Characteristic Function

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Problem GATE 2015 (MA), Q. 8

Question

Let $X \sim B\left(5, \frac{1}{2}\right)$ and $Y \sim U(0, 1)$. The the value of:

$$\frac{\Pr\left(X + Y \leq 2\right)}{\Pr\left(X + Y \geq 5\right)}$$

is equal to? (X and Y are independent)

Characteristic function

Definition

1. It is an alternative way to define the probability mass function or probability density function for a random variable.
2. It completely defines the probability distribution for the random variable.

Expression

For discrete random variable

$$C_X(t) = \sum_k \Pr(X = k)\, e^{itk} \tag{1}$$

For continuous random variable

$$C_X(t) = \int_{-\infty}^{\infty} f_X(x) e^{itx} dx \tag{2}$$

Characteristic function

Properties

1. It exists for all real value random variables
2. It is continuous over entire space
3. It is bounded $|C_X(t)| \leq 1$
4. If X and Y are independent then

$$Z = X + Y \tag{3}$$

$$C_Z(t) = C_X(t)C_Y(t) \tag{4}$$

5. Gil Pelaez Formula:

$$\Pr(X \leq x) = F_X(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\left(e^{-itx}C_X(t)\right)}{t}dt \tag{5}$$

Solution

Characteristic function of X

For $X \sim B\left(5, \frac{1}{2}\right)$ will be(binomial distribution):

$$C_X(t) = \sum_k \Pr\left(X = k\right) e^{itk} \tag{6}$$

$$= \sum_k \binom{5}{k} \left(\frac{1}{2}\right)^5 e^{itk} \tag{7}$$

$$= \left(\frac{e^{it} + 1}{2}\right)^5 \tag{8}$$

Solution

Characteristic function of Y

For $Y \sim U(0,1)$ will be:

$$C_Y(t) = \int_{-\infty}^{\infty} f_Y(y) e^{ity} dy \tag{9}$$

$$= \int_0^1 \frac{1}{1-0} e^{ity} dy \tag{10}$$

$$C_Y(t) = \frac{e^{it} - 1}{it} \tag{11}$$

Solution

Characteristic function for Z

Since both X and Y are independent we can take:

$$Z = X + Y \tag{12}$$

$$C_Z(t) = C_X(t)C_Y(t) \tag{13}$$

$$C_Z(t) = \frac{(e^{it} + 1)^5(e^{it} - 1)}{32it} \tag{14}$$

Solution

Characteristic function for Z

Applying Gil-Pelaez formula:

$$F_Z(z) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{\text{Im}\left(e^{-itz}C_Z(t)\right)}{t}dt \tag{15}$$

$$F_Z(z) = \frac{1}{2} - \int_0^{\infty} \frac{1}{2i\pi t}\left(\frac{(e^{it}+1)^5(e^{it}-1)e^{-itz} + (e^{-it}+1)^5(e^{-it}-1)e^{itz}}{32it}\right)dt \tag{16}$$

Solution

$\Pr(X + Y \leq 2)$ or $\Pr(Z \leq 2)$

Substituting $z = 2$, the value for $\Pr(Z \leq 2)$:

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{8\cos 2t + 2\cos 4t}{64t^2}dt + \frac{1}{\pi}\int_0^\infty \frac{+8\cos 3t - 8\cos t - 10}{64t^2}dt \tag{17}$$

Solution

Integration

Finding a general expression for integrating:

$$\int \frac{\cos ax}{x^2} dx = -\frac{\cos ax}{x} - a \int \frac{\sin ax}{x} dx + C \tag{18}$$

By applying integration by parts. Now finding the value of other integral, by substituting $u = ax$ for limits as 0 and ∞:

$$\int_0^\infty \frac{a \sin ax}{x} dx = \int_0^\infty \frac{a \sin u}{u} du \tag{19}$$

$$= \frac{a\pi}{2} \tag{20}$$

Solution

$\Pr(X + Y \le 2)$ or $\Pr(Z \le 2)$

Now using the above general expressions to calculate (17) and simplifying the expression after putting the limits we get

$$
\begin{aligned}
&= \frac{-1}{8\pi}\left(\int_0^\infty \frac{\sin 4t + 3\sin 3t + 2\sin 2t - \sin t}{t} dt\right) \\
&\quad - \frac{2(\cos t - 1)(\cos t + 1)^3}{8\pi t}\bigg|_0^\infty + \frac{1}{2} \qquad (21) \\
&= \frac{1}{2} + \frac{-1}{8\pi} \times \frac{5\pi}{2} + 0 \qquad (22) \\
&= \frac{3}{16} \qquad (23)
\end{aligned}
$$

Solution

$\Pr(X + Y \geq 5)$ or $\Pr(Z \geq 5)$

Similarly on substituting $z = 5$, the value for $\Pr(Z \leq 5)$:

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{-10\cos 3t - 8\cos 4t}{64t^2}dt + \frac{1}{\pi}\int_0^\infty \frac{-2\cos 5t + 12\cos t + 8}{64t^2}dt \tag{24}$$

$$= \frac{1}{\pi}\left(\int_0^\infty \frac{5\sin 5t + 16\sin 4t + 15\sin 3t - 6\sin t}{32}dt\right) + \frac{1}{2} + \frac{1}{\pi}\left(\left.\frac{16(\cos t - 1)(\cos t)(\cos t + 1)^3}{32t}\right|_0^\infty\right) \tag{25}$$

$$= \frac{1}{2} + \frac{1}{\pi} \times \frac{15\pi}{32} + 0 \tag{26}$$

$$= \frac{31}{32} \tag{27}$$

Solution

$\Pr(X+Y \geq 5)$ or $\Pr(Z \geq 5)$

The value for $\Pr(Z \geq 5)$:

$$\Pr(Z > 5) = 1 - \Pr(Z \leq 5) \tag{28}$$

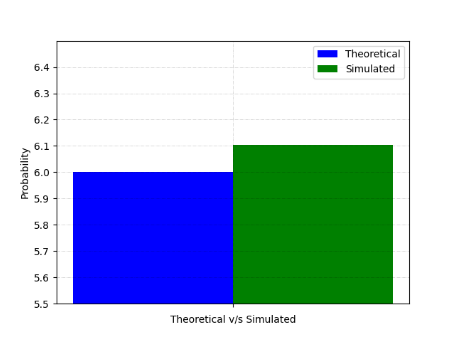
$$= 1 - \frac{31}{32} = \frac{1}{32} \tag{29}$$

Solution

Final Answer

Upon substituting (23)and (29), we get:

$$\frac{\Pr(X+Y \leq 2)}{\Pr(X+Y \geq 5)} = 6 \tag{30}$$



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