Prime numbers, Factorization and GCD-LCM

Course on Mathematics & Puzzles for Interview Preparation

Arjun Arul • Lesson 1 • Feb 9, 2021

Maths and Puzzles

Lecture 1

Prime Numbers and Prime Factorisation

Prime No -> 1 & n -> divisors.

$a \mid b \rightarrow$ a divides b.

$10/5 = 2$

$a \nmid b = 0$

$5/10 = 1/2$

$a/b \Rightarrow a \mid b$

$5 \mid 10$

$x = \frac{b}{a}$

$\exists$ $a \mid b \rightsquigarrow b = xa$

$b \mid c \rightsquigarrow c = yb$ (2)

$\Rightarrow a \mid c$ $\quad c = yxa$

$= (yx)a$

$a \mid c$

$a \mid b$

$a \mid c$

$a \mid mb + nc$

$a \mid mb + nc$

$mb + nc$

a

$\frac{mb}{a} + \frac{nc}{a}$

$a \mid b$ $\quad$ $a \mid c$

$\Downarrow$ $\quad$ $\Downarrow$

$\frac{b}{a} = n$ $\quad$ $\frac{c}{a} = m$

$a \mid xb + yc$

$\frac{xb + yc}{a}$ $\quad$ $\frac{xb}{a} + \frac{yc}{a}$

$= xn + ym$

$3 \mid 12 \qquad 3 \mid 9$

$\Rightarrow 3 \mid 12 \times 7 + 8 \times 9$

$a \mid b$ & $c \mid d$

$\frac{b}{a} = n$

$\frac{d}{c} = m$

$ac \mid bd$

$2 \mid 8$

$\frac{8}{2}$ is integer

$\frac{bd}{ac} = \frac{b}{a} \cdot \frac{d}{c}$

$= n \cdot m$

$3 \mid 21$

↳ $\frac{21}{3}$ is an integer

$2 \mid 6$ $\quad$ $7 \mid 21$

$\Rightarrow 14 \mid 126$

63

~~125~~

~~4~~

>

$= 9$

$a \mid b \;\&\; b \mid c \Rightarrow a \mid c$

$a \mid b \;\&\; a \mid c \Rightarrow a \mid xb + yc$

$a \mid b \;\&\; c \mid d \Rightarrow ac \mid bd$

1) $2 \mid 6 \;\&\; c \mid 18 \Rightarrow 2 \mid 18$

2) $2 \mid c \;\&\; 2 \mid 4 \Rightarrow 2 \mid 6x + 4y \Rightarrow 2 \mid 12 + 2y$

$ac \mid bd$

~~$\Rightarrow$~~ $a \mid b$ & $c \mid d$?

$3 \cdot 7 \mid 6 \cdot 49$

$\Rightarrow 3 \mid 6$ & $7 \mid 49$

$2 \times 3 \mid 3 \times 2$

$2 \nmid 3$ $\quad$ $3 \nmid 2$

$a \mid c$ & $b \nmid c$

$\not\Rightarrow$ $a \mid b$?

$2 \mid 6$ & $3 \mid 6$

$a \mid c$ & $a \nmid b$

$\Rightarrow$ $b \mid c$?

$2 \mid 10$ & $2 \mid 14$

$14 \mid 10$

$10 \mid 14$

Prime factorization

$\frac{84}{2} = 42$ $\quad\quad$ $\frac{21}{3} = 7$

$\frac{42}{2} = 21$

$84 = 2 \times 42 = 2 \times (2 \times 21) = 2 \times 2 \times 3 \times 7$

$= 2^2 \times 3 \times 7$

$84 = 11 \times 7$

non-1

1) Smallest factor of any integer is a prime.

