T1) $y_2 = \alpha y_1 + w_1$

$y_1 = \alpha y_0 + w_0$

$w_0, w_1 \sim \mathcal{N}(0, \sigma^2)$

$y_0 \sim \mathcal{N}(0, \lambda)$

$$P(\underbrace{y_2, y_1}_{A}, \underbrace{y_0}_{B} \mid \alpha) = P(\underbrace{y_2, y_1}_{A} \mid \underbrace{y_0}_{B}, \alpha) P(\underbrace{y_0}_{B} \mid \alpha)$$

from hint:

$P(y_2 \mid y_1, y_0)$

$= P(y_2 \mid y_1)$

and $P(A, B) = P(A \mid B) P(B)$

$\therefore P(A, B \mid \alpha) = P(A \mid B, \alpha) P(B \mid \alpha)$

to α!

$$= P(y_2 \mid y_1, y_0, \alpha) P(y_1 \mid y_0, \alpha) P(y_0 \mid \alpha)$$

from hint

$$= P(y_2 \mid y_1, \alpha) P(y_1 \mid y_0, \alpha) \underbrace{P(y_0 \mid \alpha)}_{\text{independent } = P(y_0)}$$

$$= \mathcal{N}(\alpha y_1, \lambda) \mathcal{N}(\alpha y_0, \lambda) \mathcal{N}(0, \sigma^2)$$

$$\mathcal{N}(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\therefore P(y_2, y_1, y_0 \mid \alpha) = \underbrace{\frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(y_2 - \alpha y_1)^2}{2\lambda}}}_{\mathcal{N}(\alpha y_1, \lambda)} \cdot \underbrace{\frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(y_1 - \alpha y_0)^2}{2\lambda}}}_{\mathcal{N}(\alpha y_0, \lambda)} \cdot \underbrace{\frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{y_0^2}{2\sigma^2}}}_{\mathcal{N}(0, \sigma^2)}$$

$$= \frac{1}{2\pi\lambda\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_2 - \alpha y_1)^2}{2\lambda} - \frac{(y_1 - \alpha y_0)^2}{2\lambda} - \frac{y_0^2}{2\sigma^2}\right)$$

$$l(\alpha) = \log \frac{1}{2\pi\sigma^2\sqrt{2\pi\sigma^2}} - \frac{(y_2 - \alpha y_1)^2}{2\lambda} - \frac{(y_1 - \alpha y_0)^2}{2\lambda} - \frac{y_0^2}{2\sigma^2}$$

$$\frac{\partial}{\partial \alpha} l(\alpha) = -\frac{1}{2\lambda} \frac{\partial}{\partial \alpha} \left((y_2 - \alpha y_1)^2 + (y_1 - \alpha y_0)^2\right)$$

$$0 = -\frac{1}{2\lambda} \left(2(y_2 - \alpha y_1)(-y_1) - 2(y_1 - \alpha y_0)(y_0)\right)$$

$$= \frac{1}{\lambda} \left((y_2 - \alpha y_1)(y_1) + (y_1 - \alpha y_0)(y_0)\right)$$

$$= y_2 y_1 - \alpha y_1^2 + y_1 y_0 - \alpha y_0^2$$

$$\alpha(y_1^2 + y_0^2) = y_2 y_1 + y_1 y_0$$

$$\alpha = \boxed{\frac{y_2 y_1 + y_1 y_0}{y_1^2 + y_0^2}}$$ #

OT1) $y_{n+1} = \alpha y_n + W_n \; , n = 0,1,2,\ldots$

$$p(y_{N+1}, y_N, \ldots, y_0 \mid \alpha) = p(y_{N+1}, y_N, \ldots, y_1 \mid y_0 ; \alpha)\, p(y_0 \mid \alpha)$$

$$= p(y_{N+1}, y_N, \ldots, y_2 \mid y_0 ; \alpha)\, p(y_1 \mid y_0 ; \alpha)\, p(y_0 \mid \alpha)$$

$$\vdots$$

p_{y_0}, α are independent $= p(y_0)$

$$= p(y_{N+1} \mid y_N, \ldots, y_0 ; \alpha)\, p(y_N \mid y_{N-1}, \ldots, y_0 ; \alpha) \cdots p(y_1 \mid y_0 ; \alpha)\, p(y_0 \mid \alpha)$$

since a process is Markov, $p(y_n \mid y_{n-1}, y_{n-2}, \ldots) = p(y_n \mid y_{n-1})$

$$= p(y_{N+1} \mid y_N ; \alpha)\, p(y_N \mid y_{N-1} ; \alpha) \cdots p(y_1 \mid y_0, \alpha)\, p(y_0)$$

