2

Equations of First Order and First Degree

2·1. Differential equation of the first order and first degree. A differential equation of the type

$$M+N\frac{dy}{dx}=0,$$

where M and N are functions of x and y or constants, is called *a differential equation of the first order and first degree.*

We give below some methods of solving such equations.

2·2. Solution of the differential equation when variables are separable.

If an equation can be written in such a way that dx and all the terms containing x are on one side and dy and all the terms containing y on the other side, then this is an equation in which variables are separable. Such equations can therefore be written as $f_1(x)\,dx=f_2(y)\,dy$ and can be solved by integrating directly and adding a constant on either side.

Ex. 1. *Solve* $\dfrac{dy}{dx}=\dfrac{1+y^2}{1+x^2}$.

Solution Separating the variables the equation becomes

$$\frac{dy}{1+y^2}=\frac{dx}{1+x^2}$$

Integrating, we get $\tan^{-1} y=\tan^{-1} x+A$

or $\tan^{-1} y-\tan^{-1} x=A$ *i.e.*, $\tan^{-1}\dfrac{y-x}{1+xy}=A=\tan^{-1} C$ (say).

$\therefore\ y-x=C(1+xy)$

which is the solution.

Ex. 2. *Solve* $\dfrac{dy}{dx}=e^{x-y}+x^2e^{-y}$.

[Gorakhpur 59 ; Andhra 60 ; Sagar 54]

Solution. The given equation can be written as

$$e^y\,dy=(e^x+x^2)\,dx.$$

Integrating, $\quad e^y=e^x+\frac{1}{3}x^3+C.$

Ex. 3. *Solve* $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy=0$

[Nagpur T.D.C. 61 ; Delhi 51]

Solution. Separating the variables, we get

$$\frac{\sec^2 x}{\tan x}dx+\frac{\sec^2 y}{\tan y}dy=0.$$

Integrating, $\log \tan x+\log \tan y=A$

or $\tan x \tan y=e^A=C.$

Ex. 4. *Solve* $(y-px)\,x=y.$ **[Saugar 62]**

Solution. Equation is $px^2=y\,(x-1)$, *i.e.*, $\dfrac{dy}{dx}=\dfrac{y\,(x-1)}{x^2}$,

i.e., $$\frac{dy}{y}=\frac{x-1}{x^2}dx=\left(\frac{1}{x}-\frac{1}{x^2}\right)dx.$$

Integrating, $\log y=\log x+\dfrac{1}{x}+\log A$ or $\dfrac{y}{x}=Ae^{1/x}.$

Ex. 5. *Solve* $y-x\dfrac{dy}{dx}=a\left(y^2+\dfrac{dy}{dx}\right).$ **[Saugar 63]**

Solution. The equation can be written as

$$\frac{dx}{x+a}=\frac{dy}{y\,(1-ay)}=\left(\frac{1}{y}+\frac{a}{1-ay}\right)dy.$$

Integrating, $x+a=C\dfrac{y}{1-ay}.$

Ex. 6. *Solve*

(i) $(3+2\sin x+\cos x)\,dy=(1+2\sin y+\cos y)\,dx.$

(ii) $(e^y+1)\cos x\,dx+e^y\sin x\,dx=0.$ **[Poona 64]**

2·3. Equations reducible to the form in which variables are separable.

Equations of the form

$$\frac{dy}{dx}=f(ax+cy+c)$$

can be reduced to an equation in which variables can be separated What is required is that we put

$$ax+by+c=v,$$

so that $a+b\dfrac{dy}{dx}=\dfrac{dv}{dx}$, *i.e.*, $\dfrac{dy}{dx}=\dfrac{1}{b}\left[\dfrac{dv}{dx}-a\right).$

Then the equation becomes

$$\frac{1}{b}\left(\frac{dv}{dx}-a\right)=f(v) \text{ or } \frac{dv}{dx}=a+bf(v),$$

in which variables are separable.

Ex. 1. *Solve* $\dfrac{dy}{dx}=(4x+y+1)^2.$

[Raj. 61 ; Agra 54 ; Gujrat 65, 58]

Solution. Put $4x+y+1=v$, so that $4+\dfrac{dy}{dx}=\dfrac{dv}{dx}.$

The equation then reduces to

$$\frac{dv}{dx}-4=v^2 \quad \text{or} \quad \frac{dv}{dx}=v^2+4.$$

The variables are now separable and we can write $\frac{dv}{v^2+4}=dx.$

Integrating $\frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right)=x+C$

or $\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right)=x+C$ is the solution.

* **Ex. 2.** *Solve* $\frac{dy}{dx}=\sin(x+y)+\cos(x+y).$ [Agra B Sc. 67]

Solution. Put $x+y=v$, $1+\frac{dy}{dx}=\frac{dv}{dx}.$

$\therefore$ equation is $\frac{dv}{dx}-1=\sin v+\cos v$ or $\frac{dv}{dx}=1+\sin v+\cos v$

or $dx=\frac{dv}{1+\sin v+\cos v}=\frac{dv}{2\cos^2\frac{1}{2}v+2\sin\frac{1}{2}v\cos\frac{1}{2}v}$

or $\frac{dv}{2\cos^2\frac{1}{2}v\,(1+\tan\frac{1}{2}v)}=dx$ or $\frac{\frac{1}{2}\sec^2\frac{1}{2}v\,dv}{1+\tan\frac{1}{2}v}=dx.$

Integrating, $\log(1+\tan\frac{1}{2}v)=x+C$, where $v=x+y.$

$\therefore$ $\log[1+\tan\frac{1}{2}(x+y)]=x+C$ is the required solution.

Ex. 3. *Solve* $(x-y)^2\frac{dy}{dx}=a^2.$

[Calcutta Hons. 63; Bihar 61; Vikram 65]

Solution. Put $x-y=v$, so that $1-\frac{dy}{dx}=\frac{dv}{dx}.$

$\therefore$ equation is $v^2\left[1-\frac{dv}{dx}\right]=a^2$ or $\frac{dv}{dx}=\frac{v^2-a^2}{v^2}$

or $dx=\frac{v^2}{v^2-a^2}dv=\left(1+\frac{a^2}{v^2-a^2}\right)dv.$

Integrating, $x+C=v+a^2\frac{1}{2a}\log\frac{v-a}{v+a}$

or $x+C=(x-y)+\frac{1}{2}a\log\frac{x-y-a}{x-y+a}$ is the solution.

Ex 4. *Solve* $(x+y)^2\frac{dy}{dx}=a^2$

[Poona 64; Raj. 63; Delhi Hons. 60; Alld 60]

Solution. Put $x+y=v$, so that $1+\frac{dy}{dx}=\frac{dv}{dx}.$

$\therefore$ $v^2\left(\frac{dv}{dx}-1\right)=a^2$, $\frac{dv}{dx}=1+\frac{a^2}{v^2}=\frac{a^2+v^2}{v^2}$

$$\therefore \quad dx = \frac{v^2}{a^2+v^2}\,dv = \left(1 - \frac{a^2}{a^2+v^2}\right) dv.$$

Integrating, $x + C = v - a \tan^{-1} \dfrac{v}{a}$

or $x + C = (x+y) - a \tan^{-1} \dfrac{x+y}{a}$

or $y = C + a \tan^{-1} \dfrac{x+y}{a}$ is the solution.

***Ex. 5.** *Solve* $\dfrac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\left(\dfrac{a^2 - x^2 - y^2}{x^2+y^2}\right)}$.

[Delhi Hons. 62; Agra B.Sc. 55]

Solution. Here we change to polar co-ordinates by putting

$x = r\cos\theta,\ y = r\sin\theta,\ x^2+y^2 = r^2,\ x\,dx + y\,dy = r\,dr.$

$\dfrac{y}{x} = \tan\theta, \quad \therefore \quad \dfrac{x\,dy - y\,dx}{x^2} = \sec^2\theta\,d\theta$ or $x\,dy - y\,dx = r^2\,d\theta.$

$\therefore$ the equation becomes $\dfrac{1}{r}\dfrac{dr}{d\theta} = \sqrt{\left(\dfrac{a^2 - r^2}{r^2}\right)}$

Separating the variables, $\dfrac{dr}{\sqrt{(a^2 - r^2)}} = d\theta.$

Integrating, $\sin^{-1}(r/a) = \theta + C$ or $r = a\sin(\theta + C)$,

i.e., $\sqrt{(x^2+y^2)} = a\sin[\tan^{-1}(y/x) + C]$.

Ex. 6. *Solve* $x\dfrac{dy}{dx} - y = x\sqrt{(x^2+y^2)}$. **[Bombay 61; Agra 56]**

Solution. The equation can be put as

$x\,dy - y\,dx = x\sqrt{(x^2+y^2)}\,dx$ or $\dfrac{x\,dy - y\,dx}{x^2} = \sec^2\theta\,d\theta.$

Changing to polars as above, the equation becomes

$x^2\sec^2\theta\,d\theta = xr\,dx$

or $x\sec^2\theta\,d\theta = r\,dx$ or $r\cos\theta\sec^2\theta\,d\theta = r\,dx$

or $\sec\theta\,d\theta = dx$, variables separated.

Integrating, $\log(\sec\theta + \tan\theta) = x + \log C$.

$\therefore$ $\sec\theta + \tan\theta = ce^x$ or $\sqrt{(1 + y^2/x^2)} + y/x = ce^x$.

