FINAL JEE(Advanced) EXAMINATION - 2019

(Held On Monday 27th MAY, 2019)

PAPER-2 | TEST PAPER WITH ANSWER & SOLUTION

PART-3 : MATHEMATICS

SECTION-1 : (Maximum Marks: 32)

- This section contains **EIGHT (08)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s)
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	If only (all) the correct option(s) is (are) chosen.
Partial Marks	:	+3	If all the four options are correct but ONLY three options are chosen.
Partial Marks	:	+2	If three or more options are correct but ONLY two options are chosen and both of which are correct.
Partial Marks	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correct option.
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered).
Negative Marks	:	–1	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

choosing ONLY (A), (B) and (D) will get +4 marks;
choosing ONLY (A) and (B) will get +2 marks;
choosing ONLY (A) and (D) will get +2 marks;
choosing ONLY (B) and (D) will get +2 marks;
choosing ONLY (A) will get +1 marks;
choosing ONLY (B) will get +1 marks;
choosing ONLY (D) will get +1 marks;
choosing no option (i.e. the question is unanswered) will get 0 marks, and
choosing any other combination of options will get –1 mark.

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1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (x - 1)(x - 2)(x - 5)$. Define $F(x) = \int_0^x f(t)\,dt$, $x > 0$. Then which of the following options is/are correct ?

(1) F has a local minimum at $x = 1$

(2) F has a local maximum at $x = 2$

(3) $F(x) \neq 0$ for all $x \in (0, 5)$

(4) F has two local maxima and one local minimum in $(0, \infty)$

Ans. (1,2,3)

Sol. $f(x) = (x-1)(x-2)(x-5)$

$$F(x)=\int_0^x f(t)dt, x>0$$

$$F'(x)=f(x)=(x-1)(x-2)(x-5), x>0$$

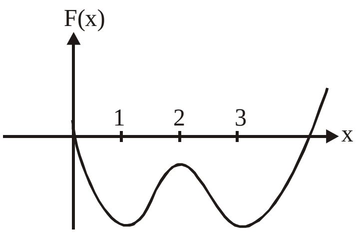
clearly F(x) has local minimum at x = 1,5

F(x) has local maximum at x = 2

$f(x) = x^3 - 8x^2 + 17x - 10$

$$\Rightarrow F(x)=\int_0^x (t^3-8t^2+17t-10)dt$$

$$F(x)=\frac{x^4}{4}-\frac{8x^3}{3}+\frac{17x^2}{2}-10x$$



from the graph of y = F(x), clearly $F(x) \neq 0 \ \forall \ x \in (0,5)$

2. For $a \in \mathbb{R}$, $|a| > 1$, let $\lim_{n\to\infty}\left(\frac{1+\sqrt[3]{2}+.....+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+.......+\frac{1}{(an+n)^2}\right)}\right)=54$. Then the possible value(s) of a is/are :

(1) 8 (2) –9 (3) –6 (4) 7

Ans. **(1,2)**

Sol. $$\lim_{n\to\infty}\frac{n^{1/3}\left(\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}\right)}{n^{7/3}\left(\sum_{r=1}^{n}\frac{1}{(an+r)^2}\right)}=54 \Rightarrow \lim_{n\to\infty}\left(\frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}}{\frac{1}{n}\sum_{r=1}^{n}\frac{1}{(a+r/n)^2}}\right)=54 \Rightarrow \frac{\int_0^1 x^{1/3}dx}{\int_0^1 \frac{1}{(a+x)^2}dx}=54 \Rightarrow \frac{\frac{3}{4}}{\frac{1}{a(a+1)}}=54$$

$\Rightarrow \quad a(a+1) = 72 \quad \Rightarrow \quad a^2 + a - 72 = 0 \quad \Rightarrow \quad a = -9, 8$

3. Three lines

$$L_1 : \vec{r} = \lambda\hat{i},\ \lambda \in \mathbb{R},$$

$$L_2 : \vec{r} = \vec{k} + \mu\hat{j},\ \mu \in \mathbb{R} \text{ and}$$

$$L_3 : \vec{r} = \hat{i} + \hat{j} + v\hat{k},\ v \in \mathbb{R}$$

are given. For which point(s) Q on L_2 can we find a point P on L_1 and a point R on L_3 so that P, Q and R are collinear ?

