Angular Momentum Complete

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1 Formalization

In this frame we will be formalizing the the angular momentum opertors in 3D. First, we know that in general

$$L = R \times P$$

From this we formalize the equations for our angular momentum to be

$$L_x = Yp_z \times Zp_y L_y = Zp_x \times Xp_z L_z = Xp_y \times Yp_x \tag{1}$$

also we define the total angular momentum to be

$$\langle L \rangle^2 = \langle x \rangle^2 + \langle y \rangle^2 + \langle z \rangle^2 \tag{2}$$

The commution relations are as follows

$$[L^2, L_z] = 0[L_x, L_y] = i\hbar \tag{3}$$

From the above, we conclude that there exists common eigen functions for $\langle L \rangle^2$ and $\langle L_z \rangle$ Let Y_{lm} be the common eigen function of them both. The commutation relations of Y_{lm} is

$$[Y_{lm}, \langle L \rangle^2] = 0 \tag{4}$$

Also we define two non Hermitian ladder operators due to the commutation relations as follows

$$L_\pm = L_x \pm L_y \tag{5}$$

Now the commutation relation of Y_{lm} with our ladder operators are

$$[Y_{lm}, L_\pm] = \hbar m \tag{6}$$

Before moving on lets layout the eigenValues of Y_{lm} when acted upon by the opertors

$$L^2 Y_{lm} = \hbar^2 l(l+1) \tag{7}$$

$$L_z Y_{lm} = \hbar m Y_{lm} \tag{8}$$

from this we define the followling relations

$$\text{Let } \psi = L_\pm Y_{lm} \tag{9}$$

Hence from here we define the following

$$\langle L \rangle^2 \psi = L^2 L_\pm Y_{lm} \tag{10}$$

$$[[L^2, L_\pm] + L\pm] Y_{lm} \tag{11}$$

$$[0] + L_\pm \hbar^2 l(l+1) Y_l m \tag{12}$$

From this we see that itL^2 has not effect on the wavefunction if its acted on by $Y_\pm$. Now the next thing is

$$L_z \psi = L_z L_\pm Y_{lm} \tag{13}$$

$$[[L_z, L_\pm] + L_\pm L_z] \, Y_{lm} \tag{14}$$

$$[\hbar L_\pm + L_\pm \hbar m] \, Y_{lm} \tag{15}$$

$$\hbar[m+1] L_\pm Y_{lm} \tag{16}$$

$$\hbar[m+1] \psi(x) \tag{17}$$

2 The Ladder Structure

Now we would like to find the possible values of m. To do this we use the expectation values of the opertors.

$$\langle \psi | \, L^2 \, | \psi \rangle = \langle \psi | \, L_x^2 \, | \psi \rangle + \langle \psi | \, L_y^2 \, | \psi \rangle + \langle \psi | \, L_z^2 \, | \psi \rangle \tag{18}$$

This yeilds

$$\hbar^2 l(l+1) > \hbar^2 m^2 \tag{19}$$

Solving this quadratic equation yields the following

$$m_+ = l_+ \tag{20}$$

$$m_- = l_- \tag{21}$$

This means the maximum value of m is l_+ and minimum is l_- We hence find out that in the ladder of Momentum eigen values, the number of states N is

$$N = 2l + 1 \tag{22}$$

which means that l can only take integer or half int values

$$l = 1, 1/2, 2/3, 2, ... \tag{23}$$

NOTE : say$L = 1$ and $L_z = 1$. This still does not mean that L_x and L_y will be zero. Lets say in general that $l = m$. Finding the expected values we get

$$\hbar^2 l(l+1) = \langle\psi| L_x^2 |\psi\rangle + \langle\psi| L_y^2 |\psi\rangle + \hbar^2 m^2 \tag{24}$$

Now as $l = m$

$$\langle\psi| L_x^2 |\psi\rangle + \langle\psi| L_y^2 |\psi\rangle = \hbar^2 l^2 + \hbar^2 l - \hbar^2 m^2 \tag{25}$$

$$\langle\psi| L_x^2 |\psi\rangle + \langle\psi| L_y^2 |\psi\rangle = \hbar^2 l \tag{26}$$

