JEE Main Exam 2023 - Session 1

1 Feb 2023 - Shift 2 (Memory-Based Questions)

Section A: Physics

Q.1. Ratio of acceleration due to gravity on the surface of planet 1 to the surface of planet 2 is x, while the ratio of radii of respective planets is y. The ratio of respective escape velocity on the surface of planet 1 and planet 2 is equal to

A) $\sqrt{\frac{x}{y}}$ B) $\frac{x}{y}$ C) $\sqrt{xy}$ D) xy

Answer: $\sqrt{xy}$

Solution: The escape velocity from the surface of a planet is

$$v_e = \sqrt{\frac{2GM}{R}} = \sqrt{2gR} \quad (\text{since } g = G\frac{M}{R^2})$$

where M is the mass of planet, R is its radius and g is the acceleration due to gravity at the surface of planet.

If v_1 is the escape velocity from the surface of planet 1 and v_2 is the escape velocity from the surface of planet 2, then

$$\frac{v_1}{v_2} = \sqrt{\frac{2g_1R_1}{2g_2R_2}} = \sqrt{xy}$$

Hence, option C is correct.

Q.2. In a hydrogen atom, an electron makes a transition from 3^{rd} excited state to ground state. Find the energy of the photon emitted.

A) $10.8\ \text{eV}$ B) $13.6\ \text{eV}$ C) $12.75\ \text{eV}$ D) $8.6\ \text{eV}$

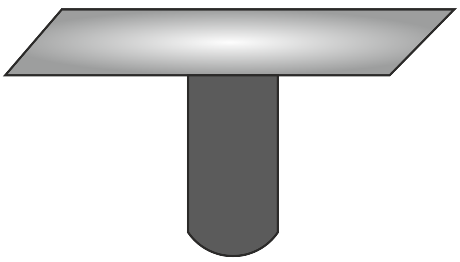
Answer: $12.75\ \text{eV}$

Solution: The energy of photon emitted E is

$$E = 13.6\left(\frac{1}{{n_2}^2} - \frac{1}{{n_1}^2}\right) eV$$
$$\Rightarrow E = 13.6\left(\frac{1}{1^2} - \frac{1}{4^2}\right) = 13.6 \times \frac{15}{16} = 12.75\ \text{eV}$$

Hence, option C is correct.

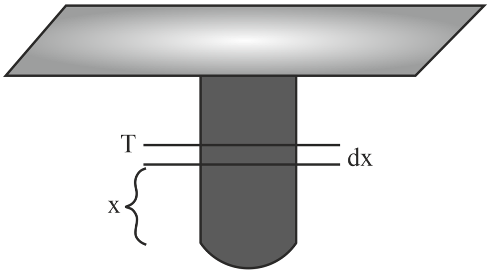
Q.3. A uniform rod of mass $10\ \text{kg}$ and length $6\ \text{m}$ is hanged from the ceiling as shown. Given that the area of cross-section of rod is $3\ \text{mm}^2$ and Young's modulus is $2 \times 10^{11}\ \text{N m}^{-2}$. Find extension in the rod's length. [use $g = 10\ \text{m s}^{-2}$]



A) $1\ \text{mm}$ B) $0.5\ \text{mm}$ C) $0.25\ \text{mm}$ D) $1.2\ \text{mm}$

Answer: $0.5\ \text{mm}$

Solution:



Consider an infinitesimal element of thickness dx at a height of x above the lower end of the rod. Tension T at height x is,

$T = \frac{m}{L} x g.$

If $d\varepsilon$ is the extension of element of thickness dx,

$Y = \frac{\frac{mgx}{AL}}{\frac{d\varepsilon}{dx}} \Rightarrow d\varepsilon = \frac{mgxdx}{YAL}$

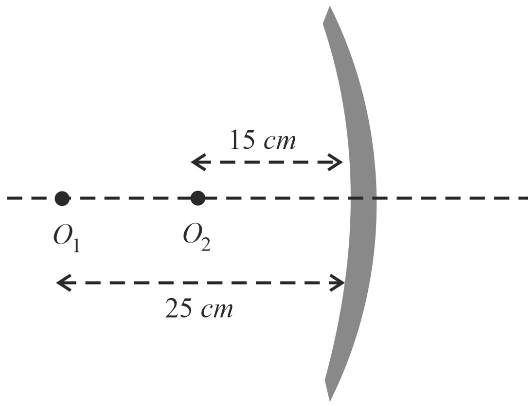
Therefore, the total extension ε is

$\varepsilon = \int_0^L \frac{mgxdx}{YAL} = \frac{mgL}{2YA}$

$\Rightarrow \varepsilon = \frac{10 \times 10 \times 6}{2 \times 2 \times 10^{11} \times 3 \times 10^{-6}} = 0.5 \text{ mm}$

Hence, option B is correct.

Q.4. Two point objects O_1 and O_2 are placed on principle axis of concave mirror of radius of curvature 40 cm. Find the distance between the two images.



A) 160 cm

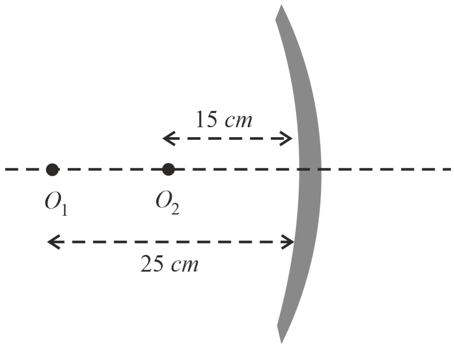
B) 40 cm

C) 100 cm

D) 80 cm

Answer: 160 cm

Solution:



Using mirror formula for object O_2

$$\frac{1}{v_2}+\frac{1}{u_2}=\frac{1}{f}$$
$$\Rightarrow \frac{1}{v_2}+\frac{1}{-15}=\frac{1}{-20}$$
$$\Rightarrow \frac{1}{v_2}=\frac{1}{15}+\frac{1}{-20}=\frac{1}{60}$$
$$\Rightarrow v_2=60 \text{ cm}$$

Using mirror formula for object O_1

$$\frac{1}{v_1}+\frac{1}{u_1}=\frac{1}{f}$$
$$\Rightarrow \frac{1}{v_1}+\frac{1}{-25}=\frac{1}{-20}$$
$$\Rightarrow \frac{1}{v_1}=\frac{1}{25}+\frac{1}{-20}=\frac{-1}{100}$$
$$\Rightarrow v_1=-100 \text{ cm}$$

Therefore, the distance between the two images is

$$v_2-v_1=60-(-100)=\ 160 \text{ cm}$$

Hence, option A is correct.

