Puzzles - Programming and Non-Programming

Course on Mathematics & Puzzles for Interview Preparation

Arjun Arul • Lesson 6 • Feb 18, 2021

Maths and Puzzles

Lesson 6

CONCEPT

Catalan Numbers

Catalan Sequence

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, ...

1. ()()()
2. (())()
3. ()(())
4. ((()))
5. (()())

$C_0 = 1$

$C_1 = 1$

$C_2 = 2$

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}.$$

$$F_n = F_{n-1} + F_{n-2}$$

$$C_0$$

$$C_1$$

$$C_2 = \sum_{i=0}^{1} C_i C_{1-i} = C_0 C_1 + C_1 C_0$$

$$C_3 = \sum_{i=0}^{2} C_i C_{2-i}$$

$$= C_0 C_2 + C_1 C_1 + C_2 C_0$$

$$C_4 = C_0 C_3 + C_1 C_2 + C_2 C_1 + C_3 C_0$$

CFG

Balanced Paranthesis

(( ) ( ))

(( ))) (

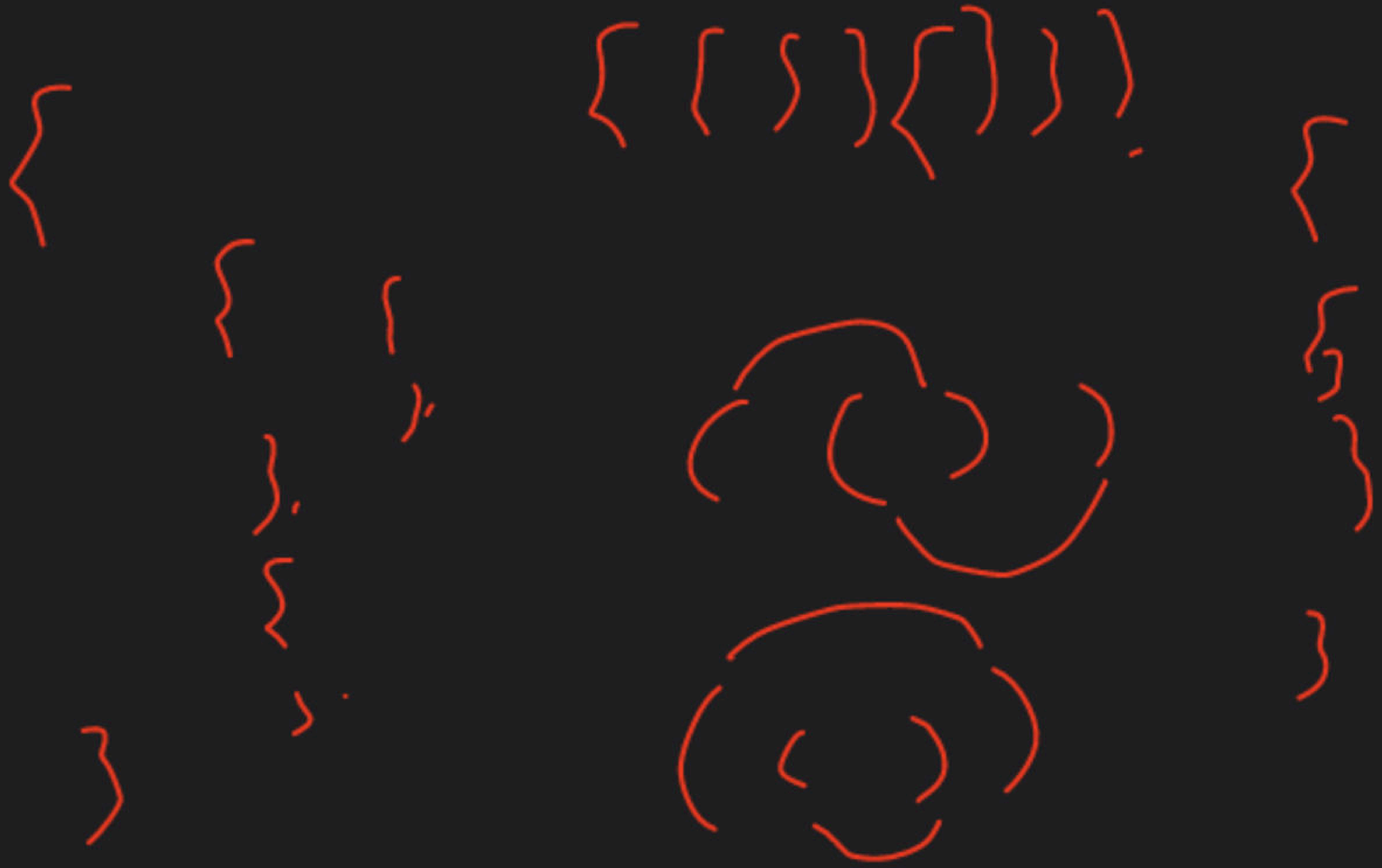
( )   ( )

(( ) ( ))

1. ( ) → Balanced

2. If A is balanced then (A)

2. If A & B are balanced then AB



($\rightarrow$ +1

1.) $\rightarrow$ −1

( ) ((( ) ( ) . . . . . ( )

× 1. Every prefix sum should be ≥ 0

2. Number of (= Number of)

(( ))) (

~~$n = 4$~~ $n = 2$

( )

$\underline{(}\ \underline{)}$

$\underline{(}\ \underline{)}\ \underline{(}\ \underline{)}$ $\underline{)}\ \underline{)}\ \underline{(}\ \underline{(}$

(( ))

