PART-III : MATHEMATICS

SECTION - 1 (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	: +4	If only (all) the correct option(s) is(are) chosen;
Partial Marks	: +3	If all the four options are correct but ONLY three options are chosen;
Partial Marks	: +2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks	: +1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks	: 0	If unanswered;
Negative Marks	: – 2	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  choosing ONLY (A), (B) and (D) will get +4 marks;
  choosing ONLY (A) and (B) will get +2 marks;
  choosing ONLY (A) and (D) will get +2 marks;
  choosing ONLY (B) and (D) will get +2 marks;
  choosing ONLY (A) will get +1 mark;
  choosing ONLY (B) will get +1 mark;
  choosing ONLY (D) will get +1 mark;
  choosing no option(s) (i.e. the question is unanswered) will get 0 marks and
  choosing any other option(s) will get “2 marks.

1. Let $S_1 = \{(i,j,k) : i,j,k \in \{1,2,...,10\}\}$,

$S_2 = \{(i,j) : 1 \le i < j+2 \le 10, i,j \in \{1,2,...,10\}\}$,

$S_3 = \{(i,j,k,l) : 1 \le i < j < k < l, i,j,k,l \in \{1,2,...,10\}\}$,

and $S_4 = \{(i,j,k,l) :$ i, j, k and l are distinct elements in $\{1,2,...,10\}\}$.

If the total number of elements in the set S_r is n_r, r = 1, 2,3,4, then which of the following statements is (are) TRUE?

(A) $n_1 = 1000$

(B) $n_2 = 44$

(C) $n_3 = 220$

(D) $\frac{n_4}{12} = 420$

Answer (A,B,D)

Sol. Number of elements in S_1 = 10 × 10 × 10 = 1000

Number of elements in S_2 = 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 = 44

Number of elements in S_3 = ${}^{10}C_4$ = 210

Number of elements in S_4 = ${}^{10}P_4$ = 210 × 4! = 5040

2. Consider a triangle PQR having sides of lengths p,q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) TRUE ?

(A) $\cos P \geq 1 - \dfrac{p^2}{2qr}$

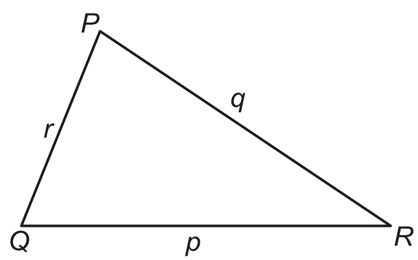
(B) $\cos R \geq \left(\dfrac{q-r}{p+q}\right)\cos P + \left(\dfrac{p-r}{p+q}\right)\cos Q$

(C) $\dfrac{q+r}{p} < 2\dfrac{\sqrt{\sin Q \sin R}}{\sin P}$

(D) If $p < q$ and $p < r$, then $\cos Q > \dfrac{p}{r}$ and $\cos R > \dfrac{p}{q}$

Answer (A,B)

Sol.



$$\cos P = \frac{q^2+r^2-P^2}{2qr} \quad \text{and} \quad \frac{q^2+r^2}{2} \geq \sqrt{q^2.r^2} \quad (\text{AM} \geq \text{GM})$$

$\Rightarrow$ $q^2 + r^2 \geq 2qr$

So, $\cos P \geq \dfrac{2qr - p^2}{2qr}$

$$\cos P \geq 1 - \frac{p^2}{2qr} \qquad \text{(A)}$$

(B) $\dfrac{(q-r)\cos P + (p-r)\cos Q}{p+q} = \dfrac{(q\cos P + p\cos Q) - r(\cos P + \cos Q)}{p+q}$

$$= \frac{r(1-\cos P - \cos Q)}{p+q} = \frac{r(q - p\cos R) - (p - q\cos R)}{p+q} = \frac{(r-p-q)+(p+q)\cos R}{p+q}$$

$$= \cos R + \frac{r-q-p}{p+q} \leq \cos R \ \ (\because r < p+q)$$

(C) $\dfrac{q+r}{p} = \dfrac{\sin Q + \sin R}{\sin P} \geq \dfrac{2\sqrt{\sin Q \cdot \sin R}}{\sin P}$

(D) If $p < q$ and $q < r$

So, p is the smallest side, therefore one of Q or R can be obtuse

So, one of $\cos Q$ or $\cos R$ can be negative

Therefore $\cos Q > \dfrac{p}{r}$ and $\cos R > \dfrac{p}{q}$ cannot hold always.

