0)
        return 1
    let res = binpow(a, b / 2)
    if (b % 2)
        return res * res * a
    else
        return res * res
}
```

Iterative Implementation

```
function binpow(a, b) {
    let res = 1
    while (b > 0) {
        if (b % 2 != 0)
            res = res * a
        a = a * a
        b = b / 2
    }
    return res
}
```

$$3^{13} = 3^{1101_2} = 3^8 \cdot 3^4 \cdot 3^1$$

QUIZ

Binary Exponentiation

1. Best Time complexity of A^N Exponentiation that I can code is -

A. O(N)
B. O(sqrt(N))
C. O(log(N))
D. O(1)

2. Binary Exponentiation of A^B will fail in which case?

A. When A is Prime number
B. When B is Odd power
C. When GCD(A,B) = 1
D. None of the above

CONCEPT

Modular Arithmetic

Properties of Modular Arithmetic

$$(a + b) \mod n = ((a \mod n) + (b \mod n)) \mod n$$

$$(a \cdot b) \mod n = ((a \mod n) \cdot (b \mod n)) \mod n$$

$$(a - b) \mod n = ((a \mod n) - (b \mod n) + n) \mod n$$

$$(a / b) \mod n \neq ((a \mod n) / (b \mod n)) \mod n$$

QUIZ

Modular Arithmetic

3. The value of A mod B always lies in the interval -

A. [0,A-1]
B. [A,B-1]
C. [0,B]
D. [0,B-1]

4. If A belongs to the interval [220,330]. Range of (A mod 100) + 10, will be -

A. [0,99]
B. [0,0]
C. [10,109]
D. [10,99]

5. We can apply chaining modulo for the following calculation -

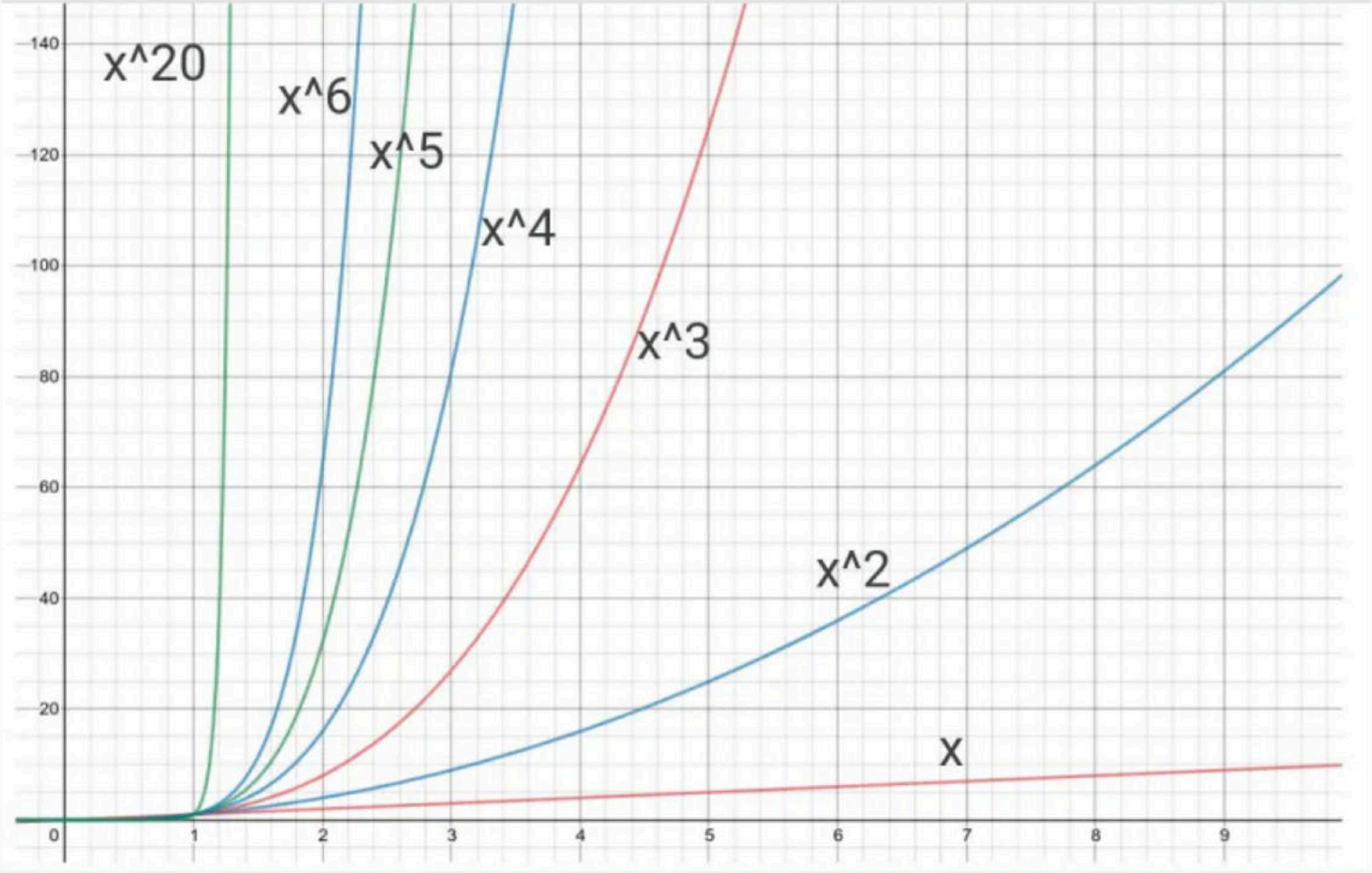
$$(10 - 20 + 24 - 13 - 43) \bmod (17)$$

A. True
B. False

CONCEPT

Modular Exponentiation

Exponential growth comparison



Modular Exponentiation

```
long long binpow(long long a, long long b, long long m) {
    a %= m;
    long long res = 1;
    while (b > 0) {
        if (b % 2 != 0)
            res = (res * a) % m;
        a = (a * a) % m;
        b = b / 2;
    }
    return res;
}
```

CONCEPT

Fermat’s little theorem

Fermat's Little Theorem

If M is a prime number, $m | (a^m - a)$ $\longleftrightarrow$ $a^m \equiv a \mod m$

Which can be written as $a^m \equiv a \mod m$

$\gcd(a, m) = 1$

Therefore, $a^{m-1} \equiv 1 \mod m$

Fermat's Little Theorem is special case of Euler's Theorem, for prime numbers

$$a^{\varphi(m)} \equiv 1 \mod m$$

$\gcd(a, m) = 1$

a

$5, \quad 6$

$\varphi(6)$

$a \equiv 1 \bmod 6$

$a = 5$

$m = 6$

$5^2 \equiv 1 \bmod 6$ ✓

$$\begin{cases} a = 2 \\ m = 5 \end{cases}$$

$a = 3$ $\rightarrow 3^5 = \underline{24}1 \equiv 3 \bmod 5$

$m = 5$ $\quad 3^4 = 81 \equiv 1 \bmod 5$

FLT $\Rightarrow$ $a^m \equiv a \bmod m$

$$2^5 = 32 \equiv 2 \bmod 5$$

$$a^{m-1} \equiv 1 \bmod 5 \rightarrow 2^4 = 16 \equiv 1 \bmod 5$$

$a = 4$

$m = 5$

$a = 10$

$m = 5$

$10^5 \equiv 10 \bmod 5$

$4^5 = 1024 \equiv 4 \bmod 5$

$4^4 = 256 \equiv 1 \bmod 5$

$\rightarrow m = 5$

$a = 2$

a, a^2, a^3, a^4, a^5

$2, 4, 8, 16, 32$

$2, 4, 3, 1, 2$

$m = 7$

$a = 3$

$3, 9, 27, 81, 243, 729,$

2187

$3, 2, 6, 4, 5, 1, 3$

$a = 11$

$m = 5$

$$11^{5} \equiv 11 \bmod 5 \equiv 1 \bmod 5$$

Euler Totient Function

$\varphi(n)$ = The number of integers x in $[1, n]$, such that $\gcd(x, n) = 1$.

