GATE EC 2010- Q.42

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Question

The transfer function for a discrete time LTI system is given by:

$$H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \tag{1}$$

Consider the following statements:
S1: The system is stable and causal for ROC: $|z| > \frac{1}{2}$
S2: The system is stable but not causal for ROC: $|z| < \frac{1}{4}$
S3: The system is neither stable nor causal for ROC: $\frac{1}{4} < |z| < \frac{1}{2}$
Which one of the following statement are valid?

1. Both S1 and S2 are true
2. Both S2 and S3 are true
3. Both S1 and S3 are true
4. S1, S2 and S3 are all true

Stable and Causal Systems

Definition

We say that a system is **stable** if it produces a bounded output for every possible bounded input, i.e it satisfies the BIBO(Bounded-input-Bounded-output) condition.

Definition

We say that a system is **Causal** if the output of a system at a given time instance is independent of the future input values, i.e the output at a particular instance only depends on the present and past input values.

Lemma

Lemma

A system is said to be BIBO stable if and only if the ROC consists of the unit circle in the Z plane.

Lemma

Lemma

A system is causal if and only if the transfer function $h[n]$ *satisfies* $h[n] = 0, n < 0$

Proof.

Let the input signal be given by $x[n]$ and the output signal be given by $y[n]$, then, we know in an LTI system:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{2}$$

Since, $y[n]$ is causal, it should be independent of future values of n. If $k < 0$, then $n - k > n$, which is undesirable, and thus, to keep $y[n]$ independent of future values, $h[k] = 0, k < 0$ □

Another Lemma on Causality of System

Lemma

A system is said to be causal if and only if the ROC of the impulse function lies outside the outermost pole.

Z-transform Lemma

Lemma

If $x[n] = a^n u[n]$, *where*

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & otherwise \end{cases} \tag{3}$$

then $x[n] \overset{\mathcal{Z}}{\rightleftharpoons} X[z] = \frac{1}{1-az^{-1}}$ *with* $ROC = |z| > a$

Z-transform Lemma proof

Proof.

Using the formula for the sum of an infinite GP, we get:

$$x[n] = \begin{cases} a^n & n \geq 0 \\ 0 & otherwise \end{cases} \tag{4}$$

$$\mathcal{Z}\{x[n]\} = X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{5}$$

$$= \sum_{n=-\infty}^{0} 0 \times z^{-n} + \sum_{n=0}^{\infty} (az^{-1})^n \tag{6}$$

$$= \frac{1}{1 - az^{-1}}, ROC = \left|az^{-1}\right| < 1 \tag{7}$$

$$= \frac{1}{1 - az^{-1}}, ROC = |z| > a \tag{8}$$

□

Z-transform Lemma when ROC condition is violated

Lemma

If

$$X[z] = \frac{1}{1 - az^{-1}} \tag{9}$$

and the region of convergence is

$$Z \setminus (ROC \cup |a|) \tag{10}$$

where Z is the entire Z plane and ROC is the region of convergence mentioned in (6), *then*

$$x[n] = -a^n u[-n - 1] \tag{11}$$

Proof

Proof.

ROC $= Z \setminus (ROC \cup |a|) \implies |z| < |a|$. From (6), we see that we cannot apply the formula for the sum of an infinite GP directly as the conditions are not satisfied. Thus, we manipulate the function.

$$|z| < |a| \implies \left|\frac{z}{a}\right| < 1 \tag{12}$$

$$\frac{1}{1 - az^{-1}} = \frac{-z}{a}\frac{1}{1 - \frac{z}{a}}, \left|\frac{z}{a}\right| < 1 \tag{13}$$

$$= \sum_{n=0}^{\infty} \frac{-z}{a}\left(\frac{z}{a}\right)^n \tag{14}$$

$$= -\sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^{n+1} \tag{15}$$

$$= -\sum_{n=-\infty}^{\infty} \left(\frac{z}{a}\right)^{n+1} u[n] \tag{16}$$

Proof continued

Proof.

