CHAPTER – 1

COMPLEX NUMBERS

1. Classification of Numbers :

(i) Natural numbers : **D. U. H. T, 75.**

The set of all natural numbers (or counting numbers or the positive integers) is denoted by N.

$N = \{1, 2, 3, \ldots \ldots\}$.

(ii) Integers : The set of all integers are those numbers whose elements are the positive and negative whole numbers and the zero and it is denoted by I (or Z) :

$I = \{\ldots \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \ldots\}$.

(iii) Rational numbers : **D. U. H. T. 75.**

The set of all rational numbers are those numbers which can be expressed as the ratio of two integers a/b where $b \neq 0$ and it is denoted by Q

$Q = \{a/b : a, b \in I, b \neq 0\}$.

Example 1 : $1, -1, 2, -2, 0, 3/2, -7/9$ etc are rational numbers.

(iv) Irrational numbers : The set of all irrational numbers are those numbers whose decimal representations are non-terminating and non-repeating and it can not be repesented in the form p/q where $p, q \in I$, $q \neq 0$. It is denoted by Q'

Example 2 : $2+\sqrt{3}$, $3-\sqrt{5}$, $\sqrt{2}$, π, e etc are irrational numbers.

(v) Real numbers : **D. U. C. T. 75.**

The set of all rational and all irrational numbers is called the set of all real numbers and it is denoted by $\boldsymbol{R}$.

$\boldsymbol{R}=\{x\,;\, x \in \boldsymbol{Q} \text{ or } x \in \boldsymbol{Q}'\}=\boldsymbol{Q} \cup \boldsymbol{Q}'$.

Here it is clear that $\boldsymbol{Q} \cap \boldsymbol{Q}'=\phi$.

Example 3 : $0, 5, -2, 2/3, -8/7, \pi, e, 2+\sqrt{3}, -5+\sqrt{7}$ etc are real numbers.

D. U. C. T. 75.

vi) Complex numbers :

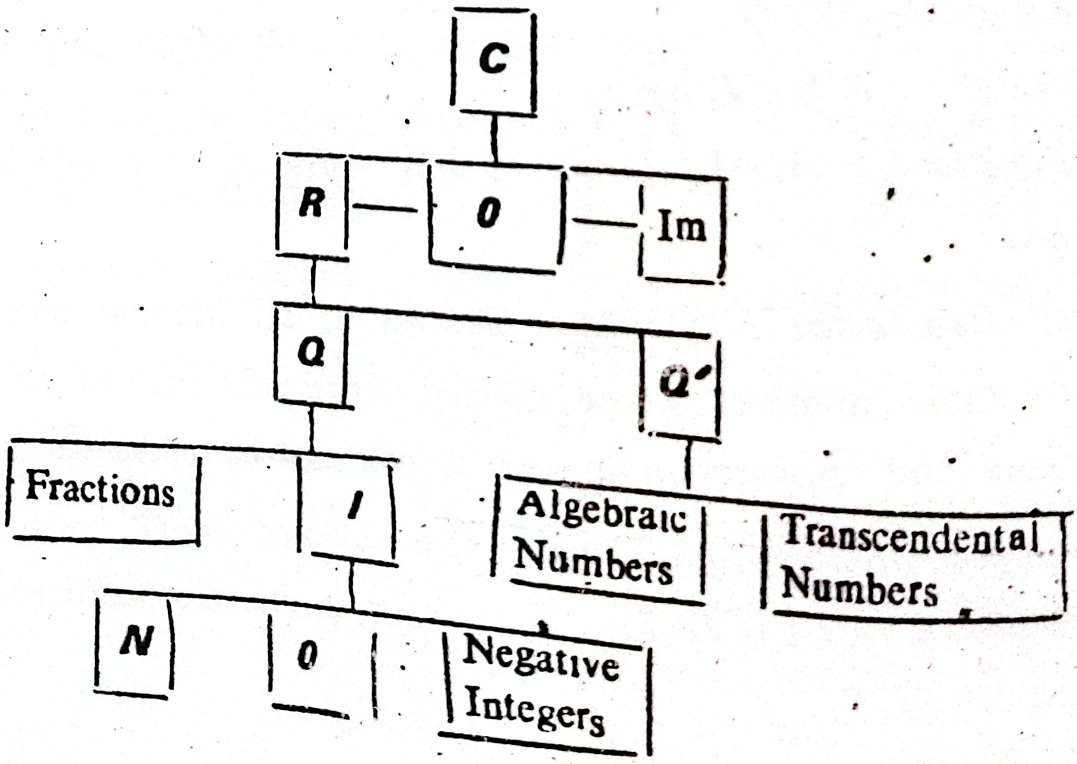
The set of all complex numbers are those numbers which can be represented in the form $a+ib$ where $a, b \in \boldsymbol{R}$ and $i=\sqrt{(-1)}$. It is denoted by $\boldsymbol{C}$:

$\boldsymbol{C}=\{a+ib : a, b \in \boldsymbol{R}, i=\sqrt{(-1)}\}=\boldsymbol{R} \cup \text{Im}$, where Im denotes the set of all imaginary numbers.

Here $\text{Im}=\{0\} \cup \{x+iy : x, y \in R, y \neq 0, i=\sqrt{(-1)}\}$.

Example 4 ; $0, 3, -5, 7/8, -27/19, 3+4i, -2-5i, 7-9i$ etc are complex numbers.

2. *A diagram* :



N.B. The definition of Algebraic and Transcendental numbers are given after some steps in th s chapter.

3. *Some definitions*:

(i) Set. Any collection of objects or any well defined list is called a set and its objects are called elements or members.

***(ii) Null set (or void set or empty set)*:**

A set that contains no elements is called the empty set or void set or null set and it is denoted by the symbol ϕ.

***(iii) Sub set*:** A set A is said to be a subset of a set B if each element of A is also an element of B and is written $A \subset B$.

***(iv) Union of sets*:** The union of two sets A and B is denoted by $A \cup B$ and it is defined by:

$A \cup B = \{x : x \in A \text{ or } x \in B\}$.

The union of A and B is also denoted by $A+B$.

***(v) Intersection of sets*:** The intersection of two sets A and B is denoted by $A \cap B$ and it is demated by AB

$A \cap B = \{x : x \in A \text{ and } x \in B\}$.

The intersection of A and B is also defined by: AB.

***(vi) Complement*:** The complement of a set A is denoted by A^c and it is defined by: $A^c = \{x : x \notin A\}$.

(vii) Countability of a set*:** If there exists a one to one correspondence, to the members or elements of a set S with the set of integers $N = \{1, 2, 3, \dots\dots\}$, then the set S is called ***Countable or ***denumerable***. If it is not countable, then it is call..

non-countable or ***non-denumerable***.

4. ***Introduction***; We know in the real field R, the following types of equations have no solutions;

$x^2+1=0$, $x^2+2=0$, $x^2+7=0$, $x^2+1/3=0$ etc. To solve these types of equations, the complex number system was introduced.

Historical Note: Swiss mathematician Leonard Euler (1707–83) introduced the ***imaginary unit*** i in 1748.

Definition: $i=\sqrt{(-1)}$ where $i^2=-1$.

Definition: If $a>0$, then $\sqrt{(-a)}=i\sqrt{a}$.

Definition: If $a, b>0$, then $\sqrt{(-a)}\sqrt{(-b)}=(i\sqrt{a})(i\sqrt{b})$
$=i^2\sqrt{a}\sqrt{b}=-\sqrt{(ab)}$.

N. B. If $a, b>0$, then $\sqrt{(-a)}\sqrt{(-b)}\neq\sqrt{\{(-a)(-b)\}}$.

Example: If $n\in I$, then $i^n\in\{1,-1,i,-i\}$.

5. ***Complex Number***: The number having the form $a+ib$ is called a complex number where $a, b\in R$. We can also define a complex number $a+ib$ in an ordered pair of real numbers (a, b) i, e. $a+ib=(a, b)$.

Historical Note: The German mathematician Carl Friedrich Gauss (1777–1855), one of the greatest mathematician of the world, introduced the complex number system in 1832. The ordered pair system of complex number was introduced by the Irish mathematician William Rowan Hamilton (1805–65) in 1835.

6. ***Real and Imaginary parts of a complex number***;
Let $z=(x, y)=x+iy$. Then x is called the ***real part*** of z

and it is denoted by Re (z) *i. e.* Re $(z)=x$. Again the ***imaginary part*** of z is y and it is denoted by Im (z) *i. e.* Im $(z)=y$.

7. The fundamental operations of complex numbers :

The following four definitions are given as like as the fundamental laws of algebra of real numbers using $i^2=-1$:

(i) Definition of sum : Let $z_1=(x_1, y_1)$ and $z_2=(x_2, y_2)$ then $z_1+z_2=(x_1, y_1)+(x_2, y_2)=(x_1+iy_1)+(x_2+iy_2)=(x_1+x_2)$ $+i(y_1+y_2)=(x_1+x_2, y_1+y_2)$.

(ii) Definition of difference : Let $z_1=(x_1, y_1)$ and $z_2=(x_2, y_2)$, then $z_1-z_2=(x_1, y_1)-(x_2, y_2)=(x_1+iy_1)-(x_2+iy_2)$ $=(x_1-x_2)+i(y_1-y_2)=(x_1-x_2, y_1-y_2)$.

(iii) Definition of $-z$; Let $z=(x, y)$, then $-z=(-x,-y)$. It is clear that $0-z=(0,0)-(x, y)=(0-x, 0-y)=(-x,-y)$ $=-z$.

(iv) Definition of product ;

Let $z_1=(x_1, y_1)$, and $z_2=(x_2, y_2)$, then $z_1 z_2=(x_1, y_1)(x_2, y_2)$ $=(x_1+iy_1)(x_2+iy_2)=(x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)$ $=(x_1x_2-y_1y_2, x_1y_2+y_1x_2)$.

8. Addition properties of complex numbers : The following properties hold in **C**.

(i) Closure law for addition : $\forall z_1, z_2 \in \mathbf{C} \Rightarrow z_1+z_2 \in \mathbf{C}$.

(ii) Commutative law for addition ; $\forall z_1, z_2 \in C \Rightarrow$ $z_1+z_2=z_2+z_1$.

(iii) Associative law for addition : $\forall z_1, z_2, z_3 \in C$ $\Rightarrow z_1+(z_2+z_3)=(z_1+z_2)+z_3$.

(iv) ***Identity for addition*** : There exists exactly one c omp ex number, $0=0+i0 \in \mathbf{C} \Rightarrow z+0=0+z=z, \forall z \in \mathbf{C}$. Here o is called the additive identity of **C**:

(v) ***Inverse for addition*** : For each $z \in \mathbf{C}$, there exists exactly one $-z \in \mathbf{C}$ snch that $z+(-z)=(-z)+z=0$. Here $-z$ is called the inverse of z.

9. ***Multiplication properties of complex numbers*** ;

The following properties hold in **C** :

(i) ***Closure law for multiplication*** : $\forall z_1, z_2 \in \mathbf{C} \Rightarrow z_1 z_2 \in \mathbf{C}$.

(ii) ***Commutative law for multiplication*** : $\forall z_1, z_2 \in \mathbf{C} \Rightarrow z_1 z_2 = z_2 z_1$.

(iii) ***Associative law for multiplication*** : $\forall z_1, z_2, z_3 \in \mathbf{C} \Rightarrow z_1 (z_2 z_3) = (z_1 z_2) z_3$.

(iv) ***Identity for multiplication*** : There exists exactly one complex number, $1=1+i0 \in \mathbf{C} \Rightarrow z.1=1.z=z, \forall z \in \mathbf{C}$. Here 1 is called the multiplicative identity of **C**.

(v) ***Inverse for multiplication*** : Far each $z \in \mathbf{C}$, where $z \neq 0$, there exists exactly one $\frac{1}{z} \in \mathbf{C}$ such that $z.\frac{1}{z}=1$. Here $\frac{1}{z}$ is called the inverse of z.

10. ***The definition of division*** :

Let $z_1=(x_1, y_1)$ and $z_2=(x_2, y_2)$, then $\frac{z_1}{z_2} = \frac{(x_1, y_1)}{(x_2, y_2)}$

$$= \frac{(x_1, y_1)}{(x_2, y_2)} \cdot \frac{(x_2, -y_2)}{(x_2, -y_2)} = \frac{x_1 + i\, y_1}{x_2 + i y_2} \cdot \frac{x_2 - i\, y_2}{x_2 - i\, y_2}$$

$$= \frac{x_1 x_2 + y_1\, y_2 + i(y_1 x_2 - x_1\, y_2)}{x_2^2 + y_2^2} \quad \text{where } x_2^2 + y_2^2 \neq 0.$$

Example 5 : Express $\frac{(1+2i)^2}{(2+i)^2}$ in the form $A+iB$. Also find its modulus and argument. ***D. U. 83.***

Solution : $\frac{(1+2i)^2}{(2+i)^2} = \frac{1+4i+4i^2}{4+4i+i^2} = \frac{1+4i-4}{4+4i-1} = \frac{-3+4i}{3+4i}$

$$= \frac{(-3+4i)(3-4i)}{(3+4i)(3-4i)} = \frac{-9+24i+16}{3^2+4^2} = \frac{7+24i}{25} = 7/25 + 24/25i.$$

Here $\left|\frac{(1+2i)^2}{(2+i)^2}\right| = \left|7/25 + 24/25i\right| = \frac{\sqrt{(7^2+24^2)}}{25} = \frac{25}{25} = 1,$

its principal argument $= \tan^{-1} \frac{24/25}{7/25} = \tan^{-1} 24/7$ and general argument $= \tan^{-1} \frac{24}{7} + 2n\pi,$

where $n = 0, \pm 1, \pm 2, \pm 3, \quad \ldots \quad \ldots$

11 Equality of two complex numbers :

(i) $a + ib = 0 \Rightarrow a = 0,\ b = 0,$

(ii) $a + ib = c + ib \Rightarrow a = c,\ b = d.$

Two complex numbers are said to be equal if and only if their real and imaginary parts are equal.

12. *The distributive law :* The distributive law holds in **C** : $\forall z_1, z_2, z_3 \in C,\ z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$. Here it is clear that $z_1(z_2+z_3) = (z_2+z_3) z_1$.

13. Absolute value or modulus :

J. U. H. 86 ; R. U. H. 85.

Let $z = x + iy$ be a complex number, then the absolute value or modulus of z is denoted by $|z|$ and is given by

$|z| = |x+iy| = \sqrt{(x^2+y^2)}$. Here it is clear that $|z|$ is the distance from the origin and it is a non negative real number.

Example 6 : $|3+4i| = \sqrt{(3^2+4^2)} = 5$, $|3-4i| = \sqrt{(3^2+(-4)^2)} = 5$, $|-3+4i| = 5$, $|-3-4i| = 5$.

14. *Order relation* : Order relation exists only in the real number system and it does not exist in the complex number system. Therefore, ***inequality*** can be applied in the ordered real number system. Also it can be applied in the set of moduluses of complex numbers since the set of moduluses of complex numbers are non-negative real numbers.

Thus in complex variable "greater than" and "less than" have no meaning.

Example 7 : Which is the greater $3+4i$ or $6-8i$.