(1) $\hat{k} + \hat{j}$ (2) $\hat{k}$ (3) $\hat{k} + \frac{1}{2}\hat{j}$ (4) $\hat{k} - \frac{1}{2}\hat{j}$

Ans. (3,4)

Sol. Let P(λ, 0, 0), Q(0, μ, 1), R(1, 1, ν) be points. L_1, L_2 and L_3 respectively

Since P, Q, R are collinear, $\overrightarrow{PQ}$ iscollinear with $\overrightarrow{QR}$

Hence $\frac{-\lambda}{1} = \frac{\mu}{1-\mu} = \frac{1}{\nu - 1}$

For every $\mu \in R - \{0, 1\}$ there exist unique $\lambda, \nu \in R$

Hence Q cannot have coordinates (0, 1, 1) and (0, 0, 1).

4. Let F : $\mathbb{R} \to \mathbb{R}$ be a function. We say that f has

PROPERTY 1 if $\lim_{h\to 0} \frac{f(h) - f(0)}{\sqrt{|h|}}$ exists and is finite, and

PROPERTY 2 if $\lim_{h\to 0} \frac{f(h) - f(0)}{h^2}$ exists and is finite.

Then which of the following options is/are correct ?

(1) f(x) = x|x| has PROPERTY 2 (2) f(x) = $x^{2/3}$ has PROPERTY 1

(3) f(x) = sin x has PROPERTY 2 (4) f(x) = |x| has PROPERTY 1

Ans. (2,4)

Sol. P -1 :

$$\lim_{h\to 0} \frac{f(h) - f(0)}{\sqrt{|h|}} = \text{exist and finite}$$

(B) $f(x) = x^{2/3}, \lim_{h\to 0} \frac{h^{2/3} - 0}{\sqrt{|h|}} = \lim_{h\to 0} \frac{|h|^{2/3}}{\sqrt{|h|}} = 0$

(D) $f(x)=|x|, \lim_{h\to 0}\frac{|h|-0}{\sqrt{|h|}} \Rightarrow \lim_{h\to 0}\sqrt{|h|}=0$

P-2 :

$\lim_{h\to 0}\frac{f(h)-f(0)}{h^2}$ = exist and finite

(A) $f(x) = x|x|$, $\lim_{h\to 0}\frac{h|h|-0}{h^2} = \left[\begin{array}{l} RHL = \lim_{h\to 0}\frac{h^2}{h^2}=1 \\ LHL = \lim_{h\to 0}\frac{-h^2}{h^2}=-1 \end{array}\right.$

(C) $f(x) = \sin x$ $\lim_{h\to 0}\frac{\sin h-0}{h^2} = \text{DNE}$

5. For non-negative integers n, let

$$f(n)=\frac{\sum_{k=0}^{n}\sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n}\sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming $\cos^{-1} x$ takes values in $[0, \pi]$, which of the following options is/are correct ?

(1) $\sin(7\cos^{-1} f(5)) = 0$

(2) $f(4)=\frac{\sqrt{3}}{2}$

(3) $\lim_{n\to\infty} f(n)=\frac{1}{2}$

(4) If $\alpha = \tan(\cos^{-1} f(6))$, then $\alpha^2 + 2\alpha - 1 = 0$

Ans. (1,2,4)

