Fluid dynamics of relativistic blast waves

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A fluid dynamical treatment of an ultra-relativistic spherical blast wave enclosed by a strong shock is presented. A simple similarity solution describing the explosion of a fixed amount of energy in a uniform medium is derived, and this is generalized to include cases in which power is supplied by a central source and the density of the external medium varies with radius. Radiative shocks, in which the escaping photons carry away momentum as well as energy, are also discussed. Formulas that interpolate between the non- and ultra-relativistic limits are proposed.

I. INTRODUCTION

When a large amount of energy is released within a small volume, an explosion will result and a strong shock wave will expand supersonically into the surrounding medium. In many circumstances there will be a phase when the motion of the shock and the variation of the pressure, velocity, etc., behind the shock do not depend significantly on the precise details of the explosion but only on the amount of energy involved and the nature of the undisturbed medium. Such a phase will then be well described by a similarity solution. Appropriate similarity solutions when the motion is nonrelativistic were discovered independently by Sedov[1] and Taylor[2] and are also discussed in Refs. 3 and 4 and in greater mathematical detail by Stanyukovich.[5]

However, when the energy release E is so large that $E \gg (M+\rho V)c^2$, where M is the mass of the explosion products, ρ is the ambient density, and V is the volume swept out by the shock, then the motion of the shock front will be relativistic, which introduces some novel kinematic and dynamical features. In this paper, we demonstrate that similarity solutions can also be found in the ultra-relativistic limit when the explosions are spherically symmetric, and we analyze the physical properties of these solutions, contrasting their behavior with that of their nonrelativistic counterparts. We also discuss strong shocks and blast waves in the mildly relativistic case.

There is evidence for the existence of relativisitic explosions within the nuclei of active galaxies occurring with dynamical time scales of the order of several months,[6] and this is the principal motivation for this work. The application to this and other astrophysical situations will be presented elsewhere.

In Sec. II we review the properties of strong relativistic shocks and present some new relations appropriate for the mildly relativistic case. The fundamental dynamical equations are derived in Sec. III. In Sec. IV similarity solutions are presented that are appropriate when radiative cooling can be ignored. These are generalized to include the possibility that power be supplied continuously by a central source and that the external density vary with radius. Radiative and mildly relativistic blast waves are discussed in Sec. V.

II. RELATIVISTIC STRONG SHOCKS

The jump conditions across an arbitrary gas dynamic shock follow from the continuity of the energy $(w\gamma^2\beta)$, momentum $(w\gamma^2\beta^2+p)$, and particle $(n\gamma\beta)$ flux densities in the shock frame.[3,7] Here, the energy density e, the pressure p, the enthalpy $w=e+p$, and the density n are all measured in the fluid frame. We have set the speed of light c equal to 1. The Lorentz factor γ is related to the velocity β by $\gamma=(1-\beta^2)^{-1/2}$. Note that because of the possibility of particle creation, n must be taken as the density of a set of conserved particles, such as baryons.

For the case of a strong shock, the jump conditions can be solved without making any assumption about the shock velocity. In a strong shock the random kinetic energy per particle behind the shock is much greater than that ahead

$$p_2/n_2 \gg p_1/n_1 , \tag{1}$$

where the subscript 2 denotes the shocked gas and 1 denotes the unshocked gas. Define the quantity $\hat{\gamma}$ (not a Lorentz factor) by the relation

$$p=(\hat{\gamma}-1)(e-\rho) , \tag{2}$$

where ρ is the rest mass density. In simple cases, $\hat{\gamma}$ is just the ratio of specific heats. When the fluid is semirelativistic or when the fluid has two components, one nonrelativistic and the other relativistic (e.g., gas plus radiation or nonrelativistic protons and relativistic electrons), this interpretation is no longer correct; nonetheless, $\hat{\gamma}$ generally lies between $\frac{4}{3}$ and $\frac{5}{3}$. Let γ_2 be the Lorentz factor of the shocked gas measured in the frame of the unshocked gas, and let Γ be the Lorentz factor of the shock itself, also measured in this frame. The jump conditions for an arbitrary strong shock can then be cast into the surprisingly simple form

$$\frac{e_2}{n_2}=\gamma_2\frac{w_1}{n_1} , \tag{3}$$

$$\frac{n_2}{n_1}=\frac{\hat{\gamma}_2\gamma_2+1}{\hat{\gamma}_2-1} , \tag{4}$$

$$\Gamma^2=\frac{(\gamma_2+1)[\hat{\gamma}_2(\gamma_2-1)+1]^2}{\hat{\gamma}_2(2-\hat{\gamma}_2)(\gamma_2-1)+2} . \tag{5}$$

These solutions are appropriate for describing mildly relativistic shocks as well as ultra-relativistic shocks and can be used in conjunction with the interpolated solutions described in Sec. V. Equation (3) has a simple interpretation when $w_1=\rho_1$: as viewed from the frame of the shocked fluid, the energy per particle is constant across the shock.[8]

In the nonrelativistic case the jump conditions for a strong shock can be obtained by setting $p_1=0$. In the relativistic case it is possible for $p_1 \gg \rho_1$, so that the unshocked gas is itself relativistic. In such a case the shock would have to be extremely relativistic in order to be strong, and, in general, one can show that the strong shock condition (1) implies

$$\gamma_2 - 1 \gg \frac{1}{\hat{\gamma}_2 - 1} \frac{p_1}{\rho_1} \left(1 + \frac{\hat{\gamma}_1 p_1}{(\hat{\gamma}_1 - 1)\rho_1}\right)^{-1} . \tag{6}$$

Following the analysis of de Hoffman and Teller,[9] we note that the relations (3)–(5) remain valid in the presence of a transverse magnetic field B if the replacements

$$p \to p + B^2/8\pi, \quad e \to e + B^2/8\pi \tag{7}$$

are made.

We shall be primarily interested in the ultra-relativistic case in which $\hat{\gamma} = \frac{4}{3}$ and $\Gamma \gg 1$. To an accuracy $O(\Gamma^{-1})$ we have

$$p_2 = \tfrac{1}{3} e_2 = \tfrac{2}{3} \Gamma^2 w_1 , \tag{8}$$

$$n_2' = 2\Gamma^2 n_1 , \tag{9}$$

$$\gamma_2^2 = \tfrac{1}{2}\Gamma^2 , \tag{10}$$

where $n_2' = \gamma_2 n_2$ is the density measured in the frame of the unshocked gas. If the gas ahead of the shock is cold, then $w_1 = \rho_1$; if it is relativistic, $w_1 = 4p_1$.