$$= -\sum_{n=-\infty}^{\infty} a^{-n-1} z^{n+1} u[n] \tag{18}$$

$$= -\sum_{k=-\infty}^{\infty} a^{k} z^{-k} u[-k-1] \tag{19}$$

by substituting $n+1 = -k$
Finally, on comparing with the general z-transform formula of $x[n] \overset{\mathcal{Z}}{\rightleftharpoons} X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$, we get:

$$x[n] = -a^{n} u[-n-1] \tag{20}$$

$\square$

Decomposition of Transform Function

We are given the transfer function:

$$H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \tag{21}$$

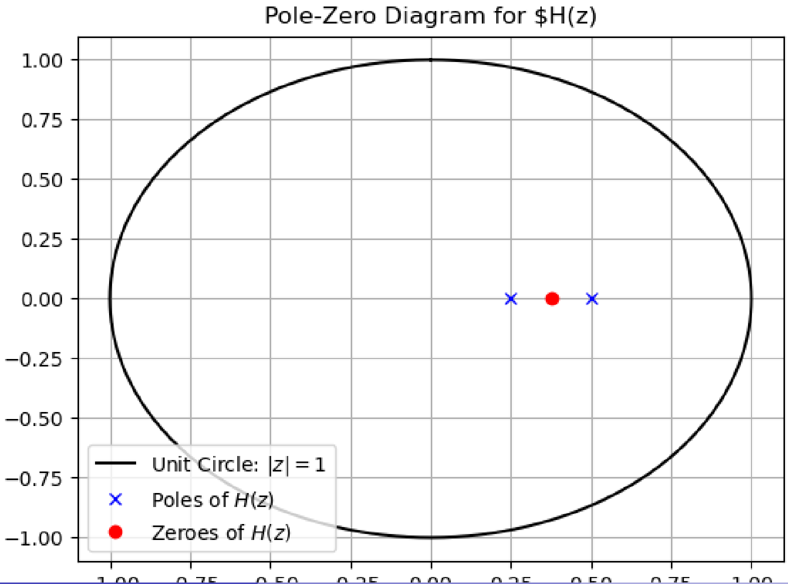
To find the inverse Z-Transform, we would decompose the function using partial fractions:

$$\begin{aligned} H(z) &= \frac{16 - 6z^{-1}}{8 - 6z^{-1} + z^{-2}} && (22) \\ &= \frac{16 - 6z^{-1}}{(4 - z^{-1})(2 - z^{-1})} && (23) \\ &= \frac{4}{4 - z^{-1}} + \frac{2}{2 - z^{-1}} && (24) \\ &= \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}} && (25) \end{aligned}$$

Pole Zero plot

From the above decomposition, we find that the poles of $H(z)$ are $z = \frac{1}{4}, \frac{1}{2}$, and the zeroes are $z = \frac{3}{8}$, as shown in the plot-zero diagram given below. Thus, we can also say the outermost pole is $z = \frac{1}{2}$, and thus, from (5), the system is Causal iff $|z| > \frac{1}{2}$