Sol. $$f(n)=\frac{\sum_{k=0}^{n}\left(\cos\left(\frac{\pi}{n+2}\right)-\cos\left(\frac{2k+3}{n+2}\right)\pi\right)}{\sum_{k=0}^{n}\left(1-\cos\left(\frac{2k+2}{n+2}\right)\pi\right)}$$

$$f(n)=\frac{(n+1)\cos\left(\frac{\pi}{n+2}\right)-\left(\sum_{k=0}^{n}\cos\left(\frac{2k+3}{n+2}\right)\pi\right)}{(n+1)-\left(\sum_{k=0}^{n}\cos\left(\frac{2k+2}{n+2}\right)\pi\right)}$$

$$f(n)=\frac{(n+1)\cos\frac{\pi}{n+2}-\left(\frac{\sin\left(\frac{(n+1)\pi}{n+2}\right)}{\sin\left(\frac{\pi}{n+2}\right)}.\cos\left(\frac{n+3}{n+2}\right)\pi\right)}{(n+1)-\left(\frac{\sin\left(\frac{(n+1)\pi}{n+2}\right)}{\sin\left(\frac{\pi}{n+2}\right)}.\cos\left(\frac{2(n+2)\pi}{2(n+2)}\right)\right)}$$

$$f(n)=\frac{(n+1)\cos\left(\frac{\pi}{n+2}\right)+\cos\left(\frac{\pi}{n+2}\right)}{(n+1)+1}\Rightarrow g(x)=\cos\left(\frac{\pi}{n+2}\right)$$

(A) $\sin\left(7\cos^{-1}\cos\frac{\pi}{7}\right)=\sin\pi=0$

(B) $f(4)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$

(C) $\lim\limits_{n\to\infty}\cos\left(\frac{\pi}{n+2}\right)=1$

(D) $\alpha=\tan\left(\cos^{-1}\cos\frac{\pi}{8}\right)=\sqrt{2}-1\Rightarrow\alpha+1=\sqrt{2}$

$\alpha^2+2\alpha-1=0$

6. Let $P_1=I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $P_2=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$, $P_3=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$, $P_4=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$, $P_5=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$,

$P_6=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ and $X=\sum\limits_{k=1}^{6}P_K\begin{bmatrix}2&1&3\\1&0&2\\3&2&1\end{bmatrix}P_K^T$

where P_K^T denotes the transpose of the matrix P_K. Then which of the following options is/are correct?

(1) $X-30I$ is an invertible matrix

(2) The sum of diagonal entries of X is 18

(3) If $X\begin{bmatrix}1\\1\\1\end{bmatrix}=\alpha\begin{bmatrix}1\\1\\1\end{bmatrix}$, then $\alpha = 30$

(4) X is a symmetric matrix

Ans. (2,3,4)

Sol. Let $Q = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

$$X = \sum_{k=1}^{6} \left(P_k Q P_K^T \right)$$

$$X^T = \sum_{k=1}^{6} \left(P_k Q P_K^T \right)^T = X.$$

X is symmetric

Let $R = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$XR = \sum_{k=1}^{6} P_K Q P_K^T R. \ [\because \ P_K{}^T R = R]$$

$$= \sum_{K=1}^{6} P_K QR. = \left(\sum_{K=1}^{6} P_K \right) QR$$

$$\sum_{K=1}^{6} P_K = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \qquad QR = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

$$\Rightarrow XR = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix} = 30R$$

$\Rightarrow \alpha = 30.$

Trace X = Trace $\left(\sum_{K=1}^{6} P_K Q P_K^T \right)$

$$= \sum_{K=1}^{6} \text{Trace}\left(P_K Q P_K^T\right) = 6\left(\text{Trace}\, Q\right) = 18$$

$$X\begin{bmatrix}1\\1\\1\end{bmatrix} = 30\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\Rightarrow \left(X - 30I\right)\begin{bmatrix}1\\1\\1\end{bmatrix} = O \Rightarrow \left|X - 30I\right| = 0$$

$\Rightarrow X - 30I$ is non-invertible

7. Let $x \in \mathbb{R}$ and let $P = \begin{bmatrix}1 & 1 & 1\\0 & 2 & 2\\0 & 0 & 3\end{bmatrix}$, $Q = \begin{bmatrix}2 & x & x\\0 & 4 & 0\\x & x & 6\end{bmatrix}$ and $R = PQP^{-1}$.

Then which of the following options is/are correct ?

