Gate Assignment 2

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Download all python codes from

```
https://github.com/tanmaygoyal258/EE3900−Linear
    −Systems−and−Signal−processing/blob/main/
    GateAssignment2/code.py
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https://github.com/tanmaygoyal258/EE3900−Linear
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    GateAssignment2/main.tex
```

1 Problem

(EC-2010/Q.42)The transfer function for a discrete time LTI system is given by:

$$H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \tag{1.0.1}$$

Consider the following statements:
S1: The system is stable and causal for ROC: $|z| > \frac{1}{2}$
S2: The system is stable but not causal for ROC: $|z| < \frac{1}{4}$
S3: The system is neither stable nor causal for ROC: $\frac{1}{4} < |z| < \frac{1}{2}$
Which one of the following statement are valid?

1) Both S1 and S2 are true
2) Both S2 and S3 are true
3) Both S1 and S3 are true
4) S1, S2 and S3 are all true

2 Solution

Definition 1. *We say that a system is **stable** if it produces a bounded output for every possible bounded input, i.e it satisfies the BIBO(Bounded-input-Bounded-output) condition.*

Definition 2. *We say that a system is **Causal** if the output of a system at a given time instance is independent of the future input values, i.e the output at a particular instance only depends on the present and past input values.*

Lemma 2.1. *A system is said to be BIBO stable if and only if the ROC consists of the unit circle in the Z plane.*

Lemma 2.2. *A system is causal if and only if the transfer function $h[n]$ satisfies $h[n] = 0, n < 0$*

Proof. Let the input signal be given by $x[n]$ and the output signal be given by $y[n]$, then, we know in an LTI system:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{2.0.1}$$

Since, $y[n]$ is causal, it should be independent of future values of n.
If $k < 0$, then $n - k > n$, which is undesirable, and thus, to keep $y[n]$ independent of future values, $h[k] = 0, k < 0$ □

Lemma 2.3. *A system is said to be causal if and only if the ROC of the impulse function lies outside the outermost pole.*

Lemma 2.4. *If $x[n] = a^n u[n]$, where*

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & otherwise \end{cases} \tag{2.0.2}$$

then $x[n] \overset{\mathcal{Z}}{\rightleftharpoons} X[z] = \frac{1}{1-az^{-1}}$ with $ROC = |z| > a$

Proof. Using the formula for the sum of an infinite GP, we get:

$$x[n] = \begin{cases} a^n & n \geq 0 \\ 0 & otherwise \end{cases} \tag{2.0.3}$$

$$\mathcal{Z}\{x[n]\} = X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{2.0.4}$$

$$= \sum_{n=-\infty}^{0} 0 \times z^{-n} + \sum_{n=0}^{\infty} (az^{-1})^n \tag{2.0.5}$$

$$= \frac{1}{1 - az^{-1}}, ROC = \left|az^{-1}\right| < 1 \tag{2.0.6}$$

$$= \frac{1}{1 - az^{-1}}, ROC = |z| > a \tag{2.0.7}$$

$\square$

Lemma 2.5. *If* $X[z] = \frac{1}{1-az^{-1}}$ *and the region of convergence* = $Z \setminus (ROC \cup |a|)$ *where* Z *is the entire* Z *plane and* ROC *is the region of convergence mentioned in* (2.4), *then* $x[n] = -a^n u[-n-1]$

Proof. ROC $= Z \setminus (ROC \cup |a|) \implies |z| < |a|$. From (2.4), we see that we cannot apply the formula for the sum of an infinite GP directly as the conditions are not satisfied. Thus, we manipulate the function.

