

# Gate Assignment 1

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Download all python codes from

<https://github.com/tanmaygoyal258/EE3900-Linear-Systems-and-Signal-processing/blob/main/GateAssignment1/code.py>

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<https://github.com/tanmaygoyal258/EE3900-Linear-Systems-and-Signal-processing/blob/main/GateAssignment1/main.tex>

## 1 PROBLEM

(EC 2017- Q.7) The input  $x(t)$  and output  $y(t)$  of a continous time signal are related as:

$$y(t) = \int_{t-T}^t x(u) du \quad (1.0.1)$$

The system is:

- 1) Linear and Time-variant
- 2) Linear and Time-invariant
- 3) Non-Linear and Time-variant
- 4) Non-Linear and Time-invariant

## 2 SOLUTION

**Definition 1.** We say that a system is **linear** if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.

**Definition 2.** A system is said to be **time invariant** if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.

**Lemma 2.1.** The system relating the input signal  $x(t)$  and output signal  $y(t)$ , given by

$$y(t) = \int_{t-T}^t x(u) du \quad (2.0.1)$$

is linear and time invariant in nature.

**Proof.** 1) **Linearity and Time invariance**

From (1), we can say the system is linear if it follows both the laws of Additivity and

Homogeneity.

Law of Additivity:

Let the two input signals be  $x_1(t)$  and  $x_2(t)$ , and their corresponding output signals be  $y_1(t)$  and  $y_2(t)$ , then:

$$y_1(t) = \int_{t-T}^t x_1(u) du \quad (2.0.2)$$

$$y_2(t) = \int_{t-T}^t x_2(u) du \quad (2.0.3)$$

$$y_1(t) + y_2(t) = \int_{t-T}^t [x_1(u) + x_2(u)] du \quad (2.0.4)$$

Now, consider the input signal of  $x_1(t) + x_2(t)$ , then the corresponding output signal is given by  $y'(t)$ :

$$y'(t) = \int_{t-T}^t [x_1(u) + x_2(u)] du \quad (2.0.5)$$

Clearly, from (2.0.4) and (2.0.5):

$$y'(t) = y_1(t) + y_2(t) \quad (2.0.6)$$

Thus, the Law of Additivity holds.

Law of Homogeneity:

Consider an input signal  $kx(t)$ , where  $k$  is any constant. Let the corresponding output be given by  $y'(t)$ , then:

$$y'(t) = \int_{t-T}^t kx(u) du \quad (2.0.7)$$

$$= k \int_{t-T}^t x(u) du \quad (2.0.8)$$

$$= ky(t) \quad (2.0.9)$$

Clearly, from (2.0.9),

$$y'(t) = ky(t) \quad (2.0.10)$$

Thus, the Law of Homogeneity holds.

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

From (2) , to check for time-invariance, we would introduce a delay of  $t_0$  in the output and input signals.

Delay in output signal:

$$y(t - t_0) = \int_{t-t_0-T}^{t-t_0} x(u) du \quad (2.0.11)$$

Now, we consider an input signal with a delay of  $t_0$ , given by  $x(t - t_0)$ , and let the corresponding output signal be given by  $y'(t)$ , then:

$$y'(t) = \int_{t-T}^t x(u - t_0) du \quad (2.0.12)$$

Substituting  $a = u - t_0$ :

$$y'(t) = \int_{t-t_0-T}^{t-t_0} x(a) da \quad (2.0.13)$$

Clearly, from (2.0.11) and (2.0.13):

$$y'(t) = y(t - t_0) \quad (2.0.14)$$

Thus, the system is **time-invariant**.