(1) For x = 1, there exists a unit vector $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ for which $R\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(2) There exists a real number x such that PQ = QP

(3) det R = det $\begin{bmatrix}2 & x & x\\0 & 4 & 0\\x & x & 5\end{bmatrix}$ + 8, for all $x \in \mathbb{R}$

(4) For x = 0, if $R\begin{bmatrix}1\\a\\b\end{bmatrix} = 6\begin{bmatrix}1\\a\\b\end{bmatrix}$, then a + b = 5

Ans. (3,4)

Sol. $\det(R) = \det(PQP^{-1}) = (\det P)(\det Q)\left(\dfrac{1}{\det P}\right)$

$= \det Q$

$= 48 - 4x^2$

Option-1 :

for x = 1 det (R) = 44 $\neq$ 0

$\therefore$ for equation $R\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

We will have trivial solution

$\alpha = \beta = \gamma = 0$

Option-2 :

$PQ = QP$

$PQP^{-1} = Q$

$R = Q$

No value of x.

Option-3 :

$$\det\begin{bmatrix}2 & x & x\\0 & 4 & 0\\x & x & 5\end{bmatrix}+8$$

$= (40 - 4x^2) + 8 = 48 - 4x^2 = \det R \ \forall\ x \in R$

Option-4 :

$$R=\begin{bmatrix}2 & 1 & 2/3\\0 & 4 & 4/3\\0 & 0 & 6\end{bmatrix}$$

$$(R-6I)\begin{bmatrix}1\\a\\b\end{bmatrix}=O$$

$$\Rightarrow -4+a+\frac{2b}{3}=0$$

$$-2a+\frac{4b}{3}=0$$

$\Rightarrow a = 2 \qquad b = 3$

$a + b = 5$

8. Let $f(x)=\dfrac{\sin \pi x}{x^2}, x>0$

Let $x_1 < x_2 < x_3 < ... < x_n < ...$ be all the points of local maximum of f and $y_1 < y_2 < y_3 < ... < y_n < ...$ be all the points of local minimum of f. Then which of the following options is/are correct ?

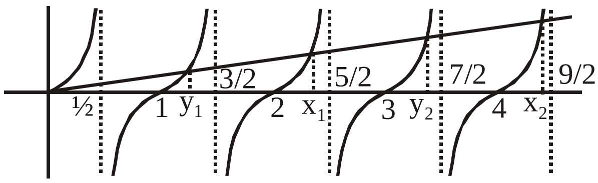
(1) $|x_n - y_n| > 1$ for every n (2) $x_1 < y_1$

(3) $x_n \in \left(2n, 2n+\dfrac{1}{2}\right)$ for every n (4) $x_{n+1} - x_n > 2$ for every n

Ans. **(1,3,4)**

Sol. $f(x)=\dfrac{\sin \pi x}{x^2}$

$$f'(x)=\frac{2x\cos \pi x\left(\dfrac{\pi x}{2}-\tan \pi x\right)}{x^4}$$



$\Rightarrow$ $|x_n - y_n| > 1$ for every n

$x_1 > y_1$

$x_n \in (2n, 2n + 1/2)$

$x_{n+1} - x_n > 2.$

SECTION-2 : (Maximum Marks: 18)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **Two** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If ONLY the correct numerical value is entered.

Zero Marks : 0 In all other cases.

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1. The value of $\sec^{-1}\left(\dfrac{1}{4}\sum_{k=0}^{10}\sec\left(\dfrac{7\pi}{12}+\dfrac{k\pi}{2}\right)\sec\left(\dfrac{7\pi}{12}+\dfrac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\dfrac{\pi}{4},\dfrac{3\pi}{4}\right]$ equals

