Mathematics MA1059 / MA1055 Calculus

§4. Limits and Continuity

§4.1 Continuity

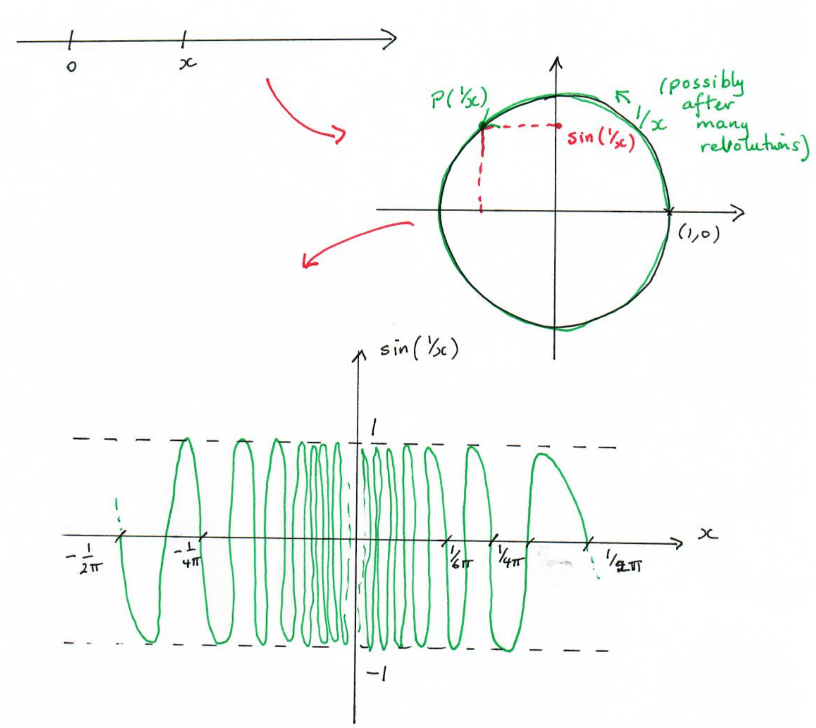
Suppose that f is a real-valued function of a real variable. That the function is continuous is sometimes described by the idea that the graph of the function can be drawn without 'taking one's pen off the paper'. While this is a good intuitive definition, it isn't really precise enough for us to work with from a mathematical point of view.

Let's look at some examples.

Example 1. First, the function f given by

$$f(x) = \sin\left(\frac{1}{x}\right), \quad x \neq 0.$$

As x decreases towards 0, $1/x$ grows large very quickly. Thinking of $1/x$ as the distance travelled by a point around the unit circle from the point $(1, 0)$ we see that, as x decreases gradually to 0, this point travels faster and faster around the unit circle: and so $\sin(1/x)$ changes over and over, more and more rapidly, from 1 to -1 and back to 1 again. For example, at each of $x_n = 1/(2n\pi)$, $n \in \mathbb{N}$, we have $\sin(1/x_n) = 0$. Between each of these points, $1/x$ changes by 2π making a full revolution of the unit circle, and so $\sin x$ changes from 0 to 1 to -1 and back to 0 again. The graph oscillates infinitely often from 1 to -1 and back to 1 again as x decreases to 0. Exactly the same occurs as x approaches 0 from the left.

