Assignment 8

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Download all latex-tikz codes from

https://github.com/tanmaygoyal258/AI1103---Probability/blob/main/Assignment8/main.tex

1 Problem

A fair coin is tossed repeatedly. Let X be the number of tails before the first heads occurs. Let Y denote the number of tails between the first and second heads. Let $X + Y = N$. Then which of the following are true?

1) X and Y are independent random variables with
$$\Pr(X = k) = \Pr(Y = k) = \begin{cases} 2^{-(k+1)} & k = 0, 1, 2 \dots \\ 0 & otherwise \end{cases} \tag{1.0.1}$$
2) N has a probability mass function given by
$$\Pr(N = k) = \begin{cases} (k-1)2^{-k} & k = 2, 3, 4 \dots \\ 0 & otherwise \end{cases} \tag{1.0.2}$$
3) Given $N = n$, the conditional distribution of X and Y are independent
4) Given $N = n$
$$\Pr(X = k) = \begin{cases} \frac{1}{n+1} & n = 0, 1, 2 \dots \\ 0 & otherwise \end{cases} \tag{1.0.3}$$

2 Solution

Let us consider Tails as a failure and Heads as a success, then X and Y, both, can be seen to be geometric random variables, with $p = \frac{1}{2}$ and:

$$\Pr(X = k) = \Pr(Y = k) = (1-p)^k p \tag{2.0.1}$$

Substituting the value of p in (2.0.1):

$$\Pr(X = k) = \Pr(Y = k) = \frac{1}{2^{k+1}} = 2^{-(k+1)} \tag{2.0.2}$$

To test for independence of X and Y, we calculate $\Pr(X = k, Y = k)$, which means obtaining k tails, 1 head, k tails, and one head, in order. Thus,

$$\Pr(X = k, Y = k) = (1-p)^k p \times (1-p)^k p \tag{2.0.3}$$
$$= \Pr(X = k)\Pr(Y = k) \tag{2.0.4}$$

Thus, X and Y are independent, and hence, **Option 1 is correct**
From (2.0.1) and (2.0.4), we get:

$$\Pr(N = k) = \Pr(X + Y = k) \tag{2.0.5}$$
$$= \sum_{i=0}^{k} \Pr(X = i, Y = k - i) \tag{2.0.6}$$
$$= \sum_{i=0}^{k} \Pr(X = i)\Pr(Y = k - i) \tag{2.0.7}$$
$$= \sum_{i=0}^{k} (1-p)^i p (1-p)^{k-i} p \tag{2.0.8}$$
$$= p^2 (1-p)^k (k+1) \tag{2.0.9}$$
$$= (k+1)2^{-(k+2)} \tag{2.0.10}$$

Hence, **Option 2 is incorrect**
We know, if a conditional distribution is independent, then:

$$\Pr(X = x, Y = y | Y = y) = \Pr(X = x) \tag{2.0.11}$$

Thus, the conditional distribution of X given $N = n$:

$$\Pr(X = k | N = n) = \frac{\Pr(X = k, X + Y = n)}{\Pr(N = n)} \tag{2.0.12}$$
$$= \frac{\Pr(X = k, Y = n - k)}{\Pr(N = n)} \tag{2.0.13}$$
$$= \frac{2^{-(k+1)}2^{-(n-k+1)}}{(n+1)2^{-(n+2)}} \tag{2.0.14}$$
$$= \frac{1}{n+1} \tag{2.0.15}$$
$$\neq \Pr(X = k) \tag{2.0.16}$$

Similarly,

$$\Pr(Y = k | N = n) = \frac{1}{n+1} \tag{2.0.17}$$
$$\neq \Pr(Y = k) \tag{2.0.18}$$

Thus, the condition for independence fails and **Option 3 is incorrect**, however, **Option 4 is correct**

The correct options are **(1)** and **(4)**