Combinatorics and Probability

Course on Mathematics & Puzzles for Interview Preparation

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Maths and Puzzles

Lesson 5

CONCEPT

Binomial Coefficient

Binomial Coefficient

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + \binom{n}{n}b^n$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$^nC_k = {}^nC_{n-k}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\boxed{\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n}$$

$^nC_k\, a^k b^{n-k}$

$a^{n-k}\, b^k$

$^nC\, a^{n-k}\, b^n$

$a = 1$

$x = 1$

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$$

$$ {}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r} $$

✓

$$ \begin{matrix} A_1 \\ = \\ r \end{matrix} \quad \underbrace{A_2 \; \cdots \cdots \; A_n}_{k\,7} \qquad {}^{n}C_{r} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k} $$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^n = \underbrace{(a+b)(a+b)\cdots\cdots(a+b)}_{n \text{ times}}$$

$$= a^k b^{n-k} \qquad {}^nC_k$$

$a\ a\ b\ a$ $\qquad$ $a\ b\ a\ a$ $\qquad$ $a^3 b$

$^{n}C_{r}$ $\quad \binom{n}{r}$ $\quad \boxed{{}^{n}C_{r} = \dfrac{n!}{(n-r)!\,r!}}$

$\{A, B, C, D\}$

$^{4}C_{2} = \left|\{AB, AC, AD, BC, BD, CD\}\right| = 6$

$A_1, A_2, \ldots, A_n$

$4 \times 3 \times 2 \times 1 = 24$

A B C D

A B D C

A C B D

A C D B

A D B C

B A C D

B A D C

$n = 5,\ r = 2$

A B C D E

$$\frac{5!}{2!\,3!}$$

$12 = 2! \times 3!$

C E A B C

C E A C B

C E B A C

C E B C A

C E C A B

C E C B A

E C .

$$\frac{n!}{r!\,(n-r)!}$$

QUIZ

Binomial Coefficient

1. What is the value of x?

$$\binom{18}{x} = \binom{18}{x+2}$$

A. 2

B. 10

C. 8

D. 18

$${}^{n}C_{r} = {}^{n}C_{n-r}$$

$$r + (n-r) = n$$

$$x + (x+2) = n$$

$$= 18$$

$$2x + 2 = 18$$

$$2x = 16$$

$$\boxed{x = 8}$$

2. Which one is the correct relation?

A. $\binom{n}{k} = \binom{n}{k-1} + \binom{n-1}{k-1}$

B. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k}$

C. $\binom{n}{k} = \binom{n}{k-1} + \binom{n-1}{k}$

D. $\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$

3. The coefficient for x^k in expansion of $(1+x)^n$ is -

$(a+r)^n$ $\quad {}^nC_k\, a^k r^{n-k}$

A. $\binom{n}{k-1}$

B. $\binom{k}{n}$

C. $\binom{n}{k+1}$

D. $\binom{n}{k}$ ✓

$${}^nC_k = {}^nC_{n-k}$$

CONCEPT

Pascal’s Triangle

Pascal's Triangle

$${}^{n-1}C_{r-1} + {}^{n-1}C_{r} = {}^{n}C_{r}$$

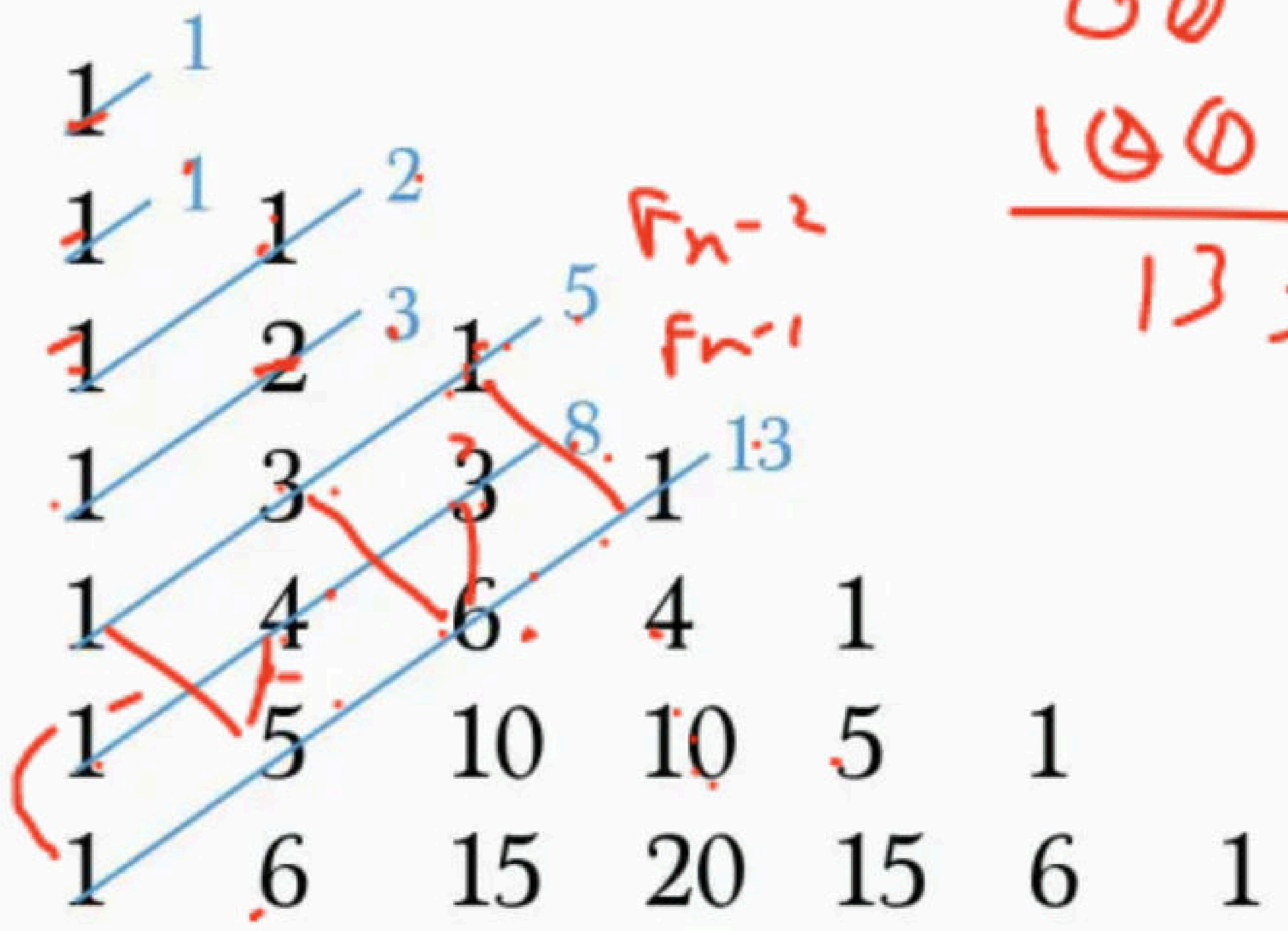
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

