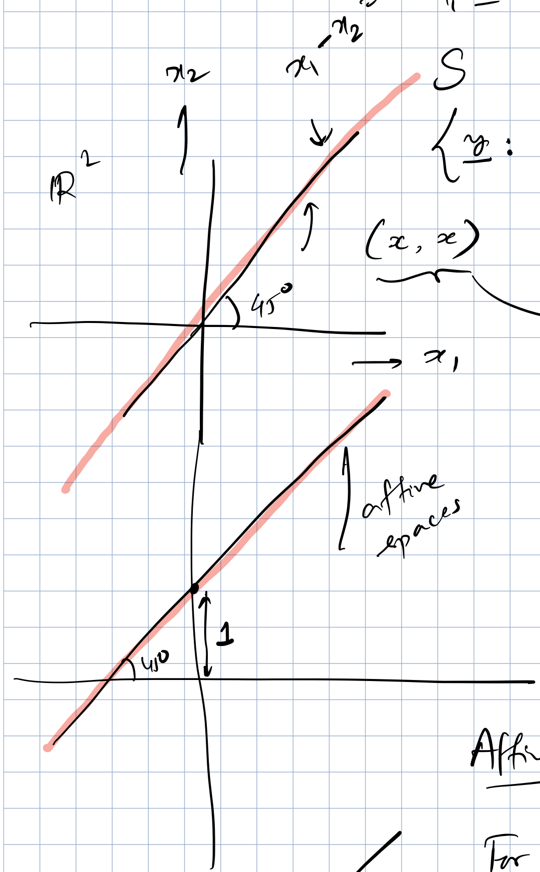
Recap: Subspaces — For any $\underline{x_1}, \underline{x_2} \in S$, $\alpha_1 \underline{x_1} + \alpha_2 \underline{x_2} \in S$ — Any subspace can be represented

linear comb

"span"

$S \to \subseteq \mathbb{R}^n$



S $\{\underline{y} :$

(x, x)

generator matrix.

① $\{\underline{y} = G\underline{x}\}$ every vector in S written as a linear combination of columns of G.

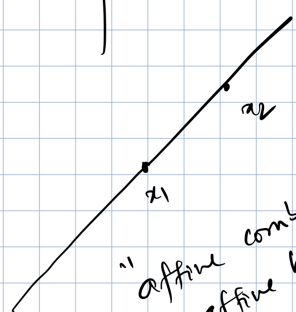
$$\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{2\times 1} \underbrace{x}_{1\times 1}$$

② Every vector in S satisfies $H\underline{y} = 0$

$$S = \{\underline{y} : H\underline{y} = 0\}$$

Affine spaces are translated subspaces.

For any $\underline{x_1}, \underline{x_2} \in S$, the line joining $\underline{x_1}$ & $\underline{x_2}$ is also in S.



For any $\underline{x_1}, \underline{x_2} \in S$

$\theta \underline{x_1} + (1-\theta)\underline{x_2} \in S$ for any θ

"affine combination"

affine hull.

$\{\underline{y} : H\underline{y} = b\}$ or $\{\underline{y} = G\underline{x} + b'\}$

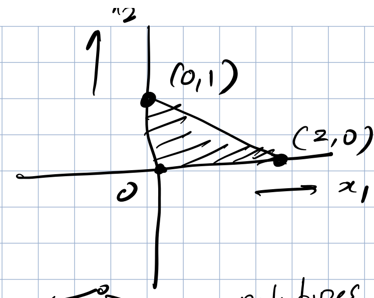
Convex: A set is convex if for any subset of points in S, $x_1, x_2, \ldots, x_k$

$$\sum_{i=1}^{k} \theta_i x_i \in S \quad \text{for any } \theta_1, \theta_2 \to \theta_k \geq 0, \quad \sum \theta_i = 1.$$

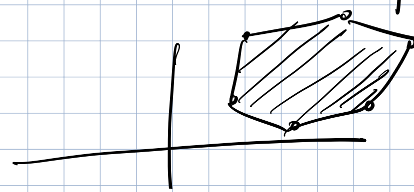
"convex combination"

convex hull.

Q: Example of finite sets S such that Convex hull(S)



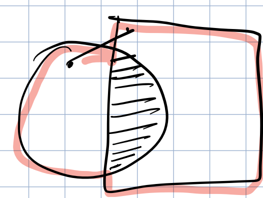
$$S = \{(0,0),\ (0,1),\ (2,0)\}$$



polytopes

polyhedra

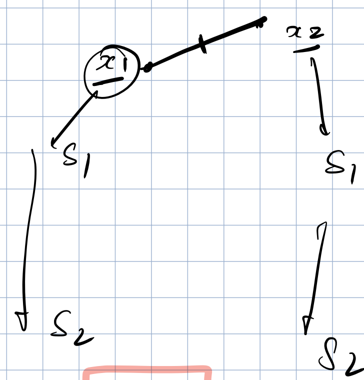
Operations preserving convexity



Intersection: Given sets S_1, S_2

If S_1, S_2 are convex, so is $S_1 \cap S_2$.