$n \rightarrow$ Smallest factor $= x \leq \sqrt{x \times y}$

$c \mid n \equiv xy \mid n \Rightarrow x \mid n$

$84 = \cdots\cdots\cdots$

$2^{2} \times 3 \times 7$

$\frac{84}{2} = 42$

$\frac{42}{2} = 21$

$\frac{21}{3} = 7$

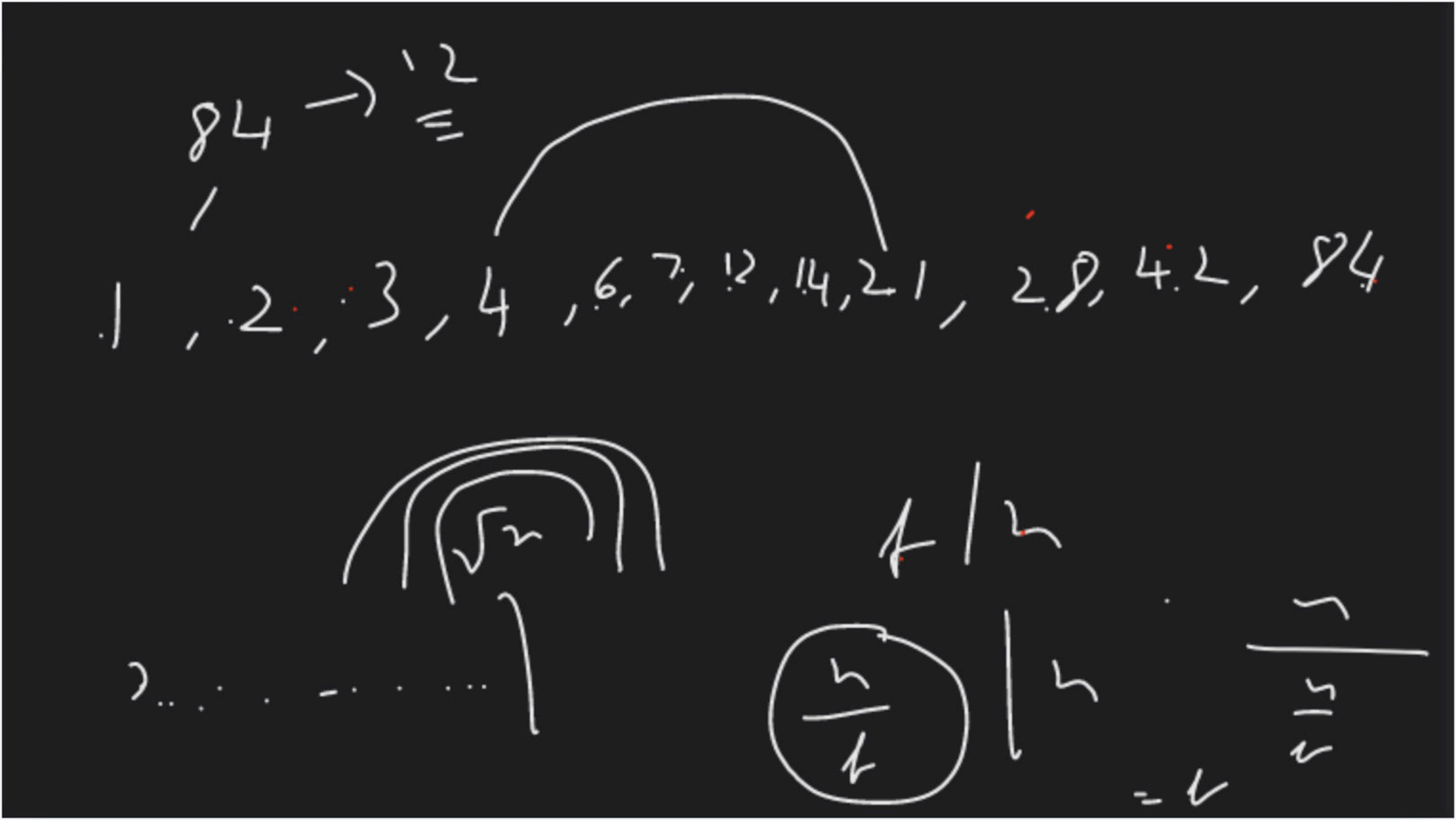
$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots p_k^{\alpha_k}$$