Ex. 7. *Solve* $\left(\dfrac{x+y-a}{x+y-b}\right)\dfrac{dy}{dx} = \left(\dfrac{x+y+a}{x+y+b}\right)$

[Delhi Hons. 63; Nagpur 55]

Solution. Put $x + y = v$, so that $1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$.

i.e., $\dfrac{dv}{dx} = 1 + \left(\dfrac{v+a}{v-b}\right)\left(\dfrac{v-b}{v-a}\right) = \dfrac{2(v^2 - ab)}{v^2 + (b-a)v - ab}$

or $2\,dx = \left(1 + \dfrac{b-a}{2}\,\dfrac{2v}{v^2 - ab}\right) dv.$

Integrating, $2x+C=v+\frac{b-a}{2}\log(v^2-ab)$

or $2x+C=x+y+\frac{1}{2}(b-a)\log[(x+y)^2-ab]$ etc.

Ex. 8. $\frac{dy}{dx}=(x+y)^2$. [Gauhati 62; Delhi 62; Raj. 62]

Hint. Put $x+y=v$ etc.

2.4 Homogeneous Differential Equations. [Poona 61 (S)]

An equation of the form $\frac{dy}{dx}=\frac{f_1(x, y)}{f_2(x, y)}$ in which $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions* of x and y of the same degree can be reduced to an equation in which variables are separable by putting $y=vx$, $\frac{dy}{dx}=v+x\frac{dv}{dx}$.

The following few examples will illustrate the method.

Ex. 1 *Solve* $(x^2+y^2)\,dx+2xy\,dy=0$.

Solution. We have $\frac{dy}{dx}=-\frac{x^2+y^2}{2xy}$ (homogeneous).

Putting $y=vx$, $\frac{dy}{dx}=v+x\frac{dv}{dx}$, the equation becomes

$$v+x\frac{dv}{dx}=-\frac{x^2+x^2v^2}{2x.vx}=-\frac{1+v^2}{2v}$$

or $$x\frac{dv}{dx}=-\frac{1+v^2}{2v}-v=-\frac{1+3v^2}{2v}$$ (variable separable).

$$\therefore \quad \frac{dx}{x}=-\frac{2v}{1+3v^2}\,dv.$$

Integrating, $\log x+\frac{1}{3}\log(1+3v^2)=\log C$

or $x(1+3v^2)^{1/3}=C$ or $x(1+3y^2/x^2)^{1/3}=C$.

Ex. 2. *Solve* $x^2y\,dx-(x^3+y^3)\,dy=0$. [Agra B Sc. 54]

Solution. We have $\frac{dy}{dx}=\frac{x^2y}{x^3+y^3}$ (homogeneous).

Putting $y=vx$, $\frac{dy}{dx}=v+x\frac{dv}{dx}$, the equation becomes

$$v+x\frac{dv}{dx}=\frac{v}{1+v^3} \text{ or } x\frac{dv}{dx}=\frac{v}{1+v^3}-v=-\frac{v^4}{1+v^3}$$

or $$\frac{dx}{x}=-\frac{1+v^3}{v^4}\,dv=-\left[\frac{1}{v^4}+\frac{1}{v}\right]dv.$$

Integrating, $\log x=\frac{1}{3v^3}-\log v+C$; $\log vx=\frac{1}{3v^3}+C$

*A function $f(x, y)$ is called homogeneous of degree n, if $f(tx, ty)=t^nf(x, y)$.

or $\log y = \frac{x^3}{3y^3} + C$ as $v = \frac{y}{x}$.

Ex. 3. *Solve* $\frac{dy}{dx} = \frac{y^3 + 3x^2y}{x^3 + 3xy^2}$. [Lucknow Pass 60]

Solution. Putting $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$x\frac{dv}{dx} = \frac{dy}{dx} - v = \frac{v^3 + 3v}{1 + 3v^2} - v = \frac{2v(1 - v^2)}{1 + 3v^2}$$

or $$\frac{2\,dx}{x} = \frac{1 + 3v^2}{2v(1 - v^2)}\,dv = \left(\frac{1}{v} - \frac{2}{1+v} + \frac{2}{1-v}\right)dv.$$

Integrating,

$$2\log x = \log v - 2\log(1 - v) - 2\log(1 + v) + \log C$$

or $x^2(1 - v)^2(1 + v)^2 = Cv$. Put $v = y/x$ etc.

Ex. 4. *Solve* $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

[Delhi Hons. 66; Cal. Hons. 61, 56; Osmania 60; Gujrat 61]

Solution. The equation is $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ [homogeneous].

Putting $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = \frac{v^2}{v - 1} \quad \text{or} \quad x\frac{dv}{dx} = \frac{v^2}{v - 1} - v$$

or $x\frac{dv}{dx} = \frac{v}{v - 1}$ or $\frac{dx}{x} = \frac{v - 1}{v}\,dv$

or $\frac{dx}{x} = \left(1 - \frac{1}{v}\right)dv$.

Integrating, $\log x = v - \log v + \log c$

or $\log xv = v + \log c$ or $xv = ce^v$

or $y = ce^{y/x}$ as $y = vx$.

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Ex. 5. *Solve* $(x^2 + y^2)\,dy = xy\,dx$. [Nagpur T.D.C. 1961]

Hint. Homogeneous. Put $y = vx$. Ans. $y = Ce^{x^2/2y^2}$

Ex. 6. *Solve the following homogeneous equations:*

(i) $y(y^2 - 2x^2)\,dx + x(2y^2 - x^2)\,dy = 0$.

[Karnatak B.Sc. (Sub.) 1960]

(ii) $\frac{1}{2x}\frac{dy}{dx} + \frac{x + y}{x^2 + y^2} = 0$. [Lucknow Pass 1955]

(iii) $\frac{dy}{dx} + \frac{y(x + y)}{x^2} = 0$. Ans. $x^2y = c^2(y + 2x)$

[Poona 1964; Nag. 58; Kerala 61; Vikram 61]

(iv) $x^2y\,dx - x^3\,dy = y^3\,dy$. Ans. $\log y = \frac{x^2}{3y^2} + C$.

(v) $(x^2-y^2)\dfrac{dy}{dx}=xy.$

(vi) $(x+y)^2=xy\dfrac{dy}{dx}.$ [Poona 1964]

(vii) $x\dfrac{dy}{dx}-y=\sqrt{(x^2+y^2)}.$ [Sagar 1963; Cal. Hons. 62; Raj. 56]

(Cf. Ex. 6 P. 10) Ans. $x^2+y^2=(Cx^2-y)^2.$

Ex. 7. $\left(x\cos\dfrac{y}{x}+y\sin\dfrac{y}{x}\right)y=\left(y\sin\dfrac{y}{x}-x\cos\dfrac{y}{x}\right)x\dfrac{dy}{dx}$

[Cal. Hons 1962]

or $x\cos\dfrac{y}{x}(y\,dx+x\,dy)=y\sin\dfrac{y}{x}(x\,dy-y\,dx).$

[Raj. 1959; Cal. Hons. 61, 55; Delhi 68, 61]

Solution. The equation is $\dfrac{dy}{dx}=\dfrac{y\,(y\sin y/x+x\cos y/x)}{x\,(y\sin y/x-x\cos y/x)}.$

Putting $y=vx$, $x\dfrac{dv}{dx}=\dfrac{dy}{dx}-v=\dfrac{2v\cos v}{v\sin v-\cos v}$

or $\left(\tan v-\dfrac{1}{v}\right)dv=2\dfrac{dx}{x}$, i.e., $\log\dfrac{\sec v}{v}=\log C+2\log x$

or $\sec(y/x)=Cxy$ is the solution.

Ex. 8. *Solve* $\left(x\sin\dfrac{y}{x}\right)\dfrac{dy}{dx}=\left(y\sin\dfrac{y}{x}-x\right)$ [Delhi Pass 67]

Solution. Equation is $\dfrac{dy}{dx}=\dfrac{y}{x}-\operatorname{cosec}\dfrac{y}{x}.$

Putting $y=vx$. $\dfrac{dy}{dx}=x\dfrac{dv}{dx}+v.$

Equation reduces to $\sin v\;dv=-\dfrac{dx}{x}.$

Integrating, $-\cos v=-\log Cx$

or $\cos\dfrac{y}{x}=\log Cx$ is the solution.

Ex. 9. *Solve* $(x^2+2xy-y^2)\,dx+(y^2+2xy-x^2)\,dy=0.$

[Gujrat B.Sc. (Prin.) 1961]

Solution. $\dfrac{dy}{dx}=-\dfrac{x^2+2xy-y^2}{y^2+2xy-x^2}.$ Put $y=vx$.

$\therefore\quad v+x\dfrac{dv}{dx}=-\dfrac{1+2v-v^2}{v^2+2v-1}.$

$$x\frac{dv}{dx}=-\frac{1+2v-v^2}{v^2+2v-1}-v=-\frac{v^3+v^2+v+1}{v^2+2v-1}.$$

$$\therefore\quad \frac{dx}{x}=-\frac{v^2+2v-1}{v^3+v^2+v+1}\,dv=-\frac{v^2+2v-1}{(v+1)(v^2+1)}\,dv$$

$$=\left(\frac{1}{v+1}-\frac{2v}{v^2+1}\right)dv.$$

Integrating, $\log x=\log(v+1)-\log(v^2+1)+\log C$

or $\dfrac{x}{v^2+1}=C(v+1)$ or $\dfrac{x}{y^2/x^2+1}=C\left(\dfrac{y}{x}+1\right)$

Ex. 10. *Solve* $2y^3\,dx+(x^2-3y^2)\,x\,dy=0.$

[Bombay B.Sc. (Sub.) 1962]

Solution. Proceed yourself.

2·5. Equation Reducible to Homogeneous Form.

