Solution to TA # 04, EE 250 (Control System Analysis) - Spring 2025*

DEPARTMENT OF ELECTRICAL ENGINEERING, IIT KANPUR

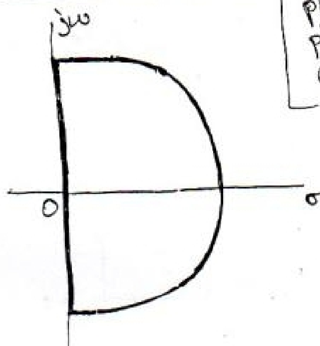
Discuss the stability of the unity-feedback CL counterparts of the following transfer functions using Nyquist Stability Theory.

1. $\frac{K}{(s+1)^2}$
2. $\frac{K}{s(s+1)(\frac{s}{10}+1)}$
3. $\frac{K}{s(s+1)^2}$
4. $\frac{K}{s^2(s+1)^2}$
5. $\frac{K}{s^3(s+1)^2}$
6. $\frac{K(s+1)}{s(\frac{s}{10}-1)}$

BODE PLOT (BP) method of sketching Nyquist plots

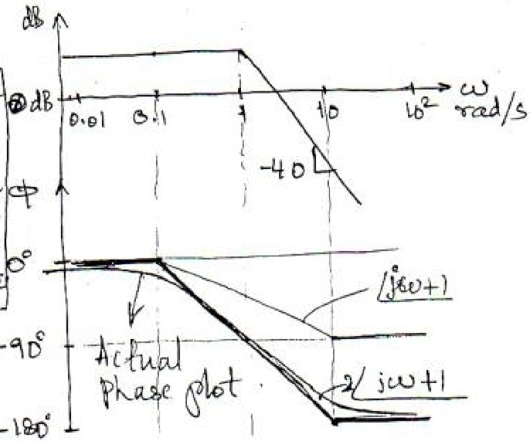
1. $\frac{K}{(s+1)^2}$

step 1: s-plane contour:

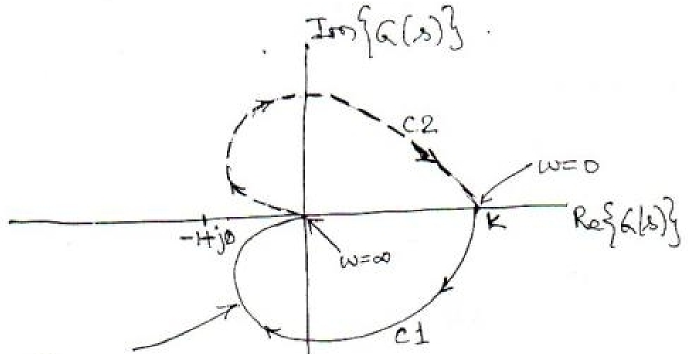


step 2: Bode plot:

This maps the +ve part of the jω axis to the complex plane (G(s)-plane) via $G(s)=\frac{K}{(s+1)^2}$



step 3: Nyquist plot:



This section (C1) sketched ~~drawn~~ approximately using the Asymptotic Bode Plot sketched in Step 1.

Section C2 is drawn as a mirror image of C1 about the Re{G(s)}-axis.

Step 4:

Is the CL sys

stable? If so, then for what values of $K>0$? (We are looking only @ $K>0$ in this course; the student can explore $K\leq 0$ on his own).

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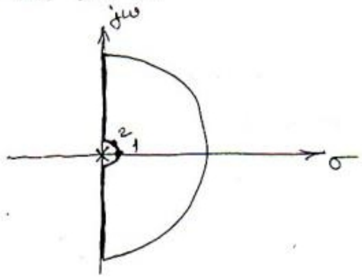
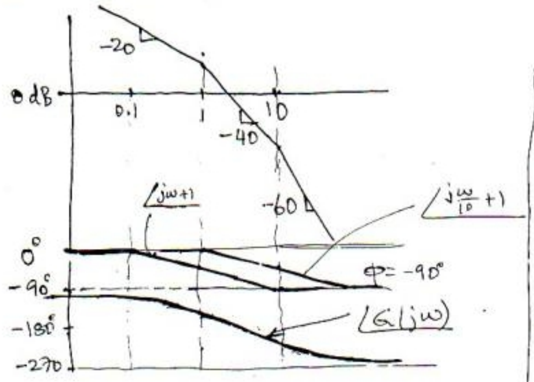
We see that $\forall K>0$, the NP will never encircle the point $-1+j0$. So, $N=0$. Also, $P=0$ as all poles of $G(s)$ were in RH s-plane. So, $N=Z-P$ gives $Z=0$. So, the given CL system is stable for all $K>0$.

