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THE EVOLUTION OF GALAXIES. I. FORMULATION AND MATHEMATICAL BEHAVIOR OF THE ONE-ZONE MODEL

RAYMOND J. TALBOT, JR., AND W. DAVID ARNETT*†
Department of Space Science, Rice University, Houston
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ABSTRACT

A procedure is discussed for calculating the evolution of a closed system of gas and stars. Numerical and analytical solutions are given for a simple set of prescriptions for the stellar birth rate and the evolutionary end state of stars. The various features of the numerical results are easily understood through the analytic solutions. More significantly, the analytic solutions clearly display how the results will change when the astrophysical input is varied.

A fundamental result, which previous investigators have not stressed, is that the metal content Z of the gas in a galaxy need not be a monotonically increasing function of time *even if the system is homogeneous in space.* We show that under conditions which lead to only a small fraction of mass in the form of gas, there generally may exist an early period with Z much larger than the present value. We propose that the very old super-metal-rich stars under investigation by Spinrad and others could have been formed during such a period.

I. INTRODUCTION

A theoretical description of the evolution of a galaxy requires the specification of initial conditions for the various constituents, prescriptions for star formation and the ejecta from stars, and a scheme for following the properties of the system in time and space. Several attempts toward solution of this problem have been made (Salpeter 1955, 1959; Schmidt 1959, 1963; Limber 1960; Fowler and Hoyle 1960; Truran, Hansen, and Cameron 1965; Partridge and Peebles 1967*a*, *b;* Tinsley 1968; Reddish and Wickramasinghe 1969; Matsuda 1970; Fowler 1970; Cameron and Truran 1971). All of these investigations have considered the case of an initial mixture of gas (or gas plus stars) which remains homogeneous in space throughout the calculation. This simplification, which we shall call the one-zone model, allows formulation of the problem with time as a single independent variable. Some of these investigations have attempted a physical description of the stellar birth rate through a parametrization in terms of physical variables (such as the gas density). Others have assumed simple analytic prescriptions which possess the qualitative features of the investigator's physical picture of the process. The accuracy of the nucleosynthesis aspects of the early papers was greatly hampered by wide gaps in the theory of stellar evolution.

In this paper we will carefully reconsider the formulation of the one-zone model, and examine its mathematical behavior by numerical and by analytical means. In particular we will demonstrate the existence of an effect which has not been emphasized by previous investigators. In some cases we find that *the metal abundance in the interstellar gas does not increase monotonically with time.* This paper will deal with the mathematical aspects of this effect for relatively simple astrophysical assumptions; the intent is to provide simple analytical solutions with which to understand the mathematical behavior of the one-zone model. These analytical solutions will aid in the interpretation of the numerical results for the more realistic astrophysical assumptions that we will employ in future work.

* Alfred P. Sloan Foundation Fellow.
† Present address: University of Texas, Austin.

Section II of this paper describes the mathematical framework for the one-zone computations, and § III discusses the numerical techniques that we employ. This framework is general and without reference to a specific rate of star formation or prescription for the evolution of stars. With a view toward providing a simple conceptual picture to aid in understanding future numerical results, § IV presents a simplified set of astrophysical assumptions together with analytical and numerical solutions. Section V briefly discusses these results.

II. MATHEMATICAL FRAMEWORK

The one-zone model of galactic evolution follows the general principles set forth in the references listed in the introduction. A constant total mass $\mathfrak{M}$ is contained within a volume V which is not necessarily constant. The mass is considered to be distributed between gas and stars according to

$$\mathfrak{M}_{\rm gas}(t) = \mathfrak{M}\mathcal{G}(t) \tag{1}$$

and

$$\mathfrak{M}_{\rm stars}(t) = \mathfrak{M}[1 - \mathcal{G}(t)] = \mathfrak{M}[\mathcal{S}(t) + \mathcal{D}(t)]\,, \tag{2}$$

where $\mathcal{G}$ is the mass fraction of the gas, $\mathcal{S}$ is the mass fraction of stars burning nuclear fuel, and $\mathcal{D}$ is the mass fraction of "dead stars." The latter is our terminology for the stellar remnants of stars having lived out their nuclear-burning lifetimes and ejected (perhaps) some of their mass back into the interstellar medium. Specifically, we place white dwarfs, neutron stars, and black holes in the category of dead stars and denote the fraction of total mass in each subcategory by $\mathcal{D}_{\rm WD}$, $\mathcal{D}_n$, and $\mathcal{D}_{\rm BH}$.

The gas is considered to be homogeneous (the volume well mixed) and to consist of a mixture of elemental or isotopic species specified by the set of abundance mass fractions $\{X_i,\ i = 1,\ i_{\rm max}\}$, $\Sigma_i X_i = 1$, where Σ_i will denote summation from $i = 1$ to $i_{\rm max}$. Our computations employ as their fundamental dependent variables the members of the set $\{\mathcal{G}_i\}$, where $\mathcal{G}_i = \mathcal{G}X_i$. The mass fraction of stars $\mathcal{S}$ and dead stars $\mathcal{D}$ are derived quantities. We also define the mass fraction of stars which possess individual masses in the stellar mass interval m to $m + dm$; this is denoted by $\mathcal{S}_m dm$ or $\mathcal{D}_m dm$.

Throughout these investigations there is the possibility of confusion between quantities measured per unit mass *of the system* and those measured per unit *stellar* mass. To minimize this we adopt the symbol m for the mass of an individual star and the symbol $\mathfrak{M}$ for the mass of a system of gas and/or stars (all masses will always be measured in solar units). Furthermore, all variables (and only those) which denote quantities measured per unit total mass $\mathfrak{M}$ are given script symbols.

The rate of change of $\mathcal{G}_i$ is given by

$$\frac{d\mathcal{G}_i}{dt} = -\mathcal{B}(t)X_i(t) + \int_0^\infty \mathcal{B}_m(t - \tau_m)R_{mi}(t - \tau_m)dm\,, \tag{3}$$

where τ_m is the lifetime of a star of mass m, $\mathcal{B}_m dm$ is the birth rate of stars in the mass interval m to $m + dm$,

$$\mathcal{B}(t) = \int_0^\infty \mathcal{B}_m(t)dm \tag{4}$$

is the birth rate of stars of all masses, and R_{mi} is the fraction of the mass of a star of mass m which is ejected back into the interstellar medium in the form of species i. In general one expects $\mathcal{B}_m$ to be a function of the gas composition, density, temperature, magnetic field, etc. The function τ_m depends upon the initial composition of the star, as does R_{mi}. From the definition of $\mathcal{G}$, it is clear that the mass per unit time converted from gas into stars is $\mathfrak{M}\mathcal{B}$. If multiplied by $\mathfrak{M}$, the integrand of the second term on the right-hand side of equation (3) would be the mass per unit time returned to the inter-

stellar gas in the form of species i by stars of mass m. The death rate of stars of mass m at t is equated to their birth rate at $t - \tau_m$. For $t < 0$, $\mathfrak{B}(t)$ is defined to be zero.

