Gate Assignment 4

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Download all latex codes from

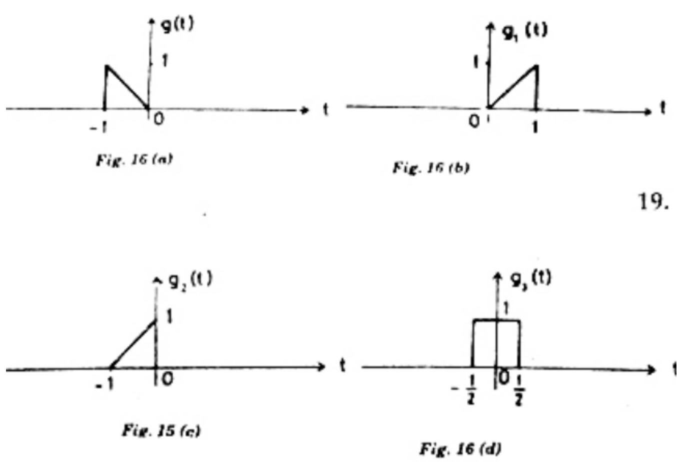
https://github.com/tanmaygoyal258/EE3900−Linear−Systems−and−Signal−processing/blob/main/GateAssignment4/main.tex

1 Problem

(EC-2001/Q.16) The Fourier Transform $G(\omega)$ of the signal $g(t)$ is given by

$$G(\omega) = \frac{1}{\omega^2}(e^{j\omega} - j\omega e^{j\omega} - 1) \tag{1.0.1}$$

Using this information, find the Fourier Transforms of the signals $g_1(t)$, $g_2(t)$ and $g_3(t)$.



2 Solution

Lemma 2.1. *If*

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(\omega) \tag{2.0.1}$$

then,

$$g(t \pm t_0) \overset{\mathcal{F}}{\rightleftharpoons} G(\omega)e^{\pm j\omega t_0} \tag{2.0.2}$$

Proof. We know,

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\, dt \tag{2.0.3}$$

Let

$$g(t + t_0) \overset{\mathcal{F}}{\rightleftharpoons} G'(\omega) \tag{2.0.4}$$

Then,

$$G'(\omega) = \int_{-\infty}^{\infty} g(t + t_0)e^{-j\omega t}\, dt \tag{2.0.5}$$

Substituting $t + t_0 = T$, we get:

$$G'(\omega) = \int_{-\infty}^{\infty} g(T)e^{-j\omega(T-t_0)}\, dT \tag{2.0.6}$$

$$= \int_{-\infty}^{\infty} g(T)e^{-j\omega T}e^{j\omega t_0}\, dT \tag{2.0.7}$$

$$= e^{j\omega t_0}\int_{-\infty}^{\infty} g(T)e^{-j\omega T}\, dT \tag{2.0.8}$$

$$= e^{j\omega t_0}G(\omega) \tag{2.0.9}$$

Similarly, it can be proved:

$$g(t - t_0) \overset{\mathcal{F}}{\rightleftharpoons} e^{-j\omega t_0}G(\omega) \tag{2.0.10}$$

□

Lemma 2.2. *If*

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(\omega) \tag{2.0.11}$$

then,

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|}G\left(\frac{\omega}{\alpha}\right) \tag{2.0.12}$$

Proof. Consider $\alpha > 0$. Then, we know,

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\, dt \tag{2.0.13}$$

Let

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} G'(\omega) \tag{2.0.14}$$

Then,

$$G'(\omega) = \int_{-\infty}^{\infty} g(\alpha t)e^{-j\omega t}\, dt \tag{2.0.15}$$

Making the substitution $T = \alpha t$, we get:

$$G'(\omega) = \frac{1}{\alpha}\int_{-\infty}^{\infty} g(T)e^{-j\frac{\omega T}{\alpha}}\,dT \tag{2.0.16}$$

$$= \frac{1}{\alpha}G\left(\frac{\omega}{\alpha}\right) \tag{2.0.17}$$

Similarly, it can be proved for $\alpha < 0$

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{-1}{\alpha}G\left(\frac{-\omega}{\alpha}\right) \tag{2.0.18}$$

