দ্বিতীয় অধ্যায় [CHAPTER-2]
দ্বিতীয় পরিচ্ছেদ [SECTION-2]
কতিপয় প্রামাণ্য আকার
[SOME STANDARD FORMS]

প্রমাণ কর [Prove that] : $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\frac{x}{a} + c.$

oof : We put $I = \int \frac{dx}{a^2 + x^2}$ [and] $x = a\tan\theta$ then $\theta = \tan^{-1}\frac{x}{a}$

$$\therefore dx = a\sec^2\theta\, d\theta$$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2(1 + \tan^2\theta)} = \frac{1}{a}\int \frac{\sec^2\theta\, d\theta}{\sec^2\theta} = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c$$

$$\frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$$

প্রমাণ কর [Prove that] : $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} ln\frac{x - a}{x + a} + c$

oof : [ধরি] We put $I = \int \frac{dx}{x^2 - a^2} = \int \frac{1}{(x - a)(x + a)} dx$

বা $I = \int \frac{1}{2a}\left(\frac{1}{x - a} - \frac{1}{x + a}\right) dx$

বা $I = \frac{1}{2a}\left[\int \frac{dx}{x - a} - \int \frac{dx}{x + a}\right]$

বা $I = \frac{1}{2a}[ln(x - a) - ln(x + a)] + c$

$$\therefore \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} ln\frac{x - a}{x + a} + c.$$

প্রমাণ কর [Prove that] : $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} ln\frac{a + x}{a - x} + c$

oof : [ধরি] We put $I = \int \frac{dx}{a^2 - x^2} = \int \frac{1\ dx}{(a + x)(a - x)}$

বা $I = \frac{1}{2a}\int\left[\frac{1}{a + x} + \frac{1}{a - x}\right] dx$

বা $I = \frac{1}{2a}\left[\int \frac{dx}{a + x} + \int \frac{dx}{a - x}\right]$

$$\text{বা } I = \frac{1}{2a}[\ln(a+x) - \ln(a-x)] + c$$

$$\therefore \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \frac{a+x}{a-x} + c.$$

4. প্রমাণ কর [Prove that] ঃ $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c.$

Proof ঃ [ধরি] We put $I = \int \frac{dx}{\sqrt{a^2 - x^2}}$ and $x = a\sin\theta$ then $\theta = \sin^{-1}\frac{x}{a}$

$$\therefore dx = a\cos\theta \, d\theta$$

$$\text{or } I = \int \frac{a\cos\theta \, d\theta}{\sqrt{a^2\cos^2\theta}} = \int d\theta = \theta + c$$

$$\therefore \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c.$$

5. প্রমাণ কর [Prove that] ঃ $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} = \ln(x + \sqrt{x^2 + a^2})$

Proof ঃ [ধরি] We put $I = \int \frac{dx}{\sqrt{x^2 + a^2}}$ and $x = a\sinh\theta$ then $\theta = \sinh^{-1}\frac{x}{a}$

$$\therefore dx = a\cosh\theta \, d\theta$$

$$\text{or } I = \int \frac{a\cosh\theta \, d\theta}{\sqrt{a^2(\sinh^2\theta + 1)}} = \int \frac{a\cosh\theta \, d\theta}{a\cosh\theta} = \int d\theta = \theta + c$$

$$\therefore \int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} = \ln(x + \sqrt{x^2 + a^2}).$$

6. প্রমাণ কর [Prove that] ঃ $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} = \ln(x + \sqrt{x^2 - a^2})$

Proof ঃ [ধরি] We put $I = \int \frac{dx}{\sqrt{x^2 - a^2}}$ and $x = a\cosh\theta$ then $\theta = \cosh^{-1}\frac{x}{a}$

$$\therefore dx = a\sinh\theta \, d\theta$$

$$\text{or } I = \int \frac{a\sinh\theta \, d\theta}{\sqrt{a^2(\cosh^2\theta - 1)}} = \int \frac{a\sinh\theta \, d\theta}{\sqrt{a^2\sinh^2\theta}} = \int d\theta = \theta + c$$

$$\therefore \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} = \ln(x + \sqrt{x^2 - a^2}).$$

7. প্রমাণ কর [Prove that] ঃ $\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

Proof ঃ [ধরি] We put $I = \int \sqrt{a^2 - x^2}\, dx$ and $x = a\sin\theta$ then $\theta = \sin^{-1}\frac{x}{a}$

$$\therefore dx = a\cos\theta \, d\theta$$

বা $I = \int \sqrt{a^2(1-\sin^2\theta)}\,.\,a\cos\theta\,d\theta = a^2\int \cos^2\theta\,d\theta$

বা $I = \frac{a^2}{2}\int (1+\cos 2\theta)\,d\theta = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$

বা $I = \frac{a^2}{2}[\theta + \sin\theta.\cos\theta] + c = \frac{a^2}{2}[\theta + \sin\theta\sqrt{1-\sin^2\theta}] + c$

বা $I = \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\cdot\sqrt{1-\frac{x^2}{a^2}}\right] + c$

বা $I = \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{a^2}\right] + c$

$\therefore \int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c.$

Prove that ঃ $\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c.$

ঃ [ধরি] We put $I = \int \sqrt{a^2+x^2}\,dx$ and $x = a\sinh\theta$ then $\theta = \sinh^{-1}\frac{x}{a}$

$\therefore dx = a\cosh\theta\,d\theta$

বা $I = \int \sqrt{a^2(1+\sinh^2\theta)}.\,a\cosh\theta\,d\theta = a^2\int \cosh^2\theta\,d\theta$

বা $I = \frac{a^2}{2}\int (\cosh 2\theta + 1)\,d\theta = \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2} + \theta\right] + c$

বা $I = \frac{a^2}{2}[\sinh\theta.\cosh\theta + \theta] + c$

বা $I = \frac{a^2}{2}[\sinh\theta.\sqrt{1+\sinh^2\theta} + \theta] + c$

বা $I = \frac{a^2}{2}\left[\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}} + \sinh^{-1}\frac{x}{a}\right] + c$

বা $I = \frac{a^2}{2}\left[\frac{x\sqrt{a^2+x^2}}{a^2} + \sinh^{-1}\frac{x}{a}\right] + c$

$\therefore \int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$

$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln(x+\sqrt{a^2+x^2}) + c.$

9. Prove that : $\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a}$.

Proof : [ধরি] We put $I = \int \sqrt{x^2 - a^2}\, dx$ and $x = a \cosh\theta$ then $\theta = \cosh^{-1}$

$$\therefore dx = a \sinh\theta\, d\theta$$

$$\text{বা } I = \int \sqrt{a^2 (\cosh^2 \theta - 1)} \,.\, a \sinh\theta\, d\theta = a^2 \int \sinh^2\theta\, d\theta$$

$$\text{বা } I = \frac{a^2}{2} \int (\cosh 2\theta - 1)\, d\theta = \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2} - \theta\right] + c$$

$$\text{বা } I = \frac{a^2}{2} [\sinh\theta \,.\, \cosh\theta - \theta] + c$$

$$\text{বা } I = \frac{a^2}{2} [\cosh\theta. \sqrt{\cosh^2 \theta - 1} - \theta] + c$$

$$\text{বা } I = \frac{a^2}{2}\left[\frac{x}{a} \sqrt{\frac{x^2}{a^2} - 1} - \cosh^{-1} \frac{x}{a}\right] + c$$

$$\text{বা } I = \frac{a^2}{2}\left[\frac{x\sqrt{x^2 - a^2}}{a^2} - \cosh^{-1} \frac{x}{a}\right] + c$$

$$\therefore \int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a} + c$$

$$= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln (x + \sqrt{x^2 - a^2}) + c$$

সুতরাং আমরা পাইলাম [So we obtained] :

1. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

2. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \frac{x - a}{x + a}$

3. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \frac{a + x}{a - x}$

4. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$

5. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} = \ln (x + \sqrt{x^2 + a^2})$

6. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} = \ln (x + \sqrt{x^2 - a^2})$

7. $\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$

8. $\int \sqrt{x^2 + a^2}\, dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a}$

$$= \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \ln (x + \sqrt{x^2 + a^2})$$

$$\int \sqrt{x^2 - a^2}\,dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$
$$= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\ln\left(x + \sqrt{x^2 - a^2}\right).$$

te-1 ঃ উপরের সূত্রের প্রত্যেক ফলাফলে আমরা দেখি যে [We see that in esult of the above formula]

একটি চলক এবং একটি ধ্রুবক আছে [There is one variable and one ant]

চলকের সহগ 1 [Coefficient of variable is 1]

) চলকের ঘাত 2 [Power of variable is 2]

তরাং কোন ইনটিগ্র্যালের মান নির্ণয় করিবার সময় উপরের 3টি শর্ত সিদ্ধ করিতে সূত্রের ন্যায় হইবে। অতঃপর ইনটিগ্রেশন করিতে হইবে।

o at the time of integration, if we satisfy the above 3 itions then it will becomes as formula. Then to be integrate