An equation of the type $\dfrac{dy}{dx}=\dfrac{ax+by+c}{a'x+b'y+c'}$, when $\dfrac{a}{a'}\neq\dfrac{b}{b'}$ can be reduced to homogeneous form as follows :

Put $x=X+h$, $y=Y+k$; then $\dfrac{dy}{dx}=\dfrac{dY}{dX}$, where X, Y are new variables and h, k are arbitrary constants. The equation now becomes

$$\frac{dY}{dX}=\frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}.$$

We choose the constants h and k in such a way that

$$ah+bk+c=0,\quad a'h+b'k+c'=0.$$

With this substitution the differential equation reduces to $\dfrac{dY}{dX}=\dfrac{aX+bY}{a'X+b'Y}$ which is a homogeneous equation in X, Y and can be solved by putting $Y=vX$ as earlier.

Special Case. When $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{1}{m}$ (say), then the differential equation can be written as

$$\frac{dy}{dx}=\frac{ax+by+c}{m\,(ax+by)+c}.$$

Put $ax+by=v$, so that $a+b\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

(1) then becomes $\dfrac{1}{b}\left(\dfrac{dv}{dx}-a\right)=\dfrac{v+c}{mv+c}$ in which variables can be separated.

Ex. 1. *Solve* $\dfrac{dy}{dx}=\dfrac{x+2y-3}{2x+y-3}$.

[Vikram 60]

Solution. Put $x=X+h$, $y=Y+k$, where h, k are some constants; then $\dfrac{dy}{dx}=\dfrac{dY}{dX}$. The given equation then becomes

$$\frac{dY}{dX}=\frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)}.$$

Now choose h, k such that $h+2k-3=0$ and $2h+k-3=0$. Solving these we get $h=1, k=1$.

$$\therefore \quad \frac{dY}{dX}=\frac{X+2Y}{2X+Y} \text{ homogeneous in } X \text{ and } Y.$$

Put $Y=vX$, so that $\dfrac{dY}{dX}=v+X\dfrac{dv}{dX}$.

$$\therefore \quad v+X\frac{dv}{dX}=\frac{X+2vX}{2X+vX}=\frac{1+2v}{2+v}, \quad i.e., \quad X\frac{dv}{dX}=\frac{1+2v}{2+v}-v$$

or $$\frac{dX}{X}=\frac{2+v}{1-v^2}\,dv=\left(\frac{1}{1-v^2}+\frac{v}{1-v^2}\right)dv.$$

Integrating, $\log X=2.\frac{1}{2}\log\dfrac{1+v}{1-v}-\frac{1}{2}\log(1-v^2)+\log C$

or $$X=C\,\frac{1+v}{1-v}\cdot\frac{1}{\sqrt{(1-v^2)}}=\frac{C\sqrt{(1+v)}}{(1-v)^{3/2}}$$

or $$X^2(1-v)^3=C^2(1+v)$$

or $$X^2\left(1-\frac{Y}{X}\right)^3=C^2\left(1+\frac{Y}{X}\right) \text{ as } v=\frac{Y}{X}$$

or $$(X-Y)^3=C^2(X+Y) \text{ but } x=X+1,\ y=Y+1.$$

$\therefore \quad (x-y)^3=C^2(x+y-2)$ is the required solution.

Ex. 2. *Solve* $(3x-7y-3)\dfrac{dy}{dx}=3y-7x+7.$ **[Raj. M.Sc. 61]**

Solution. $\dfrac{dy}{dx}=\dfrac{3y-7x+7}{3x-7y-3}.$

Put $x=X+h$, $y=Y+k$, where h, k are some constants. Then $\dfrac{dy}{dx}=\dfrac{dY}{dX}$. And the given equation becomes

$$\frac{dY}{dX}=\frac{3Y-7X+(3k-7h+7)}{3X-7Y+(3h-7k-3)}.$$

Choose h, k such that $3h-7k-3=0$ and $3k-7h+7=0$, which give $h=1$, $k=0$.

$$\therefore \quad \frac{dY}{dX}=\frac{3Y-7X}{3X-7Y} \text{ [homogeneous].}$$

Put $Y=vX$, $\dfrac{dY}{dX}=v+X\dfrac{dv}{dX}$.

$$\therefore \quad v+X\frac{dv}{dX}=\frac{3vX-7X}{3X-7vX}=\frac{3v-7}{3-7v}$$

or $$X\frac{dv}{dX}=\frac{3v-7}{3-7v}-v=\frac{7(v^2-1)}{3-7v}$$

or $$7\frac{dX}{X}=\frac{3-7v}{(v^2-1)}\,dv=-\left(\frac{2}{v-1}+\frac{5}{v+1}\right)dv.$$

Integrating, $7 \log X = -2 \log (v-1) - 5 \log (v+1) + \log C$

or $\quad X^7 (v-1)^2 (v+1)^5 = C$

or $\quad X^7 \left(\dfrac{Y}{X}-1\right)^2 \left(\dfrac{Y}{X}+1\right)^5 = C$ as $Y = vX$

or $\quad (Y-X)^2 (Y+X)^5 = C$

or $\quad (y-x+1)^2 (y+x-1)^5 = C$ as $x = X+1,\ y = Y+0.$

Ex. 3. *Solve* $(2x+y+3)\dfrac{dy}{dx} = x+2y+3.$

[Karnatak B.Sc. (Princ.) 60]

Solution. $\dfrac{dy}{dx} = \dfrac{x+2y+3}{2x+y+3}.$

Put $x = X+h,\ y = Y+k$, where h, k are constants.

$$dx = dX, \quad dy = dY; \quad \therefore \quad \frac{dy}{dx} = \frac{dY}{dX}$$

$$\therefore \quad \frac{dY}{dX} = \frac{X+2Y+(h+2k+3)}{2X+Y+(2h+k+3)}.$$

Choose h, k such that $h+2k+3=0,\ 2h+k+3=0$. Solving these, we get $h=-1,\ k=-1$.

$$\therefore \quad \frac{dY}{dX} = \frac{X+2Y}{2X+Y}. \quad \text{Put } Y = vX, \ \frac{dY}{dX} = v + X\frac{dv}{dX}.$$

$$\therefore \quad v + X\frac{dv}{dX} = \frac{X+2vX}{2X+vX} \quad \text{or} \quad X\frac{dv}{dX} = \frac{1+2v}{2+v} - v$$

or $\quad \dfrac{dX}{X} = \dfrac{2+v}{1-v^2}\,dv = \left(\dfrac{\frac{3}{2}}{1-v} + \dfrac{\frac{1}{2}}{1+v}\right) dv.$

Integrating, $2 \log X = -3 \log (1-v) + \log (1+v) + \log C$

or $\quad X^2 \dfrac{(1-v)^3}{1+v} = C$ or $X^2 \dfrac{(1-Y/X)^3}{(1+Y/X)} = C$

or $\quad (X-Y)^3 = C(X+Y)$; where $x = X-1,\ y = Y-1$

or $\quad (X-y)^3 = C(x+y-2)$ is the solution.

Ex. 4. *Solve* $(2x-2y+5)\dfrac{dy}{dx} = x-y+3.$

[Sagar 63; Agra B.Sc. 61, 52]

Solution. The equation is $\dfrac{dy}{dx} = \dfrac{x-y+3}{2(x-y)+5}.$

Put $x-y=v$, so that $1-\dfrac{dy}{dx} = \dfrac{dv}{dx}$ or $\dfrac{dy}{dx} = 1 - \dfrac{dv}{dx}.$

$\therefore$ The equation becomes

$$1 - \frac{dv}{dx} = \frac{v+3}{2v+5} \quad \text{or} \quad \frac{dv}{dx} = 1 - \frac{v+3}{2v+5} = \frac{v+2}{2v+5}$$

or $\quad dx = \dfrac{2v+5}{v+2}\,dv = \left(2 + \dfrac{1}{v+2}\right) dv$, separating the variables.

Integrating, $x=2v+\log(v+2)+C$,

$$x=2(x-y)+\log(x-y+2)+C \text{ as } v=x-y$$

or $2y-x=\log(x-y+2)+C$ is the required solution.

Ex. 5. *Solve* $\dfrac{dy}{dx}=\dfrac{6x-4y+3}{3x-2y+1}$

[Poona 64; Karnatak B.Sc. (Princ.) 61]

Solution. Put $3x-2y=v$, *i.e.*, $3-2\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

$$\therefore \quad \frac{dv}{dx}=3-2\,\frac{2v+3}{v+1}=-\frac{v+3}{v+1}.$$

$$\therefore \quad dx=-\frac{v+1}{v+3}\,dv=-\left(1-\frac{2}{v+3}\right)dv.$$

Integrating, $x=-v+2\log(v+3)+C$

or $\quad x=(2y-3x)+2\log(3x-2y+3)+C$

or $\quad 2x-y=\log(3x-2y+3)+\frac{1}{2}C$ is the solution.

Ex. 6. *Solve* $(6x-4y+1)\dfrac{dy}{dx}=(3x-2y+1)$.

[Karnatak B.Sc. (Sub.) 61]

Solution. $\dfrac{dy}{dx}=\dfrac{3x-2y+1}{2(3x-2y)+1}$. Put $3x-2y=v$.

$$\therefore \quad \frac{dv}{dx}=3-2\,\frac{dy}{dx}=3-2\,\frac{v+1}{2v+1}=\frac{4v+1}{2v+1}$$

or $\quad dx=\dfrac{2v+1}{4v+1}\,dv$ or $2\,dx=\left(1+\dfrac{1}{4v+1}\right)dv$ etc.