***Solution* :** Inequality can not be applied there since they are complex numbers. But $|6-8i| = 10 > |3+4i| = 5$.

15. *Complex plane or Argand diagram or Argand plane or Gaussian plane or diagram* :

We know a complex number $x+iy$ can be considered as an ordered pair (x,y) where $x,y \in R$ and it canbe represented by points in the xy-plane which is called the complex plane or Argand diagram or Gaussian

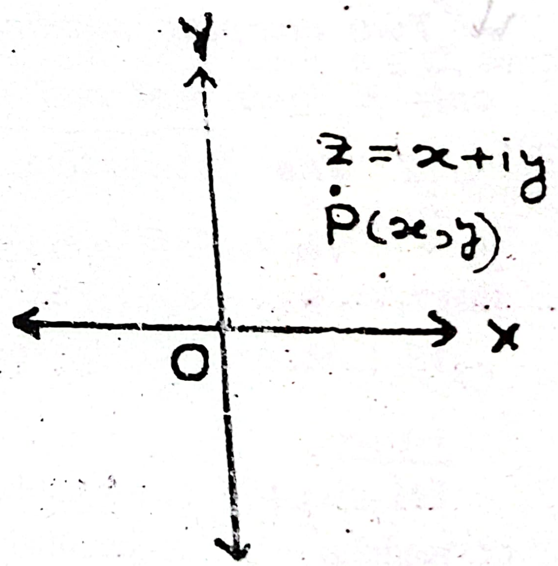


diagram. Thus the plane on which the complex numbers are represented is called the complex plane. In complex plane, to each complex number there exists one and conversely to each point in the complex plane there exists one and only one complex number.

Real axis : The points on the x–axis $= \{(x, 0) : x \in \boldsymbol{R}\}$ which are real numbers since $(x, 0) = x + i0 = x \in \boldsymbol{R}$. For this reason in the complex plane x–axis is called the real axis.

Imaginary axis : The points on the y–axis $= \{(0, y) : y \in \boldsymbol{R}\}$ which are pure imaginary numbers since $(0, y) = 0 + iy = iy \in \text{Im}$. For this reason in the complex plane y–axis is called the imaginary axis.

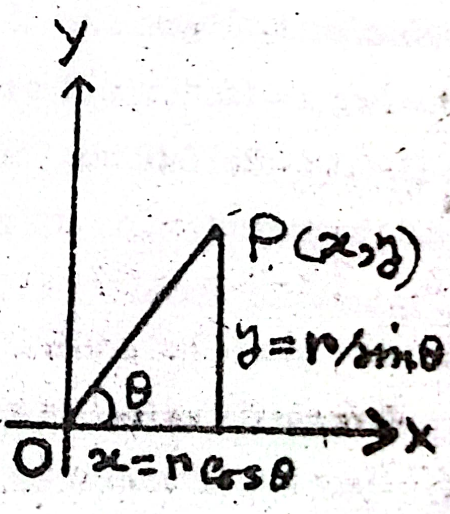
Zero : We have $(0, 0) = 0 + i0 = 0$ which is a complex number. It is clear that zero is the only number which is at a time real and imaginary since it lie on the real and imaginary axis at the origin.

Historical Note : French mathematician Jean Robert Argand (1768 – 1822) first explained the complex plane in 1806.

16. *Polar (or trigonometric) form of complex numbers :*

Let $z = x + iy$ be a complex number which is represected by the vector ***OP***.

Let OP $= |\,\boldsymbol{OP}\,| = r$ and any angle θ (positive which the vector makes with the positive x–axis, then we have $x = r\cos\theta,\ y = r\sin\theta,$



Here $OP = |OP| = \sqrt{(x^2+y^2)} = r$ is called the ***modulus*** or ***absolute value*** of $z=x+iy$ and it is denoted by $|z|$ or mod z. Again $\theta=\tan^{-1}y/x+2n\pi,\ n\in I$ is called the ***amplitude*** or ***argument*** or ***Phase*** of $z=x+iy$ and it is denoted by amp z or arg z. The form $z=x+iy=r\ (\cos\theta+i\sin\theta)$ is called the ***Polar form*** of the complex number, where r and θ is called the ***Polar coordinates***.

17. *Modulus*: ***J. U. H. 87 ; R. U. H. 85.***

Let (r, θ) be the polar coordinates corresponding to the complex number $z=x+iy=(x, y)$, then $r=\sqrt{(x^2+y^2)}$ is called the ***modulus*** or ***absolute value*** of z where $r=|z|=\sqrt{(x^2+y^2)}$.

18. Argument or amplitude or phase :

J. U. H. 85 ; R. U. H. 85.

Let (r, θ) be the polar coordinates corresponding to the complex number $z=(x, y)=x+iy$, then $\theta=\tan^{-1} y/x+2n\pi$ ... (1) is called the ***argument*** or ***amplitude*** or ***phase*** of z where $n\in I$ and $-\pi<\theta\leq\pi$. If $n=0$, then (1) $\Rightarrow \theta=\tan^{-1} y/x$ is called the ***principal argument*** of z and it is denoted by Arg z *i. e.* $\theta=\text{Arg } z=\tan^{-1} y/x$. It is clear that $\arg z=\text{Arg } z+2\pi n,\ n\in I$.

N. B. In the following chapters, by arg z we will mean the principal argument, *i. e.* $\arg z=\theta=\tan^{-1} y/x$, where $-\pi<\theta\leq\pi$, unless otherwise stated.

$\theta=\tan^{-1} y/x+2n\pi,\ n\in I$ is

sometimes called the ***general argument.***

19. Principal value of the argument :

Let $\theta=\arg z$, where $z\neq0$. Then the particular value of θ such that $\theta \in]-\pi, \pi]$ or $-\pi<\theta\leq\pi$ is called the principal value of the argument of the complex number z and it is denoted by Arg z.

Example 8 : Let $z=1+\sqrt{3}i$, then $\arg z=\tan^{-1}\sqrt{3}/1+2n\pi$ $=\pi/3+2n\pi$, where $n=0, \pm1, \pm2, \pm3, \cdots \cdots$ and $\text{Arg } z=\theta=\pi/3$.

Example 9 : Let $z=1-i$, then $\arg z=\tan^{-1}\left(\frac{-1}{1}\right)+2n\pi$ $=-\pi/4+2n\pi$, where $n=0, \pm1, \pm2, \pm3, \cdots \cdots$ and $\text{Arg } z=\theta$ $=-\pi/4$.

Example 10 : Let $z=-2-2i$, then $\arg z=\tan^{-1}\frac{-2}{-2}$ $+2n\pi=-3\pi/4+2n\pi$, where $n=0, \pm1, \pm2, \pm3, \cdots \cdots$ and $\text{Arg } z=\theta=-3\pi/4$.

Example 11 : $\arg i=\pi/2+2n\pi$, $n \in I$; $\text{Arg } i=\pi/_2$; $\arg(-1)$ $=\pi+2n\pi$, $n \in I$; $\text{Arg }(-1)=\pi$. n ; $\arg(-1+i)=3\pi/_4+2n\pi$, $n \in I$ and $\text{Arg }(-1+i)=\pi/_4$

N. B. (i) arg z is not unique but Arg z is unique.

(ii) if $z=0$, then arg z and Arg z do not exist since $z=0 \Rightarrow x=0, y=0$, then $\tan^{-1} y/x$ does not exists.

20. The distance between two points :

Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$.

Then $z_1-z_2=(x_1-x_2)+i(y_1-y_2)$ and $|z_1-z_2|$ $=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$. Here $|z_1-z_2|$ denotes the distance between the points z_1 and z_2.

21. Complex conjugate :

The complex conjugate of the number $z=x+iy$ is denoted by $\bar{z}$ and is defined by $\bar{z}=\overline{x+iy}=x-iy$.

Again the complex conjugate of the number $z = x - iy$ is $\bar{z} = x + iy$. The complex conjugate of z is called the ***reflection*** or ***image*** of z with respect to the real axis.

Historical Note : The French mathematician Augustin Cauchy (1789–1857) introduced the name "Conjugate" in 1821 in his Cours d' Aralyse algébrique.

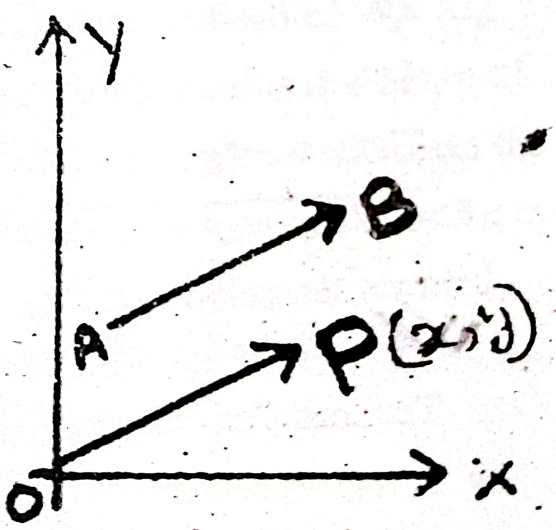
22. *Complex conjugate coordinates :* We have if $z = x + iy$, then

$$\bar{z} = x - iy, \quad x = \frac{1}{2}(z + \bar{z}) \text{ and } y = \frac{1}{2i}(z - \bar{z}).$$

Now if the coordinates $(z, \bar{z})$ denote a point, then these coordinates are called ***complex conjugate coordinates*** or simply ***conjugate coordinates*** of this point.

23. *Complex numbers in vector form :*

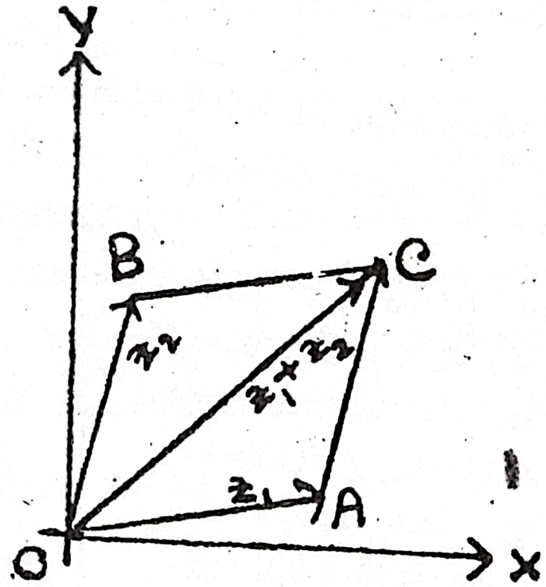
The complex number $z = x + iy$ can be represented by the directed line segment or vector ***OP*** whose initial Point O is called the origin and whose terminal Point P is the point (x, y). If the vectors ***OP*** and ***AB*** have the same length (or magnitude) and direction,



then they are considered equal *i. e.* ***OP*** = ***AB*** = $x + iy$. (see in the Fig.)

24. Geometrical representation of the sum, difference product and quotient of two complex numbers :

(1) Addition or sum : **D. U. H. T. 90.**



Let A and B be two points $z_1=(x_1, y_1)=x_1+iy_1$ and $z_2=(x_2, y_2)$ $=x_2+iy_2$ in the Argand plane. Now we complete the parallelogram OACB.

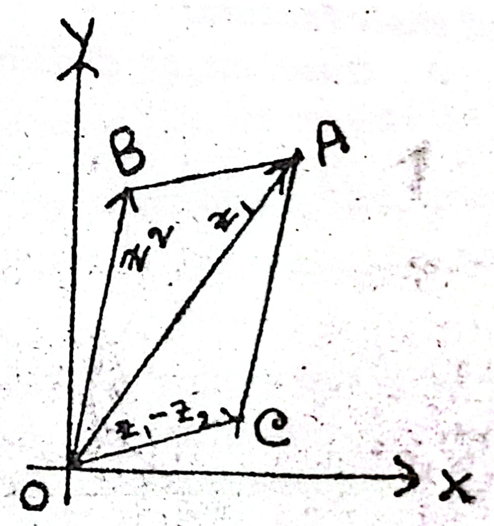
Then $z_1+z_2=(x_1+x_2)+i(y_1+y_2)$ represents the point C, see in the Fig. In vector form :

$$\vec{OC}=\vec{OA}+\vec{AC}=\vec{OA}+\vec{OB}=\vec{OB}+\vec{BC}.$$

2. Subtraction or difference :

Let A and B be the points $z_1=(x_1, y_1)=x_1+iy_1$ and $z_2=(x_2, y_2)$ $=x_2+iy_2$ in the Argand plane.

Now we complete the parallelogram OCAB.

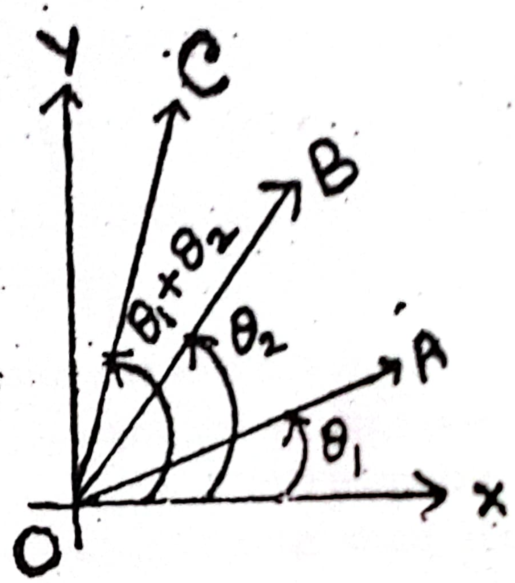


Then $z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)i$ represents the point C, see in the Fig. In vector form $\vec{OC} = \vec{OA} - \vec{CA} = \vec{OA} - \vec{OB} = \vec{BA}$.

3. *Product or multiplication :*

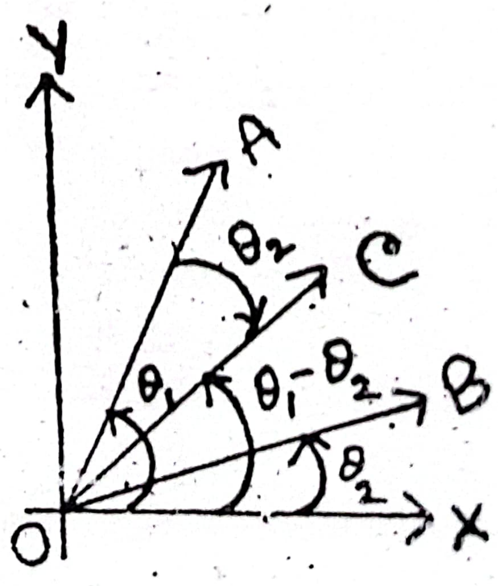
D. U. H. T. 90 ; C. U. 68.

Let A and B be the points $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$ where $OA = r_1$, $OB = r_2$ and $\angle XOA = \theta_1$, $\angle XOB = \theta_2$, see in the Fig. Then $z_1 z_2 = r_2 r_1 \{\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\}$ represents the point C where $OC = r_1 r_2$ and $\angle XOC = \theta_1 + \theta_2$, see in the Fig.



Hence, the modulus and amplitude of the product of two complex nnmbers are equal respectively to the product of their moduli and the sum of the amplitudes of their factors.

