

FINAL JEE(Advanced) EXAMINATION - 2019

(Held On Monday 27th MAY, 2019)

PAPER-1

TEST PAPER WITH ANSWER & SOLUTION

PART-3 : MATHEMATICS

SECTION-1 : (Maximum Marks : 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks : +3 If **ONLY** the correct option is chosen.

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered)

Negative Marks : -1 In all other cases

1. Let $M = \begin{bmatrix} \sin^4 \theta & -1 - \sin^2 \theta \\ 1 + \cos^2 \theta & \cos^4 \theta \end{bmatrix} = \alpha I + \beta M^{-1}$,

where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real number, and I is the 2×2 identity matrix. If

α^* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi)\}$ and

β^* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$,

then the value of $\alpha^* + \beta^*$ is

- (1) $-\frac{37}{16}$ (2) $-\frac{29}{16}$ (3) $-\frac{31}{16}$ (4) $-\frac{17}{16}$

Ans. (2)

Sol. Given $M = \alpha I + \beta M^{-1}$

$$\Rightarrow M^2 - \alpha M - \beta I = O$$

By putting values of M and M^2 , we get

$$\alpha(\theta) = 1 - 2\sin^2 \theta \cos^2 \theta = 1 - \frac{\sin^2 2\theta}{2} \geq \frac{1}{2}$$

$$\text{Also, } \beta(\theta) = -(\sin^4 \theta \cos^4 \theta + (1 + \cos^2 \theta)(1 + \sin^2 \theta))$$

$$= -(\sin^4 \theta \cos^4 \theta + 1 + \cos^2 \theta + \sin^2 \theta + \sin^2 \theta \cos^2 \theta)$$

$$= -(t^2 + t + 2), \quad t = \frac{\sin^2 2\theta}{4} \in \left[0, \frac{1}{4}\right]$$

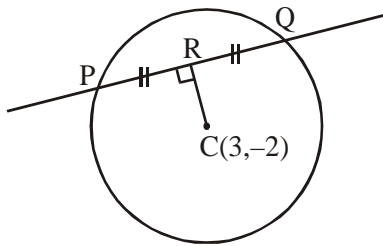
$$\Rightarrow \beta(\theta) \geq -\frac{37}{16}$$

2. A line $y = mx + 1$ intersects the circle $(x - 3)^2 + (y + 2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct ?

- (1) $6 \leq m < 8$ (2) $2 \leq m < 4$ (3) $4 \leq m < 6$ (4) $-3 \leq m < -1$

Ans. (2)

Sol.



$$R \equiv \left(-\frac{3}{5}, \frac{-3m}{5} + 1 \right)$$

$$\text{So, } m \left(\frac{-\frac{3m}{5} + 3}{-\frac{3}{5} - 3} \right) = -1$$

$$\Rightarrow m^2 - 5m + 6 = 0$$

$$\Rightarrow m = 2, 3$$

3. Let S be the set of all complex numbers z satisfying $|z - 2 + i| \geq \sqrt{5}$. If the complex number z_0 is such that

$\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{ \frac{1}{|z - 1|} : z \in S \right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is

- (1) $\frac{\pi}{4}$ (2) $-\frac{\pi}{2}$ (3) $\frac{3\pi}{4}$ (4) $\frac{\pi}{2}$

Ans. (2)

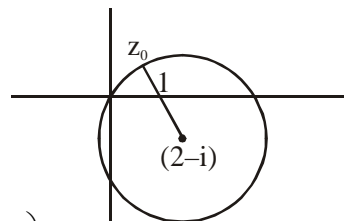
Sol. $\arg \left(\frac{4 - (z_0 - \bar{z}_0)}{(z_0 - \bar{z}_0) + 2i} \right)$

$$= \arg \left(\frac{4 - 2 \operatorname{Re} z_0}{2i \operatorname{Im} z_0 + 2i} \right) = \arg \left(\frac{2 - \operatorname{Re} z_0}{(1 + \operatorname{Im} z_0)i} \right)$$

$$= \arg \left(- \left(\frac{2 - \operatorname{Re} z_0}{1 + \operatorname{Im} z_0} \right) i \right)$$

$$= \arg(-ki) ; k > 0 \quad (\text{as } \operatorname{Re} z_0 < 2 \text{ \& } \operatorname{Im} z_0 > 0)$$