Pole Zero plot



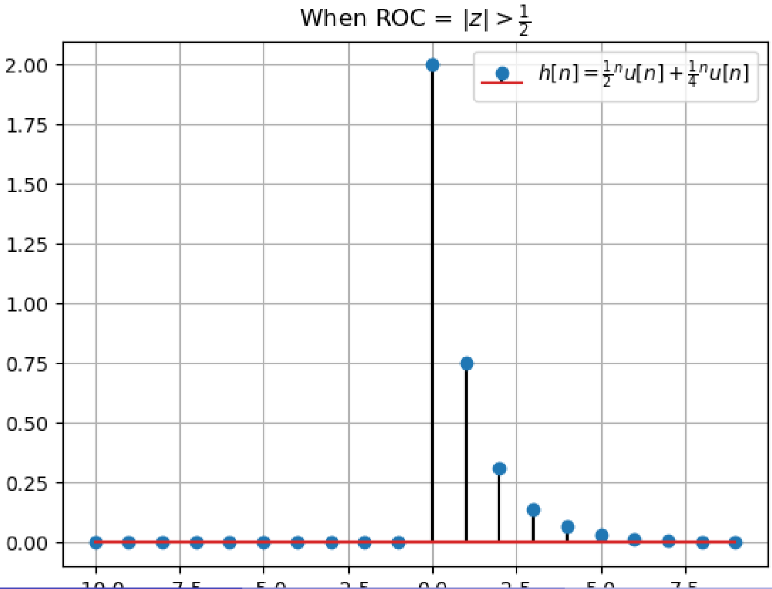
When ROC: $|z| > \frac{1}{2}$

If ROC $= |z| > \frac{1}{2}$, this automatically implies $|z| > \frac{1}{4}$, thus, from (6), we can say:

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n] \tag{26}$$

Since ROC $= |z| > \frac{1}{2}$ includes the unit circle, the system is stable.
Moreover, we see $h[n] = 0$ for $n < 0$, since $u[n] = 0$ for $n < 0$. Thus, the system is Causal as well.
Hence, S1 is correct

When ROC: $|z| > \frac{1}{2}$



When ROC $= \frac{1}{4} < |z| < \frac{1}{2}$

Since the unit circle is not included in the ROC, the system cannot be stable. Moreover, the ROC condition for only one of the two fractions in (25) is satisfied, i.e

$$\left(\frac{1}{4}\right)^n u[n] \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-\frac{1}{4}z^{-1}}, |z| > \frac{1}{4} \tag{27}$$

Since, the ROC condition is not satisfied for the other term, from (7), we get:

$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-\frac{1}{2}z^{-1}}, |z| < \frac{1}{2} \tag{28}$$

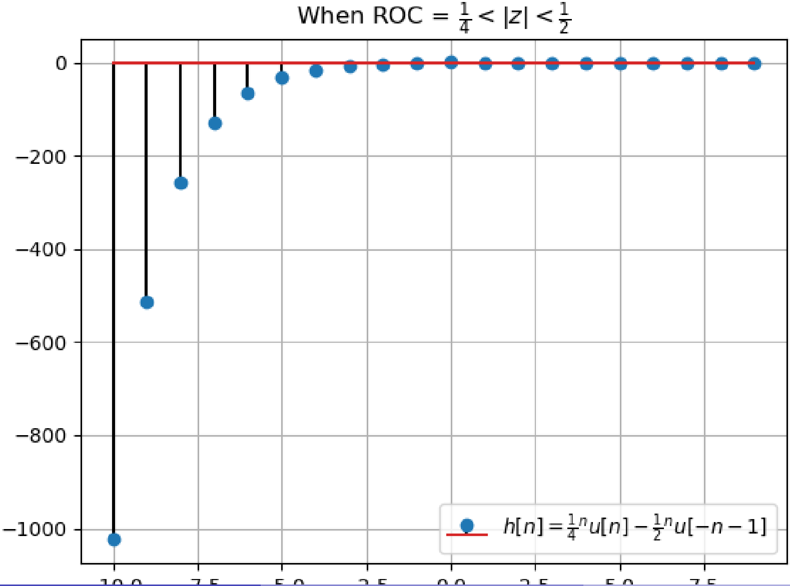
When ROC $= \frac{1}{4} < |z| < \frac{1}{2}$ continued

Thus, for the ROC $= \frac{1}{4} < |z| < \frac{1}{2}$, we get:

$$h[n] = \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1] \tag{29}$$

Clearly, for $n < -1$, $-n-1 > 0$ and hence, $h[n] \neq 0$ for $n < 0$, and thus, the system is non-causal.
S3 is also correct.