Q.5. For a heat engine based on carnot cycle, source is at temperature 600 K. Now if source temperature is doubled then efficiency also gets doubled while keeping the sink temperature same at $x \text{ K}$. Value of x is equal to

A) 400 K B) 600 K C) 200 K D) 300 K

Answer: 400 K

Solution: The efficiency of heat engine

$$\eta=\left(1-\frac{T_{\sin k}}{T_{source}}\right)$$

Since the efficiency gets doubled when the source temperature is doubled,

$$2\left(1-\frac{x}{600}\right)=1-\frac{x}{1200}$$

$$\Rightarrow 2-\frac{x}{300}=1-\frac{x}{1200}$$

$$\Rightarrow\ 1=\frac{x}{300}-\frac{x}{1200}$$

$$\Rightarrow 1=\frac{3x}{1200}$$

$$\Rightarrow x=400 \text{ K}$$

Hence, option A is correct.

Q.6. If universal gravitational constant (G) , Planck's constant (h) and speed of light in vacuum (c) are taken as fundamental quantities, then the dimension of mass is equal to

A) $\sqrt{\frac{Gh}{c}}$

B) $\sqrt{\frac{h}{Gc}}$

C) $\sqrt{\frac{G}{hc}}$

D) $\sqrt{\frac{hc}{G}}$

Answer: $\sqrt{\frac{hc}{G}}$

Solution: If E is energy, then

$[E]=\left[\frac{Gm_1m_2}{r}\right]$

$[E]\,[r]=[Gm_1m_2]$

Since $E=\frac{hc}{\lambda}$

$\left[\frac{hc}{\lambda}\right][r]=[Gm_1m_2]$

$\Rightarrow \frac{[hc]}{[L]}[L]=[G]\left[M^2\right]$

(both λ and r have dimensions of length)

$\therefore \left[\sqrt{\frac{hc}{G}}\right]=[M]$

Hence, option D is correct.

Q.7. For a photoelectric setup, the threshold frequency is f_0. For an incident frequency of $2f_0$, the stopping potential is V_1 and for an incident frequency of $5f_0$, the stopping potential is V_2. Find $\frac{V_1}{V_2}$.

A) $\frac{1}{5}$

B) $\frac{1}{2}$

C) $\frac{1}{3}$

D) $\frac{1}{4}$

Answer: $\frac{1}{4}$

Solution: The stopping potential V is related to the incident frequency f and the threshold frequency $f_\circ$ as:

$eV=hf-hf_\circ$

$\therefore eV_1=h\times(2f_0)-hf_0=hf_0$ ...(1)

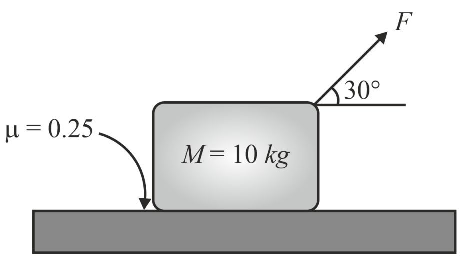
and $eV_2=h\times(5f_0)-hf_0=4hf_0$ ...(2)

Dividing equation (1) by (2), we get

$\frac{V_1}{V_2}=\frac{1}{4}$

Hence, D is the correct option.

Q.8. A block is acted upon by a force F as shown. If $M=10\ \mathrm{kg}$ and the coefficient of friction between the surface and the block is 0.25, find minimum magnitude of F so that the block slides.



A) $\frac{200}{4\sqrt{3}+1}$ N

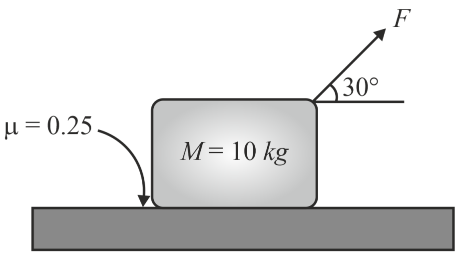
B) $\frac{200}{4\sqrt{3}-1}$ N

C) $\frac{100}{4\sqrt{3}+1}$ N

D) 50 N

Answer: $\frac{200}{4\sqrt{3}+1}$ N

Solution:



The normal reaction R between the surface and the block is

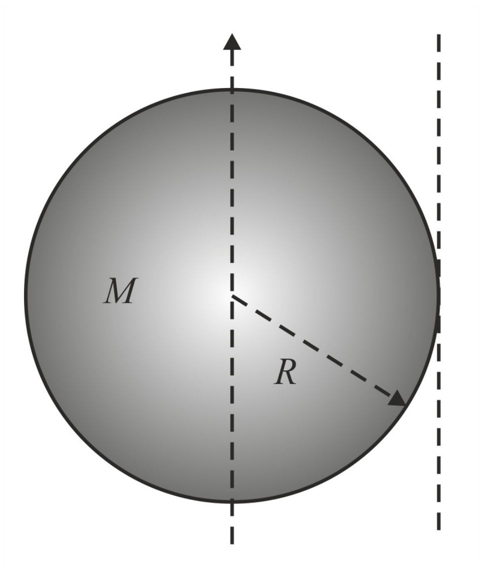
$R = mg - F\ \sin\ 30^{\circ}$

If force F is just enough to make the block slide,

$$F\cos 30^{\circ} - \mu R = 0$$
$$\Rightarrow F\cos 30^{\circ} + \mu F\sin 30^{\circ} - \mu mg = 0$$
$$\Rightarrow F = \frac{\mu mg}{\cos 30^{\circ}+\mu\sin 30^{\circ}}$$
$$= \frac{0.25\times10\times10}{\frac{\sqrt{3}}{2}+\frac{1}{4}\times\frac{1}{2}}$$
$$= \frac{200}{4\sqrt{3}+1}\ N$$

Hence, option A is correct.

Q.9. Moment of inertia of a uniform disc about diameter is $\frac{MR^2}{4}$, where M is mass and R is radius of disc. Find the moment of inertia about tangent parallel to diameter.



A) $\frac{3}{4}MR^2$

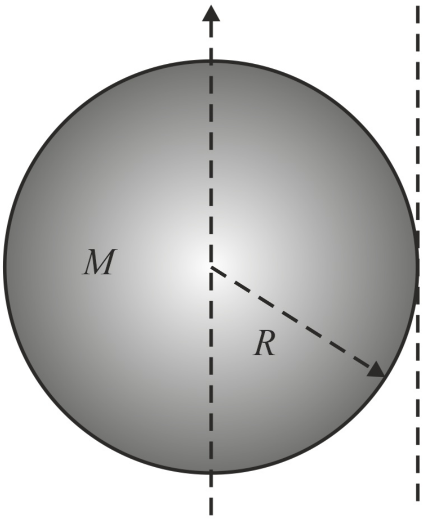
B) $\frac{5}{4}MR^2$

C) $\frac{3}{2}MR^2$

D) $\frac{5}{2}MR^2$

Answer: $\frac{5}{4}MR^2$

Solution:



Using parallel axis theorem, we can write

$I_{\tan gent}=\frac{MR^2}{4}+MR^2$

$=\frac{5}{4}MR^2$

Q.10. Which of the following is correct for zener diode.