The correct option is **2) Linear and Time-invariant**

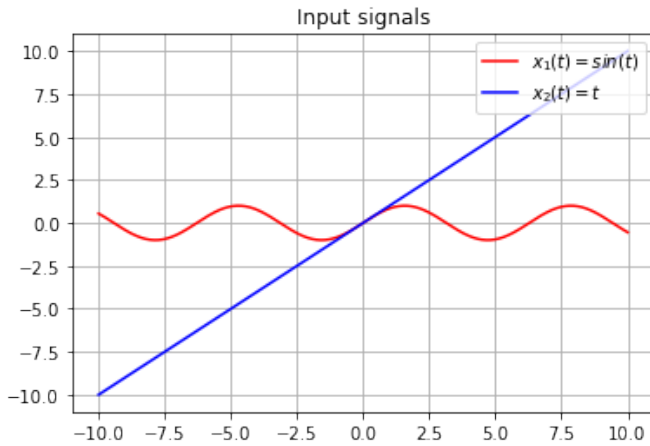


Fig. 1:  $x_1(t) = \sin t$  and  $x_2(t) = t$

## 2) Calculating impulse response of LTI system

Since the given system is an LTI system, it would possess an impulse response  $h(t)$ , which is the output of the system when the input signal is the Impulse function, given by  $\delta(t)$ . Thus,

