Introductory Functional Analysis and Applications

Solutions to Exercises

1 Metric Spaces

1.1 Metric Space

1. *Show that the real line is a metric space.*

Proof. To show that the real line is a metric space, we wish to show that there is a valid metric on this space. Obviously, the choice of the metric is $d(x, y) = |x - y|$. We now wish to show the following properties:

1. d is non-negative.
2. $d(x, y) = 0 \iff x = y$.
3. $d(x, y) = d(y, x)$.
4. $d(x, y) \leq d(x, z) + d(z, y)$

Properties (1) , (2), and (3) are obvious. We now show property (4) for the choice of $d(x, y) = |x - y|$. Consider

$$\begin{aligned}
(x - y)^2 &= x^2 - 2xy + y^2 \\
&= x^2 - 2xz + z^2 + z^2 - 2zy + y^2 + (2xz + 2zy - 2xy - 2z^2) \\
&= (x - z)^2 + (z - y)^2 + 2x(z - y) + 2z(y - z) \\
&= |x - z|^2 + |z - y|^2 + 2(x - z)(z - y) \\
&\leq (|x - z| + |y - z|)^2
\end{aligned}$$

where the inequality follows since $x - y \leq |x - y|$. Taking a square root on both sides finishes the proof. ∎

2. *Does $d(x, y) = (x - y)^2$ define a metric on the set of all real numbers?*

Proof. To show that $d(x, y)$ is a valid metric, we must ensure that it is non-negative, symmetric, satisfies the triangle inequality, and satisfies the condition that $(x - y)^2 = 0 \iff x = y$. We see that the first, second, and last conditions are always true. We only need to check for the triangle inequality. Notice, from the answer of 1, we have

$$(x - y)^2 = (x - z)^2 + (y - z)^2 + 2(x - z)(z - y)$$

WLOG, assume $x \leq y$. If $z \in [x, y]$, then we have $(x - z) \leq 0$ and $(z - y) \leq 0$, and hence, $(x - z)(z - y) \geq 0$, which results in

$$d(x, y) \geq d(x, z) + d(z, y)$$

which is a contradiction of the triangle inequality. Hence, $d(x, y) = (x - y)^2$ is not a metric. ∎

3. *Show that $d(x, y) = \sqrt{|x - y|}$ defines a metric on the set of all real numbers.*

Proof. Once again, we can check that $d(x, y)$ satisfies non-negativity, symmetry, and the property that $d(x, y) = 0 \iff x = y$. We wish to show d satisfies the triangle inequality. Since $|x - y|$ is a valid metric (refer 1), using the triangle inequality for this along with the fact that $\sqrt{.}$ is monotonically increasing, we get

$$\sqrt{|x - y|} \leq \sqrt{|x - z| + |z - y|} \leq \sqrt{|x - z|} + \sqrt{|z - y|}$$

where the last inequality follows from the fact that if $a, b \geq 0$, then

$$\sqrt{a + b} \leq \sqrt{a + 2\sqrt{ab} + b} = \sqrt{(\sqrt{a} + \sqrt{b})^2} = \sqrt{a} + \sqrt{b}$$

. ∎

4. *Find all metrics on a set X consisting of two points and only one point.*

Proof. If X consists of only one point, then the only metric that can be defined keeping in mind all properties (note that the triangle inequality does not apply since we do not have a second point for this) is $d(x, x) = 0$.

If X consists of two points, once again, the triangle inequality does not apply, and hence, it is easy to check that

$$d(x, y) = \begin{cases} 0 & x = y \\ c & x \neq y \end{cases}.$$

■

5. *Let d be a metric on X. Determine all constants k such that kd and $d + k$ is a metric on X.*

Proof. Note that for kd to remain a metric, all we need is $k \geq 0$ to ensure it is non-negative. All other properties continue to hold as it is.

For $d+k$ to be a metric, the simplest starting point is that $(d+k)(x, y) = 0 \iff x = y$. However, we know that $d(x, y) = 0 \iff x = y$, and since, k is a constant, this property is satisfied only if $k(x, y) = 0$. Hence, no constant k can be added such that $d + k$ is also a metric. ■

6. *Show that d in 1.1-6 satisfies the triangle inequality.*

Proof. The metric in 1.1-6 is for a sequence space ℓ^∞ such that for $x = (\xi_i)$ and $y = (\eta_i)$ such that $\xi \leq c_x$ and $\eta_i \leq c_y$, we have

$$d(x, y) = \sup_{i \in \mathbb{N}} |\xi_i - \eta_i|$$

Note that this is always non-negative and symmetric, owing to the absolute difference of the terms (since absolute difference is always non-negative, the least upper bound is also non-negative). Also, if $x = y$,. i.e, $\xi_i = \eta_i$ for all i, then the supremum is 0. Finally, we show the triangle inequality as follows. For some $z = (\zeta_i)$, we have

$$d(x, y) = \sup_{i \in \mathbb{N}} |\xi_i - \eta_i| \leq \sup_{i \in \mathbb{N}} (|\xi_i - \zeta_i| + |\zeta_i - \eta_i|) \leq \sup_{i \in \mathbb{N}} |\xi_i - \zeta_i| + \sup_{i \in \mathbb{N}} |\zeta_i - \eta_i| = d(x, z) + d(z, y)$$

where we use the triangle inequality for $|x - y|$ (check 1) and the fact that $\sup(f+g)(x) \leq \sup f(x) + \sup g(x)$. ■

7. *If A is the subspace of ℓ^∞ consisting of all sequences of zeros and ones, then what is the induced metric on A?*

Proof. If A is the subspace consisting of all sequences only containing zeros and ones, then the coordinate-wise absolute difference of any two sequences can only take on two values; 0, when the coordinates are the same, and 1 when the coordinates are different. Thus, the supremum of this absolute difference is 1. In other words, the metric can be expressed as:

$$d(x, y) = \mathbb{1}\{x = y\}$$

which is exactly the discrete metric. ■

8. *Show that another metric $\tilde{d}$ on the space X in 1.1-7 is defined by*

$$\tilde{d}(x, y) = \int_a^b |x(t) - y(t)| \, dt$$

Proof. We wish to show that the metric $d(x, y)$ is a defined on the space of continuous functions $C[a, b]$, i.e x and y are now continuous functions. Note that $d(x, y)$ is clearly non-negative and symmetric owing to the point-wise absolute difference of the functions. Further, if $x(c) = y(c)$ $\forall c \in [a, b]$, then, $d(x, y) = \int_a^b 0 dt = 0$. Thus, we wish to show the triangle inequality now. Using the fact that the absolute difference is a metric (check 1), using the triangle inequality, we have

$$\int_a^b |x(t) - y(t)| \, dt \leq \int_a^b (|x(t) - z(t)| + |z(t) - y(t)|) \, dt = \int_a^b |x(t) - z(t)| \, dt + \int_a^b |z(t) - y(t)| \, dt$$

which proves that $d(x, y)$ satisfies the triangle inequality. ■

9. *Show that d in 1.1-8 is a metric.*

Proof. We wish to show that the discrete metric, defined as

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

is a metric. Clearly, this is non-negative and symmetric. Further, we have that $d(x, y) = 0 \iff x = y$. Finally, we also have

1. If $x \neq y$, then $d(x, y) = 1$. In such a case, if $x = z$ or $y = z$, then $d(x, z) + d(y, z) = 1$. However, if $x \neq z \neq y$, then $d(x, z) + d(y, z) = 2$. Thus, $d(x, y) \leq d(x, z) + d(z, y)$.

2. If $x = y$, then $d(x, y) = 0$. Now, if $x = y \neq z$, then $d(x, z) + d(z, y) = 2$, and if $x = y = z$, then $d(x, z) + d(z, y) = 0$, and in both cases, $d(x, z) + d(z, y) \geq d(x, y)$.

Thus, the triangle inequality holds for d, and hence, it is a valid metric. ■

10. *Let X be the set of all ordered triples of zeros and ones. Show that X consists of eight elements and a metric d on X is defined by $d(x, y) =$number of places where x and y have different entries.*

Proof. Clearly, X has $2^3 = 8$ elements. We wish to show that

$$d(x, y) = \sum_{i=1}^{3} \mathbb{1}\{x_i \neq y_i\}$$

is a metric on X where x_i represents the i^{th} coordinate of x. Clearly, this metric is non-negative, symmetric, and satisfies the property that $d(x, y) = 0 \iff x = y$. Also, we have

$$\mathbb{1}\{x_i \neq y_i\} \leq \mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i}$$

This is easy to see using a case-by-case analysis.

1. If $x_i = y_i$, then $\mathbb{1}\{x_i \neq y_i\} = 0$. If $z_i = x_i = y_i$, we have $\mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i} = 0$, while if $z_i \neq x_i = y_i$, then $\mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i} = 2$, and hence, $\mathbb{1}x_i \neq y_i \leq \mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i}$.

2. If $x_i \neq y_i$, then $\mathbb{1}\{x_i \neq y_i\} = 1$. If $z_i = x_i$ or $z_i = y_i$, we have $\mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i} = 1$, while if $z_i \neq x_i \neq y_i$, then $\mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i} = 2$, and hence, $\mathbb{1}x_i \neq y_i \leq \mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i}$.

Thus, the Hamming distance satisfies all the properties of a metric. ■

11. *Prove (1).*

Proof. We wish to prove the statement:

$$d(x_1, x_n) \leq \sum_{i=2}^{n} d(x_{i-1}, x_i)$$

We can do this by induction on n. The base case $n = 3$ holds due to the triangle inequality. Assuming it holds for $n = k$, we wish to show that it holds for $n = k + 1$. This is easy to show as follows:

$$d(x_1, x_{k+1}) \leq d(x_1, x_k) + d(x_k, x_{k+1}) \leq \sum_{i=2}^{k} d(x_{i-1}, x_i) + d(x_k, x_{k+1}) = \sum_{i=2}^{k+1} d(x_{i-1}, x_i)$$

where the first inequality follows by the triangle inequality and the second follows from the induction hypothesis. ■

12. *Using (1), show that*

$$|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w)$$

Proof. By the repeated application of triangle inequality, we have

$$d(x, y) \leq d(x, z) + d(z, y) \leq d(x, z) + d(z, w) + d(w, y)$$

which gives us

$$d(x, y) - d(z, w) \leq d(x, z) + d(y, w)$$

Similarly, we have

$$d(z, w) \leq d(z, x) + d(x, z) \leq d(z, x) + d(x, y) + d(y, w)$$

which gives us

$$d(z, w) - d(x, y) \leq d(x, z) + d(y, w)$$

Combining both these facts and using the symmetry property of metrics gives us the required result. ■

13. *Using the triangle inequality, show that*

$$|d(x,z) - d(y,z)| \leq d(x,y)$$

Proof. Note that $d(x,z) \leq d(x,y) + d(y,z)$ and hence, $d(x,z) - d(y,z) \leq d(x,y)$. Similarly, $d(y,z) \leq d(x,y) + d(x,z)$ and hence, $d(y,z) - d(x,z) \leq d(x,y)$. Combining both results finishes the proof. ■

14. *Show that (M3) and (M4) could be obtained from (M2) and*

$$d(x,y) \leq d(z,x) + d(z,y)$$

Proof. We wish to show that the axioms $d(x,y) = d(y,x)$ and $d(x,y) \leq d(x,z) + d(z,y)$ can be obtained from the axioms $d(x,y) = 0 \iff x = y$ and $d(x,y) \leq d(z,x) + d(z,y)$.

To prove symmetry, we have the following two equations:

$$d(x,y) \leq d(z,x) + d(z,y)$$

$$d(y,x) \leq d(z,y) + d(z,x)$$

Subtracting both equations results in

$$d(x,y) \leq d(y,x) \text{ and } d(y,x) \leq d(x,y) \implies d(x,y) = d(y,x).$$

Now, simply using symmetry, we have

$$d(x,y) \leq d(z,x) + d(z,y) = d(x,z) + d(z,y)$$

since $d(z,x) = d(x,z)$. ■

15. *Show that non-negativity of a metric follows from (M2) to (M4).*

Proof. We wish to show that the non-negativity of a metric follows from the following axioms:

1. $d(x,y) = d(y,x)$.
2. $d(x,y) = 0 \iff x = y$.
3. $d(x,y) \leq d(x,z) + d(z,y)$.

Substituting $y = x$ in the triangle inequality, we have

$$d(x,x) \leq d(x,z) + d(z,x)$$

Using the first and second axiom in tandem gives us $d(x,z) \geq 0$, which completes the proof. ■

1.2 Further Examples of Metric Spaces

1. *Show that in 1.2-1, we can obtain another metric by replacing* $1/2^i$ *with* μ_i *such that* $\sum \mu_i$ *converges.*

Proof. We wish to show that for some $\mu_j > 0$ such that $\sum mu_j$ converges, and two sequences $x = (\xi_i)$ and $y = (\eta_i)$, we have

$$d(x,y) = \sum_{j=1}^{\infty} \mu_j \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}$$

is a valid metric defined on the sequence space.

The proof follows on the same lines as the one given for 1.2-1. It is clear that $d(x,y)$ is non-negative, symmetric, and satisfies $d(x,y) = 0 \iff x = y$. However, we wish to show this is bounded, i.e

$$d(x,y) = \sum_{j=1}^{\infty} \mu_j \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \sup_{j \in \mathbb{N}} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \sum_{j=1}^{\infty} \mu_j$$

Note that the RHS is bounded if and only if $\sum \mu_j$ is convergent (since the supremum term can be upper bounded by 1). Further, the proof for the triangle inequality follows in the same fashion, define $f(t) = \frac{t}{1+t}$. Then, $f(t)$ is monotonically increasing, and hence, using the triangle inequality for numbers, we get

$$f(|a+b|) \leq f(|a| + |b|)$$

and hence, substituting $a = \xi_j - \zeta_j$ and $b = \zeta_j - \eta_j$, we get

$$\frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \frac{|\xi_j - \zeta_j| + |\zeta_j - \eta_j|}{1 + |\xi_j - \zeta_j| + |\zeta_j - \eta_j|} \leq \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} + \frac{|\zeta_j - \eta_j|}{1 + + |\zeta_j - \eta_j|}$$

Multiplying by μ_j and summing over all $j \in \mathbb{N}$ finishes the proof. ■

2. *Using (6) show that the geometric mean of two positive numbers does not exceed the arithmetic mean.*

Proof. We wish to show that the geometric mean does not exceed the arithmetic mean using the equation:

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

where $p^{-1} + q^{-1} = 1$.

Substituting $x = \sqrt{a}, y = \sqrt{b}, p = q = 2$, we get

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

■

3. *Show that the Cauchy Schwarz inequality implies*

$$\left(\sum_{i=1}^{n} |\xi_i| \right)^2 \leq n \sum_{i=1}^{n} |\xi_i|^2$$

Proof. The Cauchy-Schwarz inequality states that

$$\left(\sum_{i=1}^{n} |\xi_i| \, |\eta_i| \right)^2 \leq \left(\sum_{i=1}^{n} |\eta_i|^2 \right) \left(\sum_{i=1}^{n} |\xi_i|^2 \right)$$

Putting $\eta_i = 1$ for all $i \in [n]$ completes the proof. ■

4. *Find a sequence which converges to 0, but is not in any space ℓ^p, $1 \leq p < \infty$.*

Proof. Recall that the ℓ^p space consists of sequences $x = (\xi_i)$ such that $\sum |\xi_i|^p$ converges. The sequence $\xi_i = \frac{1}{\log(i+1)}$ converges to 0 but is not part of any ell^p space. To see this, note that $\log(i+1)$ can be bounded above by $i^{1/p}$ and hence,

$$\frac{1}{\log(i+1)} \geq i^{-1/p}.$$

Thus, we have

$$\sum_i \left(\frac{1}{\log(i+1)} \right)^p \geq \sum_i \left(\frac{1}{i^{1/p}} \right)^p \to \infty$$

■

5. *Find a sequence which is in ℓ^p with $p > 1$, but $x \notin \ell^1$.*

Proof. We wish to find a sequence $x = (\xi_i)$ such that $\sum |xi_i|^p$ converges for all $p > 1$ but does not converge for $p = 1$. The simplest example is $\xi_i = i^{-1}$, the sum of which diverges. ■

6. *The diameter $\delta(A)$ of a nonempty set A in a metric space (X, d) is defined to be*

$$\delta(A) = \sup_{x,y \in A} d(x, y)$$

A is said to be bounded if $\delta(A) < \infty$. Show that $A \subset B$ implies $\delta(A) \leq \delta(B)$.

Proof. First, by the definition of supremum, we have

$$\sup X = s \implies \forall x \in X \ , \ x \leq s$$

Thus, over the set B, if $\delta(B)$ is the diameter of B, we have that

$$\forall (x, y) \in B \times B, d(x, y) \leq \delta(B).$$

Since, $A \subset B$, we have that all points in A also belong to B and hence, satisfy the property above, i.e

$$\forall (x,y) \in A \times A \ , \ d(x,y) \leq \delta(B)$$

and hence, taking the supremum on both sides, we get

$$\delta(A) = \sup_{x,y \in A} d(x,y) \leq \sup_{x,y \in A} \delta(B) = \delta(B).$$

■

7. *Show that $\delta(A) = 0$ if and only if A consists of a single point.*

Proof. We wish to show

$$\delta(A) = 0 \iff |A| = 1.$$

We first show RHS implies LHS. Suppose $\delta(A) = 0$, which means

$$\sup_{x,y \in A} d(x,y) = 0 \implies d(x,y) \leq 0 \ \forall x, y \in A.$$

However, since $d(x,y)$ is always non-negative, we have that

$$d(x,y) = 0 \ \forall x, y \in A \implies x = y \ \forall x, y \in A.$$

This tells us that there is only one unique point in A.