2. $G(s) = \dfrac{K}{s(s+1)\left(\frac{s}{10}+1\right)}$.

step1:



step2: Bode plot:



Step4: Need to determine the section at infinity of the NP. For this use points 1 & 2 on the s-plane contour.

s	① $r.e^{j0}$	② $r.e^{j45°}$
$G(s)$	$\simeq R.e^{j0}$ ①'	$\simeq R.e^{-j45°}$ ②'

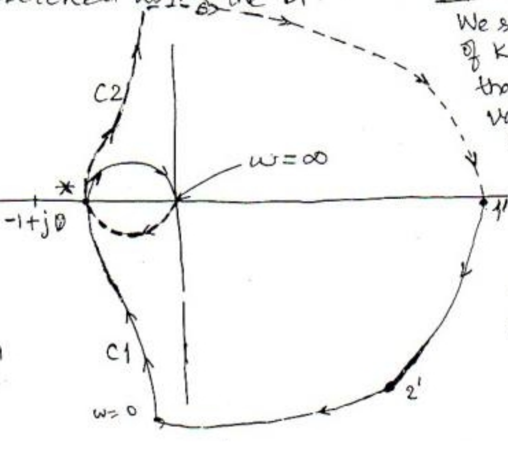
Mark 1' & 2' on the NP and complete the NP.

step3: Sketch the part of the NP that can be sketched using the BP.

Section C1 follows from the BP

Section C2 is mirror image of C1

The arrows on C2 have been placed to show traversal of ~~NP in clockwise~~ C2 as a continuation of traversal of C1.



Step5: Is the CL sys. stable?

We see that for "small" values of $K>0$, $N=0$. For K larger than a certain minimum value, $N=2$. In both cases $P=0$. So, for this second K, $Z=2$, and the CL sys. will have 2 poles in the RHP.

What is this value of K? →

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In order to answer this question, we observe that the point marked "*" on the NP occurs at the frequency at which the phase of $G(j\omega) = -180°$:

$$-90° - \arctan\omega - \arctan\frac{\omega}{10} = -180°$$

$$\Rightarrow \arctan\omega + \arctan\frac{\omega}{10} = 90°$$

$$\Rightarrow \arctan\left(\frac{\omega + \frac{\omega}{10}}{1 - \omega\cdot\frac{\omega}{10}}\right) = 90°$$

$$\Rightarrow \frac{\omega + \frac{\omega}{10}}{1 - \frac{\omega^2}{10}} = \tan 90° = \infty$$

$$\Rightarrow \frac{\omega^2}{10} = 1 \Rightarrow \omega = \sqrt{10} \text{ rad/s}.$$

$$\boxed{\omega = 3.162 \text{ rad/s}}$$

Now we have to determine the value of K for which $|G(j\omega)| = 1$ @ $\omega = 3.162$ rad/s.

$$\frac{K}{\omega\sqrt{\omega^2+1}\sqrt{\left(\frac{\omega}{10}\right)^2+1}} = 1$$

$$\Rightarrow K = \omega\sqrt{\omega^2+1}\cdot\sqrt{\left(\frac{\omega}{10}\right)^2+1}$$

