Introductory Functional Analysis and Applications

Solutions to Exercises

1 Metric Spaces

1.1 Metric Space

1. *Show that the real line is a metric space.*

Proof. To show that the real line is a metric space, we wish to show that there is a valid metric on this space. Obviously, the choice of the metric is $d(x, y) = |x - y|$. We now wish to show the following properties:

1. d is non-negative.
2. $d(x, y) = 0 \iff x = y$.
3. $d(x, y) = d(y, x)$.
4. $d(x, y) \leq d(x, z) + d(z, y)$

Properties (1) , (2), and (3) are obvious. We now show property (4) for the choice of $d(x, y) = |x - y|$. Consider

$$\begin{aligned}(x-y)^2 &= x^2 - 2xy + y^2 \\ &= x^2 - 2xz + z^2 + z^2 - 2zy + y^2 + (2xz + 2zy - 2xy - 2z^2) \\ &= (x-z)^2 + (z-y)^2 + 2x(z-y) + 2z(y-z) \\ &= |x-z|^2 + |z-y|^2 + 2(x-z)(z-y) \\ &\leq (|x-z| + |y-z|)^2\end{aligned}$$

where the inequality follows since $x - y \leq |x - y|$. Taking a square root on both sides finishes the proof. ∎

2. *Does $d(x, y) = (x - y)^2$ define a metric on the set of all real numbers?*

Proof. To show that $d(x, y)$ is a valid metric, we must ensure that it is non-negative, symmetric, satisfies the triangle inequality, and satisfies the condition that $(x - y)^2 = 0 \iff x = y$. We see that the first, second, and last conditions are always true. We only need to check for the triangle inequality. Notice, from the answer of 1, we have

$$(x-y)^2 = (x-z)^2 + (y-z)^2 + 2(x-z)(z-y)$$

WLOG, assume $x \leq y$. If $z \in [x, y]$, then we have $(x - z) \leq 0$ and $(z - y) \leq 0$, and hence, $(x - z)(z - y) \geq 0$, which results in

$$d(x, y) \geq d(x, z) + d(z, y)$$

which is a contradiction of the triangle inequality. Hence, $d(x, y) = (x - y)^2$ is not a metric. ∎

3. *Show that $d(x, y) = \sqrt{|x - y|}$ defines a metric on the set of all real numbers.*

Proof. Once again, we can check that $d(x, y)$ satisfies non-negativity, symmetry, and the property that $d(x, y) = 0 \iff x = y$. We wish to show d satisfies the triangle inequality. Since $|x - y|$ is a valid metric (refer 1), using the triangle inequality for this along with the fact that $\sqrt{.}$ is monotonically increasing, we get

$$\sqrt{|x-y|} \leq \sqrt{|x-z| + |z-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|}$$

where the last inequality follows from the fact that if $a, b \geq 0$, then

$$\sqrt{a+b} \leq \sqrt{a + 2\sqrt{ab} + b} = \sqrt{(\sqrt{a} + \sqrt{b})^2} = \sqrt{a} + \sqrt{b}$$

. ∎

4. *Find all metrics on a set X consisting of two points and only one point.*

Proof. If X consists of only one point, then the only metric that can be defined keeping in mind all properties (note that the triangle inequality does not apply since we do not have a second point for this) is $d(x, x) = 0$.

If X consists of two points, once again, the triangle inequality does not apply, and hence, it is easy to check that

$$d(x, y) = \begin{cases} 0 & x = y \\ c & x \neq y \end{cases}.$$

■

5. *Let d be a metric on X. Determine all constants k such that kd and $d + k$ is a metric on X.*

Proof. Note that for kd to remain a metric, all we need is $k \geq 0$ to ensure it is non-negative. All other properties continue to hold as it is.

For $d + k$ to be a metric, the simplest starting point is that $(d + k)(x, y) = 0 \iff x = y$. However, we know that $d(x, y) = 0 \iff x = y$, and since, k is a constant, this property is satisfied only if $k(x, y) = 0$. Hence, no constant k can be added such that $d + k$ is also a metric. ■

6. *Show that d in 1.1-6 satisfies the triangle inequality.*

Proof. The metric in 1.1-6 is for a sequence space ℓ^∞ such that for $x = (\xi_i)$ and $y = (\eta_i)$ such that $\xi \leq c_x$ and $\eta_i \leq c_y$, we have

$$d(x, y) = \sup_{i \in \mathbb{N}} |\xi_i - \eta_i|$$

Note that this is always non-negative and symmetric, owing to the absolute difference of the terms (since absolute difference is always non-negative, the least upper bound is also non-negative). Also, if $x = y$,. i.e, $\xi_i = \eta_i$ for all i, then the supremum is 0. Finally, we show the triangle inequality as follows. For some $z = (\zeta_i)$, we have

$$d(x, y) = \sup_{i \in \mathbb{N}} |\xi_i - \eta_i| \leq \sup_{i \in \mathbb{N}} (|\xi_i - \zeta_i| + |\zeta_i - \eta_i|) \leq \sup_{i \in \mathbb{N}} |\xi_i - \zeta_i| + \sup_{i \in \mathbb{N}} |\zeta_i - \eta_i| = d(x, z) + d(z, y)$$

where we use the triangle inequality for $|x - y|$ (check 1) and the fact that $\sup(f+g)(x) \leq \sup f(x) + \sup g(x)$. ■

7. *If A is the subspace of ℓ^∞ consisting of all sequences of zeros and ones, then what is the induced metric on A?*

Proof. If A is the subspace consisting of all sequences only containing zeros and ones, then the coordinate-wise absolute difference of any two sequences can only take on two values; 0, when the coordinates are the same, and 1 when the coordinates are different. Thus, the supremum of this absolute difference is 1. In other words, the metric can be expressed as:

$$d(x, y) = \mathbb{1}\{x = y\}$$

which is exactly the discrete metric. ■

8. *Show that another metric $\tilde{d}$ on the space X in 1.1-7 is defined by*

$$\tilde{d}(x, y) = \int_a^b |x(t) - y(t)| \; dt$$

Proof. We wish to show that the metric $d(x, y)$ is a defined on the space of continuous functions $C[a, b]$, i.e x and y are now continuous functions. Note that $d(x, y)$ is clearly non-negative and symmetric owing to the point-wise absolute difference of the functions. Further, if $x(c) = y(c) \; \forall c \in [a, b]$, then, $d(x, y) = \int_a^b 0 dt = 0$. Thus, we wish to show the triangle inequality now. Using the fact that the absolute difference is a metric (check 1), using the triangle inequality, we have

$$\int_a^b |x(t) - y(t)| \; dt \leq \int_a^b (|x(t) - z(t)| + |z(t) - y(t)|) \; dt = \int_a^b |x(t) - z(t)| \; dt + \int_a^b |z(t) - y(t)| \; dt$$

which proves that $d(x, y)$ satisfies the triangle inequality. ■

9. *Show that d in 1.1-8 is a metric.*

Proof. We wish to show that the discrete metric, defined as

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

is a metric. Clearly, this is non-negative and symmetric. Further, we have that $d(x,y) = 0 \iff x = y$. Finally, we also have

