AI1103: Challenge Problem Mixture

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Download all python codes from

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https://github.com/tanmaygar/AI−Course/blob/main
/challenge%20mixture/codes/challenge_mix.py
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and latex-tikz codes from

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https://github.com/tanmaygar/AI−Course/blob/main
/challenge%20mixture/Challenge_Mixed.tex
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Challenge Problem Mixed

Let $X \sim B\left(5, \frac{1}{2}\right)$ and $Y \sim U(0,1)$. The the value of:

$$\frac{\Pr(X+Y \le 2)}{\Pr(X+Y \ge 5)}$$

is equal to? (X and Y are independent)

Solution:

It is given that X is random variable which follows binomial distribution with $n = 5$ and $p = \frac{1}{2}$. Y is a uniform distribution in the interval $(0,1)$. Both X and Y are independent.

$$\Pr(X = k) = \binom{5}{k}\left(\frac{1}{2}\right)^5 \tag{0.0.1}$$

$$F_Y(y) = \Pr(Y \le y) = \begin{cases} 0 & y < 0 \\ y & 0 \le y \le 1 \\ 1 & y > 1 \end{cases} \tag{0.0.2}$$

Calculating:

$$\Pr(X+Y \ge 5) = \sum_{k=0}^{5} \Pr(X = k, Y \ge 5-k) \tag{0.0.3}$$

$$= \sum_{k=0}^{5} \Pr(X = k)\Pr(Y \ge 5-k) \tag{0.0.4}$$

$$= 0 + \Pr(X = 5)\Pr(Y \ge 0) \tag{0.0.5}$$

$$= \binom{5}{5}\left(\frac{1}{2}\right)^5 = \frac{1}{32} \tag{0.0.6}$$

$$\Pr(X+Y \le 2) = \Pr(X = 0, Y \le 2) + \Pr(X = 1, Y \le 1) + \Pr(X = 2, Y \le 0) \tag{0.0.7}$$

$$= 1 \times \binom{5}{0}\left(\frac{1}{2}\right)^5 + 1 \times \binom{5}{1}\left(\frac{1}{2}\right)^5 + 0 \times \binom{5}{2}\left(\frac{1}{2}\right)^5 \tag{0.0.8}$$

$$= \frac{3}{16} \tag{0.0.9}$$

Substituting (0.0.9) and (0.0.6)

$$\frac{\Pr(X+Y \le 2)}{\Pr(X+Y \ge 5)} = 6 \tag{0.0.10}$$

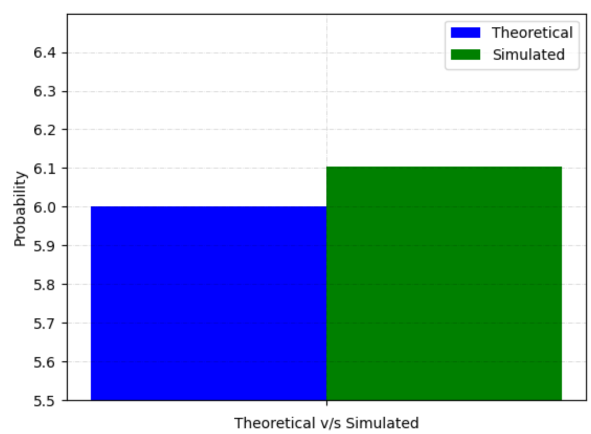

Fig. 0

Characteristic function approach

Characteristic function for $X \sim B\left(5, \frac{1}{2}\right)$ will be:

$$C_X(t) = \left(\frac{e^{it} + 1}{2}\right)^5 \tag{0.0.11}$$

Characteristic function for $Y \sim U(0,1)$ will be:

$$C_Y(t) = \frac{e^{it} - 1}{it} \tag{0.0.12}$$

Since both X and Y are independent we can take:

$$Z = X + Y \tag{0.0.13}$$

$$C_Z(t) = C_X(t)C_Y(t) \tag{0.0.14}$$

Upon substituting:

$$C_Z(t) = \frac{(e^{it} + 1)^5(e^{it} - 1)}{32it} \tag{0.0.15}$$

Applying Gil-Pelaez formula:

$$F_Z(z) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\left(e^{-itz}C_Z(t)\right)}{t}dt \tag{0.0.16}$$

$$\begin{aligned} F_Z(z) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty & \frac{1}{2it}\left(\frac{(e^{it} + 1)^5(e^{it} - 1)e^{-itz}}{32it}\right) \\ &+ \frac{1}{2it}\left(\frac{(e^{-it} + 1)^5(e^{-it} - 1)e^{itz}}{32it}\right)dt \end{aligned}$$

Substituting $z = 2$, the value for $\Pr(Z \le 2)$:

$$\begin{aligned} &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{8\cos 2t + 2\cos 4t}{64t^2}dt \\ &\quad + \frac{1}{\pi}\int_0^\infty \frac{+8\cos 3t - 8\cos t - 10}{64t^2}dt \qquad (0.0.17) \\ &= \frac{1}{2} + \frac{1}{\pi} \times \frac{-5\pi}{16} = \frac{3}{16} \qquad (0.0.18) \end{aligned}$$

Similarly on substituting $z = 5$, the value for $\Pr(Z \le 5)$:

$$\begin{aligned} &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{-10\cos 3t - 8\cos 4t}{64t^2}dt \\ &\quad + \frac{1}{\pi}\int_0^\infty \frac{-2\cos 5t + 12\cos t + 8}{64t^2}dt \qquad (0.0.19) \\ &= \frac{1}{2} + \frac{1}{\pi} \times \frac{15\pi}{32} = \frac{31}{32} \qquad (0.0.20) \end{aligned}$$

The value for $\Pr(Z \ge 5)$:

$$\begin{aligned} \Pr(Z > 5) &= 1 - \Pr(Z \le 5) \qquad (0.0.21) \\ &= 1 - \frac{31}{32} = \frac{1}{32} \qquad (0.0.22) \end{aligned}$$

Upon substituting (0.0.18)and (0.0.22), we get:

$$\frac{\Pr(X + Y \le 2)}{\Pr(X + Y \ge 5)} = 6 \tag{0.0.23}$$