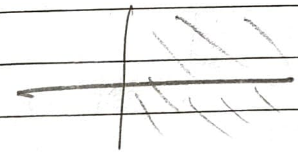
Tanmay Garg CS20BTECH11063

TOPIC Convex Assignment 1 DATE

1. A) C is not a subspace

counter eg

$S = \{(a,0)\}$ $(a > 0)$



$\therefore ax_1 \geq 0$ $\therefore x_1 \geq 0$

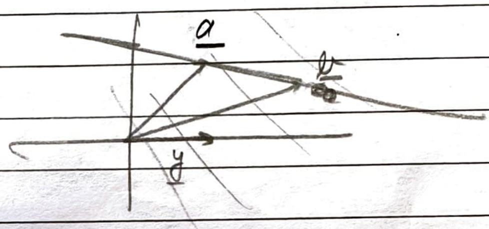
$x_2 \in \mathbb{R}$

if $a < 0$ then $x_1 \leq 0$, $x_2 \in \mathbb{R}$

C is not subspace for any S

B) C is not an affine set

$S = \{(1,0)\}$



the part where x_1 coordinate < 0 doesn't lie in C

$\therefore$ not affine

C) Let us take $\underline{a}, \underline{b} \in C$

$\therefore \underline{a}^T y \geq 0$ & $\underline{b}^T \underline{y} \geq 0$ $\forall y \in S$

$\theta \underline{a}^T \underline{y} \geq 0$, $(1-\theta)\underline{b}^T \underline{y} \geq 0$ $\forall y \in S$ $\forall \theta \in [0,1]$

$(\theta \underline{a} + (1-\theta)\underline{b})^T \underline{y} = \theta \underline{a}^T y + (1-\theta)\underline{b}^T y \geq 0$

$\therefore \theta \underline{a} + (1-\theta)\underline{b} \in C$

$\therefore$ C is convex

D) let $\underline{a} \in C$

$\underline{a}^T y \geq 0$ $\forall y \in S$

$\theta \underline{a}^T y \geq 0$ $\forall y \in S$ $\forall \theta \geq 1$

$\therefore \theta \underline{a} \in C$

$\therefore$ C is a cone

2. To prove: f_1 is convex

Proof:

$\because$ we already know that $f(x) = \|\underline{x}\|_2$ is a convex f^n

$\because$ $\theta\|\underline{a}\|_2 + (1-\theta)\|\underline{b}\|_2 \geq \|\theta\underline{a} + (1-\theta)\underline{b}\| \quad \forall \theta \in [0,1]$

(from triangle inequality)

$\because$ $\underline{y} - A\underline{x}$ is an affine Transformation of $\underline{x}$

$\because$ composition of convex f^n is also convex

$\therefore$ $\|\underline{y} - A\underline{x}\|$ is also convex

To prove: f_2 is convex

Proof:

$$f(\underline{x}) = \|\underline{x}\|^2$$

if $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$\therefore$ $f(\underline{x}) = x_1^2 + \dots + x_n^2$

$\therefore$ $\nabla^2 f(\underline{x}) = 2\underline{I}$

& this is +ve semi definite hence it is convex

Affine maps of convex f^n are also convex

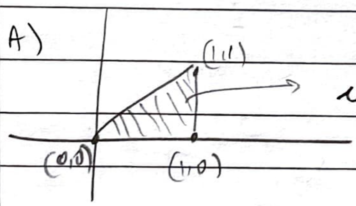
$\therefore$ $\|\underline{y} - A\underline{x}\|^2$ is also convex

$$f_2(\underline{x}) = \|y\|^2 - 2yA\underline{x} + x^T A^T A x$$

$$\frac{\partial f_2(\underline{x})}{\partial \underline{x}} = -2yA + x^T A^T A + A^T A x = -2yA + 2A^T A x$$

$$\frac{\partial^2 f_2(\underline{x})}{\partial \underline{x}^2} = 2A^T A \quad (\text{+ve semi definite})$$

3. A)



convex hull $= \sum \theta_i x_i$

$0 \leq \theta_i \leq 1 \quad \forall i$

$\sum \theta_i = 1$

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8) To prove: $f(\underline{x}) \le \max\{ f(\underline{x}_1), f(\underline{x}_2), \ldots, f(\underline{x}_m)\}$

for $\underline{x} \in \mathrm{conv}(S)$

Proof:

let us take some $\underline{x} = \sum \alpha_i \underline{x}_i$ $\quad (\sum \alpha_i = 1)$