We now show LHS implies RHS. Suppose A has only one point x, then,

$$\delta(A) = \sup_{A} d(x,y) = \sup\{d(x,x)\} = \sup\{0\} = 0$$

This finishes the proof. ■

8. *The distance $D(A,B)$ between two non-empty sets of a metric space (X,d) is defined to be*

$$D(A,B) = \inf_{\substack{a \in A \\ b \in B}} d(a,b)$$

Show that D does not define a metric on the power set of X.

Proof. An easy way to see that this does not define a metric on the power set of X (denoted by $\mathcal{P}(X)$) is as follows: clearly, the metric is non-negative and symmetric. However, let $A, B \in \mathcal{P}(X)$. If $A \cap B \neq \emptyset$, then $\exists x \in X$ such that $x \in A$ and $x \in B$. In such a case,

$$D(A,B) = \inf_{\substack{a \in A \\ b \in B}} d(a,b) = d(x,x) = 0.$$

However, $A \cap B \neq \emptyset \nRightarrow A = B$, and hence, the condition

$$D(A,B) = 0 \iff A = B$$

fails. ■

9. *If $A \cap B \neq \emptyset$, show that $D(A,B) = 0$. What about the converse?*

Proof. Let $x \in A$ and $x \in B$. Then,

$$\{d(a,b)\}_{\substack{a \in A \\ b \in B}} = \{d(x,x)\} \cup \{d(a,b)\}_{(a,b) \in A \times B \setminus \{(x,x)\}} = \{0\} \cup \{d(a,b)\}_{(a,b) \in A \times B \setminus \{(x,x)\}}$$

and hence,

$$\inf_{\substack{a \in A \\ b \in B}} d(a,b) = 0 = D(A,B).$$

The converse may not be true. The idea comes from accumulation points (defined in the next section). x is said to be an accumulation point of a set A if every neighborhood of x contains atleast one point from A distinct from x. Let $x \in B$ and $x \notin A$ be an accumulation point of A. Then, every $\epsilon-$neighborhood of x will contain $a(\epsilon) \in A$ and hence, for every $\epsilon > 0$, you can find $a(\epsilon)$ such that $d(a(\epsilon), x) = \epsilon$. Since ϵ can be arbitrarily small, the infimum becomes zero. However, $A \cap B = \emptyset$. Thus, the converse may not be true. ■

10. *The distance $D(x)$ from a point x to a non-empty subset B of (X, d) is defined as*

$$D(x, B) = \inf_{b \in B} d(x, b)$$

Show that for any $x, y \in X$,

$$|D(x, B) - D(y, B)| \leq d(x, y).$$

Proof. First, consider the case when $x \in B$ and $y \in B$. Clearly, $D(x, B) = D(y, B) = 0$ and hence,

$$|D(x, B) - D(y, B)| = 0 \leq d(x, y).$$

by the properties of d. Now, consider the case when $x \notin B$ and $y \in B$. Clearly, $D(y, b) = 0$ and hence,

$$|D(x, B)| = \inf_{b \in B} d(x, b) \leq d(x, y)$$

since $y \in B$. Finally, consider the case when $x \notin B$ and $y \notin B$. Then, we have

$$D(x, B) = \inf_{b \in B} d(x, b) \leq \inf_{b \in B} (d(x, y) + d(y, b)) = d(x, y) + \inf_{b \in B} d(y, b) = d(x, y) + D(y, B).$$

where the first inequality follows from the triangle inequality applied on d. Rearranging the terms gives us

$$D(x, B) - D(y, B) \leq d(x, y).$$

By symmetry, interchanging x and y results in

$$D(y, B) - D(x, B) \leq d(x, y).$$

This finishes the proof. ∎

11. *If (X, d) is any metric space, show that another metric on X is defined by*

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

and X is bounded in the metric $\tilde{d}$.

Proof. First, the metric $\tilde{d}$ is non-negative, symmetric and satisfies $\tilde{d}(x, y) = 0 \iff x = y$ from the properties of d. Also

$$\lim_{d(x,y) \to \infty} \tilde{d}(x, y) = 1$$

and hence, it does remain bounded. Also, notice that

$$f(t) = \frac{t}{1 + t}$$

is always increasing and hence, by the non-negativity of the metric d, we have

$$\begin{aligned} f(d(x, y)) \leq f(d(x, z) + d(z, y)) = \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)} &= \frac{d(x, z)}{1 + d(x, z) + d(z, y)} + \frac{d(z, y)}{1 + d(x, z) + d(z, y)} \\ &\leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(x, z)} \end{aligned}$$

which finishes the proof. ∎

12. *Show that the union of two bounded sets is also bounded.*

Proof. Let A and B be two bounded sets such that $\delta(A) < \infty$ and $\delta(B) < \infty$. Then, using the definition of $\delta(A)$ from 6, we have

$$\forall a_1, a_2 \in A, d(a_1, a_2) \leq \delta(A) < \infty \text{ and } \forall b_1, b_2 \in B, d(b_1, b_2) \leq \delta(B) < \infty$$

Thus, for any $a_1 \in A, b_1 \in B$, we have

$$d(a_1, b_1) \leq d(a_1, a_2) + d(a_2, b_2) + d(b_1, b_2) \leq \delta(A) + \delta(B) + d(a_2, b_2)$$

and since d is a metric, it is also bounded, resulting in $d(a_1, b_1)$ being bounded. ∎

13. *The cartesian product $X = X_1 \times X_2$ of two metric spaces (X_1, d_1) and (X_2, d_2) can be made into a metric space (X, d) in many ways. For example, show that a metric d is defined by*

$$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Proof. First, note that d is bounded, non-negative, and symmetric owing to the metric properties of d_1 and d_2. Also, due to the metric properties of d_1 and d_2, we have

$$d(x, y) = 0 \iff d_1(x_1, y_1) = 0 \text{ and } d_2(x_2, y_2) = 0 \iff x = (x_1, x_2) = (y_1, y_2) = y.$$

Finally, we show the triangle inequality for some $z = (z_1, z_2)$ as follows:

$$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2) \leq d_1(x_1, z_1) + d_1(z_1, y_1) + d_2(x_2, z_2) + d_2(z_2, y_2) = d(x, z) + d(z, y).$$

This finishes the proof. ■

14. *Show that another metric on X is defined by*

$$\tilde{d}(x, y) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$$

Proof. Once again, note that $\tilde{d}$ is bounded, non-negative, symmetric and satisfies $\tilde{d}(x, y) = 0 \iff x = y$ owing to the metric properties of d_1 and d_2. Finally, we show the triangle inequality for some $z = (z_1, z_2)$ as follows:

$$\begin{aligned}
\tilde{d}(x, y) &= \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2} \\
&\leq \sqrt{(d_1(x_1, z_1) + d_1(z_1, y_1))^2 + (d_2(x_2, z_2) + d_2(z_2, y_2))^2} \\
&\leq \sqrt{d_1(x_1, z_1)^2 + d_2(x_2, z_2)^2 + d_1(z_1, y_1)^2 + d_2(z_2, y_2)^2 + 2d_1(x_1, z_1)d_1(z_1, y_1) + 2d_2(x_2, z_2)d_2(z_2, y_2)}
\end{aligned}$$

Using Hölder's inequality, which says that for $a_i, b_i \geq 0$ and p, q such that $p^{-1} + q^{-1} = 1$,

$$\sum_{i=1}^{N} a_i b_i \leq \left(\sum_{i=1}^{N} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{N} b_i^q\right)^{\frac{1}{q}}$$

we get

$$d_1(x_1, z_1)d_1(z_1, y_1) + d_2(x_2, z_2)d_2(z_2, y_2) \leq \sqrt{d_1(x_1, z_1)^2 + d_2(x_2, z_2)^2}\sqrt{d_1(z_1, y_1)^2 + d_2(z_2, y_2)^2}$$

and hence, we get

$$\tilde{d}(x, y) \leq \sqrt{d_1(x_1, z_1)^2 + d_2(x_2, z_2)^2}\sqrt{d_1(z_1, y_1)^2 + d_2(z_2, y_2)^2} = \tilde{d}(x, z) + \tilde{d}(z, y).$$

This finishes the proof. ■

15. *Show that another metric on X is defined by*

$$\tilde{\tilde{d}}(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

Proof. Once again, note that $\tilde{\tilde{d}}$ is bounded, non-negative, symmetric and satisfies $\tilde{\tilde{d}}(x, y) = 0 \iff x = y$ owing to the metric properties of d_1 and d_2. Finally, we show the triangle inequality for some $z = (z_1, z_2)$ as follows:

$$\begin{aligned}
\tilde{\tilde{d}}(x, y) &= \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \\
&\leq \max\{d_1(x_1, z_1) + d_1(z_1, y_1), d_2(x_2, z_2) + d_2(z_2, y_2)\} \\
&= d_i(x_i, z_i) + d_i(z_i, y_i) \; i \in \{1, 2\} \\
&\leq \max\{d_1(x_1, z_1), d_2(x_2, z_2)\} + \max\{d_1(z_1, y_1), d_2(z_2, y_2)\} \\
&= \tilde{\tilde{d}}(x, z) + \tilde{\tilde{d}}(z, y)
\end{aligned}$$

This finishes the proof. ■

1.3 Open set, Closed set, Neighborhood

1. *Justify the terms* open ball *and* closed ball *by showing that an open ball is an open set and a closed ball is a closed set.*

Proof. Let $B(x_0, r)$ be a ball of radius r around x_0. If we wish to show that $B(x_0, r)$ is open, it suffices to show that we can draw a ball around any point of $B(x_0, r)$. In other words, choose any point $y_0 \in B(x_0, r)$. For some radius $r', B(y_0, r') \subseteq B(x_0, r)$. Let $r' \leq r - d(x_0, y_0)$. Then, for some $x \in B(y_0, r')$, we have

$$d(x, x_0) \leq d(x, y_0) + d(y_0, x_0) \leq r$$

and hence, $x \in B(x_0, r)$. This shows that an open ball is also an open set.

Showing that a closed ball is a closed set is equivalent to showing the complement of a closed ball is open. Let X be a set and $x_0 \in X$. Let $\tilde{B}(x_0, r)$ denote a closed ball of radius r in X and let $\tilde{B}(x_0, r)^C$ denote the complement of $\tilde{B}(x_0, r)$. Choose some point $y_0 \in \tilde{B}(x_0, r)^C$ and $r' \leq d(x_0, y_0) - r$. Clearly, $d(x_0, y_0) \geq r$. Then, $B(y_0, r')$ is a ball about y_0 with radius r'. Let $x \in B(y_0, r'$. Then,

$$d(x_0, y_0) \leq d(x_0, x) + d(y_0, x) \implies d(x_0, x) \geq d(x_0, y_0) - d(y_0, x) \geq r - d(y_0, x) \geq r.$$

and hence, $x \in \tilde{B}(x_0, r)^C$. Thus, the complement of the closed ball is open. This finishes the proof. ■

2. *What is an open ball $B(x_0, 1)$ in $\mathbb{R}$ and $\mathbb{C}$? What is an open ball in $C[a, b]$?*

Proof. In $\mathbb{R}$, the open ball is the set of points $B(x_0, 1) = \{x_0 \pm \epsilon \mid |\epsilon| < 1\}$. In $\mathbb{C}$, let $x_0 = a + \iota b$, then the open ball is defined as $B(x_0, 1) = \{(a + \epsilon_1) + \iota(b + \epsilon_2) \mid \sqrt{\epsilon_1^2 + \epsilon_2^2} < 1\}$. Finally, let $x_0 \in C[a, b]$ be a function of t. Then, under the metric

$$d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|$$

we have the open ball to be defined as

$$B(x_0, 1) = \left\{ x_0(t) \pm \epsilon(t) \mid \left| \max_{t \in [a,b]} \epsilon(t) \right| < 1 \right\}.$$

■

3. *Consider $C[0, 2\pi]$ and determine the smallest r such that $y \in \tilde{B}(x, r)$ where $x(t) = \sin t$ and $y(t) = \cos t$.*

Proof. We wish to find the smallest value of r such that $y(t) \in \tilde{B}(x(t), r)$. Let the metric on $C[0, 2\pi]$ be defined as

$$d(x, y) = \max_{t \in [0,2\pi]} |x(t) - y(t)|.$$

Then, we wish to find the smallest value of r such that $d(x, y) \leq r$. Substituting the values of $x(t)$ and $y(t)$, we get

$$\max_{t \in [0,2\pi]} |\sin t - \cos t| \leq r$$

Differentiating $\sin t - \cos t$ w.r.t t and setting to 0, we find that the maximum value of the expression is obtained at $t = 3\pi/4$ and the maximum value is $\sqrt{2}$. Thus, setting $r = \sqrt{2}$ ensures $y \in B(x, r)$. ■

4. *Show that any nonempty set $A \subset (X, d)$ is open if and only if it is a union of open balls.*

Proof. We first assume that a nonempty set $A \subset (X, d)$ is open and wish to show it is the union of open balls. Choose any point $x \in A$. Then, there exists a ball B around x such that $B \subset A$. Since x is arbitrary, at each point in A, there exists a ball comprised entirely in A and hence, A is the union of all such open balls.

Now we assume A is a union of open balls. For some open ball B, $\exists x \in B$ such that we can draw another ball B_0 around x and $B_0 \subset B$. Since A is the union of open balls, $x \in A$ and $B_0 \subset B \subset A$. Since x is arbitrary, A is an open set. ■

5. *It is important to realize that some sets may be open and closed at the same time. Show that this is always the case for X and $\emptyset$. Show that in a discrete metric space X, every subset is open and closed.*

Proof. Note that $\emptyset$ is always open since there are no points in it. This also ensures that the set X is closed (since it's complement $\emptyset$ is open). On the other hand, X is open because it consists of the entire space and all limit points, and hence, it's complement $\emptyset$ is closed.

In a discrete metric space, let $A \subset X$. A is open if we can draw a ball around each point in A. Let $r < 1$. Then, for each $a \in A$, $B(a, r) = \{a\} \subset A$, and hence, the subsets are open. In a similar fashion, let A^C be the complement of A, and let $a \in A^C$ be some point. Then, choosing $r < 1$, we have a ball $B(a, r) = \{a\} \subset A^C$, and hence, A^C is also open, resulting in A being closed. ■

6. *If x_0 is an accumulation point of $A \subset (X, d)$, show that any neighborhood of x_0 contains infinitely many points of A.*

Proof. We shall prove the claim by contradiction. By the definition of an accumulating point, the neighborhood of x_0 contains at least one point distinct from x_0 from A. Let $B(x_0, \epsilon)$ be an $\epsilon-$neighborhood of x_0, which consists of finitely many points ($\neq x_0$) $\{a_1, \ldots, a_k\}$. Let $r \leq \min_{j \in [k]} d(x_0, a_j)$. Then, the $r-$neighborhood of x_0 consists of no points other than x_0, which is a contradiction since x_0 is an accumulation point. Hence, there should be another point in the $r-$neighborhood, which contradicts our assumption of finitely many points. ■

7. *Describe the closure of each of the sets: the integers on $\mathbb{R}$, the rational numbers on $\mathbb{R}$, the complex numbers with rational real and imaginary parts in $\mathbb{C}$, and the disk $\{z \mid |z| < 1\} \subset \mathbb{C}$.*

Proof. The closure of a set includes all points of the set and all it's limit points. The limit points of a set are points such *every* neighborhood of the point consists of at least one element of the set distinct from the point. Clearly, for the set of integers on $\mathbb{R}$, the closure is the set of integers itself. In other words, there are no limit points for the set of integers. Let z be an integer, then drawing an $\epsilon-$neighborhood around z with $\epsilon < 1$ ensures that there is no integer apart from z in this neighborhood. Same is the case for any rational number q, any $\epsilon-$neighborhood around q may not consist of an integer.

For the set of rationals on $\mathbb{R}$, the closure is $\mathbb{R}$. This is because choosing any point in $\mathbb{R}$ will have atleast one rational number in it's neighborhood. Same is the case for the set of complex numbers with rational real and imaginary parts. You can always find a new complex number with rational real and imaginary parts in a neighborhood.

Finally, for the open disk of radius 1 in the complex numbers, defined as

$$\{z \mid |z| < 1\} \subset \mathbb{C}$$

the accumulation points will be

$$\{z \mid |z| \leq 1\}.$$

Since any neighborhood of a "boundary point" will intersect with the disk, they also are accumulation points, and hence, the closure is simply the closed disk of radius 1. ■

8. *Show that the closure $\overline{B(x_0, r)}$ of an open ball $B(x_0, r)$ in a metric space can differ from the closed ball $\tilde{B}(x_0, r)$.*

Proof. TODO. ■

9. *Show that $A \subset \overline{A}$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$, and $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.*

Proof. $A \subset \overline{A}$ follows from the definition of a closure. Now, we show $\overline{A \cup B} = \overline{A} \cup \overline{B}$. To show this, we show that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. First, let $x \in \overline{A \cup B}$. Then,

1. $x \in A \cup B$. This means that $x \in A$ or $x \in B$. WLOG, assume $x \in A$, then by the definition of closure, $x \in \overline{A}$, and hence, $x \in \overline{A} \cup \overline{B}$.

2. $x \notin A \cup B$ and x is an accumulation point of $A \cup B$. This means that x is also an accumulation point of either A or B. This is clear if every neighborhood of x contains some point of A or if every neighborhood contains a point of B. Now consider the case where some ϵ_A-neighborhood of x contains only points of A and ϵ_B-neighborhood contains points of only B. WLOG, let $\epsilon_A \geq \epsilon_B$. Then, ϵ_A-neighborhood also contains points from B. Thus, x would be a accumulation point for either A or B, which implies that $x \in \overline{A}$ or $x \in \overline{B}$, resulting in $x \in \overline{A} \cup \overline{B}$.

This shows that $x \in \overline{A} \cup \overline{B}$.

