Unit IV
Higher Order Ordinary Differential Equations

Overview:

Please write an overview of the unit

Outcome:

After completion of this unit, students would be able to:

use effective mathematical tools for the solutions of ordinary differential that model physical processes

Include

1. *Prerequisite to the topic/unit:*
   Derivatives and solving equations
2. *Formulae*

4.1 SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The equation $a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ .\ .\ .\ .\ .\ .\ .\ .\ .+a_ny=Q$ … (1)

Where all the 'a's are constants and Q is a function of x alone is called a linear differential equation of order n with constant coefficients in the variable x and y.

Since D^ny stands for $\frac{d^ny}{dx^n}$, the equation (1) can be written as

$(a_0\ D^n+a_1\ D^{n-1}+\ .\ .\ .\ .\ .\ .\ .+a_n)y=Q$ or $F(D)y = Q$

where F(D) stands for $(a_0D^n + a_1D^{n-1} + \ldots\ldots + a_n)$

$$\boxed{\text{Complete Solution = Complementary Function + Particular Integral}}$$

C.S. = C.F. + P.I.

Methods of Finding Complementary Function (C.F.) :

To find the complementary function put F(D) = 0. This equation is called Auxiliary equation. Solve this equation. Let its roots be $m_1, m_2, m_3, \ldots..$; D being treated like an ordinary algebraic quantity.

Roots	Nature	Complimentary functions
m_1, m_2, m_3, m_4 ……	Real and unequal	C.F.=

		$C_1e^{m_1x}+C_2e^{m_2x}+C_3e^{m_3x}+C_4e^{m_4x}+....$
$m_1 = m_2, m_3, m_4,$ $m_3 \neq m_4$	Some are real and equal	C.F.$=(C_1+C_2x)\ e^{m_1x}+C_3e^{m_3x}+C_4e^{m_4x}+$ ....
$m_1, m_2 = a \pm ib,$ m_3, m_4	Some are imaginary	C.F.$= e^{ax}\ (C_1\ \cos bx+C_2\ \sin bx)$ $+C_3\ e^{m_3x}+C_4\ e^{m_4x}+$ ....
$m_1, m_2 = a \pm ib$ $m_3, m_4 = a \pm ib$	Imaginary and equal	C.F. $= e^{ax}$ $[(C_1 + C_2x) \cos bx +$ $+ (C_3 + C_4x) \sin bx]$

Operator Method:

Particular Integral (P.I.)

Let the given equation be $F(D)y = Q$.

Case I: If $Q = e^{(ax+b)}$

then P.I $= \frac{1}{F(D)}Q = \frac{1}{F(D)}e^{(ax+b)}$

$= \frac{1}{F(a)}e^{(ax+b)}$ [Provided $F(a) \neq 0$]. [i.e. for D write 'a' in F(D)].

if $F(a)= 0$ then P.I.$= \frac{1}{F(D)}e^{(ax+b)} = x.\ \frac{1}{F'(D)}e^{(ax+b)}$

where $F'(D)$ is the differentiation of $F(D)$ with respect to D

$= x.\ \frac{1}{F'(a)}e^{(ax+b)}$ [Provided $F'(a) \neq 0$] and so on.

Case II: If $Q = \sin(ax + b)$ or $\cos(ax + b)$

To obtain P.I. $= \frac{1}{F(D)}Q = \frac{1}{F(D)}\sin(ax + b)$

Write $-a^2$ for D^2 provided $F(D) \neq 0$

If $F(D) = 0$ when $D^2 = -a^2$

then P.I. $= \frac{1}{F(D)}\sin(ax + b) = x\frac{1}{F'(D)}\sin(ax + b)$

operate on sin(ax + b) with $\frac{1}{F'(D)}$ by replacing D^2 by $-a^2$

The process may be repeated if F ′(D) = 0 when $D^2 = -a^2$.

Note : The rule for evaluating $\frac{1}{F(D)}$ cos (ax + b) is the same.