Ex. 7. *Solve the following equatians :*

(i) $(2x+y+1)\,dx+(4x+2y-1)\,dy=0.$

[Gujrat B.Sc. (Princ.) 61]

(ii) $\dfrac{dy}{dx}=\dfrac{6x-2y-7}{3x-y+4}$. [Luck. Pass 56]

(iii) $(2x-5y+3)\,dx-(2x+4y-6)\,dy=0.$ [Delhi Hons. 61]

(iv) $\dfrac{dy}{dx}=\dfrac{y-x+1}{y-x-5}$ [Poona 62; Nag. 62]

(v) $\dfrac{dy}{dx}=\dfrac{3x-4y-2}{3x-4y-3}$. [Cal. Hons 63]

(vi) $(3y+2x+4)\,dx-(4x+6y+5)\,dx=0.$ [Karnatak 63]

(vii) $(2x-5y+3)\,dx-(2x+4y-6)\,dy=0.$ [Delhi Hons. 65]

(viii) $(x-y-2)\,dx+(x-2y-3)\,dy=0.$ [All. 66]

(ix) $(4x+2y+1)\,dy=(2x+y+3)\,dx.$ [Delhi Pass 67]

Hint. In (*i*) put $2x+y=v$, in (*ii*) put $3x-y=v$ and (*iii*) can be reduced to homogeneous form as usual. In (*ix*) putting $v=2x+y$, variables can be separated.

Ex. 8. *Solve* $2y\dfrac{dy}{dx}=\dfrac{x+y^2}{x+4y^2}$ **[Bombay B.Sc. 61]**

Solution. Put $y^2=v$, $2y\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

$\therefore\ \dfrac{dv}{dx}=\dfrac{x+v}{x+4v}$ [homogeneous]. Now put $v=xz$ etc.

2·6. A particular case

A differential equation of the form

$$\frac{dy}{dx}=\frac{ax+by+c}{-bx+hy+k},$$

in which coefficient of y in the numerator is equal to the coefficient of x in the denominator with sign changed, can be integrated as follows :

The equation (1) can be written as

$$-b\,(x\,dy+y\,dx)+(hy+k)\,dy-(ax+c)\,dx=0.$$

Integrating, we get $-bxy+(\tfrac{1}{2}hy^2+ky)-(\tfrac{1}{2}ax^2+cx)=A.$

Ex. 1. *Solve* $\dfrac{dy}{dx}+\dfrac{ax+hy+g}{hx+by+f}=0.$

[Raj. B.Sc. 66; Agra B.Sc. 57; Delhi B.A. 57; Raj. M.Sc. 62]

Solution. The equation can be written as

$$(hx+by+f)\,dy+(ax+hy+g)\,dx=0$$

or $\quad h\,(x\,dy+y\,dx)+(by+f)\,dy+(ax+g)\,dx=0.$

Integrating, $hxy+\tfrac{1}{2}by^2+fy+\tfrac{1}{2}ax^2+gx=A$

or $\quad ax^2+2hxy+by^2+2fy+2gx+c=0$, writing $c=-2A$.

Ex. 2. *Solve* $\dfrac{dy}{dx}=\dfrac{2x-y+1}{x+2y-3}$ **[Agra B Sc. 59; Nag. 53 (S)]**

Solution. Here coefficient of y in numerator is equal to coefficient of x in the denominator with sign changed. Hence write it as

$$(x+2y-3)\,dy-(2x-y+1)\,dx=0$$

or $\quad (x\,dy+y\,dx)+(2y-3)\,dy-(2x+1)\,dx=0.$

Integrating, $xy+y^2-3y-x^2-x=C.$

Ex. 3. *Solve* $(2x-y+1)\,dx+(2y-x-1)\,dy=0.$

[Bombay B Sc. (Sub.) 61; Poona 61]

Solution. The equation is of above type. Hence on regrouping, we have

$$(2x+1)\,dx+(2y-1)\,dy-(y\,dx+x\,dy)=0$$

Integrating, $(x^2+x)+(y^2-y)-xy=c$,

which is the solution.

Ex. 4. *Solve* $\dfrac{dy}{dx}+\dfrac{2x+3y+1}{3x+4y-1}=0.$ **[Delhi Hons. 60]**

$3x\frac{2}{3}$

Solution The equation is of the above type and can be written as
$$(3x+4y-1)\,dy+(2x+3y+1)\,dx=0,$$
i.e., $3(x\,dy+y\,dx)+(4y-1)\,dy+(2x+1)\,dx=0.$

Integrating, $3xy+2y^2-y+x^2+x=C$ is the solution.

2.7. Linear Differential Equations

[Poona 63, 61; Nagpur 62, 61; Guj 61]

A differential equation of the form
$$\frac{dy}{dx}+Py=Q,$$
where P, Q are functions of x or constants, is called the *linear differential equation of the first order.*

To solve this equation, multiply both the sides by $e^{\int P\,dx}$

Then it becomes $e^{\int P\,dx}\dfrac{dy}{dx}+Py\,e^{\int P\,dx}=Qe^{\int P dx}$.

or $$\frac{d}{dx}\left[ye^{\int P\,dx}\right]=Q\,e^{\int P\,dx}.$$

Integrating both the sides, w.r.t. x, we get
$$ye^{\int P\,dx}=\int\left[Qe^{\int P dx}\right]dx+C,$$
which is the required solution.

Integrating factor (I.F.). It will be noticed that for solving (1), we multiplied it by a factor $e^{\int P\,dx}$ and the equation became readily (directly) integrable. Such a factor is called the integrating factor.

Note. Sometimes a differential equation takes linear form if we *regard x as dependent variable and y as independent variable.* The equation can then be put as $\dfrac{dx}{dy}+Px=Q$, where P, Q are functions of y or constants.

The integrating factor in this case is $e^{\int P\,dy}$ and solution is
$$xe^{\int P\,dy}=\int\left[Qe^{\int P\,dy}\right]dy+C.$$

(See Ex. 1 to 4 pages 21 and 22).

Ex. 1. *Solve* $(1-x^2)\dfrac{dy}{dx}-xy=1.$

[Delhi 68 : Nag. 61]

Solution. The equation can be written as
$$\frac{dy}{dx}-\frac{x}{1-x^2}\,y=\frac{1}{1-x^2}$$
This is now expressed in the linear form

$$P=-\frac{x}{1-x^2},\ \text{I.F.}=e^{\int P\,dx}=e^{\int\frac{-x}{1-x^2}dx}=e^{\frac{1}{2}\log(1-x^2)}$$
$$=\sqrt{(1-x^2)}.$$

Hence the solution is

$$y.\sqrt{(1-x^2)}=\int\frac{1}{1-x^2}\sqrt{(1-x^2)}\,dx+C.$$

Ex. 2. (a) *Solve* $x\dfrac{dy}{dx}+2y=x^2\log x.$ **[Lucknow 52]**

Solution. The equation is $\dfrac{dy}{dx}+\dfrac{2}{x}y=x\log x.$

$$\text{I.F.}=e^{\int(2/x)\,dx}=e^{2\log x}=x^2.$$

Hence the solution is

$$y.x^2=C+\int x^2.x\log x\,dx=C+\int x^3\log x\,dx$$
$$=C+\log x.\frac{x^4}{4}-\int\frac{1}{x}.\frac{x^4}{4}dx$$
$$=C+\tfrac{1}{4}x^4\log x-\tfrac{1}{16}x^4$$

or $$y=Cx^{-2}+\tfrac{1}{4}x^2(\log x-\tfrac{1}{4}).$$

Ex. 2. (b) *Solve* $x\dfrac{dy}{dx}+2y=x^4.$

[Bombay B.Sc. 61]

Solution. Equation is $\dfrac{dy}{dx}+\dfrac{2}{x}y=x^3$. I.F. $=x^2$ as above.

Solution is $y.x^2=C+\int x^3.x^2\,dx=C+\frac{1}{6}x^6.$

Ex. 3. *Solve* $(x^3-x)\dfrac{dy}{dx}-(3x^2-1)\,y=x^5-2x^3+x.$

[Gujrat B.Sc. (Sub.) 1961]

Solution. The equation is

$$\frac{dy}{dx}-\frac{3x^2-1}{x^3-x}\,y=(x^2-1).$$

$$\text{I.F.}=e^{-\int\frac{(3x^2-1)}{x^3-x}}=e^{-\log(x^3-x)}=\frac{1}{x^3-x}.$$

$\therefore$ Solution is $y.\dfrac{1}{x^3-x}=C+\displaystyle\int\frac{x^2-1}{x^3-1}dx$

$$=C+\int\frac{1}{x}\,dx=C+\log x.$$

Ex 4. *Solve* $xp+y=ax^2+bx+c,\ p=\dfrac{dy}{dx}.$

[Delhi Hons. 1957]

Solution. The equation can be written as

$$\frac{dy}{dx}+\frac{1}{x}y=ax+b+\frac{c}{x}\text{ [linear]}.$$

$$\text{I.F.}=e^{\int\frac{1}{x}dx}=e^{\log x}=x.$$

$$\therefore\quad y.x=C+\int\left(ax+b+\frac{c}{x}\right)x\,dx=C+\int(ax^2+bx+c)\,dx$$

$$=C+\tfrac{1}{3}ax^3+\tfrac{1}{2}bx^2+cx.$$

Ex. 5. *If* $\frac{dy}{dx}+2y\tan x=\sin x$ *and if* $y=0$ *when* $x=\frac{1}{3}\pi$, *express* y *in terms of* x. **[Poona 1964 ; Nagpur 61]**

Solution. The equation is linear.

$$\text{I.F.}=e^{\int 2\tan x\,dx}=e^{-2\log\cos x}=\sec^2 x.$$

Hence general solution is

$$y.\sec^2 x=C+\int\sin x\sec^2 x\,dx=C+\int\sec x\tan x\,dx$$

or $y\sec^2 x=C+\sec x.$

When $y=0,\ x=\frac{1}{3}\pi$, $\therefore\ 0=C+\sec\frac{1}{3}\pi$ or $C+2=0,\ C=-2.$

