Draft version May 4, 2025
Typeset using LATEX **twocolumn** style in AASTeX631

Stellar Model Report and Summary

Tanner O'Dwyer[1]

[1] *Johns Hopkins University, Baltimore, MD*

ABSTRACT

This document summarizes and describes the process used to create and model a star's Zero Age Main Sequence (ZAMS). The structure of the star (hydrogen, helium, and metallicity) is used in tandem with a chosen solar mass ratio to create a model showing the characteristics of the star compared to that of a standard model used in the Stellar Astrophysics community.

1. INTRODUCTION

In the study of Stellar Astronomy, the ability to create and use an accurate model is crucial in conducting studies of the interiors of stars. However, the development of such a model involves multiple complex systems working together to achieve accurate and meaningful results. Thankfully, access to the physics behind these systems and programming languages capable of creating such a model is widely available.

The goal of the project outlined in this document is to create an accurate model of the Zero Age Main Sequence (ZAMS) of a sun-like star, given an adjustable solar mass ratio. The Successful creation of this model returns characteristics of the star's constitution, which are then compared to a current high-accuracy stellar structure software, Modules for Experiments in Stellar Astrophysics or MESA Paxton et al. (2010).

2. CODING ENVIRONMENT AND EXTERNAL TOOLS

The development of this model was created in a Python coding environment and utilizes a collection of libraries within such as numpy, pandas, matplotlib, and SciPy. The formulas used in the code to create individual aspects of the stellar model described below were pulled from two main texts, Kippenhahn et al. (2012) & Hansen et al. (2004). Opacity values used within the code are pulled from the Opacity Project (OP) and from table #73 within. For any uncertain constants used in this document, see A.

3. MODEL OVERVIEW

The models presented in this document are made with the starting conditions of 1.2 $M_\odot$ and solar composition of X=0.7, T=0.28, Z=0.02. In the following sections, I will introduce and briefly discuss the key formulas used in creating a stellar model of my chosen star. Some equations can be complex and have more than one internal equation. My goal is to highlight those that are more important and are specifically addressed in the code used for my model.

3.1. *Energy Generation*

Different masses of a modeled star involve various levels of both Carbon-Nitrogen-Oxygen or CNO and Proton-Proton or PP-chain powered energy generation. Both CNO and PP-chain cycles have a strong temperature dependence, which is crucial in determining which form of nuclear fusion dominates in a star.

PP chain depends on a few variables, one being $T_9 = 1 * 10^{-9} * T$. A factor g_{11} which accounts for the generation efficiency of the reaction, a ψ term which we will set as 1 since I am only using PP1 energy generation. A weak screening factor (3) was also used to express how particle interactions increase with T. For more information, I refer the reader to chapter 18 of Kippenhahn et al. (2012).

The energy generation rate for the PP-chain is given by:

$$\epsilon_{\rm pp} = 2.57 \times 10^4 \psi f_{11} g_{11} X^2 T_9^2 \left(e^{-3.381/T_9^{1/3}}\right) \tag{1}$$

$$g_{11} = \left(1 + 3.82 T_9 + 1.51 T_9^2 + 0.144 T_9^3 - 0.0114 T_9^4\right) \tag{2}$$

$$f = e^{E_D/kT}; \quad \frac{E_d}{kT} = 5.92 \times 10^{-3} Z_1 Z_2 \sqrt{\frac{\rho\zeta}{T_7^3}} \tag{3}$$

The CNO cycle dominates at higher temperatures than the PP-chain but similarly depends on T_9, as well as the density ρ, metallicity Z, and a efficiency term $g_{14,1}$

The energy generation rate for the CNO cycle is given by:

$$\epsilon_{\rm CNO} = 8.24 \times 10^{25} g_{14,1} \left(0.7 \times Z\right) X \rho T_9^{-2/3} \left(e^{-15.231 \times T_9^{-1/3} - (T_9/0.8)^2}\right) \tag{4}$$

The function $g_{14,1}$ is given by:

$$g_{14,1} = 1 - 2.00 T_9 + 3.41 T_9^2 - 2.43 T_9^3 \tag{5}$$

To ensure the PP-chain (1) and CNO (4) equations are correct, I evaluate the two-generation rates over temperature and plot them both individually and together in Figure 1.