3. Let $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$ be a continuous function such that $f(0)=1$ and $\int_0^{\frac{\pi}{3}} f(t)dt=0$

Then which of the following statements is (are) TRUE?

(A) The equation $f(x) - 3\cos 3x = 0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(B) The equation $f(x) - 3\sin 3x = -\frac{6}{\pi}$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(C) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=-1$

(D) $\lim\limits_{x\to 0}\frac{\sin x\int_0^x f(t)dt}{x^2}=-1$

Answer (A,B,C)

Sol. $f(0)=1$, $\int_0^{\frac{\pi}{3}} f(t)dt=0$

(A) Consider a function $g(x)=\int_0^x f(t)dt-\sin 3x$

$g(x)$ is continuous and differentiable function

and $g(0) = 0$

$$g\left(\frac{\pi}{3}\right)=0$$

By Rolle's theorem $g'(x) = 0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

$$f(x)-3\cos 3x=0 \text{ for some } x\in\left(0,\frac{\pi}{3}\right)$$

(B) Consider a function

$$h(x)=\int_0^x f(t)dt+\cos 3x+\frac{6}{\pi}x$$

$h(x)$ is continuous and differentiable function

and $h(0) = 1$

$$h\left(\frac{\pi}{3}\right)=1$$

By Rolle's theorem $h'(x) = 0$ for at least one $x\in\left(0,\frac{\pi}{3}\right)$

$$f(x)-3\sin 3x+\frac{6}{\pi}=0 \text{ for some } x\in\left(0,\frac{\pi}{3}\right)$$

(C) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}$, $\left(\frac{0}{0}\text{form}\right)$

By L' Hopital rule

$$\lim_{x\to 0} \frac{xf(x)+\int_0^x f(t)dt}{-2xe^{x^2}}, \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0} \frac{xf'(x)+f(x)+f(x)}{-4x^2e^{x^2}-2e^{x^2}} = \frac{0+2f(0)}{-0-2} = -1$$

(D) $$\lim_{x\to 0} \frac{\sin x \int_0^x f(t)dt}{x^2}, \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0} \frac{\sin x\cdot f(x)+\cos x\int_0^x f(t)dt}{2x}$$

$$= \lim_{x\to 0} \frac{\left(\cos x.f(x)+\sin x.f'(x)+\cos x.f(x)-\sin x.\int_0^x f(t)dt\right)}{2}$$

$$= \frac{1+0+1-0}{2}$$

$$= 1$$

4. For any real numbers α and β, let $y_{\alpha,\beta}(x)$, $x \in \mathbb{R}$, be the solution of the differential equation $\frac{dy}{dx}+\alpha y = xe^{\beta x}, y(1)=1$

Let $S = \{y_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R}\}$. Then which of the following functions belong(s) to the set S?

(A) $f(x) = \frac{x^2}{2}e^{-x} + \left(e-\frac{1}{2}\right)e^{-x}$

(B) $f(x) = -\frac{x^2}{2}e^{-x} + \left(e+\frac{1}{2}\right)e^{-x}$

(C) $f(x) = \frac{e^x}{2}\left(x-\frac{1}{2}\right) + \left(e-\frac{e^2}{4}\right)e^{-x}$

(D) $f(x) = \frac{e^x}{2}\left(\frac{1}{2}-x\right) + \left(e+\frac{e^2}{4}\right)e^{-x}$

Answer (A, C)

Sol. $\frac{dy}{dx}+\alpha y = xe^{\beta x}$

Integrating factor (I.F.) = $e^{\int \alpha dx} = e^{\alpha x}$

So, the solution is $y\cdot e^{\alpha x} = \int xe^{\beta x}\cdot e^{\alpha x}dx$

$$ye^{\alpha x} = \int xe^{(\alpha+\beta)x}dx$$

If $\alpha+\beta \neq 0$

$$ye^{\alpha x} = x\frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)} - \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2} + C$$

$$y = \frac{xe^{\beta x}}{(\alpha+\beta)} - \frac{e^{\beta x}}{(\alpha+\beta)^2} + Ce^{-\alpha x}$$

$$y = \frac{e^{\beta x}}{(\alpha+\beta)}\left(x - \frac{1}{\alpha+\beta}\right) + Ce^{-\alpha x} \quad \text{... (i)}$$