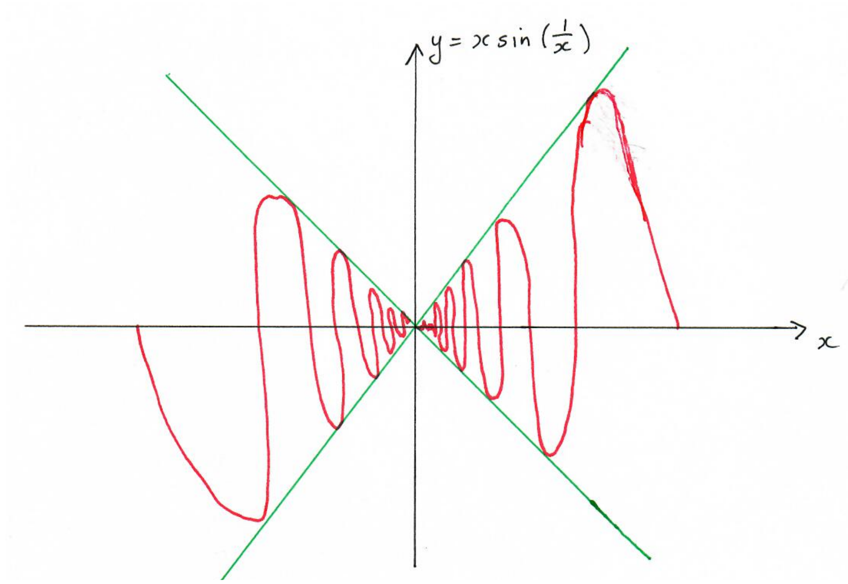


The function as it stands is not defined at $x = 0$ as '$\sin(+\infty)$' makes no sense. In fact, there is no reasonable value that one could assign to the function at 0. The function doesn't have a limiting value at $x = 0$. Furthermore, the graph of f is infinitely long as one approaches 0 from the right or the left.

Example 2. Let f be the function given by

$$f(x) = x \sin\left(\frac{1}{x}\right), \quad x \neq 0.$$

We can obtain the graph of this function from the graph in the previous example. For x positive, we compute $\sin(1/x)$ as above, and then multiply by x – this has the effect of scaling the graph of $\sin(1/x)$ in the vertical direction so that it varies between a minimum of $-x$ (when $\sin(1/x) = -1$) and a maximum of x (when $\sin(1/x) = 1$). The same happens for $x < 0$. The graph therefore looks like ...



This time we see that the function does have a limiting value at 0, and that it makes sense to set $f(0)$ to be 0, in which case f becomes continuous at 0. Again, the length of the graph as 0 is approached from either the left or the right is infinite - so if we attempt to draw the graph of f, between -1 and 1 say, we can't make it through 0 as this would require drawing a curve of infinite length.

In modern analysis, continuity of a function is first defined pointwise, that is continuity of a function at a point, and then the function is said to be continuous on an interval if it is continuous at each point of the interval. This approach is not particularly intuitive and certainly not along the lines of our intuitive definition above. This is remedied in a later course (probably MA2051) where a function is shown to be continuous if 'the preimage of each open set is open'.

⇛ Limit of a function at a point Suppose that f is defined near a on the real line, though not necessarily *at* a. We want to define the limit of the function f at a. Calling this limit L, we would want '*$f(x)$ to be close to L if x is close to, but not equal to, a*'.
How do we measure how close two numbers x and y on the real line are to one another? This we know: the distance between x and y is $|x-y|$, the absolute value of $x-y$. We can therefore rephrase the previous statement as '*$|f(x)-L|$ is small if $|x-a|$ is small, but not equal to 0*'.
The next question is '*how small*', as in how small should $|f(x)-L|$ be, and how small should $|x-a|$ be? The error in the output is $|f(x)-L|$. The goal should be to make this error as small as we wish. We can't reasonably expect this error to be 0, as this would correspond to a perfect hit: $f(x)$ *equal to* L. So we have to allow for the possibility of some error. The amount of error that we permit is traditionally given the variable name ε (the fifth letter of the Greek alphabet 'epsilon'). So we suppose that a positive maximum allowable error in the output, ε, has been specified in advance. The next task is to find a range of input x that will guarantee that the error in the output does not exceed this maximum error. That is, we need to produce a number, traditionally called δ (the fourth letter of the Greek alphabet 'delta'), such that

$$\boxed{\text{if the accuracy in the input is at least } \delta \text{ then the error in the output is at most } \varepsilon.}$$

We now turn this into a formal mathematical definition.

§4.2 Formal definitions

Definition. *Let $a \in \mathbb{R}$ and let $L \in \mathbb{R}$. Let f be a real-valued function defined in some interval about a, though not necessarily at a itself. We say that f has limit L at a if, to each positive number ε, there corresponds a positive number δ such that*

$$\text{if } (|x-a| \leq \delta \text{ and } x \neq a) \text{ then } |f(x)-L| \leq \varepsilon.$$

⇛ Notation If the function f has limit L at a, we write

$$\lim_{x \to a} f(x) = L$$

or '$f(x) \to L$ as $x \to a$'.
The concept of limit then allows us to define continuity of a function at a point.