$y_0 \sim N(0, \lambda)$, $\omega_0, \omega_1 \sim N(0, 6^2)$

$$= N(\alpha y_N, \lambda)\, N(\alpha y_{N-1}, \lambda) \cdots N(\alpha y_0, \lambda)\, N(0, 6^2)$$

constant

$$p(y_{N+1}, y_N, \ldots, y_1, y_0 \mid \alpha) = k \exp\left[\left[-(y_{N+1} - \alpha y_N)^2 - (y_N - \alpha y_{N-1})^2 - \ldots - (y_1 - \alpha y_0)^2\right] / 2\lambda - y_0^2 / 26^2\right]$$

$$L(\alpha) = \log k - \left((y_{N+1} - \alpha y_N)^2 + (y_N - \alpha y_{N-1})^2 + \ldots + (y_1 - \alpha y_0)^2\right) / 2\lambda) - y_0^2 / 26^2$$

$$\frac{\partial}{\partial \alpha} L(\alpha) = \left[(y_{N+1} - \alpha y_N) y_N + (y_N - \alpha y_{N-1}) y_{N-1} + \ldots + (y_1 - \alpha y_0) y_0\right] / \lambda$$

$$0 = (y_{N+1} y_N + y_N y_{N-1} + \ldots + y_1 y_0) - \alpha (y_N^2 + y_{N-1}^2 + \ldots + y_0^2)$$

$$\alpha = \frac{y_{N+1} y_N + y_N y_{N-1} + \ldots + y_1 y_0}{y_N^2 + y_{N-1}^2 + \ldots + y_0^2}$$

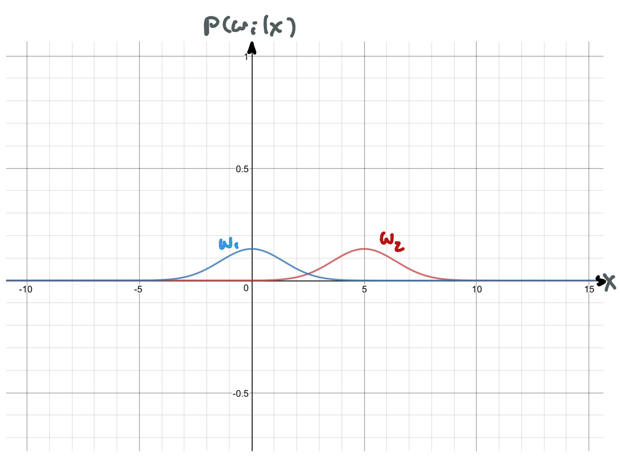
$$\alpha = \boxed{\frac{\sum_{i=1}^{n} y_{i+1} y_i}{\sum_{i=1}^{n} y_i^2}} \quad \#$$

T2) Posterior

$$P(\omega_1 \mid x) = P(x \mid \omega_1) P(\omega_1) = N(5, 2) \cdot \tfrac{1}{2}$$
$$P(\omega_2 \mid x) = P(x \mid \omega_2) P(\omega_2) = N(0, 2) \cdot \tfrac{1}{2}$$

$$P(\omega_1 \mid x) = \frac{1}{2\sqrt{4\pi}} \exp\left(-\frac{(\omega_1 - 5)^2}{4}\right)$$

$$P(\omega_2 \mid x) = \frac{1}{2\sqrt{4\pi}} \exp\left(-\frac{\omega_2^2}{4}\right)$$



LRT:

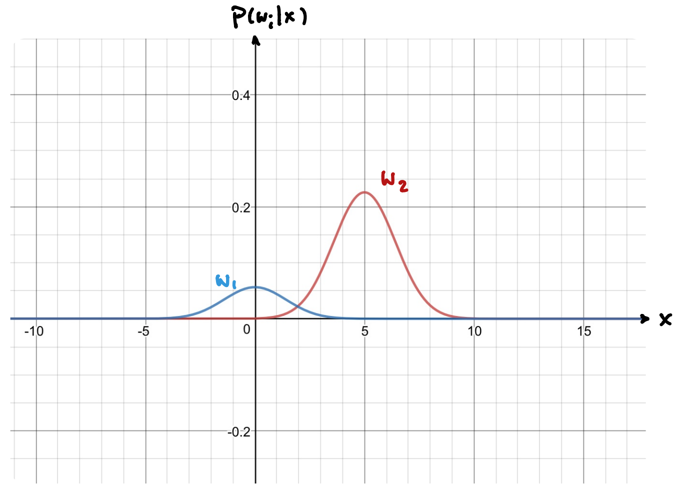
$$\frac{P(x \mid \omega_1)}{P(x \mid \omega_2)} \quad ? \quad 1$$

decision boundary at $(2.5, 0.0290)$

$x = 2.5$ #

T3) $P(\omega_1|x) = N(5,2)(0.8)$

$P(\omega_2|x) = N(0,2)(0.2)$



decision boundary at (1.945, 0.0249)

x=1.945 #

OT2) $P(x|\omega_1) = N(\mu_1, \sigma^2)$, $P(x|\omega_2) = N(\mu_2, \sigma^2)$, $P(\omega_1) = P(\omega_2) = 0.5$

$$P(\omega_1|x) = P(x|\omega_1)P(\omega_1) = \frac{1}{2}N(\mu_1, \sigma^2)$$

$$= \frac{1}{2\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu_1)^2}{2\sigma^2}\right]$$

$$P(\omega_2|x) = P(x|\omega_2)P(\omega_2) = \frac{1}{2}N(\mu_2, \sigma^2)$$

$$= \frac{1}{2\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu_2)^2}{2\sigma^2}\right]$$

Let $P(\omega_1|x) = P(\omega_2|x)$

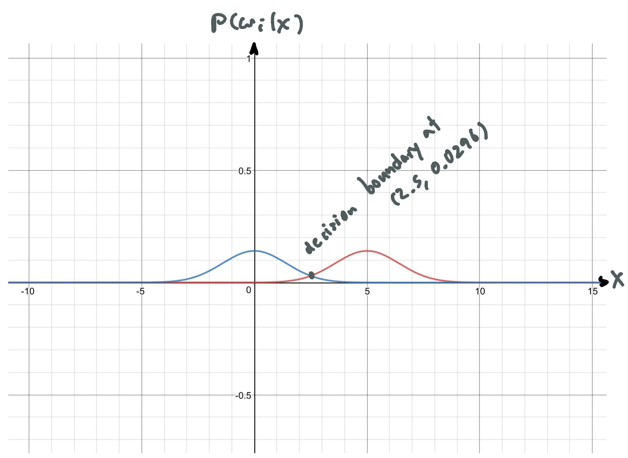
$$\cancel{\frac{1}{2\sqrt{2\pi\sigma^2}}} \exp\left[-\frac{(x-\mu_1)^2}{2\sigma^2}\right] = \cancel{\frac{1}{2\sqrt{2\pi\sigma^2}}} \exp\left[-\frac{(x-\mu_2)^2}{2\sigma^2}\right]$$ //take log both sides

$$(x-\mu_1)^2 = (x-\mu_2)^2$$

$$(x-\mu_1-x+\mu_2)(x-\mu_1+x-\mu_2) = 0$$

$$\cancel{(\mu_2-\mu_1)}(2x-\mu_1-\mu_2) = 0$$

$$\boxed{x = \frac{\mu_1+\mu_2}{2}}$$ #



$N(0,2)$
$N(5,2)$ } decision boundary $= \frac{5}{2} = 2.5$