$p_1, p_2, p_3 \ldots, p_k$ are primes

$\alpha_i \geq 1$

$$5 = 7^0 \times 11^0 \times 5^1$$

$$3^0$$


84
1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84

n

t , o $\frac{n}{t} \leq \sqrt{n}$

$t > \sqrt{n}$

$\frac{n}{n} > \sqrt{n}$

$t * \frac{n}{t} > \sqrt{n} * \sqrt{n}$

$n > n$ ⊥

$\min\left(t, \frac{n}{t}\right) \leq \sqrt{n}$ ->

$\max\left(t, \frac{n}{n}\right) \geq \sqrt{n}$

Claim

$\min\left(f, \frac{n}{f}\right) \leq \sqrt{n}$

Fix $\frac{n}{f} > \sqrt{n} \times \sqrt{n}$

$n > n$

Proof

→ Assume the contrary.

Assume $\min\left(f, \frac{n}{f}\right) > \sqrt{n}$

$\Rightarrow f > \sqrt{n}$ & $\frac{n}{f} > \sqrt{n}$

$xy \mid n \Rightarrow \frac{n}{xy} = k$

$\Rightarrow n = kxy$

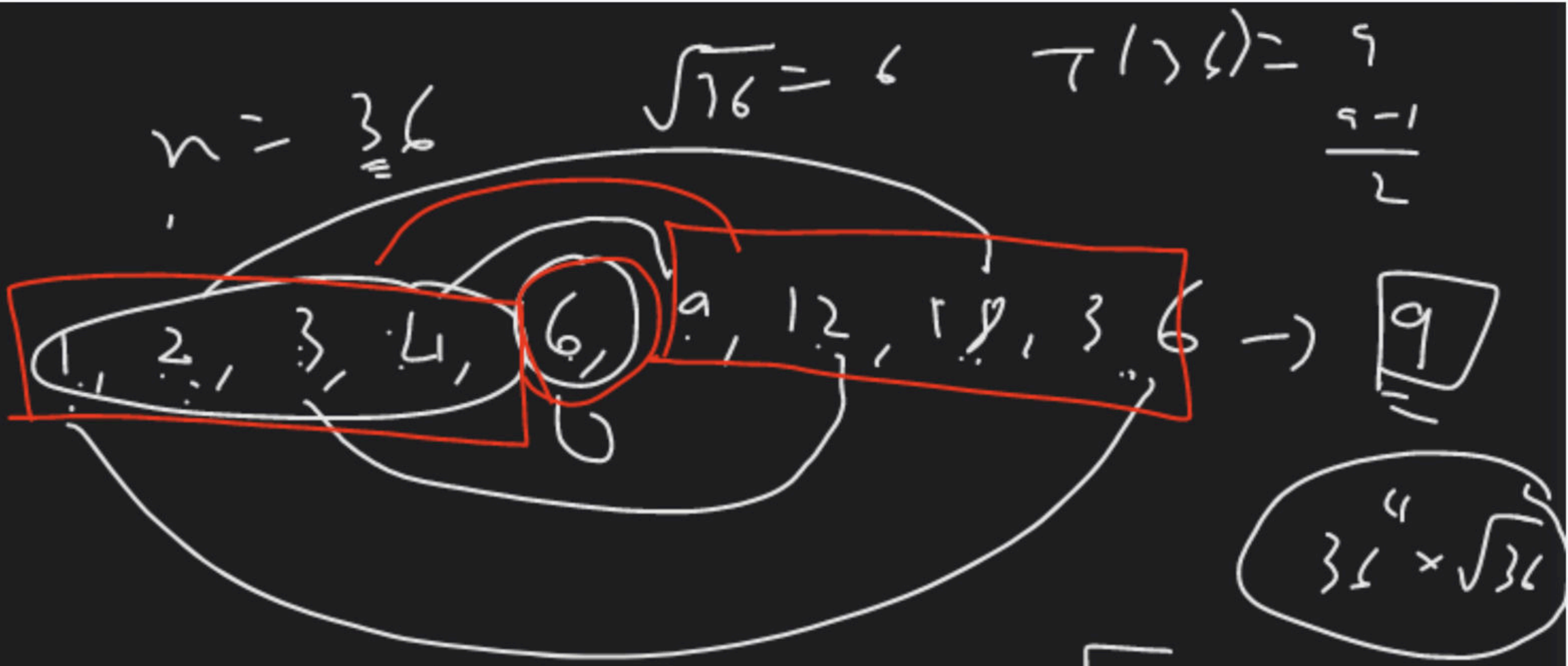
$= x(ky)$

$\Rightarrow x \mid n \;\&\; y \mid n$

$\frac{n}{x} = ky$

$n = y(kx)$

$\frac{n}{y} = kx$



Number of factors of $n \leq 2\sqrt{n}$

$\frac{9}{2}$

$36 =$

$\tau(n)$

$$\frac{\tau(n)-1}{2}$$

n

$$\times \sqrt{} \; n^{\frac{1}{2}}$$

$$\frac{\tau(n)-1}{2} + \frac{1}{2} = \frac{\tau(n)}{2}$$

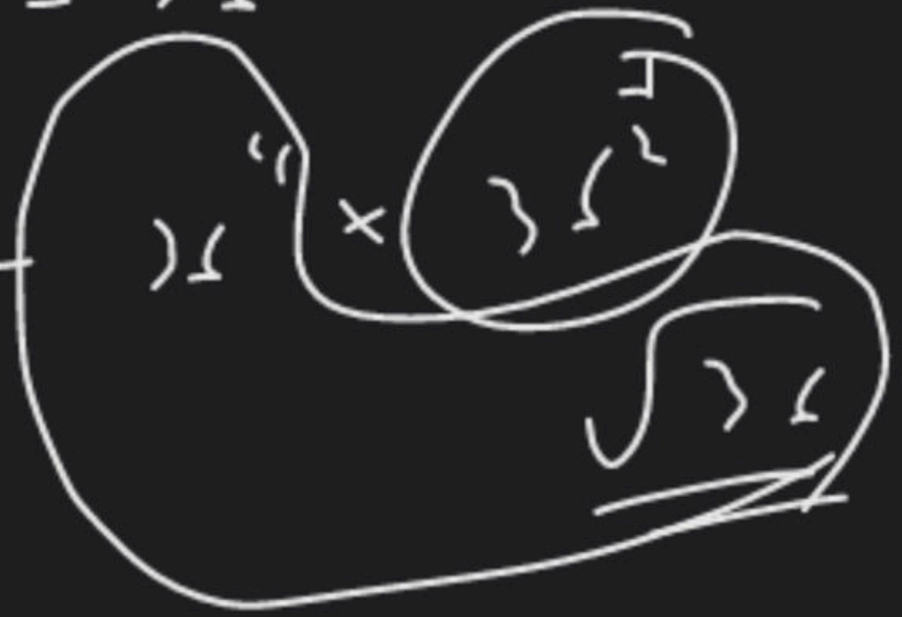
n

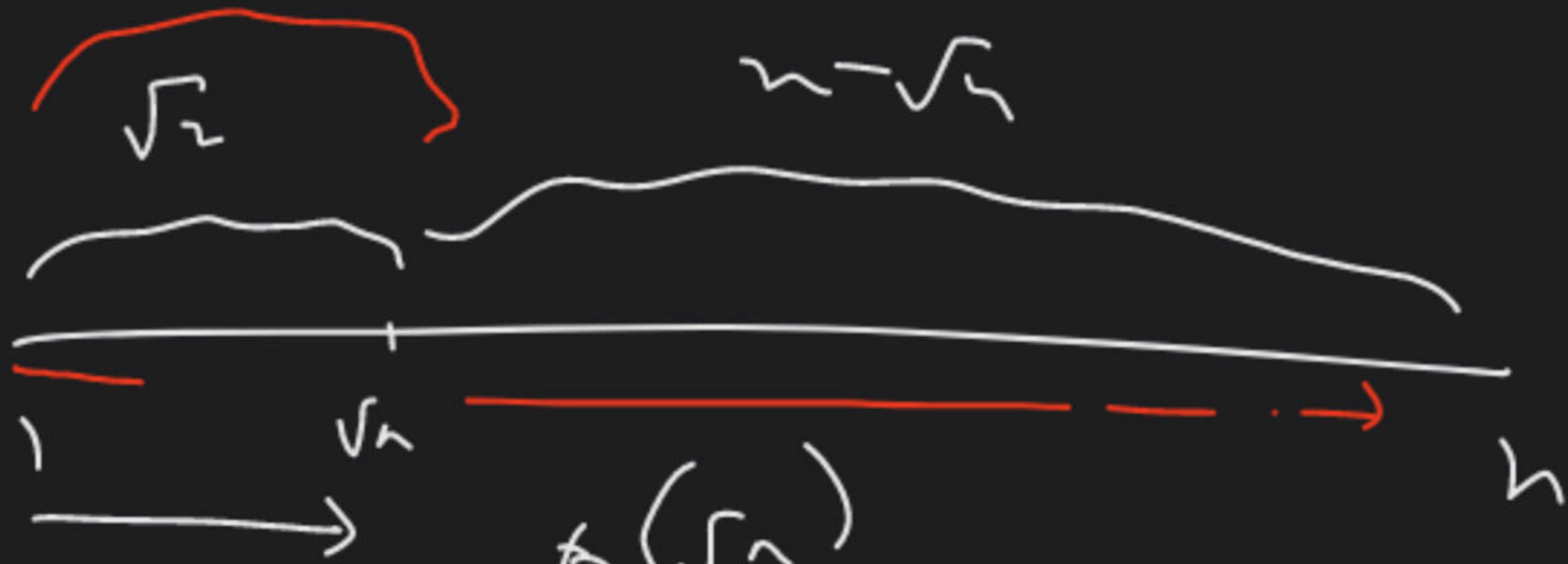
n

$$36^{\frac{9}{2}} = 36^{\frac{8+1}{2}} = 36^{4+\frac{1}{2}}$$

$$= 36^{4+\frac{1}{2}}$$

Perfect square → odd #





$O(\sqrt{n})$

$n = 100$

$\frac{n}{2} = 50$

$\sqrt{n} = 10$

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

$\tau(n)$ = The number of factors of n $= (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$

$84 = 2^2 \times 3^1 \times 7^1$

$\tau(84) = (2+1)(1+1)(1+1) = 3 \times 2 \times 2 = \underline{12}$

$$36 = 2^2 \times 3^2$$

$$\tau(36) = (2+1)(2+1)$$

$$= 3 \times 3$$

$$= 9$$

$42 = 2^1 \times 3^1 \times 7^1$

$\tau(42) = 2 \times 2 \times 2$

$= 8$

1, 2, 3, 6, 7, 14, 21, 42

n

$[2, \sqrt{n}) - x$

$2 > \sqrt{n}$

$\frac{n}{2} < \frac{\sqrt{2} \times \sqrt{n}}{2} < \frac{\sqrt{n} \times \sqrt{n}}{\sqrt{n}}$

