TODO:

- Fix and then produce PDF to archive this bitch off
- Maybe a numerical simulation where forcce laws are used to run simulation. Show that closed form analysis agrees with direct simulation using newtwons laws

1 Motivation

Simply consider two objects floating in space with respective masses and known intial positions and velocities. These two bodies follow Newtons Law of graviatation. These will be our only assumptions (along with conservation of energy). Specifically we start by defining the force between two point paritcles in 3+1 flat euclian space with positions, velocities, and masses $x_1(t)$, $x_2(t)$, $\dot{x}_1(t)$, $\dot{x}_2(t), M_1, M_2$. Each particle exerts an equal and opposite force on each other along the vector $r(t)$ i.e. the line connecting the two particles . The force F_1denotes the force of particle 2 on 1, and vice versa for F_1 acts in the direction $r(t)$ which we define as the vector pointing from $x_2(t)$ to $x_1(t)$ (note we also define $\hat{r}$to denote the unit vector where as r encodes the distant between the two particles as well as the direction). Formally,

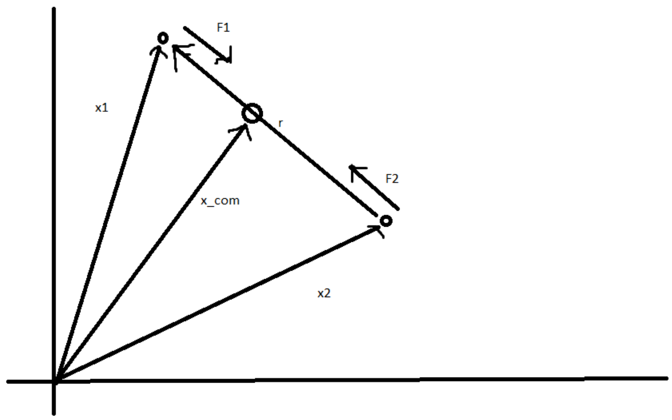
$$\begin{cases} F_1 = -G\frac{M_1 M_2}{\|r(t)\|^2}\hat{r} \quad F_2 = G\frac{M_1 M_2}{\|r(t)\|^2}\hat{r} \end{cases} \tag{1}$$

$$\begin{cases} r(t) = x_1(t) - x_2(t) \quad \hat{r} = \frac{x_1(t)-x_2(t)}{\|x_1(t)-x_2(t)\|} \end{cases} \tag{2}$$

Applying netwons second law ($F = m\ddot{x}$) to this yields the following highly non linear ODE:

$$\begin{cases} \ddot{x}_1 = -G\frac{M_2}{\|r(t)\|^2}\hat{r} \quad \ddot{x}_2 = G\frac{M_1}{\|r(t)\|^2}\hat{r} \end{cases} \tag{3}$$

To help visualize:



2 Center of Mass, Reduction to 1-D Problem, and Linear Momentum Conserved

(3) Is very compact and intuitive but is highly non linear and does imediatly suggest a solution. So we know the center of mass is an important point in gravitation problems so lets see the dynamics of the center of mass $x_{com}(t)$. By definition $x_{com}(t) = \frac{1}{M_1+M_2}(M_1x_1(t) + M_2x_2(t))$. Now simply differentiate twice with respect to time:$\ddot{x}_{com}(t) = \frac{\|r(t)\|^2}{M_1+M_2}(F_1 + F_2) = 0$ since $F_1 = -F_2$.

Thus we can see that $x_{com}(t)$ is non accelerating and thus is an interial frame and so we will continue our analysis in this frame. As a quick detour, this allows us to show that momentum is conserved. To see this, consider that $\dot{x}_{com}(t)$ is constant by the above reasoning. Thus $\dot{x}_{com}(t) = \frac{1}{M_1+M_2}(p_1(t) + p_2(t))$ is constant implying the total momentum is constant. Now continuing, we can rewrite $x_2(t)$ and $x_1(t)$ in terms of the center of mass. Foreshadowing, lets define $\mu = \frac{M_1M_2}{M_1+M_2}$

