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UNIT-2

RELATIONS

Let A and B be 2 sets, the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the cartesian product or cross product or product set of A and B, denoted by

$$\boxed{A \times B = \{(a,b) \mid a \in A \ \& \ b \in B\}}$$

$$A \times B \neq B \times A$$

$$B \times A = \{(b,a) \mid b \in B \ \& \ a \in A\}$$

Ex: $A = \{1, 2, 3\}$ $B = \{4, 5\}$

$$A \times B = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$$

$$B \times A = \{(4,1), (4,2), (4,3), (5,1), (5,2), (5,3)\}$$

Note → $A \times B$ can be defined even when $B = A$

$\therefore$ $\boxed{A^2 = A \times A = \{(a,a) \mid a \in A \ \& \ a \in A\}}$

→ If A and B are finite sets with $|A| = m$ and $|B| = n$ then $A \times B$ is a finite set with $|A \times B| = mn$

$A = \{1, 2, 3, 4, 2, 5, 6, 7, 5\}$

$|A| = 7$

⇒ $A = \{1, 0\}$ $B = \{2, -2\}$. $C = \{0, -1\}$

$A \times B \times C = ?$

$A \times B = \{(1,2); (1,-2), (0,2), (0,-2)\}$

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$A \times B \times C = \{(1,2,0), (1,-2,0), (0,2,0), (0,-2,0),$
$(1,2,-1), (1,-2,-1), (0,2,-1), (0,-2,-1)\}$

Q.) find x and y in each of the following cases.

(i) $(2x, x+y) = (6,1)$

(ii) $(y-2, 2x+1) = (x-1, y+2)$

(i) $2x = 6$ $\quad$ $3+y = 1$

$\boxed{x=3}$ $\quad$ $\boxed{y=-2}$

(ii) $y-2 = x-1$ $\quad$ $2x+1 = y+2$

$x-y = -1$ $\quad$ $2x-y = 1$

$y-x = 1$

$2x-y = 1$ $\quad$ $2-y = -1$

$x = 2$ $\quad$ $\boxed{y=3}$

$\boxed{x=2}$

Q.) let $A = \{1,3,5\}$ $B = \{2,3\}$, $C = \{4,6\}$
write down the following.

(i) $A \times B$ $\quad$ (iv) $(A \cup B) \times C$ $\quad$ [HW]

(ii) $A \times A$ $\quad$ (v) $(A \times B) \cap (A \times C)$

(iii) $B \times C$

(i) $A \times B = \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3)\}$

(ii) $A \times A = \{(1,1),$

Q) Let $A = \{1, 2, 3, 4, 6\}$ Let R be relation on set A $(a,b) \in R$ iff a is multiple of b write down R as a set of ordered pair.

$A \times A =$ $R = \{(1,1), (2,1), (3,1), (4,1), (6,1),$
$(2,2), (3,3), (4,4), (6,6),$
$(4,2), (6,2), (6,2) (6,3)\}$

Q: If A is set with m elements, B is set with n elements find the no. of relations from A to B.

Ans) Since $|A| = m$ $|B| = n$
$|A \times B| = mn$
$\therefore$ $A \times B$ has 2^{mn} no. of subsets

Suppose, $|A| = 3$ $|B| = 2$

(64) 2^6 relations are possible.

Q.) Let A and B be finite sets with $|B| = 3$ If there are 4096 relations from A to B what is $|A|$?

$\boxed{n=3}$ $\boxed{m=?}$

$4096 = 2^{3 \times m}$

$4096 = 16 \, 2^m$

$2^m = \frac{4096}{16}$

$2^m = 256$

$2^m = 2^9$

$\boxed{m=9}$

$\boxed{m=4}$

$3m \log 2 = \log 4096$

$$m = \frac{\log 4096}{3 \log 2} = 4 \quad \boxed{m = 4}$$

Q.) Determine $A = \{1, 2, 3\}$
$B = \{2, 4, 5\}$

(i) $(A \times B)$ (ii) No. of relations A to B
(iii) No. of binary relations on set A.
(iv) No. of relations from A to B that contain $(1, 2)$ and $(1, 5)$
(v) No. of relations from A to B that contain exactly 5 ordered pairs.
(vi) No. of binary relations on set A that contain atleast 7 ordered pair.

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Zero-one Matrices

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

$$a_i b_j = \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$$

$A = \{a_1, a_2, a_3 \ldots a_m\}$ m

$B = \{b_1, b_2, b_3 \ldots b_n\}$ n

$A \times B = \{(a_1 b_1), (a_2 b_2) \ldots (a_m, b_n)\}$

order $= m \times n$

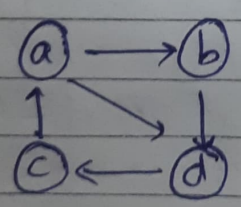
M_R or $M(R)$ ⇒ Matrix Relation

Directed Graph. $G = (V, E)$

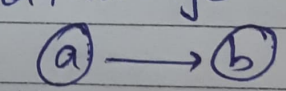
V – set of vertices

E – set of edges

All the edges should have direction is termed as Directed graph.



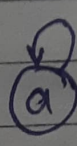
for this edge



Source vertex — terminus vertex.

e

→ for e which is neither source vertex nor terminus vertex is called as Isolated vertex.

→  If the the vertex has source and terminus are same is termed as LOOP.

~~→ No. of the vertex towards~~

→ Indigree: (a) = 1 (b) = 1 (c) = 1 (d) = 2 ⇒ 5

→ Outdigree: Total no. of edges that are leaving the vertex.

(a) = 2
(b) = 1
(c) = 1
(d) = 1
5

Sum of the Indigrees is equal to sum of the outdegree is equal to total no. of edges in the graph.

[If this is not satisfied then the graph is wrong]

---

* Consider the sets $A = \{a_1, a_2, a_3 \ldots a_m\}$ and $B = \{b_1 b_2 b_3 \ldots b_n\}$ of orders m and n respectively then $A \times B$ consists of all the ordered pairs of the form $(a_i b_j)$, $1 \le i \le m$, $1 \le j \le n$ which are $m \times n$ in number.

Let R be the relation from A to B so that R is a subset of $A \times B$.

Now let us put $m_{ij} = (a_i, b_j)$ and assign the values 1 or 0 to m_{ij} according to the following rule

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

The $m \times n$ formed by these m_{ij}'s is called Matrix of Relation R or Relation Matrix for R

Denoted by M_R or $M(R)$.
Since $M(R)$ contains 0 and 1 as its elements $M(R)$ also called **Zero one Matrix** for relation R.

Diagram of Relation

Let R be the relation on a finite set $A = \{a, b, c, d\}$ then R can be represented pictorially as described below.
Draw a small circle or a bullet for each element of A. A vertex from which an edge leaves is called source vertex and vertex where an edge ends is called Terminus vertex for that edge.

→ A vertex which is neither a source or terminus is called Isolated vertex. An edge for which the source & terminus are one and the same vertex is called Loop.

→ The no. of edges incident towards the vertex is called Indegree of vertex.

→ No. of edges leaving the vertex is called Outdegree of the vertex.

① Let $A = \{1, 2\}$ & $B = \{p, q, r, s\}$ relation R from A to B be defined by
$R = \{(1, q), (1, r), (2, p), (2, q), (2, s)\}$
write down the matrix R.

Sol^n $m = 2$ $n = 4$
2×4 matrix

$$M_R = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

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Q2.) Let $A = \{1, 2, 3, 4\}$ and let R be the relation on set A defined by

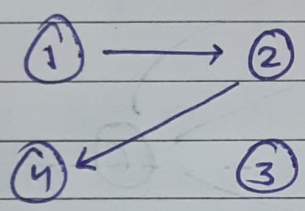
$x R y$ iff $y = 2x$

x is related to y — if and only if

(i) Write down R as a set of ordered pair.
(ii) Diagraph of R defined on set A.
(iii) Determine the indegree and outdegree of the vertices in the digraph.

(i) $R = \{(1,2), (2,4)\}$

(ii)



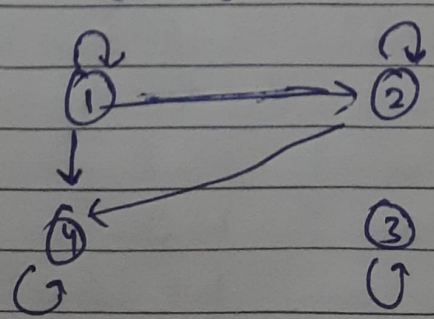
(iii) indegree $(1) = 0$
$(2) = 1$
$(3) = 0$
$(4) = 1$
$\overline{\quad 2 \quad}$

outdegree $(1) = 1$
$(2) = 1$
$(3) = 0$
$(4) = 0$
$\overline{\quad 2. \quad}$

Q.) Let $A = \{1, 2, 3, 4\}$ and R be relation on set A defined by set A $x R y$ iff $x | y$.
[x divides y]

(i)
$R = \{(1,2), (1,3), (1,1), (1,4), (2,4), (2,2), (3,3), (4,4)\}$

(ii)



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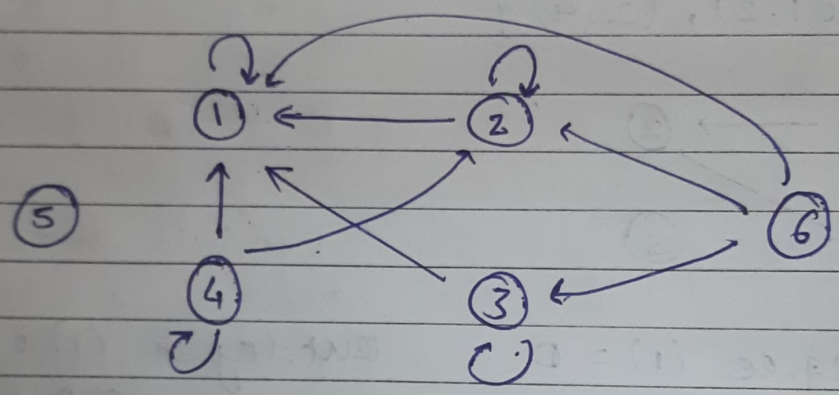
Indegree (1) = 1
(2) = 2
(3) = 1
(4) = 3

Outdegree (1) = 3
(2) = 2
(3) = 1
(4) = 1

Q.) Let $A = \{1, 2, 3, 4, 6\}$ and R be the relation on set A defined by aRb iff a is multiple of b.

(i) $R = \{(1,1), (2,1), (3,1), (4,1), (6,1), (4,2), (6,3), (6,2), (2,2), (3,3), (4,4), (6,6)\}$

(ii)



(iii) Indegree (1) = 5
(2) = 3
(3) = 2
(4) = 1
(5) = 0
(6) = 0

Outdegree (1) = 1
(2) = 2
(3) = 2
(4) = 3
(5) = 0
(6) = 3

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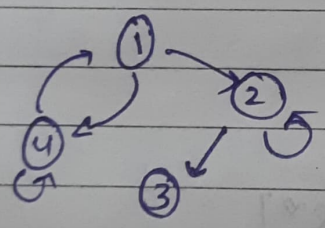
Q.) Determine the relation R from set A to set B as described by the following.

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} a & b & c \end{array} \end{array}$$

Sol^n

$R = \{(a,a), (a,c), (b,a), (b,b), (c,c)$
$(d,a)\}$

Q.) find the relation represented by the digraph given below also write down its matrix.



Sol^n

$R = \{(1,2), (2,2), (2,3), (1,4), (4,4), (4,1)\}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Q.) Let $A = \{a, b, c, d\}$ and R be a relation on set A that has the matrix.