$$h(t) = \int_{t-T}^t \delta(u) du \quad (2.0.15)$$

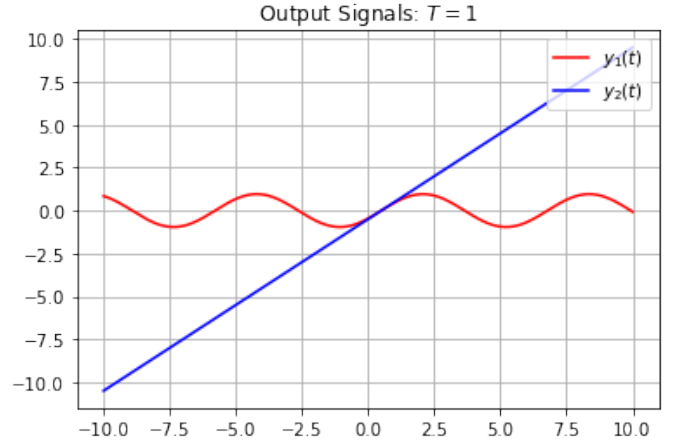


Fig. 1:  $y_1(t)$  and  $y_2(t)$

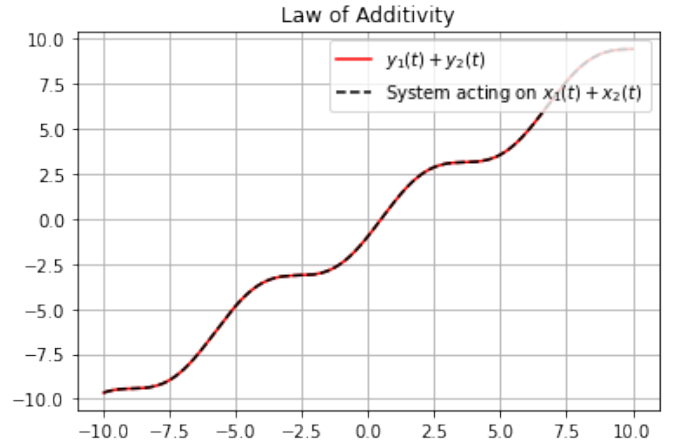


Fig. 1: Law of Additivity

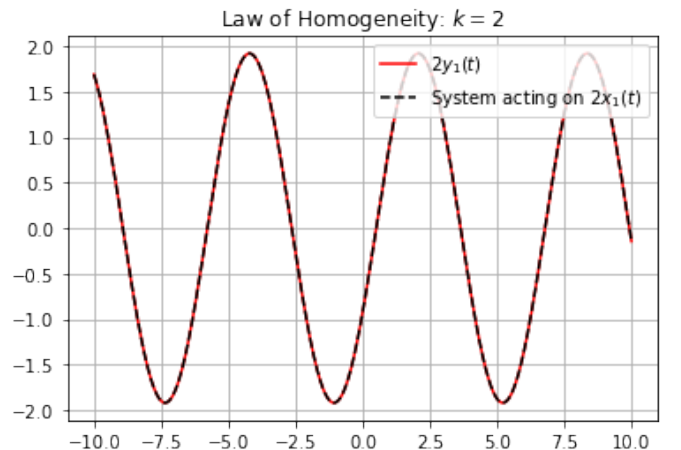


Fig. 1: Law of Homogeneity

The Impulse function can be loosely defined

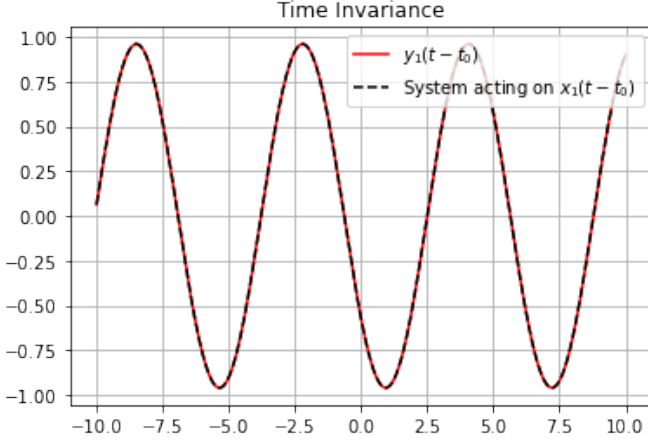


Fig. 1: Time invariance

as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (2.0.16)$$

Since the Impulse function is zero everywhere aside from  $t = 0$ , the non-zero value of integration is a result of  $\delta(0)$ . Thus, we can say  $h(t)$  will be non-zero only if the limits of integration would include  $t = 0$ , i.e:

$$h(t) = \begin{cases} \int_{t-T}^t \delta(u) du & t - T < 0; t > 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.0.17)$$

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \quad (2.0.18)$$

### 3) Expressing the impulse function in terms of $u(t)$

The unit step signal,  $u(t)$ , is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.0.19)$$

On time-shifting  $u(t)$  by  $T$ , we get:

$$u(t - T) = \begin{cases} 1 & t - T \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & t \geq T \\ 0 & \text{otherwise} \end{cases} \quad (2.0.20)$$

On subtracting (2.0.19) and (2.0.20), we get our impulse response  $h(t)$  in terms of the unit step signal:

$$h(t) = u(t) - u(t - T) \quad (2.0.21)$$

### 4) Expressing the impulse function in terms of $rect(t)$

The unit rectangular signal,  $rect(t)$  is given by:

$$rect(t) = \begin{cases} 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (2.0.22)$$

We can obtain the impulse response  $h(t)$  in terms of  $rect(t)$  using time scaling and shifting as follows:

$$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1 & -\frac{1}{2} \leq \frac{t}{\tau} \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases} \quad (2.0.23)$$

Substituting  $\tau = T$ :

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1 & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \quad (2.0.24)$$

Now, we want to right-shift the signal by  $\frac{T}{2}$ :

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} = h(t) \quad (2.0.25)$$

Since the time shifting is to be performed on the variable  $t$  and not  $\frac{1}{T}$

### 5) Calculating the Fourier Transform of $h(t)$

Let the Fourier Transform of  $h(t)$  be given by  $H(f)$  and of the rectangular signal,  $rect(t)$  be given by  $Y(f)$ .

$$h(t) \xrightarrow{\mathcal{F}} H(f) \quad (2.0.26)$$

$$rect(t) \xrightarrow{\mathcal{F}} Y(f) \quad (2.0.27)$$

Then,

$$Y(f) = \int_{-\infty}^{\infty} rect(t) e^{-j2\pi ft} dt \quad (2.0.28)$$

From (2.0.22), we can write (2.0.28) as:

$$Y(f) = \int_{-\infty}^{-\frac{1}{2}} 0 dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi ft} dt + \int_{\frac{1}{2}}^{\infty} 0 dt \quad (2.0.29)$$

$$= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} \quad (2.0.30)$$

$$= \frac{2j \sin \pi f}{j2\pi f} \quad (2.0.31)$$

$$= \frac{\sin(\pi f)}{\pi f} \quad (2.0.32)$$

$$= \text{sinc}(f) \quad (2.0.33)$$

where  $\text{sinc}(t)$ , the sampling function is defined as:

$$\text{sinc}(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & \text{otherwise} \end{cases} \quad (2.0.34)$$

