GATE EC 2017- Q.7

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Question

The input $x(t)$ and output $y(t)$ of a continous time signal are related as:

$$y(t) = \int_{t-T}^{t} x(u)\,du \quad (1)$$

The system is:

1. Linear and Time-variant
2. Linear and Time-invariant
3. Non-Linear and Time-variant
4. Non-Linear and Time-invariant

Linear Systems and Time Invariant Systems

Definition

We say that a system is **linear** if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.

Definition

A system is said to be **time invariant** if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.

Lemma

Lemma

The system relating the input signal $x(t)$ and output signal $y(t)$, given by

$$y(t) = \int_{t-T}^{t} x(u)\,du \tag{2}$$

is linear and time invariant in nature.

Proof: Law of Additivity

Let the input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{t-T}^{t} x_1(u)\,du \tag{3}$$

$$y_2(t) = \int_{t-T}^{t} x_2(u)\,du \tag{4}$$

$$y_1(t) + y_2(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\,du \tag{5}$$

Proof: Law of Additivity

Now, consider the input signal of $x_1(t) + x_2(t)$, then the corresponding output signal is given by $y'(t)$:

$$y'(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\, du \tag{6}$$

Clearly, from (5) and (6):

$$y'(t) = y_1(t) + y_2(t) \tag{7}$$

Thus, the Law of Additivity holds.

Proof: Law of Homogeneity

Consider an input signal $kx(t)$, where k is any constant. Let the corresponding output be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^{t} kx(u)\,du \tag{8}$$

$$= k\int_{t-T}^{t} x(u)\,du \tag{9}$$

$$= ky(t) \tag{10}$$

Clearly, from (10),

$$y'(t) = ky(t) \tag{11}$$

Thus, the Law of Homogeneity holds.

Proof

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

Proof: Time Invariance

To check for time-invariance, we would introduce a delay of t_0 in the output and input signals.
Delay in output signal:

$$y(t - t_0) = \int_{t-t_0-T}^{t-t_0} x(u)\, du \tag{12}$$

Proof: Time Invariance

Now, we consider an input signal with a delay of t_0, given by $x(t - t_0)$, and let the corresponding output signal be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^{t} x(u - t_0)\, du \tag{13}$$

Substituting $a = u - t_0$:

$$y'(t) = \int_{t-t_0-T}^{t-t_0} x(a)\, da \tag{14}$$

Clearly, from (12) and (14):

$$y'(t) = y(t - t_0) \tag{15}$$

Thus, the system is **time-invariant**.
Thus, **2) Linear and Time- invariant** is the correct answer.

Impulse response

Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given by $\delta(t)$. Thus,

$$h(t) = \int_{t-T}^{t} \delta(u)du \tag{16}$$

Impulse function

The Impulse function can be loosely defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & otherwise \end{cases} \; and \int_{-\infty}^{\infty} \delta(t) dt = 1 \tag{17}$$

Impulse response

Since the Impulse function is zero everywhere aside from $t = 0$, the non-zero value of integration is a result of $\delta(0)$. Thus, we can say $h(t)$ will be non-zero only if the limits of integration would include $t = 0$, i.e:

$$h(t) = \begin{cases} \int_{t-T}^{t} \delta(u)du & t - T < 0; t > 0 \\ 0 & otherwise \end{cases} \tag{18}$$

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & otherwise \end{cases} \tag{19}$$

Impulse response in terms of unit step signal

The unit step signal, $u(t)$, is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & otherwise \end{cases} \tag{20}$$

On time-shifting $u(t)$ by T, we get:

$$u(t-T) = \begin{cases} 1 & t-T \geq 0 \\ 0 & otherwise \end{cases} = \begin{cases} 1 & t \geq T \\ 0 & otherwise \end{cases} \tag{21}$$

On subtracting (20) and (21), we get our impulse response $h(t)$ in terms of the unit step signal:

$$h(t) = u(t) - u(t-T) \tag{22}$$

Impulse response in terms of the unit rectangular function

The unit rectangular signal, $rect(t)$ is given by:

$$rect(t) = \begin{cases} 1 & \frac{-1}{2} \geq t \geq \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{23}$$