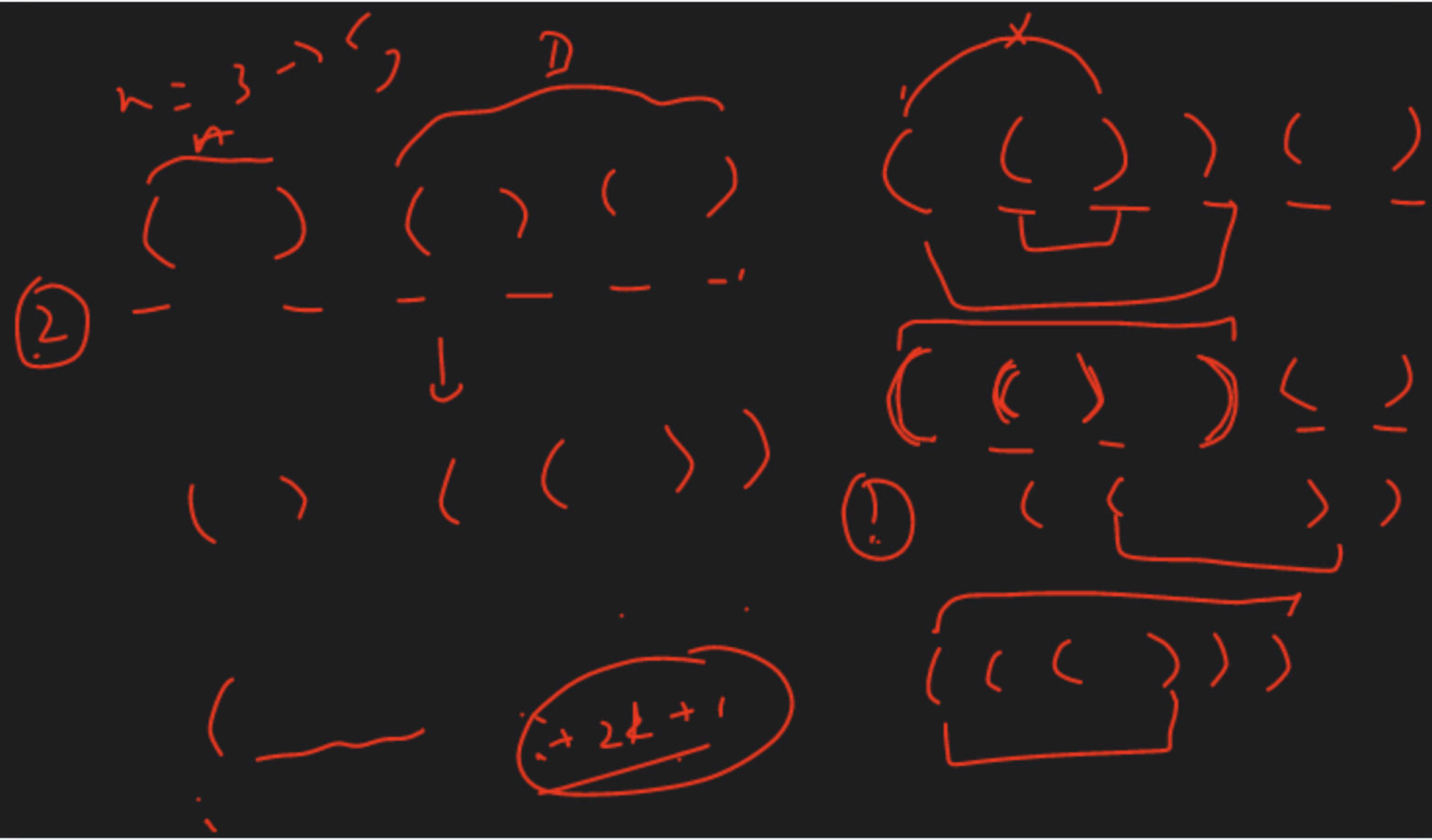
$\underline{\underline{n = 5}}$

$\underline{\underline{n}} \rightarrow \underline{\underline{2n}}$ $\underline{\underline{2n}}$

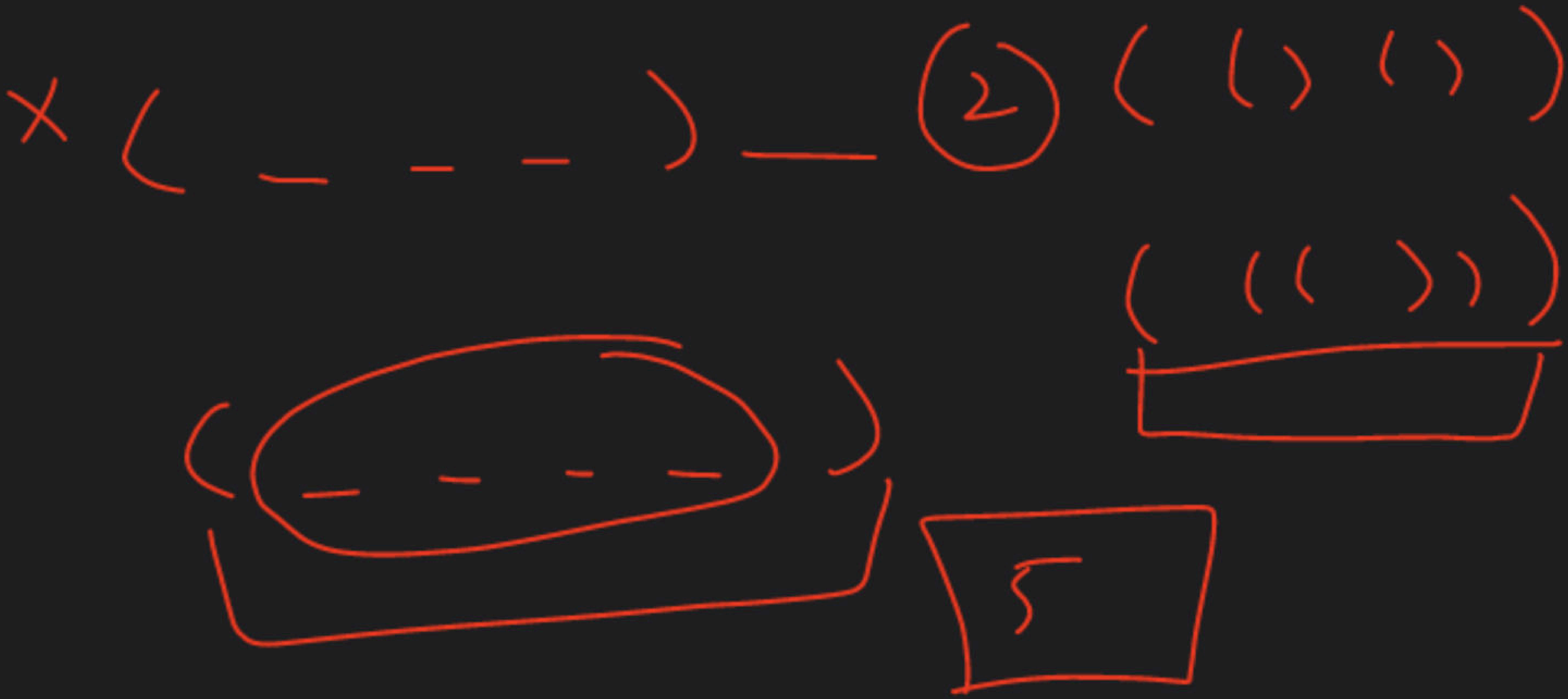
$n = 2 \rightarrow 2$

$n = 1$ $\frac{1}{\cdot} \rightarrow 1 \rightarrow$ (,

$n = 2 \rightarrow 2 \rightarrow$ ((



n = 3 -> ( )
A
D
2
1
+ 2k + 1


x
2
5

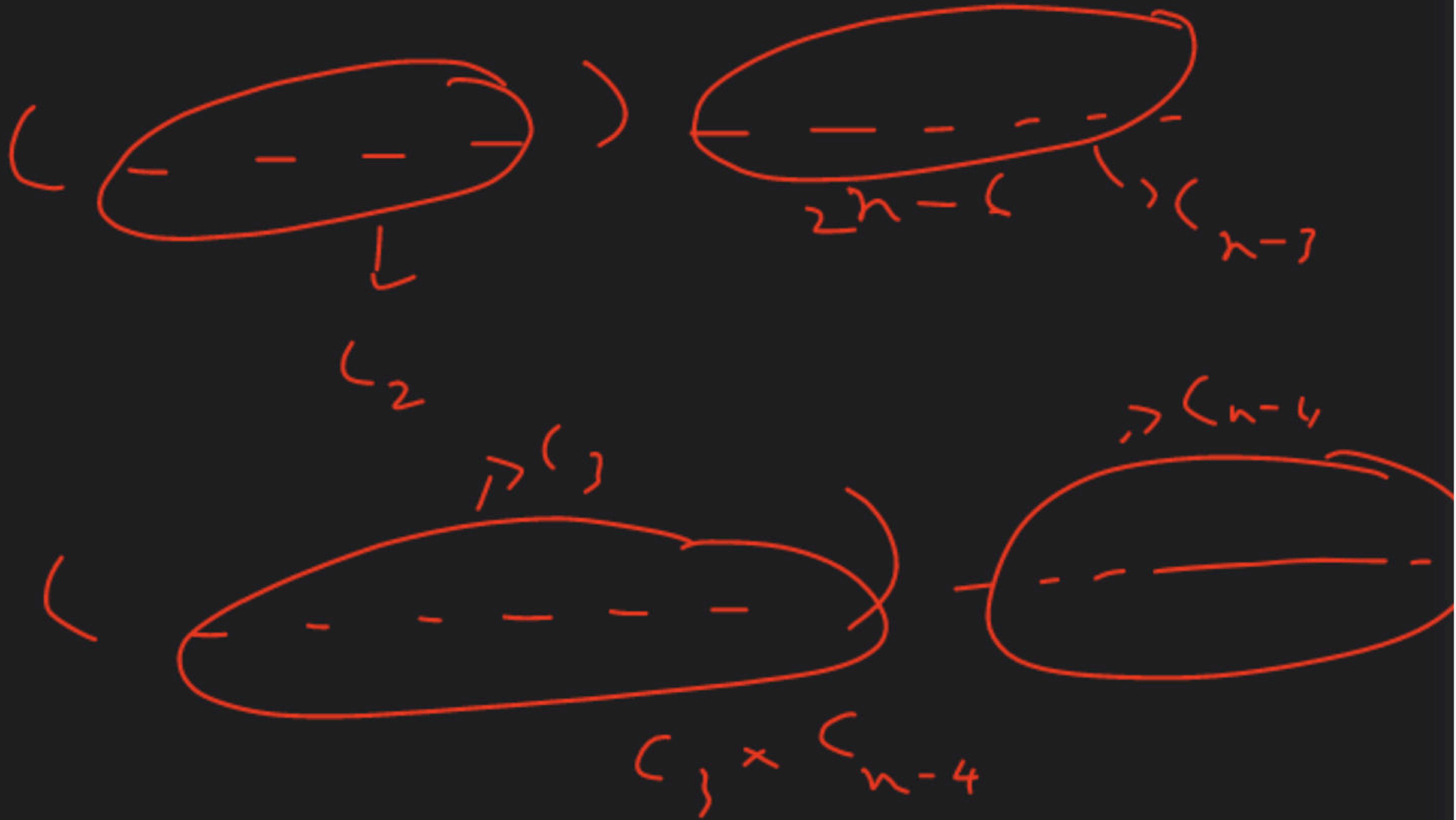
$$C_n = C_{n-1} + (1 \times \underline{\underline{C_{n-2}}}) + (C_2 \times C_{n-3}) + C_{n-1}$$

$\underline{\underline{n}}$

(( )) _ _ _ _ _ _ _ _ → C_{n-1}

( ) _ _ _ _ _ _ _ _

2n blanks.



