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TOPIC: Convex Assignment 1 DATE:

1\. A) C is not a subspace

counter eg

$S = \{(a,0)\}$ $(a>0)$

$\therefore a x_1 \geq 0$ $\therefore x_1 \geq 0$

$x_2 \in \mathbb{R}$

if $a<0$ then $x_1 \leq 0$, $x_2 \in \mathbb{R}$

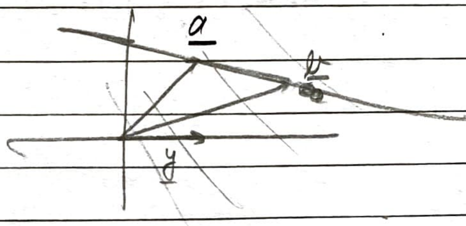
C is not subspace for any S

B) C is not an affine set

$S = \{(1,0)\}$

the part where x_1 coordinate <0 doesn't lie in C

$\therefore$ not affine



C) Let us take $\underline{a}, \underline{b} \in C$

$\therefore \underline{a}^T \underline{y} \geq 0$ & $\underline{b}^T \underline{y} \geq 0$ $\forall y \in S$

$\theta \underline{a}^T \underline{y} \geq 0$, $(1-\theta)\underline{b}^T \underline{y} \geq 0$ $\forall y \in S$ $\forall \theta \in [0,1]$

$\therefore (\theta \underline{a} + (1-\theta)\underline{b})^T \underline{y} = \theta \underline{a}^T y + (1-\theta)\underline{b}^T y \geq 0$

$\therefore \theta \underline{a} + (1-\theta)\underline{b} \in C$

$\therefore$ C is convex

D) let $\underline{a} \in C$

$\underline{a}^T y \geq 0$ $\forall y \in S$

$\theta \underline{a}^T y \geq 0$ $\forall y \in S$ $\forall \theta \geq 1$

$\therefore \theta \underline{a} \in C$

$\therefore$ C is a cone

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2. To prove: f_1 is convex

Proof:

$\because$ we already know that $f(x) = \|\underline{x}\|_2$ is a convex f^n

$\therefore$ $\theta\|\underline{a}\|_2 + (1-\theta)\|\underline{b}\|_2 \geq \|\theta \underline{a} + (1-\theta)\underline{b}\| \quad \forall \theta \in [0,1]$

(from triangle inequality)

$\because$ $\underline{y} - A\underline{x}$ is an affine Transformation of $\underline{x}$

$\therefore$ composition of convex f^n is also convex

$\therefore$ $\|\underline{y} - A\underline{x}\|$ is also convex

To prove: f_2 is convex

Proof:

$f(x) = \|\underline{x}\|^2$

if $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$\therefore f(\underline{x}) = x_1^2 + \cdots + x_n^2$

$\therefore \nabla^2 f(\underline{x}) = 2\underline{I}$

& this is +ve semi definite hence it is convex

Affine maps of convex f^n are also convex

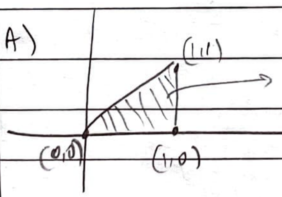
$\therefore \|y - Ax\|^2$ is also convex

$f_2(\underline{x}) = \|y\|^2 - 2yAx + x^T A^T A x$

$\dfrac{\partial f_2(\underline{x})}{\partial \underline{x}} = -2yA + x^T A^T A + A^T A x = -2yA + 2A^T A x$

$\dfrac{\partial^2 f_2(\underline{x})}{\partial \underline{x}^2} = 2A^T A$ (+ve semi definite)

3. A)



convex hull $= \sum \theta_i x_i$

$0 \leq \theta_i \leq 1 \quad \forall i$

$\sum \theta_i = 1$

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3) To prove: $f(x) \le \max\{f(x_1), f(x_2), \dots, f(x_m)\}$

for $x \in \text{conv}(S)$

Proof:

let us take some $x = \sum \alpha_i x_i$ $\quad (\sum \alpha_i = 1)$

$(\alpha_i \in [0,1])$

$\because f(x)$ is a convex f^n

$$\therefore f(x) = f\left(\sum \alpha_i x_i\right)$$
$$\le \sum \alpha_i f(x_i)$$
$$\le \left(\sum \alpha_i\right) \max_{i=1}^{m} \{f(x_i)\}$$
$$= \max_{i=1}^{m} \{f(x_i)\}$$