We further define

$$R_m = \Sigma_i R_{mi} , \tag{5}$$

the total mass fraction ejected by stars of mass m; $\mathfrak{S}_m$ and $\mathfrak{S}$ are given by

$$\mathfrak{S}_m(t) = \int_{t-\tau_m}^{t} \mathfrak{B}_m(t')dt' \quad \text{and} \quad \mathfrak{S}(t) = \int_0 \mathfrak{S}_m(t)dm . \tag{6}$$

The mass fraction $\mathfrak{D}$ is most easily found from $\mathfrak{D} = 1 - \mathfrak{G} - \mathfrak{S}$; alternatively, $\mathfrak{D}_m$ or the various subcategories of $\mathfrak{D}$ may be expressed as integrals of $\mathfrak{B}_m$.

Time will always be measured in units of 10^9 years.

III. NUMERICAL METHOD

The set of nonlinear integro-differential equations (3) may be rewritten as

$$\mathfrak{G}_i(t_2) = \mathfrak{G}_i(t_1) \exp [-\chi(t_1, t_2)] + \int_{t_1}^{t_2} \mathfrak{W}_i(t) \exp [-\chi(t, t_2)]dt \tag{7}$$

where

$$\chi(t_1, t_2) \equiv \int_{t_1}^{t_2} \tilde{\nu}(t)dt , \tag{8}$$

$$\tilde{\nu}(t) \equiv \mathfrak{B}(t)/\mathfrak{G}(t) , \tag{9}$$

and

$$\mathfrak{W}_i(t) \equiv \int_0^\infty \mathfrak{B}_m(t - \tau_m)R_{mi}(t - \tau_m)dm$$

$$= \int_0^{t-\tau_\infty} [\mathfrak{B}_m(t')R_{mi}(t')(-dm/d\tau_m)]_{m(t-t')}dt' , \tag{10}$$

where $m(t)$ is the mass of a star with lifetime t. By τ_∞ we denote the lifetime of a star of infinite mass. The integration over a time step $\Delta t = t^{n+1} - t^n$ is accomplished by considering $\tilde{\nu}$ and $\mathfrak{W}_i$ to be constant over that time interval; they are each evaluated at $t^{n+1/2} = t^n + \frac{1}{2}\Delta t$, and for this reason an iteration process is required. With this approximation

$$\mathfrak{G}_i(t^{n+1}) = \mathfrak{G}_i(t^n) \exp (-\tilde{\nu}\Delta t) + \mathfrak{W}_i\tilde{\nu}^{-1}[1 - \exp (-\tilde{\nu}\Delta t)] . \tag{11}$$

This displays an important property of the system which will be emphasized in the next section, to wit, the gas loses "memory" of preceding conditions and tends toward a semi-equilibrium state in which

$$\mathfrak{G}_i(t) \simeq \mathfrak{W}_i(t)/\tilde{\nu}(t) . \tag{12}$$

It is possible to improve upon the accuracy of equation (11) by assuming $\tilde{\nu}(t) = \tilde{\nu}(t_1) + [\tilde{\nu}(t_2) - \tilde{\nu}(t_1)](t - t_1)/(t_2 - t_1)$, a similar expression for $\mathfrak{W}_i$, and analytically integrating the resulting expression. However, the conditions under which significant improvements would be obtained are the same as those for which small time steps are desired for other reasons; consequently, the simpler formulation has been satisfactory.

The iterative solution of equation (11) may be performed in one of two ways: (*a*) iterate upon $\tilde{\nu}$ alone or (*b*) iterate upon $\tilde{\nu}$ and $\mathfrak{W}_i$ together. Iteration (*b*) provides very

little improvement over (*a*) for two reasons: $\tilde{\nu}$ enters in the exponentials whereas $\mathcal{W}_i$ enters only linearly; and $\tilde{\nu}$ involves $\mathcal{G}$ at only t^n and t^{n+1} whereas $\mathcal{W}_i$ is an integral over all past values of $\mathcal{G}$ and hence is less sensitive to $\mathcal{G}(t^{n+1})$ than is $\tilde{\nu}$. Numerical experiments have shown that iteration (*a*) is adequate.

The solution is obtained by defining $\mathcal{G}_i^{(k)}$ as the value of the best estimate for $\mathcal{G}_i(t^{n+1})$ at the kth iteration and calculating successive corrections $\delta_i^{(k+1)}$ where $\mathcal{G}_i(t^{n+1}) = \mathcal{G}_i^{(k)}[1 + \delta_i^{(k+1)}]$; we may take $\mathcal{G}_i^{(0)}$ to be $\mathcal{G}_i(t^n)$. From equation (11) one may derive

$$\delta_i^{(k+1)} = (\mathcal{G}_i^{(k)} - A_i)/(B_i - \mathcal{G}_i^{(k)}) ,$$

where

$$A_i = \exp(-\tilde{\nu}'\Delta t)[\mathcal{G}_i(t^n) - \mathcal{W}'_i/\tilde{\nu}'] + \mathcal{W}'_i/\tilde{\nu}' ,$$

$$B_i = \tfrac{1}{2}(\partial \ln \tilde{\nu}/\partial \ln \mathcal{G})\{[\exp(-\tilde{\nu}'\Delta t) - 1]\mathcal{W}'_i/\tilde{\nu}' - \tilde{\nu}'\Delta t \exp(-\tilde{\nu}'\Delta t)[\mathcal{G}_i(t^n) - \mathcal{W}'_i/\tilde{\nu}']\} ,$$

and the current midpoint values of $\mathcal{W}_i$ and $\tilde{\nu}$ are denoted by primes. With these $\delta_i^{(k+1)}$ values, the $\mathcal{G}_i^{(k)}$ are updated to $\mathcal{G}_i^{(k+1)} = \mathcal{G}_i^{(k)}[1 + \delta_i^{(k+1)}]$. The iteration continues until each $\delta_i^{(k+1)}$ is less than some preassigned value $\delta_{\max}$.