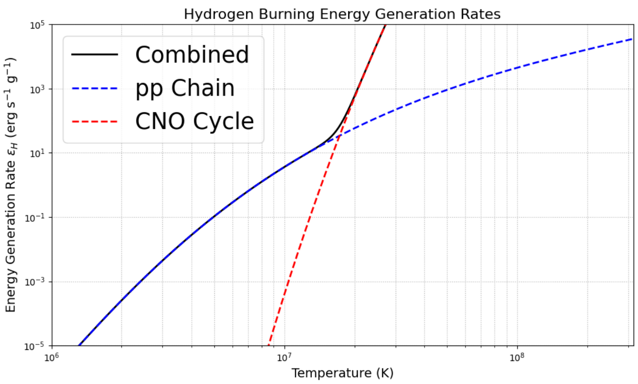

Figure 1. A recreation of figure 18.7 from Kippenhahn et al. (2012)

3.2. *Dependent Variables*

A star's interior can be described by four coupled differential equations with respect to M_r, the mass coordinate, equation(6):

$$M_r = \int_0^r 4\pi r^2 \rho \, dr \tag{6}$$

The following differential equations(7-10) are used to get values of Radius (R), Luminosity (L), Pressure (P), and Temperature (T), all as a function of mass as it is convenient to take M_r as the independent variable. These equations neglect external forces like magnetic fields or non-sphericity and operate under the assumption that the star is in hydrostatic equilibrium and thermal balance. This "equilibrium" implies that time-dependent processes are ignored, see Hansen et al. (2004) section 7.1 for an in-depth explanation.

$$\frac{dL}{dm} = \epsilon_{tot} \tag{7}$$

Where ϵ_{tot} is the sum of ϵ_{pp} (equation 1) and ϵ_{CNO} (equation 4)

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \tag{8}$$

$$\frac{dR}{dm} = \frac{1}{4\pi R^2 \rho} \tag{9}$$

$$\frac{dT}{dm} = -\frac{GmT}{4\pi r^4 P} \nabla \tag{10}$$

3.3. *Extra Equations*

While the primary method used to generate the resulting plots of R, L, P, and T is produced by integrating equations (7-10), there are a few equations used within these differential equations that are worth mentioning.

The pressure defined in a radiative zone is well defined and is dependent on the radiative constant and temperature:

$$P_{rad} = \frac{1}{3} a_{rad} * T^4 \tag{11}$$

This can then be used in the calculation of ρ (density) using:

$$\rho = (P_{tot} - P_{rad}) * \frac{\mu}{T * k_B * N_A} \tag{12}$$

Where:

$$\mu = \frac{4}{3 + X} \approx 0.6 \tag{13}$$

P_{tot} is an independent variable defined when integrating our star's interior, its value is subject to change based on the composition and mass ratio of our modeled star.

The final equations of significance are the temperature gradients denoted by ∇. The value used for the gradients depends on whether the region of integration of the modeled star is a radiative (∇_{rad}) or convective (∇_{ad}) zone. Its use in equation (10) is defined by:

$$\nabla = \nabla_{rad} \quad \text{if} \quad \nabla_{rad} \leq \nabla_{ad}$$

or

$$\nabla = \nabla_{ad} \quad \text{if} \quad \nabla_{rad} > \nabla_{ad}$$

The radiative temperature gradient is defined by:

$$\nabla_{\rm rad} = \frac{3}{64\pi\sigma} \frac{P\kappa}{T^4} \frac{L}{GM_r} \tag{14}$$

The opacity κ is a critical factor that informs us on how easily photons travel through the star. Such a factor is incredibly complicated; thus, calculating it personally in our code would be difficult. To circumvent this, I take advantage of the Opacity Projects (OP) publicly listed opacity tables for various stellar compositions. With a table of our modeled stars' composition, I can utilize the SciPy.Interpolate library; specifically, the Rect-Bivariate-Spline command compared to Regular-Grid-Interpolator for better results, to retrieve specific values of κ given changing ρ and temperature values.