If let: The expectation values of L_x and L_y be equal. Then

$$\langle\psi| L_x^2 |\psi\rangle = \frac{\hbar^2 l}{2} \text{which is} > 0 \tag{27}$$

Hence even if $L_z = L^2$ there is still non zero expectation values in both L_x and L_y

3 The 3 Dimensions

Now let us move on to discussing the Spherical Harmonics part of angular momentum which is exciting. In here Y_{lm} is a funtion of ϕ and θ . ϕ is the angle made at the $x - y$plane and θ with the z axis. Now our operstors in 3D are

$$L_z = \frac{\hbar}{i}\partial_\phi \text{or} L_\phi = -i\hbar\partial_\phi \tag{28}$$

$$L_\pm = \pm\hbar e^{\pm i\phi}(\pm\partial_\theta + \cot\theta\partial_\phi) \tag{29}$$

Now if we use the fact that

$$L_z Y_{lm} = \hbar m Y_{lm} \tag{30}$$

Substituting the value of L_z we get

$$Y_{lm}(\theta, \phi) = e^{im\phi} P_l(\theta) \tag{31}$$

Where

$$P_l(\theta) = \text{some unknow dependence on } \theta \tag{32}$$

Now to proceed from here we establish the fact that at $\phi = 2\pi$ and $\phi = 0$ hence

$$Y_{lm}\theta, 0 = Y_{lm}(\theta, 2\pi) \tag{33}$$

This is because after one complete rotation it must come back to the same point. Now

$$Y_{lm}\theta, 0) = e^{im0} P_l(\theta) = P_l(\theta \tag{34}$$

Similarly

$$Y_{lm}(\theta, 2\pi) = \begin{cases} P_l(\theta) \text{if } m = int \\ -P_l(\theta) \text{if } m = \text{half ints} \end{cases} \tag{35}$$

Now the second case should notbe possible. Hence we conculde that

$$m \text{ cannot have half int values. Hence } l \text{ is strictly an integer} \tag{36}$$

4 The WaveFunction In 3D

Having established this fact we know that L_+ acting on m_+ will be equal to 0. Hence by substitution

$$\pm e^{im\phi}(\partial_\theta \pm \cot\theta \partial_\phi)(m_+) = 0 \tag{37}$$

We find the solutions of this to be associated with legendre polynomials. Hence the properlt NORMALIZED wvefunction is

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(|l| - |m|)!}{(|l| + |m|)!}} e^{im\phi} P_l^m(\cos\theta) \tag{38}$$

where $P_l^m(x) =$ associated legendre polynomial and in that $P_l(x) =$ the legendre polynomial solutions and $x = \cos\theta$

$$P_l^m(x) = (1 - x^2)^{\frac{|m|}{2}} \frac{\partial^m}{\partial_x^m} p_l(x) \tag{39}$$

$$P_l(x) = \frac{1}{2^x x!} \frac{\partial^x}{\partial_x^x} (x^2 - 1)^x \tag{40}$$

And the normalization conditon used is

$$\int_0^\pi \int_0^{2\pi} \int_{-\infty}^{\infty} |Y_{lm}(\theta, \phi)|^2 = 1 \tag{41}$$

5 The hadronic and leptonic tensor

This is a formal introduction to the leptonic and hadronic tensor.

$$d\sigma = \frac{1}{4ME} \frac{d^3k}{2\pi^3 2E} \frac{d^3p}{2\pi^3 2p'} \left\{ \frac{e^4}{q^4 L_e^{\mu\nu}} L_{\mu\nu}^{\text{muon}} \right\} (2\pi)^4 \sigma^4(p + q - p^{'}) \tag{42}$$

This implies that the hadronic tensor, which is of a covariant nature can be written as :

$$W_{\mu\nu} = \frac{1}{4\pi M} \left(\frac{1}{2} \sum_s \sum_{s^{'}} \right) \int \frac{d^3p'}{(2\pi)^3 2p'} \langle p, s| J_\mu^+ |p^{'}, s^{'}\rangle \, \mathbf{X} \, \langle p^{'}, s^{'}| J_\mu^+ |p, s\rangle \, (2\pi)^4 (p + q - p^{'})$$