

GATE EC 2017- Q.8

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Question

The input $x(t)$ and output $y(t)$ of a continuous time signal are related as:

$$y(t) = \int_{t-T}^t x(u) du \quad (1)$$

The system is:

- ① Linear and Time-variant
- ② Linear and Time-invariant
- ③ Non-Linear and Time-variant
- ④ Non-Linear and Time-invariant

Linear Systems and Time Invariant Systems

Definition

We say that a system is **linear** if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.

Definition

A system is said to be **time invariant** if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.

Lemma

Lemma

The system relating the input signal $x(t)$ and output signal $y(t)$, given by

$$y(t) = \int_{t-T}^t x(u) du \quad (2)$$

is linear and time invariant in nature.

Proof: Law of Additivity

Let the input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{t-T}^t x_1(u) du \quad (3)$$

$$y_2(t) = \int_{t-T}^t x_2(u) du \quad (4)$$

$$y_1(t) + y_2(t) = \int_{t-T}^t [x_1(u) + x_2(u)] du \quad (5)$$

Proof: Law of Additivity

Now, consider the input signal of $x_1(t) + x_2(t)$, then the corresponding output signal is given by $y'(t)$:

$$y'(t) = \int_{t-T}^t [x_1(u) + x_2(u)] du \quad (6)$$

Clearly, from (5) and (6):

$$y'(t) = y_1(t) + y_2(t) \quad (7)$$

Thus, the Law of Additivity holds.

Proof: Law of Homogeneity

Consider an input signal $kx(t)$, where k is any constant. Let the corresponding output be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^t kx(u) du \quad (8)$$

$$= k \int_{t-T}^t x(u) du \quad (9)$$

$$= ky(t) \quad (10)$$

Clearly, from (10),

$$y'(t) = ky(t) \quad (11)$$

Thus, the Law of Homogeneity holds.

Proof

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

Proof: Time Invariance

To check for time-invariance, we would introduce a delay of t_0 in the output and input signals.

Delay in output signal:

$$y(t - t_0) = \int_{t-t_0-T}^{t-t_0} x(u) du \quad (12)$$

Proof: Time Invariance

Now, we consider an input signal with a delay of t_0 , given by $x(t - t_0)$, and let the corresponding output signal be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^t x(u - t_0) du \quad (13)$$

Substituting $a = u - t_0$:

$$y'(t) = \int_{t-t_0-T}^{t-t_0} x(a) da \quad (14)$$

Clearly, from (12) and (14):

$$y'(t) = y(t - t_0) \quad (15)$$

Thus, the system is **time-invariant**.

Thus, **2) Linear and Time- invariant** is the correct answer.

Impulse response

Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given by $\delta(t)$. Thus,

$$h(t) = \int_{t-T}^t \delta(u) du \quad (16)$$

Impulse function

The Impulse function can be loosely defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (17)$$

Impulse response

Since the Impulse function is zero everywhere aside from $t = 0$, the non-zero value of integration is a result of $\delta(0)$. Thus, we can say $h(t)$ will be non-zero only if the limits of integration would include $t = 0$, i.e:

$$h(t) = \begin{cases} \int_{t-T}^t \delta(u) du & t - T < 0; t > 0 \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \quad (19)$$

Impulse response in terms of unit step signal

The unit step signal, $u(t)$, is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (20)$$

On time-shifting $u(t)$ by T , we get:

$$u(t - T) = \begin{cases} 1 & t - T \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & t \geq T \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

On subtracting (20) and (21), we get our impulse response $h(t)$ in terms of the unit step signal:

$$h(t) = u(t) - u(t - T) \quad (22)$$

Impulse response in terms of the unit rectangular function

The unit rectangular signal, $rect(t)$ is given by:

$$rect(t) = \begin{cases} 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

We can obtain the impulse response $h(t)$ in terms of $rect(t)$ using time scaling and shifting as follows:

$$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1 & -\frac{1}{2} \leq \frac{t}{\tau} \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases} \quad (24)$$

Substituting $\tau = T$:

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1 & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \quad (25)$$

Impulse response in terms of the unit rectangular function

Now, we want to right-shift the signal by $\frac{T}{2}$:

$$\text{rect}\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} = h(t) \quad (26)$$

Since the time shifting is to be performed on the variable t and not $\frac{t}{T}$

Fourier Transform of rectangular function

$$\text{rect}(t) \xrightarrow{\mathcal{F}} Y(f) \quad (27)$$

$$Y(f) = \int_{-\infty}^{\infty} \text{rect}(t) e^{-j2\pi ft} dt \quad (28)$$

From (23), we can write (28) as:

$$Y(f) = \int_{-\infty}^{-\frac{1}{2}} 0 dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi ft} dt + \int_{\frac{1}{2}}^{\infty} 0 dt = \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} \quad (29)$$

$$= \frac{\sin(\pi f)}{\pi f} = \text{sinc}(\pi f) \quad (30)$$

where $\text{sinc}(t)$ is defined as:

$$\text{sinc}(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(t)}{t} & \text{otherwise} \end{cases} \quad (31)$$

