JEE Main 2020 Paper

Date: **7th January 2020 (Shift 1)**
Time: **9:30 A.M. to 12:30 P.M.**
Subject: **Mathematics**

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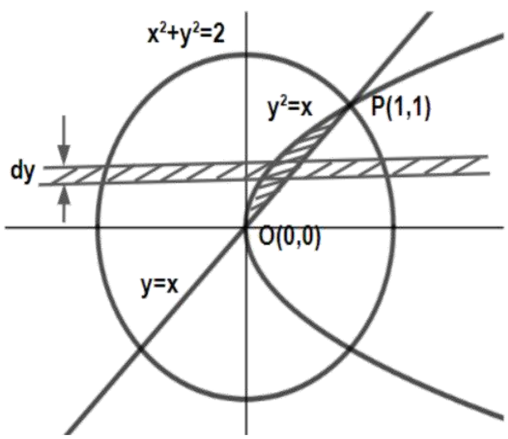
1. The area of the region enclosed by the circle $x^2 + y^2 = 2$ which is not common to the region bounded by the parabola $y^2 = x$ and the straight line $y = x$, is

   a. $\frac{1}{12}(24\pi - 1)$
   b. $\frac{1}{6}(12\pi - 1)$
   c. $\frac{1}{12}(6\pi - 1)$
   d. $\frac{1}{12}(12\pi - 1)$

   Answer: (b)

   Solution:

   Required area = area of the circle – area bounded by given line and parabola



   Required area = $\pi r^2 - \int_0^1 (y - y^2)\, dy$

   Area $= 2\pi - \left(\frac{y^2}{2} - \frac{y^3}{3}\right)_0^1 = 2\pi - \frac{1}{6}$ sq. units

2. If $g(x) = x^2 + x - 1$ and $(gof)(x) = 4x^2 - 10x + 5$, then $f\left(\frac{5}{4}\right)$ is

   a. $\frac{1}{2}$
   b. $-\frac{1}{2}$
   c. $-\frac{1}{3}$
   d. $\frac{1}{3}$

   Answer: (b)

   Solution:

   $g(x) = x^2 + x - 1$

$gof(x) = 4x^2 - 10x + 5$

$g(f(x)) = 4x^2 - 10x + 5$

$f^2(x) + f(x) - 1 = 4x^2 - 10x + 5$

Putting $x = \frac{5}{4}$ & $f\left(\frac{5}{4}\right) = t$

$$t^2 + t + \frac{1}{4} = 0$$

$$t = -\frac{1}{2}$$

$f\left(\frac{5}{4}\right) = -\frac{1}{2}$

3. If $y = y(x)$ is the solution of the differential equation $e^y\left(\frac{dy}{dx} - 1\right) = e^x$ such that $y(0) = 0$, then $y(1)$ is equal to

a. $\ln 2$
b. $2 + \ln 2$
c. $1 + \ln 2$
d. $3 + \ln 2$

Answer: (c)

Solution:

$e^y(y' - 1) = e^x$

$$\Rightarrow \frac{dy}{dx} = e^{x-y} + 1$$

Let $x - y = t$

$1 - \frac{dy}{dx} = \frac{dt}{dx}$

So, we can write

$$\Rightarrow 1 - \frac{dt}{dx} = e^t + 1$$

$\Rightarrow -e^{-t}\, dt = dx$

$\Rightarrow e^{-t} = x + c$

$\Rightarrow e^{y-x} = x + c$

$1 = 0 + c$

$\Rightarrow e^{y-x} = x + 1$

at $x = 1$

$\Rightarrow e^{y-1} = 2$

$\Rightarrow y = 1 + \ln 2$

4. If $y = mx + 4$ is a tangent to both the parabolas, $y^2 = 4x$ and $x^2 = 2by$, then b is equal to

a. -64
b. -32
c. -128
d. 16

Answer: (c)

Solution:

Any tangent to the parabola $y^2 = 4x$ is $y = mx + \frac{a}{m}$

Comparing it with $y = mx + 4$, we get $\frac{1}{m} = 4 \Rightarrow m = \frac{1}{4}$

