GATE EC 2017- Q.8

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Question

The input $x(t)$ and output $y(t)$ of a continous time signal are related as:

$$y(t) = \int_{t-T}^{t} x(u)\,du \tag{1}$$

The system is:

1. Linear and Time-variant
2. Linear and Time-invariant
3. Non-Linear and Time-variant
4. Non-Linear and Time-invariant

Linear Systems and Time Invariant Systems

Definition

We say that a system is **linear** if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.

Definition

A system is said to be **time invariant** if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.

Lemma

Lemma

The system relating the input signal $x(t)$ and output signal $y(t)$, given by

$$y(t) = \int_{t-T}^{t} x(u)\,du \tag{2}$$

is linear and time invariant in nature.

Proof: Law of Additivity

Let the input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{t-T}^{t} x_1(u)\,du \tag{3}$$

$$y_2(t) = \int_{t-T}^{t} x_2(u)\,du \tag{4}$$

$$y_1(t) + y_2(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\,du \tag{5}$$

Proof: Law of Additivity

Now, consider the input signal of $x_1(t) + x_2(t)$, then the corresponding output signal is given by $y'(t)$:

$$y'(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\, du \tag{6}$$

Clearly, from (5) and (6):

$$y'(t) = y_1(t) + y_2(t) \tag{7}$$

Thus, the Law of Additivity holds.

Proof: Law of Homogeneity

Consider an input signal $kx(t)$, where k is any constant. Let the corresponding output be given by $y'(t)$, then:

$$\begin{aligned} y'(t) &= \int_{t-T}^{t} kx(u)\,du &(8)\\ &= k\int_{t-T}^{t} x(u)\,du &(9)\\ &= ky(t) &(10) \end{aligned}$$

Clearly, from (10),

$$y'(t) = ky(t) \tag{11}$$

Thus, the Law of Homogeneity holds.

Proof

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

Proof: Time Invariance

To check for time-invariance, we would introduce a delay of t_0 in the output and input signals.
Delay in output signal:

$$y(t - t_0) = \int_{t-t_0-T}^{t-t_0} x(u)\, du \tag{12}$$

Proof: Time Invariance

Now, we consider an input signal with a delay of t_0, given by $x(t - t_0)$, and let the corresponding output signal be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^{t} x(u - t_0)\, du \tag{13}$$

Substituting $a = u - t_0$:

$$y'(t) = \int_{t-t_0-T}^{t-t_0} x(a)\, da \tag{14}$$

Clearly, from (12) and (14):

$$y'(t) = y(t - t_0) \tag{15}$$

Thus, the system is **time-invariant**.
Thus, **2) Linear and Time- invariant** is the correct answer.

Impulse response

Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given by $\delta(t)$. Thus,

$$h(t) = \int_{t-T}^{t} \delta(u)du \tag{16}$$

Impulse function

The Impulse function can be loosely defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & otherwise \end{cases} \; and \int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{17}$$

Impulse response

Since the Impulse function is zero everywhere aside from $t = 0$, the non-zero value of integration is a result of $\delta(0)$. Thus, we can say $h(t)$ will be non-zero only if the limits of integration would include $t = 0$, i.e:

$$h(t) = \begin{cases} \int_{t-T}^{t} \delta(u)du & t - T < 0; t > 0 \\ 0 & otherwise \end{cases} \tag{18}$$

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & otherwise \end{cases} \tag{19}$$

Impulse response in terms of unit step signal

The unit step signal, $u(t)$, is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & otherwise \end{cases} \tag{20}$$

On time-shifting $u(t)$ by T, we get:

$$u(t-T) = \begin{cases} 1 & t-T \geq 0 \\ 0 & otherwise \end{cases} = \begin{cases} 1 & t \geq T \\ 0 & otherwise \end{cases} \tag{21}$$

On subtracting (20) and (21), we get our impulse response $h(t)$ in terms of the unit step signal:

$$h(t) = u(t) - u(t-T) \tag{22}$$

Impulse response in terms of the unit rectangular function

The unit rectangular signal, $rect(t)$ is given by:

$$rect(t) = \begin{cases} 1 & \frac{-1}{2} \leq t \leq \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{23}$$

We can obtain the impulse response $h(t)$ in terms of $rect(t)$ using time scaling and shifting as follows:

$$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \frac{-1}{2} \leq \frac{t}{\tau} \leq \frac{1}{2} \\ 0 & otherwise \end{cases} = \begin{cases} 1 & \frac{-\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & otherwise \end{cases} \tag{24}$$

Subsituting $\tau = T$:

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1 & \frac{-T}{2} \leq t \leq \frac{T}{2} \\ 0 & otherwise \end{cases} \tag{25}$$

Impulse response in terms of the unit rectangular function

Now, we want to right-shift the signal by $\frac{T}{2}$:

$$rect\left(\frac{1}{T}\left(t-\frac{T}{2}\right)\right)=\begin{cases}1 & 0\leq t\leq T\\ 0 & otherwise\end{cases}=h(t) \tag{26}$$

Since the time shifting is to be performed on the variable t and not $\frac{t}{T}$

Fourier Transform of rectangular function

$$rect(t) \overset{\mathcal{F}}{\rightleftharpoons} Y(f) \tag{27}$$

$$Y(f) = \int_{-\infty}^{\infty} rect(t)e^{-j2\pi ft}\, dt \tag{28}$$

From (23), we can write (28) as:

$$Y(f) = \int_{-\infty}^{\frac{-1}{2}} 0\, dt + \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-j2\pi ft}\, dt + \int_{\frac{1}{2}}^{\infty} 0\, dt = \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} \tag{29}$$

$$= \frac{\sin(\pi f)}{\pi f} = sinc(f) \tag{30}$$

where $sinc(t)$ is defined as:

$$sinc(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & otherwise \end{cases} \tag{31}$$

Properties of Fourier Transform

Let the Fourier Transform of a signal $x(t)$ be $X(f)$.

$$x(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) \tag{32}$$

When the signal $x(t)$ is time shifted by t_0, the resultant Fourier Transform is given by:

$$x(t \pm t_0) \overset{\mathcal{F}}{\rightleftharpoons} X(f)e^{\pm j2\pi f t_0} \tag{33}$$

And when the signal $x(t)$ is time scaled by α, the resulting Fourier Transform is given by:

$$x(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|}X\left(\frac{f}{\alpha}\right) \tag{34}$$

Fourier Transform of impulse response

$$rect(t) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f) \tag{35}$$

Using (33):

$$rect\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f)e^{-j(2\pi f)\frac{T}{2}} \tag{36}$$

$$rect\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f)e^{-j\pi fT} \tag{37}$$

Using (34),

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{\frac{1}{|T|}} sinc\left(\frac{f}{\frac{1}{T}}\right) e^{\frac{-j\pi fT}{T}} \tag{38}$$

$$h(t) \overset{\mathcal{F}}{\rightleftharpoons} Tsinc\left(fT\right) e^{-j\pi f} \tag{39}$$

$$\therefore H(f) = Tsinc\left(fT\right) e^{-j\pi f} \tag{40}$$

Cosine input signal

Consider an input signal of $x(t) = \cos 2\pi f_0 t$. The Fourier Transform of $x(t)$ is given by:

$$x(t) = \cos 2\pi f_0 t \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \tag{41}$$

using the fact that

$$\cos 2\pi f_0 t = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \tag{42}$$

and the Fourier Transform of $e^{\pm j2\pi f_0 t}$ is given by:

$$e^{\pm j2\pi f_0 t} \overset{\mathcal{F}}{\rightleftharpoons} \delta(f \mp f_0) \tag{43}$$

Output Signal

The output signal will be given by:

$$y(t) = \int_{t-T}^{t} \cos 2\pi f_0 u \, du \tag{44}$$

$$= \frac{1}{2\pi f_0}\left[\sin 2\pi f_0 t - \sin 2\pi f_0 (t - T)\right] \tag{45}$$

$$= \frac{\sin \pi f_0 T}{\pi f_0}\left[\cos 2\pi f_0 \left(t - \frac{T}{2}\right)\right] \tag{46}$$

$$= T sinc(f_0 T) \cos 2\pi f_0 \left(t - \frac{T}{2}\right) \tag{47}$$

Fourier Transform of output signal

The Fourier transform of $\cos 2\pi f_0 \left(t - \frac{T}{2}\right)$ can be obtained using (34) and (33) as follows:

$$\cos t = \frac{1}{2}\left[e^{jt} + e^{-jt}\right] \tag{48}$$

$$\cos t \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{49}$$

$$\cos\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] e^{j\pi fT} \tag{50}$$

$$\cos 2\pi f_0\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2\pi f_0}\,\frac{\delta(\frac{f}{2\pi f_0} - \frac{1}{2\pi}) + \delta(\frac{f}{2\pi f_0} + \frac{1}{2\pi})}{2} e^{j\pi\frac{f}{2\pi f_0}T} \tag{51}$$

$$\cos 2\pi f_0\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{4\pi f_0}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) e^{j\pi\frac{f}{2f_0}T} \tag{52}$$

Fourier Transform of output signal

The Fourier Transform of the output signal $y(t)$ from (47) is given by:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{Tsinc(f_0T)}{4\pi f_0} e^{j\pi \frac{f}{2f_0}T} \left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right) \right) \tag{53}$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} ke^{j\pi \frac{f}{2f_0}T} \left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right) \right) \tag{54}$$

where $k = \frac{Tsinc(f_0T)}{4\pi f_0}$.