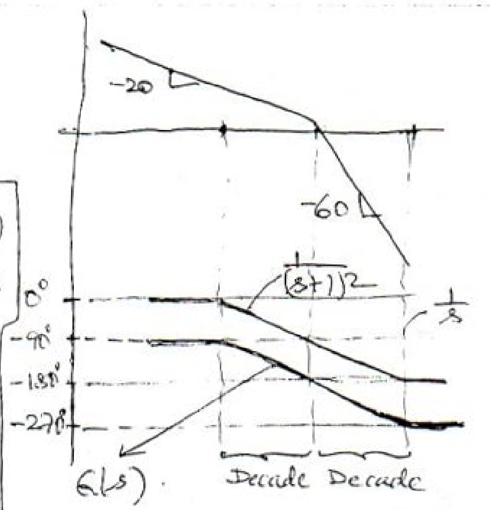
$$= 3.162\sqrt{11}\sqrt{1+\frac{1}{10}} = \sqrt{10 * 11 * 1.1} = \sqrt{11^2}$$

$$= 11$$

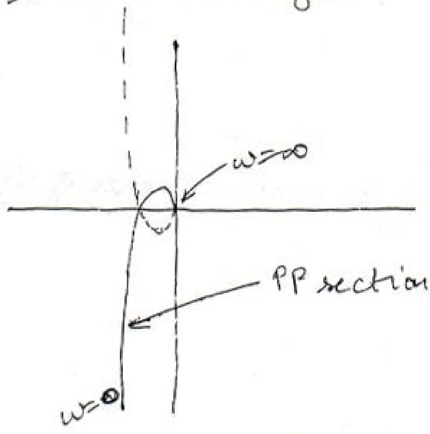
So, for $K \geq 11$, the unity feedback CL sys. is unstable

3. $G(s) = \frac{K}{s(s+1)^2}$

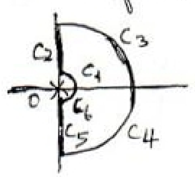
Step 1: Sketch the BP:



Step 2: Draw the polar plot (PP) section of the NP and the PP's mirror image:



Step 3: Identify points on the s-plane contour that will help us complete the NP:



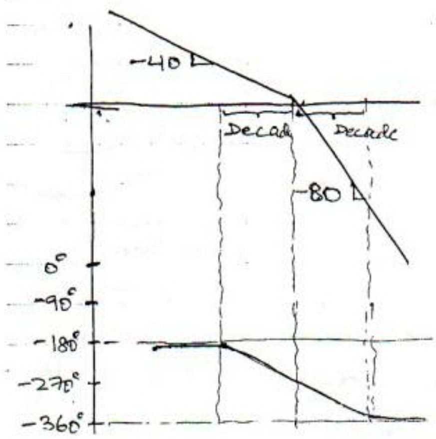
The sections at infinity (C_3 & C_4) map to an infinitisimally small region around the origin of the $G(s)$-plane, since this region does not affect the count of the encirclements of $-1+j0$ by the NP, we show it as coinciding with the origin of the $G(s)$-plane. So, we need to only consider C1 of the s-plane contour.

Consider the following two points of the s-plane contour: $r.e^{j0}$ and $r.e^{j45°}$, $r \to 0$. These map to $R.e^{j0}$ and $R.e^{-j45°}$, $R \to \infty$ of the $G(s)$-plane. These two points allow us to sketch section C1′ of the NP. Section C6′ is the mirror image of C1′ above the $Re\{G(s)\}$ axis. The NP is as follows:

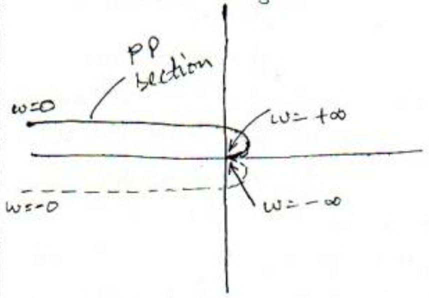
The stability of the CL system can be evaluated as done in the case of $G(s) = \frac{K}{s(s+1)(\frac{s}{10}+1)}$

4. $G(s) = \dfrac{K}{s^2(s+1)^2}$

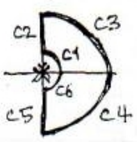
Step 1: Sketch the BP.



Step 2: Sketch the PP section of the NP and the PP's mirror image.