$$x_{com}(t) = \frac{\mu}{M_2}x_1(t) + \frac{\mu}{M_1}x_2(t)$$

$$x_1(t) = \frac{\mu}{M_2}x_1(t) + \frac{\mu}{M_1}x_2(t) - \frac{\mu}{M_1}x_2(t) - (\frac{\mu}{M_2} - 1)x_1(t)$$

$$x_1(t) = x_{com}(t) - (\frac{\mu}{M_1}x_2(t) + \frac{-M_2}{M_1 + M_2}x_1(t))$$

Finally we get the following, where the posistion of particle 2 comes from symmetry:

$$\begin{cases} x_1(t) = x_{com}(t) + \frac{\mu}{M_1}r(t) \quad x_2(t) = x_{com}(t) - \frac{\mu}{M_2}r(t) \end{cases} \tag{4}$$

Selecting the center of mass as our interial frame shows $x_2(t)$ and $x_1(t)$ only depend on the vector defining their displacement $r(t)$. So what are the dynamics of $r(t)$? Well first

$$\ddot{r}(t) = \ddot{x}_1(t) - \ddot{x}_2(t)$$

Aplying (3)

$$\ddot{r} = -G\frac{M_2}{\|r(t)\|^2} - G\frac{M_1}{\|r(t)\|^2}\hat{r} = -(M_1 + M_2)\frac{G}{\|r(t)\|^2}\hat{r}$$

A little maniputlation:

$$\frac{M_1 M_2}{M_1 + M_2}\ddot{r} = F_1 = -G\frac{M_1 M_2}{\|r(t)\|^2}\hat{r}$$

Cleaning up

$$\mu\ddot{r} = F_1$$

So we see that our two particle system can be broken into a simplier one body problem, whosw position is defined by $r(t)$, with mass $\mu = \frac{M_1 M_2}{M_1+M_2}$, subject to a net force F_1. Note that F_1points in the opposite direction of $r(t)$, impling the force is trying to pull the bodies together as expected, but the pull weakens as $r(t)$ increases.

3 Angular Momentum is Conserved and Reduction to 2+1 Space

Lets looks at the angular momentum of this 1 particle system. $L = r \times p = r \times \mu\dot{r}$. Now simply take the time derivative this expression. $\dot{L} = (\dot{r} \times \mu\dot{r}) + (r \times \mu\ddot{r})$. Now the first term is 0, because the cross product of a vector and its scalar multiple is zero. Now the second term is just $r \times \mu\ddot{r} = r \times F_1$. These two vectors are anti parallel by defintion and this is 0. This implies the rate at which angular momentum is changing is zero. This implies angular momentum is constant. So we let L be the intial and only angular momentum of this system.

With angular momentum being conserved, we can find an interial frame where are 1 particle system rotates in a 2D plane. So from here on out we will assume that we are in that we are in that interial frame and will assume the third component of our 3D system is 0. So, $r(t) = r_x(t)\hat{i} + r_y(t)\hat{j}$ in the 2D cartesian coordinate plane.

4 Energy and Polar Coordinates

Now let us look at the energy of this, 2D 1 body problem. The total energy is given by $E = K+U$. Where $K = \frac{1}{2}\mu\|\dot{r}\|^2$ and $U = -G\frac{M_1 M_2}{\|r(t)\|}$. So before continuing let us define polar cordinates. $r_y(t) = R(t)\sin(\theta(t))$ and $r_x = R(t)\cos(\theta(t))$.

This is motivated by the fact that our problem now exists in the catesian plane, and we know that these equations describe graviataion and thus usually exhibit an orbitting behavior. So $U(R) = -G\frac{M_1 M_2}{R}$. Now finding the kinetic energy is a bit more complicated, but we have the following transformation

$$\dot{r}_x = \dot{R}\cos(\theta) - R\sin(\theta)\dot{\theta}$$

$$\dot{r}_y = \dot{R}\sin(\theta) + R\cos(\theta)\dot{\theta}$$

So we already have our potentail energy in terms of our new polar coordinate variables, (R, θ). Now let us find an expression in these coordinates for our kinetic energy. We know $K = \frac{1}{2}\mu(\dot{r}_x^2 + \dot{r}_y^2)$. We can plug our transformation from above into this expression and one gets:

$$K = \frac{1}{2}\mu(\dot{R}^2 + R^2\dot{\theta}^2)$$

However we can simplify this further. Since we know that angular momentum is constant, we can rewrite the angular velocity term above using this constant. We know $L = I\dot{\theta}$ where $I = \mu R^2$ is the moment of inertia. Solving this for the angular velocity leads to $w = \dot{\theta} = \frac{L}{\mu R^2}$. Pluging this into kinetic energy gives: $K = \frac{1}{2}\mu\dot{R}^2 + \frac{L^2}{2\mu R^2}$. So we get an expression for the total energy of this system $E = K + U$

$$E = \frac{1}{2}\mu\dot{R}^2 + \frac{L^2}{2\mu R^2} - G\frac{M_1 M_2}{R} \tag{5}$$

In the next section we show how to solve this using conservation of energy.

5 Solution

To solve (5) we must assume that energy is conserved and thus total energy E is constant. It would nice to derive this from the defintion of the forces acting our system, and it possibly falls out from the fact that the force is conservative, but its simply easier to assume its conserved and move on. So we take the time derivative of (5) and get the following differential equations (we also include the dynamics of the angular velocity as the below will now fully describe the time depedant dynamics of our system):

$$\left\{ \frac{dE}{dt} = \mu\ddot{R} - \frac{L^2}{\mu R^3} + G\frac{M_1 M_2}{R^2} = 0 \quad w = \dot{\theta} = \frac{l}{\mu R^2} \right. \tag{6}$$

This is a system of non-linear second order differential equationa and thus we will need to play with it a bit to solve it. The above is a time depedant equation, i.e. $R = R(t)$. We can simplyify the above by using the angular velocity to find a single time indepandt differential equation. So, we can do a change of variables as follows:

$$\frac{d\theta}{dt} = \frac{L}{\mu R^2}$$

$$\frac{dR}{dt} = \frac{dR}{d\theta}\frac{d\theta}{dt} = \frac{dR}{d\theta}\frac{L}{\mu R^2}$$

$$\frac{d^2R}{dt^2} = (\frac{d}{dt}\frac{dR}{d\theta})\frac{d\theta}{dt} + \frac{dR}{d\theta}(\frac{d^2\theta}{dt^2}) = (\frac{d^2R}{d\theta^2})(\frac{L}{\mu R^2})^2 + (\frac{dR}{d\theta})^2\frac{-2L}{\mu R^3}\frac{L}{\mu R^2}$$

Now we can subsitute in the above into (6) giving us a time indepedant differential equation describing our system

$$\frac{d^2R}{d\theta^2}\frac{1}{R^2} - (\frac{dR}{d\theta})^2\frac{2}{R^3} - \frac{1}{R} + \mu\frac{GM_1M_2}{L^2} = 0 \tag{7}$$

Now lets try the subsitution $R = \frac{1}{u}$ and $\frac{dR}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$ and $\frac{d^2R}{d\theta^2} = \frac{2}{u^3}\frac{du}{d\theta} - \frac{1}{u^2}\frac{d^2u}{d\theta^2}$. Plugging this subsittion into our time indepedant equation gives:

$$\frac{d^2u}{d\theta^2} = \mu\frac{GM_1M_2}{L^2} - u$$

We can easily solve this. To make that apparent we let $v = u - \mu\frac{GM_1M_2}{L^2}$. Which gives

$$\frac{d^2v}{d\theta^2} = -v$$

This has solutions $v = c_1\sin(\theta) + c_2\cos(\theta)$. So $u = \frac{1}{R} = c_1\sin(\theta) + c_2\cos(\theta) + \mu\frac{GM_1M_2}{L^2}$. Now we can choose our coordinates so that the sin term goes away. This yields the following explicit soltion

$$R = \frac{L^2}{\mu GM_1M_2}\frac{1}{1+e\cos(\theta)} \tag{8}$$

Keep in mind we still have have the following relation $w = \dot{\theta} = \frac{L}{\mu R^2}$ which allows us to convert from the angular domain to the time domain. But as we can see from (8) the general geometry of the solution is that off conic sections. Thus the possible set of solutions is a cirlce, ellipse, parabola, or hyberbola (and possibly others). For a given set of initial conditions, determining which set the dynamics fall into is simply a matter of finding the constant e in (8). Now we can plug the radius above into the formulas for Kinetic Energy, Total energy, and angular velocity so that we have a set of equations that sufficently describe the dynamics of our two body and reduced 1 body system. These are:

Radial:

$$R = \frac{L^2}{\mu GM_1M_2}\frac{1}{1+e\cos(\theta)}$$

Energy:

$$K = \frac{1}{2}\mu\dot{R}^2 + \frac{L^2}{2\mu R^2} = \frac{\mu G^2 M_1^2 M_2^2}{2L^2}(e^2 + 1 + 2e\cos(\theta))$$

$$E = K + U = \frac{\mu G^2 M_1^2 M_2^2}{L^2}(\frac{e^2 - 1}{2})$$

Angular:

$$w = \frac{L}{\mu R^2} = \frac{\mu G^2 M_1^2 M_2^2}{L^3}(1 + e\cos(\theta))^2$$

Time:

$$t(\theta) = \frac{L^3}{\mu G^2 M_1^2 M_2^2} \int_0^\theta \frac{ds}{(1 + e\cos(s))^2}$$

The above fully describe the dynamics of our system. For a given angle we can describe the radius, angular velocity, and kinetic energy. If one is interested in answering questions such as how long it takes to do X or at what point will X happen, then one can use numeric integration to calculate how long it takes to sweep a certain angle and use the other formulas to compute the desired quantity. The total energy, and thus the eccentricity of the orbit is determined by the initial conditions. As well as the angular momentum.

6 Characterization of Solutions

In this section we further analyze the dyamics of the solution found in the previous section. We use the energy, the geometric shape of the orbit, and other parameters to investigate the physical reality of the initial two body problem.

6.1 Calculating Eccentricity

Just use the above relation for total energy.

$$e = \sqrt{1 + \frac{2EL^2}{\mu G^2 M_1^2 M_2^2}}$$

The above has a "phase" transistion at the following values, inspired by the fact that this value describes the eccentricity of our orbit:

- If $E = -\frac{\mu G^2 M_1^2 M_2^2}{2L^2}$, then $e = 0$. So circle.
- If $E > -\frac{\mu G^2 M_1^2 M_2^2}{2L^2}$,but $E < 0$, then $0 < e < 1$. So ellipse.
- If $E > 0$, then $e > 1$. So hyperbola.
- If $E < -\frac{\mu G^2 M_1^2 M_2^2}{2L^2}$, then e is imagnary?? Predict this is the case where one body falls into the other.

6.2 Breaking it down by case

We found our solution for the reduced 1-body problem and observe there are several cases based on the initial conditions. Thus we break down these cases and charactize the resulting solution in each case. Our initial conditions allow us to use the masses, relative initial positions i.e. $r(0)$, and relative intial velocites. From these we can compute the conserved quantites such as total energy and angular momentum, which we tabulate below as a reminder of what can be taken as given.

Initial Condtions	Value (2-Body)	Value (Reduced 1-Body)
Mass	M_1, M_2	$\mu = \frac{M_1 M_2}{M_1+M_2}$
Position	$x_1(0)$, $x_2(0)$	$r(0) = x_1(0) - x_2(0)$
Angular Momentum	$\mu(r(0) \times v_1(0) - v_2(0))$	$L = r(0) \times \mu(v_1(0) - v_2(0))$
Total Energy	$\frac{1}{2}M_1 v_1^2(0) + \frac{1}{2}M_2 v_2^2(0) - \frac{GM_1M_2}{\|x_1(0)-x_2(0)\|}$	$\frac{1}{2}\mu\|v_1(0) - v_2(0)\|^2 - \frac{GM_1M_2}{r(0)}$

In the analysis that follows we consider only the 1 - body problem and treat the solutions as describing a single body subject to an external force which yields closed orbits in some cases and the single particle just flying off to infinity in others.

