Gate Assignment 1

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Download all python codes from

https://github.com/tanmaygoyal258/EE3900−Linear−Systems−and−Signal−processing/blob/main/GateAssignment1/code.py

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https://github.com/tanmaygoyal258/EE3900−Linear−Systems−and−Signal−processing/blob/main/GateAssignment1/main.tex

1 Problem

(EC 2017- Q.7) The input $x(t)$ and output $y(t)$ of a continous time signal are related as:

$$y(t) = \int_{t-T}^{t} x(u)\,du \tag{1.0.1}$$

The system is:

1) Linear and Time-variant
2) Linear and Time-invariant
3) Non-Linear and Time-variant
4) Non-Linear and Time-invariant

2 Solution

Definition 1. *We say that a system is* ***linear*** *if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.*

Definition 2. *A system is said to be* ***time invariant*** *if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.*

Lemma 2.1. *The system relating the input signal $x(t)$ and output signal $y(t)$, given by*

$$y(t) = \int_{t-T}^{t} x(u)\,du \tag{2.0.1}$$

is linear and time invariant in nature.

Proof. 1) **Linearity and Time invariance**
From (1), we can say the system is linear if it follows both the laws of Additivity and Homogeneity.

Law of Additivity:
Let the two input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{t-T}^{t} x_1(u)\,du \tag{2.0.2}$$

$$y_2(t) = \int_{t-T}^{t} x_2(u)\,du \tag{2.0.3}$$

$$y_1(t) + y_2(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\,du \tag{2.0.4}$$

Now, consider the input signal of $x_1(t) + x_2(t)$, then the corresponding output signal is given by $y'(t)$:

$$y'(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\,du \tag{2.0.5}$$

Clearly, from (2.0.4) and (2.0.5):

$$y'(t) = y_1(t) + y_2(t) \tag{2.0.6}$$

Thus, the Law of Additivity holds.

Law of Homogeneity:
Consider an input signal $kx(t)$, where k is any constant. Let the corresponding output be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^{t} kx(u)\,du \tag{2.0.7}$$

$$= k\int_{t-T}^{t} x(u)\,du \tag{2.0.8}$$

$$= ky(t) \tag{2.0.9}$$

Clearly, from (2.0.9),

$$y'(t) = ky(t) \tag{2.0.10}$$

Thus, the Law of Homogeneity holds.

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

From (2) , to check for time-invariance, we would introduce a delay of t_0 in the output and input signals.
Delay in output signal:

$$y(t-t_0) = \int_{t-t_0-T}^{t-t_0} x(u)\,du \tag{2.0.11}$$

Now, we consider an input signal with a delay of t_0, given by $x(t-t_0)$, and let the corresponding output signal be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^{t} x(u-t_0)\,du \tag{2.0.12}$$

Substituting $a = u - t_0$:

$$y'(t) = \int_{t-t_0-T}^{t-t_0} x(a)\,da \tag{2.0.13}$$

Clearly, from (2.0.11) and (2.0.13):

$$y'(t) = y(t-t_0) \tag{2.0.14}$$

Thus, the system is **time-invariant**.
The correct option is **2) Linear and Time-invariant**

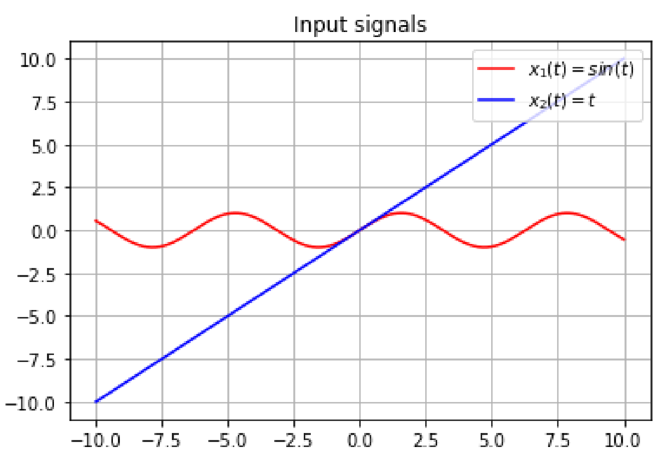

Fig. 1: $x_1(t) = \sin t$ and $x_2(t) = t$

2) **Calculating impulse response of LTI system**
Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given by $\delta(t)$. Thus,