Now we show that $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Let $x \in \overline{A} \cup \overline{B}$. WLOG, assume $x \in \overline{A}$. Then,

1. $x \in A$, which means that $x \in A \cup B$ and hence, $x \in \overline{A \cup B}$.

2. $x \notin A$ and x is an accumulation point for A. This means that every neighborhood of x contains some point from A. That means that every neighborhood of x also contains some point from $A \cup B$, and hence, x is an accumulation point for $A \cup B$. Thus, $x \in \overline{A \cup B}$.

Thus, $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. This finishes the proof.

Finally, we show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Let $x \in \overline{A \cap B}$. Then,

1. $x \in A \cap B$. This means that $x \in A$ and $x \in B$, and hence, $x \in \overline{A}$ and $x \in \overline{B}$, resulting in $x \in \overline{A} \cap \overline{B}$.

2. $x \notin A \cap B$ but x is an accumulating point of $A \cap B$. That means that every neighborhood of x has a point from $A \cap B$, which means the point in the neighborhood belongs to both A and B. Thus, every neighborhood contains a point from A and B, making x an accumulation point for both A and B. Hence, $x \in \overline{A}$ and $x \in \overline{B}$, resulting in $x \in \overline{A} \cap \overline{B}$.

This finishes the proof. ∎

10. *A point x not belonging to a closed set $M \subset (X, d)$ always has a nonzero distance from M. To prove this, show that $x \in \overline{A}$ if and only if $D(x, A) = 0$, where A is any nonempty subset.*

Proof. First, we define

$$D(x, A) = \inf_{a \in A} d(x, a).$$

We show that $x \in \overline{A} \implies D(x, A) = 0$. If $x \in A$, then by definition of D, we have $D(x, A) = 0$. However, let $x \notin A$, then for every $\epsilon-$neighborhood of x, $B(x, \epsilon) \cap A \neq \emptyset$, which means that for any $\epsilon > 0$, we can find a point distinct from x belonging to A. As ϵ can be arbitrarily small, the infimum would be 0.

Next we show that $D(x, A) = 0 \implies x \in \overline{A}$. If $x \in A$, then it is trivial. However, if $x \notin A$, then we have that for all $\epsilon > 0$, there exists some point $a(\epsilon) \in A$ such that $d(a(\epsilon), x) = \epsilon$. Since ϵ can be made arbitrarily small, the infimum results in 0. However, this also means that for every $\epsilon-$neighborhood of x, there exists some point $a(\epsilon) \in A$ and hence, x is an accumulation point of A. Thus, $x \in \overline{A}$. This finishes the proof. ∎

11. *A boundary point of $A \subset (X, d)$ is a point of X (which may or may not be in A) such that every neighborhood of x contains points of A as well as points not belonging to A. The boundary of A is defined to be the set of all boundary points of A. Find the boundaries of $(-1, 1)$, $[-1, 1)$, $[-1, 1]$ on $\mathbb{R}$, the set of rationals on $\mathbb{R}$, and the disks $\{z \mid |z| < 1\}$ and $\{z \mid |z| \leq 1\} \subset \mathbb{C}$.*

Proof. The boundary points of $(-1, 1)$ are $\{-1, 1\}$. This is because any $\epsilon-$neighborhood of -1 will contain $-1+\epsilon \in (-1, 1)$ and $-1-\epsilon \notin (-1, 1)$. Note that $x \in (-1, 1)$ is not a boundary point because every neighborhood will not contain any point which does not belong to $(-1, 1)$. The boundary points for $(-1, 1]$ and $[-1, 1]$ are also $\{-1, 1\}$.

The boundary points for the set of rationals on $\mathbb{R}$ is $\mathbb{R}$. This is because for any $q \in \mathbb{R}$, the neighborhood of q will contain rationals as well as irrationals.

Finally, the boundary points for the disks $\{z \mid |z| < 1\}$ and $\{z \mid |z| \leq 1\} \subset \mathbb{C}$ is simply the set $\{z \mid |z| = 1\}$. ∎

12. *Show that $B[a, b]$ is not separable.*

Proof. TODO ∎

13. *Show that a metric space X is separable if and only if X has a countable subset Y with the following property: For every $\epsilon > 0$ and every $x \in X$, there is a $y \in Y$ such that $d(x, y) < \epsilon$.*

Proof. We first show that if a metric space X is separable, then it contains a countable subset with the property mentioned above. First, if X is separable it would contain separable dense subsets Y. Since Y is dense, we have that $\overline{Y} = X$, which means that every point in $x \in X$ either belongs to Y or is an accumulation point of Y. In such a case, if $x \in Y$, then it is trivial. However, if $x \notin Y$ and is an accumulation point, then by the definition of an accumulation point, every $\epsilon-$neighborhood of x would contain some $y \in Y$, and hence, $d(x, y) < \epsilon$. This finishes the proof.

We now prove the converse, i.e if there is a countable subset Y that satisfies the above mentioned property, then X is separable. In such a case, the property mentions that for every $\epsilon > 0$ and for every $x \in X$, there exists $y \in Y$ such that $d(x, y) < 0$. If $x \in Y$, then this is trivially true. However, if $x \notin Y$, then, by the above mentioned property, x is an accumulation point for Y, which means that all points in X are either in Y or are accumulation points for Y. Thus, $\overline{Y} = X$ and hence, Y is dense in X. Since Y is countable, we have that X is a separable set. This finishes the proof. ∎

14. *Show that a mapping $T : X \mapsto Y$ is continuous if and only if the inverse image of any closed set $M \subset Y$ is a closed set in X.*

Proof. We instead show that if T is a continuous mapping if and only if the inverse image of any open set in Y is open in X.

Suppose T is a continuous mapping. Then for every $\epsilon > 0, \exists \delta > 0$ such that

$$d(x_0, x) \leq \delta \implies d(Tx_0, Tx) \leq \epsilon.$$

Thus, consider the point $Tx_0 \in Y$ and the set $B(Tx_0, \epsilon)$ which is open. Clearly, due to the definition of T, $B(Tx_0, \epsilon)$ maps back to $B(x_0, \delta)$ which is again open, and hence, if T is continuous, then the inverse image of an open set is an open set.

Now, we show the converse. Suppose we take the set $B(Tx_0, \epsilon) \subset Y$ which is open. Then, we have that the inverse image of $B(Tx_0, \epsilon)$ is also open, say some set $N \in X$ such that $x_0 \in N$. Also, since N is a neighborhood, $B(x_0, \delta) \subset N$, and hence, the definition of continuity is fulfilled.

Now, we shall show the statement. Suppose T is continuous and $M \subset Y$ is closed. This means that M^C is open and hence, the inverse image of M, say $A \subset X$ is open. Thus, A^C is closed. Similarly, let $M \subset Y$ is closed and it's inverse image $A \subset X$ is also closed. This means that M^C and A^C is open and hence, T is continuous. ■

15. *Show that the image of an open set under a continuous mapping need not be open.*

Proof. Consider a function f on the set (a, b) that maps to $[c, d]$ such that for some $a < a_1, b_1 < b$, we have $f(a_1) = c$ and $f(b_1) = d$. Any such function will satisfy the above claim. ■

1.4 Convergence, Cauchy Sequences, Completeness

1. *If a sequence (x_n) in a metric space X is convergent and has limit x show that every subsequence (x_{n_k}) of (x_n) is also convergent and has the same limit x.*

Proof. Since (x_n) is convergent, we have that for every $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $n \geq N(\epsilon)$, we have that $d(x, x_n) < \epsilon$. This also means that for all $n_k \geq N(\epsilon)$, $d(x_{n_k}, x) \leq \epsilon$. Hence, the subsequence (x_{n_k}) also follows the definition of convergence. ■

2. *If (x_n) is Cauchy and has a convergent subsequence, say $x_{n_k} \to x$, show that (x_n) is convergent with the limit x.*

Proof. Since (x_n) is Cauchy, we have that for every $\epsilon > 0$, there exists $N_1(\epsilon)$ such that for all $m, n \geq N_1(\epsilon)$, we have $d(x_m, x_n) \leq \epsilon$. Now, since the subsequence (x_{n_k}) is converging to x, we have that for every $\epsilon > 0$, there exists $N_2(\epsilon)$ such that for all $n_k \geq N_2(\epsilon)$, $d(x_{n_k}, x) \leq \epsilon$. Thus, fixing the value of ϵ, and choosing the value of $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$, we have that for some $n, n_k \geq N(\epsilon)$

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \leq 2\epsilon.$$

This finishes the proof. ■

3. *Show that $x_n \to x$ if and only if for every neighborhood V of x there is an integer n_0 such that $x_n \in V$ for all $n > n_0$.*

Proof. Suppose $x_n \to x$. Then, for every $\epsilon > 0$, there exists some $N(\epsilon)$ such that for all $n \geq N(\epsilon)$, $d(x_n, x) \leq \epsilon$. This means that for every $\epsilon > 0$, there exists $N(\epsilon)$ such that for $n \geq N(\epsilon)$, $x_n \in B(x, \epsilon)$. Thus, for every neighborhood V, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset V$, and hence, for $n \geq N(\epsilon) = n_0$, $x_n \in V$.

We now show the converse. Suppose, for every neighborhood V, there is an integer n_0 such that $x_n \in V$ for $n > n_0$. Assume that $B(x, \epsilon) \subset V$. Thus, setting $n_0 = N(\epsilon)$, for all $n > N(\epsilon)$, $x_n \in B(x, \epsilon)$, and hence, the definition of convergence is fulfilled. ■

4. *Show that a Cauchy sequence is bounded.*

Proof. A Cauchy sequence (x_n) is defined as follows: for every $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $m, n \geq N(\epsilon)$, we have that $d(x_m, x_n) \leq \epsilon$. Fix an ϵ and $x_N = x_{N(\epsilon)}$ as our reference point. Now we know that for $n \geq N$, $d(x_n, x_N) \leq \epsilon$. Thus, choose

$$r = \max_{i \in [N]} d(x_i, x_N)$$

and $r_0 = \max\{r, \epsilon\}$. Then, for all $n > 0$, we have $d(x_n, x_N) \leq r_0$ which shows that the entire sequence is bounded in a ball of radius r_0 centered at x_N. ■

5. *Is boundedness of a sequence in a metric space sufficient for the sequence to be Cauchy or convergent?*

Proof. No, the boundedness property is not sufficient. An example is $x_n = \sin \frac{n\pi}{2}$, which is always bounded by 1 but is neither Cauchy nor convergent. ■

6. *If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d), show that (a_n), where $a_n = d(x_n, y_n)$ is convergent.*

Proof. **Note: this question seems incorrect, and should only be true for complete spaces.**

Since (x_n) is Cauchy, for every $\epsilon > 0$, there exists $N_x(\epsilon)$ such that for every $m, n > N_x(\epsilon)$, we have that $d(x_m, x_n) < \epsilon$. Similarly, since (y_n) is Cauchy, for every $\epsilon > 0$, there exists $N_y(\epsilon)$ such that for every $m, n > N_y(\epsilon)$, we have that $d(y_m, y_n) < \epsilon$. Now, fix ϵ and let $N(\epsilon) = \max\{N_x(\epsilon), N_y(\epsilon)\}$. Then for every $m, n > N(\epsilon)$, we have

$$d(x_n, y_n) \le d(x_n, x_m) + d(y_n, y_m) + d(x_m, y_m) < 2\epsilon + d(x_m, y_m)$$

and hence,

$$d(x_n, y_n) - d(x_m, y_m) \le 2\epsilon$$

Similarly, by symmetry

$$d(x_m, y_m) - d(x_n, y_n) \le 2\epsilon$$

and hence,

$$|d(x_n, y_n) - d(x_m, y_m)| \le 2\epsilon.$$

Thus, the sequence (a_n) is Cauchy, and since the space should be complete, the sequence is also convergent. ■

7. *Give an indirect proof of Lemma 1.4-2(b).*

Proof. What an indirect proof is seems unclear. We wish to prove that if a sequence $(x_n) \to x$ and $(y_n) \to y$, then $d(x_n, y_n) \to d(x, y)$. By the properties of convergence, we have

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ and } \lim_{n\to\infty} d(y_n, y) = 0.$$

Thus, using the triangle inequality and non-negativity of the metric, we have

$$0 \le d(x_n, y_n) \le d(x_n, x) + d(y_n, y) + d(x, y).$$

As n grows, we have $0 \le \lim_{n\to\infty} d(x_n, y_n) \le d(x, y)$. Similarly, we can also obtain $0 \le d(x, y) \le \lim_{n\to\infty} d(x_n, y_n)$, which gives us the result that

$$\lim_{n\to\infty} d(x_n, y_n) = d(x, y).$$

■

8. *If d_1 and d_2 are metrics on the same set X and there are positive numbers a and b such that for all $x, y \in X$*

$$ad_1(x, y) \le d_2(x, y) \le bd_1(x, y),$$

show that the Cauchy sequences in (X, d_1) and (X, d_2) are the same.

Proof. Let (x_n) be a Cauchy sequence on (X, d_1). Then, for every $\epsilon > 0$, we have some $N(\epsilon)$ such that for all $n, m > N(\epsilon)$, $d_1(x_m, x_n) < \epsilon$. Using the relation between d_1 and d_2, we have that for every $m, n > N(\frac{1}{b}(b\epsilon))$, $d_2(x_m, x_n) < b\epsilon$ and hence, every sequence (x_n) that is Cauchy in (X, d_1) is also Cauchy in (X, d_2). Similarly, if a sequence (y_n) is Cauchy in (X, d_2), for some $\epsilon > 0$, we have some $N(\epsilon)$ such that for all $m, n > N(\epsilon)$, $d_2(y_m, y_n) < \epsilon$. Using the relation once again, we have that for $m, n > N(a(\frac{\epsilon}{a}))$, $d_1(y_m, y_n) < \frac{\epsilon}{a}$ and hence, (y_n) is Cauchy in (X, d_1). This finishes the proof. ■

9. *Using 8, show that the metric spaces in 13, 14, 15 from section 1.2 have the same Cauchy sequences.*

Proof. Define

$$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$$
$$\tilde{d}(x, y) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$$
$$\tilde{\tilde{d}}(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

on the metric space $X = X_1 \times X_2$. It is easy to see the following:

1. $\tilde{d} \le \sqrt{2}\tilde{\tilde{d}}$.
2. $d \le 2\tilde{\tilde{d}}$.
3. $d \le \tilde{d}$.
4. $\tilde{\tilde{d}} \le d$.

Combining these gives us

$$\tilde{\tilde{d}} \leq d \leq \tilde{d} \leq 2\tilde{\tilde{d}}.$$

This finishes the proof. ■

10. *Using the completeness of* $\mathbb{R}$ *,prove the completeness of* $\mathbb{C}$.

Proof. Since $\mathbb{R}$ is complete, every Cauchy sequence converges to some point in $\mathbb{R}$. Now, every complex number has a real part and an imaginary part. Let (c_n) be some cauchy sequence of complex numbers such that $c_n = a_n + \iota b_n$. Clearly, the sequence (a_n) is Cauchy and convergent since $\mathbb{R}$ is complete. Also, multiplying every element of a sequence does not change the convergence properties and hence, (ιb_n) is also cauchy and convergent. Thus, $(a_n + \iota b_n) = (c_n)$ is also cauchy and convergent. ■

1.5 Examples: Completeness Proofs

1. *Let* $a, b \in \mathbb{R}$ *and* $a < b$. *Show that the open interval* (a, b) *is an incomplete subspace of* $\mathbb{R}$ *while the closed interval is* $[a, b]$ *is complete.*

Proof. First, due to Theorem 1.4-7, we have that a closed subspace is complete and hence, $[a, b]$ is complete. We shall prove this theorem again which states that *A subspace* M *of a metric space* X *is complete if and only if the set* M *is closed.* First, let M be complete, this means that for any Cauchy sequence $(x_n) \in M$, $x_n \to x \in M$. This means that all the limit points (accumulation points) of M belong to M and hence, $\overline{M} = M$. Thus, M is closed. Now, we show the converse. Let M be closed. This means that all the limit points (accumulation points) of M belong to M, since $\overline{M} = M$, and hence, any Cauchy sequence $(x_n) \in M$ will converge to some $x \in M$, and hence, the space is complete.