The evaluation of $\mathcal{W}_i(t)$ involves an integration over a specific path in the (m, t)-plane. This is easily facilitated by keeping every past value of $\mathcal{B}_m(t^n)$ in the computer memory throughout the calculation. Also, in general, R_{mi} depends upon the composition of the gas at time $t - \tau_m$; consequently it is necessary to have every past value of $\mathcal{G}_i(t^n)$ in memory throughout the calculation. [It is not necessary to store $\mathcal{B}_m(t)$ since it is a function of $\mathcal{G}_i(t)$ and other quantities; however, if machine storage is available, it saves repetitive recomputions of the function.]

We have divided the mass range into a set of mass intervals; each interval is denoted by the index μ which runs from 1 to $\mu_{\max}$. Integrals over m are replaced by sums:

$$\int_0^\infty f(m)dm \to \sum_\mu f_\mu \equiv \sum_{\mu=1}^{\mu_{\max}} f_\mu . \tag{13}$$

The upper boundary of each mass interval is m_μ; the lower boundary of the $\mu = 1$ interval, m_0, is taken to be zero. The indexed function f_μ is defined by

$$f_\mu = \int_{m_{\mu-1}}^{m_\mu} f(m)dm . \tag{14}$$

Some specific quantities of interest which are defined in this fashion are $\mathcal{B}_\mu(t)$, the birth rate of stars in the μth mass interval; and

$$\mathcal{S}_\mu(t) = \int_{t-\tau_{\mu-1}}^{t} \mathcal{B}_\mu(t')dt' , \tag{15}$$

the fraction of total mass which resides in the stars in the μth mass interval [$\tau_{\mu-1} = \tau(m_{\mu-1})$].

In performing the $\mathcal{W}_i$ and $\mathcal{S}$ integrals (summations), care must be taken that the integrands are properly calculated by interpolation in the (m_μ, t^n)-grid; otherwise unphysical discontinuities and other errors appear in the results.

The size of the time-integration errors is controlled by the step-size criteria. Our criteria is that the initial value of Δt is the minimum of $0.3\epsilon/\tilde{\nu}$ and $\epsilon\tau_\infty$, and thereafter Δt is set such that the maximum fractional change of any nonzero $\mathcal{G}_i$ is ϵ; however, Δt is never allowed to increase by more than a factor of 1.5 per time step. We have empirically found that with this criterion (and our specific set of mass intervals and other assump-

tions) $\epsilon = 0.08$ yields results which differ by less than 1 percent from results computed with $\epsilon = 0.04$. As the core-storage requirements and computing time increase with the number of time steps, we have employed $\epsilon = 0.08$ in this initial set of investigations.

The set of mass zones $\{m_\mu\}$ we have employed is likewise a compromise between computational tractability and accuracy. As we are concerned with a range of at least a factor of 10^3 in the stellar mass (more significant, at least a factor of 10^4 variation in stellar lifetimes), a set of 30 mass zones provides rather sparse coverage of the range; however, experimentation with finer mass zoning in one of the cases most sensitive to these errors showed that a wisely chosen 30-zone configuration gave fractional errors of less than a few percent in all cases and generally less than 1 percent. The most appropriate choice of mass zones depends sensitively upon the variation of $\mathfrak{B}_m$ and R_{mi} with m. In these initial investigations we have accepted these errors of a few percent; higher numerical precision in $\mathfrak{B}_\mu$ and $R_{\mu i}$ is superfluous owing to the physical uncertainties in the prescriptions employed for $\mathfrak{B}_m$ and R_{mi}. These errors are systematic and in no way affect any of the qualitative features of the solutions. Table 1 gives the set of mass zones employed in the numerical examples in this paper.

IV. SOLUTIONS AND APPROXIMATIONS

In order to study the behavior of the system of equations (3), it is convenient to assume the following specific forms for the functions $\mathfrak{B}_m(t)$ and $R_{mi}(t)$:

$$\mathfrak{B}_m(t) \equiv \nu \mathcal{G}^\kappa(t) \Psi_m \tag{16}$$

(except for notation, the power-law representation is the form investigated by Schmidt 1959) and

$$R_{mi}(t) \equiv \Sigma_j Q_{mij} X_j(t) , \tag{17}$$

where ν and κ are constants and Ψ_m is the initial mass function which is taken to be time independent and is normalized to

$$\int_0^\infty \Psi_m dm = 1 .$$

The quantity Q_{mij} is a time-independent matrix for each stellar mass which specifies the mass fraction of the star in which material initially in the form of species j is ejected as species i (diagonal elements represent the mass fraction in which species i is unprocessed and ejected).

With these definitions we may rewrite equations (3) as

$$\frac{d\mathcal{G}}{dt} + \nu \mathcal{G}^\kappa(t) = \mathfrak{W}(t) \tag{18}$$

and

$$\frac{dX_i}{dt} = \frac{1}{\mathcal{G}(t)} [\mathcal{P}_i(t) + \mathfrak{U}_i(t) - X_i(t)\mathfrak{W}(t)] , \tag{19}$$

TABLE 1

BOUNDARIES OF MASS ZONES (Solar Masses)

0.20	1 17	2 35	7.0	50
0.50	1.31	2 70	8 4	70
0.70	1.45	3.20	10.0	120
0.85	1 60	3 80	12 5	260
0 94	1.80	4.60	18.0	500
1 05	2 00	5 80	30.0	10^6

where

$$\mathcal{W}(t) \equiv \int_{m(t)}^{\infty} \nu \mathcal{G}^{\kappa}(t - \tau_m) \Psi_m R_m dm \,, \tag{20}$$

$$\mathcal{P}_i(t) \equiv \int_{m(t)}^{\infty} \nu \mathcal{G}^{\kappa}(t - \tau_m) \Psi_m \sum_{j=1, i \neq j}^{i_{\max}} Q_{mij} X_j(t - \tau_m) dm \,, \tag{21}$$

$$\mathcal{U}_i(t) \equiv \int_{m(t)}^{\infty} \nu \mathcal{G}^{\kappa}(t - \tau_m) \Psi_m Q_{mij} X_i(t - \tau_m) dm \,, \tag{22}$$

and

$$R_m \equiv \Sigma_i R_{mi}(t) = 1 - d_m \,; \tag{23}$$

and d_m is the fraction of its mass which a star of mass m retains as a stellar remnant upon its death and it is assumed to be independent of the initial stellar composition.