The adiabatic convection gradient is defined by:

$$\nabla_{\rm ad} = \frac{d \ln T}{d \ln P} = \frac{\gamma - 1}{\gamma} = \frac{2}{5} = 0.4 \tag{15}$$

4. BOUNDARY CONDITIONS

With important equations now defined, I can comment on the boundary conditions of our model. As both the center and surface of our star result in singularity points for our listed equations, approximations are made to ensure the integration does not break before the model reaches a convergence point. The boundary conditions will be made in two groups, the first is used to allow integration starting outside the core to a midpoint $(1.2 * M_{\odot}/2)$. The second will be from below the photosphere or surface of the star to the same midpoint.

4.1. *The Center*

To avoid singularities, a lower limit mass at the center is set to be $m_{core} = 10^{-10} M_{\odot}$. The temperature at the core is also a hard set value in this boundary group based on the sun's temperature at its core ≈15 million, Hathaway (2022), and will be defined as $T_{core} = 1.5{*}10^7$. The other variables in the center integrate are as follows:

$$r_0 = (\frac{3 * m_{core}}{4\pi \rho_{core}})^{1/3} \tag{16}$$

$$l_0 = \epsilon_{core} * m_{core} \tag{17}$$

$$P = P_{core} - [\frac{3G}{8\pi} * (\frac{4\pi\rho_{core}}{3})^{4/3} * m_{core}^{2/3}] \tag{18}$$

4.2. *The Surface*

The surface integration depends on pressure and temperature; however, I will set the opacity value $\kappa_s = 0.32$, which is based on Thomson scattering opacity under the assumption of a fully ionized region in the outer layers of the star.

The pressure and temperature equations are given by:

$$P_R = \frac{2GM}{3R^2}\frac{1}{\kappa_s} \tag{19}$$

$$T_{\text{eff}} = \left(\frac{L}{4\pi\sigma R^2}\right)^{1/4} \tag{20}$$

5. INTEGRATION AND OPTIMIZATION

As mentioned, in order to model our star, I need to integrate out four differential equations and then perform some form of optimization of our initial parameters. The method used to perform integration was the Runge-Kutta (RK45) through SciPy's library using the solve$_{ivp}$ function in a shootf algorithm. This integration is performed over the two sets of boundary conditions, starting at the midpoint defined above and integrating outward.

Using initial parameters based on solar values of Radius, Luminosity, Pressure, and Temperature, I attach a set of scaled values allowing me to adjust these four guesses in an attempt to improve the resulting plots displayed in Figure 2. Once I have found a decent scale, I perform optimization to refine these parameters using a least-squares method. This method conducts multiple attempts to decrease the discrepancy between our inner and outer models. The models reach a convergence point after several iterations, resulting in a higher accuracy outcome shown in Figure 3.

6. RESULTS AND COMPARISON

I present first the unoptimized initial parameters, scale, and resulting values in Table 1, along with the plots corresponding to these guesses in Figure 2.

Table 1. Initial Parameters

	Guess	Scale	Scaled Guess
R (cm)	$6.957 * 10^{10}$	1.6	$\approx 1.11 * 10^{11}$
L (erg/s)	$3.828 * 10^{33}$	2.4	$\approx 9.19 * 10^{33}$
P (dyne/cm^2)	$1.92 * 10^{17}$	1	$\approx 1.92 * 10^{17}$
T (K)	$1.57 * 10^{7}$	1.05	$\approx 1.65 * 10^{7}$

As Figure 2 shows, the results of our model are rather promising. Radius and Luminosity match greatly, while Pressure and Temperature are slightly off in their convergence match. Next I perform the Optimization routine on these parameters shown in Table 2 and the plots in Figure 3

