# Assignment

Homework 2

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A MEGN544A Homework Assignment

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**Problem 1**

Given the Euler angle set Roll($\psi$), Pitch($\theta$), Yaw($\phi$): RotZ($\phi$)RotY($\theta$)RotX($\psi$)

(a) Derive the resulting rotation matrix in terms of $(\phi, \theta, \psi)$

**Solution.**

$$RotZ(\phi) = \begin{bmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, RotY(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix},$$

$$RotX(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\psi & -sin\psi \\ 0 & sin\psi & cos\psi \end{bmatrix}$$

$$RotZ(\phi)RotY(\theta)RotX(\psi) = \begin{bmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\psi & -sin\psi \\ 0 & sin\psi & cos\psi \end{bmatrix}$$

$$RotZ(\phi)RotY(\theta)RotX(\psi) = \begin{bmatrix} cos\phi cos\theta & -sin\phi cos\psi + cos\phi sin\theta sin\psi & sin\phi sin\psi + cos\phi sin\theta cos\phi \\ sin\phi cos\theta & cos\phi cos\psi + sin\phi sin\theta sin\psi & -cos\phi sin\psi + sin\theta cos\phi sin\psi \\ -sin\theta & cos\theta sin\psi & cos\theta cos\psi \end{bmatrix}$$

(b) Provide the inverse solution. That is given a rotation matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

determine the the formulas for $\phi, \theta$ and $\psi$ in terms of the entries in R.

**Solution.** So now that we have our rotation matrix from above we can generate a inverse solution.

$$\theta\pm = atan2(-r_{31}, \pm\sqrt{r_{11}^2 + r_{21}^2}),$$
$$\phi\pm = atan2(r_{21}/\pm\cos\theta, r_{11}/\pm\cos\theta),$$
$$\psi\pm = atan2(r_{32}/\pm\cos\theta, r_{33}/\pm\cos\theta)$$

(c) What value of $\phi, \theta$ and $\theta$ corresponds to a degeneracy in the inverse solution?

**Solution.** So if we look at the equation we get $\theta$ as $\pm\pi/2$ which will generate degeneracy. We cannot find $\phi$ and $\psi$ from the above equation.

(d) What does the full rotation matrix look like at the degeneracy?

**Solution.** If we put the value of $\theta = +\pi/2$ in equation we get in part (a) we get the matrix.

$$RotZ(\phi)RotY(+\pi/2)RotX(\psi) = \begin{bmatrix} 0 & -sin\phi cos\psi + cos\phi sin\psi & sin\phi sin\psi + cos\phi cos\phi \\ 0 & cos\phi cos\psi + sin\phi sin\psi & -cos\phi sin\psi + cos\phi sin\psi \\ -1 & 0 & 0 \end{bmatrix}$$

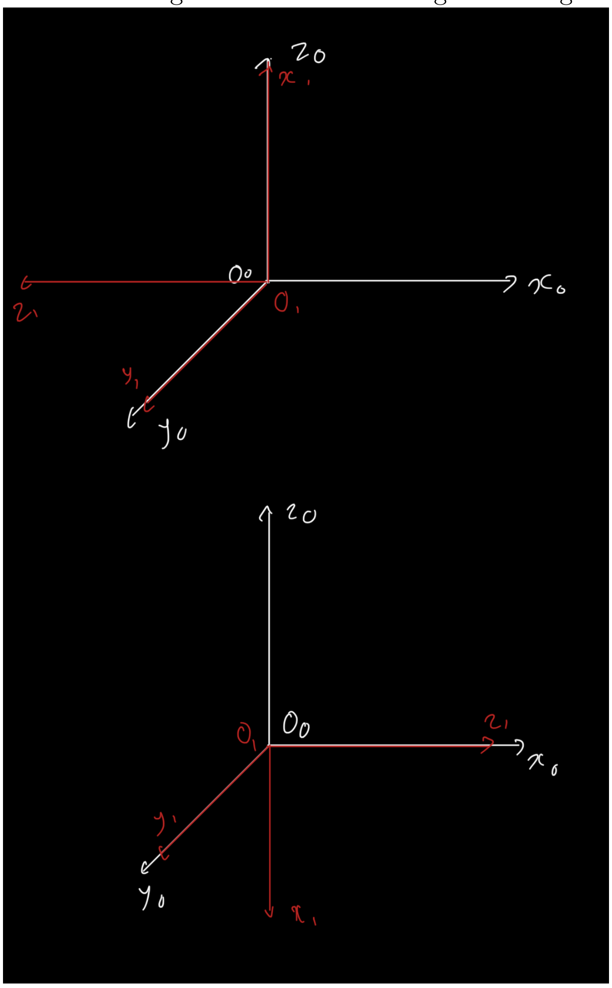
If we put the value of $\theta = +\pi/2$ in equation we get in part (a) we get the matrix.

