Assignment

Homework 3

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A MEGN544A Homework Assignment

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Problem 1

Given the following twist ξ, what is the homogenous transformation matrix?

(a) $\xi = [v^T, \Omega^T] = [3/\pi\ 0\ 0\ \pi/3\ 0\ 0]^T$

Solution.

$$\text{Transformation Matrix is given by, } T = \begin{bmatrix} R & d \\ 0^T & 1 \end{bmatrix}$$

$$\text{R} = \cos(\theta)I + (1 - \cos(\theta)\hat{k}\hat{k}^T + \sin(\theta)[\hat{k}_x]$$

$$\text{Using Matlab we get R as} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -0.86 \\ 0 & 0.86 & 0.5 \end{bmatrix}$$

$$\text{The displacement vector is given by d} = (I - R)[\hat{k}_x] + \Omega\hat{k}^T$$

$$\text{Using Matlab we get d as} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{So the final transformation matrix is} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0.5 & -0.86 & 0 \\ 0 & 0.86 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) $\xi = [v^T, \Omega^T] = [0\ 0.5\ 0\ 0\ \pi/2\ 0]^T$

Solution.

$$\text{Transformation Matrix is given by, } T = \begin{bmatrix} R & d \\ 0^T & 1 \end{bmatrix}$$

$$\text{R} = \cos(\theta)I + (1 - \cos(\theta)\hat{k}\hat{k}^T + \sin(\theta)[\hat{k}_x]$$

$$\text{Using Matlab we get R as} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\text{The displacement vector is given by d} = (I - R)[\hat{k}_x] + \Omega\hat{k}^T$$

$$\text{Using Matlab we get d as} \begin{bmatrix} 0 \\ \pi/4 \\ 0 \end{bmatrix}$$

$$\text{So the final transformation matrix is} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \pi/4 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) $\xi = [v^T, \Omega^T] = [1\ 0\ 0\ 0\ 0\ \pi/3]^T$

Solution.

$$\text{Transformation Matrix is given by, } T = \begin{bmatrix} R & d \\ 0^T & 1 \end{bmatrix}$$

$$R = \cos(\theta)I + (1 - \cos(\theta)\hat{k}\hat{k}^T + \sin(\theta)[\hat{k}_x]$$

Using Matlab we get R as $\begin{bmatrix} 0.5 & -0.86 & 0 \\ 0.86 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The displacement vector is given by d = $(I - R)[\hat{k}_x] + \Omega\hat{k}^T$

Using Matlab we get d as $\begin{bmatrix} 0.86 \\ 0.5 \\ 0 \end{bmatrix}$

So the final transformation matrix is $\begin{bmatrix} 0.5 & -0.86 & 0 & 0.86 \\ 0.86 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Problem 2

Given the homogenous transformation matrix:

$$T = \begin{bmatrix} 0.5 & -0.6124 & 0.6124 & 0.8415 \\ 0.6214 & 0.75 & 0.25 & 0.3251 \\ -0.6214 & 0.25 & 0.75 & 0.3251 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) What is the displacement?

Solution.

$$d = \begin{bmatrix} 0.8415 \\ 0.3251 \\ -0.3251 \end{bmatrix}$$

(b) What is the Quaternion that encodes the rotation?

Solution.

$$\text{The Rotation Matrix is} \begin{bmatrix} 0.5 & -0.6214 & 0.6214 \\ 0.6214 & 0.75 & 0.25 \\ -0.6214 & 0.25 & 0.75 \end{bmatrix}$$

Let's calculate the trace of this matrix

Tr(R) = 0.5 + 0.75 + 0.75 = 2

The quaternions are given by

$$q_0 = \sqrt{1 + Tr(R)}/2 = \sqrt{3}/2$$

$$q_1 = r_{32} - r_{23}/4q_0 = 0$$

$$q_2 = r_{13} - r_{31}/4q_0 = 0.3535$$

$$q_3 = r_{21} - r_{12}/4q_0 = 0.3535$$

The quaternion that we get is $[0.8660, [0\ 0.3535\ 0.3535]^T]$

(c) What is the Angle-Axis that encodes the rotation?

Solution.

