Gate Assignment 4

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Download all latex codes from

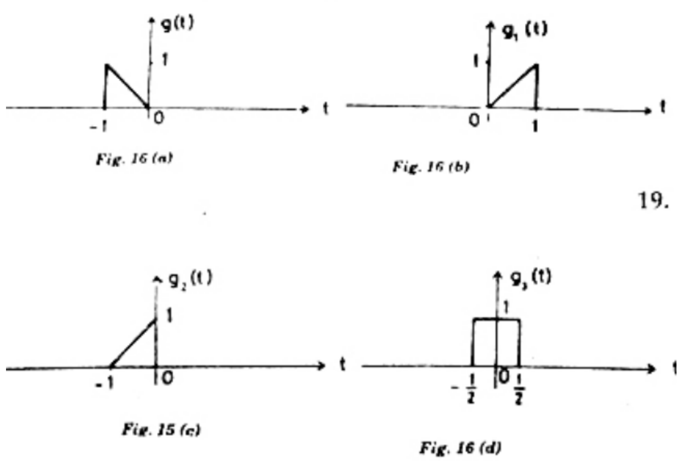
https://github.com/tanmaygoyal258/EE3900−Linear−Systems−and−Signal−processing/blob/main/GateAssignment4/main.tex

1 Problem

(EC-2001/Q.16) The Fourier Transform $G(\omega)$ of the signal $g(t)$ is given by

$$G(\omega) = \frac{1}{\omega^2}(e^{j\omega} - j\omega e^{j\omega} - 1) \tag{1.0.1}$$

Using this information, find the Fourier Transforms of the signals $g_1(t)$, $g_2(t)$ and $g_3(t)$.



2 Solution

We replace ω by $2\pi f$. Then,

$$G(f) = \frac{1}{4\pi^2 f^2}(e^{2\pi jf} - 2\pi jf e^{2\pi jf} - 1) \tag{2.0.1}$$

Lemma 2.1. *If*

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f) \tag{2.0.2}$$

then,

$$g(t \pm t_0) \overset{\mathcal{F}}{\rightleftharpoons} G(f)e^{\pm 2\pi jft_0} \tag{2.0.3}$$

Proof. We know,

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi jft}\, dt \tag{2.0.4}$$

Let

$$g(t + t_0) \overset{\mathcal{F}}{\rightleftharpoons} G'(f) \tag{2.0.5}$$

Then,

$$G'(f) = \int_{-\infty}^{\infty} g(t + t_0)e^{-2\pi jft}\, dt \tag{2.0.6}$$

Substituting $t + t_0 = T$, we get:

$$G'(f) = \int_{-\infty}^{\infty} g(T)e^{-2\pi jf(T-t_0)}\, dT \tag{2.0.7}$$

$$= \int_{-\infty}^{\infty} g(T)e^{-2\pi jfT}e^{2\pi jft_0}\, dT \tag{2.0.8}$$

$$= e^{2\pi jft_0}\int_{-\infty}^{\infty} g(T)e^{2\pi jfT}\, dT \tag{2.0.9}$$

$$= e^{-2\pi jft_0}G(f) \tag{2.0.10}$$

Similarly, it can be proved:

$$g(t - t_0) \overset{\mathcal{F}}{\rightleftharpoons} e^{-2\pi jft_0}G(f) \tag{2.0.11}$$

□

Lemma 2.2. *If*

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f) \tag{2.0.12}$$

then,

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|}G\left(\frac{f}{\alpha}\right) \tag{2.0.13}$$

Proof. Consider $\alpha > 0$. Then, we know,

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi jft}\, dt \tag{2.0.14}$$

Let

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} G'(f) \tag{2.0.15}$$

Then,

$$G'(f) = \int_{-\infty}^{\infty} g(\alpha t) e^{-2\pi j f t}\, dt \tag{2.0.16}$$