Ans. **(0.00)**

Sol. $$\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{1}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{12}\right)\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$$

$$=\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{\sin\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}-\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right).\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$$

$$=\sec^{-1}\left(\frac{1}{4}\left(\sum_{k=0}^{10}\tan\left(\frac{7\pi}{12}+(k+1)\frac{\pi}{2}\right)-\tan\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)\right)$$

$$=\sec^{-1}\left(\frac{1}{4}\left(\tan\left(\frac{11\pi}{2}+\frac{7\pi}{12}\right)-\tan\left(\frac{7\pi}{12}\right)\right)\right)$$

$$=\sec^{-1}\left(\frac{1}{4}\left(-\cot\frac{7\pi}{12}-\tan\frac{7\pi}{12}\right)\right)$$

$$=\sec^{-1}\left(\frac{1}{4}\left(-\frac{1}{\sin\frac{7\pi}{12}\cos\frac{7\pi}{12}}\right)\right)$$

$$=\sec^{-1}\left(-\frac{1}{2}\times\frac{1}{\sin\frac{7\pi}{6}}\right)=\sec^{-1}(1)=0.00$$

2. Let |X| denote the number of elements in set X. Let S = {1,2,3,4,5,6} be a sample space, where each element is equally likely to occur. If A and B are indepenent events associated with S, then the number of ordered pairs (A,B) such that $1 \leq |B| < |A|$, equals

Ans. (422.00)

Sol. $$P\left(\frac{B}{A}\right)=P(B)$$

$\Rightarrow$ $$n\frac{(A\cap B)}{n(A)}=\frac{n(B)}{n(s)} \qquad \text{.......... (1)}$$

$\Rightarrow$ n(A) should have 2 or 3 as prime factors

$\Rightarrow$ n(A) can be 2, 3, 4 or 6 as n(A) > 1 n(A) = 2 does not satisfy the constraint (1).

for $n(A) = 3$. $n(B) = 2$ and $n(A \cap B) = 1$

$\Rightarrow$ No. of ordered pair = ${}^6C_4 \times \frac{4!}{2!} = 180$

for $n(A) = 4$. $n(B) = 3$ and $n(A \cap B) = 2$

$\Rightarrow$ No. of ordered pairs = ${}^6C_5 \times \frac{5!}{2!2!} = 180$

for $n(A) = 6$. $n(B)$ can be 1, 2, 3, 4, 5.

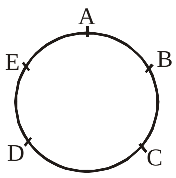
$\Rightarrow$ No. of ordered pairs = $2^6 - 2 = 62$

Total ordered pair = 180 + 180 + 62 = 422.

3. Five person A,B,C,D and E are seated in a ciruclar arrangement. If each of them is given a hat of one of the three colours red, blue and green ,then the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats is

Ans. (30.00)

Sol.



When 1R, 2B, 2G

$5C_1 \times 2 = 10$

Other possibilities

1B, 2R, 2G

or 1G, 2R, 2B

So total no. of ways = 3 × 10 = 30

4. Suppose

$$\det\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$$, holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals

Ans. (6.20)

Sol. Suppose

$$\begin{vmatrix} \frac{n(n+1)}{2} & n(n-1).2^{n-2}+n.2^{n-1} \\ n.2^{n-1} & 4^n \end{vmatrix} = 0$$

$$\frac{n(n+1)}{2}.4^n - n^2(n-1).2^{2n-3} - n^2 2^{2n-2} = 0$$

$$\frac{n(n+1)}{2} - \frac{n^2(n-1)}{8} - \frac{n^2}{4} = 0$$

$n^2 - 3n - 4 = 0$

$n = 4$

Now $\sum_{k=0}^{4} \frac{{}^4C_k}{k+1} = \sum_{k=0.}^{4} \frac{k+1}{5}.\,{}^5C_{k+1}\frac{1}{k+1}$

$$= \frac{1}{5}.\left[{}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5\right] \;= \frac{1}{5}\left[2^5 - 1\right] = \frac{31}{5} = 6.20$$

5. The value of the integral $\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} d\theta$ equals

Ans. (0.50)

Sol. $I = \int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} d\theta$

$$= \int_0^{\pi/2} \frac{3\sqrt{\sin\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sec\theta}\right)} d\theta$$

$$2I = \int_0^{\pi/2} \frac{3d\theta}{\left(\sqrt{\cos\theta}+\sqrt{\sec\theta}\right)^4}$$

$$= 3\int_0^{\pi/2} \frac{\sec^2\theta d\theta}{\left(1+\sqrt{\tan\theta}\right)^4}$$