When ROC $= \frac{1}{4} < |z| < \frac{1}{2}$



When ROC $= |z| < \frac{1}{4}$

this automatically implies $|z| < \frac{1}{2}$. Since the unit circle is not included, the system is unstable. Moreover, from (6), since both the ROC conditions are violated, from (7), we get:

$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-\frac{1}{2}z^{-1}}, |z| < \frac{1}{2} \tag{30}$$

$$-\left(\frac{1}{4}\right)^n u[-n-1] \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-\frac{1}{4}z^{-1}}, |z| < \frac{1}{4} \tag{31}$$

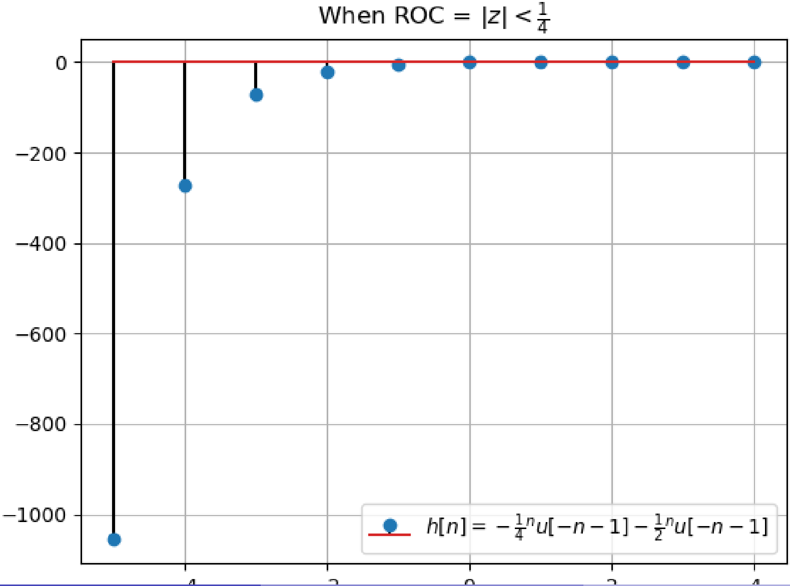
Thus, we get:

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \left(\frac{1}{4}\right)^n u[-n-1] \tag{32}$$

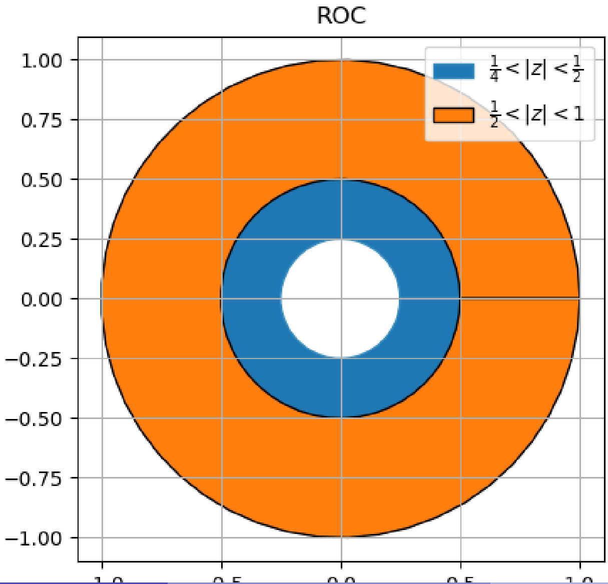
$$h[n] = -u[-n-1]\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] \tag{33}$$

Clearly, for $n < -1$, $-n-1 > 0$, and thus, $h[n] \neq 0, n < 0$. The system is non-causal. **Hence, S2 is incorrect**

When ROC $= |z| < \frac{1}{4}$



ROC


ROC
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
$\frac{1}{4} < |z| < \frac{1}{2}$
$\frac{1}{2} < |z| < 1$