$${}^{6}C_{3} = \frac{6!}{3!\,3!}$$

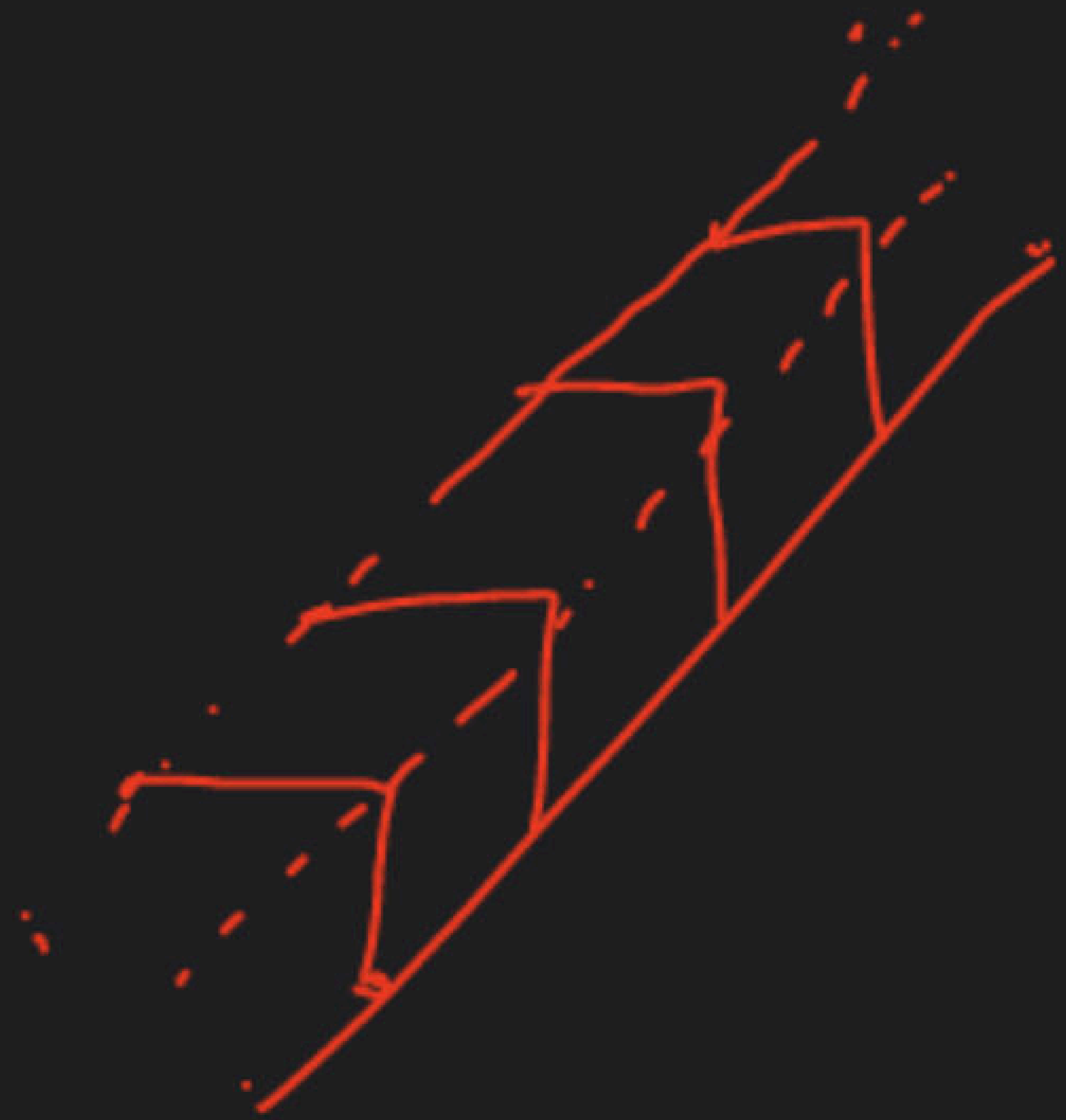
$${}^{2}C_{0} \quad {}^{2}C_{1} \quad {}^{2}C_{2}$$

$${}^{6}C_{0} \quad {}^{6}C_{1} \quad \dots \quad {}^{6}C_{6}$$

Fibonacci Numbers



1 1
1 1 0
1 2 1



Hockey Stick Identity

$$n, r \in \mathbb{N}, n > r, \sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

$$\binom{r}{r} \quad \binom{r+1}{r} \quad \binom{r+2}{r} \quad \cdots \quad \binom{n}{r} = \binom{n+1}{r+1}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

$A_1, \; A_2, \; \dots, \; A_{k+1}, \; \dots, \; A_{n-1}, \; A_n, \; A_{n+1}$

$${}^{n+1}C_{k+1} = {}^{n}C_{k} + {}^{n-1}C_{k} + \dots + {}^{k}C_{k}$$

QUIZ

Pascal's Triangle

4. The sum of n^{th} row in 0-based indexing in The Pascal's Triangle is

A. 2^n

B. 2^{n+1}

C. 2^{2n}

D. 2^{n-1}

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}
$$

CONCEPT

Calculating Binomial Coefficient

Using Analytical Formula

```
int binomialCoeff(int n, int k) {
    int res = 1;
    for (int i = n - k + 1; i <= n; ++i)
        res *= i;
    for (int i = 2; i <= k; ++i)
        res /= i;
    return res;
}
```

$${}^{n}C_{k} = \frac{n!}{(n-k)!\,k!}$$

$${}^{n}C_{k}$$

$$\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$$

Likely to overflow

$$g \times l = ab$$

$$lcm = \frac{ab}{gcd}$$

$$(a \times b) / gcd = \left(a / gcd\right) \times b$$

Using Analytical Formula - Improved

$$\frac{(n-k+1)(n-k+2)\ldots n}{k!}$$

```
int binomialCoeff(int n, int k) {
    double res = 1;
    for (int i = 1; i <= k; ++i)
        res = res * (n - k + i) / i;
    return (int)(res + 0.01);
}
```

