

AI1103: Assignment 8

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PROBLEM CSIR UGC NET EXAM (DEC 2016),
Q.109:

X_1, X_2, \dots, X_n are independent and identically distributed as $N(\mu, \sigma^2)$, $-\infty < \mu < \infty$, $\sigma^2 > 0$. Then

- 1) $\sum_1^n \frac{(X_i - \bar{X})^2}{n-1}$ is the Minimum Variance Unbiased Estimate of σ^2
- 2) $\sqrt{\sum_1^n \frac{(X_i - \bar{X})^2}{n-1}}$ is the Minimum Variance Unbiased Estimate of σ
- 3) $\sum_1^n \frac{(X_i - \bar{X})^2}{n}$ is the Maximum Likelihood Estimate of σ^2
- 4) $\sqrt{\sum_1^n \frac{(X_i - \bar{X})^2}{n}}$ is the Maximum Likelihood Estimate of σ

SOLUTION:

The pdf for each random variable is same as they are all identical and independent Normal Distributions with same μ and σ^2 .

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(x - \mu)^2}{2\sigma^2} \quad (0.0.1)$$

Let us take our maximum likelihood function for given random variable X_i

$$L(\mu; \sigma | X_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(X_i - \mu)^2}{2\sigma^2} \quad (0.0.2)$$

Since all the random variables are i.i.d

$$L(\mu; \sigma | X_1, X_2, \dots, X_n) = \prod_{i=1}^n L(\mu; \sigma | X_i) \quad (0.0.3)$$

Let us denote:

$$L_m : L(\mu; \sigma | X_1, X_2, \dots, X_n) \quad (0.0.4)$$

Substituting (0.0.2) for each Random Variable in (0.0.3)

$$L_m = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(X_i - \mu)^2}{2\sigma^2} \quad (0.0.5)$$

Taking natural log on both sides and simplifying

$$\ln L_m = \frac{-n}{2} \ln 2\pi - n \ln \sigma - \sum_{i=1}^n \frac{(X_i - \mu)^2}{2\sigma^2} \quad (0.0.6)$$

In order to find Maximum Likelihood we need to maximise μ and σ w.r.t. all Random variables. Taking partial derivative w.r.t μ and taking σ as constant

$$\frac{\partial \ln L_m}{\partial \mu} = \sum_{i=1}^n \frac{(X_i - \mu)}{\sigma^2} \quad (0.0.7)$$

The value for μ at which L_m achieves maximum value is same in $\ln L_m$

$$\therefore \frac{\partial \ln L_m}{\partial \mu} = 0 \quad (0.0.8)$$

$$\therefore \sum_{i=1}^n \frac{(X_i - \mu)}{\sigma^2} = 0 \quad (0.0.9)$$

On simplifying the expression we get:

$$n\mu = \sum_{i=1}^n X_i \quad (0.0.10)$$

$$\mu = \frac{1}{n} \sum_{i=1}^n X_i \quad (0.0.11)$$

Let us denote the value achieved in (0.0.11) as \bar{X} . Taking partial derivative w.r.t σ and taking μ as constant

$$\frac{\partial \ln L_m}{\partial \sigma} = \frac{-n}{\sigma} + \sum_{i=1}^n \frac{(X_i - \mu)^2}{\sigma^3} \quad (0.0.12)$$

The value for σ at which L_m achieves maximum value is same in $\ln L_m$

$$\frac{\partial \ln L_m}{\partial \sigma} = 0 \quad (0.0.13)$$

$$\frac{-n}{\sigma} + \sum_{i=1}^n \frac{(X_i - \mu)^2}{\sigma^3} = 0 \quad (0.0.14)$$

Upon simplifying the expression

$$\frac{n}{\sigma} = \sum_{i=1}^n \frac{(X_i - \mu)^2}{\sigma^3} \quad (0.0.15)$$

$$\sigma^2 = \sum_{i=1}^n \frac{(X_i - \mu)^2}{n} \quad (0.0.16)$$

Substituting (0.0.11) in (0.0.16)

$$\sigma^2 = \sum_{i=1}^n \frac{(X_i - \bar{X})^2}{n} \quad (0.0.17)$$

$$\sigma = \sqrt{\sum_{i=1}^n \frac{(X_i - \bar{X})^2}{n}} \quad (0.0.18)$$

Hence **Option 3** and **Option 4** are correct