4. *Quotient or division :*

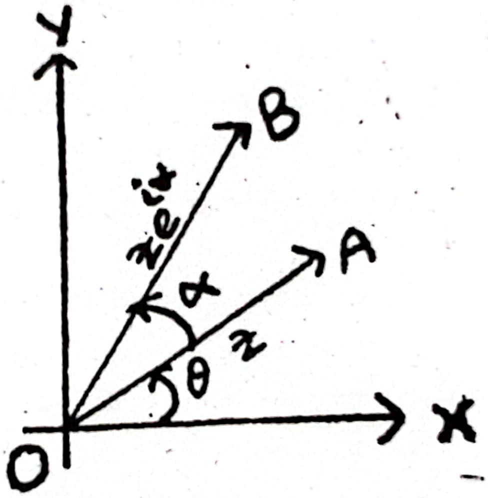


Let A and B be the points $z_1 = r_1$ $(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2$ $(\cos\theta_2 + i\sin\theta_2)$ where $OA = r_1$, $OB = r_2$ and $\angle XOA = \theta_1$, $\angle XOB = \theta_2$, see in the Fig.

Then $z_1/z_2 = r_1/r_2$ $\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$ represents the point C where $OC = r_1\, r_2$ and $\angle XOC = \theta_1 - \theta_2$, see in the Fig.

Hence, the modulus and amplitude of the quotient of two complex numbers are equal respectively to the quotient of their moduli and the difference of the amplitudes of the numerator and denominator.

25. *Interpretation of* $ze^{i\alpha}$:

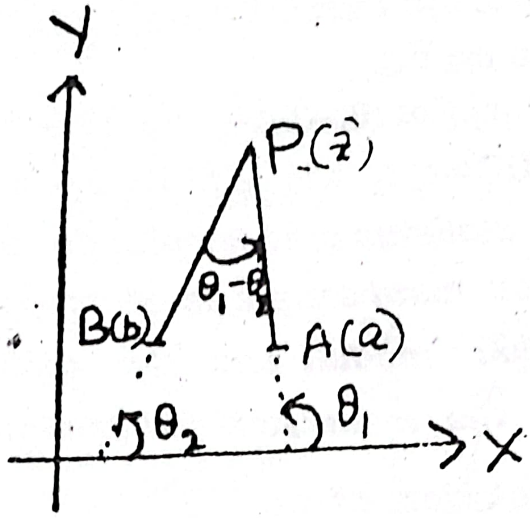


Let $z = re^{i\theta}$ be represented by the vector $\vec{OA}$, see in the Fig.

Then $ze^{i\alpha} = re^{i\theta} \cdot e^{i\alpha} = re^{i(\theta+\alpha)}$ is represented by the vector $\vec{OB}$ where θ and α are real.

Henee $ze^{i\alpha}$ *i. e.* multiplication of a vector z by $e^{i\alpha}$ amounts the rotation of z through an angle α in the positive direction Here $z\,i \Rightarrow$ the rotation of z through an angle $\pi/2$ since $zi = ze^{\pi i/2}$.

26. *Interpretation of* arg $\frac{z-a}{z-b}$:



Let z, a, b represent the points P, A, B respectively on the Argand plane. Then $\vec{AP}=z-a$, $\vec{BP}=z-b$, $\arg \vec{AP}=\theta_1$, $\arg \vec{BP}=\theta_2$ and $\angle BPA=\theta_1-\theta_2=\arg(z-a)-\arg(z-b)=\arg\frac{z-a}{z-b}$ see in the Fig. where we have considered only the principal argument. Thus $\arg\frac{z-a}{z-b}$ gives the angle between the lines AP and BP in the positive sense. Similarly, $\arg\frac{z-b}{z-a}$ gives the angle in the negative sense.

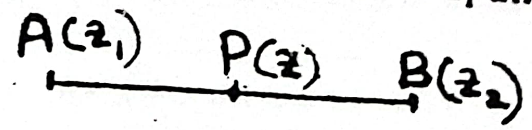
(i) *Condition for perpendicularity*:

If the lines AP and BP are perpendicular, then $\arg\frac{z-a}{z-b}=\pi/2$ & $\arg\frac{z-b}{z-a}=-\pi/2 \Rightarrow \frac{z-a}{z-b}$ and $\frac{z-b}{z-a}$ are purely imaginaries.

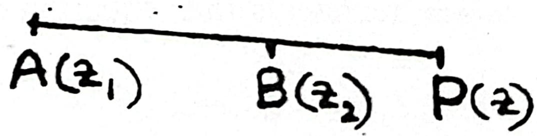
(ii) Condition for collinearity ;

If the points A, B, P are collinear, then $\arg \dfrac{z-a}{z-b}=0$ or $\pi \Rightarrow \dfrac{z-a}{z-b}$ is purely real.

27. Equation of a straight line joining the points z_1 and z_2 in the Argand plane :



Let P (z) be any point on the line AB where z_1 and z_2 are points of A and B respectively, see in the Fig.



If P (z) lies inside, then $\arg \dfrac{z-z_1}{z-z_2}=\pi \Rightarrow \dfrac{z-z_1}{z-z_2}$ is purely real. Again if P (z) lies outside, then $\arg \dfrac{z-z_1}{z-z_2}=0 \Rightarrow$ $\dfrac{z-z_1}{z-z_2}$ is purely real. Thus in both cases $\dfrac{z-z_1}{z-z_2}$ is purely real $\Rightarrow$

$$\Rightarrow \frac{z-z_1}{z-z_2}=\overline{\left(\frac{z-z_1}{z-z_2}\right)}=\frac{\bar z-\bar z_1}{\bar z-\bar z_2} \Rightarrow$$

$$(z-z_1)\left(\bar z-\bar z_2\right)-(z-z_2)\left(\bar z-\bar z_1\right)=0.$$

$$\Rightarrow \left(\bar z_1-\bar z_2\right)z-(z_1-z_2)\,\bar z+\left(z_1\bar z_2-z_2\bar z_1\right)=0.$$

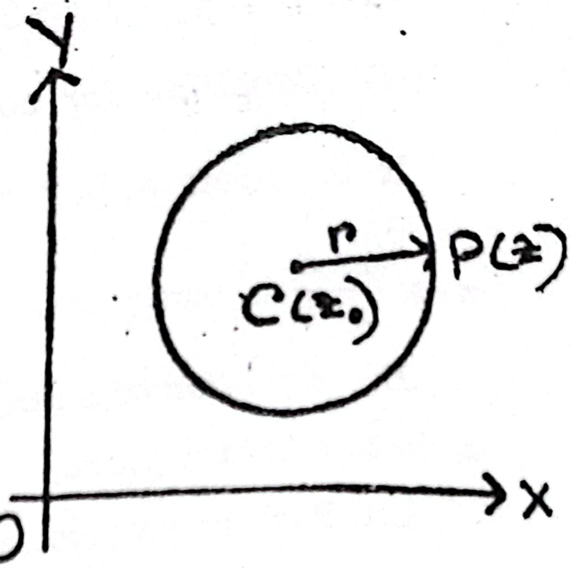
$$\Rightarrow \bar a z-a\bar z+c=0$$

which represents the equation of a straight line where

$\bar a=\bar z_1-\bar z_2$ $\therefore$ $a=z_1-z_2$ and $c=z_1\bar z_2-z_2\bar z_1$

28. *Equation of a circle with centre at z_0 and radius r :*

Let P (z) be any point on the circle with centre C (z_0) and r be its radius.



$|z-z_0|=r \Rightarrow |z-z_0|^2=r^2$

$\Rightarrow (z-z_0)\overline{(z-z_0)}=r^2$

$\Rightarrow (z-z_0)(\bar{z}-\bar{z}_0)-r^2$

$\Rightarrow z\bar{z}-\bar{z}_0z-z_0\bar{z}+(z_0\bar{z}_0-r^2)=0$

$\Rightarrow z\bar{z}+\bar{a}z+a\bar{z}+\lambda=0,$

which represents the equation of a circle where $a=-z_0,\ \bar{a}=-\bar{z}_0$ anb $z_0\bar{z}_0-r^2=\lambda$ (real).

29. *Dot product of two complex numbers :*

The dot product or scalar product of two complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ is defined by $z_1 \circ z_2=|z_1|\,|z_2|\cos\theta$

$=x_1x_2+y_1y_2=\mathrm{Re}\{\bar{z}_1 z_2\}=\frac{1}{2}\{\bar{z}_1 z_2+z_1\bar{z}_2\}$ where θ is the angle between z_1 and z_2 and $0\leqslant\theta\leqslant\pi$.

(*i*) If $z_1\neq0,\ z_2\neq0$, then $z_1 \circ z_2=0 \Rightarrow z_1$ and z_2 are perpendicular.

(*ii*) The projection of z_1 on z_2 is $\dfrac{|z_1 \circ z_2|}{|z_2|}$ and the projecton of z_2 on z_1 is $\dfrac{|z_1 \circ z_2|}{|z_1|}$.

Example 12 : If $z_1=4-3i$ and $z_2=-3+4i$, then $z_1 \circ z_2$ $=\mathrm{Re}(\bar{z}_1 z_2)=\mathrm{Re}\{(4+3i)(-3+4i)\}=\mathrm{Re}\{-24+7i\}=-24$ and

$$\cos\theta=\frac{z_1 \circ z_2}{|z_1|\,|z_2|}=\frac{-24}{|4-3i|\,|-3+4i|}=\frac{-24}{(5)(5)}=\frac{-24}{25}.$$

Example 13 : Show that $z_1 \circ z_2=z_2 \circ z_1$.

Solntion : Try yourself.

Example 14 : If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 \circ z_2$

$= \mathrm{Re}\{\bar{z}_1 z_2\} = \mathrm{Re}\{(r_1 e^{-i\theta_1})(r_2 e^{i\theta_2})\} = \mathrm{Re}\{r_1 r_2 e^{i(\theta_2-\theta_1)}\}$

$= r_1 r_2 \cos(\theta_2 - \theta_1)$.

30. Cross product of two complex numbers :

The cross product or vector product of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is defined by $z_1 \times z_2 =$

$|z_1|\,|z_2| \sin\theta = x_1 y_2 - y_1 x_2 = \mathrm{Im}\{\bar{z}_1 z_2\}$

$= \frac{1}{2i}\{\bar{z}_1 z_2 - z_1 \bar{z}_2\}$, where θ is the angle between z_1 and z_2 and $0 \leq \theta \leq \pi$.

(*i*) If $z_1 \neq 0$, $z_2 \neq 0$, then $z_1 \times z_2 = 0 \Rightarrow z_1$ and z_2 are parallel.

(*ii*) The area of a parallelogram is $|z_1 \times z_2|$ where z_1 and z_2 are its sides.

Example 15 : If $z_1 = 4 - 3i$ and $z_2 = -3 + 4i$, then $z_1 \times z_2 =$

$\mathrm{Im}\{\bar{z}_1 z_2\} = \mathrm{Im}\{(4+3i)(-3+4i)\} = \mathrm{Im}\{-24+7i\} = 7$.

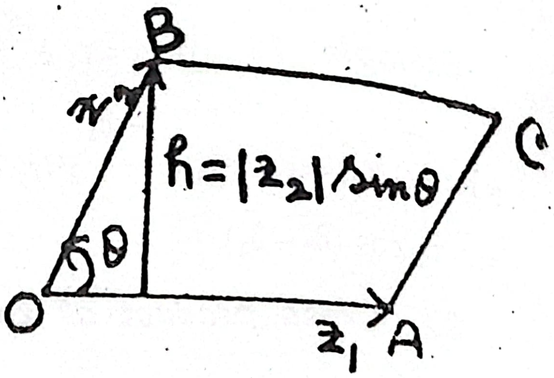
Example 16 : Show that $z_1 \times z_2 = -z_2 \times z_1$.

Solution : Try yourself.

Example 17 : If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 \times z_2$

$= \mathrm{Im}\{\bar{z}_1 z_2\} = \mathrm{Im}\{r_1 r_2 e^{i(\theta_2-\theta_1)}\} = r_1 r_2 \sin(\theta_2 - \theta_1)$.

31. *Area of a parallelogram :*



The area of a parallelogram having sides z_1 and z_2 =(base) (height) (see in the Fig) = (OA) (h) = $(|z_1|)(|z_2|)\sin\theta = |z_1||z_2|\sin\theta = |z_1 \times z_2|$

32. *Area of a triangle :*

The area of a triangle OAB is $\frac{1}{2}|z_1 \times z_2|$ where z_1 and z_2 are the sides of a parallelogram OACB, see in the above Fig. Here $\triangle OAB = \frac{1}{2}|(x_1 y_2 - y_1 x_2)|$.

The area of a trianle having vertices $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ is

$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

33 (*i*) *Theorem 1* : If $z_p = r_p(\cos\theta_p + i\sin\theta_p)$ where $p = 1, 2, \cdots n$, then $z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n\{\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)\}$.

***Proof* :** Try yourself.

33. (*i*) *Theorem 2* : If $z = r(\cos\theta + i\sin\theta)$, then show that $z^n = r^n(\cos n\theta + i\sin n\theta)$ where $n \in \{2, 3, \cdots, n\}$.

***Proof* :** Putting $z_1 = z_2 = \cdots = z_n = z$, $r_1 = r_2 = \cdots = r_n = r$ and $\theta_1 = \theta_2 = \cdots = \theta_n = \theta$. then the above theorem $\Rightarrow z^n = r^n(\cos n\theta + i\sin n\theta)$.

N. B. The above theorem is true for every integral values of n.

34. *De moivres theorem for positive integral values :*

***Theorem 3* :** $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \ \forall\, n \in \boldsymbol{N}$.

Proof : We have if $z=r\ (\cos\theta+i\sin\theta)$, then $z^n=r^n(\cos n\theta+i\sin n\theta)$ where $n\in\mathbb{N}$.

That is, $r^n(\cos\theta+i\sin\theta)^n=r^n(\cos n\theta+i\sin n\theta)$

$\Rightarrow (\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$

35. Deo moivre's theorem for negative Integers.

Theorem 4 : For all negative integral values of n,

$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$

Proof ; Try yourself.

36. Deo moivre's theorem for rational numbers :

Theorem 5 : $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.\ \forall\, n\in\mathbb{Q}$

Proof : Try yourself.

37. General Deo moivre's theorem (Theorem 5) :

$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,\ \forall\, n\in\mathbb{R}.$

Proof : Try yourself.

38. Euler Formula ; Show that : $e^{i\theta}=\cos\theta+i\sin\theta$, which is called Euler's formula. We have

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^2}{3!}+\ \cdots\ \cdots\ \cdots\ \cdots \quad (1)$$

$$\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ \cdots\ \cdots\ \cdots\ \cdots \quad (2)$$

$$\text{and } \sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}\ \cdots\ \cdots\ \cdots\ \cdots \quad (3)$$

Now putting $x=i\theta$ in (1) we get

$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)}{4!}+\ \cdots\ \cdots\ \cdots$$

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots\cdots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\cdots\right)$$

$=\cos\theta+i\sin\theta$, by (2) and (3). Similarly, $e^{-i\theta}=\cos\theta-i\sin\theta$.

It is clear that $e^z=e^{x+iy}=e^x(\cos y+i\sin y)$.

39. Deo moivre's theorem in Euler's Form :

$$(\cos\theta+i\sin\theta)^n=\left(e^{i\theta}\right)^n=e^{in\theta}$$

N.B. cis θ : It is sometimes written cis θ for $\cos\theta+i\sin\theta$.

Example 18 : Show that : (i) $\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$,

(ii) $\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$, (iii) $\tan\theta=\frac{e^{i\theta}-e^{-i\theta}}{i\,(e^{-i\theta}+e^{-i\theta})}$.