উদাহরণমালা *[EXAMPLES]*

xample-1 ঃ [Evaluate] $\int \frac{dx}{a^2x^2 + b^2}$ এর মান নির্ণয় কর।

olution ঃ ধরি [We put] $I = \int \frac{dx}{a^2x^2 + b^2}$

নোট ঃ x^2 এর সহগ 1 করার জন্য a^2 common নেওয়া হইয়াছে।

$$= \int \frac{dx}{a^2(x^2 + b^2/a^2)}$$
$$= \frac{1}{a^2}\int \frac{dx}{x^2 + (b/a)^2}$$
$$= \frac{1}{a^2}\cdot\frac{a}{b}\tan^{-1}\frac{x}{b/a} + c$$
$$= \frac{1}{ab}\tan^{-1}\frac{ax}{b} + c.$$

xample-2 ঃ (i). [Evaluate] $\int \frac{\sqrt{x}\,dx}{\sqrt{a^3 - x^3}}$ (ii). $\int x^5\sqrt{4 + 9x^{12}}\,dx$ এর নির্ণয় কর।

Solution-(i) ঃ ধরি [We put] $I = \int \frac{\sqrt{x}\,dx}{\sqrt{a^3 - x^3}} = \int \frac{x^{1/2}dx}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}$

Again we put $x^{3/2} = z$ and $a^{3/2} = k$

তবে [then] $\frac{3}{2} x^{1/2}\, dx = dz \Rightarrow x^{1/2}\, dx = \frac{2}{3} dz.$

$$\therefore I = \frac{2}{3} \int \frac{dz}{\sqrt{k^2 - z^2}}$$

$$= \frac{2}{3} \sin^{-1} \frac{z}{k} + c = \frac{2}{3} \sin^{-1} \left(\frac{x^{3/2}}{a^{3/2}}\right) + c$$

$$= \frac{2}{3} \sin^{-1} \left(\frac{x}{a}\right)^{3/2} + c.$$

Solution–(ii) ঃ We put $I = \int x^5 \sqrt{4 + 9x^{12}}\, dx = \int x^5 \sqrt{4 + (3x^6)^2}\, dx$

and $3x^6 = z$ then $18x^5\, dx = dz \Rightarrow x^5\, dx = \frac{1}{18} dz$

$$\therefore I = \frac{1}{18} \int \sqrt{2^2 + z^2}\, dz$$

$$= \frac{1}{18} \left[\frac{z\sqrt{2^2 + z^2}}{2} + \frac{2^2}{2} \sinh^{-1} \frac{z}{2}\right] + c$$

$$= \frac{1}{18} \left[\frac{3x^6 \sqrt{4 + 9x^{12}}}{2} + 2 \sinh^{-1} \left(\frac{3x^6}{2}\right)\right] + c$$

$$= \frac{x^6 \sqrt{4 + 9x^{12}}}{12} + \frac{1}{9} \sinh^{-1} \left(\frac{3x^6}{2}\right) + c.$$

Example–3 ঃ (i). [Evaluate] $\int \frac{\cos x\, dx}{a^2 + b^2 \sin^2 x}$

(ii). $\int \frac{dx}{(1 + x^2) \sqrt{1 - (\tan^{-1} x)^2}}$

(iii). $\int \frac{dx}{x \sqrt{4 - 9 (\ln x)^2}}$ এর মান নির্ণয় কর। **[D. U. H. '88]**

Solution–(i) ঃ ধরি [We put] $I = \int \frac{\cos x\, dx}{a^2 + b^2 \sin^2 x}$

and] $b \sin x = z$, then $b \cos x\, dx = dz \Rightarrow \cos x\, dx = \frac{1}{b} dz$

$$\therefore I = \frac{1}{b} \int \frac{dz}{a^2 + z^2}$$

$$= \frac{1}{b} \cdot \frac{1}{a} \tan^{-1} \frac{z}{a} + c$$

$$= \frac{1}{ab} \tan^{-1} \left(\frac{b \sin x}{a}\right) + c.$$

...lution–(ii) ঃ ধরি [We put] $I = \int \frac{dx}{(1 + x^2)\sqrt{1 - (\tan^{-1} x)^2}}$

...[and] $\tan^{-1}x = z$, তবে [then] $\frac{dx}{1 + x^2} = dz$

$$I = \int \frac{dz}{\sqrt{1 - z^2}}$$

$$= \sin^{-1} z + c$$

$$= \sin^{-1} (\tan^{-1}x) + c.$$

...lution–(iii) ঃ ধরি [We put] $I = \int \frac{dx}{x\sqrt{4 - 9\,(ln\, x)^2}}$

...[and] $3\, ln x = z$, তবে [then] $\frac{3}{x} dx = dz \Rightarrow \frac{1}{x} dx = \frac{1}{3} dz$

$$I = \frac{1}{3} \int \frac{dz}{\sqrt{2^2 - z^2}}$$

$$= \frac{1}{3} \cdot \sin^{-1} \frac{z}{2} + c$$

$$= \frac{1}{3} \sin^{-1} \left(\frac{3\, ln x}{2}\right) + c.$$

...xample–4 ঃ [Evaluate] $\int \frac{(x^2 + 1)\, dx}{x^4 + 1}$ এর মান নির্ণয় কর।

[N. U. H-04, D. U. H-95]

...olution– ঃ ধরি [We put] $I = \int \frac{(x^2 + 1)\, dx}{x^4 + 1} = \int \frac{x^2(1 + 1/x^2)\, dx}{x^2(x^2 + 1/x^2)}$

বা $I = \int \frac{(1 + 1/x^2)\, dx}{x^2 + 1/x^2} = \int \frac{(1 + 1/x^2)\, dx}{(x - 1/x)^2 + 2}$

...রি [We put] $x - 1/x = z$ তবে [then] $(1 + 1/x^2)\, dx = dz$

$$I = \int \frac{dz}{z^2 + (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{z}{\sqrt{2}} + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x - 1/x}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{x\sqrt{2}}\right) + c.$$

প্রশ্নমালা [EXERCISE]–2(B)

নিম্নলিখিত ইনটিগ্র্যাল সমূহ নির্ণয় কর [Evaluate the following integrals] :

1(i). $\int \frac{dx}{\sqrt{4-16x^2}}$ (ii). $\int \sqrt{5+2x^2}\, dx$

(iii). $\int \frac{dx}{a^4 - x^4}$

2(i). $\int \frac{3x^2\, dx}{1+x^6}$

(ii). $\int \frac{x\, dx}{x^4 + a^4}$ (iii). $\int \frac{x\, dx}{x^4 - 1}$

(iv). $\int \frac{x^3\, dx}{\sqrt{a^8 - x^8}}$ (v). $\int \frac{x\, dx}{\sqrt{a^4 + x^4}}$

(vi). $\int \frac{x^3\, dx}{\sqrt{x^8 + 1}}$ (vii). $\int x^2 \sqrt{x^6 - 1}\, dx$

(viii). $\int x^4 \sqrt{a^2 - x^{10}}\, dx$

3(i). $\int \frac{\cos x\, dx}{1 + \sin^2 x}$

(ii). $\int \frac{\cos x\, dx}{5 + 7\cos^2 x}$ (iii). $\int \frac{e^x\, dx}{\sqrt{1 + e^{2x}}}$

(iv). $\int \frac{dx}{(1+x^2)\{9 + (\cot^{-1} x)^2\}}$ (v). $\int \frac{dx}{\sqrt{1-x^2}\,.\,\sqrt{1 + (\sin^{-1} x)^2}}$

(vi). $\int \frac{2\, dx}{x[1 + (\ln x)^2]}$ (vii). $\int \frac{3\tan^2 x \sec^2 x\, dx}{1 + \tan^6 x}$

(viii). $\int \frac{dx}{e^x + e^{-x}}$

4(i). $\int \frac{(x^2 - 1)\, dx}{x^4 + 1}$

(ii). $\int \frac{dx}{x^4 + 1}$ (iii). $\int \frac{x^2\, dx}{x^4 + a^4}$

(iv). $\int \frac{(x^2 + \sin^2 x)\sec^2 x\, dx}{1 + x^2}$ (v). $\int \sqrt{1 + \sec x}\, dx$ **[D. U. H. '93]**

(vi). $\int \frac{\sin x \cos x\, dx}{\sin^4 x + \cos^4 x}$ **[D. U. H. '90]**

উত্তরমালা [ANSWERS]

1(i). $\frac{1}{4}\sin^{-1}(2x)$ (ii). $\frac{x\sqrt{5+2x^2}}{2} + \frac{5\sqrt{2}}{4}\sinh^{-1}\left(\frac{x\sqrt{2}}{\sqrt{5}}\right)$

