- Vectors in underline, matrices in capital. Assume all matrices have real entries unless stated otherwise.
- For a vector $\underline{x}$, we denote by x_i the i^{th} entry of $\underline{x}$.
- The inequality $A \geq 0$ means that the matrix A is positive semi definite. The inequality $\underline{x} \geq 0$ means that all the entries of the vector $\underline{x}$ are non negative.

1. *(5 pts)* Given an (arbitrary, non empty) set $S \subset \mathbb{R}^n$, define the set of all vectors that make an acute angle with every point in S
$$C = \{\underline{x} : \underline{x}^\mathsf{T}\underline{y} \geq 0 \quad \text{for every } \underline{y} \in S\}.$$
Identify whether C is
A. a subspace B. an affine set C. a convex set D. a cone .
Do the answers to any of the above depend on the specific structure of S ?

2. *(4 pts)* Suppose you are given $n \times n$ matrix A and a vector $\underline{y} \in \mathbb{R}^n$. Note that A may have both positive and negative eigen values. Is the function $f_1(\underline{x}) = \left\|\underline{y} - A\underline{x}\right\|_2$ convex ? Answer the same for the function $f_2(\underline{x}) = \left\|\underline{y} - A\underline{x}\right\|_2^2$.

3. *(4 pts)* Given a finite set of points $S = \{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n\}$, consider the convex hull $\text{conv}(S)$ of S.
   (a) For $S = \{(0,0), (1,1), (1,0)\}$ sketch the convex hull $\text{conv}(S)$.

   Given a **convex** function $f : \mathbb{R}^n \mapsto \mathbb{R}$, we are interested in maximising $f(\underline{x})$ with the variable $\underline{x}$ constrained to be in $\text{conv}(S)$.

   (b) Prove that
$$f(x) \leq \max\left\{f(\underline{x}_1), f(\underline{x}_2), \ldots, f(\underline{x}_m)\right\}, \text{for } \underline{x} \in \text{conv}(S).$$
   This means that the maximum value of f over $\text{conv}(S)$ occours at one of the *vertices* of $\text{conv}(S)$.

4. *(3 pts)* You are given a vector $\underline{b} \in \mathbb{R}^n$. Consider the following sets consisting of the pair (A, z) for $n \times n$ symmetric matrices A and scalars z
$$C_1 = \left\{(A, z) : A \in \mathbb{S}^n, z \in \mathbb{R}, \begin{pmatrix} A & \underline{b} \\ \underline{b}^\mathsf{T} & z \end{pmatrix} \geq 0\right\}, \quad C_2 = \left\{(A, z) : A \in \mathbb{S}^n_{++}, z \in \mathbb{R}, z \geq \underline{b}^\mathsf{T} A^{-1}\underline{b}\right\}.$$
Are C_1 and C_2 convex ? (*Hint:* Argue that $C_1 = C_2$)

5. *(5 pts)* (a) Suppose $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is a convex symmetric function, i.e. f is convex and $f(x_1, x_2) = f(x_2, x_1)$ for all x_1, x_2. Consider the problem $\min_{\underline{x} \in C} f(\underline{x})$ where C is a symmetric convex set. Argue why there exists an optimum $\underline{x}$ with both the co-ordinates equal.
   (b) Find the optimal value of the following problem
$$\begin{aligned} \max_{\underline{x}} \quad & x_1 x_2 x_3 \ldots x_n \\ \text{s.t.} \quad & \sum x_i = 1, \quad x_i \geq 0 \text{ for } i = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

6. *(6 pts)* Show the following
   (a) The function $f(\underline{x}) = \underline{x}^\mathsf{T}\underline{x}/t$, with $\text{dom} f = \{(\underline{x}, t) | t > 0\}$ is convex.
   (b) The function $f(\underline{x}) = \underline{x}^\mathsf{T}\underline{x}/t^2$, with $\text{dom} f = \{(\underline{x}, t) | t > 0\}$ is quasi-convex.
   (c) The function f defined below is convex.
$$f(\underline{x}) = \begin{cases} \left\|\underline{x} - \frac{\underline{x}}{\|\underline{x}\|_2}\right\|_2 & \text{if } \|\underline{x}\|_2 \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$