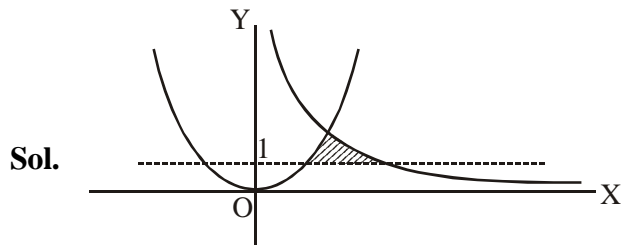
$$= -\frac{\pi}{2}$$



4. The area of the region $\{(x, y) : xy \leq 8, 1 \leq y \leq x^2\}$ is

- (1) $8\log_e 2 - \frac{14}{3}$ (2) $16\log_e 2 - \frac{14}{3}$ (3) $16\log_e 2 - 6$ (4) $8\log_e 2 - \frac{7}{3}$

Ans. (2)



For intersection, $\frac{8}{y} = \sqrt{y} \Rightarrow y = 4$

$$\begin{aligned} \text{Hence, required area} &= \int_1^4 \left(\frac{8}{y} - \sqrt{y} \right) dy \\ &= \left[8\log_e y - \frac{2}{3}y^{3/2} \right]_1^4 = 16\log_e 2 - \frac{14}{3} \end{aligned}$$

Remark : The question should contain the phrase "area of the bounded region in the first quadrant". Because, in the 2nd quadrant, the region above the line $y = 1$ and below $y = x^2$, satisfies the region, which is unbounded.

SECTION-2 : (Maximum Marks: 32)

- This section contains **EIGHT (08)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s)
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If only (all) the correct option(s) is (are) chosen.

Partial Marks : +3 If all the four options are correct but **ONLY** three options are chosen.

Partial Marks : +2 If three or more options are correct but **ONLY** two options are chosen and both of which are correct.

Partial Marks : +1 If two or more options are correct but **ONLY** one option is chosen and it is a correct option.

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered).

Negative Marks : -1 In all other cases.
- For example, in a question, if (A), (B) and (D) are the **ONLY** three options corresponding to correct answers, then

choosing **ONLY** (A), (B) and (D) will get +4 marks;

choosing **ONLY** (A) and (B) will get +2 marks;

choosing **ONLY** (A) and (D) will get +2 marks;

choosing **ONLY** (B) and (D) will get +2 marks;

choosing **ONLY** (A) will get +1 marks;

choosing **ONLY** (B) will get +1 marks;

choosing **ONLY** (D) will get +1 marks;

choosing no option (i.e. the question is unanswered) will get 0 marks, and

choosing any other combination of options will get -1 mark.

1. There are three bags B_1 , B_2 and B_3 . The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls, Bags B_1 , B_2 and B_3 have probabilities $\frac{3}{10}$, $\frac{3}{10}$ and $\frac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct ?

- (1) Probability that the selected bag is B_3 and the chosen ball is green equals $\frac{3}{10}$
- (2) Probability that the chosen ball is green equals $\frac{39}{80}$
- (3) Probability that the chosen ball is green, given that the selected bag is B_3 , equals $\frac{3}{8}$
- (4) Probability that the selected bag is B_3 , given that the chosen balls is green, equals $\frac{5}{13}$

Ans. (2,3)

Sol.

Ball	Balls composition	$P(B_i)$
B_1	5R + 5G	$\frac{3}{10}$
B_2	3R + 5G	$\frac{3}{10}$
B_3	5R + 3G	$\frac{4}{10}$

$$(1) \quad P(B_3 \cap G) = P\left(\frac{G}{B_3}\right)P(B_3)$$

$$= \frac{3}{8} \times \frac{4}{10} = \frac{3}{20}$$

$$(2) \quad P(G) = P\left(\frac{G}{B_1}\right)P(B_1) + P\left(\frac{G}{B_2}\right)P(B_2) + P\left(\frac{G}{B_3}\right)P(B_3)$$

$$= \frac{3}{20} + \frac{3}{16} + \frac{3}{20} = \frac{39}{80}$$

$$(3) \quad P\left(\frac{G}{B_3}\right) = \frac{3}{8}$$

$$(4) \quad P\left(\frac{B_3}{G}\right) = \frac{P(G \cap B_3)}{P(G)} = \frac{3/20}{39/80} = \frac{4}{13}$$