Definition. *Let $a \in \mathbb{R}$. Let f be a real-valued function defined in some interval about a. We say that f is continuous at a if*

$$\lim_{x \to a} f(x) \text{ exists, and } \lim_{x \to a} f(x) = f(a).$$

In other words, to each positive number ε there corresponds a positive number δ such that

$$\text{if } |x-a| \leq \delta \text{ then } |f(x)-f(a)| \leq \varepsilon.$$

§4.3 Examples

We now work through the existence of limits and the continuity of functions in a number of examples. We begin with the trigonometric or circular functions introduced in the last section.

Example 3. We saw in the **video lectures** that the function $f(t) = \sin t$ is continuous at each a in $\mathbb{R}$. We argued, in fact, that for any t and a, the following inequality holds:

$$|\sin t - \sin a| \leq |t - a|.$$

Thus, given a positive number ε, we choose $\delta = \varepsilon$. If $|t - a| \leq \varepsilon$ then

$$|\sin t - \sin a| \leq |t - a| \leq \varepsilon.$$

This shows that $\sin t \to \sin a$ as $t \to a$, which is continuity of the sine function at a.
A similar argument shows that the cosine function is continuous at each point on the real line.

Example 4. Let's return to the example

$$f(x) = x \sin\left(\frac{1}{x}\right), \quad x \neq 0.$$

We saw at the time that this function possibly has limit 0 at 0. Let's establish this formally. Since $|\sin t| \leq 1$ for every t, we see that (here $L = 0$)

$$|x \sin(1/x) - 0| = |x|\,|\sin(1/x)| \leq |x|, \text{ for } x \neq 0.$$

So, given ε positive, let's choose $\delta = \varepsilon$. If $(|x - 0| = |x| \leq \delta$ and $x \neq 0)$ (here $a = 0$) then

$$|x \sin(1/x) - 0| \leq |x| \leq \delta = \varepsilon.$$

Thus, $\lim_{x\to 0} x \sin(1/x) = 0$. *If* we define $f(0) = 0$ (note that f has not been defined at 0 up to now) then f becomes continuous at 0 by definition.

Example 5. Let $f(x) = x^2$, $x \in \mathbb{R}$. We will show that this quadratic function is continuous at each point a in $\mathbb{R}$. First, we need some roughwork.

> R.W. We need to make $|x^2 - a^2|$ small by making $|x - a|$ small. Since $x^2 - a^2 = (x-a)(x+a)$, we will need to control the size of $|x + a|$. If x is very far from a then $|x + a|$ could be very large. On the other hand, x is going to end up being close to a. So let's imagine that the distance from x to a is at most 1 (say). Then the largest $|x|$ can be is $|a| + 1$, and the largest $|x + a|$ can be is $2|a| + 1$. We'd end up with
>
> $$|x^2 - a^2| = |x + a| \times |x - a| \leq (2|a| + 1) \times |x - a|.$$
>
> We can make this less than ε if we make $|x - a| < \varepsilon/(2|a| + 1)$. Now back to the formal solution ...

Solution Let a positive number ϵ be given. Choose

$$\delta = \min\left\{1, \varepsilon/(1+2|a|)\right\},$$

so that $\delta \leq 1$ and $\delta \leq \varepsilon/(2|a|+1)$.[1] Suppose that $|x-a| \leq \delta$. In particular, $|x-a| \leq 1$ and we deduce that

$$|x| = |(x-a)+a| \leq |x-a| + |a| \leq 1 + |a|,$$

and so

$$|x+a| \leq |x| + |a| \leq 1 + 2|a|. \tag{1}$$

Finally,

$$\begin{aligned} |f(x) - f(a)| &= |x^2 - a^2| \\ &= |x+a|\,|x-a| \\ &\leq (1+2|a|)\,|x-a| \\ &\leq (1+2|a|)\,\frac{\varepsilon}{1+2|a|} = \varepsilon. \end{aligned}$$

Here the first inequality comes from (1) and the second from $|x-a| \leq \delta \leq \varepsilon/(2|a|+1)$. Since, given a maximum error ε, we can find an accuracy δ such that $|x^2 - a^2| \leq \varepsilon$ if $|x-a| \leq \delta$, we have

$$\lim_{x \to a} x^2 = a^2,$$

and the function $f(x) = x^2$ is continuous at a.

Example 6. We just worked through calculations showing continuity of the function $f(x) = x^2$. We skipped the functions $f(x) = C$, C a constant, and the function $f(x) = x$. These are much easier but, in some ways, can be confusing.