Making the substitution $T = \alpha t$, we get:

$$G'(f) = \frac{1}{\alpha}\int_{-\infty}^{\infty} g(T) e^{-2\pi j \frac{fT}{\alpha}}\, dT \tag{2.0.17}$$

$$= \frac{1}{\alpha} G\left(\frac{f}{\alpha}\right) \tag{2.0.18}$$

Similarly, it can be proved for $\alpha < 0$

$$g(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{-1}{\alpha} G\left(\frac{-f}{\alpha}\right) \tag{2.0.19}$$

□

Lemma 2.3. *If*

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f) \tag{2.0.20}$$

then,

$$g(-t) \overset{\mathcal{F}}{\rightleftharpoons} G(-f) \tag{2.0.21}$$

Proof. Put $\alpha = -1$ in (2.2) to obtain the result.

$$g(-t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|-1|} G\left(\frac{f}{-1}\right) \tag{2.0.22}$$

$$g(-t) \overset{\mathcal{F}}{\rightleftharpoons} G(-f) \tag{2.0.23}$$

□

Now, from the figure:

$$g(t) = \begin{cases} -t & -1 \le t \le 0 \\ 0 & otherwise \end{cases} \tag{2.0.24}$$

$$g_1(t) = \begin{cases} t & 0 \le t \le 1 \\ 0 & otherwise \end{cases} \tag{2.0.25}$$

$$g_2(t) = \begin{cases} 1+t & -1 \le t \le 0 \\ 0 & otherwise \end{cases} \tag{2.0.26}$$

$$g_3(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{2.0.27}$$

Clearly, $g_1(t) = g(-t)$, and using (2.3), we get:

$$G_1(f) = G(-f) = \frac{1}{4\pi^2 f^2}(e^{-2\pi j f} + 2\pi j f e^{-2\pi j f} - 1) \tag{2.0.28}$$

Also, $g_2(t) = g(-t-1)$. Thus, from (2.1) and (2.3), we get:

$$g(t-1) \overset{\mathcal{F}}{\rightleftharpoons} e^{-2\pi j f} G(f) \tag{2.0.29}$$

$$g(-t-1) \overset{\mathcal{F}}{\rightleftharpoons} e^{2\pi j f} G(-f) \tag{2.0.30}$$

$$g_2(t) \overset{\mathcal{F}}{\rightleftharpoons} e^{2\pi j f} G(-f) \tag{2.0.31}$$

$$g_2(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{4\pi^2 f^2}(1 + 2\pi j f - e^{2\pi j f}) \tag{2.0.32}$$

$$\implies G_2(\omega) = \frac{1}{4\pi^2 f^2}(1 + 2\pi j f - e^{2\pi j f}) \tag{2.0.33}$$

Also, $g_3(t) = g\left(t - \frac{1}{2}\right) + g\left(-t - \frac{1}{2}\right)$. Thus,

$$g\left(t - \frac{1}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} e^{-j\pi f} G(f) \tag{2.0.34}$$

$$g\left(-t - \frac{1}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} e^{j\pi f} G(-f) \tag{2.0.35}$$

$$g_3(t) \overset{\mathcal{F}}{\rightleftharpoons} e^{-j\pi f} G(f) + e^{j\pi f} G(-f) \tag{2.0.36}$$

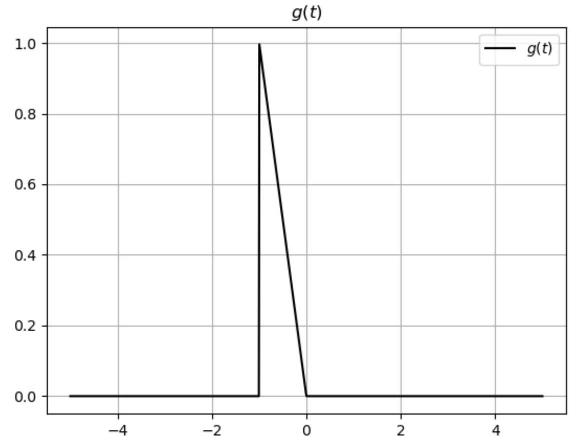
$$G_3(f) = \frac{e^{-j\pi f}}{4\pi^2 f^2}\left[e^{2\pi j f} - 2\pi j f e^{2\pi j f} - 1\right] + \frac{e^{j\pi f}}{4\pi^2 f^2}\left[e^{-2\pi j f} + 2\pi j f e^{-2\pi j f} - 1\right] \tag{2.0.37}$$