2. Define the collections $\{E_1, E_2, E_3, \dots\}$ of ellipses and $\{R_1, R_2, R_3, \dots\}$ of rectangles as follows :

$$E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1 ;$$

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1 ;

E_n : ellipse $\frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1$ of largest area inscribed in R_{n-1} , $n > 1$;

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n , $n > 1$.

Then which of the following options is/are correct ?

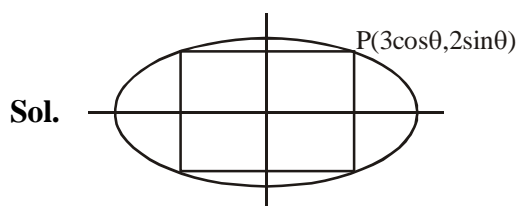
(1) The eccentricities of E_{18} and E_{19} are NOT equal

(2) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

(3) The length of latus rectum of E_9 is $\frac{1}{6}$

(4) $\sum_{n=1}^N (\text{area of } R_n) < 24$, for each positive integer N

Ans. (3,4)



Area of $R_1 = 3\sin 2\theta$; for this to be maximum

$$\Rightarrow \theta = \frac{\pi}{4} \Rightarrow \left(\frac{3}{\sqrt{2}}, \frac{2}{\sqrt{2}} \right)$$

Hence for subsequent areas of rectangles R_n to be maximum the coordinates will be in GP with common

$$\text{ratio } r = \frac{1}{\sqrt{2}} \Rightarrow a_n = \frac{3}{(\sqrt{2})^{n-1}} ; b_n = \frac{2}{(\sqrt{2})^{n-1}}$$

Eccentricity of all the ellipses will be same

$$\text{Distance of a focus from the centre in } E_9 = a_9 e_9 = \sqrt{a_9^2 - b_9^2} = \frac{\sqrt{5}}{16}$$

$$\text{Length of latus rectum of } E_9 = \frac{2b_9^2}{a_9} = \frac{1}{6}$$

$$\therefore \sum_{n=1}^{\infty} \text{Area of } R_n = 12 + \frac{12}{2} + \frac{12}{4} + \dots = 24$$

$$\Rightarrow \sum_{n=1}^N (\text{area of } R_n) < 24, \text{ for each positive integer } N$$

3. Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and $\text{adj}M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ where a and b are real numbers. Which of the following

options is/are correct ?

(1) $a + b = 3$

(2) $\det(\text{adj}M^2) = 81$

(3) $(\text{adj}M)^{-1} + \text{adj}M^{-1} = -M$

(4) If $M \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

Ans. (1,3,4)

Sol. $(\text{adj}M)_{11} = 2 - 3b = -1 \Rightarrow b = 1$

Also, $(\text{adj}M)_{22} = -3a = -6 \Rightarrow a = 2$

Now, $\det M = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = -2$

$\Rightarrow \det(\text{adj}M^2) = (\det M^2)^2$
 $= (\det M)^4 = 16$

Also $M^{-1} = \frac{\text{adj}M}{\det M}$

$\Rightarrow \text{adj}M = -2M^{-1}$

$\Rightarrow (\text{adj}M)^{-1} = \frac{-1}{2}M$

And, $\text{adj}(M^{-1}) = (M^{-1})^{-1} \det(M^{-1})$

$= \frac{1}{\det M} M = \frac{-M}{2}$

Hence, $(\text{adj}M)^{-1} + \text{adj}(M^{-1}) = -M$

Further, $MX = b$

$\Rightarrow X = M^{-1}b = \frac{-\text{adj}M}{2}b$

$= \frac{-1}{2} \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= \frac{-1}{2} \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\Rightarrow (\alpha, \beta, \gamma) = (1, -1, 1)$

4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \leq x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \leq x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \geq 3 \end{cases}$$

Then which of the following options is/are correct ?