Properties of Fourier Transform

Let the Fourier Transform of a signal $x(t)$ be $X(f)$.

$$x(t) \xrightarrow{\mathcal{F}} X(f) \quad (32)$$

When the signal $x(t)$ is time shifted by t_0 , the resultant Fourier Transform is given by:

$$x(t \pm t_0) \xrightarrow{\mathcal{F}} X(f)e^{\pm j2\pi ft_0} \quad (33)$$

And when the signal $x(t)$ is time scaled by α , the resulting Fourier Transform is given by:

$$x(\alpha t) \xrightarrow{\mathcal{F}} \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right) \quad (34)$$

Fourier Transform of impulse response

$$\text{rect}(t) \xrightarrow{\mathcal{F}} \text{sinc}(\pi f) \quad (35)$$

Using (33):

$$\text{rect}\left(t - \frac{T}{2}\right) \xrightarrow{\mathcal{F}} \text{sinc}(\pi f) e^{-j(2\pi f)\frac{T}{2}} \quad (36)$$

$$\text{rect}\left(t - \frac{T}{2}\right) \xrightarrow{\mathcal{F}} \text{sinc}(\pi f) e^{-j\pi f T} \quad (37)$$

Using (34),

$$\text{rect}\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) \xrightarrow{\mathcal{F}} \frac{1}{\left|\frac{1}{T}\right|} \text{sinc}\left(\frac{\pi f}{\frac{1}{T}}\right) e^{\frac{-j\pi f T}{T}} \quad (38)$$

$$h(t) \xrightarrow{\mathcal{F}} T \text{sinc}(\pi f T) e^{-j\pi f T} \quad (39)$$

$$\therefore H(f) = T \text{sinc}(\pi f T) e^{-j\pi f T} \quad (40)$$

Cosine input signal

Consider an input signal of $x(t) = \cos 2\pi f_0 t$. The Fourier Transform of $x(t)$ is given by:

$$x(t) = \cos 2\pi f_0 t \xrightarrow{\mathcal{F}} \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)] \quad (41)$$

using the fact that

$$\cos 2\pi f_0 t = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \quad (42)$$

and the Fourier Transform of $e^{\pm j2\pi f_0 t}$ is given by:

$$e^{\pm j2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f \mp f_0) \quad (43)$$

Output Signal

The output signal will be given by:

$$y(t) = \int_{t-T}^t \cos 2\pi f_0 u \, du \quad (44)$$

$$= \frac{1}{2\pi f_0} [\sin 2\pi f_0 t - \sin 2\pi f_0 (t - T)] \quad (45)$$

$$= \frac{\sin \pi f_0 T}{\pi f_0} \left[\cos 2\pi f_0 \left(t - \frac{T}{2} \right) \right] \quad (46)$$

$$= T \operatorname{sinc}(\pi f_0 T) \cos 2\pi f_0 \left(t - \frac{T}{2} \right) \quad (47)$$

Fourier Transform of output signal

The Fourier transform of $\cos 2\pi f_0 \left(t - \frac{T}{2}\right)$ can be obtained using (34) and (33) as follows:

$$\cos t = \frac{1}{2} [e^{jt} + e^{-jt}] \quad (48)$$

$$\cos t \xrightarrow{\mathcal{F}} \frac{1}{2} \left[\delta \left(f - \frac{1}{2\pi} \right) + \delta \left(f + \frac{1}{2\pi} \right) \right] \quad (49)$$

$$\cos \left(t - \frac{T}{2} \right) \xrightarrow{\mathcal{F}} \frac{1}{2} \left[\delta \left(f - \frac{1}{2\pi} \right) + \delta \left(f + \frac{1}{2\pi} \right) \right] e^{j\pi f T} \quad (50)$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2} \right) \xrightarrow{\mathcal{F}} \frac{1}{2\pi f_0} \frac{\delta \left(\frac{f}{2\pi f_0} - \frac{1}{2\pi} \right) + \delta \left(\frac{f}{2\pi f_0} + \frac{1}{2\pi} \right)}{2} e^{j\pi \frac{f}{2\pi f_0} T} \quad (51)$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2} \right) \xrightarrow{\mathcal{F}} \frac{1}{4\pi f_0} \left(\delta \left(\frac{f - f_0}{2\pi f_0} \right) + \delta \left(\frac{f + f_0}{2\pi f_0} \right) \right) e^{j\pi \frac{f}{2f_0} T} \quad (52)$$