(1) It acts as voltage regulator.

(2) It is used in forward bias.

(3) It is used in reverse bias.

(4) It is used as switch in series.

A) (1) and (4) B) (2) and (3) C) (1) and (3) D) (2) and (4)

Answer: (1) and (3)

Solution: Zener diode is a silicon semiconductor with a p-n junction that is specifically designed to work in the reverse biased condition. When forward biased, it behaves like a normal signal diode, but when the reverse voltage is applied to it, the voltage remains constant for a wide range of currents.

Q.11. If a force F applied on a object moving along y-axis varies with y-coordinate as, $F=5+3y^2$. The work done in displacing the body from $y=2\text{ m}$ to $y=5\text{ m}$ is

Answer: 132

Solution: Given: $F=5+3y^2$

As force is variable, we can write work done as

$w=\int_2^5 F\mathrm{d}y$

$=\int_2^5\left(5+3y^2\right)\mathrm{d}y$

$=5\left[5-2\right]+\left[y^3\right]_2^5$

$=5\times 3+\left[125-8\right]$

$=15+117$

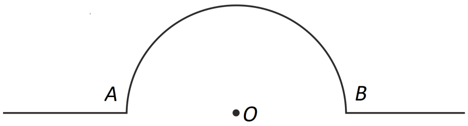
$=132$

Q.12. The electromagnetic wave, the ratio of energy(average) carried by electric field to that by magnetic field is

Answer: 1

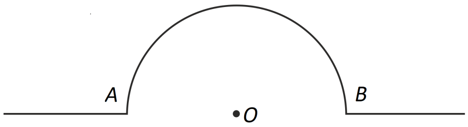
Solution: The energy in an electromagnetic wave is divided on average equally between magnetic and electric fields. Therefore, required ratio is $1:1$.

Q.13. An infinite wire is bent in the shape as shown in the figure with portion AOB being semi-circular of radius R. If current i flows through the wire then magnetic field at the centre O is equal to $\frac{\mu_0 i}{kR}$. Value of k is equal to



Answer: 4

Solution:



Magnetic field due to straight part of the wire will be zero at point O as it lies in the same line.

So magnetic field will be only due to the semi-circular part of the wire,

$B = \frac{\mu_0 i}{4R}$.

Therefore, $k = 4$.

Section B: Chemistry

Q.1. Which of the following option contains the Nessler's reagent?

A) $K_2[HgI_4]$ B) $K_2Cr_2O_7$ C) $K_4[Fe(CN)_6]$ D) $K_3[Cu(CN)_4]$

Answer: $K_2[HgI_4]$

Solution: Nessler's reagent, $K_2[HgI_4]$ is used to detect ammonia. It is a pale solution which gets darker yellow when some amount of ammonia is added. A brown precipitate may form at higher concentration. Nessler's reagent can be crystallized from a concentrated solution of mercury iodide with potassium iodide

Q.2. Find out depression in freezing point (T_f) for CH_3COOH $(\alpha = 20\%)$ dissolved in aqueous solution having 10% (w/w) CH_3COOH in solution.
Given K_f of water $= 1.86 \frac{\text{Kkg}}{\text{mole}}$

A) 4.13 K B) 2.13 K C) 1.13 K D) 0.13 K

Answer: 4.13 K

Solution: molality of solution $= \frac{\text{moles of solute}}{\text{mass of solvent in kg}}$

molality(m) $= \frac{10 \times 1000}{60 \times 90} = \frac{100}{54}$

Now, the expression for depression in freezing point is

$\Delta T_f = i \times K_f \times m$

For dissociation, Van't Hoff factor $i = 1 + (n - 1)\alpha$

where $n =$Number of ions and $\alpha =$Degree of ionisation.

$i = 1.2$

$\Delta T_f = 1.2 \times 1.86 \times \frac{100}{54} = 4.13$ K

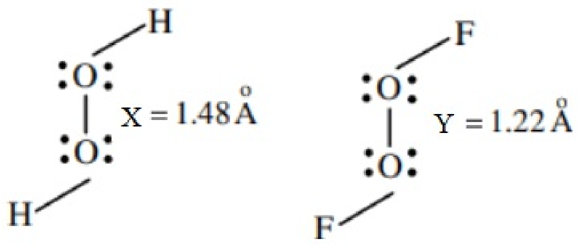
Q.3. Consider the H_2O_2 and O_2F_2 molecules where X and Y are bond lengths in H_2O_2 and O_2F_2 respectively. Compare X and Y.

A) $X > Y$ B) $X < Y$ C) $X = Y$

D) X and Y cannot be compared

Answer: $\mathrm{X} > \mathrm{Y}$

Solution: O_2F_2 has the similar structure as that of H_2O_2 but the $\mathrm{O-O}$ bond length is shorter in O_2F_2 than in H_2O_2. This is due to the high electronegativity of attached F-atoms.
H_2O_2 is a covalent compound.



Q.4. The spin only magnetic moment of Mn^{2+} in $[Mn(H_2O)_6]^{2+}$

A) 2.87 BM

B) 3.87 BM

C) 5.91 BM

D) 1.73 BM

Answer: 5.91 BM

Solution: Step 1: The electronic configuration of Mn:

- First, write the electronic configuration of Mn (the atomic number is 25):

Mn: $1s^2 2s^2 2p^6 3s^2 3p^6 3d^5 4s^2$

Step 2: The electronic configuration of Mn^{2+}

- Now, after removing 2 electrons from. Mn^{2+} will be formed. Its electronic configuration will be

Mn^{2+}: $1s^2 2s^2 2p^6 3s^2 3p^6 4s^0 3d^5$

- The 2 electrons will be removed from the outermost shell of Mn which is the 4th shell respectively.