Hence solution is $y\sec^2 x=\sec x-2,$

$$y=\cos x-2\cos^2 x.$$

Ex. 6. *Solve* $x(x-1)\frac{dy}{dx}-y=x^2(x-1)^2$. **[Luck. Pass 1958]**

Solution. Equation is $\frac{dy}{dx}-\frac{1}{x(x-1)}y=x(x-1).$

$$\text{I.F.}=e^{-\int\frac{1}{x(x-1)}dx}=e^{\int\left(\frac{1}{x}-\frac{1}{x-1}\right)dx}=\frac{x}{x-1}.$$

Hence $y.\frac{x}{x-1}=C+\int x(x-1).\frac{x}{x-1}dx=C+\int x^2\,dx$

or $$y\frac{x}{x-1}=C+\tfrac{1}{3}x^3.$$

Ex. 7. *Solve* $(1+x)\frac{dy}{dx}+3y=\frac{1+x+x^2}{(1+x)^3}$. **[Lucknow Pass 1957]**

Solution. Equation is $\frac{dy}{dx}+\frac{3}{1+x}y=\frac{1+x+x^2}{(1+x)^4}.$

$$\text{I.F.}=e^{\int\frac{3}{1+x}dx}=e^{3\log(1+x)}=(1+x)^3$$

$$\therefore\quad y(1+x)^3=C+\int\frac{(1+x+x^2)}{(1+x)^4}(1+x)^3.dx$$

$$=C+\int\frac{1+x+x^2}{1+x}=C+\int\left(\frac{1}{1+x}+x\right)dx$$

$$=C+\log(1+x)+\tfrac{1}{2}x^2.$$

Ex. 8. *Solve* $x\frac{dy}{dx}+2y=\frac{dy}{dx}+4.$ [Nagpur T.D.C. 1961 (S)]

Solution. The equation can be written as

$$(x-1)\frac{dy}{dx}+2y=4 \quad \text{or} \quad \frac{dy}{dx}+\frac{2}{x-1}y=\frac{4}{x-1}.$$

Linear, $\text{I.F.}=e^{\int\frac{2}{x-1}dx}=e^{2\log(x-1)}=(x-1)^2.$

$$y(x-1)^2=\int\frac{4}{x-1}(x-1)^2\,dx+C$$

$y(x-1)^2=2(x-1)^2+C$, which is the solution.

Ex. 9. *Solve* $x\frac{dy}{dx}-2y=x^2+\sin\frac{1}{x^2}.$

[Bombay B.A. (Sub.) 1958]

Solution. The equation is $\frac{dy}{dx}-\frac{2}{x}y=x+\frac{1}{x}\sin\frac{1}{x^2}.$

$$\text{I.F.}=e^{-\int\frac{2}{x}dx}=e^{-2\log x}=\frac{1}{x^2}.$$

$$\therefore\quad y.\frac{1}{x^2}=C+\int x\frac{1}{x^2}dx+\int\frac{1}{x^3}\sin\frac{1}{x^2}dx.$$

$=C+\log x-\tfrac{1}{2}\int\sin t\,dt$, where $\frac{1}{x^2}=t$, $\frac{-2}{x^3}dx=dt$

$$=C+\log x+\tfrac{1}{2}\cos t$$

$$=C+\log x+\tfrac{1}{2}\cos\frac{1}{x^2}.$$

Ex. 10. *Solve* $\frac{dy}{dx}-2y\cos x=-2\sin 2x.$

[Vikram 65; Gujrat B.Sc. (Sub.) 61]

Solution. $\text{I.F.}=e^{-2\int\cos x\,dx}=e^{-2\sin x}.$

$\therefore$ Solution is

$$ye^{-2\sin x}=C-2\int\sin 2x e^{-2\sin x}\,dx$$

$=C-4\int\sin x\cos x e^{-2\sin x}\,dx$; put $-2\sin x=t$

$$=C-\int te^t\,dt=C-e^t(t-1).$$

$\therefore\quad y=Ce^{2\sin x}+(2\sin x+1)$ is the solution.

Equations which become linear when x is treated as dependent variable.

Ex. 1. *Solve* $y\log y\,dx+(x-\log y)\,dy=0.$

[Poona T.D.C. 61(S)]

Solution. Write the equation as

$$\frac{dx}{dy}+\frac{1}{y\log y}x=\frac{1}{y}.$$

$$\text{I.F.}=e^{\int\frac{1}{y}\frac{1}{\log y}dy}=e^{\log(\log y)}=\log y.$$

$$\therefore\quad x\log y=C+\int\frac{1}{y}\log y\,dy$$

$$=C+\tfrac{1}{2}(\log y)^2 \text{ is the solution.}$$

Ex 2. *Solve $dx+x\,dy=e^{-y}\log y\,dy$.* [Poona 61]

Solution. The equation can be written as

$$\frac{dx}{dy}+x=e^{-y}\log y,\ \text{I.F.}=e^{y}.$$

$$\therefore\quad xe^{y}=C+\int e^{-y}\log y.e^{y}\,dy$$

$$=C+\int\log y\,dy=C+\log y.y-\int y.\frac{1}{y}dy$$

$$=C+y\log y-y.$$

Ex. 3. *Solve $(1+y^2)\,dx+(x-\tan^{-1}y)\,dy=0$.* [Gujrat 65; Delhi Hons. 65 ; Pb. 62; Cal. Hons. 62; Agra 67, 58]

Solution. The equation can be written as

$$\frac{dx}{dy}+\frac{1}{1+y^2}x=\frac{\tan^{-1}y}{1+y^2}.$$

$$\text{I.F.}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}.$$

$$\therefore\quad xe^{\tan^{-1}y}=\int\frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}dy+C$$

$$=\int te^{t}\,dt+C \text{ where } t=\tan^{-1}y$$

$$=e^{t}(t-1)+C=e^{\tan^{-1}y}(\tan^{-1}y-1)+C.$$

Hence $x=(\tan^{-1}y-1)+Ce^{-\tan^{-1}y}$ is the solution.

Ex. 4. *Solve $(x+2y^3)\dfrac{dy}{dx}=y$.*

[Agra B.Sc. 1956 ; Raj B.Sc. 56]

Hint. The equation can be written as

$$\frac{dx}{dy}=x+2y^3 \text{ [linear].}$$

Ans. $x=y^3+Cy$.

2.8. Equations reducible to linear form

***I. Bernoulli Equation*.** $\dfrac{dy}{dx}+Py=Qy^n$.

*Known after James Bernoulli. The method of solution was discovered by Leibnitz.

where P and Q are functions of x or constants.

[Nag. I.D.C. 1961; Poona T.D.C. 61 ; Gujrat B.Sc. (Prin.) 58; Poona B.A. (Gen.) 60]

Dividing both the sides by y^n we have

$$y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q. \qquad ...(1)$$

Now put $y^{-n+1}=v$ so that $(1-n)\,y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$.

Then (1) becomes $\frac{1}{1-n}\frac{dv}{dx}+Pv=Q$

or $\frac{dv}{dx}+P\,(1-n)\,v=(1-n)\,Q$

which is a linear equation in v and x.

II. Equation $f'(y)\frac{dy}{dx}+Pf(y)=Q$,

where P and Q are functions of x or constants.

Put $f(y)=v$ so that $f'(y)\frac{dy}{dx}=\frac{dv}{dx}$.

$\therefore$ equation becomes $\frac{dv}{dx}+Pv=Q$,

which is a linear equation in v and x.

Note. In each of these equations, single out Q (function of on the right) and then make suitable substitution to reduce the equation in linear form.

Ex. 1. *Solve* $\frac{dy}{dx}=x^3y^3-xy$.

[Karnatak B Sc. (Prin.) 1960, 62; Agra 61; Bihar 62; Gujrat B.Sc. (Sub.) 61]

Solution. The equation is $\frac{dy}{dx}+xy=x^3y^3$.

Dividing by y^3; $\frac{1}{y^3}\frac{dy}{dx}+x\frac{1}{y^2}=x^3$.

Put $\frac{1}{y^2}=v$, so that $-\frac{2}{y^3}\frac{dy}{dx}=\frac{dv}{dx}$, *i.e.*, $\frac{1}{y^3}\frac{dy}{dx}=-\frac{1}{2}\frac{dv}{dx}$.

$\therefore$ equation becomes $-\frac{1}{2}\frac{dv}{dx}+xv=x^3$

or $\frac{dv}{dx}-2x.v=-2x^3$.

Linear, I. F. $=e^{\int -2x\,dx}=e^{-x^2}$.

Hence $ve^{-x^2}=\int -2x^3e^{-x^2}\,dx+C$

$=\int x^2\,(-2x)\,e^{-x^2}\,dx+C$

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3

Equations of First Order and First Degree

Exact Differential Equations and Reduction to Exact Equations

3·1. Exact Differential Equations. **[Bombay 61 ; Karnatak 60]**

Study the following two differential eqnations :

1. $x\,dy+y\,dx=0$. Solution is $xy=C$.
2. $\sin x \cos y\,dy+\cos x \sin y\,dx=0$.

Solution is $\sin x \sin y=C$.

We see that these differential equations can be obtained by directly differentiating their solutions. Differential equations of this type are called exact equations and bear the following property :

An exact differential equation can always be obtained from its primitive directly by differentiation, without any subsequent multiplication, elimination etc.

3·2. Necessarry and Sufficient Condition

To find the necessary and sufficient condition for a differential equation of first degree being exact.

[Poona 63, 61 ; Delhi Hons. 57, 55 ; Nag. 63 ; Gujrat 59; Bombay 61]

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Let the equation be $\mathbf{M}+\mathbf{N}\dfrac{dy}{dx}=0$. ...(1)

Let $u=C$ be its primitive. ...(2)

If (1) is exact, it can be obtained by directly differentiating its primitive.

