Math - 680

HW - 1

(1)

(i) To show → $\min_a E(x-a)^2 = E(X-EX)^2$.

Solution →

$$E(X-a)^2 = E(X+EX-EX-a)^2 = E(X-EX)^2 + (EX-a)^2$$

OR → $= E[(X-EX)^2 + 2(X-EX)(EX-a) + (EX-a)^2]$

$$E(X-EX)^2 \leq E(X-EX)^2 + (EX-a)^2$$

⇒

Which is minimum when $\boxed{a = EX}$

[This was a problem of squared loss function]

(ii) To show → $\min_a E|X-a| = E|X-m|$

where m is the median of X.

$$E|X-a| = \int_{-\infty}^{\infty} |X-a| f(x)\,dx \quad \text{(derivative)}$$

$$= \int_{-\infty}^{a} (a-x) f(x)\,dx + \int_{a}^{\infty} (x-a) f(x)\,dx$$

{ ~~...~~ ... }

$$= \int_{-\infty}^{a} f(x)\,dx - \int_{a}^{\infty} f(x)\,dx$$

$$= 2F(a) - 1$$

$2F(a) - 1 = 0$

$\Rightarrow \boxed{F(a) = 1/2}$

$\boxed{\text{So } a = \text{median}}$

(2) (i) Posterior distribution of p.

$$f(p|n) \propto \pi(p|\alpha,\beta)\, f(x|p)$$

$$= \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1} \Rightarrow \prod_{i=1}^{n} \theta_{x_i} (1-p)^{n-\Sigma n}$$

$$= \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1} p^{\sum_{i=1}^{n} x_i} (1-p)^{n-\Sigma n}$$

$$\Rightarrow \propto p^{\sum_{i=1}^{n} x_i+\alpha-1}(1-p)^{n-\Sigma n}$$

$$\Rightarrow \sim Beta\left(\sum_{i=1}^{n} x_i+\alpha,\ n-\sum_{i=1}^{n} x_i+\beta\right).$$

(ii) To show $\rightarrow$ $\hat{p}_{bayes} = \frac{\sum_{i=1}^{n} x_i+\alpha}{\alpha+\beta+n}$

Bayesian estimator of $p \Rightarrow \hat{p}_{bayes}$

Posterior mean $\Rightarrow$ $E(p|n) \Rightarrow$

$\Rightarrow$ Which is equal to $\Rightarrow \frac{\sum_{i=1}^{n} x_i+\alpha}{\alpha+\beta+n}$

Which can be further classified as.

$$\Rightarrow \frac{n}{\alpha+\beta+n} \cdot \frac{\Sigma x_i}{n} + \frac{\alpha+\beta}{\alpha+\beta+n} \cdot \frac{\alpha}{\alpha+\beta}$$

So $\hat{p}_{bayes} \Rightarrow \frac{x_i+\alpha}{\alpha+\beta+n}$

(iii) To show → $\hat{p}_{bayes}$ is given by

$$R(p, \hat{p}_{bayes}) = \frac{np(1-p)}{(\alpha+\beta+n)^2} + \left(\frac{np+\alpha}{\alpha+\beta+n} - p\right)^2$$

$$R(p, \hat{p}_{bayes}) = Var(p) + Bias^2(p)$$

$$= Var\left(\frac{\sum_{i=1}^{n} x_i + \alpha}{\alpha+\beta+n}\right) + \left(\frac{1}{\alpha+\beta+n} E\left(\sum_{i=1}^{n} x_i\right) + \frac{\alpha}{\alpha+\beta+n} - p\right)^2$$

$$= \frac{np(1-p)}{(\alpha+\beta+n)^2} + \left(\frac{np+n}{\alpha+\beta+n} - p\right)^2$$

$$\Rightarrow \frac{(\alpha(p-1) + \beta p)^2 + np(1-p)}{(\alpha+\beta+n)^2}.$$

Risk function is given by →

$$R(p, \hat{p}_{bayes}) = \frac{np(1-p) + (\alpha(p-1) + \beta p)^2}{(\alpha+\beta+n)^2}$$

(iv) $\alpha + \beta = \sqrt{n}/4$

(a) for $\hat{p}_{bayes}$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i + \alpha}{\alpha+\beta+n}$$

$$= \frac{\sum_{i=1}^{n} x_i + \sqrt{n}/4}{\sqrt{n}/4 + \sqrt{n}/4 + n}$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i + \sqrt{n/4}}{2\left(\sqrt{n/2}\right) + n} \Rightarrow \boxed{\frac{\sum_{i=1}^{n} x_i + \sqrt{n/4}}{n + \sqrt{n}}}$$

