AI1103: Challenge Problem Mixture

Tanmay Garg
CS20BTECH11063 EE20BTECH11048

Download all python codes from

https://github.com/tanmaygar/AI−Course/blob/main/challenge%20mixture/codes/challenge_mix.py

and latex-tikz codes from

https://github.com/tanmaygar/AI−Course/blob/main/challenge%20mixture/Challenge_Mixed.tex

Challenge Problem Mixed

Let $X \sim B\left(5, \frac{1}{2}\right)$ and $Y \sim U(0,1)$. The the value of:

$$\frac{\Pr(X+Y \le 2)}{\Pr(X+Y \ge 5)}$$

is equal to? (X and Y are independent)

Solution: Characteristic function approach

Characteristic function for $X \sim B\left(5, \frac{1}{2}\right)$ will be:

$$C_X(t) = \left(\frac{e^{it}+1}{2}\right)^5 \tag{0.0.1}$$

Characteristic function for $Y \sim U(0,1)$ will be:

$$C_Y(t) = \frac{e^{it}-1}{it} \tag{0.0.2}$$

Since both X and Y are independent we can take:

$$Z = X + Y \tag{0.0.3}$$

$$C_Z(t) = C_X(t)C_Y(t) \tag{0.0.4}$$

$$C_Z(t) = \frac{(e^{it}+1)^5(e^{it}-1)}{32it} \tag{0.0.5}$$

Applying Gil-Pelaez formula:

$$F_Z(z) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\left(e^{-itz}C_Z(t)\right)}{t}dt \tag{0.0.6}$$

$$F_Z(z) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{2it}\left(\frac{(e^{it}+1)^5(e^{it}-1)e^{-itz}}{32it}\right) + \frac{1}{2it}\left(\frac{(e^{-it}+1)^5(e^{-it}-1)e^{itz}}{32it}\right)dt$$

Substituting $z = 2$, the value for $\Pr(Z \le 2)$:

$$\begin{aligned} &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{8\cos 2t + 2\cos 4t}{64t^2}dt \\ &+ \frac{1}{\pi}\int_0^\infty \frac{+8\cos 3t - 8\cos t - 10}{64t^2}dt \end{aligned} \tag{0.0.7}$$

Finding a general expression for integrating:

$$\int \frac{\cos ax}{x^2}dx = -\frac{\cos ax}{x} - a\int \frac{\sin ax}{x}dx + C \tag{0.0.8}$$

By applying integration by parts. Now finding the value of other integral, by substituting $u = ax$ for limits as 0 and ∞:

$$\int_0^\infty \frac{a\sin ax}{x}dx = \int_0^\infty \frac{a\sin u}{u}du \tag{0.0.9}$$

$$= \frac{a\pi}{2} \tag{0.0.10}$$

Now using the above general expressions to calculate (0.0.7) and simplifying the expression after putting the limits we get

$$= \frac{-1}{8\pi}\left(\int_0^\infty \frac{\sin 4t + 3\sin 3t + 2\sin 2t - \sin t}{t}dt\right) \tag{0.0.11}$$

$$- \left.\frac{2(\cos t - 1)(\cos t + 1)^3}{8\pi t}\right|_0^\infty + \frac{1}{2} \tag{0.0.12}$$

$$= \frac{1}{2} + \frac{-1}{8\pi} \times \frac{5\pi}{2} + 0 \tag{0.0.13}$$

$$= \frac{3}{16} \tag{0.0.14}$$

Using (0.0.10) and (0.0.8) to calculate for our second case Similarly on substituting $z = 5$, the value for $\Pr(Z \leq 5)$:

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{-10\cos 3t - 8\cos 4t}{64t^2}dt$$
$$+ \frac{1}{\pi}\int_0^\infty \frac{-2\cos 5t + 12\cos t + 8}{64t^2}dt \tag{0.0.15}$$
$$= \frac{1}{\pi}\left(\int_0^\infty \frac{5\sin 5t + 16\sin 4t + 15\sin 3t - 6\sin t}{32}dt\right)$$
$$+ \frac{1}{2} + \frac{1}{\pi}\left(\frac{16(\cos t - 1)(\cos t)(\cos t + 1)^3}{32t}\bigg|_0^\infty\right) \tag{0.0.16}$$
$$= \frac{1}{2} + \frac{1}{\pi} \times \frac{15\pi}{32} + 0 \tag{0.0.17}$$
$$= \frac{31}{32} \tag{0.0.18}$$

The value for $\Pr(Z \geq 5)$:

$$\Pr(Z > 5) = 1 - \Pr(Z \leq 5) \tag{0.0.19}$$
$$= 1 - \frac{31}{32} = \frac{1}{32} \tag{0.0.20}$$

Upon substituting (0.0.14)and (0.0.20), we get:

$$\frac{\Pr(X + Y \leq 2)}{\Pr(X + Y \geq 5)} = 6 \tag{0.0.21}$$

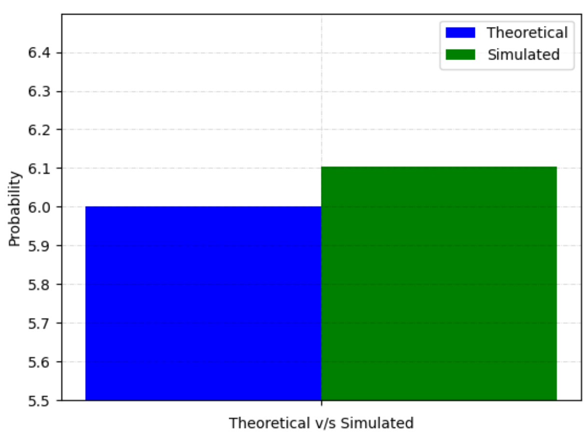

Fig. 0