III. BASIC EQUATIONS WITH SPHERICAL GEOMETRY

The basic equations for the motion of an isotropic perfect fluid in the absence of external (including gravitational) forces, dissipation and heat flow are obtained by setting the divergence of the relativistic energy-momentum tensor equal to zero (e.g., Ref. 3). For spherically symmetric outflow, the pressure, energy density, and velocity satisfy

$$\frac{\partial}{\partial t} \frac{(e + \beta^2 p)}{(1-\beta^2)} + \frac{1}{r^2} \frac{\partial}{\partial r} \frac{r^2(e+p)\beta}{(1-\beta^2)} = 0 \tag{11}$$

$$\frac{\partial}{\partial t} \frac{(e+p)\beta}{(1-\beta^2)} + \frac{1}{r^2} \frac{\partial}{\partial r} \frac{r^2(e+p)\beta^2}{(1-\beta^2)} + \frac{\partial p}{\partial r} = 0 \tag{12}$$

in the fixed (r, t) frame. In general, these equations must be solved in conjunction with the equation of particle continuity and an equation of state. However, we immediately specialize by introducing the ultra-relativistic equation of state

$$p = \tfrac{1}{3} e . \tag{13}$$

This will be appropriate as long as the internal energy of the moving fluid is composed mainly of either relativistic particles or radiation and the particle rest masses can be ignored, as is the case in most of what follows. The replacement of Eq. (13) by an equation of state explicitly involving the particle density [e.g., that proposed in Eq. (2)] significantly complicates the analysis. The requirement that the pressure be isotropic precludes the presence of an ordered magnetic field with an energy density comparable to the particle energy density.

Re-arrangment of Eqs. (11) and (12) then leads to the relations

$$\frac{d}{dt}(p\gamma^4) = \gamma^2 \frac{\partial p}{\partial t} , \tag{14}$$

$$\frac{d}{dt} \ln(p^3\gamma^4) = \frac{-4}{r^2} \frac{\partial}{\partial r}(r^2\beta) , \tag{15}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial r}$$

is the convective derivative.

If there are no sources or sinks of particles, the density n' measured in the fixed frame satisfies

$$\frac{\partial n'}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 n' \beta) = 0 . \tag{16}$$

Equations (14), (15), and (16) can be combined to yield

$$\frac{d}{dt}\left(\frac{p}{n^{4/3}}\right) = 0 , \tag{17}$$

where

$$n = \gamma^{-1} n' \tag{18}$$

is the density measured in the co-moving frame. This confirms that entropy is conserved along a flow line, as the relevant specific heat ratio for an ultra-relativistic fluid is $\frac{4}{3}$.

The total energy density (in the fixed frame) is given by $(4\gamma^2 - 1)p$ [assuming Eq. (13)]. Thus, the energy of the blast wave instantaneously contained between radii $R_0(t)$, $R_1(t)$ is given by

$$E(R_0, R_1, t) = \int_{R_0}^{R_1} 16\pi p \gamma^2 r^2 dr \tag{19}$$

to lowest contributing order in γ^{-2}. The expression used in Ref. 10 is in error.

There is, in fact, an exact time dependent solution satisfying energy and momentum conservation. By direct substitution we can confirm that the relations

$$\beta = r/t \tag{20}$$

$$p \propto (\gamma/t)^4 \tag{21}$$

obey Eqs. (14) and (15). We then find that the density must be given by

$$n = t^{-3}\phi(\beta) , \tag{22}$$

where $\phi(\beta)$ is an arbitrary function. If $\phi(\beta) \propto \gamma^3$, the entropy is the same on all flow lines. Equation (20) shows that in this solution the speed of individual fluid elements remains constant with time and that in the co-moving frame, the expansion appears isotropic, i.e., there is no pressure gradient. In general, the establishment of this solution requires specialized initial conditions. However, as we shall show, it can be approximately compatible with boundary conditions imposed by an expanding shock.

IV. ADIABATIC BLAST WAVES

A. Approximate adiabatic similarity solutions

We now consider some approximate similarity solutions appropriate to relativistic blast waves in which there is no flux of energy outward through the shock surface representing the boundary with the undisturbed medium. We do, however, include within the definition "adiabatic" solutions in which there is an internal supply of energy.

In the well-known Sedov similarity solution[1] for a nonrelativistic blast wave of energy E into an external medium of density ρ_0 it is possible to construct a characteristic velocity at time t from the combination $(E/\rho_0 t^3)^{1/5}$. In the corresponding relativistic problem, an additional velocity is introduced naturally into the problem, the speed of light. The derivation of an appropriate similarity solution is therefore less straightforward. One approach, investigated by Eltgroth[10] is to treat β as the similarity variable and search for solutions for the pressure p that are separable in the independent variables β, r. The compatibility of these solutions with the shock jump conditions can then be determined. The approach we shall follow is to choose a slightly different similarity variable that is directly related to the shock conditions and to derive solutions accurate to the lowest contributing order in γ^{-2}.

Three distinct arguments can be advanced to determine the approximate scale height in the shocked fluid. First, from Eq. (9) we see that the density in the shocked fluid exceeds that in the unshocked fluid (for the moment restricted to be uniform) by a factor $\sim\Gamma^2$. Thus, if most of the particles have been swept up at the current shock radius R, the effective thickness of the blast wave will be approximately R/Γ^2. In addition, we observe that if the Lorentz factor of the shocked fluid is maintained at approximately $\Gamma/\sqrt{2}$, a fluid element will fall a distance approximately equal to R/Γ^2 behind the shock in the time it takes the shock radius to double. Finally, from Eqs. (8) and (9), we see that the mean energy per particle in the shocked fluid varies as Γ^2, one factor of Γ arising from the increase in energy measured in the co-moving frame and the second arising from a Lorentz transformation into the fixed frame. If most of the energy is in recently shocked particles, the total energy is proportional to $\Gamma^2 R^3$ and from Eqs. (8) and (19) the thickness of the shell of shocked particles is approximately R/Γ^2.

