JEE Main 2020 Paper

Date: 9th January 2020 (Shift 2)
Time: 2:30 P.M. to 5:30 P.M.
Subject: Mathematics

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1. If $f(x) = \begin{vmatrix} x+a & x+2 & x+1 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix}$ and $a - 2b + c = 1$ then

   a. $f(-50) = -1$ b. $f(50) = 1$

   c. $f(50) = -501$ d. $f(50) = 501$

   Answer: (a)

   Solution:

   Given $f(x) = \begin{vmatrix} x+a & x+2 & x+1 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix}$

   $a - 2b + c = 1$

   Applying $R_1 \to R_1 - 2R_2 + R_3$

   $f(x) = \begin{vmatrix} a-2b+c & 0 & 0 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix}$

   Using $a - 2b + c = 1$

   $\therefore f(x) = (x+3)^2 - (x+2)(x+4)$

   $\Rightarrow f(x) = 1$

   $\Rightarrow f(50) = 1$

   $\Rightarrow f(-50) = 1$

2. If $f(x) = \begin{cases} x & 0 < x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ 1 - x & \frac{1}{2} < x < 1 \end{cases}$

   $g(x) = \left(x - \frac{1}{2}\right)^2$ then find the area bounded by $f(x)$ and $g(x)$ from $x = \frac{1}{2}$ to $x = \frac{\sqrt{3}}{2}$.

   a. $\frac{\sqrt{3}}{2} - \frac{1}{3}$ b. $\frac{\sqrt{3}}{4} + \frac{1}{3}$

   c. $2\sqrt{3}$ d. $3\sqrt{3}$

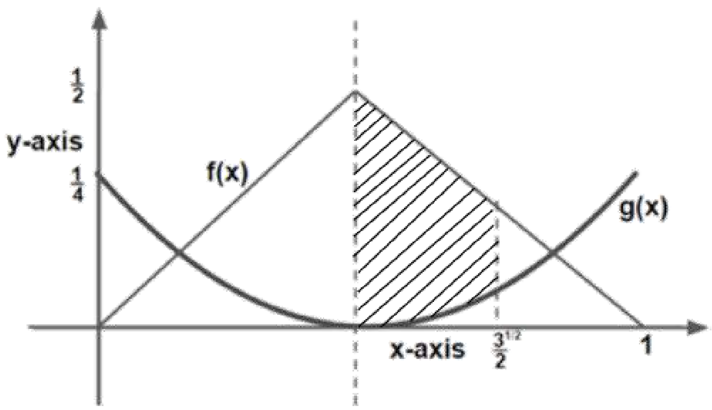
Answer: (a)

Solution:

Given $f(x) = \begin{cases} x & 0 < x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ 1 - x & \frac{1}{2} < x < 1 \end{cases}$

$g(x) = (x - \frac{1}{2})^2$

The area between $f(x)$ and $g(x)$ from $x = \frac{1}{2}$ to $= \frac{\sqrt{3}}{2}$:



Points of intersection of $f(x)$ and (x) :

$1 - x = \left(x - \frac{1}{2}\right)^2$

$\Rightarrow x = \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$

Required area $= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} (f(x) - g(x))dx$

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \left(1 - x - \left(x - \frac{1}{2}\right)^2\right) dx$$

$$= x - \frac{x^2}{2} - \frac{1}{3}\left(x - \frac{1}{2}\right)^3 \Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\sqrt{3}}{4} - \frac{1}{3}$$

3. If $p \to (p \wedge \sim q)$ is false. Truth value of p and q will be

a. TF

b. FT

c. TT

d. FF

Answer: (c)

Solution:

Given $p \to (p \wedge \sim q)$

Truth table:

p	q	$\sim q$	$(p \wedge \sim q)$	$p \to (p \wedge \sim q)$
T	T	F	F	F
T	F	T	T	T
F	T	F	F	T
F	F	T	F	T

$p \to (p \wedge \sim q)$ is false when p is true and q is true.

