

Gate Assignment 1

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Download all python codes from

<https://github.com/tanmaygoyal258/EE3900-Linear-Systems-and-Signal-processing/blob/main/GateAssignment1/code.py>

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<https://github.com/tanmaygoyal258/EE3900-Linear-Systems-and-Signal-processing/blob/main/GateAssignment1/main.tex>

1 PROBLEM

(EC 2017- Q.7) The input $x(t)$ and output $y(t)$ of a continous time signal are related as:

$$y(t) = \int_{t-T}^t x(u) du \quad (1.0.1)$$

The system is:

- 1) Linear and Time-variant
- 2) Linear and Time-invariant
- 3) Non-Linear and Time-variant
- 4) Non-Linear and Time-invariant

2 SOLUTION

Definition 1. We say that a system is **linear** if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.

Definition 2. A system is said to be **time invariant** if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.

Lemma 2.1. The system relating the input signal $x(t)$ and output signal $y(t)$, given by

$$y(t) = \int_{t-T}^t x(u) du \quad (2.0.1)$$

is linear and time invariant in nature.

Proof. From (1), we can say the system is linear if it follows both the laws of Additivity and Homogeneity.

Law of Additivity:

Let the two input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{t-T}^t x_1(u) du \quad (2.0.2)$$

$$y_2(t) = \int_{t-T}^t x_2(u) du \quad (2.0.3)$$

$$y_1(t) + y_2(t) = \int_{t-T}^t [x_1(u) + x_2(u)] du \quad (2.0.4)$$

Now, consider the input signal of $x_1(t) + x_2(t)$, then the corresponding output signal is given by $y'(t)$:

$$y'(t) = \int_{t-T}^t [x_1(u) + x_2(u)] du \quad (2.0.5)$$

Clearly, from (2.0.4) and (2.0.5):

$$y'(t) = y_1(t) + y_2(t) \quad (2.0.6)$$

Thus, the Law of Additivity holds.

Law of Homogeneity:

Consider an input signal $kx(t)$, where k is any constant. Let the corresponding output be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^t kx(u) du \quad (2.0.7)$$

$$= k \int_{t-T}^t x(u) du \quad (2.0.8)$$

$$= ky(t) \quad (2.0.9)$$

Clearly, from (2.0.9),

$$y'(t) = ky(t) \quad (2.0.10)$$

Thus, the Law of Homogeneity holds.

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

From (2), to check for time-invariance, we would introduce a delay of t_0 in the output and input

signals.

Delay in output signal:

$$y(t - t_0) = \int_{t-t_0-T}^{t-t_0} x(u) du \quad (2.0.11)$$

Now, we consider an input signal with a delay of t_0 , given by $x(t - t_0)$, and let the corresponding output signal be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^t x(u - t_0) du \quad (2.0.12)$$

Substituting $a = u - t_0$:

$$y'(t) = \int_{t-t_0-T}^{t-t_0} x(a) da \quad (2.0.13)$$

Clearly, from (2.0.11) and (2.0.13):

$$y'(t) = y(t - t_0) \quad (2.0.14)$$

Thus, the system is **time-invariant**.

The correct option is **2) Linear and Time-invariant**

Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given by $\delta(t)$. Thus,

$$h(t) = \int_{t-T}^t \delta(u) du \quad (2.0.15)$$

The Impulse function can be loosely defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (2.0.16)$$

Since the Impulse function is zero everywhere aside from $t = 0$, the non-zero value of integration is a result of $\delta(0)$. Thus, we can say $h(t)$ will be non-zero only if the limits of integration would include $t = 0$, i.e:

$$h(t) = \begin{cases} \int_{t-T}^t \delta(u) du & t - T < 0; t > 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.0.17)$$

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \quad (2.0.18)$$

The unit step signal, $u(t)$, is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.0.19)$$

On time-shifting $u(t)$ by T , we get:

$$u(t - T) = \begin{cases} 1 & t - T \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & t \geq T \\ 0 & \text{otherwise} \end{cases} \quad (2.0.20)$$

On subtracting (2.0.19) and (2.0.20), we get our impulse response $h(t)$ in terms of the unit step signal:

$$h(t) = u(t) - u(t - T) \quad (2.0.21)$$

The unit rectangular signal, $rect(t)$ is given by:

$$rect(t) = \begin{cases} 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (2.0.22)$$

We can obtain the impulse response $h(t)$ in terms of $rect(t)$ using time scaling and shifting as follows:

$$rect(Tt) = \begin{cases} 1 & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \quad (2.0.23)$$

$$rect(Tt + \frac{T}{2}) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} = h(t) \quad (2.0.24)$$

Let the Fourier Transform of $h(t)$ be given by $H(f)$, i.e

$$h(t) \xrightarrow{\mathcal{F}} H(f) \quad (2.0.25)$$

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt \quad (2.0.26)$$

From (2.0.18), we can write (2.0.26) as:

$$H(f) = \int_{-\infty}^0 0 dt + \int_0^T e^{-j2\pi ft} dt + \int_T^{\infty} 0 dt \quad (2.0.27)$$

$$= \frac{1 - e^{-j2\pi fT}}{j2\pi f} \quad (2.0.28)$$

$$= \frac{1 - \cos 2\pi fT + j \sin 2\pi fT}{j2\pi f} \quad (2.0.29)$$

$$= \frac{2 \sin^2(\pi fT) + 2j \sin(\pi fT) \cos(\pi fT)}{j2\pi f} \quad (2.0.30)$$

$$= \frac{2 \sin(\pi fT)}{2\pi f} \left(\frac{\sin(\pi fT) + j \cos(\pi fT)}{j} \right) \quad (2.0.31)$$

$$= T e^{-j\pi fT} \times \frac{\sin(\pi fT)}{\pi fT} \quad (2.0.32)$$

$$= T e^{-j\pi fT} \text{sinc}(\pi fT) \quad (2.0.33)$$

where $\text{sinc}(t)$, the sampling function is defined as:

$$\text{sinc}(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(t)}{t} & \text{otherwise} \end{cases} \quad (2.0.34)$$