$n = p_1^{\alpha_1}$

$n = 5$

$p_1^0, p_1^1, p_1^2, p_1^3, \ldots\ldots, p_1^{\alpha_1}$

$a \mid b$

$\alpha_1 + 1$

$\exists t \qquad a \mid \mu$

$p_1 > \alpha_1$

$a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

$t \to p_1^{\beta_1}$

$\frac{\mu}{a}$ $\quad p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

$n = p_1^{\alpha_1}$

$7^5 = 6$

$1, 7, 49, 343, \ldots$

$1, p_1, p_1^2, \ldots\ldots p_1^{\alpha_1}$

$\alpha_1 + 1$

$n = p_1^{\alpha_1} p_2^{\alpha_2}$

$\beta_1 \in \{0, 1, 2 \ldots \alpha_1\}$

$\beta_2 \in \{0, 1, 2, \ldots \alpha_2\}$

$\beta_1 \leq \alpha_1$

$(\alpha_1 + 1)(\alpha_2 + 1)$

↳ $p_1^{\beta_1} p_2^{\beta_2}$

$\beta_1, \beta_2 \geq 0$

$(\alpha_1 + 1)(\alpha_2 + 1)$

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

$$d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$$

$$\beta_i \in \{0, 1, \ldots, \alpha_i\} \quad \alpha_i + 1$$

$$(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1) = n$$

Given n

Find number of integers x where $\tau(x) = n$

& prime factors of x are $[2, 3, 5, 7, 11]$

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

Find Sum of all factors.

$n = 12 = 2^2 \times 3^1$

$\tau(12) = 3 \times 2 = 6$

1, 2, 3, 4, 6, 12

Sum $= 1+2+3+4+6+12$

$= \boxed{28}$

$$\left(1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1}\right)\left(1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}\right)$$

$$\left(1 + \cdots\right) \cdots$$

$$\left(1 + p_k + p_k^2 + \cdots + p_k^{\alpha_k}\right)$$

$$n = 12 = 2^2 \times 3^1$$

$$\text{Sum of factors} = (2^0 + 2^1 + 2^2)(3^0 + 3^1)$$

$$= (1 + 2 + 4)(1 + 3)$$

$$= 7 \times 4$$

$$= 28$$

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2}$$

$$\left(1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1}\right)\left(1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}\right)$$

$$= p_1^{\beta_1} p_2^{\beta_2}$$

$$= p_1^{\beta_1} p_2^{\beta_2}$$

$$(a+b)(c+d)$$

$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

$f \times \frac{n}{f} = n$

Product of all factors?

$n = 12$

$12 \times 12 \times 12 = 12^3$

1, 2, 3, 4, 6, 12

product $= 24 \times 72 = \underline{1728}$

$9 = 3^2$

$\tau(9) = 3$

1, 3, 9

$\frac{\tau(9)}{2}$ $\quad$ $\frac{3}{2}$

$9^{\frac{\tau(9)}{2}} \quad 9^{\frac{3}{2}} \quad 9^2$

$$n^{\frac{\tau(n)}{2}}$$

1. I understand what is Prime Factorisation

A. Very Clear
B. Somewhat Clear
C. Please explain again

2. In the given Prime Factorisation, p1 can be same as p2.

$$n = p_1^{a_1} \cdot p_2^{a_2}$$

A. True
B. False

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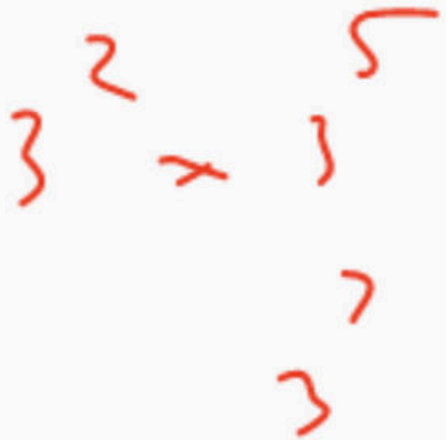
$$n = p_1^{a_1} \cdot p_2^{a_2}$$