$\varphi(6) = \underline{\underline{2}}$

$1, 2, 3, 4, 5, 6$

$\varphi(11) = 10$

$\varphi(n) = \underline{\underline{n-1}}$

n is prime

QUIZ

Fermat’s little theorem

6. Find the value of x

$$a^{18} \equiv x \mod 7$$

A. 0
B. 1
C. a
D. Can't determine

$a^{m-1} \equiv 1 \mod m$

$\boxed{\gcd(a,7)=1}$

$a^{m} \equiv a \mod m$

$a^{n-1} \equiv 1 \mod m$

$a^{7} \equiv a \mod 7$

$\boxed{a^{6} \equiv 1 \mod 7}$

$$a^{12} = a^{6} \times a^{6}$$

$$a^{12} \bmod) = \Big(\underbrace{(a^{6} \bmod)}_{1} \times \underbrace{(a^{6} \bmod)}_{1} \Big) \bmod ()$$

$$= 1 \bmod)$$

$$a^{12} = 1 \times 1 = 1$$

7. Find the value of x

$$p^{(p-1)} \equiv x \mod p$$

A. 0
B. 1
C. p
D. Can't determine

$a^{m-1} \equiv 1 \bmod m$ ✗

$a^{m} \equiv a \bmod m$

CONCEPT

MMI

Modulo Multiplicative Inverse and Calculation

Modulo Multiplicative Inverse

If, $b \cdot x \equiv 1(\mod m)$

x is an inverse of b
wrt m if

x b ≡ 1 mod m

Then, $(a \div b) \mod m \equiv (a \cdot x) \mod m$

If M is a prime number, then MMI of x is $x^{\varphi(m)-1}$

$2 \times \frac{1}{2} = 1$

For a prime 'm', we have $\varphi(m) = m - 1$

$7 \times \frac{1}{7} = 1$

7^{-1}

Therefore, MMI of X under M is x^{m-2}

if m is prime

Inverse of x wrt m is x^{m-2}

$$x \times \boxed{x^{m-2}} = x^{m-1} \equiv 1 \bmod m$$

$$x^{\phi(m)} \equiv 1 \bmod m$$

$x^{\phi(m)-1}$

$x \times x$

$$\boxed{x^{-1} = x^{\varphi(m)-1}}$$

$\gcd(x, m) = 1$

$$\frac{a}{b}$$

$a \times (b^{-1})$

$b^{m-2} \to \log(n)$

$$x^{n-1} \equiv 1 \bmod n$$

$r = 2$

$m = 5$