(iii). $\frac{1}{4a^3}\left[2\tan^{-1}\frac{x}{a} + \ln\frac{a+x}{a-x}\right]$

2(i). $\tan^{-1}(x^3)$ (ii). $\frac{1}{2a^2}\tan^{-1}\left(\frac{x^2}{a^2}\right)$

(iii). $\frac{1}{4}\ln\frac{x^2-1}{x^2+1}$ (iv). $\frac{1}{4}\sin^{-1}\left(\frac{x^4}{a^4}\right)$

(v). $\frac{1}{2}\sinh^{-1}\left(\frac{x^2}{a^2}\right)$ (vi). $\frac{1}{4}\sinh^{-1}(x^4)$

(vii). $\frac{x^3\sqrt{x^6-1}}{6}-\frac{1}{6}\cosh^{-1}(x^3)$

(viii). $\frac{x^5\sqrt{a^2-x^{10}}}{10}+\frac{a^2}{10}\sin^{-1}\left(\frac{x^5}{a}\right)$

3(i). $\tan^{-1}(\sin x)$

(ii). $\frac{1}{4\sqrt{21}}\ln\left(\frac{2\sqrt{3}+\sqrt{7}\,.\,\sin x}{2\sqrt{3}-\sqrt{7}\,.\,\sin x}\right)$

(iii) $\sinh^{-1}(e^x)$

(iv). $\frac{1}{3}\cot^{-1}\left(\frac{1}{3}\cot^{-1}x\right)$ (v). $\sinh^{-1}(\sin^{-1}x)$

(vi). $2\tan^{-1}(\ln x)$ (vii). $\tan^{-1}(\tan^3 x)$

(viii). $\tan^{-1}(e^x)$

4(i). $\frac{1}{2\sqrt{2}}\ln\left(\frac{x^2+1-x\sqrt{2}}{x^2+1+x\sqrt{2}}\right)$

(ii). $\frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)-\frac{1}{2\sqrt{2}}\ln\left(\frac{x^2+1-x\sqrt{2}}{x^2+1+x\sqrt{2}}\right)\right]$

(iii). $\frac{1}{2a\sqrt{2}}\left[\tan^{-1}\left(\frac{x^2-a^2}{ax\sqrt{2}}\right)+\frac{1}{2}\ln\left(\frac{x^2+a^2-ax\sqrt{2}}{x^2+a^2+ax\sqrt{2}}\right)\right]$

(iv). $\tan x-\tan^{-1}x$

(v). $2\sin^{-1}(\sqrt{2}\sin x/2)$

(vi). $\frac{1}{2}\tan^{-1}(\tan^2 x)$.

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দ্বিতীয় অধ্যায় [CHAPTER–2]
তৃতীয় পরিচ্ছেদ [SECTION–3]
বিশেষ আকারের ইনটিগ্র্যাল
[SPECIAL TYPE OF INTEGRAL]

নিয়ম–1 ঃ যদি কোন ইনটিগ্র্যাল

$$\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \sqrt{ax^2+bx+c}\,dx$$

এর যে কোন এক আকারে থাকে, তবে ax^2+bx+c কে দুই বর্গের সমষ্টি অথবা অন্তররূপে প্রকাশ করিতে হয়। ইহার জন্য প্রথমে x^2 এর সহগ 1 করিতে হইবে। দ্বিতীয়তঃ চলক বিশিষ্ট দুইটি পদকে একটি পদে পরিণত করিতে হইবে। অর্থাৎ $(x+\frac{1}{2}\cdot x$ এর সহগ$)^2$ লিখিয়া ধ্রুব সংখ্যা Balance করিতে হয়। অতঃপর প্রামান্য আকার [Standard form] এর সূত্র অনুসারে ইনটিগ্রেট করিতে হয়।

[If the integral is any one of the following form

$$\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\,dx$$

then reduce ax^2+bx+c into sum of two squares or difference of two squares. For this at first reduce the coefficient of x^2 to be 1 and then two terms involving variables will be reduce to one term. Finally integrate as formula of standard form.]

যেমন ঃ $ax^2+bx+c = a\left[x^2+\frac{bx}{a}+\frac{c}{a}\right]$

$$= a\left[\left(x+\frac{1}{2}\frac{b}{a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

ধরি [We put] $I = \int \frac{dx}{ax^2+bx+c}$

$$= \frac{1}{a}\int \frac{dx}{x^2+\frac{bx}{a}+\frac{c}{a}}$$

$$= \frac{1}{a}\int \frac{dx}{\left(x+\frac{1}{2}\frac{b}{a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}}$$

$$I = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}} \cdots (1)$$

$$= \frac{1}{a}\int \frac{dx}{(x+b/2a)^2+\{(\sqrt{4ac-b^2})/2a\}^2}$$ যদি $b^2<4ac$

ধরি [We put] $\frac{\sqrt{4ac-b^2}}{2a} = k$ এবং [and] $x+\frac{b}{2a} = z$ তবে [then] $dx = dz$

$$\therefore I = \frac{1}{a}\int \frac{dz}{z^2+k^2} = \frac{1}{a}\cdot\frac{1}{k}\tan^{-1}\frac{z}{k}+c_1$$

আবার (1) নং হইতে পাই [Again from (1) we get]

$$I = \frac{1}{a}\int \frac{dx}{(x+b/2a)^2-(b^2-4ac)/4a^2},$$ যদি [If] $b^2>4ac$

$$= \frac{1}{a}\int \frac{dx}{(x+b/2a)^2-\{(\sqrt{b^2-4ac})/2a\}^2}$$

ধরি [We put] $\frac{\sqrt{b^2-4ac}}{2a} = k$ এবং [and] $x+\frac{b}{2a} = z$ তবে [then] $dx = dz$

$$\therefore I = \frac{1}{a}\int \frac{dz}{z^2-k^2}$$

$$= \frac{1}{a}\cdot\frac{1}{2k}\,ln\,\frac{z-k}{z+k}+c_2.$$

উদাহরণমালা [EXAMPLES]

Example–1 ঃ (a) [Evaluate] $\int \frac{dx}{2x^2+x+1}$ (b). $\int \frac{dx}{\sqrt{4x^2-12x+7}}$

(c). $\int \sqrt{4-3x-2x^2}\,dx$ এর মান নির্ণয় কর।

Solution– (a) ঃ ধরি [We put] $I = \int \frac{dx}{2x^2+x+1} = \frac{1}{2}\int \frac{dx}{x^2+x/2+1/2}$

$$= \frac{1}{2}\int \frac{dx}{(x+1/4)^2+1/2-1/16}$$

$$= \frac{1}{2}\int \frac{dx}{(x+1/4)^2+(\sqrt{7}/4)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{7}/4}\tan^{-1}\frac{x+1/4}{\sqrt{7}/4}+c$$

$$= \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right)+c.$$

Solution-(b) ঃ ধরি [We put] $I = \int \frac{dx}{\sqrt{4x^2 - 12x + 7}} = \int \frac{dx}{\sqrt{4(x^2 - 3x + 7/4)}}$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{(x - 3/2)^2 + 7/4 - 9/4}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{(x - 3/2)^2 - (1/\sqrt{2})^2}}$$

$$= \frac{1}{2} \cdot \cosh^{-1}\left(\frac{x - 3/2}{1/\sqrt{2}}\right) + c$$

$$= \frac{1}{2} \cosh^{-1}\left(\frac{2x - 3}{2/\sqrt{2}}\right) + c$$

$$= \frac{1}{2} \cosh^{-1}\left(\frac{2x - 3}{\sqrt{2}}\right) + c.$$

Solution-(c) ঃ ধরি [We put] $I = \int \sqrt{4 - 3x - 2x^2}\, dx$

$$= \int \sqrt{2\left(2 - \frac{3x}{2} - x^2\right)}\, dx$$

$$= \sqrt{2}\int \sqrt{2 + \frac{9}{16} - (x + 3/4)^2}\, dx$$

$$= \sqrt{2}\int \sqrt{(\sqrt{41}/4)^2 - (x + 3/4)^2}\, dx$$

$$= \sqrt{2}\left[\left(x + \frac{3}{4}\right)\frac{\sqrt{(\sqrt{41}/4)^2 - (x + 3/4)^2}}{2} + \frac{(\sqrt{41}/4)^2}{2}\sin^{-1}\frac{x + 3/4}{\sqrt{41}/4}\right] + c$$

$$= \sqrt{2}\left[\frac{(4x + 3)}{4}\frac{\sqrt{4 - 3x - 2x^2}}{2\sqrt{2}} + \frac{41/16}{2}\sin^{-1}\frac{4x + 3}{\sqrt{41}}\right] + c$$

$$= \frac{(4x + 3)}{8}\sqrt{4 - 3x - 2x^2} + \frac{41\sqrt{2}}{32}\sin^{-1}\frac{4x + 3}{\sqrt{41}} + c.$$