Step 3: Calculate the spin-only magnetic moment

- The spin magnetic moment is a magnetic moment caused due to the spin of particles.
- The formula to calculate spin only magnetic moment is
- $\mu = \sqrt{n(n+2)}$

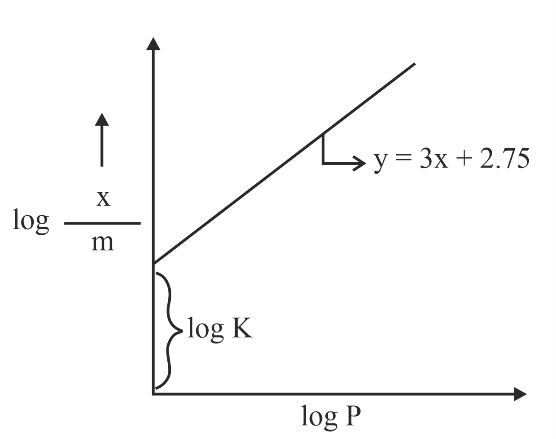
Where $n =$ number of unpaired electrons.
The unpaired electrons of Mn^{2+}, by the shell configuration $\uparrow\uparrow\uparrow\uparrow$ will be 5 .
So, $n = 5$.
The spin magnetic moment will be:

Spin magnetic moment $= \sqrt{5(5+2)}$

$= \sqrt{5 \times 7}$

$= 5.916\ \mathrm{B.M}$

Q.5. Consider the given graph. Find the value of $\frac{1}{n} + \log\ K$



A) 2.75 B) 3.75 C) 6.75 D) 5.75

Answer: 5.75

Solution: The expression for the Freundlich isotherm can be represented by the following equation

$\frac{x}{m} = KP^{1/n}$

Taking log on both sides

$\log \frac{x}{m} = \log K + \frac{1}{n} \log P$

On comparing with given euation of straight line, we get:

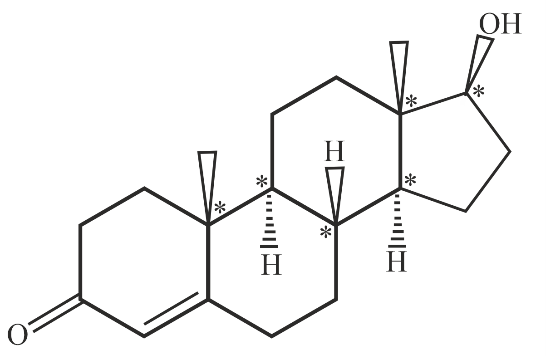
$\log K = 2.75$ and $\frac{1}{n} = 3$

On adding, we get, 2.75 + 3 = 5.75

Q.6. Number of chiral carbons in one molecule of testosterone is?

Answer: 6

Solution: The chiral carbons are marked in the structure that have all four different groups attached to that particular carbon



Testosterone

Q.7. Which of the following acts as a tranquilizer?

A) Aminoglycoside B) Chloramphenicol C) Asprin D) Valium

Answer: Valium

Solution: The barbituric acid derivatives are tranquillizers. They are used to treat mental illness, anxieties, tension and sleeplessness. They are often called depressants because they suppress the central nervous system and slow the body down.The examples are diazepam (Valium), chlordiazepoxide (Librium), and alprazolam (Xanax).

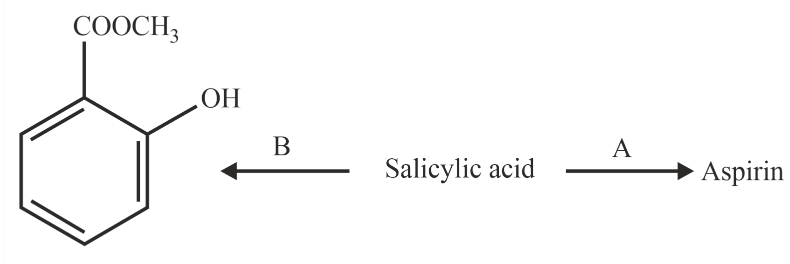
Q.8. Which of the following order is correct regarding magnitude of first electron gain enthalpy.

A) $Cl < F$ B) $O < S$ C) $Te < O$ D) $S < Se$

Answer: $O < S$

Solution: As the size of the atom increases, the electron gain enthalpy decreases, and thus the electron gain enthalpies of the elements decrease down the group.
But exceptionally, sulphur is known to have a higher negative electron gain enthalpy compared to oxygen, which is due to the smaller structure of oxygen, which contributes to the interelectronic repulsion between valence electrons and any other electrons that approach.

Q.9. Consider the given reaction. Find the correct statement regarding A and B.



A) A. Methanol/ H^+
B. Ethanoic anhydride

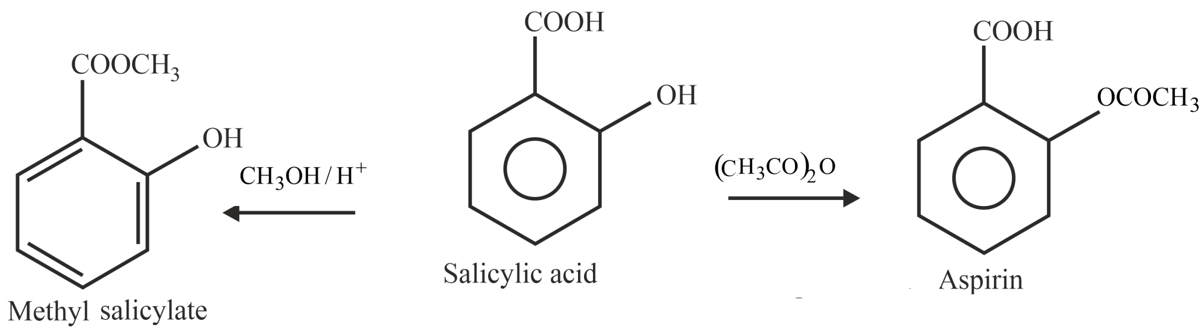
B) A. Ethanol/H^+
B. Ethanoic anhydride

C) A. Ethanoic anhydride
B. Methanol/ H^+

D) A. Ethanoic anhydride
B. Ethanol/ H^+

Answer: A. Ethanoic anhydride
B. Methanol/ H^+

Solution: When salicylic acid is heated with methyl alcohol, the carboxyl group of salicylic acid is esterified producing a strong-smelling liquid ester (methyl salicylate). Aspirin is prepared by chemical synthesis from salicylic acid, through acetylation with acetic anhydride.



Q.10. Which of the following given complex has 2 isomers.

A) $[Co(NH_3)_5NO_2]^{2+}$ B) $[Co(NH_3)_5Cl]^{2+}$ C) $[Co(NH_3)_5I]^{2+}$ D) $[Co(NH_3)_5Br]^{2+}$

Answer: $[Co(NH_3)_5NO_2]^{2+}$

Solution: The complex shows linkage isomerism in the presence of ambidentate ligans.

$[Co(NH_3)_5NO_2]^{2+}$ shows linkage isomerism, because NO_2^- ligand is ambidentate ligand. The other linkage isomer of the above complex is $[Co(NH_3)_5ONO]^{2+}$.