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

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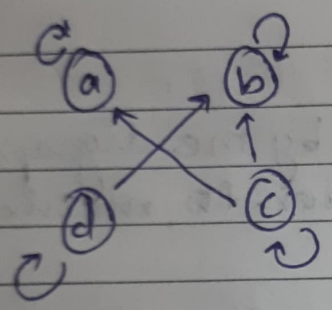
Construct the digraph of R and list the indegrees and outdegrees of all vertices.

$$M_R = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} a \\ b \\ c \\ d \end{matrix} \\ \begin{matrix} a & b & c & d \end{matrix} \end{array}$$

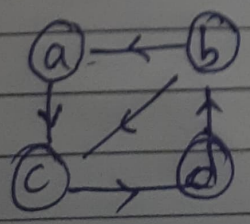
$R = \{(a,a), (b,b), (c,a), (c,b), (c,c), (d,b), (d,d)\}$



CONNECTIVITY RELATION $[R^\infty]$

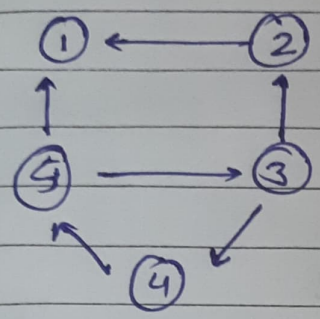
* A Relation R is said to be connectivity relation if and only if there is a path from one vertex to another vertex and it is denoted by $(x,y) \in R^\infty$ if and only if there is a path from x to y.

A relation is said to be strongly connected if $(x,y) \in R^\infty$ for all distinct vertices x & y



$$R^\infty = \begin{bmatrix} (a,a), (a,b), (a,c), (a,d) \\ (b,b), (b,a), (b,c), (b,d) \\ (c,c), (c,a), (c,b), (c,d) \\ (d,d), (d,a), (d,b), (d,c) \end{bmatrix}$$

Q.) Obtain R^∞ for the following diagraph & determine whether graph is strongly or weakly connected.



$R = \{(2,1), (3,2), (3,4), (4,5), (5,3), (5,1)\}$

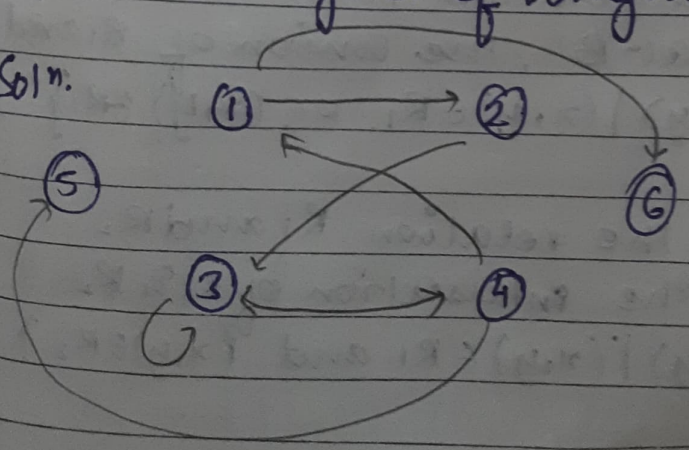
$R^\infty = \{(3,4), (3,5), (3,3), (3,1), (3,2)$
$(2,1), (4,5), (4,1), (4,3), (4,2), (4,4)$
$(5,1), (5,2), (5,3), (5,4), (5,5)\}$

The given relation is weakly connected. Bcoz there is no path from vertex 1 to other vertices.

Q.) Let $A = \{1,2,3,4,5,6\}$
$R = \{(1,2), (1,6), (2,3), (3,3), (3,4), (4,1), (4,3), (4,5), (6,4)\}$

(i) List All paths of length 3. originating at vertex 1.
(ii) List all cycles of length 4

Soln.



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Ans ~~$R_1 = \{ (1,2)$~~ ~~1, 2,~~

(i) $R_1 = \{(1,2), (2,3), (3,3)\}$

$R_2 = \{(1,6), (6,4), (4,3)\}$

$R_3 = \{(1,2), (2,3), (3,4)\}$

$R_4 = \{(1,6), (6,4), (4,1)\}$

$R_5 = \{(1,6), (6,4), (4,5)\}$

(ii) cycles of length 4.

$R_1 = \{(1,2), (2,3), (3,4), (4,1)\}$

$R_2 = \{(2,3), (3,4), (4,1), (1,2)\}$

$R_3 = \{(3,4), (4,1), (1,2), (2,3)\}$

$R_4 = \{(4,1), (1,2), (2,3), (3,4)\}$

Operations on Relations:

1. Union: Given the relation R_1 and R_2 from a set A to set B, the union of R_1 and R_2 is $R_1 \cup R_2 = \{(x,y) \mid (x,y) \in R_1 \text{ or } (x,y) \in R_2\}$

2. Intersection: Given the relation R_1 and R_2 from set A to B the intersection of R_1 & R_2 is $R_1 \cap R_2 = \{(x,y) \mid (x,y) \in R_1 \text{ and } (x,y) \in R_2\}$

3. <u>Compliment</u>: Given a relation R from set A to B then compliment of R is denoted

$$\boxed{\bar{R} = A \times B - R}$$

4. <u>Converse</u>: Given a relation from set A to B the converse of R is denoted by R^c.

$R^c = \{(x,y) \in R^c \mid (y,x) \in R\}$

8.) Consider the sets ~~A = {a, b, c}~~ ~~and~~ $A = \{1,2,3\}$ $B = \{1,2,3,4\}$ the relation R and S from set A to B are represented by the following matrices. Determine the relations $\bar{R}$, $R \cup S$, $R \cap S$, R^c, S^c and their matrix representation. ~~is shown below~~

$$M_R = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \end{array} \qquad M_S = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \end{array}$$

$R = \{(1,1), (1,3), (2,4), (3,1), (3,2), (3,3)\}$

$S = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,2), (3,4)\}$

$R \cup S = \{(1,1), (1,3), (2,4), (3,1), (3,2), (3,3), (1,2), (1,4), (3,4)\}$

$R \cap S = \{(1,1), (1,3), (2,4), (3,2)\}$

$R^c = \{(1,1), (3,1), (4,2), (1,3), (2,3), (3,3)\}$

$A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4\}$

$$A \times B = \left\{ \begin{matrix} (1,1), (1,2), (1,3), (1,4) \\ (2,1), (2,2), (2,3), (2,4) \\ (3,1), (3,2), (3,3), (3,4) \end{matrix} \right\}$$

$\bar{R} = A \times B - R$

$$= \left\{ \begin{matrix} (1,2), (1,4), (2,1), (2,2), (2,3), \\ (3,4) \end{matrix} \right\}$$

$$S^c = \{ (1,1), (2,1), (3,1), (4,1), (4,2), (2,3) \\ (4,3) \}$$

$$M_{R \cup S} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \\ \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \end{array}$$

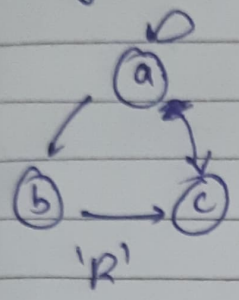
$$M_{R \cap S} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \\ \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \end{array}$$

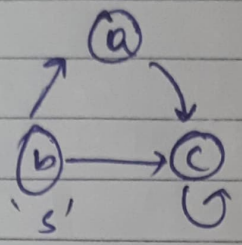
$$M_{\bar{R}} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} 1 & 2 & 3 & 4 \end{matrix}$$

$$M_{R^c} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \\ \begin{matrix} 0 & 1 & 0 & 0 \end{matrix}$$

$$M_{S^c} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Q.) The diagraphs of two relations on the set $A = \{a, b, c\}$ are given below. Draw the diagraphs of $\bar{R}$, $R \cup S$, $R \cap S$, R^c.



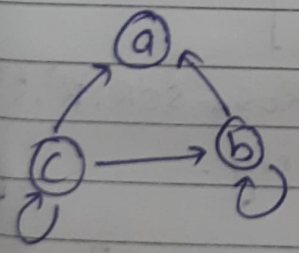


$R = \{(a,a), (a,b), (a,c), (b,c)\}$

$S = \{(a,c), (b,a), (c,c)\}$

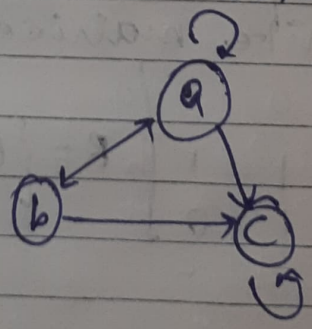
$A \times A = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$

$\bar{R} = \{(b,a), (b,b), (c,a), (c,b), (c,c)\}$

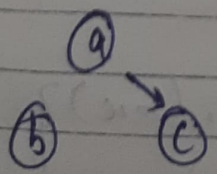


$\bar{R}$

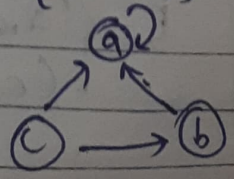
$R \cup S = \{(a,a), (a,b), (a,c), (b,c), (b,a), (c,c)\}$



$R \cap S = \{(a,c)\}$



$R^c = \{(a,a), (b,a), (c,a), (c,b)\}$



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Composition of Relations.

Defined as $R \circ S = \{(a,c) \mid a \in A \text{ and } c \in C$
$\exists b \in B \ (a,b) \in A,B$
$(b,c) \in B,C\}$

Q) Let $A = \{a,b,c\}$ and R & S be the relations on set A whose matrices are as given below.

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

find the composition relations $R \circ S$, $S \circ R$, R^2, S^2. and also write matrices respectively.

$$M_R = \begin{array}{c|c} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{matrix} a \\ b \\ c \end{matrix} \\ \begin{matrix} a & b & c \end{matrix} & \end{array} \quad R = \{(a,a), (a,c), (b,a), (b,b), (b,c), (c,b)\}$$

$$M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad S = \{(a,a), (b,b), (b,c), (c,a), (c,c)\}$$

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$R \circ S = \{(a,a), (a,c), (b,a), (b,b), (b,c)$
$(c,c)\}$

$S \circ R = \{(a,a), (a,c), (b,a), (b,b),$
$(b,c), (c,b), (c,a), (c,c)\}$

$R^2 = R \circ R = \{(a,a), (a,c), (a,b), (b,a),$
$(b,c), (b,b), (c,a), (c,b)$
$(c,c)\}$

$S^2 = S \circ S = \{(a,a), (b,b), (b,c), (c,a), (c,b)$
$(a,c),$

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very imp.

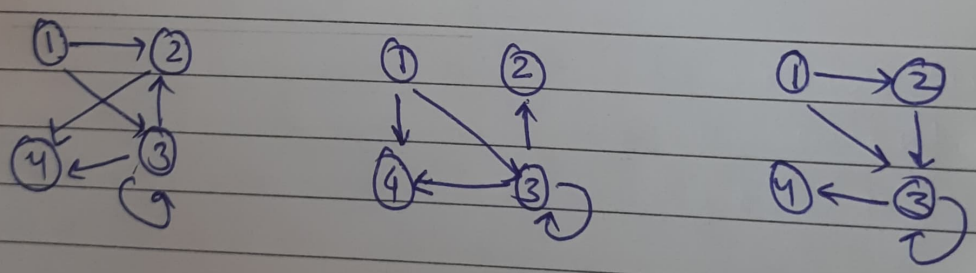
Q.) If $A = \{1, 2, 3, 4\}$ and 'R' 'A'

$R = \{(1,2), (1,3), (2,4), (3,2), (3,3), (3,4)\}$

(i) find R^2 and R^3 write their diagraphs
(ii) find $M(R)$, $M(R^2)$, $M(R^3)$.
(iii) verify that $M(R^2) = [(M(R))]^2$
(iv) verify that $M(R^3) = [M(R)]^3$

(i) ~~$R^2 = \{(1,1,2), (1,1,3), (1,2,4), (1,3,2), (1,3,3), (1,3,4),$
$(2,1,2), (2,1,3), (2,2,4), (2,3,2), (2,3,3), (2,3,4),$
$(3,1,2), (3,1,3), (3,2,4), (3,3,2), (3,3,3), (3,3,4),$
$(4,1,2), (4,1,3), (4,2,4), (4,3,2), (4,3,3), (4,3,4)\}$~~

$R = \{(1,2), (1,3), (2,4), (3,2), (3,3), (3,4)\}$
$R^2 = \{(1,4), (1,3), (3,4), (3,2), (3,3)\}$
$R^3 = \{(1,2), (1,3), (3,4), (2,3), (3,3)\}$



(ii)

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[M(R)]^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ & & & \end{bmatrix}$$

$R^2 = \{(1,4), (1,2), (1,3), (3,4), (3,2)\}$

$M(R^2)$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R = \{(1,2), (1,3), (2,4), (3,2), (3,3), (3,4)\}$

$R^3 = \{(1,4), (1,2), (1,3), (3,4)\}$

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Properties of Relation

⇒ **REFLEXIVE RELATION:**

Relation R on a set A is said to be reflexive if $(a,a) \in R$ $\forall a \in A$.

