

Characteristic Function

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Question

Let $X \sim B\left(5, \frac{1}{2}\right)$ and $Y \sim U(0, 1)$. The the value of:

$$\frac{\Pr(X + Y \leq 2)}{\Pr(X + Y \geq 5)}$$

is equal to? (X and Y are independent)

Characteristic function

Definition

- 1 It is an alternative way to define the probability mass function or probability density function for a random variable.
- 2 It completely defines the probability distribution for the random variable.

Expression

For discrete random variable

$$C_X(t) = \sum_k \Pr(X = k) e^{itk} \quad (1)$$

For continuous random variable

$$C_X(t) = \int_{-\infty}^{\infty} f_X(x) e^{itx} dx \quad (2)$$

Characteristic function

Properties

- 1 It exists for all real value random variables
- 2 It is continuous over entire space
- 3 It is bounded $|C_X(t)| \leq 1$
- 4 If X and Y are independent then

$$Z = X + Y \quad (3)$$

$$C_Z(t) = C_X(t)C_Y(t) \quad (4)$$

- 5 Gil Pelaez Formula:

$$\Pr(X \leq x) = F_X(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \frac{\operatorname{Im}(e^{-itx} C_X(t))}{t} dt \quad (5)$$

Solution

Characteristic function of X

For $X \sim B\left(5, \frac{1}{2}\right)$ will be (binomial distribution):

$$C_X(t) = \sum_k \Pr(X = k) e^{itk} \quad (6)$$

$$= \sum_k \binom{5}{k} \left(\frac{1}{2}\right)^5 e^{itk} \quad (7)$$

$$= \left(\frac{e^{it} + 1}{2}\right)^5 \quad (8)$$

Solution

Characteristic function of Y

For $Y \sim U(0, 1)$ will be:

$$C_Y(t) = \int_{-\infty}^{\infty} f_Y(y) e^{ity} dy \quad (9)$$

$$= \int_0^1 \frac{1}{1-0} e^{ity} dy \quad (10)$$

$$C_Y(t) = \frac{e^{it} - 1}{it} \quad (11)$$

Solution

Characteristic function for Z

Since both X and Y are independent we can take:

$$Z = X + Y \quad (12)$$

$$C_Z(t) = C_X(t)C_Y(t) \quad (13)$$

$$C_Z(t) = \frac{(e^{it} + 1)^5(e^{it} - 1)}{32it} \quad (14)$$

Applying Gil-Pelaez formula:

$$F_Z(z) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}(e^{-itz} C_Z(t))}{t} dt \quad (15)$$

$$F_Z(z) = \frac{1}{2} - \int_0^\infty \frac{1}{2i\pi t} \left(\frac{(e^{it} + 1)^5(e^{it} - 1)e^{-itz} + (e^{-it} + 1)^5(e^{-it} - 1)e^{itz}}{32it} \right) dt \quad (16)$$

Solution

$\Pr(X + Y \leq 2)$ or $\Pr(Z \leq 2)$

Substituting $z = 2$, the value for $\Pr(Z \leq 2)$:

$$\begin{aligned} &= \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{8 \cos 2t + 2 \cos 4t}{64t^2} dt \\ &\quad + \frac{1}{\pi} \int_0^{\infty} \frac{+8 \cos 3t - 8 \cos t - 10}{64t^2} dt \end{aligned} \quad (17)$$

Solution

Integration Method

Finding a general expression for integrating:

$$\int \frac{\cos ax}{x^2} dx = -\frac{\cos ax}{x} - a \int \frac{\sin ax}{x} dx + C \quad (18)$$

By applying integration by parts. Now finding the value of other integral, by substituting $u = ax$ for limits as 0 and ∞ :

$$\int_0^{\infty} \frac{a \sin ax}{x} dx = \int_0^{\infty} \frac{a \sin u}{u} du \quad (19)$$

$$= \frac{a\pi}{2} \quad (20)$$

Solution

$\Pr(X + Y \leq 2)$ or $\Pr(Z \leq 2)$

Now using the above general expressions to calculate (17) and simplifying the expression after putting the limits we get

$$\begin{aligned} &= \frac{-1}{8\pi} \left(\int_0^\infty \frac{\sin 4t + 3 \sin 3t + 2 \sin 2t - \sin t}{t} dt \right) \\ &\quad - \frac{2(\cos t - 1)(\cos t + 1)^3}{8\pi t} \Bigg|_0^\infty + \frac{1}{2} \end{aligned} \quad (21)$$

$$= \frac{1}{2} + \frac{-1}{8\pi} \times \frac{5\pi}{2} + 0 \quad (22)$$

$$= \frac{3}{16} \quad (23)$$

Solution

$\Pr(X + Y \geq 5)$ or $\Pr(Z \geq 5)$

Similarly on substituting $z = 5$, the value for $\Pr(Z \leq 5)$:

$$\begin{aligned} &= \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{-10 \cos 3t - 8 \cos 4t}{64t^2} dt \\ &\quad + \frac{1}{\pi} \int_0^{\infty} \frac{-2 \cos 5t + 12 \cos t + 8}{64t^2} dt \end{aligned} \quad (24)$$

$$\begin{aligned} &= \frac{1}{\pi} \left(\int_0^{\infty} \frac{5 \sin 5t + 16 \sin 4t + 15 \sin 3t - 6 \sin t}{32} dt \right) \\ &\quad + \frac{1}{2} + \frac{1}{\pi} \left(\frac{16(\cos t - 1)(\cos t)(\cos t + 1)^3}{32t} \Big|_0^{\infty} \right) \end{aligned} \quad (25)$$

$$= \frac{1}{2} + \frac{1}{\pi} \times \frac{15\pi}{32} + 0 \quad (26)$$

$$= \frac{31}{32} \quad (27)$$

Solution

$\Pr(X + Y \geq 5)$ or $\Pr(Z \geq 5)$

The value for $\Pr(Z \geq 5)$:

$$\Pr(Z > 5) = 1 - \Pr(Z \leq 5) \quad (28)$$

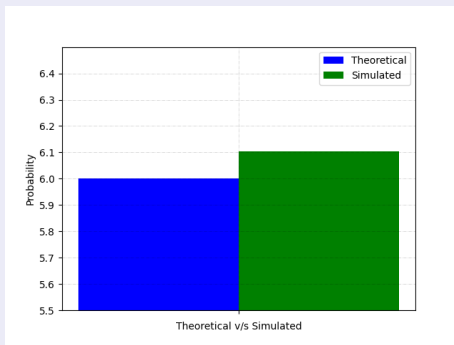
$$= 1 - \frac{31}{32} = \frac{1}{32} \quad (29)$$

Solution

Final Answer

Upon substituting (23) and (29), we get:

$$\frac{\Pr(X + Y \leq 2)}{\Pr(X + Y \geq 5)} = 6 \quad (30)$$



Figure