Q.11. An atom forms two lattices FCC and BCC. The edge length of FCC lattice is 2.5Å and edge length of BCC lattice is 2Å . Then find the ratio of density of FCC to density of BCC. Round of to the nearest integer.

Answer: 1

Solution: The density of crystal, $\rho = \frac{nM}{N_0 a^3}$ g/ cm^3

n = The number of effective atoms

M = Molar mass

N_0 = Avogadro's number

a = Edge length of the unit cell

For FCC, $n = 4$

Given that $a = 2.5 \times 10^{-8}$ cm

So, $\rho_1 = \frac{4 \times M}{N_0 \times (2.5)^3 \times \left(10^{-8}\right)^3}$ g/ cm^3 $\quad \cdots$(i)

For BCC, $n = 2$

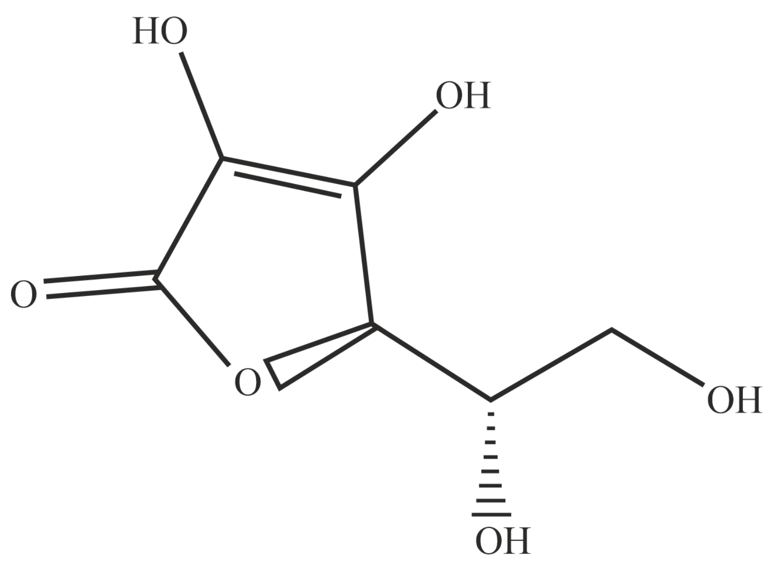
Given that $a = 2 \times 10^{-8}$ cm

So, $\rho_2 = \frac{2 \times M}{N_0 \times (2)^3 \times \left(10^{-8}\right)^3}$ g/ cm^3 $\quad \cdots$(ii)

so, (i)/ (ii)

$\Rightarrow \frac{\rho(\text{fcc})}{\rho(\text{bcc})} = \frac{4 \times (2)^3}{2 \times (2.5)^3} = \frac{32}{31.25} = 1.024$

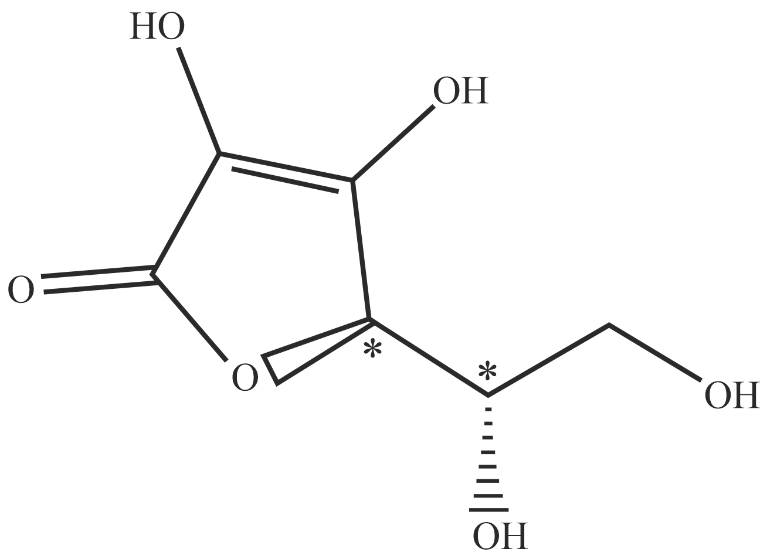
Q.12. Find the number of asymmetric carbon in the structure of Vitamin C



Vitamin C

Answer: 2

Solution: An asymmetric carbon atom (chiral carbon) is a carbon atom that is attached to four different types of atoms or groups of atoms. The carbons marked with * are assymmetric carbons.



Vitamin C

Q.13. For a first order reaction, half life $\left(t_{1/2}\right)$ is $50\,\mathrm{min}$, find $t_{3/4}$ (in minutes) of the reaction.

Answer: 100

Solution: The half life is the time required to completion of 50% of the reaction.

$\mathrm{t}_{50\%} = 50\ \mathrm{min}$

$\mathrm{t}_{3/4}$ means the time required to completion of 75% of the reaction.

$\mathrm{t}_{75\%} = 2 \times \mathrm{t}_{50\%}$(The half life of the first order reaction is independent of initial amount of the substance).

$\mathrm{t}_{75\%} = 2 \times 50 = 100\ \mathrm{min}$

Section C: Mathematics

Q.1. Term independent of x in $\left(x^{\frac{2}{3}} + \frac{\alpha}{x^3}\right)^{22}$ is 7315, then $|\alpha| =$

A) 1 B) 2 C) 4 D) 6

Answer: 1

Solution:

General term in the binomial expansion of $\left(x^{\frac{2}{3}}+\frac{\alpha}{x^{3}}\right)^{22}$ is

$T_{r+1}={}^{22}C_{r}\left(x^{\frac{2}{3}}\right)^{22-r}\left(\frac{\alpha}{x^{3}}\right)^{r}$

$\Rightarrow T_{r+1}={}^{22}C_{r}(x)^{\frac{44-11r}{3}}\alpha^{r}$

For term independent of x,

$\frac{44-11r}{3}=0$

$\Rightarrow r=4$

So,

$T_{5}={}^{22}C_{4}\alpha^{4}=7315$

$\Rightarrow\left(\frac{22!}{4!\times18!}\right)\alpha^{4}=7315$

$\Rightarrow 7315\alpha^{4}=7315$

$\Rightarrow \alpha^{4}=1$

$\Rightarrow |\alpha|=1$

Q.2.