C_2
$\to C_3$
$C_3 \times C_{n-4}$
$\to C_{n-4}$
$2n-6$
$\to C_{n-3}$

$$(\;) \; \underbrace{\overbrace{\text{— — — —}}^{C_i}}_{2i} \;) \; \underbrace{\overbrace{\qquad\qquad}}_{C_{n-i-1}} \;)$$

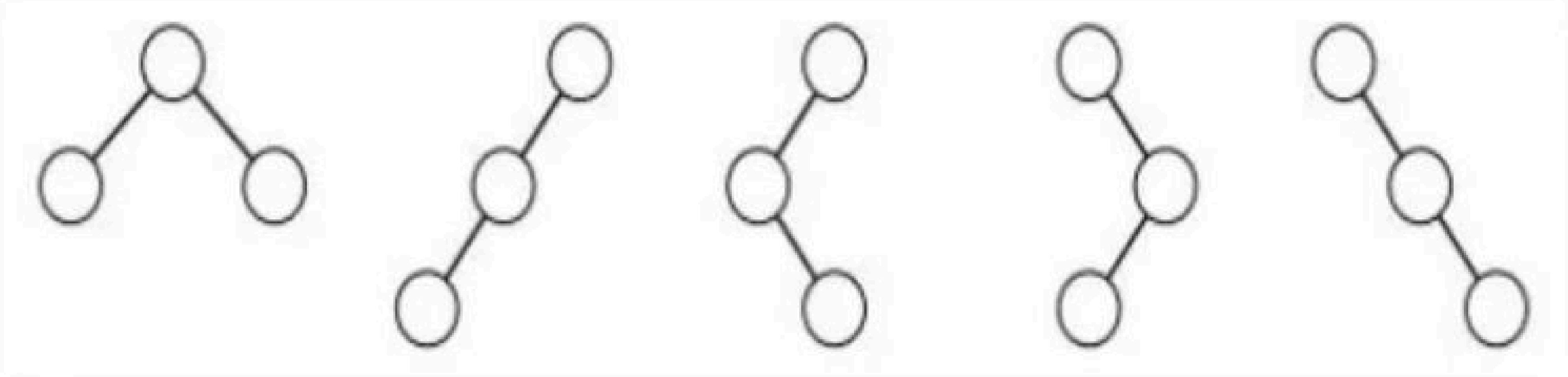
$$\boxed{C_i \times C_{n-i-1}} \qquad 2n-2-2i$$