For a constant function $f(x) = C$ and $a \in \mathbb{R}$, given ε positive, choose $\delta = 1$. If $|x-a| \leq \delta$ then

$$|f(x) - f(a)| = |C - C| = 0 \leq \varepsilon.$$

(In effect, there's never any error in the output.)

For the function $f(x) = x$ and $a \in \mathbb{R}$, given ε positive, choose $\delta = \epsilon$. If $|x-a| \leq \delta$ then

$$|f(x) - f(a)| = |x-a| \leq \delta = \varepsilon.$$

(In effect, the error in the output is exactly the same as the error in the input.)

Exercise 1. *Let $f(x) = ax+b$, $x \in \mathbb{R}$, where a and b are fixed. Show directly from the definition that the function f is continuous at each a in $\mathbb{R}$.*

[1] We need to do this, or something like it, in case ε happens to be a big number and $\varepsilon/(2|a|+1) > 1$ - in that case the estimate for $|x+a|$ wouldn't hold as stated.

Example 7. Set $f(x) = \sqrt{x}$ for $x > 0$. Let $a > 0$. We show that $f(x)$ is continuous at a.

R.W. We need to make $|\sqrt{x} - \sqrt{a}|$ small by making $|x - a|$ small. Notice that

$$|\sqrt{x} - \sqrt{a}| = \left|\frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x} + \sqrt{a}}\right| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}}$$

Here, $\sqrt{x}$ and $\sqrt{a}$ are positive (assuming that x is close to the positive number a so that it too is positive). So if we drop the '$\sqrt{x}$' in the denominator, this makes the fraction bigger, and we end up with $|\sqrt{x} - \sqrt{a}| \leq |x - a|/\sqrt{a}$. If we choose $\delta = \varepsilon\sqrt{a}$, it should work. Now back to the formal solution again . . .

Let a positive number ε be given. Choose

$$\delta = \min\{a/2, \varepsilon\sqrt{a}\}.$$

Suppose that $|x - a| \leq \delta$. Since $\delta \leq a/2$, we have $|x - a| \leq a/2$ and so

$$a - \frac{a}{2} \leq x \leq a + \frac{a}{2}$$

so that $x \geq a/2 > 0$ and $\sqrt{x}$ is perfectly well-defined.
Next, since $|x - a| \leq \varepsilon\sqrt{a}$,

$$\begin{aligned} |\sqrt{x} - \sqrt{a}| &= \left|\frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x} + \sqrt{a}}\right| \\ &= \frac{|x - a|}{\sqrt{x} + \sqrt{a}} \\ &\leq \frac{|x - a|}{\sqrt{a}} \leq \frac{\varepsilon\sqrt{a}}{\sqrt{a}} = \varepsilon, \end{aligned}$$

showing that the definition of continuity is satisfied.

Not all functions have limits, so the phrase *if the limit exists* is not vacuous. A classic situation in which a function doesn't have a limit is when there is a jump in the value of the function - this is sometimes called a *discontinuity of the first kind*. Here is the canonical example.

Example 8. Set

$$f(x) = \begin{cases} -1, & \text{if } x < 0; \\ 1, & \text{if } x > 0. \end{cases}$$

Then, $\lim_{x \to 0} f(x)$ does not exist.
We argue by contradiction (even if the result is *obvious* from the graph of f).

Suppose that $\lim_{x\to 0} f(x)$ does, indeed, exist and has the value L. Take $\varepsilon = 1/2$ in the definition of limit. There is then, by assumption, a positive number δ such that

$$\text{if } (|x| \le \delta \text{ and } x \ne 0) \text{ then } |f(x) - L| \le \frac{1}{2},$$

that is

$$\text{if } (-\delta \le x \le \delta \text{ and } x \ne 0) \text{ then } |f(x) - L| \le \frac{1}{2}. \tag{2}$$

If we choose $x = -\delta$ then $f(-\delta) = -1$ and so we deduce from (2) that $|-1-L| \le \frac{1}{2}$.
If we choose $x = \delta$ then $f(\delta) = 1$ and so we deduce from (2) that $|1-L| \le \frac{1}{2}$.
Together these imply that

$$2 = |(1-L) + (-1-L)| \le |1-L| + |-1-L| \le \frac{1}{2} + \frac{1}{2} = 1,$$

so that $2 \le 1$!! Thus, the existence of the limit of f at 0 leads to an impossibility and we are forced to conclude that the limit does not, in fact, exist. $\square$