This suggests that an appropriate choice of similarity variable is

$$\xi = (1 - r/R)\Gamma^2 \geq 0. \tag{23}$$

If the total energy contained in the shocked fluid remains constant, then

$$\Gamma^2 \propto t^{-3}. \tag{24}$$

We shall consider the more general case

$$\Gamma^2 \propto t^{-m}, \quad m > -1, \tag{25}$$

allowing us to treat the case when the energy is supplied continuously at a rate proportional to a power of the time.

From Eq. (25), we find that to $O(\Gamma^{-2}t)$ the shock radius is given by

$$R = t\{1 - [2(m+1)\Gamma^2]^{-1}\}. \tag{26}$$

It is convenient to change the similarity variable to

$$\chi = 1 + 2(m+1)\xi = [1 + 2(m+1)\Gamma^2](1 - r/t) \tag{27}$$

where we have expanded to second contributing order in (Γ^{-2}), which we shall need in evaluating the partial derivatives.

In terms of this variable, we can write the pressure, velocity, and density in the shocked fluid as

$$p = \tfrac{2}{3} w_1 \Gamma^2 f(\chi), \tag{28}$$

$$\gamma^2 = \tfrac{1}{2}\Gamma^2 g(\chi), \tag{29}$$

$$n' = 2n_1 \Gamma^2 h(\chi), \tag{30}$$

where $\chi \geq 1$ and where

$$f(1) = g(1) = h(1) = 1 \tag{31}$$

insures that the shock jump conditions, Eqs. (8)–(10) are satisfied.

Treating Γ^2, χ as new independent variables in place of r, t we find

$$\frac{\partial}{\partial \ln t} \equiv -m\frac{\partial}{\partial \ln\Gamma^2} + [(m+1)(2\Gamma^2 - \chi) + 1]\frac{\partial}{\partial\chi}, \tag{32}$$

$$t\frac{\partial}{\partial r} \equiv -[1 + 2(m+1)\Gamma^2]\frac{\partial}{\partial\chi}, \tag{33}$$

$$\frac{d}{d\ln t} \equiv -m\frac{\partial}{\partial\ln\Gamma^2} + (m+1)\left(\frac{2}{g} - \chi\right)\frac{\partial}{\partial\chi}. \tag{34}$$

It must be emphasized that these relations hold only as long as both the shock and the fluid velocities are ultrarelativistic so that the expansions in Γ^{-2} and γ^{-2} can be truncated at the first contributing term.

After substituting in Eqs. (14)–(16) and some algebra, we arrive at the following equations for f, g, h:

$$\frac{1}{g}\frac{d\ln f}{d\chi} = \frac{8(m-1) - (m-4)g\chi}{(m+1)(4 - 8g\chi + g^2\chi^2)}, \tag{35}$$

$$\frac{1}{g}\frac{d\ln g}{d\chi} = \frac{(7m-4) - (m+2)g\chi}{(m+1)(4 - 8g\chi + g^2\chi^2)}, \tag{36}$$

$$\frac{1}{g}\frac{d\ln h}{d\chi} = \frac{2(9m-8) - 2(5m-6)g\chi + (m-2)g^2\chi^2}{(m+1)(4 - 8g\chi + g^2\chi^2)(2 - g\chi)}. \tag{37}$$

We require the solutions to these equations for $\chi \geq 1$ subject to the boundary conditions (31). However, we must also determine the maximum value of χ for which the solutions are physically relevant. This is most easily done by observing that for a particular fluid element,

$$d\chi/dt \geq 0 \tag{38}$$

in order that it originate in the shock front and be part of a self-similar flow. Equation (34) then gives

$$\chi \leq \chi_c = 2/g(\chi_c) \tag{39}$$

which effectively determines an upper limit for χ.

B. Adiabatic impulsive solution

There is a very simple analytic solution of Eqs. (35)–(37) and boundary conditions (31) when $m = 3$, corresponding to an "impulsive" injection of energy into the blast wave on a time scale short compared with R, namely,

$$f = \chi^{-17/12} , \tag{40}$$

$$g = \chi^{-1} , \tag{41}$$

$$h = \chi^{-7/4} . \tag{42}$$

It is worth examining this solution in a little detail as it illustrates some physical aspects of the more general solution. The functions $f(\chi)$, $g(\chi)$, $h(\chi)$ are displayed in Figs. 1, 2, and 3. In a frame moving radially outward with the shock front, the shocked fluid appears to be moving toward the source of the explosion with a speed increasing with distance from the shock, i.e., being accelerated by the pressure gradient. As $g\chi = 1$ throughout the solution, Inequality (38) is satisfied and the solution remains valid until the bulk motion becomes subrelativistic (in the fixed frame), that is, when χ becomes comparable to Γ^2. The total energy, given by Eq. (19) is

$$E = 8\pi w_1 t^3 \Gamma^2/17 \tag{43}$$

which gives the constant of proportionality in (24).

It is possible to derive Eq. (41) without recourse to Eqs. (35)–(37) using a generalization of the argument applied in Ref. 3 to the nonrelativistic blast wave. In the similarity solution, the energy associated with some interval in χ must remain constant. Therefore, we can equate the energy flowing through a stationary sphere at radius r in a time interval dt to the energy contained in the element of volume swept out by a sphere of constant χ moving with speed β_n in the same time interval. This gives

$$4\pi r^2 w\gamma^2 \beta\, dt = 4\pi r^2 (w\gamma^2 - p)\beta_n\, dt . \tag{44}$$

Now, from Eq. (27),

$$\beta_n = 1 - \chi/2\Gamma^2 + O(\Gamma^{-4}) , \tag{45}$$

and so if we substitute $w = 4p$, we obtain Eq. (41). We

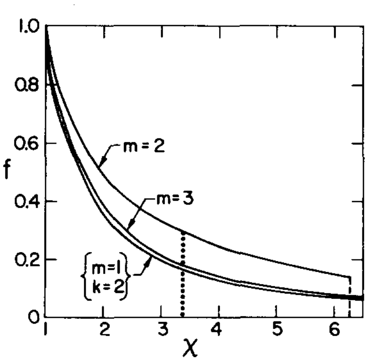

FIG. 1. The normalized pressure f behind the shock front ($\chi = 1$) for three cases: (1) $m = 3$, impulsive energy injection (i.e., constant energy explosion) in a uniform medium; (2) $m = 2$, energy injection $L \propto t_e^{-2/3}$ in a uniform medium; and (3) $m = 1$, $k = 2$, impulsive energy injection in a nonrelativistic, constant velocity wind. The normalized pressure f is unity just behind the shock [Eq. (28)] and the distance behind the shock ($\chi - 1$) is measured in the fixed frame in units of $R/[2(m+1)\Gamma^2]$ [Eq. (27)]. The dotted line in the $m = 2$ curve indicates the position of the contact discontinuity between the swept-up fluid and the injected, relativistic fluid; the dashed line represents the inner shock in the flow of the injected fluid.