Eg:

$A = \{1,2,3,4\}$

$R = \{(1,1),(2,2),(3,3),(4,4)\}$

⇒ **SYMMETRIC RELATION:**

A Relation R on a set A said to be symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ $\forall (a,b) \in A$

Eg:

$A = \{1,2,3\}$

$R = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$

[Both symmetric & reflexive.]

⇒ **Irreflexive relations:**

Relations R on set A said to be irreflexive if $(a,a) \notin R$ $\exists a \in A$. i.e, if no element of set A is related to itself by R.

Eg:

$A = \{1,2,3\}$

$R = \{(1,1),(1,2)\}$ [not an Irreflexive relation]

$R = \{(1,2),(2,1),(3,1),(1,3)\}$ [Irreflexive R]

⇒ **Transitive Relation:**

A relation R on a set A is said to be Transitive if $(a,b) \in R$, $(b,c) \in R$ then $(a,c) \in R$ $\forall a,b,c \in A$.

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Eg: $A = \{1,2,3\}$ $R_1 = \{(1,1), (1,2), (2,3), (1,3), (3,1), (3,2)\}$

$R_2 = \{(1,2), (2,3), (1,3), (3,1)\}$

→ In R_1 is not transitive bcoz $(2,2), (2,1)$ is not.
→ R_2 is " " bcoz $(1,1)$ is not present.

$A = \{1,2,3,4\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$
R is transitive relation.

⇒ <u>Equivalence Relation</u>
A relation R on a set A is said to be equivalence relation set A if
- R is reflexive.
- R is symmetric.
- R is transitive.

$A = \{1,2,3,4\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$

⇒ <u>Asymmetric relation:</u>
A relation R on set A is said to be Asymmetric if whenever a related b, aRb then bRa.
It then follows that R is not symmetric if we have some $(a,b) \in A$ with both (aRb) & (bRa).

$A = \{1,2,3\}$
$R_1 = \{(1,1), (1,2), (2,1)\}$ not Asymmetric.

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$R_2 = \{(1,2),(2,1),(1,3)\}$ not asymmetric relation.

$A = \{1,2,3,4\}$

$R = \{(1,2),(3,4),(4,1)\}$ It is asymmetric.

⇒ <u>Antisymmetric relation</u>

A relation R on set A is said to be antisymmetric whenever aRb and bRa when $a=b$.
It follows that R is not antisymmetric if there exists $(a,b) \in A$ such that $(a,b) \in R$ and $(b,a) \in R$ but $a \neq b$.

$R_1 = \{(1,1),(1,2),(2,1)\}$ [~~Anti~~ symmetric]

$A = \{1,2,3\}$
$R_1 = \{(1,1),(2,2)\}$

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* All Assymetric relations are Anti-symmetric But not vice versa.

Q) Determine the nature of the following relations.
Let $A = \{1,2,3\}$.

(i) $R_1 = \{(1,2),(2,1),(1,3),(3,1)\}$

(ii) $R_2 = \{(1,1),(2,2),(3,3),(2,3)\}$

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(iii) $R_3 = \{(1,1), (2,2), (3,3)\}$

(iv) $R_4 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$

(v) $R_5 = \{(1,1), (2,3), (3,3)\}$

(vi) $R_6 = \{(2,3), (3,4), (2,4)\}$

(vii) $R_7 = \{(1,3), (3,2)\}$

Ans (i) • R_1 is not reflexive. $(a,a) \notin R$
• R_1 is symmetric $(a,b) \in R$ & $(b,a) \in R$
• R_1 is not transitive.
Bcz $(1,2) \in R$, $(2,1) \in R$ but $(1,1) \notin R$
• R_1 is not asymmetric, anti-symmetric.
• R_1 is an Irreflexive.

(ii) R_2 is reflexive $(a,a) \in R$
$(1,1), (2,2), (3,3) \in R$
R_2 is not symmetric relation.
R_2 is transitive.
R_2 is Anti-symmetric relation.
R_2 is not irreflexive relation.

This is a partially ordered Relation. [Reflexive, transitive, anti-symmetric]

(iii) R_3 is: Reflexive
Symmetric
Transitive
R_3 is equivalence relation.

(iv) R_4 is Reflexive, Symmetric, Transitive, R_4 is equivalence.

(v) R_5 is transitive, Antisymmetric

(vi) ~~R_6 is irreflexive, Asymmetric~~
R_4 is not a relation because 4 is not in A.

(vii) R_7 is Asymmetric, Anti-symmetric, irreflexive.

Q.) find the nature of the relations, represented by the following matrices.

(i)
$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

(ii)
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

(iii)
$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(i) $R_1 = \{(1,2), (1,3), (2,1), (2,2), (3,1), (3,3), (3,4), (4,3), (4,4)\}$

- R is not reflexive because $\forall a \in A$, $(a,a) \notin R$.
- R is symmetric $(1,2) \in R$ & $(2,1) \in R$
- R is not transitive $(1,2) \in R$ $(2,1) \in R$ but $(1,1) \notin R$.

(ii) $R_2 = \{(1,1), (1,3), (2,2), (2,4), (3,1), (3,3), (4,2), (4,4)\}$

- R is reflexive. $(1,1), (2,2), (3,3), (4,4) \in R$

- R is ~~not~~ symmetric. $(1,3) \in R$ & $(3,1) \in R$.
- R is transitive.

$\therefore$ R is an Equivalence Relation.

(iii) $R = \{(1,3), (1,4), (2,3), (3,4), (4,1)\}$

- R is not reflexive. $\forall a \in A$, $(a,a) \notin R$.
- R is not symmetric because $(1,3) \in R$ $(3,1) \notin R$.
- R is not transitive.

Q.) On the set Z^+ a relation R is defined by aRb if and only if $a \div b$
Prove that R is reflexive, Transitive, Antisymmetric but not symmetric.

(i) $\forall a \in Z^+ aRa$ $(a,a) \in R$ $a|a$ Reflexive

(ii) $a,b \in Z^+$ $a|b$ $b \nmid a$
$2,4 \in A$ $2|4$ $4|2$ Not Symmetric
T F

(iii) $a,b,c \in Z^+$ $a|b$ $b|c$ then $a|c$ Transitive.
$2,4,8$ $2|4$ $4|8$ then $2|8$

(iv) $a,b \in Z^+$ $a|b$ $b|a$ iff $a=b$
$2,2$
Anti-symmetric

[Important]

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Explanation:

(i) for any $\exists a \in Z^+$, the statement $a|a$ is true
thus $\forall a \in Z^+$, hence R is Reflexive.
Give example.

(ii) $\exists a, b \in Z^+$ $a|b$ need not imply that $b|a$.
thus aRb doesnot always imply bRa
Hence R is not symmetric.

$2|4 \Rightarrow T$ but $4|2 \Rightarrow F$

(iii) further $a|b$ & $b|a$ imply that $a=b$
thus aRb & bRa is Anti-symmetric.
Eg: $2|4$, $4 \nmid 2$, $2|2$ when $2=2$.

(iv) Lastly we note that for any $a, b, c \in Z^+$
$a|b$ and $b|c$ imply that $a|c$.
thus aRb, bRc, $\Rightarrow a|c$.

Ex: $2|4$, $4|8$ $\Rightarrow 2|8$

Partition of universal set S

Q) Let S be a universal set. on $P(S)$ define the relation by $(a,b) \in R$ iff $A \subseteq B$
Prove that is reflexive, transitive, Anti-symmetric but not symmetric.

Soln

- for any subset A of S we have $A \subseteq A$
This means that $(A, A) \in R$ hence R is reflexive.

- Next we note that for any $(A,B) \in P(S)$
$A \subseteq B$ and $B \subseteq A$ imply $A = B$, i.e, $(A,B) \in R$
and $(B,A) \in R$ implies $A = B$

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Hence R is Anti-symmetric on P(S).

- further, $A, B, C \in P(S)$ if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$, i.e, if $(A,B) \in R$, $(B,C) \in R$ then $(A,C) \in R$. Hence R is transitive on P(S).

- finally note that $\exists A, B \in P(S)$, $A \subseteq B$ does not necessarily imply that $B \subseteq A$. i.e, $(A,B) \in R$ doesnot always imply $(B,A) \in R$, i.e. R is not symmetric on P(S).

Q.) Let R be a non-empty relation on a set A. Prove that if R satisfies any two of the properties: irreflexive, symmetric and transitive then it can satisfy the third. [HW]

Equivalence Relation

A relation R on a set A is said to be equivalence relation on a set A if

→ R is reflexive.
→ R is symmetric.
→ R is transitive.

① Let $A = \{1, 2, 3, 4\}$
$R = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,3), (1,3), (4,1), (4,4)\}$
Is R an Equivalence Relation?

Soln

R is reflexive because $(1,1), (2,2), (3,3), (4,4) \in R$
$\forall a \in A$, $(a,a) \in R$. (write the definitions. Carrys marks :)
$(1,1), (2,2), (3,3), (4,4) \in R$

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(2) for any $a, b \in A$ $(a, b) \in R$ then $(b, a) \in R$
$(4,1) \in R$
$(1,4) \notin R$ $\therefore$ R is not symmetric.

$\therefore$ R is not an Equivalence Relation.

Q.) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
On this set define the relation R by $(x, y) \in R$
iff ~~(x-5~~ $(x-y)$ is a multiple of 5.
Verify that R is an equivalence Relation.

~~R = {(6,1), (10,5), (7,2), (8,3), (1,1), (2,2),~~
~~(3,~~

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6),$
$(7,7), (8,8), (9,9), (10,10), (11,11), (1,11), (11,1)$
$(12,12), (1,6), (6,1), (2,7), (7,2),$
$(3,8), (8,3), (4,9), (9,4), (5,10), (10,5),$
$(6,11), (11,6), (7,12), (12,7)\}, (2,12), (12,2)\}$

Reflexive: $\forall a \in A$, $(a,a) \in R$
$\therefore$ R is a reflexive relation.
$\{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (8,8), (9,9),$
$(10,10), (11,11), (12,12)\} \in R$

Symmetric: $\exists a, b \in A$, $(a,b) \in R$ then $(b,a) \in R$
$\therefore$ R is symmetric relation.

Transitive: $\exists a, b, c \in A$ $(a,b) \in R$, $(b,c) \in R$ then
$(a,c) \in R$
$(1,11), (11,1), (1,1) \in R$

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Same answer different way.

for any $x \in A$ we have $x - x = 0$ which is a multiple of 5.
$\therefore (x, x) \in R$. So, R is Reflexive.

for any $x, y \in A$ if $(x, y) \in R$ ~~then and (y, x) ∈ R~~
then $x - y = 5k$, for some integer k.
Consequently $y - x = -5k$ so that R is symmetric

for any $x, y, z \in A$ if $(x, y) \in R$ and $(y, z) \in R$
then $x - y = 5k_1$, $y - z = 5k_2$
$x - y + y - z = 5(k_1 + k_2)$
$x - z = 5(k_1 + k_2)$
$\therefore (x, z) \in R$.
so that R is transitive.

R is Equivalence Relation.