We can obtain the impulse response $h(t)$ in terms of $rect(t)$ using time scaling and shifting as follows:

$$rect(Tt) = \begin{cases} 1 & \frac{-T}{2} \geq Tt \geq \frac{T}{2} \\ 0 & otherwise \end{cases} \tag{24}$$

$$rect(Tt + \frac{T}{2}) = \begin{cases} 1 & 0 \geq Tt + \frac{T}{2} \geq T \\ 0 & otherwise \end{cases} = h(t) \tag{25}$$

Fourier Transform of impulse response

$$h(t) \overset{\mathcal{F}}{\rightleftharpoons} H(\omega) \tag{26}$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt \tag{27}$$

From (19), we can write (27) as:

$$H(\omega) = \int_{-\infty}^{0} 0\, dt + \int_{0}^{T} e^{-j\omega t}\, dt + \int_{T}^{\infty} 0\, dt \tag{28}$$

$$= \frac{1 - e^{-j\omega T}}{j\omega} \tag{29}$$

$$\tag{30}$$

Fourier Transform of impulse response

$$H(\omega) = \frac{1 - \cos\omega T + j\sin\omega T}{j\omega} \tag{31}$$

$$= \frac{2\sin^2(\frac{\omega T}{2}) + 2j\sin(\frac{\omega T}{2})\cos(\frac{\omega T}{2})}{j\omega} \tag{32}$$

$$= \frac{2\sin(\frac{\omega T}{2})}{\omega}\frac{\sin(\frac{\omega T}{2}) + j\cos(\frac{\omega T}{2})}{j} \tag{33}$$

$$= Te^{-j\omega\frac{T}{2}} \times \frac{\sin(\frac{\omega T}{2})}{\frac{\omega T}{2}} \tag{34}$$

$$= Te^{-j\omega\frac{T}{2}} sinc(\frac{wT}{2}) \tag{35}$$

where $sinc(t)$, the sampling function is defined as:

$$sinc(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(t)}{t} & otherwise \end{cases} \tag{36}$$

Graphs: Input and Output signals

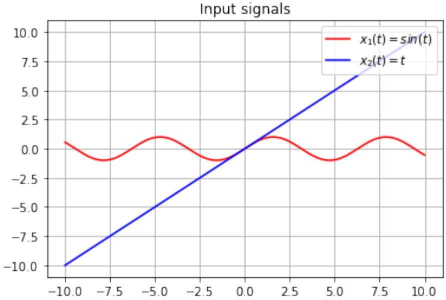

Figure: $x_1(t) = \sin t$ and $x_2(t) = t$

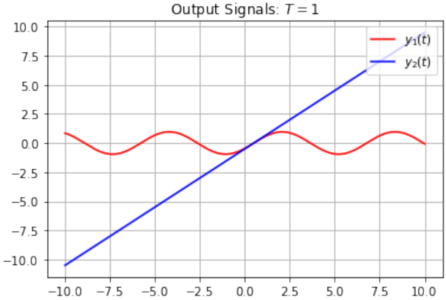

Figure: $y_1(t)$ and $y_2(t)$

Graphs: Laws of Additivity and Homogeneity

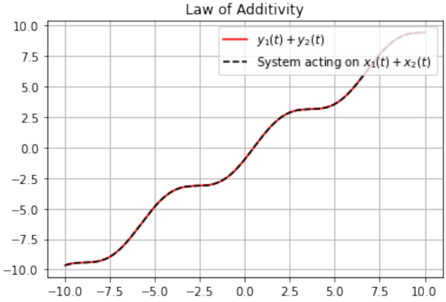

Figure: Law of Additivity

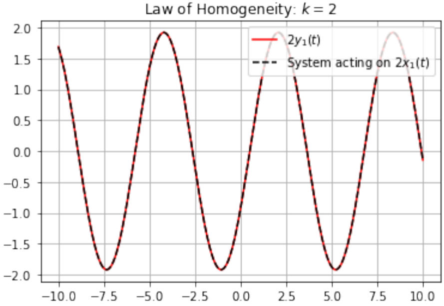

Figure: Law of Homogeneity

Graphs: Time Invariance