Proof: clear from definition.



Q: If S_1 and S_2 are convex, is $S_1 \cup S_2$ convex?

If S_1 & S_2, S_3 are convex so is $\boxed{S_1 \cap S_2} \cap S_3$

Suppose we have convex sets $S_1, S_2 - - -$

$$\bigcap_{i=1}^{\infty} S_i \qquad \bigcap_{\alpha} S_\alpha$$

is convex if S_α is convex.

Example: pick coefficients

$$\{a_{-10},\ a_{-9},\ - - -\ a_{-1},\ a_0,\ a_1,\ a_2 - - -,\ a_{10}$$

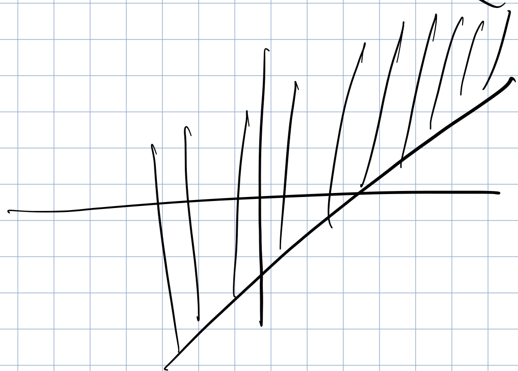
21 dec $\mathbb{R}^{21}$

$$\left|\sum_{k=-10}^{10} a_k \cos kt\right| \le 1$$

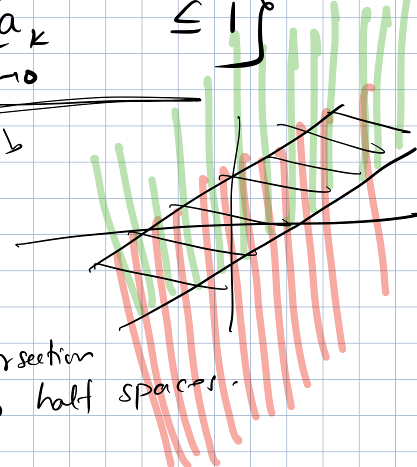
Consider the set $S \subseteq \mathbb{R}^{21}$ given by

$$\rightarrow S = \left\{ a : -1 \le \sum a_k \cos kt \le 1 \text{ for all } t \right\}.$$

Let $S_0 = \left\{ a : -1 \le \sum_{k=-10}^{10} a_k \le 1 \right\}$



slab



Intersection of two half spaces.

$$S_{0.1} = \left\{ a : -1 \le \sum a_k \cos 0.1k \le 1 \right\}$$

S_t is convex for each t.

(S_t is in fact a slab / intersection of two half spaces).

$$S = \bigcap_t S_t$$

Consider the following set of matrices

Problem: set of all symmetric $n \times n$ matrices $\mathbb{S}^n$

$$S_+^n = \left\{ A : x^T A x \ge 0 \text{ for any } x \right\}$$

$\mathbb{R}^2, \mathbb{R}^3 \ldots \mathbb{R}^n$

quadratic form

$S = \bigcap_x S_x \qquad S_x = \{A : x^T A x \geq 0\}$.

$S_{\underline{1}} = \{A : 1^T A 1 \geq 0\}$ "halfspace".

$\theta(1^T A_1 1 \geq 0)$

$1-\theta(1^T A_2 1 \geq 0)$

$1^T(\theta A_1 + (1-\theta) A_2) 1 \geq 0$

S_x is convex (halfspace) for each x.

S_2 : 2×2 symmetric matrices $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Example:

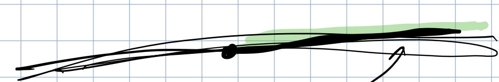
$S_+^2 = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a \geq 0,\ ac - b^2 \geq 0 \right\}$ is convex,

$\underbrace{S_{++}^n}$ $= \{A : x^T A x > 0 \text{ for all } x \neq 0\}$.

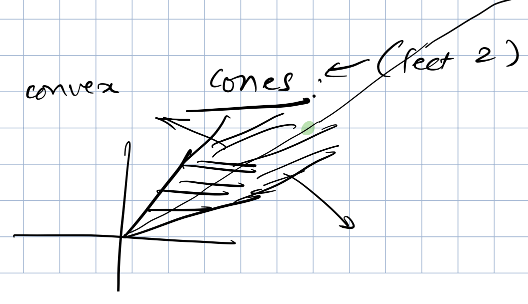
set of all $n\times n$ symmetric positive definite matrices.

Exercise: convex?.

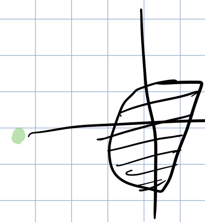
open, closed



S_+^n, S_{++}^n are actually convex cones. ← (lect 2).



Property 2:



1) Translation

2) Rotation.

3) Dilation.