নিয়ম-2 ঃ যদি কোন ইনটিগ্র্যাল $\int \frac{(px + q)\, dx}{ax^2 + bx + c}$, $\int \frac{(px + q)\, dx}{\sqrt{ax^2 + bx + c}}$, $\int (px + q)\sqrt{ax^2 + bx + c}\, dx$ এর যে কোন এক আকারে থাকে, তবে একঘাত রাশিমালার স্থলে দ্বিঘাত রাশিমালার অন্তরক সহগ লিখিয়া যত **x** ছিল তত দিয়া গুণ এবং যত **x** লিখিত হইয়াছে তত দিয়া ভাগ করিতে হয়। অতঃপর ধ্রুব সংখ্যা Balance করিয়া পৃথক করার পর ইনটিগ্রেশন করিতে হয়।

$$\text{বা } I = \frac{1}{2}\int (x^2-2x+5)^{1/2}\, d(x^2-2x+5) + 2\int \sqrt{(x-1)^2+2^2}\, dx$$

$$= \frac{1}{2}\frac{(x^2-2x+5)^{3/2}}{3/2} + 2\left[\frac{(x-1)\sqrt{x^2-2x+5}}{2} + \frac{2^2}{2}\sinh^{-1}\frac{(x-1)}{2}\right] + c$$

$$= \frac{1}{3}(x^2-2x+5)^{3/2} + (x-1)\sqrt{x^2-2x+5} + 4\sinh^{-1}\frac{(x-1)}{2} + c.$$

নিয়ম-3 : (i). যদি কোন ইনটিগ্র্যাল এর লব এবং হর উভয়ই এক জাতীয় চলক বিশিষ্ট হয়; লবের চলকের ঘাত, হরের চলকের ঘাতের সমান [অথবা হরের ঘাত হইতে বেশী] হয়, তবে লবকে হর দ্বারা ভাগ করার পর ইনটিগ্রেশন করিতে হয়।

(ii). ভাগ করার নিয়ম : লবের জায়গায় হুবহু হর লিখিয়া Balance করার পর ভাগ করিতে হয়।

(iii). যদি কোন ইনটিগ্র্যাল $\int \frac{dx}{ae^{kx}+b}$ এই আকারে থাকে, অর্থাৎ কেবলমাত্র হরে $ae^{kx}+b$ থাকে, তবে লব এবং হর উভয়কে e^{-kx} দ্বারা গুণ করার পর "হর = z" ধরিয়া সরলীকরণ করার পর ইনটিগ্রেশন করিতে হয়।

Example-3 : (a). [Evaluate] $\int \frac{(x^2+x+1)dx}{\sqrt{x^2+2x+3}}$

(b). $\int \frac{(ae^{3x}+b)\,dx}{ce^{3x}+k}$ এবং [and]

(c). $\int \frac{(x^3+2x^2+x+7)\,dx}{\sqrt{x^2+2x+3}}$ এর মান নির্ণয় কর।

Solution-(a) :

ধরি [We put] $I = \int \frac{(x^2+x+1)\,dx}{\sqrt{x^2+2x+3}} = \int \frac{(x^2+2x+3)-x-2}{\sqrt{x^2+2x+3}}\,dx$

$$= \int \sqrt{x^2+2x+3}\,dx - \int \frac{(x+2)\,dx}{\sqrt{x^2+2x+3}}$$

$$= \int \sqrt{x^2+2x+3}\,dx - \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}\,dx$$

$$= \int \sqrt{(x+1)^2+(\sqrt{2})^2}\,dx - \frac{1}{2}\int \frac{(2x+2)\,dx}{\sqrt{x^2+2x+3}} - \int \frac{dx}{\sqrt{(x+1)^2+(\sqrt{2})^2}}$$

$$= \frac{(x+1)\sqrt{x^2+2x+3}}{2} + \frac{(\sqrt{2})^2}{2}\sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) - \frac{1}{2}\cdot 2\sqrt{x^2+2x+3} - \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$\text{বা } I = \left(\frac{x+1}{2} - 1\right)\sqrt{x^2+2x+3} + c$$

$$= \frac{1}{2}(x-1)\sqrt{x^2+2x+3} + c.$$

Solution-(b) : ধরি [We put] $I = \int \frac{(ae^{3x}+b)\,dx}{ce^{3x}+k}$

নোট : লব এবং হরে এক জাতীয় চলক রাশি, কাজেই নিয়ম 3(i) দ্বারা।

$$= \int \frac{\frac{a}{c}(ce^{3x}+k) + b - \frac{ak}{c}}{c\,e^{3x}+k}\,dx$$

$$= \frac{a}{c}\int dx + \left(b - \frac{ak}{c}\right)\int \frac{dx}{c\,e^{3x}+k}$$

$$= \frac{a}{c}\int dx + \frac{(bc-ak)}{c}\int \frac{e^{-3x}\,dx}{c+k\,e^{-3x}}$$; নিয়ম 3(iii) দ্বারা।

ধরি $c+k\,e^{-3x} = z$ তবে $-3ke^{-3x}\,dx = dz$, বা $e^{-3x}\,dx = -\frac{dz}{3k}$

$$\therefore I = \frac{a}{c}\int dx - \frac{(bc-ak)}{c}\cdot\frac{1}{3k}\int\frac{dz}{z}$$

$$= \frac{ax}{c} - \frac{(bc-ak)}{3ck}\ln z + c_1$$

$$= \frac{ax}{c} - \frac{(bc-ak)}{3ck}\ln(c+k\,e^{-3x}) + c_1.$$

Solution-(c) : ধরি [We put] $I = \int \frac{(x^3+2x^2+x+7)\,dx}{\sqrt{x^2+2x+3}}$

$$= \int \frac{x(x^2+2x+3) - 2x + 7}{\sqrt{x^2+2x+3}}\,dx$$

$$= \int x\sqrt{x^2+2x+3}\,dx - \int \frac{(2x-7)\,dx}{\sqrt{x^2+2x+3}}$$

$= I_1 - I_2$, [ধরি].

Now $I_1 = \int x\sqrt{x^2+2x+3}\,dx = \int\left\{\frac{1}{2}(2x+2) - 1\right\}\sqrt{x^2+2x+3}\,dx$

$$= \frac{1}{2}\int (2x+2)(x^2+2x+3)^{1/2}\,dx - \int \sqrt{(x+1)^2+(\sqrt{2})^2}\,dx$$

$$= \frac{1}{2}\frac{(x^2+2x+3)^{3/2}}{3/2} - \left[\frac{(x+1)\sqrt{x^2+2x+3}}{2} + \frac{(\sqrt{2})^2}{2}\sinh^{-1}\frac{(x+1)}{\sqrt{2}}\right] + c_1$$

$$= \frac{1}{3}(x^2+2x+3)^{3/2} - \frac{1}{2}(x+1)\sqrt{x^2+2x+3} - \sinh^{-1}\frac{(x+1)}{\sqrt{2}} + c_1$$

$$\text{বা } I = \frac{1}{6}(2x^2 + 4x + 6 - 3x - 3)\sqrt{x^2+2x+3} - \sinh^{-1}\{(x+1)/\sqrt{2}\} + c_1$$

$$= \frac{1}{6}(2x^2 + x + 3)\sqrt{x^2+2x+3} - \sinh^{-1}\{(x+1)/\sqrt{2}\} + c_1$$

$$I_2 = \int \frac{(2x-7)\,dx}{\sqrt{x^2+2x+3}} = \int \frac{(2x+2)-9}{\sqrt{x^2+2x+3}}\,dx$$

$$= \int \frac{(2x+2)\,dx}{\sqrt{x^2+2x+3}} - 9\int \frac{dx}{\sqrt{(x+1)^2+(\sqrt{2})^2}}$$

$$= 2\sqrt{x^2+2x+3} - 9\sinh^{-1}\{(x+1)/\sqrt{2}\} + c_2$$

কিন্তু আমাদের আছে [But we have] $I = I_1 - I_2$

$$\text{বা } I = \frac{1}{6}(2x^2 + x + 3)\sqrt{x^2+2x+3} - \sinh^{-1}\{(x+1)/\sqrt{2}\}$$
$$- 2\sqrt{x^2+2x+3} + 9\sinh^{-1}\{(x+1)/\sqrt{2}\} + c$$

$$\text{বা } I = \frac{1}{6}(2x^2 + x - 9)\sqrt{x^2+2x+3} + 8\sinh^{-1}\{(x+1)/\sqrt{2}\} + c.$$

বিবিধ উদা–4 : [Evaluate] $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ এর মান নির্ণয় কর।

[N. U. H. '94]

Solution : ধরি [We put] $I = \int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

এবং $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\therefore\ dx = \alpha.2\cos\theta(-\sin\theta)\,d\theta + \beta.\,2\sin\theta.\cos\theta\,d\theta$