- (1) f' has a local maximum at $x = 1$ (2) f is onto
(3) f is increasing on $(-\infty, 0)$ (4) f' is NOT differentiable at $x = 1$

Ans. (1,2,4)

Sol. $f(x) = \begin{cases} (x+1)^5 - 2x, & x < 0; \\ x^2 - x + 1, & 0 \leq x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \leq x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \geq 3 \end{cases}$

for $x < 0$, $f(x)$ is continuous

& $\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow 0^-} f(x) = 1$

Hence, $(-\infty, 1) \subset \text{Range of } f(x) \text{ in } (-\infty, 0)$

$f'(x) = 5(x+1)^4 - 2$, which changes sign in $(-\infty, 0)$

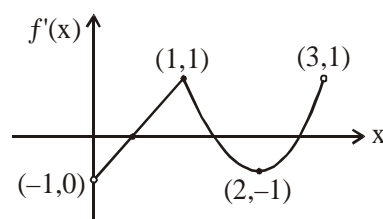
$\Rightarrow f(x)$ is non-monotonic in $(-\infty, 0)$

For $x \geq 3$, $f(x)$ is again continuous and $\lim_{x \rightarrow \infty} f(x) = \infty$ and $f(3) = \frac{1}{3}$

$\Rightarrow \left[\frac{1}{3}, \infty \right) \subset \text{Range of } f(x) \text{ in } [3, \infty)$

Hence, range of $f(x)$ is \mathbb{R}

$f'(x) = \begin{cases} 2x - 1, & 0 \leq x < 1 \\ 2x^2 - 8x + 7, & 1 \leq x < 3 \end{cases}$



Hence f' has a local maximum at $x = 1$ and f' is NOT differentiable at $x = 1$.

5. Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n , define

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1$$

$$b_1 = 1 \text{ and } b_n = a_{n-1} + a_{n+1}, \quad n \geq 2.$$

Then which of the following options is/are correct ?

(1) $a_1 + a_2 + a_3 + \dots + a_n = a_{n+2} - 1$ for all $n \geq 1$

(2) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(3) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

(4) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

Ans. (1,2,4)

Sol. α, β are roots of $x^2 - x - 1$

$$a_{r+2} - a_r = \frac{(\alpha^{r+2} - \beta^{r+2}) - (\alpha^r - \beta^r)}{\alpha - \beta} = \frac{(\alpha^{r+2} - \alpha^r) - (\beta^{r+2} - \beta^r)}{\alpha - \beta}$$

$$= \frac{\alpha^r(\alpha^2 - 1) - \beta^r(\beta^2 - 1)}{\alpha - \beta} = \frac{\alpha^r\alpha - \beta^r\beta}{\alpha - \beta} = \frac{\alpha^{r+1} - \beta^{r+1}}{\alpha - \beta} = a_{r+1}$$

$$\Rightarrow a_{r+2} - a_{r+1} = a_r$$

$$\Rightarrow \sum_{r=1}^n a_r = a_{n+2} - a_2 = a_{n+2} - \frac{\alpha^2 - \beta^2}{\alpha - \beta} = a_{n+2} - (\alpha + \beta) = a_{n+2} - 1$$

$$\text{Now } \sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{\sum_{n=1}^{\infty} \left(\frac{\alpha}{10}\right)^n - \sum_{n=1}^{\infty} \left(\frac{\beta}{10}\right)^n}{\alpha - \beta}$$

$$= \frac{\frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} - \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}}}{\alpha - \beta} = \frac{\frac{\alpha}{10 - \alpha} - \frac{\beta}{10 - \beta}}{(\alpha - \beta)} = \frac{10}{(10 - \alpha)(10 - \beta)} = \frac{10}{89}$$

$$\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \sum_{n=1}^{\infty} \frac{a_{n-1} + a_{n+1}}{10^n} = \frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} + \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}} = \frac{12}{89}$$

Further, $b_n = a_{n-1} + a_{n+1}$

$$= \frac{(\alpha^{n-1} - \beta^{n-1}) + (\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta}$$

(as $\alpha\beta = -1 \Rightarrow \alpha^{n-1} = -\alpha^n\beta$ & $\beta^{n-1} = -\alpha\beta^n$)

$$= \frac{\alpha^n(\alpha - \beta) + (\alpha - \beta)\beta^n}{\alpha - \beta} = \alpha^n + \beta^n$$

6. Let Γ denote a curve $y = y(x)$ which is in the first quadrant and let the point $(1, 0)$ lie on it. Let the tangent to Γ at a point P intersect the y -axis at Y_p . If PY_p has length 1 for each point P on Γ , then which of the following options is/are correct ?