General analytic solutions of equations (18) and (19) are not possible; however, the qualitative behavior of the solutions, and much of the detailed quantitative behavior, are revealed by some simple approximations. The approximations are chosen in order to facilitate analytic integrations; however, they possess the essential features of the more realistic astrophysical assumptions employed in our future numerical work. We assume:

$$\begin{aligned} \Psi_m &= 0 && \text{for} \quad m < m_L \,, \\ &= (\mu - 1) m_L^{\mu-1} m^{-\mu} && \text{for} \quad m \geq m_L \quad (\mu > 1) \,; \end{aligned} \tag{24}$$

$$\begin{aligned} R_m &= 0 && \text{for} \quad m < m_R \,, \\ &= 1 - m_R/m && \text{for} \quad m \geq m_R \,; \end{aligned} \tag{25}$$

$$\tau_m = \tau_\odot / m^2 + \tau_\infty \,; \tag{26}$$

and

$$\begin{aligned} Q_{mZX} &= 0 && \text{for} \quad m < m_S \,, \\ &= q_Z(1 - m_S/m) && \text{for} \quad m \geq m_S \,. \end{aligned} \tag{27}$$

The power-law form for Ψ_m approximates the observed mass function in the solar neighborhood. The specification of R_m approximates the view that stars below some mass m_R evolve into white dwarfs with little mass loss whereas those with $m > m_R$ leave a stellar remnant (white dwarf or neutron star) of approximately the same mass, m_R. The function Q_{mZX} is defined to be the fraction of mass which a star of mass m has converted from species X into species Z and which is eventually ejected from the star. Our prescription approximates the view that heavy elements (carbon, oxygen, iron, etc.; collectively denoted by Z) are produced only in massive stars, specifically in those for which $m > m_S$. The parameter q_Z represents the asymptotic maximum mass fraction ejected. In this simple model, hydrogen and helium are not distinguished from one another and are collectively denoted by X. From conservation of mass, $X + Z = 1$. The other elements of the Q_{mij} matrix are $Q_{mXX} = R_m - Q_{mZX}$, $Q_{mXZ} = 0$, and $Q_{mZZ} = R_m$.

For specific numerical values we choose $q_Z = 0.25$, $m_R = 0.7$, $m_S = 10$, $m_L = 0.1$, $\mu = 4/3$, $\tau_\odot = 11.7$, and $\tau_\infty = 10^{-3}$.

a) Instantaneous Recycling

We will first consider the limit in which τ_m may be regarded as zero in the above integrals. This is a good approximation provided τ_m is small compared with the characteristic time for $\mathcal{G}$ and X_i to change.

Under these circumstances the factor $\nu\mathcal{G}^{\kappa}(t-\tau_m)$ and the compositions $X_i(t-\tau_m)$ may be brought out of the integrals defining $\mathcal{W}$, $\mathcal{P}_i$, and $\mathcal{U}_i$; hence we write

$$\mathcal{W}(t) = \nu\mathcal{G}^{\kappa}(t)f(t) \tag{28}$$

and

$$\mathcal{P}_Z(t) = \nu\mathcal{G}^{\kappa}(t)X(t)q_Z p(t)\,, \tag{29}$$

where $f(t) = H[m(t), m_R]$, $p(t) = H[m(t), m_S]$, and

$$H(m, m_0) = \frac{1}{\mu}\left(\frac{m_L}{m_0}\right)^{\mu-1} \quad \text{for} \quad m < m_0\,,$$

$$= \frac{1}{\mu}\left(\frac{m_L}{m}\right)^{\mu-1}\left[\mu - (\mu-1)\frac{m_0}{m}\right] \quad \text{for} \quad m \geq m_0\,. \tag{30}$$

The equations for $\mathcal{G}$ and X become

$$\frac{1}{\mathcal{G}^{\kappa}}\frac{d\mathcal{G}}{dt} = -\nu[1-f(t)] \tag{31}$$

and

$$\frac{1}{X}\frac{dX}{dt} = -\nu\mathcal{G}^{\kappa-1}q_Z p(t)$$

$$= \frac{1}{\mathcal{G}}\frac{d\mathcal{G}}{dt}\frac{q_Z p(t)}{1-f(t)}\,. \tag{32}$$

The solution of equation (31) is

$$\mathcal{G}(t) = \exp[-\chi(t)] \quad \text{if} \quad \kappa = 1\,,$$

$$= [1 + (\kappa-1)\chi(t)]^{1/(1-\kappa)} \quad \text{if} \quad \kappa > 1\,, \tag{33}$$

where

$$\chi(t) = \nu\int_0^t [1-f(t')]dt' \equiv \nu t[1-f'(t)]\,. \tag{34}$$

The latter equation defines $f'(t)$, the effective average value of f; one may show that

$$f'(t) = \frac{2m_L}{\mu+1}\left(\frac{\tau_\odot}{t}\right)\left(\frac{t-\tau_\infty}{\tau_\odot}\right)^{(\mu+1)/2}\left[1 - \frac{(\mu-1)(\mu+1)}{\mu(\mu+2)}m_R\left(\frac{t-\tau_\infty}{\tau_\odot}\right)^{1/2}\right]$$

$$= 0.264(t-0.001)^{7/6}t^{-1}\,[1 - 0.0358(t-0.001)^{1/2}] \tag{35}$$

for $0.001 < t < 23.9$; $f' = 0$ for $t < 0.001$. That f' is a slowly varying function of time is evident from the functional form and the representative values listed in Table 2. The form for $f'(t)$ changes when $m(t) \leq m_R$, which occurs for $t \geq \tau_R = 23.9$. Schmidt (1959) gave the analytic approximations of equation (33) for the case of *no* recycling, i.e., $f' \equiv 0$. Furthermore, he noted that the effects of recycling upon $\mathcal{G}$ could be approximated by modifying the multiplicative constant in the birth-rate function—in our notation this corresponds to using $\nu' = \nu(1-f')$ in place of ν, where f' is taken to be some appropriately chosen constant value. It is the slowly varying nature of $f'(t)$ which makes this approximation a good one.