(iv) $\sec\theta=\frac{2}{e^{i\theta}+e^{-i\theta}}$, (v) $\operatorname{cosec}\theta=\frac{2i}{e^{i\theta}-e^{-i\theta}}$

(vi) $\cot\theta=\frac{i(e^{i\theta}+e^{-i\theta})}{e^{i\theta}-e^{-i\theta}}$

Solution : We have $e^{i\theta}=\cos\theta+i\sin\theta$ and $e^{-i\theta}=\cos\theta-i\sin\theta$, then : Try yourself.

40. *Complex polynomials and complex polynomial equations :* An expression of the form

$$a_0z^n+a_1z^{n-1}+a_2z^{n-2}+\ \cdots\cdots\ +a_{n-1}z+a_n \quad \cdots\cdots \quad (1)$$

is called a ***Complex polynomial*** of degree n, where $a_0 \neq 0, a_1, a_2, \cdots, a_n \in \boldsymbol{C}$ and $n \in \boldsymbol{N}$.

If $a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0 \cdots$ (2),

then it is called the ***Complex polynomial equation*** of degree n, where $a_0 \neq 0, a_1, a_2, \cdots, a_n \in \boldsymbol{C}$ and $n \in \boldsymbol{N}$.

If $z_1, z_2, \cdots, z_n$ are the n roots of the equation (2),

then (2) $\Rightarrow a_0 (z - z_1)(z - z_2) \cdots (z - z_n) = 0 \cdots \cdots$ (3).

The equation (3) is called the ***factor form*** of the polynomial equation.

41. Algebraic number : A number is called an ***algebraic number*** if it is a solution of any polynomial equation of the form

$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$, where $a_0, a_1, a_2, \cdots a_n \in I$.

Example 19 : $\sqrt{2}$ in an algebraic number since it satisfies the polynomial equation $z^2 - 2 = 0$ of integarl coefficients. similarly, $\sqrt{2} + \sqrt{3}$ is an algebraic number since it satisfies the polynomial equation $z^4 - 10z^2 + 1 = 0$ of integral coefficients.

42. Transcendental* number : A number is called a ***transcendetal number*** if it is not a solution of any polynomial equation with integral coefficients *i. e.* if it is not algebraic.

Example 20 : e and π are transcendental numbers. But it is not known numbers such as $e\pi$ and $e + \pi$ are transcendental or not.

43. The roots of complex numbers :

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Let z be a complex number. Then a number w is called an nth root of z if $w^n = z \Rightarrow w = z^{1/n}$ ... (1). Let $z = r\cos\theta + i\sin\theta$, then (1) $\Rightarrow w = r^{1/n}\left(\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n}\right)$ ... (2).

where $k = 0, 1, 2, \cdots, n-1 \Rightarrow$ there are n different values of $z^{1/n}$ where $z \neq 0$. (2) $\Rightarrow$

$$w = r^{1/n} e^{\left(\frac{2k\pi+\theta}{n}\right)i},$$

where $k = 0, 1, 2, \cdots, n-1$.

Example 21 : Show that : $e^{i\theta} = e^{2n\pi i + i\theta}$, where $n \in I$.

Solution : Try yourself.

44. ***The nth roots of unity*** : Let $z^n = 1$.

Then $z = (1)^{1/n} = \left\{\frac{(\cos 2k\pi + i\sin 2k\pi)}{n}\right\}^{1/n}$

$$= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{\frac{2k\pi}{n}}.$$

where $k = 0, 1, 2, \cdots, n-1$. Let $w = e^{\frac{2k\pi i}{n}}$, then the n roots are : $1, w, w^2, \cdots w^{n-1}$. The ***principal nth root of unity is 1.***

Geometrically these n roots represent the n vertices of a regular polygon of n sides inscribed in a circle of radius 1 (one) where centre is at the point (0, 0).

Here $|z| = \left|\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right|$

$= \sqrt{\cos^2\frac{2k\pi}{n} + \sin^2\frac{2k\pi}{n}} = 1$, which is the equation of a unit circle.

Example 22 : Show that : $e^{2n\pi i}=1$ where $n \in I$.

Solution : We have $e^{2n\pi i}=\cos 2n\pi+i \sin 2n\pi=1+i0=1$, where $n \in I$.

Example 23 ; If $w, w^2, w^3, \cdots, w^{n-1}, w^n=1$ are the n roots of $(1)^{1/n}$, where n is a positive integer then show that $1+w+w^2+\cdots+w^{n-1}=0$.

Solution : Try yourself.

45. The nth roots of (−1) : Let $z^n=-1$.

Then $z=(-1)^{1/n}=\{\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)\}^{1/n}$

$$=\cos\frac{(2k+1)\pi}{n}+i\sin\frac{(2k+1)\pi}{n}=e^{\frac{(2k+1)\pi i}{n}},$$

where $k=0, 1, 2, 3, \cdots, n-1$. The ***principal nth root of*** (-1) is $e^{\frac{\pi i}{n}}$ if $k=0$. If n is odd, then the principal root is (-1) if $k=\frac{1}{2}(n-1)$.

46. Factors of z^n-1 : The n factors of z^n-1 are :

$z-e^{\frac{2k\pi i}{n}}$ where $k=0, 1, 2, 3, \cdots, n-1$.

(*i*) If n is even, then

$$z^n-1=(z-1)(z+1)\prod_{k=1}^{(n-2)/2}\left(z^2-2z\cos\frac{2k\pi}{n}+1\right).$$

(*ii*) If n is odd, then

$$z^n-1=(z-1)\prod_{k=1}^{(n-1)/2}\left(z^2-2z\cos\frac{2k\pi}{n}+1\right).$$

47. Factors of z^n+1 : The n factors of z^n+1 are :

$$z=e^{\frac{(2k+1)\pi i}{n}} \quad \text{where } k=0, 1, 2, 3, \cdots, n-1.$$

(*i*) If n is even, then

$$z^n+1=\prod_{k=0}^{k=(n-2)/n}\left\{z^2-2z\cos\frac{(2k+1)\pi}{n}+1\right\}.$$

(*ii*) If n is odd, then

$$z^n+1=(z+1)\prod_{k=0}^{(n-3)/2}\left\{z^2-2z\cos\frac{(2k+1)\pi}{n}+1\right\}.$$

48. Point sets in the Argand plane :

(*i*) ***Neighbourhood*** : A neighbour hood of a point z_0 in the Argand plane is the set of all points z such that $|z-z_0|<\delta$ where $\delta>0$. It is also called a ***delta*** or ***δ–neighbourhood*** of z_0. A ***deleted*** or ***δ–neighbourhood*** of a point z_0 in the Argand plane is the set of all points z such that $0<|z-z_0|<\delta$, where $\delta>0$. In this case z_0 is omitted.

(*ii*) ***Limit point or cluster point or accumulation point :*** A point z_0 is called a limit point for a set S in the Argand plane if every deleted δ-neighbourhood of z_0 contains points of S other than z_0.

(*iii*) ***Closed set :*** A set S in the Argand plane is called closed if every limit point of the set S belongs to the set S.

(*iv*) ***Bounded set :*** A set S in the Argand plane is called bounded if there exists a positive constant M such that $|z|<M$ for every point z of S.

(v) ***Unbounded set*** : A set S in the Argand plane is called unbounded it it is not bounded.

(vi) ***Compact set*** : A set S in the Argand plane is called compact if it is both bounded and closed.

(vii) ***Interior point*** : A point z_0 is said to be an interior point of a set S in the Argand plane if there exists a δ-neighbourhood of z_0 all of whose points belonging to S.

(viii) ***Boundary (or frontier) point*** : A point z_0 is called a boundary point of a set S in the Argand plane if every δ-neighbourhood of z_0 contains points belonging to S and also points not belonging to S.

(ix) ***Exterior point*** : A point is called an exterior point of a set S in the Argand plane if it is not an interior point and also not a boundary point of the set S.

(x) ***Open set*** : A set S in the Argand plane is called an open set if it is a set which consists entirely of interior points.

(xi) ***Connected set*** : An open set S in the Argand plane is said to be connected if each pair of its point can be joined by some continuous chain of finite number of line segments all points of which lie in the set S.

(xii) ***Domain or open region*** : An open connected set S in the Argand plane is called a domain or open region.

(xiii) ***Interior of a set***. The set of all interior points of a set S in the Argand plane is called the interior of the set and it is deonted by S_i.

(xiv) *Frontier (or boundary) of a set :*
The set of all frontier points of a set in the Argand plane is called the frontier of the set and it is denoted by S_f.

(xv) *Exterior of a set ;* The set of all exterior points of a set in the Argand plane is called the exterior of the set and it is denotod by S_e.

(xvi) *Derived sets :* The set of all the limit points of a set S in the Argand plane is called the derived set and it is denoted by $\bar{S}$:

(xvii) *Closure of a set :* The union $S_i \cup S_f$ is called the closure of the set S and it is denoted by $\bar{S}$ where $\bar{S} = S_i \cap S_f$, S_i = Interior of the set S and S_f = frontier of the set S.

(xviii) *Closed region* The closure of an open region or domain in the Argand plane is called a closed region.

(xix) *Region :* Let S be an open region or domain. If we take some or all or none of its limit, then we obtain a set R (say) which is called a region.

If all the limit points are added, then R is called a closed region. If none limit points are added, then R is called an open region

N. B. In this book, by a region we will mean the open region unless otherwise any other region is stated.

(xx) ***Two important theorems :***

(1) ***Bolzano-weierstrass theorem*** (***Theorem – 6***) :

Every infinite bounded set on the Argand plane has at least one limit point.

(2) ***Heine-Borel theorem*** (***Theorem – 7***) :

Let S be a compact set in the Argand plane.

If to each point of S is contained in one or more of the open sets $A_1, A_2, A_3, \cdots$, then there exists a finite number of sets $A_1, A_2, A_3, \cdots$, which will cover S.

49. *Theorem 8*. If $z_1, z_2, \cdots, z_n \in \mathbf{C}$, then show that :

(*i*) $\operatorname{Re}(z_1 \pm z_2) = \operatorname{Re}(z_1) \pm \operatorname{Re}(z_2)$;

(*ii*) $\operatorname{Re}(z_1 \pm z_2 \pm \cdots \pm z_n) = \operatorname{Re}(z_1) \pm \operatorname{Re}(z_2) \pm \cdots \pm \operatorname{Re}(z_n)$;

(*iii*) $\operatorname{Im}(z_1 \pm z_2) = \operatorname{Im}(z_1) \pm \operatorname{Im}(z_2)$;

(*iv*) $\operatorname{Im}(z_1 \pm z_2 \pm \cdots \pm z_n) = \operatorname{Im}(z_1) \pm \operatorname{Im}(z_2) \pm \cdots \pm \operatorname{Im}(z_n)$.

Proof : Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2) \Rightarrow \operatorname{Re}(z_1 \pm z_2) = x_1 \pm x_2 = \operatorname{Re} z_1 \pm \operatorname{Re}(z_2)$

(*ii*), (*iii*) and (*iv*) : Try yourself.

50. *Theorem 9* ; Show that :

(*i*) $\operatorname{Re}(z_1 z_2) = \operatorname{Re}(z_1)\operatorname{Re}(z_2) - \operatorname{Im}(z_1)\operatorname{Im}(z_2)$,

(*ii*) $\operatorname{Im}(z_1 z_2) = \operatorname{Re}(z_1)\operatorname{Im}(z_2) + \operatorname{Im}(z_1)\operatorname{Re}(z_2)$.

Proof ; Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$.

(*i*) $\operatorname{Re}(z_1 z_2) = x_1 x_2 - y_1 y_2 = \operatorname{Re}(z_1)\operatorname{Re}(z_2) - \operatorname{Im}(z_1)\operatorname{Im}(z_2)$

and (*ii*) $\operatorname{Im}(z_1 z_2) = x_1 y_2 + y_1 x_2 = \operatorname{Re}(z_1)\operatorname{Im}(z_2) + \operatorname{Im}(z_1)\operatorname{Re}(z_2)$.

51. *Theorem 10* : Let z be a complex number, then show that :

(*i*) z is real if $z = \bar{z}$ and

(*ii*) z is purely imaginary if $z = -\bar{z}$.

Proof ; Let $z = x + iy$, then $\bar{z} = x - iy$.

(*i*) Now if $z = \bar{z} \Rightarrow x + iy = x - iy \Rightarrow 2iy = 0 \Rightarrow y = 0 \Rightarrow$
$z = x \Rightarrow z$ is real.

(*ii*) Now if $z = -\bar{z} \Rightarrow x + iy = -$

$\Rightarrow z = iy \Rightarrow z$ is purely imaginary

52. ***Theorem 11*** : Show that .

(*i*) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$;

(*ii*) $\bar{\bar{z}} = z$;

(*iii*) $\overline{z + \bar{z}} = z + \bar{z}$ and

(*iv*) $\overline{z - \bar{z}} = -(z - \bar{z})$.

Proof : (*i*) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.
Then $\overline{z_1 + z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)} = \overline{(x_1 + x_2) + i(y_1 + y_2)}$

$= (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \bar{z}_1 - \bar{z}_2$.

(*ii*) Let $z = x + iy$, then $\bar{z} = x - iy \Rightarrow \bar{\bar{z}} = x + iy = z$.

(*iii*) and (*iv*) : Try yourself.

63. ***Theorem 12*** : Show that :

(*i*) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$;

(ii) $\overline{z^2}=\left(\overline{z}\right)^2$

(iii) $\overline{z_1 z_2 \cdots z_n}=\overline{z}_1\,\overline{z}_2 \cdots \overline{z}_n$;

(iv) $\overline{z^n}=\left(\overline{z}\right)^n$.

Froof : (*i*) Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$.

Then $z_1 z_2=(x_1 x_2-y_1 y_2)+i\,(x_1 y_2+y_1 x_2) \Rightarrow$

$\overline{z_1 z_2}=(x_1 x_2-y_1 y_2)-i\,(x_1 y_2-y_1 x_2)=(x_1-iy_1)\,(x_2-iy_2)$

$=\overline{z}_1\,\overline{z}_2$.

(*ii*) If $z_1=z_2=z$, then (*i*) $\Rightarrow \overline{z^2}=\left(\overline{z}\right)^2$

(*iii*) using (*i*) again and again we have

$\overline{z_1 z_2 \cdots z_n}=\overline{z_1 (z_2 z_3 \cdots z_n)}=\overline{z}_1\,\overline{z_2 z_3 \cdots z_n}$

$=\overline{z}_1\,\overline{z_2 (z_3 z_4 \cdots z_n)}=\overline{z}_1\,\overline{z}_2\,\overline{z_3 z_4 \cdots z_n}= \cdots \cdots$

$=\overline{z}_1\,\overline{z}_2 \cdots \overline{z}_n$.

(*iv*) If $z_1=z_2=\cdots=z_n=z$, (*iii*) $\Rightarrow \overline{z^n}=\left(\overline{z}\right)^n$.

54. *Theorem 13 :* Show that :

(*i*) $|z|^2=|\overline{z}|^2=z\overline{z}$; (*ii*) $|z_1 z_2 \cdots z_n|^2=\overline{|z_1 z_2 \cdots z_n|}^2=(z_1 z_2 \cdots z_n)\,(\overline{z_1 z_2 \cdots z_n})$.

Proof: (*i*) Let $z=x+iy$, then $z\overline{z}=(x+iy)\,(x-iy)=x^2+y^2=|z|^2$. Again

$|\overline{z}|^2=|x-iy|^2=x^2+y^2 \Rightarrow |z|^2=|\overline{z}|^2=z\overline{z}$.