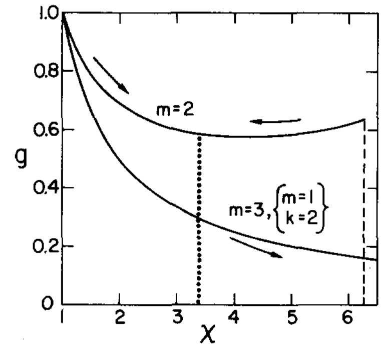

FIG. 2. Velocity variation behind the shock front for the cases discussed in the caption to Fig. 1; g is defined in Eq. (29). The arrows indicate the direction of flow: towards the contact discontinuity when one is present, away from the shock when the contact discontinuity is absent.

therefore see that the velocity field is determined by the assumption of self-similar motion and the relativistic definitions of energy density and flux.

The total number of swept-up particles obtained by integrating n' through the shell using Eqs. (30) and (42) equals $\frac{4}{3}\pi R^3 n_1$, as it must. The mean energy per particle, ϵ, measured in the co-moving frame, is given by

$$\epsilon = \frac{3p\gamma}{n'} = \frac{w_1 \Gamma}{2^{1/2} n_1 \chi^{1/6}} . \tag{46}$$

As long as $\chi \ll \Gamma^2$ and $\Gamma^2 \gg 1$, we see that $\epsilon \gg p/n$, justifying our choice of the ultra-relativistic equation of state, Eq. (13). However, the particles do cool as they flow downstream, expanding and compressing the freshly shocked fluid and keeping the energy stored close to the shock. A further consequence of this cooling is that when the shock becomes mildly relativistic, so will all of the nucleons. This cooling of the particles behind the adiabatic shock is in contrast to the behavior displayed in the nonrelativistic blast wave where the temperature rises quite steeply with distance from the shock.[4]

A second important contrast is that in the nonrelativistic solution, the thermal energy is distributed fairly uniformly throughout the sphere, which provides a pressure that can accelerate a shell containing most of the matter.[4] In the relativistic case, the energy is concentrated in a shell of constant momentum.

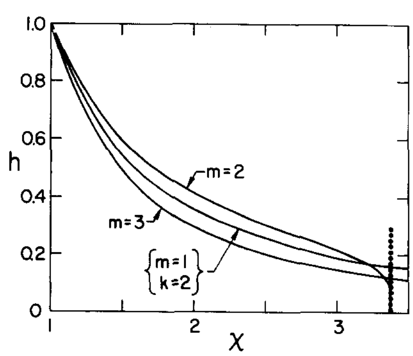

FIG. 3. The normalized density h of the shocked external fluid [see Eq. (30)] for the cases discussed in the caption to Fig. 1.

The solution given here is, in fact, identical to a solution found by Eltgroth[10] (his case III with $f_{21}=\frac{1}{8}$, $a=\frac{13}{4}$, $b=\frac{17}{4}$). This is because the functions $f(\chi)$, $g(\chi)$, $h(\chi)$ are power laws so that the pressure can be written as a product of functions of β, r. As we shall see, in general, this is not the case.

C. Blast wave with energy supply

When $m<3$, the total energy associated with the explosion increases with time, suggesting that the blast wave be matched onto a central power supply. Physically, this power could take a variety of forms, e.g., a cold piston, the absorption of wave modes, or highly relativistic particles. One possibility, that is fairly plausible in an astrophysical context, is that the energy is supplied in a fluid form with a bulk Lorentz factor significantly exceeding Γ so that in the frame of the moving shell, the energy and momentum fluxes of the unshocked fluid are effectively equal. We expect that there will be a second interior shock moving with Γ_s where the incident power is declerated to the speed of the externally shocked fluid.

If this part of the motion is also self-similar, the inner shock will be located at a constant value of $\chi=\chi_s$ which is straightforwardly calculated. From Eq. (45) we see that

$$\Gamma_s^2=\Gamma^2/\chi_s \ . \tag{47}$$

However, for a strong internal shock we know that the shocked internal fluid moves with a speed $c/3$ relative to the internal shock, and therefore with a Lorentz factor in the fixed frame given by

$$\gamma^2=2\Gamma_s^2 \ . \tag{48}$$

From Eq. (29), χ_s is therefore determined by

$$g(\chi_s)\chi_s=4 \ . \tag{49}$$

Thus, the interior shock lies within the contact discontinuity separating the two fluids given by Eq. (39).

We must also balance energy and momentum fluxes across the interior shock. In the frame of the shock (assumed strong and relativistic), the energy flux F_s, which is continuous, is given by

$$F_s=\Gamma_s^2(1-\beta_s)^2\frac{L}{4\pi t^2}\simeq\frac{L\chi_s}{16\pi\Gamma^2t^2} \tag{50}$$

in front of the shock [using Eq. (47)] and by $\frac{3}{2}p(\chi_s)$ behind the shock. Here, L is the power supplied by the central source, measured in the fixed frame. Thus,

$$L=16\pi w_1\Gamma^4t^2f_s\chi_s^{-1} \ , \tag{51}$$

where $f_s=f(\chi_s)$, etc.