$= (\beta-\alpha)\,2\sin\theta\cos\theta\,d\theta$

এখন $x - \alpha = \alpha\cos^2\theta + \beta\sin^2\theta - \alpha$

বা $x - \alpha = \beta\sin^2\theta - \alpha(1-\cos^2\theta)$

বা $x - \alpha = \beta\sin^2\theta - \alpha\sin^2\theta$

বা $x - \alpha = (\beta-\alpha)\sin^2\theta$

বা $\sqrt{x-\alpha} = \sqrt{\beta-\alpha}\,\sin\theta$

বা $\sin\theta = \frac{\sqrt{x-\alpha}}{\sqrt{\beta-\alpha}} \Rightarrow \theta = \sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}$

এবং $\beta - x = \beta - \alpha\cos^2\theta - \beta\sin^2\theta$

বা $\beta - x = \beta(1-\sin^2\theta) - \alpha\cos^2\theta$

বা $\beta - x = (\beta-\alpha)\cos^2\theta$

(ii). $\int \frac{(2x+5)\,dx}{\sqrt{x^2-2x+2}}$

(iii). $\int \frac{(x+1)\,dx}{\sqrt{x^2-x+1}}$

(iv). $\int \frac{(x-1)\,dx}{\sqrt{4+x^2-2x}}$

(v). $\int \frac{(2x-1)\,dx}{\sqrt{4x^2+4x+2}}$.

c(i). $\int (x-1)\sqrt{x^2-x+1}\,dx$

(ii). $\int (x+4)\sqrt{x^2+2x+7}\,dx$

(iii). $\int (x+a)\sqrt{x^2+a^2}\,dx$

(iv). $\int (2x-3)\sqrt{x^2-4x+13}\,dx$

(v). $\int (x+2)\sqrt{2x^2+2x+1}\,dx$

(vi). $\int (2x+3)\sqrt{4x^2+8x+5}\,dx$

(vii). $\int (x-1)\sqrt{x^2-1}\,dx$.

3(i). $\int \frac{(ax+b)\,dx}{cx-d}$

(ii). $\int \frac{(x^2-a^2)\,dx}{x^2+a^2}$

(iii). $\int \frac{x^2\,dx}{x^2-4}$

(iv). $\int \frac{(x^2+x+1)\,dx}{x^2-x+1}$ **[N. U.H–04]**

(v). $\int \frac{2x^2\,dx}{\sqrt{x^2+1}}$

(vi). $\int \frac{(x^2+2x+3)\,dx}{\sqrt{x^2+x+1}}$

(vii). $\int \frac{(x^2-a^2)\,dx}{\sqrt{x^2+a^2}}$

(viii). $\int \frac{(2x^2-x+1)\,dx}{\sqrt{x^2+2x+5}}$

(ix). $\int \frac{(x^2+x+1)\,dx}{\sqrt{1-x^2}}$

(x). $\int \frac{(e^x+1)\,dx}{e^x-1}$

(xi). $\int \frac{(e^{3x}-1)\,dx}{e^{3x}+1}$.

4(i). $\int \frac{dx}{\sqrt{(x-1)(2-x)}}$ **[D. U. '85, D. U. H. '90]**

(ii). $\int \frac{2x\,dx}{x^4+2x^2+2}$

(iii). $\int \frac{\sec^2 x\,dx}{\tan^2 x+4\tan x+3}$

(iv). $\int \frac{e^x\,dx}{e^{2x}+2e^x+10}$

(v). $\int \frac{x^2\,dx}{x^6-6x^3+5}$

(vi). $\int \frac{dx}{x\{(\ln x)^2+7\ln x+10\}}$

(vii). $\int \frac{\cos x\,dx}{\sqrt{5\sin^2 x-12\sin x+4}}$

(v). $\frac{1}{2}\ln(2x^2-2x+1)+8\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$

(vi). $\frac{3}{2}\ln(x^2-x+2)+\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right)$

(vii). $\frac{5}{6}\ln(3x^2+2x+1)-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)$

b(i). $\frac{1}{2}\sqrt{2x^2-8x+5}$

(ii). $2\sqrt{x^2-2x+2}+7\sinh^{-1}(x-1)$

(iii). $\sqrt{x^2-x+1}+\frac{3}{2}\sinh^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$

(iv). $\sqrt{4+x^2-2x}$,

(v). $\frac{1}{2}\sqrt{4x^2+4x+2}-\sinh^{-1}(2x+1)$

c(i). $\frac{1}{24}(8x^2-14x+11)\sqrt{x^2-x+1}-\frac{3}{16}\sinh^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$

(ii). $\frac{1}{6}(2x^2+13x+23)\sqrt{x^2+2x+7}+9\sinh^{-1}\left(\frac{x+1}{\sqrt{6}}\right)$

(iii). $\frac{1}{3}(x^2+a^2)^{3/2}+\frac{ax}{2}\sqrt{x^2+a^2}+\frac{a^3}{2}\sinh^{-1}\frac{x}{a}$

(iv). $\frac{1}{6}(4x^2-13x+20)\sqrt{x^2-4x+13}+\frac{9}{2}\sinh^{-1}\left(\frac{x-2}{3}\right)$

(v). $\frac{1}{24}(8x^2+26x+13)\sqrt{2x^2+2x+1}+\frac{3\sqrt{2}}{16}\sinh^{-1}(2x+1)$

(vi). $\frac{1}{6}(4x^2+11x+8)\sqrt{4x^2+8x+5}+\frac{1}{4}\sinh^{-1}2(x+1)$

(vii). $\frac{1}{3}(x^2-1)^{3/2}-\frac{1}{2}x\sqrt{x^2-1}+\frac{1}{2}\cosh^{-1}x$

3(i). $\frac{ax}{c}+\frac{(bc+ad)}{c^2}\ln(cx+d)$

(ii). $x-2a\tan^{-1}\frac{x}{a}$

(iii). $x+\ln\frac{x-2}{x+2}$

দ্বিতীয় অধ্যায় [CHAPTER-2]
চতুর্থ পরিচ্ছেদ [SECTION-4]
বিশেষ আকারের ইনটিগ্র্যাল
[SPECIAL FORM OF INTEGRAL]

নিয়ম-1 ঃ যদি কোন ইনটিগ্র্যাল $\int \frac{dx}{(ax+b)\sqrt{cx+d}}$ এই আকারে থাকে, অর্থাৎ ইনটিগ্র্যালের হরে $\sqrt{\ }$ এর ভিতরে একঘাত রাশিমালা এবং $\sqrt{\ }$ এর বাহিরে ও একঘাত রাশিমালা থাকে, তবে $\sqrt{\ }$ এর ভিতরের রাশিমালা $cx + d = z^2$ ধরিয়া সরলীকরণ করার পর ইনটিগ্রেশন করিতে হয়।

যেমন $I = \int \frac{dx}{(ax+b)\sqrt{cx+d}}, a \neq 0, c \neq 0.$

যেহেতু হরে $\sqrt{\ }$ এর ভিতরে এবং বাহিরে একঘাত রাশিমালা, কাজেই $\sqrt{\ }$ এর ভিতরের রাশিমালা $cx + d = z^2$ ধরি।

[**Rule-1 :** If the integral is of the form $\int \frac{dx}{(ax+b)\sqrt{cx+d}}$ that is in the denominator of integral, there are linear expressions inside $\sqrt{\ }$ and linear expression outside $\sqrt{\ }$ then putting linear expression inside $\sqrt{\ }$, $cx + d = z^2$. After simplification to be integrate it

i.e. $I = \int \frac{dx}{(ax+b)\sqrt{cx+d}}, a \neq 0, c \neq 0.$

Since in the denominator, there are linear expressions inside and outside $\sqrt{\ }$

So putting linear expression inside $\sqrt{\ }$, $cx + d = z^2$]

বা $cx = z^2 - d$

বা $x = (z^2 - d)/c$

$\therefore dx = 2z\,dz/c$

$$\therefore I = \frac{1}{c}\int \frac{2z\,dz}{\left\{\frac{a}{c}(z^2 - d) + b\right\} z}$$

$$I = \frac{2}{c}\int \frac{c\,dz}{az^2 - ad + bc}$$

$$= \frac{2}{a}\int \frac{dz}{z^2 + (bc - ad)/a};$$

যা প্রামাণ্য আকারের ইনটিগ্র্যাল। [This is the integral of standard form]

Example-1 ঃ [Evaluate] (i). $\int \frac{dx}{(1-x)\sqrt{1+x}}$ [রাঃ বিঃ '82]

(ii). $\int \frac{(x+1)\,dx}{(x-1)\sqrt{x+2}}$ এর মান নির্ণয় কর।

Solution : ধরি [We put] $I = \int \frac{dx}{(1-x)\sqrt{1+x}}$ এবং [and]