Let the Fourier Transform of a signal  $x(t)$  be  $X(f)$ .

$$x(t) \xrightarrow{\mathcal{F}} X(f) \quad (2.0.35)$$

When the signal  $x(t)$  is time shifted by  $t_0$ , the resultant Fourier Transform is given by:

$$x(t \pm t_0) \xrightarrow{\mathcal{F}} X(f)e^{\pm j2\pi f t_0} \quad (2.0.36)$$

And when the signal  $x(t)$  is time scaled by  $\alpha$ , the resulting Fourier Transform is given by:

$$x(\alpha t) \xrightarrow{\mathcal{F}} \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right) \quad (2.0.37)$$

Since we have already derived the Fourier Transform of  $\text{rect}(t)$ , we would use the properties mentioned above to find the Fourier Transform of  $h(t)$ :

$$\text{rect}(t) \xrightarrow{\mathcal{F}} \text{sinc}(f) \quad (2.0.38)$$

Using (2.0.36):

$$\text{rect}\left(t - \frac{T}{2}\right) \xrightarrow{\mathcal{F}} \text{sinc}(f)e^{-j(2\pi f)\frac{T}{2}} \quad (2.0.39)$$

$$\text{rect}\left(t - \frac{T}{2}\right) \xrightarrow{\mathcal{F}} \text{sinc}(f)e^{-j\pi f T} \quad (2.0.40)$$

Using (2.0.37),

$$\text{rect}\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) \xrightarrow{\mathcal{F}} \frac{1}{|T|} \text{sinc}\left(\frac{f}{T}\right)e^{-j\pi f T} \quad (2.0.41)$$

$$h(t) \xrightarrow{\mathcal{F}} T \text{sinc}\left(\frac{f}{T}\right)e^{-j\pi f} \quad (2.0.42)$$

$$\therefore H(f) = T \text{sinc}\left(\frac{f}{T}\right)e^{-j\pi f} \quad (2.0.43)$$

## 6) An example

Consider an input signal of  $x(t) = \cos 2\pi f_0 t$ . The Fourier Transform of  $x(t)$  is given by:

$$x(t) = \cos 2\pi f_0 t \xrightarrow{\mathcal{F}} \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)] \quad (2.0.44)$$

using the fact that

$$\cos 2\pi f_0 t = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \quad (2.0.45)$$

and the Fourier Transform of  $e^{\pm j2\pi f_0 t}$  is given by:

$$e^{\pm j2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f \mp f_0) \quad (2.0.46)$$

The output signal will be given by:

$$y(t) = \int_{t-T}^t \cos 2\pi f_0 u \, du \quad (2.0.47)$$

$$= \frac{1}{2\pi f_0} [\sin 2\pi f_0 t - \sin 2\pi f_0 (t - T)] \quad (2.0.48)$$

$$= \frac{\sin \pi f_0 T}{\pi f_0} \left[ \cos 2\pi f_0 \left(t - \frac{T}{2}\right) \right] \quad (2.0.49)$$

$$= T \text{sinc}(f_0 T) \cos 2\pi f_0 \left(t - \frac{T}{2}\right) \quad (2.0.50)$$

The Fourier transform of  $\cos 2\pi f_0 \left(t - \frac{T}{2}\right)$  can be obtained using (2.0.37) and (2.0.36) as follows:

$$\cos t = \frac{1}{2} [e^{jt} + e^{-jt}] \quad (2.0.51)$$

$$\cos t \xrightarrow{\mathcal{F}} \frac{1}{2} \left[ \delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right) \right] \quad (2.0.52)$$

$$\cos\left(t - \frac{T}{2}\right) \xrightarrow{\mathcal{F}} \frac{e^{j\pi f T}}{2} \left[ \delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right) \right] \quad (2.0.53)$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \xrightarrow{\mathcal{F}} \frac{e^{j\pi \frac{f}{2f_0} T}}{2\pi f_0} \frac{\delta\left(\frac{f}{2\pi f_0} - \frac{1}{2\pi}\right) + \delta\left(\frac{f}{2\pi f_0} + \frac{1}{2\pi}\right)}{2} \quad (2.0.54)$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \xrightarrow{\mathcal{F}} \frac{e^{j\pi \frac{f}{2f_0} T}}{4\pi f_0} \left( \delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right) \right) \quad (2.0.55)$$