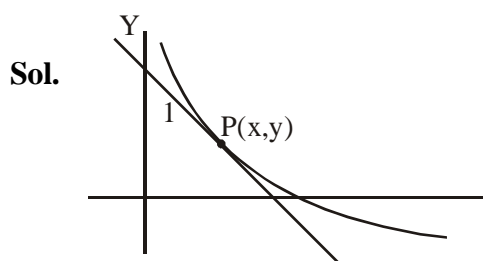
$$(1) \ y = \log_e \left(\frac{1 + \sqrt{1 - x^2}}{x} \right) - \sqrt{1 - x^2}$$

$$(2) \ xy' - \sqrt{1 - x^2} = 0$$

$$(3) \ y = -\log_e \left(\frac{1 + \sqrt{1 - x^2}}{x} \right) + \sqrt{1 - x^2}$$

$$(4) \ xy' + \sqrt{1 - x^2} = 0$$

Ans. (1,4)



$$Y - y = y'(X - x)$$

$$\text{So, } Y_p = (0, y - xy')$$

$$\text{So, } x^2 + (xy')^2 = 1 \Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1 - x^2}{x^2}}$$

$\left[\frac{dy}{dx} \right]$ can not be positive i.e. $f(x)$ can not be increasing in first quadrant, for $x \in (0, 1)$

$$\text{Hence, } \int dy = -\int \frac{\sqrt{1 - x^2}}{x} dx$$

$$\Rightarrow y = -\int \frac{\cos^2 \theta d\theta}{\sin \theta} ; \text{ put } x = \sin \theta$$

$$\Rightarrow y = -\int \operatorname{cosec} \theta d\theta + \int \sin \theta d\theta$$

$$\Rightarrow y = \ln(\operatorname{cosec} \theta + \cot \theta) - \cos \theta + C$$

$$\Rightarrow y = \ln \left(\frac{1 + \sqrt{1 - x^2}}{x} \right) - \sqrt{1 - x^2} + C$$

$$\Rightarrow y = \ln \left(\frac{1 + \sqrt{1 - x^2}}{x} \right) - \sqrt{1 - x^2} \quad (\text{as } y(1) = 0)$$

7. In a non-right-angled triangle ΔPQR , let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S , the perpendicular from P meets the side QR at E , and RS and PE intersect at O . If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

(1) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$

(2) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}(2 - \sqrt{3})$

(3) Length of $RS = \frac{\sqrt{7}}{2}$

(4) Length of $OE = \frac{1}{6}$

Ans. (2,3,4)

Sol. $\frac{\sin P}{\sqrt{3}} = \frac{\sin Q}{1} = \frac{1}{2R} = \frac{1}{2}$

$$\Rightarrow P = \frac{\pi}{3} \text{ or } \frac{2\pi}{3} \text{ and } Q = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

Since $p > q \Rightarrow P > Q$

So, if $P = \frac{\pi}{3}$ and $Q = \frac{\pi}{6} \Rightarrow R = \frac{\pi}{2}$ (not possible)

Hence, $P = \frac{2\pi}{3}$ and $Q = R = \frac{\pi}{6}$

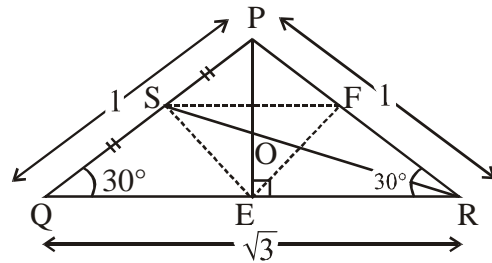
$$r = \frac{\Delta}{s} = \frac{\frac{1}{2}(1)(\sqrt{3})\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}+2}{2}\right)} = \frac{\sqrt{3}}{2}(2 - \sqrt{3})$$

