AI1103: Challenge Problem Mixture

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Download all python codes from

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https://github.com/tanmaygar/AI−Course/blob/main
/challenge%20mixture/codes/challenge_mix.py
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and latex-tikz codes from

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https://github.com/tanmaygar/AI−Course/blob/main
/challenge%20mixture/Challenge_Mixed.tex
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Challenge Problem Mixed

Let $X \sim B\left(5, \frac{1}{2}\right)$ and $Y \sim U(0, 1)$. The the value of:

$$\frac{\Pr(X + Y \leq 2)}{\Pr(X + Y \geq 5)}$$

is equal to? (X and Y are independent)

Solution: Characteristic function approach

Characteristic function for $X \sim B\left(5, \frac{1}{2}\right)$ will be:

$$C_X(t) = \left(\frac{e^{it} + 1}{2}\right)^5 \tag{0.0.1}$$

Characteristic function for $Y \sim U(0, 1)$ will be:

$$C_Y(t) = \frac{e^{it} - 1}{it} \tag{0.0.2}$$

Since both X and Y are independent we can take:

$$Z = X + Y \tag{0.0.3}$$

$$C_Z(t) = C_X(t)C_Y(t) \tag{0.0.4}$$

$$C_Z(t) = \frac{(e^{it} + 1)^5(e^{it} - 1)}{32it} \tag{0.0.5}$$

Applying Gil-Pelaez formula:

$$F_Z(z) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{\operatorname{Im}\left(e^{-itz}C_Z(t)\right)}{t}dt \tag{0.0.6}$$

$$F_Z(z) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{1}{2it}\left(\frac{(e^{it} + 1)^5(e^{it} - 1)e^{-itz}}{32it}\right) + \frac{1}{2it}\left(\frac{(e^{-it} + 1)^5(e^{-it} - 1)e^{itz}}{32it}\right)dt$$

Substituting $z = 2$, the value for $\Pr(Z \leq 2)$:

$$\begin{aligned} &= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \frac{8\cos 2t + 2\cos 4t}{64t^2}dt \\ &+ \frac{1}{\pi}\int_0^{\infty} \frac{+8\cos 3t - 8\cos t - 10}{64t^2}dt \end{aligned} \tag{0.0.7}$$

$$= \frac{1}{2} + \frac{1}{\pi} \times \frac{-5\pi}{16} = \frac{3}{16} \tag{0.0.8}$$

Similarly on substituting $z = 5$, the value for $\Pr(Z \leq 5)$:

$$\begin{aligned} &= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \frac{-10\cos 3t - 8\cos 4t}{64t^2}dt \\ &+ \frac{1}{\pi}\int_0^{\infty} \frac{-2\cos 5t + 12\cos t + 8}{64t^2}dt \end{aligned} \tag{0.0.9}$$

$$= \frac{1}{2} + \frac{1}{\pi} \times \frac{15\pi}{32} = \frac{31}{32} \tag{0.0.10}$$

The value for $\Pr(Z \geq 5)$:

$$\Pr(Z > 5) = 1 - \Pr(Z \leq 5) \tag{0.0.11}$$

$$= 1 - \frac{31}{32} = \frac{1}{32} \tag{0.0.12}$$

Upon substituting (0.0.8)and (0.0.12), we get:

$$\frac{\Pr(X + Y \leq 2)}{\Pr(X + Y \geq 5)} = 6 \tag{0.0.13}$$

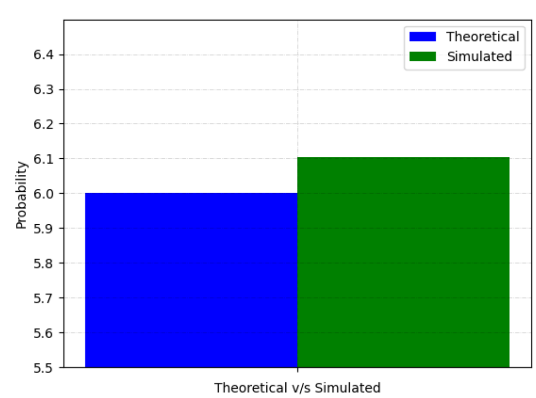

Fig. 0