$$RotZ(\phi)RotY(-\pi/2)RotX(\psi) = \begin{bmatrix} 0 & -sin\phi cos\psi - cos\phi sin\psi & sin\phi sin\psi - cos\phi cos\phi \\ 0 & cos\phi cos\psi - sin\phi sin\psi & -cos\phi sin\psi - cos\phi sin\psi \\ 1 & 0 & 0 \end{bmatrix}$$

(e) Using a sketch of the intermediate rotations to assist, describe why this is a degeneracy.

**Solution.**

So if we turn $\theta$ by $\pi/2$ or $-\pi/2$ we can see that axis X aligns with axis Z which suggests that we lose one degree of freedom which gives us degeneracy.



**Problem 2**

Given the following rotation matrix:

$$R = \begin{bmatrix} 0.6533 & 0.6403 & 0.4040 \\ -0.3488 & 0.7282 & -0.5900 \\ -0.6719 & 0.2445 & 0.6991 \end{bmatrix}$$

(a) Prove that it is a rotation matrix.

**Solution.** In order to know if the matrix is a rotation matrix we have to check if it is orthonormal i.e. the rows and column elements will unit vectors and dot product of columns should be zero suggesting they are orthogonal.

$$\text{Row 1: } \sqrt{(0.6533)^2 + (0.6403)^2 + (0.4040)^2} = 1,$$
$$\text{Row 2: } \sqrt{(-0.3488)^2 + (0.7282)^2 + (-0.5900)^2} = 1,$$
$$\text{Row 3: } \sqrt{(-0.6719)^2 + (0.2445)^2 + (0.6991)^2} \approx 1$$

Column 1 ×Column 2: 0.6533×0.6403 + (-0.3488) ×0.7282 + (-0.6719) ×0.2445 ≈ 0

Column 2 ×Column 3: 0.6403×0.4040 + 0.7282 ×(-0.5900) + 0.2445 ×0.6991 = 0

Column 3 ×Column 1: 0.4040×0.6533 + (-0.5900) ×(-0.3488) + 0.6991 ×(-0.6719) = 0

Finally we calculate the determinant which should be equal to 1. I used Matlab for

det = 1

So as we see that the determinant is +1 which suggests that the above matrix is a rotation matrix.

(b) What are the ZYZ angles for this rotation matrix?

**Solution.** In order to get the ZYZ angles from this rotation matrix we can calculate a inverse solution using ZYZ inverse equations.

$\theta = atan2(\pm\sqrt{r_{13}^2 + r_{23}^2}, r_{33})$

$\phi = atan2(r_{23}/\pm\sin\theta, r_{13}/\pm\sin\theta)$

$\psi = atan2(r_{32}/\pm\sin\theta, -r_{31}/\pm\sin\theta)$

If we substitute all the values and calculate $atan2$ for $\theta$ we get values.

$\theta = \pm 0.7966 rad$

Now lets calculate $\phi$ and $\psi$ for $\theta = 0.7966 rad$

$\phi = -0.9704$ and $\psi = 0.3490$

Now lets calculate $\phi$ and $\psi$ for $\theta = -0.7966 rad$

$\phi = 2.1712$ and $\psi = -2.7926$

(c) What is the angle and axis representation for this rotation matrix?

**Solution.** For getting the angle and axis representation for this rotation matrix we can calculate Trace first.

Tr(R) = 2.0806

Next we calculate the angle $\theta$ using the equation.

$\theta = atan2[1/2|| \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} ||, Tr(R) - 1/2]$

$\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = \begin{bmatrix} 0.8345 \\ 1.0759 \\ -0.9891 \end{bmatrix}$

Now lets calculate $\theta$ $\theta = atan2(0.8414, 0.5403)$

$\theta = 1 rad$

Lets calculate $\hat{k}$ which gives us the axis.

$\hat{k} = 1/2 \sin\theta * \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$

We get $\hat{k} = \begin{bmatrix} 0.4958 \\ 0.6392 \\ -0.5877 \end{bmatrix}$

The angle-axis representation is $\Omega = \theta\hat{k}$

$\Omega = \begin{bmatrix} 0.4958 \\ 0.6392 \\ -0.5877 \end{bmatrix}$

(d) What is the Quaternion representation?

