

Gate Assignment 1

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Download all python codes from

<https://github.com/tanmaygoyal258/EE3900-Linear-Systems-and-Signal-processing/blob/main/GateAssignment1/code.py>

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<https://github.com/tanmaygoyal258/EE3900-Linear-Systems-and-Signal-processing/blob/main/GateAssignment1/main.tex>

1 PROBLEM

(EC 2017- Q.7) The input $x(t)$ and output $y(t)$ of a continous time signal are related as:

$$y(t) = \int_{t-T}^t x(u) du \quad (1.0.1)$$

The system is:

- 1) Linear and Time-variant
- 2) Linear and Time-invariant
- 3) Non-Linear and Time-variant
- 4) Non-Linear and Time-invariant

2 SOLUTION

Definition 1. We say that a system is **linear** if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.

Definition 2. A system is said to be **time invariant** if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.

Lemma 2.1. The system relating the input signal $x(t)$ and output signal $y(t)$, given by

$$y(t) = \int_{t-T}^t x(u) du \quad (2.0.1)$$

is linear and time invariant in nature.

Proof. 1) **Linearity and Time invariance**

From (1), we can say the system is linear if it follows both the laws of Additivity and

Homogeneity.

Law of Additivity:

Let the two input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{t-T}^t x_1(u) du \quad (2.0.2)$$

$$y_2(t) = \int_{t-T}^t x_2(u) du \quad (2.0.3)$$

$$y_1(t) + y_2(t) = \int_{t-T}^t [x_1(u) + x_2(u)] du \quad (2.0.4)$$

Now, consider the input signal of $x_1(t) + x_2(t)$, then the corresponding output signal is given by $y'(t)$:

$$y'(t) = \int_{t-T}^t [x_1(u) + x_2(u)] du \quad (2.0.5)$$

Clearly, from (2.0.4) and (2.0.5):

$$y'(t) = y_1(t) + y_2(t) \quad (2.0.6)$$

Thus, the Law of Additivity holds.

Law of Homogeneity:

Consider an input signal $kx(t)$, where k is any constant. Let the corresponding output be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^t kx(u) du \quad (2.0.7)$$

$$= k \int_{t-T}^t x(u) du \quad (2.0.8)$$

$$= ky(t) \quad (2.0.9)$$

Clearly, from (2.0.9),

$$y'(t) = ky(t) \quad (2.0.10)$$

Thus, the Law of Homogeneity holds.

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

From (2) , to check for time-invariance, we would introduce a delay of t_0 in the output and input signals.

Delay in output signal:

$$y(t - t_0) = \int_{t-t_0-T}^{t-t_0} x(u) du \quad (2.0.11)$$

Now, we consider an input signal with a delay of t_0 , given by $x(t - t_0)$, and let the corresponding output signal be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^t x(u - t_0) du \quad (2.0.12)$$

Substituting $a = u - t_0$:

$$y'(t) = \int_{t-t_0-T}^{t-t_0} x(a) da \quad (2.0.13)$$

Clearly, from (2.0.11) and (2.0.13):

$$y'(t) = y(t - t_0) \quad (2.0.14)$$

Thus, the system is **time-invariant**.