$(\alpha_i \in [0,1])$

$\because f(\underline{x})$ is a convex f^n

$$\begin{aligned} \therefore f(\underline{x}) &= f(\sum \alpha_i \underline{x}_i) \\ &\le \sum \alpha_i f(x_i) \\ &\le (\sum \alpha_i) \max_{i=1}^{m} \{ f(x_i)\} \\ &= \max_{i=1}^{m} \{ f(x_i)\} \end{aligned}$$

<u>Hence proved</u>

4 To prove: C_1, C_2 are ~~convex~~ & $C_1 = C_2$

Proof:

$$C_1 = \left\{ (A, z) : A \in S^n, z \in \mathbb{R}, \begin{pmatrix} A & \underline{b} \\ \underline{b}^T & z \end{pmatrix} \succeq 0 \right\}$$

let us take $(A_1, z_1) \in C$

$(A_2, z_2) \in C$

$$\therefore x^T \begin{pmatrix} A_1 & \underline{b} \\ \underline{b}^T & z_1 \end{pmatrix} x \ge 0 \qquad \forall \|x\| = 1$$

similarly $x^T \begin{pmatrix} A_2 & \underline{b} \\ \underline{b}^T & z_2 \end{pmatrix} x \ge 0$

$$\therefore x^T \left(\theta \begin{pmatrix} A_1 & \underline{b} \\ \underline{b}^T & z_1 \end{pmatrix} + (1-\theta) \begin{pmatrix} A_2 & \underline{b} \\ \underline{b}^T & z_2 \end{pmatrix} \right) x \ge 0$$

$$\therefore \theta \begin{pmatrix} A_1 & \underline{b} \\ \underline{b}^T & z_1 \end{pmatrix} + (1-\theta) \begin{pmatrix} A_2 & \underline{b} \\ \underline{b}^T & z_2 \end{pmatrix} \succeq 0$$

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4. To prove: C_1, C_2 are convex $\& C_1 = C_2$

Proof:

$$C_1 = \left\{ (A, z) : A \in S^n, z \in \mathbb{R}, \begin{pmatrix} A & b \\ b^T & z \end{pmatrix} \succeq 0 \right\}$$

let us take $(A_1, z_1) \in C_1$

$(A_2, z_2) \in C_2$

$$\begin{pmatrix} A_1 & b \\ b^T & z_1 \end{pmatrix} \succeq 0 \quad \& \quad \begin{pmatrix} A_2 & b \\ b^T & z_2 \end{pmatrix} \succeq 0$$

$$\theta \begin{pmatrix} A_1 & b \\ b^T & z_1 \end{pmatrix} + (1-\theta) \begin{pmatrix} A_2 & b \\ b^T & z_2 \end{pmatrix} \succeq 0$$

$$\therefore \begin{pmatrix} \theta A_1 + (1-\theta) A_2 & b \\ b^T & \theta z_1 + (1-\theta) z_2 \end{pmatrix} \succeq 0$$

$\therefore (\theta A_1 + (1-\theta) A_2, \theta z_1 + (1-\theta) z_2) \in C$

$\therefore C_1$ is convex

$\begin{pmatrix} A_1 & b \\ b^T & z_1 \end{pmatrix} \succeq 0 \quad \therefore A_1 \succeq 0 \quad \& \quad z_1 - b^T A_1^{-1} b > 0$

Schur's complement of A_1 in M

$M / A_1 > 0$

$\therefore z \geq b^T A^{-1} b$

$C_1 \& C_2$ are the same set

$\therefore C_1 = C_2$

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$\therefore$ C_1 ~~is convex~~

~~M is the semidefinite if $A > 0$~~

~~$\therefore$ $(A,z) \in C_1$ & $(A,z) \in C_2$ $\forall (A,z)$~~

~~$\therefore$ $C_1 = C_2$~~

~~$\because$ C_1 is convex $\therefore$ C_2 is convex~~

5(a) Given: $f(\underline{x})$ is symmetric convex

To prove: $\exists\, \underline{x}$ with both the co-ordinates equal

Proof: let us take $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the optimal value of $\underline{x} \in C$

we know that $f((x_1, x_2)) = f((x_2, x_1))$

$\therefore$ $\underline{y} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$ is also optimal

