

AI1103: Challenge Problem Mixture

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CS20BTECH11063 EE20BTECH11048

Download all python codes from

https://github.com/tanmaygar/AI-Course/blob/main/challenge%20mixture/codes/challenge_mix.py

and latex-tikz codes from

https://github.com/tanmaygar/AI-Course/blob/main/challenge%20mixture/Challenge_Mixed.tex

CHALLENGE PROBLEM MIXED

Let $X \sim B\left(5, \frac{1}{2}\right)$ and $Y \sim U(0, 1)$. The the value of:

$$\frac{\Pr(X + Y \leq 2)}{\Pr(X + Y \geq 5)}$$

is equal to? (X and Y are independent)

SOLUTION: CHARACTERISTIC FUNCTION APPROACH

Characteristic function for $X \sim B\left(5, \frac{1}{2}\right)$ will be:

$$C_X(t) = \left(\frac{e^{it} + 1}{2}\right)^5 \quad (0.0.1)$$

Characteristic function for $Y \sim U(0, 1)$ will be:

$$C_Y(t) = \frac{e^{it} - 1}{it} \quad (0.0.2)$$

Since both X and Y are independent we can take:

$$Z = X + Y \quad (0.0.3)$$

$$C_Z(t) = C_X(t)C_Y(t) \quad (0.0.4)$$

$$C_Z(t) = \frac{(e^{it} + 1)^5(e^{it} - 1)}{32it} \quad (0.0.5)$$

Applying Gil-Pelaez formula:

$$F_Z(z) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}\left(e^{-itz}C_Z(t)\right)}{t} dt \quad (0.0.6)$$

$$F_Z(z) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{1}{2it} \left(\frac{(e^{it} + 1)^5(e^{it} - 1)e^{-itz}}{32it} + \frac{1}{2it} \left(\frac{(e^{-it} + 1)^5(e^{-it} - 1)e^{itz}}{32it} \right) \right) dt$$

Substituting $z = 2$, the value for $\Pr(Z \leq 2)$:

$$= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{8 \cos 2t + 2 \cos 4t}{64t^2} dt + \frac{1}{\pi} \int_0^\infty \frac{+8 \cos 3t - 8 \cos t - 10}{64t^2} dt \quad (0.0.7)$$

$$= \frac{-1}{8\pi} \left(\int_0^\infty \frac{\sin 4t + 3 \sin 3t + 2 \sin 2t - \sin t}{t} dt \right) - \frac{2(\cos t - 1)(\cos t + 1)^3}{8\pi t} \Big|_0^\infty + \frac{1}{2} \quad (0.0.8)$$

$$= \frac{1}{2} + \frac{-1}{8\pi} \times \frac{5\pi}{2} + 0 \quad (0.0.9)$$

$$= \frac{3}{16} \quad (0.0.10)$$

Similarly on substituting $z = 5$, the value for $\Pr(Z \leq 5)$:

$$= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{-10 \cos 3t - 8 \cos 4t}{64t^2} dt + \frac{1}{\pi} \int_0^\infty \frac{-2 \cos 5t + 12 \cos t + 8}{64t^2} dt \quad (0.0.11)$$

$$= \frac{1}{\pi} \left(\int_0^\infty \frac{5 \sin 5t + 16 \sin 4t + 15 \sin 3t - 6 \sin t}{32} dt \right) + \frac{1}{2} + \frac{1}{\pi} \left(\frac{16(\cos t - 1)(\cos t)(\cos t + 1)^3}{32t} \Big|_0^\infty \right) \quad (0.0.12)$$

$$= \frac{1}{2} + \frac{1}{\pi} \times \frac{15\pi}{32} + 0 \quad (0.0.13)$$

$$= \frac{31}{32} \quad (0.0.14)$$

The value for $\Pr(Z \geq 5)$:

$$\Pr(Z > 5) = 1 - \Pr(Z \leq 5) \quad (0.0.15)$$

$$= 1 - \frac{31}{32} = \frac{1}{32} \quad (0.0.16)$$

Upon substituting (0.0.10) and (0.0.16), we get:

$$\frac{\Pr(X + Y \leq 2)}{\Pr(X + Y \geq 5)} = 6 \quad (0.0.17)$$

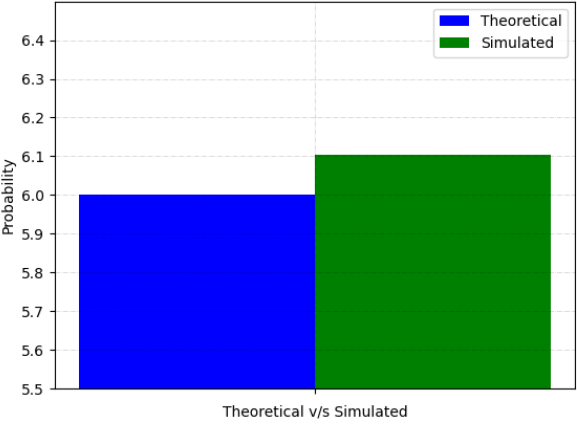


Fig. 0