24/02/2022

- How unfair is this algorithm?
- Does the output changes depending on the order in which men proposes?

NO!

Defn:- Call a woman w a valid partner of a man m if there exists a stable matching containing the pair (m, w)

Defn:- Call her a best valid partner, if no woman of higher rank on his preference list is a valid partner. Let us denote this by best(m).

Define $S^* := \{(m, \text{best}(m)) : m \in M\}$

Lemma: Every execution of the algorithm produces the set S^*.

Proof:- Suppose not. Then $\exists$ a man m s.t. he is paired with someone who is not the best valid partner.

(m, w) exists and $w \neq best(m)$.

This implies $best(m)$ has rejected m at certain point.

[Let this rejection be the first rejection in the execution by a valid partner.

Let m' be the person engaged with $best(m)$.

$S \leftarrow$ output matching

(m, w)

$(m', best(m))$

Since $best(m)$ is a valid partner $\exists$ a stable matching S' s.t. $(m, best(m)) \in S'$.

$S' \leftarrow$ a stable matching

$(m, best(m))$

(m', w') := w' be the match of m' in S'.

Our assumption implies m' has not been rejected by a valid partner.

which further implies that

m' prefers best(m) over w'

best(m) prefers m' over m

m' prefers best(m) over w'.

$\Rightarrow$ that S' is not a stable matching.

$\Rightarrow$ a contradiction.

From the perspective of women

this is the worst valid partner.

<u>Defn</u>:- Worst valid partner is a valid partner s.t. no men lower in rank is a valid partner for her.

$$\underset{\text{best}}{m_1} > m_2 \underbrace{> \cdots >}_{\text{valid partners}} \underset{\text{worst}}{m_n}$$

Claim: $S^* = \{ (\text{worst}(\omega), \omega) : \omega \in W \}$

Proof:- Suppose not. $\exists$ a woman who is not engaged with worst (ω).

say (m, ω) is engaged.

so $m \neq \text{worst}(\omega) \Rightarrow m > \text{worst}(\omega)$.

$\Rightarrow \exists$ a valid partner, say m', of ω of rank lower than m.

$\Rightarrow \exists$ a stable matching S' s.t. $(m', \omega) \in S'$.

In S',

m — ω'	(m, ω) belongs to the output of the algorithm
m' — ω	$\Rightarrow$ best$(m) = \omega$.

m prefer w over w'

w prefers m over m'

The above two things implies that S' is not stable.

$\Rightarrow \Leftarrow$

$\blacksquare$

<u>Second proof</u>:-

<u>Output</u>. m — $best(m) := w$: in S-output

choose any other valid partner of w, say m'.

$(m', w) \in S'$:= stable

$(m, w') \in S'$

$\blacksquare$