The correct option is **2) Linear and Time-invariant**

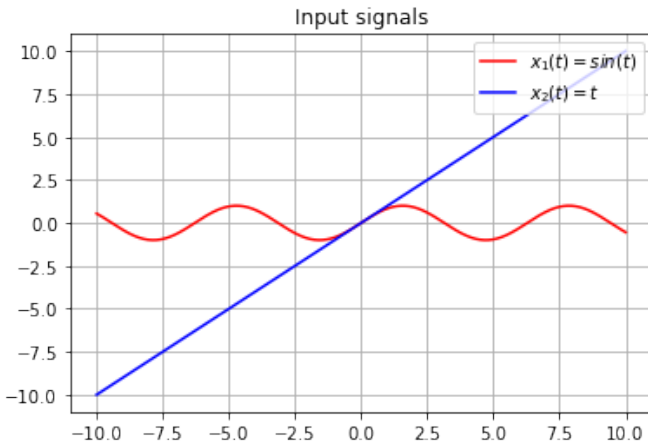


Fig. 1: $x_1(t) = \sin t$ and $x_2(t) = t$

2) Calculating impulse response of LTI system

Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given by $\delta(t)$. Thus,

$$h(t) = \int_{t-T}^t \delta(u) du \quad (2.0.15)$$

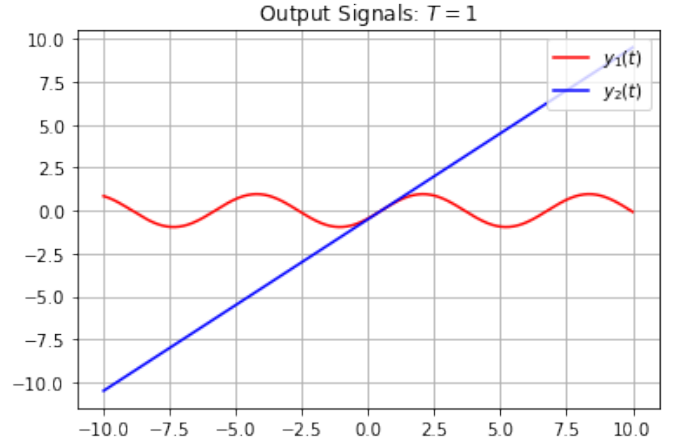


Fig. 1: $y_1(t)$ and $y_2(t)$

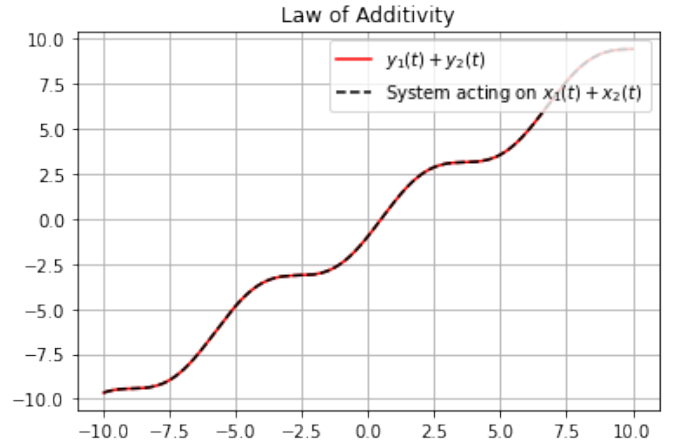


Fig. 1: Law of Additivity

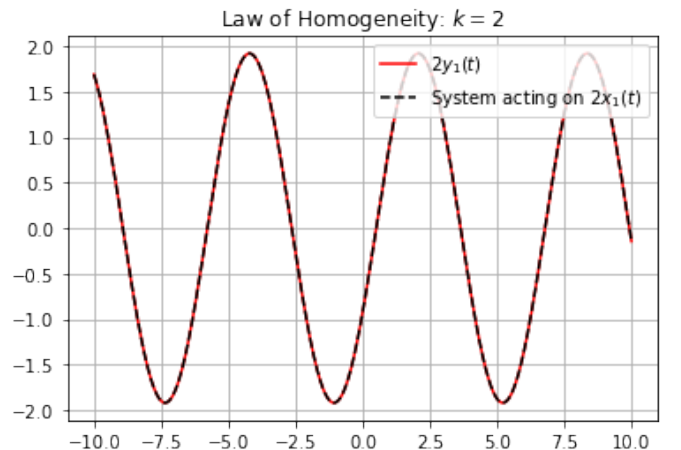


Fig. 1: Law of Homogeneity

The Impulse function can be loosely defined

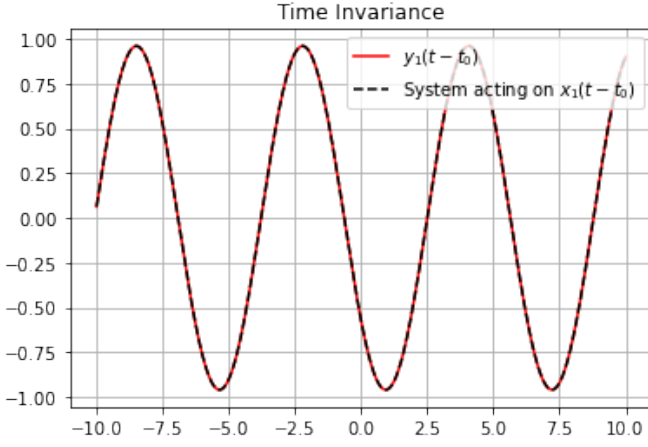


Fig. 1: Time invariance

as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (2.0.16)$$

Since the Impulse function is zero everywhere aside from $t = 0$, the non-zero value of integration is a result of $\delta(0)$. Thus, we can say $h(t)$ will be non-zero only if the limits of integration would include $t = 0$, i.e:

$$h(t) = \begin{cases} \int_{t-T}^t \delta(u) du & t - T < 0; t > 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.0.17)$$

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \quad (2.0.18)$$

3) Expressing the impulse function in terms of $u(t)$

The unit step signal, $u(t)$, is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.0.19)$$

On time-shifting $u(t)$ by T , we get:

$$u(t - T) = \begin{cases} 1 & t - T \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & t \geq T \\ 0 & \text{otherwise} \end{cases} \quad (2.0.20)$$

On subtracting (2.0.19) and (2.0.20), we get our impulse response $h(t)$ in terms of the unit step signal:

$$h(t) = u(t) - u(t - T) \quad (2.0.21)$$

4) Expressing the impulse function in terms of $rect(t)$

The unit rectangular signal, $rect(t)$ is given by:

$$rect(t) = \begin{cases} 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (2.0.22)$$

We can obtain the impulse response $h(t)$ in terms of $rect(t)$ using time scaling and shifting as follows:

$$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1 & -\frac{1}{2} \leq \frac{t}{\tau} \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases} \quad (2.0.23)$$

Substituting $\tau = T$:

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1 & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \quad (2.0.24)$$

Now, we want to right-shift the signal by $\frac{T}{2}$:

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} = h(t) \quad (2.0.25)$$

Since the time shifting is to be performed on the variable t and not $\frac{t}{T}$

5) Calculating the Fourier Transform of $h(t)$

Let the Fourier Transform of $h(t)$ be given by $H(f)$ and of the rectangular signal, $rect(t)$ be given by $Y(f)$.

$$h(t) \xrightarrow{\mathcal{F}} H(f) \quad (2.0.26)$$

$$rect(t) \xrightarrow{\mathcal{F}} Y(f) \quad (2.0.27)$$

Then,

$$Y(f) = \int_{-\infty}^{\infty} rect(t) e^{-j2\pi ft} dt \quad (2.0.28)$$

From (2.0.22), we can write (2.0.28) as:

$$Y(f) = \int_{-\infty}^{-\frac{1}{2}} 0 dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi ft} dt + \int_{\frac{1}{2}}^{\infty} 0 dt \quad (2.0.29)$$

$$= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} \quad (2.0.30)$$

$$= \frac{2j \sin \pi f}{j2\pi f} \quad (2.0.31)$$

$$= \frac{\sin(\pi f)}{\pi f} \quad (2.0.32)$$

$$= \text{sinc}(f) \quad (2.0.33)$$

where $\text{sinc}(t)$, the sampling function is defined as:

$$\text{sinc}(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & \text{otherwise} \end{cases} \quad (2.0.34)$$

Let the Fourier Transform of a signal $x(t)$ be $X(f)$.