Find the value of the integral $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{x+\frac{\pi}{4}}{2-\cos 2x}\,\mathrm{d}\,x$

A) $\frac{3\pi^{2}}{\sqrt{6}}$

B) $\sqrt{3}\pi^{2}$

C) $\frac{\pi^{2}}{6\sqrt{3}}$

D) $\frac{6\pi^{2}}{\sqrt{3}}$

Answer: $\frac{\pi^{2}}{6\sqrt{3}}$

Solution:

Let $I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{x+\frac{\pi}{4}}{2-\cos 2x}\right)dx$

$\Rightarrow I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{x}{2-\cos 2x}dx+\frac{\pi}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{2-\cos 2x}dx$

$\Rightarrow I=0+\frac{\pi}{2}\int_{0}^{\frac{\pi}{4}}\frac{1}{2-\cos 2x}dx\quad\left\{\text{here }\frac{x}{2-\cos 2x}\text{ is odd function and }\int_{-a}^{a}f(x)dx=0\text{ when }f(x)\text{ is odd function}\right\}$

So, $I=\frac{\pi}{2}\int_{0}^{\frac{\pi}{4}}\frac{1}{2-\cos 2x}dx\ \left\{\text{as }\int_{-a}^{a}f(x)dx=2\int_{0}^{a}f(x)dx\text{ when }f(x)\text{ is even}\right\}$

$\Rightarrow I=\frac{\pi}{2}\int_{0}^{\frac{\pi}{4}}\frac{1}{2-\frac{1-\tan^2 x}{1+\tan^2 x}}dx$

$\Rightarrow I=\frac{\pi}{2}\int_{0}^{\frac{\pi}{4}}\frac{\sec^2 \mathrm{x}}{2+2\tan^2 x-1+\tan^2 x}dx$

$\Rightarrow I=\frac{\pi}{2}\int_{0}^{\frac{\pi}{4}}\frac{\sec^2 \mathrm{x}}{1+3\tan^2 x}dx$

Now let $\tan x=t\Rightarrow\sec^2 x dx=dt$

$\Rightarrow I=\frac{\pi}{2}\int_{0}^{1}\frac{1}{1+3t^2}dt$

$\Rightarrow I=\frac{\pi}{2\times 3}\int_{0}^{1}\frac{1}{t^2+\left(\frac{1}{\sqrt{3}}\right)^2}dt$

$\Rightarrow I=\frac{\pi}{6}\times\sqrt{3}\left[\tan^{-1}\sqrt{3}t\right]_0^1$

$\Rightarrow I=\frac{\pi}{6}\times\sqrt{3}\left[\tan^{-1}\sqrt{3}-\tan^{-1}0\right]$

$\Rightarrow I=\frac{\pi}{6}\times\sqrt{3}\left[\frac{\pi}{3}\right]$

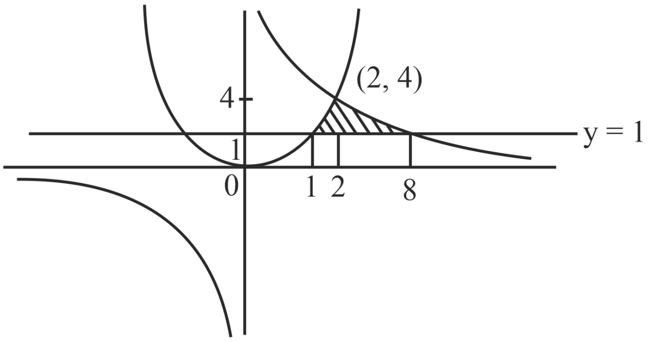
$\Rightarrow I=\frac{\pi^2}{6\sqrt{3}}$

Q.3. Area enclosed by $xy<8,\ y<x^2$ and $y>1$ is

A) $4\ln 2-\frac{14}{3}$ B) $4\ln 2+\frac{20}{3}$ C) $8\ln 4-\frac{14}{3}$ D) $8\ln 4-\frac{20}{3}$

Answer: $8\ln 4-\frac{14}{3}$

Solution: We have,



Required area

$= \int_1^2 \left(x^2 - 1\right)dx + \int_2^8 \left(\frac{8}{x} - 1\right)dx$

$= \int_1^2 x^2 dx + \int_2^8 \frac{8}{x}dx - \int_1^8 dx$

$= \left[\frac{x^3}{3}\right]_1^2 + 8[\ln x]_2^8 - [x]_1^8$

$= 8\ln 4 - \frac{14}{3}$ sq. units

Q.4. If $f(x) = x^x;\ x > 0$, then $f''(2) + f'(2) =$

A) $10 + 12\ln 2 + 4(\ln 2)^2$ B) $10 + 4(\ln 2)^2$ C) $10 + 12\ln 2$ D) $2^{\ln 2} + (\ln 2)^2$

Answer: $10 + 12\ln 2 + 4(\ln 2)^2$

Solution: Given:

$f(x) = x^x$

$\Rightarrow \ln f(x) = x\ln x$

$\Rightarrow \frac{f'(x)}{f(x)} = \ln x + 1$

$\Rightarrow f'(x) = x^x(\ln x + 1)$

$\Rightarrow f''(x) = (\ln x + 1)\frac{d}{dx}(x^x) + x^x\left(\frac{1}{x}\right)$

$\Rightarrow f''(x) = x^x\left[(\ln x + 1)^2 + \left(\frac{1}{x}\right)\right]$

So,

$f''(x) + f'(x) = x^x\left[(\ln x + 1)^2 + (\ln x + 1) + \left(\frac{1}{x}\right)\right]$

$\Rightarrow f''(2) + f'(2) = 2^2\left[(\ln 2 + 1)^2 + (\ln 2 + 1) + \left(\frac{1}{2}\right)\right]$

$\Rightarrow f''(2) + f'(2) = 10 + 12\ln 2 + 4(\ln 2)^2$

Q.5. For some values of λ, system of equations

$\alpha x + y + z = 1$
$x + \alpha y + z = 1$
$x + y + \alpha z = \beta$

has infinitely many solutions, then

A) $\alpha = 1,\ \beta = 1$ B) $\alpha = 1,\ \beta = -1$ C) $\alpha = -1,\ \beta = -1$ D) $\alpha = -1,\ \beta = 1$

Answer: $\alpha = 1,\ \beta = 1$

Solution: System of equations

$\alpha x + y + z = 1$
$x + \alpha y + z = 1$
$x + y + \alpha z = \beta$

has infinitely many solutions, so

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \alpha+2 & 2+\alpha & 2+\alpha \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

$$\Rightarrow (2+\alpha) \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

$$\Rightarrow (2+\alpha) \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha-1 & 0 \\ 1 & 0 & \alpha-1 \end{vmatrix} = 0$$

$$\Rightarrow (2+\alpha)(\alpha-1)^2 = 0$$

$$\Rightarrow \alpha = 1, \ -2$$

For $\alpha = 1$, we get infinite solutions if $\beta = 1$, so option (A) is correct.