A. True

B. False

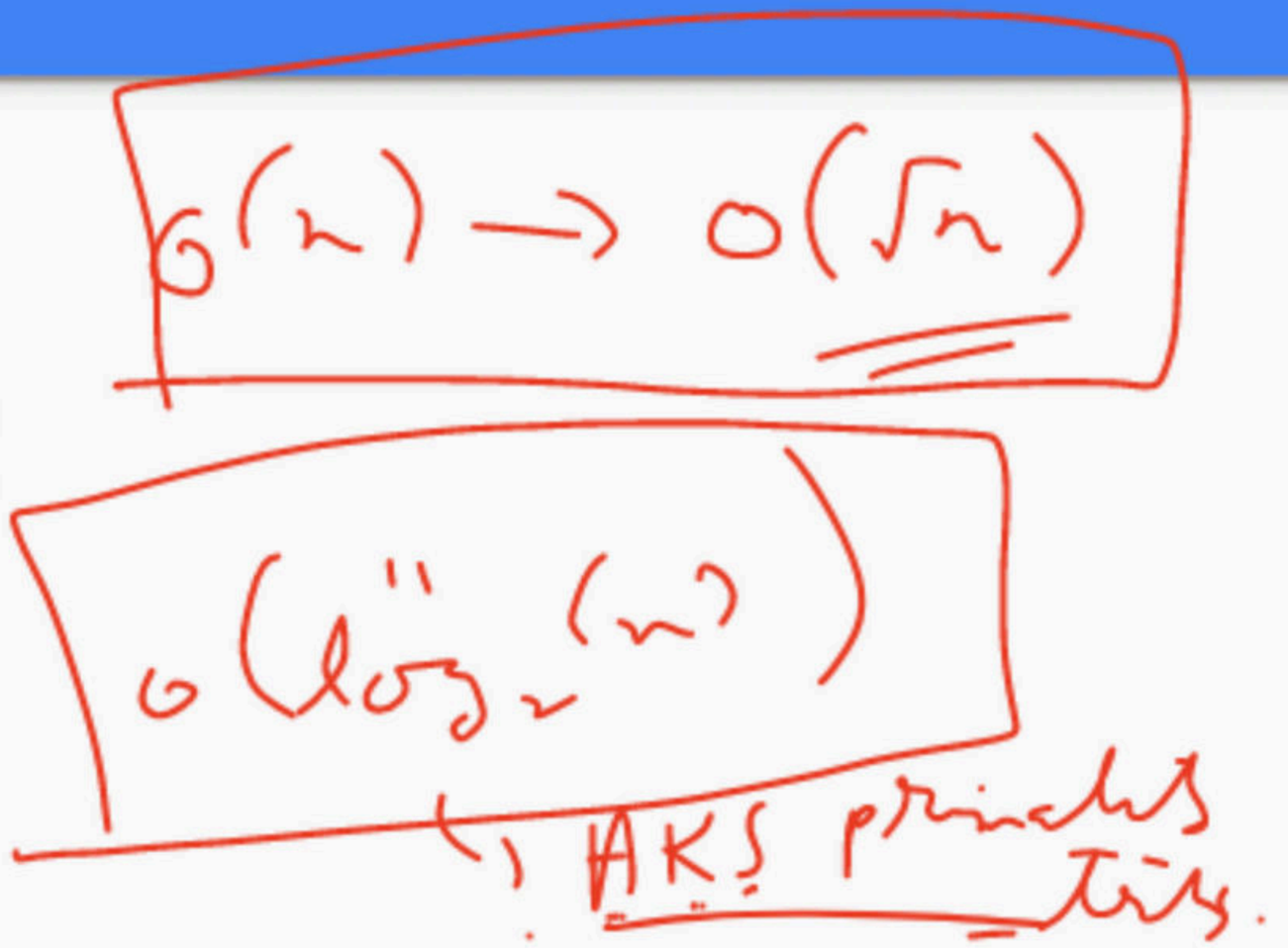
Bonus: Can a_1 be same as a_2 ?

Checking Primality and Sieve of Eratosthenes

```
prime(n) :
  if n < 2 :
    return false
  loop x=2 upto x=n-1 :
    if x divides n :
      return false
  //return outside loop
  return true
```

O(n)

```
prime(n) :
 if n < 2 :
  return false
loop x=2, x*x<=n, x++ :
   if x divides n:
      return false;
//outside loop
return true
```



```
int n;
vector<char> is_prime(n+1, true);
is_prime[0] = is_prime[1] = false;
for (int i = 2; i * i <= n; i++) {
    if (is_prime[i]) {
        for (int j = i * i; j <= n; j += i)
            is_prime[j] = false;
    }
}
```