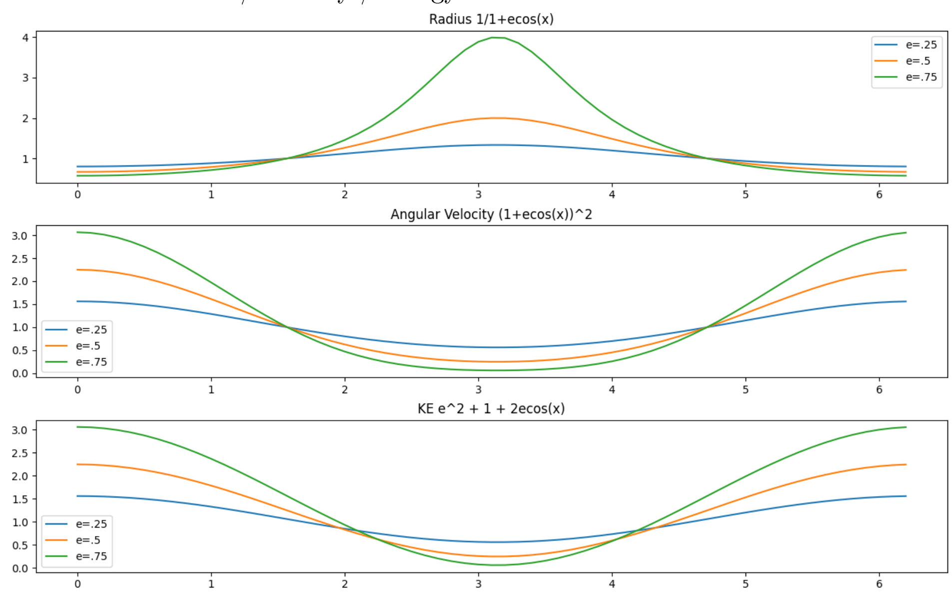
6.2.1 Circle

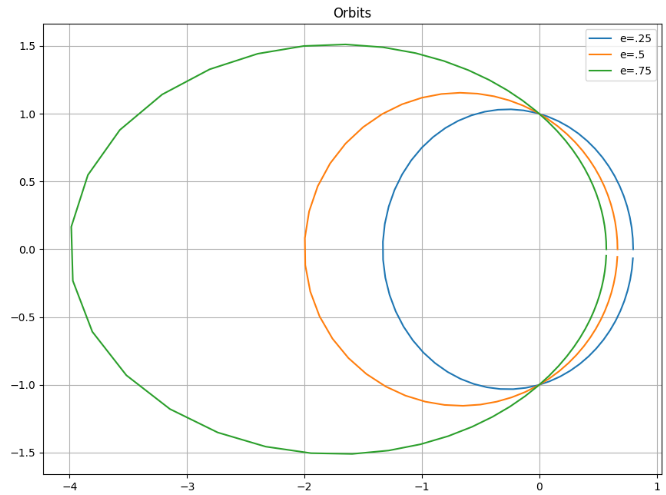
Paramters (Reduced 1-Body)	Solution
Eccentricity	$e = 0$
Radius	$R = \frac{L^2}{\mu G M_1 M_2}$
Angular Velocity	$w = \dot{\theta} = \frac{\mu G^2 M_1^2 M_2^2}{L^3}$
Angular Position	$\theta = \dot{\theta} t$
Angular Domain	$\theta \epsilon \mathbb{R}$
Period of Orbit	$T = \frac{2\pi L^3}{\mu G^2 M_1^2 M_2^2}$
Kinetic Energy	$K = \frac{1}{2}\mu \frac{G^2 M_1^2 M_2^2}{L^2}$
Potential Energy	$U = -\frac{GM_1M_2}{R} = -\frac{\mu G^2 M_1^2 M_2^2}{L^2} = -2K$
Total Energy	$E = K + U = -K$

The case of the circle is very easy to analyze as it is just uniform circle motion. The key take away here is that the defining dynamics are such that potential energy and kinetic energy do no depend on time and thus do not trade off. This implies the defining inital conditions that lead to the case of a circular orbit are those such that the overall kinetic energy of the motion of the particle is exactly half the magnitude of the gravitational potential energy. The particle will then orbit the origin with constant radius, contant angular velocity, and constant kinetic energy i.e. uniform circular motion. Thus, as shown in the above table, we can simplify the general formulas into time and angle indepedant ones. Moreover, we can compute the period of this orbit rather simply, and its value is again shown above. Finally we note that the values the angle can take on are not limited and can be any real number.

