Gate Assignment 1

Tanmay Goyal - AI20BTECH11021

Download all python codes from

```
https://github.com/tanmaygoyal258/EE3900−Linear
    −Systems−and−Signal−processing/blob/main/
    GateAssignment1/code.py
```

Download all latex codes from

```
https://github.com/tanmaygoyal258/EE3900−Linear
    −Systems−and−Signal−processing/blob/main/
    GateAssignment1/main.tex
```

1 Problem

(EC 2017- Q.7) The input $x(t)$ and output $y(t)$ of a continous time signal are related as:

$$y(t) = \int_{t-T}^{t} x(u)\, du \tag{1.0.1}$$

The system is:

1) Linear and Time-variant
2) Linear and Time-invariant
3) Non-Linear and Time-variant
4) Non-Linear and Time-invariant

2 Solution

Definition 1. *We say that a system is* ***linear*** *if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.*

Definition 2. *A system is said to be* ***time invariant*** *if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.*

Lemma 2.1. *The system relating the input signal $x(t)$ and output signal $y(t)$, given by*

$$y(t) = \int_{t-T}^{t} x(u)\, du \tag{2.0.1}$$

is linear and time invariant in nature.

Proof. From (1), we can say the system is linear if it follows both the laws of Additivity and Homogeneity.

Law of Additivity:

Let the two input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{t-T}^{t} x_1(u)\, du \tag{2.0.2}$$

$$y_2(t) = \int_{t-T}^{t} x_2(u)\, du \tag{2.0.3}$$

$$y_1(t) + y_2(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\, du \tag{2.0.4}$$

Now, consider the input signal of $x_1(t) + x_2(t)$, then the corresponding output signal is given by $y'(t)$:

$$y'(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\, du \tag{2.0.5}$$

Clearly, from (2.0.4) and (2.0.5):

$$y'(t) = y_1(t) + y_2(t) \tag{2.0.6}$$

Thus, the Law of Additivity holds.

Law of Homogeneity:

Consider an input signal $kx(t)$, where k is any constant. Let the corresponding output be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^{t} kx(u)\, du \tag{2.0.7}$$

$$= k\int_{t-T}^{t} x(u)\, du \tag{2.0.8}$$

$$= ky(t) \tag{2.0.9}$$

Clearly, from (2.0.9),

$$y'(t) = ky(t) \tag{2.0.10}$$

Thus, the Law of Homogeneity holds.

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

From (2) , to check for time-invariance, we would introduce a delay of t_0 in the output and input

signals.
Delay in output signal:

$$y(t-t_0)=\int_{t-t_0-T}^{t-t_0} x(u)\,du \tag{2.0.11}$$

Now, we consider an input signal with a delay of t_0, given by $x(t-t_0)$, and let the corresponding output signal be given by $y'(t)$, then:

$$y'(t)=\int_{t-T}^{t} x(u-t_0)\,du \tag{2.0.12}$$

Substituting $a=u-t_0$:

$$y'(t)=\int_{t-t_0-T}^{t-t_0} x(a)\,da \tag{2.0.13}$$

Clearly, from (2.0.11) and (2.0.13):

$$y'(t)=y(t-t_0) \tag{2.0.14}$$

Thus, the system is **time-invariant**.
The correct option is **2) Linear and Time-invariant**

Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given by $\delta(t)$. Thus,

$$h(t)=\int_{t-T}^{t} \delta(u)du \tag{2.0.15}$$

The Impulse function can be loosely defined as:

$$\delta(t)=\begin{cases}\infty & t=0\\ 0 & otherwise\end{cases} \; and \int_{-\infty}^{\infty}\delta(t)dt=1 \tag{2.0.16}$$

Since the Impulse function is zero everywhere aside from $t=0$, the non-zero value of integration is a result of $\delta(0)$. Thus, we can say $h(t)$ will be non-zero only if the limits of integration would include $t=0$, i.e:

$$h(t)=\begin{cases}\int_{t-T}^{t}\delta(u)du & t-T<0; t>0\\ 0 & otherwise\end{cases} \tag{2.0.17}$$

$$h(t)=\begin{cases}1 & 0<t<T\\ 0 & otherwise\end{cases} \tag{2.0.18}$$

The unit step signal, $u(t)$, is given by:

$$u(t)=\begin{cases}1 & t\geq 0\\ 0 & otherwise\end{cases} \tag{2.0.19}$$

On time-shifting $u(t)$ by T, we get:

$$u(t-T)=\begin{cases}1 & t-T\geq 0\\ 0 & otherwise\end{cases}=\begin{cases}1 & t\geq T\\ 0 & otherwise\end{cases} \tag{2.0.20}$$