3. Best time complexity that I know, to check if a number is prime is -

A. O(n^2)
B. O(n)
C. $O(\log_2 n)$
D. $O(\sqrt{n})$

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A. O(n^2)
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C. $O(\log_2 n)$
D. $O(\sqrt{n})$ ✓

4. Best way to generate list of first 10,000 primes is

A. Using the Sieve of Eratosthenes
B. Using O(N) Primality check
C. Using O(√N) Primality check

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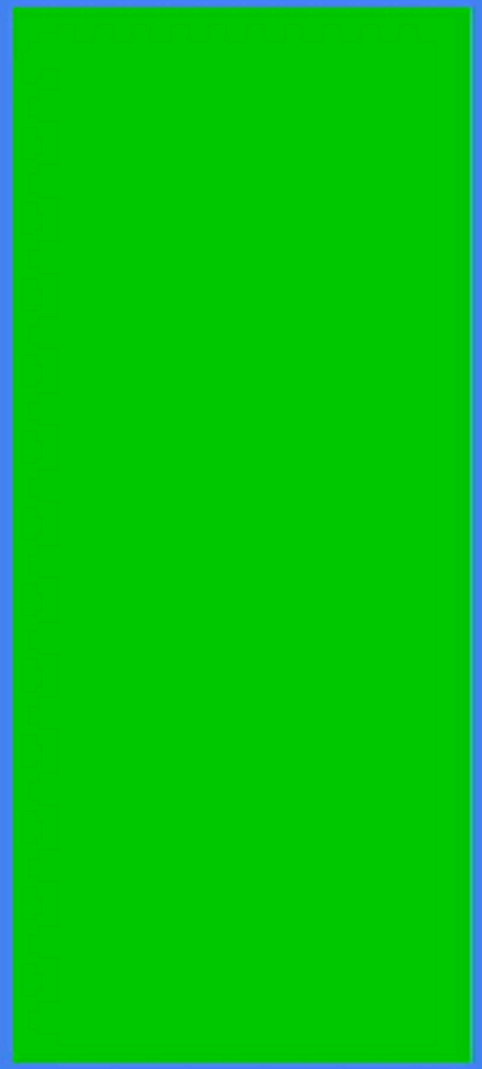
GCD and LCM



```
gcd(a, b) :
    if b == 0 : return a
    else return gcd(b, a%b)
```

```
int gcd(int a, int b, int& x, int& y) {
    if (b == 0) {
        x = 1;
        y = 0;
        return a;
    }
    int x1, y1;
    int d = gcd(b, a % b, x1, y1);
    x = y1;
    y = x1 - y1 * (a / b);
    return d;
}
```

5. What is the value of y in the following equation?

$$\gcd(a \cdot x, b \cdot x) = y \cdot \gcd(a, b)$$

A. x
B. x*x
C. a*b*x
D. x/(a*b)

5. What is the value of y in the following equation?

$$\gcd(a \cdot x, b \cdot x) = y \cdot \gcd(a, b)$$

A. x
B. x*x
C. a*b*x
D. x/(a*b)

6. What is the output, if GCD(A,B) = 1.

```
bool *marked = new bool[B+1];
memset(marked,0,B);
for(int i=A; i<B+1; i+=A){
        marked[i] = true;
}
cout<<marked[B]<<endl;
```

A. 0
B. 1
C. Can't Determine

6. What is the output, if GCD(A,B) = 1.

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}
cout<<marked[B]<<endl;
```

A. 0 ✓

B. 1

C. Can't Determine

Euclid’s Algorithms

7. Using Euclid's Algorithm we can check if two numbers are Coprime.

A. True
B. False

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A. True
B. False

Bonus : What is GCD(A,B) if A and B are coprime?

8. Time complexity for finding the GCD of an array of integers is

A. $O(N^2)$
B. $O(N \log_2 M)$, where M is maximum value of array
C. $O(N \log_2 N)$
D. $O(N)$

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