Let $1+\sqrt{\tan\theta} = t$

$$\frac{\sec^2\theta}{2\sqrt{\tan\theta}}d\theta = dt$$

$\sec^2\theta d\theta = 2(t-1)dt$

$$= 3\int_1^{\infty}\frac{2(t-1)dt}{t^4}$$

$$= 6\int_1^{\infty}\left(t^{-3}-t^{-4}\right)dt$$

$$2I = 6\left(\frac{t^{-2}}{-2}-\frac{t^{-3}}{-3}\right)_1^{\infty} = 6\left[0-0-\left\{-\frac{1}{2}+\frac{1}{3}\right\}\right]$$

$I = 0.50$

6. Let $\vec{a} = 2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+\hat{k}$ be two vectors. Consider a vector $\vec{c} = \alpha\vec{a}+\beta\vec{b}$, $\alpha,\beta \in \mathbb{R}$. If the projection of $\vec{c}$ on the vector $\left(\vec{a}+\vec{b}\right)$ is $3\sqrt{2}$, then the minimum value of $\left(\vec{c}-\left(\vec{a}\times\vec{b}\right)\right).\vec{c}$ equals

Ans. (18.00)

Sol. $\vec{c} = (2\alpha+\beta)\hat{i}+\hat{j}(\alpha+2\beta)+\hat{k}(\beta-\alpha)$

$$\frac{\vec{c}.(\vec{a}+\vec{b})}{|\vec{a}+\vec{b}|} = 3\sqrt{2}$$

$\Rightarrow \quad \alpha + \beta = 2 \quad$ .... (1)

$(\vec{c}-(\vec{a}\times\vec{b})).(\alpha\vec{a}+\beta\vec{b})$

$= |\vec{c}|^2 = \alpha^2|\vec{a}|^2 + \beta^2|b|^2 + 2\alpha\beta(\vec{a}.\vec{b})$

$= 6(\alpha^2 + \beta^2 + \alpha\beta)$

$= 6(\alpha^2 + (2-\alpha)^2 + \alpha(2-\alpha))$

$= 6((\alpha-1)^2 + 3)$

$\Rightarrow \quad$ Min. value = 18

SECTION–3 : (Maximum Marks : 12)

- This section contains **TWO (02)** List-Match sets.
- Each List-Match set has **Two (02)** Multiple Choice Questions.
- Each List-Match set has two lists : **List-I** and **List-II**
- **List-I** has **Four** entries (I), (II), (III) and (IV) and **List-II** has **Six** entries (P), (Q), (R), (S), (T) and (U)
- **FOUR** options are given in each Multiple Choice Question based on **List-I** and **List-II** and **ONLY ONE** of these four options satisfies the condition asked in the Multiple Choice Question.
- Answer to each question will be evaluated according to the following marking scheme :

  Full Marks : +3 If ONLY the option corresponding to the correct combination is chosen.

  Zero Marks : 0 If none of the options is chosen (i.e., the question is unanswered);

  Negative Marks : –1 In all other cases

1. **Answer the following by appropriately matching the lists based on the information given in the paragraph**

Let $f(x) = \sin(\pi \cos x)$ and $g(x) = \cos(2\pi \sin x)$ be two functions defined for $x > 0$. Define the following sets whose elements are written in the increasing order :

$$X = \{x : f(x) = 0\}, \quad Y = \{x : f'(x) = 0\}$$
$$Z = \{x : g(x) = 0\}, \quad W = \{x : g'(x) = 0\}.$$

List-I contains the sets X,Y,Z and W. List -II contains some information regarding these sets.

List-I		**List-II**	
(I)	X	(P)	$\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II)	Y	(Q)	an arithmetic progression
(III)	Z	(R)	NOT an arithmetic progression
(IV)	W	(S)	$\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
		(T)	$\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
		(U)	$\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

Which of the following is the only CORRECT combination ?