1. If $x \neq y$, then $d(x,y) = 1$. In such a case, if $x = z$ or $y = z$, then $d(x,z) + d(y,z) = 1$. However, if $x \neq z \neq y$, then $d(x,z) + d(y,z) = 2$. Thus, $d(x,y) \leq d(x,z) + d(z,y)$.
2. If $x = y$, then $d(x,y) = 0$. Now, if $x = y \neq z$, then $d(x,z) + d(z,y) = 2$, and if $x = y = z$, then $d(x,z) + d(z,y) = 0$, and in both cases, $d(x,z) + d(z,y) \geq d(x,y)$.

Thus, the triangle inequality holds for d, and hence, it is a valid metric. ■

10. *Let X be the set of all ordered triples of zeros and ones. Show that X consists of eight elements and a metric d on X is defined by $d(x,y) =$number of places where x and y have different entries.*

Proof. Clearly, X has $2^3 = 8$ elements. We wish to show that

$$d(x,y) = \sum_{i=1}^{3} \mathbb{1}\{x_i \neq y_i\}$$

is a metric on X where x_i represents the i^{th} coordinate of x. Clearly, this metric is non-negative, symmetric, and satisfies the property that $d(x,y) = 0 \iff x = y$. Also, we have

$$\mathbb{1}\{x_i \neq y_i\} \leq \mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i}$$

This is easy to see using a case-by-case analysis.

1. If $x_i = y_i$, then $\mathbb{1}\{x_i \neq y_i\} = 0$. If $z_i = x_i = y_i$, we have $\mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i} = 0$, while if $z_i \neq x_i = y_i$, then $\mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i} = 2$, and hence, $\mathbb{1}x_i \neq y_i \leq \mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i}$.
2. If $x_i \neq y_i$, then $\mathbb{1}\{x_i \neq y_i\} = 1$. If $z_i = x_i$ or $z_i = y_i$, we have $\mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i} = 1$, while if $z_i \neq x_i \neq y_i$, then $\mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i} = 2$, and hence, $\mathbb{1}x_i \neq y_i \leq \mathbb{x_i} \neq \mathbb{z_i} + \mathbb{z_i} \neq \mathbb{y_i}$.

Thus, the Hamming distance satisfies all the properties of a metric. ■

11. *Prove (1).*

Proof. We wish to prove the statement:

$$d(x_1, x_n) \leq \sum_{i=2}^{n} d(x_{i-1}, x_i)$$

We can do this by induction on n. The base case $n = 3$ holds due to the triangle inequality. Assuming it holds for $n = k$, we wish to show that it holds for $n = k + 1$. This is easy to show as follows:

$$d(x_1, x_{k+1}) \leq d(x_1, x_k) + d(x_k, x_{k+1}) \leq \sum_{i=2}^{k} d(x_{i-1}, x_i) + d(x_k, x_{k+1}) = \sum_{i=2}^{k+1} d(x_{i-1}, x_i)$$

where the first inequality follows by the triangle inequality and the second follows from the induction hypothesis. ■

12. *Using (1), show that*

$$|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w)$$

Proof. By the repeated application of triangle inequality, we have

$$d(x,y) \leq d(x,z) + d(z,y) \leq d(x,z) + d(z,w) + d(w,y)$$

which gives us

$$d(x,y) - d(z,w) \leq d(x,z) + d(y,w)$$

Similarly, we have

$$d(z,w) \leq d(z,x) + d(x,z) \leq d(z,x) + d(x,y) + d(y,w)$$

which gives us

$$d(z,w) - d(x,y) \leq d(x,z) + d(y,w)$$

Combining both these facts and using the symmetry property of metrics gives us the required result. ■

13. *Using the triangle inequality, show that*

$$|d(x,z) - d(y,z)| \leq d(x,y)$$

Proof. Note that $d(x,z) \leq d(x,y) + d(y,z)$ and hence, $d(x,z) - d(y,z) \leq d(x,y)$. Similarly, $d(y,z) \leq d(x,y) + d(x,z)$ and hence, $d(y,z) - d(x,z) \leq d(x,y)$. Combining both results finishes the proof. ∎

14. *Show that (M3) and (M4) could be obtained from (M2) and*

$$d(x,y) \leq d(z,x) + d(z,y)$$

Proof. We wish to show that the axioms $d(x,y) = d(y,x)$ and $d(x,y) \leq d(x,z) + d(z,y)$ can be obtained from the axioms $d(x,y) = 0 \iff x = y$ and $d(x,y) \leq d(z,x) + d(z,y)$.

To prove symmetry, we have the following two equations:

$$d(x,y) \leq d(z,x) + d(z,y)$$

$$d(y,x) \leq d(z,y) + d(z,x)$$

Subtracting both equations results in

$$d(x,y) \leq d(y,x) \text{ and } d(y,x) \leq d(x,y) \implies d(x,y) = d(y,x).$$

Now, simply using symmetry, we have

$$d(x,y) \leq d(z,x) + d(z,y) = d(x,z) + d(z,y)$$

since $d(z,x) = d(x,z)$. ∎

15. *Show that non-negativity of a metric follows from (M2) to (M4).*

Proof. We wish to show that the non-negativity of a metric follows from the following axioms:

1. $d(x,y) = d(y,x)$.
2. $d(x,y) = 0 \iff x = y$.
3. $d(x,y) \leq d(x,z) + d(z,y)$.

Substituting $y = x$ in the triangle inequality, we have

$$d(x,x) \leq d(x,z) + d(z,x)$$

Using the first and second axiom in tandem gives us $d(x,z) \geq 0$, which completes the proof. ∎

1.2 Further Examples of Metric Spaces

1. *Show that in 1.2-1, we can obtain another metric by replacing* $1/2^i$ *with* μ_i *such that* $\sum \mu_i$ *converges.*

Proof. We wish to show that for some $\mu_j > 0$ such that $\sum mu_j$ converges, and two sequences $x = (\xi_i)$ and $y = (\eta_i)$, we have

$$d(x,y) = \sum_{j=1}^{\infty} \mu_j \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}$$

is a valid metric defined on the sequence space.