Further, we can construct a counterexample to show that (a, b) is incomplete. Consider the sequence $x_n = a + \frac{1}{n}$. Clearly, as $n \to \infty$, $x_n \to a$, however, $x \notin (a, b)$. Thus, the set (a, b) is incomplete. We can also see that a is an accumulation point of (a, b) and hence, the closure of (a, b) is $[a, b]$. Since the closure is not equal to the set, it is incomplete. ■

2. *Let* X *be the space of all ordered* $n-$*tuples* $x = (\xi_1, \ldots, \xi_n)$ *of real numbers and*

$$d(x, y) = \max_i |\xi_i - \eta_i|$$

where $y = (\eta_j)$. *Show that* (X, d) *is complete.*

Proof. First, assume that $(x_n) = (\xi_j^{(n)})$ is a Cauchy sequence. This means that for every $\epsilon > 0$ there exists $N(\epsilon)$ such that for all $m, n > N(\epsilon)$, we have

$$d(x_n, x_m) < \epsilon.$$

This means that

$$\max_i \left|\xi_i^{(n)} - \xi_i^{(m)}\right| < \epsilon \implies \forall i \in [n] \ \left|\xi_i^{(n)} - \xi_i^{(m)}\right| < \epsilon$$

By the completeness of real numbers, we have that the sequences $(\xi_i^{(1)}, \xi_i^{(2)}, \ldots)$ is Cauchy, and hence, converges to some value, say ξ_i. Hence, define $x = (\xi_i)$. Clearly, $x \in X$. We now wish to show that the sequence (x_n) converges. Recall that for $\epsilon > 0$ and $m, n > N(\epsilon)$, we have that $d(x_m, x_n) < \epsilon$. As $m \to \infty$, we have

$$d(x_n, x) < \epsilon$$

and hence, x is a limit point for x_n. Thus, the space is complete. ■

3. *Let* $M \subset \ell^\infty$ *be the subspace of all sequences* $x = (\xi_j)$ *with at most finitely many nonzero terms. Find a Cauchy sequence in* M *that does not converge in* M *so that* M *is not complete.*

Proof. The easiest way to show that the space is not complete is to find a Cauchy sequence that converges to an all-zero sequence. Consider the sequence

$$x_n = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, \ldots, 0\right)$$

Then, choose $\epsilon > 0$ and $N = \frac{1}{\epsilon}$ and for some $n > m > N$,

$$d(x_n, x_m) = \sup\left\{\frac{1}{m+1}, \ldots, \frac{1}{n}\right\} = \frac{1}{m+1} \leq \frac{1}{m} \leq \frac{1}{N} = \epsilon.$$

Thus, it is a Cauchy sequence. However, note that the sequence will converge to $x = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right) \notin M$. ■

4. *Show that the subspace M in the previous question 3 is not complete using Theorem1.4-7.*

Proof. Theorem 1.4-7 says that a subspace is complete if and only if the subspace is closed. We wish to show that the subspace is not closed, i.e $M \neq \overline{M}$, or in other words, there exists some x which is an accumulation point and $x \notin M$. This is exactly what we showed in the previous question. ■

5. *Show that the set X of all integers with metric $d(x, y) = |x - y|$ is a complete metric space.*

Proof. Recall that the set of integers is closed in $\mathbb{R}$ (Section 1.3 7) and hence, it is a complete metric space. We shall also attempt to show it using Cauchy sequences. Consider some sequence (x_n). Then for every $\epsilon > 0$, there exists $N(\epsilon)$ such that for every $n, m > N(\epsilon)$

$$d(x_m, x_n) < \epsilon.$$

However, if we choose $\epsilon < 1$, then at some point, all these points would become equal, i.e

$$d(x_m, x_n) < 1 \implies x_m = x_n$$

and hence, the sequence converges to some integer x, which shows that the space is complete. ■

6. *Show that the set of all real numbers is incomplete if we choose*

$$d(x, y) = |\arctan x - \arctan y|$$

Proof. Consider the sequence $x_n = \tan\left(\frac{\pi}{2} - \frac{1}{n}\right)$. Clearly,

$$\lim_{n\to\infty} x_n = \infty$$

and hence, the sequence is diverging. However, for $m > n$,

$$d(x_n, x_m) = \left|\frac{1}{m} - \frac{1}{n}\right| = \frac{1}{n} - \frac{1}{m}.$$

Clearly, for some $\epsilon > 0$, $N(\epsilon) = \frac{1}{\epsilon}$, and $m > n > N(\epsilon)$, we have that $d(x_n, x_m) < \epsilon$ and hence, the sequence is Cauchy. Thus, the space is not complete. ■

7. *Let X be the set of all positive integers and $d(m, n) = \left|m^{-1} - n^{-1}\right|$. Show that this is not complete.*

Proof. Consider the sequence $(x_n) = n$, then for some $\epsilon > 0$ and $N(\epsilon) = \frac{1}{\epsilon}$, choosing $n > m > N$, we have

$$d(x_m, x_n) = \frac{1}{m} - \frac{1}{n} < \frac{1}{N} = \epsilon$$

and hence, the sequence is Cauchy. However, clearly, it is not convergent. ■

8. *Show that the subspace $Y \subset C[a, b]$ consisting of all $x \in C[a, b]$ such that $x(a) = x(b)$ is complete.*

Proof. Suppose there is some Cauchy sequence x_n such that for some $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $m, n > N(\epsilon)$, we have $d(x_n, x_m) < \epsilon$, where $d(x, y)$ is defined as

$$d(x, y) = \max_{t\in[a,b]} |x(t) - y(t)|.$$

Suppose $x_n \to x$. To show that Y is complete, we wish to show $x \in Y$. Since $x_n \to x$, we have that for some $\epsilon > 0$ and $n > N(\epsilon)$,

$$d(x_n, x) < \epsilon \implies \max_{t\in[a,b]} |x_n(t) - x(t)| < \epsilon \implies |x_n(t) - x(t)| < \epsilon \; \forall t \in [a, b].$$

Thus,

$$d(x(a), x(b)) < d(x(a), x_n(a)) + d(x_n(a), x_n(b)) + d(x_n(b), x(b)) < 2\epsilon$$

since $x_n(a) = x_n(b)$. Thus, we have

$$0 \leq d(x(a), x(b)) < 2\epsilon$$

for all $\epsilon > 0$. As ϵ becomes arbitrarily small, we have $d(x(a), x(b)) = 0 \implies x(a) = x(b)$ and hence, $x \in Y$. ■

9. *Prove that if a sequence (x_m) of continuous functions on $[a,b]$ converges on $[a,b]$ and the convergence is uniform, then the limit function x is also continuous on $[a,b]$.*

Proof. First, since x_m is continuous, we have for a fixed $t_0 \in [a,b]$ and for some $\epsilon > 0$, there exists some $\delta > 0$ such that

$$d(t,t_0) \leq \delta \implies d(x_m(t), x_m(t_0)) \leq \epsilon \; \forall m.$$

Also, since the sequence (x_m) converges pointwise to x, we have for some $\epsilon > 0$, there exists $N(\epsilon)$ such that for $n > N(\epsilon)$,

$$d(x_n(t), x(t)) \leq \epsilon \; \forall t \in [a,b].$$

Thus, for a fixed $t_0 \in [a,b]$ and $\epsilon > 0$, corresponding $\delta > 0$, and $n > N(\epsilon)$, we have that if $d(t,t_0) < \delta$

$$d(x(t), x(t_0)) \leq d(x(t), x_n(t)) + d(x_n(t), x_n(t_0)) + d(x(t_0), x_n(t_0)) \leq 3\epsilon.$$

This shows that x is continuous at t_0. Since t_0 was an arbitrary point, it shows that x is continuous. ■

10. *Show that the discrete metric space is complete.*

Proof. Recall the discrete metric space is defined as $d(x,x) = 0$ and $d(x,y) = 1$ if $x \neq y$. Recall that this space has no accumulation points, and hence, the space is closed. Thus, the only Cauchy sequences are constant sequences, and these definitely converge. This also makes sense since setting $\epsilon < 1$ ensures a constant sequence, while $\epsilon \geq 1$ brings the whole space into play, and thus, no sequence will ever converge. Thus, convergence only occurs with $\epsilon < 1$, which results in a constant sequence. Also, the only Cauchy sequences are constant. Thus, the space is complete. ■

11. *Show that in the space s, we have $x_n \to x$ if and only if $\xi_j^{(n)} \to \xi_j$ for all j.*

Proof. Let $x_n = \xi_j^{(n)}$ and the metric is defined as

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}$$

Suppose $x_n \to x$. This means

$$\lim_{n\to\infty} d(x, x_n) = 0 \implies \lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\left|\xi_j^{(n)} - \xi_j\right|}{1 + \left|\xi_j^{(n)} - \xi_j\right|} = 0$$

However, since the sum comprises of non-negative terms, the sum is zero only if each term is zero, or

$$\lim_{n\to\infty} \left|\xi_j^{(n)} - \xi_j\right| = 0 \implies \lim_{n\to\infty} d(\xi_j^{(n)}, \xi_j) = 0 \implies \xi_j^{(n)} \to \xi_j$$

On the other hand, if $\xi_j^{(n)} \to \xi_j$, then

$$\lim_{n\to\infty} \left|\xi_j^{(n)} - \xi_j\right| = 0.$$

Thus,

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

This finishes the proof. ■

12. *Using 11 show that the sequence space s is complete.*

Proof. Suppose (x_n) is a Cauchy sequence where $x_n = (\xi_1^{(n)}, \ldots)$. Then, for some $\epsilon > 0$, there exists some $N(\epsilon)$ and some $m, n > N(\epsilon)$ such that

$$d(x_m, x_n) < \epsilon \implies \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\left|\xi_j^{(n)} - \xi_j^{(m)}\right|}{1 + \left|\xi_j^{(n)} - \xi_j^{(m)}\right|} < \epsilon \implies \left|\xi_j^{(n)} - \xi_j^{(m)}\right| < \frac{\epsilon}{2}$$

or, in other words, $(\xi_j^{(1)}, \xi_j^{(2)}, \ldots)$ is Cauchy, and hence, converges. Since the space of complex numbers is complete, we also have $\xi^{(k)_j} \to \xi_j$(say), and hence, from 11, we can say $x_n \to x$. This shows that x is complete. ■

13. *Show that another Cauchy sequence is given by*

$$x_n(t) = n \text{ if } 0 \leq t \leq \frac{1}{n^2} \text{ and } x_n(t) = t^{-\frac{1}{2}} \text{ if } \frac{1}{n^2} \leq t \leq 1$$

under the metric

$$d(x, y) = \int_0^1 |x(t) - y(t)| \ dt.$$

Proof. Clearly, for some $m > n$,

$$\begin{aligned} d(x_m, x_n) &= \int_0^{\frac{1}{m^2}} |m - n| \ dt + \int_{\frac{1}{m^2}}^{\frac{1}{n^2}} \left| t^{-\frac{1}{2}} - n \right| \ dt \\ &= \frac{m-n}{m^2} + \int_{\frac{1}{m^2}}^{\frac{1}{n^2}} t^{-\frac{1}{2}} - n \ dt \\ &= \frac{m-n}{m^2} + 2\left(\frac{1}{n} - \frac{1}{m}\right) - n\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \\ &= \frac{1}{n} - \frac{1}{m} \\ &\leq \epsilon \end{aligned}$$

for some $\epsilon > 0$ and choosing $N(\epsilon) = \frac{1}{\epsilon}$ and $m > n > N(\epsilon)$. Thus, the sequence is Cauchy. ■

14. *Show that the Cauchy sequence in 13 does not converge.*

Proof. Note, from 13, we have for some $m > n$,

$$d(x_m, x_n) = \int_0^{\frac{1}{m^2}} |m - n| \ dt + \int_{\frac{1}{m^2}}^{\frac{1}{n^2}} \left| t^{-\frac{1}{2}} - n \right| \ dt$$

and hence,

$$\lim_{m \to \infty} d(x_m, x_n) = d(x, x_n) = \int_0^{\frac{1}{n^2}} \left| t^{-\frac{1}{2}} - n \right| \ dt$$

Clearly, x would be unbounded at $t = 0$ while the function would then be defined as

$$x(t) = t^{-\frac{1}{2}} \ \forall t \in (0, 1]$$

Hence, there is no convergence at $t = 0$. ■

15. *Let X be the metric space of all real sequences that has only finitely many nonzero terms, and $d(x, y) = \sum |\xi_j - \eta_j|$, where $y = (\eta_j)$. Show that with $x_n = (\eta_j^{(n)})$,*

$$\eta_j^{(n)} = j^{-2} \text{ for } n \in [j] \text{ and } \eta_j^{(n)} = 0 \text{ for } j > n$$

is Cauchy but does not converge.

Proof. Let $\epsilon > 0$ be fixed. Then, there exists $N(\epsilon)$ such that for $m > n > N(\epsilon)$, we have

$$d(x_m, x_n) = \sum_{j=n}^{m} \frac{1}{j^2} \leq \frac{m-n}{n^2} < \epsilon.$$

However, say we fix some N, then for $n > N$, we have

$$d(x_n, x) = |1 - \xi_1| + \left| \frac{1}{4} - \xi_2 \right| + \dots \frac{1}{(N+1)^2} + \dots \frac{1}{n^2} \geq \frac{1}{(N+1)^2}$$

which will never be arbitrarily small since N is fixed. This equation uses the fact that if $x \in X$, then, after some N, all it's terms would be zero. ■

1.6 Completion of Metric Spaces

1. *Show that if a subspace Y of a metric space consists of finitely many points, then Y is complete.*

Proof. First, suppose there is a Cauchy sequence x_n. This means that for every $\epsilon > 0$, there exists $N(\epsilon)$ and some $m, n > N(\epsilon)$ such that $d(x - m, x_n) < \epsilon$. However, due to a finite number of points available in the set, the sequence has finitely many terms, and thus, we can express $\epsilon_0 = \min_{a,b \in Y} d(a, b)$. Thus, if we choose $\epsilon = \epsilon_0$, the sequence will have to become constant, which always converges. Thus, all Cauchy sequences become constant eventually, which converge, and hence, the space is complete. ■

2. *What is the completion of (X, d) where X is the set of all rational numbers and $d(x, y) = |x - y|$?*

Proof. The completion would be the set of all reals, $\mathbb{R}$. ■

3. *What is the completion of a discrete metric space?*

Proof. Recall that a discrete metric space is always complete. Hence, the completion is the entire space X itself. ■

4. *If X_1 and X_2 are isometric and X_1 is complete, show that X_2 is complete.*

Proof. X_1 and X_2 are isometric spaces if there exists some bijective isometry $T : X_1 \mapsto X_2$. By an isometry, we mean that for some $x, y \in X_1$

$$d_1(x, y) = d_2(Tx, Ty).$$

If X_1 is complete, then X_1 is also closed, i.e, $\overline{X_1} = X_1$. Suppose $x \in X_1$ is an accumulation point of X_1. This means that for every $\epsilon > 0$, there exists some $x_\epsilon \in X_1$ such that

$$d_1(x, x_\epsilon) < \epsilon.$$

Now, since X_1 and X_2 are isometric spaces

$$d_2(Tx, Tx_\epsilon) < \epsilon$$

which means that $Tx \in X_2$ is also an accumulation point for X_2 (since x_ϵ is arbitrary). Thus, all accumulation points of X_2 lie in X_2 and hence, X_2 is also complete. ■

5. *A homeomorphism is a continuous bijective mapping $T : X \mapsto Y$ whose inverse is continuous, the metric spaces X and Y are said to homeomorphic. Show that is X and Y are isometric, then they are also homeomorphic.*

Proof. First, an isometry implies the existence of a bijective map, so we already have a bijection. We need to ensure both the map and it's inverse is continuous. The definition of continuity is as follows: for every $\epsilon > 0$, there exists $\delta > 0$, such that $d_1(x, x_0) < \delta \implies d_2(Tx, Tx_0) < \epsilon$ ensures that the map T is continuous as x_0. Now, from the condition of isometric spaces, we have that

$$d_1(x, x_0) = d_2(Tx, Tx_0)$$

and hence, if we set $\delta = \epsilon$, we see that the definition of continuity is always fulfilled. Thus, the mapping from X to Y is always continuous. Making a similar argument for T^{-1} results in the conclusion that T^{-1} is continuous and hence, X and Y are also homeomorphic. ■

6. *Show that $C[0, 1]$ and $C[a, b]$ are isomorphic.*

Proof. To show that $C[0, 1]$ and $C[a, b]$ is isometric, we wish to find some mapping $T : C[0, 1] \mapsto C[a, b]$ such that for some $x, y \in C[0, 1]$,

$$d_1(x, y) = d_2(Tx, Ty),$$

where d_1 and d_2 are metrics on $C[0, 1]$ and $C[a, b]$ respectively.