Options

(1) (II), (R), (S) (2) (I), (P), (R)

(3) (II), (Q), (T) (4) (I), (Q), (U)

Ans. (3)

2. **Answer the following by appropriately matching the lists based on the information given in the paragraph**

Let $f(x) = \sin(\pi \cos x)$ and $g(x) = \cos(2\pi \sin x)$ be two functions defined for $x > 0$. Define the following sets whose elements are written in the increasing order :

$$X = \{x : f(x) = 0\}, \quad Y = \{x : f'(x) = 0\}.$$
$$Z = \{x : g(x) = 0\}, \quad W = \{x : g'(x) = 0\}.$$

List-I contains the sets X,Y,Z and W. List -II contains some information regarding these sets.

List-I		**List-II**	
(I)	X	(P)	$\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II)	Y	(Q)	an arithmetic progression
(III)	Z	(R)	NOT an arithmetic progression
(IV)	W	(S)	$\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
		(T)	$\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
		(U)	$\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

Which of the following is the only CORRECT combination ?

Options

(1) (IV), (Q), (T) (2) (IV), (P), (R), (S) (3) (III), (R), (U) (4) (III), (P), (Q), (U)

Ans. (2)

Solution Q.1 and Q.2

Q.1 Ans. (3)

Q.2 Ans. (2)

Sol. $f(x) = \sin(\pi \cos x)$

$X : \{x : f(x) = 0\}$

$f(x) = 0 \Rightarrow \sin(\pi \cos x) = 0 \Rightarrow \cos x = n \Rightarrow \cos x = 1, -1, 0 \Rightarrow x = \frac{n\pi}{2}$

$$X = \left\{\frac{n\pi}{2} : n \in N\right\} = \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, ...\right\}$$

$g(x) = \cos(2\pi \sin x)$

$Z = \{x : g(x) = 0\}$

$\cos(2\pi \sin x) = 0 \Rightarrow 2\pi \sin x = (2n + 1)\frac{\pi}{2} \Rightarrow \sin x = \frac{(2n+1)}{4}$

$$\sin x = -\frac{1}{4}, \frac{1}{4}, \frac{-3}{4}, \frac{3}{4}$$

$$Z = \left\{n\pi \pm \sin^{-1}\left(\frac{1}{4}\right), n\pi \pm \sin^{-1}\left(\frac{3}{4}\right), n \in I\right\}$$

$Y = \{x : f'(x) = 0\}$

$f(x) = \sin(\pi \cos x) \Rightarrow f'(x) = \cos(\pi \cos x) . (-\pi \sin x) = 0$

$\sin x = 0 \Rightarrow x = n\pi.$

$\cos(\pi \cos x) = 0 \Rightarrow \pi \cos x = (2n + 1)\frac{\pi}{2} \Rightarrow \cos x = \frac{(2n+1)}{2} \Rightarrow \cos x = -\frac{1}{2}, \frac{1}{2}$

$$Y = \left\{n\pi, n\pi \pm \frac{\pi}{3}\right\} = \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi, ...\right\}$$

$W = \{x : g'(x) = 0\}$

$g(x) = \cos(2\pi \sin x) \Rightarrow g'(x) = -\sin(2\pi \sin x).(2\pi \cos x) = 0$

$\cos x = 0 \Rightarrow x = (2n + 1)\frac{\pi}{2}$

$\sin(2\pi \sin x) = 0 \Rightarrow 2\pi \sin x = n\pi \Rightarrow \sin x = \frac{n}{2} = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$

$$W = \left\{\frac{n\pi}{2}, n\pi \pm \frac{\pi}{6}, n \in I\right\} = \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, ...\right\}$$

Now check the options

3. **Answer the following by appropriately matching the lists based on the information given in the paragraph**

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions :

(i) centre of C_3 is collinear with the centres of C_1 and C_2

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expression given in the List-I whose values are given in List-II below:

	List-I		**List-II**
(I)	$2h + k$	(P)	6
(II)	$\dfrac{\text{Length of ZW}}{\text{Length of XY}}$	(Q)	$\sqrt{6}$
(III)	$\dfrac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	(R)	$\dfrac{5}{4}$
(IV)	α	(S)	$\dfrac{21}{5}$
		(T)	$2\sqrt{6}$
		(U)	$\dfrac{10}{3}$

Which of the following is the only INCORRECT combination ?

