

1. A)  $C$  is not a subspace

counter eg

$$S = \{(a, 0)\} \quad (a > 0)$$

$$\therefore ax_1 \geq 0$$

$$\therefore x_1 \geq 0$$

$$x_2 \in \mathbb{R}$$



if  $a < 0$  then  $x_1 \leq 0, x_2 \in \mathbb{R}$

$C$  is not subspace for any  $S$

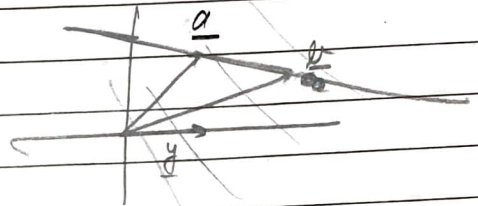
B)  $C$  is not an affine set

$$S = \{(1, 0)\}$$

the part where  $x_1$  coordinate

$< 0$  doesn't lie in  $C$

$\therefore$  not affine



C) let us take  $a, b \in C$

$$\therefore a^T y \geq 0 \text{ \& } b^T y \geq 0 \quad \forall y \in S$$

$$\theta a^T y \geq 0, (1-\theta)b^T y \geq 0 \quad \forall y \in S \quad \forall \theta \in [0, 1]$$

$$(\theta a + (1-\theta)b)^T y = \theta a^T y + (1-\theta)b^T y \geq 0$$

$$\therefore \theta a + (1-\theta)b \in C$$

$\therefore C$  is convex

D) let  $a \in C$

$$a^T y \geq 0 \quad \forall y \in S$$

$$\theta a^T y \geq 0 \quad \forall y \in S \quad \forall \theta \geq 1$$

$$\therefore \theta a \in C$$

$\therefore C$  is a cone

2. To prove:  $f_1$  is convex

Proof:

$\therefore$  we already know that  $f(x) = \|x\|_2$  is a convex  $f^n$

$$\theta \|x\|_2 + (1-\theta) \|y\|_2 \geq \|\theta x + (1-\theta)y\| \quad \forall \theta \in [0,1]$$

(from triangle inequality)

$\therefore y - Ax$  is an affine transformation of  $x$

$\therefore$  composition of convex  $f^n$  is also convex  
 $\|y - Ax\|$  is also convex

To prove:  $f_2$  is convex

Proof:

$$f(x) = \|x\|^2$$

$$\text{if } x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\therefore f(x) = x_1^2 + \dots + x_n^2$$

$$\therefore \nabla^2 f(x) = 2I$$

& this is +ve semi definite hence it is convex

Affine maps of convex  $f^n$  are also convex

$\therefore \|y - Ax\|^2$  is also convex

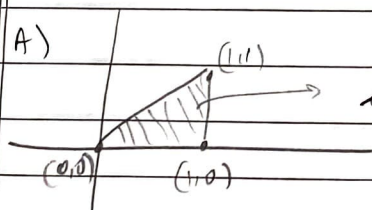
$$f_2(x) = \|y\|^2 - 2y^T Ax + x^T A^T A x$$

$$\frac{\partial f_2(x)}{\partial x} = -2y^T A + x^T A^T A + A^T A x = -2y^T A + 2A^T A x$$

$$\frac{\partial^2 f_2(x)}{\partial x^2} = 2A^T A \quad (\text{+ve semi definite})$$

3.

A)



convex hull

$$= \sum \theta_i x_i$$

$$0 \leq \theta_i \leq 1 \quad \forall i$$

$$\sum \theta_i = 1$$

8) To Prove:  $f(x) \in \max \{ f(x_1), f(x_2), \dots, f(x_m) \}$

Proof:

for  $x \in \text{conv}(S)$

$$\text{let us take some } x = \sum \alpha_i x_i \quad (\sum \alpha_i = 1) \\ (\alpha_i \in [0, 1])$$

$\therefore f(x)$  is a convex  $f^n$

$$\begin{aligned} f(x) &= f\left(\sum \alpha_i x_i\right) \\ &\leq \sum \alpha_i f(x_i) \\ &\leq \left(\sum \alpha_i\right) \max_{i=1}^m \{ f(x_i) \} \\ &= \max_{i=1}^m \{ f(x_i) \} \end{aligned}$$