(2) If $A = A_1 \cup A_2 \cup A_3$ where $A_1 = \{1, 2\}$
$A_2 = \{2, 3, 4\}$ $A_3 = \{5\}$
Define the Relation R on set A by xRy iff x & y are in the same set. A_i, $i = 1, 2, 3 \ldots$
Is R an Equivalence Relation?

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$A = \{1, 2, 3, 4, 5\}$

~~R = {~~ (i) Reflexive: $\{(1,1), (2,2), (3,3), (4,4), (5,5)\}$

(ii) Symmetric: $\{(1,2), (2,1), (2,3), (3,2), (3,4), (4,3)\}$
$(2,4), (4,2)$,

(iii) Not transitive. R is not Equivalence Relation.

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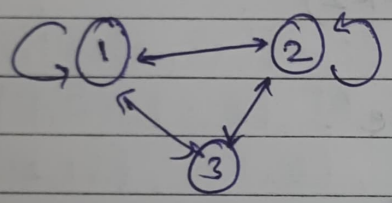
We know that (xRx) for every x in A_i.
Because $(x,x) \in A_i$.
$\therefore$ R is Reflexive. for eg: $(1,1), (2,2)$ etc.
further, $(x,y) \in A_i$ then $(y,x) \in A_i$ for any $(x,y) \in A_i$
$\therefore$ R is symmetric. for eg: $(1,2), (2,1) \in A_1$.
for any $(x,y) \in A_i$, i.e, $(1,2) \in A_1$ & $(y,z) \in A_i$
i.e, $(2,3) \in A_2$ then $(1,3) \notin A_i$. [1, and 3 are not in same set.]
$\therefore$ R is not a transitive.

$\Rightarrow$ R is not an Equivalence Relation.

Q.) A Relation R on set $A = \{1,2,3\}$ determine R is an equivalence relation.



Soln.

$R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (1,3), (3,1)\}$

Reflexive:
We know xRx for every $x \in A$., $(x,x) \notin A$.
$\therefore$ R is not reflexive. for eg: $(3,3) \notin A$.

Symmetric: for any $(a,b) \in A$ then $(b,a) \in R$ for all $(a,b) \in A$.
$\therefore$ R is a symmetric relation.
$(1,2), (2,1), (2,3), (3,2), (1,3), (3,1) \in R$.

Transitive: ~~for any (a,b) ∈ R, (b,c)~~
for any $(a,b,c) \in A$, $(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) \in R$
R is not transitive. $(3,3) \notin R$.
$\therefore$ R is not an Equivalence Relation

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**** Very very imp.

Q.) Let S be set of non-zero integers. and $A = S \times S$.
On set A define the relation R by $(a,b) R (c,d)$ iff $ad = bc$.
Show that R is an Equivalence Relation.

Soln ~~Reflexive $\forall a \in A$, $(a,a) \in R$~~
~~$(a,a) R (a,a)$ $aa = aa$~~
~~$\therefore$ The relation R is re~~

We note that, $(a,a) R (a,a)$ because $aa = aa$.
for all $a \in S$, S is a set of all non-zero integers.
$\therefore$ R is Reflexive.

further ~~$(x,y) \in$~~ .

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Q.) for a fixed integer $n > 1$, Prove that the relation "congruent modulo n" is an equivalence Relation on the set of all Integers Z.

Soln for $(a, b) \in Z$

$a \equiv b \mod n$

$(a-b)$ is multiple of n

$n \mid (a-b)$.

Reflexive: $\forall a \in Z$, $n \mid (a-a)$

$n \mid 0$

$a \equiv a \mod n$

$\therefore$ R is Reflexive

Symmetric: $\exists (a,b) \in Z$

$n \mid (a-b)$

$n \mid -(a-b)$

$a \equiv b \mod n$

$b \equiv a \mod n$

$\therefore$ R is Symmetric.

Transitive: $\exists (a,b,c) \in Z$

$a \equiv b \mod n$ $\quad$ $b \equiv c \mod n$

$n \mid (a-b)$ $\quad$ $n \mid (b-c)$

then

$n \mid (a - \cancel{b} + \cancel{b} - c)$ $\quad$ [$(a-b)$ & $(b-c)$ both are multiples of n.]

$n \mid (a-c)$

$\Rightarrow a \equiv c \mod n$

$\therefore$ R is transitive.

R is an Equivalence Relation.

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PARTITION.

Let R be an equivalence Relation on a set A & $a \in A$ then the set of all those elements 'x' of A which are related to 'a' by R is called an Equivalence class of 'a'. wrt R. This equivalence class is denoted by $R(a)$ or $\bar{a}$ or $[a]$.

$$\boxed{\bar{a} = [a] = R(a) = \{x \in A \mid (x,a) \in R\}}$$

$R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$

$[1] = \{1,3\}$
$[2] = \{2\}$

$[3] = \{1,3\}$

PARTITION OF A SET.

Let A be ~~an~~ a non-empty set. Suppose there exists non-empty subsets, $A_1, A_2, A_3 \ldots A_k$ of 'A'. such that the following 2 conditions hold:

① $A = A_1 \cup A_2 \cup A_3 \ldots A_k$

② Any two of the subsets $A_1, A_2, A_3 \ldots A_k$ are disjoint, i.e, $A_i \cap A_j = \phi$, $i \neq j$.
then the set $P = \{A_1, A_2, A_3, \ldots A_k\}$ is called a partition of A.

Q.) for the Equivalence Relation
$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$
defined on set $A = \{1,2,3,4\}$. Determine the partition induced. [Compute A/R]

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$[1] = \{1, 2\}$ $[2] = \{1, 2\}$

$[3] = \{4, 3\}$ $[4] = \{3, 4\}$

$[1] \cup [3] = \{1, 2\} \cup \{3, 4\}$

$= \{1, 2, 3, 4\} = A$

$[1] \cap [3] = \phi$

$P = \{\{1, 2\}, \{3, 4\}\}$

② Determine the partition of A induced by R.
$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and

$R = (x, y) \in R$ iff ~~(x+5)~~ $(x-y)$ is multiple of 5.

Soln

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7),$
$(8,8), (9,9), (10,10), (11,11), (12,12),$
$(1,11), (11,1), (2,12), (12,2), (1,6), (6,1),$
$(7,2), (2,7), (3,8), (8,3), (4,9), (9,4), (5,10),$
$(10,5), (6,11), (11,6), (7,12), (12,7)\}$

$[1] = \{1, 6, 11\}$ $[2] = \{2, 12, 7\}$
$[6] = \{1, 6, 11\}$ $[12] = \{2, 12, 7\}$
$[11] = \{1, 6, 11\}$ $[7] = \{2, 7, 12\}$

$[3] = \{3, 8\}$ $[4] = \{4, 9\}$ $[5] = \{5, 10\}$
$[8] = \{3, 8\}$ $[9] = \{4, 9\}$ $[10] = \{5, 10\}$

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$[1] \cup [2] \cup [3] \cup [4] \cup [5] = A$

$[1] \cap [2] \cap [3] \cap [4] \cap [5] = \phi$

~~P = {~~

$P = \{\{1,6,11\}, \{2,12,7\}, \{3,8\}, \{4,9\}, \{5,10\}\}$

(3) Let $A = \{a, b, c, d, e\}$
$P = \{\{a,b\}, \{c,d\}, \{e\}\}$ of set A. find the equivalence relation inducing this partition.

~~R = {~~ $A_1 = \{a,b\}$ $A_2 = \{c,d\}$ $A_3 = \{e\}$

$R = \{(a,a), (b,b), (c,c), (d,d), (e,e),$
$(a,b), (b,a), (c,d), (d,c)\}$ ~~, (d, e),~~
~~(e, d)}~~

(4) Let $A = \{1,2,3,4,5,6,7\}$ and R be the equivalence relation on set A that induces the partition $P = \{1,2\} \cup \{3\}, \cup \{4,5,7\} \cup \{6\}$.
[HW]

$A_1 = \{1,2\}$ $A_2 = \{3\}$ $A_3 = \{4,5,7\}$ $A_4 = \{6\}$

$R = \{(1,1), (2,2), (1,2), (2,1), (3,3), (4,5), (5,4),$
$(4,4), (5,5), (7,7), (4,7), (7,4), (5,7), (7,5),$
$(6,6)\}$

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Q) On the set Z of all integers a relation R is defined by aRb iff $a^2 = b^2$. Verify that R is an equivalence relation.
Determine the partition induced by this relation.

Soln Reflexive:

$\forall a \in Z \quad (a,a) \in Z, \quad a^2 = a^2$

R is reflexive.

Symmetric:

for any $a, b \in Z$ $(a,b) \in R$ & $(b,a) \in R$

$a^2 = b^2 \Rightarrow b^2 = a^2$

R is symmetric.

Transitive.

for any $a, b, c \in Z$ $(a,b) \in R$, $(b,c) \in R$ then $(a,c) \in R$.

$a^2 = b^2$, $b^2 = c^2 \Rightarrow a^2 = c^2$

R is transitive.

R is an Equivalence Relation.

$[a] = \{x \in Z \mid (x,a) \in R\}$

$x^2 = a^2$

$x = \pm a$

$[0] = \{0\}$

$[n] = \{n, -n\}$

$P = \{\{0\}, \{n, -n\}\}$

*** 3a,b,c,d

Partial Orders.

A Relation R on a set A is said to be partially ordered relation or a partial order on A if
→ R is Reflexive
→ R is Anti-symmetric
→ R is Transitive.

A set A with partial order R defined on it is called partially ordered set. (or) Poset. (A, R)
↓
denoted by pair.

Example: (i) The relation is $\leq$ is a partial order on Z.
(ii) The relation is $\geq$ is a partial order on Z.
(iii) The divisibility relation on the set Z^+ defined by $a|b$ for any $(a,b) \in Z^+$.
(iv) $\subseteq$ also satisfies.

$A = \{1, 2, 3, 4\}$
for $\leq$ $R = \{(1,1), (2,2), (3,3), (4,4)$
$(1,2), (1,3), (1,4), (2,3), (2,4)$
$(3,4)\}$

Total Order

Let R be a partial order on a set A, then R is called total order on set A if for all $(x,y) \in A$ either xRy or yRx. In this case, poset (A, R) is called a Totally ordered set.
[Good examples for totally ordered set is $\geq$ or $\leq$]

Ex: Prove that set of all Z^+ is not totally ordered by the relation of divisibility
Ans ⇒ $x = 2$ $y = 3$ $2 \nmid 3$ or $3 \nmid 2$

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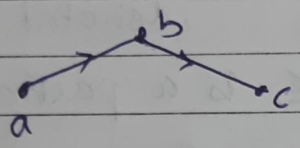
Hasse Diagram. [only for Poset]

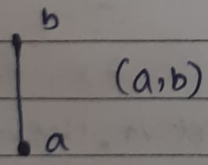
In the diagraph of a partial order there is an edge from vertex 'A' to a vertex 'b'. and there is an edge from vertex 'b' to vertex 'c' then there should be an edge from vertex 'a' to vertex 'c'.

As such, we need not exhibit an edge from 'a' to 'c' explicitly, it will be automatically understood



To simplify the format of a diagraph of a partial order, we represent vertices by '•' (dot) and draw the diagraph in such a way that all the edges point upward, with this convention we need put arrows (→) in the edges.

In the diagraph of a partial order drawn by adopting the conventions indicated above is called a Poset diagram or Hasse Diagram.



(a,b)

Problems

① Let $A = \{1,2,3,4\}$ $R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$

Verify that R is a partial order on set A, also draw Hasse diagram of R.

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Ans) $\forall a \in A$, $(a,a) \in R$

$(1,1), (2,2), (3,3), (4,4) \in R$

R is Reflexive.

for any $(a,b) \in A$ $\quad (a,b) \in R$ & $(b,a) \notin R$

$(a,b) \in R$ & $(b,a) \in R$ iff $a=b$

$\{(1,1), (2,2), (3,3), (4,4), (1,3), (3,4), (1,4)\} \in R$

R is anti-symmetric.