$\begin{array}{cccc} 1 & 2 & & \\ 3, & 8, & 13, & 18 \end{array}$

$x = 3, 8, 13, 18$

$2x \equiv 1 \mod 5$

$1 \cdot 1$

$4 \times 4 = 16 = 1$

$$7 \times \frac{1}{7} = 1$$

$$7 \times 7^{-1} = 7^{1} \times 7^{-1} = 7^{1+-1} = 7^{0} = 1$$

$$\frac{p}{q} \qquad \frac{q}{p}$$

$$\frac{a}{k}$$

$$a \times \frac{1}{k}$$

$$= a \times k^{-1}$$

ak

$$2x \equiv 1 \bmod 6$$

~~mod~~

$m = 6$

$t = 2$

$\gcd(2, 6) \neq 1$

$2x = 6y + 1$

$2 \times 1 = 2$

1

$2 \times 2 = 4$

$2 \times 3 = 6 = 0$

$2 \times 4 = 8 = 2$

$2 \times 5 = 10 = 4$

$2 \times 6 = 12 = 0$

MMI Implementation

```
Function mmi(a, m) {
    //m here should be prime
    let b = m-2;
    let res = 1;
    while (b > 0) {
        if (b & 1)
            res = res * a;
        a = a * a;
        b = b / 2;
    }
    return res;
}
```

QUIZ

MMI and Calculation

8. Using the procedure explained, we can find MMI of ‘x’ under modulo ‘m’ in following cases

A. When x and m both are prime
B. When x is composite and m is prime
C. When only x is prime
D. When m is prime

9. Time complexity to find the MMI of ‘x’ under modulo ‘m’ will be

A. O(log(m-2))
B. O(m log(m))
C. O(log(x))
D. O(log(x-2))

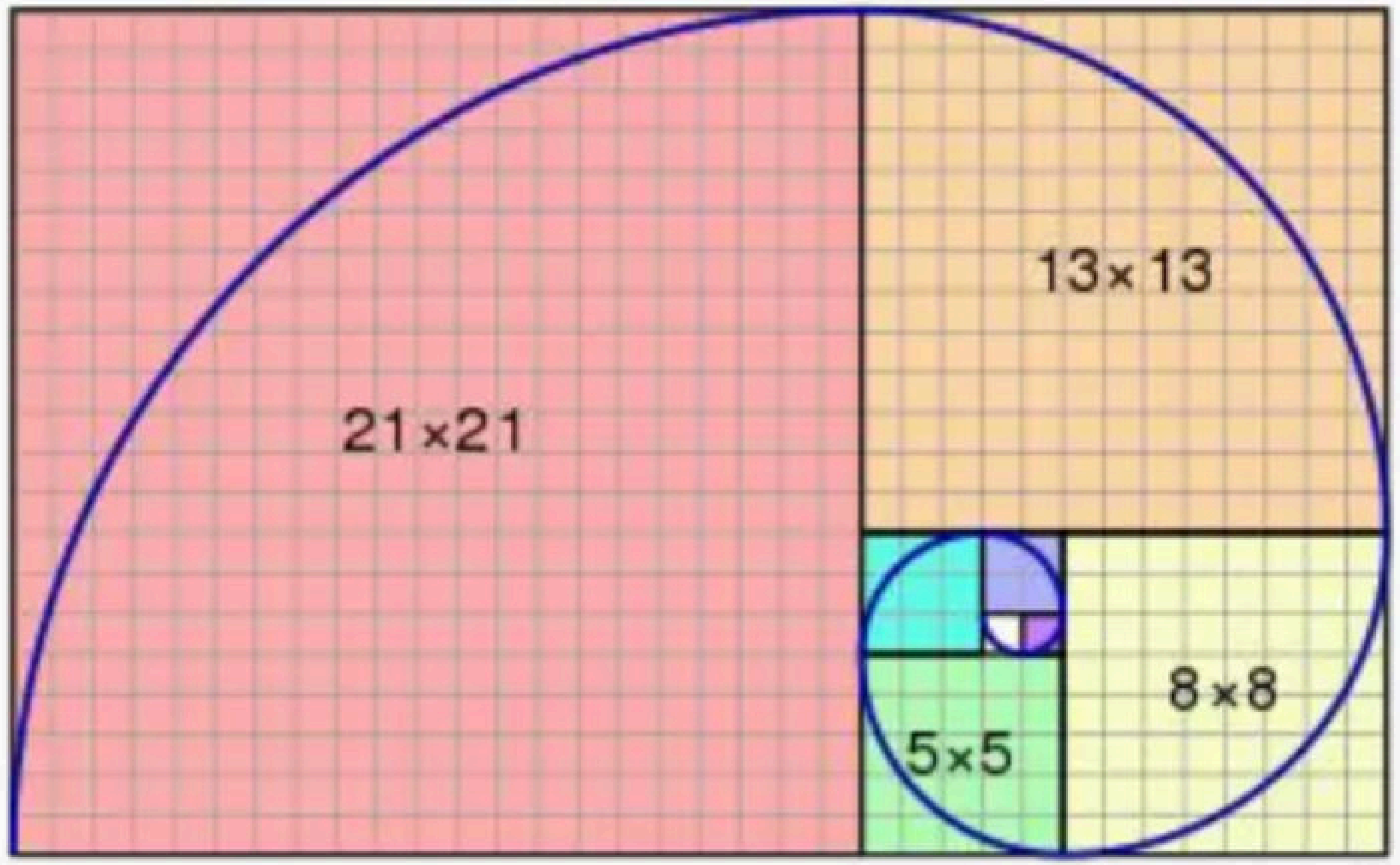
Maths and Puzzles

Lecture 3

CONCEPT

Fibonacci Numbers

Fibonacci Numbers



$$f(0) = 0$$

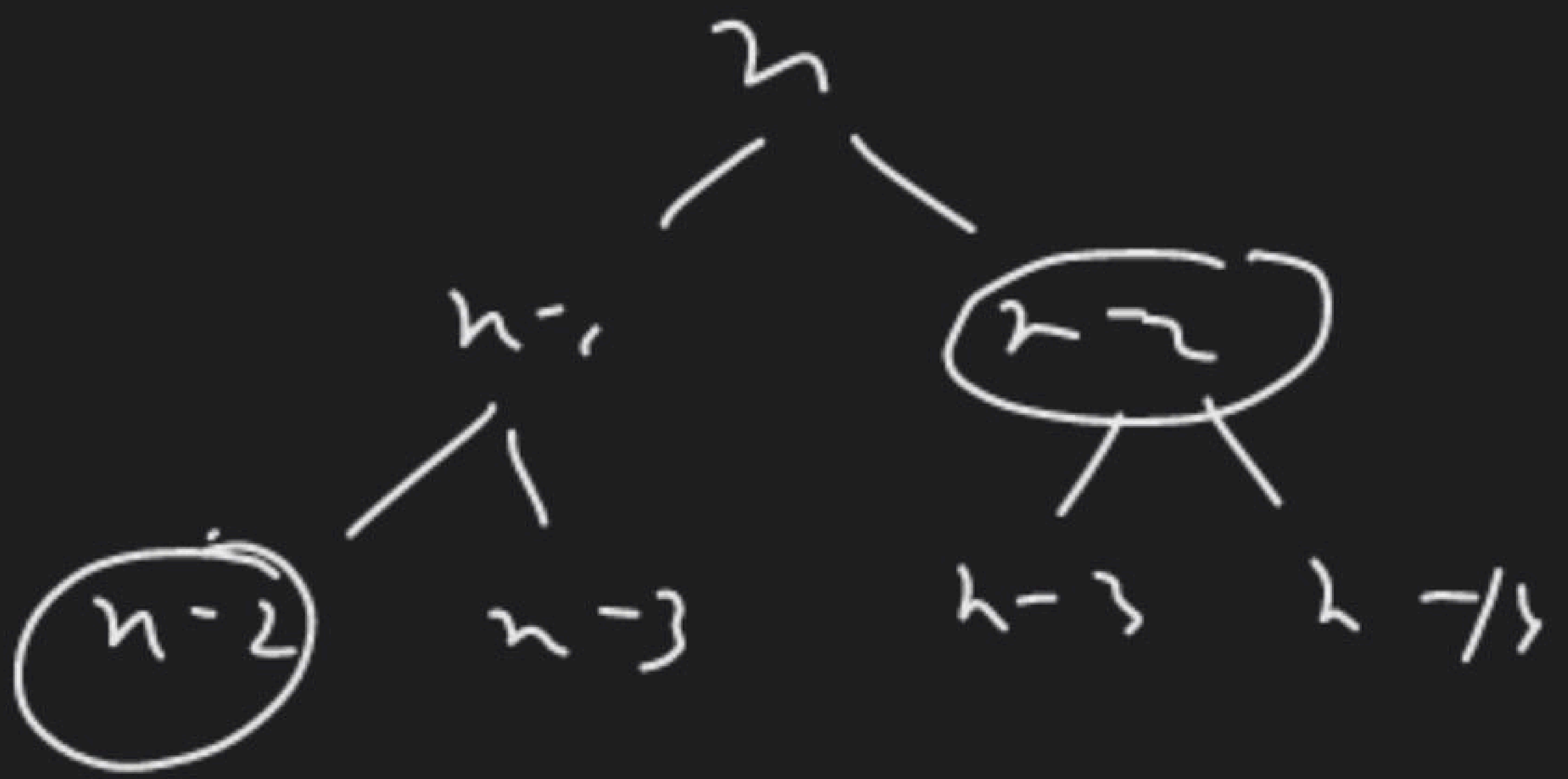
$$f(1) = 1$$

$$f(n) = f(n-1) + f(n-2)$$

0,1,1,2,3,5,8,13,21,34,55,...

Naive Implementation

```
fib(n) :
  if n<=1 :
    return n
  else :
    return fib(n)+fib(n-1)
```



$O(2^n)$

Optimised Implementation

```
fib(n) :