As we have quaternion now we can just get the angle axis using

$\theta = atan2(||q||, q_0) = 1.0470$ and

$$\hat{k} = q/||q|| = \begin{bmatrix} 0 \\ 0.7071 \\ 0.7071 \end{bmatrix}$$

$$\text{Finally } \Omega = \theta\hat{k} = \begin{bmatrix} 0 \\ 0.7403 \\ 0.7403 \end{bmatrix}$$

(d) What is the twist ξ that encodes the full transform?

Solution. For ξ we need to find v and we have Ω from the previous sub-part.
Formula for v is given by,

$$v = [\sin\theta/2(1-\cos\theta)I + (2(1-\cos\theta) - \theta\sin\theta)/2\theta(1-\cos\theta) * \hat{k}\hat{k}^T - 1/2[\hat{k}_x]]d$$

$$\sin\theta/2(1-\cos\theta)I = \begin{bmatrix} 0.8660 & 0 & 0 \\ 0 & 0.8660 & 0 \\ 0 & 0 & 0.8660 \end{bmatrix}$$

$$(2(1-\cos\theta) - \theta\sin\theta)/2\theta(1-\cos\theta) * \hat{k}\hat{k}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.0445 & 0.0445 \\ 0 & 0.0445 & 0.0445 \end{bmatrix}$$

$$1/2[\hat{k}]_x = \begin{bmatrix} 0 & -0.3535 & 0.3535 \\ 0.3535 & 0 & 0 \\ -0.3535 & 0 & 0 \end{bmatrix}$$

Adding all the above matrices and multiplying them with d,

$$v = \begin{bmatrix} 0.9587 \\ -0.0160 \\ 0.0160 \end{bmatrix}$$

Finally the twist representation is
$\xi = [v^T \ \Omega^T] = [0.9587 \ -0.0160 \ 0.0160 \ 0 \ 0.7403 \ 0.7403]$

Problem 3

For the following give one pro and one con for using it as a representation:

(a) Rotation and displacement separately

Solution.
Pro: Having Rotation and displacement separately makes it easy to update either of them without affecting the other i.e. modularity.
Cons: Having these separate adds extra complexity. Also storing them separately requires more storage.

(b) Homogenous Transformation Matrices

Solution. Pro: The representation of transformation matrices have rotation and translation in one matrix which helps for chaining.
Cons: Storage is one of the big issues transformation matrices face. Computationally expensive.

(c) Twist coordinates

Solution. Pro: The twist coordinates encodes transformations in a single vector (angular velocity and linear velocity) which simplifies calculations and requires less memory.
Cons: Involve complex mathematical operations.

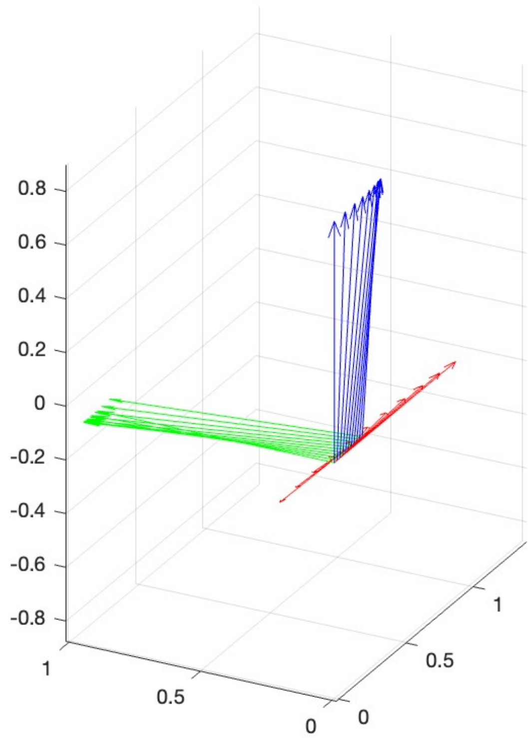
Problem 4

Given the following two homogeneous transforms, use Matlab to plot a linear interpolation between the two. Your plot should contain a minimum of 5 inter- mediate points and should have the orthogonal axis (columns of the rotation matrix) drawn at each point (hint: use the quiver3 command).

$$T_{initial} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{final} = \begin{bmatrix} 0.5 & -0.6124 & 0.6124 & 0.8415 \\ 0.6214 & 0.75 & 0.25 & 0.3251 \\ -0.6214 & 0.25 & 0.75 & 0.3251 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a) Perform the interpolation using ZYZ angles and position.