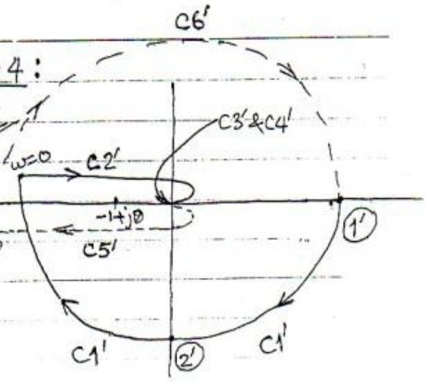


Step 3: Identify points on the s-plane contour that will help us complete the NP:



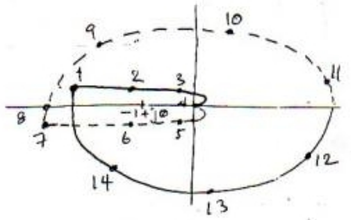
s	$r.e^{j0}$, $r \to 0$	$r.e^{j45°}$
$G(s)$	$R.e^{j0}$, $R \to \infty$	$R\,e^{-j90°}$
	1'	2'

Step 4: sketch the NP



Step 5: Discussion of stability.

Count the encirclements of $-1+j0$ as follows:



1^{st} encirclement: $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8$

This completes one 360° turn around $-1+j0$ in the clockwise direction.

2^{nd} encirclement: $8 \to 9 \to 10 \to 11 \to 12 \to 13 \to 14 \to 1$

This completes another 360° turn around $-1+j0$ in the clockwise sense.

So, $N = 2$.

$P = 0$, we know already.

So, $Z = N + P = 2$.

So, unity feedback CL sys. built around this $G(s)$ is unstable $\forall K > 0$.

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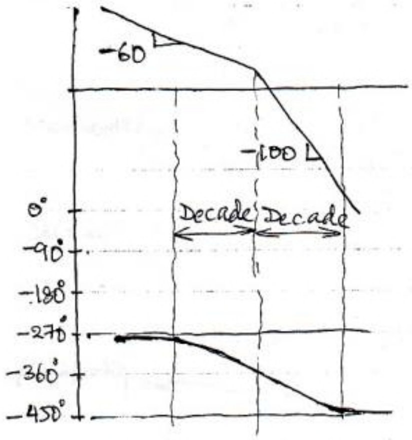
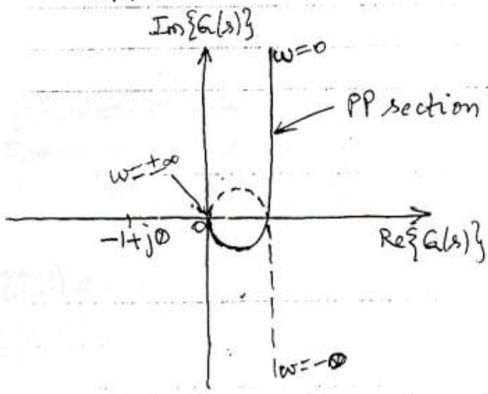
5. $G(s) = \dfrac{K}{s^3(s+1)^2}$

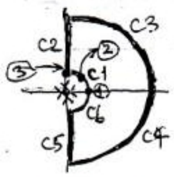
Step 1: Sketch the Bode Plot



Step 2: Sketch the PP section of the NP:

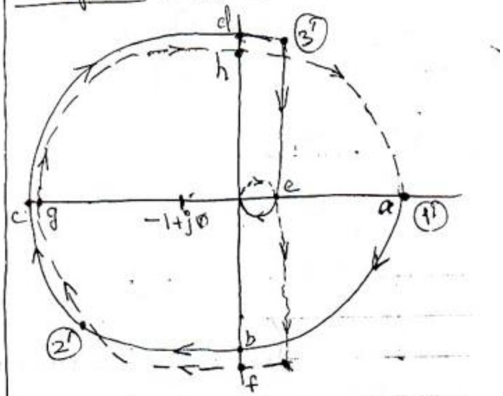


Step 3: Identify points on the s-plane contour that will help us complete the NP:



	①	②
s	$r.e^{j0}$, $r \to 0$	$r.e^{j45°}$
$G(s)$	$R.e^{j0}$, $R \to \infty$	$R.e^{-j135°}$
	①'	②'

Step 4: Sketch the NP.