The solution to the first version of equation (32) may now be expressed as

$$X(t) = X(0)\exp\left[-\nu q_Z\int_0^t \mathcal{G}^{\kappa-1}(t')p(t')dt'\right] \tag{36}$$

TABLE 2

FUNCTIONS FOR ANALYTIC APPROXIMATIONS TO THE ONE-ZONE GALAXY

t	f'	$q_Z p'$	γ	$h(t)$	Q_{mZX}/R_m	Z^e
1.34(−3)	1.78(−2)	4.42(−3)	4.50(−3)	1.031	0.241	0.249
3.56(−3)	7.01(−2)	1.71(−2)	1.84(−2)	0.509	0.216	0.248
1.42(−2)	0.119	2.81(−2)	3.18(−2)	0.510	0.171	0.226
7.61(−2)	0.168	3.64(−2)	4.37(−2)	0.621	0.053	0.133
0.240	0.203	3.93(−2)	4.94(−2)	0.712	0.0	9.19(−2)
0.811	0.246	4.01(−2)	5.32(−2)	0.788	0.0	9.85(−2)
1.606	0.273	4.03(−2)	5.54(−2)	0.802	0.0	0.102
3.612	0.305	4.04(−2)	5.81(−2)	0.756	0.0	0.107
6.819	0.329	4.04(−2)	6.03(−2)	0.640	0.0	0.110
13.24	0.353	4.04(−2)	6.25(−2)	0.391	0.0	0.114
16.19	0.359	4.04(−2)	6.41(−2)	0.278	0.0	0.117

which may be easily integrated only for $\kappa = 1$; in that case

$$X(t) = X(0) \exp [-\nu t q_Z p'(t)] , \tag{37}$$

where $p'(t)$ has the same functional form as $f'(t)$ given in equation (35), with m_R replaced by m_S. For $m > m_S$ ($t < \tau_S = 0.118$) the numerical solution is

$$p'(t) = \frac{1}{\mu}\left(\frac{m_L}{m_S}\right)^{\mu-1}\left[1 - \frac{\mu(\mu-1)}{(\mu+1)(\mu+2)}\frac{\tau_S}{t}\right] = 0.1618[1 - 6.74 \times 10^{-3}/t] . \tag{38}$$

Some numerical values are listed in Table 2.

The second line of equation (32), coupled with the fact that $f(t)$ and $p(t)$ are weak functions of time, suggests a solution by assuming that

$$\gamma = q_Z \frac{p(t)}{1 - f(t)} \tag{39}$$

is constant; one obtains

$$X(t) = X(0)\mathcal{G}^{\gamma}(t)$$

and

$$Z(t) = Z(0)\mathcal{G}^{\gamma}(t) + [1 - \mathcal{G}^{\gamma}(t)] \tag{40}$$

independent of κ and ν. For our numerical example $\gamma \simeq 0.0 \rightarrow 0.06$ (see Table 2).

b) Initial-Burst Approximation: Semiequilibrium

If ν is larger than 1/(age of galaxy), the solutions of the previous section show that $\mathcal{G}$ will decrease appreciably; hence the approximation of setting $\tau_m = 0$ in the integrals of $\mathcal{W}$, $\mathcal{O}_i$, and $\mathcal{U}_i$ will *underestimate* these quantities. Since it will be important later, we repeat: *Because $\mathcal{G}(t - \tau_m) > \mathcal{G}(t)$, $f'(t)$ is an underestimate of the correct value.*

In this limit it is useful to consider the general properties of integrals of the form

$$F(t) = \int_{m(t)}^{\infty} \nu \mathcal{G}^{\kappa}(t - \tau_m) a(m) dm . \tag{41}$$

This may be rewritten as

$$F(t) = \int_{0}^{t-\tau_{\infty}} \nu \mathcal{G}^{\kappa}(t') A(t - t') dt' , \tag{42}$$

where $A(\tau_m) \equiv a(m)(-dm/d\tau_m)$. Our power-law assumptions for Ψ_m and Q_{mij} mean that we will be interested in functions $a(m)$ proportional to $m^{-\mu}$ and $m^{-\mu-1}$; consequently, $A(\tau)$ is of the form $A_0\tau^{-\lambda}$, where $\lambda = (3-\mu)/2$ or $(2-\mu)/2$.

Because $\lambda < 1$ for all μ's of interest, the singularity of $A(\tau)$ at $\tau = 0$ is integrable; nevertheless, the singular nature indicates that an appreciable fraction of the integral is from that region. The very sharp decreases of $\mathcal{G}(t')$ near $t' = 0$ suggests that this region also may make a major contribution to the integral. Over most of the region between $t' = 0$ and $t' = t$, the integrand is small. To take advantage of these properties, we consider the integral for $F(t)$ to be the sum of an integral from $t' = 0$ to $t' = T$, F_1, and an integral from $t' = T$ to $t' = t$, F_2, where T is midway between 0 and t.

By employing equation (31) to describe $\mathcal{G}(t')$ near $t' = 0$, and integrating by parts, $F_1(t)$ is expressible as

$$F_1(t) \simeq A(t)/(1-f^*) + O(1/\nu t) \tag{43}$$

where we have written f^* rather than f to emphasize that the effective value of f is somewhat larger than $f(t)$. We have assumed the case $\mathcal{G}(0) = 1$. In a similar spirit, and to the same order of approximation, the integral for $F_2(t)$ is

$$F_2(t) \simeq \nu\mathcal{G}^\kappa(t) \int_{m(t)}^{\infty} a(m)dm \simeq f^*\nu\mathcal{G}^\kappa . \tag{44}$$

F_1 is the contribution of the original burst of star formation. F_2 is the rapid-recycling approximation for the later generations and is based upon the fact that $\mathcal{G}$ does not vary appreciably over the region in which the singularity in A makes most of its contribution.