Differentiating (2), we have $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dx}=0$. ...(3)

Comparing (1) and (3) we get $M=\dfrac{\partial u}{\partial x}$ and $N=\dfrac{\partial u}{\partial y}$, so that

$$\frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y \partial x},\quad \frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x \partial y}.$$

Hence the condition is $\dfrac{\partial \mathbf{M}}{\partial \mathbf{y}}=\dfrac{\partial \mathbf{N}}{\partial \mathbf{x}}$.

That the condition is necessary has been proved. Now we prove that it is sufficient also, *i.e.* if $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, then we show that

$M+N\dfrac{dy}{dx}=0$ or $M\,dx+N\,dy=0$ is an exact equation.

Let $\int M\,dx=U$, then $\dfrac{\partial U}{\partial x}=M$, so that

$$\frac{\partial^2 U}{\partial y\,\partial x}=\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \text{ as } \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$$

i.e. $\dfrac{\partial N}{\partial x}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial U}{\partial y}\right)$

Integrating, $N=\dfrac{\partial U}{\partial y}+f(y)$, where $f(y)$ is a function of y free from x.

$$\therefore \quad M+N\frac{dy}{dx}=\frac{\partial U}{\partial x}+\left[\frac{\partial U}{\partial y}+f(y)\right]\frac{dy}{dx}$$

$$=\frac{d}{dx}\left[U+\int f(y)\,\frac{dy}{dx}\,dx\right]$$

$$=\frac{d}{dx}[U+F(y)].$$

This shows that $M+N\dfrac{dy}{dx}=0$ is an exact equation.

3.3. Working Rule (Remember it).

If the equation $M\,dx+N\,dy=0$ satisfies the condition

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$$

then it is exact. To integrate it,

(*i*) integrate M with regard to x regarding y as constant;

(*ii*) find out those terms in N which are free from x and integrate them with regard to y;

(*iii*) add the two expressions so obtained and equate the sum to an arbitrary constant.

This gives the general solution of the given exact equation.

Ex. 1. $(y^4+4x^3y+3x)\,dx+(x^4+4xy^3+y+1)\,dy=0$

[Karnatak 60]

Solution Here $M=y^4+4x^3y+3x$ and $N=x^4+4xy^3+y+1$.

$$\frac{\partial M}{\partial y}=4y^3+4x^3 \text{ and } \frac{\partial N}{\partial x}=4x^3+4y^3.$$

Since these are equal, the equation is exact.

To find solution of the differential equation, integrating M *i.e.* y^4+4x^3y+3x w.r.t. x, keeping y as constant, we get $y^4x+x^4y+\frac{3}{2}x^2$.

In x^4+4xy^3+y+1, terms free from x are $y+1$ whose integral with respect to y is $\frac{1}{2}y^2+y$.

Therefore the general solution is

$$y^4x+x^4y+\frac{2}{3}x^2+\frac{1}{2}y^2+y=C.$$

Ex. 2. *Solve* $x(x^2+y^2-a^2)\,dx+y(x^2-y^2-b^2)\,dy=0.$ [Nag. 63; Poona 61]

Solution. Here $M=x^3+xy^2-a^2x$, $N=yx^2-y^3-b^2y$.

$$\frac{\partial M}{\partial y}=2xy \text{ and } \frac{\partial N}{\partial x}=2xy.$$

Since these are equal, the equation is exact,

Integrating M w.r.t. x keeping y as constant, we get

$$\frac{1}{4}x^4+\frac{1}{2}x^2y^2-\frac{1}{2}a^2x^2.$$

In N, terms free from x are $-y^3-b^2y$ whose integral is

$$-\frac{1}{4}y^4-\frac{1}{2}b^2y^2.$$

Hence the general solution is

$$\frac{1}{4}x^4+\frac{1}{2}x^2y^2-\frac{1}{2}a^2x^2-\frac{1}{4}y^4-\frac{1}{2}b^2y^2=\text{const.}$$

or $$x^4-y^4+2x^2y^2-2a^2x^2-2b^2y^2=C.$$

Ex. 3. *Solve* $(x^2-2xy+3y^2)\,dx+(4y^3+6xy-x^2)\,dy=0.$ [Delhi Hons. 55]

Solution. Here $\frac{\partial M}{\partial y}=-2x+6y$, $\frac{\partial N}{\partial x}=6y-2x$.

Since these are equal the equation is exact.

Integrating M, *i.e.* $x^2-2xy+3y^2$ w.r.t. x keeping y as constant, we get

$$\frac{1}{3}x^3-x^2y+3y^2x$$

In N, term free from x is $+4y^3$ whose integral is y^4.

Hence the solution is $\frac{1}{3}x^3-x^2y+3y^2x+y^4=C.$

Ex. 4. *Solve* $(x-2e^y)\,dy+(y+x\sin x)\,dx=0.$ [Gujrat 61]

Solution. Here $M=y+x\sin x$, $N=x-2e^y$.

$\therefore$ $\frac{\partial M}{\partial y}=1$, $\frac{\partial N}{\partial x}=1$; therefore equation is exact.

Integrating $y+x\sin x$ with respect to x keeping y as constant, we get

$$xy+\int x\sin x\,dx=xy-x\cos x+\sin x.$$

In N, term free from x is $-2e^y$ whose integral with respect to y is $-2e^y$.

Hence the complete solution is

$$xy-x\cos x+\sin x-2e^y=C.$$

***Ex. 5.** (a) *Solve* $x\,dx+y\,dy=\dfrac{a^2(x\,dy-y\,dx)}{x^2+y^2}$. — imidiate batch

[Delhi Hons. 62

Solution. The equation can be put as

$$\left(x+\frac{a^2y}{x^2+y^2}\right)dx+\left(y-\frac{a^2x}{x^2+y^2}\right)dy=0.$$

Here $M=x+\dfrac{a^2y}{x^2+y^2}$ and $N=y-\dfrac{a^2x}{x^2+y^2}$.

$$\therefore\quad \frac{\partial M}{\partial y}=\frac{(x^2+y^2)\,a^2-a^2y\,.\,2y}{(x^2+y^2)^2}=\frac{a^2\,(x^2-y^2)}{(x^2+y^2)^2}$$

and $$\frac{\partial N}{\partial x}=\frac{-a^2\,(x^2+y^2)+2a^2x^2}{(x^2+y^2)^2}=\frac{a^2\,(x^2-y^2)}{(x^2+y^2)^2}.$$

$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$

Since $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, the equation is exact.

Integrating M w.r.t. x regarding y as constant, we get

$$\tfrac{1}{2}x^2+a^2y\frac{1}{y}\tan^{-1}\frac{x}{y}\quad\text{or}\quad \tfrac{1}{2}x^2+a^2\tan^{-1}\frac{x}{y}.$$

In N, term free from x is y whose integral is $\tfrac{1}{2}y^2$.

Hence the solution is $\tfrac{1}{2}x^2+a^2\tan^{-1}\dfrac{x}{y}+\tfrac{1}{2}y^2=\text{const.}$

or $\quad x^2+y^2+2a^2\tan^{-1}\dfrac{x}{y}=C.$

Ex. 5. (b) *Solve* $x\,dx+y\,dy+\dfrac{x\,dy-y\,dx}{x^2+y^2}=0.$

The equation is exact; proceed as in the above example. imidiate Batch

*****Ex. 6.** *Solve* $(1+e^{x/y})\,dx+e^{x/y}\,(1-x/y)\,dy=0.$ →

[Karnatak 61; Bombay 50; Gujrat 59; Poona 61]

Solution. Here $M=1+e^{x/y}$ and $N=e^{x/y}\,(1-x/y)$

$$\frac{\partial M}{\partial y}=e^{x/y}\left(-\frac{x}{y^2}\right)$$

and $$\frac{\partial N}{\partial x}=e^{x/y}\frac{1}{y}\cdot\left(1-\frac{x}{y}\right)+e^{x/y}\left(-\frac{1}{y}\right)=e^{x/y}\left(-\frac{x}{y^2}\right).$$

Since $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, the equation is exact.

Now integrating $1+e^{x/y}$ with respect to x keeping y as constant,

we get $\quad x+\dfrac{e^{x/y}}{1/y}$ *i.e.*, $x+ye^{x/y}$

In N *i.e.*, in $e^{x/y}\,(1-x/y)$ there is no term free from x.

Hence the required solution is $x+ye^{x/y}=C$.

Ex. 7. $[\cos x\tan y+\cos(x+y)]\,dx$

$$+[\sin x\sec^2 y+\cos(x+y)]\,dy=0.$$

[Bombay 61; Gujrat 61]

Solution. Here $M=\cos x\tan y+\cos(x+y)$,

and $\quad N=\sin x\sec^2 y+\cos(x+y)$.

Now $\quad \dfrac{\partial M}{\partial y}=\cos x\sec^2 y-\sin(x+y),$

$$\frac{\partial N}{\partial x}=\cos x \sec^2 y-\sin (x+y).$$

Since these are equal, the equation is exact.

Now integrating M, *i.e.* $\cos x \tan y+\cos (x+y)$ with respect to x keeping y as constant, we get

$$\sin x \tan y+\sin (x+y)$$

In N, there is no term free from x.

Hence the general solution is

$$\sin x \tan y+\sin (x+y)=C.$$

Ex. 8. $(\cos x \tan y-\sin x \sec y)\, dx$
$$+(\sin x \sec^2 y+\cos x \tan^2 y \operatorname{cosec} y)\, dy=0.$$

[Bombay B. A. (Sub.) 58]

Solution. We have $M=\cos x \tan y-\sin x \sec y$,

and $N=\sin x \sec^2 y+\cos x \tan^2 y \operatorname{cosec} y$.

$$\therefore \quad \frac{\partial M}{\partial y}=\cos x \sec^2 y-\sin x \sec y \tan y$$

$$\frac{\partial N}{\partial x}=\cos x \sec^2 y-\sin x \tan y \sec y.$$

as $\tan^2 y \operatorname{cosec} y=\tan y \sec y$.