4. $\int \frac{d\theta}{\cos^2\theta\ (\sec 2\theta + \tan 2\theta)} = \lambda \tan\theta + 2\log f(x) + c$ then ordered pair $(\lambda,\ f(x))$ is

a. $(1, 1+\tan\theta)$

b. $(1, 1-\tan\theta)$

c. $(-1, 1+\tan\theta)$

d. $(-1, 1-\tan\theta)$

Answer: (c)

Solution:

Let $I = \int \frac{d\theta}{\cos^2\theta(\sec 2\theta + \tan 2\theta)}$

$$I = \int \frac{\sec^2\theta d\theta}{\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right) + \left(\frac{2\tan\theta}{1-\tan^2\theta}\right)}$$

$$I = \int \frac{(1-\tan^2\theta)(\sec^2\theta)d\theta}{(1+\tan\theta)^2}$$

Let $\tan\theta = k \Rightarrow \sec^2\theta\ d\theta = \mathrm{d}k$

$$I = \int \frac{(1-k^2)}{(1+k)^2}dk = \int \frac{(1-k)}{(1+k)}\ dk$$

$$I = \left(\frac{2}{1+k} - 1\right)dk$$

$$I = 2\ln|1+k| - k + c$$

$$I = 2\ln|1+\tan\theta| - \tan\theta + c$$

Given $I = \lambda \tan\theta + 2\log f(x) + c$

$\therefore \lambda = -1, f(x) = |1 + tan\theta|$

5. Let a_n is a positive term of GP and $\sum_{n=1}^{100} a_{2n+1} = 200$, $\sum_{n=1}^{100} a_{2n} = 100$, then the value of $\sum_{n=1}^{200} a_n$ is

a. 150
b. 225
c. 300
d. 175

Answer: (a)

Solution:

a_n is a positive term of GP.

Let GP be $a, ar, ar^2, \ldots..$

$\sum_{n=1}^{100} a_{2n+1} = a_3 + a_5 + \ldots\ldots + a_{201}$

$200 = ar^2 + ar^4 + \ldots\ldots + ar^{201}$

$200 = \frac{ar^2(r^{200}-1)}{r^2-1} \quad \ldots(1)$

Also, $\sum_{n=1}^{100} a_{2n} = 100$

$100 = a_2 + a_4 + \ldots \ldots + a_{200}$

$100 = ar + ar^3 + \ldots\ldots + ar^{199}$

$100 = \frac{ar(r^{200}-1)}{r^2-1} \quad \ldots(2)$

From (1) and (2), r = 2

And $\sum_{n=1}^{100} a_{2n+1} + \sum_{n=1}^{100} a_{2n} = 300$

$\Rightarrow a_2 + a_3 + a_4 \ldots \ldots + a_{200} + a_{201} = 300$

$\Rightarrow ar + ar^2 + ar^3 + \ldots\ldots + ar^{200} = 300$

$\Rightarrow r(a + ar + ar^2 + \ldots + ar^{199}) = 300$

$\Rightarrow 2(a_1 + a_2 + a_3 + \ldots\ldots + a_{200}) = 300$

$\sum_{n=1}^{200} a_n = 150$

6. z is a complex number such that $|\text{Re}(z)| + |\text{Im}(z)| = 4$, then $|z|$ cannot be equal to

a. $\sqrt{8}$
b. $\sqrt{7}$
c. $\sqrt{\frac{17}{2}}$
d. $\sqrt{10}$

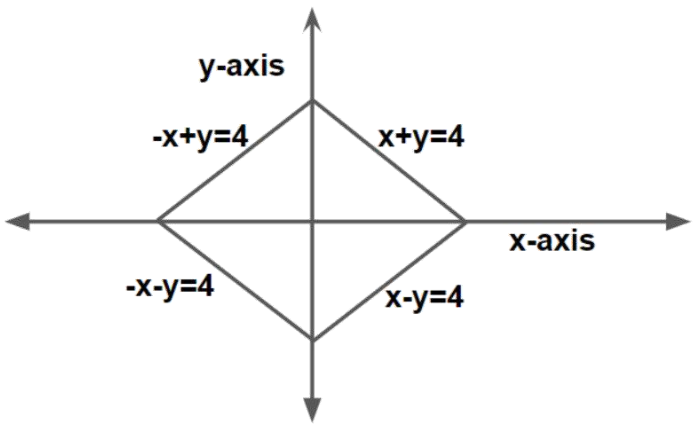
Answer: (b)

Solution:

$|\text{Re}(z)| + |\text{Im}(z)| = 4$

Let $z = x + iy$

$\Rightarrow |x| + |y| = 4$



$\therefore$ z lies on the rhombus.