Consequently we may write

$$\frac{d\mathcal{G}}{dt} + \nu\mathcal{G}^\kappa = \mathcal{W} = \frac{\mathcal{W}'(t)}{1-f^*} + f^*\nu\mathcal{G}^\kappa , \tag{45}$$

where

$$\mathcal{W}'(t) = \mathcal{W}_0 h(t)/t = \Psi_m R_m(-dm/d\tau_m)\,|_{m(t)} ,$$

$$\mathcal{W}_0 = \tfrac{1}{2}(\mu-1)m_L{}^{\mu-1}\tau_\odot{}^{(1-\mu)/2} = 0.0514 , \tag{46}$$

and

$$h(t) = t(t-\tau_\infty)^{(\mu-3)/2}[1 - m_R(t-\tau_\infty)^{1/2}\tau_\odot{}^{-1/2}]$$

$$= t(t-0.001)^{-5/6}[1 - 0.205(t-0.001)^{1/2}] . \tag{47}$$

The solution of interest is the asymptotic solution obtained by solving equation (45) for $\mathcal{G}$:

$$\mathcal{G}(t) = \left[\frac{\mathcal{W}_0 h(t)}{\nu t(1-f^*)^2} - \frac{d\mathcal{G}/dt}{\nu(1-f^*)}\right]^{1/\kappa} \approx \left[\frac{\mathcal{W}_0 h(t)}{\nu t(1-f^*)^2}\right]^{1/\kappa}[1 + O(1/\nu t)] . \tag{48}$$

That $h(t)$ is a slowly varying function of t is seen from equation (47) and the values tabulated in Table 2. In this approximation the solution for $\mathcal{B}$ is essentially identical with the situation assumed by Partridge and Peebles (1967*a*, *b*) and $\mathcal{B}$ is *independent of the functional relationship between* $\mathcal{B}$ *and* $\mathcal{G}$.

In the same spirit of approximation, we may write

$$Z(t) = \frac{\mathcal{P}_Z(t) + \mathcal{U}_Z(t)}{\mathcal{W}(t)} - \frac{\mathcal{G}(t)}{\mathcal{W}(t)}\frac{dZ}{dt} \tag{49}$$

$$\approx \left[\int_{m(t)}^{\infty} \nu\mathcal{G}^\kappa(t-\tau_m)\Psi_m[Q_{mZX} + (R_m - Q_{mZX})Z(t-\tau_m)]dm \Big/ \right.$$

$$\left. \int_{m(t)}^{\infty} \nu\mathcal{G}^\kappa(t-\tau_m)\Psi_m R_m dm\right] + O\left(\frac{1}{\nu t}\right) . \tag{50}$$

We make the same approximations as made to obtain equations (43) and (44). In doing so, equations (31) and (40) are used to approximate $\mathcal{G}(t - \tau_m)$ and $Z(t - \tau_m)$ near $t = \tau_m$; $Z(0) = 0$ will be assumed. The solution may be expressed as

$$Z(t) \approx Z^e(t) = Z^p(t) + Z^l(t) , \tag{51}$$

where

$$\gamma^*(t) = q_Z p^*(t)/[1 - f^*(t)] , \tag{52}$$

$$Z^p(t) = Q_{m(t)ZX}/[R_{m(t)}(1 + \gamma^*)] , \tag{53}$$

and

$$Z^l(t) = \gamma^*(2 + \gamma^*)/(1 + \gamma^*)^2 . \tag{54}$$

The function $Z^e(t)$ as given by equation (51) is called the *semiequilibrium* value; *it is independent of* $\mathcal{G}$ *and* $\mathcal{R}$ and varies with time as $Q_{m(t)ZX}$, $R_{m(t)}$, $f^*(t)$, and $p^*(t)$ change.

Provided $t < \tau_S$, $Z^e(t) \simeq Z^p(t) \simeq Q_{m(t)ZX}$. For $t > \tau_S$, $Z^e \simeq Z^l$; $Z^l(t)$ will be called the *limiting semiequilibrium* value. It is a slow function of time since it depends upon f^* and p^* only. At $t = \tau_R$, Z^l ceases to change. However, at this time the approximations that lead to equation (51) fail since of the stars ejecting mass none were formed at $t = 0$.

Generally γ^* is small; consequently, $Z^l \simeq 2\gamma^*$. By employing p' and f' for p^* and f^* and considering only $\tau_S \ll t < \tau_R$, one finds

$$Z^l(t) = \frac{2}{\mu} q_Z \left(\frac{m_L}{m_S}\right)^{\mu-1} \left\{1 - \left[\frac{m_L}{m(t)}\right]^{\mu-1}\left[1 - \frac{\mu - 1}{\mu}\frac{m_R}{m(t)}\right]\right\}^{-1}$$

$$\simeq \frac{2}{\mu} q_Z \left(\frac{m_L}{m_S}\right)^{\mu-1} \left\{1 + \left[\frac{m_L}{m(t)}\right]^{\mu-1}\left[1 - \frac{\mu - 1}{\mu}\frac{m_R}{m(t)}\right]\right\} . \tag{55}$$

The latter expression employs the fact that in most cases of interest $m_L \ll m(t)$. This expression clearly displays how $Z^l(t)$ depends upon the various parameters (recall that $\mu > 1$): Z^l is directly proportional to q_Z; Z^l increases if m_S decreases (and decreases if m_S increases)—proportional to $m_S^{1-\mu}$; Z^l increases if m_L increases (and decreases if m_L decreases)—approximately proportional to $m_L{}^{\mu-1}$; Z^l increases if m_R decreases (and decreases if m_R increases); Z^l depends upon μ through a function which is dominated by $\exp(-\alpha\mu)/\mu$, $\alpha = \ln(m_S/m_L)$.

For the numerical values in our example, the variation of either m_S or m_L by a factor of 2 changes Z^l by only a factor of $2^{1/3} = 1.26$. However, $\alpha = \ln(10/0.1) = 4.6$, which means that changing μ from 4/3 to 5/3 decreases Z^l by a factor of 5.8.

c) *Comparison with Numerical Solutions*

The set of equations (3) have been numerically integrated with the simplified astrophysical assumptions presented in this section and the numerical technique described in § III.

Figure 1 displays the function $\mathcal{G}(t)$ for the case $\kappa = 1$ and several values of ν. For $\nu t < 1$, the numerical solution is well approximated by the instantaneous recycling approximation, equation (33). For $\nu t > 10$, $\mathcal{G}$ has diverged from the instantaneous recycling approximation and closely follows the semiequilibrium approximation, equation (48). Owing to the fact that $f^*(t)$ is not well defined, we have employed $f'(t)$ in computing the curve shown. As was emphasized previously, f^* is greater than f'; consequently, the semiequilibrium approximation should be above the dashed curve shown. This is consistent with the numerical results. If one wished, this calculation could be considered a calibration for f^*. By employing $1.4/(1 - f')^2$ in place of $1/(1 - f^*)^2$ in equation (48), the errors in the semiequilibrium approximation for $\mathcal{G}$ may be reduced to about 10 percent.