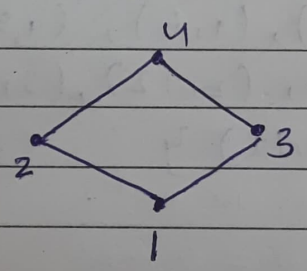
for any $(a,b,c) \in A$ $\quad (a,b) \in R$ & $(b,c) \in R$ then $(a,c) \in R$

$(1,2) \in R$, $(2,3) \in R \Rightarrow (1,3) \in R$

R is transitive.

Hence, R is partial order.

HASSE DIAGRAM.



[• dot also represents (1,1), connection to itself]

② Let R be a relation on the set $A = \{1,2,3,4\}$ defined by xRy iff x divides y. Prove that (A,R) is a Poset. Draw its Hasse Diagram.

Ans

$R = \{(1,1), (1,2), (1,3), (1,4), (2,4), (2,2), (3,3), (4,4)\}$

$\forall a \in A$, $(a,a) \in R$

$(1,1) \in R$ R is Reflexive.

for any $(a,b) \in A$ $\quad (a,b) \in R$ but $(b,a) \notin R$

$(1,2) \in R$ $\quad (2,1) \notin R$

R is anti-symmetric.

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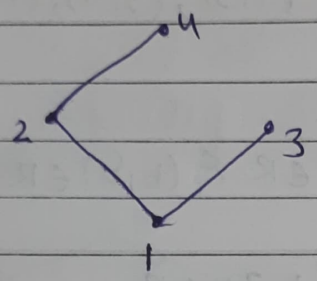
For any $(a,b,c) \in A$, $(a,b) \in R$ $(b,c) \in R$, $(a,c) \in R$

$(1,2) \in R$ $(2,4) \in R \Rightarrow (1,4) \in R$

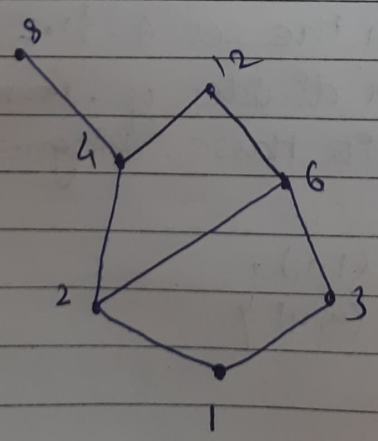
R is transitive.

$\therefore$ R is poset.

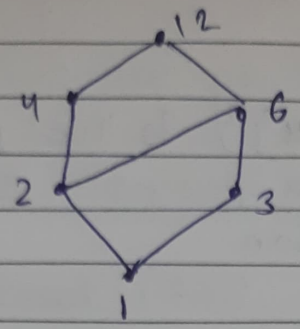


(ii) $A = \{1, 2, 3, 4, 6, 8, 12\}$

$R = \{ (1,1), (2,2), (3,3), (4,4), (6,6), (8,8), (12,12),$
$(1,2), (1,3), (1,4), (1,6), (1,8), (1,12)$
$(2,4), (2,6), (2,8), (2,12), (3,6), (3,12),$
$(4,8), (4,12), (6,12) \}$

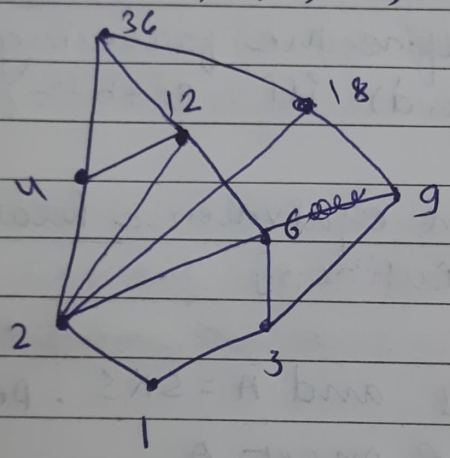


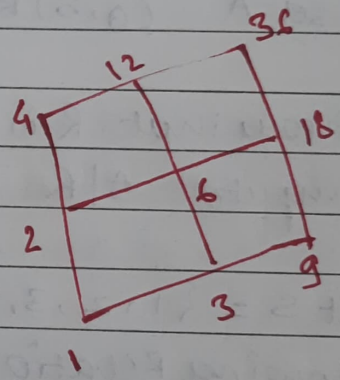
(iii) $A = \{1, 2, 3, 4, 6, 12\}$



(iv) Positive divisors of 36.

$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$





10/01/23

Pg-169 [29 coproblems] [Ex-5]
Pg-200 [solutions]

Q.) If $\{\{a,i\},\{e,o\},\{u\}\}$ is the partition of the set $A=\{a,e,i,o,u\}$
Determine the Relation.

Ans) $R=\{(a,a),(i,i),(a,i),(e,e),(o,o),(i,a),(e,o),(o,e),(u,u)\}$
Equivalence Relation.

[Homework]

Q.) Let $A=R\times R$, Define the following relation R on set A $(a,b)R(c,d)$ iff $a^2+b^2=c^2+d^2$

(i) Show that R is an equivalence Relation.
(ii) Compute A/R [Partition]

Q.) Let $S=\{1,2,3,4\}$ and $A=S\times S$. Define the following Relation R on set A.
$(a,b)R(a',b')$ iff $a+b=a'+b'$.
(i) Show that R is an Equivalence Relation
(ii) Compute A/R.

Q.) Let $S=\{1,2,3,4,5\}$ and $A=S\times S$, Define the following relation R on set A, $(a,b)R(a',b')$ iff $ab'=a'b$.
(i) Show that R is an equivalence relation.
(ii) Compute A/R.

Q.) Let set $A=\{1,2,3,4,5\}$ Define R on $A\times A$ by $(x_1,y_1)R(x_2,y_2)$ iff $x_1+y_1=x_2+y_2$.
Verify that R is an Equivalence relation.

Second element of ordered pair.

(ii) Determine the equivalence classes $[(1,3)]$, $[(2,4)]$ and $[(1,1)]$.

(iii) Determine the partition of $A \times A$ induced by R.

Extremal Elements in Posets - [01/01/23]

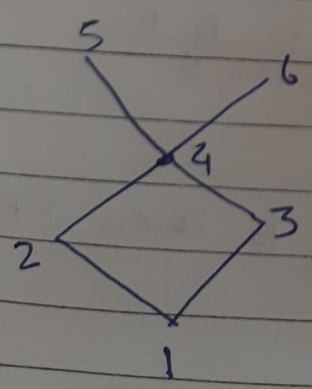
Consider a Poset (A,R). We define some special elements ~~called~~ (Extremal elements) that may exist in set A.

- **MAXIMAL ELEMENT** - This means 'a' is a maximal ~~to~~ element of A iff in the Hasse Diagram of relation R no edge starts at A.

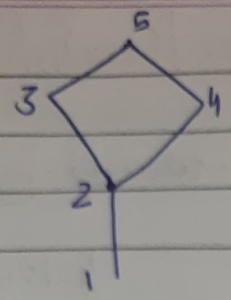
MINIMAL ELEMENT - An element $a \in A$ is called a minimal element of set A if an only if in the Hasse diagram of relation R no edge terminates at 'a'.

GREATEST ELEMENT - An element $a \in A$ is called a greatest element of A if $xRa \ \forall x \in A$.

LEAST ELEMENT - An element $a \in A$ is called a least element of A if $aRx \ \forall x \in A$.



Maximal element = $\{5,6\}$
Minimal element = $\{1\}$
Greatest element = ~~$\{1\}$~~ No element
Least element = $\{1\}$



Minimal element = {1}
Maximal element = {5}
Greatest element = {5}
Least element = {1}

Upper bound of a Subset - [look upward]

An element $a \in A$ is called an upper bound of a subset 'B' of 'A' if $xRa \ \forall x \in B$.

Lower bound of a Subset - [look downward]

An element $a \in A$ is called a lower bound of a subset 'B' of 'A' if $aRx \ \forall x \in B$.

Least Upper Bound (LUB)

An element $a \in A$ is called Least Upper bound (LUB) of a subset B of A. if the following 2 condition hold:

(i) a is an upper bound of 'B'.
(ii) If a' is an upper bound of 'B' then aRa'.

Also called as Supremum LUB.

Greatest Lower Bound (GLB)

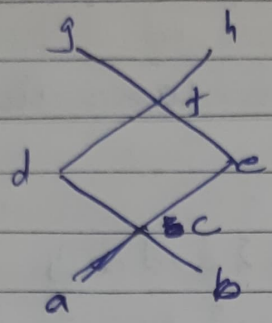
An element $a \in A$ is called the Greatest lower bound (GLB) of a subset 'B' of 'A' if the following conditions hold:

(i) a is a lower bound of B.
(ii) If 'a' is a lower bound of B then a'Ra.

Also called as Infimum GLB.

PROBLEMS.

Q.) Consider a Hasse diagram of a Poset (A, R) given below.



If $B = \{c, d, e\}$ find

(i) All upper bounds of B.
(ii) All lower " " ".
(iii) The least upper bound.
(iv) Greates lower bound.

[Maximal element = {g, h} Greatest = No element
Minimal element = {a, c} Least = No "]

Ans) (i) $B = \{c, d, e\}$

for c	for d	for e
cRc	dRd	eRe
cRd	dRf	eRf
cRe	dRg	eRg
cRf	dRh	eRh
cRg		
cRh		

upper bound = {f, g, h} of subset B of A.

(ii)

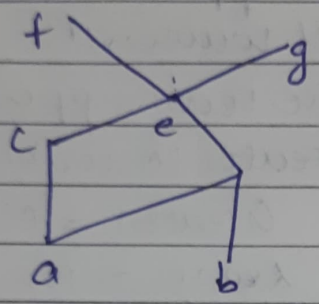
for c	for d	for e
cRc	dRd	eRe
aRc	cRd	cRe
bRc	aRd	aRe
	bRd	bRe

Lower bound = {a, b, c} of subset B of A.

(iii) Least upper bound is f [first reaching vertex]

(iv) Greatest lower bound is c [first reaching downward]

Q.) Consider the poset whose hasse diagram is shown below. Find

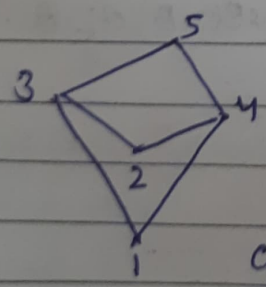


$B = \{c, d, e\}$

(i) Greatest lower bound - a
(ii) Least upper bound - e.

* In any hasse diagram where LUB & GLB exists then it is called as LATTICE.

LATTICE - Let A, R be a poset, is called a lattice if $\forall x, y \in A$ the elements LUB $\{x, y\}$ & GLB $\{x, y\}$ exist in 'A'.



$B = \{3, 4\}$

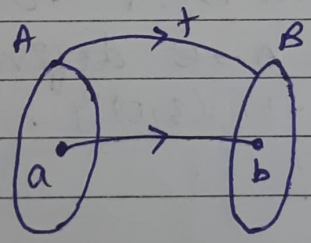
It's not a lattice.
Because there is more than 1 Greatest lower bound.
LUB = 5
GLB = $\{1, 2\}$ [Doesn't make sense]

UNIT-FUNCTIONS

function:

Let A and B be two non-empty sets. Then a function f from $A \to B$ is a relation from $A \to B$ such that for each $a \in A$ there is a unique image $b \in B$ such that $(a, b) \in f$ then we write $b = f(a)$.

- b is called image of a. and a is called pre-image of b under f.
- The element 'a' is called an argument of a function f and $b = f(a)$ is then called the value of the function f for the argument a.
- A function f from A to B is denoted by $f: A \to B$.
- The pictorial representation of f is as shown below:



- for the function $f: A \to B$, A is called the domain of f and B is called co-domain of f.
- The subset of B consisting of the images of all elements of set A under f is called the range of the function f and it is denoted by $f(A)$.
- The following observations are immediate consequences of the function $f: A \to B$

(i) Every $a \in A$ belongs to some pair $(a, b) \in f$ and if $(a, b_1) \in f$ and $(a, b_2) \in f$ then f is not a function.