Put $\alpha = \beta = 1$ in (i)

$$y = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + Ce^{-x}$$

$y(1) = 1$

$$1 = \frac{e}{2} \times \frac{1}{2} + \frac{C}{e} \Rightarrow C = e - \frac{e^2}{4}$$

So, $y = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$

If $\alpha + \beta = 0$ & $\alpha = 1$

$$\frac{dy}{dx} + y = xe^{-x}$$

I.F. $= e^x$

$$ye^x = \int x\,dx$$

$$ye^x = \frac{x^2}{2} + C$$

$$y = \frac{x^2}{2}e^{-x} + Ce^{-x}$$

$y(1) = 1$

$$1 = \frac{1}{2e} + \frac{C}{e} \Rightarrow C = e - \frac{1}{2}$$

$$y = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$$

5. Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$ for some $\lambda > 0$. If $\left|\overrightarrow{OB} \times \overrightarrow{OC}\right| = \frac{9}{2}$, then which of the following statements is (are) TRUE ?

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

Answer (A,B,C)

Sol. $OA = 2\hat{i} + 2\hat{j} + \hat{k}$

$\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$

$\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$

$\overrightarrow{OB} \times \overrightarrow{OC} = \overrightarrow{OB} \times \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA}) = -\frac{\lambda}{2}\overrightarrow{OB} \times \overrightarrow{OA} = \frac{\lambda}{2}(\overrightarrow{OA} \times \overrightarrow{OB})$

Now, $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = 6\hat{i} - 3\hat{j} - 6\hat{k}$

So, $\overrightarrow{OB} \times \overrightarrow{OC} = \frac{3\lambda}{2}(2\hat{i} - \hat{j} - 2\hat{k})$

$\left|\overrightarrow{OB} \times \overrightarrow{OC}\right| = \left|\frac{9\lambda}{2}\right| = \frac{9}{2}$

So, $\lambda = 1 \qquad (\because \lambda > 0)$

$\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})$

$\overrightarrow{OC} = \frac{1}{2}(-\hat{i} - 4\hat{j} + \hat{k})$

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ = $\frac{\overrightarrow{OC} \cdot \overrightarrow{OA}}{\left|\overrightarrow{OA}\right|} = \frac{\frac{1}{2}(-2-8+1)}{3} = -\frac{3}{2}$

(B) Area of the triangle OAB = $\frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = \frac{9}{2}$

(C) Area of the triangle ABC is = $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -4 & 1 \\ -\frac{5}{2} & -4 & -\frac{1}{2} \end{vmatrix} \right\| = \frac{1}{2}\left|6\hat{i} - 3\hat{j} - 6\hat{k}\right| = \frac{9}{2}$

(D) Acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ $= \theta$

$$\frac{(\overrightarrow{OA} + \overrightarrow{OC}) \cdot (\overrightarrow{OA} - \overrightarrow{OC})}{\left|\overrightarrow{OA} + \overrightarrow{OC}\right| \left|\overrightarrow{OA} - \overrightarrow{OC}\right|} = \cos\theta$$

$$\cos\theta = \frac{\left(\frac{3}{2}\hat{i} + \frac{3}{2}\hat{k}\right) \cdot \left(\frac{5}{2}\hat{i} + 4\hat{j} + \frac{1}{2}\hat{k}\right)}{\frac{3}{2}\sqrt{2} \times \sqrt{\frac{90}{4}}} = \frac{18}{3\sqrt{2}\sqrt{90}}$$

$\theta \neq \frac{\pi}{3}$

6. Let E denote the parabola $y^2 = 8x$. Let $P = (-2, 4)$, and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) TRUE ?

(A) The triangle PFQ is a right-angled triangle

(B) The triangle QPQ' is a right-angled triangle

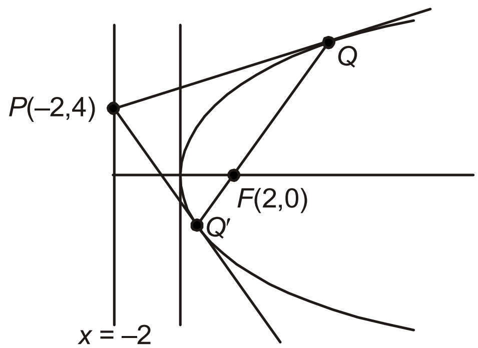
(C) The distance between P and F is $5\sqrt{2}$

(D) F lies on the line joining Q and Q'

Answer (A,B,D)

Sol. $E : y^2 = 8x$

$P : (-2, 4)$



Point P (–2, 4) lies on directrix ($x = -2$) of parabola $y^2 = 8x$

So, $\angle QPQ' = \frac{\pi}{2}$ and chord QQ' is a focal chord and segment PQ subtends right angle at the focus.

