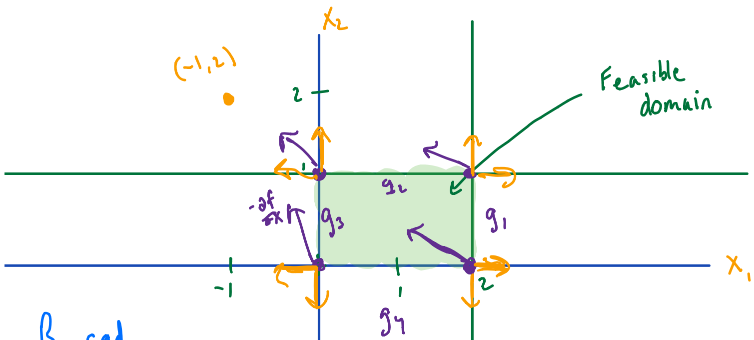
1. (10 points) Sketch graphically the problem

$$\min\ f(\mathbf{x}) = (x_1+1)^2 + (x_2-2)^2$$
$$\text{subject to } g_1 = x_1 - 2 \le 0, \quad g_3 = -x_1 \le 0,$$
$$g_2 = x_2 - 1 \le 0, \quad g_4 = -x_2 \le 0.$$

$x_1 \ge 0$
$x_1 \le 2$
$x_2 \ge 0$
$x_2 \le 1$

Find the optimum graphically. Determine directions of feasible descent at the corner points of the feasible domain. Show the gradient directions of f and g_is at these points. Verify graphical results analytically using the KKT conditions.

If f were unconstrained the min = 0 at $(-1,2)$



Based on the graph, the solution should be $(0,1)$.

$$\hat{g}_1 = x_1 - 2 = 0$$
$$\hat{g}_2 = x_2 - 1 = 0$$
$$\hat{g}_3 = -x_1 = 0$$
$$\hat{g}_4 = -x_2 = 0$$

$$\frac{\partial \hat{g}_1}{\partial x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\frac{\partial \hat{g}_2}{\partial x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\frac{\partial \hat{g}_3}{\partial x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$\frac{\partial \hat{g}_4}{\partial x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix}$$

$$-\frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \qquad -\frac{\partial f}{\partial x}\Big|_{(2,0)} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

$$-\frac{\partial f}{\partial x}\Big|_{(0,1)} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \qquad -\frac{\partial f}{\partial x}\Big|_{(2,1)} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

KKT

$$L(x_1, x_2, \mu_1, \mu_2, \mu_3, \mu_4) = (x_1+1)^2 + (x_2-2)^2 + \mu_1(x_1-2) + \mu_2(x_2-1) + \mu_3(-x_1) + \mu_4(-x_2)$$

