Lecture 9: Continuous Random Variables and Their Distributions

Announcements and reminders

Previously, on CSCI 3022…

Definition: A **discrete random variable** (r.v.) X is a function that maps the elements of the sample space Ω to a finite number of values $a_1, a_2, \dots, a_n$ or an infinite number of values $a_1, a_2, \dots$

Definition: A **probability mass function** (pmf) is the map between the random variable's values and the probabilities of those values.

$$f(a) = P(X=a)$$

Definition: A **cumulative distribution function** (cdf) is a function whose value at a point *a* is the cumulative sum of probability masses up until *a*

$$F(a) = P(X \leq a) = \sum_{x \leq a} f(x)$$

Continuous random variables

Many real-life random processes must be modeled by random variables that can take on continuous (i.e., not discrete) values. Some examples:

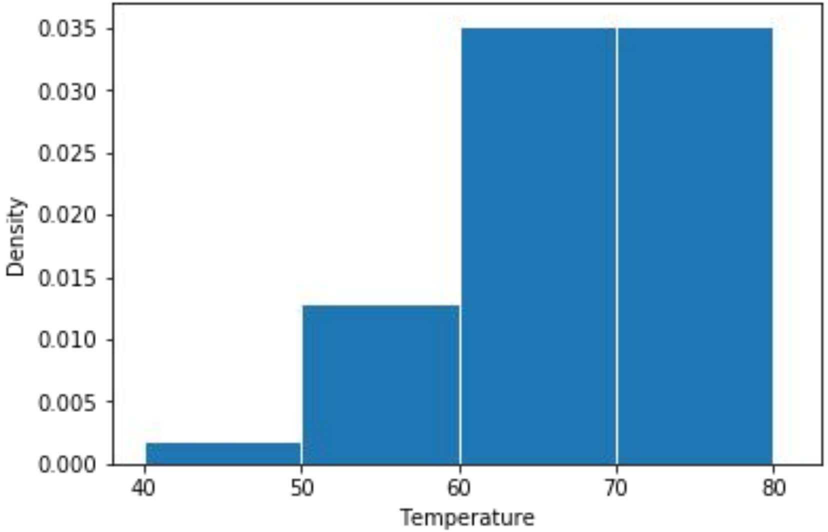
- Peoples' heights: $X \in$ __________
- Final grades in a class: $X \in$ __________
- Time between people checking out in a line at the store: $X \in$ __________

Other examples?

Continuous from discrete

Example: S'pose your friend asks you what the temperature will be like today. Specifically, they want to know what the probability that the temperature is between 70 and 80º F, so they can decide whether or not to wear shorts.

How would you calculate your response?



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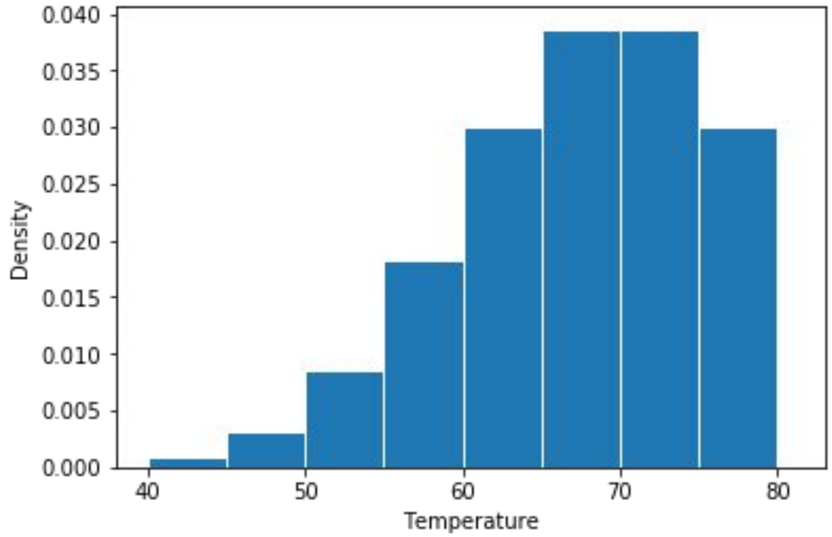