Hence Proved

4 To Prove:  $C_1, C_2$  are convex &  $C_1 = C_2$

Proof:

$$C_1 = \left\{ (A, z) : A \in S^n, z \in \mathbb{R}, \begin{pmatrix} A & b \\ b^T & z \end{pmatrix} \succeq 0 \right\}$$

let us take  $(A_1, z_1) \in C$

$(A_2, z_2) \in C$

$$\therefore x^T \begin{pmatrix} A_1 & b \\ b^T & z_1 \end{pmatrix} x \geq 0 \quad \forall \|x\| = 1$$

$$\text{similarly } x^T \begin{pmatrix} A_2 & b \\ b^T & z_2 \end{pmatrix} x \geq 0$$

$$\therefore x^T \left( \theta \begin{pmatrix} A_1 & b \\ b^T & z_1 \end{pmatrix} + (1-\theta) \begin{pmatrix} A_2 & b \\ b^T & z_2 \end{pmatrix} \right) x \geq 0$$

$$\therefore \theta \begin{pmatrix} A_1 & b \\ b^T & z_1 \end{pmatrix} + (1-\theta) \begin{pmatrix} A_2 & b \\ b^T & z_2 \end{pmatrix} \succeq 0$$



4. To Prove:  $C_1, C_2$  are convex &  $C_1 = C_2$

Proof:

$$C_1 = \left\{ (A, z) : A \in S^n, z \in \mathbb{R}, \begin{pmatrix} A & \underline{b} \\ \underline{b}^T & z \end{pmatrix} \succeq 0 \right\}$$

let us take  $(A_1, z_1) \in C_1$

$(A_2, z_2) \in C_2$

$$\begin{pmatrix} A_1 & \underline{b} \\ \underline{b}^T & z_1 \end{pmatrix} \succeq 0 \quad \& \quad \begin{pmatrix} A_2 & \underline{b} \\ \underline{b}^T & z_2 \end{pmatrix} \succeq 0$$

$$\theta \begin{pmatrix} A_1 & \underline{b} \\ \underline{b}^T & z_1 \end{pmatrix} + (1-\theta) \begin{pmatrix} A_2 & \underline{b} \\ \underline{b}^T & z_2 \end{pmatrix} \succeq 0$$

$$\therefore \begin{pmatrix} \theta A_1 + (1-\theta)A_2 & \underline{b} \\ \underline{b}^T & \theta z_1 + (1-\theta)z_2 \end{pmatrix} \succeq 0$$

$$\therefore (\theta A_1 + (1-\theta)A_2, \theta z_1 + (1-\theta)z_2) \in C$$

$\therefore C_1$  is convex

$$\begin{pmatrix} A_1 & \underline{b} \\ \underline{b}^T & z_1 \end{pmatrix} \succeq 0 \quad \therefore A_1 \succeq 0 \quad \& \quad z_1 - \underline{b}^T A_1^{-1} \underline{b} > 0$$

Schur's complement of  $A_1$  in  $M$

$$M / A_1 > 0$$

$$\therefore z > \underline{b}^T A^{-1} \underline{b}$$

$C_1$  &  $C_2$  are the same set

$$\therefore C_1 = C_2$$

$\therefore C_1$  is convex

$M$  is the ~~semidefinite~~ if  $A > 0$

$\therefore (A, z) \in C_1$  &  $(A, z) \in C_2 \quad \forall (A, z)$

$\therefore C_1 = C_2$

$\therefore C_1$  is convex  $\therefore C_2$  is convex

5(c) Given:  $f(x)$  is symmetric convex

To prove:  $f(x)$  with both the co-ordinates equal

Proof: let us take  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  be the optimal

value of  $x \in C$

we know that  $f(x_1, x_2) = f(x_2, x_1)$

$\therefore \underline{y} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$  is also optimal

let us take a point on the line joining  $x$  &  $y$

$$\therefore \underline{p} = \theta x + (1-\theta) \underline{y}$$

$$\begin{aligned} f(\theta x + (1-\theta) \underline{y}) &\leq \theta f(x) + (1-\theta) f(\underline{y}) \\ &= \theta f(x) + (1-\theta) f(x) \\ &= f(x) \end{aligned}$$