Maximum value of $|z| = 4$ when $z = 4, -4, 4i, -4i$

Minimum value of $|z| = 2\sqrt{2}$ when $z = 2 \pm 2i,\ \pm 2 + 2i$

$|z| \in [2\sqrt{2}, 4]$

$|z| \in [\sqrt{8}, \sqrt{16}]$

$|z| \neq \sqrt{7}$

7. $f(x) : [0,5] \to R, F(x) = \int_0^x x^2\, g(x) dx$, $f(1) = 3, g(x) = \int_1^x f(t) dt$ then correct choice is

   a. $F(x)$ has no critical point
   b. $F(x)$ has local minimum at $x = 1$
   c. $F(x)$ has local maximum at $x = 1$
   d. $F(x)$ has point of inflection at $x = 1$

Answer: (b)

Solution:

$F(x) = x^2 g(x)$

Put $x = 1$

$\Rightarrow F(1) = g(1) = 0$ ... (1)

Now $F''(x) = 2xg(x) + g'(x)x^2$

$F''(1) = 2g(1) + g'(1)$ $\{\because g'(x) = f(x)\}$

$F''(1) = f(1) = 3$ ... (2)

From (1) and (2), $F(x)$ has local minimum at $x = 1$

8. Let $x = 2\sin\theta - \sin 2\theta$ and $y = 2\cos\theta - \cos 2\theta$, then the value of $\frac{d^2y}{dx^2}$ at $\theta = \pi$ is

a. $\frac{3}{8}$

b. $\frac{5}{8}$

c. $\frac{7}{8}$

d. $\frac{3}{2}$

Answer: (a)

Solution:

$$\frac{dx}{d\theta} = 2\cos\theta - 2\cos 2\theta$$

$$\frac{dy}{d\theta} = -2\sin\theta + 2\sin 2\theta$$

$$\frac{dy}{dx} = \frac{2\cos\frac{3\theta}{2}\sin\frac{\theta}{2}}{2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}}$$

$$\frac{dy}{dx} = \cot\frac{3\theta}{2}$$

$$\frac{d^2y}{dx^2} = -\frac{3}{2}\ cosec^2\frac{3\theta}{2}\ \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \left(-\frac{3}{2}\ cosec^2\frac{3\theta}{2}\right)\frac{1}{(2\cos\theta - 2\cos 2\theta)}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi} = \frac{3}{8}$$

9. If $f(x)$ and $g(x)$ are continuous functions, fog is identity function, $g'(b) = 5$ and $g(b) = a$, then $f'(a)$ is

a. $\frac{3}{5}$

b. 5

c. $\frac{2}{5}$

d. $\frac{1}{5}$

Answer: (d)

Solution:

$f(g(x)) = x$

$f'(g(x))g'(x) = 1$

Put $x = b$

$f'(g(b))g'(b) = 1$

$f'(a) \times 5 = 1$

$f'(a) = \frac{1}{5}$

10. Let $x + 6y = 8$ is tangent to standard ellipse where minor axis is $\frac{4}{\sqrt{3}}$, then eccentricity of ellipse is

a. $\frac{1}{4}\sqrt{\frac{11}{12}}$

b. $\frac{1}{4}\sqrt{\frac{11}{3}}$

c. $\sqrt{\frac{5}{6}}$

d. $\sqrt{\frac{11}{12}}$

Answer: (d)

Solution:

If $2b = \frac{4}{\sqrt{3}}$

$b = \frac{2}{\sqrt{3}}$

Comparing $y = -\frac{x}{6} + \frac{8}{6}$ with $y = mx \pm \sqrt{a^2m^2 + b^2}$

$m = -\frac{1}{6}$ and $a^2m^2 + b^2 = \frac{16}{9}$

$\frac{a^2}{36} + \frac{4}{3} = \frac{16}{9}$

$\Rightarrow \frac{a^2}{36} = \frac{16}{9} - \frac{4}{3}$

$\Rightarrow a^2 = 16$

$e = \sqrt{1 - \frac{b^2}{a^2}}$

$\Rightarrow e = \sqrt{\frac{11}{12}}$

11. If one end of focal chord of parabola $y^2 = 8x$ is $\left(\frac{1}{2}, -2\right)$, then the equation of tangent at the other end of this focal chord is

a. $x + 2y + 8 = 0$
b. $x - 2y = 8$
c. $x - 2y + 8 = 0$
d. $x + 2y = 8$

Answer: (c)

Solution:

Let PQ be the focal chord of the parabola $y^2 = 8x$

$\Rightarrow P(t_1) = (2t_1^2, 4t_1)$ & $Q(t_2) = (2t_2^2, 4t_2)$

$\Rightarrow t_1 t_2 = -1$

$\because \left(\frac{1}{2}, -2\right)$ is one of the ends of the focal chord of the parabola

Let $\left(\frac{1}{2}, -2\right) = (2t_2^2, 4t_2)$

$\Rightarrow t_2 = -\frac{1}{2}$

$\Rightarrow$ Other end of focal chord will have parameter $t_1 = 2$

$\Rightarrow$ The co-ordinate of the other end of the focal chord will be $(8, 8)$

$\therefore$ The equation of the tangent will be given as $\rightarrow$ $8y = 4(x + 8)$

$\Rightarrow 2y - x = 8$

12. If $7x + 6y - 2z = 0$, $3x + 4y + 2z = 0$ & $x - 2y - 6z = 0$, then the system of equations has

a. No solution
b. Infinite non-trivial solution for $(x = 2z)$
c. Infinite non-trivial solution for $(y = 2z)$
d. Only trivial solution

Answer: (b)

Solution:

$7x + 6y - 2z = 0$

$3x + 4y + 2z = 0$

$x - 2y - 6z = 0$

As the system of equations are Homogeneous ⇒ the system is consistent.

$$\Rightarrow \begin{vmatrix} 7 & 6 & -2 \\ 3 & 4 & 2 \\ 1 & -2 & -6 \end{vmatrix} = 0$$

⇒ Infinite solutions exist (both trivial and non-trivial solutions)

When $y = 2z$

Let's take $y = 2, z = 1$

When $(x, 2,1)$is substituted in the system of equations

$\Rightarrow 7x + 10 = 0$

$3x + 10 = 0$

$x - 10 = 0$ (which is not possible)

$\therefore y = 2z \Rightarrow$ Infinite non-trivial solutions does not exist.

For $x = 2z$, lets take $x = 2, z = 1, y = y$

Substitute $(2, y, 1)$in system of equations

$\Rightarrow y = -2$

$\therefore$For each pair of (x, z), we get a value of y.

Therefore, for $x = 2z$ infinite non-trivial solution exists.

13. If both the roots of the equation $ax^2 - 2bx + 5 = 0$ are α and of the equation $x^2 - 2bx - 10 = 0$ are α and β. Then the value of $\alpha^2 + \beta^2$

a. 15
b. 20
c. 25
d. 30

Answer: (c)

Solution:

$ax^2 - 2bx + 5 = 0$ has both roots as α

$\Rightarrow 2\alpha = \frac{2b}{a} \Rightarrow \alpha = \frac{b}{a}$

And $\alpha^2 = \frac{5}{a}$

$\Rightarrow b^2 = 5a (a \neq 0)$ ... (1)

$\Rightarrow \alpha + \beta = 2b$ & $\alpha\beta = -10$

$\alpha = \frac{b}{a}$ is also a root of $x^2 - 2bx - 10 = 0$

$\Rightarrow b^2 - 2ab^2 - 10a^2 = 0$

$\because b^2 = 5a \Rightarrow 5a - 10a^2 - 10a^2 = 0$

$\Rightarrow a = \frac{1}{4} \Rightarrow b^2 = \frac{5}{4}$

$\Rightarrow \alpha^2 = 20, \beta^2 = 5$

$\Rightarrow \alpha^2 + \beta^2 = 25$

14. If $A = \{x: |x| < 2$ and $B = \{x: |x - 2| \geq 3\}$ then

a. $A \cap B = [-2, -1]$
b. $B - A = \mathbf{R} - (-2,5)$
c. $A - B = [-1,2)$
d. $A \cup B = \mathbf{R} - (2,5)$

Answer: (b)

Solution:

$A = \{x: x \in (-2,2)\}$

$B = \{x: x \in (-\infty, -1] \cup [5, \infty)\}$

$A \cap B = \{x: x \in (-2, -1]\}$

$B - A = \{x: x \in (-\infty, -2] \cup [5, \infty)\}$

$A - B = \{x: x \in (-1,2)\}$

$A \cup B = \{x: x \in (-\infty, 2) \cup [5, \infty)\}$

15. The value of $P(x_i > 2)$ for the given probability distribution is

x_i	1	2	3	4	5
P_i	k^2	$2k$	k	$2k$	$5k^2$

a. $\frac{1}{36}$
b. $\frac{23}{36}$
c. $\frac{1}{6}$
d. $\frac{7}{12}$

Answer: (b)