§4.4 Rules of Limits Already in the previous section, we are building up a collection of functions that we know to be continuous (or at least to have limits at certain points). Working directly from the definition is not straightforward - for each positive number ε, one has to produce an explicit number δ for which the definition of limit is satisfied. Thankfully, we don't need to continue to work from *first principles* in this manner as we have general results that automatically produce new continuous functions from functions that we already know to be continuous. In particular, the sum, product, quotient (quotient with one caveat) of continuous functions is automatically continuous. We won't include full proofs of these results - just that for the sum (which is easiest) to give an idea of the direction the proofs of the other cases might follow.

Theorem. *Suppose that the functions f and g are both defined near a in $\mathbb{R}$, though not necessarily at a. Suppose that the limit of f at a and the limit of g at a both exist. Then*

(i) ***Sum Rule*** *The limit of the function $(f+g)(x)$ at a exists and*

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

(ii) ***Product Rule*** *The limit of the function $(f \times g)(x)$ at a exists and*

$$\lim_{x\to a}(f \times g)(x) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x).$$

(iii) ***Quotient Rule*** *If $\lim_{x\to a} g(x) \ne 0$, the limit of the function $(f/g)(x)$ at a exists and*

$$\lim_{x\to a}(f/g)(x) = \lim_{x\to a} f(x) / \lim_{x\to a} g(x).$$

(iv) ***Sum, Product, Quotient of Continuous Functions*** *In particular, if the functions f and g are both continuous at a then so are the functions $f+g$, $f\times g$ and, if $g(a)\neq 0$, f/g.*

Proof of (i). Suppose that $L_1 = \lim_{x\to a} f(x)$ exists and that $L_2 = \lim_{x\to a} g(x)$ exists. We aim to show that $\lim_{x\to a}(f(x)+g(x))$ exists and equals $L_1 + L_2$.
To this end, let a positive number ε be given. Then $\varepsilon/2$ is also positive. Since $L_1 = \lim_{x\to a} f(x)$ exists there is, by definition, a positive number δ_1 such that

$$\text{if } (|x-a|\leq \delta_1,\ x\neq a) \text{ then } |f(x)-L_1| \leq \frac{\varepsilon}{2}. \tag{3}$$

Similarly, since $L_2 = \lim_{x\to a} g(x)$ exists there is, by definition, a positive number δ_2 such that

$$\text{if } (|x-a|\leq \delta_2,\ x\neq a) \text{ then } |g(x)-L_2| \leq \frac{\varepsilon}{2}. \tag{4}$$

Set δ to be the smaller of δ_1 and δ_2, and suppose that $|x-a|\leq\delta$, $x\neq a$. Then, $|x-a|\leq\delta_1$ and so (3) holds. Similarly, $|x-a|\leq\delta_2$ and so (4) holds. Since both (3) and (4) hold if $x\neq a$ and $|x-a|\leq\delta$, we have

$$\begin{aligned}
|(f(x)+g(x))-(L_1+L_2)| &= |(f(x)-L_1)+(g(x)-L_2)| \\
&\leq |f(x)-L_1| + |g(x)-L_2| \\
&\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

showing that the definition of limit is satisfied.

Example 9. Suppose that a function $f(x)$ is continuous at a. Then so is the function $f^n(x)$ where

$$f^n(x) = (f(x))^n = f(x)\times f(x)\times\ldots\times f(x) \quad (n \text{ times}).$$

One can see this by induction. The result is true for $n=1$ by assumption. Assuming that the function $f^{n-1}(x)$ is continuous at a, and knowing that the function $f(x)$ is continuous at a, the **Product Rule for Continuous Functions** shows that the function $f^n(x) = f^{n-1}(x)\times f(x)$ is continuous at a.

Example 10. Every polynomial

$$P(x) = c_0 + c_1x + c_2x^2 + \ldots c_nx^n, \quad x\in\mathbb{R},$$

where the c_i are constants, is continuous everywhere.

Solution Let a be in $\mathbb{R}$. The function $f(x)=x$ is continuous at a by Example 6. By the previous Example 9, we see that $f^n(x) = (f(x))^n = x^n$ is continuous at a for each natural number power n.
We can now multiply by a constant function $g(x)=c$, which is also continuous at a by Example 6, showing that the function $g(x)\times f^n(x) = cx^n$ is continuous at a for any choice of constant c.