$1 + x = z^2$, বা $x = z^2 - 1$ তবে [then] $dx = 2z\,dz$

$$I = \int \frac{2z\,dz}{(1 - z^2 + 1)z} = 2\int \frac{dz}{(\sqrt{2})^2 - z^2}$$

$$= \frac{2}{2\sqrt{2}} \ln \frac{\sqrt{2} + z}{\sqrt{2} - z} + c$$

$$= \frac{1}{\sqrt{2}} \ln \frac{\sqrt{2} + \sqrt{1+x}}{\sqrt{2} - \sqrt{1+x}} + c.$$

Solution-(ii) ঃ ধরি [We put] $I = \int \frac{(x+1)\,dx}{(x-1)\sqrt{x+2}}$

$$= \int \frac{\{(x-1) + 2\}\,dx}{(x-1)\sqrt{x+2}}$$

$$= \int \frac{dx}{\sqrt{x+2}} + \int \frac{2\,dx}{(x-1)\sqrt{x+2}}$$

$$= 2\sqrt{x+2} + I_1 \cdots (1)$$

যখন [When] $I_1 = 2\int \frac{dx}{(x-1)\sqrt{x+2}}$

ধরি [We put] $x + 2 = z^2$, বা $x = z^2 - 2$ তবে [then] $dx = 2z\,dz$

$$\therefore I_2 = 2\int \frac{2z\,dz}{(z^2-2-1)z} = 4\int \frac{dz}{z^2-(\sqrt{3})^2}$$

$$= \frac{4}{2\sqrt{3}} \ln \frac{z-\sqrt{3}}{z+\sqrt{3}} + c$$

$$= \frac{2}{\sqrt{3}} \ln \frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}} + c.$$

নিয়ম-2 : যদি কোন ইনটিগ্র্যাল $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$ এই আকারে থাকে অর্থাৎ ইনটিগ্র্যালের হরে $\sqrt{\ }$ এর ভিতরে দ্বিঘাত রাশিমালা এবং $\sqrt{\ }$ এর বাহিরে একঘাত রাশিমালা থাকে, তবে একঘাত রাশিমালা $px+q=\frac{1}{z}$ ধরিয়া সরলীকরণ করার পর ইনটিগ্রেশন করিতে হয়।

[**Rule-2 :** If the integral is of the form $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$ that is, in the denominator, there is quadratic expression inside $\sqrt{\ }$ and there is linear expression outside $\sqrt{\ }$. Then putting linear expression $px+q=\frac{1}{z}$. After simplification to be integrate it]

Example-2 : [Evaluate] $\int \frac{dx}{(x+1)\sqrt{1+2x-x^2}}$ এর মান নির্ণয় কর।

Solution : ধরি [We put] $I = \int \frac{dx}{(x+1)\sqrt{1+2x-x^2}}$

এবং [and] $x+1=\frac{1}{z}$, বা $x=\frac{1}{z}-1$ তবে [then] $dx=-\frac{1}{z^2}dz$

$$\therefore I = -\int \frac{dz}{z^2(1/z)\sqrt{1+2(1/z-1)-(1/z-1)^2}}$$

$$= -\int \frac{dz}{z\sqrt{1+2/z-2-1/z^2+2/z-1}}$$

$$= -\int \frac{dz}{z\sqrt{-1/z^2+4/z-2}}$$

$$= -\int \frac{dz}{\sqrt{-1+4z-2z^2}}$$

$$= -\int \frac{dz}{\sqrt{2(-1/2+2z-z^2)}} = -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{-1/2+1-(z-1)^2}}$$

$$I = -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{(1/\sqrt{2})^2-(z-1)^2}} = -\frac{1}{\sqrt{2}}\sin^{-1}\frac{z-1}{1/\sqrt{2}}+c$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{2}(z-1)+c = -\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{2}\left(\frac{1}{x+1}-1\right)+c$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{2}\left(\frac{-x}{x+1}\right)+c = \frac{1}{\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{x+1}+c.$$

Example-3 : [Evaluate] $\int \frac{\sqrt{1+x+x^2}\,dx}{1+x}$ এর মান নির্ণয় কর।

Solution : ধরি [We put] $I = \int \frac{\sqrt{1+x+x^2}\,dx}{1+x}$

$$\text{বা } I = \int \frac{(1+x+x^2)\,dx}{(1+x)\sqrt{1+x+x^2}}$$

$$= \int \frac{\{1+x(1+x)\}\,dx}{(1+x)\sqrt{1+x+x^2}}$$

$$= \int \frac{dx}{(1+x)\sqrt{1+x+x^2}} + \int \frac{x\,dx}{\sqrt{1+x+x^2}}$$

$= I_1 + I_2$ ধরি [say]

Now $I_1 = \int \frac{dx}{(1+x)\sqrt{1+x+x^2}}$

ধরি [We put] $1+x=\frac{1}{z}$, বা $x=\frac{1}{z}-1$ তবে [then] $dx=-\frac{1}{z^2}dz$

$$\therefore I_1 = -\int \frac{dz}{z^2(1/z)\sqrt{1/z+(1/z-1)^2}}$$

$$= -\int \frac{dz}{z\sqrt{1/z+1/z^2-2/z+1}} = -\int \frac{dz}{z\sqrt{1/z^2-1/z+1}}$$

$$= -\int \frac{dz}{\sqrt{1-z+z^2}} = -\int \frac{dz}{\sqrt{1-1/4+(z-1/2)^2}}$$

$$= -\int \frac{dz}{\sqrt{(\sqrt{3}/2)^2+(z-1/2)^2}} = -\sinh^{-1}\frac{z-1/2}{\sqrt{3}/2}+c_1$$

$$I_1 = -\sinh^{-1}\left(\frac{2z-1}{\sqrt{3}}\right) + c_1$$

$$= -\sinh^{-1}\frac{1}{\sqrt{3}}\left(\frac{2}{1+x} - 1\right) + c_1$$

$$= -\sinh^{-1}\frac{1}{\sqrt{3}}\left(\frac{1-x}{1+x}\right) + c_1$$

And $I_2 = \int \frac{x\,dx}{\sqrt{1+x+x^2}} = \int \frac{\left\{\frac{1}{2}(2x+1) - 1/2\right\}dx}{\sqrt{1+x+x^2}}$

$$= \frac{1}{2}\int \frac{(2x+1)\,dx}{\sqrt{1+x+x^2}} - \frac{1}{2}\int \frac{dx}{\sqrt{(x+1/2)^2 + (\sqrt{3}/2)^2}}$$

$$= \frac{1}{2}.2\sqrt{1+x+x^2} - \frac{1}{2}\sinh^{-1}\frac{x+1/2}{\sqrt{3}/2} + c_2$$

$$= \sqrt{1+x+x^2} - \frac{1}{2}\sinh^{-1}\frac{2x+1}{\sqrt{3}} + c_2$$

কিন্তু আমাদের আছে [But we have] $I = I_1 + I_2$

বা $I = -\sinh^{-1}\frac{1}{\sqrt{3}}\left(\frac{1-x}{1+x}\right) + \sqrt{1+x+x^2} - \frac{1}{2}\sinh^{-1}\frac{2x+1}{\sqrt{3}} + c.$

নিয়ম–3 ঃ যদি কোন ইনটিগ্র্যাল $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$ এই আকারে থাকে, অর্থাৎ ইনটিগ্র্যালের হরে $\sqrt{\ }$ এর ভিতরে একঘাত রাশিমালা এবং $\sqrt{\ }$ এর বাহিরে দ্বিঘাত রাশিমালা থাকে, তবে একঘাত রাশিমালা $px + q = z^2$ ধরিয়া সরলীকরণ করার পর ইনটিগ্রেশন করিতে হয়।

[**Rule–3 :** If the integral is of the form $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$ that is, in the denominator, there is linear expression inside $\sqrt{\ }$ and there is quadratic expression outside $\sqrt{\ }$, then putting linear expression $px + q = z^2$. After simplification to be integrate it.]