Solution.



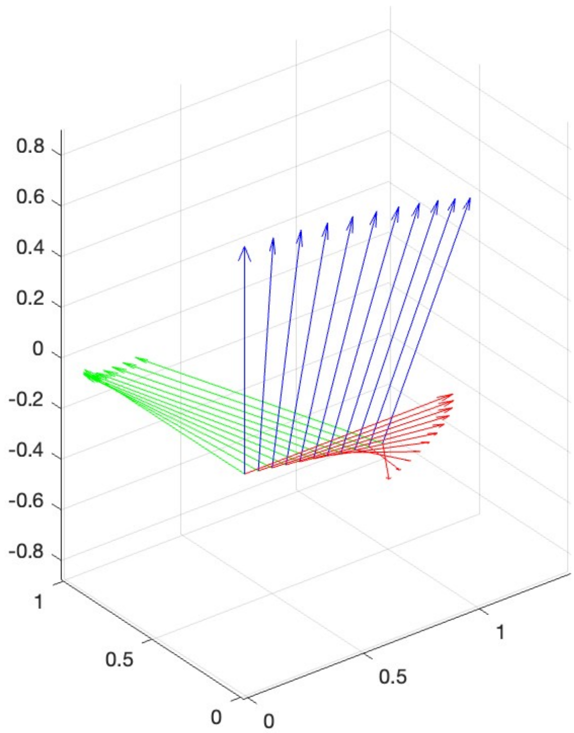

```
    T_init = [1 0 0 0;
    0 1 0 0;
    0 0 1 0;
    0 0 0 1];
T_final = [0.5 -0.6124 0.6124 0.84515;
```

```
           0.6124 0.75 0.25 0.3251;
           -0.6124 0.25 0.75 -0.3251;
           0 0 0 1];

%Let's create a vector with angles and vector positions.

theta_init = atan2(sqrt(0^2+0^2),1);
phi_init = atan2(0,0)0>;
psi_init = atan2(0,0)0>;
position_init = [0 0 00]>;

theta_final = atan2(sqrt((0.6124)^2+(0.25)^2),0.75);
phi_final = atan2(0.25/sin(theta_final),0.6214/sin(theta_final));
psi_final = atan2(0.25/sin(theta_final),0.6214/sin(theta_final));
position_final = [0.8415 0.3251 0-0.3251]>;

v_int = [phi_init theta_init psi_init, position_init0]>;

v_final = [phi_final theta_final psi_final, position_final0]>;

quiver3(position_init(1), position_init(2), position_init(3), T_init(1,1), T_init(2,1), T_init(3,1),"Color","red");
axis equal;
hold on;
quiver3(position_init(1), position_init(2), position_init(3), T_init(1,2), T_init(2,2), T_init(3,2),"Color","green");
quiver3(position_init(1), position_init(2), position_init(3), T_init(1,3), T_init(2,3), T_init(3,3),"Color","blue");
for i=1:10
    r = i/10;
 inter_v = v_int + (v_final - v_int)0>*r;

inter_phi = inter_v(1)0>;
inter_theta = inter_v(2)0>;
inter_psi = inter_v(3)0>;

R = rprRot(-inter_phi,inter_theta)0>;

position_d = inter_v(4:6)0>;
quiver3(position_d(1), position_d(2), position_d(3),R(1,1), R(2,1), R(3,1),"Color","red");
axis equal;
hold on;
```

```
quiver3(position_d(1), position_d(2), position_d(3),R(1,2), R(2,2)
    , R(3,2),"Color","green");
quiver3(position_d(1), position_d(2), position_d(3),R(1,3), R(2,3)
    , R(3,3),"Color","blue");
end


```

(b) Perform the interpolation using Angle-Axis and position.

Solution.