The total energy contained in $1\leq\chi\leq\chi_s$ is given by Eq. (19) as

$$E=8\pi w_1\alpha\Gamma^2t^3/3(m+1) \ , \tag{52}$$

with

$$\alpha=\int_1^{\chi_s}fg\,d\chi \ .$$

It is possible to relate the rate of increase of this energy to the power flowing through the interior shock (as measured in the fixed frame). This gives

$$L(1-\beta_s)\simeq\frac{L}{2\Gamma_s^2}=\frac{dE}{dt} \ , \tag{53}$$

and so

$$\alpha=[3(m+1)/(3-m)]f_s \ . \tag{54}$$

If the power is supplied by a stationary source located at $r=0$ and varies with emitted time, t_e, as a power law, i.e.,

$$L=L_0t_e^q \ , \tag{55}$$

then if it propagates effectively with speed c (i.e., with Lorentz factor much greater than Γ_s), we obtain, from Eq. (27),

$$t_e=\tfrac{1}{2}t\chi_s/\Gamma^2(m+1) \tag{56}$$

and so $\Gamma^2\propto t^{(q-2)/(q+2)}$, or

$$m=(2-q)/(2+q) \ . \tag{57}$$

Using Eq. (53), we then obtain

$$\Gamma^2=K\left(\frac{3L_0}{16\pi w_1}\right)^{1/(q+2)}t^{(q-2)/(q+2)} \ , \tag{58}$$

where

$$K=[2^q(m+1)^q(q+1)\alpha]^{-1/(q+2)}\chi_s^{(q+1)/(q+2)} \ . \tag{59}$$

Equation (58) is appropriate as long as the energy, E, increases as a power of time (i.e., $q>-1$). This then restricts m to the range $3>m>-1$. If L decays faster with t_e than t_e^{-1}, then most of the energy will have been supplied at early times and the adiabatic impulsive solution is appropriate. Somewhat different assumptions can be made about the absorption of energy, in which case Eq. (58) will still be valid, but, of course, the constant K will be altered.

Equations (35)–(37) can, in fact, be integrated analytically, and this is demonstrated in the appendix. However, the solutions obtained do not give f, g, h explicitly as functions of χ. Numerical integration is therefore more convenient, and the results for different values of q are displayed in Figs. 1–6. The corresponding values of α are given in Table I.

There are two convenient checks on the accuracy of both the analysis and the computations. One such check is Eq. (54). The second check is provided by requiring that the total number of particles contained between the contact discontinuity and the exterior shock be equal to the total number of particles swept up. This gives

TABLE I. Parameters of similarity solutions for adiabatic blast waves.

m	k	q	χ_c	χ_s	f_c	f_s	g_c	g_s	α	K
3	0	···	···	···	···	···	···	···	0.706	···
2	0	−2/3	3.37	6.28	0.297	0.138	0.593	0.637	1.24	7.50
1	0	0	1.81	2.70	0.711	0.449	1.10	1.48	1.35	1.42
1	2	···	···	···	···	···	···	···	0.667	···
0	0	2	1.27	1.54	1.17	0.977	1.57	2.60	0.977	0.747
0	2	0	1.77	2.51	0.645	0.379	1.13	1.59	1.14	0.858

$$\int_1^{\chi_c} h\, d\chi = (m+1)/3 \; . \tag{60}$$

Particles that have been shocked at very early times will at later times reside close to the contact discontinuity, and for $m > 0$ we see from Eq. (37) that $n' \to 0$ as $g\chi \to 2$, so that the individual baryon energy ϵ becomes infinite there. In practice, of course, the individual particle energies will be limited by radiative losses and deviations from self-similar motion, but it is of interest to calculate the distribution function of the baryons within this portion of the shock. This we can achieve by expanding either Eqs. (35)–(37) or the analytic solution in the appendix about $g\chi = 2$. A straightforward calculation gives, for the distribution function N_ϵ,

$$N_\epsilon \propto \epsilon^{-(m+12)/m} \; . \tag{61}$$

The constant of proportionality decreases with Γ, and we would expect the relationship to break down as $\Gamma \to 1$. Nevertheless, Eq. (61) does demonstrate that as a decelerating shock becomes mildly relativistic, it should be possible to preserve a small fraction of the energy of the shock in the form of ultra-relativistic particles. The slope of this distribution function is always much steeper than that associated with cosmic rays.

D. Blast waves in an external density gradient

A further generalization that is of interest for astrophysical applications is possible, and this is to assume that the external density varies as a power of the radius. This is particularly appropriate for a blast wave propagating through a spherically symmetric wind. We confine our attention to a nonrelativistic external medium that is cold and pressureless with density satisfying $n_1 \propto r^{-k}$. The extension to a relativistic exterior is straightforward.

In place of Eqs. (35)–(37), we then obtain

$$\frac{1}{g}\,\frac{d\ln f}{d\chi} = \frac{4[2(m-1)+k]-(m+k-4)g\chi}{(m+1)(4-8g\chi+g^2\chi^2)} \;, \tag{62}$$

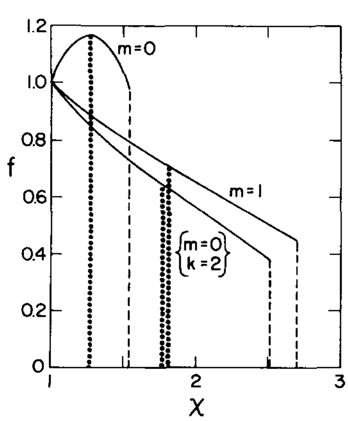

FIG. 4. The normalized pressure behind the shock front for three cases: (1) $m=0$, energy injection $L \propto t_e^2$ in a uniform medium; (2) $m=1$, steady energy injection (L = const) in a uniform medium; and (3) $m=0$, $k=2$, steady energy injection in a nonrelativistic, constant velocity wind. See Fig. 1 for the meaning of the dotted and dashed lines.