Q.6. If $f(x) + f\left(\frac{1}{1-x}\right) = 1 - x$, then the value of $f(2)$ is

A) $\frac{1}{4}$ B) $\frac{-5}{4}$ C) $\frac{3}{4}$ D) $\frac{-3}{4}$

Answer: $\frac{-5}{4}$

Solution: Given,

$$f(x) + f\left(\frac{1}{1-x}\right) = 1 - x \ .....(1)$$

Now putting $x = 2$ in above equation we get,

$$f(2) + f(-1) = -1 \ .....(2)$$

Now putting $x = -1$ in equation (1) we get,

$$f(-1) + f\left(\frac{1}{2}\right) = 2 \ ....(3)$$

Now taking $x = \frac{1}{2}$ in equation (1) we get,

$$f\left(\frac{1}{2}\right) + f(2) = \frac{1}{2} \ ....(4)$$

Now applying operation in equation $(2) - (3) \ + (4)$ we get,

$$f\left(\frac{1}{2}\right) + f(2) - f(-1) - f\left(\frac{1}{2}\right) + f(2) + f(-1) \ = -1 - 2 + \frac{1}{2}$$

$$\Rightarrow 2f(2) \ = \frac{-5}{2}$$

$$\Rightarrow f(2) \ = \frac{-5}{4}$$

Q.7. If $\frac{dy}{dx} = \frac{x^2+3y^2}{3x^2+y^2}$; $y(1) = 0$, then

A) $\frac{2x^2}{(x-y)^2} = \frac{2x}{x-y} + \ln|x-y|$

B) $\frac{2x^2}{(x-y)^2} = 1 + \ln|x-y|$

C) $\frac{2x^2}{(x-y)^2} = \frac{y}{x-y} + \ln|x-y|$

D) $\frac{2x}{(x-y)^2} = \frac{y}{x-y} + \ln|x-y|$

Answer: $\frac{2x^2}{(x-y)^2} = \frac{2x}{x-y} + \ln|x-y|$

Solution: Given:

$$\frac{dy}{dx} = \frac{x^2+3y^2}{y^2+3x^2}$$

This is a homogenous differential equation.

Put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

So,

$$\frac{dy}{dx} = \frac{x^2+3y^2}{y^2+3x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+3v^2}{v^2+3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+3v^2-v^3-3v}{v^2+3}$$

$$\Rightarrow \int\left(\frac{v^2+3}{v^3-3v^2+3v-1}\right)dv = -\int\frac{dx}{x}$$

$$\Rightarrow \int\left(\frac{v^2+3}{(1-v)^3}\right)dv = \int\frac{dx}{x}$$

$$\Rightarrow \int\left(\frac{1}{1-v}\right)dv - \int\left(\frac{2}{(1-v)^2}\right)dv + \int\left(\frac{4}{(1-v)^3}\right)dv = \int\frac{dx}{x}$$

$$\Rightarrow -\ln|1-v| - \frac{2}{1-v} + \frac{2}{(1-v)^2} = \ln|x| + C$$

$$\Rightarrow -\ln|x-y| + \ln|x| - \frac{2x}{x-y} + \frac{2x^2}{(x-y)^2} = \ln|x| + C$$

$$\Rightarrow -\ln|x-y| - \frac{2x}{x-y} + \frac{2x^2}{(x-y)^2} = C$$

Since, $y(0) = 1 \Rightarrow C = 0$

$$\frac{2x^2}{(x-y)^2} = \frac{2x}{x-y} + \ln|x-y|$$

Q.8. Which of the following is tautology ?

A) $p \to (\sim p \wedge q)$

B) $p \to (p \vee q)$

C) $p \to (\sim p \vee q)$

D) $p \to (\sim p \wedge \sim q)$

Answer: $p \to (p \vee q)$

Solution: We know that $A \to B \equiv {\sim}A \vee B$

Now solving option (A) $p \to ({\sim}p \wedge q)$ we get,

$\equiv {\sim}p \vee ({\sim}p \wedge q)$ which is not a tautology as $A^c \cup (A^c \cap B) \not\equiv \mathrm{U}$ $\{\text{where U is universal set}\}$

Now solving option (B) $p \to (p \vee q)$

$\equiv {\sim}p \vee (p \vee q)$

$\equiv ({\sim}p \vee p) \vee q$

$\equiv \mathrm{U} \vee q \equiv \mathrm{U}$ which is a tautology,

Hence, $p \to (p \vee q)$ is a tautology

Solving option (C) $p \to ({\sim}p \vee q)$

$\equiv {\sim}p \vee ({\sim}p \vee q) \equiv {\sim}p \vee q \not\equiv \mathrm{U}$ which is not a tautology,

Now solving option (D) $p \to ({\sim}p \wedge q)$

$\equiv {\sim}p \vee ({\sim}p \wedge q) \equiv {\sim}p \wedge q \not\equiv \mathrm{U}$ which is not a tautology.

Q.9. If $A = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$, then which of the following is true:

A) $A^{30} = A^{25}$ B) $A^{30} + A^{25} + A = I$ C) $A^{30} - A^{25} + A = I$ D) $A^{30} = A^{25} + A$

Answer: $A^{30} - A^{25} + A = I$

Solution: Given,

$$A = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

Now its characteristic equation is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} \frac{1}{2} - \lambda & \frac{\sqrt{3}}{2} - \lambda \\ -\frac{\sqrt{3}}{2} - \lambda & \frac{1}{2} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + \frac{1}{4} - \lambda + \frac{3}{4} = 0$$

$$\Rightarrow \lambda^2 - \lambda + 1 = 0$$

$$\Rightarrow A^2 - A + I = 0 \quad ....(1)$$

$$\Rightarrow A^3 - A^2 + A = 0 \;....(2)$$

Now finding $A^4 = \left(A^2\right)^2 = (A - I)^2$ $\{\text{from equation (1)}\}$

$$\Rightarrow A^4 = A^2 + I - 2A$$

$$\Rightarrow A^4 = A - 2A = -A \;\{\text{again using the equation (1)}\}$$

Now finding $A^{30} = \left(A^4\right)^7 \cdot A^2 = -A^9 = -\left(A^4\right)^2 \cdot A = -A^3$

Now using equation (2) we get,

$A^{30} = -A^3 = A - A^2 = A - A + I = I$

Now finding $A^{25} = \left(A^4\right)^6 \cdot A = A^7 = A^4 \cdot A^3 = -A \cdot A^3 = -A^4 = A$

So, $A^{30} - A^{25} + A = I - A + A = I$

Hence, $A^{30} - A^{25} + A = I$ is correct option.