(*ii*) Try yourself.

55. ***Theorem 14*** : Show that :

D. U. M. SC. P.89.

(*i*) $|z_1 z_2| = |z_1|\,|z_2|$;

(*ii*) $|z_1 z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|$. **D.U.H.S.T.85.**

Salution : (i) We have $|z_1 z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)}$

$= (z_1 z_2)(\bar{z}_1 \bar{z}_2) = (z_1 \bar{z}_1)(z_2 \bar{z}_2) = |z_1|^2 |z_2|^2 \Rightarrow$

$|z_1 z_2| = |z_1|\,|z_2|$.

(*ii*) We have $|z_1 z_2 \cdots z_n|^2 = (z_1 z_2 \cdots z_n)\overline{(z_1 z_2 \cdots z_n)}$.

$= (z_1 z_2 \cdots z_n)(\bar{z}_1 \bar{z}_2 \cdots \bar{z}_n) = (z_1 \bar{z}_1)(z_2 \bar{z}_2) \cdots (z_n \bar{z}_n) =$

$|z_1|^2 |z_2|^2 \cdots |z_n|^2 \Rightarrow |z_1 z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|$

56 ***Theorem 15*** : Show that for any two complex numbers z_1 and z_2 :

(*i*) $|z_1+z_2| \leqslant |z_1| + |z_2|$;

D. U. H. T. 87 ; D. U. H. 86 ; D. U, M. SC. P. 88, 89 ; R. U. 67 ; C. U. 68 , J. U. H. 87.

(*ii*) $|z_1+z_2+z_3| \leqslant |z_1| + |z_2| + |z_3|$;

(*ii*) $|z_1+z_2+\cdots+z_n| \leqslant |z_1| + |z_2| + \cdots + |z_n|$.

D. U. H. S.T. 85.

Proof : (*i*) We have $|z|^2 = z\bar{z}$, $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, $z + \bar{z} = 2\text{Re}(z) \leqslant 2|z|$, $|z_1 z_2| = |z_1|\,|z_2|$ and $|z| = |\bar{z}|$

Now using these we have $|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$

$=(z_1+z_2)\overline{(z_1+z_2)}=z_1\bar{z}_1+z_2\bar{z}_2+\bar{z}_1z_2+z_1\bar{z}_2=|z_1|^2+|z_2|^2+$

$\bar{z}_1z_2+\overline{(\bar{z}_1z_2)}=|z_1|^2+|z_2|^2+2\text{Re}(z_1\bar{z}_2)$

$\leqslant|z_1|^2+|z_2|^2+2|z_1\bar{z}_2|=|z_1|^2+|z_2|^2+2|z_1||\bar{z}_2|$

$=|z_1|^2+|z_2|^2+2|z_1||z_2|=(|z_1|+|z_2|)^2$

$\Rightarrow|z_1+z_2|\leqslant|z_1|+|z_2|$

(*ii*) Using (*i*) we have $|z_1+z_2+z_3|=|(z_1+z_2)+z_3|$

$\leqslant|z_1+z_2|+|z_3|\leqslant|z_1|+|z_2|+|z_3|$

(*iii*) Using (*i*) we have $|z_1+z_2+\cdots+z_n|$

$=|(z_1+z_2+\cdots+z_{n-1})+z_n|\leqslant|z_1+z_2+\cdots+z_{n-1}|+|z_n|$

$\leqslant|z_1+z_2+\cdots+z_{n-2}|+|z_{n-1}|+|z_n|\leqslant\cdots\ \cdots\ \cdots$

$\leqslant|z_1|+|z_2|+\cdots+|z_n|\Rightarrow|z_1+z_2+\cdots+z_n|$

$\leqslant|z_1|+|z_2|+\cdots+|z_n|$

Thus the theorem is proved.

56. ***Theorem 16*** **:** Show that for any two complex numbers z_1 and z_2 :

$$|z_1-z_2|\leqslant|z_1|+|z_2|\text{ .D. U. H. 89.}$$

Proof (First method) : We have $|z_1+z_2|\leqslant|z_1|+|z_2|$

Now putting $-z_2$ for z_2, we get $|z_1-z_2|\leqslant|z_1|+|-z_2|$

$=|z_1|+|z_2|$ since $|-z_2|=|z_2|$.

(Second method) : We have $|z_1-z_2|^2=(z_1-z_2)\overline{(z_1-z_2)}$

$=(z_1-z_2)(\bar{z}_1-\bar{z}_2)=z_1\bar{z}_1+z_2\bar{z}_2-(\bar{z}_1z_2+z_1\bar{z}_2)=|z_1|^2$

$+|z_2|^2-(\bar{z}_1z_2+\overline{\bar{z}_1z_2})=|z_1|^2+|z_2|^2-2\text{Re}(z_1\bar{z}_2)$

$$\leqslant |z_1|^2+|z_2|^2+2\left|z_1\bar{z}_2\right| \quad [\because \ -\text{Re}(z)\leqslant |z|]$$

$$=|z_1|^2+|z_2|^2+2|z_1||z_2|=(|z_1|+|z_2|)^2$$

$$\Rightarrow |z_1-z_2|\leqslant |z_1|+|z_2|.$$

57. Theorem 17 : Show that for any two complex numbers z_1 and z_2 ;

(*i*) $|z_1-z_2|\geqslant \big||z_1|-|z_2|\big|\geqslant |z_1|-|z_2|$.

D. U. H. T. 87 ; R. U. H. 90 ; D. U. H. 86. 89 ; C. U. H. 81 ; D. U. M. SC. P. T. 90 ; D. U. M. SC. P. 88, 89 ; J. U. H.87.

(*ii*) $|z_1+z_2|\geqslant \big||z_1|-|z_2|\big|\geqslant |z_1|-|z_2|$.

Proof : (*i*) We have $|z_1-z_2|^2=(z_1-z_2)\overline{(z_1-z_2)}$

$$=(z_1-z_2)(\bar{z}_1-\bar{z}_2)=z_1\bar{z}_1+z_2\bar{z}_2-(z_1\bar{z}_2+\bar{z}_1z_2)$$

$$=|z_1|^2+|z_2|^2-2\text{Re}\,(z_1\bar{z}_2)\geqslant |z_1|^2+|z_2|^2-2\left|z_1\bar{z}_2\right|$$

$$=|z_1|^2+|z_2|^2-|z_1|\left|\bar{z}_2\right|=|z_1|^2+|z_2|^2-2|z_1||z_2|$$

$$=(|z_1|-|z_2|)^2=(\big||z_1|-|z_2|\big|)^2$$

$$\Rightarrow |z_1-z_2|\geqslant \big||z_1|-|z_2|\big|$$

Again we have $\big||z_1|-|z_2|\big|\geqslant |z_1|-|z_1|$. Thus

$$|z_1-z_2|\geqslant \big||z_1|-|z_2|\big|\geqslant |z_1|-|z_2|.$$

(ii) Now replacing z_2 by $-z_2$ in (*i*), we have

$$|z_1+z_2|\geqslant \big||z_1|-|-z_2|\big|\geqslant |z_1|-|-z_2|\Rightarrow$$

$$|z_1+z_2|\geqslant \big||z_1|-|z_2|\big|\geqslant |z_1|-|z_2| \text{ since}$$

$$|-z_2|=|z_2|.$$

58. Theorem 18 : Show that :

$$|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2.$$

D. U. H. 85; D. U. H. T. 87, 90 ; D. U. M. SC. P. 84 ;

D. U. 63 ; R. U. 73.

Proof : We have $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\,\text{Re}\left(z_1\,\overline{z_2}\right)$

and $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2\,\text{Re}\left(z_1\overline{z_2}\right)$. Then

$$|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2.$$

Example 24: Show that :

$$|\alpha+\sqrt{(\alpha^2-\beta^2)}|^2+|\alpha-\sqrt{(\alpha^2-\beta^2)}|^2=|\alpha+\beta|\,|\alpha-\beta|.$$

D. U. H. T. 87 ; D. U. H. 85.

Solution : We have $(|\alpha+\sqrt{(\alpha^2-\beta^2)}|+|\alpha-\sqrt{(\alpha^2-\beta^2)}|)^2$

$=|\alpha+\sqrt{(\alpha^2-\beta^2)}|^2+|\alpha-\sqrt{(\alpha^2-\beta^2)}|^2$

$+2|\alpha+\sqrt{(\alpha^2-\beta^2)}|\,|\alpha-\sqrt{(\alpha^2-\beta^2)}|$

$=2|\alpha|^2+2|\sqrt{(\alpha^2-\beta^2)}|^2+2|(\alpha+\sqrt{(\alpha^2-\beta^2)})(\alpha-\sqrt{(\alpha^2-\beta^2)})|$

$=2|\alpha|^2+2|\alpha^2-\beta^2|+2|\alpha^2-(\alpha^2-\beta^2)|$

$=2|\alpha|^2+2|(\alpha+\beta)(\alpha-\beta)|+2|\beta|^2$

$=2|\alpha|^2+2|\beta|^2+2|\alpha+\beta|\,|\alpha-\beta|$

$=|\alpha+\beta|^2+|\alpha-\beta|^2+2|\alpha+\beta||\alpha-\beta|$

$=(|\alpha+\beta|+|\alpha-\beta|)^2$

$\Rightarrow |\alpha+\sqrt{(\alpha^2-\beta^2)}|+|\alpha-\sqrt{(\alpha^2-\beta^2)}|=|\alpha+\beta|+|\alpha-\beta|.$

59. Theorem 19. : Show that :

$\left|\dfrac{z_1}{z_2+z_3}\right|\leq\dfrac{|z_1|}{||z_2|-|z_3||}$, where z_1, z_2, z_3 are complex numbers with $|z_2|\neq|z_3|$.

D. U. H. 87 ; D. U. H. T. 88.

Proof : We have $|z_2+z_3| \geq ||z_2|-|z_3||$, then

$$\frac{1}{|z_2+z_3|} \leq \frac{1}{||z_2|-|z_3||} \quad \ldots (1) \text{ if } |z_2| \neq |z_3|.$$

But $|z_1| \geq 0$ ... (2). Then by (1) and (2) we have

$$\frac{|z_1|}{|z_2+z_3|} \leq \frac{|z_1|}{||z_2|-|z_3||} \text{ where } |z_2| \neq |z_3|.$$

60. ***Theorem 20 :*** Show that :

$$\left|\frac{z_1+z_2}{z_3+z_4}\right| \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||} \text{ where } |z_3| \neq |z_4|$$

D. U. H. 88, 90.

Proof : We have $|z_1+z_2| \leq |z_1|+|z_2|$ ...... (1).

Again we have $|z_3+z_4| \geq ||z_3|-|z_4||$, then

$$\frac{1}{|z_3+z_4|} \leq \frac{1}{||z_3|-|z_4||} \quad \ldots (2) \text{ where } |z_3| \neq |z_4|.$$

Then by (1) and (2) we have

$$|z_1+z_2| \cdot \frac{1}{|z_3+z_4|} \leq (|z_1|+|z_2|) \cdot \frac{1}{||z_3|-|z_4||}$$

$$\Rightarrow \frac{|z_1+z_2|}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$$

$$\Rightarrow \left|\frac{z_1+z_2}{z_3+z_4}\right| \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}.$$

61. ***Theorem 21 :*** Show that ,

$|z|\sqrt{2} \geq |\text{Re}(z)|+|\text{Im}(z)|$, where z is any complex number.

D.U.H.88 ; D.U.H.T.88, 90.

Proof ; Let $z=x+iy$, then $\text{Re}(z)=x$ and $\text{Im}(z)=y$.

But we know

$$\frac{|x|^2+|y|^2}{2} \geq \left(\frac{|x|+|y|}{2}\right)^2 \Rightarrow \frac{x^2+y^2}{2} \geq \left(\frac{|x|+|y|}{2}\right)$$

$$\Rightarrow \frac{\sqrt{(x^2+y^2)}}{\sqrt{2}} \geqslant \frac{|x|+|y|}{2} \Rightarrow \sqrt{2(x^2+y^2)} \geqslant |x|+|y|.$$

$$\Rightarrow \sqrt{2}\,|z| \geqslant |\text{Re}(z)| + |\text{Im}(z)|.$$

62. ***Theorem 22*** : If $|z_1| = |z_2|$ and amp z_1+amp $z_2=0$ then $z_2=\bar{z}_1$. ***R.U.H. 85, 40 ; J.U.H 86.***

Proof : Let $z_1=r_1e^{i\theta_1}$ and $z_2=r_2e^{i\theta_2}$. Then $|z_1| = |z_2|$

$\Rightarrow |r_1 e^{i\theta_1}| = |r_2 e^{i\theta_2}| \Rightarrow r_1=r_2 \cdots (1)$ since $|e^{i\theta_1}| = |e^{i\theta_2}|=1$.

Again amp z_1 +amp $z_2=0 \Rightarrow$ amp $z_1 z_2=0 \Rightarrow \theta_1+\theta_2=2n\pi$

$\Rightarrow \theta_2=2n\pi-\theta_1 \cdots (2)$ where $n=0, \pm 1, \pm 2, \pm 3, \cdots \cdots$

Now by (1) and (2) we have $z_2=r_2 e^{i\theta_2}=r_1 e^{i(2n\pi-\theta_1)}$

$=r_1 e^{2n\pi i} e^{-i\theta_1}=r_1 e^{-i\theta_1}=\bar{z}_1 \qquad [\because e^{2n\pi i}=1]$.

63. ***Theorem 23*** : Show that the modulus of the quotient of two conjugate complex numbers is 1.

R. U. H. 85 : J. U. H. 86.

Proof ; Let $z=x+iy$ be a complex number, then its conjugate is $\bar{z}=x-iy$. Now

$$\left|\frac{z}{\bar{z}}\right| = \frac{|z|}{|\bar{z}|} = \frac{|x+iy|}{|x-iy|} = \frac{\sqrt{(x^2+y^2)}}{\sqrt{\{x^2+(-y)\}^2}} = \frac{\sqrt{(x^2+y^2)}}{\sqrt{(x^2+y^2)}} = 1.$$

Hence the theorem is proved.

Second proof : Let $z=re^{i\theta}$, then $\bar{z}=re^{-i\theta}$

$$\text{Now} \left|\frac{z}{\bar{z}}\right| = \left|\frac{re^{i\theta}}{re^{-i\theta}}\right| = |e^{2i\theta}| = |\cos 2\theta + i \sin 2\theta|$$

$$=\sqrt{(\cos^2 2\theta+\sin^2 2\theta)}=1.$$

64. *Theorem 24*: Show that :

(*i*) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ and (*ii*) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ where $z_2 \neq 0$.

***Proof*:** Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

(*i*) $$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{r_1}{r_2} e^{-i(\theta_1 - \theta_2)} = \frac{r_1 e^{-i\theta_1}}{r_2 e^{-i\theta_2}} = \frac{\bar{z}_1}{\bar{z}_2}$$

(*ii*) $$\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \text{ since } \left|e^{i(\theta_1 - \theta_2)}\right| = 1.$$

***Second Proof*:** $$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right)\overline{\left(\frac{z_1}{z_2}\right)} = \frac{z_1 \bar{z}_1}{z_2 \bar{z}_2}$$

$$= \frac{|z_1|^2}{|z_2|^2} \Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

65. *Theorem 25*: Show that :

(*i*) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$;

(*ii*) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$;

(*iii*) $\arg(z_1 z_2 \cdots z_n) = \arg z_1 + \arg z_2 + \cdots + \arg z_n$;

(*iv*) $\arg z^n = n \arg z$;

(*v*) $\arg \bar{z} = -\arg z$.