6.2.2 Ellipese

The ellipse's dynamics are more complicated than that of the circle, obviously. More specifically, the dynamic of the radius, angular velocity, energy, etc. now vary according to the general formulas given in the previous section. However, we can characterize the intial conditions that lead to this orbit. Moreover, given the geometry of the orbit, we know there are points at which the paramters (i.e. radius) are at extremes and we can calcualate these extremal values as well as where they occur in the orbit. Below we plot the general form of the radius, angular velocity, and kinetic energy as a function of the angular position. Note that we do not scale the graphs in accordance to some choosen initial conditions, it is simply the form of the general solutions given in the previous section with a few choosen eccentricity parameters less than 1. We also plot the elliptical orbit associated with each radius / velocity / energy curve shown.





So the orbit, radius, angular velocity, and energy are fairly well described by the general formulas and the above graphs. We can see that at $\theta = 0$, the radius is at its mininum and its kinetic energy is at its max. This flips at $\theta = \pi$. So in the reduced 1-body system, when the object to closest to the orgin its moving fast. It gets kicked away from the orgin, slowing down. It then accelerates towards back towards the orgin until it reaches its closest point, maximum kinetic energy, and the cycle repeats. We can see this orbit is, obviously an ellipse, is stable and closed. Thus our angle can take on value. Finally, this orbit is predicted by having energy greater than that of the circular orbit, but less than 0.

Paramters (Reduced 1-Body)	Solution
Eccentricity	$0 < e < 1$
Radius	$R_{max} = \frac{L^2}{\mu G M_1 M_2} \frac{1}{1-e}$,$R_{min} = \frac{L^2}{\mu G M_1 M_2} \frac{1}{1+e}$
Angular Velocity	$w_{min} = \frac{\mu G^2 M_1^2 M_2^2}{L^3}(1-e)^2$, $w_{max} = \frac{\mu G^2 M_1^2 M_2^2}{L^3}(1+e)^2$
Angular Domain	$\theta \epsilon \mathbb{R}$
Kinetic Energy	$\mathrm{K}_{min} = \frac{\mu G^2 M_1^2 M_2^2}{2L^2}(1-e)^2$,$K_{max} = \frac{\mu G^2 M_1^2 M_2^2}{2L^2}(1+e)^2$
Total Energy	$E < 0$, such that $E \to 0$ as $e \to 1$

6.2.3 Non Capturing Orbits (Parabola / Hyperbola)

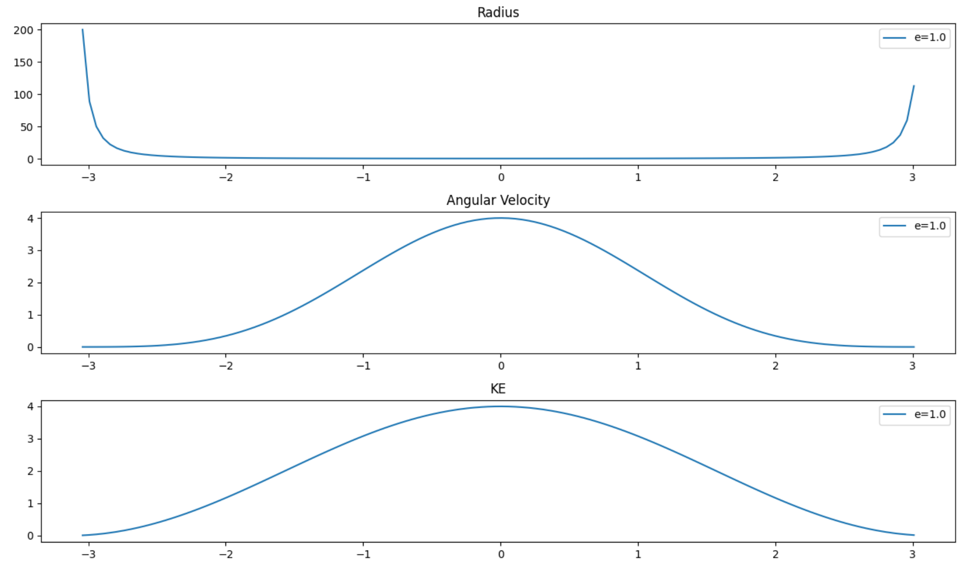
Predicted by positive total energy.