$$\boxed{\sum_{i=0}^{n-1} C_i \times C_{n-i-1}} = 2(n-i-1)$$

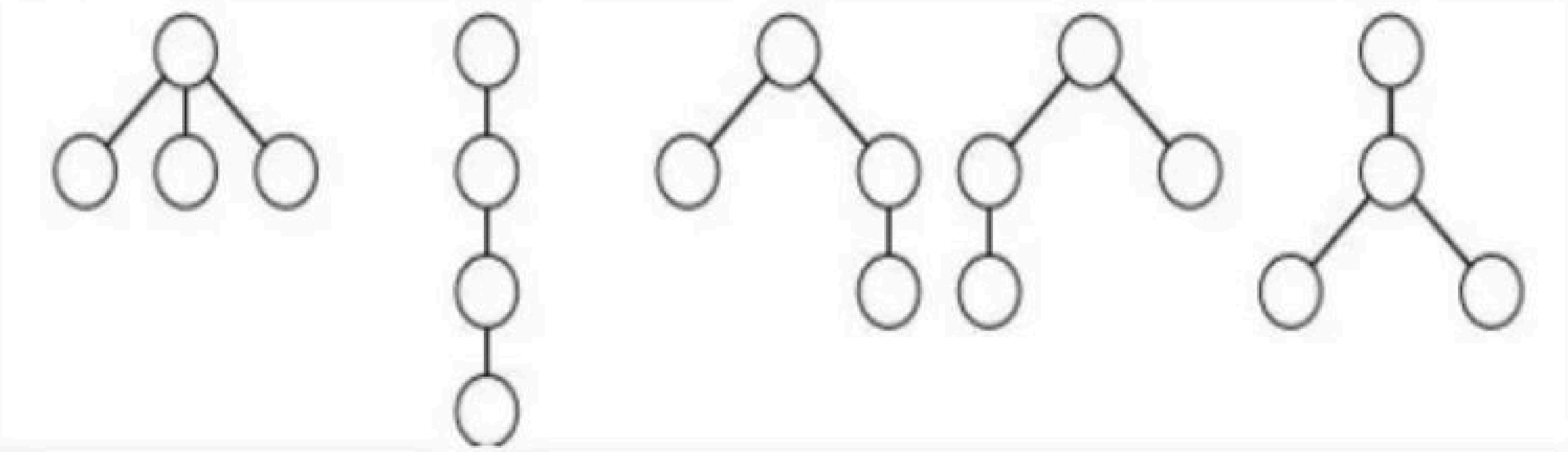
$$= C_n$$

Applications

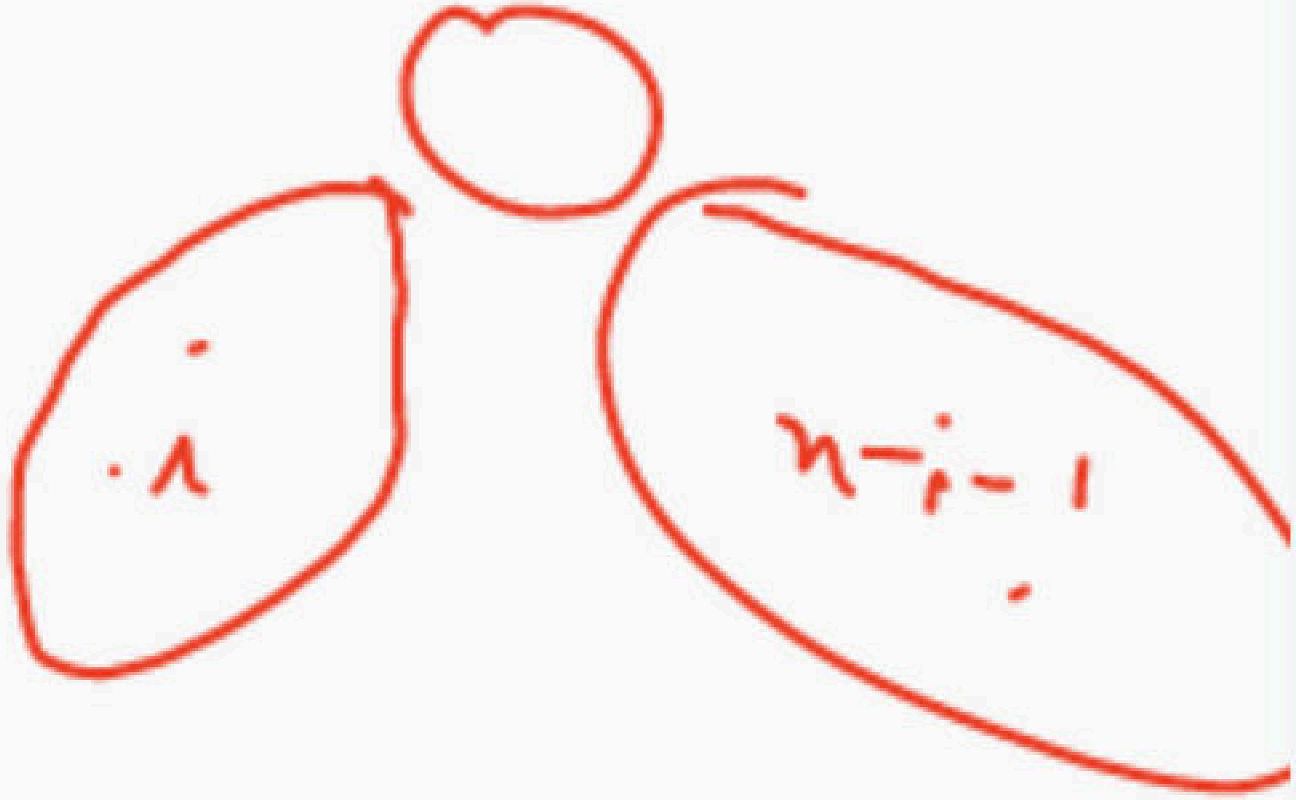
1. C_n Binary Trees of N nodes



2. C_{n-1} Rooted Trees of N nodes



N



Applications

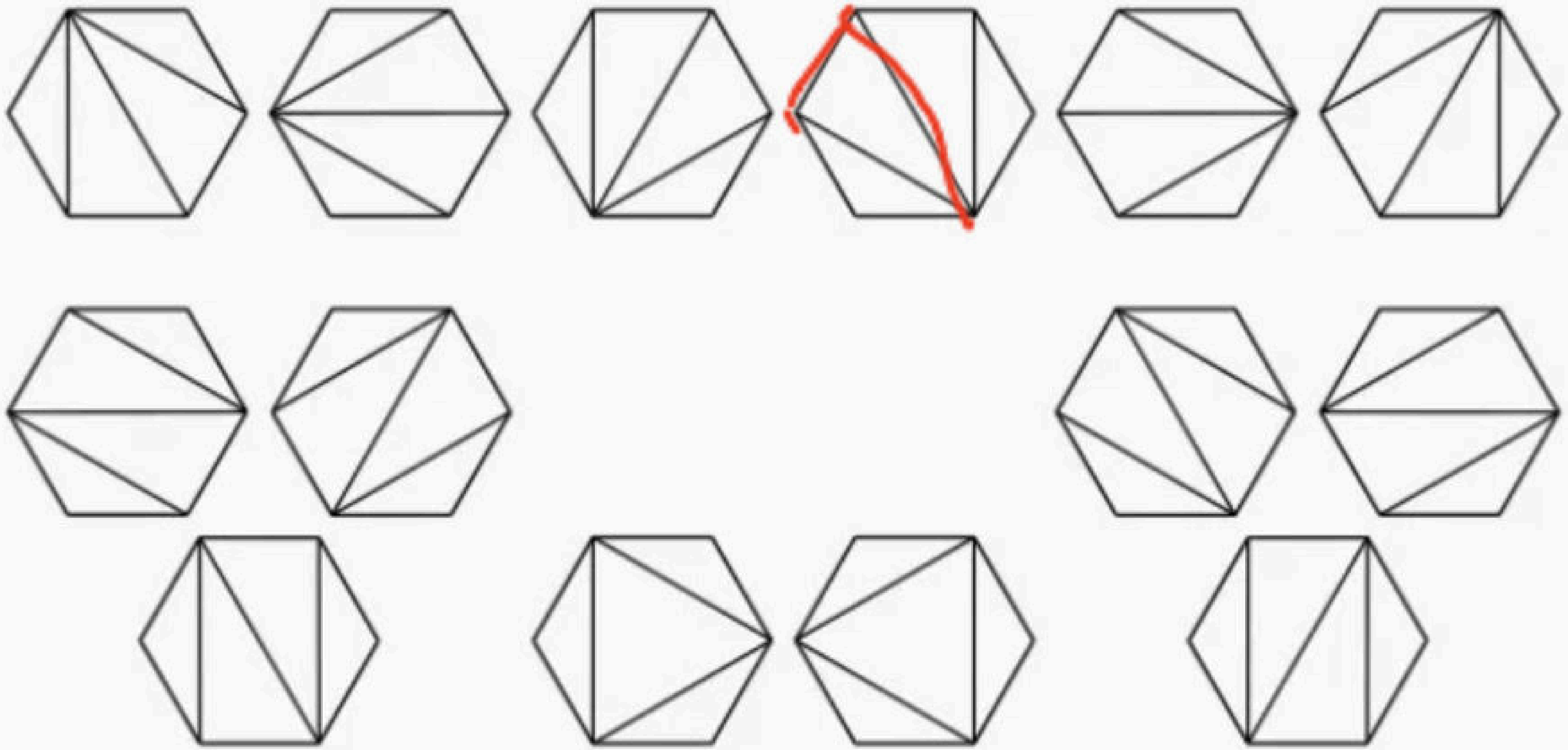
3 Cn is the number of different ways n + 1 factors can be completely parenthesized.

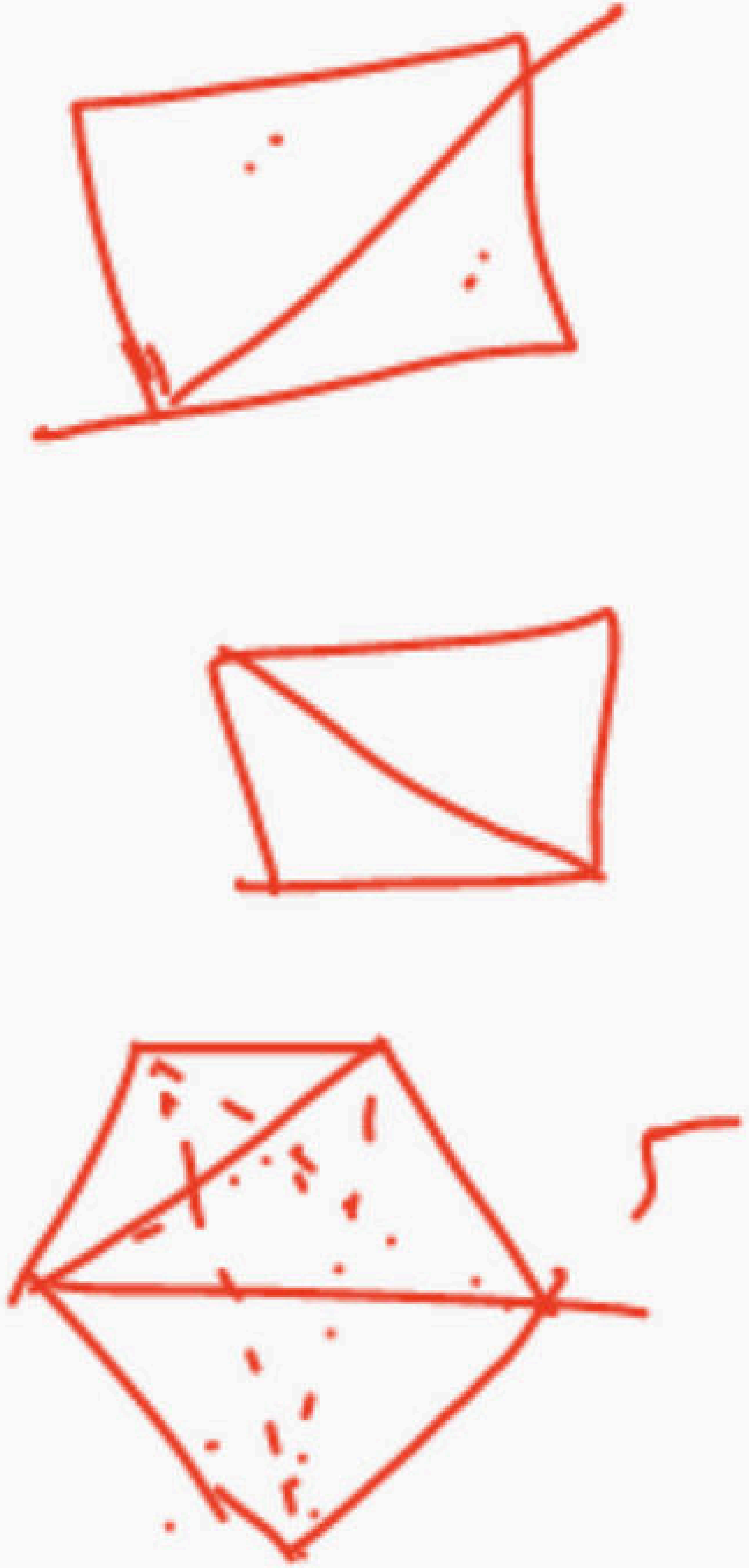
((ab)c)d   (a(bc))d   (ab)(cd)   a((bc)d)   a(b(cd))

4 Cn is the number of permutations of {1, ..., n} that avoid the permutation pattern 123 (or, alternatively, any of the other patterns of length 3).
For n = 3, these permutations are 132, 213, 231, 312 and 321.