$$|z| < |a| \implies \left|\frac{z}{a}\right| < 1 \tag{2.0.8}$$

$$\frac{1}{1-az^{-1}} = \frac{-z}{a}\frac{1}{1-\frac{z}{a}}, \left|\frac{z}{a}\right| < 1 \tag{2.0.9}$$

$$= \sum_{n=0}^{\infty} \frac{-z}{a}\left(\frac{z}{a}\right)^n \tag{2.0.10}$$

$$= -\sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^{n+1} \tag{2.0.11}$$

$$= -\sum_{n=-\infty}^{\infty} \left(\frac{z}{a}\right)^{n+1} u[n] \tag{2.0.12}$$

$$= -\sum_{n=-\infty}^{\infty} a^{-n-1} z^{n+1} u[n] \tag{2.0.13}$$

$$= -\sum_{k=-\infty}^{\infty} a^k z^{-k} u[-k-1] \tag{2.0.14}$$

by substituting $n+1 = -k$

Finally, on comparing with the general z-transform formula of $x[n] \overset{\mathcal{Z}}{\rightleftharpoons} X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$, we get:

$$x[n] = -a^n u[-n-1] \tag{2.0.15}$$

$\square$

We are given the transfer function:

$$H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \tag{2.0.16}$$

To find the inverse Z-Transform, we would decompose the function using partial fractions:

$$H(z) = \frac{16 - 6z^{-1}}{8 - 6z^{-1} + z^{-2}} \tag{2.0.17}$$

$$= \frac{16 - 6z^{-1}}{(4 - z^{-1})(2 - z^{-1})} \tag{2.0.18}$$

$$= \frac{4}{4 - z^{-1}} + \frac{2}{2 - z^{-1}} \tag{2.0.19}$$

$$= \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}} \tag{2.0.20}$$

From the above decomposition, we find that the poles of $H(z)$ are $z = \frac{1}{4}, \frac{1}{2}$, and the zeroes are $z = \frac{3}{8}$, as shown in the plot-zero diagram given below. Thus, we can also say the outermost pole is $z = \frac{1}{2}$, and thus, from (2.3), the system is Causal iff $|z| > \frac{1}{2}$

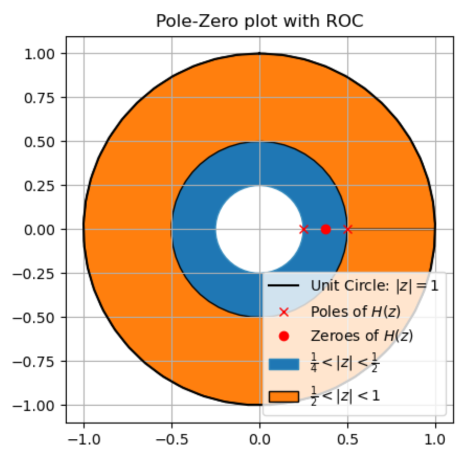

Fig. 4: Pole-Zero plot for $H(z)$ along with the various ROCs shaded

Now, if ROC $= |z| > \frac{1}{2}$, this automatically implies $|z| > \frac{1}{4}$, thus, from (2.4), we can say:

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n] \tag{2.0.21}$$

Since ROC $= |z| > \frac{1}{2}$ includes the unit circle, the system is stable.

Moreover, we see $h[n] = 0$ for $n < 0$, since $u[n] = 0$ for $n < 0$. Thus, the system is Causal as well.

Hence, S1 is correct

When ROC $= \frac{1}{4} < |z| < \frac{1}{2}$, since the unit circle is not included in the ROC, the system cannot be stable. Moreover, the ROC condition for only one of the