The proof follows on the same lines as the one given for 1.2-1. It is clear that $d(x,y)$ is non-negative, symmetric, and satisfies $d(x,y) = 0 \iff x = y$. However, we wish to show this is bounded, i.e

$$d(x,y) = \sum_{j=1}^{\infty} \mu_j \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \sup_{j \in \mathbb{N}} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \sum_{j=1}^{\infty} \mu_j$$

Note that the RHS is bounded if and only if $\sum \mu_j$ is convergent (since the supremum term can be upper bounded by 1). Further, the proof for the triangle inequality follows in the same fashion, define $f(t) = \frac{t}{1+t}$. Then, $f(t)$ is monotonically increasing, and hence, using the triangle inequality for numbers, we get

$$f(|a+b|) \leq f(|a| + |b|)$$

and hence, substituting $a = \xi_j - \zeta_j$ and $b = \zeta_j - \eta_j$, we get

$$\frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \frac{|\xi_j - \zeta_j| + |\zeta_j - \eta_j|}{1 + |\xi_j - \zeta_j| + |\zeta_j - \eta_j|} \leq \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} + \frac{|\zeta_j - \eta_j|}{1 + + |\zeta_j - \eta_j|}$$

Multiplying by μ_j and summing over all $j \in \mathbb{N}$ finishes the proof. ■

2. *Using (6) show that the geometric mean of two positive numbers does not exceed the arithmetic mean.*

Proof. We wish to show that the geometric mean does not exceed the arithmetic mean using the equation:

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

where $p^{-1} + q^{-1} = 1$.

Substituting $x = \sqrt{a}, y = \sqrt{b}, p = q = 2$, we get

$$\sqrt{ab} \leq \frac{a + b}{2}.$$

■

3. *Show that the Cauchy Schwarz inequality implies*

$$\left(\sum_{i=1}^{n} |\xi_i| \right)^2 \leq n \sum_{i=1}^{n} |\xi_i|^2$$

Proof. The Cauchy-Schwarz inequality states that

$$\left(\sum_{i=1}^{n} |\xi_i| \, |\eta_i| \right)^2 \leq \left(\sum_{i=1}^{n} |\eta_i|^2 \right) \left(\sum_{i=1}^{n} |\xi_i|^2 \right)$$

Putting $\eta_i = 1$ for all $i \in [n]$ completes the proof. ■

4. *Find a sequence which converges to 0, but is not in any space ℓ^p, $1 \leq p < \infty$.*

Proof. Recall that the ℓ^p space consists of sequences $x = (\xi_i)$ such that $\sum |\xi_i|^p$ converges. The sequence $\xi_i = \frac{1}{\log(i+1)}$ converges to 0 but is not part of any ell^p space. To see this, note that $\log(i + 1)$ can be bounded above by $i^{1/p}$ and hence,

$$\frac{1}{\log(i + 1)} \geq i^{-1/p}.$$

Thus, we have

$$\sum_i \left(\frac{1}{\log(i + 1)} \right)^p \geq \sum_i \left(\frac{1}{i^{1/p}} \right)^p \to \infty$$

■

5. *Find a sequence which is in ℓ^p with $p > 1$, but $x \notin \ell^1$.*

Proof. We wish to find a sequence $x = (\xi_i)$ such that $\sum |xi_i|^p$ converges for all $p > 1$ but does not converge for $p = 1$. The simplest example is $\xi_i = i^{-1}$, the sum of which diverges. ■

6. *The diameter $\delta(A)$ of a nonempty set A in a metric space (X, d) is defined to be*

$$\delta(A) = \sup_{x,y \in A} d(x, y)$$

A is said to be bounded if $\delta(A) < \infty$. Show that $A \subset B$ implies $\delta(A) \leq \delta(B)$.

Proof. First, by the definition of supremum, we have

$$\sup X = s \implies \forall x \in X \ , \ x \leq s$$

Thus, over the set B, if $\delta(B)$ is the diameter of B, we have that

$$\forall (x, y) \in B \times B, d(x, y) \leq \delta(B).$$

Since, $A \subset B$, we have that all points in A also belong to B and hence, satisfy the property above, i.e

$$\forall (x,y) \in A \times A \ , \ d(x,y) \le \delta(B)$$

and hence, taking the supremum on both sides, we get

$$\delta(A) = \sup_{x,y \in A} d(x,y) \le \sup_{x,y \in A} \delta(B) = \delta(B).$$

■

7. *Show that $\delta(A) = 0$ if and only if A consists of a single point.*

Proof. We wish to show

$$\delta(A) = 0 \iff |A| = 1.$$

We first show RHS implies LHS. Suppose $\delta(A) = 0$, which means

$$\sup_{x,y \in A} d(x,y) = 0 \implies d(x,y) \le 0 \ \forall x,y \in A.$$

However, since $d(x,y)$ is always non-negative, we have that

$$d(x,y) = 0 \ \forall x,y \in A \implies x = y \ \forall x,y \in A.$$

This tells us that there is only one unique point in A.

We now show LHS implies RHS. Suppose A has only one point x, then,

$$\delta(A) = \sup_{A} d(x,y) = \sup\{d(x,x)\} = \sup\{0\} = 0$$

This finishes the proof. ■

8. *The distance $D(A,B)$ between two non-empty sets of a metric space (X,d) is defined to be*

$$D(A,B) = \inf_{\substack{a \in A \\ b \in B}} d(a,b)$$

Show that D does not define a metric on the power set of X.