On subtracting (2.0.19) and (2.0.20), we get our impulse response $h(t)$ in terms of the unit step signal:

$$h(t)=u(t)-u(t-T) \tag{2.0.21}$$

The unit rectangular signal, $rect(t)$ is given by:

$$rect(t)=\begin{cases}1 & \frac{-1}{2}\geq t\geq \frac{1}{2}\\ 0 & otherwise\end{cases} \tag{2.0.22}$$

We can obtain the impulse response $h(t)$ in terms of $rect(t)$ using time scaling and shifting as follows:

$$rect(Tt)=\begin{cases}1 & \frac{-T}{2}\geq Tt\geq \frac{T}{2}\\ 0 & otherwise\end{cases} \tag{2.0.23}$$

$$rect(Tt+\frac{T}{2})=\begin{cases}1 & 0\geq Tt+\frac{T}{2}\geq T\\ 0 & otherwise\end{cases}=h(t) \tag{2.0.24}$$

Let the Fourier Transform of $h(t)$ be given by $H(\omega)$,i.e

$$h(t)\overset{\mathcal{F}}{\rightleftharpoons}H(\omega) \tag{2.0.25}$$

$$H(\omega)=\int_{-\infty}^{\infty}h(t)e^{-j\omega t}\,dt \tag{2.0.26}$$

From (2.0.18), we can write (2.0.26) as:

$$H(\omega)=\int_{-\infty}^{0}0\,dt+\int_{0}^{T}e^{-j\omega t}\,dt+\int_{T}^{\infty}0\,dt \tag{2.0.27}$$

$$=\frac{1-e^{-j\omega T}}{j\omega} \tag{2.0.28}$$

$$=\frac{1-\cos\omega T+j\sin\omega T}{j\omega} \tag{2.0.29}$$

$$=\frac{2\sin^2(\frac{\omega T}{2})+2j\sin(\frac{\omega T}{2})\cos(\frac{\omega T}{2})}{j\omega} \tag{2.0.30}$$

$$=\frac{2\sin(\frac{\omega T}{2})}{\omega}\left(\frac{\sin(\frac{\omega T}{2})+j\cos(\frac{\omega T}{2})}{j}\right) \tag{2.0.31}$$

$$=Te^{-j\omega\frac{T}{2}}\times\frac{\sin(\frac{\omega T}{2})}{\frac{\omega T}{2}} \tag{2.0.32}$$

$$=Te^{-j\omega\frac{T}{2}}sinc(\frac{wT}{2}) \tag{2.0.33}$$

where $sinc(t)$, the sampling function is defined as:

$$sinc(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(t)}{t} & otherwise \end{cases} \tag{2.0.34}$$

