

# Angular Momentum Complete

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## 1 Formalization

In this frame we will be formalizing the the angular momentum opertors in 3D. First, we know that in general

$$L = R \times P$$

From this we formalize the equations for our angular momentum to be

$$L_x = Yp_z \times Zp_y L_y = Zp_x \times Xp_z L_z = Xp_y \times Yp_x \quad (1)$$

also we define the total angular momentum to be

$$\langle L \rangle^2 = \langle x \rangle^2 + \langle y \rangle^2 + \langle z \rangle^2 \quad (2)$$

The commutation relations are as follows

$$[L^2, L_z] = 0 [L_x, L_y] = i\hbar \quad (3)$$

From the above, we conclude that there exists common eigen functions for  $\langle L \rangle^2$  and  $\langle L_z \rangle$  Let  $Y_{lm}$  be the common eigen function of them both. The commutation relations of  $Y_{lm}$  is

$$[Y_{lm}, \langle L \rangle^2] = 0 \quad (4)$$

Also we define two non Hermitian ladder operators due to the commutation relations as follows

$$L_{\pm} = L_x \pm L_y \quad (5)$$

Now the commutation relation of  $Y_{lm}$  with our ladder operators are

$$[Y_{lm}, L_{\pm}] = \hbar m \quad (6)$$

Before moving on lets layout the eigenValues of  $Y_{lm}$  when acted upon by the opertors

$$L^2 Y_{lm} = \hbar^2 l(l+1) \quad (7)$$

$$L_z Y_{lm} = \hbar m Y_{lm} \quad (8)$$

from this we define the followling relations

$$\text{Let } \psi = L_{\pm} Y_{lm} \quad (9)$$

Hence from here we define the following

$$\langle L \rangle^2 \psi = L^2 L_{\pm} Y_{lm} \quad (10)$$

$$[[L^2, L_{\pm}] + L_{\pm}] Y_{lm} \quad (11)$$

$$[0] + L_{\pm} \hbar^2 l(l+1) Y_{lm} \quad (12)$$

From this we see that it  $L^2$  has not effect on the wavefunction if its acted on by  $Y_{\pm}$ . Now the next thing is

$$L_z \psi = L_z L_{\pm} Y_{lm} \quad (13)$$

$$[[L_z, L_{\pm}] + L_{\pm} L_z] Y_{lm} \quad (14)$$

$$[\hbar L_{\pm} + L_{\pm} \hbar m] Y_{lm} \quad (15)$$

$$\hbar [m+1] L_{\pm} Y_{lm} \quad (16)$$

$$\hbar [m+1] \psi(x) \quad (17)$$

## 2 The Ladder Structure

Now we would like to find the possible values of  $m$ . To do this we use the expectation values of the opertors.

$$\langle \psi | L^2 | \psi \rangle = \langle \psi | L_x^2 | \psi \rangle + \langle \psi | L_y^2 | \psi \rangle + \langle \psi | L_z^2 | \psi \rangle \quad (18)$$

This yeilds

$$\hbar^2 l(l+1) > \hbar^2 m^2 \quad (19)$$

Solving this quadratic equation yields the following

$$m_{+} = l_{+} \quad (20)$$

$$m_- = l_- \quad (21)$$

This means the maximum value of  $m$  is  $l_+$  and minimum is  $l_-$ . We hence find out that in the ladder of Momentum eigen values, the number of states  $N$  is

$$N = 2l + 1 \quad (22)$$

which means that  $l$  can only take integer or half int values

$$l = 1, 1/2, 2/3, 2, \dots \quad (23)$$

NOTE : say  $L = 1$  and  $L_z = 1$ . This still does not mean that  $L_x$  and  $L_y$  will be zero. Lets say in general that  $l = m$ . Finding the expected values we get

$$\hbar^2 l(l+1) = \langle \psi | L_x^2 | \psi \rangle + \langle \psi | L_y^2 | \psi \rangle + \hbar^2 m^2 \quad (24)$$

Now as  $l = m$

$$\langle \psi | L_x^2 | \psi \rangle + \langle \psi | L_y^2 | \psi \rangle = \hbar^2 l^2 + \hbar^2 l - \hbar^2 m^2 \quad (25)$$

$$\langle \psi | L_x^2 | \psi \rangle + \langle \psi | L_y^2 | \psi \rangle = \hbar^2 l \quad (26)$$

If let: The expectation values of  $L_x$  and  $L_y$  be equal. Then

$$\langle \psi | L_x^2 | \psi \rangle = \frac{\hbar^2 l}{2} \text{ which is } > 0 \quad (27)$$

Hence even if  $L_z = L^2$  there is still non zero expectation values in both  $L_x$  and  $L_y$

### 3 The 3 Dimensions

Now let us move on to discussing the Spherical Harmonics part of angular momentum which is exciting. In here  $Y_{lm}$  is a funtion of  $\phi$  and  $\theta$ .  $\phi$  is the angle made at the  $x - y$  plane and  $\theta$  with the  $z$  axis. Now our operstors in 3D are

$$L_z = \frac{\hbar}{i} \partial_\phi \text{ or } L_\phi = -i\hbar \partial_\phi \quad (28)$$

$$L_\pm = \pm \hbar e^{\pm i\phi} (\pm \partial_\theta + \cot \theta \partial_\phi) \quad (29)$$

Now if we use the fact that

$$L_z Y_{lm} = \hbar m Y_{lm} \quad (30)$$

Substituting the value of  $L_z$  we get

$$Y_{lm}(\theta, \phi) = e^{im\phi} P_l(\theta) \quad (31)$$

Where

$$P_l(\theta) = \text{some unknow dependence on } \theta \quad (32)$$

Now to proceed from here we establish the fact that at  $\phi = 2\pi$  and  $\phi = 0$  hence

$$Y_{lm}(\theta, 0) = Y_{lm}(\theta, 2\pi) \quad (33)$$

This is because after one complete rotation it must come back to the same point. Now

$$Y_{lm}(\theta, 0) = e^{im\phi} P_l(\theta) = P_l(\theta) \quad (34)$$

Similarly

$$Y_{lm}(\theta, 2\pi) = \begin{cases} P_l(\theta) & \text{if } m = \text{int} \\ -P_l(\theta) & \text{if } m = \text{half ints} \end{cases} \quad (35)$$

Now the second case should not be possible. Hence we conclude that

$$m \text{ cannot have half int values. Hence } l \text{ is strictly an integer} \quad (36)$$

## 4 The WaveFunction In 3D

Having established this fact we know that  $L_+$  acting on  $m_+$  will be equal to 0. Hence by substitution

$$\pm e^{im\phi} (\partial_\theta \pm \cot \theta \partial_\phi)(m_+) = 0 \quad (37)$$

We find the solutions of this to be associated with legendre polynomials. Hence the properlt NORMALIZED wvefunction is

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(|l|-|m|)!}{(|l|+|m|)!}} e^{im\phi} P_l^m(\cos \theta) \quad (38)$$

where  $P_l^m(x)$  = associated legendre polynomial and in that  $P_l(x)$  = the legendre polynomial solutions and  $x = \cos \theta$

$$P_l^m(x) = (1-x^2)^{\frac{|m|}{2}} \frac{\partial^m}{\partial x^m} P_l(x) \quad (39)$$

$$P_l(x) = \frac{1}{2^l l!} \frac{\partial^l}{\partial x^l} (x^2-1)^l \quad (40)$$

And the normalization conditon used is

$$\int_0^\pi \int_0^{2\pi} \int_{-\infty}^\infty |Y_{lm}(\theta, \phi)|^2 = 1 \quad (41)$$