Now, area of $\Delta SEF = \frac{1}{4}$ area of ΔPQR

$$\Rightarrow \text{area of } \Delta SOE = \frac{1}{3} \text{ area of } \Delta SEF = \frac{1}{12} \text{ area of } \Delta PQR = \frac{1}{12} \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{48}$$

$$RS = \frac{1}{2} \sqrt{2(3) + 2(1) - 1} = \frac{\sqrt{7}}{2}$$

$$OE = \frac{1}{3} PE = \frac{1}{3} \cdot \frac{1}{2} \sqrt{2(1)^2 + 2(1)^2 - 3} = \frac{1}{6}$$



8. Let L_1 and L_2 denotes the lines

$$\vec{r} = \hat{i} + \lambda(-\hat{i} + 2\hat{j} + 2\hat{k}), \lambda \in \mathbb{R} \text{ and}$$

$$\vec{r} = \mu(2\hat{i} - \hat{j} + 2\hat{k}), \mu \in \mathbb{R}$$

respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3 ?

$$(1) \vec{r} = \frac{1}{3}(2\hat{i} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R}$$

$$(2) \vec{r} = \frac{2}{9}(2\hat{i} - \hat{j} + 2\hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R}$$

$$(3) \vec{r} = t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R}$$

$$(4) \vec{r} = \frac{2}{9}(4\hat{i} + \hat{j} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R}$$

Ans. (1,2,4)

Sol. Points on L_1 and L_2 are respectively $A(1 - \lambda, 2\lambda, 2\lambda)$ and $B(2\mu, -\mu, 2\mu)$

$$\text{So, } \overline{AB} = (2\mu + \lambda - 1)\hat{i} + (-\mu - 2\lambda)\hat{j} + (2\mu - 2\lambda)\hat{k}$$

and vector along their shortest distance $= 2\hat{i} + 2\hat{j} - \hat{k}$.

$$\text{Hence, } \frac{2\mu + \lambda - 1}{2} = \frac{-\mu - 2\lambda}{2} = \frac{2\mu - 2\lambda}{-1}$$

$$\Rightarrow \lambda = \frac{1}{9} \text{ \& } \mu = \frac{2}{9}$$

$$\text{Hence, } A \equiv \left(\frac{8}{9}, \frac{2}{9}, \frac{2}{9}\right) \text{ and } B \equiv \left(\frac{4}{9}, -\frac{2}{9}, \frac{4}{9}\right)$$

$$\Rightarrow \text{Mid point of AB} \equiv \left(\frac{2}{3}, 0, \frac{1}{3}\right)$$

SECTION-3 : (Maximum Marks: 18)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
 - For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **Two** decimal places.
 - Answer to each question will be evaluated according to the following marking scheme:
Full Marks : +3 If **ONLY** the correct numerical value is entered.
Zero Marks : 0 In all other cases.
-

1. If

$$I = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \frac{dx}{(1 + e^{\sin x})(2 - \cos 2x)}$$

then $27I^2$ equals _____

Ans. (4.00)

Sol. $2I = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \left[\frac{1}{(1 + e^{\sin x})(2 - \cos 2x)} + \frac{1}{(1 + e^{-\sin x})(2 - \cos 2x)} \right] dx$ (using King's Rule)

$$\Rightarrow I = \frac{1}{\pi} \int_{-\pi/4}^{\pi/4} \frac{dx}{2 - \cos 2x}$$

$$\Rightarrow I = \frac{2}{\pi} \int_0^{\pi/4} \frac{dx}{2 - \cos 2x} = \frac{2}{\pi} \int_0^{\pi/4} \frac{\sec^2 x dx}{1 + 3 \tan^2 x}$$

$$= \frac{2}{\sqrt{3}\pi} \left[\tan^{-1}(\sqrt{3} \tan x) \right]_0^{\pi/4} = \frac{2}{3\sqrt{3}}$$

$$\Rightarrow 27I^2 = 27 \times \frac{4}{27} = 4$$

2. Let the point B be the reflection of the point A(2, 3) with respect to the line $8x - 6y - 23 = 0$. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is _____