Case III : If Q = x^n then P.I. = $\frac{1}{F(D)}x^n$

Expand $\frac{1}{F(D)}$ by Binomial Series in ascending powers of D uptoD^n in the form a + bD + cD^2 + ... and operate on x^n by each term of the expansion.

Case IV : If Q = e^{ax} V. where V is a function of x.

then P.I.= $\frac{1}{F(D)}(e^{ax}V) = e^{ax}\frac{1}{F(D+a)}V$

To operate on e^{ax} V (where V is a function of x) by $\frac{1}{F(D)}$ operate on V by $\frac{1}{F(D+a)}$ and multiply the result by e^{ax}.

Case V: If Q = xV where V is a function of x then

P.I.= $\frac{1}{F(D)}(xV) = \left[x - \frac{1}{f(D)}f'(D)\right]\frac{1}{f(D)}V$

4.2 Method of Undermine Coefficient

To find P.I of $f(D)y = X$, we assume P.I as $y = Af_1(x) + Bf_2(x) + ... + Jf_r(x)$ where $f_1, f_2, ...$ are functions of x and $A, B, ...$ are constants. P.I depends on nature of X as,

i) If $X = ae^{mx}$ then P.I. $= Ae^{mx}$
ii) If $X = ae^{mx} + be^{nx}$ then P.I. $= Ae^{mx} + Be^{nx}$
iii) If $X = a\sin mx$ or $a\cos mx$ then P.I. $= A\sin mx + B\cos mx$
iv) If $X = ax^m$ then P.I. $= Ax^m + Bx^{m-1} + ... + Mx + N$
v) If $X = ae^{mx}\sin nx$ then P.I. $= Ae^{mx}\sin nx + Be^{mx}\cos nx$

Substituting this assumed P.I. in $f(D)y = X$, equating coefficient, we get constants.

4.3 METHOD OF VARIATION OF PARAMETERS TO FIND PARTICULAR INTEGRAL

Consider the linear differential equation of the second order with constant coefficients.

$$\frac{d^2y}{dx^2} + K_1\frac{dy}{dx} + K_2 y = X \qquad \dots (1)$$

where X is a function of ‘ x ’
Let its complementary function be

$C.F. = c_1y_1 + c_2y_2$ … (2)

such that y_1 and y_2 satisfy the equation (1).
Let us assume that the particular integral of (1) as $y = uy_1 + vy_2$ … (3)
Obtained by replacing c_1 and c_2 by u and v are known function of 'x'.

where $u = -\int \frac{y_2 X}{w} dx$ and $v = \int \frac{y_1 X}{w} dx$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

4.4 CAUCHY EULER'S LINEAR DIFFERENTIAL EQUATIONS

An equation of the type

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + ... + a_{n-1} x \frac{dy}{dx} + a_n y = Q$$

where Q is a function of x and a_0, a_1, …. ,a_n are constants.

The above equation is called Cauchy's linear differential equation or nth order linear differential equation with variable coefficients.

Let $Dy = \frac{dy}{dx}$; $D^2 y = \frac{d^2 y}{dx^2}$; $D^n y = \frac{d^n y}{dx^n}$

Then the above equation will take the form

$$a_0 x^n D^n y + a_1 x^{n-1} D^{n-1} y + a_2 x^{n-2} D^{n-2} y + \; .... \; + a_{n-1} x.Dy + a_n y = Q \qquad \ldots(2)$$

i.e. $(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \; .... \; + a_{n-1} x D + a_n) y = Q$

It is converted to linear differential equation of order n with constant coefficients by substituting $x = e^z$.

4.5 POWER SERIES SOLUTIONS

Standard form of homogenous differential equation**:** $\frac{d^2 y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$

Analytic:
A point x_0 is said to be a **<u>ordinary point (O.P)</u>** of differential equation if both $P(x)$ and $Q(x)$ are analytic at x_0.
A point x_0 is said to be a **<u>Singular point</u>** of differential equation if either $P(x)$ or $Q(x)$ or both are not analytic at x_0.

