Gate Assignment 4

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Download all latex codes from

https://github.com/tanmaygoyal258/EE3900−Linear−Systems−and−Signal−processing/blob/main/Quiz1/main.tex

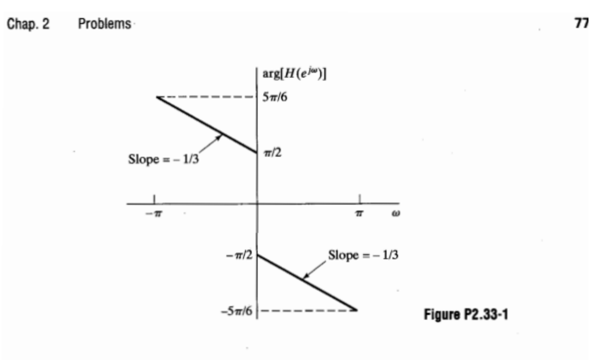
https://github.com/tanmaygoyal258/EE3900−Linear−Systems−and−Signal−processing/blob/main/Quiz1/code.py

1 Problem

(Oppenheim/2.33) Consider an LTI system with $\left|H(e^{j\omega})\right| = 1$ and let $arg[H(e^{j\omega})]$ be shown in the figure. If the input is:

$$x[n] = \cos\left(\frac{3\pi}{2}n + \frac{\pi}{4}\right) \tag{1.0.1}$$

find $y[n]$



2 Solution

Lemma 2.1. *If*

$$x(t) \overset{\mathcal{F}}{\rightleftharpoons} X(j\omega) \tag{2.0.1}$$

then,

$$x^*(t) \overset{\mathcal{F}}{\rightleftharpoons} X^*(j\omega) \tag{2.0.2}$$

Proof.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \tag{2.0.3}$$

$$X^*(j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}\,dt \tag{2.0.4}$$

$$X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}\,dt \tag{2.0.5}$$

□

Lemma 2.2. *If $x(t)$ is real, then*

$$X(-j\omega) = X^*(j\omega) \tag{2.0.6}$$

Proof.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \tag{2.0.7}$$

$$X^*(j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}\,dt \tag{2.0.8}$$

$$X^*(j\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt \tag{2.0.9}$$

$$X(-j\omega) = X^*(j\omega) \tag{2.0.10}$$

using the fact that x is real and hence, $x^*(t) = x(t)$

□

Lemma 2.3. *If $x[n] = A\cos(\omega_0 n + \phi)$, and if $h[n]$ is real, i.e $H(e^{-j\omega}) = H^*(e^{j\omega})$, then*

$$y[n] = A\left|H(e^{j\omega_0})\right|\cos\left(\omega_0 n + \phi + arg(H(e^{j\omega_0})\right) \tag{2.0.11}$$

Now, from the figure, we can write

$$arg[H(e^{j\omega})] = \begin{cases} -\frac{w}{3} + \frac{\pi}{2} & -\pi < \omega < 0 \\ -\frac{w}{3} - \frac{\pi}{2} & 0 < \omega < \pi \end{cases} \tag{2.0.12}$$

Thus,

$$H(e^{j\omega}) = \begin{cases} e^{j\left(\frac{-w}{3}+\frac{\pi}{2}\right)} & -\pi < \omega < 0 \\ e^{j\left(\frac{-w}{3}-\frac{\pi}{2}\right)} & 0 < \omega < \pi \end{cases} \tag{2.0.13}$$

$$H(e^{j\omega}) = \begin{cases} je^{-\frac{j\omega}{3}} & -\pi < \omega < 0 \\ -je^{-\frac{j\omega}{3}} & 0 < \omega < \pi \end{cases} \tag{2.0.14}$$

$$\tag{2.0.15}$$

Now,

$$H(e^{-j\omega}) = \begin{cases} je^{\frac{j\omega}{3}} & 0 < \omega < \pi 0 \\ -je^{\frac{j\omega}{3}} & -\pi < \omega < 0 \end{cases} \tag{2.0.16}$$

$$H^*(e^{j\omega}) = \begin{cases} -je^{\frac{j\omega}{3}} & -\pi < \omega < 0 \\ je^{\frac{j\omega}{3}} & 0 < \omega < \pi \end{cases} \tag{2.0.17}$$

$$\implies H(e^{-j\omega}) = H^*(e^{j\omega}) \tag{2.0.18}$$

Thus, using (2.3), we can say:

$$y[n] = 1 \times \left|H\left(e^{\frac{j3\pi}{2}}\right)\right| \cos\left(\frac{3\pi}{2}n + \frac{\pi}{4} + arg\left[H\left(e^{\frac{j3\pi}{2}}\right)\right]\right) \tag{2.0.19}$$

$$= \cos\left(\frac{3\pi}{2}n + \frac{\pi}{4} + arg\left[H\left(e^{\frac{j3\pi}{2}}\right)\right]\right) \tag{2.0.20}$$