$2 = \text{int}(1.9999)$

$\underline{0.01}$

+0.01 done to take care of accumulated errors

Using Pascal's Triangle

$C[n][k] = {}^{n}C_{k}$

```
int C[maxn + 1][maxn + 1];
C[0][0] = 1;
for (int n = 1; n <= maxn; ++n) {
    C[n][0] = C[n][n] = 1;
    for (int k = 1; k < n; ++k)
        C[n][k] = C[n - 1][k - 1] + C[n - 1][k];
}
```

Time complexity $O(n^2)$

$a \cdot r \,\%\, m = ((a \,\%\, m) \cdot (r \,\%\, m)) \bmod m.$

Fast computation Under modulo M

→ Prime

```
//Precompute factorials till MAXN
factorial[0] = 1;
for (int i = 1; i <= MAXN; i++) {
   factorial[i] = factorial[i - 1] * i % m;
}

//Compute Bin-Coeff in O(log m) time
long long binomial_coefficient(int n, int k) {
   return factorial[n] * inverse(factorial[k] * factorial[n - k] % m) % m;
}
```

$^nC_k \bmod M$

$^nC_r = \frac{n!}{(n-r)!\, r!}$

$\lg(M)$

Can be done even faster if we precompute inverse too

CONCEPT

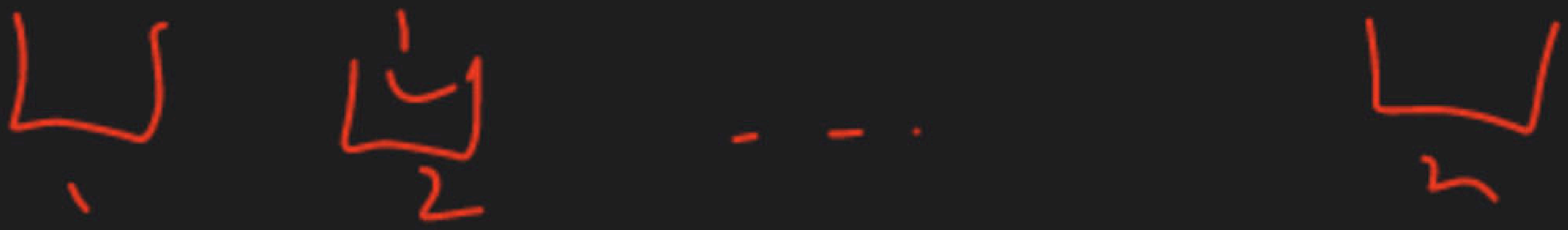
Combinatorics

Combinatorics

1. Rule of sum
2. Rule of product
3. Combinations → ^nC_r
4. Permutations → ^nP_r

$^nP_r = \frac{n!}{(n-r)!}$

Boxes & Balls

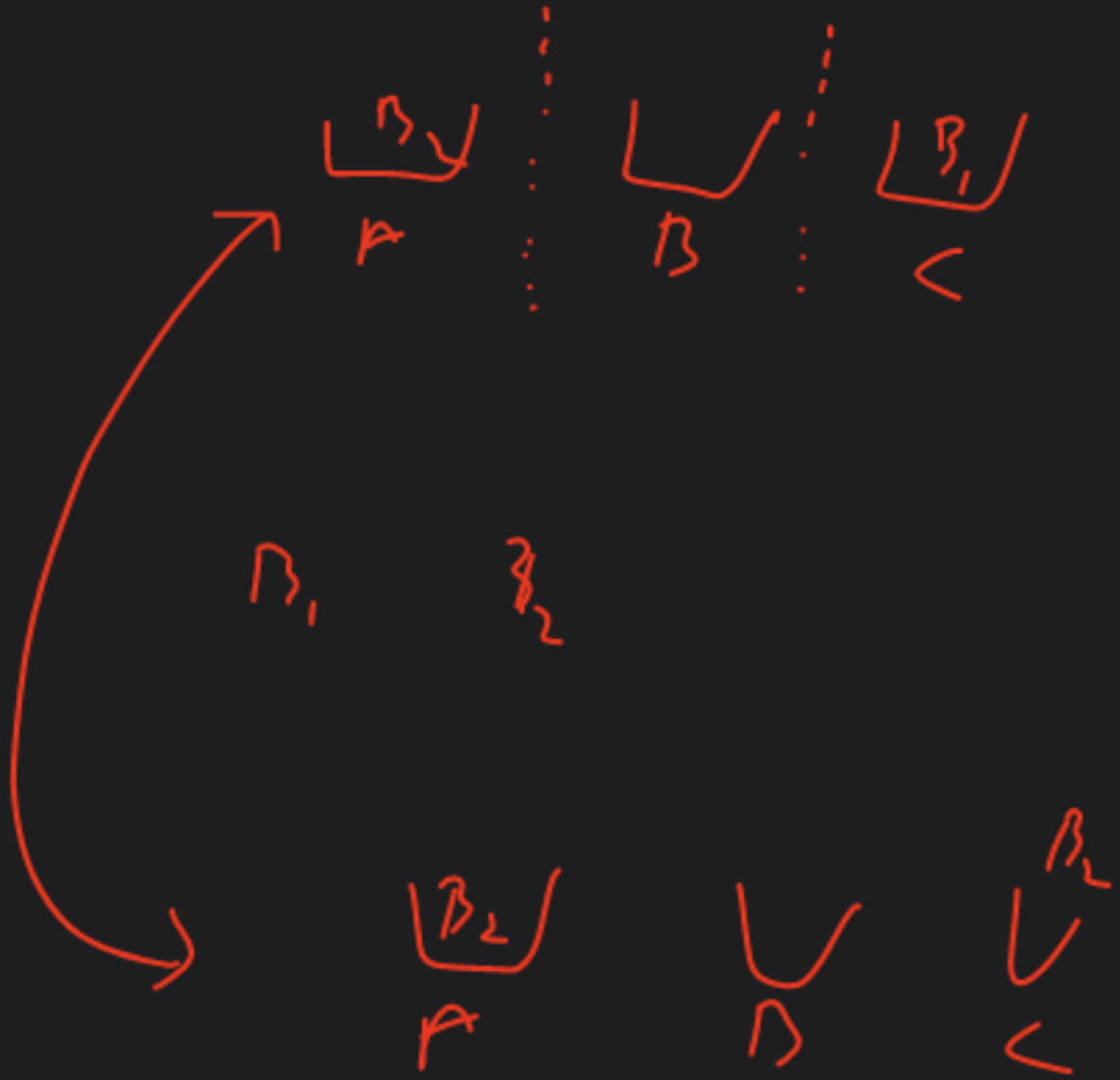


k balls

$n < k$

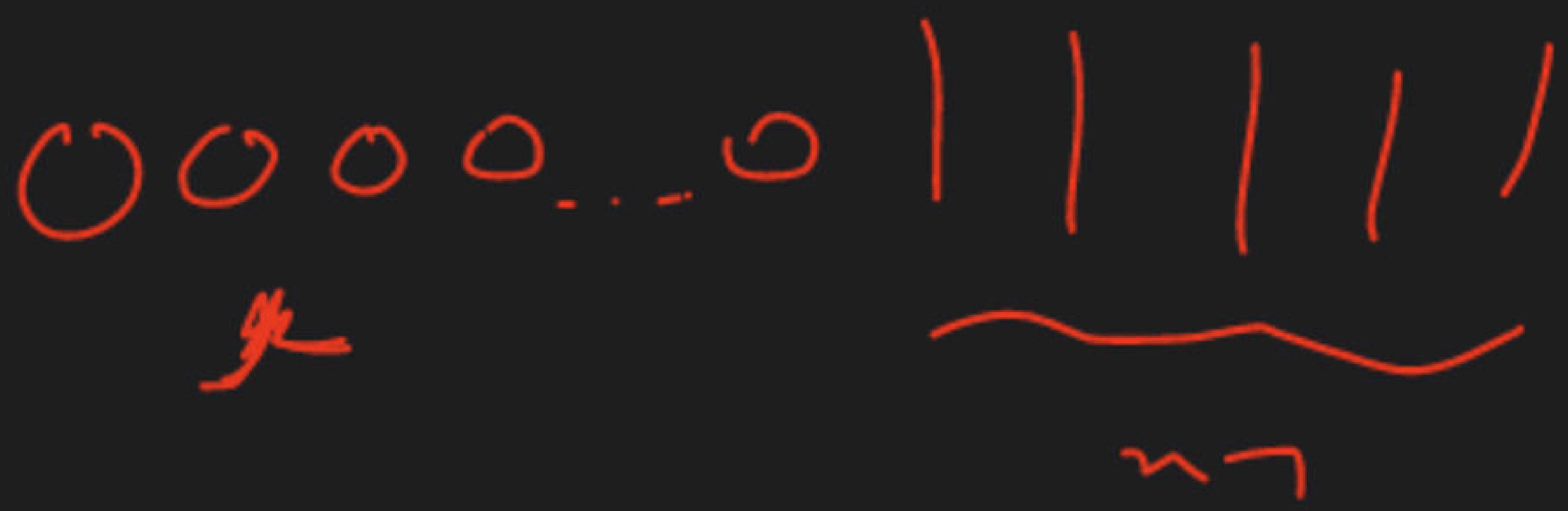
$n = 3$

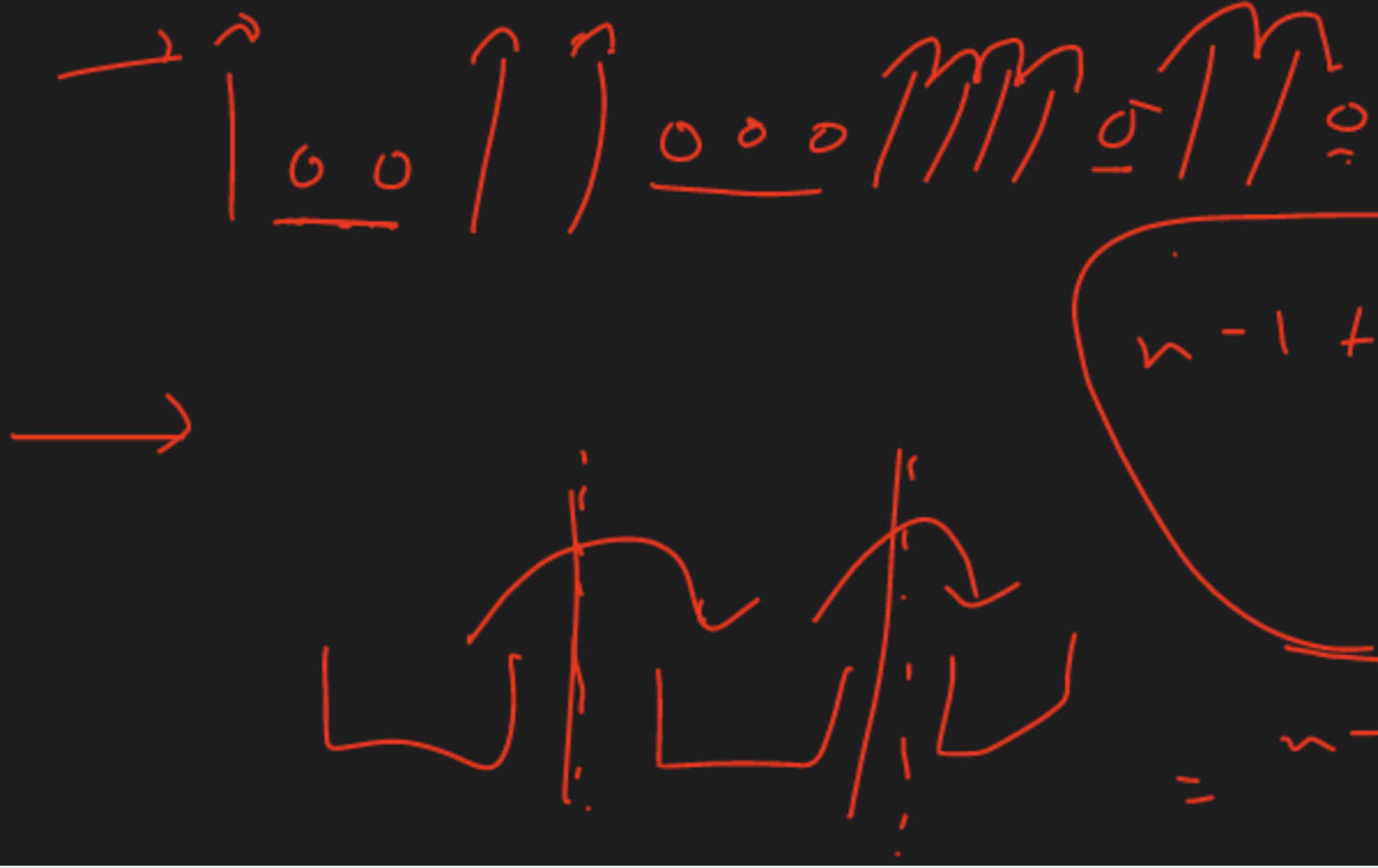
$p = 2$



Sticks & Balls.