Slope of $QQ' = \frac{2}{t_1 + t_2} = 1$

Slope of $PF = -1$

$PF = 4\sqrt{2}$

SECTION - 2 (Maximum Marks : 12)

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+2	If ONLY the correct numerical value is entered.
Zero Marks	:	0	In all other cases.

Question Stem for Question Nos. 7 and 8

Consider the region $R = \{(x, y) \in R \times R : x \geq 0 \text{ and } y^2 \leq 4 - x\}$. Let F be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in F. Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$.

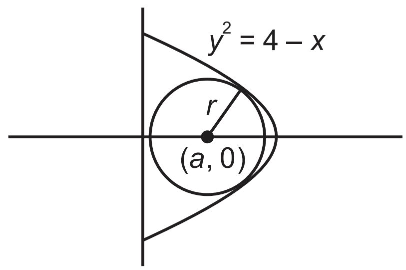
7. The radius of the circle C is _____.

Answer (1.50)

8. The value of α is _____.

Answer (2.00)

Sol. For comprehension Q7 & Q8



Let the circle be,

$(x - a)^2 + y^2 = r^2$

Solving it with parabola

$y^2 = 4 - x$ we get

$(x - a)^2 + 4 - x = r^2$

$\Rightarrow \; x^2 - x(2a + 1) + (a^2 + 4 - r^2) = 0$ ...(1)

$D = 0$

$\Rightarrow \; 4r^2 + 4a - 15 = 0$

Clearly $a \geq r$

So $4r^2 + 4r - 15 \leq 0$

$$\Rightarrow \; r_{\max} = \frac{3}{2} = a$$

Radius of circle C is $\dfrac{3}{2}$

From (1) $x^2 - 4x + 4 = 0$

$\Rightarrow \; x = 2 = \alpha$

Question Stem for Question Nos. 9 and 10

Let $f_1 : (0, \infty) \to \mathrm{R}$ and $f_2 : (0, \infty) \to R$ be defined by $f_1(x) = \int_0^x \prod_{j=1}^{21} (t - j)^j \, dt, \; x > 0$

and $f_2(x) = 98(x - 1)^{50} - 600(x - 1)^{49} + 2450$, $x > 0$, where, for any positive integer n and real number $a_1, a_2, \ldots a_n$, $\prod_{i=1}^{n} a_i$ denotes the product of $a_1, a_2, \ldots a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i = 1, 2$, in the interval $(0, \infty)$.

Solution for Q9 and 10

$$f_1'(x)=\prod_{j=1}^{21}(x-j)^j$$

$$f_1'(x)=(x-1)(x-2)^2(x-3)^3,\dots,(x-20)^{20}(x-21)^{21}$$

Checking the sign scheme of $f_1'(x)$ at x = 1, 2, 3, ..., 21, we get

$f_1(x)$ has local minima at x = 1, 5, 9, 13, 17, 21 and local maxima at x = 3, 7, 11, 15, 19

$\Rightarrow$ $m_1 = 6$, $n_1 = 5$

$f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450$

$$f_2'(x)=98\times 50(x-1)^{49}-600\times 49\times(x-1)^{48}$$

$= 98 \times 50 \times (x-1)^{48}(x-7)$

$f_2(x)$ has local minimum at x = 7 and no local maximum.

$\Rightarrow$ $m_2 = 1$, $n_2 = 0$

9. The value of $2m_1 + 3n_1 + m_1n_1$ is _____.

Answer (57.00)

Sol. $2m_1 + 3n_1 + m_1n_1$

$= 2 \times 6 + 3 \times 5 + 6 \times 5$

$= 57$

10. The value of $6m_2 + 4n_2 + 8m_2n_2$ is _____.

Answer (06.00)

