GATE EC 2001 - Q.16

Tanmay Goyal - AI20BTECH11021

Question

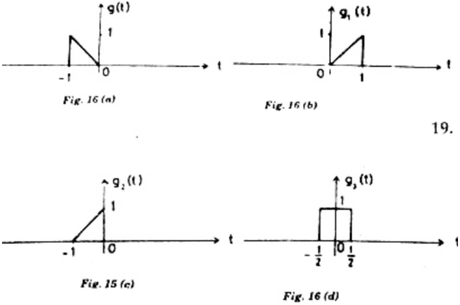

Fig. 16 (a) Fig. 16 (b) Fig. 15 (c) Fig. 16 (d)

The Fourier Transform $G(\omega)$ of the signal $g(t)$ is given by

$$G(\omega) = \frac{1}{\omega^2}(e^{j\omega} - j\omega e^{j\omega} - 1) \quad (1)$$

Using this information, find the Fourier Transforms of the signals $g_1(t)$, $g_2(t)$ and $g_3(t)$.

Result of Time Shifting on Fourier Transform

Lemma

If

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(\omega) \tag{2}$$

then,

$$g(t \pm t_0) \overset{\mathcal{F}}{\rightleftharpoons} G(\omega)e^{\pm j\omega t_0} \tag{3}$$

Proof.

We know,

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\, dt \tag{4}$$

□

Proof continued

Proof.

Let

$$g(t + t_0) \overset{\mathcal{F}}{\rightleftharpoons} G'(\omega) \tag{5}$$

Then,

$$G'(\omega) = \int_{-\infty}^{\infty} g(t + t_0) e^{-j\omega t}\, dt \tag{6}$$

Substituting $t + t_0 = T$, we get:

$$G'(\omega) = \int_{-\infty}^{\infty} g(T) e^{-j\omega(T - t_0)}\, dT \tag{7}$$

$$= e^{j\omega t_0} \int_{-\infty}^{\infty} g(T) e^{-j\omega T}\, dT \tag{8}$$

$$= e^{j\omega t_0} G(\omega) \tag{9}$$

Result of Time scaling on Fourier Transform

Lemma

If

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(\omega) \tag{10}$$

then,

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|} G\left(\frac{\omega}{\alpha}\right) \tag{11}$$

Proof.

Consider $\alpha > 0$. Then, we know,

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t}\, dt \tag{12}$$

□

Proof Continued

Proof.

Let

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} G'(\omega) \tag{13}$$

Then,

$$G'(\omega) = \int_{-\infty}^{\infty} g(\alpha t) e^{-j\omega t}\, dt \tag{14}$$

Making the substitution $T = \alpha t$, we get:

$$G'(\omega) = \frac{1}{\alpha} \int_{-\infty}^{\infty} g(T) e^{-j\frac{\omega T}{\alpha}}\, dT \tag{15}$$

$$= \frac{1}{\alpha} G\left(\frac{\omega}{\alpha}\right) \tag{16}$$

□

Effect of Time Reversal on Fourier Transform

Corollary

If

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(\omega) \tag{17}$$

then,

$$g(-t) \overset{\mathcal{F}}{\rightleftharpoons} G(-\omega) \tag{18}$$

Differentiation of Fourier Transform in Time Domain

Lemma

If

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(\omega) \tag{19}$$

then,

$$\frac{dg(t)}{dt} \overset{\mathcal{F}}{\rightleftharpoons} (j\omega)G(\omega) \tag{20}$$

Proof

Proof.

Using the formula for Inverse Fourier transform, we know:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}\,d\omega \tag{21}$$

$$\frac{dg(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}(j\omega)\,d\omega \tag{22}$$

$$= \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}\,d\omega \tag{23}$$

$$= (j\omega)G(\omega) \tag{24}$$

□

The Various Signals

Now, from the figure:

$$g(t) = \begin{cases} -t & -1 \geq t \geq 0 \\ 0 & otherwise \end{cases} \tag{25}$$

$$g_1(t) = \begin{cases} t & 0 \geq t \geq 1 \\ 0 & otherwise \end{cases} \tag{26}$$

$$g_2(t) = \begin{cases} 1+t & -1 \geq t \geq 0 \\ 0 & otherwise \end{cases} \tag{27}$$

$$g_3(t) = \begin{cases} 1 & -\frac{1}{2} \geq t \geq \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{28}$$

$G_1(\omega)$

Clearly, $g_1(t) = g(-t)$, and using (3), we get:

$$G_1(\omega) = G(-\omega) = \frac{1}{\omega^2}(e^{-j\omega} + j\omega e^{-j\omega} - 1) \tag{29}$$

$G_2(\omega)$

Also, $g_2(t) = g_1(t+1)$. Thus, from (1), we get:

$$G_2(\omega) = G_1(\omega)e^{j\omega.1} \tag{30}$$

$$= \frac{1}{\omega^2}(e^{-j\omega} + j\omega e^{-j\omega} - 1) \times e^{j\omega} \tag{31}$$

$$= \frac{1}{\omega^2}(1 + j\omega - e^{j\omega}) \tag{32}$$

$g_3(t)$

Finally, $g_3(t)$ is non-zero between $\frac{-1}{2}$ and $\frac{1}{2}$. Thus, we can shift $g_1(t)$ and take it's derivative wrt time:

$$g_1(t) = \begin{cases} t & 0 \geq t \geq 1 \\ 0 & otherwise \end{cases} \tag{33}$$

$$g_1\left(t + \frac{1}{2}\right) = \begin{cases} t + \frac{1}{2} & -\frac{1}{2} \geq t \geq \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{34}$$

$$\frac{dg_1\left(t + \frac{1}{2}\right)}{dt} = \begin{cases} 1 & -\frac{1}{2} \geq t \geq \frac{1}{2} \\ 0 & otherwise \end{cases} = g_3(t) \tag{35}$$

$G_3(\omega)$

Using (1) and (4), we get:

$$g_1(t) \overset{\mathcal{F}}{\rightleftharpoons} G_1(\omega) \tag{36}$$

$$g_1\left(t+\frac{1}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} e^{\frac{j\omega}{2}} G_1(\omega) \tag{37}$$

$$g_1\left(t+\frac{1}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{e^{-\frac{j\omega}{2}}}{\omega^2}(1+j\omega-e^{j\omega}) \tag{38}$$

$$\frac{dg_1\left(t+\frac{1}{2}\right)}{dt} \overset{\mathcal{F}}{\rightleftharpoons} \frac{j\omega e^{-\frac{j\omega}{2}}}{\omega^2}(1+j\omega-e^{j\omega}) \tag{39}$$

$$g_3(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{je^{-\frac{j\omega}{2}}}{\omega}(1+j\omega-e^{j\omega}) \tag{40}$$