n-1 sticks & k balls.





$${}^{n-1+k}C_{k}$$

$$= {}^{n-1+k}C_{n-1}$$

$$A^{-n} = \lambda_1 + \lambda_2 + \lambda_3 - - \dots + \lambda_n$$

$$\lambda_i \geq 0$$

$$A = c_1 + c_2 + \dots + c_n$$

$$c_i \geq 1 \qquad B_i \geq 0$$

$$B_i = c_i - 1$$

$$\sum B_i = \sum c_i - n$$

$$= \boxed{A - n}$$

Combinatorics - Problem Solving

QUIZ

Combinatorics

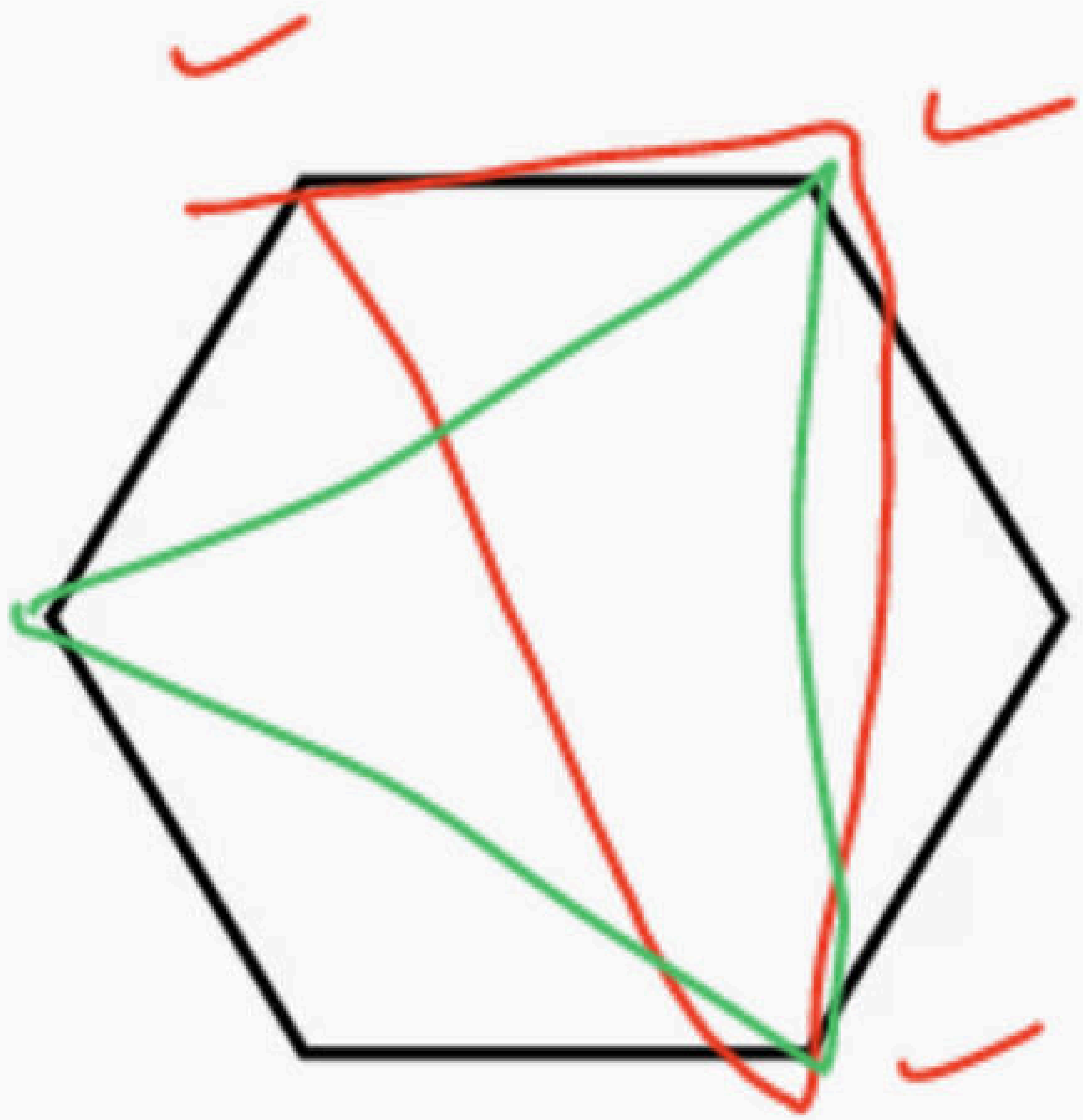
5. How many triangles can be formed using the vertices of hexagon?

A. $\binom{6}{3}$

B. $\binom{7}{4}$

C. $\binom{6}{4}$

D. $\binom{7}{3}$



6. Number of ways to fill 4 indistinguishable buckets from 6 waterfalls, such that we can have multiple buckets under one waterfall is -