Applications

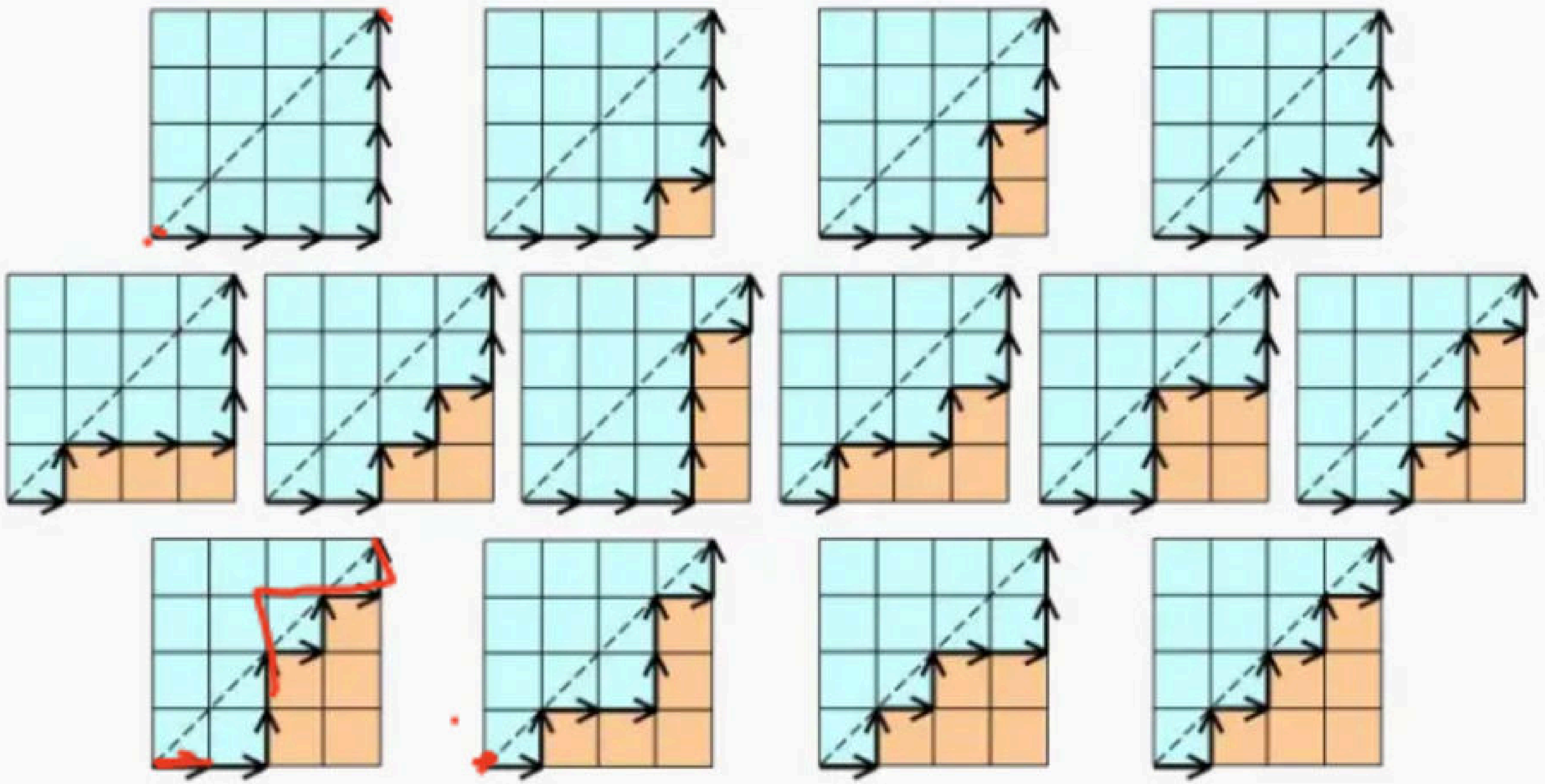
5. A convex polygon with n + 2 sides can be cut into triangles by connecting vertices with non-crossing line segments. The number of triangles formed is n and the number of different ways that this can be achieved is Cn





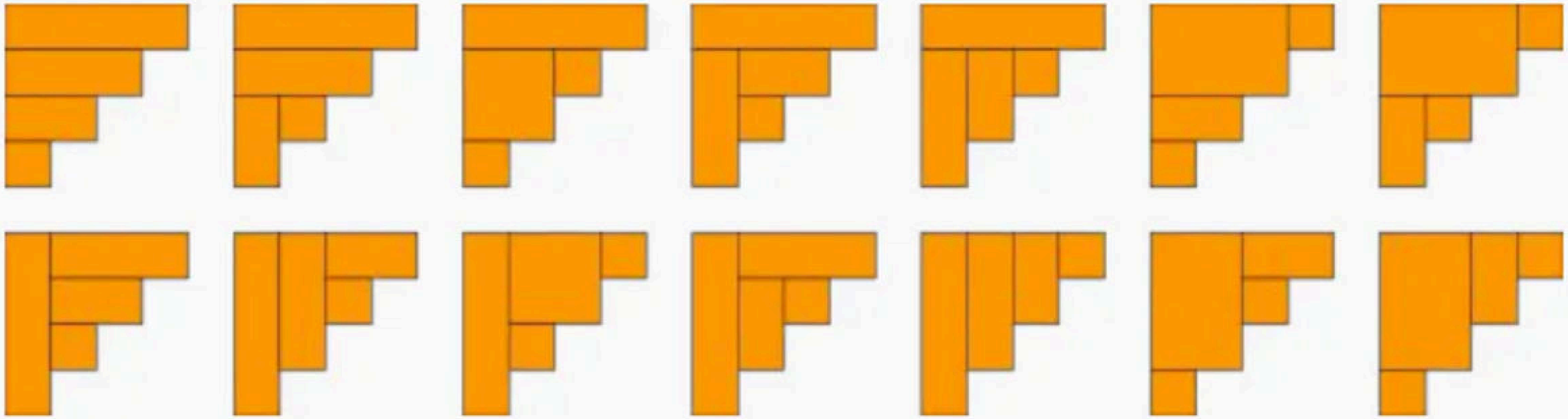
Applications

6. Cn is the number of monotonic lattice paths along the edges of a grid with n × n square cells, which do not pass above the diagonal.