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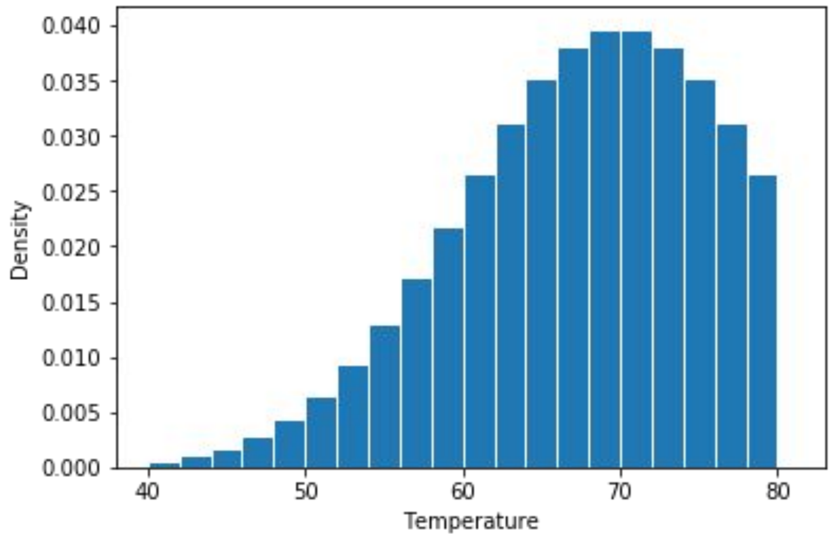

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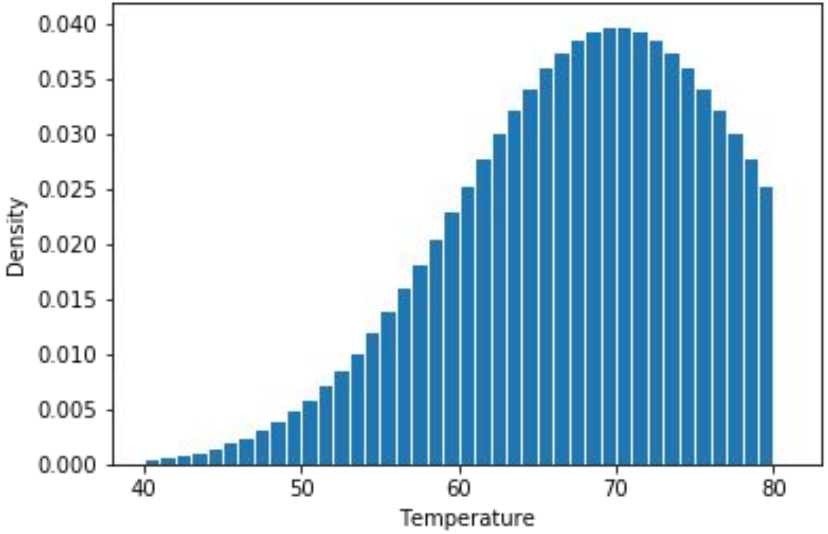

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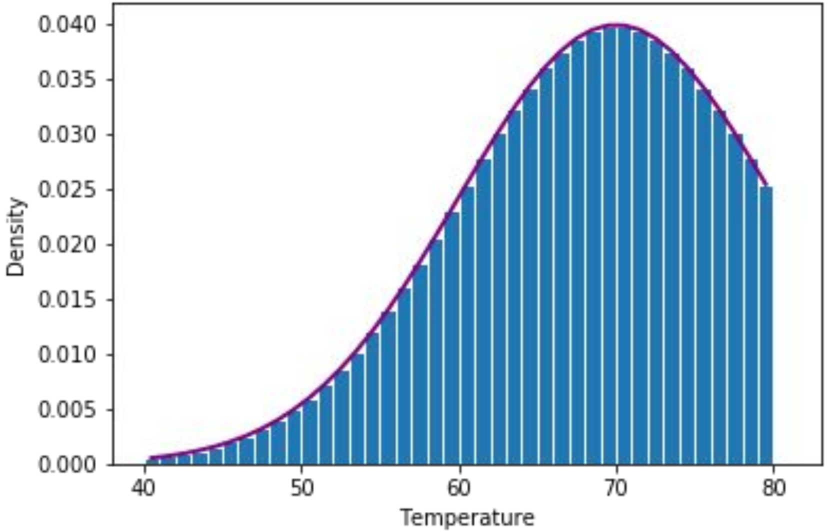


Continuous random variables

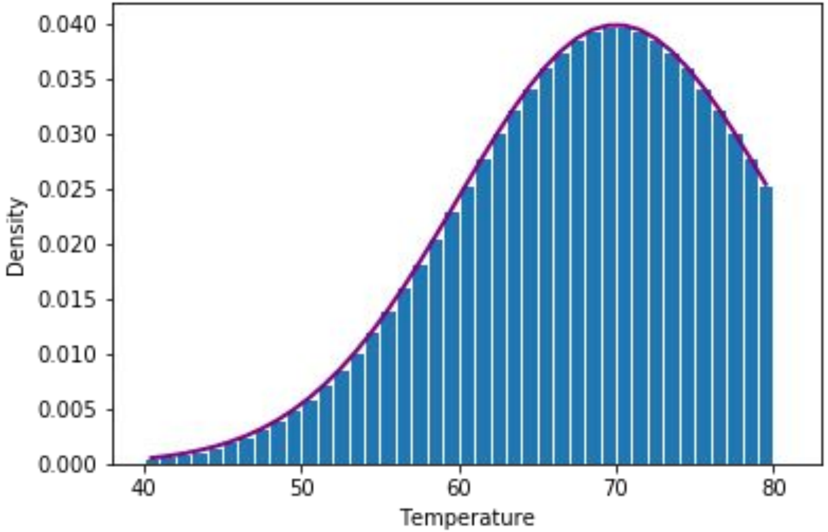
Definition: A random variable X is **continuous** if for some function $f : \mathbb{R} \to \mathbb{R}$ and for any numbers a and b with $a \le b$,