A. $\binom{10}{6}$

B. $\binom{9}{6}$

C. $\binom{10}{4}$

D. $\binom{9}{4}$ ✓

$n = 6$

$r = 4$

$^{n-1+R}C_{r} \qquad ^{n-1+k}C_{n-1}$

$^{9}C_{4}$ ${}^{9}C_{6}$

7. The following equation is -

$$^nP_k = {}^nC_k \cdot k!$$

A. True

B. False

$$\frac{n!}{(n-k)!} = \frac{n!}{(n-k)!\,\cancel{k!}} \times \cancel{k!}$$

PERMUTATIONS

→ MISSISSIPPI

$$\frac{11!}{4!\,4!\,2!}$$

ABC
ACB
BAC
BCA
CAB
CBA

A A B

A B A $= \dfrac{3!}{2!}$

B A A

CONCEPT

Catalan Numbers

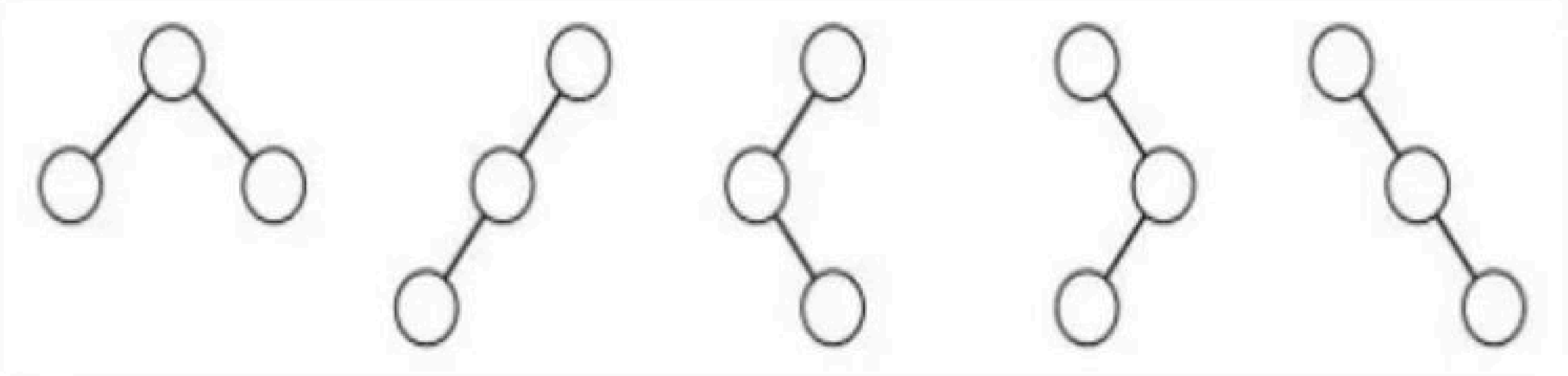
Catalan Sequence

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, ...

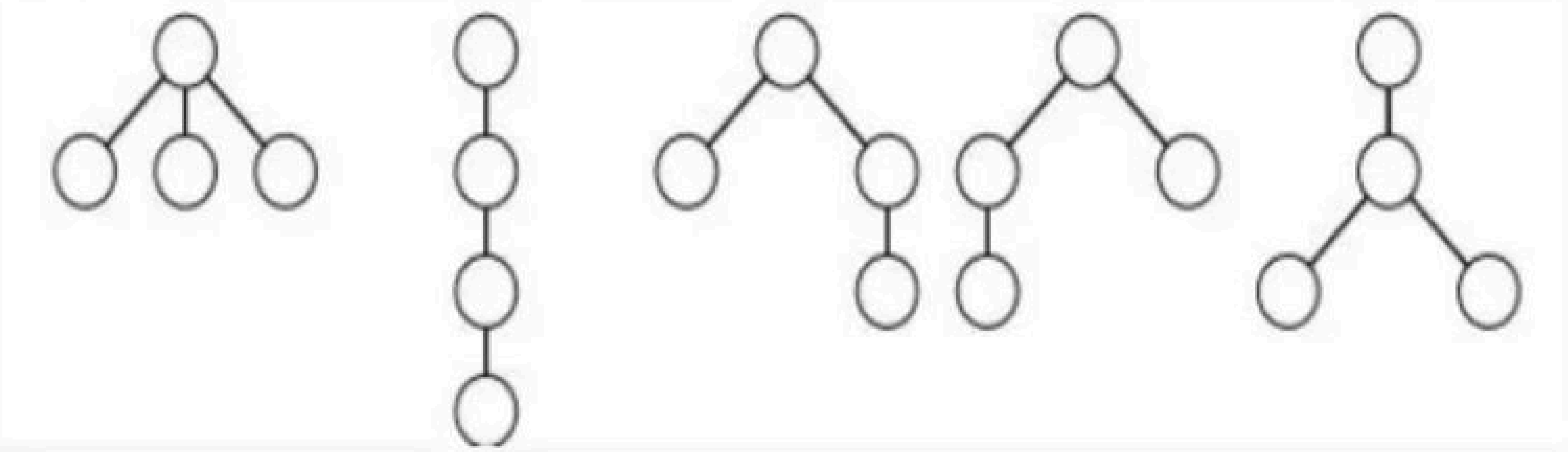
1. ()()()
2. (())()
3. ()(())
4. ((()))
5. (()())

Applications

1. C_n Binary Trees of N nodes



2. C_{n-1} Rooted Trees of N nodes



Applications

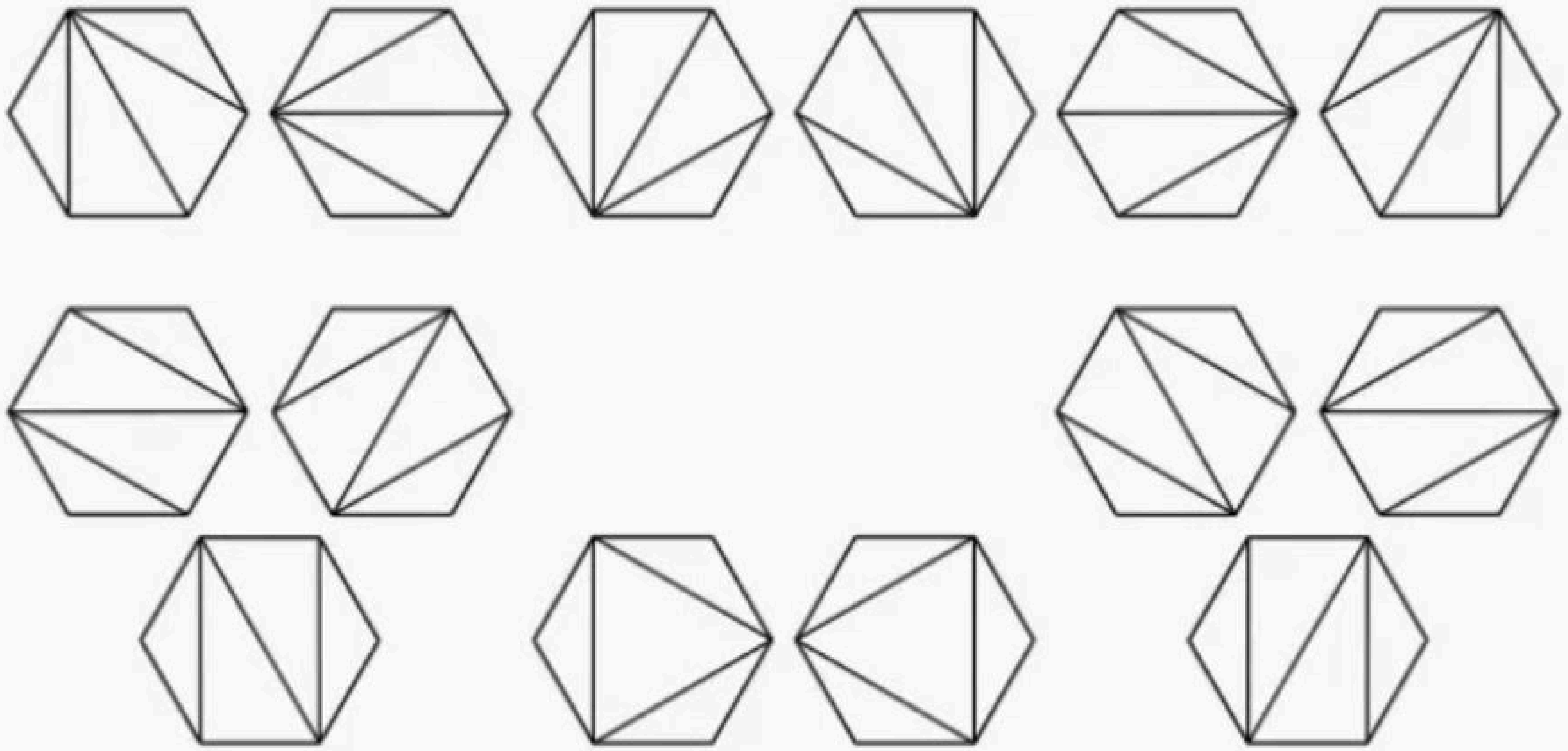
3 Cn is the number of different ways n + 1 factors can be completely parenthesized.