(ii) An element $b \in B$ need not have a pre-image in set A under f.

(iii) Two different elements of A can have same image in B.

(iv) The statements $(a,b) \in f$, afb and $b = f(a)$ are equivalent.

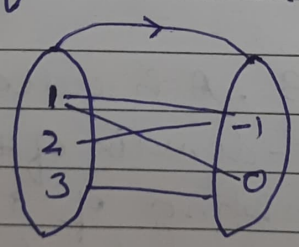
(v) The Range of f from $A \to B$ is given by $f(A) = \{f(x) \mid x \in A\}$

(vi) for the function $f: A \to B$ if $b \in B$ and $f^{-1}(b)$ is defined by $f^{-1}(b) = \{x \in A \mid f(x) = b\}$
$f^{-1}(b)$ is called the pre-image of set B.

(vii) ~~A relation~~ Every function is a relation but a relation ~~needed~~ need not be a function.

Q.) Let $A = \{1,2,3\}$ and $B = \{-1, 0\}$ and R be relation from $A \to B$ defined by $R = \{(1,-1), (1,0), (2,-1), (3,0)\}$. Is R a function from $A \to B$?

Soln:



1 is mapped to two different elements.
$\therefore$ R is not a function.

Q.) Let $A = \{1,2,3,4\}$. Determine whether or not the following relations on set A are functions.

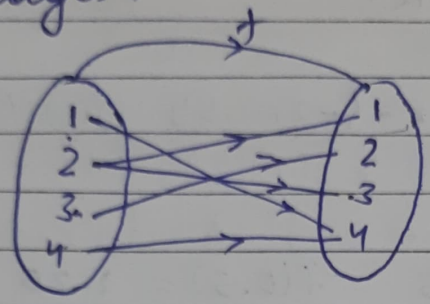
$f = \{(2,3), (1,4), (2,1), (3,2), (4,4)\}$

$g = \{(3,1), (4,2), (1,1)\}$

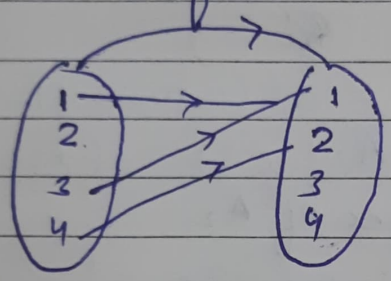
$h = \{(2,1), (3,4), (4,4), (1,4), (2,1)\}$

Ans:

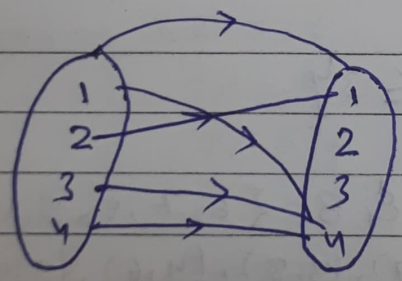
f is not a function because 2 has 2 images.



g is not a function.



h is a function



Q.) Let $A = \{0, \pm 1, \pm 2, 3\}$. $f: A \to R$ (R is set of Real no.s) $f(x) = x^3 - 2x^2 + 3x + 1$ for $x \in A$. Find the range of 'f'.

$f(0) = 1$

$f(-1) = (-1)^3 - 2(-1)^2 - 3 + 1$

$= \cancel{1} - 2 - 3 \cancel{+1} = -5$

$f(1) = \cancel{1} - \cancel{2} + 3 \cancel{+1}$

$= 3$

$f(2) = \cancel{8} - \cancel{8} + 6 + 1$

$= 7$

$f(-2) = -8 - 8 - 6 + 1$

$=$

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Q.) Let A function $f: R \to R$ be defined by $f(x) = x^2+1$. Determine the images of the following subsets of R:

(i) $A_1 = \{2, 3\}$ (ii) $A_2 = \{-2, 0, 3\}$ (iii) $A_3 = \{0, 1\}$
(iv) $A_4 = [-6, 3]$

$f(2) = 5$ $f(-2) = 5$ $f(0.5) = 1.25$
$f(3) = 10$ $f(0) = 1$ $f(-6) = 37$
$f(3) = 10$

$\{5, 10\}$ are images of A_1
$\{5, 1, 10\}$ are images of A_2

$f(A_3) = \{f(x) \mid 0 < x < 1\}$

$f(A_4) = \{f(x) \mid -6 \le x \le 3\}$

Q.) Let $A = \{1, 2, 3, 4, 5, 6\}$ & $B = \{6, 7, 8, 9, 10\}$
$f: A \to B$ $f = \{(1,7), (2,7), (3,8), (4,6), (5,9), (6,9)\}$
Determine
$f^{-1}(6)$ & $f^{-1}(9)$. If $B_1 = \{7, 8\}$ $B_2 = \{8, 9, 10\}$
find $f^{-1}(B_1)$ & $f^{-1}(B_2)$

Ans) $f^{-1}(6) = 4$
$f^{-1}(9) = \{5, 6\}$
$f^{-1}(B_1) = \{1, 2, 3\}$
$f^{-1}(B_2) = \{3, 5, 6\}$

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PYQ

Q.) Let $f: R \to R$

$$f(x) = \begin{cases} 3x-5 & \text{for } x > 0 \\ -3x+1 & \text{for } x \le 0 \end{cases}$$

(i) Determine $f(0), f(-1), f(5/3), f(-5/3)$

(ii) find $f^{-1}(0), f^{-1}(1), f^{-1}(-1), f^{-1}(3), f^{-1}(-3), f^{-1}(-6)$

(iii) $f^{-1}([-5,5]), f^{-1}([-6,5])$

Ans) (i) $f(0) = -3(0)+1$
$= 1$

$f(-1) = -3(-1)+1 = 4$

$f(5/3) = \cancel{3}\left(\frac{5}{\cancel{3}}\right) - 5 = 0$

$f\left(-\frac{5}{3}\right) = -\cancel{3}\left(\frac{5}{\cancel{3}}\right) + 1 = -5+1 = -6$

(ii) $3x-5=0$ $\qquad$ $-3x+1=0$
$3x=5$ $\qquad$ $1=3x$
$x=5/3$ $\qquad$ $x=1/3$ ✗

$f^{-1}(0) = \{5/3\}$

$3x-5=1$ $\qquad$ $-3x+\cancel{1}=\cancel{1}$
$3x=6$ $\qquad$ $\boxed{x=0}$
$\boxed{x=2}$

$f^{-1}(1) = \{0,2\}$

$3x-5=-1$ $\qquad$ $-3x+1=-1$
$\boxed{x=4/3}$ $\qquad$ $+3x=+2$
$\boxed{x=2/3}$ ✗

$f^{-1}(-1) = \{4/3\}$

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~~Q) Let A = {~~

$3x-5=2$ $\qquad$ $-3x+1=2$

$3x=7$ $\qquad$ $-3x=1$

$\boxed{x=7/3}$ $\qquad$ $\boxed{x=-1/3}$

$f^{-1}(2)=\{\frac{7}{3},-\frac{1}{3}\}$

$3x-5=3$ $\qquad$ $-3x+1=3$

$3x=8$ $\qquad$ $-3x=2$

$\boxed{x=8/3}$ $\qquad$ $\boxed{x=-2/3}$

$f^{-1}(3)=\{\frac{8}{3},\frac{-2}{3}\}$

$3x-5=-3$ $\qquad$ $-3x+1=-3$

$3x=2$ $\qquad$ $-3x=-4$

$\boxed{x=2/3}$ $\qquad$ $\boxed{x=4/3}$

$f^{-1}(-3)=\{2/3\}$

$3x-5=-6$ $\qquad$ $-3x+1=-6$

$3x=-1$ $\qquad$ $-3x=-7$

$\boxed{x=-1/3}$ $\qquad$ $x=7$

$f^{-1}(-6)=\phi$

(iii) ~~$f^{-1}([-5,5])=\{\frac{5}{3},0,2,\frac{4}{3},\frac{8}{3},\frac{-2}{3},\frac{2}{3}\}$~~

$f^{-1}([-5,5])=\{x\in R \mid -5\le f(x)\le 5\}$

$=-5\le 3x-5\le 5$

$=0\le x\le 10/3$

$=\{x\in R \mid -5\le f(x)\le 5\}$

$=2\le x\le \frac{-4}{3}$ $\qquad$ $x \Rightarrow 2\geq x\geq \frac{-4}{3}$

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$f^{-1}([-5,5]) = \left[-\frac{4}{3}, \frac{10}{3}\right]$

$f^{-1}([-6,5]) = \{x \in R \mid -6 \le f(x) \le 5\}$

$= -6 \le 3x-5 \le 5$

$= -1 \le 3x \le 10$

$-\frac{1}{3} \le x \le \frac{10}{3}$

$= -6 \le 3x+1 \le 5$

$= -6-1 \le -3x \le 4$

$= -7 \le -3x \le 4$

$= \frac{+7}{+3} \le x \le \frac{-4}{3}$

$= -\frac{4}{3} \le x \le \frac{7}{3}$

$= \left[-\frac{4}{3}, \frac{10}{3}\right]$

Q.) Let 'A' & 'B' be finite sets with (cardinality) $|A| = m$ & $|B| = n$. find how many ~~possible~~ func^ns are possible from $A \to B$? If they are 2187 func^ns from $A \to B$ and $|B| = 3$ what is $|A| = ?$

~~$2^{mn} = 2187$~~

~~$2^{3m} = 2187$~~

Let set $A = \{a_1, a_2 \cdots a_m\}$ & $B = \{b_1, b_2, b_3 \ldots b_n\}$ then $f: (A \to B)$ is of the form $f = \{(a_1, x), (a_2, x) \ldots (a_m, x)\}$ $x \in B$; for some j. Since there are n of b_js, there are n choices for x in each of the m ordered pair belonging to f.

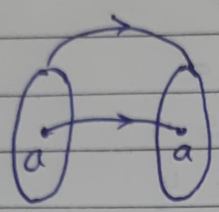
$\therefore$ The total possible no. of choices for x is $n \times n \times n \ldots \times n$ (m factors) $= n^m = |B|^{|A|}$

$f(A) \to$ Range

~~m = 5.235~~ $m = 7$

TYPES OF FUNCTION :

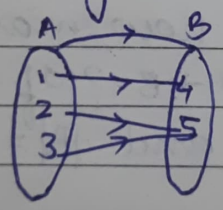
① **Identity functions** – A function $f: A \to A$ such that $f(a) = a$ $\forall a \in A$ is called identity function on set A.



$I_A \to$ denoted

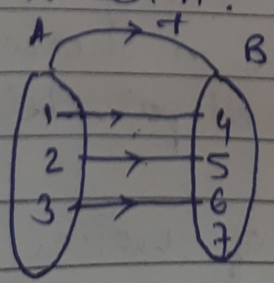
② **Constant function** – A function $f: A \to B$ such that $f(a) = c$ $\forall a \in A$ where c is fixed element set B. is called constant function.

③ **Onto function** – A function $f: A \to B$ is said to be onto function if every element of B has a pre-image in set A.



Also called as Surjective function.

④ **One-one function** – A function $f: A \to B$ is said to be one-one function if every element of A has a unique image in set B and every element of ~~f(A)~~ $f(A)$ has a unique pre-image in set A.



$f(A) = \{4, 5, 6\}$

Also called Injective

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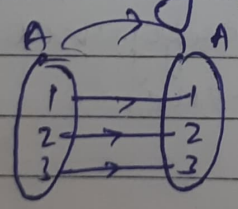
* If $f(a_1) = f(a_2)$ for $a_1, a_2 \in A$ then $a_1 = a_2$.

* The functions that satisfy both onto & one-one is called Bijective / one-one correspondance.