Figure 2 compares the analytic approximation for $Z(t)$ with the numerical results.

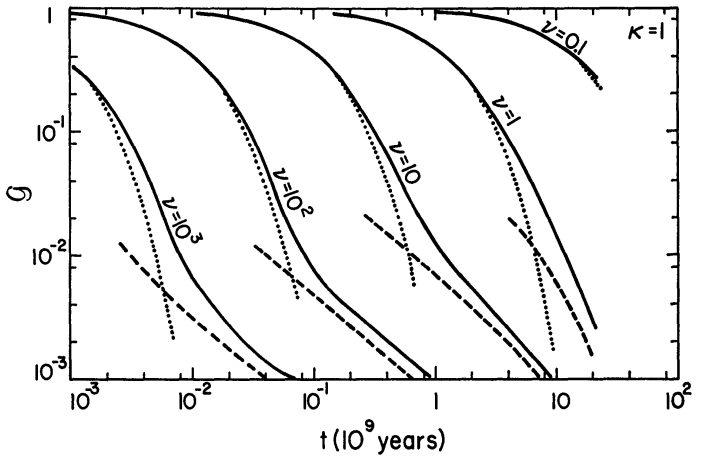

FIG. 1.—Numerical solutions and analytic approximations for the gas mass fraction $\mathcal{G}$. *Solid curves*, numerical solutions. The instantaneous-recycling approximation assumes that conditions in the gas do not change between the times of birth and death of any star; it is displayed by dotted curves. The semi-equilibrium approximation is approximately equivalent to assuming that the initial birth rate is a δ-function and thereafter equating the birth rate to the death rate of that initial generation; it is displayed with dashed curves. Curves are calculated with $\kappa = 1$ and are labeled with ν.

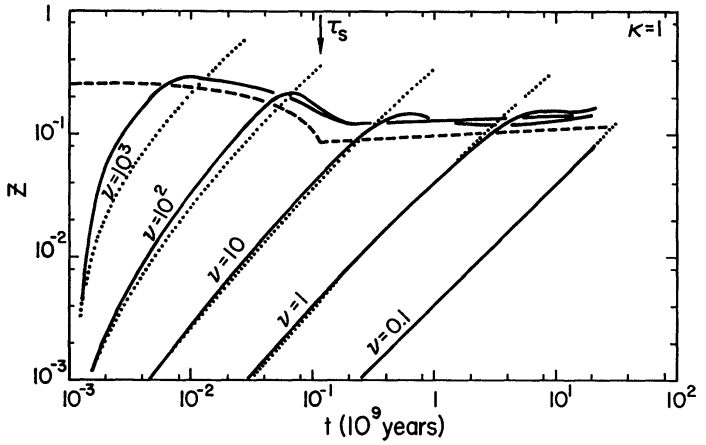

FIG. 2.—Numerical solutions and analytic approximations for the "heavy element" abundance Z. See legend to Fig. 1.

In this calculation $Z(0) = 0$. It is clear that for $\nu t \lesssim 3$ the instantaneous recycling approximation describes the numerical results very accurately. For $\nu t \gtrsim 8$ the numerical curves approximately follow the semiequilibrium solution. Again the solution shown is calculated with f' and p', both of which are smaller than f^* and p^*. From the definition of Z^e it is clear that employing p^* and f^* would increase Z^e; this is consistent with the numerical values being larger than those given by the semiequilibrium curve shown.

From these figures we see that over a large range of ν the two analytic approximations are excellent in their respective regimes, and that a relatively simple smooth transition from one to the other in the regime $3 \lesssim \nu t \lesssim 8$ should be satisfactory. Similar test calculations for $\kappa = 4$ show equally good agreement between the analytical approximations and numerical results.

It should be noted that when Schmidt (1959) stated that his analytical approximation

(our eq. [33]) gives errors of the order of 10 percent compared with exact numerical solutions, the comparison was made for cases which yielded $\mathcal{G} \simeq 0.1$ at the present time. We see from Figure 1 that our calculations are in agreement with Schmidt's conclusion; however, $\mathcal{G} \simeq 0.1$ delimits the end of the range of validity for the rapid-recycling approximation. For calculations which are to be applied to circumstances (e.g., elliptical galaxies and galactic nuclei) where $\mathcal{G}$ is less than this transition value of 0.1, the rapid-recycling approximation can give very large errors, and the initial-burst approximation should be employed.

Figure 3 shows $Z(t)$ calculated with some other values of the numerical parameters. All curves are calculated with $\mu = 5/3$ and $\kappa = 1$; the values of ν are shown.

Curve A differs from the $\nu = 10^2$ curve in Figure 2 only in its value for μ. The values of $Z(t = 10) = Z^l(t = 10)$ for $\mu = 4/3$ and $5/3$ are 0.14 and 0.018, respectively. Equation (55) predicts a ratio of 5.8 using only its leading term; the numerical ratio is 7.7.

Curve B differs from curve A by almost precisely a factor of 2 for all t. B employs $q_Z = 0.5$ rather than the standard value $q_Z = 0.25$ used for A; this confirms that $Z \propto q_Z$ whether $\nu t < 1$ (instantaneous recycling) or $\nu t \gtrsim 10$ (semiequilibrium) and whether $t < \tau_S(Z \sim Z^p)$ or $t > \tau_S(Z \sim Z^l)$.

The curve labeled C employs $m_S = 5$ rather than the standard $m_S = 10$. There are two consequences. First, Z attains Z^p at approximately the same value of t as before, but since m_S is smaller for curve C, Z^p is larger at that time. The changeover to $Z \sim Z^l$ occurs later for C. The second consequence is that Z^l for curve C is larger by a factor of 1.55. Equation (51) predicts that Z^l should be larger by $2^{\mu-1} = 2^{2/3} = 1.59$.

It is clear that our analytic statements for Z give excellent approximations over a wide range of conditions.

V. DISCUSSION

From Figure 1 one visually observes that $\mathcal{G}$ is a function of time primarily through the combination νt only. This is verified analytically by equations (34) and (48) where t enters through νt and the functions $f'(t), f^*(t)$, and $h(t)$; the latter are very weak functions of t. Consequently, an observed value of $\mathcal{G}$ would give information about the product νt, but no information about each individual factor. If the age of a system is known, ν could be determined from $\mathcal{G}$. (These statements suppose that a fixed value of κ is known, or at least assumed.)