**Solution.** The quaternion representation from angle-axis representation is given by, $q_0 = \cos(||\Omega||/2)$ and $\vec{q} = \sin(||\Omega||/2)(\Omega/||\Omega||)$

$\Omega = \hat{k}$ as $\theta = 1$ so $||\Omega|| \approx 1$

$q_0 = 0.8776$ rad and $\vec{q} = \begin{bmatrix} 0.2376 \\ 0.3064 \\ -0.2817 \end{bmatrix}$

**Problem 3**

What is the rotation matrix and angle-axis ($\hat{k}\theta$) hat is described by the following quaternions

(a) $[1, [0\ 0\ 0]^T]$

**Solution.** So if we look at the vector $\vec{q}$ part in the above representation we can say that there is no rotation in any axes. This gives us the rotation matrix to be an identity matrix.

$$\mathrm{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\hat{k} = \vec{q}/||q||$ but as $\vec{q} = [\ 0\ 0\ 0\ ]$ $\hat{k} = 0$

Angle-Axis representation is given by $\Omega = \theta\hat{k}$. I did not calculate $\theta$ as $\hat{k}$ is 0.

$$\Omega = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) $[1, [\ 0\ 0\ 1]^T]$

**Solution.** So if we look at the vector $\vec{q}$ part in the above representation we can say that there is rotation in Z axis and $\theta = \pi$ as $\vec{q} = \sin(\theta/2)$ . So the rotation matrix is

$$\mathrm{R} = Rot_Z(\pi) = \begin{bmatrix} \cos\pi & -\sin\pi & 0 \\ \sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Angle $\theta$ is given by $2 * atan2(||q||, q_0)$ which is $\theta = 3.1416$

$$\text{Axis } \hat{k} = \vec{q}/||q|| = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Omega = \theta\hat{k} = \begin{bmatrix} 0 \\ 0 \\ 3.1416 \end{bmatrix}$$

(c) $[1, [1\ 0\ 0]^T]$

**Solution.** So if we look at the vector $\vec{q}$ part in the above representation we can say that there is rotation in X axis and $\theta = \pi$ as $\vec{q} = \sin(\theta/2)$ . So the rotation matrix is

$$\mathrm{R} = Rot_X(\pi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi \\ 0 & \sin\pi & \cos\pi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Angle $\theta$ is given by $2 * atan2(||q||, q_0)$ which is $\theta = 3.1416$

$$\text{Axis } \hat{k} = \vec{q}/||q|| = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Omega = \theta\hat{k} = \begin{bmatrix} 3.1416 \\ 0 \\ 0 \end{bmatrix}$$

(d) $[0.5332, [0.5928\ 0.08311\ 0.5978]^T]$

**Solution.** For this question we will have a different approach, we will calculate angle-axis first.

$$\theta = 2 * atan2(||q||, q_0) \text{ and } \hat{k} = q/||q||$$
$$||q|| = \sqrt{(0.5928)^2 + (0.08311)^2 + (0.5978)^2} = 0.8459$$
$$\theta = 2 * atan2(0.8459, 0.5332) = 2.0167$$
$$\hat{k} = \begin{bmatrix} 0.7007 \\ 0.0982 \\ 0.7067 \end{bmatrix}$$
$$\Omega = \begin{bmatrix} 1.4731 \\ 0.1980 \\ 1.4252 \end{bmatrix}$$

Now in order to get rotation matrix we can use the equation:

$$R = (q_0^2 - q^T q)I + 2q_0[q]_x + 2qq^T \quad [q]_x = \begin{bmatrix} 0 & -0.5978 & 0.08311 \\ 0.5978 & 0 & -0.5928 \\ -0.08311 & 0.5928 & 0 \end{bmatrix}$$

I used matlab to calculate R which is $\begin{bmatrix} 0.2714 & -0.5390 & 0.7974 \\ 0.7360 & -0.4176 & -0.5328 \\ 0.6201 & 0.7315 & 0.2833 \end{bmatrix}$

**Problem 4**

An object is spinning with some constant angular velocity ${}^0\vec{\omega}$ such that after some time $t$ it has rotated $\theta = \|\vec{\omega}\|t$ radians about the ${}^0\hat{k} = {}^0\vec{\omega}/\|\vec{\omega}\|$ direction. At this point, its new orientation is given by ${}^0R_f = {}^0R_i(\theta, \hat{k})^iR_f$. Using angle axis, derive the time derivative of a rotation matrix given an arbitrary angular velocity. That is compute: $dR/d\imath = \left(\frac{d}{d\theta}{}^0R_i(\theta, \hat{k})^iR_f\right)\frac{d\theta}{d\ell} + \left(\frac{d}{d\hat{k}}{}^0R_i(\theta, \hat{k})^iR_f\right)\frac{d\hat{k}}{dl}$ and evaluate at $\theta = t = 0$, i.e. before the orientation is modified by the rotation. Hint: what is $\frac{d\hat{k}}{d\ell}$ for a constant angular velocity?