$$x(t) \xrightarrow{\mathcal{F}} X(f) \quad (2.0.35)$$

When the signal $x(t)$ is time shifted by t_0 , the resultant Fourier Transform is given by:

$$x(t \pm t_0) \xrightarrow{\mathcal{F}} X(f)e^{\pm j2\pi f t_0} \quad (2.0.36)$$

And when the signal $x(t)$ is time scaled by α , the resulting Fourier Transform is given by:

$$x(\alpha t) \xrightarrow{\mathcal{F}} \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right) \quad (2.0.37)$$

Since we have already derived the Fourier Transform of $\text{rect}(t)$, we would use the properties mentioned above to find the Fourier Transform of $h(t)$:

$$\text{rect}(t) \xrightarrow{\mathcal{F}} \text{sinc}(f) \quad (2.0.38)$$

Using (2.0.36):

$$\text{rect}\left(t - \frac{T}{2}\right) \xrightarrow{\mathcal{F}} \text{sinc}(f)e^{-j(2\pi f)\frac{T}{2}} \quad (2.0.39)$$

$$\text{rect}\left(t - \frac{T}{2}\right) \xrightarrow{\mathcal{F}} \text{sinc}(f)e^{-j\pi f T} \quad (2.0.40)$$

Using (2.0.37),

$$\text{rect}\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) \xrightarrow{\mathcal{F}} \frac{1}{|T|} \text{sinc}\left(\frac{f}{T}\right)e^{-\frac{j\pi f T}{T}} \quad (2.0.41)$$

$$h(t) \xrightarrow{\mathcal{F}} T \text{sinc}\left(\frac{f}{T}\right)e^{-j\pi f} \quad (2.0.42)$$

$$\therefore H(f) = T \text{sinc}\left(\frac{f}{T}\right)e^{-j\pi f} \quad (2.0.43)$$

6) An example

Consider an input signal of $x(t) = \cos 2\pi f_0 t$. The Fourier Transform of $x(t)$ is given by:

$$x(t) = \cos 2\pi f_0 t \xrightarrow{\mathcal{F}} \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)] \quad (2.0.44)$$

using the fact that

$$\cos 2\pi f_0 t = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \quad (2.0.45)$$

and the Fourier Transform of $e^{\pm j2\pi f_0 t}$ is given by:

$$e^{\pm j2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f \mp f_0) \quad (2.0.46)$$

The output signal will be given by:

$$y(t) = \int_{t-T}^t \cos 2\pi f_0 u \, du \quad (2.0.47)$$

$$= \frac{1}{2\pi f_0} [\sin 2\pi f_0 t - \sin 2\pi f_0 (t - T)] \quad (2.0.48)$$

$$= \frac{\sin \pi f_0 T}{\pi f_0} \left[\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \right] \quad (2.0.49)$$

$$= T \text{sinc}(f_0 T) \cos 2\pi f_0 \left(t - \frac{T}{2}\right) \quad (2.0.50)$$

The Fourier transform of $\cos 2\pi f_0 \left(t - \frac{T}{2}\right)$ can be obtained using (2.0.37) and (2.0.36) as follows:

$$\cos t = \frac{1}{2} [e^{jt} + e^{-jt}] \quad (2.0.51)$$

$$\cos t \xrightarrow{\mathcal{F}} \frac{1}{2} \left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right) \right] \quad (2.0.52)$$

$$\cos\left(t - \frac{T}{2}\right) \xrightarrow{\mathcal{F}} \frac{e^{j\pi f T}}{2} \left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right) \right] \quad (2.0.53)$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \xrightarrow{\mathcal{F}} \frac{e^{j\pi \frac{f}{2f_0} T}}{2\pi f_0} \frac{\delta\left(\frac{f}{2\pi f_0} - \frac{1}{2\pi}\right) + \delta\left(\frac{f}{2\pi f_0} + \frac{1}{2\pi}\right)}{2} \quad (2.0.54)$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \xrightarrow{\mathcal{F}} \frac{e^{j\pi \frac{f}{2f_0} T}}{4\pi f_0} \left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right) \right) \quad (2.0.55)$$