□

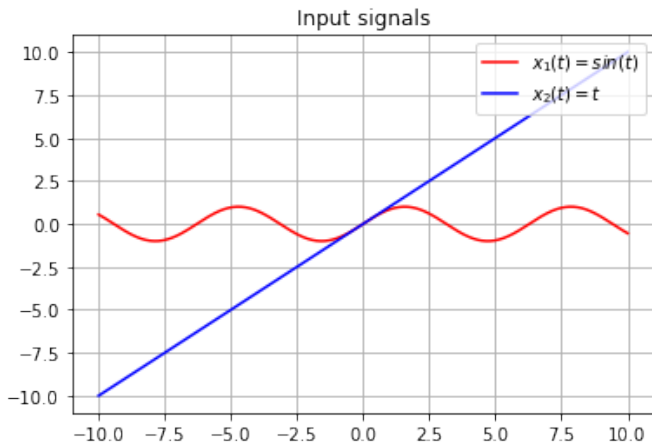


Fig. 4: $x_1(t) = \sin t$ and $x_2(t) = t$

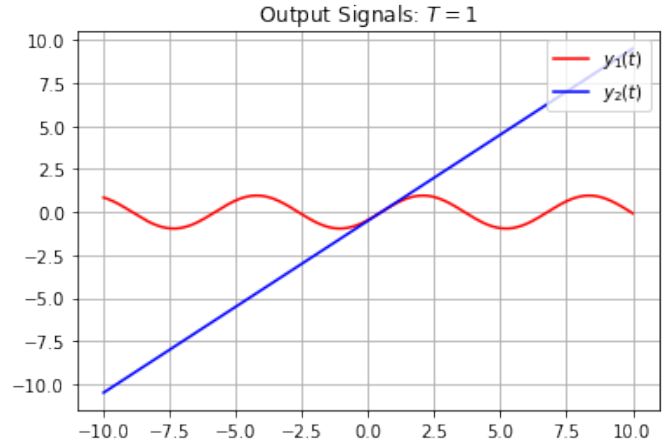


Fig. 4: $y_1(t)$ and $y_2(t)$

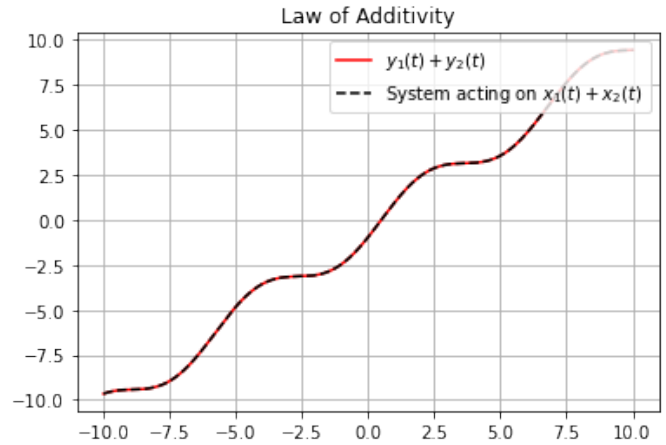


Fig. 4: Law of Additivity

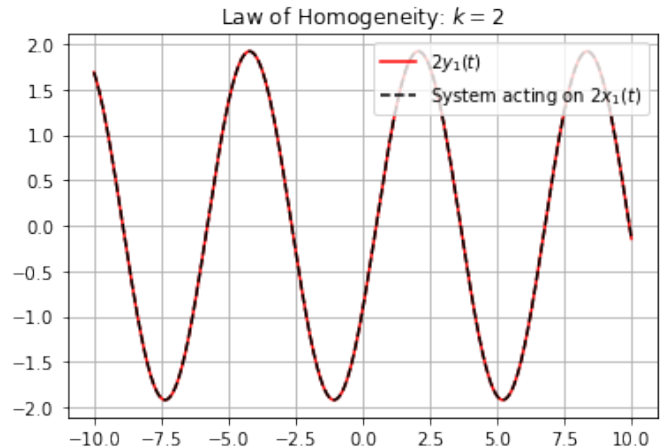


Fig. 4: Law of Homogeneity

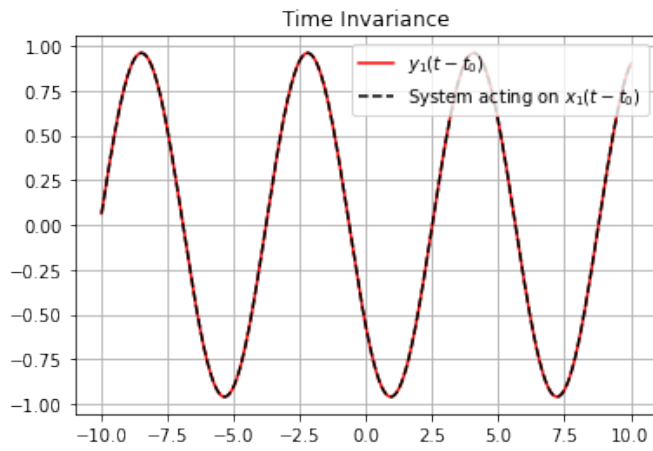


Fig. 4: Time invariance