Proof. An easy way to see that this does not define a metric on the power set of X (denoted by $\mathcal{P}(X)$) is as follows: clearly, the metric is non-negative and symmetric. However, let $A, B \in \mathcal{P}(X)$. If $A \cap B \neq \emptyset$, then $\exists x \in X$ such that $x \in A$ and $x \in B$. In such a case,

$$D(A,B) = \inf_{\substack{a \in A \\ b \in B}} d(a,b) = d(x,x) = 0.$$

However, $A \cap B \neq \emptyset \nRightarrow A = B$, and hence, the condition

$$D(A,B) = 0 \iff A = B$$

fails. ■

9. *If $A \cap B \neq \emptyset$, show that $D(A,B) = 0$. What about the converse?*

Proof. Let $x \in A$ and $x \in B$. Then,

$$\{d(a,b)\}_{\substack{a \in A \\ b \in B}} = \{d(x,x)\} \cup \{d(a,b)\}_{(a,b) \in A \times B \setminus \{(x,x)\}} = \{0\} \cup \{d(a,b)\}_{(a,b) \in A \times B \setminus \{(x,x)\}}$$

and hence,

$$\inf_{\substack{a \in A \\ b \in B}} d(a,b) = 0 = D(A,B).$$

The converse may not be true. The idea comes from accumulation points (defined in the next section). x is said to be an accumulation point of a set A if every neighborhood of x contains atleast one point from A distinct from x. Let $x \in B$ and $x \notin A$ be an accumulation point of A. Then, every $\epsilon-$neighborhood of x will contain $a(\epsilon) \in A$ and hence, for every $\epsilon > 0$, you can find $a(\epsilon)$ such that $d(a(\epsilon), x) = \epsilon$. Since ϵ can be arbitrarily small, the infimum becomes zero. However, $A \cap B = \emptyset$. Thus, the converse may not be true. ■

10. *The distance $D(x)$ from a point x to a non-empty subset B of (X, d) is defined as*

$$D(x, B) = \inf_{b \in B} d(x, b)$$

Show that for any $x, y \in X$,

$$|D(x, B) - D(y, B)| \leq d(x, y).$$

Proof. First, consider the case when $x \in B$ and $y \in B$. Clearly, $D(x, B) = D(y, B) = 0$ and hence,

$$|D(x, B) - D(y, B)| = 0 \leq d(x, y).$$

by the properties of d. Now, consider the case when $x \notin B$ and $y \in B$. Clearly, $D(y, b) = 0$ and hence,

$$|D(x, B)| = \inf_{b \in B} d(x, b) \leq d(x, y)$$

since $y \in B$. Finally, consider the case when $x \notin B$ and $y \notin B$. Then, we have

$$D(x, B) = \inf_{b \in B} d(x, b) \leq \inf_{b \in B} (d(x, y) + d(y, b)) = d(x, y) + \inf_{b \in B} d(y, b) = d(x, y) + D(y, B).$$

where the first inequality follows from the triangle inequality applied on d. Rearranging the terms gives us

$$D(x, B) - D(y, B) \leq d(x, y).$$

By symmetry, interchanging x and y results in

$$D(y, B) - D(x, B) \leq d(x, y).$$

This finishes the proof. ∎

11. *If (X, d) is any metric space, show that another metric on X is defined by*

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

and X is bounded in the metric $\tilde{d}$.

Proof. First, the metric $\tilde{d}$ is non-negative, symmetric and satisfies $\tilde{d}(x, y) = 0 \iff x = y$ from the properties of d. Also

$$\lim_{d(x,y) \to \infty} \tilde{d}(x, y) = 1$$

and hence, it does remain bounded. Also, notice that

$$f(t) = \frac{t}{1 + t}$$

is always increasing and hence, by the non-negativity of the metric d, we have

$$\begin{aligned} f(d(x, y)) \leq f(d(x, z) + d(z, y)) = \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)} &= \frac{d(x, z)}{1 + d(x, z) + d(z, y)} + \frac{d(z, y)}{1 + d(x, z) + d(z, y)} \\ &\leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(x, z)} \end{aligned}$$

which finishes the proof. ∎

12. *Show that the union of two bounded sets is also bounded.*

Proof. Let A and B be two bounded sets such that $\delta(A) < \infty$ and $\delta(B) < \infty$. Then, using the definition of $\delta(A)$ from 6, we have

$$\forall a_1, a_2 \in A, d(a_1, a_2) \leq \delta(A) < \infty \text{ and } \forall b_1, b_2 \in B, d(b_1, b_2) \leq \delta(B) < \infty$$

Thus, for any $a_1 \in A, b_1 \in B$, we have

$$d(a_1, b_1) \leq d(a_1, a_2) + d(a_2, b_2) + d(b_1, b_2) \leq \delta(A) + \delta(B) + d(a_2, b_2)$$

and since d is a metric, it is also bounded, resulting in $d(a_1, b_1)$ being bounded. ∎

13. *The cartesian product $X = X_1 \times X_2$ of two metric spaces (X_1, d_1) and (X_2, d_2) can be made into a metric space (X, d) in many ways. For example, show that a metric d is defined by*

$$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Proof. First, note that d is bounded, non-negative, and symmetric owing to the metric properties of d_1 and d_2. Also, due to the metric properties of d_1 and d_2, we have

$$d(x, y) = 0 \iff d_1(x_1, y_1) = 0 \text{ and } d_2(x_2, y_2) = 0 \iff x = (x_1, x_2) = (y_1, y_2) = y.$$

Finally, we show the triangle inequality for some $z = (z_1, z_2)$ as follows:

$$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2) \leq d_1(x_1, z_1) + d_1(z_1, y_1) + d_2(x_2, z_2) + d_2(z_2, y_2) = d(x, z) + d(z, y).$$

This finishes the proof. ∎

14. *Show that another metric on X is defined by*

$$\tilde{d}(x, y) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$$

Proof. Once again, note that $\tilde{d}$ is bounded, non-negative, symmetric and satisfies $\tilde{d}(x, y) = 0 \iff x = y$ owing to the metric properties of d_1 and d_2. Finally, we show the triangle inequality for some $z = (z_1, z_2)$ as follows:

$$\begin{aligned}
\tilde{d}(x, y) &= \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2} \\
&\leq \sqrt{(d_1(x_1, z_1) + d_1(z_1, y_1))^2 + (d_2(x_2, z_2) + d_2(z_2, y_2))^2} \\
&\leq \sqrt{d_1(x_1, z_1)^2 + d_2(x_2, z_2)^2 + d_1(z_1, y_1)^2 + d_2(z_2, y_2)^2 + 2d_1(x_1, z_1)d_1(z_1, y_1) + 2d_2(x_2, z_2)d_2(z_2, y_2)}
\end{aligned}$$

Using Hölder's inequality, which says that for $a_i, b_i \geq 0$ and p, q such that $p^{-1} + q^{-1} = 1$,

$$\sum_{i=1}^{N} a_i b_i \leq \left(\sum_{i=1}^{N} a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{N} b_i^q \right)^{\frac{1}{q}}$$

we get

$$d_1(x_1, z_1)d_1(z_1, y_1) + d_2(x_2, z_2)d_2(z_2, y_2) \leq \sqrt{d_1(x_1, z_1)^2 + d_2(x_2, z_2)^2}\sqrt{d_1(z_1, y_1)^2 + d_2(z_2, y_2)^2}$$

and hence, we get

$$\tilde{d}(x, y) \leq \sqrt{d_1(x_1, z_1)^2 + d_2(x_2, z_2)^2}\sqrt{d_1(z_1, y_1)^2 + d_2(z_2, y_2)^2} = \tilde{d}(x, z) + \tilde{d}(z, y).$$