***Proof*:** (*i*) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then we have $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$.

Now we have $z_1 z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$\Rightarrow \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$.

(*ii*) Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 =$ $r_2 (\cos \theta_2 + i \sin \theta_2)$, then $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$.

Now $\dfrac{z_1}{z_2} = \dfrac{r_1 (\cos \theta_1 + i \sin \theta_1)}{r_2 (\cos \theta_2 + i \sin \theta_2)}$

$= \dfrac{r_1}{r_2} \{\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)\}$

$\Rightarrow \arg \left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2.$

(*iii*), (*iv*) and (*v*) : Try yourself.

N. B. The above theorem is true for some of $\arg z_1$ and $\arg z_2$ and they are not true for all cases since $\arg z_1$ and $\arg z_2$ have many values and also they may not hold even if we use the principal values.

Example 25 : Show that : $\arg z + \arg \bar{z} = 2n\pi$, where $n \in I$.

Solution : We have $\arg z = \text{Arg } z + 2p\pi$ and $\arg \bar{z}$ $= \text{Arg } \bar{z} + 2q\pi = -\text{Arg } z + 2q\pi$ where $p, q \in I$.

Then $\arg z + \arg \bar{z} = \text{Arg } z + 2p\pi - \text{Arg } z + 2q\pi = 2(p+q)\pi$ $= 2n\pi$, where $n = p + q \in I$.

Example 26 : Using Euler's formula show that :

$\arg (z_1 z_2) = \arg z_1 + \arg z_2$ and $\arg \left(\dfrac{z_1}{z_1}\right) = \arg z_1 - \arg z_2$.

Solution ; Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$.

Again $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \Rightarrow \arg(z_1 z_2) = \theta_1+\theta_2 = \arg z_1 +$ $\arg z_2$ and $z_1/z_2 = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \Rightarrow \arg(z_1/z_2) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$.

66. ***Theorem 26*** **:** If P divides AB in the ratio m : n and O is any point, then show that : $(m+n)\,\mathbf{OP} = n\,\mathbf{OA} + m\,\mathbf{OB}$.

Proof **:** Try yourself.

67. ***Theorem 27*** **:** If $P(z)$ divides the straight line joining $A(z_1)$ and $B(z_2)$ in the ratio $m_1 : m_2$, then show that :

$$z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}.$$

proof **;** Try yourself.

68. ***Theorem 28*** **:** If z is the centriod of particles of masses $m_1, m_2, m_3, \cdots$ are located at points $z_1, z_2, z_3, \cdots$ respectively, then show that : $(m_1+m_2+m_3+\cdots)z = m_1 z_1 + m_2 z_2 + m_3 z_3 + \cdots$.

Proof **:** Try Tourself.

69. ***Theorem 29*** **:** Show that any three complex numbers z_1, z_2, z_3 are connected by a relation of the form $a z_1 + b z_2 + c z_3 = 0$ where a, b, c are real numbers.

Proof **:** Try yourself.

Example 27 **:** If the vectors $\mathbf{OA}, \mathbf{OB}, \mathbf{OC}$ are connected by the relation $l\,\mathbf{OA} + m\,\mathbf{OB} + n\,\mathbf{OC} = \mathbf{0}$, where $l+m+n=0$, then ***A, B, C*** are collinear.

Solution **:** Try yourself.

70 ***Theorem 30*** **:** If the three points P, Q, R are collinear and O is any point, then show that :

$OA.\ BC + OB.\ CA + OC.\ AB = 0.$

Proo. : Try yourself.

Example 28 : If the points z_1, z_2, z_3 are collinear, then show that :

$$z_1 \mid z_2 - z_3 \mid \pm z_2 \mid z_3 - z_1 \mid \pm z_3 \mid z_1 - z_2 \mid = 0.$$

Solution ; Try yourself.

71 ***Theorem 31*** : Show that the medians of a triangle with vertices at z_1, z_2, z_3 intersect in the point $\frac{1}{3}(z_1 + z_2 + z_3)$.

Proof ; Try yourself.

72 ***Theorem 32*** : If z_1, z_2, z_3, are the three vertices of an equilateral triangle, then show that

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Proof : Try yourself.

73 ***Theorem 33*** : If A, B, C are complex constants and $\bar{A}, \bar{B}, \bar{C}$ their conjugates, then show that the equation

$$(A + \bar{A}) z \bar{z} + \bar{B} z + B \bar{z} + C + \bar{C} = 0 \quad (1) \text{ will}$$

represent a circle if $B\bar{B} > (A + \bar{A})(C + \bar{C})$ ... ... (2).

J. U. H. 86 ; R. U. H. 85.

proof : Let $A = a_1 + ia_2$, $B = b_1 + ib_2$, $C = c_1 + ic_2$ and $z = x + iy$ ... (3), Then $\bar{A} = a_1 - ia_2$, $\bar{B} = b_1 - ib_2$, $\bar{C} = c_1 - ic_2$ and $\bar{z} = x - iy$ ... (4). Now by (1), (3) and (4) we have $2a_1(x^2 + y^2) + 2b_1 x + 2b_2 y + 2c_1 = 0$

$\Rightarrow a_1(x^2+y^2)+b_1x+b_2y+c_1=0$ which is the equation of a circle. The radius of this circle

$$=\left(\frac{b_1}{2a_1}\right)^2+\left(\frac{b_2}{2a_1}\right)^2-\frac{c_1}{a_1} \text{ which is greater than zero } \Rightarrow$$

$$b_1^2+b_2^2-4a_1c_1>0 \Rightarrow b_1^2+b_2^2>4a_1c_1 \Rightarrow$$

$$B\bar{B}>(A+\bar{A})(C+\bar{C}), \text{ by (3) and (4).}$$

74. ***Theorem 34*:** Show that the equation of a circlo or line in the Argand plane can be written as

$Az\bar{z}+B\bar{z}+\bar{B}z+D=0$ where A and C are real constants and B may be a complex constant.

Proof : The general equation of a circle in the xy-plane can be written $a(x^2+y^2)+bx+cy+d=0$ ... ... (1)

But we have $x=\frac{z+\bar{z}}{2}$ and $y=\frac{z-\bar{z}}{2i}$ ... ... (2)

Then by (1) and (2) we have

$$az\bar{z}+b\frac{z+\bar{z}}{2}+c\frac{z-\bar{z}}{2i}+d=0 \qquad [\because z\bar{z}=x^2+y^2]$$

$$\Rightarrow az\bar{z}+\left(\frac{b}{2}+\frac{c}{2i}\right)z+\left(\frac{b}{2}-\frac{c}{2i}\right)\bar{z}+d=0$$

$\Rightarrow Az\bar{z}+Bz+\bar{B}\bar{z}+D=0$ ... ... (3) where $A=a$,

$$B=\frac{b}{2}+\frac{c}{2i}, \qquad \bar{B}=\frac{b}{2}-\frac{c}{2i}.$$

and $D = d$, The equation (3) represents the general equation of a circle. If $a = A = 0$, then (3) represents the general equation of a straight line.

75 Theorem 35 : Show that

$$\left(\frac{z-z_1}{z-z_2}\right) \Big/ \left(\frac{z_3-z_1}{z_3-z_2}\right) = \left(\frac{\overline{z}-\overline{z_1}}{\overline{z}-\overline{z_2}}\right) \Big/ \left(\frac{\overline{z_3}-\overline{z_1}}{\overline{z_3}-\overline{z_2}}\right)$$

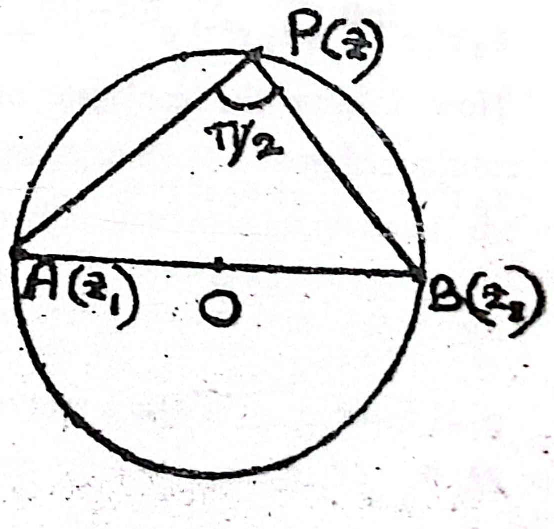
represents the equation of a circle through the points z_1, z_2, z_3.

Proof : Try yourself.

76 Theorem 36 : Show that :

$(z-z_1)\left(\overline{z}-\overline{z_2}\right)+(z-z_2)\left(\overline{z}-\overline{z_1}\right)=0$ represents the equation of a circle where A (z_1) and B (z_2) are the extremities of a diameter.

Proof : Let P (z) be a point on the circle. Here $\angle APB = \pi/2 \Rightarrow \arg \frac{z-z_1}{z-z_2} = \pi/2 \Rightarrow \frac{z-z_1}{z-z_2}$ is purely imaginary $\Rightarrow \text{Re}\left(\frac{z-z_1}{z-z_2}\right) = 0 \Rightarrow 2\,\text{Re}\left(\frac{z-z_1}{z-z_2}\right) = 0$



$$\Rightarrow \frac{z-z_1}{z-z_2}+\overline{\left(\frac{z-z_1}{z-z_2}\right)}=0 \Rightarrow \frac{z-z_1}{z-z_2}+\frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}=0$$

$$\Rightarrow (z-z_1)\left(\bar{z}-\bar{z}_2\right)+(z-z_2)\left(\bar{z}-\bar{z}_1\right)=0.$$ Hence the theorem is proved.

77. ***Theorem 37*** : In an equation with real coefficients complex roots occur in conjugate pairs.

Proof : Let $a_0 z^n+a_1 z^{n-1}+\dots+a_n=0$ ... (1) where $a_0 \neq 0$, $a_1, \dots, a_n \in R$. Let $p+iq$ be a root of (1) where $p, q \in R$. Now we will show that $p-iq$ is also a root of (1).

Let $p+iq=re^{i\theta}$ .... (3) in polar form. Then by (1) and (2) we get

$$a_0 r^n e^{in\theta}+a_1 r^{n-1} e^{i(n-1)\theta}+\dots+a_{n-1} re^{i\theta}+a_n=0 \quad \dots (3)$$

Now taking the conjugate of both sides of (3), we get

$$a_0 r^n e^{-in\theta}+a_1 r^{n-1} e^{-i(n-1)\theta}+\dots+a_{n-1} re^{-i\theta}+a_n=0 \quad \dots \quad \dots \quad (4)$$

By (1) and (4) we see that conjugate of (2), *i. e.* $p-iq=re^{-i\theta}$ is also a root of (1). Thus the theorem is proved.

N. B. If $a_0, a_1, \dots\dots, a_n$ are not all real, then the above theorem is not correct.

Example 29 : Show that the conjugate pairs $\frac{1}{2}(1 \pm i\sqrt{3})$ and $\frac{1}{2}(-1 \pm i\sqrt{3})$ are the roots of $z^4+z^2+1=0$.

Solution : Try yourself.

$= -\text{Im}\left(\bar{z}\right)$, *(iv)* $\text{Im } z = y = \frac{2y}{2} = \frac{z-\bar{z}}{2}$

$= -\text{Im}\left(\bar{z}\right)$, (v) $-\sqrt{(x^2+y^2} \leqslant x \leqslant \sqrt{(x^2+y^2)}$

$\Rightarrow -|z| \leqslant \text{Re}(z) \leqslant |z|$, (vi) $-\sqrt{(x^2+y^2)} \leqslant y \leqslant \sqrt{(x^2+y^2)}$

$\Rightarrow -|z| \leqslant \text{Im}(z) \leqslant |z|$; **(vii), (viii), (ix)** and **(x)** Try yourself.

Example 34 : ***Solve*** :

(i) $az^2+bx+c=0, a \neq 0$, *(ii)* $z^3=1$.

Solution : *(i)* $az^2+bz+c= \Rightarrow az^2+bz=-c$

$\Rightarrow 4a^2z^2+4abz=-4ac \Rightarrow 4a^2z^2+4abz+b^2=b^2-4ac$

$\Rightarrow (2az+b)^2=b^2-4ac \Rightarrow 2az+b=\pm\sqrt{(b^2-4ac)}$

$\Rightarrow 2az=-b\pm\sqrt{(b^2-4ac)} \Rightarrow z=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$

(ii) $z^3=1 \Rightarrow z^3-1=0 \Rightarrow (z-1)(z^2+z+1)= \Rightarrow$

$z=1, \frac{-1\pm\sqrt{(-3)}}{2}$.

Example 35 ; Show that if the amplitude of a complex number is $\pi/2$, then the complex number is purely imaginary but if the amplitude is o or π, then the complex number is purely real.

Solution : Let $z=r(\cos\theta+i\sin\theta)$ where $\text{amp} z=\theta$.

Now we have :

(i) if $\theta=\pi/2$, then $z=r(\cos\pi/2+i\sin\pi/2)=r\,i \Rightarrow$ z is purely imaginary.

(ii) if $\theta=0$, then $z=r(\cos 0+i\sin 0)=r \Rightarrow$ z is purely real.

(iii) if $\theta=\pi$, then $z=r(\cos\pi+i\sin\pi)=-r \Rightarrow$

z is purely real.

Example 36 : If $|z_1+z_2|=|z_1-z_2|$, then show that z_1/z_2 and z_2/z_1 are purely imaginary numbers.

Solution : We have $|z_1+z_2|=|z_1-z_2|$

$\Rightarrow \text{Re}\left(z_1\,\bar{z}_2\right)=\text{Re}\left(\bar{z}_1\,z_2\right)=0 \Rightarrow z_1\,\bar{z}_2$ and $\bar{z}_1 z_2$ are imaginary numbers.

$$\text{Here } z_1/z_2=\frac{z_1\,\bar{z}_2}{z_2\,\bar{z}_2}=\frac{z_1\,\bar{z}_2}{|z_2|^2} \text{ and } \frac{z_2}{z_1}=\frac{\bar{z}_1\,z_2}{z_1\,\bar{z}_1}=\frac{\bar{z}_1\,z_2}{|z_1|^2}$$

$\Rightarrow$ they are imaginry numbers since $z_1\,\bar{z}_2$, $\bar{z}_1\,z_2$ are imaginary and $|z_1|^2$, $|z_2|^2$ are real.

Example 37 : Show that the sum of the products of all the nth roots of unity taken 2, 3, 4, ... , $(n-1)$ at a time is zero.