Solution:

We know that $\sum_{x_i=1}^{5} P_i = 1$

$\Rightarrow k^2 + 2k + k + 2k + 5k^2 = 1$

$\Rightarrow k = -1, \frac{1}{6} \therefore k = \frac{1}{6}$

$P(x_i > 2) = P(x_i = 3) + P(x_i = 4) + P(x_i = 5)$

$= k + 2k + 5k^2 = \frac{23}{36}$

16. Let the distance between the plane passing through lines $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+1}{8}$ and $\frac{x+3}{2} = \frac{y+2}{1} = \frac{z-1}{\lambda}$ and the plane $23x - 10y - 2z + 48 = 0$ is $\frac{k}{\sqrt{633}}$, then the value of k is

a. 4
b. 3
c. 2
d. 1

Answer: (b)

Solution:

We find the point of intersection of the two lines, and the distance of given plane from the two lines is the distance of plane from the point of intersection.

$\therefore (2p - 1, 2p + 3, 8p - 1) = (2q - 3, q - 2, \lambda q + 1)$

$p = -2$ and $q = -1$

$\lambda = 18$

Point of intersection is $(-5, -3, -17)$

$$\therefore \frac{k}{\sqrt{633}} = \left|\frac{-115 + 30 + 34 + 48}{\sqrt{633}}\right| \Rightarrow k = 3$$

17. Let $x = \sum_{n=0}^{\infty}(-1)^n(\tan\theta)^{2n}$ and $y = \sum_{n=0}^{\infty}(\cos\theta)^{2n}$, where $\theta \in \left(0, \frac{\pi}{4}\right)$, then

a. $y(x-1) = 1$

b. $y(1-x) = 1$

c. $x(y+1) = 1$

d. $y(1+x) = 1$

Answer: (b)

Solution:

$y = 1 + \cos^2\theta + \cos^4\theta + \cdots$

$\Rightarrow y = \dfrac{1}{1-\cos^2\theta} \Rightarrow \dfrac{1}{y} = \sin^2\theta$

$x = 1 - \tan^2\theta + \tan^4\theta - \cdots$

$\Rightarrow x = \dfrac{1}{1-(-\tan^2\theta)} = \cos^2\theta$

$\therefore x + \dfrac{1}{y} = 1 \Rightarrow y(1-x) = 1$

18. If $\lim_{x\to 0} x\left[\frac{4}{x}\right] = A$, then the value of x at which $f(x) = [x^2]\sin\pi x$ is discontinuous at (where [.] denotes greatest integer function)

a. $\sqrt{A+5}$

b. $\sqrt{A+1}$

c. $\sqrt{A+21}$

d. $\sqrt{A}$

Answer: (b)

Solution:

$f(x) = [x^2]\sin\pi x$

It is continuous $\forall x \in \mathbf{Z}$ as $\sin\pi x \to 0$ as $x \to \mathbf{Z}$.

$f(x)$ is discontinuous at points where $[x^2]$ is discontinuous i.e. $x^2 \in \mathbf{Z}$ with an exception that $f(x)$ is continuous as x is an integer.

$\therefore$ Points of discontinuity for $f(x)$ would be at

$x = \pm\sqrt{2}, \pm\sqrt{3}, \pm\sqrt{5}, \dots\dots$

Also, it is given that $\lim_{x\to 0} x\left[\frac{4}{x}\right] = A$ (indeterminate form $(0\times\infty)$)

$\Rightarrow \lim_{x\to 0} x\left(\frac{4}{x} - \left\{\frac{4}{x}\right\}\right) = A$

$\Rightarrow 4 - \lim_{x\to 0}\left\{\frac{4}{x}\right\} = A$

$\Rightarrow A = 4$

$\sqrt{A+5} = 3$

$\sqrt{A+1} = \sqrt{5}$

$\sqrt{A+21} = 5$

$\sqrt{A} = 2$

$\therefore$ Points of discontinuity for $f(x)$ is $x = \sqrt{5}$

19. Circles $(x-0)^2 + (y-4)^2 = k$ and $(x-3)^2 + (y-0)^2 = 1^2$ touch each other. The maximum value of k is __________.