This finishes the proof. ∎

15. *Show that another metric on X is defined by*

$$\tilde{\tilde{d}}(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

Proof. Once again, note that $\tilde{\tilde{d}}$ is bounded, non-negative, symmetric and satisfies $\tilde{\tilde{d}}(x, y) = 0 \iff x = y$ owing to the metric properties of d_1 and d_2. Finally, we show the triangle inequality for some $z = (z_1, z_2)$ as follows:

$$\begin{aligned}
\tilde{\tilde{d}}(x, y) &= \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \\
&\leq \max\{d_1(x_1, z_1) + d_1(z_1, y_1), d_2(x_2, z_2) + d_2(z_2, y_2)\} \\
&= d_i(x_i, z_i) + d_i(z_i, y_i) \; i \in \{1, 2\} \\
&\leq \max\{d_1(x_1, z_1), d_2(x_2, z_2)\} + \max\{d_1(z_1, y_1), d_2(z_2, y_2)\} \\
&= \tilde{\tilde{d}}(x, z) + \tilde{\tilde{d}}(z, y)
\end{aligned}$$

This finishes the proof. ∎

1.3 Open set, Closed set, Neighborhood

1. *Justify the terms* open ball *and* closed ball *by showing that an open ball is an open set and a closed ball is a closed set.*

Proof. Let $B(x_0, r)$ be a ball of radius r around x_0. If we wish to show that $B(x_0, r)$ is open, it suffices to show that we can draw a ball around any point of $B(x_0, r)$. In other words, choose any point $y_0 \in B(x_0, r)$. For some radius r', $B(y_0, r') \subseteq B(x_0, r)$. Let $r' \le r - d(x_0, y_0)$. Then, for some $x \in B(y_0, r')$, we have

$$d(x, x_0) \le d(x, y_0) + d(y_0, x_0) \le r$$

and hence, $x \in B(x_0, r)$. This shows that an open ball is also an open set.

Showing that a closed ball is a closed set is equivalent to showing the complement of a closed ball is open. Let X be a set and $x_0 \in X$. Let $\tilde{B}(x_0, r)$ denote a closed ball of radius r in X and let $\tilde{B}(x_0, r)^C$ denote the complement of $\tilde{B}(x_0, r)$. Choose some point $y_0 \in \tilde{B}(x_0, r)^C$ and $r' \le d(x_0, y_0) - r$. Clearly, $d(x_0, y_0) \ge r$. Then, $B(y_0, r')$ is a ball about y_0 with radius r'. Let $x \in B(y_0, r'$. Then,

$$d(x_0, y_0) \le d(x_0, x) + d(y_0, x) \implies d(x_0, x) \ge d(x_0, y_0) - d(y_0, x) \ge r - d(y_0, x) \ge r.$$

and hence, $x \in \tilde{B}(x_0, r)^C$. Thus, the complement of the closed ball is open. This finishes the proof. ■

2. *What is an open ball $B(x_0, 1)$ in $\mathbb{R}$ and $\mathbb{C}$? What is an open ball in $C[a, b]$?*

Proof. In $\mathbb{R}$, the open ball is the set of points $B(x_0, 1) = \{x_0 \pm \epsilon \mid |\epsilon| < 1\}$. In $\mathbb{C}$, let $x_0 = a + \iota b$, then the open ball is defined as $B(x_0, 1) = \{(a + \epsilon_1) + \iota(b + \epsilon_2) \mid \sqrt{\epsilon_1^2 + \epsilon_2^2} < 1\}$. Finally, let $x_0 \in C[a, b]$ be a function of t. Then, under the metric

$$d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|$$

we have the open ball to be defined as

$$B(x_0, 1) = \left\{ x_0(t) \pm \epsilon(t) \mid \left| \max_{t \in [a,b]} \epsilon(t) \right| < 1 \right\}.$$

■

3. *Consider $C[0, 2\pi]$ and determine the smallest r such that $y \in \tilde{B}(x, r)$ where $x(t) = \sin t$ and $y(t) = \cos t$.*

Proof. We wish to find the smallest value of r such that $y(t) \in \tilde{B}(x(t), r)$. Let the metric on $C[0, 2\pi]$ be defined as

$$d(x, y) = \max_{t \in [0,2\pi]} |x(t) - y(t)|.$$

Then, we wish to find the smallest value of r such that $d(x, y) \le r$. Substituting the values of $x(t)$ and $y(t)$, we get

$$\max_{t \in [0,2\pi]} |\sin t - \cos t| \le r$$

Differentiating $\sin t - \cos t$ w.r.t t and setting to 0, we find that the maximum value of the expression is obtained at $t = 3\pi/4$ and the maximum value is $\sqrt{2}$. Thus, setting $r = \sqrt{2}$ ensures $y \in B(x, r)$. ■

4. *Show that any nonempty set $A \subset (X, d)$ is open if and only if it is a union of open balls.*

Proof. We first assume that a nonempty set $A \subset (X, d)$ is open and wish to show it is the union of open balls. Choose any point $x \in A$. Then, there exists a ball B around x such that $B \subset A$. Since x is arbitrary, at each point in A, there exists a ball comprised entirely in A and hence, A is the union of all such open balls.

Now we assume A is a union of open balls. For some open ball B, $\exists x \in B$ such that we can draw another ball B_0 around x and $B_0 \subset B$. Since A is the union of open balls, $x \in A$ and $B_0 \subset B \subset A$. Since x is arbitrary, A is an open set. ■

5. *It is important to realize that some sets may be open and closed at the same time. Show that this is always the case for X and $\emptyset$. Show that in a discrete metric space X, every subset is open and closed.*

Proof. Note that $\emptyset$ is always open since there are no points in it. This also ensures that the set X is closed (since it's complement $\emptyset$ is open). On the other hand, X is open because it consists of the entire space and all limit points, and hence, it's complement $\emptyset$ is closed.