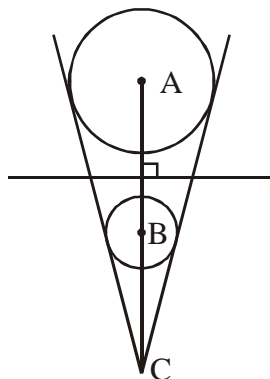
Ans. (10.00)

Sol. Distance of point A from given line = $\frac{5}{2}$

$$\frac{CA}{CB} = \frac{2}{1}$$

$$\Rightarrow \frac{AC}{AB} = \frac{2}{1}$$

$$\Rightarrow AC = 2 \times 5 = 10$$



3. Let $AP(a; d)$ denote the set of all the terms of an infinite arithmetic progression with first term a and common difference $d > 0$. If $AP(1; 3) \cap AP(2; 5) \cap AP(3; 7) = AP(a; d)$ then $a + d$ equals ____

Ans. (157.00)

Sol. We equate the general terms of three respective

$$\text{A.P.'s as } 1 + 3a = 2 + 5b = 3 + 7c$$

$$\Rightarrow 3 \text{ divides } 1 + 2b \text{ and } 5 \text{ divides } 1 + 2c$$

$$\Rightarrow 1 + 2c = 5, 15, 25 \text{ etc.}$$

So, first such terms are possible when $1 + 2c = 15$ i.e. $c = 7$

Hence, first term = $a = 52$

$$d = \text{lcm}(3, 5, 7) = 105$$

$$\Rightarrow a + d = 157$$

4. Let S be the sample space of all 3×3 matrices with entries from the set $\{0, 1\}$. Let the events E_1 and E_2 be given by

$$E_1 = \{A \in S : \det A = 0\} \text{ and}$$

$$E_2 = \{A \in S : \text{sum of entries of } A \text{ is } 7\}.$$

If a matrix is chosen at random from S , then the conditional probability $P(E_1|E_2)$ equals ____

Ans. (0.50)

Sol. $n(E_2) = {}^9C_2$ (as exactly two cyphers are there)

Now, $\det A = 0$, when two cyphers are in the same column or same row

$$\Rightarrow n(E_1 \cap E_2) = 6 \times {}^3C_2.$$

$$\text{Hence, } P\left(\frac{E_1}{E_2}\right) = \frac{n(E_1 \cap E_2)}{n(E_2)} = \frac{18}{36} = \frac{1}{2} = 0.50$$

5. Three lines are given by

$$\vec{r} = \lambda \hat{i}, \lambda \in \mathbb{R}$$

$$\vec{r} = \mu(\hat{i} + \hat{j}), \mu \in \mathbb{R} \text{ and}$$

$$\vec{r} = \nu(\hat{i} + \hat{j} + \hat{k}), \nu \in \mathbb{R}.$$

Let the lines cut the plane $x + y + z = 1$ at the points A, B and C respectively. If the area of the triangle ABC is Δ then the value of $(6\Delta)^2$ equals ____

Ans. (0.75)

Sol. $A(1, 0, 0), B\left(\frac{1}{2}, \frac{1}{2}, 0\right) \& C\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Hence, $\overrightarrow{AB} = -\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j}$ & $\overrightarrow{AC} = -\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{3}\hat{k}$

So, $\Delta = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \sqrt{\frac{1}{2} \times \frac{2}{3} - \frac{1}{4}}$

$$= \frac{1}{2 \times 2\sqrt{3}}$$

$$\Rightarrow (6\Delta)^2 = \frac{3}{4} = 0.75$$

6. Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set

$$\{ |a + b\omega + c\omega^2|^2 : a, b, c \text{ distinct non-zero integers} \}$$

equals _____

Ans. (3.00)

Sol. $|a + b\omega + c\omega^2|^2 = (a + b\omega + c\omega^2) \overline{(a + b\omega + c\omega^2)}$

$$= (a + b\omega + c\omega^2) (a + b\omega^2 + c\omega)$$

$$= a^2 + b^2 + c^2 - ab - bc - ca$$

$$= \frac{1}{2} [(a-b)^2 + (b-c)^2 + (c-a)^2]$$

$$\geq \frac{1+1+4}{2} = 3 \text{ (when } a = 1, b = 2, c = 3)$$