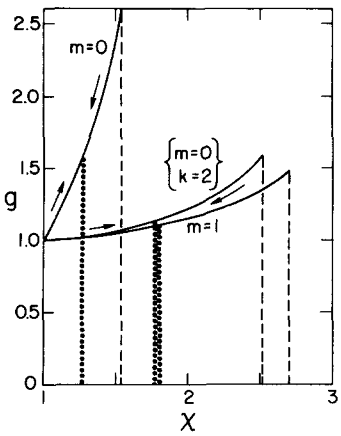

FIG. 5. Velocity variation for the cases discussed in the caption to Fig. 4. As in Fig. 2, the arrows indicate the flow direction.

$$\frac{1}{g}\,\frac{d\ln g}{d\chi} = \frac{(7m+3k-4)-(m+2)g\chi}{(m+1)(4-8g\chi+g^2\chi^2)} \;, \tag{63}$$

$$\frac{1}{g}\,\frac{d\ln h}{d\chi} = \frac{2(9m+5k-8)-2(5m+4k-6)g\chi+(m+k-2)g^2\chi^2}{(m+1)(2-g\chi)(4-8g\chi+g^2\chi^2)} \, . \tag{64}$$

Again, these equations have a very simple analytic solution generalizing Eqs. (40)–(42) appropriate to adiabatic impulsive blast waves:

$$f = \chi^{-(17-4k)/(12-3k)} \;, \tag{65}$$

$$g = \chi^{-1} \;, \tag{66}$$

$$h = \chi^{-(7-2k)/(4-k)} \;, \tag{67}$$

for

$$m = 3 - k > -1 \; . \tag{68}$$

The total energy is therefore given by

$$E = 8\pi\rho_1\Gamma^2 t^3/(17-4k) \;, \tag{69}$$

which is constant (to lowest order). Here, ρ_1 is to be interpreted as the external density at the position of the shock. When $k \geq 4$, i.e., $m \leq -1$, Eq. (26) is invalid and χ is no longer an appropriate similarity variable.

For $-1 < m < 3 - k$, the solution to Eqs. (62)–(64) can again be found analytically as shown in the appendix. The generalization of Eq. (57) is

$$m = (2-q-k)/(2+q) \; . \tag{70}$$

One case of special interest is $k=2$, $q=0$, $m=0$, which corresponds to a constant power source feeding a blast wave in a constant velocity wind. The shock then expands with constant velocity. The solutions for this case are displayed in Figs. 4–6. The relationship between L and Γ^2 is

$$\Gamma^2 = (L\chi_s/16\pi\rho_1 t^2\alpha)^{1/2} \tag{71}$$

or

$$K = (\chi_s/\alpha)^{1/2} \; . \tag{72}$$

More generally, Eqs. (58) and (59) have been written in a form so as to be applicable in the presence of an external density gradient.

From Figs. 4–6, it can be seen that with $q = m = 0$

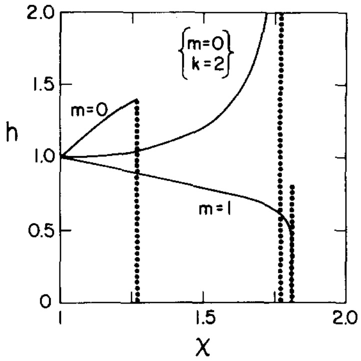

FIG. 6. The normalized density in the shocked external fluid for the cases discussed in the caption to Fig. 4.

the most energetic particles are those that have just been shocked, and the particles cool as they approach the contact discontinuity; in general, this will be true if $q<(2-3k)/(1+k)$.

Table I summarizes the values of various parameters of the similarity solutions for adiabatic blast waves for several cases of interest. In particular, $m=3$ and $k=0$ corresponds to an impulsive blast wave in a uniform medium; $m=1$, $k=0$, and $q=0$ corresponds to a blast wave with a constant power supply in a uniform medium; $m=1$, $k=2$ corresponds to an impulsive blast wave in a constant velocity wind; and, as mentioned, $m=0$, $k=2$, and $q=0$ corresponds to a blast wave with a constant power supply in a constant velocity wind.

Finally, we note that the exact analytic solution [Eqs. (20)–(22)] is appropriate when $k=7/2$, $m=1/2$, and the explosion is implusive. However, this is no more accurate than the ultra-relativistic solutions since it satisfies the shock jump conditions only to lowest order in Γ^{-1}. Thus, when the energy is constant and the external density falls as $r^{-7/2}$, the pressure of the shocked particles remains constant and there is no pressure gradient. The density n' compatible with the shock jump conditions is given by Eq. (22) with $\phi(\beta)=\gamma^{-1}$.

V. RADIATIVE AND SEMI-RELATIVISTIC BLAST WAVES

In the similarity solutions obtained here, we have assumed that the flow is adiabatic and extremely relativistic. We now evaluate the effect of changing either or both of these assumptions, since they may not be satisfied in applications. After passage through the shock, the gas may radiate all its internal energy (the radiative case), and even if the explosion has an extreme relativistic, self-similar phase, it may eventually decelerate so that the shock velocity is only mildly relativistic. Rather than attempt to discover similarity solutions to cover these cases, our approach is to find approximate expressions for the shock velocity as a function of time which are correct in both the nonrelativistic and ultra-relativistic limits; the semi-relativistic results can then be estimated by interpolation. For simplicity, we shall restrict attention to a uniform, cold external medium and consider only two types of energy injection, impulsive and steady. The generalization to include power law luminosities and density gradients is straightforward using the methods of the previous section. We continue to assume that the fluid is described by a constant ratio of specific heats $\hat{\gamma}$, although as discussed in Sec. II in reality this is not always true.

Remarkably enough, three of the resulting four cases can be described by a single equation based on energy conservation: the adiabatic impulsive, adiabatic steady injection, and radiative steady injection cases. An exact solution can be found for the remaining radiative impulsive case.

A. Energy conservation analysis

The dynamics of the blast wave in all but the radiative impulsive case can be treated approximately by requiring that the total energy contained in *or* radiated by the expanding fluid E_e be proportional to the total energy supplied E_s. An estimate of E_e valid in both nonrelativistic and ultra-relativistic limits is

$$E_e=w_1\Gamma^2\beta^2V\ , \tag{73}$$

where $\beta\equiv(1-\Gamma^{-2})^{1/2}$ is the shock velocity and $V=\frac{4}{3}\pi R^3$ is the volume swept out by the shock. In the impulsive case, E_s is simply the energy of the explosion E.