In a discrete metric space, let $A \subset X$. A is open if we can draw a ball around each point in A. Let $r < 1$. Then, for each $a \in A$, $B(a, r) = \{a\} \subset A$, and hence, the subsets are open. In a similar fashion, let A^C be the complement of A, and let $a \in A^C$ be some point. Then, choosing $r < 1$, we have a ball $B(a, r) = \{a\} \subset A^C$, and hence, A^C is also open, resulting in A being closed. ■

6. *If x_0 is an accumulation point of $A \subset (X, d)$, show that any neighborhood of x_0 contains infinitely many points of A.*

Proof. We shall prove the claim by contradiction. By the definition of an accumulating point, the neighborhood of x_0 contains at least one point distinct from x_0 from A. Let $B(x_0, \epsilon)$ be an $\epsilon-$neighborhood of x_0, which consists of finitely many points ($\neq x_0$) $\{a_1, \ldots, a_k\}$. Let $r \leq \min_{j \in [k]} d(x_0, a_j)$. Then, the $r-$neighborhood of x_0 consists of no points other than x_0, which is a contradiction since x_0 is an accumulation point. Hence, there should be another point in the $r-$neighborhood, which contradicts our assumption of finitely many points. ■

7. *Describe the closure of each of the sets: the integers on $\mathbb{R}$, the rational numbers on $\mathbb{R}$, the complex numbers with rational real and imaginary parts in $\mathbb{C}$, and the disk $\{z \mid |z| < 1\} \subset \mathbb{C}$.*

Proof. The closure of a set includes all points of the set and all it's limit points. The limit points of a set are points such *every* neighborhood of the point consists of at least one element of the set distinct from the point. Clearly, for the set of integers on $\mathbb{R}$, the closure is the set of integers itself. In other words, there are no limit points for the set of integers. Let z be an integer, then drawing an $\epsilon-$neighborhood around z with $\epsilon < 1$ ensures that there is no integer apart from z in this neighborhood. Same is the case for any rational number q, any $\epsilon-$neighborhood around q may not consist of an integer.

For the set of rationals on $\mathbb{R}$, the closure is $\mathbb{R}$. This is because choosing any point in $\mathbb{R}$ will have atleast one rational number in it's neighborhood. Same is the case for the set of complex numbers with rational real and imaginary parts. You can always find a new complex number with rational real and imaginary parts in a neighborhood.

Finally, for the open disk of radius 1 in the complex numbers, defined as

$$\{z \mid |z| < 1\} \subset \mathbb{C}$$

the accumulation points will be

$$\{z \mid |z| \leq 1\}.$$

Since any neighborhood of a "boundary point" will intersect with the disk, they also are accumulation points, and hence, the closure is simply the closed disk of radius 1. ■

8. *Show that the closure $\overline{B(x_0, r)}$ of an open ball $B(x_0, r)$ in a metric space can differ from the closed ball $\tilde{B}(x_0, r)$.*

Proof. TODO. ■

9. *Show that $A \subset \overline{A}$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$, and $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.*

Proof. $A \subset \overline{A}$ follows from the definition of a closure. Now, we show $\overline{A \cup B} = \overline{A} \cup \overline{B}$. To show this, we show that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. First, let $x \in \overline{A \cup B}$. Then,

1. $x \in A \cup B$. This means that $x \in A$ or $x \in B$. WLOG, assume $x \in A$, then by the definition of closure, $x \in \overline{A}$, and hence, $x \in \overline{A} \cup \overline{B}$.

2. $x \notin A \cup B$ and x is an accumulation point of $A \cup B$. This means that x is also an accumulation point of either A or B. This is clear if every neighborhood of x contains some point of A or if every neighborhood contains a point of B. Now consider the case where some ϵ_A-neighborhood of x contains only points of A and ϵ_B-neighborhood contains points of only B. WLOG, let $\epsilon_A \geq \epsilon_B$. Then, ϵ_A-neighborhood also contains points from B. Thus, x would be a accumulation point for either A or B, which implies that $x \in \overline{A}$ or $x \in \overline{B}$, resulting in $x \in \overline{A} \cup \overline{B}$.

This shows that $x \in \overline{A} \cup \overline{B}$.

Now we show that $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Let $x \in \overline{A} \cup \overline{B}$. WLOG, assume $x \in \overline{A}$. Then,

1. $x \in A$, which means that $x \in A \cup B$ and hence, $x \in \overline{A \cup B}$.

2. $x \notin A$ and x is an accumulation point for A. This means that every neighborhood of x contains some point from A. That means that every neighborhood of x also contains some point from $A \cup B$, and hence, x is an accumulation point for $A \cup B$. Thus, $x \in \overline{A \cup B}$.

Thus, $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. This finishes the proof.

Finally, we show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Let $x \in \overline{A \cap B}$. Then,

1. $x \in A \cap B$. This means that $x \in A$ and $x \in B$, and hence, $x \in \overline{A}$ and $x \in \overline{B}$, resulting in $x \in \overline{A} \cap \overline{B}$.

2. $x \notin A \cap B$ but x is an accumulating point of $A \cap B$. That means that every neighborhood of x has a point from $A \cap B$, which means the point in the neighborhood belongs to both A and B. Thus, every neighborhood contains a point from A and B, making x an accumulation point for both A and B. Hence, $x \in \overline{A}$ and $x \in \overline{B}$, resulting in $x \in \overline{A} \cap \overline{B}$.

This finishes the proof. ■

10. *A point x not belonging to a closed set $M \subset (X, d)$ always has a nonzero distance from M. To prove this, show that $x \in \overline{A}$ if and only if $D(x, A) = 0$, where A is any nonempty subset.*

Proof. First, we define

$$D(x, A) = \inf_{a \in A} d(x, a).$$

We show that $x \in \overline{A} \implies D(x, A) = 0$. If $x \in A$, then by definition of D, we have $D(x, A) = 0$. However, let $x \notin A$, then for every $\epsilon-$neighborhood of x, $B(x, \epsilon) \cap A \neq \emptyset$, which means that for any $\epsilon > 0$, we can find a point distinct from x belonging to A. As ϵ can be arbitrarily small, the infimum would be 0.

