Gate Assignment 4

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1 Problem

(EC-2001/Q.16) The Fourier Transform $G(\omega)$ of the signal $g(t)$ is given by

$$G(\omega) = \frac{1}{\omega^2}(e^{j\omega} - j\omega e^{j\omega} - 1) \qquad (1.0.1)$$

Using this information, find the Fourier Transforms of the signals $g_1(t)$, $g_2(t)$ and $g_3(t)$.
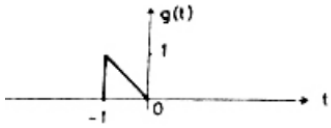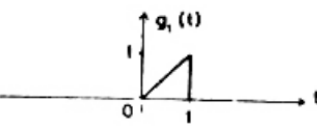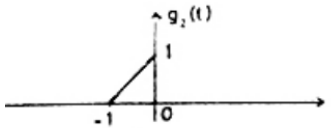

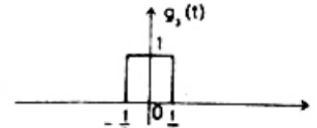Fig. 16 (a)

Fig. 16 (b)

19.

Fig. 15 (c)

Fig. 16 (d)

2 Solution

We replace ω by $2\pi f$. Then,

$$G(f) = \frac{1}{4\pi^2 f^2}(e^{2\pi jf} - 2\pi jf e^{2\pi jf} - 1) \qquad (2.0.1)$$

Lemma 2.1. *If*

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f) \qquad (2.0.2)$$

then,

$$g(t \pm t_0) \overset{\mathcal{F}}{\rightleftharpoons} G(f)e^{\pm 2\pi jft_0} \qquad (2.0.3)$$

Proof. We know,

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi jft}\,dt \qquad (2.0.4)$$

Let

$$g(t + t_0) \overset{\mathcal{F}}{\rightleftharpoons} G'(f) \qquad (2.0.5)$$

Then,

$$G'(f) = \int_{-\infty}^{\infty} g(t + t_0)e^{-2\pi jft}\,dt \qquad (2.0.6)$$

Substituting $t + t_0 = T$, we get:

$$G'(f) = \int_{-\infty}^{\infty} g(T)e^{-2\pi jf(T-t_0)}\,dT \qquad (2.0.7)$$

$$= \int_{-\infty}^{\infty} g(T)e^{-2\pi jfT}e^{2\pi jft_0}\,dT \qquad (2.0.8)$$

$$= e^{2\pi jft_0}\int_{-\infty}^{\infty} g(T)e^{2\pi jfT}\,dT \qquad (2.0.9)$$

$$= e^{-2\pi jft_0}G(f) \qquad (2.0.10)$$

Similarly, it can be proved:

$$g(t - t_0) \overset{\mathcal{F}}{\rightleftharpoons} e^{-2\pi jft_0}G(f) \qquad (2.0.11)$$

$\square$

Lemma 2.2. *If*

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f) \qquad (2.0.12)$$

then,

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|}G\left(\frac{f}{\alpha}\right) \qquad (2.0.13)$$

Proof. Consider $\alpha > 0$. Then, we know,

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi jft}\,dt \qquad (2.0.14)$$

Let

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} G'(f) \qquad (2.0.15)$$

Then,

$$G'(f) = \int_{-\infty}^{\infty} g(\alpha t)e^{-2\pi jft}\, dt \qquad (2.0.16)$$

Making the substitution $T = \alpha t$, we get:

$$G'(f) = \frac{1}{\alpha} \int_{-\infty}^{\infty} g(T)e^{-2\pi j\frac{fT}{\alpha}}\, dT \qquad (2.0.17)$$

$$= \frac{1}{\alpha}G\left(\frac{f}{\alpha}\right) \qquad (2.0.18)$$

Similarly, it can be proved for $\alpha < 0$

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{-1}{\alpha}G\left(\frac{-f}{\alpha}\right) \qquad (2.0.19)$$

$\square$

Lemma 2.3. *If*

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f) \qquad (2.0.20)$$

then,

$$g(-t) \overset{\mathcal{F}}{\rightleftharpoons} G(-f) \qquad (2.0.21)$$

Proof. Put $\alpha = -1$ in (2.2) to obtain the result.

$$g(-t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|-1|}G\left(\frac{f}{-1}\right) \qquad (2.0.22)$$

$$g(-t) \overset{\mathcal{F}}{\rightleftharpoons} G(-f) \qquad (2.0.23)$$

$\square$

Now, from the figure:

$$g(t) = \begin{cases} -t & -1 \le t \le 0 \\ 0 & otherwise \end{cases} \qquad (2.0.24)$$

$$g_1(t) = \begin{cases} t & 0 \le t \le 1 \\ 0 & otherwise \end{cases} \qquad (2.0.25)$$

$$g_2(t) = \begin{cases} 1+t & -1 \le t \le 0 \\ 0 & otherwise \end{cases} \qquad (2.0.26)$$

$$g_3(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & otherwise \end{cases} \qquad (2.0.27)$$

Clearly, $g_1(t) = g(-t)$, and using (2.3), we get:

$$G_1(f) = G(-f) = \frac{1}{4\pi^2 f^2}(e^{-2\pi jf} + 2\pi jfe^{-2\pi jf} - 1) \qquad (2.0.28)$$

Also, $g_2(t) = g(-t-1)$. Thus, from (2.1) and (2.3), we get:

$$g(t-1) \overset{\mathcal{F}}{\rightleftharpoons} e^{-2\pi jf}G(f) \qquad (2.0.29)$$

$$g(-t-1) \overset{\mathcal{F}}{\rightleftharpoons} e^{2\pi jf}G(-f) \qquad (2.0.30)$$

$$g_2(t) \overset{\mathcal{F}}{\rightleftharpoons} e^{2\pi jf}G(-f) \qquad (2.0.31)$$

$$g_2(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{4\pi^2 f^2}(1 + 2\pi jf - e^{2\pi jf}) \qquad (2.0.32)$$

$$\implies G_2(\omega) = \frac{1}{4\pi^2 f^2}(1 + 2\pi jf - e^{2\pi jf}) \qquad (2.0.33)$$

Also, $g_3(t) = g\left(t - \frac{1}{2}\right) + g\left(-t - \frac{1}{2}\right)$. Thus,

$$g\left(t - \frac{1}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} e^{-j\pi f}G(f) \qquad (2.0.34)$$

$$g\left(-t - \frac{1}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} e^{j\pi f}G(-f) \qquad (2.0.35)$$

$$g_3(t) \overset{\mathcal{F}}{\rightleftharpoons} e^{-j\pi f}G(f) + e^{j\pi f}G(-f) \qquad (2.0.36)$$

$$G_3(f) = \frac{e^{-j\pi f}}{4\pi^2 f^2}\left[e^{2\pi jf} - 2\pi jfe^{2\pi jf} - 1\right] + \frac{e^{j\pi f}}{4\pi^2 f^2}\left[e^{-2\pi jf} + 2\pi jfe^{-2\pi jf} - 1\right] \qquad (2.0.37)$$

$$G_3(f) = \frac{j}{2\pi f}\left[e^{-\pi jf} - e^{\pi jf}\right] \qquad (2.0.38)$$

$$G_3(f) = \frac{\sin \pi f}{\pi f} = sinc(f) \qquad (2.0.39)$$

where $sinc(t)$, the sampling function is defined as:

$$sinc(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & otherwise \end{cases} \qquad (2.0.40)$$

$g(t)$

$G_1(f)$

$G(f)$

$g_2(t)$

$g_1(t)$

$G_2(f)$

$g_3(t)$

$G_3(f)$