  if n<=1 :
    return n
  else :
    fib_{n-2} <- 0
    fib_{n-1} <- 1
    loop i=2, i<=n, i++:
        fib_n <- fib_{n-1} + fib_{n-2}
        fib_{n-2} <- fib_{n-1}
        fib_{n-1} <- fib_n
    return fib_n
```

$\Theta(n)$

0 1 1

$f_{n-2} = f_{n-1}$

$\underline{O(\log n)}$

$$F(n) = F(n-1) + F(n-2)$$

$$F(n+1) = F(n) + F(n-1)$$

$$\begin{bmatrix} F(n) \\ F(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F(n-1) \\ F(n) \end{bmatrix}$$

$$F(n) = 0 \times F(n-1) + 1 \times F(n)$$

$$\begin{bmatrix} F(1) \\ F(2) \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}} \begin{array}{c} \begin{bmatrix} F(0) \\ F(1) \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \end{array}$$

$$\begin{bmatrix} F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F(1) \\ F(2) \end{bmatrix}$$

$$\begin{bmatrix} F(n) \\ F(n+1) \end{bmatrix} = A \begin{bmatrix} F(n-1) \\ F(n) \end{bmatrix}$$

$$= A\left(A \begin{bmatrix} F(n-2) \\ F(n-1) \end{bmatrix} \right)$$

$$= A^2 \rightarrow \left(A \times \begin{bmatrix} F(n-3) \\ F(n-2) \end{bmatrix} \right) \cdots$$

$$\begin{bmatrix} F(n) \\ F(n+1) \end{bmatrix} = A^n \begin{bmatrix} F(0) \\ F(1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$O(8 \lg n)$$

$$F(n) = a_1 F(n-1) + a_2 F(n-2) + \dots + a_k F(n-k) + C$$

$$F(n) = \begin{bmatrix} 0 & 1 & 0 & - & - & \cdot \\ 0 & 0 & 1 & - & - & - \\ - & - & & & - & - \\ a_1 & a_2 & \dots & \dots & a_k & C \end{bmatrix} \begin{bmatrix} F(n-1) \\ \\ \\ F(n+k-1) \end{bmatrix}$$

$F(n+k)$

$$F(n) = F(n-1) + F(n-2) + \underline{\underline{3}}\ a(n)$$