Therefore, the Fourier Transform of the output signal $y(t)$ from (2.0.50) is given by:

$$y(t) \xrightarrow{\mathcal{F}} \frac{T \text{sinc}(f_0 T)}{4\pi f_0} e^{j\pi \frac{f}{2f_0} T} \left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right) \right) \quad (2.0.56)$$

$$y(t) \xrightarrow{\mathcal{F}} k e^{j\pi \frac{f}{2f_0} T} \left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right) \right) \quad (2.0.57)$$

where $k = \frac{T \text{sinc}(f_0 T)}{4\pi f_0}$. Substituting $2\pi f_0 = 1$ and $T = 1$:

$$y(t) \xrightarrow{\mathcal{F}} k e^{j\pi^2 f} \left(\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right) \right) \quad (2.0.58)$$

$$y(t) \xrightarrow{\mathcal{F}} k e^{j\frac{\pi}{2}} \delta\left(f - \frac{1}{2\pi}\right) + k e^{j\frac{\pi}{2}} \delta\left(f + \frac{1}{2\pi}\right) \quad (2.0.59)$$

using the multiplication property of the Delta function:

$$x(t)\delta(t - t_1) = x(t_1)\delta(t - t_1) \quad (2.0.60)$$

Since, $e^{j\frac{\pi}{2}} = j$ and $e^{-j\frac{\pi}{2}} = -j$, we finally get:

$$y(t) \xrightarrow{\mathcal{F}} k j \left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right) \right] \quad (2.0.61)$$

Clearly, the Fourier transform of $y(t)$ can be manipulated

to represent a sinusoidal wave, which is given by:

$$\sin(t) \stackrel{\mathcal{F}}{\Leftrightarrow} \frac{-j}{2} \left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right) \right] \quad (2.0.62)$$

The attenuation happens for the same values of f , as depicted in the graphs of the Fourier Transforms given below.

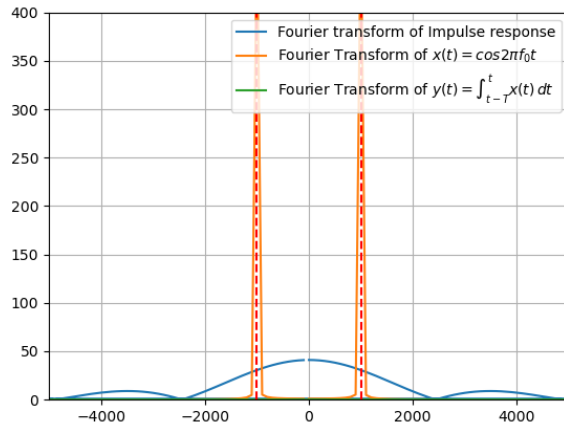


Fig. 6: Fourier Transform of Impulse response $h(t)$

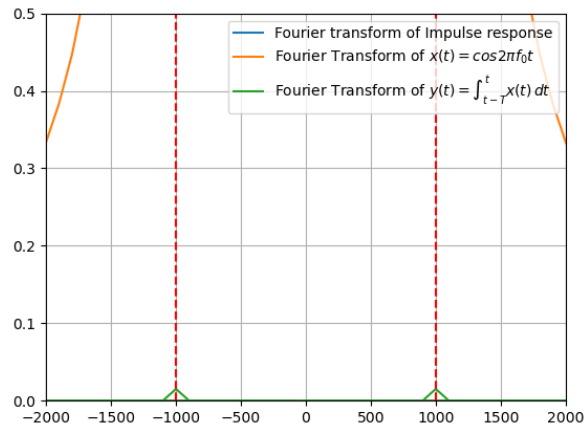


Fig. 6: (6) zoomed in

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