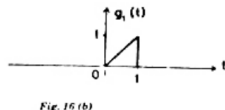
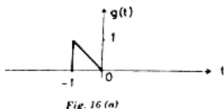


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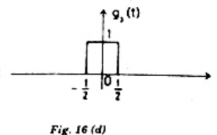
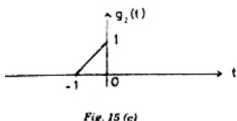
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Question



The Fourier Transform $G(\omega)$ of the signal $g(t)$ is given by

$$19. \quad G(\omega) = \frac{1}{\omega^2}(e^{j\omega} - j\omega e^{j\omega} - 1) \quad (1)$$



Using this information, find the Fourier Transforms of the signals $g_1(t)$, $g_2(t)$ and $g_3(t)$.

Note: We replace ω by $2\pi f$. Then,

$$G(f) = \frac{1}{4\pi^2 f^2}(e^{2\pi jf} - 2\pi jfe^{2\pi jf} - 1) \quad (2)$$

Result of Time Shifting on Fourier Transform

Lemma

If

$$g(t) \xrightarrow{\mathcal{F}} G(f) \quad (3)$$

then,

$$g(t \pm t_0) \xrightarrow{\mathcal{F}} G(f)e^{\pm 2\pi jft_0} \quad (4)$$

Proof.

We know,

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi jft} dt \quad (5)$$



Proof continued

Proof.

Let

$$g(t + t_0) \xrightarrow{\mathcal{F}} G'(f) \quad (6)$$

Then,

$$G'(f) = \int_{-\infty}^{\infty} g(t + t_0) e^{-2\pi jft} dt \quad (7)$$

Substituting $t + t_0 = T$, we get:

$$G'(f) = \int_{-\infty}^{\infty} g(T) e^{-2\pi jf(T-t_0)} dT \quad (8)$$

$$= e^{2\pi jft_0} \int_{-\infty}^{\infty} g(T) e^{2\pi jfT} dT \quad (9)$$

$$= e^{-2\pi jft_0} G(f) \quad (10)$$

Result of Time scaling on Fourier Transform

Lemma

If

$$g(t) \stackrel{\mathcal{F}}{\rightleftharpoons} G(f) \quad (11)$$

then,

$$g(\alpha t) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|} G\left(\frac{f}{\alpha}\right) \quad (12)$$

Proof.

Consider $\alpha > 0$. Then, we know,

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-2\pi jft} dt \quad (13)$$



Proof Continued

Proof.

Let

$$g(\alpha t) \stackrel{\mathcal{F}}{\rightleftharpoons} G'(f) \quad (14)$$

Then,

$$G'(f) = \int_{-\infty}^{\infty} g(\alpha t) e^{-2\pi j f t} dt \quad (15)$$

Making the substitution $T = \alpha t$, we get:

$$G'(f) = \frac{1}{\alpha} \int_{-\infty}^{\infty} g(T) e^{-2\pi j \frac{fT}{\alpha}} dT \quad (16)$$

$$= \frac{1}{\alpha} G\left(\frac{f}{\alpha}\right) \quad (17)$$



Effect of Time Reversal on Fourier Transform

Lemma

If

$$g(t) \stackrel{\mathcal{F}}{\rightleftharpoons} G(f) \quad (18)$$

then,

$$g(-t) \stackrel{\mathcal{F}}{\rightleftharpoons} G(-f) \quad (19)$$

Proof.

Put $\alpha = -1$ in (2) to obtain the result.

$$g(-t) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{1}{|-1|} G\left(\frac{f}{-1}\right) \implies g(-t) \stackrel{\mathcal{F}}{\rightleftharpoons} G(-f) \quad (20)$$



The Various Signals

Now, from the figure:

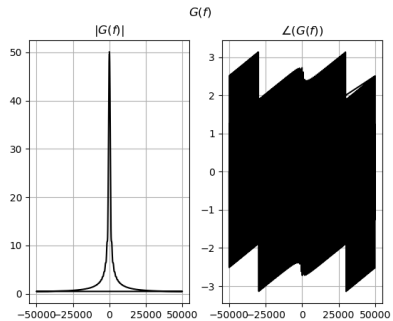
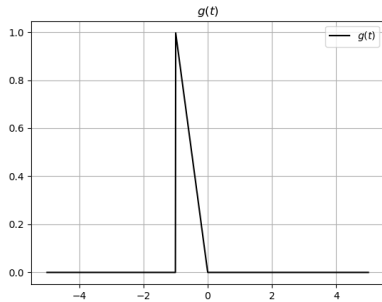
$$g(t) = \begin{cases} -t & -1 \leq t \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