Now, to find $arg\left[H\left(e^{\frac{j3\pi}{2}}\right)\right]$, we have to realise that $H(e^{j\omega})$ is periodic with period 2π, and thus from (2.0.12), we get:

$$arg\left[H\left(e^{\frac{j3\pi}{2}}\right)\right] = arg\left[H\left(e^{\frac{j3\pi}{2}} + j2n\pi\right)\right] \tag{2.0.21}$$

$$arg\left[H\left(e^{\frac{j3\pi}{2}}\right)\right] = arg\left[H\left(e^{\frac{-j\pi}{2}}\right)\right] \tag{2.0.22}$$

$$arg\left[H\left(e^{\frac{j3\pi}{2}}\right)\right] = \frac{2\pi}{3} \tag{2.0.23}$$

Thus,

$$y[n] = \cos\left(\frac{3\pi}{2}n + \frac{\pi}{4} + \frac{2\pi}{3}\right) \tag{2.0.24}$$

$$y[n] = \cos\left(\frac{3\pi}{2}n + \frac{11\pi}{12}\right) \tag{2.0.25}$$

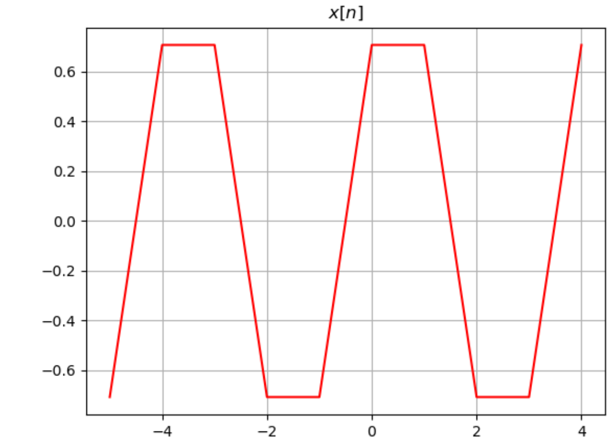


Fig. 0: $x[n]$

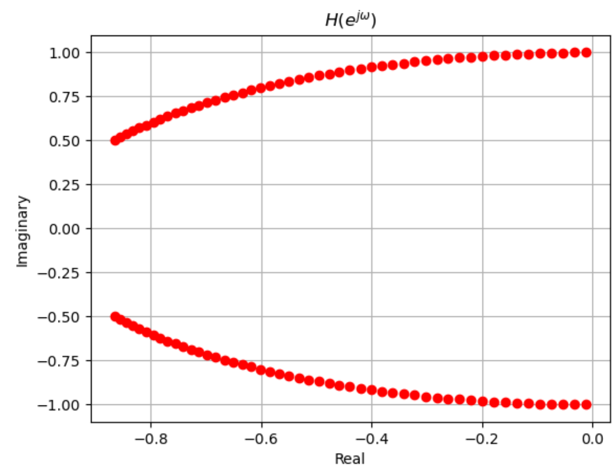


Fig. 0: $H(e^{j\omega})$

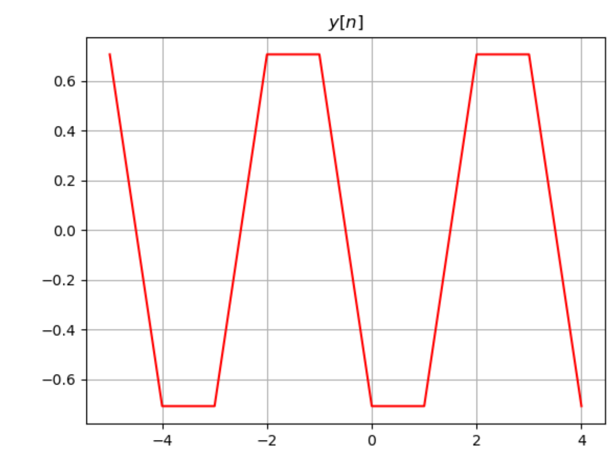


Fig. 0: $y[n]$