$$P(a \leq X \leq b) = \int_a^b f(x)\ dx$$

The function f must satisfy:

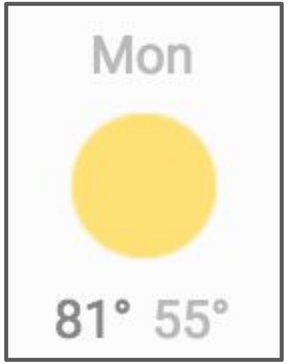
1) $f(x) \ge 0$ for all x, and
2) $$\int_{-\infty}^{\infty} f(x)\ dx = 1$$

We call f the **probability density function** (pdf) of X.



Continuous random variables

Example: S'pose you have some reason to believe that the temperature is equally likely to be anywhere between 55 and 81º F. Find the probability density function (pdf) for X.



Continuous random variables

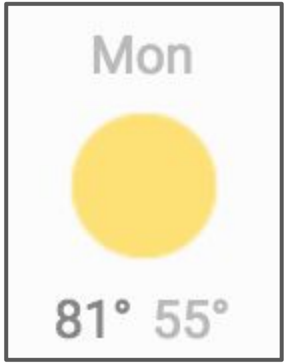
Example: S'pose you have some reason to believe that the temperature is equally likely to be anywhere between 55 and 81º F. Find the probability density function (pdf) for X.

Equally likely ⇒ $f(x) = \begin{cases} C & 55 \le x \le 81 \\ 0 & \text{otherwise} \end{cases}$

But the "sum to 1" condition means: $1 \overset{\heartsuit}{=} \displaystyle\int_{-\infty}^{\infty} f(x)\ dx = \int_{55}^{81} C\ dx = (81 - 55)C = 26C$

$\Rightarrow \quad C = \dfrac{1}{26}$

and: $f(x) = \begin{cases} \dfrac{1}{26} & 55 \le x \le 81 \\ 0 & \text{otherwise} \end{cases}$



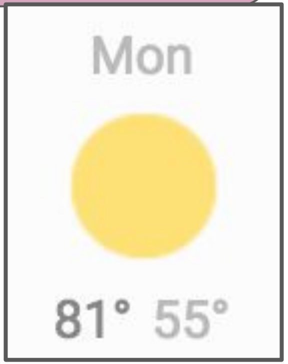
Continuous random variables

Example: S'pose you have some reason to believe that the temperature is equally likely to be anywhere between 55 and 81º F. Find the probability density function (pdf) for X.

Definition: A continuous random variable has a **uniform distribution** on the interval [α, β] if its probability density function *f* is given by *f*(*x*) = 0 if *x* is not in [α, β] and

$$f(x) = \frac{1}{\beta - \alpha} \qquad \text{for } \alpha \leq x \leq \beta$$

We say $X \sim U(\alpha, \beta)$

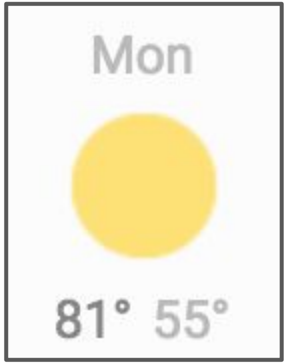


Continuous random variables

Example: S'pose you have some reason to believe that the temperature is equally likely to be anywhere between 55 and 81^{o} F. Find the probability density function (pdf) for X.

Question: What distribution does temperature follow in the example?

Follow-up question, getting to what we might actually care about:
What is the probability that temperature is between 75 and 81^{o} F?



Continuous random variables

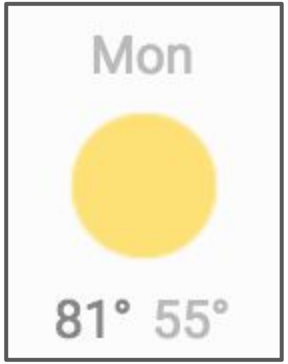
Example: S'pose you have some reason to believe that the temperature is equally likely to be anywhere between 55 and 81º F. Find the probability density function (pdf) for X.