Let us take a point on the line joining $\underline{x}$ & $\underline{y}$

$\therefore$ $\underline{P} = \theta \underline{x} + (1-\theta)\, \underline{y}$

$$
\begin{aligned}
f(\theta \underline{x} + (1-\theta)\underline{y}) &\leq \theta f(\underline{x}) + (1-\theta)\, f(\underline{y}) \\
&= \theta f(\underline{x}) + (1-\theta)\, f(\underline{x}) \\
&= f(\underline{x})
\end{aligned}
$$

$\because$ for every convex optimization every local minima is a global minima

$\therefore$ $f(\underline{P}) = f(\underline{x})$

let us put $\theta = 1/2$

$$f\left(\frac{\underline{x}}{2} + \frac{\underline{y}}{2}\right) = f(\underline{x})$$

$$= f\left(\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}\right)\right) = f(\underline{x})$$

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$\therefore$ we proved $\exists$ a point equal co-ordinates

(b) To find: $\max_{\underline{x}} \; x_1 x_2 \cdots x_n$

s.t. $\sum x_i = 1 \;;\; x_i \geq 0 \quad \forall i = 1, \dots n$

Let us apply AM-GM as $x_i \in \mathbb{R}$ & $x_i \geq 0$

$$\therefore \frac{\sum x_i}{n} \geq \left(\prod x_i\right)^{1/n}$$

$$\Rightarrow \frac{1}{n} \geq \left(\prod x_i\right)^{1/n}$$

$$\Rightarrow \left(\frac{1}{n}\right)^n = \prod x_i$$

$\therefore$ the maximum achievable value is $\left(\frac{1}{n}\right)^n$

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6 A) To prove: $f(x) = \frac{x^T x}{t}$ is convex

Proof: $\because$ if $g(x)$ is convex

then $f(x, t) = t\, g\left(\frac{x}{t}\right)$ is also convex

$\forall \; t > 0$

$\because$ we proved before that $\|x\|^2$ is convex

$\therefore f(x, t) = t\, g\left(\frac{x}{t}\right)$

$= t \frac{x^T x}{t^2}$ $\quad (g(x) = x^T x)$

$= \frac{x^T x}{t}$

$\therefore f(x,t)$ is also convex

B) To prove: $f(x,t) = \frac{x^T x}{t^2}$ is quasi convex

Proof: For quasi convex all sublevels are convex

$\therefore C = \{ (x,t) : f(x) \leq \alpha \} = \{ (x,t) : \|x\|^2 \leq \alpha t^2 \}$

$= \left\{ (x,t) \,\middle|\, \frac{\sum x_i^2}{t^2} \leq \alpha \right\}$ (Perspective $P(S_\alpha)$)

$= \{ y \mid \|y\|^2 \leq \alpha \}$ if $\alpha \geq 0$ then ball of radius $\sqrt{\alpha}$

$\therefore P(S_\alpha)$ is convex $\quad \therefore f(\bar{x}, t)$ is convex

c) To prove: $f(x) = \left\| x - \frac{x}{\|x\|} \right\|_2$ if $\|x\| \geq 1$

Proof: $f(x) = \|x\| \left(\frac{\|x\| - 1}{\|x\|} \right) = \|x\| - 1$

$\therefore f(x) = \begin{cases} \|x\| - 1 & \|x\| \geq 1 \\ 0 & \text{else} \end{cases}$

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let us take $\underline{x}_1, \underline{x}_2$ st $\|x_1\| \geq 1$ & $\|x_2\| \geq 1$

$f(\theta \underline{x}_1 + (1-\theta)\underline{x}_2) = \|\theta \underline{x}_1 + (1-\theta)\underline{x}_2\| - 1$ — ①

$\theta f(\underline{x}_1) + (1-\theta) f(x_2) = \theta\|x_1\| - \cancel{\theta} + (1-\theta)\|x_2\|$
$- 1 + \cancel{\theta}$

$= \theta\|x_1\| + (1-\theta)\|x_2\| - 1$ — ②

$\therefore$ ① < ② hence $f(\underline{x}) = \|\underline{x}\| - 1$
is convex

$f(\underline{x}) = 0$ is also convex when $\|x\| < 1$
$\therefore$ max of 2 convex f^n is also convex

$f(\underline{x}) = \begin{cases} \|x\| - 1 & \text{if } \|x\| \geq 1 \\ 0 & \text{else} \end{cases}$

is also convex
as $f(x)$ can be written as
$\max\{\|x-1\|, 0\}$