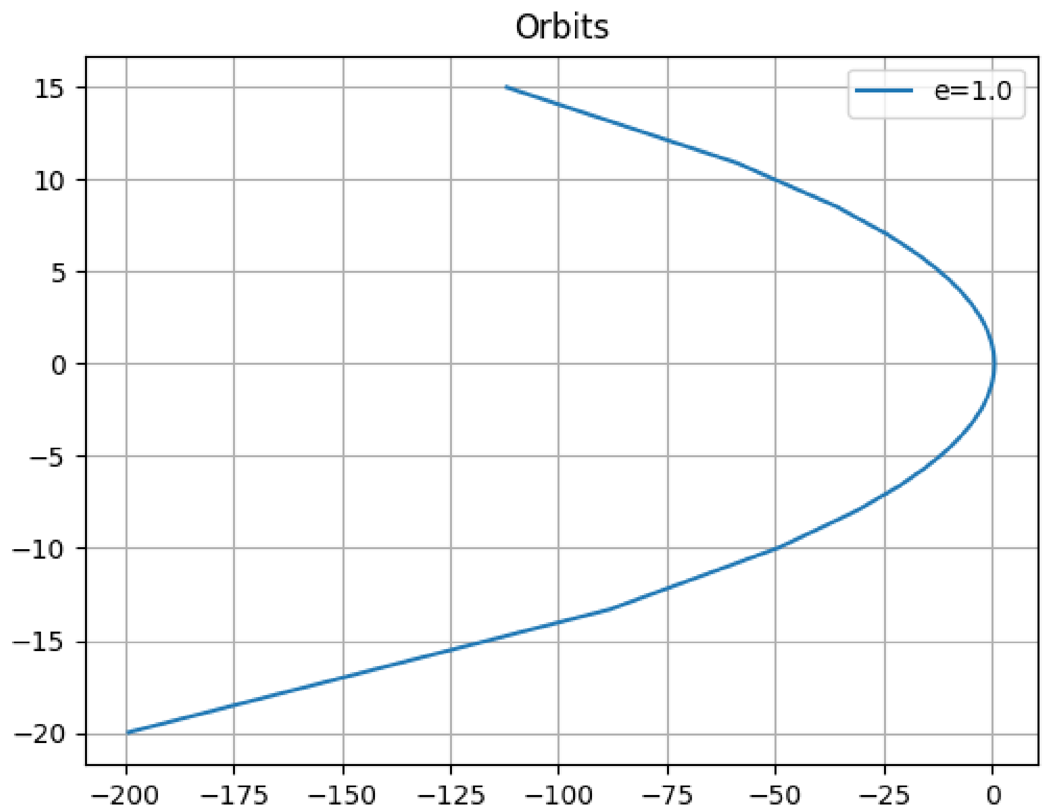
Orbit: Now the orbit in this case is a hyperbola or parabola. Thus we can expect that the mass starts and some position and as time progresses, $R \to \infty$. This orbit is not periodic and thus it does not make sense to talk about a period. None the less, we can still talk about the minimum radius and mininum kinetic energy. We can also talk about the angular velocity and position and their extrema, and we can also limit the values of the angular position that makes sense, since its not periodic it doesn't make sense to allow our angle to take any value. And we can talk about what happens as the radius goes to infinity and the resulting dynamics. Thus this orbit is described by a point closest to the body followed by the object drifting away to infinity.