$$h(t) = \int_{t-T}^{t} \delta(u)du \tag{2.0.15}$$

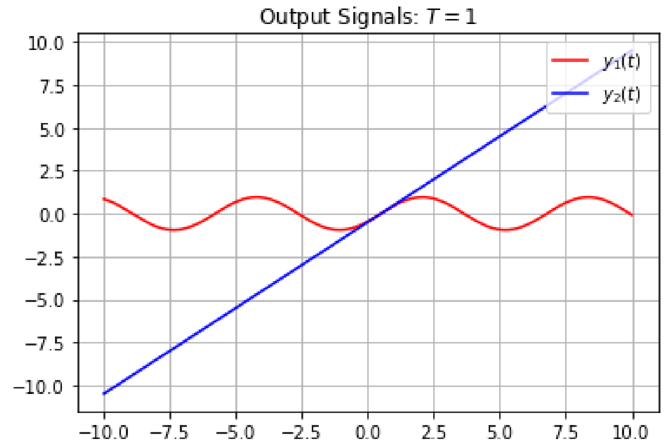

Fig. 1: $y_1(t)$ and $y_2(t)$

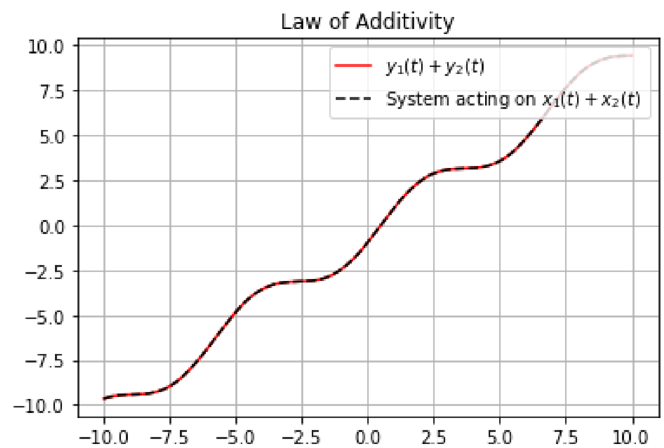

Fig. 1: Law of Additivity

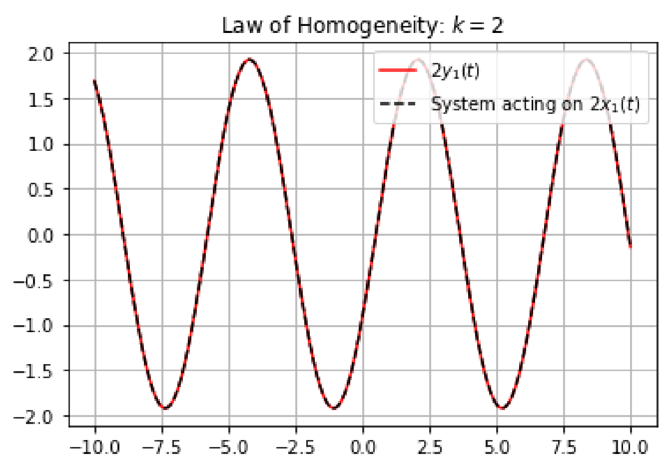

Fig. 1: Law of Homogeneity

The Impulse function can be loosely defined

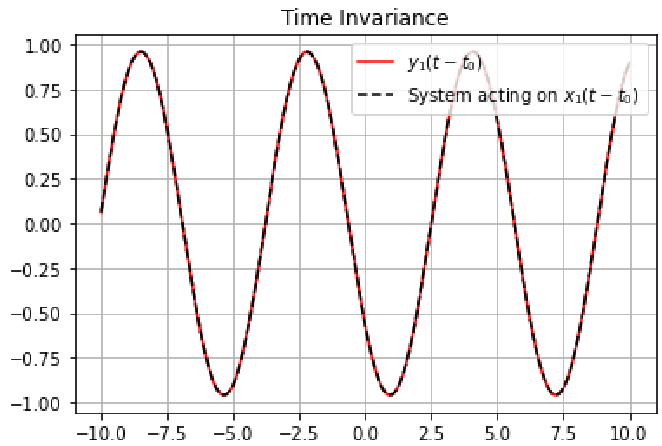

Fig. 1: Time invariance

as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & otherwise \end{cases} and \int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{2.0.16}$$

Since the Impulse function is zero everywhere aside from $t = 0$, the non-zero value of integration is a result of $\delta(0)$. Thus, we can say $h(t)$ will be non-zero only if the limits of integration would include $t = 0$, i.e:

$$h(t) = \begin{cases} \int_{t-T}^{t} \delta(u)du & t - T < 0; t > 0 \\ 0 & otherwise \end{cases} \tag{2.0.17}$$

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & otherwise \end{cases} \tag{2.0.18}$$

3) **Expressing the impulse function in terms of $u(t)$**
The unit step signal, $u(t)$, is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & otherwise \end{cases} \tag{2.0.19}$$

On time-shifting $u(t)$ by T, we get:

$$u(t-T) = \begin{cases} 1 & t - T \geq 0 \\ 0 & otherwise \end{cases} = \begin{cases} 1 & t \geq T \\ 0 & otherwise \end{cases} \tag{2.0.20}$$