Equation of tangent becomes $y = \frac{x}{4} + 4$

$y = \frac{x}{4} + 4$ is a tangent to $x^2 = 2by$

$\Rightarrow x^2 = 2b\left(\frac{x}{4} + 4\right)$

Or $2x^2 - bx - 16b = 0$,

$D = 0$

$b^2 + 128b = 0$,

$\Rightarrow b = 0$ (not possible),

$b = -128$

5. If α and β are two real roots of the equation $(k+1)\tan^2 x - \sqrt{2}\,\lambda\tan x = 1 - k$, where $(k \neq 1)$ and λ are real numbers. If $\tan^2(\alpha+\beta) = 50$, then value of λ is

a. 10
b. 5
c. 7
d. 12

Answer: (a)

Solution:

$(k+1)\tan^2 x - \sqrt{2}\lambda\tan x = 1 - k$

$\tan^2(\alpha+\beta) = 50$

Now,

$$\tan\alpha + \tan\beta = \frac{\sqrt{2}\lambda}{k+1}, \qquad \tan\alpha\tan\beta = \frac{k-1}{k+1}$$

$$\Rightarrow \left(\frac{\frac{\sqrt{2}\lambda}{k+1}}{1 - \frac{k-1}{k+1}}\right)^2 = 50$$

$$\Rightarrow \frac{2\lambda^2}{4} = 50$$

$\Rightarrow \lambda^2 = 100$

$\Rightarrow \lambda = 10$

6. If the distance between the foci of an ellipse is 6 and the distance between its directrices is 12, then the length of its latus rectum is

a. 4
b. $3\sqrt{2}$
c. 9
d. $2\sqrt{2}$

Answer: (b)

Solution:

Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ (a > b)$

Now $2ae = 6$ & $\frac{2a}{e} = 12$

$\Rightarrow ae = 3$ & $\frac{a}{e} = 6$

$\Rightarrow a^2 = 18$

$\Rightarrow a^2e^2 = c^2 = a^2 - b^2 = 9$

$\Rightarrow b^2 = 9$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 9}{\sqrt{18}} = 3\sqrt{2}$

7. The logical statement $(p \to q) \wedge (q \to \sim p)$ is equivalent to

a. $\sim p$
b. p
c. $p \wedge q$
d. $p \vee q$

Answer: (a)

Solution:

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \to q}$	$\boldsymbol{\sim p}$	$\boldsymbol{q \to \sim p}$	$\boldsymbol{(p \to q) \wedge (q \to \sim p)}$
T	T	T	F	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	T	T

Clearly $(p \to q) \wedge (q \to \sim p)$ is equivalent to $\sim p$

8. If the system of equations

$2x + 2ay + az = 0$

$2x + 3by + bz = 0$

$2x + 4cy + cz = 0,$

where $a, b, c \in \mathbf{R}$ are non-zero and distinct, has non zero solution then

a. $a + b + c = 0$
b. a, b, c are in A.P
c. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P
d. a, b, c are in G.P

Answer: (c)

Solution:

$$\begin{vmatrix} 2 & 2a & a \\ 2 & 3b & b \\ 2 & 4c & c \end{vmatrix} = 0$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\Rightarrow \begin{vmatrix} 2 & 2a & a \\ 0 & 3b - 2a & b - a \\ 0 & 4c - 2a & c - a \end{vmatrix} = 0$$

$\Rightarrow (3b - 2a)(c - a) - (4c - 2a)(b - a) = 0$

$\Rightarrow 3bc - 2ac - 3ab + 2a^2 - [4bc - 4ac - 2ab + 2a^2] = 0$

$\Rightarrow -bc + 2ac - ab = 0$

$\Rightarrow ab + bc = 2ac$

$\Rightarrow \frac{1}{c} + \frac{1}{a} = \frac{2}{b}$

9. 5 numbers are in A.P whose sum is 25 and product is 2520. If one of these 5 numbers is $-\frac{1}{2}$, then the greatest number amongst them is

a. $\frac{21}{2}$
b. 16
c. 27
d. 7

Answer: (b)