$$\begin{bmatrix} F(n) \\ F(n+1) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & 0 & L(n) \end{bmatrix} \begin{bmatrix} F(n-1) \\ F(n) \\ 1 \end{bmatrix}$$

Using Matrix Exponentiation

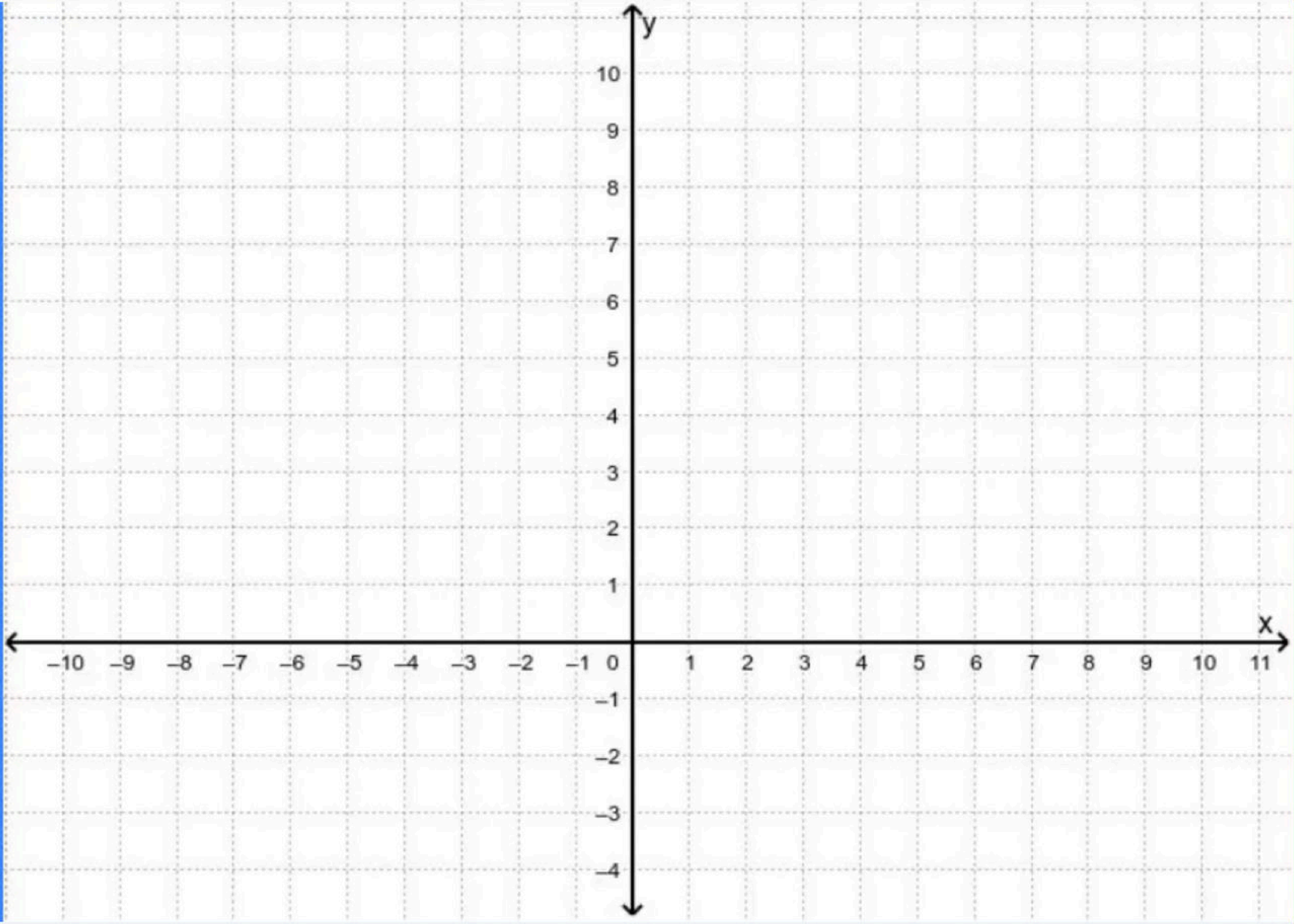
$$\begin{pmatrix} F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} F_{n-2} & F_{n-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} F_0 & F_1 \end{pmatrix} \cdot P^n$$

CONCEPT

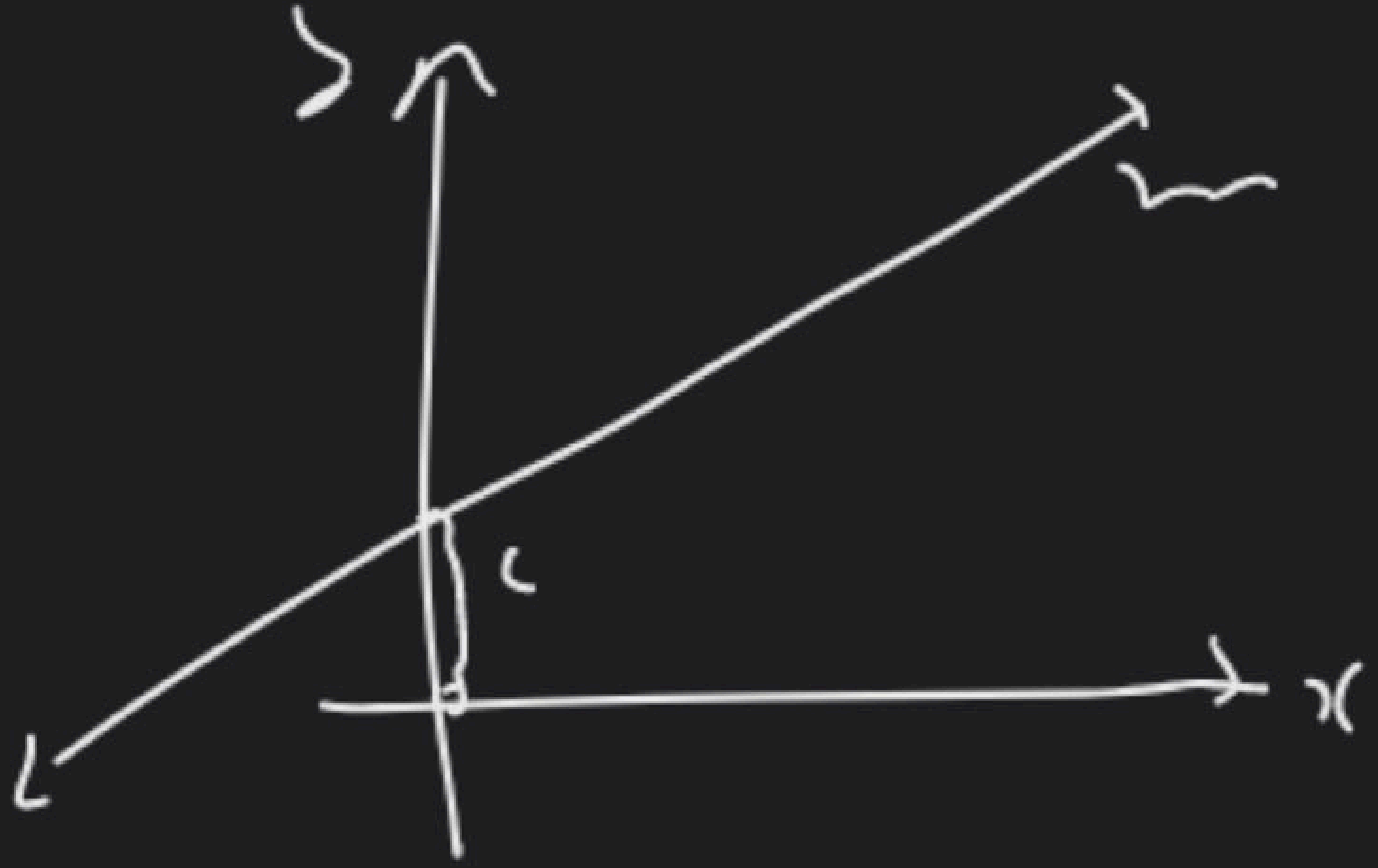
Geometry


y
x
10
9
8
7
6
5
4
3
2
1
0
−1
−2
−3
−4
−10
−9
−8
−7
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6
7
8
9
10
11

Equation of a line

From -

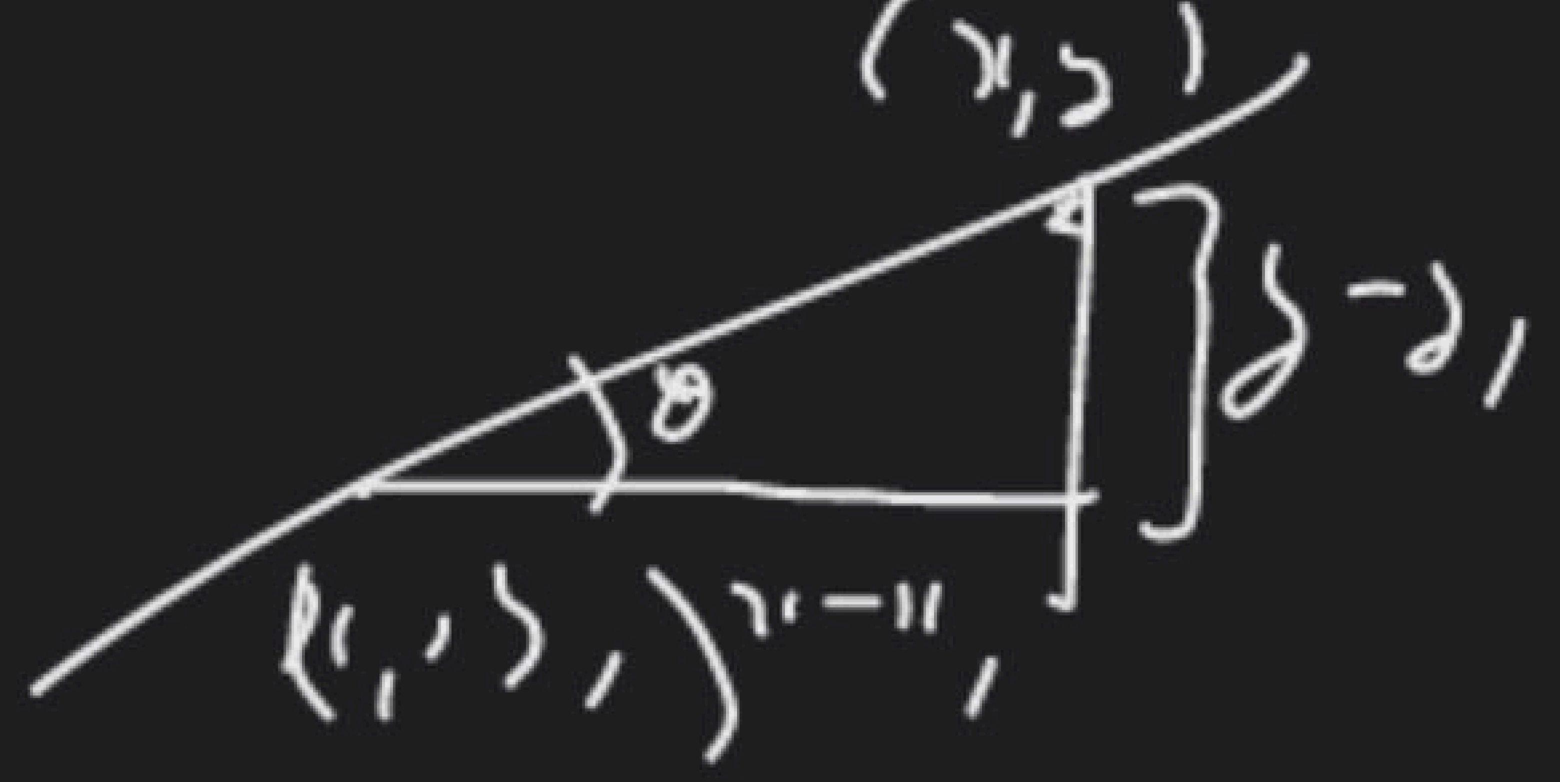
1. Slope & Y Intercept
2. Slope & Point
3. Two points
4. Both intercepts
5. Parallel to ax+by+c=0
6. Perpendicular to ax+by+c=0



$(0, c)$

$$\underline{\underline{y = mx + c}}$$

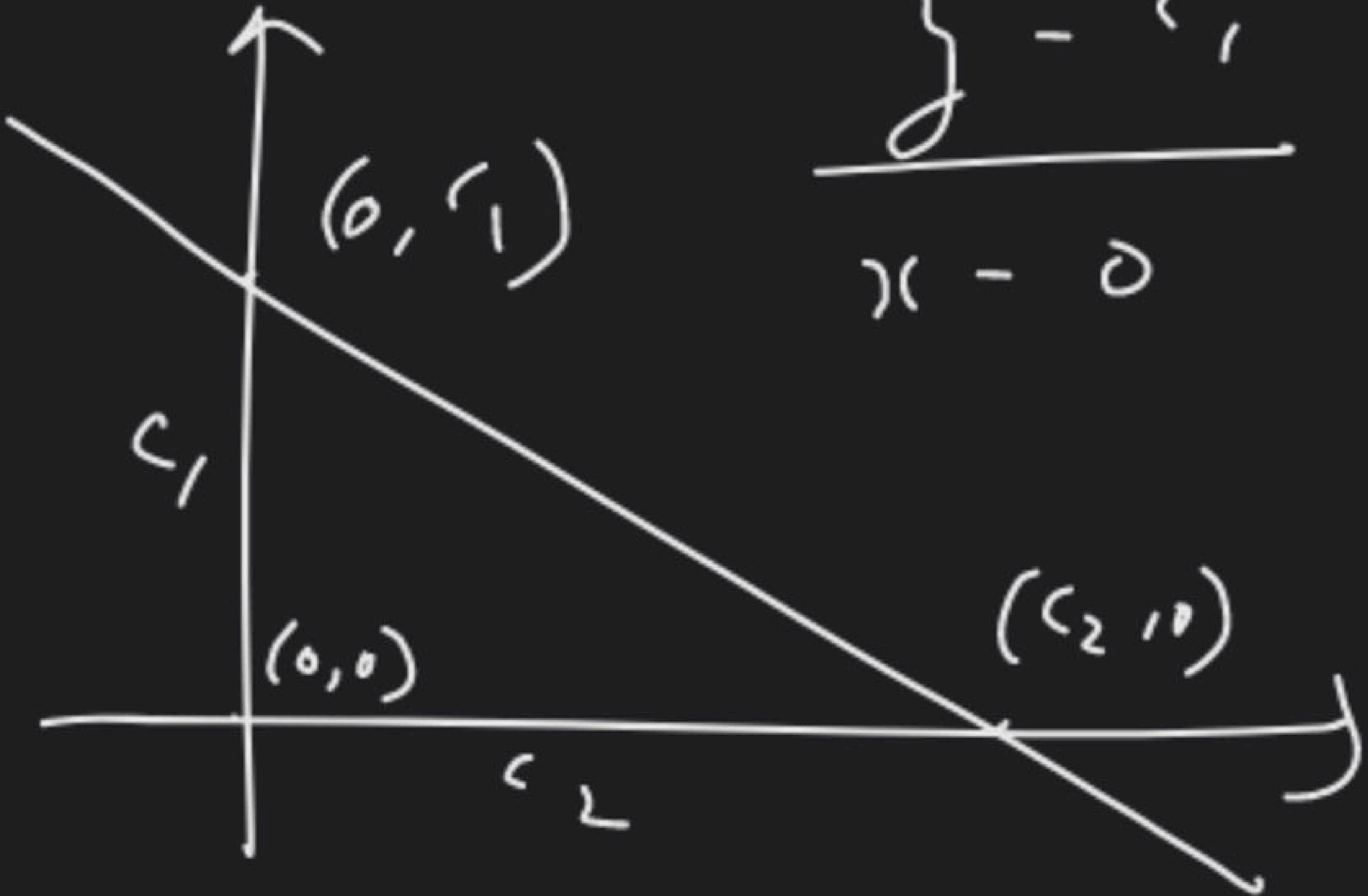
(x_1, y_1)



m

$$\frac{y - y_1}{x - x_1} = m = \frac{y_2 - y_1}{x_2 - x_1}$$


(x_1, y_1)
(x_2, y_2)



$$\frac{y - c_1}{x - 0} = \frac{-c_1}{c_2}$$

$$-\frac{y}{c_1} + 1 = \frac{x}{c_2}$$

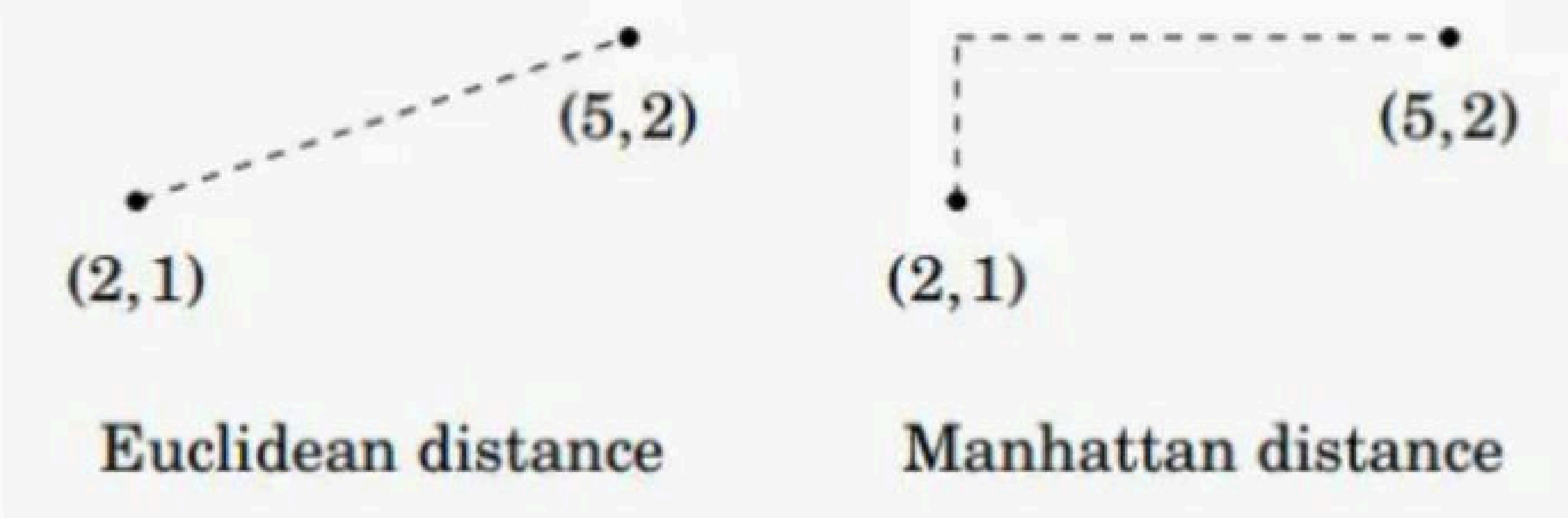
$$\frac{x}{c_2} - \frac{y}{c_1} + 1 = 0 \qquad y - c_1 = \frac{-c_1 x}{c_2}$$

Distances

1. Between two points
   a. Euclidean Distance
   b. Manhattan Distance

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

$$|x_1 - x_2| + |y_1 - y_2|.$$



2. Between line and point

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

Intersection of two lines

Solving simultaneous equation in two variables.