Q.10. If 2 unbiased die are thrown independently, A is the event such that the number on the first die is less than second die, B is the event, such that number on the first is even and number on the second die is odd, C is the event such that first die show odd number and second die shows even number, then

A) $n((A \cup B) \cap C) = 6$

B) A & B are mutually exclusive events

C) A & B are independent events

D) $n(A) = 18,\ n(B) = 6$ & $n(C) = 6$

Answer: $n((A \cup B) \cap C) = 6$

Solution: Given,

2 unbiased die are thrown independently, A is the event such that the number on the first die is less than second die,

So event A will be $\{(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}$

Hence, $n(A) = 15$

Now B is the event, such that number on the first is even and number on the second die is odd,

So, event B will be $\{(2,1),(2,3),(2,5),(4,1),(4,3),(4,5),(6,1),(6,3),(6,5)\}$

Hence, $n(B) = 9$

Now C is the event such that first die show odd number and second die shows even number,

So, event C will be $\{(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\}$

Hence, $n(C) = 9$

Now elements in $((A \cup B) \cap C)$ will be $\{(1,2),(1,4),(1,6),(3,4),(3,6),(5,6)\}$

Hence, $n((A \cup B) \cap C) = 6$

Q.11. Let 3 $\mathrm{A.P}$'s be

$S_1 : 2, 5, 8, 11, \ldots, 394$

$S_2 : 1, 3, 5, 7, \ldots, 397$

$S_3 : 2, 7, 12, \ldots, 397$

Then, sum of common terms of these three AP's is__________

Answer: 2561

Solution: We have,

$S_1 : 2, 5, 8, 11, \ldots, 394$

Common difference is $d_1 = 3$

$S_2 : 1, 3, 5, 7, \ldots, 397$

Common difference is $d_2 = 2$

S_2 is the series of odd terms.

$S_3 : 2, 7, 12, \ldots, 397$

Common difference is $d_3 = 5$

Common difference of AP formed by common terms of S_1 & S_3 is

$d = \text{LCM}\,(d_1, d_3) = \text{LCM}\,(3, 5) = 15$

So, series is

$2, 17, 32, 47, \ldots$

So, series of common terms of $S_1,\ S_2,\ S_3$ is

$S_4 : 17,\ 47,\ 87,\ \ldots.,\ 377$

Here, last term we have calculated using general observation that 377 is the last common term of S_1 & S_3.

Let the number of terms in S_4 be n, then

$377 = 17 + 30\,(n - 1)$

$\Rightarrow n = 13$

Sum of common terms of these three AP's is

$S = \frac{13}{2}(17 + 377) = 2561$

Q.12. Find number of total 6 digit numbers which are formed using the digits $4,\ 5,\ 9$ which are divisible by 6 are

Answer: 81

Solution: Given,

We have to form 6 digit number using the digits $4,\ 5$ & 9 and for number should be divisible by 6 which means that the addition of number should be divisible by 3 and unit place should be even, so we will make following cases:

Case (1) The number is 444444 which can be arranged in 1 way,

Case (2) when the number is formed using the digits $4, 4, 4, 5, 9$ and last place is fixed with 4, that can be arranged in $\frac{5!}{3!} = 20 \text{ ways}$

Case (3) when number is formed using the digits $4, 4, 5, 5, 5$ and last place is fixed with 4, that can be arranged in $\frac{5!}{3!2!} = 10 \text{ ways}$

Case (4) when number is formed using the digits $4, 4, 9, 9, 9$ and last place is fixed by 4, that can be arranged in $\frac{5!}{3!2!} = 10 \text{ ways}$

Case (5) when number is formed using the digits $4, 5, 5, 9, 9$ and last place is fixed with 4, that can be arranged in $\frac{5!}{2!2!} = 30 \text{ ways}$

Case (6) when number is formed using the digits $5, 9, 9, 9, 9$ and last place is fixed with 4, that can be arranged in $\frac{5!}{4!} = 5 \text{ ways}$

Case (7) when number is formed using the digits $5, 5, 5, 5, 9$ and last place is fixed with 4, that can be arranged in $\frac{5!}{4!} = 5 \text{ ways}$

Now adding all the cases we get, $1 + 20 + 10 + 10 + 30 + 5 + 5 = 81 \text{ ways.}$

Q.13. Number of non-negative integral solutions of $x + y + z = 21$ where $x \geq 1,\ y \geq 3,\ z \geq 6$ is

Answer: 78

Solution: We have,

$x \geq 1,\ y \geq 3,\ z \geq 6$

$\Rightarrow x-1 \geq 0,\ y-3 \geq 0,\ z-6 \geq 0$

So,

$x+y+z=21$

$\Rightarrow (x-1)+(y-3)+(z-6)=11$

So, number of non-negative integral solutions is

$= {}^{11+3-1}C_{3-1} = {}^{13}C_2 = \frac{13 \times 12}{2} = 78$

Q.14. If there is point (x_0, y_0) on the curve $3x^2-4y^2=36$ and line $3x+2y=1$ is at minimum distance from (x_0, y_0) then find the value of $|\sqrt{2}(x_0-y_0)|$

Answer: 9

Solution: Given,

There is point (x_0, y_0) on the curve $3x^2-4y^2=36$ and line $3x+2y=1$ which is at minimum distance from (x_0, y_0)

Now $3x^2-4y^2=36$

$\Rightarrow \frac{x^2}{12}-\frac{y^2}{9}=1$ is equation of hyperbola,

Now any point which will be closer to $3x+2y=1$ and on curve $3x^2-4y^2=36$ will lie on tangent to given hyperbola and will be parallel to the line $3x+2y=1$

Now any line parallel to $3x+2y=1 \Rightarrow y=\frac{-3}{2}x+1$ is given by $y=\frac{-3}{2}x+c$

Now by condition of tangent to hyperbola we get, $c=\pm\sqrt{12\left(\frac{-3}{2}\right)^2-9}=\pm\sqrt{27-9}=\pm 3\sqrt{2}$

So, the equation of tangent will be $y=\frac{-3}{2}x+3\sqrt{2}\ \{\text{taking} + \text{sign}\}$

Now solving $3x^2-4y^2=36$ and $y=\frac{-3}{2}x+3\sqrt{2}$ we get points as $x=3\sqrt{2}$, $y=\frac{-3}{\sqrt{2}}$,

So $(x_0, y_0) \equiv \left(3\sqrt{2}, \frac{-3}{\sqrt{2}}\right)$

Hence, the value of $|\sqrt{2}(x_0-y_0)| = \left|\sqrt{2}\left(3\sqrt{2}+\frac{3}{\sqrt{2}}\right)\right| = |6+3| = 9$