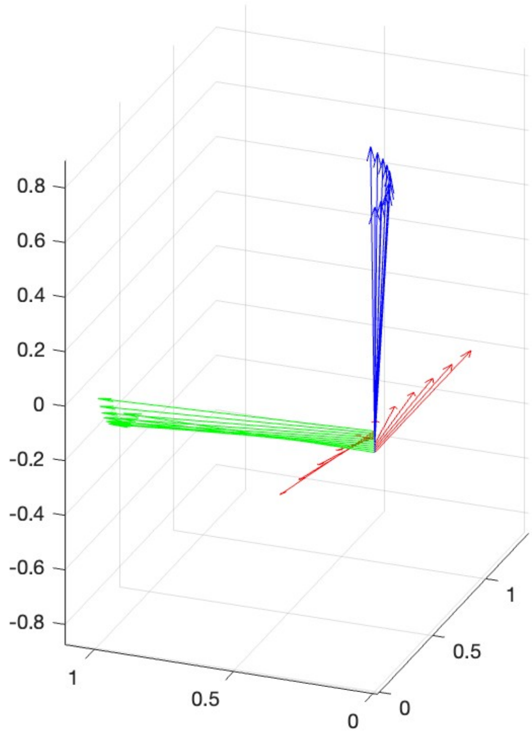



```
        T_init = [1 0 0 0;
    0 1 0 0;
    0 0 1 0;
    0 0 0 1];
T_final = [0.5 -0.6124 0.6124 0.84515;
           0.6124 0.75 0.25 0.3251;
           -0.6124 0.25 0.75 -0.3251;
           0 0 0 1];

```

```
R_init = T_init(1:3,1:3)0>;
R_final = T_final(1:3,1:3)0>;

Omega_init = rot2AngleAxis(R_init)0>;
Omega_final = rot2AngleAxis(R_final)0>;

position_init = T_init(1:3,4)0>;
position_final = T_final(1:3,4)0>;

v_int = [Omega_init, position_init0]>;

v_final = [Omega_final, position_final0]>;

quiver3(position_init(1), position_init(2), position_init(3), T_init(1,1), T_init(2,1), T_init(3,1),"Color","red");
axis equal;
hold on;
quiver3(position_init(1), position_init(2), position_init(3), T_init(1,2), T_init(2,2), T_init(3,2),"Color","green");
quiver3(position_init(1), position_init(2), position_init(3), T_init(1,3), T_init(2,3), T_init(3,3),"Color","blue");

for i=1:10
    r=i/10;
    inter_v = v_int + (v_final - v_int)0>*r;
    position_d = inter_v(4:6)0>;
    R = angleAxis2Rot(inter_v(1:3,1));
quiver3(position_d(1), position_d(2), position_d(3),R(1,1), R(2,1), R(3,1),"Color","red");
axis equal;
hold on;
quiver3(position_d(1), position_d(2), position_d(3),R(1,2), R(2,2), R(3,2),"Color","green");
quiver3(position_d(1), position_d(2), position_d(3),R(1,3), R(2,3), R(3,3),"Color","blue");
end

```

(c) Perform the interpolation using Quaternion and Position space (remember to enforce the length constraint on the quaernion before calculating the intermediate orientations)

Solution.

