DE-CONVOLUTION OF BARELY RESOLVED RADIO SOURCES MAPPED WITH AERIAL BEAMS OF ELLIPTICAL CROSS SECTION†

By J. P. Wild‡

Although it is formally impossible to improve the angular resolution of an observed radio brightness distribution beyond the limits imposed by the resolving power of the radio telescope, it is often useful in the case of barely resolved sources to obtain an estimate of the angular size of the object in two dimensions on the assumption that it possesses simple structure. Such estimates are straightforward when the aerial beam is circular in cross section, but less so when it is elliptical—the case relevant to mapping with large arrays. The purpose of this note is to give a simple and exact solution of this problem with the assumption that both the observed brightness distribution F_0 and the power polar diagram of the radio telescope F_2 can be adequately represented by elliptical–Gaussian functions, i.e. functions with elliptical isophotes and Gaussian profiles. We shall calculate the distribution F_1 (also elliptical–Gaussian) which, when convolved with the instrumental function F_2, yields the observed distribution F_0. The statement and solution of this problem are summarized in Figures 1 and 2 respectively.

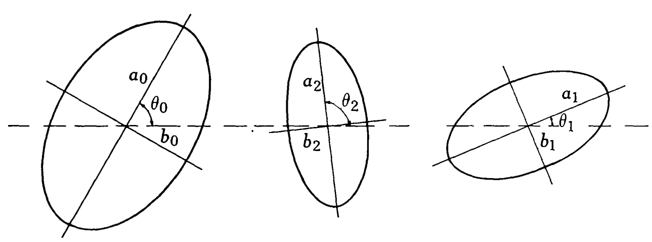

Fig. 1.—Specification of the problem of simple de-convolution. The ellipses represent contours of the observed distribution F_0, the aerial beam F_2, and the de-convolved distribution F_1 to be determined.

The functions F_0, F_1, and F_2 are related by the equation

$$F_0 = F_1 * F_2\,, \tag{1}$$

where the asterisk denotes the process of two-dimensional convolution. Their Fourier transforms $\bar{F}_0$, $\bar{F}_1$, and $\bar{F}_2$ are related by the equation

$$\bar{F}_0 = \bar{F}_1 \bar{F}_2\,. \tag{2}$$

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‡ Division of Radiophysics, CSIRO, P.O. Box 76, Epping, N.S.W. 2121.

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Equations (1) and (2) are special cases of the more general equations

$$F_0 = F_1 * F_2 * F_3 \ldots * F_n$$

and

$$\bar{F}_0 = \bar{F}_1 \bar{F}_2 \bar{F}_3 \ldots \bar{F}_n\,, \tag{3}$$

which we prefer to consider since doing so involves no extra work.

The general function F_j may be written

$$F_j(x,y) \propto \exp\left(-\frac{(x\cos\theta_j + y\sin\theta_j)^2}{a_j^2} - \frac{(x\sin\theta_j - y\cos\theta_j)^2}{b_j^2}\right),$$

where a_j and b_j denote the major and minor semi-axes of the elliptical isophote representing $\mathrm{e}^{-1}\times$(maximum intensity) and θ_j is the inclination of the major axis. The Fourier transform $\bar{F}_j$ may simply be determined by writing down the transform for the case $\theta = 0$ and then rotating the axes:

$$\bar{F}_j(\xi,\eta) \propto \exp[-\tfrac{1}{4}\{a_j^2(\xi\sin\theta_j - \eta\cos\theta_j)^2 + b_j^2(\xi\cos\theta_j + \eta\sin\theta_j)^2\}]. \tag{4}$$

Substituting (4) in (3) and equating in turn the coefficients of ξ^2, η^2, and $\xi\eta$, we obtain

$$a_0^2\sin^2\theta_0 + b_0^2\cos^2\theta_0 = \sum_{j=1}^{n} (a_j^2\sin^2\theta_j + b_j^2\cos^2\theta_j)\,, \tag{5}$$

$$a_0^2\cos^2\theta_0 + b_0^2\sin^2\theta_0 = \sum_{j=1}^{n} (a_j^2\cos^2\theta_j + b_j^2\sin^2\theta_j)\,, \tag{6}$$

$$(a_0^2 - b_0^2)\sin 2\theta_0 = \sum_{j=1}^{n} (a_j^2 - b_j^2)\sin 2\theta_j\,. \tag{7}$$

Taking the sum and difference of equations (5) and (6) we have

$$a_0^2 + b_0^2 = \sum_{j=1}^{n} (a_j^2 + b_j^2) \tag{8}$$

and

$$(a_0^2 - b_0^2)\cos 2\theta_0 = \sum_{j=1}^{n} (a_j^2 - b_j^2)\cos 2\theta_j\,. \tag{9}$$

Equations (7) and (9) may be combined by multiplying the former by $\mathrm{i} = \sqrt{-1}$ and adding it to the latter, giving

$$\boldsymbol{D}_0 = \sum_{j=1}^{n} \boldsymbol{D}_j\,, \tag{10}$$

where the $\boldsymbol{D}_j$ are vectors defined by

$$\boldsymbol{D}_j = (a_j^2 - b_j^2)\exp(2\mathrm{i}\theta_j)\,.$$

Thus if any n of the $n+1$ vectors are given the remaining one can be determined from a vector diagram. With the aid of equation (8) the coefficients a, b, and θ for the unknown function can then be calculated.

In the special case of the radio astronomical problem ($n = 2$) we need to find (a_1, b_1, θ_1) given (a_0, b_0, θ_0) and (a_2, b_2, θ_2) (see Fig. 1). The vector relation (10) reduces to the triangle shown in Figure 2. It allows θ_1 to be determined directly while a_1

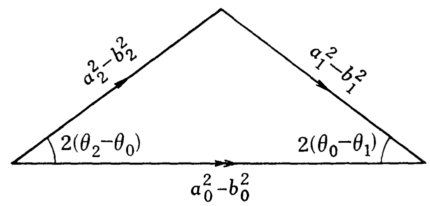

Fig. 2.—Diagram which, with the aid of equations (11), gives the solution to the problem defined in Figure 1.

and b_1 are determined by measuring the third side, $D_1 = a_1^2-b_1^2$, and substituting in equation (8), which gives

$$a_1^2 = \tfrac{1}{2}(a_0^2+b_0^2-a_2^2-b_2^2+D_1)\,, \qquad b_1^2 = \tfrac{1}{2}(a_0^2+b_0^2-a_2^2-b_2^2-D_1)\,. \tag{11}$$

Analytically, solution of the triangle gives

$$D_1^2 = D_0^2+D_2^2-2D_0\,D_2\cos 2(\theta_0-\theta_2)$$

and

$$\tan 2\theta_1 = (D_0\sin 2\theta_0 - D_2\sin 2\theta_2)/(D_0\cos 2\theta_0 - D_2\cos 2\theta_2)\,,$$

where $D_j = a_j^2-b_j^2$.

It should be remarked that although we have chosen to define the ellipse parameters a_j, b_j, and θ_j in terms of the isophotes representing $e^{-1}\times$(maximum intensity), the same results are valid if we consistently define these parameters in terms of isophotes representing $N^{-1}\times$(maximum intensity), where N is any number greater than unity.