Sol. $6m_2 + 4n_2 + 8m_2n_2$

$= 6 \times 1 + 4 \times 0 + 8 \times 1 \times 0 = 6$

Question Stem for Question Nos. 11 and 12

Let $g_i:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to \mathbb{R},\ i=1,2,$ and $f:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to \mathbb{R}$ be functions such that $g_1(x) = 1$, $g_2(x) = |4x - \pi|$ and $f(x) = \sin^2 x$, for all $x\in\left[\frac{\pi}{8},\frac{3\pi}{8}\right]$. Define $S_i=\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x)\cdot g_i(x)\,dx,\ i=1,2$

11. The value of $\frac{16S_1}{\pi}$ is _____.

Answer (02.00)

Sol. $$S_1=\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}}\sin^2 x.1dx$$

$$=\frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}}(1-\cos 2x)\,dx$$

$$=\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)\Bigg]_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$S_1=\frac{1}{2}\left(\frac{\pi}{4}-0\right)=\frac{\pi}{8}$$

$$\Rightarrow \quad \frac{16S_1}{\pi}=2$$

12. The value of $\frac{48S_2}{\pi^2}$ is _____.

Answer (01.50)

Sol. $S_2=\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}}\sin^2 x.|4x-\pi|\,dx$

$$=\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}}4\sin^2 x\left|x-\frac{\pi}{4}\right|dx$$

Let $x-\frac{\pi}{4}=t \Rightarrow \; dx=dt$

$$S_2=\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}4\sin^2\left(\frac{\pi}{4}+t\right)|t|\,dt$$

$$=\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}2\left(1-\cos 2\left(\frac{\pi}{4}+t\right)\right)|t|\,dt$$

$$=\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}(2+2\sin 2t)|t|\,dt$$

$$=2\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}|t|\,dt+2\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}|t|\sin(2t)\,dt$$

$$=4\int_{0}^{\frac{\pi}{8}}t\,dt+0$$

$$S_2=2t^2\Big]_0^{\frac{\pi}{8}}=\frac{\pi^2}{32}$$

$$\Rightarrow \quad \frac{48S_2}{\pi^2}=\frac{3}{2}$$

SECTION - 3 (Maximum Marks : 12)

- This section contains **TWO (02) paragraphs**. Based on each paragraph, there are **TWO (02) questions.**
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+3	If ONLY the correct option is chosen;
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	:	-1	In all other cases.

Paragraph

Let $M = \{(x, y) \in R \times R : x^2 + y^2 \leq r^2\}$, where $r > 0$. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}$, $n = 1, 2, 3, ...$ Let $S_0 = 0$ and, for $n \geq 1$, let S_n denote the sum of the first n terms of this progression. For $n \geq 1$, let C_n denote the circle with center $(S_{n-1}, 0)$ and radius a_n, and D_n denote the circle with center (S_{n-1}, S_{n-1}) and radius a_n.

13. Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then

(A) $k + 2l = 22$

(B) $2k + l = 26$

(C) $2k + 3l = 34$

(D) $3k + 2l = 40$

Answer (D)

Sol. $\because$ $a_n = \frac{1}{2^{n-1}}$ and $S_n = 2\left(1 - \frac{1}{2^n}\right)$

For circles C_n to be inside M.

$$S_{n-1} + a_n < \frac{1025}{513}$$

$$\Rightarrow \quad S_n < \frac{1025}{513}$$

$$\Rightarrow \quad 1 - \frac{1}{2^n} < \frac{1025}{1026} = 1 - \frac{1}{1026}$$

$\Rightarrow$ $2^n < 1026$

$\Rightarrow$ $n \leq 10$

$\therefore$ Number of circles inside be $10 = K$

Clearly alternate circles do not intersect each other *i.e.*, C_1, C_3, C_5, C_7, C_9 do not intersect each other as well as C_2, C_4, C_6, C_8 and C_{10} do not intersect each other hence maximum 5 set of circles do not intersect each other.

$\therefore$ $l = 5$

$\therefore$ $3K + 2l = 40$

$\therefore$ Option (D) is correct

14. Consider M with $r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is

(A) 198 (B) 199

(C) 200 (D) 201

Answer (B)