Therefore, the Fourier Transform of the output signal  $y(t)$  from (2.0.50) is given by:

$$y(t) \xrightarrow{\mathcal{F}} \frac{T \text{sinc}(f_0 T)}{4\pi f_0} e^{j\pi \frac{f}{2f_0} T} \left( \delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right) \right) \quad (2.0.56)$$

$$y(t) \xrightarrow{\mathcal{F}} k e^{j\pi \frac{f}{2f_0} T} \left( \delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right) \right) \quad (2.0.57)$$

where  $k = \frac{T \text{sinc}(f_0 T)}{4\pi f_0}$ . Substituting  $2\pi f_0 = 1$  and  $T = 1$ :

$$y(t) \xrightarrow{\mathcal{F}} k e^{j\pi^2 f} \left( \delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right) \right) \quad (2.0.58)$$

$$y(t) \xrightarrow{\mathcal{F}} k e^{j\frac{\pi}{2}} \delta\left(f - \frac{1}{2\pi}\right) + k e^{j\frac{\pi}{2}} \delta\left(f + \frac{1}{2\pi}\right) \quad (2.0.59)$$

using the multiplication property of the Delta function:

$$x(t)\delta(t - t_1) = x(t_1)\delta(t - t_1) \quad (2.0.60)$$

Since,  $e^{j\frac{\pi}{2}} = j$  and  $e^{-j\frac{\pi}{2}} = -j$ , we finally get:

$$y(t) \xrightarrow{\mathcal{F}} k j \left[ \delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right) \right] \quad (2.0.61)$$

Clearly, the Fourier transform of  $y(t)$  can be manipulated

to represent a sinusoidal wave, which is given by:

$$\sin(t) \stackrel{\mathcal{F}}{\Leftrightarrow} \frac{-j}{2} \left[ \delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right) \right] \quad (2.0.62)$$

The attenuation happens for the same values of  $f$ , as depicted in the graphs of the Fourier Transforms given below.

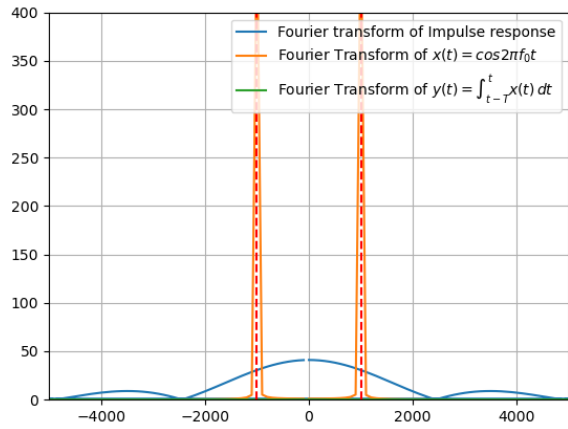


Fig. 6: Fourier Transform of Impulse response  $h(t)$

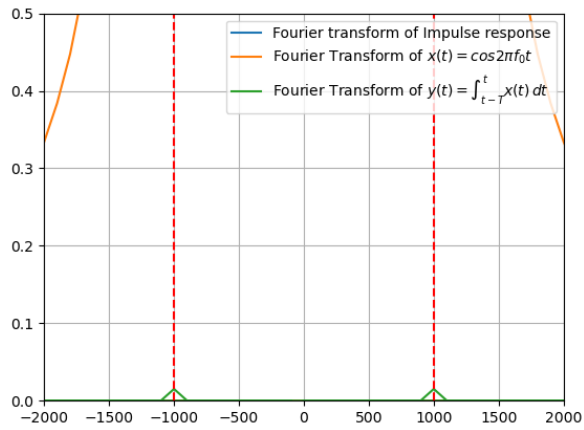


Fig. 6: Fourier Transform of Impulse response  $h(t)$  zoomed in

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