Example–4 ঃ $\int \frac{(x+2)\,dx}{(x^2+3x+3)\sqrt{x+1}}$

Solution : ধরি [We put] $I = \int \frac{(x+2)\,dx}{(x^2+3x+3)\sqrt{x+1}}$ and

$x + 1 = z^2$, বা $x = z^2 - 1 \Rightarrow dx = 2z\,dz$

$$\therefore I = \int \frac{(z^2-1+2)\,2z\,dz}{\{(z^2-1)^2 + 3(z^2-1) + 3\}z}$$

$$= 2\int \frac{(z^2+1)\,dz}{z^4+z^2+1} = 2\int \frac{z^2(1+1/z^2)\,dz}{z^2(z^2+1+1/z^2)}$$

$$I = 2\int \frac{(1+1/z^2)\,dz}{z^2+1+1/z^2} = 2\int \frac{(1+1/z^2)\,dz}{(z-1/z)^2+3}$$

ধরি [We put] $z - 1/z = t$ তবে [then] $(1 + 1/z^2)\,dz = dt$

$$\therefore I = 2\int \frac{dt}{t^2+(\sqrt{3})^2} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + c$$

$$= \frac{2}{\sqrt{3}}\tan\frac{z-1/z}{\sqrt{3}} + c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{z^2-1}{\sqrt{3}\,z} + c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3x+3}} + c$$

নিয়ম– 4 ঃ যদি কোন ইনটিগ্র্যাল $\int \frac{\sqrt{ax+b}}{\sqrt{cx+d}}dx$ এই আকারে থাকে অর্থাৎ ইনটিগ্র্যালের লবে $\sqrt{\ }$ এর ভিতরে একঘাত রাশিমালা এবং হরে ও $\sqrt{\ }$ এর ভিতরে একঘাত রাশিমালা থাকে, তবে লবকে $\sqrt{\ }$ মুক্ত করিয়া সরলীকরণ করার পর ইনটিগ্রেশন করিতে হয়।

[**Rule–4 :** If the integral is of the form $\int \frac{\sqrt{ax+b}}{\sqrt{cx+d}}dx$ that is in the numerator, there is linear expression inside $\sqrt{\ }$ and in the denominator, there is linear expression inside $\sqrt{\ }$. Then at first to make the numerator $\sqrt{\ }$ free. After simplification to be integrate it.]

Example–5 ঃ [Evaluate] $\int \sqrt{\frac{x-1}{x+1}}dx$ এর মান নির্ণয় কর। [ঢাঃ বিঃ '86]

Solution : ধরি [We put] $I = \int \frac{\sqrt{x-1}}{\sqrt{x+1}}dx = \int \frac{\sqrt{x-1}\sqrt{x-1}}{\sqrt{x+1}\sqrt{x-1}}dx$

বা $I = \int \frac{(x-1)\,dx}{\sqrt{x^2-1}} = \int \frac{x\,dx}{\sqrt{x^2-1}} - \int \frac{dx}{\sqrt{x^2-1}}$

$= \frac{1}{2}\int \frac{2x\,dx}{\sqrt{x^2-1}} - \int \frac{dx}{\sqrt{x^2-1}}$

$= \frac{1}{2} . 2\sqrt{x^2-1} - \cosh^{-1}x + c$

$= \sqrt{x^2-1} - \cosh^{-1}x + c.$

নিয়ম–5 **:** যদি কোন ইনটিগ্র্যাল $\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$ এই আকারে থাকে, অর্থাৎ ইনটিগ্রালের হরে $\sqrt{\ }$ এর ভিতরে দ্বিঘাত রাশিমালা এবং $\sqrt{\ }$ এর বাহিরেও দ্বিঘাত রাশিমালা থাকে, তবে প্রথমে $x = 1/z$ ধরিয়া সরলীকরণ করার পর " $\sqrt{\ }$ এর ভিতরের রাশিমালা $= u^2$" ধরিতে হয় এবং পুনরায় সরলীকরণ করার পর ইনটিগ্রেশন করিতে হয়।

[**Rule–5 :** If the integral is of the form $\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$ that is, in the denominator, there are quadratic expressions inside and outside $\sqrt{\ }$. Then at first putting $x = 1/z$. After simplification, putting expression inside $\sqrt{\ } = u^2$. Again after simplification to be integrate it.]

Example–6 : [Evaluate] $\int \frac{dx}{(x^2+1)\sqrt{x^2+4}}$ এর মান নিণয় কর।

[ঢাঃ বিঃ সঃ '85]

Solution : ধরি [We put] $I = \int \frac{dx}{(x^2+1)\sqrt{x^2+4}}$

এবং [and] $x = \frac{1}{z}$ তবে [then] $dx = -\frac{dz}{z^2}$

$\therefore I = -\int \frac{dz}{z^2(1/z^2+1)\sqrt{1/z^2+4}} = -\int \frac{z\,dz}{(1+z^2)\sqrt{(1+4z^2)}}$

ধরি [We put] $1 + 4z^2 = u^2$, বা $4z^2 = u^2 - 1$

বা $z^2 = (u^2-1)/4 \Rightarrow 2z\,dz = (2u\,du)/4$

$\therefore I = -\frac{1}{4}\int \frac{u\,du}{\{1 + (u^2-1)/4\}u} = -\frac{1}{4}\int \frac{4\,du}{4 + u^2 - 1}$

$= -\int \frac{du}{(\sqrt{3})^2 + u^2} = -\frac{1}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + c$

$= -\frac{1}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}\sqrt{1+4z^2} + c = -\frac{1}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}\sqrt{1+4/x^2} + c$

$= -\frac{1}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}\sqrt{(x^2+4)/x^2} + c = -\frac{1}{\sqrt{3}}\tan^{-1}\frac{\sqrt{x^2+4}}{x\sqrt{3}} + c$

Example–7 : [Evaluate] $\int \frac{x\sqrt{a^2-x^2}}{a^2+x^2}\,dx$ এর মান নির্ণয় কর।

Solution : ধরি [We put] $I = \int \frac{x\sqrt{a^2-x^2}\,dx}{a^2+x^2}$

and $a^2 - x^2 = z^2$, বা $x^2 = a^2 - z^2 \Rightarrow 2x\,dx = -2z\,dz$

$\therefore I = -\int \frac{z \,.\, z\,dz}{a^2 + a^2 - z^2} = -\int \frac{z^2\,dz}{2a^2 - z^2}$

$= -\int \frac{-(2a^2-z^2) + 2a^2}{2a^2 - z^2}\,dz = \int dz - 2a^2\int \frac{dz}{(\sqrt{2}\,a)^2 - z^2}$

$= z - \frac{2a^2}{2\sqrt{2}a}\ln\frac{\sqrt{2}a + z}{\sqrt{2}a - z} + c$

$= \sqrt{a^2-x^2} - \frac{a}{\sqrt{2}}\ln\frac{a\sqrt{2} + \sqrt{a^2-x^2}}{a\sqrt{2} - \sqrt{a^2-x^2}} + c.$

প্রশ্নমালা [EXERCISE]–2(D)

নিম্নলিখিত ইনটিগ্র্যাল সমূহ নির্ণয় কর [Evaluate the following integral] **:**

(i). $\int \frac{dx}{(x+1)\sqrt{x+3}}$ (ii). $\int \frac{dx}{(x+3\sqrt{x-4})}$

(iii). $\int \frac{dx}{(1-x)\sqrt{x+8}}$ (iv). $\int \frac{dx}{x\sqrt{x+9}}$

(v). $\int \frac{dx}{(1+x)\sqrt{x}}$ (vi). $\int \frac{dx}{(3x+7)\sqrt{x-2}}$

(vii). $\int \frac{dx}{(2x+3)\sqrt{x+5}}$ (viii). $\int \frac{dx}{(2x+1)\sqrt{4x+3}}$

(ix). $\int \frac{dx}{(x-2)\sqrt{x+1}}$

(xi). $\int \frac{dx}{(2x+1)\sqrt{x+5}}$

2(i). $\int \frac{dx}{(1+x)\sqrt{1-x^2}}$

(iii). $\int \frac{dx}{(x-1)\sqrt{2x^2-8x-1}}$

(v). $\int \frac{dx}{(x-1)\sqrt{x^2+2x+2}}$

(vii). $\int \frac{dx}{(x-2)\sqrt{4x^2-5x+3}}$

(ix). $\int \frac{dx}{(3x+1)\sqrt{9x^2+3x-1}}$

(xi). $\int \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

(xiii). $\int \frac{dx}{x\sqrt{x^2+2x-1}}$

3(i). $\int \frac{\sqrt{a^2-x^2}\,dx}{x}$

(iii). $\int \frac{\sqrt{x^2-2}\,dx}{x}$

(v). $\int \frac{x\,dx}{(x+1)\sqrt{x^2+1}}$

(ii). $\int \frac{dx}{(x^2-4)\sqrt{x+1}}$

(iv). $\int \frac{(x^2+1)\,dx}{(x^2+2x+2)\sqrt{x+1}}$

(ii). $\int \sqrt{\frac{x-a}{x+a}}\,dx$

(iv). $\int \sqrt{\frac{2x+1}{x+2}}\,dx$

(x). $\int \frac{dx}{(2+x)\sqrt{1+x}}$

(xii). $\int \frac{(x+1)\,dx}{x\sqrt{x+2}}$

(ii). $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

(iv). $\int \frac{dx}{x\sqrt{2+x-x^2}}$

(vi). $\int \frac{dx}{x\sqrt{1+x^2}}$

(viii). $\int \frac{dx}{x\sqrt{x^2+a^2}}$

(x). $\int \frac{dx}{(x+3)\sqrt{x^2+1}}$

(xii). $\int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$

(xiv). $\int \frac{dx}{(x-3)\sqrt{x^2-6x+8}}$

(ii). $\int \frac{\sqrt{x^2+a^2}\,dx}{x}$

(iv). $\int \frac{2x\,dx}{(1-x^2)\sqrt{x^4-1}}$

4(i). $\int \frac{dx}{x^2\sqrt{x+1}}$

(iii). $\int \frac{x\,dx}{(x^2+2x+2)\sqrt{x+1}}$

5(i). $\int \sqrt{\frac{x+2}{x-1}}\,dx$

(iii). $\int \sqrt{\frac{1+x}{1-x}}\,dx$

(v). $\int \sqrt{\frac{x+2}{x+5}}\,dx$

(vi). $\int \sqrt{\frac{2x+1}{3x+2}}\,dx$

(viii). $\int \sqrt{\frac{a+x}{x}}\,dx$

(x). $\int \frac{x\sqrt{1+x}}{\sqrt{1-x}}\,dx$ [W.B.H.S. '90]