Solution:

Let 5 numbers be $a - 2d, a - d, a, a + d, a + 2d$

$5a = 25$

$a = 5$

$(a - 2d)(a - d)a(a + d)(a + 2d) = 2520$

$(25 - 4d^2)(25 - d^2) = 504$

$4d^4 - 125d^2 + 121 = 0$

$4d^4 - 4d^2 - 121d^2 + 121 = 0$

$d^2 = 1 \text{ } or \text{ } d^2 = \frac{121}{4}$

$d = \pm\frac{11}{2}$

For $d = \frac{11}{2}$, $a + 2d$ is the greatest term, $a + 2d = 5 + 11 = 16$

10. If α is a root of the equation $x^2 + x + 1 = 0$ and $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$ then A^{31} equal to

a. A
b. A^2
c. A^3
d. A^4

Answer: (c)

Solution:

The roots of equation $x^2 + x + 1 = 0$ are complex cube roots of unity.

$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$$

$$A^2 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$$

$$A^2 = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$A^4 = I$

$A^{28} = I$

Therefore, we get

$A^{31} = A^{28}A^3$

$A^{31} = IA^3$

$A^{31} = A^3$

11. Let $x^k + y^k = a^k$ where $a, k > 0$ and $\frac{dy}{dx} + \left(\frac{y}{x}\right)^{\frac{1}{3}} = 0$, then k is

a. $\frac{1}{3}$

b. $\frac{2}{3}$

c. $\frac{4}{3}$

d. 2

Answer: (b)

Solution:

$x^k + y^k = a^k$

$kx^{k-1} + ky^{k-1}\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\left(\frac{x}{y}\right)^{k-1}$

$\Rightarrow \frac{dy}{dx} + \left(\frac{y}{x}\right)^{1-k} = 0$

$\Rightarrow 1 - k = \frac{1}{3}$

$\Rightarrow k = \frac{2}{3}$

12. If real part of $\left(\frac{z-1}{2z+i}\right) = 1$ where $z = x + iy$, then the point (x, y) lies on

a. straight line with slope 2

b. straight line with slope $\frac{1}{2}$

c. circle with diameter $\frac{\sqrt{5}}{2}$

d. circle with diameter $\frac{1}{2}$

Answer: (c)

Solution:

$z = x + iy$

$$\frac{x + iy - 1}{2x + 2iy + i} = \frac{(x-1) + iy}{2x + i(2y+1)}\left(\frac{2x - i(2y+1)}{2x - i(2y+1)}\right) = 1$$

$$\frac{2x(x-1) + y(2y+1)}{4x^2 + (2y+1)^2} = 1$$

$$2x^2 + 2y^2 - 2x + y = 4x^2 + 4y^2 + 4y + 1$$

$$x^2 + y^2 + x + \frac{3}{2}y + \frac{1}{2} = 0$$

Circle's centre will be $\left(-\frac{1}{2}, -\frac{3}{4}\right)$

Radius $= \sqrt{\frac{1}{4} + \frac{9}{16} - \frac{1}{2}} = \frac{\sqrt{5}}{4}$

Diameter$= \frac{\sqrt{5}}{2}$

13. If $y(\alpha) = \sqrt{\frac{2(\tan\alpha + \cot\alpha)}{1+\tan^2\alpha} + \frac{1}{\sin^2\alpha}}$ where $\alpha \in \left(\frac{3\pi}{4}, \pi\right)$ then find $\frac{dy}{d\alpha}$ at $\alpha = \frac{5\pi}{6}$

a. 4

b. 2

c. 3

d. -4

Answer: (a)