Let $x, y \in C[0, 1]$. Let $T : C[0, 1] \mapsto C[a, b]$. Then, we can define $Tx \in C[a, b]$ as

$$(Tx)(t) = x\left(\frac{t - a}{b - a}\right).$$

In such a case,

$$d(Tx, Ty) = \max_{t \in [a,b]} |(Tx)(t) - (Ty)(t)| = \max_{t \in [a,b]} \left| x\left(\frac{t - a}{b - a}\right) - y\left(\frac{t - a}{b - a}\right) \right| = \max_{t \in [0,1]} |x(t) - y(t)| = d(x, y)$$

and hence, the map is distance preserving. We can similarly define for $x, y \in C[a,b]$, $T^{-1} : C[a,b] \mapsto C[0,1]$ such that $T^{-1}x \in C[0,1]$

$$(T^{-1}x)(t) = x((b-a)t + a)$$

and hence, we have

$$d(T^{-1}(x), T^{-1}y) = \max_{t\in[0,1]} \left|(T^{-1}x)(t) - (T^{-1}y)(t)\right| = \max_{t\in[a,b]} |x(t) - y(t)| = d(x,y)$$

and hence, T^{-1} is also distance preserving. Further, because of the linear mapping, the mapping is bijective. Hence, the spaces are isometric. ∎

7. *If (X,d) is complete, show that $(X,\tilde{d})$ where $\tilde{d} = \frac{d}{1+d}$ is complete.*

Proof. Let x_m be some Cauchy sequence in the space (X,d), such that for every $\epsilon > 0$, there exists some $N(\epsilon)$, such that for all $m, n > N(\epsilon)$, $d(x_m, x_n) < \epsilon$. Now, consider the same sequence under the metric $\tilde{d}$. Thus, we have

$$\tilde{d}(x_m, x_n) = \frac{d(x_m, x_n)}{1 + d(x_m, x_n)} < \epsilon$$

and hence, x_m is a Cauchy sequence in $(X,\tilde{d})$ too. Since (X,d) is complete, we have that every Cauchy sequence converges to some point in X. Let $x_m \to x$, then we have that for some $\epsilon > 0$, there exists $N(\epsilon)$ such that for $n > N(\epsilon)$, $d(x_n, x) < \epsilon$. Similarly, we have for the same sequence x_n,

$$\tilde{d}(x, x_n) = \frac{d(x, x_n)}{1 + d(x, x_n)} < \epsilon$$

and hence, the sequence converges in $(X,\tilde{d})$. Thus, this space is also complete. ∎

8. *Show that in 7, completeness of $(X,\tilde{d})$ implies completeness of (X,d).*

Proof. Since $(X,\tilde{d})$ is complete, we have that all accumulation points of X belong to X itself. Let $x \in X$ be an accumulation point. Then, for every $\epsilon > 0$, there exists x_ϵ such that

$$\tilde{d}(x, x_\epsilon) < \epsilon \implies \frac{d(x, x_\epsilon)}{1 + d(x, x_\epsilon)} < \epsilon \implies d(x, x_\epsilon) < \frac{\epsilon}{1-\epsilon}$$

Thus, if $\epsilon < 1$, x_ϵ is also an accumulation point for X under the metric d. On the other hand, if $\epsilon > 1$, then this is trivially true. ∎

9. *If (x_n) and (x'_n) om (X,d) are such that* (1) *holds and $x_n \to l$, show that (x'_n) converges and has the limit l.*

Proof. (1) is the following:

$$\lim_{n\to\infty} d(x_n, x'_n) = 0$$

Since x_n converges to l, we have that

$$\lim_{n\to\infty} d(x_n, l) = 0$$

Thus, we have

$$0 \le \lim_{n\to\infty} d(x'_n, l) \le \lim_{n\to\infty} d(x'_n, x_n) + \lim_{n\to\infty} d(x_n, l) = 0$$

and hence, x'_n converges to l. ∎

10. *If (x_n) and (x'_n) are convergent sequences in a metric space (X,d) and have the same limit l, show that the satisfy (1).*

Proof. (1) is the following:

$$\lim_{n\to\infty} d(x_n, x'_n) = 0$$

Since, both (x_n) and (x'_n) converge to the same limit l, we have that

$$\lim_{n\to\infty} d(x_n, l) = 0$$

$$\lim_{n\to\infty} d(x'_n, l) = 0$$

Thus, we have

$$0 \le \lim_{n\to\infty} d(x_n, x'_n) \le \lim_{n\to\infty} d(x_n, l) + \lim_{n\to\infty} d(l, x'_n) = 0$$

This finishes the proof. ∎

11. *Show that (1) defines an equivalence relation on the set of all Cauchy sequences of elements of X.*

Proof. (1) claims that two Cauchy sequences (x_n) and (x'_n) are equivalent (represented by $x_n \sim x'_n$) if

$$\lim_{n\to\infty} d(x_n, x'_n) = 0$$

An equivalence relation $R \subset X \times X$ satisfies the following three claims:

1. $R(x,x) \forall x \in X$.
2. $R(x,y) = R(y,x) \forall x, y \in X$.
3. $R(x,y)$ and $R(y,z)$ implies $R(x,y)$ $\forall x, y, z \in X$.

Clearly, $\lim_{n\to\infty} d(x_n, x_n) = 0$ and also $\lim_{n\to\infty} d(x_n, x'_n) = \lim_{n\to\infty} d(x'_n, x_n) = 0$. Suppose, $\lim_{n\to\infty} d(x_n, z_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$, then

$$0 \leq \lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) = 0 \implies \lim_{n\to\infty} d(x_n, y_n) = 0$$

Thus, the relation satisfies all the conditions of being an equivalence relation. ■

12. *If (x_n) is Cauchy in (X, d) and (x'_n) in X satisfies (1), show that (x'_n) is Cauchy in X.*

Proof. Since x_n is Cauchy in X, we have that for every $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $m, n > N(\epsilon)$, $D(x_m, x_n) < \epsilon$. Also, (1) claims that

$$\lim_{n\to\infty} d(x_n, x'_n) = 0.$$

Thus, for some $\epsilon > 0$ and $m, n > N(\epsilon)$, we have

$$d(x'_n, x'_m) \leq d(x'_n, x_n) + d(x_n, x_m) + d(x'_m, x_m) \leq 3\epsilon.$$

Thus, (x'_n) is also Cauchy. ■

13. *A finite psuedometric on a set X is a function $d : X \times X \mapsto \mathbb{R}$ satisfying (M1) , (M3), (M4) and*

$$\text{(M2*)} d(x,x) = 0.$$

What is the difference between a metric and pseudfometric. Show that $d(x,y) = |\xi_1 - \eta_1|$ defines a pseudometric on the set of all ordered pairs of real numbers where $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$.

Proof. Let us reinterpret the original condition M2 for a metric

$$d(x,y) = 0 \iff x = y$$

This is a two sided condition, i.e if $d(x,y) = 0$, then $x = y$. On the other hand, if $x = y$, then $d(x,y) = 0$. However the condition for a psuedometric seems to be one sided, in other words the distance between the same point is 0. However, if the distance between two points is 0, it does not necessarily mean it is the same point. In other words, $x = y \implies d(x,y) = 0$. However, $d(x,y) = 0 \not\Rightarrow x = y$. This is clear from the example. Suppose $x = y = (\xi_1, \xi_2)$. Then, $d(x,y) = |\xi_1 - \xi_1| = 0$. However, $d(x,y) = |\xi_1 - \eta_1| = 0 \implies \xi_1 = \eta_1$. However, this does not mean $\xi_2 = \eta_2$ and hence, x and y may not be equal points. ■

14. *Does*

$$d(x,y) = \int_a^b |x(t) - y(t)| \; dt$$

define a metric or pseudometric on X if X is the set of all real-valued continuous functions on $[a,b]$ and if X is the set of all real-values Riemann integrable functions on $[a,b]$?

Proof. If X is the set of all real-valued continuous functions, then it is evident that if $x = y$, then $d(x,y) = 0$. However, if $d(x,y) = 0$, since the metric is the integral of positive quantities, the only way the sum is zero is if each of the quantities is zero, i.e $x(t) = y(t)$ for all $t \in [a,b]$.

On the other hand, if X is the set of all real-valued Riemann integrable functions, then it is once again clear that if $x = y$, $d(x,y) = 0$. However, $d(x,y) = 0$ does not imply $x = y$. For example, consider the function $x(t) = c_1$ and $y(t) = c_2$ if $t \in C$ where C is a finite set, and c_1 otherwise. Then, the pointwise difference does not matter, and the integral shall remain 0. ■

15. *If (X, d) is a pseudometric space, we call a set*

$$B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$$

an open ball in X with center x_0 and radius r. What are open balls of radius 1 in 13?

Proof. In problem 13, we have X to be the set of ordered real value pairs such that $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$, then, $d(x, y) = |\xi_1 - \eta_1|$. Suppose, $x_0 = (a, b)$. Then, any point within distance 1 of x_0 has first coordinate $a \pm 1$, while the second coordinate does not matter. Thus, the open ball would be of the form

$$B((a, b), 1) = \{(a \pm 1, r) \mid r \in \mathbb{R}\}$$

which results in vertical strips of width 2 centered at a. ■

2 Normed Spaces. Banach Spaces.

2.1 Vector Space

1. *Show that the set of all real numbers with the usual addition and multiplication constitutes a one-dimensional real vector space and the set of all complex numbers constitutes a one-dimensional complex vector space.*

Proof. In the set of all real numbers, it is easy to see that that addition satisfies associativity, commutativity, and consists of an identity (0), and an inverse $(-x) \in \mathbb{R}$. Also, multiplication satisfies associativity, has an identity (1), and satisfies distributivity. Thus, it is a vector space.

For the set of complex numbers, addition satisfies associativity, commutativity, and consists of an identity $(0 + \iota 0)$, and an inverse $(-x) \in \mathbb{C}$. Also, multiplication satisfies associativity, has an identity $(1 + \iota 0)$, and satisfies distributivity. Thus, it is a vector space. ■

2. *Prove (1) and (2).*

Proof. Denote the zero vector as θ. Then, we wish to prove:

$$0x = \theta$$

$$\alpha\theta = \theta$$

$$(-1)x = -x$$

For the first statement, notice that for some $\alpha \in K$ (where K is the scalar field), we have

$$\theta = \alpha x + (-\alpha)x = (\alpha + (-\alpha)x) = 0x$$

where the first equality follows from the properties of addition on the vector space, the second equality follows from the distributive property for vector spaces, and the third follows from the properties of addition on K (there exists an additive inverse).

For the second statement, substituting $x = \theta$ in the first statement gives us the result.

For the third statement, we have

$$(1 + (-1))x = \theta \implies (-1)x = -x$$

where the implication follows from distributive property. ■

3. *Describe the span of $M = \{(1, 1, 1), (0, 0, 2)\} \in \mathbb{R}^3$.*

Proof. The span of M is described as all possible linear combinations of the vectors in M. Let the coefficients be $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ for $(1, 1, 1)$ and $(0, 0, 2)$ respectively. Then, we have

$$\text{span } M = \{(\alpha, \alpha, \alpha + 2\beta)\}$$

which represents a plane passing through the line $x = y$. ■

4. *Which of the following subsets of $\mathbb{R}^3$ constitute a subspace of $\mathbb{R}^3$? Here, $x = (\xi_1, \xi_2, \xi_3)$.*

1. *All x with $\xi_1 = \xi_2$ and $\xi_3 = 0$?*
2. *All x with $\xi_1 = \xi_2 + 1$.*
3. *All x with positive ξ_1, ξ_2, ξ_3.*
4. *All x with $\xi_1 - \xi_2 + \xi_3 = k$.*

Proof. Denote the subsets by X.

1. Consider two points $x_1 = (a_1, a_1, 0) \in X$ and $x_2 = (a_2, a_2, 0) \in X$. Then, we have $x_1 + x_2 = (a_1 + a_2, a_1 + a_2, 0) \in X$. Also, let α be some scalar, then, $\alpha x_1 = (\alpha a_1, \alpha a_1, 0) \in X$. Hence, X is a subspace.
2. Consider two points $x_1 = (1 + a_1, a_1, b_1) \in X$ and $x_2 = (1 + a_2, a_2, b_2) \in X$. Then, $x_1 + x_2 = (2 + a_1 + a_2, a_1 + a_2, b_1 + b_2) \notin X$ since $2 + a_1 + a_2 \neq 1 + (a_1 + a_2)$. Thus, X is not a subspace.
3. Let $x_1 = (a_1, b_1, c_1) \in X$, i.e, $a_1, b_1, c_1 > 0$. Consider some $\alpha < 0$, then, $\alpha x_1 \notin X$, since the coordinates now become negative. Hence, X is not a subspace.

4. Consider $x_1 = (a_1, b_1, c_1) \in X$ and let α be some scalar. Then, $\alpha x_1 = (\alpha a_1, \alpha b_1, \alpha c_1)$. However, $\alpha(a_1 - b_1 + c_1) \neq k$ and hence, X is not a subspace.

■

5. *Show that $\{x_1, \ldots, x_n\}$ where $x_j(t) = t^j$ is a linearly independent set in the space $C[a, b]$.*

Proof. We wish to show that

$$\sum_{j=1}^{n} \alpha_j x_j = 0 \implies \alpha_j = 0 \;\forall j \in [n].$$

Note that this convergence has to be pointwise. Consider the function $\sum_{j=1}^{n} \alpha_j x_j$. Then,

$$\left(\sum_{j=1}^{n} \alpha_j x_j\right)(t) = \sum_{j=1}^{n} \alpha_j t^j = 0$$

Clearly, if $t > 0$, then this only holds if all the coefficients $\alpha_j = 0$. ■

6. *Show that in a $n-$dimensional vector space X, the representation of any x as a linear combination of given basis vector is unique.*

Proof. Suppose not. Then, let the two sets of coefficients be $\{\alpha_i\}$ and $\{\beta_i\}$. We have that

$$x = \sum_{i=1}^{n} \alpha_i e_i = \sum_{i=1}^{n} \beta_i e_i.$$

Thus, we also have that

$$\sum_{i=1}^{n} (\alpha_i - \beta_i) e_i = 0.$$

But since the set of vectors e_i are linearly independent, this means that for all i, $\alpha_i - \beta_i = 0 \implies \alpha_i = \beta_i$. This finishes the proof. ■

7. *Let $\{e_1, \ldots, e_n\}$ be a basis for a complex vector space X. Find a basis for X regarded as a real vector space. What is the dimension of X in either case?*

Proof. Suppose X were a real space. Then, we would need a separate basis for the real parts and the imaginary parts. This leads to the basis of $\{e_1, \ldots, e_n, \iota e_1, \ldots, \iota e_n\}$, resulting in a dimension of $2n$. ■

8. *If M is linearly dependent set in a complex vector space X, is M linearly dependent in X, regarded as a real vector space?*

Proof. Suppose $M = \{a_i + \iota b_i\}_{i=1}^{n}$ and let

$$a_i + \iota b_r = \sum_{i=1 i \neq r}^{n} \frac{\alpha_i}{\alpha_r}(a_i + \iota b_i)$$

Equating the real and imaginary parts results in

$$\alpha_r = \sum_{i=1 i \neq r}^{n} \alpha_r a_i \text{ and } b_r = \sum_{i=1 i \neq r}^{n} \alpha_r b_i.$$

Thus, regarding X as real vector space, each of the coefficients can still be expressed as a linear combination of the other vectors. Thus, it is still linearly dependent. ■

9. *On a fixed interval $[a, b] \subset \mathbb{R}$, consider the set X consisting of all polynomials with real coefficients and of degree not exceeding a given n and the polynomial $x = 0$. Show that X with the usual addition and multiplication by real numbers is a real vector space of dimension $n+1$. Find a basis for X. Show that we can obtain a complex vector space $\overline{X}$ in a similar fashion if we let those coefficients be complex. Is X a subspace of $\overline{X}$?*

Proof. First, note that for addition, associativity and commutativity holds. Also, the zero polynomial is a part of X, and hence, there exists an additive identity. Further, the additive inverse is simply given by negating all coefficients, and hence, the additive inverse also belongs to the set X. Similarly, the associative property for multiplication also holds, and a multiplicative identity is simply the polynomial $x = 1$. Thus, it is a valid vector space. If every polynomial has degree at most n, then we require the monomials $x^n, x^{n-1} \dots, x, 1$ to build each polynomial, and hence, the degree is $n+1$. The basis is the set of monomials $\{x^j\}_{j=0}^n$. If each of the coefficients are complex, then the overall polynomial can be split into a real polynomial and an imaginary polynomial, resulting in the space of complex polynomials. However, X is not a subspace of $\overline{X}$ because allowing complex coefficients with polynomials of X would result in a complex polynomial, which will not belong to X. ■

10. *Show that if Y and Z are subspaces of a vector space X, show that $Y \cap Z$ is a subspace of X but $Y \cup Z$ need not be one.*

Proof. Let the set of vectors in $Y \cap Z$ be denoted by u_i. Then, by the definition of $u_i \in Y$ and $u_i \in Z$. Then, any linear combination $\sum \alpha_j u_j$ will belong to both Y and Z and hence, to $Y \cap Z$. Thus, $Y \cap Z$ is a subspace. On the other hand, the union of subspaces may not be a subspace. Let Y be the subspace of polynomials of the form $\alpha + \beta x$, while let Y is the subspace of polynomials of the form $\alpha + \beta x^2$. Then, the union is simply all polynomials of the form $\alpha + \beta x$ or $\alpha + \beta x^2$. However, a linear combination of the vectors in $Y \cup Z$ may result in polynomials of the form $\alpha + \beta x + \gamma x^2$, which do not belong to $Y \cup Z$. ■

11. *If $M \neq \emptyset$ is any subset of a vector space X, show that span M is a subspace of X.*

Proof. Let $M = \{x_1, \dots, x_n\}$. By the definition of span, we know that

$$\text{span } M = \{\sum_{j=1}^{n} \alpha_j x_j \mid (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n\}.$$

Suppose we take two vectors $\sum_{j=1}^n \alpha_j x_j \in \text{span } M$ and $\sum_{j=1}^n \beta_j x_j \in \text{span } M$. Representing $\gamma_j = \alpha_j + \beta_j$, we get the sum of the vectors to be $\sum_{j=1}^n \gamma_j x_j$, which belongs to M. Similarly, let $\alpha_j \beta = \gamma_j$, then multiplying the vector $\sum_{j=1}^n \alpha_j x_j$ by β results in the vector $\sum_{j=1}^n \gamma_j x_j \in \text{span } M$. Thus, M is a subspace. ■

12. *Show that the set of all real two rowed square matrices forms a vector space X. What is the zero vector in X? Determine* $\dim X$*. Find a basis for X. Give examples of subspaces for X. Do the symmetric matrices $x \in X$ form a subspace? The singular matrices?*

Proof. Clearly, addition for the two-rowed square matrices is associative and commutative, since the addition is element-wise. Further, the additive identity can be represented as

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Also, the additive inverse can be defined as follows:

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies -x = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

Also, multiplication is associative (and not commutative!), and we can define the multiplicative identity as

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, this space is a vector space. Clearly, $\dim X = 4$ and the basis is the following set:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

The set of symmetric matrices is a subspace because the addition of two symmetric matrices is also symmetric, while multiplying by a constant maintains symmetry. However, the set of all singular matrices is not a subspace. Denote the set of singular matrices by S. Then,

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in S \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S \text{ but } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin S.$$

■

13. *Show that the Cartesian product $X = X_1 \times X_2$ of two vector spaces over the same field becomes a vector space if we define the two algebraic operations by*

$$(x_1, x_2) + (y_1, y_2) = (x_1 + x_2, y_1 + y_2),$$

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$$

.