$\square$

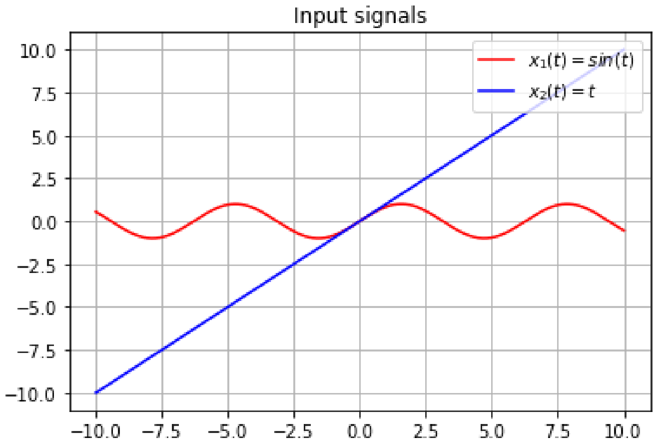

Fig. 4: $x_1(t) = \sin t$ and $x_2(t) = t$

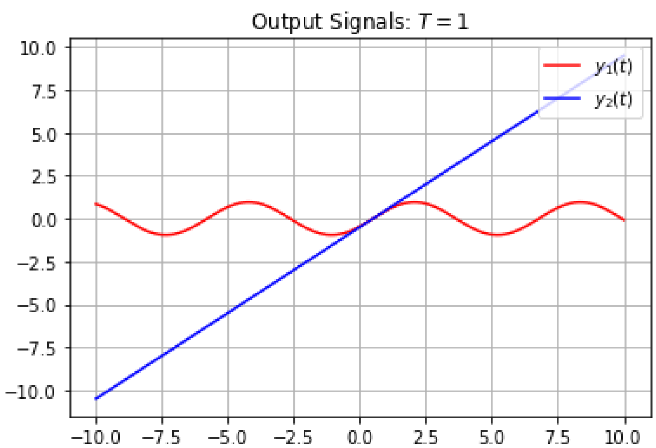

Fig. 4: $y_1(t)$ and $y_2(t)$

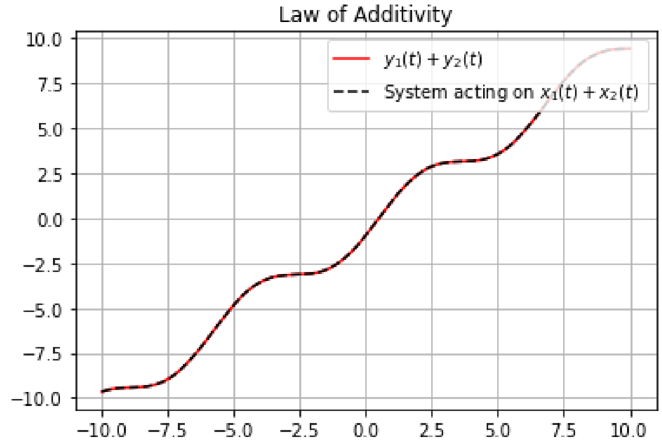

Fig. 4: Law of Additivity

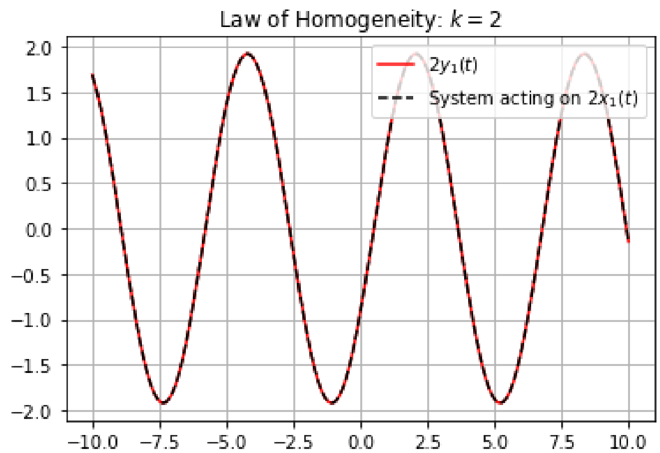

Fig. 4: Law of Homogeneity

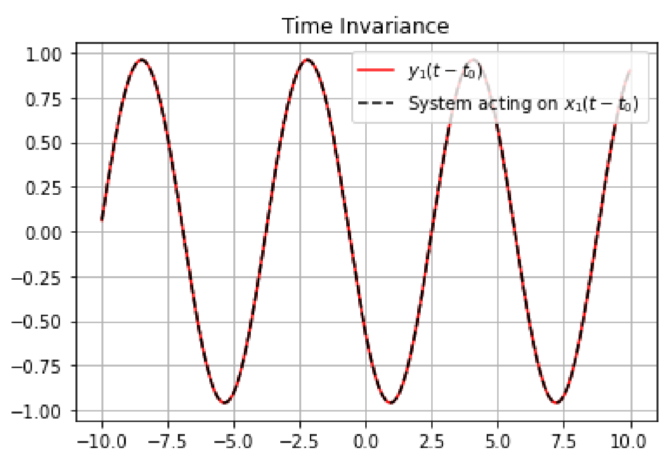

Fig. 4: Time invariance