Sol. $\because \quad r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$

Now, $\sqrt{2}\, S_{n-1} + a_n < \left(\frac{2^{199}-1}{2^{198}}\right)\sqrt{2}$

$$2\cdot\sqrt{2}\left(1-\frac{1}{2^{n-1}}\right)+\frac{1}{2^{n-1}} < \left(\frac{2^{199}-1}{2^{198}}\right).$$

$$\therefore \quad 2\sqrt{2}-\frac{\sqrt{2}}{2^{n-2}}+\frac{1}{2^{n-1}} < 2\sqrt{2}-\frac{\sqrt{2}}{2^{198}}$$

$$\frac{1}{2^{n-2}}\left(\frac{1}{2}-\sqrt{2}\right) < -\frac{\sqrt{2}}{2^{198}}$$

$$\frac{2\sqrt{2}-1}{2\cdot 2^{n-2}} > \frac{\sqrt{2}}{2^{198}}$$

$$2^{n-2} < \left(2-\frac{1}{\sqrt{2}}\right)2^{197}$$

$\therefore \quad n \le 199$

$\therefore \quad$ Number of circles = 199

Option (B) is correct.

Paragraph

Let $\psi_1 : [0, \infty) \to \mathbb{R}$, $\psi_2 : [0, \infty) \to \mathbb{R}$, $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ be functions such that $f(0) = g(0) = 0$,

$$\psi_1(x) = e^{-x} + x,\ x \ge 0,$$

$$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2,\ x \ge 0,$$

$$f(x) = \int_{-x}^{x} \left(|t| - t^2\right) e^{-t^2}\, dt,\ x > 0$$

and $$g(x) = \int_0^{x^2} \sqrt{t}\, e^{-t}\, dt,\ x > 0.$$

15. Which of the following statements is TRUE?

(A) $f\left(\sqrt{\ln 3}\right) + g\left(\sqrt{\ln 3}\right) = \frac{1}{3}$

(B) For every $x > 1$, there exists an $\alpha \in (1, x)$ such that $\Psi_1(x) = 1 + \alpha x$

(C) For every $x > 0$, there exists a $\beta \in (0, x)$ such that $\Psi_2(x) = 2x\,(\Psi_1(\beta) - 1)$

(D) f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

Answer (C)

Sol. $\because \quad g(x) = \int_0^{x^2} \sqrt{t}\, e^{-t}\, dt, \ \ x > 0$

Let $t = u^2 \Rightarrow dt = 2u\ du$

$\therefore \quad g(x) = \int_0^{x} u\, e^{-u^2} \cdot 2u\, \text{du}$

$$= 2\int_0^{x} t^2\, e^{-t^2} dt \qquad \ldots\text{(i)}$$

and $f(x) = \int_{-x}^{x} \left(|t| - t^2\right) e^{-t^2} dt, \ x > 0$

$$\therefore \quad f(x) = 2\int_0^{x} \left(t - t^2\right) e^{-t^2} dt \qquad \ldots\text{(ii)}$$

From equation (i) + (ii) : $f(x) + g(x) = \int_0^{x} 2te^{-t^2}\, dt$

Let $t^2 = P \quad \Rightarrow \quad 2t\ dt = dP$

$\therefore \quad f(x) + g(x) = \int_0^{x^2} e^{-P} dP = \left[-e^{-P}\right]_0^{x^2}$

$$\therefore \quad f(x) + g(x) = 1 - e^{-x^2} \qquad \ldots\text{(iii)}$$

$\therefore \quad f\left(\sqrt{ln\ 3}\right) + g\left(\sqrt{ln\ 3}\right) = 1 - e^{-ln 3} = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore$ Option (A) is incorrect.

From equation (ii) : $f'(x) = 2\left(x - x^2\right)e^{-x^2} = 2x(1 - x)e^{-x^2}$

$\because$ $f(x)$ is increasing in (0, 1)

$\therefore$ Option (D) is incorrect

$\because$ $\Psi_1(x) = e^{-x} + x$

$\Rightarrow$ $\Psi'_1(x) = 1 - e^{-x} < 1$ for $x > 1$

Then for $\alpha \in (1, \ x)$, $\Psi_1(x) = 1 + \alpha x$ does not true for $\alpha > 1$.

$\therefore$ Option (B) is incorrect

Now $\psi_2(x) = x^2 - 2x - 2e^{-x} + 2$

$\Rightarrow \quad \psi'_2(x) = 2x - 2 + 2e^{-x}$

$\therefore \quad \psi'_2(x) = 2\psi_1(x) - 2$

From LMVT

$\frac{\psi_2(x) - \psi_2(0)}{x - 0} = \psi'_2(\beta)$ for $\beta \in (\infty, x)$

$\Rightarrow \quad \psi_2(x) = 2x(\psi_1(\beta) - 1)$

$\therefore$ Option (C) is correct.