(ii). $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ [W.B.H.S. '90]

(iv). $\int \frac{dx}{(4+x^2)\sqrt{x^2+2}}$

(vi). $\int \frac{dx}{(x^2+1)\sqrt{2x^2-1}}$

(ii). $\int \frac{(x+1)\,dx}{(x^2+4)\sqrt{x^2+9}}$

(vi). $\int \frac{(x^2+x+1)\,dx}{(x+2)\sqrt{x-1}}$

(vii). [illegible]

(ix). [illegible]

(xi). [illegible]

(ii). $\int \frac{dx}{x^2\sqrt{1-x^2}}$

(iv). $\int \frac{dx}{(2x^2+3)\sqrt{x^2-4}}$

7(i). $\int \frac{2x\,dx}{(1-x^2)\sqrt{x^2-1}}$

(iii). $\int \frac{\sqrt{1+x^2}}{1-x^2}\,dx$

(v). $\int \frac{x^2\,dx}{(x+1)\sqrt{x-2}}$

উত্তরমালা [ANSWERS]

1(i). $\frac{1}{\sqrt{2}} \ln \frac{\sqrt{x+3}-\sqrt{2}}{\sqrt{x+3}+\sqrt{2}}+c$

(ii). $\frac{2}{\sqrt{7}} \cdot \tan^{-1}\sqrt{(x-4)/7}$

(iii). $\frac{1}{3} \ln \frac{3+\sqrt{x+8}}{3-\sqrt{x+8}}$

(iv). $\frac{1}{3} \ln \frac{\sqrt{x+9}-3}{\sqrt{x+9}+3}$

(v). $2\tan^{-1} x$

(vi). $\frac{2}{\sqrt{39}} \tan^{-1} \sqrt{\frac{3x-6}{13}}$

(vii). $\frac{1}{2\sqrt{14}} \ln \frac{\sqrt{x+5}-\sqrt{7}}{\sqrt{x+5}+\sqrt{7}}$

(viii). $\frac{1}{2} \ln \frac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1}$

(ix). $\frac{1}{2} \ln \frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}$

(x). $2\tan^{-1}\sqrt{1+x}$

(xi). $\frac{1}{3\sqrt{2}} \ln \frac{\sqrt{2x+10}-3}{\sqrt{2x+10}+3}$,

(xii). $\frac{1}{\sqrt{2}} \ln \frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$

2(i). $-\sqrt{\dfrac{1-x}{1+x}}$

(ii). $\cos^{-1}\left(\dfrac{1}{1+x}\right)$

(iii). $-\dfrac{1}{\sqrt{7}}\sin^{-1}\dfrac{5+2x}{3\sqrt{2}.\ (x-1)}$

(iv). $-\dfrac{1}{\sqrt{2}}\cosh^{-1}\dfrac{4+x}{3x}$

(v). $-\dfrac{1}{\sqrt{5}}\sinh^{-1}\dfrac{2x+3}{x-1}$

(vi). $-\sinh^{-1}(1/x)$

(vii). $-\dfrac{1}{3}\sinh^{-1}\left\{\dfrac{11x-4}{(x-2)\sqrt{23}}\right\}$

(viii). $-\dfrac{1}{a}\sinh^{-1}(a/x)$

(ix). $-\dfrac{1}{3}\sinh^{-1}\left\{\dfrac{3(x+1)}{(1+3x)\sqrt{5}}\right\}$

(x). $-\dfrac{1}{\sqrt{10}}\sinh^{-1}\dfrac{1-3x}{x+3}$

(xi). $\dfrac{1}{\sqrt{2}}\sin^{-1}\dfrac{x\sqrt{2}}{1+x}$

(xii). $\sin^{-1}\dfrac{3x+1}{(1+x)\sqrt{5}}$

(xiii). $\sin^{-1}\dfrac{x-1}{x\sqrt{2}}$

(xiv). $\sec^{-1}(x-3)$

3(i). $\sqrt{a^2-x^2}-a\cosh^{-1}(a/x)$

(ii). $\sqrt{x^2+a^2}-a\sinh^{-1}(a/x)$

(iii). $\sqrt{x^2-2}+\sqrt{2}\sin^{-1}(\sqrt{2}/x)$

(iv). $-\sqrt{(x^2+1)/(x^2-1)}$

(v). $\sinh^{-1}x+\dfrac{1}{\sqrt{2}}\sinh^{-1}\dfrac{1-x}{1+x}$

4(i). $-\dfrac{\sqrt{x+1}}{x}+\dfrac{1}{2}\ln\dfrac{\sqrt{x+1}+1}{\sqrt{x+1}-1}$

(ii). $\dfrac{1}{4\sqrt{3}}\ln\dfrac{\sqrt{x+1}-\sqrt{3}}{\sqrt{x+1}+\sqrt{3}}-\dfrac{1}{2}\tan^{-1}\sqrt{x+1}$

(iii). $\dfrac{1}{\sqrt{2}}\ln\dfrac{x+2-\sqrt{2(x+1)}}{x+2+\sqrt{2(x+1)}}$

(iv). $y-\dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{y^2-1}{y\sqrt{2}}-\dfrac{3}{4\sqrt{2}}\ln\dfrac{y^2+1-y\sqrt{2}}{y^2+1+y\sqrt{2}}$, যখন $y^2=x+1$

5(i). $\sqrt{x^2+x-2}+(3/2)\cosh^{-1}\{(2x+1)/3\}$

(ii). $\sqrt{x^2-a^2}-a\cosh^{-1}(x/a)$

(iii). $\sin^{-1}x-\sqrt{1-x^2}$

(iv). $\sqrt{2x^2+5x+2}-(3/2\sqrt{2})\cosh^{-1}\{(4x+5)/3\}$

(v). $\sqrt{x^2+7x+10}-(3/2)\cosh^{-1}\{(2x+7)/3\}$

$(1/3)\sqrt{6x^2+7x+2}-(1/6\sqrt{6})\cosh^{-1}(12x+7)$

$(1/3)\sqrt{3x^2-13x+12}-(7/3\sqrt{3})\cosh^{-1}\{(6x-13)/5\}$

$\sqrt{x^2+ax}+(a/2)\,.\cosh^{-1}\{(2x+a)/a\}$

$a\sin^{-1}(x/a)-\sqrt{a^2-x^2}$

$(1/2)(x+2)\sqrt{1-x^2}+(1/2)\sin^{-1}x$

$\cosh^{-1}x-\sec^{-1}x$

$-\dfrac{\sqrt{1-x^2}}{x}$

6(i). $-\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{\sqrt{1-x^2}}{x\sqrt{2}}$

(iii). $\dfrac{1}{4\sqrt{2}}\ln\dfrac{\sqrt{2x^2+4}-x}{\sqrt{2x^2+4}+x}$

$\dfrac{1}{2\sqrt{65}}.\ \ln\dfrac{x\sqrt{13}+\sqrt{5x^2-20}}{x\sqrt{13}-\sqrt{5x^2-20}}$

$\dfrac{1}{2\sqrt{3}}\ln\dfrac{x\sqrt{3}+\sqrt{2x^2-1}}{x\sqrt{3}-\sqrt{2x^2-1}}$

$\dfrac{2}{\sqrt{x^2-1}}$

$\dfrac{1}{2\sqrt{5}}\left[\ln\dfrac{\sqrt{x^2+9}-\sqrt{5}}{\sqrt{x^2+9}+\sqrt{5}}-\tan^{-1}\dfrac{2\sqrt{x^2+9}}{x\sqrt{5}}\right]$

$\sinh^{-1}x-\dfrac{1}{\sqrt{2}}\ln\dfrac{\sqrt{1+x^2}-x\sqrt{2}}{\sqrt{1+x^2}+x\sqrt{2}}$

$(2/3)(x-1)^{3/2}+2\sqrt{3}\tan^{-1}\sqrt{(x-1)/3}$

$2\left[\dfrac{1}{5}(x-5)^{5/2}+(x-2)^{3/2}+3\sqrt{x-2}-\dfrac{1}{\sqrt{3}}\tan^{-1}\sqrt{(x-2)/3}\right]$