Standard form of homogenous differential equation**:** $y''+P(x)y'+Q(x)y=0$ (1)

Steps to find the solution by power series method:

Step i) Assume that $y=\sum_{0}^{\infty} c_n(x-x_0)^n$ be the solution (1) where x_0 is ordinary point. (2)

Step ii) Substitute, y, y', y'' obtain by differentiating (2) term wise in (1). Collect the coefficient of like power of $(x-x_0)$. This convert the differential equation (1) in the form

$$k_0+k_1(x-x_0)+k_2(x-x_0)^2+......=0 \tag{3}$$

Here $k_i(i=0,1,2,3....)$ are function of certain coefficient of c_n .

Step iii) If (2) is the solution of (1) , all k_i's must be zero.

Solve $k_0=0,\ k_1=0,.......$ for unknown coefficient c_n 's.

Step iv) Substituting of these c_n 's in (2) gives the required power series solution of (1).

LEGENDRE POLYNOMIALS

Legendre Equation plays a vital role in many problems of Mathematical Physics and in the theory of quadratures (as applied to Numerical Integration).

DEFINITION :*The equation*

$$\left(1-x^2\right)y''-2xy'+n\left(n+1\right)y=0 \qquad -1<x<1$$

where $n\in N$ ***is called a LEGENDRE EQUATION of order*** n

The above Equation was studied by Legendre and hence the name Legendre Equation.

$$P_n(x)=\sum_{m=0}^{M}\frac{(-1)^m(2n-2m)! \quad x^{n-2m}}{2^n m!(n-m)!(n-2m)!} \quad \textit{where} \quad M=\frac{n}{2} \quad \textit{when } n \textit{ is even}$$
$$=\frac{n-1}{2} \quad \textit{when } n \textit{ is odd}$$

$When\, n=0,\ P_0(x)=1$

$When\, n=1,\ P_1(x)=x$

$When\, n=2,\ P_2(x)=\frac{1}{2}(3x^2-1)$

$When\, n=3,\ P_3(x)=\frac{1}{2}(5x^3-3x)$

$When\, n=4,\ P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$

$When\, n=5,\ P_5(x)=\frac{1}{8}(63x^5-70x^3+15x) \qquad etc$

Note:

1) $x=1,\quad P_n(x=1)=P_n(1)=1$

2) *Any polynomial* $f(x)$ *of degree* n *can be expressed in terms of* $P_n(x)$ *as*

$$f(x)=\sum_{m=0}^{n} c_m P_m(x)$$

Rodrigue's Formula

$$P_n(x)=\frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$$

BESSEL FUNCTIONS OF THE FIRST KIND AND THEIR PROPERTIES.

The differential equation is $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$, is called **Bessel's differential equation**. Its equation is given by Frobinus series $y=\sum_{m=0}^{\infty} a_m\, x^{m+n}$, where $a_0\neq 0$ and m a constant. The most general solution is given by $y=AJ_n(x)+BJ_{-n}(x)$ where A and B arbitrary constants and $\boxed{J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\lceil n+m+1}\left(\frac{x}{2}\right)^{n+2m}}$ is known as **Bessel's function** of first kind.

Session 1

Solve the following linear differential equation:

1. $(D^2-1)(D-1)^2y=0$ Ans: $y=(C_1+C_2x+C_3x^2)e^x+C_4e^{-x}$
2. $\left(D^2-2D+2\right)y=0$ Ans: $y=e^x\left[c_1\cos x+c_2\sin x\right]$

Session 2

Solve the following linear differential equation:

1. $\frac{d^4y}{dx^4}-2\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}=0$
2. $\frac{d^4y}{dx^4}+13\frac{d^2y}{dx^2}+36y=0$
3. $\left((D-1)^4\left(D^2+2D+2\right)^2\right)y=0$

Session 3

Solve the following linear Differential Equation using operator method:

1. $\left(D^2+3D+2\right)y=e^{2x}$ Ans: $y=c_1e^{-2x}+c_2e^{-x}+\frac{e^{2x}}{12}$