Radius: From the general formula, in the hyperbolic case, we see two interesting points. The first is the mininum radius at $\theta = 0$ where $R_{min} = \frac{L^2}{\mu G M_1 M_2} \frac{1}{1+e}$. The next is the point where $R \to \infty$ which happens as $\theta \to \arccos(-1/e)$. Due to even symetry of the cosine function, $R \to -\infty$ which happens as $\theta \to -\arccos(-1/e)$. As an exmple if $e = 1$, then $R \to \infty$ as $\theta \to \pi$. But if $e = 2$, then $R \to \infty$ as $\theta \to 2\pi/3$.

Angular: So from the above, we can conclude that we must limit our angualar positions to the interval $(-\arccos(-1/e), \arccos(-1/e))$. We can also say the following. The angular velocity is maxed at $\theta = 0$ with $w_{max} = \frac{\mu G^2 M_1^2 M_2^2}{L^3}(1+e)^2$. And as $\theta \to \pm \arccos(-1/e)$, then angular velocity is minimized, with $w \to 0$.

Energy: The energy is perhaps the easiest to analyze. As the $R \to \infty$, $U \to 0$, and thus we can say that kinetic energy is minned as $R \to \infty$ with the limiting energy simply being the total conserved energy, $K \to E = (e^2 - 1)\frac{\mu G^2 M_1^2 M_2^2}{2L^2}$. Likewise, the kinetic energy is maxed at the minimum radius or 0 angle and this max kinetic energy is $K_{max} = \frac{\mu G^2 M_1^2 M_2^2}{2L^2}(1+e)^2$.


Radius
e=1.0
Angular Velocity
e=1.0
KE
e=1.0



Paramters (Reduced 1-Body)	Solution
Eccentricity	$e > 1$
Radius	$R_{max} = \infty$,$R_{min} = \frac{L^2}{\mu G M_1 M_2} \frac{1}{1+e}$
Angular Velocity	$w_{min} = 0$, $w_{max} = \frac{\mu G^2 M_1^2 M_2^2}{L^3}(1+e)^2$
Angular Domain	$\theta \epsilon (-\arctan(-1/e), \arctan(-1/e))$
Kinetic Energy	$\mathrm{K}_{min} = E$,$K_{max} = \frac{\mu G^2 M_1^2 M_2^2}{2L^2}(1+e)^2$
Total Energy	$E = (e^2-1)\frac{\mu G^2 M_1^2 M_2^2}{2L^2}$

6.2.4 Falling Orbits

Predicted by energy less than that of the circle. Hypothesis is that this is a degenerate case that only happens if $L = 0$. Equivalently, this would only happen if the initial velocities both lie on the line connecting the two bodies. This can be reasoned using conservation of angular momentum. If it has angualar momentum, it must keep it. Thus the two bodies can never lie on path that intersects (given their point particles). Could try to prove this further, but that reasoning feels right.

7 Application and Verification

Have two body numerical simulation in this directory, TB_num_sym.py. Given initial conditions it simulates Newton's laws of gravitation and saves off the data. It then plots an animation showing the path of the two bodies through space and prints the conserved quantities to verify their conserved. The following can be verified using this simulator:

- Total energy and angular momentum are conserved
- Classification of orbit based on total energy
- Period of elliptical orbits
- Formulas for total energy, min/max kinetic energy, eccentricity, min/max radius, and min/max angular velocity hold
  - Note the prediction of eccentricity works in the elliptical case works, but for the hyperbolic case it does not give correct value. Will not investigate as I calculated eccentricity in a wonky manner.
  - All other values are accurately predicted though

8 What's Next

- Langranian Mechanics. How we solved this problem is a classic example of how and why the Langrange formulation of classic mechanics is used. Thus a further investigation of this topic is warranted.
- 3 Body Problem. We solved the general 2-body case. What about 3? Its not analytically solvable, but tricks exist.
- N-body problem. What about for N-bodies?
- Langrange Points. Stable 3-body configurations.
- Linear Algebra / Matrix formulation of system of diff eq.
- Newton's shell theorem
- LRL vector?
- Eccentricty vector