$$\frac{\partial L}{\partial x_1} = 2(x_1+1) + \mu_1 - \mu_3 = 0$$

$$\frac{\partial L}{\partial x_2} = 2(x_2-2) + \mu_2 - \mu_4 = 0$$

if $x_1 - 2 = 0$ the $\mu_1 > 0$
else $x_1 - 2 < 0$ then $\mu_2 = 0$ ①

if $x_2 - 1 = 0$ then $\mu_2 \geq 0$
else $x_2 - 1 < 0$ then $\mu_2 = 0$ ②

if $-x_1 = 0$ then $\mu_3 \geq 0$
else $-x_1 < 0$ then $\mu_3 = 0$ ③

if $-x_2 = 0$ then $\mu_4 \geq 0$
else $-x_2 < 0$ the $\mu_4 = 0$ ④

Case #1

$\mu_1 > 0$ and $\mu_2 > 0$, $\mu_3 = 0$, $\mu_4 = 0$

$x_1 = 2$ $\quad 2(2+1) + \mu_1 - 0 = 0$

$x_2 = 1$ $\quad \mu_1 = -6 \Rightarrow$ not possible

Case #2

$\mu_1 = 0, \quad \mu_2 > 0, \quad \mu_3 > 0, \quad \mu_4 = 0$

$x_1 = 0$

$x_2 = 1$

$$\frac{\partial L}{\partial x_1} = 2(0+1) + 0 - \mu_3 = 0$$

$$\mu_3 = 2 > 0 \quad \checkmark$$

$$\frac{\partial L}{\partial x_2} = 2(1-2) + \mu_2 - 0 = 0$$

$$\mu_2 = 2 > 0 \quad \checkmark$$

possible solution

Check For Convexity

Domain is convex

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix}$$

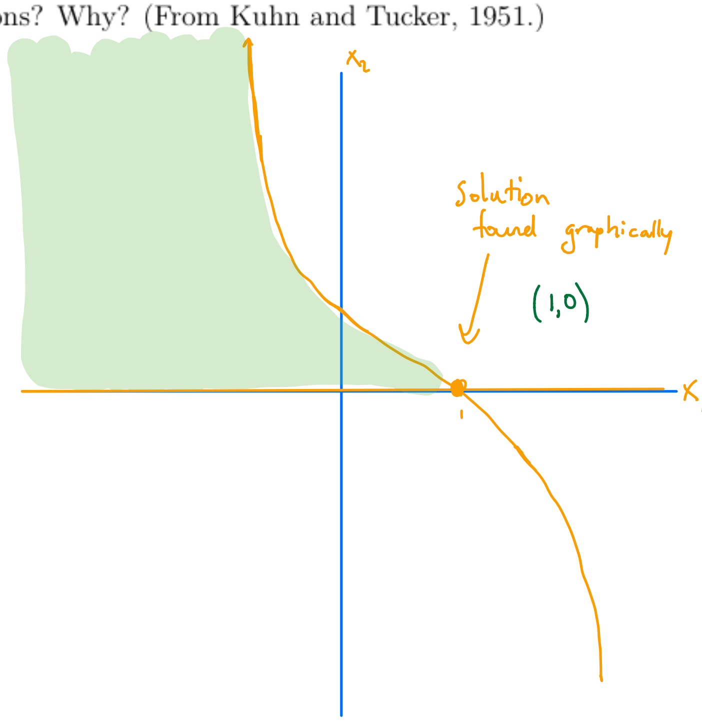
$$\frac{\partial^2 f}{\partial x^2} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow$$

convex therefore our soln found in case 2 is the global solution.

2. (10 points) Graph the problem

$$\min f = -x_1, \text{subject to}$$
$$g_1 = x_2 - (1 - x_1)^3 \le 0 \quad \text{and} \quad x_2 \ge 0.$$

Find the solution graphically. Then apply the optimality conditions. Can you find a solution based on the optimality conditions? Why? (From Kuhn and Tucker, 1951.)



$$\min_{\vec{x}} \quad -x_1$$

s.t. $g_1 = x_2 - (1-x_1)^3 \le 0$

$g_2 = -x_2 \le 0$

$$\frac{\partial g_1}{\partial \vec{x}} = \begin{bmatrix} 3(1-x_1)^2 \\ 1 \end{bmatrix}$$

$$\frac{\partial g_2}{\partial x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\left.\frac{\partial g_1}{\partial x}\right|_{(1,0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$\left.\frac{\partial g_2}{\partial x}\right|_{(1,0)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

gradients of constraint are linearly dependent so the solution (1,0) is irregular. Therefore, KKT optimality conditions would not find this solution.

3. (30 points) Find a local solution to the problem

$$\max f = x_1x_2 + x_2x_3 + x_1x_3$$
$$\text{subject to } h = x_1 + x_2 + x_3 - 3 = 0.$$

Use three methods: direct elimination, reduced gradient, and Lagrange multipliers. Compare. Is the solution global?