Next we show that $D(x, A) = 0 \implies x \in \overline{A}$. If $x \in A$, then it is trivial. However, if $x \notin A$, then we have that for all $\epsilon > 0$, there exists some point $a(\epsilon) \in A$ such that $d(a(\epsilon), x) = \epsilon$. Since ϵ can be made arbitrarily small, the infimum results in 0. However, this also means that for every $\epsilon-$neighborhood of x, there exists some point $a(\epsilon) \in A$ and hence, x is an accumulation point of A. Thus, $x \in \overline{A}$. This finishes the proof. ■

11. *A boundary point of $A \subset (X, d)$ is a point of X (which may or may not be in A) such that every neighborhood of x contains points of A as well as points not belonging to A. The boundary of A is defined to be the set of all boundary points of A. Find the boundaries of $(-1, 1)$, $[-1, 1)$, $[-1, 1]$ on $\mathbb{R}$, the set of rationals on $\mathbb{R}$, and the disks $\{z \mid |z| < 1\}$ and $\{z \mid |z| \leq 1\} \subset \mathbb{C}$.*

Proof. The boundary points of $(-1, 1)$ are $\{-1, 1\}$. This is because any $\epsilon-$neighborhood of -1 will contain $-1+\epsilon \in (-1, 1)$ and $-1-\epsilon \notin (-1, 1)$. Note that $x \in (-1, 1)$ is not a boundary point because every neighborhood will not contain any point which does not belong to $(-1, 1)$. The boundary points for $(-1, 1]$ and $[-1, 1]$ are also $\{-1, 1\}$.

The boundary points for the set of rationals on $\mathbb{R}$ is $\mathbb{R}$. This is because for any $q \in \mathbb{R}$, the neighborhood of q will contain rationals as well as irrationals.

Finally, the boundary points for the disks $\{z \mid |z| < 1\}$ and $\{z \mid |z| \leq 1\} \subset \mathbb{C}$ is simply the set $\{z \mid |z| = 1\}$. ■

12. *Show that $B[a, b]$ is not separable.*

Proof. TODO ■

13. *Show that a metric space X is separable if and only if X has a countable subset Y with the following property: For every $\epsilon > 0$ and every $x \in X$, there is a $y \in Y$ such that $d(x, y) < \epsilon$.*

Proof. We first show that if a metric space X is separable, then it contains a countable subset with the property mentioned above. First, if X is separable it would contain separable dense subsets Y. Since Y is dense, we have that $\overline{Y} = X$, which means that every point in $x \in X$ either belongs to Y or is an accumulation point of Y. In such a case, if $x \in Y$, then it is trivial. However, if $x \notin Y$ and is an accumulation point, then by the definition of an accumulation point, every $\epsilon-$neighborhood of x would contain some $y \in Y$, and hence, $d(x, y) < \epsilon$. This finishes the proof.

We now prove the converse, i.e if there is a countable subset Y that satisfies the above mentioned property, then X is separable. In such a case, the property mentions that for every $\epsilon > 0$ and for every $x \in X$, there exists $y \in Y$ such that $d(x, y) < 0$. If $x \in Y$, then this is trivially true. However, if $x \notin Y$, then, by the above mentioned property, x is an accumulation point for Y, which means that all points in X are either in Y or are accumulation points for Y. Thus, $\overline{Y} = X$ and hence, Y is dense in X. Since Y is countable, we have that X is a separable set. This finishes the proof. ■

14. *Show that a mapping $T : X \mapsto Y$ is continuous if and only if the inverse image of any closed set $M \subset Y$ is a closed set in X.*

Proof. We instead show that if T is a continuous mapping if and only if the inverse image of any open set in Y is open in X.

Suppose T is a continuous mapping. Then for every $\epsilon > 0, \exists \delta > 0$ such that

$$d(x_0, x) \leq \delta \implies d(Tx_0, Tx) \leq \epsilon.$$

Thus, consider the point $Tx_0 \in Y$ and the set $B(Tx_0, \epsilon)$ which is open. Clearly, due to the definition of T, $B(Tx_0, \epsilon)$ maps back to $B(x_0, \delta)$ which is again open, and hence, if T is continuous, then the inverse image of an open set is an open set.

Now, we show the converse. Suppose we take the set $B(Tx_0, \epsilon) \subset Y$ which is open. Then, we have that the inverse image of $B(Tx_0, \epsilon)$ is also open, say some set $N \in X$ such that $x_0 \in N$. Also, since N is a neighborhood, $B(x_0, \delta) \subset N$, and hence, the definition of continuity is fulfilled.

Now, we shall show the statement. Suppose T is continuous and $M \subset Y$ is closed. This means that M^C is open and hence, the inverse image of M, say $A \subset X$ is open. Thus, A^C is closed. Similarly, let $M \subset Y$ is closed and it's inverse image $A \subset X$ is also closed. This means that M^C and A^C is open and hence, T is continuous. ■

15. *Show that the image of an open set under a continuous mapping need not be open.*

Proof. Consider a function f on the set (a, b) that maps to $[c, d]$ such that for some $a < a_1, b_1 < b$, we have $f(a_1) = c$ and $f(b_1) = d$. Any such function will satisfy the above claim. ■

1.4 Convergence, Cauchy Sequences, Completeness

1. *If a sequence (x_n) in a metric space X is convergent and has limit x show that every subsequence (x_{n_k}) of (x_n) is also convergent and has the same limit x.*

Proof. Since (x_n) is convergent, we have that for every $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $n \geq N(\epsilon)$, we have that $d(x, x_n) < \epsilon$. This also means that for all $n_k \geq N(\epsilon)$, $d(x_{n_k}, x) \leq \epsilon$. Hence, the subsequence (x_{n_k}) also follows the definition of convergence. ■

2. *If (x_n) is Cauchy and has a convergent subsequence, say $x_{n_k} \to x$, show that (x_n) is convergent with the limit x.*

Proof. Since (x_n) is Cauchy, we have that for every $\epsilon > 0$, there exists $N_1(\epsilon)$ such that for all $m, n \geq N_1(\epsilon)$, we have $d(x_m, x_n) \leq \epsilon$. Now, since the subsequence (x_{n_k}) is converging to x, we have that for every $\epsilon > 0$, there exists $N_2(\epsilon)$ such that for all $n_k \geq N_2(\epsilon)$, $d(x_{n_k}, x) \leq \epsilon$. Thus, fixing the value of ϵ, and choosing the value of $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$, we have that for some $n, n_k \geq N(\epsilon)$

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \leq 2\epsilon.$$

This finishes the proof. ■

3. *Show that $x_n \to x$ if and only if for every neighborhood V of x there is an integer n_0 such that $x_n \in V$ for all $n > n_0$.*

Proof. Suppose $x_n \to x$. Then, for every $\epsilon > 0$, there exists some $N(\epsilon)$ such that for all $n \geq N(\epsilon)$, $d(x_n, x) \leq \epsilon$. This means that for every $\epsilon > 0$, there exists $N(\epsilon)$ such that for $n \geq N(\epsilon)$, $x_n \in B(x, \epsilon)$. Thus, for every neighborhood V, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset V$, and hence, for $n \geq N(\epsilon) = n_0$, $x_n \in V$.