$$g_1(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (22)$$

$$g_2(t) = \begin{cases} 1+t & -1 \leq t \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

$$g_3(t) = \begin{cases} 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (24)$$

$$g(t)$$

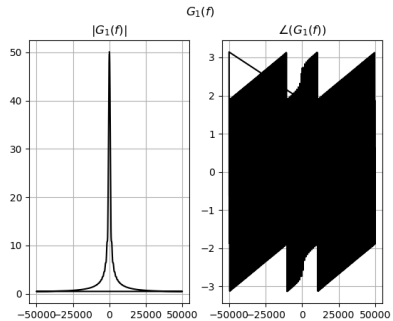
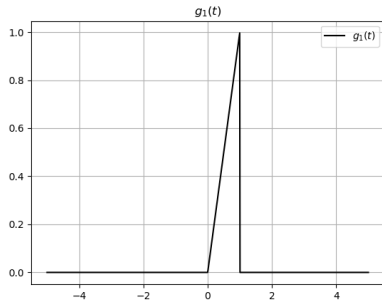


$$G_1(\omega)$$

Clearly, $g_1(t) = g(-t)$, and using (3), we get:

$$G_1(f) = G(-f) = \frac{1}{4\pi^2 f^2} (e^{-2\pi j f} + 2\pi j f e^{-2\pi j f} - 1) \quad (25)$$

$$g_1(t)$$



$$G_2(\omega)$$

Also, $g_2(t) = g(-t - 1)$. Thus, from (1) and (3), we get:

$$g(t - 1) \xrightarrow{\mathcal{F}} e^{-2\pi jf} G(f) \quad (26)$$

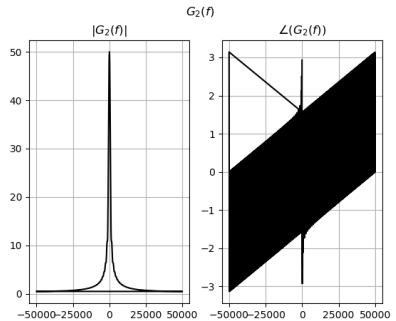
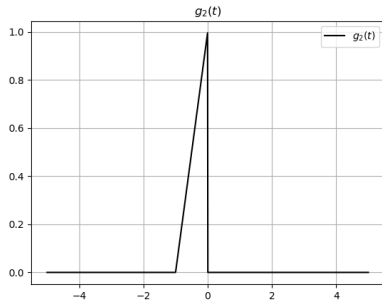
$$g(-t - 1) \xrightarrow{\mathcal{F}} e^{2\pi jf} G(-f) \quad (27)$$

$$g_2(t) \xrightarrow{\mathcal{F}} e^{2\pi jf} G(-f) \quad (28)$$

$$g_2(t) \xrightarrow{\mathcal{F}} \frac{1}{4\pi^2 f^2} (1 + 2\pi jf - e^{2\pi jf}) \quad (29)$$

$$\Rightarrow G_2(\omega) = \frac{1}{4\pi^2 f^2} (1 + 2\pi jf - e^{2\pi jf}) \quad (30)$$

$$g_2(t)$$



$$G_3(\omega)$$

Also, $g_3(t) = g(t - \frac{1}{2}) + g(-t - \frac{1}{2})$. Thus,

$$g_3(t) \xrightarrow{\mathcal{F}} e^{-j\pi f} G(f) + e^{j\pi f} G(-f) \quad (31)$$

$$G_3(f) = \frac{e^{-j\pi f}}{4\pi^2 f^2} \left[e^{2\pi jf} - 2\pi jf e^{2\pi jf} - 1 \right] + \quad (32)$$

$$\frac{e^{j\pi f}}{4\pi^2 f^2} \left[e^{-2\pi jf} + 2\pi jf e^{-2\pi jf} - 1 \right] \quad (33)$$

$$G_3(f) = \frac{j}{2\pi f} \left[e^{-\pi jf} - e^{\pi jf} \right] = \frac{\sin \pi f}{\pi f} = \text{sinc}(f) \quad (34)$$

where $\text{sinc}(t)$, the sampling function is defined as:

$$\text{sinc}(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & \text{otherwise} \end{cases} \quad (35)$$

$$g_3(t)$$