Answer: (36)

Solution:

Two circles touch each other if $C_1C_2 = |r_1 \pm r_2|$

$\sqrt{k} + 1 = 5$ or $\left|\sqrt{k} - 1\right| = 5$

$\Rightarrow k = 16$ or 36

Maximum value of k is 36

20. If ${}^{25}C_0 + 5\,{}^{25}C_1 + 9\,{}^{25}C_2 + \cdots + 101\,{}^{25}C_{25} = 2^{25}k$, then the value of k is ________.

Answer: (51)

Solution:

$S = {}^{25}C_0 + 5\,{}^{25}C_1 + 9\,{}^{25}C_2 + \cdots + 97\,{}^{25}C_{24} + 101\,{}^{25}C_{25} = 2^{25}k \qquad (1)$

Reverse and apply property ${}^{n}C_r = {}^{n}C_{n-r}$ in all coefficients

$S = 101\,{}^{25}C_0 + 97\,{}^{25}C_1 + \cdots + 5\,{}^{25}C_{24} + {}^{25}C_{25} \qquad (2)$

Adding (1) and (2), we get

$2S = 102[{}^{25}C_0 + {}^{25}C_1 + \cdots + {}^{25}C_{25}]$

$S = 51 \times 2^{25}$

$\Rightarrow k = 51$

21. Number of common terms in both the sequences $3, 7, 11, \ldots 407$ and $2, 9, 16, \ldots 905$ is ________.

Answer: (14)

Solution:

First common term is 23

Common difference $= \text{LCM}(7, 4) = 28$

$23 + (n-1)28 \leq 407$

$n - 1 \leq 13.71$

$n = 14$

22. Let $|\vec{a}| = \sqrt{3}, |\vec{b}| = 5, \vec{b}.\vec{c} = 10$ and angle between $\vec{b}$ and $\vec{c}$ is equal to $\frac{\pi}{3}$. If $\vec{a}$ is perpendicular to $\vec{b} \times \vec{c}$, then the value of $|\vec{a} \times (\vec{b} \times \vec{c})|$ is

Answer: (30)

Solution:

$|\vec{a} \times (\vec{b} \times \vec{c})| = |\vec{a}||\vec{b} \times \vec{c}| \sin\theta$ where θ is the angle between $\vec{a}$ and $\vec{b} \times \vec{c}$

$\theta = \frac{\pi}{2}$ given

$\Rightarrow |\vec{a} \times (\vec{b} \times \vec{c})| = \sqrt{3}|\vec{b} \times \vec{c}| = \sqrt{3}|\vec{b}||\vec{c}| \sin\frac{\pi}{3}$

$\Rightarrow |\vec{a} \times (\vec{b} \times \vec{c})| = \sqrt{3} \times 5 \times |\vec{c}| \times \frac{\sqrt{3}}{2}$

$\Rightarrow |\vec{a} \times (\vec{b} \times \vec{c})| = \frac{15}{2}|\vec{c}|$

Now, $|\vec{b}||\vec{c}| \cos\theta = 10$

$5|\vec{c}|\frac{1}{2} = 10$

$|\vec{c}| = 4$

23. If minimum value of term free from x for $\left(\frac{x}{\sin\theta} + \frac{1}{x\cos\theta}\right)^{16}$ is L_1 in $\theta \in \left[\frac{\pi}{8}, \frac{\pi}{4}\right]$ and L_2 in $\theta \in \left[\frac{\pi}{16}, \frac{\pi}{8}\right]$, the value of $\frac{L_2}{L_1}$ is

Answer: (16)

Solution:

$$T_{r+1} = {}^{16}C_r \left(\frac{x}{\sin\theta}\right)^{16-r} \left(\frac{1}{x\cos\theta}\right)^r$$

For term independent of x,

$16 - 2r = 0 \Rightarrow r = 8$

$$T_9 = {}^{16}C_8 \left(\frac{1}{\sin\theta\cos\theta}\right)^8 = {}^{16}C_8 2^8 \left(\frac{1}{\sin 2\theta}\right)^8$$

$L_1 = {}^{16}C_8 2^8 \quad \text{at } \theta = \frac{\pi}{4}$

$L_2 = {}^{16}C_8 \frac{2^8}{\left(\frac{1}{\sqrt{2}}\right)^8} = {}^{16}C_8 2^{12} \quad \text{at } \theta = \frac{\pi}{8}$

$\frac{L_2}{L_1} = 16$