Finally, we can now recognise the polynomial $P(x)$ as a (finite) sum of functions each continuous at a. Then $P(x)$ itself, by the **Sum Rule for Continuous Functions**, is also continuous at a.

Exercise 2. *Show that the function $f(x) = x^{3/2}$ is continuous at a for each positive a. Can you generalise this result?*

Example 11. Every rational function

$$R(x) = \frac{P(x)}{Q(x)} \quad \text{where } P,\ Q \text{ are polynomials,}$$

is continuous at each a where $Q(a) \neq 0$.

This is a direct application of the **Quotient Rule for Continuous Functions** and Example 10, bearing in mind the condition in the Quotient Rule that the limit in the denominator cannot be 0.

Example 12. Find the limit

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

if the limit exists.

Solution Remember that when taking the limit of a function at a, the value of the function at a is irrelevant. In fact, the function in question may not even be defined at a. This is the case here, as putting $x = 3$ in the quotient $(x^2 - 9)/(x - 3)$ leads to the indeterminate form $\frac{0}{0}$. That 'x equal to 3' is not relevant works to our advantage, however, in the following calculation

$$\frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{x-3} = \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = x + 3.$$

The cancelation above is allowed because, since $x \neq 3$, we're not cancelling 0!
Now, by Example 10,

$$\lim_{x \to 3}(x + 3) = 3 + 3 = 6,$$

since $P(x) = x + 3$ is a polynomial. Thus $\lim_{x \to 3} \frac{x^2-9}{x-3}$ exists and

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3}(x + 3) = 6.$$

Once one gets used to how these limit rules work, it is more convenient to effectively work backwards.

Example 13. Find the limit

$$\lim_{x \to -3} \frac{x^3 + 27}{x^2 - 9}$$

if the limit exists.

Solution Again this limit is indeterminate of the form $\frac{0}{0}$ - but this doesn't mean the limit doesn't exist!
Factoring the cubic, we find that

$$x^3 + 27 = (x+3)(x^2 - 3x + 9).$$

Then, for $x \neq -3$,

$$\frac{x^3+27}{x^2-9} = \frac{(x+3)(x^2-3x+9)}{(x+3)(x-3)} = \frac{\cancel{(x+3)}(x^2-3x+9)}{\cancel{(x+3)}(x-3)} = \frac{x^2-3x+9}{x-3}.$$

The last expression is a rational function of x (a quotient of two polynomials, $R(x) = P(x)/Q(x)$ where $P(x) = x^2 - 3x + 9$ and $Q(x) = x - 3$). Also, $Q(-3) = -6$ is *not* 0. We are therefore in the situation described in Example 11 and we conclude that the limit in question exists and equals $R(-3)$, so that

$$\lim_{x \to -3} \frac{x^3+27}{x^2-9} = \lim_{x \to -3} \frac{x^2-3x+9}{x-3} = \frac{(-3)^2 - 3(-3) + 9}{-3-3} = -\frac{9}{2}.$$

⇛ **The Squeeze Rule** The Squeeze Rule is another way of showing that the limit of a given function exists if the function is squeezed between two other functions that have the *same* limit. We used this result in the **video lectures** when proving that $(\sin t)/t \to 1$ as $t \to 0$ having established the inequalities

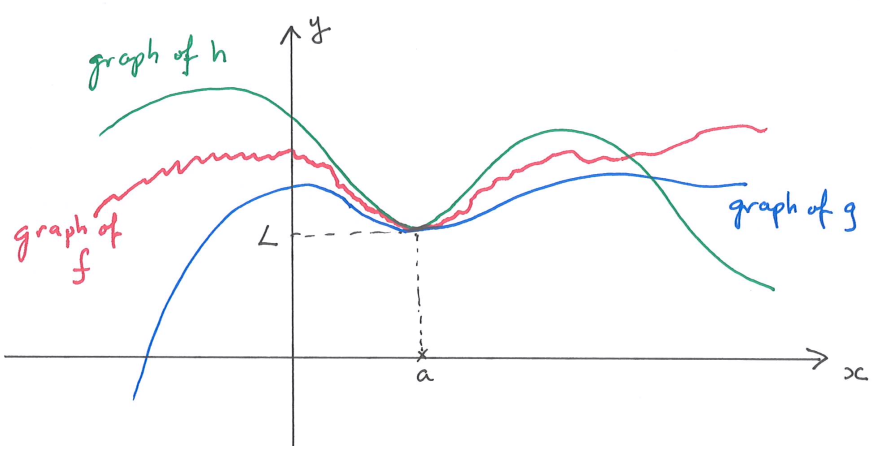
$$\cos t \leq \frac{\sin t}{t} \leq 1.$$

Here is the formal statement and a picture to show what's going on.