Direct Elimination

$$x_1 = -x_2 - x_3 + 3$$

$$\min \quad (-x_2 - x_3 + 3)x_2 + x_2x_3 + (-x_2 - x_3 + 3)x_3 = f$$

$$= -x_2^2 - \cancel{x_2x_3} + 3x_2 + \cancel{x_2x_3} - x_2x_3 - x_3^2 + 3x_3 = f$$

$$= -x_2^2 + 3x_2 - x_2x_3 - x_3^2 + 3x_3 = f$$

$$\frac{\partial f}{\partial x_2} = -2x_2 + 3 - x_3 = 0$$

$$\frac{\partial f}{\partial x_3} = -x_2 - 2x_3 + 3 = 0$$

$$\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

$$x_2 = 1$$
$$x_3 = 1$$
$$x_1 + 1 + 1 - 3 = 0$$
$$x_1 = 1$$

$$\boxed{\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

Check Hessian

$$H = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

$$\det(\lambda I - H) = 0$$

$$\begin{vmatrix} \lambda+2 & 1 \\ 1 & \lambda+2 \end{vmatrix} = 0$$

$$(\lambda+2)^2 - 1^2 = 0$$

$$\lambda^2 + 4\lambda + 4 - 1 = 0$$

$$\lambda^2 + 4\lambda + 3 = 0$$

$\lambda_1 = -3$

$\lambda_2 = -1$ $\rightarrow$ this is a global maximum

Reduced Gradient

3 variable $n=3$

1 constraint $m=1$

$$\max f = x_1x_2 + x_2x_3 + x_1x_3$$
subject to $h = x_1 + x_2 + x_3 - 3 = 0.$

$$X = \begin{bmatrix} d \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$d = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$s = x_3$$

$$\frac{\partial z}{\partial d} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d} = 0$$

$$h = 0$$

$$\frac{\partial f}{\partial d} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \end{bmatrix}$$

$$\frac{\partial f}{\partial s} = x_1 + x_2$$

$$\frac{\partial h}{\partial s} = 1 \Rightarrow \left(\frac{\partial h}{\partial s}\right)^{-1} = 1$$

$$\frac{\partial h}{\partial d} = \begin{bmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{\partial z}{\partial d} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d} = 0$$

$$= \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \end{bmatrix} - \begin{bmatrix} x_1 + x_2 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_2 + x_3 - x_1 - x_2 \\ x_1 + x_3 - x_1 - x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -x_1 + x_3 \\ -x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_1 + x_3 - 3 = 0$$

3 eqs, 3 unknowns

$\Rightarrow$ same answer as above

$$\boxed{\vec{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

Lagrange Multiplier

$$f(x_1, x_2, x_3, \lambda) = x_1 x_2 + x_2 x_3 + x_1 x_3 + \lambda(x_1 + x_2 + x_3 - 3)$$

$$\frac{\partial f}{\partial x_1} = x_2 + x_3 + \lambda = 0$$

$$\frac{\partial f}{\partial x_2} = x_1 + x_3 + \lambda = 0$$

$$\frac{\partial f}{\partial x_3} = x_1 + x_2 + \lambda = 0$$

$$\frac{\partial f}{\partial \lambda} = x_1 + x_2 + x_3 - 3 = 0$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \end{bmatrix}}$$

4. (20 points) Use reduced gradient to find the value(s) of the parameter b for which the point $x_1 = 1$, $x_2 = 2$ is the solution to the problem

$$\max\ f = 2x_1 + bx_2$$
$$\text{subject to } g_1 = x_1^2 + x_2^2 - 5 \leq 0$$
$$\text{and } g_2 = x_1 - x_2 - 2 \leq 0.$$

First, check which constraints are active at the solution

$$g_1 = 1^2 + 2^2 - 5 \leq 0$$
$$g_1 = 0 \qquad \text{active}$$

$$g_2 = x_1 - x_2 - 2 \leq 0$$
$$g_2 = 1 - 2 - 2 \leq 0$$
$$g_2 = -3 \qquad \text{not-active}$$

$$\min_{x_1, x_2} \quad 2x_1 + bx_2$$

$$x_1^2 + x_2^2 - 5 = 0$$

$n = 2 \qquad d = x_1$

$m = 1 \qquad s = x_2$

$$\frac{\partial f}{\partial d} = 2 \qquad \frac{\partial h}{\partial d} = 2x_1$$