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

$$x = -\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = -\frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

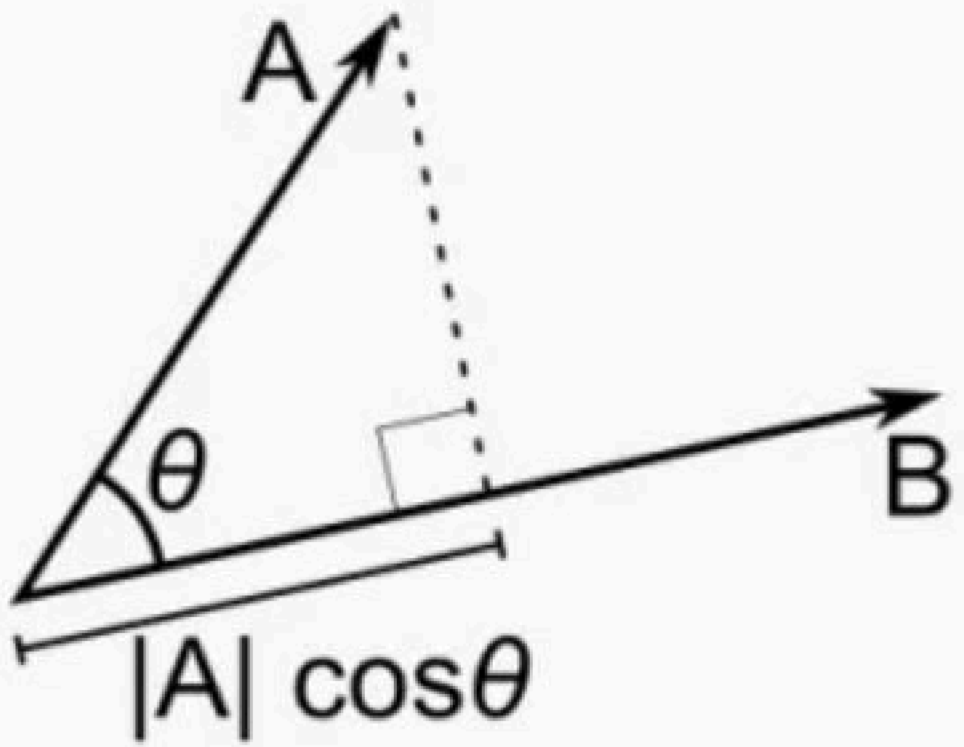
$$y = -\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = -\frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

If lines are parallel or they overlap

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = 0$$

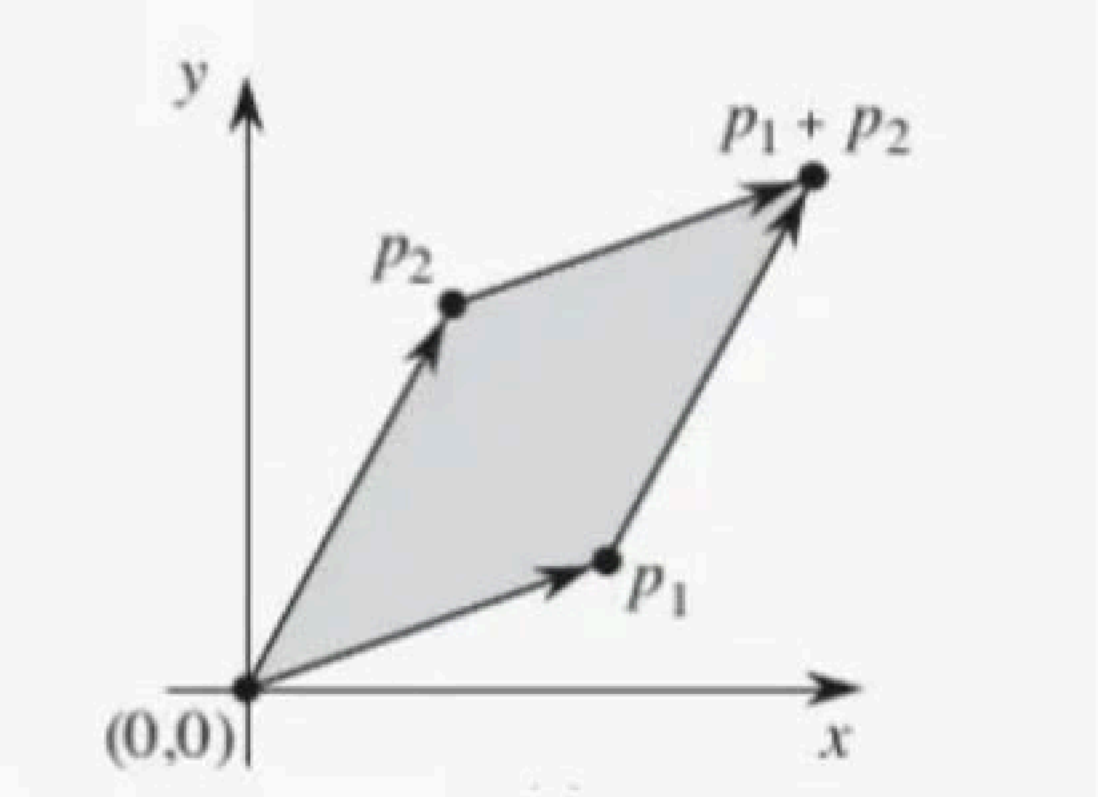
Dot and Cross Product

Dot product



$$\mathbf{a} \cdot \mathbf{b} = x_1 x_2 + y_1 y_2$$

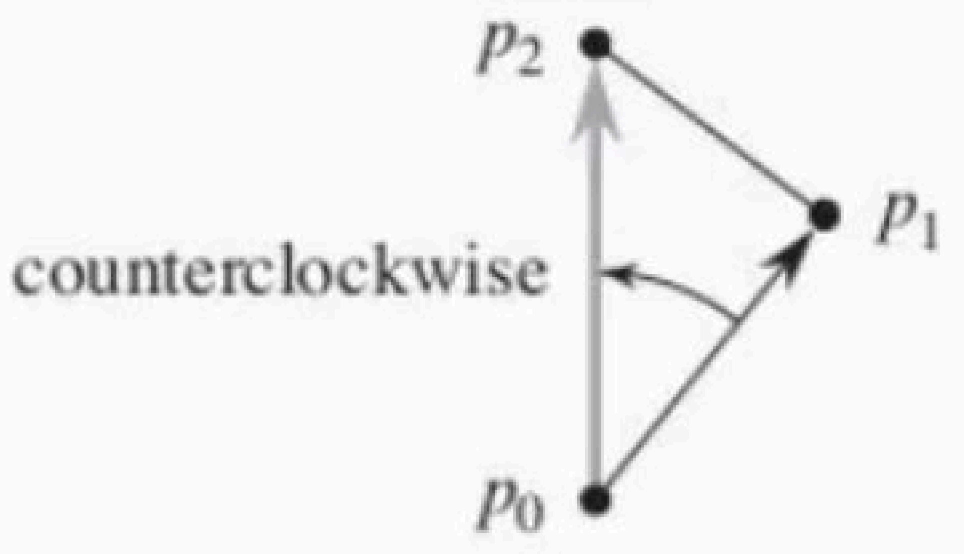
Cross product



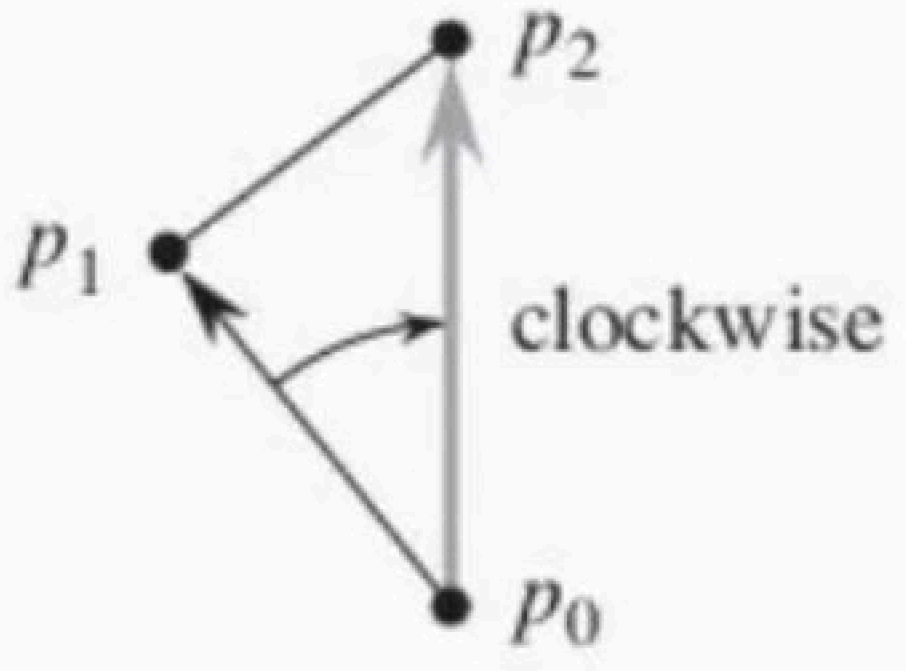