Theorem (The Squeeze Rule)**.** *Suppose that the functions g and h both have the same limit L at the point a. Suppose also that the graph of a third function f lies between those of g and h (at least near a) in that*

$$g(x) \leq f(x) \leq h(x), \qquad \textit{for all } x \textit{ near } a,\ x \neq a.$$

Then the limit of f at a also exists and equals L.
In particular, if g and h are both continuous at a with $g(a) = h(a)$, and if f is a function for which $g(x) \leq f(x) \leq h(x)$ for all x near a, then $f(x)$ is also continuous at a.



Proof Let a positive number ε be given. We need to produce a positive number δ such that

$$\big(|x-a| \le \delta,\ x \ne a\big) \implies |f(x)-L| < \varepsilon.$$

Note that $|f(x)-L| \le \varepsilon$ is equivalent to

$$L-\varepsilon \le f(x) \le L+\varepsilon.$$

We're given that $\lim_{x\to a} g(x)$ exists and equals L. Thus, there is a positive number δ_1 such that $|g(x)-L| \le \varepsilon$ if $|x-a| \le \delta_1$ and $x \ne a$, that is

$$\big(|x-a| \le \delta_1,\ x \ne a\big) \implies L-\varepsilon \le g(x) \le L+\varepsilon. \tag{5}$$

Similarly, since $\lim_{x\to a} h(x)$ exists and equals L (the *same* L), there is a positive number δ_2 such that

$$\big(|x-a| \le \delta_2,\ x \ne a\big) \implies L-\varepsilon \le h(x) \le L+\varepsilon. \tag{6}$$

Set $\delta = \min\{\delta_1, \delta_2\}$, so that δ is positive. If $\big(|x-a| \le \delta,\ x \ne a\big)$ then both (5) and (6) hold and so, for such x (using the inequalities in blue),

$$L-\varepsilon \le g(x) \le f(x) \le h(x) \le L+\varepsilon,$$

that is $L-\varepsilon \le f(x) \le L+\varepsilon$ for x in this range, which is what we wished to show. □

Exercise 3. *Explain why*

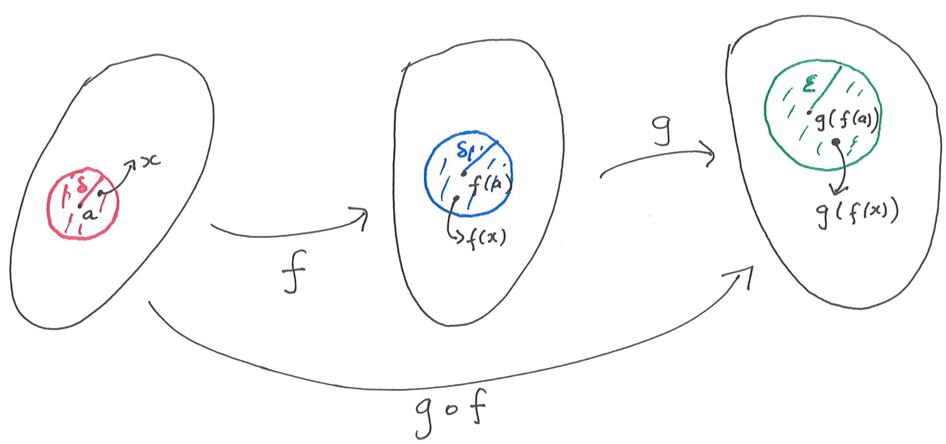
$$-x \le x\sin\left(\frac{1}{x}\right) \le x, \quad x \ne 0.$$

Deduce from the Squeeze Rule that, for the function $f(x)$ in Example 2, $\lim_{x\to 0} f(x) = 0$.