On subtracting (2.0.19) and (2.0.20), we get our impulse response $h(t)$ in terms of the unit step signal:

$$h(t) = u(t) - u(t-T) \tag{2.0.21}$$

4) **Expressing the impulse function in terms of $rect(t)$**
The unit rectangular signal, $rect(t)$ is given by:

$$rect(t) = \begin{cases} 1 & \frac{-1}{2} \leq t \leq \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{2.0.22}$$

We can obtain the impulse response $h(t)$ in terms of $rect(t)$ using time scaling and shifting as follows:

$$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \frac{-1}{2} \leq \frac{t}{\tau} \leq \frac{1}{2} \\ 0 & otherwise \end{cases} = \begin{cases} 1 & \frac{-\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & otherwise \end{cases} \tag{2.0.23}$$

Subsituting $\tau = T$:

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1 & \frac{-T}{2} \leq t \leq \frac{T}{2} \\ 0 & otherwise \end{cases} \tag{2.0.24}$$

Now, we want to right-shift the signal by $\frac{T}{2}$:

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & otherwise \end{cases} = h(t) \tag{2.0.25}$$

Since the time shifting is to be performed on the variable t and not $\frac{t}{T}$

5) **Calculating the Fourier Transform of $h(t)$**
Let the Fourier Transform of $h(t)$ be given by $H(f)$ and of the rectangular signal, $rect(t)$ be given by $Y(f)$.

$$h(t) \overset{\mathcal{F}}{\rightleftharpoons} H(f) \tag{2.0.26}$$

$$rect(t) \overset{\mathcal{F}}{\rightleftharpoons} Y(f) \tag{2.0.27}$$

Then,

$$Y(f) = \int_{-\infty}^{\infty} rect(t)e^{-j2\pi ft}\, dt \tag{2.0.28}$$

From (2.0.22), we can write (2.0.28) as:

$$Y(f) = \int_{-\infty}^{\frac{-1}{2}} 0\,dt + \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-j2\pi f t}\,dt + \int_{\frac{1}{2}}^{\infty} 0\,dt \tag{2.0.29}$$

$$= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} \tag{2.0.30}$$

$$= \frac{2j\sin \pi f}{j2\pi f} \tag{2.0.31}$$

$$= \frac{\sin(\pi f)}{\pi f} \tag{2.0.32}$$

$$= sinc(f) \tag{2.0.33}$$

where $sinc(t)$, the sampling function is defined as:

$$sinc(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & otherwise \end{cases} \tag{2.0.34}$$

Let the Fourier Transform of a signal $x(t)$ be $X(f)$.

$$x(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) \tag{2.0.35}$$

When the signal $x(t)$ is time shifted by t_0, the resultant Fourier Transform is given by:

$$x(t \pm t_0) \overset{\mathcal{F}}{\rightleftharpoons} X(f)e^{\pm j2\pi f t_0} \tag{2.0.36}$$

And when the signal $x(t)$ is time scaled by α, the resulting Fourier Transform is given by:

$$x(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right) \tag{2.0.37}$$

Since we have already derived the Fourier Transform of $rect(t)$, we would use the properties mentioned above to find the Fourier Transform of $h(t)$:

$$rect(t) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f) \tag{2.0.38}$$

Using (2.0.36):

$$rect\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f)e^{-j(2\pi f)\frac{T}{2}} \tag{2.0.39}$$

$$rect\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f)e^{-j\pi f T} \tag{2.0.40}$$

Using (2.0.37),

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{\frac{1}{|T|}} sinc\left(\frac{f}{T}\right) e^{\frac{-j\pi f T}{T}} \tag{2.0.41}$$

$$h(t) \overset{\mathcal{F}}{\rightleftharpoons} T\,sinc\left(\frac{f}{T}\right) e^{-j\pi f} \tag{2.0.42}$$

$$\therefore H(f) = T\,sinc\left(\frac{f}{T}\right) e^{-j\pi f} \tag{2.0.43}$$

6) **An example**
Consider an input signal of $x(t) = \cos 2\pi f_0 t$. The Fourier Transform of $x(t)$ is given by:

$$x(t) = \cos 2\pi f_0 t \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \tag{2.0.44}$$

using the fact that

$$\cos 2\pi f_0 t = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \tag{2.0.45}$$

and the Fourier Transform of $e^{\pm j2\pi f_0 t}$ is given by:

$$e^{\pm j2\pi f_0 t} \overset{\mathcal{F}}{\rightleftharpoons} \delta(f \mp f_0) \tag{2.0.46}$$