Table 2. Optimized Parameters

	Guess	Scale	Scaled Guess
R (cm)	$6.957 * 10^{10}$	1.42	$\approx 9.91 * 10^{1}$
L (erg/s)	$3.828 * 10^{33}$	2.01	$\approx 7.71 * 10^{33}$
P (dyne/cm^2)	$1.92 * 10^{17}$	0.99	$\approx 1.89 * 10^{17}$
T (K)	$1.5 * 10^{7}$	1.07	$\approx 1.61 * 10^{7}$

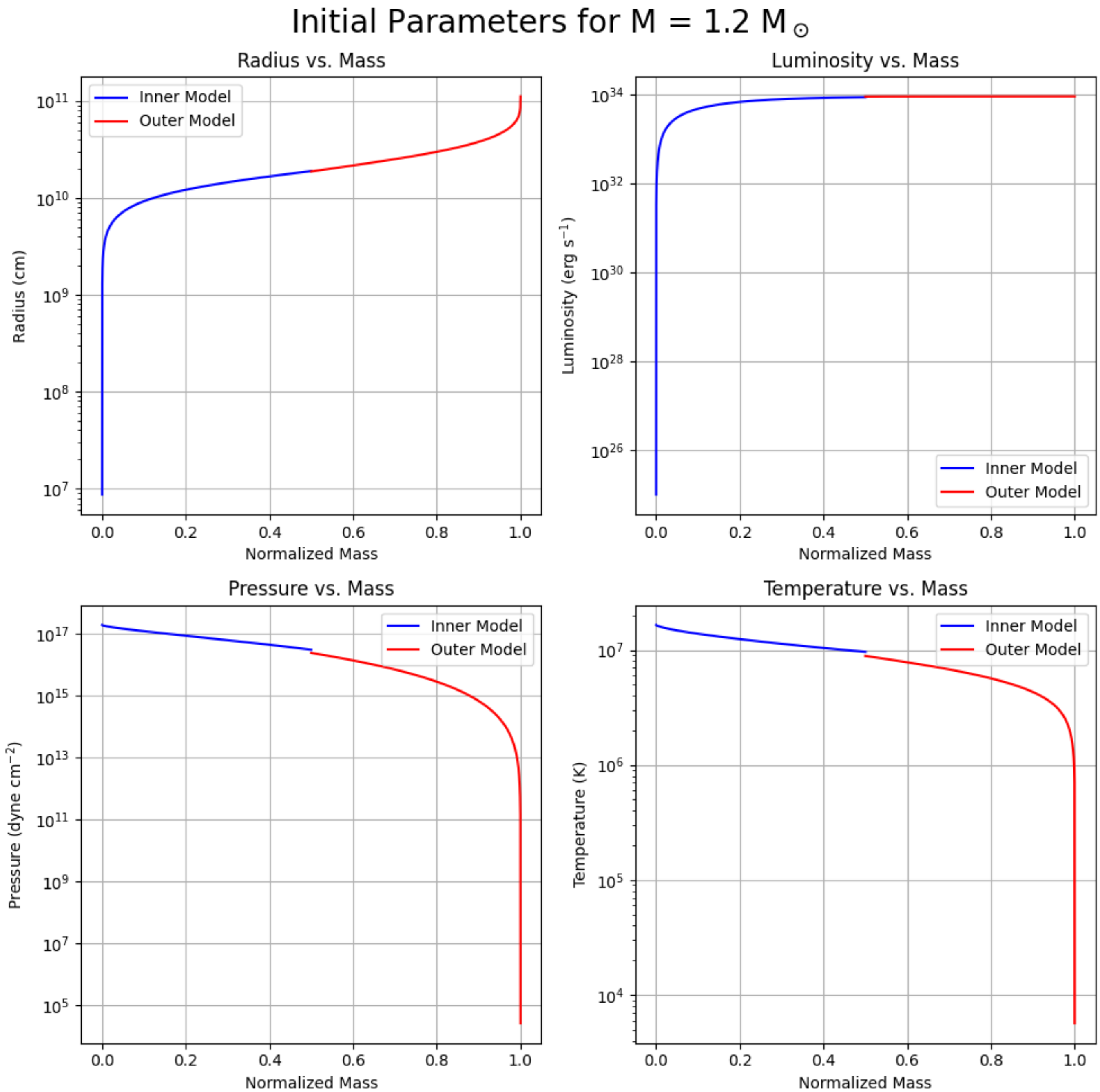

Figure 2. Output of Initial Parameters

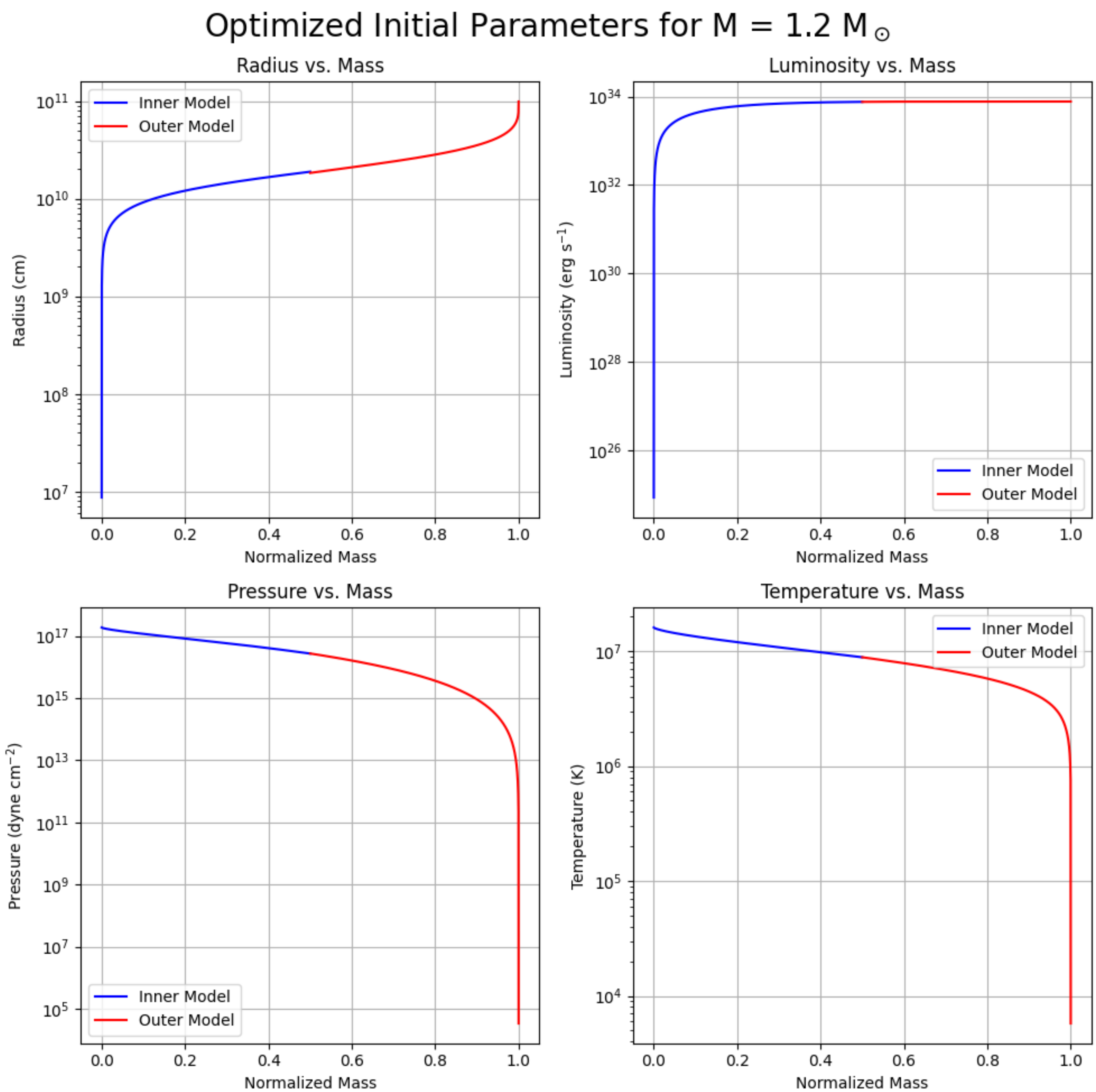

Figure 3. Output of Optimized Parameters

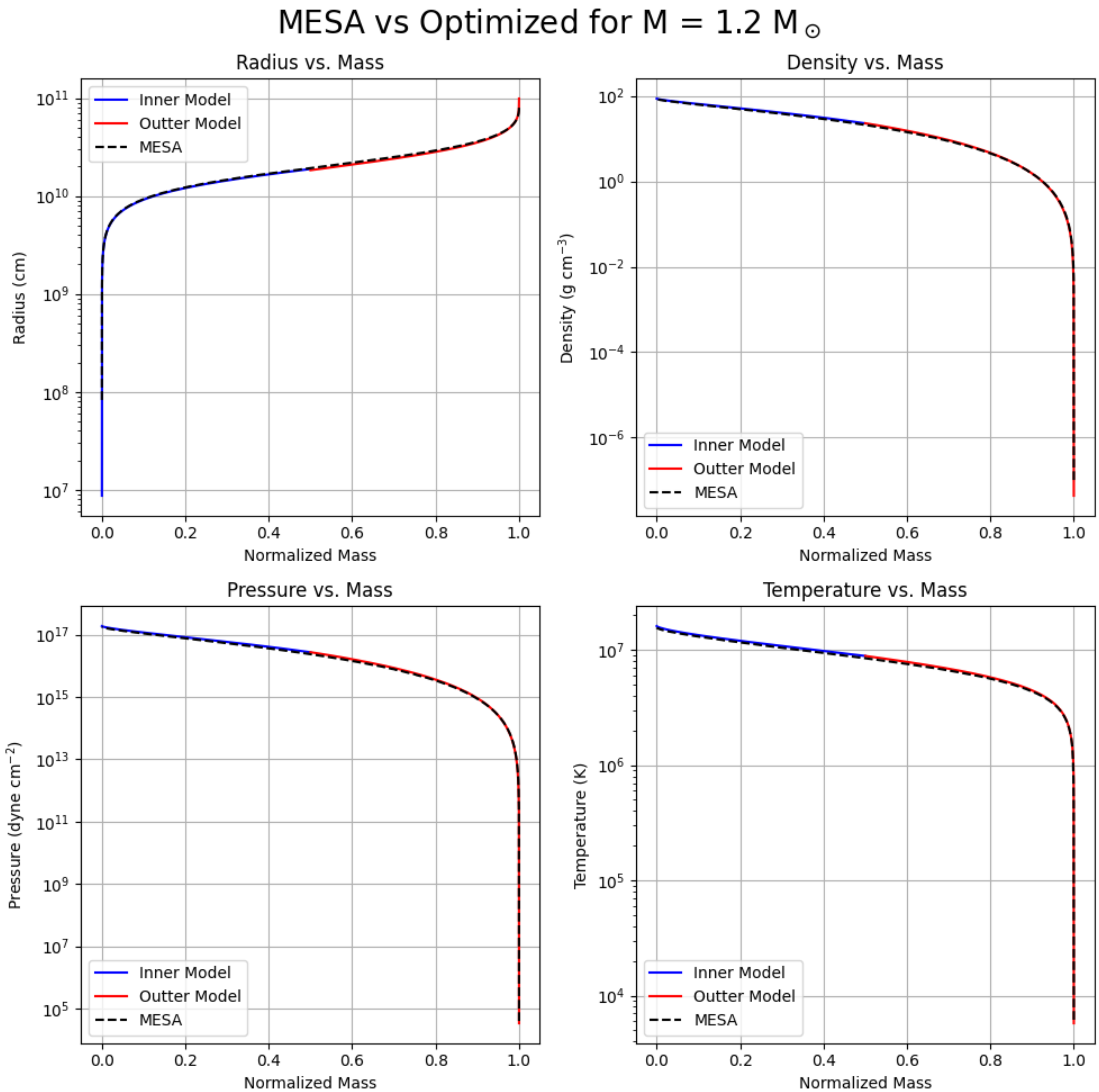

Figure 4. MESA vs Optimized Plots

7. MESA COMPARISON

With fully optimized plots generated, the last thing to do is perform a comparison to the MESA modeling software mentioned at the beginning of this document. The comparison of my model and the MESA model is shown in Figure 4. The most apparent feature is the slightly separated convergence of the Radius inner and outer models. I believe this is most likely due to approximations made in defining initial values of solar radius as well as using the least-squares method of optimization compared to something