Hence proved

4 To prove: C_1, C_2 are convex & $C_1 = C_2$

Proof:

$$C_1 = \left\{(A, z) : A \in S^n,\ z \in \mathbb{R},\ \begin{pmatrix} A & b \\ b^T & z \end{pmatrix} \succeq 0 \right\}$$

let us take $(A_1, z_1) \in C$

$(A_2, z_2) \in C$

$$\therefore x^T \begin{pmatrix} A_1 & b \\ b^T & z_1 \end{pmatrix} x \ge 0 \quad \forall \|x\| = 1$$

similarly $x^T \begin{pmatrix} A_2 & b \\ b^T & z_2 \end{pmatrix} x \ge 0$

$$\therefore x^T \left(\theta \begin{pmatrix} A_1 & b \\ b^T & z_1 \end{pmatrix} + (1-\theta) \begin{pmatrix} A_2 & b \\ b^T & z_2 \end{pmatrix} \right) x \ge 0$$

$$\therefore \theta \begin{pmatrix} A_1 & b \\ b^T & z_1 \end{pmatrix} + (1-\theta) \begin{pmatrix} A_2 & b \\ b^T & z_2 \end{pmatrix} \succeq 0$$

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4. To Prove: C_1, C_2 are convex & $C_1 = C_2$

Proof:

$$C_1 = \left\{ (A, z) : A \in S^n, z \in \mathbb{R}, \begin{pmatrix} A & b \\ b^T & z \end{pmatrix} \succeq 0 \right\}$$

let us take $(A_1, z_1) \in C_1$

$(A_2, z_2) \in C_2$

$$\begin{pmatrix} A_1 & b \\ b^T & z_1 \end{pmatrix} \succeq 0 \quad \& \quad \begin{pmatrix} A_2 & b \\ b^T & z_2 \end{pmatrix} \succeq 0$$

$$\theta \begin{pmatrix} A_1 & b \\ b^T & z_1 \end{pmatrix} + (1-\theta) \begin{pmatrix} A_2 & b \\ b^T & z_2 \end{pmatrix} \succeq 0$$

$$\therefore \begin{pmatrix} \theta A_1 + (1-\theta) A_2 & b \\ b^T & \theta z_1 + (1-\theta) z_2 \end{pmatrix} \succeq 0$$

$\therefore \theta A_1 + (1-\theta) A_2, \ \theta z_1 + (1-\theta) z_2 \in C$

$\therefore C_1$ is convex

$$\begin{pmatrix} A_1 & b \\ b^T & z_1 \end{pmatrix} \succeq 0 \qquad \therefore A_1 \succeq 0 \quad \& \quad z_1 - b^T A_1^{-1} b > 0$$

Schur's complement of A_1 in M

$M / A_1 > 0$

$\therefore z \geq b^T A^{-1} b$

C_1 & C_2 are the same set

$\therefore C_1 = C_2$

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$\therefore$ C_1 is convex

M is +ve semidefinite if $A > 0$

$\therefore$ $(A, z) \in C_1$ & $(A, z) \in C_2$ $\forall (A, z)$

$\therefore$ $C_1 = C_2$

$\because$ C_1 is convex $\therefore$ C_2 is convex

5(a) Given: $f(\underline{x})$ is symmetric convex

To prove: $\exists\, x$ with both the co-ordinates equal

Proof: let us take $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the optimal value of $\underline{x} \in C$

we know that $f((x_1, x_2)) = f((x_2, x_1))$

$\therefore$ $\underline{y} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$ is also optimal

Let us take a point on the line joining $\underline{x}$ & $\underline{y}$

$\therefore$ $\bar{p} = \theta \underline{x} + (1-\theta)\underline{y}$

$$f(\theta \underline{x} + (1-\theta)\underline{y}) \leq \theta f(\underline{x}) + (1-\theta) f(\underline{y})$$
$$= \theta f(\underline{x}) + (1-\theta) f(\underline{x})$$
$$= f(\underline{x})$$