((ab)c)d (a(bc))d (ab)(cd) a((bc)d) a(b(cd))

4 Cn is the number of permutations of {1, ..., n} that avoid the permutation pattern 123 (or, alternatively, any of the other patterns of length 3).
For n = 3, these permutations are 132, 213, 231, 312 and 321.

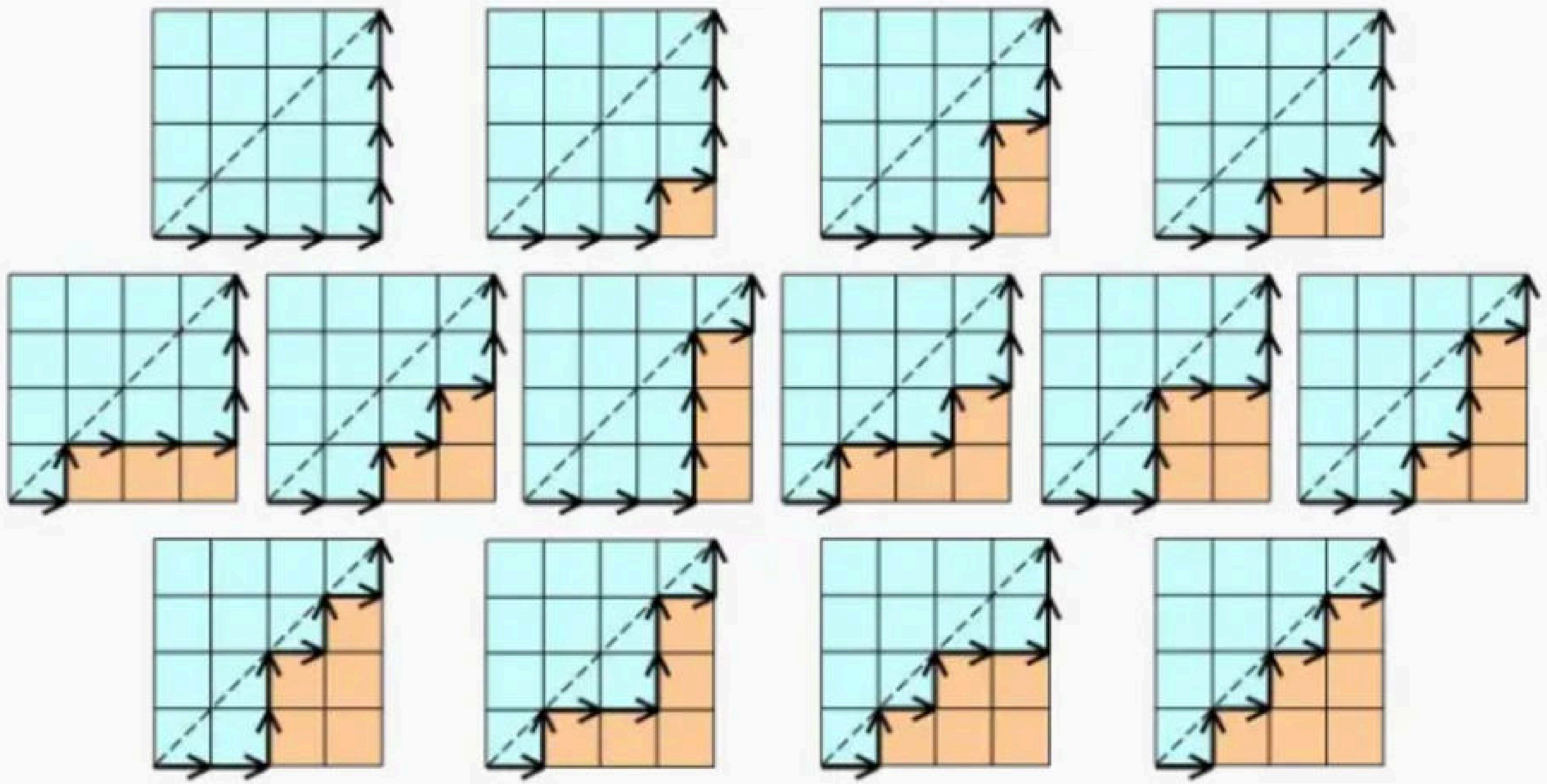
Applications

5. A convex polygon with n + 2 sides can be cut into triangles by connecting vertices with non-crossing line segments. The number of triangles formed is n and the number of different ways that this can be achieved is Cn