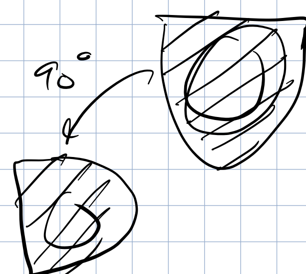
Consider a function

$$f(\underline{x}) = \underset{m\times n}{A}\underset{n\times 1}{\underline{x}} + \underset{m\times 1}{\underline{b}} \Big\} \leftarrow \text{affine map}$$

2.1 If S is convex, so is $f(S)$.

Proof:

$$\underbrace{A\underline{x}_1 + b}_{x_1 \in S} \qquad \underbrace{A\underline{x}_2 + \underline{b}}_{x_2 \in S}$$

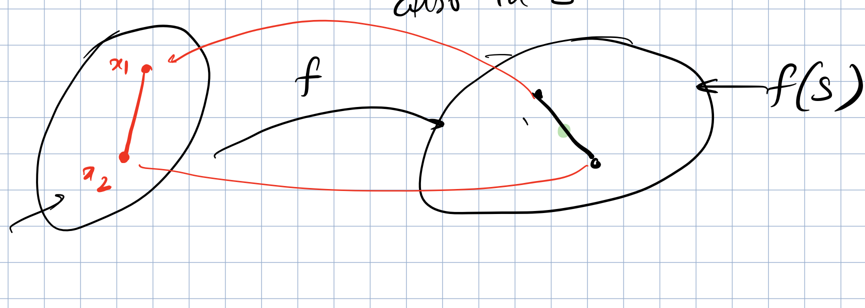


$$\theta\left(A\underline{x}_1 + b\right) + (1-\theta)\left(A\underline{x}_2 + \underline{b}\right)$$

$$= A\left(\theta \underline{x}_1 + (1-\theta)\underline{x}_2\right) + \underline{b}$$

$$= f\Big(\underbrace{\theta \underline{x}_1 + (1-\theta)\underline{x}_2}_{\text{also in } S}\Big)$$

also in S.



Cross section by slicing at $z = 1$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow[\text{affine map}]{} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x+2y \\ x-2y \end{pmatrix}$$ projection on xy plane.

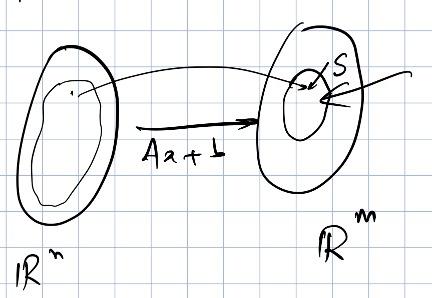
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\underline{x^2 + y^2 \leq 1} \checkmark$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

$$\frac{(x+y)^2}{a^2} + \frac{(x-10y)^2}{1^2} \leq 1$$

2.2: Consider the set of all points that potentially map to a given convex set



$$f^{-1}(S) = \{ x \in \mathbb{R}^n : Ax + b \in S \}$$

pre-image

If S is convex, so is $f^{-1}(S)$.

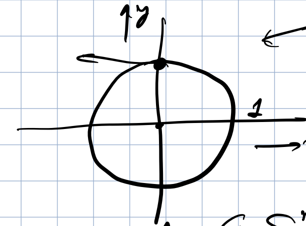
Consider the map $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}$

$\mathbb{R}^3 \qquad \mathbb{R}^2$

pre image of $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ z \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



infinite cyclinder.

$A_1, A_2 \dots A_n$

$m\times m \quad m\times m \quad m\times m$

$\{ (x_1, x_2, \dots x_n) : \underbrace{x_1 A_1 + x_2 A_2 + \dots + x_n A_n}_{\text{Linear matrix inequality}} \in S^m_{++}$

$m\times m$

$\underline{n=2}$

$S = \{ (x_1, x_2) : x_1 A_1 + x_2 A_2 \in S^m_{++} \} = f^{-1}(S^m_{++})$

Consider the map $\qquad \underline{n=2}$

$(x_1, x_2) \xrightarrow{f} x_1 A_1 + x_2 A_2$

$\mathbb{R}^2 \qquad \mathbb{R}^{m\times m}$

is an affine map.

S is the pre-image of S^m_{++} under this map.

$A_1 = A_2 = A$

$S = \{ (x_1, x_2) : (x_1 + x_2) A \in S^m_{++} \}$

If A indefinite $\quad S = \emptyset$

if A is positive definite

$S = \{ (x_1, x_2) : x_1 + x_2 > 0 \}$

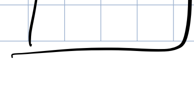
Set-additions.

$S_1 + S_2 = \left\{ x_1 + x_2 : \begin{matrix} x_1 \in S_1 \\ x_2 \in S_2 \end{matrix} \right\}$

S_1

S_2

$= \bigcup_{x_2 \in S_2} (S_1 + x_2)$



Ques: ① If S_1 and S_2 are convex, is $S_1 + S_2$ convex?.

② If S_1 and S_2 are convex, is $S_1 \times S_2$ convex?.

$$S_1 \times S_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \begin{array}{l} x \in S_1 \\ y \in S_2 \end{array} \right.$$