Options

(1) (IV), (S) (2) (IV), (U) (3) (III), (R) (4) (I), (P)

Ans. (1)

4. **Answer the following by appropriately matching the lists based on the information given in the paragraph**

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions :

(i) centre of C_3 is collinear with the centres of C_1 and C_2

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

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(III)	$\dfrac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	(R)	$\dfrac{5}{4}$
(IV)	α	(S)	$\dfrac{21}{5}$
		(T)	$2\sqrt{6}$
		(U)	$\dfrac{10}{3}$

Which of the following is the only CORRECT combination ?

Options

(1) (II), (T) (2) (I), (S)

(3) (I), (U) (4) (II), (Q)

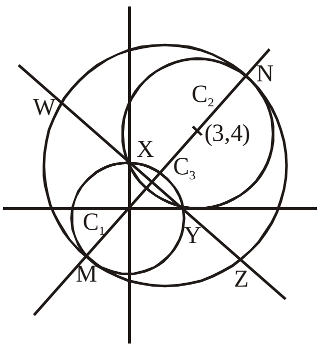
Ans. (4)

Solution Q.3 and Q.4

Q.3 Ans. (1)

Q.4 Ans. (4)

Sol.



$MC_1 + C_1C_2 + C_2N = 2r$

$\Rightarrow 3 + 5 + 4 = 2r \Rightarrow r = 6 \Rightarrow$ Radius of $C_3 = 6$

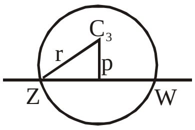
Suppose centre of C_3 be $(0 + r_4\cos\theta, 0 + r_4\sin\theta)$, $\left\{\begin{aligned} & r_4 = C_1C_3 = 3 \\ & \tan\theta = \frac{4}{3} \end{aligned}\right\}$

$C_3 = \left(\frac{9}{5}, \frac{12}{5}\right) = (h,k) \Rightarrow 2h + k = 6$

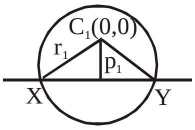
Equation of ZW and XY is $3x + 4y - 9 = 0$

(common chord of circle $C_1 = 0$ and $C_2 = 0$)

$ZW = 2\sqrt{r^2 - p^2} = \frac{24\sqrt{6}}{5}$ (where $r = 6$ and $p = \frac{6}{5}$)



$XY = 2\sqrt{r_1^2 - p_1^2} = \frac{24}{5}$ (where $r_1 = 3$ and $p_1 = \frac{9}{5}$)



$$\frac{\text{Length of ZW}}{\text{Length of XY}} = \sqrt{6}$$

Let length of perpendicular from M to ZW be λ, $\lambda = 3 + \frac{9}{5} = \frac{24}{5}$

$$\frac{\text{Area of } \Delta MZN}{\text{Area of } \Delta ZMW} = \frac{\frac{1}{2}(MN) \times \frac{1}{2}(ZW)}{\frac{1}{2} \times ZW \times \lambda} = \frac{1}{2}\frac{MN}{\lambda} = \frac{5}{4}$$

$$C_3 : \left(x - \frac{9}{5}\right)^2 + \left(y - \frac{12}{5}\right)^2 = 6^2$$

$C_1 : x^2 + y^2 - 9 = 0$

common tangent to C_1 and C_3 is common chord of C_1 and C_3 is $3x + 4y + 15 = 0$.

Now $3x + 4y + 15 = 0$ is tangent to parabola $x^2 = 8\alpha y$.

$$x^2 = 8\alpha\left(\frac{-3x - 15}{4}\right) \Rightarrow 4x^2 + 24\alpha x + 120\alpha = 0$$

$$D = 0 \Rightarrow \alpha = \frac{10}{3}$$