$\because$ for every convex optimization every local minima is a global minima

$\therefore$ $f(P) = f(\underline{x})$

let us put $\theta = 1/2$

$$f\left(\frac{\underline{x}}{2} + \frac{\underline{y}}{2}\right) = f(\underline{x})$$

$$= f\left(\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right)\right) = f(\underline{x})$$

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$\therefore$ we proved $\exists$ a point equal co-ordinates

(b) To find: $\max_{\underline{x}} \ x_1 x_2 \ldots x_n$

st $\sum x_i = 1 \ ; \ x_i \geq 0 \quad \forall i = 1, \ldots n$

Let us apply AM-GM as $x_i \in \mathbb{R}$ & $x_i \geq 0$

$$\therefore \frac{\sum x_i}{n} \geq \left(\prod x_i\right)^{1/n}$$

$$\Rightarrow \frac{1}{n} \geq \left(\prod x_i\right)^{1/n}$$

$$\Rightarrow \left(\frac{1}{n}\right)^n = \prod x_i$$

$\therefore$ the maximum achievable value is $\left(\frac{1}{n}\right)^n$

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6 A) To prove: $f(x) = \frac{x^T x}{t}$ is convex

Proof: $\because$ if $g(x)$ is convex
then $f(x, t) = t\, g\left(\frac{x}{t}\right)$ is also convex

$\forall\ t > 0$

$\because$ we proved before that $\|x\|^2$ is convex

$\therefore f(x, t) = t\, g\left(\frac{x}{t}\right)$

$= t\, \frac{x^T x}{t^2} \quad (g(x) = \|x\|)$

$= \frac{x^T x}{t}$

$\therefore f(x)$ is also convex

B) To prove: $f(x) = \frac{x^T x}{t^2}$ is quasi convex

Proof: For quasi convex all sublevels are convex

$\therefore C = \{ x : f(x) \leq \alpha \} = \{ x : \|x\|^2 \leq \alpha t^2 \}$

$= \{ x : 0 \leq \|x\| \leq t\sqrt{\alpha} \}$

let $x_1, x_2 \in C$ $\quad \therefore \|\theta x_1 + (1-\theta) x_2\| \leq \theta\|x_1\| + (1-\theta)\|x_2\|$

$\leq \theta t\sqrt{\alpha} + (1-\theta) t\sqrt{\alpha}$

$\therefore f(x)$ is quasi convex $\quad \leq t\sqrt{\alpha}$

Hence proved

c) To prove: $f(x) = \left\| x - \frac{x}{\|x\|} \right\|_2$ if $\|x\| \geq 1$

Proof: $f(x) = \|x\| \left(\frac{\|x\| - 1}{\|x\|} \right) = \|x\| - 1$

$$\therefore f(x) = \begin{cases} \|x\| - 1 & \|x\| \geq 1 \\ 0 & \text{else} \end{cases}$$

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let us take $\underline{x}_1$, $\underline{x}_2$ st $||\underline{x}_1|| \geq 1$ & $||\underline{x}_2|| \geq 1$

$$f(\theta \underline{x}_1 + (1-\theta)\underline{x}_2) = ||\theta \underline{x}_1 + (1-\theta)\underline{x}_2|| - 1 \quad -①$$

$$\theta f(\underline{x}_1) + (1-\theta) f(\underline{x}_2) = \theta ||\underline{x}_1|| - \cancel{\theta} + (1-\theta)||\underline{x}_2|| - 1 + \cancel{\theta}$$

$$= \theta ||\underline{x}_1|| + (1-\theta) ||\underline{x}_2|| - 1 \quad -②$$

$\therefore$ ① < ② hence $f(\underline{x}) = ||\underline{x}|| - 1$ is convex

$f(\underline{x}) = 0$ is also convex when $||x|| < 1$

$\therefore$ max of 2 convex f^n is also convex

$$f(\underline{x}) = \begin{cases} ||x|| - 1 & \text{if } ||x|| \geq 1 \\ 0 & \text{else} \end{cases}$$

is also convex

as $f(x)$ can be written as $\max\{||x-1||, 0\}$