In the steady injection case, we assume the energy from the central source is supplied to the shocked exterior fluid by means of an interior fluid with a high energy per particle. This interior fluid passes through a shock and pushes on the shocked exterior fluid. In the ultra-relativistic case, we saw in the previous section how the interior shock is located a distance of order R/Γ^2 from the exterior shock and that the total energies contained in the shocked interior and exterior fluids are comparable; thus in this case $E_s\approx Lt/\Gamma^2$.

In the nonrelativistic steady injection case, we assume that the interior shock is at a radius $r\ll R$ and that the temperature of the shocked interior fluid is high enough so that the fluid is effectively isobaric. If the volume enclosed by the contact discontinuity is V_c and the specific heat ratio of the shocked interior fluid is $\hat{\gamma}_i$, then the first law of thermodynamics gives

$$L=(\hat{\gamma}_i-1)^{-1}\ \frac{d(pV_c)}{dt}+p\ \frac{dV_c}{dt}\ . \tag{74}$$

From the equation of motion for the shocked exterior fluid one finds $V_c\propto t^{9/5}$ and so E_s, the work done by the shocked interior fluid, is given by

$$E_s=\int_0^{V_c}p(V_c')\,dV_c'=\tfrac{3}{8}\,\kappa(\hat{\gamma}_i)\,Lt\ , \tag{75}$$

where

$$\kappa(\hat{\gamma}_i)=24(\hat{\gamma}_i-1)/(9\hat{\gamma}_i-4)\ . \tag{76}$$

This expression is valid for both the adiabatic and radiative steady injection cases and has been normalized to unity at $\hat{\gamma}_i=\frac{4}{3}$. Thus, an *estimate* for E_s which applies in both the nonrelativistic and ultra-relativistic limits is

$$E_s\approx\kappa(\hat{\gamma}_i)Lt/\Gamma^2 \tag{77}$$

since in the ultra-relativistic limit $\hat{\gamma}_i=\frac{4}{3}$ and $\kappa=1$.

TABLE II. Values of σ.

	Nonrelativistic						Ultra-relativistic
	$\hat{\gamma}=1.2$	$\frac{4}{3}$	1.4	$\frac{13}{9}$	$\frac{5}{3}$	2	$\frac{4}{3}$
Adiabatic impulsive	2.58	1.57	1.30	1.16	0.73	0.48	0.35
Adiabatic steady injection	2.83	2.43	2.27	2.17	1.79	1.40	2.0
Radiative steady injection (independent of $\hat{\gamma}$)			···	3.73	···		6.0

We now define a quantity σ so as to make an *exact* relation between our *estimates* of E_e and E_s:

$$E=\sigma w_1 \Gamma^2 \beta^2 V \qquad \text{(impulsive)} \tag{78}$$

$$Lt=\sigma w_1 \Gamma^4 \beta^2 V/\kappa(\hat{\gamma}_i) \quad \text{(steady injection)} . \tag{79}$$

Note that in the ultra-relativistic limit, Eq. (79) can be interpreted as stating that in the shock frame the injected momentum flux $L\Gamma^2(1-\beta)^2/4\pi R^2$ is proportional to the momentum flux of the unshocked gas $w_1\Gamma^2\beta^2$. We now demonstrate that σ reduces to a constant in the non- and ultra-relativistic limits and evaluate it for several values of $\hat{\gamma}$, the ratio of specific heats of the external shocked fluid. (The value $\hat{\gamma}=13/9$ corresponds to a gas of ultra-relativistic electrons and nonrelativistic protons constrained to be at the same temperature.)

(1) Adiabatic impulsive case. The nonrelativistic limit of this case corresponds to the Sedov blast wave.[1] In this limit Eq. (78) reduces to $E=\frac{4}{3}\pi\sigma\rho_1\beta^2R^3$, which can be integrated to give

$$R=(75E/16\pi\sigma\rho_1)^{1/5}t^{2/5} . \tag{80}$$

Comparison of this expression with Sedov's results gives the values of σ in Table II.

In the ultra-relativistic limit, the similarity solution in Eq. (43) satisfies Eq. (78) with $\sigma=6/17$.

(2) Adiabatic steady injection case. The nonrelativistic limit of Eq. (79) can be integrated to give

$$R=\left(\frac{25\kappa(\hat{\gamma}_i)L}{12\pi\sigma\rho_1}\right)^{1/5}t^{3/5} . \tag{81}$$

The similarity solution for this problem (which will be described elsewhere) gives a result of this form and yields the values of σ in Table II. Note that all dependence upon $\hat{\gamma}_i$ has been incorporated into $\kappa(\hat{\gamma}_i)$ so that σ is independent of $\hat{\gamma}_i$.

The result for the ultra-relativistic case is given in Eqs. (58) and (59) with the values of α and χ_s from Table I; it agrees with Eq. (79) for $\sigma=2.0$.

(3) Radiative steady injection case. This case has recently been considered in a model for the cavity inflated by a stellar wind from a hot star.[11] Because the swept-up matter can cool, it forms a shell of negligible thickness whose momentum equals the total impulse applied at the contact discontinuity separating it from the hot interior fluid, which is isobaric. Equation (81) also applies in this case, and a straightforward calculation then gives $\sigma=56/15$. This is independent of $\hat{\gamma}$ because the internal energy of the shell is negligible. The ratio of the kinetic energy of the shell to the total energy radiated is 5/9, independent of $\hat{\gamma}_i$.

A similarity solution is unavailable for the ultra-relativistic case, but the shock dynamics can be obtained from considerations of momentum conservation. Because the shell of swept-up matter has radiated away all its internal energy, the momentum flux of the shell is of order $\rho_1\Gamma$. Comparison with the adiabatic results shows that this is smaller than the momentum flux supplied by the energy source by a factor approximately equal to Γ and so can be neglected. Hence, the injected momentum flux must balance the momentum flux of the emitted radiation and that of unshocked fluid. In the shocked fluid frame the emitted radiation is isotropic and therefore, carries off no net momentum flux. There is a slight complication in that the fluid accelerates from $2^{-1/2}\Gamma$ to Γ as it radiates. We estimate that the integrated momentum flux of radiation emitted by a fluid element is isotropic in a frame moving at approximately $2^{-1/4}\Gamma$. Equating the injected momentum flux and the momentum flux of the unshocked fluid in this frame gives

$$\Gamma^4=L/8\pi w_1 R^2 \tag{82}$$

which agrees with Eq. (79) for $\sigma=6$. Note that in this case the right-hand side of Eq. (79) represents the energy of the emitted radiation rather than that of the shell.