□

Corollary 2.1. *If*

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(\omega) \tag{2.0.19}$$

then,

$$g(-t) \overset{\mathcal{F}}{\rightleftharpoons} G(-\omega) \tag{2.0.20}$$

Lemma 2.3. *If*

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(\omega) \tag{2.0.21}$$

then,

$$\frac{dg(t)}{dt} \overset{\mathcal{F}}{\rightleftharpoons} (j\omega)G(\omega) \tag{2.0.22}$$

Proof. Using the formula for Inverse Fourier transform, we know:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}\,d\omega \tag{2.0.23}$$

$$\frac{dg(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}(j\omega)\,d\omega \tag{2.0.24}$$

$$= \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}\,d\omega \tag{2.0.25}$$

$$= (j\omega)G(\omega) \tag{2.0.26}$$

□

Now, from the figure:

$$g(t) = \begin{cases} -t & -1 \geq t \geq 0 \\ 0 & otherwise \end{cases} \tag{2.0.27}$$

$$g_1(t) = \begin{cases} t & 0 \geq t \geq 1 \\ 0 & otherwise \end{cases} \tag{2.0.28}$$

$$g_2(t) = \begin{cases} 1 + t & -1 \geq t \geq 0 \\ 0 & otherwise \end{cases} \tag{2.0.29}$$

$$g_3(t) = \begin{cases} 1 & -\frac{1}{2} \geq t \geq \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{2.0.30}$$

Clearly, $g_1(t) = g(-t)$, and using (2.1), we get:

$$G_1(\omega) = G(-\omega) = \frac{1}{\omega^2}(e^{-j\omega} + j\omega e^{-j\omega} - 1) \tag{2.0.31}$$

Also, $g_2(t) = g_1(t + 1)$. Thus, from (2.1), we get:

$$G_2(\omega) = G_1(\omega)e^{j\omega.1} \tag{2.0.32}$$

$$= \frac{1}{\omega^2}(e^{-j\omega} + j\omega e^{-j\omega} - 1) \times e^{j\omega} \tag{2.0.33}$$

$$= \frac{1}{\omega^2}(1 + j\omega - e^{j\omega}) \tag{2.0.34}$$

Finally, $g_3(t)$ is non-zero between $\frac{-1}{2}$ and $\frac{1}{2}$. Thus, we can shift $g_1(t)$ and take it's derivative wrt time:

$$g_1(t) = \begin{cases} t & 0 \geq t \geq 1 \\ 0 & otherwise \end{cases} \tag{2.0.35}$$

$$g_1\left(t + \frac{1}{2}\right) = \begin{cases} t + \frac{1}{2} & -\frac{1}{2} \geq t \geq \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{2.0.36}$$

$$\frac{dg_1\left(t + \frac{1}{2}\right)}{dt} = \begin{cases} 1 & -\frac{1}{2} \geq t \geq \frac{1}{2} \\ 0 & otherwise \end{cases} = g_3(t) \tag{2.0.37}$$

Using (2.1) and (2.3), we get:

$$g_1(t) \overset{\mathcal{F}}{\rightleftharpoons} G_1(\omega) \tag{2.0.38}$$

$$g_1\left(t + \frac{1}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} e^{\frac{j\omega}{2}}G_1(\omega) \tag{2.0.39}$$

$$g_1\left(t + \frac{1}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{e^{-\frac{j\omega}{2}}}{\omega^2}(1 + j\omega - e^{j\omega}) \tag{2.0.40}$$

$$\frac{dg_1\left(t + \frac{1}{2}\right)}{dt} \overset{\mathcal{F}}{\rightleftharpoons} \frac{j\omega e^{-\frac{j\omega}{2}}}{\omega^2}(1 + j\omega - e^{j\omega}) \tag{2.0.41}$$

$$g_3(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{je^{-\frac{j\omega}{2}}}{\omega}(1 + j\omega - e^{j\omega}) \tag{2.0.42}$$