Applications

7. Cn is the number of ways to tile a stair step shape of height n with n rectangles. The following figure illustrates the case n = 4



Calculation method 1

n^2

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}.$$

```
const int MOD = ....
const int MAX = ....
int catalan[MAX];
void init() {
    catalan[0] = catalan[1] = 1;
    for (int i=2; i<=n; i++) {
        catalan[i] = 0;
        for (int j=0; j < i; j++) {
            catalan[i] += (catalan[j] * catalan[i-j-1]) % MOD;
            if (catalan[i] >= MOD) {
                catalan[i] -= MOD;
            }
        }
    }
}
```

$\boxed{O(n^2)}$

Time complexity $O(n^2)$

$$\boxed{C_n = \frac{1}{n+1}\,{}^{2n}C_n}$$

$n < 1$

Calculation method 2

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

% prime

$$\frac{(2n)!}{n!\,n!}$$

Derivation -> $$\binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}.$$

$lg(p)$

QUIZ

Catalan Numbers

We can find the catalan number upto 'N' modulo 'M' in O(log M) time per query. All we will need is O(N) precomputation time and space.

$$\frac{1}{N+1}\binom{2\cancel{N}N}{N}$$

A. True

B. False

Please choose the correct relation.

$$\frac{1}{n+1} \, {}^{2n}C_n = \frac{1}{n+1} \frac{(2n)!}{n!\,n!} \qquad \frac{(2n)!}{(n+1)!\,n!}$$

A. $C_n = \dfrac{(2n)!}{(n)!(n)!}$

C. $C_n = \dfrac{(2n)!}{(n+1)!(n+1)!}$

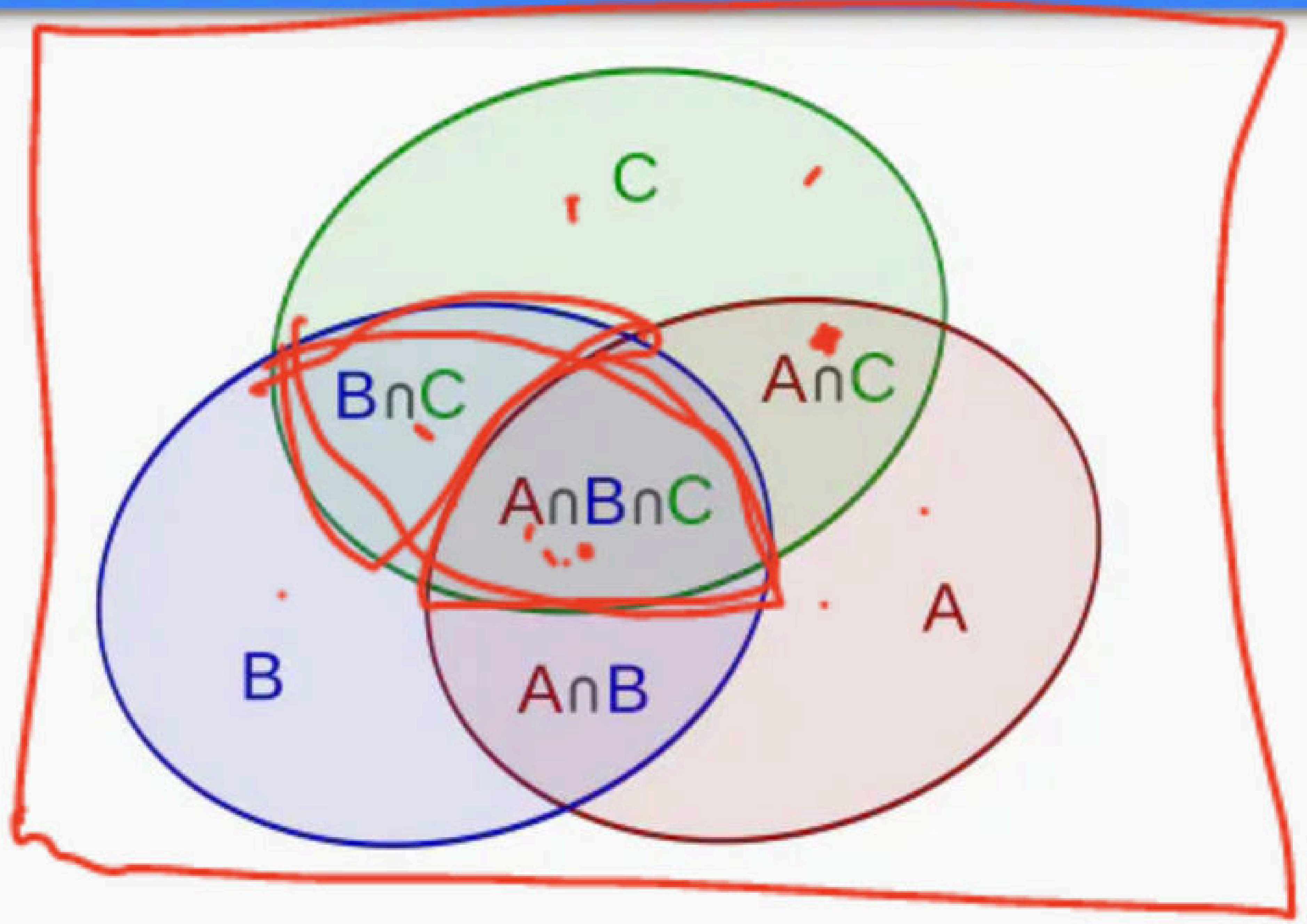
B. $C_n = \dfrac{(2n)!}{(n+1)!(n-1)!}$

D. $C_n = \dfrac{(2n)!}{(n)!(n+1)!}$

CONCEPT

Inclusion Exclusion Principle

Inclusion Exclusion Principle



$$|A \cup B \cup C| = |A| + |B| + |C| - \underline{|A \cap B|} - |A \cap C| - |B \cap C| + \underline{|A \cap B \cap C|}.$$

$$|\cup A_i| = \sum |A_i| - \sum |A_i \cap A_j|$$

$$+ \sum |\underbrace{A_i \cap A_j \cap A_k}|$$

$$(-1)^{j+1} \quad - - \qquad -- -$$

Find number of integers in [1, 100]

which are divisible by 2 or 3 or 5

50 + 33 − 16

83 − 16

$\boxed{67}$

2 a) a 5

(1, 100) 103 -

29

74

50 + 33 + 20

− 16 − 10

− 6

+ 3

Problem

Call a number prime-looking if it is composite but not divisible by 2,3 or 5.
The three smallest prime-looking numbers are 49,77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

CONCEPT

Probability

Probability

$$probability = \frac{numberofdesiredoutcomes}{totalnumberofoutcomes}$$

$$\frac{3}{6} = \frac{1}{2}$$

$$P(A) = \sum_{x \in A} p(x).$$

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

Events and Properties

$$P(\overline{A}) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - \cancel{P(A \cap B)}$$

if, $A \cap B = \emptyset$ then, $P(A \cap B) = 0$ and, $P(A \cup B) = P(A) + P(B)$

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$\frac{1}{6}$ $\frac{1}{3}$

$$P(A \cap B) = P(A)P(B|A).$$

$P(4 / \text{even}) = \frac{\frac{1}{6}}{\frac{1}{2}}$

Events A and B are independent if, $P(A|B) = P(A)$ and $P(B|A) = P(B)$,

If Events A and B are independent - $P(A \cap B) = P(A)P(B).$

$\frac{1}{6} \times 2 = \boxed{\frac{1}{3}}$

Probability - Problems

QUIZ

Probability

A locked suitcase has 4 number wheels, each labeled 0 to 9. In the passcode, no two digits are same. What is the probability of cracking it on the first try?