Proof. We first have

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) = (y_1, y_2) + (x_1, x_2)$$
$$((x_1, x_2) + (y_1, y_2)) + (z_1, z_2) = (x_1 + y_1 + z_1, x_2 + y_2 + z_2) = (x_1, x_2) + ((y_1, y_2) + (z_1, z_2))$$
$$(x_1 + x_2) + (-x_1, -x_2) = (0, 0)$$

and hence, all properties of addition are satisfied. Similarly, we can verify for scalar multiplication:

$$\alpha(\beta(x_1, x_2)) = \alpha(\beta x_1, \beta x_2) = (\alpha\beta x_1, \alpha\beta x_2) = \beta(\alpha x_1, \alpha x_2) = \beta(\alpha x)$$

$$1(x_1, x_2) = (x_1, x_2)$$

and can verify the distributive laws. Thus, this space is a vector space. ■

14. *Let Y be a subspace of a vector space X. The coset of an element $x \in X$ with respect to Y is deonted by $x + Y$ and is defined to be the set*

$$x + Y = \{v \mid v = x + y, y \in Y\}.$$

Show that the distinct cosets form a partition of X. Show that under algebraic operations defined by

$$(w + Y) + (x + Y) = (w + x) + Y$$

$$\alpha(x + Y) = \alpha x + Y$$

these cosets constitute the elements of a vector space. This space is called the quotient space of X by Y and is denoted by X/Y. It's dimension is called the co-dimension of Y and is denoted by codim $Y = \dim(X/Y)$.

Proof. We show that no two cosets can have the same element, and if they do, then they are essentially the same coset. Let $a \in X$ be some element which belongs to two different cosets: $x + Y$ and $w + Y$. Thus, there exists $y_x \in Y$ and $y_w \in Y$ such that

$$a = x + y_x = w + y_w \implies x = a - y_x \text{ and } w = a - y_w.$$

Thus, we can write the cosets as

$$x + Y = \{a - y_x + y \mid y \in Y\}$$
$$w + Y = \{a + y_w - y \mid y \in Y\}.$$

Since Y is a subspace, $-y_x + y$ for all $y \in Y$ will always result in some vector belonging to y, and similarly, $-y_w + y$ will always result in some vector belonging to Y. Thus, as we circle through all elements of Y, the sets $\{y -_x + y \mid y \in Y\}$ and $\{-y_w + y \mid y \in Y\}$ would be equal sets. Hence, the cosets $x + Y$ and $w + Y$ would be equal. Also, since X is a vector space, for any $x \in X$ and $y \in Y$, $x - y$ has to belong to X and hence, to one of the cosets. This shows that the distinct cosets form partitions of X.

Consider the sets :

$$x + Y = \{x + y \mid y \in Y\},$$
$$w + Y = \{w + y \mid y \in Y\}.$$

Then, we have

$$(w + Y) + (x + Y) = \{w + y + x + y \mid y \in Y\},$$

but since $2y \in Y$ (since Y is a subspace), denoting $2y = y'$, we have

$$(w + Y) + (x + Y) = \{w + x + y' \mid y' \in Y\} = (w + x) + Y.$$

Similarly,

$$\alpha(x + Y) = \{\alpha(x + y) \mid y \in Y\}.$$

However, since $\alpha y \in Y$ (since Y is a subspace), denoting $\alpha y = y'$, we have

$$\alpha(x + Y) = \{\alpha x + y' \mid y' \in Y\} = \alpha x + Y.$$

This finishes the proof. ■

15. *Let $X = \mathbb{R}^3$ and $Y = \{(\xi_1, 0, 0) \mid \xi_1 \in \mathbb{R}\}$. Find X/Y, X/X, and $X/\{0\}$.*

Proof. Note that $X/Y = \{\{(\xi_1, \xi_2, \xi_3) \mid \xi_2, \xi_3 \in \mathbb{R}\} \mid \xi_1 \in \mathbb{R}\}$. This means that each corset is obtained by fixing the ξ_1 coordinate and varying ξ_2 and ξ_3. Thus, each of the cosets is a plane parallel to ξ_1 axis. Now, X/X is simply $\{0\}$ because for any $x \in X$, the coset $x + X = \{x + x_0 \mid x_0 \in X\}$ is simply X and hence, there is no space of distinct cosets, i.e this quotient space is empty. Finally, $X/\{0\}$ is simply X since each point in itself would be a coset, and the space of these cosets is X itself. ∎

2.2 Normed Space. Banach Space.

1. *Show that the norm $\|x\|$ of x is the distance of x from 0.*

Proof. By the definition of a norm, we have

$$d(x, y) = \|x - y\|.$$

Subsitituting $y = 0$, we get

$$\|x\| = d(x, 0).$$

∎

2. *Verify that the usial length of a vector in the plane or in three- dimensional space has the properties (N1) to (N4).*

Proof. The properties referred to are:

1. $\|x\| \geq 0$.
2. $\|x\| = 0 \iff x = 0$
3. $\|\alpha x\| = |\alpha| \|x\|$.
4. $\|x + y\| \leq \|x\| + \|y\|$.

Now, the length of vector $x = (a, b, c)$ in three dimensions is given by

$$\|x\| = \sqrt{a^2 + b^2 + c^2}.$$

Clearly, this value is always non-negative. Also,

$$\sqrt{a^2 + b^2 + c^2} = 0 \implies a = b = c = 0.$$

Similarly, $a = b = c$ gives us $\|x\| = 0$. Finally,

$$\|\alpha x\| = \sqrt{\alpha^2 a^2 + \alpha^2 b^2 + \alpha^2 c^2} = \sqrt{\alpha^2}\sqrt{a^2 + b^2 + c^2} = |\alpha| \|x\|.$$

We now show that the triangle inequality also holds. Let $x = (a_1, b_1, c_1)$ and $y = (a_2, b_2, c_2)$. Then,

$$\|x + y\| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2 + (c_1 + c_2)^2}.$$

Recall Hölder's inequality for $a_i, b_i \geq 0$:

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}},$$

where $p^{-1} + q^{-1} = 1$. Using this, we get

$$a_1 a_2 + b_1 b_2 + c_1 c_2 \leq \sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}.$$

and hence,

$$\|x + y\| \leq \sqrt{a_1^2 + a_2^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2 + 2\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \sqrt{a_1^2 + b_1^2 + c_1^2} + \sqrt{a_2^2 + b_2^2 + c_2^2},$$

and thus,

$$\|x + y\| \leq \|x\| + \|y\|.$$

∎

3. *Prove (2).*

Proof. We wish to show

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

Using the triangle inequality, we have

$$\|x\| = \|(x-y) + y\| \leq \|x-y\| + \|y\| \implies \|x\| - \|y\| \leq \|x - y\|.$$

Similarly,

$$\|y\| - \|x\| \leq \|y - x\| = \|x - y\|$$

, and hence, combining both inequalities results in

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

■

4. *Show that we may replace (N2) by*

$$\|x\| = 0 \implies x = 0$$

without altering the concept of a norm. Show that non-negativity of a norm follows from (N3) and (N4).

Proof. First, we wish to show that we can replace

$$\|x\| = 0 \iff x = 0$$

by

$$\|x\| = 0 \implies x = 0.$$

This means, we wish to show that the condition

$$x = 0 \implies \|x\| = 0$$

is redundant. Since we have already shown that $\|x\| = d(x, 0)$, substituting $x = 0$ gives us the claim. Hence, this condition follows from the definition of a metric induced by a norm, and is hence, redundant. Another way to see this is: we know that $\|\alpha x\| = |\alpha| \|x\|$. Substituting $\alpha = 0$ gives us

$$\|0\| = 0.$$

We now wish to show that we can prove the non-negativity of a norm using the following conditions:

$$\|\alpha x\| = |\alpha| \|x\|$$

$$\|x + y\| \leq \|x\| + \|y\|.$$

We have that

$$0 = \|x - x\| \leq \|x\| + |-1| \|x\| = 2\|x\|$$

and hence, $\|x\| \geq 0$. ■

5. *Show that (3) defines a norm.*

Proof. We wish to show that

$$\|x\| = \left(\sum_{j=1}^{n} |\xi_j|^2\right)^{1/2} = \sqrt{|\xi_1|^2 + \ldots + |\xi_n|^2}$$

is a norm. We have already shown this in 2 for $n = 3$, and hence, the proof follows in exactly the same manner. We demonstrate the proof for trinagle inequality once again. Let $x = (\xi_1, \ldots, \xi_n)$ and $y = (\eta_1, \ldots, \eta_n)$. Then,

$$\begin{aligned}
\|x+y\| &= \sqrt{\sum_{i=1}^{n}(\xi_i + \eta_i)^2} \\
&= \sqrt{\sum_{i=1}^{n}\xi_i^2 + \sum_{i=1}^{n}\eta_i^2 + 2\sum_{i=1}^{n}\xi_i\eta_i} \\
&\le \sqrt{\sum_{i=1}^{n}\xi_i^2 + \sum_{i=1}^{n}\eta_i^2 + 2\sqrt{\sum_{i=1}^{n}\xi_i^2}\sqrt{\sum_{i=1}^{n}\eta_i^2}} \\
&= \sqrt{\sum_{i=1}^{n}\xi_i^2} + \sqrt{\sum_{i=1}^{n}\eta_i^2} \\
&= \|x\| + \|y\|
\end{aligned}$$

where the inequality makes use of Hölder's inequality.

■

6. *Let X be the vector space of all ordered pairs $x = (\xi_1, \xi_2)$, $y = (\eta_1, \eta_2)$ of real numbers. Show that norms on X are defined by*

$$\|x\|_1 = |\xi_1| + |\xi_2|$$
$$\|x\|_2 = \sqrt{|\xi_1|^2 + |\xi_2|^2}$$
$$\|x\|_\infty = \max\{|\xi_1|, |\xi_2|\}.$$

Proof. We first show $\|x\|_1$ is a valid norm. Clearly, it is always non-negative, and since it is the sum of non-negative terms, it is zero iff each of the terms is zero, resulting in the condition $\|x\|_1 = 0 \iff x = 0$. Now, we have

$$\|\alpha x\|_1 = |\alpha\xi_1| + |\alpha\xi_2| = |\alpha|\,|\xi_1| + |\alpha|\,|xi_2| = |\alpha|\,\|x\|_1\,.$$

Also, using the triangle inequality for the absolute values on real numbers gives:

$$\|x+y\| = |\xi_1 + \eta_1| + |\xi_2 + \eta_2| \le |\xi_1| + |\xi_2| + |\eta_1| + |\eta_2| = \|x\|_1 + \|x\|_2\,.$$

Thus, $\|x\|_1$ is a valid norm.

We have shown $\|x\|_2$ is a valid norm for three-dimensional points in 3 and for $n-$dimensional points in 5.

We now show $\|x\|_\infty$ is also a valid norm. First, by definition it is always non-negative, and the maximum of two non-negative terms can be zero iff each of the terms is zero, resulting in the condition $\|x\|_\infty = 0 \iff x = 0$. Further,

$$\|\alpha x\|_\infty = \max\{|\alpha\xi_1|, |\alpha\xi_2|\} = |\alpha| \max\{|\xi_1|, |\xi_2|\} = |\alpha|\,\|x\|_\infty\,.$$

Finally, the triangle inequality follows since:

$$\|x+y\|_\infty = \max\{|\xi_1+\eta_1|, |\xi_2+\eta_2|\} = |\xi_i + \eta_i| \le |\xi_i| + |\eta_i| \le \max\{|\xi_1|, |\xi_2|\} + \max\{|\eta_1|, |\eta_2|\} = \|x\|_\infty + \|y\|_\infty$$

Thus, $\|x\|_\infty$ is a valid norm. ■

7. *Verify that (4) satisfies (N1) to (N4).*

Proof. We wish to show that the ℓ_p norm, defined as

$$\|x\|_p = \left(\sum_{j=1}^{n}|\xi_j|^p\right)^{1/p}$$

is a valid norm. First, it is easy to see that it is the sum of non-negative terms, and hence, is always non-negative. Also, it would only be zero if each of the terms is zero, resulting in the condition that $\|x\|_p = 0 \iff x = 0$. Further,

$$\|\alpha x\|_p = \left(\sum_{j=1}^{n}|\alpha\xi_j|^p\right)^{1/p} = \left(|\alpha|^p\sum_{j=1}^{n}|\xi_j|^p\right)^{1/p} = |\alpha|\left(\sum_{j=1}^{n}|\xi_j|^p\right)^{1/p} = |\alpha|\,\|x\|_p\,.$$

Finally, the triangle inequality follows from Minkowski's inequality:

$$\|x+y\|_p = \left(\sum_{j=1}^n |\xi_j + \eta_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^n |\xi_j|^p\right)^{1/p} + \left(\sum_{j=1}^n |\eta_j|^p\right)^{1/p} = \|x\|_p + \|y\|_p$$

■

8. *Show that these norms on the vector space of ordered $n-$tuples of numbers are valid norms.*

$$\|x\|_1 = |\xi_1| + \ldots + |\xi_n|$$

$$\|x\|_2 = \sqrt{|\xi_1|^2 + \ldots + |\xi_n|^2}$$

$$\|x\|_\infty = \max\{|\xi_1|, \ldots, |\xi_n|\}$$

Proof. We first begin with $\|x\|_1$, which is clearly non-negative and can only be zero iff $x = 0$. Also, using the fact that $|\alpha x| = |\alpha|\,|x|$, we have

$$\|\alpha x\|_1 = \sum_{i=1}^n |\alpha \xi_i| = \sum_{i=1}^n |\alpha|\,|\xi_i| = |\alpha|\,\|x\|_1 .$$

Finally, the triangle inequality follows from the triangle inequality for real numbers:

$$\|x+y\|_1 = \sum_{i=1}^n |\xi_i + \eta_i| \leq \sum_{i=1}^n |\xi_i| + \sum_{i=1}^n |\eta_i| = \|x\|_1 + \|y\|_1 .$$

Thus, $\|x\|_1$ is a valid norm.

We have already shown $\|x\|_2$ is a valid norm in 5.

We now show that $\|x\|_\infty$ is a valid norm. Clearly, it is non-negative, and the maximum of non-negative terms can be zero iff each of the terms is zero , thus resulting in the condition $\|x\| = 0 \iff x = 0$. Further,

$$\|\alpha x\|_\infty = \max_{i\in[n]} |\alpha \xi_i| = |\alpha| \max_{i\in[n]} |\xi_i| = |\alpha|\,\|x\|_\infty .$$

The triangle inequality follows since:

$$\|x+y\|_\infty = \max_{i\in[n]} |\xi_i + \eta_i| = |\xi_j + \eta_j| \leq |\xi_j| + |\eta_j| \leq \max_{i\in[n]} |\xi_i| + \max_{i\in[n]} |\eta_i| = \|x\|_\infty + \|y\|_\infty .$$

Thus, $\|x\|_\infty$ is a valid norm. ■

9. *Verify that (5) defines a norm.*

Proof. We wish to show that

$$\|x\| = \max_{t\in J} |x(t)|$$

defines a norm. By definition, this is non-negative, and the only way $\|x\| = 0$ is if for each $t \in J, x(t) = 0$, or in other words, $x = 0$. Further,

$$\|\alpha x\| = \max_{t\in J} |\alpha x(t)| = |\alpha| \max_{t\in J} |x(t)| = |\alpha|\,\|x\| .$$

The triangle inequality follows since:

$$\|x+y\| = \max_{t\in J} |x(t) + y(t)| \leq |x(t_0) + y(t_0)| \leq |x(t_0)| + |y(t_0)| \leq \max_{t\in J} |x(t)| + \max_{t\in J} |y(t)| = \|x\| + \|y\| .$$

Thus, it is a valid norm. ■

10. *The sphere*

$$S(0;1) = \{x \in X \mid \|x\| = 1\}$$

in a normed space X is called the unit sphere. Show that for the norms in 6 and for the norm defined by

$$\|x\|_4 = (\xi_1^4 + \xi_2^4)^{1/4}$$

The unit spheres look as shown in Fig.16.

Proof. It is not clear on how to show this. We will attempt to show it in the following way: $\|x\|_1$, $\|x\|_2$, and $\|x\|_\infty$ in $\mathbb{R}^2$ depict a diamond, circle, and square, respectively. Then, we wish to show that if x_1 satisfies $\|x_1\|_2 = 1$, x_2 satisfies $\|x_2\|_4 = 1$, and x_3 satisfies $\|x_3\|_\infty = 1$, then ,

$$\|x_1\|_2 \leq \|x_2\|_2 \leq \|x_3\|_\infty$$

, or in other words, the distance from the origin of these points increases.

First,

$$\|x\|_1 = 1 \implies |\xi_1| + |\xi_2| = 1.$$

This results in the following 4 lines: $\pm\xi_1 \pm \xi_2 = 1$, which results in the diamond. Next,

$$\|x\|_2 = 1 \implies \xi_1^2 + \xi_2^2 = 1$$

results in the equation of a circle centered around the origin with radius 1. Finally,

$$\|x\|_\infty = 1 \implies \max\{|\xi_1|, |\xi_2|\} = 1$$

results in a square. Now, for $\|x_3\|_\infty = 1$, anuy point of the form $(\pm 1, t)$ or $(t, \pm 1)$ works, where $t\ in[-1, 1]$ and thus, $\|x_3\|_2 \leq \sqrt{1+1} = \sqrt{2}$ and $\|x_3\|_2 \geq \sqrt{1} = 1$. Finally, we see that $x_2 = (\sqrt{\cos\theta}, \sqrt{\sin\theta})$ satisfies $\|x_2\|_4 = 1$, and hence,

$$\|x_2\|_2 = \sqrt{\cos\theta + \sin\theta} = \sqrt{\sqrt{2}\sin\left(\frac{\pi}{4} + \theta\right)} \leq 2^{1/4}$$

and hence. we have

$$\|x_1\|_2 \leq \|x_2\|_2 \leq \|x_3\|_\infty$$

, which shows that the distance of points which satisfy $\|.\|_4 = 1$ is more than that of points which satisfy $\|.\|_2 = 1$ but less than that of $\|.\|_\infty = 1$. ∎

11. *A subset A of vector space X is said to be convex if $x, y \in A$ implies*

$$M = \{z \in X \mid z = \alpha x + (1-\alpha)y, 0 \leq \alpha \leq 1\} \subset A.$$

M is called a closed segment with boundary points x and y, any other $z \in M$ is called an interior point of M. Show that the closed unit ball

$$\tilde{B}(0, 1) = \{x \in X \mid \|x\| \leq 1\}$$

in a normed space is convex.