The output signal will be given by:

$$y(t) = \int_{t-T}^{t} \cos 2\pi f_0 u\,du \tag{2.0.47}$$

$$= \frac{1}{2\pi f_0}\left[\sin 2\pi f_0 t - \sin 2\pi f_0 (t - T)\right] \tag{2.0.48}$$

$$= \frac{\sin \pi f_0 T}{\pi f_0}\left[\cos 2\pi f_0\left(t - \frac{T}{2}\right)\right] \tag{2.0.49}$$

$$= T\,sinc(f_0 T)\cos 2\pi f_0\left(t - \frac{T}{2}\right) \tag{2.0.50}$$

The Fourier transform of $\cos 2\pi f_0\left(t - \frac{T}{2}\right)$ can be obtained using (2.0.37) and (2.0.36) as follows:

$$\cos t = \frac{1}{2}\left[e^{jt} + e^{-jt}\right] \tag{2.0.51}$$

$$\cos t \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{2.0.52}$$

$$\cos\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] e^{j\pi f T} \tag{2.0.53}$$

$$\cos 2\pi f_0\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2\pi f_0}\frac{\delta(\frac{f}{2\pi f_0} - \frac{1}{2\pi}) + \delta(\frac{f}{2\pi f_0} + \frac{1}{2\pi})}{2} e^{j\pi\frac{f}{2\pi f_0}T} \tag{2.0.54}$$

$$\cos 2\pi f_0\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{4\pi f_0}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) e^{j\pi\frac{f}{2f_0}T} \tag{2.0.55}$$

Therefore, the Fourier Transform of the output signal $y(t)$ from (2.0.50) is given by:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{T\,sinc(f_0 T)}{4\pi f_0} e^{j\pi\frac{f}{2f_0}T}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) \tag{2.0.56}$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} k e^{j\pi\frac{f}{2f_0}T}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) \tag{2.0.57}$$

where $k = \frac{T\,sinc(f_0 T)}{4\pi f_0}$. Substituting $2\pi f_0 = 1$ and $T = 1$:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} k e^{j\pi^2 f}\left(\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right) \tag{2.0.58}$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} k e^{j\frac{\pi}{2}}\delta\left(f - \frac{1}{2\pi}\right) + k e^{j\frac{-\pi}{2}}\delta\left(f + \frac{1}{2\pi}\right) \tag{2.0.59}$$

using the multiplication property of the Delta function:

$$x(t)\delta(t - t_1) = x(t_1)\delta(t - t_1) \tag{2.0.60}$$

Since , $e^{j\frac{\pi}{2}} = j$ and $e^{-j\frac{\pi}{2}} = -j$, we finally get:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} kj\left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{2.0.61}$$

Clearly, the Fourier transform of $y(t)$ can be manipulated to represent a sinusoidal wave, which is given by:

$$sin(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{-j}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{2.0.62}$$

The attenuation happens for the same values of f, as depicted in the graphs of the Fourier Transforms given below.

□

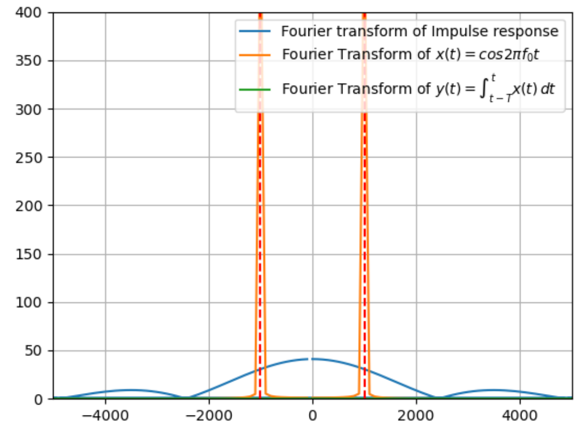

Fig. 6: Fourier Transform of Impulse response $h(t)$

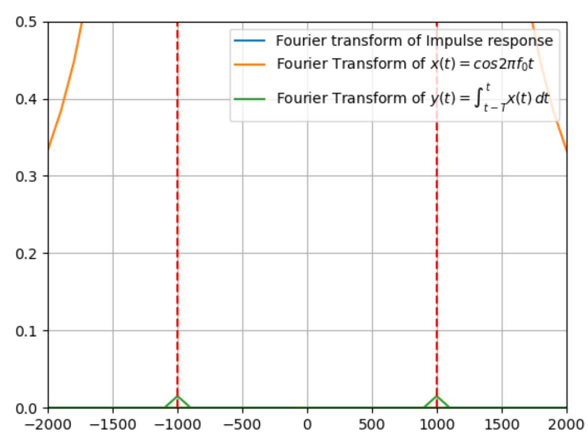

Fig. 6: Fourier Transform of Impulse response $h(t)$ zoomed in