⇛ **Composition of continuous functions are continuous** This is yet another way to produce new continuous functions from old. For example, since the functions $g(x) = \sin x$ and $f(x) = x^3+2x-1$ are both continuous on the whole real line, the composed function

$$(g\circ f)(x) = g(f(x)) = \sin(f(x)) = \sin(x^3+2x-1)$$

is also continuous everywhere on the whole real line (as is, in fact, the composed function $(f\circ g)(x) = \sin^3 x + 2\sin x - 1$). An ε-δ argument to establish continuity of either function from first principles would be an onerous task. But we can prove a general result that covers all such cases.



Theorem. *Suppose that the function f is continuous at a and that the function g is continuous at $f(a)$. Then the composed function $g \circ f$ is continuous at a.*

Proof Let a positive number ε be given. We need to produce a positive number δ such that

$$\text{if } |x-a| \le \delta \text{ then } |(g \circ f)(x) - (g \circ f)(a)| \le \varepsilon. \tag{7}$$

Since the function g is continuous at $f(a)$ there is a positive number, say δ_1, such that

$$\text{if } |y - f(a)| \le \delta_1 \text{ then } |g(y) - g(f(a))| \le \varepsilon. \tag{8}$$

Since the function f is continuous at a there is a positive number, say δ, such that

$$\text{if } |x-a| \le \delta \text{ then } |f(x) - f(a)| \le \delta_1. \tag{9}$$

(See the diagram.)
Now suppose that $|x-a| \le \delta$. Then, by (9), $|f(x) - f(a)| \le \delta_1$. This means we can apply (8) with $y = f(x)$ to conclude that $|g(f(x)) - g(f(a))| \le \varepsilon$. This establishes (7) and demonstrates the continuity of the function $g \circ f$ at a.

§4.5 Some Exercises on Limits

1 Find each of the following limits:

$$\text{(i)} \quad \lim_{x\to 3} \frac{5x^2-8x-13}{x^2-5} \qquad \text{(ii)} \quad \lim_{x\to 3} \frac{x^2+3x-18}{3x^2-10x+3} \qquad \text{(iii)} \quad \lim_{x\to a} \frac{x^3-a^3}{x^2-a^2}.$$

2 Find each of the following limits:

$$\text{(i)} \quad \lim_{x\to -2} \frac{4-x^2}{2+x} \qquad \text{(ii)} \quad \lim_{x\to 2} \frac{3x^2-x-10}{x^2-4} \qquad \text{(iii)} \quad \lim_{x\to -3} \frac{x^4-81}{2x^2+5x-3}.$$

3 Find each of the following limits:

$$\text{(i)} \quad \lim_{x\to 1} \frac{(x^2-1)^2}{x-1} \qquad \text{(ii)} \quad \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \qquad \text{(iii)} \quad \lim_{x\to -3} \frac{x^4-81}{2x^2+5x-3}.$$

4 The function f is defined by $f(x) = 5x^2 - 4x$, for x real.

(i) Show that $f(3) = 33$. Factor $f(x) - 33$ in the form $(x-3)g(x)$.

(ii) If $|x-3| \le 1$ then $|x| \le 4$. Find an explicit number M such that $|g(x)| \le M$ if $|x-3| \le 1$.

(iii) Write down an explicit positive number δ such that

$$|f(x) - 33| \le 10^{-5} \quad \text{if} \;\; |x-3| < \delta.$$

Show that this δ works.

5 (Winter 2016) Find, with proof, a positive number δ such that

$$|x^3 - 8| < 10^{-6} \quad \text{if} \quad |x - 2| < \delta.$$

(Follow the steps in the previous question.)

6 Find each of the following limits (if they exist):

(i) $\lim_{t \to 0} \frac{\sin(5t)}{t}$; (ii) $\lim_{t \to 0} \frac{\cos t}{t}$;

(iii) $\lim_{t \to 0} \frac{1 - \cos t}{t}$; (iv) $\lim_{t \to 0} \frac{1 - \cos t}{t^2}$;

(v) $\lim_{t \to 0} \frac{t}{\sin t}$; (vi) $\lim_{x \to 0} \frac{1 + \sin x}{\cos x}$;

(vii) $\lim_{t \to 0} \frac{\sin(5t)}{\sin(3t)}$; (viii) $\lim_{x \to 0} \frac{\sin(2x)}{2x^2 + 3x}$;