(b) $$R(p, \hat{p}_{bayes}) = \left(\frac{np + \alpha}{\alpha + \beta + n}\right)^2 + \frac{n(p)(1-p)}{(\alpha + \beta + n)^2}$$

$$= \frac{(p\beta - p\alpha - \alpha)^2}{(\alpha + \beta + n)^2} \quad \frac{np - np^2}{(\alpha + \beta + n)^2}$$

$$\Rightarrow \frac{(p\alpha + p\beta - \alpha)^2 + np - np^2}{(\alpha + \beta + n)^2}$$

Put α & β as $\sqrt{n/4}$

$$\Rightarrow \frac{\left(p\left(\sqrt{n/4}\right) + p\left(\sqrt{n/4}\right) - \sqrt{n/4}\right)^2 + np - np^2}{\left(\sqrt{n/4} + \sqrt{n/4} + n\right)^2}$$

[commutation as $2\sqrt{n}p$ so np^2 comes) to changing bracket.

$$\Rightarrow \frac{\left(\sqrt{n/4}\right)^2 - 2\left(\sqrt{n/4}\right)\left(\sqrt{n}p\right) + (\sqrt{n}p)^2 + np - np^2}{\left(2\sqrt{n/4} + n\right)^2}$$

$$\Rightarrow \frac{n/4 - np + np^2 + np - np^2}{(\sqrt{n} + n)^2}$$

$$\Rightarrow \boxed{\frac{4}{4(4+\sqrt{4})^2}}$$

(3)

(i) $C(0,1) = 3$, $C(1,0) = 2$

Bayes rule $\Rightarrow$
(From page 23)
Lecture-4

$$f^*(x) = \begin{cases} 1 & \text{if } \frac{g_1(x)}{g_0(x)} > \frac{\pi_0 C(0,1)}{\pi_1 C(1,0)} \\ 0 & \text{if } \frac{g_1(x)}{g_0(x)} < \frac{\pi C(0,1)}{\pi C(1,0)} \end{cases}$$

For this problem bayes rule

$$\left. \begin{aligned} f^*(x) = P(Y=1 \mid X=x) &= C(0,1) \\ P(Y=0 \mid X=x) &= C(1,0) \end{aligned} \right\}.$$

(ii) Bayes decision Boundary

(From page 24
Lecture-4)

$$\left\{ x : \frac{g_1(x)}{g_0(x)} = \frac{\pi_0}{\pi_1} \right\}$$

$$\left\{ x : \frac{g_1(x)}{g_0(x)} = \frac{3}{2} \right\} = \left\{ -1.582, 0.274 \right\}.$$

(iii) By using Newton-Raphson method

$\Rightarrow 0.65 \times \exp\left(-\frac{1}{2}(x-1)^2\right) + 0.35 \exp\left\{-\frac{(x-1)^2}{16}\right) = \frac{3}{2}$

$\Rightarrow -1.58, 0.27$

Optimal classification region $\Rightarrow$

$\Omega_1^* = (1.58, 0.27), \quad \Omega_2^* = (-\infty, -1.58)$

(4) Please refer to R code file

(5)

(i) For 0-1 loss, bayes classifier $\Rightarrow$

$X|Y=1 \sim N(\mu_1, I), \quad \mu_1 = (2,1)^T$

$X|Y=0 \sim N(\mu_0, I), \quad \mu_0 = (1,2)^T.$

This equation is satisfied by the decision boundary $\Rightarrow$

$P(Y=1|X=x) = P(Y=0|X=x)$

$\Rightarrow P(X=x|Y=1)P(Y=1) = P(X=x|Y=0) \times P(Y=0)$

$\Rightarrow \dfrac{P(X=x|Y=1)}{P(X=x|Y=0)} = \dfrac{P(Y=0)}{P(Y=1)}$

taking log.

$$= \log\left[\frac{P(X=x|Y=1)}{P(X=x|Y=0)}\right] = \log 1$$

$\left(\log\left(\frac{a}{b}\right) = \log a - \log b\right)$ $= \log P(X=x|Y=1) - \log(P(X=x|Y=0)) = 0$

log of anything $= 0$

Comparing in terms of x.

$$\Rightarrow -\frac{1}{2}(x-\mu_1)^T \Sigma^{-1}(x-\mu_1) + \frac{1}{2}(x-\mu_0)^T \Sigma^{-1}(x-\mu_0) = 0$$

$$\Rightarrow (\mu_1 + \mu_0)^T(\mu_1 - \mu_0) = 0$$

so $\boxed{x_1 - x_2 = 0}$

$$\boxed{f^*(x) = \begin{cases} 1 & \text{if } x_1 > x_2 \\ 0 & \text{if } x_1 < x_2 \end{cases}}.$$

(ii) Please refer R code file

(iii) Bayes train error $= \underline{0.295}$

Bayes test error $= \underline{0.219}$

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