16. Which of the following statements is TRUE?

(A) $\Psi_1(x) \le 1$, for all $x > 0$

(B) $\Psi_2(x) \le 0$, for all $x > 0$

(C) $f(x) \ge 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

Answer (D)

Sol. $\because \quad \Psi_1(x) = e^{-x} + x$

and for all x > 0, $\Psi_1(x) > 1$

$\therefore$ (A) is not correct

$\Psi_1(x) = x^2 + 2 - 2(e^{-x} + x) > 0$ for $x > 0$

$\therefore$ (B) is not correct

Now, $\sqrt{t}\, e^{-t} = \sqrt{t}\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + ...\infty\right)$

and $\sqrt{t}\, e^{-t} \le t^{\frac{1}{2}} - t^{\frac{3}{2}} + \frac{1}{2}t^{\frac{5}{2}}$

$$\therefore \quad \int_0^{x^2} \sqrt{t}\, e^{-t}\, dt \le \int_0^{x^2} \left(t^{\frac{1}{2}} - t^{\frac{3}{2}} + \frac{1}{2}t^{\frac{5}{2}}\right) dt$$

$$= \frac{2}{3}x^3 - \frac{2}{3}x^5 + \frac{1}{7} + \frac{1}{7}x^7$$

$\therefore$ Option (D) is correct

and $f(x) = \int_{-x}^{x} (|t| - t^2) e^{-t^2}\, dt$

$$= 2\int_0^x (t - t^2) e^{-t^2}\, dt$$

$$= \int_0^x 2te^{-t^2}\, dt - 2\int_0^x t^2 e^{-t^2}\, dt$$

$$= 1 - e^{-x^2} - 2\int_0^x t^2 e^{-t^2}\, dt$$

$$\therefore \quad f(x) \le 1 - e^{-x^2} - 2\int_0^x t^2 (1 - t^2)\, dt$$

$$= 1 - e^{-x^2} - 2\frac{x^3}{3} + \frac{2}{5}x^5 \text{ for all } x\left(0, \frac{1}{2}\right)$$

$\therefore$ Option (C) is incorrect.

SECTION - 4 (Maximum Marks : 12)

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER.**
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	If ONLY the correct integer is entered;
Zero Marks	:	0	In all other cases.

17. A number is chosen at random from the set {1, 2, 3......, 2000}. Let p be the probability that the number is a multiple of 3 or a multiple of 7. Then the value of $500p$ is ______.

Answer (214)

Sol. E = a number which is multiple of 3 or multiple of 7

$n(E)$ = (3, 6, 9, ..., 1998) + (7, 14, 21, ..., 1995) – (21, 42, 63, ... 1995)

$n(E)$ = 666+ 285 – 95

$n(E)$ = 856

$n(E)$ = 2000

$$P(E) = \frac{856}{2000}$$

$$P(E)\times 500 = \frac{856}{4} = 214$$

18. Let E be the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. For any three distinct points P, Q and Q' on E, let M (P, Q) be the mid-point of the line segment joining P and Q, and $M(P, Q')$ be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between $M(P, Q)$ and $M(P, Q')$, as P, Q and Q' vary on E, is _____.

Answer (4)

Sol. Let $P(\alpha)$, $Q(\theta)$, $Q'(\theta')$

$$M = \frac{1}{2}(4\cos\alpha + 4\cos\theta),\ \frac{1}{2}(3\sin\alpha + 3\sin\theta)$$

$$M' = \frac{1}{2}(4\cos\alpha + 4\cos\theta'),\ \frac{1}{2}(3\sin\alpha + 3\sin\theta')$$

$$MM' = \frac{1}{2}\sqrt{(4\cos\theta - 4\cos\theta')^2 + (3\sin\theta - 3\sin\theta')^2}$$

$$MM' = \frac{1}{2}\text{ distance between } Q \text{ and } Q'$$

MM' is not depending on P

Maximum of QQ' is possible when QQ' = major axis

QQ' = 2(4) = 8

$$MM' = \frac{1}{2}\cdot(QQ')$$

MM' = 4

19. For any real number x, let $[x]$ denote the largest integer less than or equal to x. If $I=\int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$, then the value of $9I$ is ________.