Solution:

$$y(\alpha) = \sqrt{2\frac{\tan\alpha + \cot\alpha}{1 + \tan^2\alpha} + \frac{1}{\sin^2\alpha}}$$

$$y(\alpha)=\sqrt{2\frac{1}{\sin\alpha\cos\alpha\times\frac{1}{\cos^2\alpha}}+\frac{1}{\sin^2\alpha}}$$

$$y(\alpha)=\sqrt{2\cot\alpha+\mathrm{cosec}^2\alpha}$$

$$y(\alpha)=\sqrt{(1+\cot\alpha)^2}$$

$$y(\alpha)=-1-\cot\alpha$$

$$\frac{dy}{d\alpha}=0+\mathrm{cosec}^2\alpha\Big|_{\alpha=\frac{5\pi}{6}}$$

$$\frac{dy}{d\alpha}=\mathrm{cosec}^2\frac{5\pi}{6}$$

$$\frac{dy}{d\alpha}=4$$

14. Find the greatest integer k for which 49^k+1 is a factor of the given sum $49^{125}+49^{124}+\cdots+49^2+49+1$

a. 63
b. 65
c. 32
d. 60

Answer: (a)

Solution:

$$1+49+49^2+\cdots+49^{125}=\frac{49^{126}-1}{49-1}$$

$$=\frac{(49^{63}+1)(49^{63}-1)}{48}$$

$$=\frac{(49^{63}+1)((1+48)^{63}-1)}{48}$$

$=\frac{(49^{63}+1)(1+48I-1)}{48}$; Where I is an integer

$=(49^{63}+1)I$

Greatest positive integer is $k=63$

15. If $A(1,1)$, $B(6,5)$, $C\left(\frac{3}{2},2\right)$ are the vertices of ΔABC. A point P is such that area of ΔPAB, ΔPAC and ΔPBC are equal, then find the length of the line the segment PQ, where Q is the point $\left(-\frac{7}{6},-\frac{1}{3}\right)$

a. 2
b. 3
c. 4
d. 5

Answer: (d)

Solution:

P is the centroid which is $\equiv \left(\frac{1+6+\frac{3}{2}}{3}, \ \frac{1+5+2}{3}\right)$

$P = \left(\frac{17}{6}, \frac{8}{3}\right)$

$Q = \left(-\frac{7}{6}, -\frac{1}{3}\right)$

$PQ = \sqrt{(4)^2 + (3)^2} = 5$

16. If $\vec{a} = \alpha\hat{\imath} + 2\hat{\jmath} + \beta\hat{k}$ lies in plane of $\vec{b}$ and $\vec{c}$ where $\overrightarrow{b} = \hat{\imath} + \hat{\jmath}$ and $\vec{c} = \hat{\imath} - \hat{\jmath} + 4\hat{k}$ and $\vec{a}$ bisects the angle between $\vec{b}$ and $\vec{c}$, then

a. $\vec{a}.\hat{k} + 2 = 0$
b. $\vec{a}.\hat{k} + 4 = 0$
c. $\vec{a}.\hat{k} - 2 = 0$
d. $\vec{a}.\hat{k} + 5 = 0$

Answer:

Solution:

More data needed to solve the question.

17. If $f(x)$ is continuous and differentiable in $x \in [-7,0]$ and $f'(x) \leq 2 \ \forall \ x \ \in [-7,0]$, also $f(-7) = -3$ then the range of $f(-1) + f(0)$ is

a. $[-5, -7]$
b. $(-\infty, 6]$
c. $(-\infty, 20]$
d. $[-5, 3]$

Answer: (c)

Solution:

$f(-7) = -3$ and $f'(x) \leq 2$

Applying LMVT in $[-7,0]$, we get

$$\frac{f(-7) - f(0)}{-7} = f'(c) \leq 2$$

$$\frac{-3 - f(0)}{-7} \leq 2$$

$$f(0) + 3 \leq 14$$

$$f(0) \leq 11$$

Applying LMVT in $[-7, -1]$, we get

$$\frac{f(-7) - f(-1)}{-7 + 1} = f'(c) \leq 2$$

$\frac{-3-f(-1)}{-6} \leq 2$

$f(-1)+3 \leq 12$

$f(-1) \leq 9$

Therefore, $f(-1)+f(0) \leq 20$

18. Find the image of the point $(2,1,6)$ in the plane containing the points $(2,1,0)$, $(6,3,3)$ and $(5,2,2)$

a. $(6,5,-2)$
b. $(6,-5,2)$
c. $(2,-3,4)$
d. $(2,-5,6)$

Answer: (a)