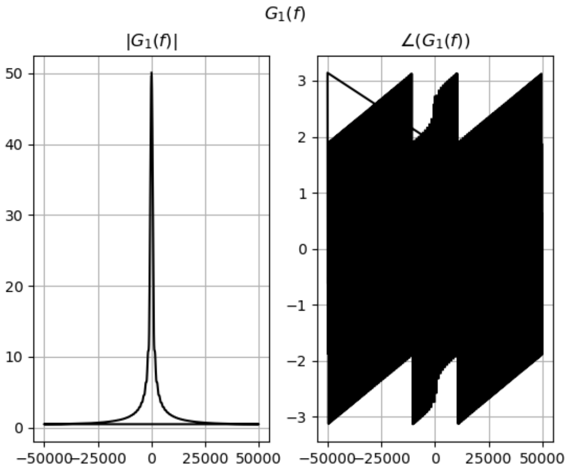
$$G_3(f) = \frac{j}{2\pi f}\left[e^{-\pi j f} - e^{\pi j f}\right] \tag{2.0.38}$$

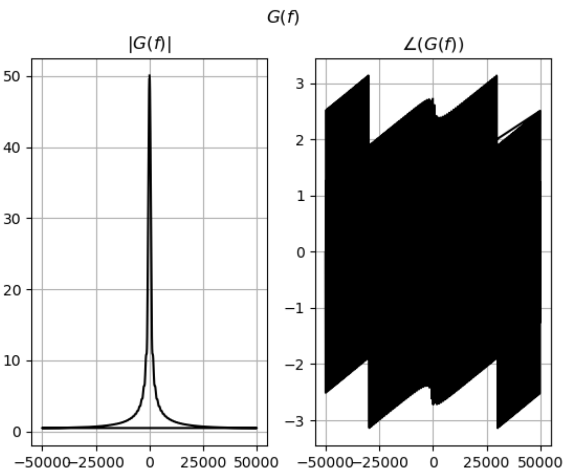
$$G_3(f) = \frac{\sin \pi f}{\pi f} = sinc(f) \tag{2.0.39}$$

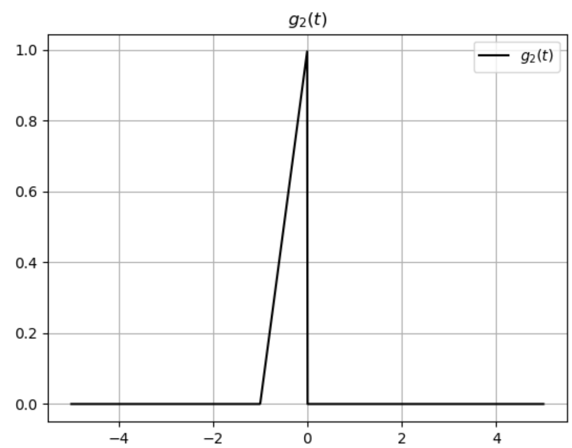
where $sinc(t)$, the sampling function is defined as:

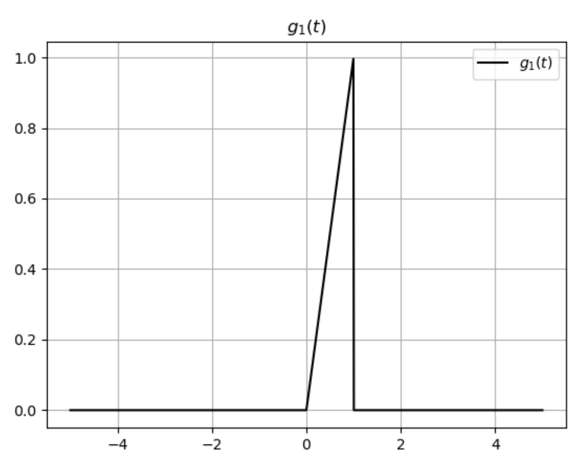
$$sinc(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & otherwise \end{cases} \tag{2.0.40}$$