$\therefore$  for every convex optimization every local minima is a global minima

$$\therefore f(p) = f(x)$$

let us put  $\theta = 1/2$

$$f\left(\frac{x}{2} + \frac{\underline{y}}{2}\right) = f(x)$$

$$= f\left(\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right)\right) = f(x)$$

∴ We proved that a point equal to ordinates

(b) To find:  $\max_x x_1 x_2 \dots x_n$

$$\text{st } \sum_{i=1}^n x_i = 1, x_i \geq 0 \quad \forall i = 1, \dots, n$$

Let us apply AM-GM as  $x_i \in \mathbb{R}$  &  $x_i \geq 0$

$$\therefore \frac{\sum x_i}{n} \geq (\prod x_i)^{1/n}$$

$$\Rightarrow \frac{1}{n} \geq (\prod x_i)^{1/n}$$

$$\Rightarrow \left(\frac{1}{n}\right)^n = \prod x_i$$

∴ the maximum achievable value

$$\text{is } \left(\frac{1}{n}\right)^n$$

6A) To prove:  $f(x) = \frac{x^T x}{t}$  is convex

Proof:  $\because$  if  $g(x)$  is convex  
then  $f(x, t) = t g\left(\frac{x}{t}\right)$  is also convex  
 $\forall t > 0$

$\because$  we proved before that  $\|x\|^2$  is convex

$$\begin{aligned} f(x, t) &= t g\left(\frac{x}{t}\right) \\ &= t \frac{x^T x}{t^2} \quad (g(x) = \|x\|^2) \\ &= \frac{x^T x}{t} \end{aligned}$$

$\therefore f(x)$  is also convex

B) To prove:  $f(x) = \frac{x^T x}{x^2}$  is quasi convex

Proof: For quasi convex all sublevels are convex

$$\begin{aligned} C &= \{x : f(x) \leq \alpha\} = \{x : \frac{x^T x}{x^2} \leq \alpha\} \\ &= \{x : 0 \leq \|x\| \leq \sqrt{\alpha} x^2\} \end{aligned}$$

Let  $x_1, x_2 \in C$   $\therefore \| \theta x_1 + (1-\theta)x_2 \| \leq \theta \|x_1\| + (1-\theta) \|x_2\|$

$$\begin{aligned} \therefore f(x) &\text{ is quasi convex} \\ &\leq \theta \sqrt{\alpha} + (1-\theta) \sqrt{\alpha} \\ &\leq \sqrt{\alpha} \end{aligned}$$

hence proved

(ball of radius  $\sqrt{\alpha}$ )

C) To prove:  $f(x) = \left\| \frac{x - \frac{x}{\|x\|}}{\|x\|} \right\|_2$  if  $\|x\| \geq 1$

Proof:  $f(x) = \|x\| \left( \frac{\|x\| - 1}{\|x\|} \right) = \|x\| - 1$

$$\therefore f(x) = \begin{cases} \|x\| - 1 & \|x\| \geq 1 \\ 0 & \text{else} \end{cases}$$



let us take  $x_1, x_2$  st  $\|x_1\| \geq 1$  &  $\|x_2\| \geq 1$

$$f(\theta x_1 + (1-\theta)x_2) = \|\theta x_1 + (1-\theta)x_2\| - 1 \quad \text{--- (1)}$$

$$\theta f(x_1) + (1-\theta)f(x_2) = \theta \|x_1\| - 1 + (1-\theta)\|x_2\| - 1 + \theta$$

$$= \theta \|x_1\| + (1-\theta)\|x_2\| - 1 \quad \text{--- (2)}$$

$\therefore \text{--- (1) < (2)}$  hence  $f(x) = \|x\| - 1$

$f(x) = 0$  is also convex when  $\|x\| < 1$  is convex  
 $\therefore$  max of 2 convex  $f$  is also convex

$$f(x) = \begin{cases} \|x\| - 1 & \text{if } \|x\| \geq 1 \\ 0 & \text{else} \end{cases}$$

is also convex

as  $f(x)$  can be written as  
 $\max\{\|x\| - 1, 0\}$