R. U. H. 86.

Solution : Let $z=(1)^{1/n} \Rightarrow z^n=1 \Rightarrow z^n-1=0$ ... ... (1)

Here $a_0=1, a_1=0, a_2=0, a_3=0, \ldots, a_{n-1}=0, a_n=-1.$

Let $z_1, z_2, \cdots, z_n$ be the roots of (1), then

$$\Sigma z_1\,z_2=\frac{a_2}{a_0}=0,\quad \Sigma z_1\,z_2\,z_3=-\frac{a_3}{a_0}=0, \ldots,\ \Sigma z_1\,z_2 \cdots z_{n-1}=(-1)^{n-1}\frac{a_{n-1}}{a_0}=0.$$

Example 38 ; Show that $\left|\frac{z-1}{z+1}\right|=$constant and amp $\left(\frac{z-1}{z+1}\right)$ $=$constant are orthogonal circles;

R. U. H. 93.

Solution : We have $\left|\frac{z-1}{z+1}\right|=$ constant $=k$ (say)

$$\Rightarrow \frac{(x-1)^2+y^2}{(x+1)^2+y^2}=k^2$$

$$\Rightarrow (1-k^2)x^2+(1-k^2)y^2-2(1+k^2)x+1-k^2=0 \quad \cdots \quad \cdots \quad (1)$$

Again we have $\operatorname{amp}\left(\frac{z-1}{z+1}\right)=$ constant $=\lambda$ (say)

$$\Rightarrow \operatorname{amp}(z-1)-\operatorname{amp}(z+1)=\lambda \Rightarrow \tan^{-1}\frac{y}{x-1}-\tan^{-1}\frac{y}{x+1}=\lambda$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\frac{y^2}{x^2-1}}\right)=\lambda \Rightarrow \frac{2y}{x^2+y^2-1}=\tan\lambda \Rightarrow$$

$\mu(x^2+y^2-1)-2y=0 \quad \cdots \quad (2)$ where $\mu=\tan\lambda$.

The eqnation (1) and (2) represent two circles. In (1) we have $g_1=-\left(\frac{1+k^2}{1-k^2}\right)$, $f_1=0$, $c_1=1$ and in (2) we have

$$g_2=0,\ f_2=-\frac{2}{\mu},\ c_2=-1$$

Here $2g_1 g_2+2f_1 f_2=0=c_1+c_2$. Hence the circles (1) and (2) cut orthogonally and they are orthogonal circles.

Example 39 : Show that the lines joining the points z_1, z_2 and z_3, z_4 are perpendicular provided that $\frac{z_1-z_2}{z_3-z_4}$ is pure imaginary. ***R. U. H. 86.***

Solution : Let θ be the angle between the lines joining the points z_1, z_2 and z_3, z_4. Then $\theta = \arg \frac{z_1 - z_2}{z_3 - z_4}$ If they are perpendicular *i. e.* $\theta = \pm\pi/2$, then $\arg \frac{z_1 - z_2}{z_3 - z_4} = \pm\pi/2$

$\Rightarrow \frac{z_1 - z_2}{z_3 - z_4}$ lies on the y-axis and therefore $\frac{z_1 - z_2}{z_3 - z_4}$ is pure imaginary.

Example 40 : Using polar coordinates measuring arguments in radians :

(*i*) show that $|z_1 + z_2| = |z_1| + |z_2|$ if $z_1 . z_2 \neq 0$ and $\arg z_2 = \arg z_1 \pm 2n\pi$, $n = 0, 1, 2, \ldots \ldots$;

(*ii*) show that $\operatorname{Re}\left(z_1 \overline{z_2}\right) = |z_1|\ |z_2|$ if $z_1 z_2 \neq 0$ and $\arg z_2 = \arg z_1 \pm 2n\pi$, $(n = 0, 1, 2, \ldots \ldots)$.

Solution : Try yourself.

Example 41 : Show that :

(*i*) $\arg\left(z_1 \overline{z_2}\right) = \arg z_1 - \arg z_2$; (*ii*) $\operatorname{Arg} \overline{z} = -\operatorname{Arg} z$, z is not real ; (*iii*) $\arg z = \operatorname{Arg} z + 2n\pi$, $n = 0, \pm 1, \pm 2, \cdots$, $z \neq 0$; (*iv*) $\operatorname{Arg}(1+i) = \pi/4$;

(*v*) $\operatorname{Arg}(\sqrt{3} - i) = -\pi/6$; (*vi*) $\operatorname{Arg}(3i) = \pi/2$.

Solution. Try yourself.

Example 42 ; Show that : if $n \in Z$, then either

$\arg \frac{z_2 - z_3}{z_1 - z_3} = \frac{1}{2} \operatorname{Arg} \frac{z_2}{z_1} + 2n\pi$

or $\arg \frac{z_2 - z_3}{z_1 - z_3} = -\frac{1}{2} \operatorname{Arg} \frac{z_2}{z_1} + 2n\pi$, where z_1, z_2 and z_3 be three distinct points that lie on the circle of radius 1 about the origin.

Solution : Try yourself.

Example 43 : Show that :

(*i*) $\text{Re}\left(\alpha \bar{\beta}\right) = \text{Re}\left(\bar{\alpha} \beta\right)$; (*ii*) $\text{Im}\left(\bar{\alpha} \beta\right) = -\text{Im}\left(\alpha\bar{\beta}\right)$ where $\alpha\ \beta \in \mathbf{C}$.

Soluton ; Let $\alpha = a+ib$ and $\beta = c+id$ then $\alpha\bar{\beta} =$

$(a, b)\ (c, -d) = (ac+bd, bc-ad)$ and $\bar{\alpha}\beta = (a, -b)\ (c, d)$

$= (ac+bd, ad-bc)$.

(*i*) $\text{Re}\left(\alpha \bar{\beta}\right) = ac+bd = \text{Re}\left(\bar{\alpha} \beta\right)$ and

(*ii*) $\text{Im}\left(\alpha \bar{\beta}\right) = bc-ad = -(ad-bc) = -\text{Im}\left(\bar{\alpha} \beta\right)$

Example 44 : Show that ;

(*i*) $|\alpha+\beta|^2 = |\alpha|^2 + |\beta|^2 + 2\text{Re}\left(\alpha\bar{\beta}\right)$;

(*ii*) $|\alpha-\beta|^2 = |\alpha|^2 + |\beta|^2 - 2\text{Re}\left(\alpha\bar{\beta}\right)$ whre $\alpha, \beta \in \mathbf{C}$.

Solution : (*i*) $|\alpha+\beta|^2 = (\alpha+\beta)\,\overline{(\alpha+\beta)}$

$= (\alpha+\beta)\left(\bar{\alpha}+\bar{\beta}\right) = \alpha\bar{\alpha} + \beta\bar{\beta} + \left(\alpha\bar{\beta} + \bar{\alpha}\beta\right)$

$= |\alpha|^2 + |\beta|^2 + 2\text{Re}\left(\alpha\bar{\beta}\right)$

(*ii*) Try yourself.

Example 45 : Show that : $\alpha\bar{\beta} + \bar{\alpha}\beta$

$= \text{Re}\left(\alpha\bar{\beta}\right) = \text{Re}\left(\bar{\alpha}\beta\right)$ where $\alpha, \beta \in \mathbf{C}$.

Solution : Try yourself.

Example 46 : Show that $|z|$ is real.

Solution : Try yourself.

Example 47 : Show that : $z\bar{z} > 0, z \in C$.

Solution : Let $z=(x, y)$ where $x, y \in R$.

Then $z\bar{z}=x^2+y^2 \geqslant 0$.

Example 48 : Let P and Q be represent the complex numbers z and $\bar{z}$ respectively, then show that PQ is perpendicular to the x-axis.

Solution : Try yourself.

Example 49 : If $n=2, 4, 6, 8, \dots \dots$, then show that

$$1+\sum_{k=0}^{n-1} \cos\frac{2k\pi}{n}=0 \text{ and } \sum_{k=1}^{n-1} \sin\frac{2k\pi}{n}=0.$$

Solution : Try yourself.

Example 50 : Show that for $m=2, 3, 4, \dots \dots$

$$\sin\frac{\pi}{m} \sin\frac{2\pi}{m} \sin\frac{3\pi}{m} \dots \sin\frac{(m-1)\pi}{m}=\frac{m}{2^{m-1}}.$$

Solution : Try yourself.

Example. 51 : Describe each of the following region geometrically :

(i) $|z+3i|>4$; ***D. U, H. T. 89.***

(ii) $|2z+3|>4$; ***D. U. H. T. 88.***

(iii) $1<|z-2i|\leqslant 2$; ***D. U. H. T. 88.***

(*iv*) $1 < |z+i| \leqslant 2$;

D. U.H. T. 89 ; D. U.M, SC. P. T. 89.

(*v*) $|z-4| > |z|$; **D. U.H. T. 88**

(*vi*) $1 < |z-2i| < 2$; **D. U. H. T. 90.**

(*xii*) $|z-i| = |z+i|$; **D. U. H. 87.**

(*viii*) $|z+2-3i| + |z-2+3i| < 10$; **D. U. H. T. 89.**

(*ix*) $|z-2| - |z+2| > 3$; **D. U. H. T. 90.**

(x) $\text{Re}\,(z^2) > 1$; **D. U. H. T. 89.**

(*xi*) $\text{Im}\,(z^2) > 0$; **D. U. T. H. 88.**

(*xii*) $\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$; **D. U. H. T 88, 90 ; D. U. H. 86.**

(*xiii*) $\text{Im}\left(\frac{1}{z}\right) < \frac{1}{2}$; **D. U. H. 87.**

(*xiv*) $|z| > 4$; **D. U. H. 86.**

(*xv*) $\text{Re}\left(\frac{-1}{z}\right) \leqslant \frac{1}{2}$; **D. U. H. 88 ; D. U. M. SC. P.88.**

(*xvi*) $\text{Im}\,(z) > 1$; **D. U. H. 88.**

(*xvii*) $|z-2+i| \leqslant 1$; **D. U. H. 88.**

(*xviii*) $\pi/3 \leqslant \arg z \leqslant \pi/2$;

(*xix*) $0 < \text{Re}\,(iz) < 1$; **D. U. H. T. 90.**

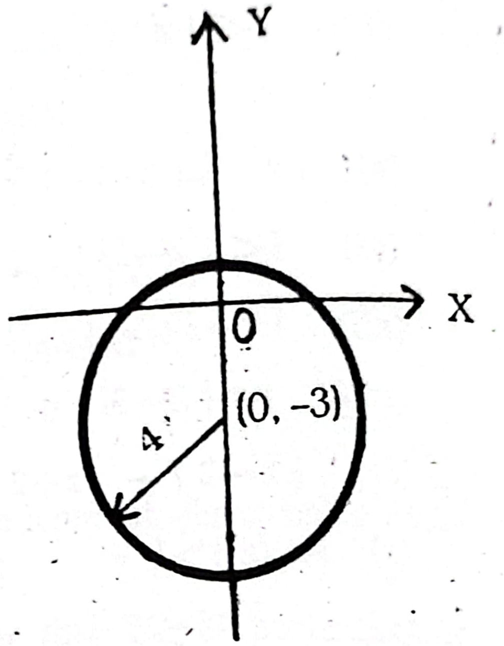
(*xx*) $-\pi < \arg z < \pi,\ |z| > 2$; **D. U.H. 88**

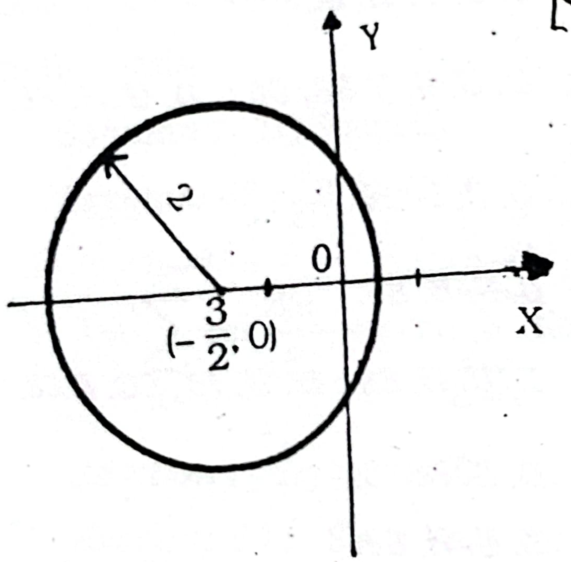
(*xxi*) $-\pi < \arg z < \pi$; **D. U, H. 86 ; D. M. SC. P, 88**

(*xxii*) $-\pi < \arg z < \pi,\ z \neq 0$; **D. U. H, 88**

(*xxiii*) $0 < \arg z < 2\pi,\ |z| > 0$; **D. U. M, SC, P, 89.**

Solution : (*i*) $|z+3i|>4 \Rightarrow$ $x^2+(y+3)^2>16 \Rightarrow$ the set of all those points external to the circle $|z+3i|=4$ with centre $(0,-3)$ and radius $=4$,

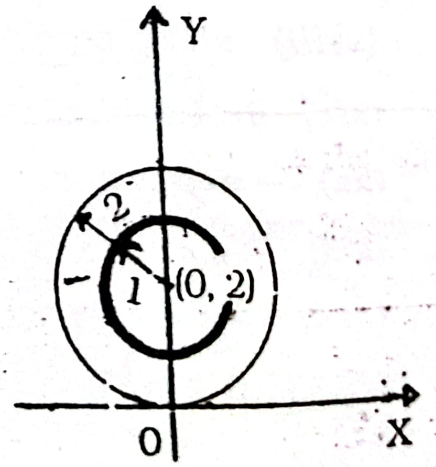




(*ii*) $|2z+3|>4$ $\Rightarrow |z+3/2|>2 \Rightarrow$ $(x+3/2)^2+y^2>4 \Rightarrow$ the set of all those points external to the circle $|z+3/2|=2$ with centre$(-3/2, 0)$ and radius $=2$

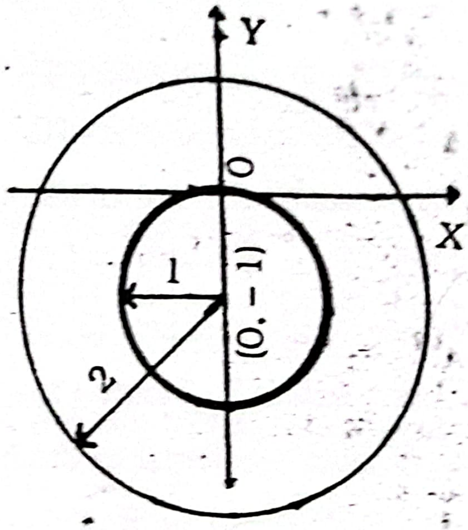
(*iii*) $1<|z-2i|\leqslant 2 \Rightarrow$ $1<x^2+(y-2)^2\leqslant 4$,

The given inequalities represent the set of all those points which are external to the circle $|z-2i|=1$ with centre $(0, 2)$ and internal to the and on the circle $|z-2i|=2$ with the same centre,



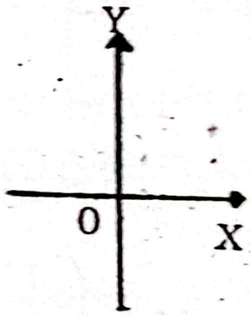
(*iv*) $1<|z+i|\leqslant 2 \Rightarrow 1<\sqrt{\{(x^2+(y+1)^2\}}\leqslant 2 \Rightarrow$
$1<x^2+(y+1)^2\leqslant 4.$

The given inequalities represent the set of all those points which are external to the circle $|z+i|=1$ with center $(0,-1)$ and internal to the and on the circle $|z+i|=2$ with the same centre.



(*v*) $|z-4|>|z| \Rightarrow |z-4|^2>|z|^2 \Rightarrow (z-4)\left(\bar{z}-4\right)$
$> z\bar{z} \Rightarrow z\bar{z}-4\left(z+\bar{z}\right)+16>z\bar{z} \Rightarrow -4\left(z+\bar{z}\right)+16>0 \Rightarrow z+\bar{z}<4$

$\Rightarrow 2x<4 \Rightarrow x<2 \Rightarrow$ the set of all points (x, y) such that $x<2$, That is, the set consists of all points (x, y) left hand side to the straight line $x=2$.