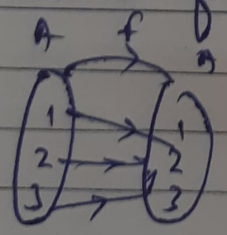
Q) Find the nature of the following functions
$A = \{1, 2, 3\}$
(i) $f = \{(1,1), (2,2), (3,3)\}$
(ii) $g = \{(1,2), (2,2), (3,2)\}$
(iii) $h = \{(1,2), (2,2), (3,1)\}$
(iv) $p = \{(1,2), (2,3), (3,1)\}$

(2) Let $A = \{1,2,3\}$. $B = \{1,2,3,4,5\}$
(i) $f = \{(1,1), (2,3), (3,4)\}$
(ii) $g = \{(1,1), (2,2), (3,3)\}$

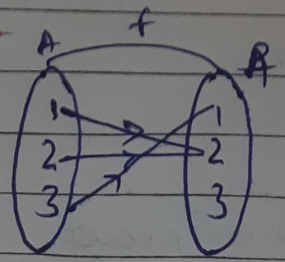
Ans (1) ~~(i) Identity function.~~
~~(ii) Constant function~~
~~(iii)~~ (i) Identity function one-one, onto, bijective.



(ii) Constant function

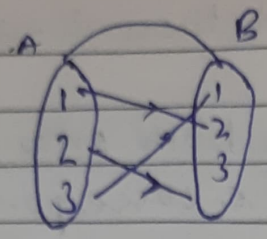


(iii)



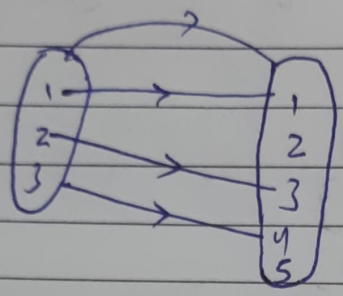
h is a function.

(iv)



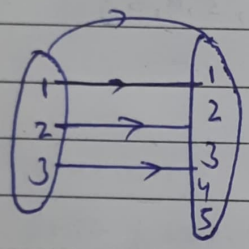
one-one, onto, bijective

(2) (i)



one-one,

(ii)



identity, one-one

Q) The functions f from $R \to R$ and $g: R \to R$ are defined by $f(x) = 3x + 7$ $\forall x \in R$ and $g(x) = x(x^3 - 1)$ $\forall x \in R$. Verify that f is one-one but g is ~~onto~~ not one-one.

$f(x_1) = 3x_1 + 7$

$f(x_2) = 3x_2 + 7$

$f(x_1) = f(x_2)$

$3x_1 + \cancel{7} = 3x_2 + \cancel{7}$

$\boxed{x_1 = x_2}$

$\therefore$ f is one-one function.

$g(0) = 0$

$g(1) = 0$

There is no unique pre-image of f(A) in A.

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Q.) Let $f: Z \to Z$ be defined by $f(a) = a+1$ for $a \in Z$ find whether f is onto or one-one.

$f(a_1) = a_1 + 1$

$f(a_2) = a_2 + 1$

$f(a_1) = f(a_2)$

$\boxed{a_1 = a_2}$

$\therefore$ f is a one-one function.

$f(0) = 1 \quad f(1) = 2 \quad \ldots \quad f(b-1) = b$.

There are unique pre-images of $f(A)$ in A.

$\therefore$ f is onto function.

$\Rightarrow$ f is Bijective function.

Proof: Take any $a_1, a_2 \in Z$ where we have

$f(a_1) = a_1 + 1$, $f(a_2) = a_2 + 1$

$a_1 + 1 = a_2 + 1$

$\boxed{a_1 = a_2}$ $\therefore$ f is one-one function.

Take any $b \in Z$, we check that b has $(b-1)$ as it's pre-image under f because

$f(a) = a + 1$

$f(b-1) = b - \cancel{1} + \cancel{1}$

$f(b-1) = b$

Thus, every element of Z has a pre-image

$\therefore$ f is onto.

$\Rightarrow$ f is both onto & one-one, f is is bijective.

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Q.) Let $A = R$ & $B = \{x \mid x \text{ is real } \& \ x \geq 0\}$
If the function $f: A \to B$ defined by $f(a) = a^2$
an onto function? a one-one function?

$\Rightarrow f(a) = a^2$

$f(a_1) = a_1^2 \quad f(a_2) = a_2^2$

$a_1^2 = a_2^2$

$\boxed{a_1 = a_2}$

$\therefore f$ is one-one function.

Take any $b \in B$, b is non-negative real no.

Here, $f(\sqrt{b}) = (\sqrt{b})^2 = b$ | $\therefore$ its roots $\pm\sqrt{b}$ exist & are real no. $\pm\sqrt{b} \in A$.

we note that $f(-\sqrt{b}) = (-\sqrt{b})^2 = b$

Thus $\pm\sqrt{b}$ are the pre-images of B under f.

f is onto function.

f is not one-one function because $\sqrt{b}, -\sqrt{b}$ both are mapped to b.

since b is an arbitrary element of B it follows that every element in B has atleast one pre-image in A.

Q.) The function $f: Z \to Z$ is defined by $f(x,y) = 2x + 3y$. Verify that f is onto but not one-one. If $A = \{(0,n) \mid n \in Z^+\}$ find $f(A)$?

Soln $f(x,y) = 2x + 3y$

Take $n \in Z$

$n = 4n - 3n$

$= 2(2n) + 3(-n)$

f is onto function

But f is not one-one function.

$f(0,0) = 0$ $\quad$ $f(0,2) = 6$

$f(-3,2) = 0$ $\quad$ $f(3,0) = 6$

$f(A) = \{3n \mid n \in Z^+\}$

$= \{3, 6, 9 \ldots\}$

NUMBER OF ONE-ONE & ONTO FUNⁿ:

If A & B are finite sets $|A| = m$ & $|B| = n$ then

① No. of one-one function possible from $A \to B$

$$P(n,m) = \frac{n!}{(n-m)!} \qquad \boxed{n \geq m}$$

② No. of onto functions possible from $A \to B$

$$P(n,m) = \sum_{k=0}^{n} (-1)^k \, {}^nC_{n-k} \, (n-k)^m$$

$$\boxed{m \geq n}$$

Q.) Let A & B finite sets. If there are 60 one-one funcs if $|A| = 3$ $|B| = ?$

$$P(n,m) = \frac{n!}{(n-m)!}$$

$$60 = \frac{n!}{(n-3)!}$$

$$60 = \frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}}$$

$$n(n^2 - 3n + 2) = 60$$

$$n^3 - 3n^2 + 2n = 60$$

$$\boxed{n = 5}$$

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Q.) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ $B = \{w, x, y, z\}$
find no. of onto funcⁿ from A to B?

$$\cancel{P(7, 4) = \frac{7!}{3!} = 7 \times 6 \times 5 \times 4 = 42 \times 20 = 840}$$

$$P(n, m) = \sum_{k=0}^{n} (-1)^k \, {}^nC_{n-k} \, (n-k)^m$$

$$= \sum_{k=0}^{4} (-1)^k \, {}^4C_{4-k} \, (4-k)^7$$

$$= (-1)^0 \, {}^4C_4 \, (4)^7 + (-1) \, {}^4C_3 \, (3)^7 +$$

$$(-1)^2 \, {}^4C_2 \, (2)^7 + (-1)^3 \, {}^4C_1 \, (1)^7$$

$$= 1 \times 4^7 - 4 \times 3^7 + 6 \times 2^7 - 4 \times 1^7$$

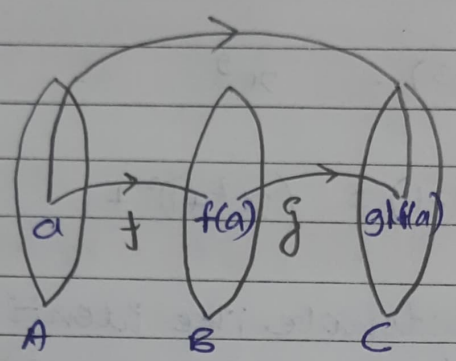
$$= 8400$$

Q.) Let $A = \{1, 2, 3, 4\}$ $B = \{1, 2, 3, 4, 5, 6\}$

(i) How many funⁿ are there from $A \to B$?
(ii) How many of these are one-one & onto?
(iii) How many funⁿ are there from B to A?
(iv) How many of these one-one & onto?

COMPOSITION OF FUNCTION.

Consider 3 non-empty sets A, B, C and the function $f:(A \to B)$ and $g:(B \to C)$. The composition (or product) of these function is defined as $gof : A \to C$ with $gof(a) = g(f(a))$
$= g(b)$
$= c$



Pictorial representation of gof is as shown.

0 for a function $f: A \to A$ fof is denoted by f^2.

① Let $A = \{1, 2, 3, 4\}$ $B = \{a, b, c\}$ $C = \{w, x, y, z\}$
with $f: A \to B$ & $g: B \to C$
$f = \{(1, a), (2, a), (3, b), (4, c)\}$. &
$g = \{(a, x), (b, y), (c, z)\}$
find gof.

② Consider the functions f & g defined by $f(x) = x^3$ & $g(x) = x^2 + 1$ $\forall x \in R$. find gof, fog, f^2, g^2.

Ans 1) $gof(1) = g(f(1)) = g(a) = x$
$gof(2) = g(f(2)) = g(a) = x$
$gof(3) = g(f(3)) = g(b) = y$
$gof(4) = g(f(4)) = g(c) = z$
$\Rightarrow \{(1, x), (2, x), (3, y), (4, z)\}$

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(2) $f(x) = x^3 \quad g(x) = x^2+1$

$gof = g(f(x)) = g(x^3) = x^6+1$

$fog = f(g(x)) = f(x^2+1) = (x^2+1)^3$

$f^2 = f(f(x)) = f(x^3) = x^9$

$g^2 = g(g(x)) = g(x^2+1) = (x^2+1)^2+1$

(3) Let I_A and I_B denote the identity functions on A & B respectively for any function $f: A \to B$ Prove that $fo I_A = f = I_B o f$

Soln

~~$fo I_A = I_A(f) = f$~~

$fo I_A = f(I_A(a)) = f(a)$

$I_B o f = I_B(f(a)) = f(a)$

$fo I_A = f = I_B o f$

(4) Let 'f' & 'g' be functions from R to R $f(x) = ax+b$ & $g(x) = 1-x+x^2$ If $gof(x) = 9x^2-9x+3$. Determine a & b.

$gof(x) = g(f(x)) = (1-(ax+b)+(ax+b)^2)$

$9x^2-9x+3 = 1-ax-b+a^2x^2+b^2+2axb$

$a^2x^2 = 9x^2$

$a = \pm 3$

$-ax+2abx = -9x$

$-3+6b = -9$

$6b = -6$

$b = -1$

$$3 - 6b = -9$$
$$-6b = -12$$
$$b = 2$$

(5) Let 'f' & 'g' be functions from $R \to R$ defined by $f(x) = ax+b$ and ~~gof~~ $g(x) = cx+d$. What relationship must be satisfied by a, b, c, d if $gof = fog$.

Given =

$$g(f(x)) = f(g(x))$$
$$g(ax+b) = f(cx+d)$$
$$c(ax+b)+d = a(cx+d)+b$$
$$\cancel{acx} + bc + d = \cancel{acx} + ad + b$$
$$\cancel{bc - ad = b - d}$$
$$\cancel{bc - b = ad - d}$$
$$\cancel{b(c-1) = d(a-1)}$$

$$\Rightarrow \boxed{ad + b = bc + d}$$

(6) Let $f: A \to B$ & $g: B \to C$ be any 2 functions. Then the following statements are true?

(i) if f and g are one-one, so is gof?
(ii) If gof is one-one, then f is one-one?
(iii) If f & g are onto, so is gof?
(iv) If gof is onto, then g is onto?

Ans) (i) Take any $a_1, a_2 \in A$

$g(f(a_1)) = g(f(a_2))$ [By def. of one-one]
$f(a_1) = f(a_2)$ [Since g is one-one]
$a_1 = a_2$ [Since f is one-one]

$\therefore$ gof is one-one func.