Proof. Choose two points $x, y \in \tilde{B}(0, 1)$ such that $\|x\| \leq 1$ and $\|y\| \leq 1$. Then, the norm of any convex combination of x and y is given by

$$\|\alpha x + (1-\alpha)y\| \leq |\alpha| \|x\| + |1-\alpha| \|y\| \leq 1$$

which shows that any convex combination of the points lies inside the closed ball. Here, the first inequality follows by using the triangle inequality and scalar multiplication property of the norm in tandem. ∎

12. *Using 11, show that*

$$\varphi(x) = \left(\sqrt{|\xi_1|} + \sqrt{|\xi_2|}\right)^2$$

does not define a norm on the vector space of all ordered pairs of real numbers.

Proof. In 11 we have shown that the closed ball of radius 1 is a convex set with respect to any norm. Suppose, $\varphi(x)$ defines a norm such that $\varphi(x) \leq 1$ and $\varphi(y) \leq 1$. Then, we have

$$\begin{aligned}
\varphi(\alpha x + (1-\alpha)y) &= \left(\sqrt{|\alpha\xi_1 + (1-\alpha)\eta_1|} + \sqrt{|\alpha\xi_2 + (1-\alpha)\eta_2|}\right)^2 \\
&\leq \left(\sqrt{\alpha|\xi_1| + (1-\alpha)|\eta_1|} + \sqrt{\alpha|\xi_2| + (1-\alpha)|\eta_2|}\right)^2 \\
&= \alpha(|\xi_1| + |\xi_2|) + (1-\alpha)(|\eta_1| + |\eta_2|) + 2\sqrt{(\alpha|\xi_1| + (1-\alpha)|\eta_1|)(\alpha|\xi_2| + (1-\alpha)|\eta_2|)}
\end{aligned}$$

Now, since $\phi(x) = (|\xi_1| + |\xi_2|) + s\sqrt{|\xi_1\xi_2|} \leq 1$, we have that $(|\xi_1| + |\xi_2|) \leq 1$. Similarly, $(|\eta_1| + |\eta_2|) \leq 1$ and hence, we get

$$\varphi(\alpha x + (1-\alpha)y) \leq 1 + 2\sqrt{(\alpha|\xi_1| + (1-\alpha)|\eta_1|)(\alpha|\xi_2| + (1-\alpha)|\eta_2|)}$$

Since each of the terms inside the square root are non-negative, we have that the the additional quantity is non-negative, and hence, $\varphi(\alpha x + (1-\alpha)y)$ may be greater than 1, leading to the fact that the subset is no longer convex. ∎

13. *Show that the discrete metric on a vector space $X \neq \{0\}$ cannot be obtained from a norm.*

Proof. Let d be a metric induced by a norm. Then, d satisfies the following:

$$d(x+a, y+a) = \|x-y\| = d(x,y),$$
$$d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = |\alpha| \, \|x-y\| = |\alpha| \, d(x,y).$$

Suppose $x \neq y$. Then, $d(x,y) = 1$. However, for $\alpha \neq \pm 1$, $d(x,y) \neq d(\alpha x, \alpha y)$, and hence, the metric is not induced from a norm. ■

14. *If d is a metric on a vector space $X \neq \{0\}$ obtained from a norm, and $\tilde{d}$ is defined by*

$$\tilde{d}(x,x) = 0 \ , \ \tilde{d}(x,y) = d(x,y) + 1 \text{ for } x \neq y.$$

Show that $\tilde{d}$ cannot be obtained from a norm.

Proof. Suppose $\tilde{d}$ was induced from a norm. Then,

$$\tilde{d}(\alpha x, \alpha y) = d(\alpha x, \alpha y) + 1 = |\alpha| \, d(x,y) + 1.$$

However, we also have

$$\tilde{d}(\alpha x, \alpha y) = |\alpha| \, \tilde{d}(x,y) = |\alpha| \, d(x,y) + |\alpha| \, .$$

Thus, if $\alpha \neq \pm 1$, then, $\tilde{d}$ is not a metric induced by a norm. Since, this should hold for all values of α, $\tilde{d}$ is not a metric induced by a norm. ■

15. *Show that a subset M in a normed space X is bounded if and only if there is a positive number c such that $\|x\| \leq c$ for every $x \in M$.*

Proof. Recall that a subset M is bounded if

$$\delta(M) = \sup_{x,y \in M} d(x,y) < \infty.$$

Suppose the set is bounded. Then, $\sup_{M \times M} d(x,y) < \infty$. Let $d(x,y) < c$ for all pairs $(x,y) \in M \times M$. Then,

$$\|x\| \leq \|x-y\| + \|y\| \leq c + \|y\|$$

which is bounded above by some other constant c'.

On the other hand, we have

$$d(x,y) = \|x-y\| \leq \|x\| + \|y\| \leq 2c$$

and hence,

$$\sup_{(x,y) \in M \times M} d(x,y) = 2c.$$

This finishes the proof. ■

2.3 Further properties of normed spaces

1. *Show that $c \subset \ell^\infty$ is a vector subspace of ℓ^∞ and so is c_0, the space of all sequences of scalars converging to zero.*

Proof. Recall that $c \subset \ell^\infty$ is the space of all convergent sequences (x_n) with the metric induced from the ℓ^∞ space. To show that c is a vector space, we need to ensure that all properties of addition and multiplication hold. First, for addition we wish to ensure that the sum of two convergent sequences is convergent, i.e if $x_n \to x$ and $y_n \to y$, then

$$0 \leq \lim_{n \to \infty} \|(x_n + y_n) - (x+y)\| \leq \lim_{n \to \infty} \|x_n - x\| + \lim_{n \to \infty} \|y_n - y\| = 0$$

and hence, $x_n + y_n \to x + y$. Clearly, the commutative and associative properties of addition hold. Also, the additive identity is simply the constant zero sequence, which is convergent.

For multiplication, if $x_n \to x$, then

$$\lim_{n \to \infty} \|\alpha x_n - \alpha x\| = |\alpha| \lim_{n \to \infty} \|x_n - x\| = 0$$

and hence, $\alpha x_n \to \alpha x$. The associativity for multiplication holds, and the multiplicative identity is simply the scalar 1.

In a similar fashion, if $x_n \to 0$ and $y_n \to 0$, then $x_n + y_n \to 0$, and the addition property satisfies associativity and commutativity. Further, the zero vector is the additive identity, and since the zero vector converges to 0, it belongs to c_0. The multiplication of a sequence $x_n \to 0$ does not change any convergence properties, and the multiplicative identity continues to remain the scalar 1. ■

2. *Show that c_0 is a closed subspace of ℓ^∞, so that c_0 is complete.*

Proof. Let $(x_1, x_2, \ldots)$ be a sequence converging to $x = (\xi_1, \xi_2, \ldots)$. Each term $x_i = (\xi_1^{(i)}, \xi_2^{(i)}, \ldots)$ it itself a sequence that converges to zero. Thus, we have that for some $\epsilon > 0$, $\exists N_1(\epsilon)$ such that for $n > N(\epsilon)$

$$d(x_n, 0) < \epsilon.$$

Now, since $(x_1, x_2, \ldots)$ also converges to x under the metric induced by the ℓ^∞ space, so we have that for some $\epsilon > 0$, $\exists N_2(\epsilon)$ such that for $n > N_2(\epsilon)$, we have

$$d(x_n, x) = \sup_i \left|\xi_i^{(n)} - \xi_i\right| < \epsilon \implies \left|\xi_i^{(n)} - \xi_i\right| < \epsilon \; \forall i.$$

We wish to show that the limit point $x = (\xi_1, \xi_2, \ldots)$ also converges to zero, so that $x \in c_0$. Thus, we have that for some $\epsilon > 0$ and $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$, for all $n > N(\epsilon)$, we have

$$d(\xi_i, 0) \leq d(\xi_i, \xi_i^{(n)}) + d(\xi_i^{(n)}, 0) \leq 2\epsilon.$$

Thus, $x \in c_0$, and hence the space is closed, which in turn also means, the space is complete. ∎

3. *In ℓ^∞, let Y be the subset of ll sequences with only finitely many non-zero terms. Show that Y is a subspace of ℓ^∞ by not a closed subspace.*

Proof. Consider the sequence of sequences $(x_1, x_2, \ldots)$ where

$$x_i = \left(1, \frac{1}{2}, \ldots, \frac{1}{i}, 0, 0, \ldots\right)$$

Then, this sequence converges to

$$\left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots\right)$$

which does not belong to Y. ∎

4. *Show that in a normed space X, vector addition and multiplication by scalars are continuous operations with respect to the norm, i.e, the mappings defined by $(x, y) \mapsto x + y$ and $(\alpha, x) \mapsto \alpha x$ are continuous.*

Proof. Recall the definition of continuity of a map $T : X \mapsto Y$ at a point x_0: for some $\epsilon > 0$, $\exists \delta > 0$ such that $\|x - x_0\| < \delta \implies \|Tx - Tx_0\| < \epsilon$.

First, for addition, consider the map $T : X \times X \mapsto X$ such that $T(x, y) = x + y$. Then, we show that

$$\|(x, y) - (x_0, y_0)\| < \delta \implies \|T(x, y) - T(x_0, y_0)\| < \epsilon.$$

First, we have

$$\|(x, y) - (x_0, y_0)\| = \|(x - x_0, y - y_0)\| < \|x - x_0\| + \|y_0\| < \delta.$$

Then,

$$\|T(x, y) - T(x_0, y_0)\| = \|x + y - x_0 - y_0\| \leq \|x - x_0\| + \|y - y_0\| \leq \delta.$$

Thus, choosing $\delta = \epsilon$ shows that addition is a continuous map.

Now, consider the map $T : K \times X \mapsto X$ such that $T(\alpha, x) = \alpha x$. Now, we show that

$$\|(\alpha, x) - (\alpha_0, x_0)\| < \delta \implies \|T(\alpha, x) - T(\alpha_0, x_0)\| < \epsilon.$$

First, we have that

$$\|(\alpha, x) - (\alpha_0, x_0)\| = \|(\alpha - \alpha_0, x - x_0)\| = \|\alpha - \alpha_0\| + \|x - x_0\| < \delta$$

Now,

$$\|T(\alpha, x) - T(\alpha_0, x_0)\| = \|\alpha x - \alpha_0 x_0\| = \|(\alpha - \alpha_0)x + \alpha_0(x - x_0)\| \leq \|\alpha - \alpha_0\| \|x\| + \|\alpha_0\| \|x - x_0\|.$$

Now, setting $\|\alpha - \alpha_0\| < \delta$ and $\|x - x_0\| < \delta$, we get

$$\|T(\alpha, x) - T(\alpha_0, x_0)\| < \delta(\|x_0\| + \delta) + \|\alpha_0\| \delta < \epsilon.$$

Thus, assuming $\delta < 1$, we get

$$\delta^2 + \delta(\|x_0\| + \|\alpha_0\|) < \delta(\|x_0\| + \|\alpha_0\| + 1) < \epsilon,$$

and solving for δ gives us

$$\delta = \min\left\{\frac{\epsilon}{2(\|x_0\| + \|a_0\|}, 1\right\}.$$

Thus, the multiplication map is also continuous. ∎

5. *Show that $x_n \to x$ and $y_n \to y$ implies $x_n + y_n \to x + y$. Show that $\alpha_n \to \alpha$ and $x_n \to x$ implies $\alpha_n x_n \to \alpha x$.*

Proof. Since $x_n \to x$ and $y_n \to y$, we have that

$$\lim_{n\to\infty} \|x_n - x\| = 0 \text{ and } \lim_{n\to\infty} \|y_n - y\| = 0,$$

and hence,

$$0 \leq \lim_{n\to\infty} \|(x_n + y_n) - (x + y)\| \leq \lim_{n\to\infty} \|x_n - x\| + \lim_{n\to\infty} \|x_n - x\| = 0$$

and hence, $x_n + y_n \to x + y$. Similarly, since $\alpha_n \to \alpha$ and $x_n \to x$, we have

$$\lim_{n\to\infty} |\alpha_n - \alpha| = 0 \text{ and } \lim_{n\to\infty} \|x_n - x\| = 0,$$

since α is a part of the scalar field. Thus,

$$\lim_{n\to\infty} \|\alpha_n x_n - \alpha x\| = \lim_{n\to\infty} \|\alpha_n(x_n - x)\| + \lim_{n\to\infty} \|(\alpha_n - \alpha)x\| = \lim_{n\to\infty} |\alpha_n| \, \|x_n - x\| + \lim_{n\to\infty} |\alpha_n - \alpha| \, \|x\| = 0.$$

■

6. *Show that the closure $\overline{Y}$ of a subspace Y of a normed space X is again a vector space.*

Proof. First, since Y is a subspace, for any $x, y \in Y$, we have

$$\alpha x + \beta y \in Y$$

. Now, consider $\overline{Y}$, which consists of Y and the accumulation points of Y. If the accumulation points of Y belong to Y itself, we are done. However, say x and y are two accumulation points which do not belong to Y. Then, there exist sequences (x_n) and (y_n) in Y that converge to x and y respectively. From 5, we have that $\alpha x_n + \beta y_n \to \alpha x + \beta y$ and hence, $\alpha x + \beta y$ is also an accumulation point. Thus, $\alpha x + \beta y$ also belongs to the set of accumulation points. One question that should arise is what if one point $y \in Y$ and $x \notin Y$ but x is an accumulation point. In such a case, the constant sequence $(y, y \ldots,) \to y$ and another sequence $x_n \to x$, and hence, $\alpha x + \beta y$ would also be an accumulation point. Thus, the set $\overline{Y}$ is also a subspace. ■

7. *Show that the convergence of $\|y_1\| + \|y_2\| + \|y_3\| \ldots$ may not imply convergence of $y_1 + y_2 + y_3 + \ldots$.*

Proof. We shall make use of the hint given in the question. Consider the space in 3, a space of sequences here each sequence consists of finitely many non-zero terms. Consider the sequence $(y_1, y_2, \ldots)$, where $y_n = (\eta_j^{(n)})$ and

$$\eta_j^{(n)} = \begin{cases} \frac{1}{n^2} & j = n \\ 0 & j \neq n \end{cases}.$$

In such a case, we know that

$$\sum_{i=1}^{\infty} \|y_i\| = \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}.$$

However, consider the series $y_1 + y_2 + y_3 + \ldots$. Consider the partial sum s_n:

$$s_n = \left(1, \frac{1}{4}, \ldots, \frac{1}{n^2}, 0, 0, \ldots\right).$$

This sequence of partial sums will converge to a term which will not have finitely many non-zero terms and hence, does not converge in the space. ■

8. *If in a normed space X, if absolute convergence of any series always implies convergence of that series, show that X is complete.*

Proof. Let (x_n) be a sequence that converges absolutely, and hence, also converges. Since it converges absolutely, we know that the partial sums of the norms converge, and hence, the sequence of the partial sums of the norms is Cauchy. Let s_n denote the n^{th} sequence of partial sums. Then, for some $\epsilon > 0$, there exists $N(\epsilon)$ such that for $m, n > N(\epsilon)$, we have

$$\|s_m - s_n\| = \left\| \sum_{i=n+1}^{m} \|x_i\| \right\| \leq \sum_{i=n+1}^{m} \|x_i\| \leq \epsilon.$$

Note that the final expression is exactly the partial sums of the sequence $(x_1, x_2, \ldots)$,

$$\|s_m - s_n\| = \left\| \sum_{i=1}^{n} x_i - \sum_{i=1}^{m} x_i \right\| = \left\| \sum_{i=n+1}^{m} x_i \right\| < \sum_{i=n+1}^{m} \|x_i\| < \epsilon$$

and hence, the sequence of partial sums is Cauchy. Since absolute convergence implies convergence, the Cauchy sequence will converge if and only if the space is complete. ■

9. *Show that in a Banach space, an absolutely convergent series is convergent.*

Proof. The answer to this is similar to the last one. A Banach space is a complete normed space. Suppose (x_n) is an absolutely convergent series. Then, the sequence of partial sums of the norm is convergent, and hence, Cauchy. Thus, representing the partial sum as s_n, we have

$$\|s_m - s_n\| = \left\| \sum_{i=n+1}^{m} \|x_i\| \right\| \leq \sum_{i=n+1}^{m} \|x_i\| \leq \epsilon.$$

This is exactly the difference of the partial sums of the series (x_n) and hence, shows that the sequence of partial sums of (x_n) is Cauchy. Since it is a Banach space, Cauchy sequences converge, and hence, the sequence of partial sums converges, and hence, the series (x_n) ■

10. *Show that if a normed space has a Schauder basis, it is separable.*

Proof. Recall the definition of a Schauder basis: a sequence (e_n) is called a Schauder basis if for every $x \in X$, there exists a unique sequence of scalars α_i such that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i e_i = x.$$

To show that X is separable, we need to show it has a countable dense subset. Let Y be a subset. Let $x \in Y$ be such that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i e_i = x.$$

Now, since α_i belong to the underlying field K, and hence, there exists a sequence β_i such that

$$|\alpha_i - \beta_i| < \epsilon.$$

Let us denote some point y such that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \beta_i e_i = y.$$

Then, we have that for some $\epsilon > 0$, $\exists N(\epsilon)$ such that for $n > N(\epsilon)$, we have

$$\begin{aligned}
\|x - y\| &\leq \left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\| + \left\| y - \sum_{i=1}^{n} \beta_i e_i \right\| + \left\| \sum_{i=1}^{n} (\alpha_i - \beta_i) e_i \right\| \\
&\leq \epsilon + \epsilon + n\epsilon \|e_i\| \\
&\leq \epsilon'
\end{aligned}$$

and hence, for every x there exists some point y such that $x \in B(y, \epsilon)$. Thus, the subset Y is dense. ■

11. *Show that (e_n) where $e_n = (\delta_{nj})$, is a Schauder basis for ℓ^p, where $1 \leq p < \infty$.*

Proof. Let $x = (\xi_1, \xi_2, \ldots) \in \ell^p$ such that

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |\xi_i|^p \right)^{1/p} < \infty.$$

In other words, $\left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p}$ converges, and hence, for some $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $n > N(\epsilon)$, we have

$$\left(\sum_{i=n+1}^{\infty} |\xi_i|^p \right)^{1/p} < \epsilon.$$

Now, consider the sequence (δ_{nj}) with the coefficients $\xi_1, \xi_2 \dots$. Represent

$$x_n = \sum_{i=1}^{n} \xi_i \delta_{ni}.$$

Then, we have

$$\|x - x_n\| = \left\| \sum_{i=n+1}^{\infty} \xi_i \delta_{ni} \right\| \leq \sum_{i=n+1}^{\infty} |\xi_i| \, \|\delta_{ni}\| \leq \epsilon.$$

and hence, $x_n \to x$. Thus, it is a Schauder basis. ■

12. *A seminorm on a vector space X is a mapping $p : X \mapsto \mathbb{R}$ satisfying (N1), (N3), (N4). Show that $p(0) = 0$ and $|p(y) - p(x)| < p(y - x)$. This also means that if $p(x) = 0 \implies x = 0$, then it is a norm.*

Proof. We are given that p satisfies the following properties:

1. $p \geq 0$
2. $p(\alpha x) = |\alpha| \, p(x)$.
3. $P(x + y) \leq p(x) + p(y)$.