Since $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, the equation is exact,

Integrating M *i.e.* $\cos x \tan y-\sin x \sec y$ with regard to x keeping y as constant we get

$$\sin x \tan y+\cos x \sec y.$$

In N there is no term free from x.

Hence the general solution is

$$\sin x \tan y+\cos x \sec y=C.$$

Ex. 9. *Solve* $(\sin x \cos y+e^{2x})\, dx$
$$+(\cos x \sin y+\tan y)\, dy=0.$$

[Poona 59]

Solution. Here $\dfrac{\partial M}{\partial y}=-\sin x \sin y$, $\dfrac{\partial N}{\partial x}=-\sin x \sin y$.

Since these are equal, the equation is exact.

Integrating M *i.e.*, $\sin x \cos y+e^{2x}$ w.r.t. x, keeping y as constant, we get $-\cos x \cos y+\frac{1}{2}e^{2x}$.

Also in N the term free from x is $\tan y$ whose integral w.r.t. y is $\log \sec y$.

Hence the solution is

$$-\cos x \cos y+\tfrac{1}{2}e^{2x}+\log \sec y=C.$$

Ex. 10. *Solve the following equations (which are exact) :*

$(2x^3+3y)\, dx+(3x+y-1)\, dy=0.$ [Poona 93]

Ans. $\frac{1}{2}x^4+3yx+\frac{1}{2}y^2-y=C$

(*ii*) $(x^2-4xy-2y^2)\,dx+(y^2-4xy-2x^2)\,dy.$

Ans. $x^3+y^3-6xy\,(x+y)=C.$

(*iii*) $\cos x\,(\cos x-\sin a \sin y)\,dx$

$+\cos y\,(\cos y-\sin a \sin x)\,dy=0.$

Ans. $2(x+y)\sin 2x+\sin 2y-4\sin a \sin x \sin y=C.$

(*iv*) $(2xy+y-\tan y)\,dx+(x^2-x\tan^2 y+\sec^2 y)\,dy=0.$

[Poona 1964]

Ans. $x^2y+xy-x\tan y+\tan y=C.$

(*v*) $(2x^2y+4x^3-12xy^2+3y^2-xe^y+e^{2x})\,dy$
$+(12x^2y+2xy^2+4x^3-4y^3+2ye^{2x}-e^y)\,dx=0.$ **[Poona 64]**

Ans. $4x^3y+x^2y^2+x^4-4y^3x+ye^{2x}-xe^y+y^3=C.$

3·4. Integrating factors.

If an equation becomes exact after it has been multiplied by a function of x and y, then such a function is called an integrating factor **[Karnatak 61]**

3·5. Number of integrating factors.

To show that there is an infinite number of integrating factors for an equatian. **[Karnatak 61]**

$$M\,dx+N\,dy=0.$$

To prove this let μ be an integrating factor: then

$$\mu\,(M\,dx+N\,dy)=du.$$

Integrating, $u=c$ is a solution.

Now multiplying both the sides by $f(u)$, a function of u,

we get $\mu f(u)\,[M\,dx+N\,dy]=f(u)\,du.$

Expression on the right is directly integrable and therefore so must be the left hand side.

Hence $\mu f(u)$ is also an integrating factor. Since $f(u)$ is an arbitrary function of u, the number of integrating factors is infinite.

3·6. Integrating factor by inspection.

Sometimes an integrating factor can be found by inspection. For this the reader should study the following results :—

Group of terms	I F.	Exact Differntial
$x\,dy-y\,dx$	$\frac{1}{x^2}$	$\frac{x\,dy-y\,dx}{x^2}=d\left(\frac{y}{x}\right)$
$x\,dy-y\,dx$	$\frac{1}{y^2}$	$\frac{y\,dx-x\,dy}{-y^2}=d\left(-\frac{x}{y}\right)$
$x\,dy-y\,dx$	$\frac{1}{xy}$	$\frac{dy}{y}-\frac{dx}{x}=d\left(\log\frac{y}{x}\right)$
$x\,dy-y\,dx$	$\frac{1}{x^2+y^2}$	$\frac{x\,dy-y\,dx}{x^2+y^2}=\frac{\frac{x\,dy-y\,dx}{x^2}}{1+\left(\frac{y}{x}\right)^2}=d\left[\tan^{-1}\frac{y}{x}\right]$

x^2+y^2

Groups of terms	I.F.	Exact Differential
$x\,dy+y\,dx$	$\dfrac{1}{(xy)^n}$	$\dfrac{x\,dy+y\,dx}{xy}=d\,[\log(xy)]$ for $n=1$
$x\,dx+y\,dy$	$\dfrac{1}{(x^2+y^2)^n}$	$\dfrac{x\,dx+y\,dy}{(x^2+y^2)^n}=d\left[-\dfrac{1}{2(n-1)(x^2+y^2)^{n-1}}\right]$
	or	$=\dfrac{x\,dx+y\,dy}{x^2+y^2}=d\,[\frac{1}{2}\log(x^2+y^2)]$ if $n=1$.

Ex. 1. *Solve* $(x+y^2)\,dy+(y-x^2)\,dx=0.$ **[Nagpur 61]**

Solution. The equation can be written as

$$x\,dy+y\,dx+y^2\,dy-x^2\,dx=0.$$

or $$d\,(xy)+y^2\,dy-x^2dx=0.$$

Integrating, $xy+\frac{1}{3}y^3-\frac{2}{3}x^3=A$ or $y^3-x^3+3xy=c.$

Ex. 2. *Solve* $y\,dx-x\,dy+3x^2y^2e^{x^3}\,dx=0.$ **[Nagpur 61]**

Solution. The equation can be written as

$$\frac{y\,dx-x\,dy}{y^2}+3x^2e^{x^3}\,dx=0,$$

$$d\left(\frac{x}{y}\right)+e^{x^3}\,d(x^3)=0.$$

Integrating, $\dfrac{x}{y}+e^{x^3}=c.$

Ex. 3. *Solve* $x\,dy-y\,dx-x\,(x^2-y^2)^{1/2}\,dx=0.$ **[Delhi Hons. 61]**

Solution. The equation can be written as

$$\frac{x\,dy-y\,dx}{(x^2-y^2)^{1/2}}-x\,dx=0$$

i.e. $\dfrac{\dfrac{x\,dy-y\,dx}{x^2}}{\left[1-\left(\dfrac{y}{x}\right)^2\right]^{1/2}}=dx$, put $\dfrac{y}{x}=t$, then $\dfrac{x\,dy-y\,dx}{x^2}=dt$

i.e., $\dfrac{dt}{\sqrt{(1-t^2)}}=dx$ or $x+c=\sin^{-1}t=\sin^{-1}\left(\dfrac{y}{x}\right).$

Ex. 4. *Solve* $a\,(x\,dy+2y\,dx)=xy\,dy.$

Solution The equation can be written as

$(a-y)\,x\,dy+2ay\,dx=0$ or $\dfrac{a-y}{y}\,dy+\dfrac{2a}{x}\,dx=0.$

Integrating. $a\log y-y+2a\log x=C_1$

or $\quad \log yx^2 = \frac{y}{a} + \log C \quad$ or $\quad yx^2 = Ce^{y/a}$,

Ex. 5. *Solve* $y\,dx - x\,dy + \log x\,dx = 0$.

Solution. The equation is $x\dfrac{dy}{dx} - y = \log x$

or $\quad \dfrac{dy}{dx} - \dfrac{1}{x}y = \dfrac{\log x}{x}$. Linear, I.F. $= e^{-\int \frac{1}{x}dx} = \dfrac{1}{x}$.

$$\therefore \quad y.\frac{1}{x} = \int \frac{1}{x^2}\log x\,dx - C$$

$$= -\frac{1}{x}(1+\log x) - C$$

or $\quad y + \log x + Cx + 1 = 0$ is the solution.

Ex. 6. *Solve* $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0$. **[Bihar 62]**

Solution. Write the equation as

$$y\,dx + x\,dy + xy\,(y\,dx - x\,dy) = 0$$

or $\quad d(xy) + xy\,(y\,dx - x\,dy) = 0.$

We readily find that $\dfrac{1}{x^2y^2}$ is the I.F. So the equation becomes

$$\frac{d(xy)}{x^2y^2} + \frac{y\,dx - x\,dy}{xy} = 0 \text{ or } \frac{d(xy)}{(xy)^2} + \left(\frac{dx}{x} - \frac{dy}{y}\right) = 0.$$

Integrating, $\quad -\dfrac{1}{xy} + \log x - \log y = C_1 \quad$ or $\quad x = Cye^{1/xy}$.

Ex. 7. *Solve* $(x^4e^x - 2mxy^2)\,dx + 2mx^2y\,dy = 0$.

Solution. Equation is $2y\dfrac{dy}{dx} - \dfrac{2y^2}{x} + \dfrac{x^2e^x}{m} = 0$.

Putting $y^2 = z$, the equation becomes $\dfrac{dz}{dx} - \dfrac{2}{x}z + \dfrac{x^2e^x}{m} = 0$.

I.F. $= e^{-\int \frac{2}{x}dx} = \dfrac{1}{x^2}$, etc.

Ex. 8. *Solve* $y\,(2xy + e^x)\,dx - e^x\,dy = 0$. **[Vikram 61]**

Solution. The equation is $e^x\dfrac{dy}{dx} = 2xy^2 + ye^x$

or $\quad -y^{-2}\dfrac{dy}{dx} + y^{-1} = -2xe^{-x}$. Put $y^{-1} = v$, $-y^{-2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$.