$\boxed{10 \times 9 \times 8 \times 7}$

$\frac{1}{10 \times 9 \times 8 \times 7}$

1 1 1 1

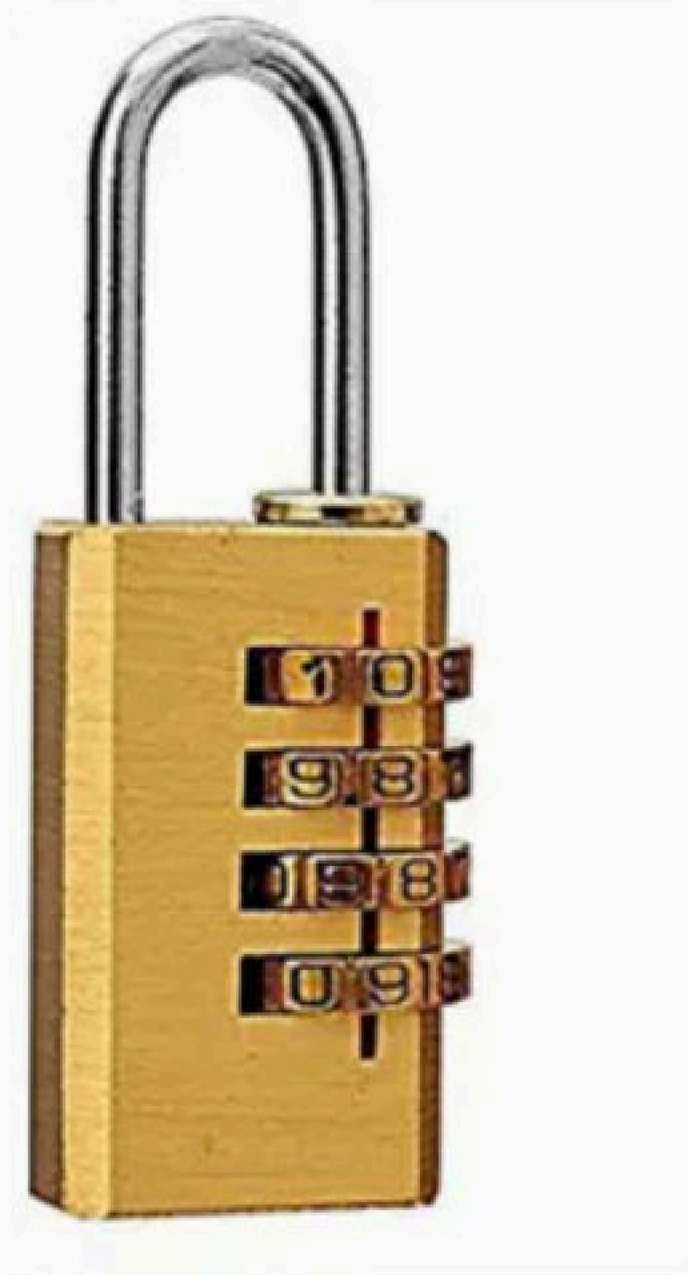
$\frac{10!}{6!} = {}^{10}P_4$

A. $\frac{1}{{}^{10}C_4}$

B. $\boxed{\frac{1}{{}^{10}P_4}}$

C. $\frac{1}{{}^{9}C_4}$

D. $\frac{1}{{}^{9}P_4}$



Consider the given statements and find the probability of a defective bulb given it is red

X = The bulb is red
Y = The bulb is defective

$$\frac{p(x \cap y)}{p(x)}$$

A. $\frac{P(X \cup Y)}{P(Y)}$

B. $\frac{P(X \cap Y)}{P(Y)}$

C. $\frac{P(X \cup Y)}{P(X)}$

D. $\frac{P(X \cap Y)}{P(X)}$

CONCEPT

Expected Value

Random Variable and Expected Value

$E[x+y] =$

$E[x] + E[y]$

1. Random Variable
2. Expected Value $\sum_x P(X = x)x,$
3. Linearity of Expectation $E[X_1 + X_2 + X_3 + \cdots] = E[X_1] + E[X_2] + E[X_3] + \cdots$
4. Expected value of a dice throw?

$$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$$

$$\frac{\cancel{6} \times 7}{2 \times \cancel{6}} = \boxed{3.5}$$

Expected Value for sum of two dice throws?

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

Expected Value - Problems

QUIZ

Expected Value

For Linearity of Expectation to hold, random variable events must be independent of each other.

A. True

B. False

The expected marks in the geometry section is 70, and 60 for the algebra section. ~~If you score more than 50 in geometry then it is 80% probable that you will score more than 70 in algebra.~~ What is the expected marks for both the sections combined?

$$E[G+A] = E(G) + E(A)$$

$$70 + 60$$

$$= 130$$

A. 106

B. 118

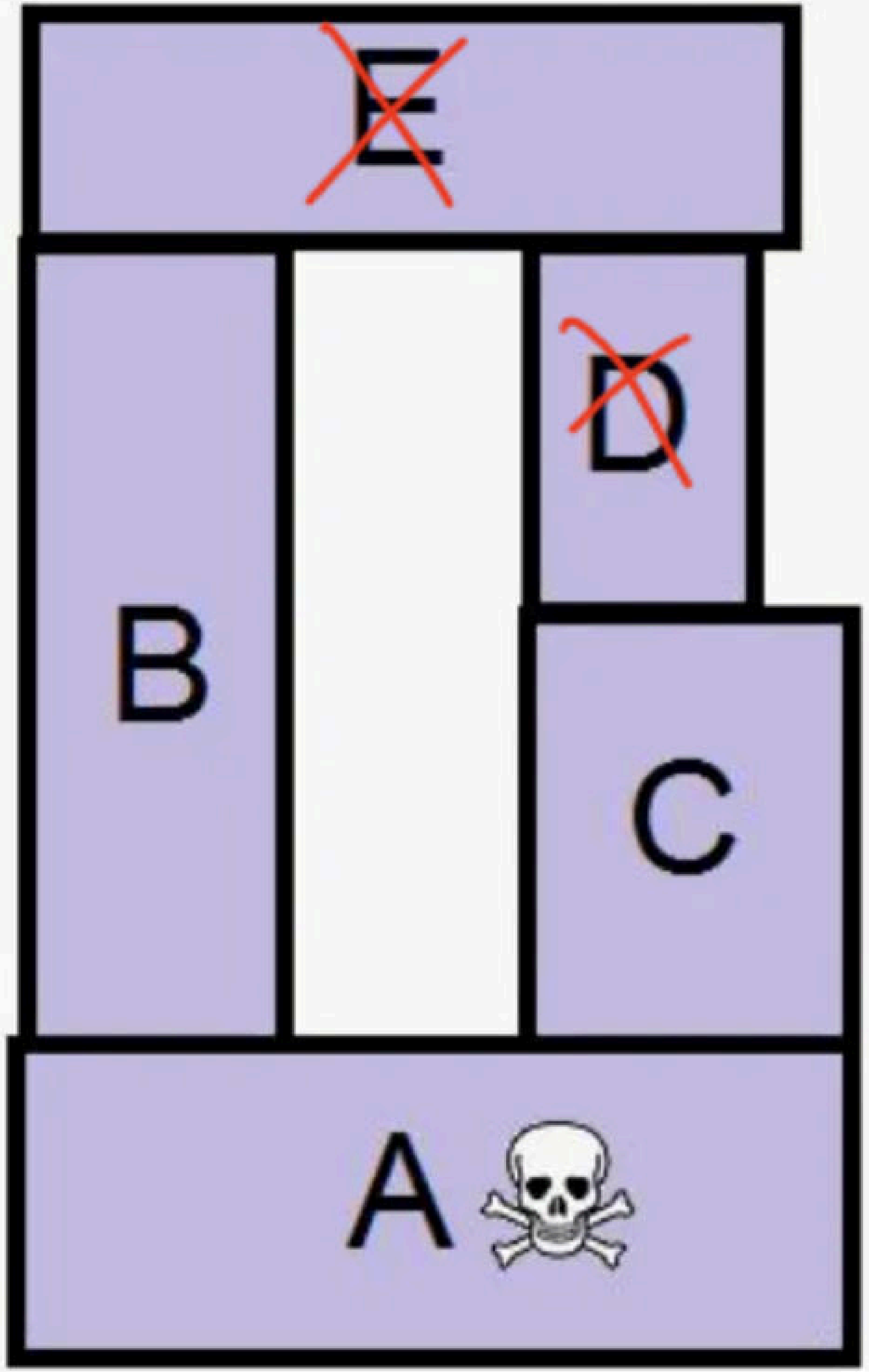
C. 126

D. 130

Puzzles!