The results for all the cases are summarized in Table II. The fact that the non- and ultra-relativistic values of σ for a given case generally differ by no more than a factor of 5 suggests that Eqs. (78) and (79) should provide reasonable approximations for semi-relativistic blast waves with σ having a value between the two limits.

B. Radiative impulsive case

The dynamics of a radiating blast wave are quite different if the energy is supplied impulsively. Treating the swept-up matter as lying in a thin, cold shell allows one to obtain an exact solution, valid for all velocities. Let E_k be the kinetic energy of the shell

$$E_k=\rho_1 V(\Gamma-1) . \tag{83}$$

In the shock frame, the entire kinetic energy flux is radiated away. Since the emitted power is Lorentz invariant, we have in the fixed frame

$$dE_k/dt=-4\pi R^2\beta(\Gamma^2\rho_1-\Gamma\rho_1) . \tag{84}$$

Because $\beta\, dt=dR$ and the swept up mass is $M=4\pi R^3\rho_1/3$, this can be written

$$d\Gamma/dM=-(\Gamma^2-1)/M , \tag{85}$$

which has the solution

$$\Gamma-1=2\left(\frac{M^2(\Gamma_0+1)}{M_0^2(\Gamma_0-1)}-1\right)^{-1} , \tag{86}$$

where Γ_0 and M_0 are the initial values. If the explosion is extremely relativistic ($\Gamma_0\gg 1$), then $\Gamma-1$ is reduced to $\frac{2}{3}$ by the time $M=2M_0$, or $R=1.26\,R_0$. Hence, the ultra-relativistic phase of a radiative impulsive blast wave is very short-lived. When the expansion is nonrelativistic, Eq. (86) reduces to

$$Mv = 2M_0\left(\frac{\Gamma_0 - 1}{\Gamma_0 + 1}\right)^{1/2}, \tag{87}$$

so that the total momentum is constant. This corresponds to the late isothermal phase of the expansion of supernova remnants.[12]

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APPENDIX: ANALYTIC SOLUTIONS FOR AN ADIABATIC BLAST WAVE

In this appendix, we derive the analytic solutions to Eqs. (62)–(64). The solutions to Eqs. (35)–(37) can be obtained by setting $k = 0$.

We introduce as a new independent variable in place of χ,

$$x = g\chi . \tag{A1}$$

Equations (62)–(64) can then be rewritten

$$\frac{d\ln f}{dx} = \frac{c_1 + d_1 x}{4 + (a_1 - 8)x - (b_1 - 1)x^2}, \tag{A2}$$

$$\frac{d\ln g}{dx} = \frac{a_1 - b_1 x}{4 + (a_1 - 8)x - (b_1 - 1)x^2}, \tag{A3}$$

$$\frac{d\ln h}{dx} = \frac{(a_2 - b_2 x + c_2 x^2)}{(2 - x)[4 + (a_1 - 8)x - (b_1 - 1)x^2]}, \tag{A4}$$

where

$$\begin{aligned} a_1 &= \frac{7m + 3k - 4}{m + 1}, \quad b_1 = \frac{m + 2}{m + 1}, \\ c_1 &= \frac{4(2m + k - 2)}{m + 1}, \quad d_1 = \frac{-(m + k - 4)}{m + 1}, \\ a_2 &= \frac{2(9m + 5k - 8)}{m + 1}, \quad b_2 = \frac{2(5m + 4k - 6)}{m + 1}, \\ c_2 &= \frac{m + k - 2}{m + 1}. \end{aligned} \tag{A5}$$

These equations can be straightforwardly integrated using $f(1) = g(1) = h(1) = 1$ to yield the expressions

$$f = \left(\frac{x^2 + 2\alpha_1 x - 8\beta_1}{1 + 2\alpha_1 - 8\beta_1}\right)^{-\alpha_2} \left(\frac{(x + \alpha_1 + \gamma_1)(1 + \alpha_1 - \gamma_1)}{(x + \alpha_1 - \gamma_1)(1 + \alpha_1 + \gamma_1)}\right)^{\beta_2/\gamma_1}, \tag{A6}$$

$$g = \left(\frac{x^2 + 2\alpha_1 x - 8\beta_1}{1 + 2\alpha_1 - 8\beta_1}\right)^{(1/2+\beta_1)} \left(\frac{(x + \alpha_1 + \gamma_1)(1 + \alpha_1 - \gamma_1)}{(x + \alpha_1 - \gamma_1)(1 + \alpha_1 + \gamma_1)}\right)^{[\alpha_1(\beta_1 - 1/2) + 8\beta_1]/\gamma_1}, \tag{A7}$$

$$h = \left(\frac{x^2 + 2\alpha_1 x - 8\beta_1}{1 + 2\alpha_1 - 8\beta_1}\right)^{-\gamma_2} \left(\frac{(x + \alpha_1 + \gamma_1)(1 + \alpha_1 - \gamma_1)}{(x + \alpha_1 - \gamma_1)(1 + \alpha_1 + \gamma_1)}\right)^{\eta/\gamma_1} (2 - x)^{-\theta}, \tag{A8}$$

where

$$\alpha_1 = \frac{m - 3k + 12}{2}, \quad \beta_1 = \frac{m + 1}{2}, \quad \gamma_1 = (\alpha_1^2 + 8\beta_1)^{1/2},$$

$$\alpha_2 = \frac{-(m + k - 4)}{2}, \quad \beta_2 = \frac{(m - 3k)(m + k) + 8(3m + 4k - 8)}{4}, \quad \gamma_2 = \frac{-(m^2 + 4mk + 3k^2 - 13m - 19k + 24)}{2(m + 3k - 12)},$$

$$\eta = \frac{2(7m^2 + 34mk - 118m + 15k^2 - 82k + 96) + (m - 3k + 12)(m^2 + 4mk + 3k^2 - 13m - 19k + 24)}{4(m + 3k - 12)}, \quad \theta = \frac{m - k}{m + 3k - 12}.$$

The analytic forms (A6)–(A8) agree with the results of the numerical integrations.

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