$$\begin{aligned} p_1 \times p_2 &= \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \\ &= x_1 y_2 - x_2 y_1 \\ &= -p_2 \times p_1 \,. \end{aligned}$$

Cross Product and Rotations

$$(p_1 - p_0) \times (p_2 - p_0) = (x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)\ .$$



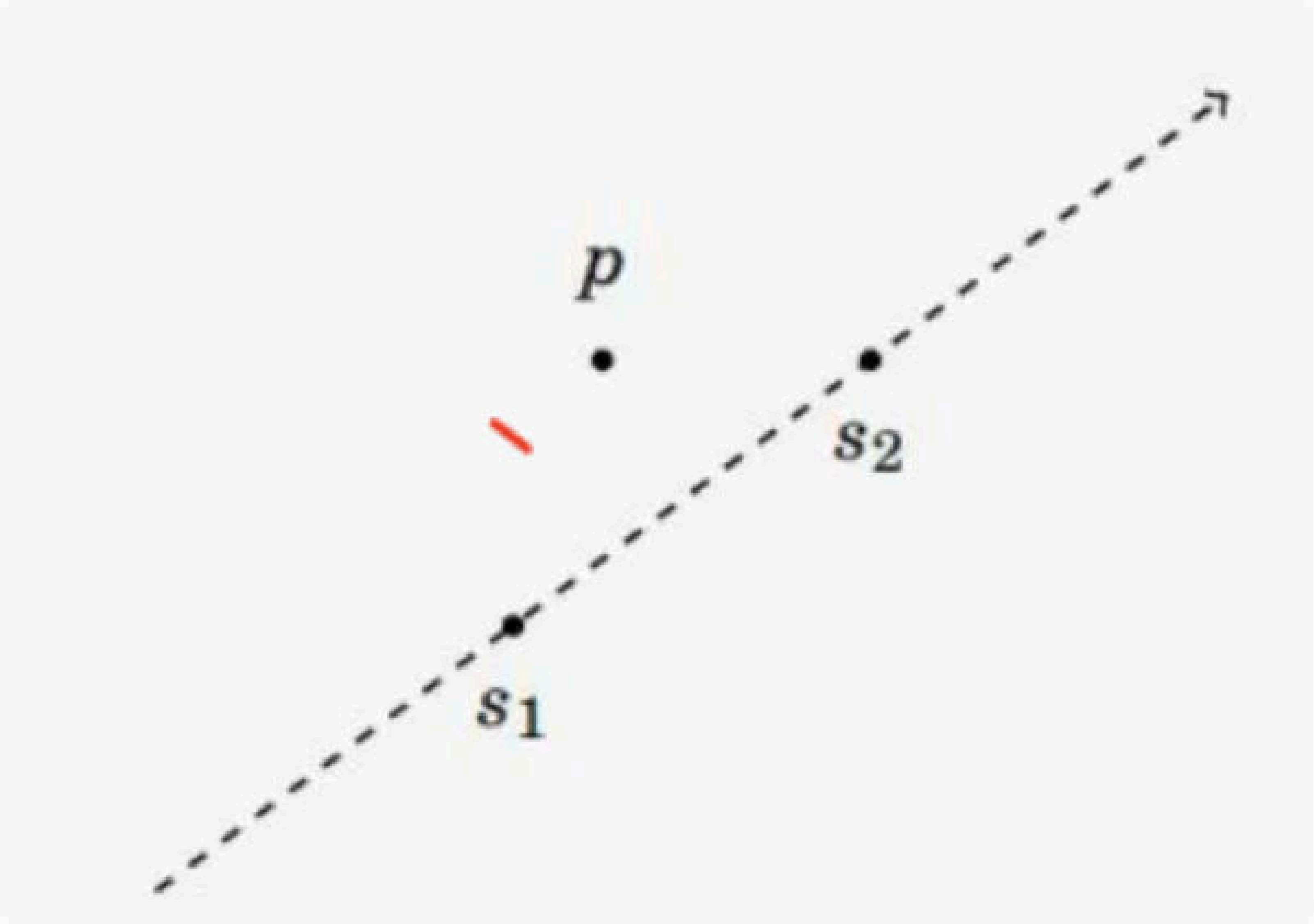
(a)



(b)

Location of a point around a line

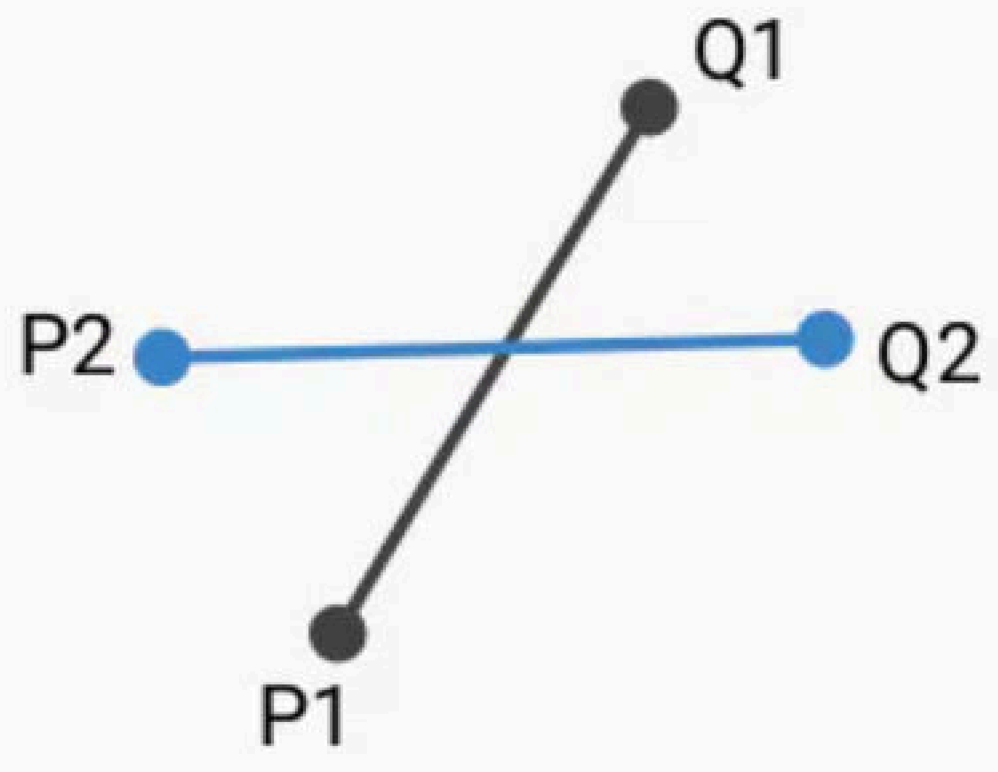
$$(p - s_1) \times (p - s_2)$$



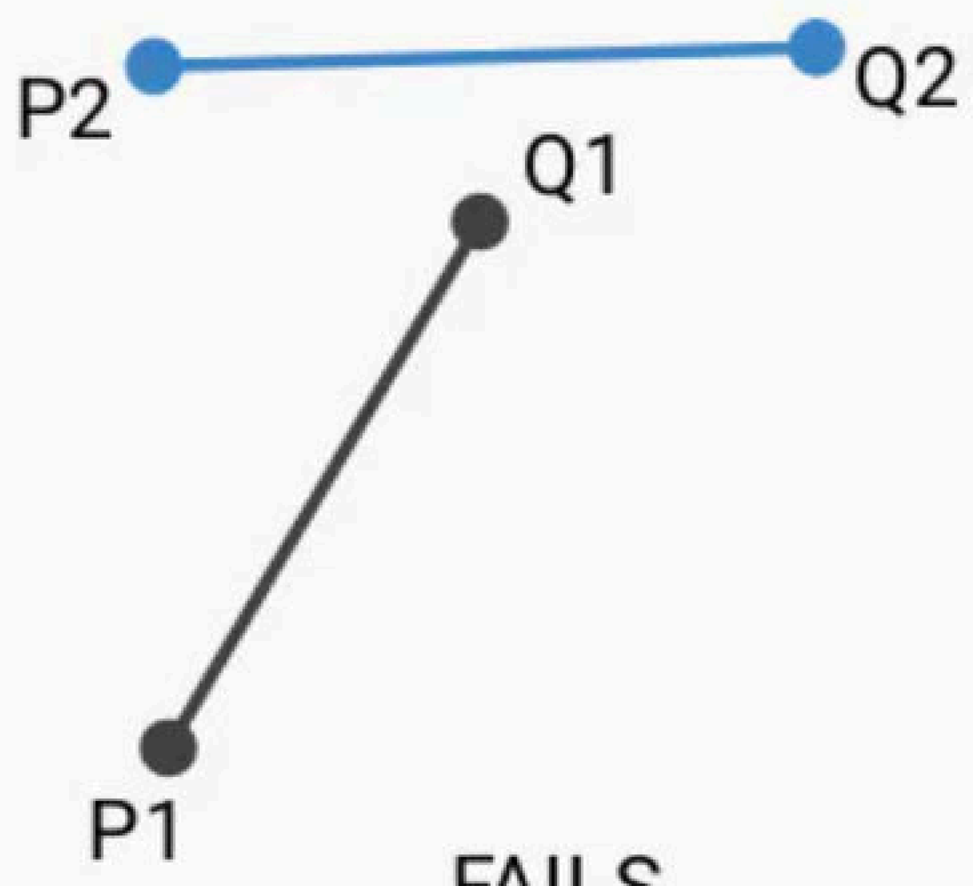
Intersection of two line SEGMENTS

GENERAL CASE

- (p1, q1, p2) and (p1, q1, q2) have different orientations and
- (p2, q2, p1) and (p2, q2, q1) have different orientations.