Solution:

Points $A(2,1,0)$, $B(6,3,3)$ $C(5,2,2)$
$\overrightarrow{AB}=(4,2,3)$
$\overrightarrow{AC}=(3,1,2)$
$\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}=(1,1,-2)$
Equation of the plane is $x+y-2z=3\ldots(1)$
Let the image of point $(2,1,6)$ is (l,m,n)
$\frac{l-2}{1}=\frac{m-1}{1}=\frac{n-6}{-2}=\frac{-2(-12)}{6}=4$
$\Rightarrow l=6, m=5, n=-2$
Hence the image of R in the plane P is $(6,5,-2)$

19. If sum of all the coefficients of even powers in $(1-x+x^2-x^3 \ldots.+x^{2n})(1+x+x^2+x^3 \ldots.+x^{2n})$ is 61 then n is equal to

a. 30
b. 32
c. 28
d. 36

Answer: (a)

Solution:

Let $(1+x+x^2+\cdots+x^{2n})(1-x+x^2+\cdots+x^{2n})$ = $a_o+a_1x+a_2x^2+\cdots$
Put $x=1$
$2n+1=a_o+a_1+a_2+a_3+\ldots\ldots\ldots\ldots$ (1)
Put $x=-1$
$2n+1=a_o-a_1+a_2-a_3+\ldots\ldots\ldots\ldots.$ (2)
Add (1) and (2)
$2(2n+1)=2(a_o+a_2+a_4+\ldots\ldots\ldots\ldots$
$2n+1=61$
$n=30$

20. An unbiased coin is tossed 5 times. Suppose that a variable X is assigned the value k when k consecutive heads are obtained for $k = 3, 4, 5$, otherwise X takes the value -1. The expected value of X is

a. $\frac{1}{8}$
b. $-\frac{1}{8}$
c. $\frac{3}{8}$
d. $-\frac{3}{8}$

Answer: (a)

Solution:

$k =$ no. of consecutive heads

$P(k = 3) = \frac{5}{32}$ (HHHTH, HHHTT, THHHT, HTHHH, TTHHH)

$P(k = 4) = \frac{2}{32}$ (HHHHT, HHHHT)

$P(k = 5) = \frac{1}{32}$ (HHHHH)

$P(\overline{3} \cap \overline{4} \cap \overline{5}) = 1 - \left(\frac{5}{32} + \frac{2}{32} + \frac{1}{32}\right) = \frac{24}{32}$

$\sum XP(X) = \left(-1 \times \frac{24}{32}\right) + \left(3 \times \frac{5}{32}\right) + \left(4 \times \frac{2}{32}\right) + \left(5 \times \frac{1}{32}\right) = \frac{1}{8}$

21. Given $f(a + b + 1 - x) = f(x)\ \forall\ x \in \mathbf{R}$ then the value of $\frac{1}{(a+b)} \int_a^b x(f(x) + f(x + 1))\ dx$ is equal to

a. $\int_{a+1}^{b+1} f(x)\ dx$
b. $\int_{a+1}^{b+1} f(x + 1)\ dx$
c. $\int_{a-1}^{b-1} f(x)\ dx$
d. $\int_{a-1}^{b-1} f(x + 1)\ dx$

Answer: (d)

Solution:

$f(a + b + 1 - x) = f(x)$ (1)

$x \to x + 1$

$f(a + b - x) = f(x + 1)$ (2)

$I = \frac{1}{a+b} \int_a^b x(f(x) + f(x + 1))dx$ (3)

From (1) and (2)

$I = \frac{1}{a+b} \int_a^b (a + b - x)(f(x + 1) + f(x))dx$ (4)