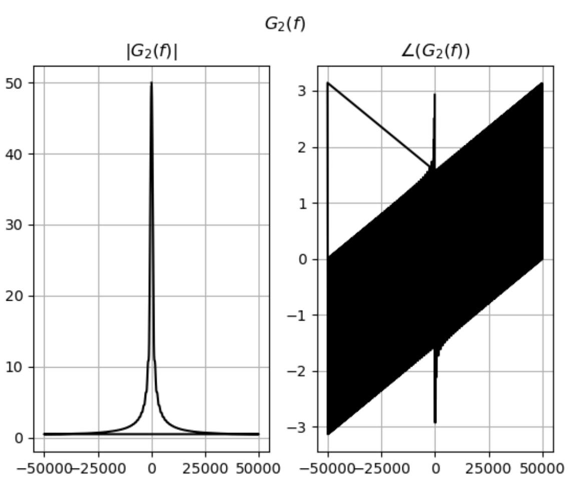

$g(t)$
$g(t)$

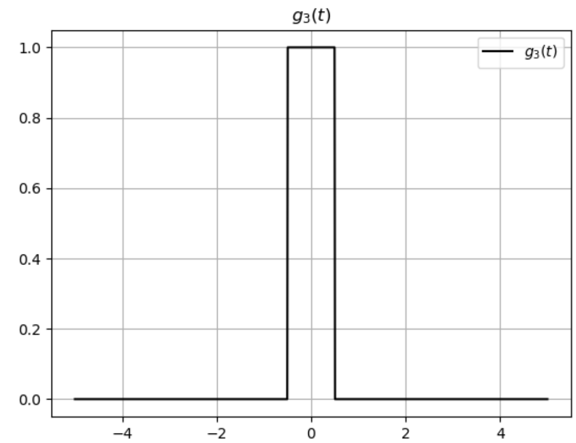

$G_1(f)$
$|G_1(f)|$
$\angle(G_1(f))$

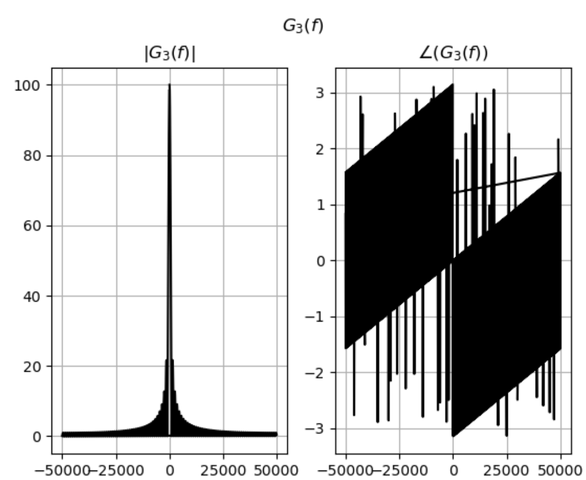

$G(f)$
$|G(f)|$
$\angle(G(f))$


$g_2(t)$
$g_2(t)$


$g_1(t)$
$g_1(t)$


$G_2(f)$
$|G_2(f)|$
$\angle(G_2(f))$


$g_3(t)$
$g_3(t)$
1.0
0.8
0.6
0.4
0.2
0.0
−4
−2
0
2
4


$G_3(f)$
$|G_3(f)|$
$\angle(G_3(f))$
100
80
60
40
20
0
−50000
−25000
0
25000
50000
3
2
1
0
−1
−2
−3