(*vi*) $1<|z-2i|<2 \Rightarrow 1<x^2+(y-2)^2<4 \Rightarrow$ the set of all those points which are external to the circle $|z-2i|=1$ with centre $(0, 2)$ and internal to the circle $|z-2i|=2$ with the same centre.

(*vii*) $|z-i|=|z+i| \Rightarrow x^2+(y-1)^2=x^2+(y+1)^2$

$\Rightarrow y^2-2y+1=y^2+2y+1 \Rightarrow y=0 \Rightarrow$ the x-axis or the real axis.

(viii) We have $|z+2-3i|+|z-2+3i|=10$ ... (1)

represents an ellipse whose foci are $(-2, 3)$ and $(2, -3)$ and the length of the major axis is 10. Hence $|z+2-3i|+|z-2+3i|<10$ represents the set of all points interior to the ellipse (1).

(ix) Here $|z-2|-|z+2|=3$ ... (1) represents a hyperbola with foci are $(2, 0)$ and $(-2, 0)$ and the length of the transverse axis is 3, Hence $|z-2|-|z+2|>3$ represents the et of all poin ts external to the hyperbola (1).

(x) $\text{Re}(z^2)>1 \Rightarrow \text{Re}(x^2-y^2+2xy\,i)>1 \Rightarrow x^2-y^2>1$

$\Rightarrow$ the set of all points external to the rectangular hyperbola $x^2-y^2=1$:

(xi) $\text{Im}(z^2)>0 \Rightarrow \text{Im}(x^2-y^2+2xyi)>0 \Rightarrow xy>0$

$\Rightarrow$ the set of all points in the first and the third quadrants.

(xii) $\text{Re}\left(\frac{1}{z}\right)<\frac{1}{2} \Rightarrow \text{Re}\left(\frac{x-iy}{x^2+y^2}\right)<\frac{1}{2} \Rightarrow \frac{x}{x^2+y^2}<\frac{1}{2}$

$\Rightarrow x^2+y^2-2x>0 \Rightarrow (x-1)^2+y^2>1 \Rightarrow$ the set of all points external to the circle $(x-1)^2+y^2=1$ with centre $(1, 0)$ and radius is 1.

(xiii) $\text{Im}\left(\frac{1}{z}\right)<\frac{1}{2} \Rightarrow \text{Im}\left(\frac{x-iy}{x^2+y^2}\right)<\frac{1}{2} \Rightarrow$

$\frac{-y}{x^2+y^2}<\frac{1}{2} \Rightarrow x^2+y^2+2y>0 \Rightarrow x^2+(y+1)^2>1$

$\Rightarrow$ the set of all points external to the circle $x^2+(y+1)^2=1$ with centrre $(0, -1)$ and radius is 1;

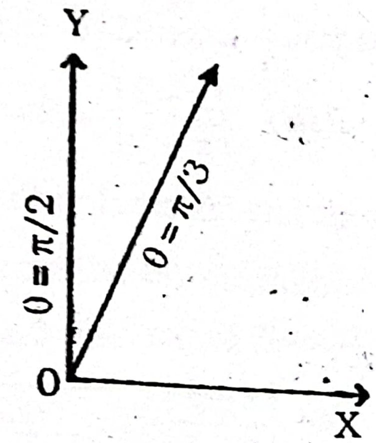
(xiv) $|z|>4 \Rightarrow x^2+y^2>16 \Rightarrow$ the set of all points external to the circle $x^2+y^2=16$ with centre $(0, 0)$ and radius $=4$.

(xv) $\text{Re}\left(\frac{1}{z}\right) \leqslant \frac{1}{2} \Rightarrow$ the set of all points external to the and on the circle $(x-1)^2+y^2=1$.

(xvi) $\text{Im}\, z>1 \Rightarrow y>1 \Rightarrow$ the set of all those points (x, y) such that $y>1$. That is, the set consists of all points (x, y) above the straight line $y=1$.

(xvii) $|z-2+i| \leqslant 1$ represents the set of all points interior to the and on the circle $|z-2+i|=1$.

(xviii) We have $\arg z=\theta$ where $z=r(\cos\theta+i\sin\theta)$ $=re^{i\theta}$. Hence $\pi/3 \leqslant \arg z \leqslant \pi/2 \Rightarrow$ the infinite region bounded by the lines $\theta=\arg z=\pi/3$ and $\theta=\arg z=\pi/2$ including these lines.



Others ; Try yourself.

Example 52 **:** Describe each of the following region geometrically :

(i) $\left|\frac{z-3}{z+3}\right|=2$; **D. U. M. SC. T. 80.**

(ii) $\left|\frac{z-3}{z+3}\right|>2$; (iii) $\left|\frac{z-3}{z+3}\right|<2$.

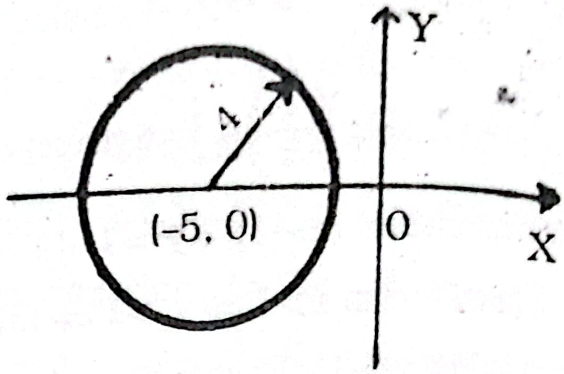
Solution : (i) $\left|\frac{z-3}{z+3}\right|=2 \Rightarrow |z-3|=2|z+3|$

$\Rightarrow |z-3|^2=4|z+3|^2$

$\Rightarrow (z-3)(\bar{z}-3)=$

$4(z+3)(\bar{z}+3)$

$\Rightarrow z\bar{z}+5(z+\bar{z})+9=0$



$\Rightarrow (z+5)\overline{(z+5)}=16$

$\Rightarrow |z+5|=16 \Rightarrow$ the equation of a circle with centre $(-5, 0)$ and radius $=4$.

(*ii*) $\left|\frac{z-3}{z+3}\right|>2 \Rightarrow |z+5|<4 \Rightarrow$ the set consisting of all the point internal to the circle $|z+5|=4$.

(*iii*) $\left|\frac{z-3}{z+3}\right|<2 \Rightarrow |z+5|>4 \Rightarrow$ the set consisting of all the points external to the circle $|z+5|=4$.

Example 53 : Describe each of the following region geometrically :

(*i*) $\left|\frac{z-3}{z+3}\right|=3$; (*ii*) $\left|\frac{z-3}{z+3}\right|>3$; (i) $\left|\frac{z-3}{z+3}\right|<3$.

D. U. M. SC. P. 89.

Solution : Try yourself.

Example 54 : Identify all the points in the complex plane determined by the following :

(*i*) $\frac{z-3}{z+3}<3$; (*ii*) $|z-1|+|z+1|=4$;

(*iii*) $0<\arg z<2\pi$, $|z|>0$.

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Solution : (*i*) $\left|\frac{z-3}{z+3}\right| < 3 \Rightarrow |z-3| < 3|z+3| \Rightarrow$

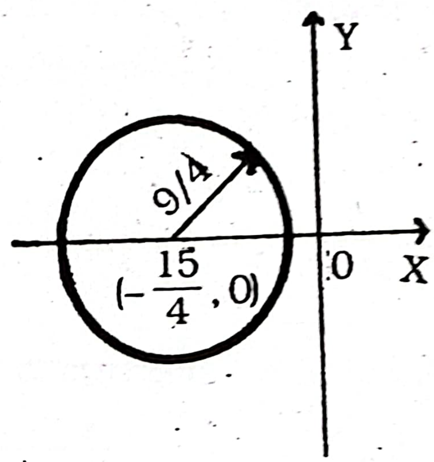
$|z-3|^2 < 9|z+3|^2 \Rightarrow (z-3)(\bar{z}-3) < 9(z+3)(\bar{z}+3)$

$\Rightarrow z\bar{z} - 3(z+\bar{z}) + 9 < 9\{z\bar{z} + 3(z+\bar{z}) + 9\} \Rightarrow$

$8z\bar{z} + 30(z+\bar{z}) + 72 > 0$

$\Rightarrow 8(x^2+y^2) + 60x + 72 > 0$

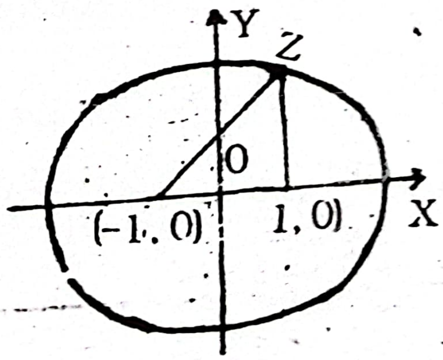
$\Rightarrow 2(x^2+y^2) + 15x + 18 > 0$



$\Rightarrow x^2+y^2+\frac{15}{2}x+9>0 \quad \ldots \quad (1)$

The equation represent a set which consists of all those points external to the circle $2(x^2+y^2)+15x+18=0$ with centre $(-15/4, 0)$ and radius $= \sqrt{\left(\frac{-15}{4}\right)^2 - \frac{18}{2}} = \sqrt{\frac{225}{16} - 9} = \sqrt{\frac{81}{16}} = \frac{9}{4}$.

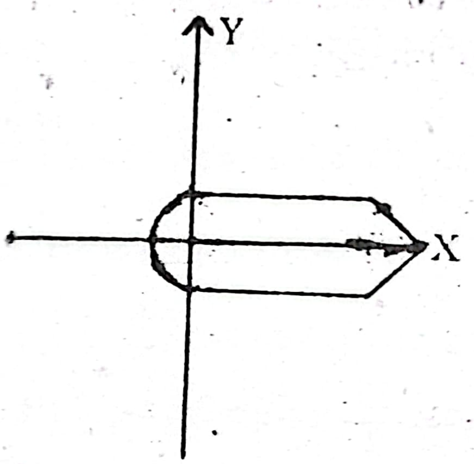
(*ii*) The equation $|z-1| + |z+1| \geq 4$ represents an ellipse whose foci are $(-1, 0)$ and $(1, 0)$ and the length of its major axis is 4



(*iii*) Here $0 < \arg z < 2\pi$, $|z| > 0$ represent the whole region excluding the origin and the positive part of the x-axis i. e. the points of the form (x, o) where $x > 0$.

Mathematically, the given set

$= \{(x, y) = x + iy : x \notin R_+ \cup \{0\}, y \in R\}$.



Example 55 : Show that the equation of a straight line through the points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is $\arg \dfrac{z - z_1}{z_1 - z_2} = 0$.

Solution : The equation of the straight line through the points (x_1, y_1) and (x_2, y_2) is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2} \Rightarrow \arg(z - z_1) = \arg(z_1 - z_2) \Rightarrow \arg(z - z_1) - \arg(z_1 - z_2) = 0 \Rightarrow \arg \dfrac{z - z_1}{z_1 - z_2} = 0$.

Example 56 : Find the equation of a circle with centre $(-3, 4)$ and radius 2.

Solution : Let $z = (x, y)$ be any point on the circle, then the equation of the circle is $|(x, y) - (-3, 4)| = 2 \Rightarrow$

$|(x+iy)-(-3+4i)| \Rightarrow |z+3-4i|=2$. In rectangular form this becomes $(x+3)^2+(y-4)^2=4$.

Example 57 **:** Find the equation of an ellipse with foci at $(0,-2)$ and $(0, 2)$ and its major axis is 10. ***D. U. 63.***

Solution **:** Let $z=(x, y)$ be any point on the ellipse. By definition, the sum of the distances from any point on the ellipse to the foei must be equal to the length of the major axis.

Then the required equation is

$$|(x, y)-(0,-2)| + |(x, y)-(0, 2)| = 10$$

$$\Rightarrow |z+2i| + |z-2i| = 10.$$

Exmple 58 **:** Find the equation of a hyperbola with foci at $(3, 0)$ and $(-3, 0)$ and its transverse axis is 4.

Solution **:** Let $z=(x, y)$ be any point on the hyperbola. By definition, the difference of the distances from any point on the hyperbola to the transverse axis. Then the required equation is

$$|(x, y)-(3, 0)| - |(x, y)-(-3, 0)| = 4$$

$$\Rightarrow |z-3| - |z+3| = 4.$$

Example 59 **:** Find the modulus and argument of the following complex nnmbers **:**

(*i*) $\dfrac{-2}{1+i\sqrt{3}}$; (*ii*) $\dfrac{1-i}{1+i}$. ***J. U. H. 87.***

Solution **:** (*i*) $\dfrac{-2}{1+i\sqrt{3}} = \dfrac{-2(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \dfrac{-2(1-i\sqrt{3})}{1+3}$

$=-1/2+i\sqrt{3}/2$. Then the requred modulus $=\frac{1}{2}\sqrt{\{(-1)^2+(\sqrt{3})^2\}}$

$=\frac{1}{2}\sqrt{4}=\frac{2}{2}=1$. The required argument (general)

$$=2n\pi+\tan^{-1}\frac{\sqrt{3}/2}{-1/2}=2n\pi+\frac{2\pi}{3},\ n=0,\pm1,\pm2,\ \dots\ \dots$$

and the principal argument $=\frac{2\pi}{3}$.

(ii) $\frac{1-i}{1+i}=\frac{(1-i)^2}{1-i^2}=\frac{1-2i+i^2}{2}=-i$. The required modulus $=|-i|=1$. The required argument (general) $=2n\pi-\pi/2$, $n=0,\pm1,\pm2,\ \cdots$ and the principal argument $=-\pi/2$.

Example 60 : Find all the fifth roots of unity.

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Solution : Let $z=(1)^{1/5}=(\cos 2n\pi+i\sin 2n\pi)^{1/5}$

$$=e^{2n\pi i/5},\ \text{where } n=0,1,2,3,4.$$

Then the required roots are $1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$

or $1, w, w^2, w^3, w^4$ where $w=e^{2\pi i/5}$.

Example 61 : Find the argument and modulus of the complex number $\left(\frac{2+i}{3-i}\right)^2$. ***D. U. H. T. 91.***

Ans. modulus $=\frac{1}{2}$, principal argument $=\pi/2$ and general argument $=\pi/2+2n\pi$, $n\in I$.

Solution ; Try yourself.

$\Rightarrow \arg (b-c)\left(\bar{a}-\bar{c}\right)=0 \Rightarrow (b-c)\left(\bar{a}-\bar{c}\right)$ is real and positive $\Rightarrow (b-c)\left(\bar{a}-\bar{c}\right)=\left|(b-c)\left(\bar{a}-\bar{c}\right)\right| \quad \ldots \ldots \quad (1)$

Again we have $|b-c|\,|a-c|=r^2 \Rightarrow$

$|b-c|\left|\bar{a}-\bar{c}\right|=r^2$ since $|z|=\left|\bar{z}\right|$

$\Rightarrow \left|(b-c)\left(\bar{a}-\bar{c}\right)\right|=r^2 \quad \ldots \quad (2)$

Now by (1) and (2), we have

$$(b-c)\left(\bar{a}-\bar{c}\right)=r^2 \Rightarrow b-c=\frac{r^2}{\bar{a}-\bar{c}} \Rightarrow b=c+\frac{r^2}{\bar{a}-\bar{c}}.$$