Now, substituting $\alpha = 0$ gives us $p(0) = 0$. Also, we have

$$p(x) \leq p(x - y) + p(y) \implies p(x) - p(y) \leq p(x - y).$$

Similarly, we have

$$p(y) \leq p(y - x) + p(x) = |-1| \, p(x - y) + p(x) \implies p(y) - p(x) \leq p(x - y).$$

Combining both these results, we get

$$|p(x) - p(y)| \leq p(x - y).$$

■

13. *Show that in 12, the elements $x \in X$ such that $p(x) = 0$ form a subspace N of X and a norm on X/N is defined by $\|\hat{x}\|_0 = p(x)$ where $x \in \hat{x}$ and $\hat{x} \in X/N$.*

Proof. Let $x, y \in N$ such that $p(x) = p(y) = 0$. We wish to show that $\alpha x + \beta y \in N$. This will only happen if $p(\alpha x + \beta y) = 0$.

$$p(\alpha x + \beta y) \leq p(\alpha x) + p(\beta y) = |\alpha| \, p(x) + |\beta| \, p(y) = 0$$

and thus, $\alpha x + \beta y \in N$. Hence, N is a subset. Recall the definitions of a coset and a quotient space as follows: for some element x, we define the coset w.r.t Y as

$$x + Y = \{x + y \mid y \in Y\}.$$

These distinct cosets partition X, and the space of these cosets is called the quotient space and is represented by X/Y.

Now, to show that $\|\hat{x}\|_0 = p(x)$ is a norm on X/N, we have to show that $p(x)$ satisfies all properties of a norm on X/N. From 13, we have that $p(x)$ satisfies non-negativity, scalar multiplication, and the triangle inequality. We have also shown that $p(0) = 0$. Thus, to show that $p(x)$ is a norm, we wish to show that $p(x) = 0 \implies x = 0$. Let $\hat{x}$ define a coset and $x \in \hat{x}$. Then,

$$\hat{x} = \{x + n \mid n \in N\}.$$

We first show that $p(x) = p(x + n)$ for any $n \in N$. We have that

$$p(x) = p((x + n) - n) \leq p(x + n) + p(n) \leq p(x) + p(n) = p(x)$$

Thus, all the inequalities have to be equalities and we get

$$p(x) = p(x + n) + p(n) = p(x + n).$$

Thus, $p(x) = 0 \implies x \in N$. But, $x \in \hat{x}$, which means that $\hat{x} = N$, and hence, $\hat{x}$ is the zero element of X/N. This is easy to see. Assume $\hat{y} = y + N$ and $\hat{x} = x + N$, where $x \in N$. Then, $\hat{x} + \hat{y} = (x + y) + N = y + N$ since $x + N = N$. ■

14. *Let Y be a closed subspace of a normed space $(X, \|.\|)$. Show that a norm $\|.\|$ on X/Y is defined by*

$$\|\hat{x}\|_0 = \inf_{x\in\hat{x}} \|x\|$$

where $\hat{x} \in X/Y$, i.e, $\hat{x}$ is any coset of Y.

Proof. Clearly, $\|\hat{x}\|_0$ is non-negative. Also, if $\hat{x} = Y$, which is the zero element of X/Y, then

$$\|\hat{x}\|_0 = \|Y\|_0 = \inf_{y\in Y} \|y\| = 0$$

since Y is a closed subspace, $0 \in Y$. Also, if $\|\hat{x}\|_0 = 0$, then

$$\|\hat{x}\|_0 = 0 = \inf_{x\in\hat{x}} \|x\| \implies \exists x \in \hat{x} \text{ such that } \|x\| = 0 \implies x = 0 \implies \hat{x} = Y.$$

where the second to last implication follows since $\|.\|$ is a valid norm. Now,

$$\|\alpha\hat{x}\|_0 = \inf_{x\in\alpha\hat{x}} \|x\| = \inf_{x\in\hat{x}} \|\alpha x\| = |\alpha| \inf_{x\in\hat{x}} \|x\| = |\alpha| \|\hat{x}\|_0 .$$

Finally, suppose $\hat{x} = x + Y$ and $\hat{y} = y + Y$, then

$$\hat{x} + \hat{y} = (x + y) + Y.$$

Thus,

$$\|\hat{x} + \hat{y}\|_0 = \inf_{a\in\hat{x}+\hat{y}} \|a\| = \inf_{x\in\hat{x},y\in\hat{y}} \|x + y\| \le \inf_{x\in\hat{x},y\in\hat{y}} \|x\| + \inf_{x\in\hat{x},y\in\hat{y}} \|y\| = \inf_{x\in\hat{x}} \|x\| + \inf_{y\in\hat{y}} \|y\| = \|\hat{x}\|_0 + \|\hat{y}\|_0 .$$

Thus, it is a valid norm. ∎

15. *If $(X_1, \|.\|_1)$ and $(X_2, \|.\|_2)$ are normed spaces, show that the product vector space $X = X_1 \times X_2$ becomes a normed space if*

$$\|x\| = \max\{\|x_1\|_1 , \|x_2\|_2\}.$$

Proof. Clearly, for $x = (x_1, x_2)$, $\|x\| \ge 0$. Also, if $x = (0, 0)$, then $\|x\| = 0$. Also, if $\|x\| = 0$, then both $\|x_1\|_1 = 0$ and $\|x_2\|_2 = 0$, resulting in $x = 0$. Now,

$$\|\alpha x\| = \max\{\|\alpha x_1\|_1 , \|\alpha x_2\|_2\} = |\alpha| \max\{\|x_1\|_1 , \|x_2\|_2\} = |\alpha| \|x\| .$$

Finally,

$$\|x + y\| = \max\{|x_1 + y_1| , |x_2 + y_2|\} = |x_j + y_j| \le |x_j| + |y_j| \le \max\{|x_1| , |x_2|\} + \max\{|y_1| + |y_2|\} = \|x\| + \|y\|$$

Thus, it is a valid norm. ∎

2.4 Finite Dimensional Normed Spaces

1. *Give examples of subspaces of ℓ^∞ and ℓ^2 which are not closed.*

Proof. Consider the subspaces with finitely many non-zero terms. For example, take the sequence

$$x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, 0, \dots\right)$$

Then, this sequence belongs to both ℓ^2 and ℓ^∞ spaces, however, the limit, does not have finitely many non-zero terms, and hence, does not belong to the subspaces. Thus, the subspaces are not closed. ∎

2. *What is the largest possible c if $X = \mathbb{R}^2$ and $x_1 = (1, 0), x_2 = (0, 1)$? If $X = \mathbb{R}^3$ and $x_1 = (1, 0, 0), x_2 = (0, 1, 0)$, and $x_3 = (0, 0, 1)$?*

Proof. We know that

$$\left\|\sum_{i=1}^n \alpha_i x_i\right\| \ge c \sum_{i=1}^n |\alpha_i| .$$

In the first case, consider

$$\|(\alpha_1, \alpha_2)\| \ge c(|\alpha_1| + |\alpha_2|).$$

If we use, $\|(x, y)\| = |x| + |y|$, since the ℓ^1 norm is maximum, then we have

$$|\alpha_1| + |\alpha_2| \ge c(|\alpha_1| + |\alpha_2|)$$

which shows $c \le 1$. The same holds for $\mathbb{R}^3$. ∎

3. *Show that in Def 2.4-4 the axioms of an equivalence relation hold.*

Proof. Def 2.4-4 says that two norms $\|.\|$ and $\|.\|_0$ are equivalent if

$$a\,\|x\|_0 \le \|x\| \le b\,\|x\|_0\,.$$

We now check if the conditions for equivalent relations hold:

1. $\|.\|$ and $\|.\|$ should be equivalent, which is true by setting $a = b = 1$.
2. If $\|.\| \sim \|.\|_0$, then $\|.\|_0 \sim \|.\|$ which is true since
$$\frac{1}{b}\,\|x\| \le \|x\|_0 \le \frac{1}{a}\,\|x\|\,.$$
3. Suppose $\|.\| \sim \|.\|_0$ and $\|.\|_0 \sim \|.\|_1$, then $\|.\| \sim \|.\|_1$. First notice that
$$a\,\|x\|_0 \le \|x\| \le b\,\|x\|_0$$
and
$$c\,\|x\|_1 \le \|x\|_0 \le d\,\|x\|_1\,.$$
Then,
$$\|x\|_1 \le \|x\| \le \|x\|_1$$
Then, we can write the following:
$$c\,\|x\|_1 \le \|x\|_0 \le \frac{1}{a}\,\|x\|_1 \implies ac\,\|x\|_1 \le \|x\|_0$$
$$\frac{1}{b}\,\|x\| \le \|x\|_0 \le d\,\|x\|_1 \implies \|x\|_0 \le bd\,\|x\|_1\,.$$

This finishes the proof. ■

4. *Show that equivalent norms on a vector space X induce the same topology on X.*

Proof. We wish to show that the set of open sets induced by equivalent norms is the same. Let $\|.\|_0$ and $\|.\|_1$ be two equivalent norms such that

$$c_1\,\|x\|_1 \le \|x\|_0 \le c_2\,\|x\|_1\,.$$

Let A be an open set under the norm $\|.\|_0$. That means that for all $x \in A$, there exists a ball of radius ϵ contained in A. In other words,

$$B(x,\epsilon) \in A \implies \forall a \in A, \|x-a\|_0 < \epsilon \implies \|x-a\|_1 < \frac{\epsilon}{c_1}.$$

This means that A is also an open set under the norm $\|.\|_1$. Thus, all the open sets induced by the metric $\|.\|_0$ is contained in the set of open sets induced by $\|.\|_1$.

In a similar way, let A be an open set under the norm $\|.\|_1$. That means that for all $x \in A$, there exists a ball of radius ϵ contained in A. In other words,

$$B(x,\epsilon) \in A \implies \forall a \in A, \|x-a\|_1 < \epsilon \implies \|x-a\|_0 < c_2\epsilon.$$

This means that A is also an open set under the norm $\|.\|_0$. Thus, all the open sets induced by the metric $\|.\|_1$ is contained in the set of open sets induced by $\|.\|_0$ and hence, the set of all open sets induced, or the topologies are equal. ■

5. *If $\|.\|$ and $\|.\|_0$ are equivalent norms on X, show that the Cauchy sequences in $(X, \|.\|)$ and $(X, \|.\|_0)$ are the same.*

Proof. Let

$$a\,\|x\|_0 \le \|x\| \le b\,\|x\|_0\,.$$

Let (x_n) be a Cauchy sequence in the space $(X, \|.\|)$, then, for some $|epsilon > 0$, we have $N(\epsilon)$ and for all $m, n > N(\epsilon)$, we have

$$\|x_m - x_n\| < \epsilon \implies \|x_m - x_n\| < \frac{\epsilon}{a}.$$

This means that (x_n) is also a Cauchy sequence in the space $(X, \|.\|_0)$.

In a similar fashion, let (x_n) be a Cauchy sequence in the space $(X, \|.\|_0)$. Then, for some $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $m, n > N(\epsilon)$, we have

$$\|x_m - x_n\|_0 < \epsilon \implies \|x_m - x_n\| < b\epsilon$$

which means that (x_n) is also a Cauchy sequence in the space $(X, \|.\|)$. Thus, the Cauchy sequences are equal. ■

6. *Show that $\|.\|_2$ and $\|.\|_\infty$ are equivalent.*

Proof. First, we know that

$$\|x\|_2 = \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{1/2}.$$

and

$$\|x\|_\infty = \max_{i\in[n]} |\xi_i|\,.$$

Thus, we have

$$\|x\|_2 \le \left(\sum_{i=1}^{n} \left(\max_{i\in[n]} |\xi_i|\right)^2\right)^{1/2} = \max_{i\in[n]} |\xi_i| \sqrt{n} = \sqrt{n}\,\|x\|_\infty$$

On the other hand, we have

$$\|x\|_\infty^2 = \left(\max_{i\in[n]} |\xi_i|\right)^2 \le \sum_{i=1}^{n} |\xi_i|^2 = \|x\|_2^2$$

and thus,

$$\|x\|_\infty \le \|x\|_2^2\,.$$

■

7. *Let $\|x\|_2$ be defined on the set of all $n-$tuples on $\mathbb{R}^2$ and let $\|.\|$ be any norm on that vector space, say X. Show directly, using that there exists $b > 0$ such that $\|x\| \le b\,\|x\|_2$ for all x.*

Proof. Let $x = \sum_{i=1}^{n} \alpha_i e_i$. Then, we have that

$$\|x\| = \left\|\sum_{i=1}^{n} \alpha_i e_i\right\| \le \sum_{i=1}^{n} |\alpha_i|\,\|e_i\|$$

and similarly,

$$\|x\|_2^2 \le \sum_{i=1}^{n} |\alpha_i|^2\,.$$

Set $\|e_i\| = \sqrt{\sum_{i=1}^{n} |\alpha_i|^2}$, then we get

$$\|x\| \le \left(\sum_{i=1}^{n} |\alpha_i|\right) \|x\|_2$$

■

8. *Show that norms $\|.\|_1$ and $\|.\|_2$ in Section 2.2, 8 satisfy*

$$\frac{1}{\sqrt{n}}\,\|x\|_1 \le \|x\|_2 \le \|x_1\|\,.$$

Proof. We have that

$$\|x\|_1 = \sum_{i=1}^{n} |\xi_i| \text{ and } \|x\|_2 = \left(\sum_{i=1}^{n} |\xi_i|^2\right).$$

By the application of Cauchy-Schwarz, we get

$$\|x\|_1 = \sum_{i=1}^{n} |\xi_i| \le \sqrt{\sum_{i=1}^{n} 1}\sqrt{\sum_{i=1}^{n} |\xi_i|^2} = \sqrt{n}\,\|x\|_2\,.$$

Also,

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |\xi_i|^2} \le \sqrt{\sum_{i=1}^{n} |\xi_i|^2 + 2\sum_{i=1}^{n} \frac{\prod_{j=1}^{n} |\xi_j|}{|\xi_i|}} \le \left|\sum_{i=1}^{n} |\xi_i|\right| = \|x\|_1\,.$$

■

9. *If two norms $\|.\|$ and $\|.\|_0$ on a vector space X are equivalent, show that $\|x_n - x\| \to 0$ implies $\|x_n - x\|_0 \to 0$ and vice-versa.*

Proof. Let
$$a \|x\|_0 \leq \|x\| \leq b \|x\|_0 .$$
Then, if $\|x_n - x\| \to 0$, we have that for every $\epsilon > 0$, there exists $N(\epsilon)$ such that for every $n > N(\epsilon)$
$$\|(x_n - x) - 0\| < \epsilon \implies \|(x_n - x) - 0\|_0 < \frac{\epsilon}{a}$$
which shows that $\|x_n - x\|_0 \to 0$. In a similar fashion, we can show that
$$\|(x_n - x) - 0\|_0 \leq \epsilon \implies \|(x_n - x) - 0\| \leq b\epsilon$$
and hence, $\|x_n - x\| \to 0$.

■

10. *Show that all complex $m \times n$ matrices $A = (a_{jk})$ with fixed m and n constitute a $mn-$ dimensional vector space Z. Show that all norms on Z are equivalent.*

Proof. Note that every $m \times n$ matrix can be written in the basis $\{E_{ij}\}_{i\in[m],j\in[n]}$ where E_{ij} is the matrix with 1 in the $(i,j)^{th}$ position and zero otherwise. Hence, the dimension is mn. Since, it is a finite dimension space, all norms are equivalent. This is easy to see since let $X = \sum_{i,j} \alpha_{i,j} E_{i,j}$. Then, we have
$$\|X\| \geq c \sum_{i,j} |\alpha_{i,j}| .$$
Also,
$$\|X\|_0 = \left\| \sum_{i,j} \alpha_{i,j} E_{i,j} \right\|_0 \leq \sum_{i,j} |\alpha_{i,j}| \|E_{i,j}\|_0 \leq \frac{\max_{i,j} \|E_{ij}\|_0}{c} \|X\| .$$
and similarly, we can exchange the norms to obtain the other side of the inequality. ■