AI1103: Assignment 8

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Download latex-tikz codes from

https://github.com/tanmaygar/AI−Course/blob/main/Assignment8/Assignment8.tex

Problem CSIR UGC NET EXAM (Dec 2016), Q.109:

$X_1, X_2, \dots, X_n$ are independent and identically distributed as $N(\mu, \sigma^2)$, $-\infty < \mu < \infty$, $\sigma^2 > 0$. Then

1) $\sum_1^n \frac{(X_i-\bar{X})^2}{n-1}$ is the Minimum Variance Unbiased Estimate of σ^2
2) $\sqrt{\sum_1^n \frac{(X_i-\bar{X})^2}{n-1}}$ is the Minimum Variance Unbiased Estimate of σ
3) $\sum_1^n \frac{(X_i-\bar{X})^2}{n}$ is the Maximum Likelihood Estimate of σ^2
4) $\sqrt{\sum_1^n \frac{(X_i-\bar{X})^2}{n}}$ is the Maximum Likelihood Estimate of σ

Solution:

The pdf for each random variable is same as they are all identical and independent Normal Distributions with same μ and σ^2.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(x-\mu)^2}{2\sigma^2} \tag{0.0.1}$$

Let us take our maximum likelihood function for given random variable X_i

$$L(\mu; \sigma | X_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(X_i-\mu)^2}{2\sigma^2} \tag{0.0.2}$$

Since all the random variables are i.i.d

$$L(\mu; \sigma | X_1, X_2, \dots, X_n) = \prod_{i=1}^{n} L(\mu; \sigma | X_i) \tag{0.0.3}$$

Let us denote:

$$L_m : L(\mu; \sigma | X_1, X_2, \dots, X_n) \tag{0.0.4}$$

Substituting (0.0.2) for each Random Variable in (0.0.3)

$$L_m = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(X_i-\mu)^2}{2\sigma^2} \tag{0.0.5}$$

Taking natural log on both sides and simplifying

$$\ln L_m = \frac{-n}{2} \ln 2\pi - n \ln \sigma - \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{2\sigma^2} \tag{0.0.6}$$

In order to find Maximum Likelihood we need to maximise μ and σ w.r.t. all Random variables. Taking partial derivative w.r.t μ and taking σ as constant

$$\frac{\partial \ln L_m}{\partial \mu} = \sum_{i=1}^{n} \frac{(X_i-\mu)}{\sigma^2} \tag{0.0.7}$$

The value for μ at which L_m achieves maximum value is same in $\ln L_m$

$$\because \frac{\partial \ln L_m}{\partial \mu} = 0 \tag{0.0.8}$$

$$\therefore \sum_{i=1}^{n} \frac{(X_i-\mu)}{\sigma^2} = 0 \tag{0.0.9}$$

On simplifying the expression we get:

$$n\mu = \sum_{i=1}^{n} X_i \tag{0.0.10}$$

$$\mu = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{0.0.11}$$

Let us denote the value achieved in (0.0.11) as $\bar{X}$. Taking partial derivative w.r.t σ and taking μ as constant

$$\frac{\partial \ln L_m}{\partial \sigma} = \frac{-n}{\sigma} + \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^3} \tag{0.0.12}$$

The value for σ at which L_m achieves maximum value is same in $\ln L_m$

$$\frac{\partial \ln L_m}{\partial \sigma} = 0 \tag{0.0.13}$$

$$\frac{-n}{\sigma} + \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^3} = 0 \tag{0.0.14}$$

Upon simplifying the expression

$$\frac{n}{\sigma} = \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^3} \tag{0.0.15}$$

$$\sigma^2 = \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{n} \tag{0.0.16}$$

Substituting (0.0.11) in (0.0.16)

$$\sigma^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n} \tag{0.0.17}$$

$$\sigma = \sqrt{\sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n}} \tag{0.0.18}$$

Hence **Option 3** and **Option 4** are correct