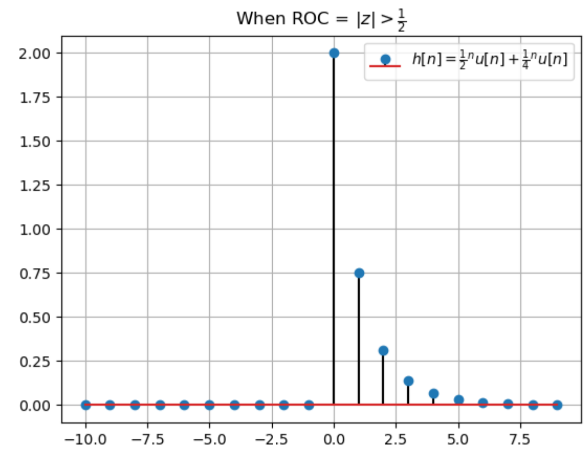

Fig. 4: $h[n]$ when $|z| > \frac{1}{2}$

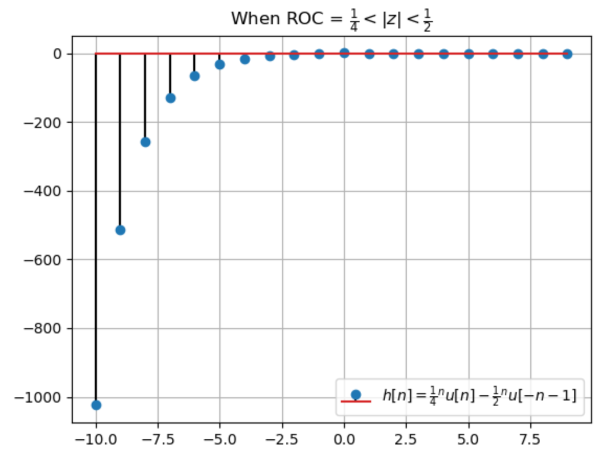

Fig. 4: $h[n]$ when $\frac{1}{4} < |z| < \frac{1}{2}$

two fractions in (2.0.20) is satisfied, i.e

$$\left(\frac{1}{4}\right)^n u[n] \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-\frac{1}{4}z^{-1}}, |z| > \frac{1}{4} \tag{2.0.22}$$

Since, the ROC condition is not satisfied for the other term, from (2.5), we get:

$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-\frac{1}{2}z^{-1}}, |z| < \frac{1}{2} \tag{2.0.23}$$

Thus, for the ROC = $\frac{1}{4} < |z| < \frac{1}{2}$, we get:

$$h[n] = \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1] \tag{2.0.24}$$

Clearly, for $n < -1$, $-n-1 > 0$ and hence, $h[n] \neq 0$ for $n < 0$, and thus, the system is non-causal.
S3 is also correct.

When ROC = $|z| < \frac{1}{4}$, this automatically implies $|z| < \frac{1}{2}$. Since the unit circle is not included, the system is unstable. Moreover, from (2.4), since both the ROC conditions are violated, from (2.5), we get:

$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-\frac{1}{2}z^{-1}}, |z| < \frac{1}{2} \tag{2.0.25}$$

$$-\left(\frac{1}{4}\right)^n u[-n-1] \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-\frac{1}{4}z^{-1}}, |z| < \frac{1}{4} \tag{2.0.26}$$

Thus, we get:

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \left(\frac{1}{4}\right)^n u[-n-1] \tag{2.0.27}$$

$$h[n] = -u[-n-1]\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] \tag{2.0.28}$$

Clearly, for $n < -1$, $-n-1 > 0$, and thus, $h[n] \neq 0, n < 0$. The system is non-causal. **Hence, S2 is incorrect**
The correct option is **3) Both S1 and S3 are correct**

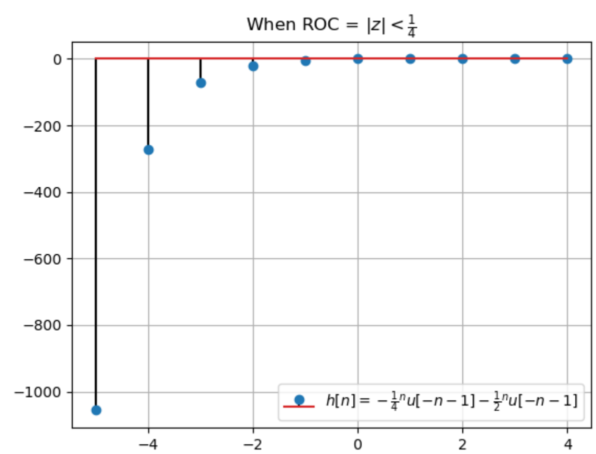

Fig. 4: $h[n]$ when $|z| < \frac{1}{4}$