like root. I decided to leave it in the least-squares method as I was able to define bounds of optimization, and when I tried to rewrite it using a root function, the plot did not converge as nicely as they do currently. When comparing my values to the MESA values, I find that both the Temperature and Pressure are under 10% difference with $T = 3.34\%$ and $P = 6.84\%$. However, Radius and Luminosity are not as close with $L = 17.04\%$ and $R = 24.09\%$; this leaves some room for improvement, specifically with the Radius values. Despite the larger difference, the plots line up extremely nicely with the MESA versions.

8. CONCLUSIONS

In conclusion, the stellar modeling code created provides accurate modeled parameters for a star of solar-like mass and composition. However, this code begins to lose accuracy the further the mass ratio moves from 1, thus, this code is only accurate for masses near that of the sun. With this document, I have described the necessary equations used in the model's output as well as summarized the initial assumptions made to create said model. While this program provides a decent, complete stellar modeling program, it is not advised to use it over something like the MESA program due to its better accuracy and range of models.

All code and files used can be found on my GitHub:https://github.com/tanner-code2000/Stellar-Model

REFERENCES

Hansen, C. J., Kawaler, S. D., & Trimble, V. 2004, Stellar interiors : physical principles, structure, and evolution

Hathaway, D. D. H. 2022, The Solar Interior, https://solarscience.msfc.nasa.gov/interior.shtml

Kippenhahn, R., Weigert, A., & Weiss, A. 2012, Stellar Structure and Evolution, doi: 10.1007/978-3-642-30304-3

(OP), T. O. P. 2009, Opacity Tables, https://cds.unistra.fr/topbase/OpacityTables.html

Paxton, B., Bildsten, L., Dotter, A., et al. 2010, ApJS, 192, 3, doi: 10.1088/0067-0049/192/1/3

APPENDIX

A. LIST OF CONSTANTS

Below is a list of constants used in this report:

- k_B (Boltzmann constant): $1.38 \times 10^{-16}\,\mathrm{erg/K}$
- m_H (Hydrogen mass): $1.67 \times 10^{-24}\,\mathrm{g}$