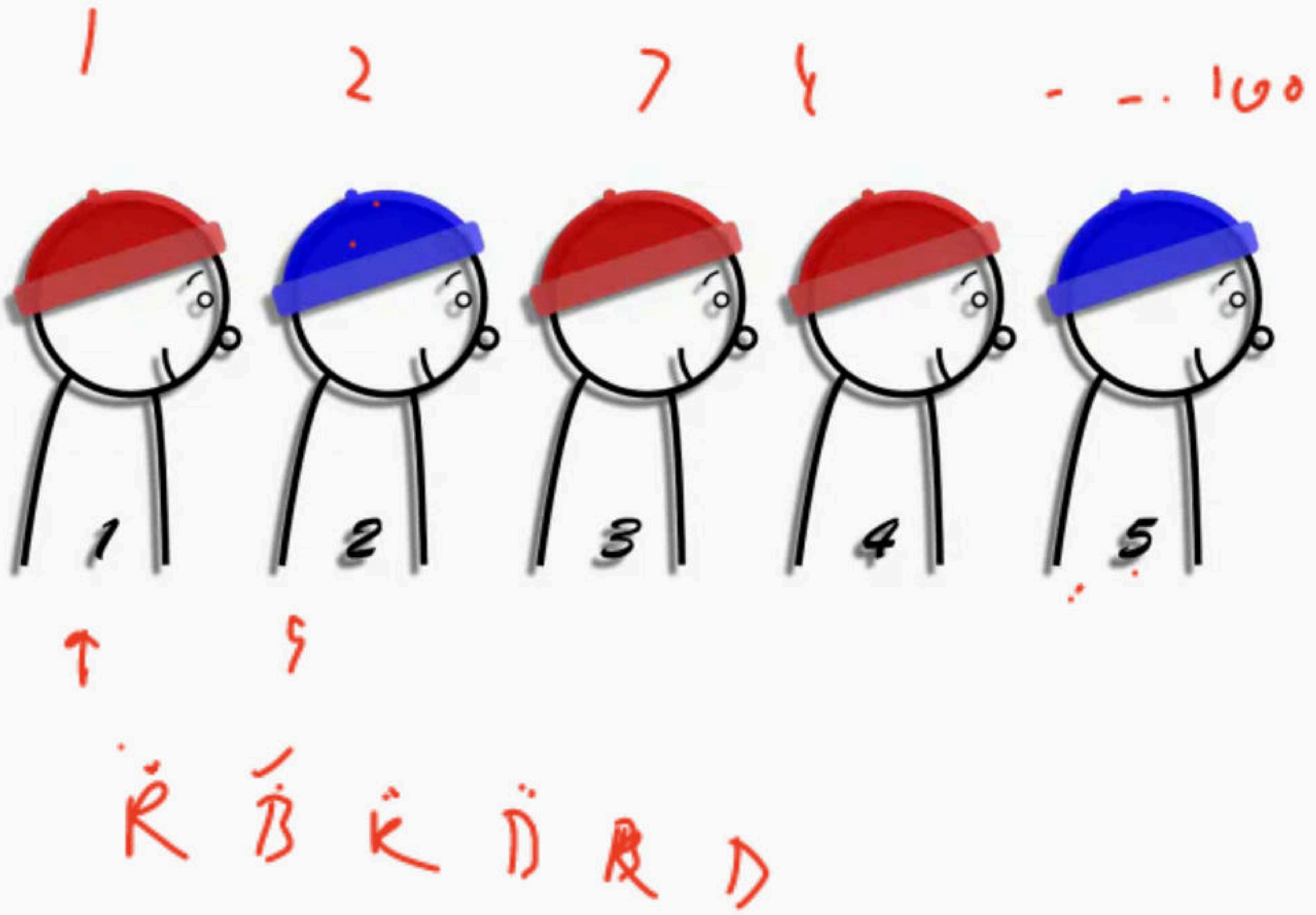
Nim Game

A. Remove B
B. Remove C
C. Remove D
D. Remove E



100 Prisoners and Coloured Hats

A. All can be saved
B. 99 saved
C. 50 saved
D. All perish



Measuring Water using Jugs

A. I found the way!
B. Can't solve it
C. It is impossible
D. Need more time

Rope Burning Puzzle

A. I found the way!
B. Can't solve it
C. It is impossible
D. Need more time

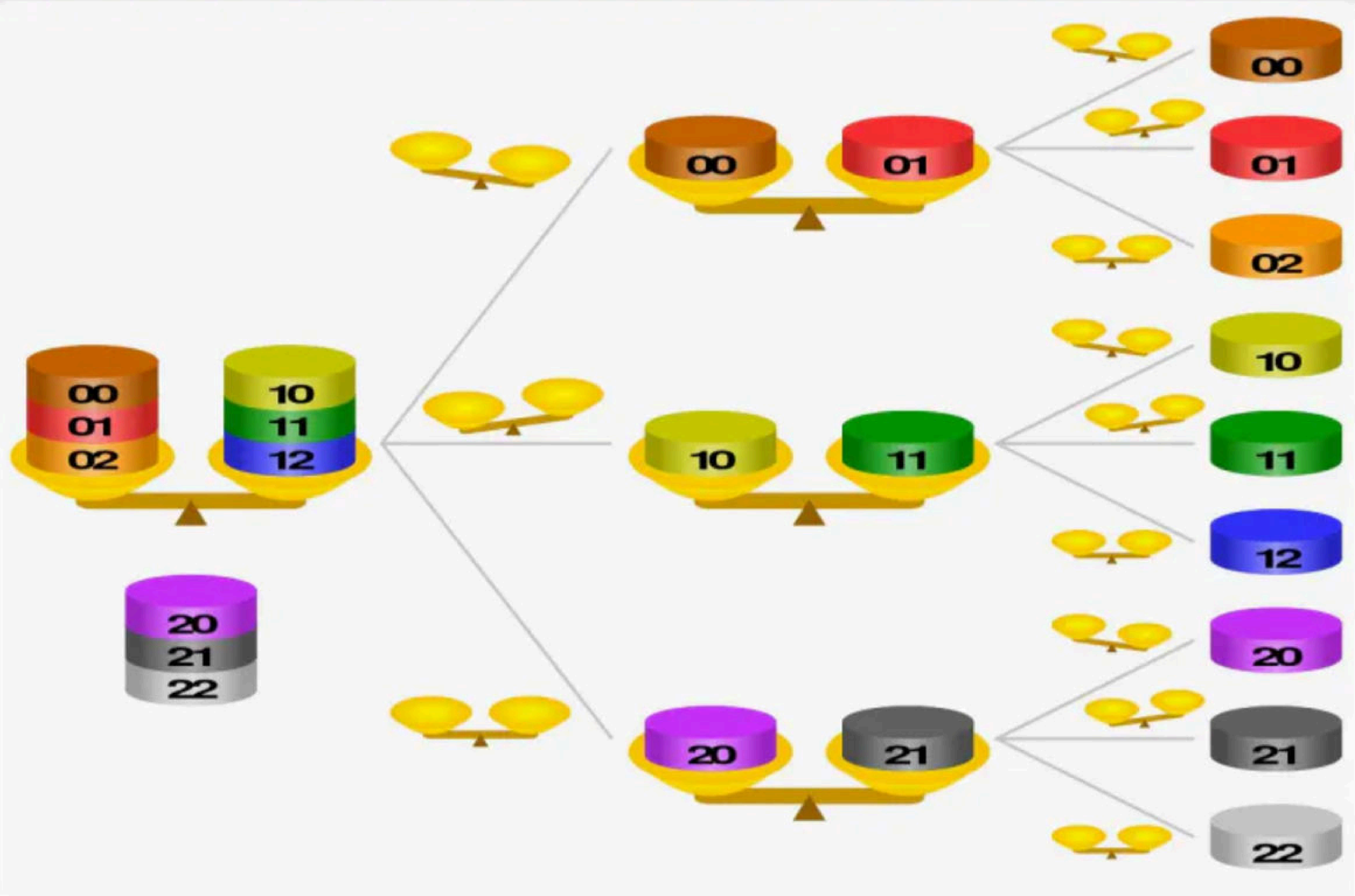
9 Coin Puzzle

A. 2
B. 3
C. 4
D. 5

Solution visualization on next slide!

Solution ahead!

9 Coin Puzzle Solution



Knights and Knaves Puzzle 1

John says, "We are both knaves."
Determine who is what.

A. Knight - John Knave - Bill
B. Knight - Bill Knave - John
C. Knight - John and Bill
D. Knave - John and Bill

Knights and Knaves Puzzle 2

John says, "We are the same kind," but Bill says, "We are of different kinds." Determine who is what.

A. Knight - John   Knave - Bill
B. Knight - Bill   Knave - John
C. Knight - John and Bill
D. Knave - John and Bill

Knights and Knaves Puzzle 3

John and Bill are standing at cross-roads. Can you determine the road to freedom by asking a yes-no question?

A. Yes
B. No
C. Impossible
D. Can’t Say

3 Doors & Monty Hall Problem

A. Select D1
B. Select D2
C. Select D3
D. Abandon the show