$$\frac{\partial f}{\partial s} = b \qquad \frac{\partial h}{\partial s} = 2x_2$$

$$\frac{\partial z}{\partial d} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d}$$

$$= 2 - b\,\frac{1}{2x_2}(2x_1) = 0$$

$$2 - b\,\frac{x_1}{x_2} = 0$$

at solution

$x_1 = 1 \qquad x_2 = 2$

$$2 - b\frac{1}{2} = 0$$

$$-b\frac{1}{2} = -2$$

$$b\frac{1}{2} = 2$$

$$\boxed{b = 4}$$

5. (30 points, MAE 598) Find the solution for

$$\min\ f = x_1^2 + x_2^2 + x_3^2$$
$$\text{subject to } h_1 = x_1^2/4 + x_2^2/5 + x_3^2/25 - 1 = 0$$
$$\text{and } h_2 = x_1 + x_2 - x_3 = 0,$$

by implementing the generalized reduced gradient method (e.g., using MATLAB). See template code here.

$z = x_1^2 + x_2^2 + x_3^2$

$s = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$ $\quad d = [x_1]$

$$h = \begin{bmatrix} \frac{x_1^2}{4} + \frac{x_2^2}{5} + \frac{x_3^2}{25} - 1 \\ x_1 + x_2 - x_3 \end{bmatrix}$$

We are solving:

$$d_{k+1} = d_k - \alpha \left(\frac{\partial z}{\partial d}\right)_k^T$$
$$s'_{k+1} = s_{k+1} + \alpha_k \left(\frac{\partial h}{\partial s}\right)_k^{-1} \left(\frac{\partial h}{\partial d}\right)_k \left(\frac{\partial z}{\partial d}\right)_k^T$$

$$\frac{\partial z}{\partial d} = \frac{\partial f}{\partial d} + \frac{\partial f}{\partial s} \left(\frac{\partial h}{\partial s}\right)^{-1} \frac{\partial h}{\partial d}$$

$$[s_{k+1}]_{j+1} = \left[s_{k+1} - \left(\frac{\partial h}{\partial s}\right)_{k+1}^{-1} h(d_{k+1}, s_{k+1})\right]_j$$

$$d_{k+1} = d_k - \alpha \left(\frac{\partial^2 z}{\partial d^2}\right)_k^{-1} \left(\frac{\partial z}{\partial d}\right)_k^T$$
$$s_{k+1} = s_k + \left(\frac{\partial s}{\partial d}\right)_k \partial d_k + \frac{1}{2} \partial d_k^T \left(\frac{\partial^2 s}{\partial d^2}\right)_k \partial d_k$$

newtons method

We need $\frac{\partial h}{\partial s}$, $\frac{\partial^2 z}{\partial d^2}$, $\frac{\partial z}{\partial d}$, $\frac{\partial s}{\partial d}$, $\frac{\partial^2 s}{\partial d^2}$, ∂d_k

$$\frac{\partial z}{\partial d} = \frac{\partial f}{\partial d} + \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d}$$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial x_1} = 2x_1$$

$$\frac{\partial f}{\partial s} = \begin{bmatrix} \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_2 & 2x_3 \end{bmatrix}$$

$$\frac{\partial h}{\partial s} = \begin{bmatrix} \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{2x_2}{5} & \frac{2x_3}{25} \\ 1 & -1 \end{bmatrix}$$

$$\frac{\partial h}{\partial d} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} = \\ \frac{\partial h_2}{\partial x_1} = \end{bmatrix} \begin{bmatrix} \frac{x_1}{2} \\ 1 \end{bmatrix}$$

$$\frac{\partial z}{\partial d} = \begin{bmatrix} 2x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 & 2x_3 \end{bmatrix} \begin{bmatrix} 2x_2 & 2x_3 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{x_1}{2} \\ 1 \end{bmatrix}$$

0.7541
1.5436
3.0873