2. $(D^2-D-6)\,y=e^{3x+8}$ Ans: $y=C_1e^{3x}+C_2e^{-2x}+\frac{x}{5}e^{3x+8}$

3. $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^x+\cos x$ Ans: $y=C_1e^x+C_2e^{2x}-xe^x+\frac{1}{10}(-3\sin x+\cos x)$

4. $(D^2-D+1)y=\cos 2x$

Session 4

Solve the following linear Differential Equation using operator method:

1. $\left(D^4+8D^2+16\right)y=\cos^2 x$ Ans: $\left(c_1+c_2x\right)\cos 2x+\left(c_3+c_4x\right)\sin 2x+\frac{1}{32}-\frac{x^2\cos 2x}{64}$

2. $\left(D^2-9D+20\right)y=20x^2$ Ans: $y=c_1e^{4x}+c_2e^{5x}+x^2+\frac{9x}{10}+\frac{61}{200}$

3. $(D^2-4D+4)y=8(x^2+\sin 2x+e^{2x})$

4. $(D^2-6D+9)y=e^{3x}(1+x)$ Ans: $y=e^{3x}\left(C_1+C_2x+\frac{x^2}{2}+\frac{x^3}{6}\right)$

Session 5

Solve the following linear Differential Equation using operator method:

1. $\left(D^2+D-6\right)y=e^{2x}\sin 3x$ Ans: $y=c_1e^{2x}+c_2e^{-3x}-\frac{e^{2x}}{102}\left(5\cos 3x+3\sin 3x\right)$

2. $\left(D^2-4D+3\right)y=e^x\cos 2x+\cos 3x$ Ans:

$c_1e^{3x}+c_2e^x-\frac{e^x}{8}\left(\sin 2x+\cos 2x\right)-\frac{1}{30}\left(2\sin 3x+\cos 3x\right)$

3. $\left(D^2+9\right)y=x\sin x$ Ans: $y=\left(c_1\cos 3x+c_2\sin 3x\right)+\frac{x\sin x}{8}-\frac{\cos x}{32}$

Session 6

Solve the following equations by the method of undermine coefficient:

1. $\left(D^2-2D+3\right)y=12\sin 3x$

2. $\left(D^2+2D+4\right)y=4x^2+3e^{-x}$

3. $(D^2-2D)y=e^x\sin 3x$

4. $(D^2-1)y=8e^x\sin 2x$

5. $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=10e^{3x}+4x^2$

Session 7

Solve the following equations by the method of variation of parameters:

1. $\left(D^2+1\right)y=x\sin x$ Ans: $y=C_1\cos x+C_2\sin x+\frac{x}{4}\sin x-\frac{x^2}{4}\cos x$

2. $\left(D^2 - 2D\right)y = e^x \sin x$ Ans: $y = C_1 + C_2 e^x - \frac{e^x}{2}\sin x$

3. $\frac{d^2y}{dx^2} + y = \sec x \tan x$ Ans: $y = C_1 \cos x + C_2 \sin x - \sin x + x\cos x + \sin x \log(\sec x)$

Session 8

Solve the following linear differential equation:

1. $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \log x$ Ans: $y = x^2\left[C_1 \cos(\log x) + C_2 \sin(\log x)\right] + x^2 \log x$
2. $x^3 \frac{d^2y}{dx^2} + 3x^2 \frac{dy}{dx} + xy = \sin(\log x)$ Ans: $y = \frac{1}{x}\left[C_1 + C_2 \log x - \sin(\log x)\right]$
3. $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin(\log x^2)$ Ans: $y = C_1 \cos(\log x) + C_2 \sin(\log x) - \frac{1}{3}\sin(\log x^2)$

Session 9

Find the power series solution in powers of x of the differential equation

1. $(1-x^2)y'' - 2xy' + 2y = 0$ Ans: $y = c_1 x + c_0\left[1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \ldots\ldots\right]$
2. $xy'' + y' + 2y = 0 \quad with\, y(1) = 2, y'(1) = 4$

   Ans: $y = 2 + 4(x-1) - 4(x-1)^2 + \frac{4}{3}(x-1)^3 - \frac{1}{3}(x-1)^4 + \ldots\ldots$