PASSES

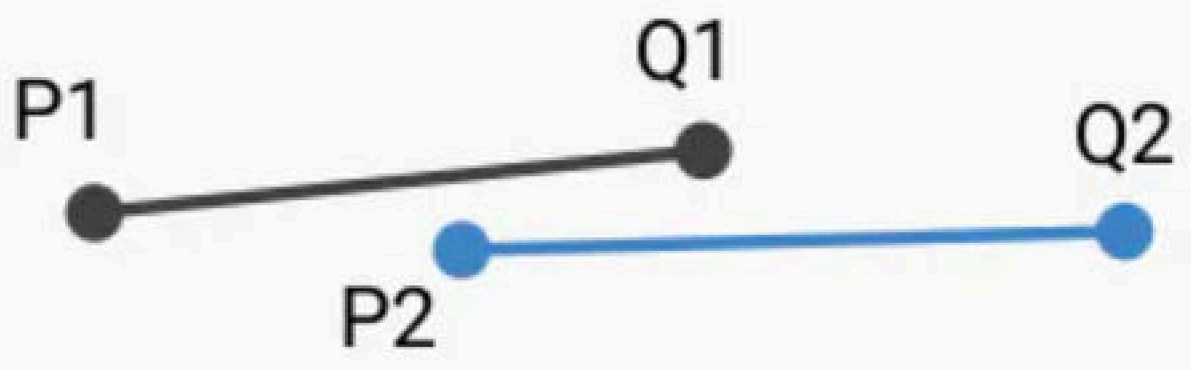


FAILS

Intersection of two line SEGMENTS - 2

Special Case

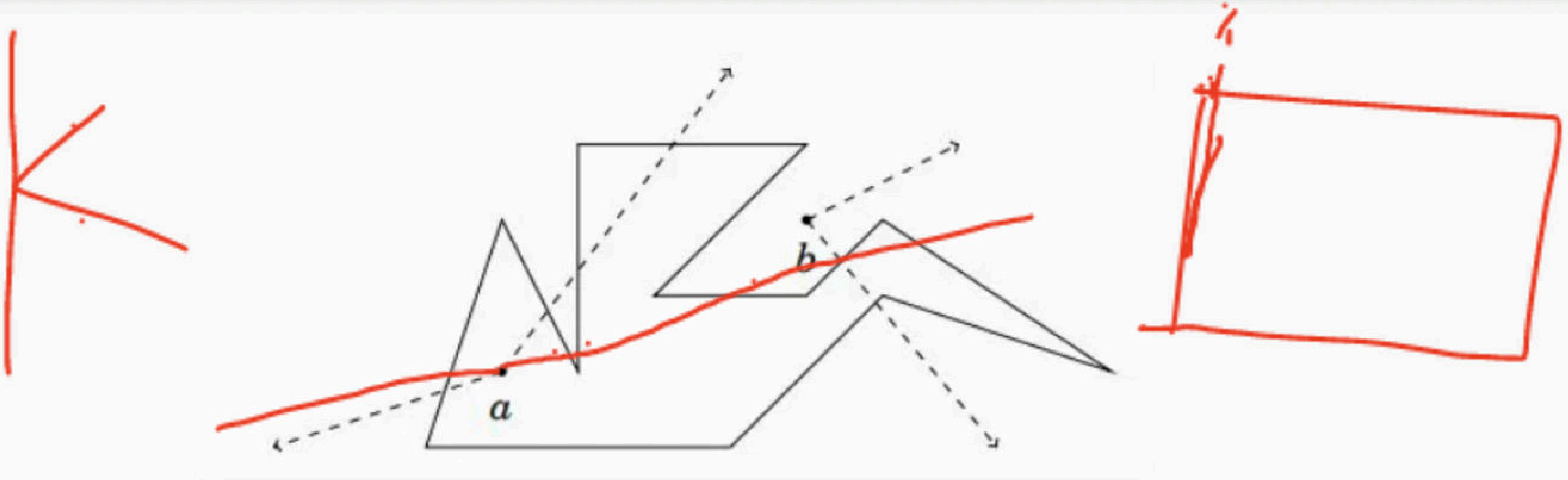
- (p1, q1, p2), (p1, q1, q2), (p2, q2, p1), and (p2, q2, q1) are all collinear and
  – the x-projections of (p1, q1) and (p2, q2) intersect
  – the y-projections of (p1, q1) and (p2, q2) intersect



PASSES

FAILS

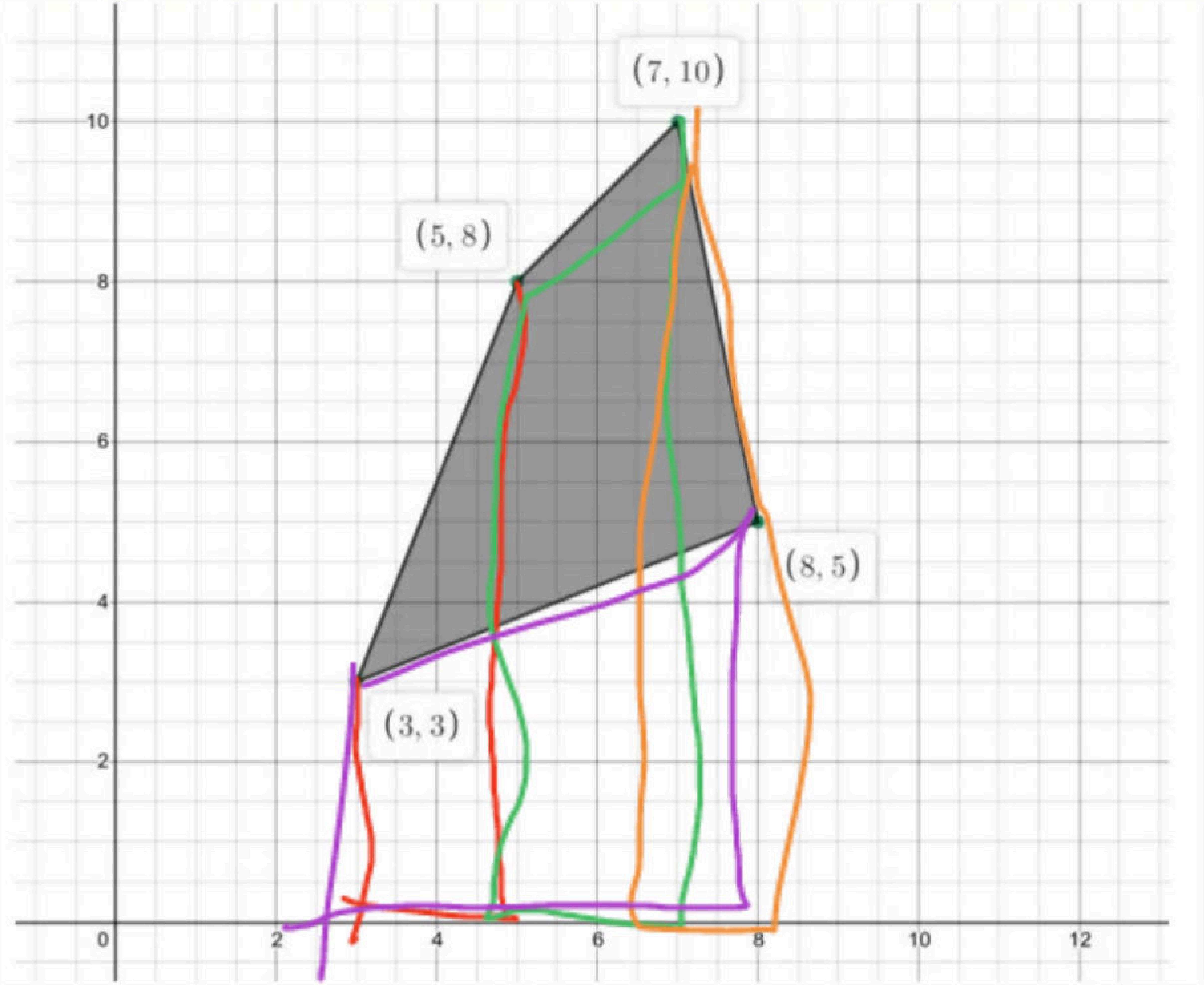
Location of point in polygon



Area of a polygon

$$A = \sum_{(p,q)\in\text{edges}} \frac{(p_x - q_x)\cdot(p_y + q_y)}{2}$$

(½)*[
(3-8)(3+5)
+ (8-7)(5+10)
+ (7-5)(10+8)
+ (5-3)(8+3)
]



Picks' Theorem

QUIZ

Geometry

1. What will be the value of “a x b” (a cross product b)

A. Positive ✓
B. Negative
C. Zero
D. Can’t say