**Solution.**

$$R\{t\} = \left(\cos(\theta\{t\})\mathcal{I} + (1 - \cos(\theta\{t\}))\hat{\mathbf{k}}\hat{\mathbf{k}} + \sin(\theta\{t\})[\hat{\mathbf{k}}]_\times\right)R$$

Lets take derivative w.r.t. t. and also derivative of $d\hat{k}/dt = 0$ as the angular velocity is constant.

$$dR\{t\}/dt = \left(-\sin(\theta\{t\})\dot{\theta}I + \sin(\theta\{t\})\dot{\theta}\hat{k}k^T + \cos(\theta\{t\})\dot{\theta}[\hat{k}]_x\right)R$$

Now if we put limit on $\theta$.

$$\lim_{\theta\to 0}\frac{d}{dt}(R\{t\})$$

$$\dot{R} = \lim_{\theta\to 0}\frac{d}{dt}(R\{t\})$$

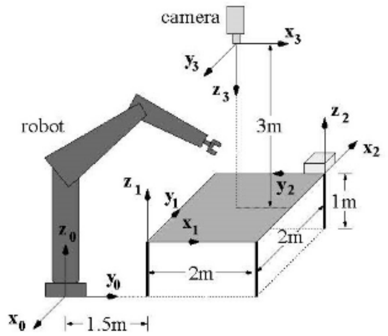
All the sin values turn to 0, cosine values are 1 and we are left with

$$\dot{R} = \left(\dot{\theta}[\hat{k}]_x\right)R \quad = [\omega]_xR$$

$$\text{So finally, } \dot{R} = \begin{bmatrix} 0 & -\omega_zR & \omega_yR \\ \omega_zR & 0 & -\omega_xR \\ -\omega_yR & \omega_xR & 0 \end{bmatrix}$$

**Problem 5**

Consider the combination of robot, table, block, and camera below, with as- sociated coordinate systems as shown. The relative locations of robot, table, block, and camera are shown.




(a) Find ${}^0T_1$, ${}^1T_2$, ${}^2T_3$ and ${}^0T_3$ by inspection.

**Solution.**

Lets calculate displacement ${}^0d_{01}$ and rotation ${}^0R_1$.

$$
{}^0d_{01} = \begin{bmatrix} 0 \\ 1.5 \\ 1 \end{bmatrix} \text{ and } {}^0R_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\text{Let's write } {}^0T_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Lets calculate displacement ${}^1d_{12}$ and rotation ${}^1R_2$.

$$
{}^1d_{12} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \text{ and } {}^1R_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\text{Let's write } {}^1T_2 = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Now when going from $O_2$ to $O_3$ we have two rotations: One in Z with angle $-\pi/2$ and one in X with angle $\pi$

$$
{}^2d_{23} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}
$$

$$
{}^2R_3 = Rot_Z(-\pi/2) * Rot_X(\pi) * Rot_Y(0) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{Let's write } {}^2T_3 = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_3 = {}^0T_1 \times {}^1T_2 \times {}^2T_3 \text{ Using Matlab } {}^0T_3 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 2.5 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a) Assume the block has moved. It is tracked by the camera and is known to exist at location ${}^3d_{32} = \begin{bmatrix} -0.5 \\ 0.5 \\ 3 \end{bmatrix}$. Where is it in the robot's coordinate system, i.e. what is ${}^0d_{02}$?

**Solution.** If we inspect the new location of the block $O_2$ with respect to $O_1$ is ${}^1d_{12} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}$ and rotation matrix ${}^1R_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let's write ${}^1T_2 = \begin{bmatrix} 0 & -1 & 0 & 0.5 \\ 1 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ In order to calculate ${}^0d_{02}$ we have to calculate

$${}^0T_2 = {}^0T_1 \times {}^1T_2 = \begin{bmatrix} -1 & 0 & 0 & -0.5 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{So we get } {}^0d_{02} = \begin{bmatrix} -0.5 \\ 2 \\ 1 \end{bmatrix}$$