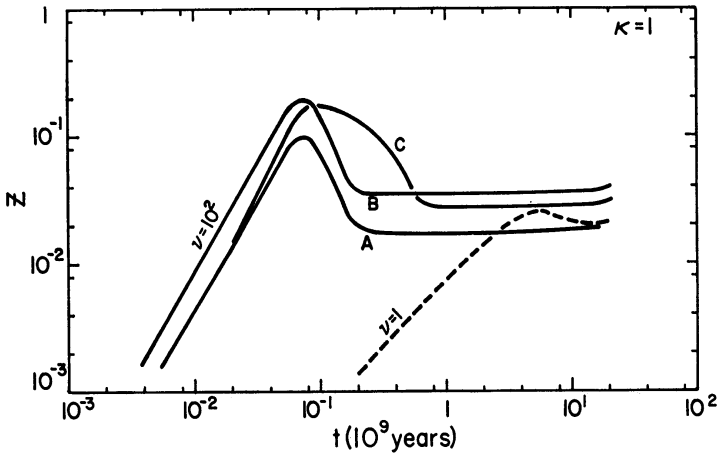

FIG. 3.—The "heavy element" abundance $Z(t)$ for several modifications of the standard set of numerical parameters. All curves are calculated with $\kappa = 1$ and $\mu = 5/3$. The dashed curve and curve A use the standard values and ν as indicated. Curve B employs $q_Z = 0.5$ rather than the standard $q_Z = 0.25$. Curve C employs $m_S = 5$ rather than the standard $m_S = 10$. For curve C, $\tau_S = 0.469$; for the others $\tau_S = 0.118$.

Figure 2 shows that the behavior of Z is somewhat different. If $\nu t < 3$, Z depends upon t (almost) solely through the combination νt. Equation (37) shows how t also enters via the slowly varying function $p'(t)$. It is the fact that Z and $\mathcal{G}$ both depend essentially upon only νt which makes possible writing Z as a function of $\mathcal{G}$ only (eq. [40]). If $\nu t \gtrsim 6$, Z is a slowly varying function of t provided $t > \tau_S$. In this regime, a measure of Z provides little information for either ν or t. This is because uncertainties in Ψ_m and Q_{mZX} far outweigh the variation of Z^l with t. The upshot is that provided $\nu t \gtrsim 6$ (for $\kappa = 1$ this means $\mathcal{G} \lesssim 0.04$), the abundances of species which are produced in stars with lifetimes appreciably shorter than t are almost independent of both t and the history of the system. This statement, though true for our analytic approximations, does not precisely describe the numerical results; for instance, the dashed curve in Figure 3 shows $Z(t)$ peaking at $t \sim 5.5$ with a peak value about 25 percent higher than $Z(t = 10)$. Slightly larger values of ν would give similar results, the peak occurring at earlier times.

This behavior, and the more dramatic behavior for $\nu > 1/\tau_S$, is a phenomenon which has not been emphasized in previous investigations of galactic evolution: *metal content is not necessarily a monotonically increasing function of time.* The phenomenon occurs dramatically if ν is sufficiently large that Z attains the semiequilibrium value Z^e before $t = \tau_S$. The maximum value of Z which can be obtained is approximately $Z^e{}_{\max} = (Q_{mZX})_{\max}$; for our example $Z^e{}_{\max} = q_Z$. The effect whereby Z attains values large compared with Z^l will be called the super-metal-rich (SMR) effect.

An interesting ratio is $Z^e{}_{\max}/Z^l \simeq \frac{1}{2}\mu(m_L/m_S)^{1-\mu}$. The factor $(m_L/m_S)^{1-\mu}$ is the inverse of the fraction of mass in a generation which goes into stars of $m \geq m_S$. If the initial generation of stars is the primary contributor to the mass-to-light ratio $\mathfrak{M}/\mathfrak{L}$, and if one approximates the luminosity of lower-main-sequence stars by $L = L_\odot m^3$, then

$$F \equiv \left(\frac{\mathfrak{M}/\mathfrak{L}}{M_\odot/L_\odot}\right) \simeq \left(\frac{3-\mu}{\mu-1}\right)\left[\frac{m_L}{m(t)}\right]^{1-\mu} m(t)^{-2} .$$

Consequently, for constant m_S and constant age t, we have the prediction that

$$Z^e{}_{\max}/Z^l \propto \tfrac{1}{2}\mu\left(\frac{\mu-1}{3-\mu}\right) F .$$

From this we predict that an SMR effect will be strongest in regions which possess the largest mass-to-light ratio. This relation for the maximum possible *strength* of an SMR effect is completely independent of the criterion for the *existence* of an SMR effect; the latter is the criterion $\nu \gtrsim 1/(\text{age of galaxy})$.

Stars formed during the phase when $Z > Z^l$ will have superhigh-Z content. The fraction of mass which ends up in such enriched stars will be of the order of 10 percent at most. This may be seen by inspection of Figures 1 and 2 which show that $\mathcal{G}$ is depleted to about 0.1 by the time Z reaches Z^p. Although the details of the process will require constructing detailed models, we propose that this general scheme may account for old SMR stars observed by Spinrad and Taylor (1969), Taylor (1970), and others. The scheme naturally explains why there would be a cutoff in the formation of SMR stars at an early epoch. In the present form of a simple one-zone model we would require that all SMR stars be old, yet they would be younger than all super-metal-poor stars. Even with this very simple one-zone model it is possible to make the SMR stage occur early enough, yet last long enough, to accommodate the differences in ages between the two SMR clusters M67 and NCG 188 (Spinrad *et al.* 1970). However, generalizations of this scheme will be needed which employ realistic models of galactic structure and dynamics and better rates of star formation before all observations can be incorporated satisfactorily. With the time-invariant function Ψ_m we can not properly fit the distribution of the relative number of stars versus metal content. That this is the best indicator

of the history of a system of this type is well known. Recognition of this formed the basis for Schmidt's second paper (Schmidt 1963) and much observational effort (e.g., Eggen, Lynden-Bell, and Sandage 1962; Dixon 1966; van den Bergh 1962; Pagel 1968; Powell 1970). This distribution function will play an important role in our future investigations.

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