Answer (182.00)

Sol. $I=\int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$

$$y=\frac{10x}{x+1}\quad,\quad 0\le x\le 10$$

$$xy+y=10x$$

$$x=\frac{y}{10-y}$$

$$0\le\frac{y}{10-y}\le 10$$

$$\frac{y}{10-y}\ge 0 \quad\text{and}\quad \frac{y}{10-y}-10\le 0$$

$$\frac{y}{y-10}\le 0 \quad\text{and}\quad \frac{11y-100}{y-10}\ge 0$$

$+\ \underset{0}{\bullet}\ -\ \underset{10}{\circ}\ +$ and $+\ \underset{\frac{100}{11}}{\bullet}\ -\ \underset{10}{\circ}\ +$

$$y\in[0,\ 10) \quad\text{and}\quad y\in\left(-\infty,\frac{100}{11}\right]\cup(10,\infty)$$

$$y\in\left[0,\frac{100}{11}\right]$$

$$\sqrt{y}\in\left[0,\frac{10}{\sqrt{11}}\right]\qquad\Rightarrow\qquad\left[\sqrt{y}\right]=\{0,1,2,3\}$$

Cose I : $0\le\frac{10x}{x+1}<1$

$$\frac{10x}{x+1}\ge 0 \quad\text{and}\quad \frac{10x}{x+1}-1<0$$

$+\ \underset{-1}{\circ}\ -\ \underset{0}{\bullet}\ +$ and $\frac{9x-1}{x+1}<0$

$+\ \underset{-1}{\circ}\ -\ \underset{\frac{1}{9}}{\circ}\ +$

$$x\in(-\infty,-1)\cup[0,\ \infty) \quad\text{and}\quad x\in\left(-1,\frac{1}{9}\right)$$

$$x\in\left[0,\frac{1}{9}\right)\quad\text{then}\quad\left[\sqrt{\frac{10x}{x+1}}\right]=0$$

Case II : $1 \le \dfrac{10x}{x+1} < 4$

$$\frac{10x}{x+1} - 1 \ge 0 \quad \text{and} \quad \frac{10x}{x+1} - 4 < 0$$

$$\frac{9x-1}{x+1} \ge 0 \quad \text{and} \quad \frac{6x-4}{x+1} < 0$$

$$\underset{-1}{\overset{+}{\circ}} \overset{-}{\text{———}} \underset{\frac{1}{9}}{\overset{+}{\bullet}} \quad \text{and} \quad \underset{-1}{\overset{+}{\circ}} \overset{-}{\text{———}} \underset{+\frac{2}{3}}{\overset{+}{\circ}}$$

$$x \in (-\infty, -1) \cup \left[\frac{1}{9}, \infty\right) \quad \text{and} \quad x \in \left(-1, \frac{2}{3}\right)$$

$$x \in \left[\frac{1}{9}, \frac{2}{3}\right) \quad , \quad \left[\sqrt{\frac{10x}{x+1}}\right] = 1$$

Case III : $4 \le \dfrac{10x}{x+1} < 9$

$$\frac{10x}{x+1} - 4 \ge 0 \quad \text{and} \quad \frac{10x}{x+1} < 9$$

$$\frac{6x-4}{x+1} \ge 0 \quad \text{and} \quad \frac{x-9}{x+1} < 0$$

$$\underset{-1}{\overset{+}{\circ}} \overset{-}{\text{———}} \underset{\frac{2}{3}}{\overset{+}{\bullet}} \quad \text{and} \quad \underset{-1}{\overset{+}{\circ}} \overset{-}{\text{———}} \underset{9}{\overset{+}{\circ}}$$

$$x \in (-\infty, -1) \cup \left[\frac{2}{3}, \infty\right) \qquad x \in (-1, 9)$$

$$x \in \left[\frac{2}{3}, 9\right) \quad ; \quad \left[\sqrt{\frac{10x}{x+1}}\right] = 2$$

Case IV : $x \in [9, 10] \quad \Rightarrow \quad \left[\sqrt{\dfrac{10x}{x+1}}\right] = 3$

$$I = \int_0^{\frac{1}{9}} 0 \cdot dx + \int_{\frac{1}{9}}^{\frac{2}{3}} 1 \cdot dx + \int_{\frac{2}{3}}^{9} 2 \cdot dx + \int_9^{10} 3 \cdot dx$$

$$I = \left(\frac{2}{3} - \frac{1}{9}\right) + 2\left(9 - \frac{2}{3}\right) + 3(10-9)$$

$$I = \frac{5}{9} + \frac{50}{3} + 3$$

$9I = 182$