Adding (3) and (4)

$$2I = \int_a^b (f(x) + f(x + 1))dx$$

$$2I = \int_a^b f(x + 1)dx + \int_a^b f(x)dx$$

$$2I = \int_a^b f(a+b-x+1)dx + \int_a^b f(x)dx$$

$$2I = 2\int_a^b f(x)dx$$

$$I = \int_a^b f(x)dx \qquad ; \quad x = t+1, dx = dt$$

$$I = \int_{a-1}^{b-1} f(t+1)dt$$

$$I = \int_{a-1}^{b-1} f(x+1)dx$$

22. Total number of six-digit numbers in which only and all the five digits 1,3,5,7 and 9 appear, is ________.

Answer: (1800)

Solution:

Selecting all 5 digits $= {}^5C_5 = 1$ way

Now, we need to select one more digit to make it a 6 digit number $= {}^5C_1 = 5$ ways

Total number of permutations $= \frac{6!}{2!}$

Total numbers $= {}^5C_5 \times {}^5C_1 \times \frac{6!}{2!} = 1800$

23. Evaluate $\lim_{x\to 2} \frac{3^x+3^{x-1}-12}{3^{\frac{-x}{2}}-3^{1-x}}$

Answer: (72)

Solution:

$$\lim_{x\to 2} \frac{3^x+\frac{3^x}{3}-12}{\frac{1}{3^{\frac{x}{2}}}+\frac{3}{3^x}}$$

$$\lim_{x\to 2} \frac{\frac{4}{3}3^x-12}{\frac{1}{3^{\frac{x}{2}}}+\frac{3}{3^x}}$$

Put $3^{\frac{x}{2}} = t$

$$\lim_{t\to 3}\frac{\frac{4t^2}{3}-12}{-\frac{3}{t^2}+\frac{1}{t}}=\lim_{t\to 3}\frac{4(t^2-9)t^2}{3(-3+t)}=\lim_{t\to 3}\frac{4t^2(3+t)}{3}=\frac{4\times 9\times 6}{3}=72$$

24. If variance of first N natural numbers is 10 and variance of first M even natural numbers is 16 then the value of $M+N$ is________.

Answer: (18)

Solution:

For N Natural number variance

$\sigma^2 = \frac{\sum {x_i}^2}{N} - (\frac{\sum x_i}{N})^2$

$\frac{\sum {x_i}^2}{N} = \frac{1^2+2^2+3^2+....N\ term}{N} = \frac{N(N+1)(2N+1)}{6N}$

$\frac{\sum x_i}{N} = \frac{1+2+3+....N\ terms}{N} = \frac{N(N+1)}{2N}$

$\sigma^2 = \frac{N^2-1}{12} = 10$ (given)

$\Rightarrow N = 11$

Variance of (2, 4, 6…) = $4\times$ variance of (1, 2, 3, 4…) = $4\times\frac{M^2-1}{12} = \frac{M^2-1}{3} = 16$ (given)

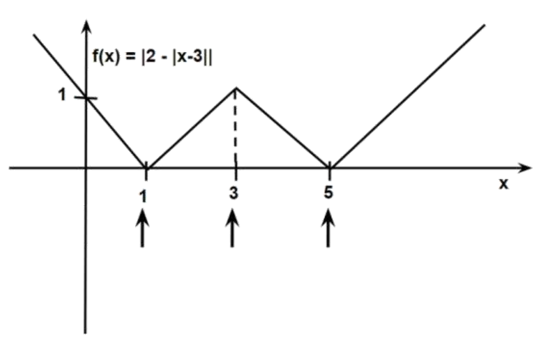
$\Rightarrow M = 7$

Therefore, $N+M = 11+7 = 18$

25. If $f(x) = \left|2-|x-3|\right|$ is non – differentiable in $x\in S$. Then, the value of $\sum_{x\in S}\left(f\left(f(x)\right)\right)$ is _________.

Answer: (3)

Solution:



There will be three points $x = 1, 3, 5$ at which $f(x)$ is non-differentiable.

So $f(f(1)) + f(f(3)) + f(f(5))$

$= f(0) + f(2) + f(0)$

$= 1 + 1 + 1$

$= 3$