Fourier Transform of output signal

The Fourier Transform of the output signal $y(t)$ from (47) is given by:

$$y(t) \xrightarrow{\mathcal{F}} \frac{T \text{sinc}(\pi f_0 T)}{4\pi f_0} e^{j\pi \frac{f}{2f_0} T} \left(\delta \left(\frac{f - f_0}{2\pi f_0} \right) + \delta \left(\frac{f + f_0}{2\pi f_0} \right) \right) \quad (53)$$

$$y(t) \xrightarrow{\mathcal{F}} k e^{j\pi \frac{f}{2f_0} T} \left(\delta \left(\frac{f - f_0}{2\pi f_0} \right) + \delta \left(\frac{f + f_0}{2\pi f_0} \right) \right) \quad (54)$$

where $k = \frac{T \text{sinc}(\pi f_0 T)}{4\pi f_0}$.

Fourier Transform of output signal

Substituting $2\pi f_0 = 1$ and $T = 1$:

$$y(t) \xrightarrow{\mathcal{F}} ke^{j\pi^2 f} \left(\delta \left(f - \frac{1}{2\pi} \right) + \delta \left(f + \frac{1}{2\pi} \right) \right) \quad (55)$$

$$y(t) \xrightarrow{\mathcal{F}} ke^{j\frac{\pi}{2}} \delta \left(f - \frac{1}{2\pi} \right) + ke^{j\frac{-\pi}{2}} \delta \left(f + \frac{1}{2\pi} \right) \quad (56)$$

using the multiplication property of the Delta function:

$$x(t)\delta(t - t_1) = x(t_1)\delta(t - t_1) \quad (57)$$

Since , $e^{j\frac{\pi}{2}} = j$ and $e^{-j\frac{\pi}{2}} = -j$, we finally get:

$$y(t) \xrightarrow{\mathcal{F}} kj \left[\delta \left(f - \frac{1}{2\pi} \right) - \delta \left(f + \frac{1}{2\pi} \right) \right] \quad (58)$$

Output signal

$$y(t) \xrightarrow{\mathcal{F}} kj \left[\delta \left(f - \frac{1}{2\pi} \right) - \delta \left(f + \frac{1}{2\pi} \right) \right] \quad (59)$$

Clearly, the Fourier transform of $y(t)$ is similar to that of a sinusoidal wave, which is given by:

$$\sin(t) \xrightarrow{\mathcal{F}} \frac{-j}{2} \left[\delta \left(f - \frac{1}{2\pi} \right) - \delta \left(f + \frac{1}{2\pi} \right) \right] \quad (60)$$

Graphs: Input and Output signals

Input signals

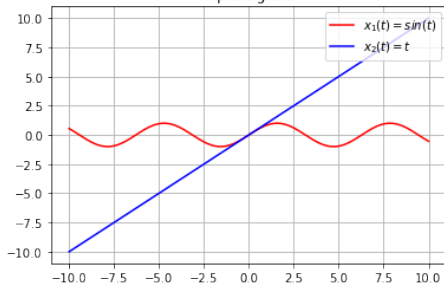


Figure: $x_1(t) = \sin t$ and $x_2(t) = t$

Output Signals: $T = 1$

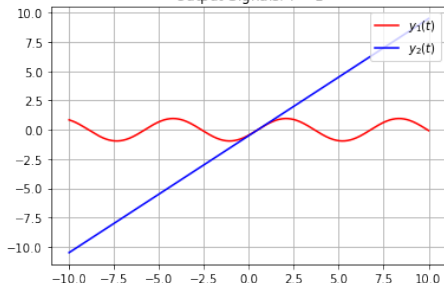


Figure: $y_1(t)$ and $y_2(t)$

Graphs: Laws of Additivity and Homogeneity

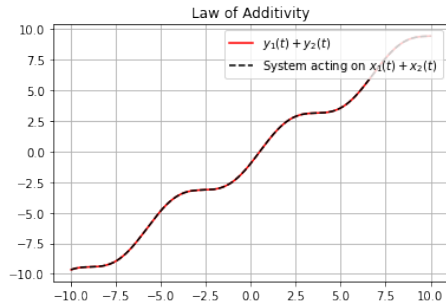


Figure: Law of Additivity

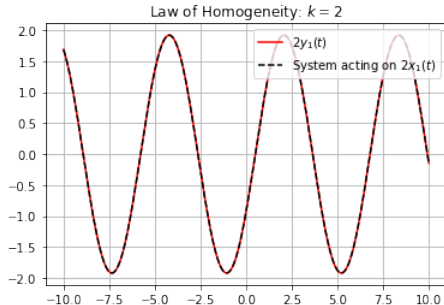


Figure: Law of Homogeneity

Graphs: Time Invariance