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(ii) Take any $a_1, a_2 \in A$

then $f(a_1), f(a_2) \in B$

By the definition, $f(a_1) = f(a_2)$

$$g(f(a_1)) = g(f(a_2))$$

$$gof(a_1) = gof(a_2)$$

$$a_1 = a_2 \quad [\because gof \text{ is one-one}]$$

$\therefore$ f is one-one.

(iii) for any element $c \in C$, for any $b \in B$,

$g(b) = c$ $\quad f(a) = b$

$$gof(a) \Rightarrow g(f(a)) = g(b) = c$$

$$gof(a) = c$$

pre-image

$\therefore$ gof is onto function.

Take any $c \in C$, since g is onto there is some $b \in B$ such that $g(b) = c$. since $b \in B$ and f is onto there is some $a \in A$ such that $f(a) = b$.

Thus for any $c \in C$ there is some $a \in A$. such that $gof(a) = c$

$\therefore$ gof is onto function.

(iv) Take any $c \in C$, since $gof: A \to C$, there is some $a \in A$ such that $g(f(a)) = c$. Since $f(a) \in b$, this means for any given $c \in C$, there is an element $f(a)$ is B. such that $g(f(a)) = c$ $\therefore$ g is onto function.

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Q.) Let $f: A \to B$, $g: B \to C$ & $h: C \to D$ ★★

$ho(gof) = (hog)of$

Proof:

$ho(gof) \Rightarrow ho(g(f(a)))$

$\Rightarrow ho(g(b))$ [WKT $f(a)=b$]

$\Rightarrow h(g(b))$

$\Rightarrow h(c)$ [$\because g(b) = c$]

$\Rightarrow d$

$(hog)of \Rightarrow (hog)f(a)$

$\Rightarrow h(g(f(a)))$

$\Rightarrow h(g(b))$ [$\because f(a) = b$]

$\Rightarrow h(c)$ [$\because g(b)=c$]

$\Rightarrow d$

Q.) Let f, g, h be functions from 'z' to 'z' ★★

$f(x) = x - 1$ $g(x) = 3x$

$$h(x) = \begin{cases} 0 & , x \text{ is even} \\ 1 & , x \text{ is odd.} \end{cases}$$

Determine $(fo(goh))(x)$ & $((fog)oh)(x)$

verify that $fo(goh) = (fog)oh$

for $h=0$

Ans) $fo(g(h(x)) = f(g(0)) = f(0)$
$= -1$

for $h=1$

$fo(g(1)) = f(3) = 2$

for $h=0$

$f(g(h(x))) = f(g(0)) = -1$

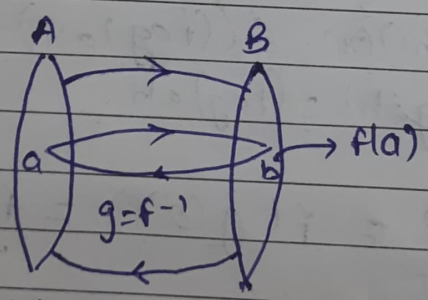
for $h=1$

$f(g(h(x))) = f(g(1)) = f(3) = 2$

Verification: $fo(goh)x$

INVERTIBLE FUNCTIONS. ★★

A function $A \to B$ is said to be invertible if there exists $g: B \to A$ such that $gof = I_A$ and $fog = I_B$ where I_A identity func on set A & I_B refers to identity func on B then g is called inverse of f and we write $g = f^{-1}$



$gof(a) = a$

Problems

① Let $A = \{1,2,3,4\}$ and f, g be functions from $A \to A$ given by $f = \{(1,4), (2,1), (3,2), (4,3)\}$
$g = \{(1,2), (2,3), (3,4), (4,1)\}$
Prove that f & g are inverses of each other.

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② Consider the function $f: R \to R$ $f(x) = 2x + 5$
$g: R \to R$ $g(x) = \frac{1}{2}(x-5)$.

P.T. g is an inverse of f.

Ans 1) $f = \{(1,4), (2,1), (3,2), (4,3)\}$
$g = \{(1,2), (2,3), (3,4), (4,1)\}$

$gof(1) = g(f(1)) = g(4) = 1$
$gof(2) = g(f(2)) = g(1) = 2$
$gof(3) = g(f(3)) = g(2) = 3$
$gof(4) = g(f(4)) = g(3) = 4$

$\because$ $gof(a) = a$
g is an inverse function.

$fog(1) = f(g(1)) = f(2) = 1$
$fog(2) = f(g(2)) = f(3) = 2$
$fog(3) = f(g(3)) = f(4) = 3$
$fog(4) = f(g(4)) = f(1) = 4$

$\because$ $fog(a) = a$
f is an inverse function.

② $f(x) = 2x + 5$ $g(x) = \frac{1}{2}(x-5)$

$gof(x) = g(f(x)) = g(2x+5) =$

$= \frac{1}{2}(2x + \cancel{5} - \cancel{5}) = x$

$gof(x) = x$
$\Rightarrow$ g is an inverse of x.

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$$fog(x) = f(g(x)) = f\left(\frac{1}{2}(x-5)\right) = \cancel{2}\left(\frac{1}{\cancel{2}}(x-5)\right)+5$$

$$= x - \cancel{5} + \cancel{5}$$

$$fog(x) = x$$

Q.) If a function $f: A \to B$ is invertible then it has a unique inverse. further if $f(a) = b$ $f^{-1}(b) = a$.

Ans) Let g & h be two inverses existing -

$\Rightarrow gof = I_A$ $\quad$ $hof = I_A$

$fog = I_B$ $\quad$ $hog = I_B$

We can say that $h \Rightarrow h o I_B$

$\Rightarrow h o (fog)$

$\Rightarrow (hof) o g$

$\Rightarrow I_A o g$

$\Rightarrow g$

$\therefore \boxed{h = g}$

We can conclude that for $f: A \to B$ there exists a unique inverse h or g.

Now suppose that $f(a) = b$

We can say that $a = I_A(a)$

$\Rightarrow gof(a)$

$\Rightarrow g(f(a))$

$\Rightarrow g(b)$

$a = g(b)$ $\quad$ $[\because f: A \to B, g: B \to A]$

$\Rightarrow f^{-1} = g$

$\Rightarrow a = f^{-1}(b)$

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a.) A function $f: A \to B$ is invertible if both one-one & onto.

<u>Proof</u>: Suppose that f is invertible then there exists a unique func^n $B \to A$ such that $gof = I_A$ $\quad fog = I_B$

Take any $a_1, a_2 \in A$ then $f(a_1) = f(a_2)$

$$g(f(a_1)) = g(f(a_2))$$
$$gof(a_1) = gof(a_2)$$
$$I(a_1) = I_A(a_2)$$
$$\boxed{a_1 = a_2}$$

$\therefore$ f is one-one function.

Any $b \in B$ then $g(b) \in A$ and $b = I_B(b)$

$$b = I_B(b)$$
$$= fog(b)$$
$$= f(g(b))$$

Thus b is the image of an element $g(b) \in A$ under f.

$\therefore$ f is an onto function.

Theorem:

If a function $f: A \to B$ and $g: B \to C$ are invertible functions then $gof: A \to C$ is an invertible function and $(gof)^{-1} = f^{-1} o g^{-1}$

$f: A \to B$ $\quad g: B \to C$ $\quad gof: A \to C$

$f^{-1}: B \to A$ $\quad g^{-1}: C \to B$

Since f and g are invertible functions, they are both onto and one-one. Consequently, gof is both one-one & onto. Therefore, gof is invertible. The inverse is a function from $B \to A$ and $g^{-1}: C \to B$

$\therefore$ if $h = f^{-1} o g^{-1}$

then: $h: C \to A$

we find that

$$(gof)oh = (gof) o (f^{-1} o g^{-1})$$
$$= go(fof^{-1}) o g^{-1}$$
$$= go I_B o g^{-1}$$
$$= go g^{-1}$$
$$= I_C$$

$$ho(gof) = (f^{-1} o g^{-1}) o (gof)$$
$$= f^{-1} o (g^{-1} o g) o f$$
$$= f^{-1} o I_B o f$$
$$= f^{-1} o f$$
$$= I_A$$

$\because$ both $(gof)oh$ and $ho(gof)$ are identity

$\Rightarrow$ $(gof)^{-1} = h = f^{-1} o g^{-1}$

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Q.) Consider a function $f: R \to R$ $f(x) = x^2 \ \forall x \in R$
Is f is invertible?

Sol^n,

for any $a \in R$, we have $f(a) = a^2$, $f(-a) = a^2$.
Thus, both (a) & $(-a)$ have the same image a^2 under f. $\therefore$ f is not one-one func.
Consequently, f is not invertible.

Q.) Let $A = \{x \mid x \text{ is real } \& \ x \geq -1\}$
$B = \{x \mid x \text{ is real } \& \ x \geq 0\}$
Consider the $f: A \to B$ defined by $f(a) = \sqrt{a+1}$ $\forall a \in A$. Show that f is invertible.
and determine f^{-1}.

Sol^n for one-one,

$$f(a_1) = f(a_2)$$
$$\sqrt{a_1+1} = \sqrt{a_2+1}$$
$$a_1 + 1 = a_2 + 1$$
$$\boxed{a_1 = a_2}$$

$\therefore$ f is one-one.

for onto, $f(a) = b$
Take any $b \in B$.

$$\sqrt{a+1} = b$$
$$a + 1 = b^2$$
$$a = b^2 - 1, \quad a \in A$$

f is onto function.
If f is satisfying both one-one & onto function.

$\therefore$ f is invertible

$$f(a) = b \Rightarrow f^{-1}(b) = a$$
$$= b^2 - 1$$

HW

Q.) find the inverse of the function, $f(x) = e^x$ defined from $R \to R^+$

Q.) Let $A = B = R$ the set of all real no.s and the functions $f: A \to B$ and $g: B \to A$ be defined $f(x) = 2x^3 - 1$, $\forall x \in A$ $g(y) = \left[\frac{1}{2}(y+1)\right]^{1/3}$ $\forall y \in B$.

Show that f and g are inverse of the other.

Soln

$$gof(x) = g(f(x)) = g(2x^3 - 1)$$

$$= \left[\frac{1}{2}\left(2x^3 \cancel{-1} \cancel{+1}\right)\right]^{1/3}$$

$$= \frac{\cancel{2^{1/3}}}{\cancel{2^{1/3}}} \times x^{\cancel{3}^{1}} = x$$

$$gof(x) = x$$

$$fog(y) = f(g(y)) = f\left(\left[\frac{1}{2}(y+1)\right]^{1/3}\right)$$

$$= \left[2\left[\left(\frac{1}{2}(y+1)\right)^{1/3}\right]^3 - 1\right.$$

$$= \cancel{2} \times \frac{1}{\cancel{2}}(y+1) - 1$$

$$= y + 1 - 1 = y$$

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Q.) Let $A = B = R$ $f: A \to B$ & $g: B \to C$ be defined by
$f(a) = 2a+1$ $g(b) = \frac{1}{3} b$ $\forall a \in A$ $\forall b \in B$.

Compute gof. Show that gof is invertible.
Determine $(gof)^{-1}$

Sol^n

$gof : A \to C$ $gof(a) = g(f(a))$
$= g(2a+1)$
$= \frac{1}{3}(2a+1)$

for one-one

$gof(a_1) = gof(a_2)$

$\frac{1}{3}(2a_1+1) = \frac{1}{3}(2a_2+1)$

$\boxed{a_1 = a_2}$

for onto,

Let $gof(a) = b$

$\frac{1}{3}(2a+1) = b$

$2a+1 = 3b$

$2a = 3b-1$

$a = \frac{3b-1}{2}$

$\therefore$ gof is one one & onto.
$\therefore$ gof is invertible.

$(gof)^{-1} = f^{-1} o g^{-1}$

$f^{-1}(b) = a$
$= \frac{b-1}{2}$

$g^{-1}(c) = b = 3c$

$f^{-1} o g^{-1} = f^{-1} o (g^{-1}(a)) = f^{-1} o (3a)$
$= f^{-1}(3a) = \frac{3a-1}{2}$