Fourier Transform of output signal

Substituting $2\pi f_0 = 1$ and $T = 1$:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} ke^{j\pi^2 f}\left(\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right) \tag{55}$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} ke^{j\frac{\pi}{2}}\delta\left(f - \frac{1}{2\pi}\right) + ke^{j\frac{-\pi}{2}}\delta\left(f + \frac{1}{2\pi}\right) \tag{56}$$

using the multiplication property of the Delta function:

$$x(t)\delta(t - t_1) = x(t_1)\delta(t - t_1) \tag{57}$$

Since , $e^{j\frac{\pi}{2}} = j$ and $e^{-j\frac{\pi}{2}} = -j$, we finally get:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} kj\left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{58}$$

Output signal

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} kj\left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{59}$$

Clearly, the Fourier transform of $y(t)$ can be manipulated to represent a sinusoidal wave, which is given by:

$$sin(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{-j}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{60}$$

Fourier Transform of Impulse response: Graph

The attenuation happens for the same value of f, as depicted in the graphs of the Fourier Transforms below.

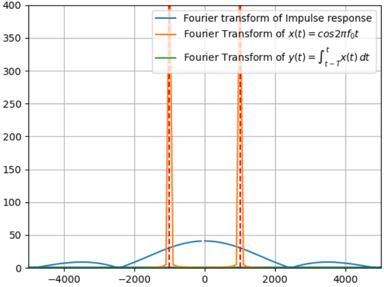

Figure: Fourier Transform of impulse response and example input and output signal

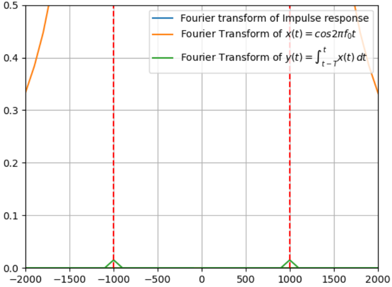

Figure: Fourier Transform of impulse response and example input and output signal zoomed in

Graphs: Input and Output signals

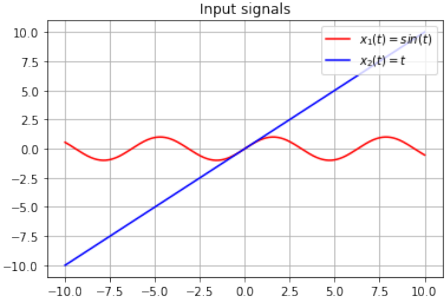

Figure: $x_1(t) = \sin t$ and $x_2(t) = t$

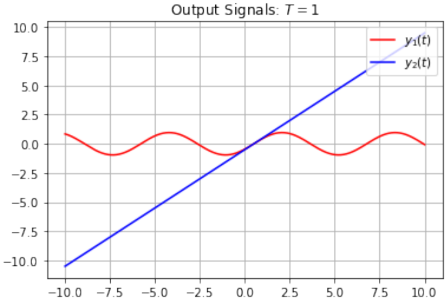

Figure: $y_1(t)$ and $y_2(t)$

Graphs: Laws of Additivity and Homogeneity

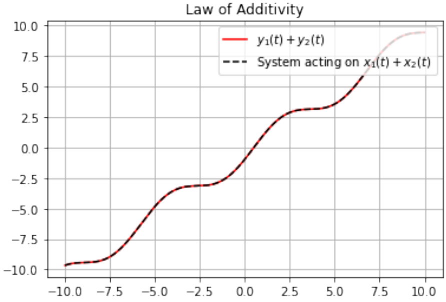

Figure: Law of Additivity

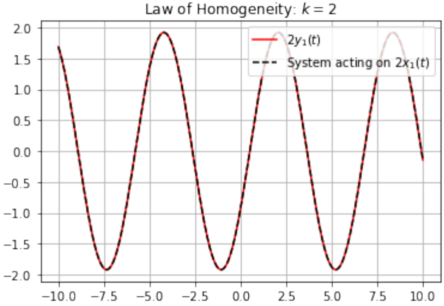

Figure: Law of Homogeneity

Graphs: Time Invariance