Step 5: Discussion of stability.

Number of encirclements:

$N = 2$

$a \to b \to c \to d \to e$.

& $e \to f \to g \to h \to a$

So, $N = 2$

$P = 0$ (we know already).

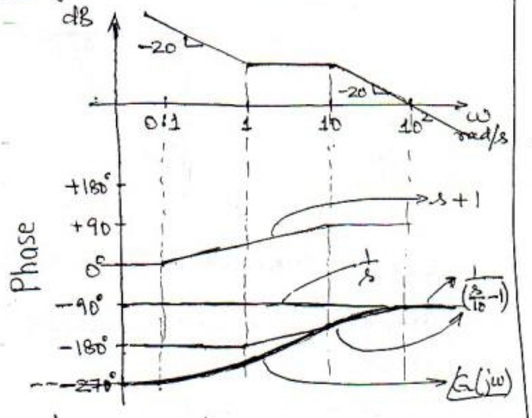
$\Rightarrow Z = N + P$

$= 2$.

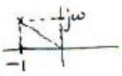
So, unity-feedback C.L sys. is unstable $\forall K > 0$.

6. $G(s) = \dfrac{K(s+1)}{s\left(\frac{s}{10}-1\right)}$

Step 1: Sketch the BP



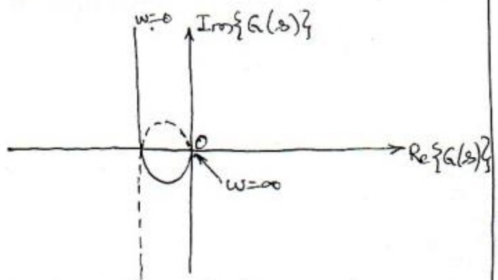
$\dfrac{1}{\frac{s}{10}-1} \rightarrow \dfrac{1}{-1+j\frac{\omega}{10}}$



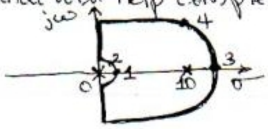
$\angle -1+j\frac{\omega}{10} = 180° - \arctan\frac{\omega}{10}$

$\Rightarrow \angle \dfrac{1}{-1+j\frac{\omega}{10}} = -180° + \arctan\frac{\omega}{10}$

Step2: Sketch the PP section of the NP.



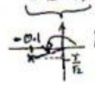
Step3: Identify points on s-plane contour that will help complete the NP:



s	① $r e^{j0}$, $r\to 0$	② $r e^{j45°}$	③ $R e^{j0°}$	④ $R e^{j45°}$
$G(s)$	$-R+j0$ ①'	$R e^{-j225°}$ ②'	$\approx 0+j0$	$\approx 0+j0$

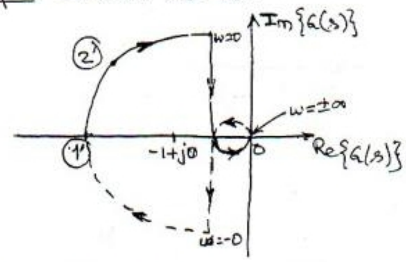
$-\dfrac{K(1+r)}{r\left(\frac{1}{10}-r\right)} = -\dfrac{K(1+r)}{r} = -R+j0$

$\left.\dfrac{K\left(1+\frac{r}{\sqrt2}+j\frac{r}{\sqrt2}\right)}{r e^{j45°}\left(-\frac{1}{10}-\frac{r}{\sqrt2}-j\frac{r}{\sqrt2}\right)}\right\}$ phase ≈ 1°



phase = 181°.

Step4: Sketch the NP.



Step5: Discussion of stability.

For $K < \hat{K}$, $N = 1 \rightarrow$ case I

For $K > \hat{K}$, $N = -1 \rightarrow$ case II

In both cases, $P = 1$

So, in case I, $Z = N + P = 2$ (CL sys. is unstable)

In case II, $Z = -1+1 = 0$ (CL sys. is stable).

So, what is the value of $\hat{K}$?

Can solve exactly as we did in Step 5 of Problem 2.

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