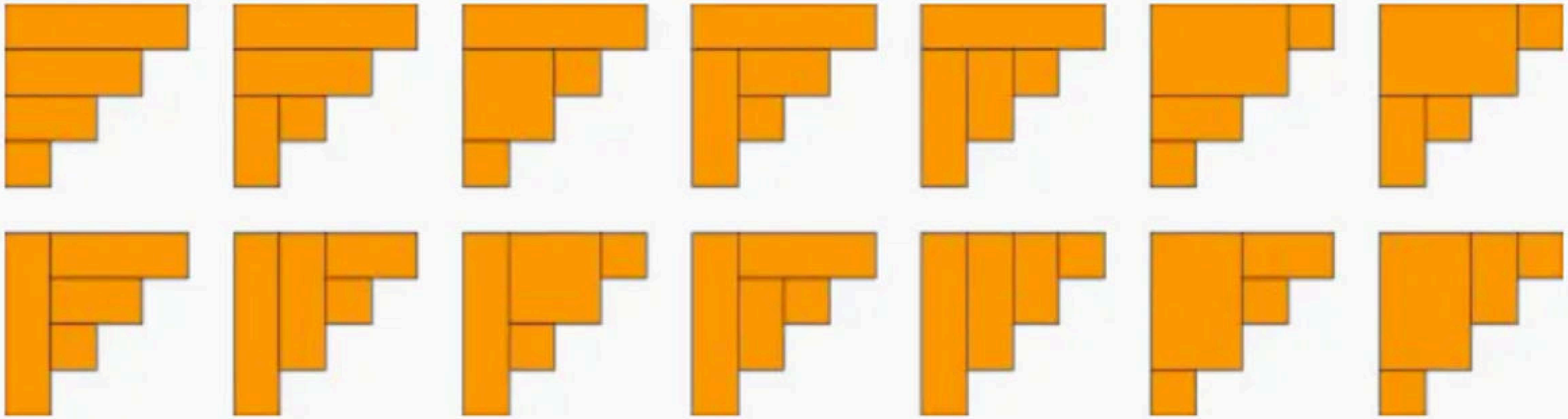
Applications

6) C_n is the number of monotonic lattice paths along the edges of a grid with n × n square cells, which do not pass above the diagonal.



Applications

7 C_n is the number of ways to tile a stair step shape of height n with n rectangles. The following figure illustrates the case n = 4



Calculation method 1

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}.$$

```
int C[maxn + 1][maxn + 1];
C[0][0] = 1;
for (int n = 1; n <= maxn; ++n) {
    C[n][0] = C[n][n] = 1;
    for (int k = 1; k < n; ++k)
        C[n][k] = C[n - 1][k - 1] + C[n - 1][k];
}
```

Calculation method 2

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

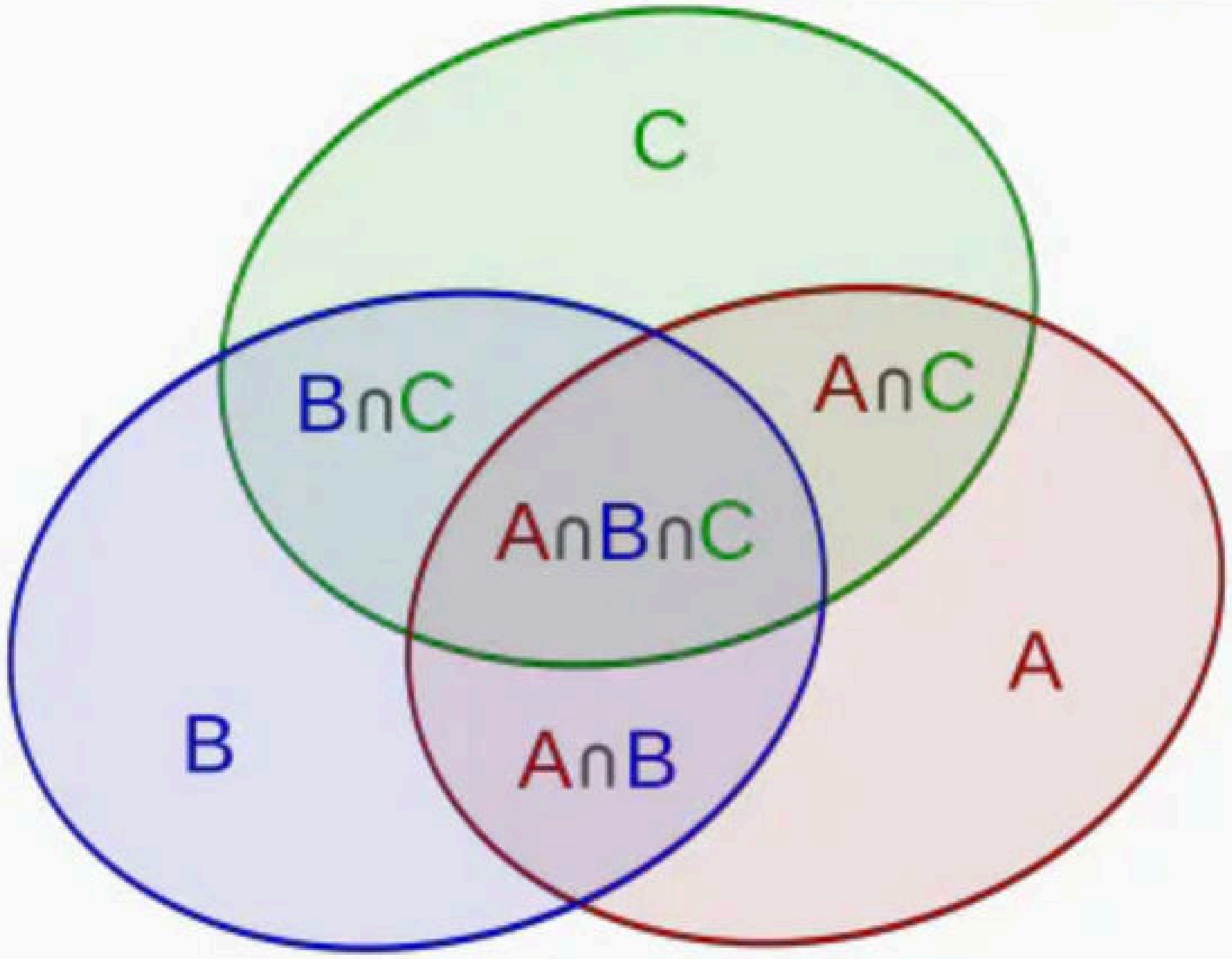
Derivation ->

$$\binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}.$$

CONCEPT

Inclusion Exclusion Principle

Inclusion Exclusion Principle



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Problem

Call a number prime-looking if it is composite but not divisible by 2,3 or 5.
The three smallest prime-looking numbers are 49,77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?