GATE EC 2001 - Q.16

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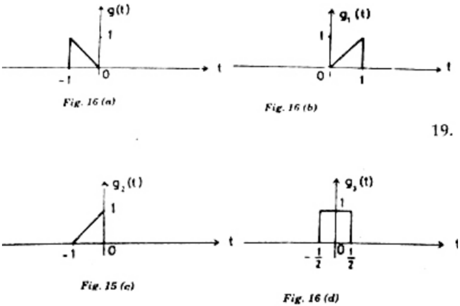
Question



The Fourier Transform $G(\omega)$ of the signal $g(t)$ is given by

$$G(\omega) = \frac{1}{\omega^2}(e^{j\omega} - j\omega e^{j\omega} - 1) \quad (1)$$

Using this information, find the Fourier Transforms of the signals $g_1(t)$, $g_2(t)$ and $g_3(t)$.

Note: We replace ω by $2\pi f$. Then,

$$G(f) = \frac{1}{4\pi^2 f^2}(e^{2\pi jf} - 2\pi jfe^{2\pi jf} - 1) \quad (2)$$

Result of Time Shifting on Fourier Transform

Lemma

If

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f) \tag{3}$$

then,

$$g(t \pm t_0) \overset{\mathcal{F}}{\rightleftharpoons} G(f)e^{\pm 2\pi j f t_0} \tag{4}$$

Proof.

We know,

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi j f t}\, dt \tag{5}$$

□

Proof continued

Proof.

Let

$$g(t + t_0) \overset{\mathcal{F}}{\rightleftharpoons} G'(f) \tag{6}$$

Then,

$$G'(f) = \int_{-\infty}^{\infty} g(t + t_0) e^{-2\pi jft}\, dt \tag{7}$$

Substituting $t + t_0 = T$, we get:

$$G'(f) = \int_{-\infty}^{\infty} g(T) e^{-2\pi jf(T - t_0)}\, dT \tag{8}$$

$$= e^{2\pi jft_0} \int_{-\infty}^{\infty} g(T) e^{2\pi jfT}\, dT \tag{9}$$

$$= e^{-2\pi jft_0} G(f) \tag{10}$$

Result of Time scaling on Fourier Transform

Lemma

If

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f) \tag{11}$$

then,

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|} G\left(\frac{f}{\alpha}\right) \tag{12}$$

Proof.

Consider $\alpha > 0$. Then, we know,

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-2\pi jft} \, dt \tag{13}$$

Proof Continued

Proof.

Let

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} G'(f) \tag{14}$$

Then,

$$G'(f) = \int_{-\infty}^{\infty} g(\alpha t) e^{-2\pi j f t} \, dt \tag{15}$$

Making the substitution $T = \alpha t$, we get:

$$G'(f) = \frac{1}{\alpha} \int_{-\infty}^{\infty} g(T) e^{-2\pi j \frac{fT}{\alpha}} \, dT \tag{16}$$

$$= \frac{1}{\alpha} G\left(\frac{f}{\alpha}\right) \tag{17}$$

□

Effect of Time Reversal on Fourier Transform

Lemma

If

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f) \tag{18}$$

then,

$$g(-t) \overset{\mathcal{F}}{\rightleftharpoons} G(-f) \tag{19}$$

Proof.

Put $\alpha = -1$ in (2) to obtain the result.

$$g(-t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|-1|} G\left(\frac{f}{-1}\right) \implies g(-t) \overset{\mathcal{F}}{\rightleftharpoons} G(-f) \tag{20}$$

$\square$

The Various Signals

Now, from the figure:

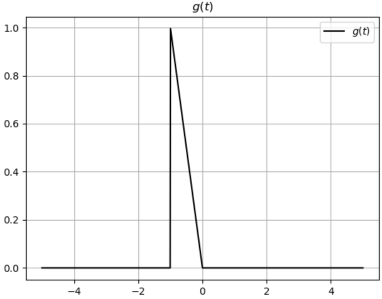
$$g(t) = \begin{cases} -t & -1 \leq t \leq 0 \\ 0 & otherwise \end{cases} \tag{21}$$

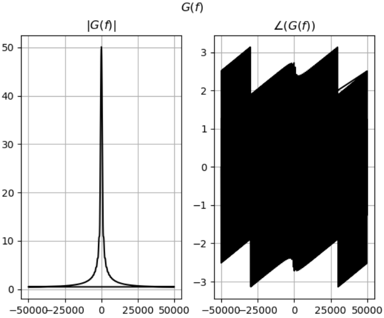
$$g_1(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 0 & otherwise \end{cases} \tag{22}$$

$$g_2(t) = \begin{cases} 1+t & -1 \leq t \leq 0 \\ 0 & otherwise \end{cases} \tag{23}$$

$$g_3(t) = \begin{cases} 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{24}$$