- $a_{\mathbf{rad}}$ (Radiation density constant): $7.5657 \times 10^{-15}\,\mathrm{erg\ cm^{-3}K^{-4}}$
- $\mathbf{G}$ (Gravitational constant): $6.6743 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}$
- $M_\odot$ (Solar mass): $1.989 \times 10^{33}\,\mathrm{g}$
- $L_\odot$ (Solar luminosity): $3.826 \times 10^{33}\,\mathrm{erg/s}$
- $R_\odot$ (Solar radius): $6.96 \times 10^{10}\,\mathrm{cm}$
- $T_\odot$ (Solar temperature): $6000\,\mathrm{K}$
- $\mathbf{c}$ (Speed of light): $2.99792458 \times 10^{10}\,\mathrm{cm/s}$
- σ (Stefan-Boltzmann constant): $5.67051 \times 10^{-5}\,\mathrm{erg\ cm^{-2}\,s^{-1}\,K^{-4}}$
- γ (Adiabatic index): $\frac{5}{3}$
- $\mathbf{N}_A$ (Avogadro's number): $6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
- ρ (density)
- κ (Rosseland mean opacity): Found using the opacity table for specific LogR and LogT values, or ≈ 0.3 when considered at the photosphere assuming Thomson scattering.
- X (mass fraction of hydrogen): 0.7
- Y (mass fraction of Helium): 0.28
- Z (metallicity mass fraction): 0.02