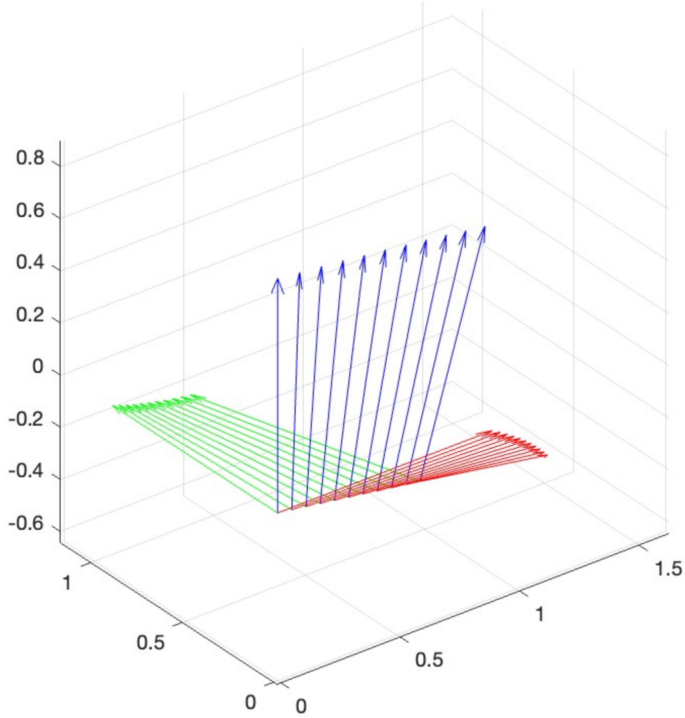



```
        T_init = [1 0 0 0;
    0 1 0 0;
    0 0 1 0;
    0 0 0 1];
T_final = [0.5 -0.6124 0.6124 0.84515;
           0.6124 0.75 0.25 0.3251;
           -0.6124 0.25 0.75 -0.3251;
           0 0 0 1];

R_init = T_init(1:3,1:3)0>;
R_final = T_final(1:3,1:3)0>;


position_init = T_init(1:3,4)0>;
position_final = T_final(1:3,4)0>;

Q_init = rot2Quat(R_init)0>;
Q_final = rot2Quat(R_final)0>;

Q_init_norm = Q_init/norm(Q_init)0>;
Q_final_norm = Q_final/norm(Q_final)0>;
```

```

Q_init_norm = transpose(Q_init_norm)0>;
Q_final_norm = transpose(Q_final_norm)0>;

quiver3(position_init(1), position_init(2), position_init(3), T_init(1,1), T_init(2,1), T_init(3,1),"Color","red");
axis equal;
hold on;
quiver3(position_init(1), position_init(2), position_init(3), T_init(1,2), T_init(2,2), T_init(3,2),"Color","green");
quiver3(position_init(1), position_init(2), position_init(3), T_init(1,3), T_init(2,3), T_init(3,3),"Color","blue");

v_int = [Q_init_norm ; position_init];

v_final = [Q_final_norm ;position_final];

for i=1:10
    r=i/10;
    inter_v = v_int + (v_final - v_int)0>*r;
    position_d = inter_v(5:7)0>;
    R = quat2Rot(inter_v(1:4));
quiver3(position_d(1), position_d(2), position_d(3),R(1,1), R(2,1), R(3,1),"Color","red");
axis equal;
hold on;
quiver3(position_d(1), position_d(2), position_d(3),R(1,2), R(2,2), R(3,2),"Color","green");
quiver3(position_d(1), position_d(2), position_d(3),R(1,3), R(2,3), R(3,3),"Color","blue");
end

```

(d) Perform the interpolation using Twist space.

Solution.




```
        T_init = [1 0 0 0;
    0 1 0 0;
    0 0 1 0;
    0 0 0 1];
T_final = [0.5 -0.6124 0.6124 0.84515;
           0.6124 0.75 0.25 0.3251;
           -0.6124 0.25 0.75 -0.3251;
           0 0 0 1];

R_init = T_init(1:3,1:3)0>;
R_final = T_final(1:3,1:3)0>;


position_init = T_init(1:3,4)0>;
position_final = T_final(1:3,4)0>;

t_init = transform2Twist(T_init)0>;
t_final = transform2Twist(T_final)0>;


quiver3(position_init(1), position_init(2), position_init(3),
```

```
    T_init(1,1), T_init(2,1), T_init(3,1),"Color","red");
axis equal;
hold on;
quiver3(position_init(1), position_init(2), position_init(3),
    T_init(1,2), T_init(2,2), T_init(3,2),"Color","green");
quiver3(position_init(1), position_init(2), position_init(3),
    T_init(1,3), T_init(2,3), T_init(3,3),"Color","blue");

v_int = [t_init; position_init];

v_final = [t_final ;position_final];

for i=1:10
    r=i/10;
    inter_v = v_int + (v_final - v_int)0>*r;
    position_d = inter_v(7:9)0>;
    R = twist2Transform(inter_v(1:6));
quiver3(position_d(1), position_d(2), position_d(3),R(1,1), R(2,1)
    , R(3,1),"Color","red");
axis equal;
hold on;
quiver3(position_d(1), position_d(2), position_d(3),R(1,2), R(2,2)
    , R(3,2),"Color","green");
quiver3(position_d(1), position_d(2), position_d(3),R(1,3), R(2,3)
    , R(3,3),"Color","blue");
end

```

(e) Comment on the similarities or differences.

Solution. The first 3 plots seem to be same and the last one is kind of offset. I plotted all of them together and found these similarity and difference.