$g(t)$


g(t)
g(t)
1.0
0.8
0.6
0.4
0.2
0.0
−4
−2
0
2
4

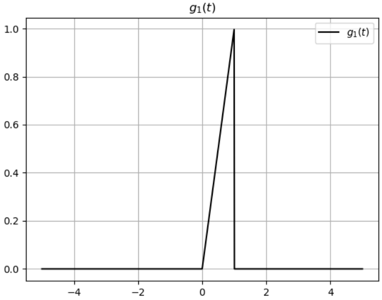

G(f)
|G(f)|
∠(G(f))
50
40
30
20
10
0
−50000−25000
0
25000
50000
3
2
1
0
−1
−2
−3

$G_1(\omega)$

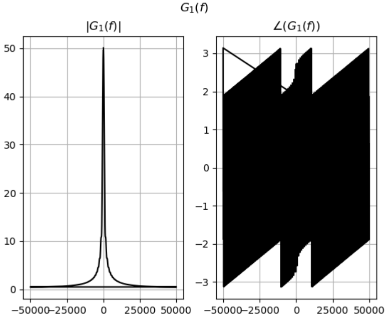
Clearly, $g_1(t) = g(-t)$, and using (3), we get:

$$G_1(f) = G(-f) = \frac{1}{4\pi^2 f^2}(e^{-2\pi jf} + 2\pi jfe^{-2\pi jf} - 1) \tag{25}$$

$g_1(t)$



$g_1(t)$
1.0
0.8
0.6
0.4
0.2
0.0
−4
−2
0
2
4



$G_1(f)$
$|G_1(f)|$
$\angle(G_1(f))$
50
40
30
20
10
0
−50000
−25000
0
25000
50000
3
2
1
0
−1
−2
−3

$G_2(\omega)$

Also, $g_2(t) = g(-t-1)$. Thus, from (1) and (3), we get:

$$g(t-1) \overset{\mathcal{F}}{\rightleftharpoons} e^{-2\pi jf} G(f) \tag{26}$$

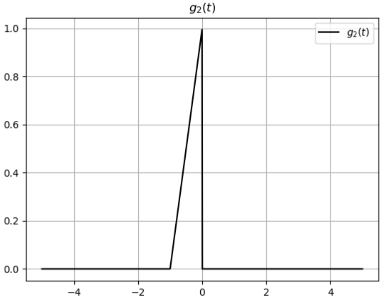
$$g(-t-1) \overset{\mathcal{F}}{\rightleftharpoons} e^{2\pi jf} G(-f) \tag{27}$$

$$g_2(t) \overset{\mathcal{F}}{\rightleftharpoons} e^{2\pi jf} G(-f) \tag{28}$$

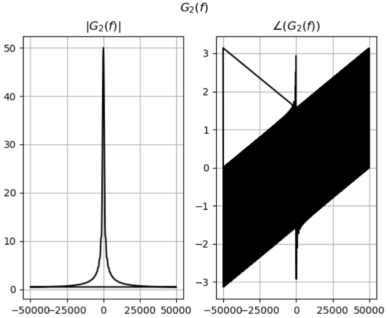
$$g_2(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{4\pi^2 f^2}(1 + 2\pi jf - e^{2\pi jf}) \tag{29}$$

$$\implies G_2(\omega) = \frac{1}{4\pi^2 f^2}(1 + 2\pi jf - e^{2\pi jf}) \tag{30}$$

$g_2(t)$



$g_2(t)$
$g_2(t)$
1.0
0.8
0.6
0.4
0.2
0.0
−4
−2
0
2
4



$G_2(f)$
$|G_2(f)|$
$\angle(G_2(f))$
50
40
30
20
10
0
3
2
1
0
−1
−2
−3
−50000
−25000
0
25000
50000

$G_3(\omega)$

Also, $g_3(t) = g\left(t - \frac{1}{2}\right) + g\left(-t - \frac{1}{2}\right)$. Thus,

$$g_3(t) \overset{\mathcal{F}}{\rightleftharpoons} e^{-j\pi f} G(f) + e^{j\pi f} G(-f) \tag{31}$$

$$G_3(f) = \frac{e^{-j\pi f}}{4\pi^2 f^2}\left[e^{2\pi jf} - 2\pi jfe^{2\pi jf} - 1\right] + \tag{32}$$

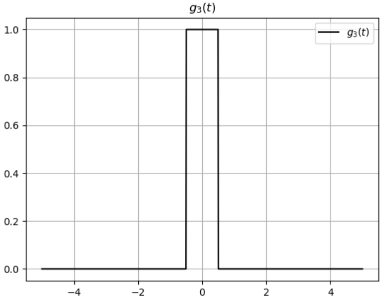
$$\frac{e^{j\pi f}}{4\pi^2 f^2}\left[e^{-2\pi jf} + 2\pi jfe^{-2\pi jf} - 1\right] \tag{33}$$

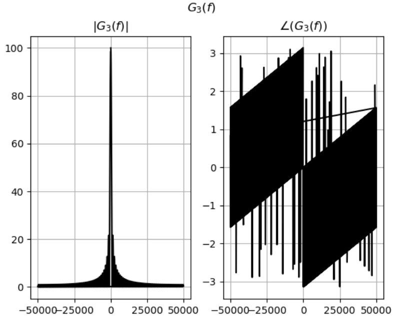
$$G_3(f) = \frac{j}{2\pi f}\left[e^{-\pi jf} - e^{\pi jf}\right] = \frac{\sin \pi f}{\pi f} = sinc(f) \tag{34}$$

where $sinc(t)$, the sampling function is defined as:

$$sinc(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & otherwise \end{cases} \tag{35}$$

$g_3(t)$


$g_3(t)$
$g_3(t)$
1.0
0.8
0.6
0.4
0.2
0.0
−4
−2
0
2
4


$G_3(f)$
$|G_3(f)|$
$\angle(G_3(f))$
100
80
60
40
20
0
3
2
1
0
−1
−2
−3
−50000
−25000
0
25000
50000