$\therefore$ the equation is $\dfrac{dv}{dx} + v = -2xe^{-x}$. I.F. $= e^x$ etc.

Solution is $\quad y^{-1}e^x = -x^2 + C$.

3·7. Rules for finding the integrating factor.

Rule. I. If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=f(x)$, a function of x only, then $e^{\int f(x)\,dx}$ is an integrating factor. [Delhi Hons. 64]

Rule II. If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}=g(y)$ is a function of y alone, then $e^{\int -g(y)\,dy}$ is an integrating factor.

We give below some examples to illustrate these rules.

Ex. 1. *Solve* $(x^2+y^2+x)\,dx+xy\,dy=0$.

Solution $M=x^2+y^2+x$, $N=xy$.

$\dfrac{\partial M}{\partial y}=2y$, $\dfrac{\partial N}{\partial x}=y$, equation is not exact.

However, $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\dfrac{2y-y}{xy}=\dfrac{1}{x}$, a function of x alone.

Hence I F. $=e^{\int \frac{1}{x}dx}=e^{\log x}=x$.

Multiplying by I F., the equation becomes

$(x^3+xy^2+x^2)\,dx+x^2y\,dy=0$, exact now (check up).

Integrating, $x^3+xy^2+x^2$ with regard to x, keeping y as constant, we get $\frac{1}{4}x^4+\frac{1}{2}x^2y^2+\frac{1}{3}x^3$.

and in x^2y^2 there is no term free from x. Therefore the solution is

$\frac{1}{4}x^4+\frac{1}{2}x^2y^2+\frac{1}{3}x^3=C'$ or $3x^4+4x^3+6x^2y^2=C$.

Ex. 2. *Solve* $(x^2+y^2+1)\,dx-2xy\,dy=0$.

Solution. $\dfrac{\partial M}{\partial y}=2y$, $\dfrac{\partial N}{\partial x}=-2y$, not exact.

However, $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\dfrac{2x+2y}{-2xy}=-\dfrac{2}{x}$ function of x alone.

$\therefore$ I.F. $=e^{-\int \frac{2}{x}dx}=e^{-2\log x}=\dfrac{2}{x^2}$.

Multiplying by $\dfrac{1}{x^2}$ the equation becomes

$\left(1+\dfrac{y^2}{x^2}+\dfrac{1}{x^2}\right)dx-\dfrac{2y}{x}\,dy=0$, exact now.

Integrating, $1+\dfrac{y^2}{x^2}+\dfrac{1}{x_2}$ with regard to x keeping y as constant.

we get $x-\dfrac{y^2}{x}-\dfrac{1}{x}$.

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and in $-\frac{2y}{x}$ there in no term free from x.

Hence the solution is

$$x-\frac{y^2}{x}-\frac{1}{x}=C \text{ or } x^2-y^2=Cx+1.$$

Ex. 3. *Solve* $(x^2+y^2)\,dx-2xy\,dy=0$.

Solution Just as in the above example, I.F. $=\frac{1}{x^2}$.

Hence multiplying by $\frac{1}{x^2}$ the equation becomes

$$\left(1+\frac{y^2}{x^2}\right)dx-\frac{2y}{x}dy=0, \text{ exact.}$$

$\therefore$ Solution is $x-\frac{y^2}{x}=c$ or $x^2-y^2=cx$.

Ex. 4. *Solve* $(x^2+y^2+2x)\,dx+2y\,dy=0$.

[Vikram 1959 ; Alld. 59].

Solution. $\frac{\partial M}{\partial y}=2y,\ \frac{\partial N}{\partial x}=0$, not exact.

However, $$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2y}{2y}=1.$$

$\therefore$ I.F. $=e^x$.

Multiplying by e^x, the equation becomes

$e^x(x^2+y^2+2x)\,dx+2ye^x\,dy=0$, now exact.

This can be written as

$(x^2+2x)\,e^x\,dx+(y^2e^x\,dx+e^x.2y\,dy)=0$

or $d(x^2e^x)+d(y^2e^x)=0$.

$\therefore$ Integrating, $x^2e^x+y^2e^x=C$ or $(x^2+y^2)\,e^x=C$

Aliter. The equation can also be written as

$$2y\frac{dy}{dx}+y^2=-(x^2+2x).$$

Putting $y^2=v$, $\frac{dv}{dx}+v=-(x^2+2x)$. Linear, I.F. $=e^x$ etc

***Ex. 5.** *Solve* $(\frac{1}{3}y^3+y+\frac{1}{2}x^2)\,dx+\frac{1}{4}(x+xy^2)\,dy=0$,

[Delhi Hons. 1965 ; Agra M.Sc. 63 : Banaras 56]

Solution. $\frac{\partial M}{\partial y}=1+y^2,\ \frac{\partial N}{\partial x}=\frac{1}{4}(1+y^2)$, not exact.

However, $$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{(1+y^2)-\frac{1}{4}(1+y^2)}{\frac{1}{4}x(1+y^2)}=\frac{3}{x},$$

a function of x alone.

$a \log b = \log b^a$

$\therefore$ I.F. $= e^{\int \frac{3}{x} dx} = e^{3 \log x} = x^3.$

Multiplying by x^3, the equation becomes

$(x^3 y + \frac{1}{3} x^3 y^3 + \frac{1}{2} x^5)\, dx + \frac{1}{4} (x^4 + x^4 y^2)\, dy = 0$, exact now.

Integrating $x^3 y + \frac{1}{3} x^3 y^3 + \frac{1}{2} x^5$ with respect to x keeping y as constant, we get $\frac{1}{4} x^4 y + \frac{1}{12} x^4 y^3 + \frac{1}{12} x^6.$

In $\frac{1}{4} (x^4 + x^4 y^2)$ there is no term free from x.

$\therefore$ the solution is $\frac{1}{4} x^4 y + \frac{1}{12} x^4 y^3 + \frac{1}{12} x^6 =$ constant

or
$$3x^4 y + y^3 x^4 + x^6 = C.$$

Ex. 6. *Is the differential equation* $(x^3 - 2y^2)\, dx + 2xy\, dy = 0$ *exact ? Solve the equation.* **[Cal. Hons. 1963]**

Solution. The equation is not exact ; however we have

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{-4y - 2y}{2xy} = -\frac{3}{x}; \quad \therefore \quad \text{I.F.} = e^{\int -3\, dx/x} = \frac{1}{x^3}.$$

Proceed as above.

Ex. 7. $(2x^3 y^2 + 4x^2 y + 2xy^2 + xy^4 + 2y)\, dx + 2(y^3 - x^2 y + x)\, dy = 0.$

Solution. Equation is not exact.

$$\frac{1}{N}\left\{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right\} = 2x. \quad \text{I.F.} = e^{\int 2x\, dx} = e^{x^2}.$$

The solution is $(2x^2 y^3 + 4xy + y^4)\, e^{x} = C.$

Ex. 8. *Solve* $(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0.$

[Cal. Hons. 1962, 61]

Solution. $\frac{\partial M}{\partial y} = 4y^3 + 2$, $\frac{\partial N}{\partial x} = y^3 - 4$, not exact.

However, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{4y^3 + 2 - (y^3 - 4)}{y^4 + 2y} = \frac{3}{y}, \text{ a function of } y \text{ alone.}$$

$$\therefore \text{I.F.} = e^{-\int \frac{3}{y} dy} = e^{-3 \log y} = \frac{1}{y^3}.$$

Multiplying by $1/y^3$, the equation becomes

$$\left(y + \frac{2}{y^2}\right) dx + \left(x + 2y - \frac{4x}{y^3}\right) dy = 0, \text{ exact now.}$$

Integrating $y + \frac{2}{y^2}$ w.r.t. x keeping y as constant, we have

$$yx + \frac{2}{y^2} x.$$

In $x + 2y - \frac{4x}{y^3}$, the term free from x is $2y$. So integrating $2y$ w.r.t. y, we get y^2.

Therefore the solution is $yx + \frac{2}{y^2}x + y^2 = C$.

Ex. 9. *Solve* $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$.

[Cal. Hons. 54, 53]

Solution. Here $\frac{\partial M}{\partial y} = 12x^2y^3 + 2x,\ \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$.

Now $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{6x^2y^3 + 4x}{y\,(3x^2y^3 + 2x)} = \frac{2}{y}$ function of y alone.

$\therefore$ I. F. $= e^{\int -(2/y)\,dy} = e^{-2\log y} = \frac{1}{y^2}$.

Multiplying by $\frac{1}{y^2}$, the equation becomes

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0, \text{ exact now.}$$

Integrating $3x^2y^2 + \frac{2x}{y}$ w.r.t. x keeping y as constant, we get

$$x^3y^2 + \frac{x^2}{y}.$$

In $2x^3y - \frac{x^2}{y^2}$, there is no term free from x.

Hence the solution is $x^3y^2 + \frac{x^2}{y} = C$

or $$x^3y^3 + x^2 = Cy.$$

Ex. 10. $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$.

Solution. We have $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{4}{y}$. $\therefore$ I.F. $= \frac{1}{y^4}$.

Solution is $$x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = C.$$

3·8. Rule III.

If $M\,dx + N\,dy = 0$ is homogeneous and $Mx + Ny \neq 0$, then $\frac{1}{Mx + Ny}$ is an integrating factor.

Rule IV. [Delhi Hons. 61]

If the equation can be written in the form

$$yf(xy)\,dx + xg(xy)\,dy = 0,\ f(xy) \neq g(xy),$$

then $\frac{1}{xy\,[f(xy) - g(xy)]} = \frac{1}{Mx - Ny}$ is an integrating factor.

Ex. 1. *Solve* $x^2y\,dx - (x^3 + y^3)\,dy = 0$.

Solution. The equation is homogeneous and