We now show the converse. Suppose, for every neighborhood V, there is an integer n_0 such that $x_n \in V$ for $n > n_0$. Assume that $B(x, \epsilon) \subset V$. Thus, setting $n_0 = N(\epsilon)$, for all $n > N(\epsilon)$, $x_n \in B(x, \epsilon)$, and hence, the definition of convergence is fulfilled. ■

4. *Show that a Cauchy sequence is bounded.*

Proof. A Cauchy sequence (x_n) is defined as follows: for every $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $m, n \geq N(\epsilon)$, we have that $d(x_m, x_n) \leq \epsilon$. Fix an ϵ and $x_N = x_{N(\epsilon)}$ as our reference point. Now we know that for $n \geq N$, $d(x_n, x_N) \leq \epsilon$. Thus, choose

$$r = \max_{i \in [N]} d(x_i, x_N)$$

and $r_0 = \max\{r, \epsilon\}$. Then, for all $n > 0$, we have $d(x_n, x_N) \leq r_0$ which shows that the entire sequence is bounded in a ball of radius r_0 centered at x_N. ■

5. *Is boundedness of a sequence in a metric space sufficient for the sequence to be Cauchy or convergent?*

Proof. No, the boundedness property is not sufficient. An example is $x_n = \sin \frac{n\pi}{2}$, which is always bounded by 1 but is neither Cauchy nor convergent. ■

6. *If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d), show that (a_n), where $a_n = d(x_n, y_n)$ is convergent.*

Proof. **Note: this question seems incorrect, and should only be true for complete spaces.**

Since (x_n) is Cauchy, for every $\epsilon > 0$, there exists $N_x(\epsilon)$ such that for every $m, n > N_x(\epsilon)$, we have that $d(x_m, x_n) < \epsilon$. Similarly, since (y_n) is Cauchy, for every $\epsilon > 0$, there exists $N_y(\epsilon)$ such that for every $m, n > N_y(\epsilon)$, we have that $d(y_m, y_n) < \epsilon$. Now, fix ϵ and let $N(\epsilon) = \max\{N_x(\epsilon), N_y(\epsilon)\}$. Then for every $m, n > N(\epsilon)$, we have

$$d(x_n, y_n) \leq d(x_n, x_m) + d(y_n, y_m) + d(x_m, y_m) < 2\epsilon + d(x_m, y_m)$$

and hence,

$$d(x_n, y_n) - d(x_m, y_m) \leq 2\epsilon$$

Similarly, by symmetry

$$d(x_m, y_m) - d(x_n, y_n) \leq 2\epsilon$$

and hence,

$$|d(x_n, y_n) - d(x_m, y_m)| \leq 2\epsilon.$$

Thus, the sequence (a_n) is Cauchy, and since the space should be complete, the sequence is also convergent. ■

7. *Give an indirect proof of Lemma 1.4-2(b).*

Proof. What an indirect proof is seems unclear. We wish to prove that if a sequence $(x_n) \to x$ and $(y_n) \to y$, then $d(x_n, y_n) \to d(x, y)$. By the properties of convergence, we have

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ and } \lim_{n\to\infty} d(y_n, y) = 0.$$

Thus, using the triangle inequality and non-negativity of the metric, we have

$$0 \leq d(x_n, y_n) \leq d(x_n, x) + d(y_n, y) + d(x, y).$$

As n grows, we have $0 \leq \lim_{n\to\infty} d(x_n, y_n) \leq d(x, y)$. Similarly, we can also obtain $0 \leq d(x, y) \leq \lim_{n\to\infty} d(x_n, y_n)$, which gives us the result that

$$\lim_{n\to\infty} d(x_n, y_n) = d(x, y).$$

■

8. *If d_1 and d_2 are metrics on the same set X and there are positive numbers a and b such that for all $x, y \in X$*

$$ad_1(x, y) \leq d_2(x, y) \leq bd_1(x, y),$$

show that the Cauchy sequences in (X, d_1) and (X, d_2) are the same.

Proof. Let (x_n) be a Cauchy sequence on (X, d_1). Then, for every $\epsilon > 0$, we have some $N(\epsilon)$ such that for all $n, m > N(\epsilon)$, $d_1(x_m, x_n) < \epsilon$. Using the relation between d_1 and d_2, we have that for every $m, n > N(\frac{1}{b}(b\epsilon))$, $d_2(x_m, x_n) < b\epsilon$ and hence, every sequence (x_n) that is Cauchy in (X, d_1) is also Cauchy in (X, d_2). Similarly, if a sequence (y_n) is Cauchy in (X, d_2), for some $\epsilon > 0$, we have some $N(\epsilon)$ such that for all $m, n > N(\epsilon)$, $d_2(y_m, y_n) < \epsilon$. Using the relation once again, we have that for $m, n > N(a(\frac{\epsilon}{a}))$, $d_1(y_m, y_n) < \frac{\epsilon}{a}$ and hence, (y_n) is Cauchy in (X, d_1). This finishes the proof. ■

9. *Using 8, show that the metric spaces in 13, 14, 15 from section 1.2 have the same Cauchy sequences.*

Proof. Define

$$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$$
$$\tilde{d}(x, y) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$$
$$\tilde{\tilde{d}}(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

on the metric space $X = X_1 \times X_2$. It is easy to see the following:

1. $\tilde{d} \leq \sqrt{2}\tilde{\tilde{d}}$.
2. $d \leq 2\tilde{\tilde{d}}$.
3. $d \leq \tilde{d}$.
4. $\tilde{\tilde{d}} \leq d$.

Combining these gives us

$$\tilde{\tilde{d}} \leq d \leq \tilde{d} \leq 2\tilde{\tilde{d}}.$$

This finishes the proof. ■

10. *Using the completeness of* $\mathbb{R}$ *,prove the completeness of* $\mathbb{C}$.

Proof. Since $\mathbb{R}$ is complete, every Cauchy sequence converges to some point in $\mathbb{R}$. Now, every complex number has a real part and an imaginary part. Let (c_n) be some cauchy sequence of complex numbers such that $c_n = a_n + \imath b_n$. Clearly, the sequence (a_n) is Cauchy and convergent since $\mathbb{R}$ is complete. Also, multiplying every element of a sequence does not change the convergence properties and hence, $(\imath b_n)$ is also cauchy and convergent. Thus, $(a_n + \imath b_n) = (c_n)$ is also cauchy and convergent. ■