Question: What distribution does temperature follow in the example?

X ~ U(55, 81)

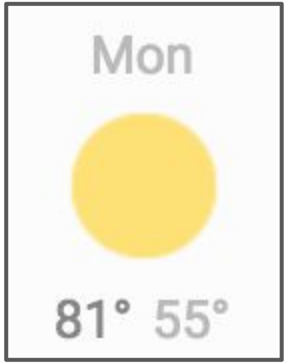
Follow-up question, getting to what we might actually care about:
What is the probability that temperature is between 75 and 81º F?

$$P(75 \leq X) = P(75 \leq X \leq 81) \qquad \text{b/c } U(55, 81)$$
$$= \int_{75}^{81} \frac{1}{26}\, dx$$
$$= \frac{81 - 75}{26} = \frac{6}{26} \approx 0.23$$



Continuous random variables

Example, follow-up: What is the probability that it is exactly 75º F?



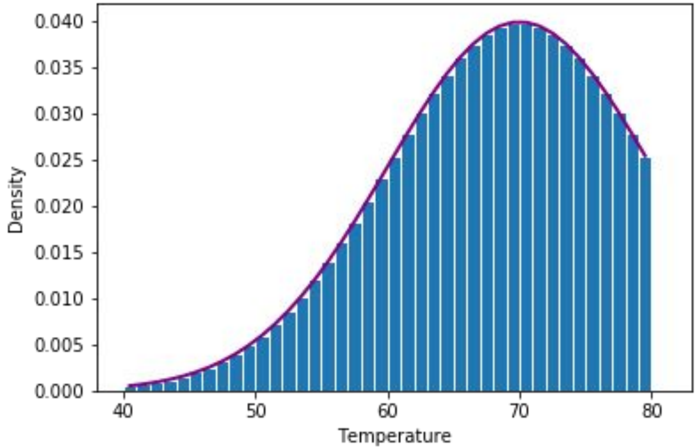
Continuous random variables

What if we want to compute things like P(X ≤ *a*)?

Is there an analog for the **cumulative distribution function** from the discrete case?

$$F(x) = P(X \leq x) = \sum_{k \leq x} f(k)$$

Question: What would the continuous analog of this sum be?



The cumulative distribution function

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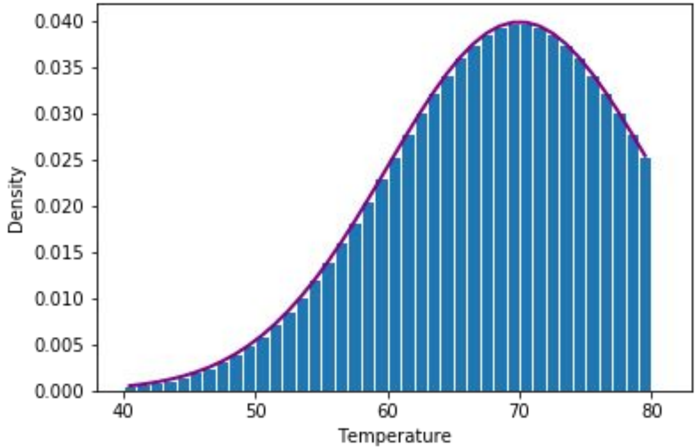
Is there an analog for the **cumulative distribution function** from the discrete case?

$$F(x) = P(X \leq x) = \sum_{k \leq x} f(k)$$

Question: What would the continuous analog of this sum be?

Answer: For continuous r.v., we ***also*** have a **cumulative distribution function**:

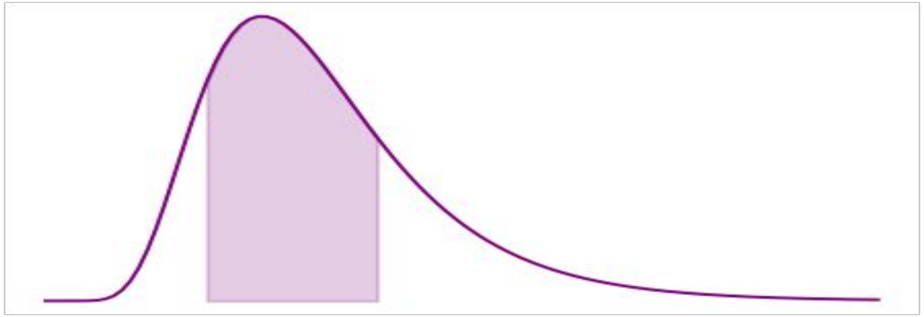
$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\ dt$$



The cumulative distribution function

Can we use the cdf to compute things like $P(a \leq X \leq b)$?

Example: What is the shaded region, in terms of the cdf?



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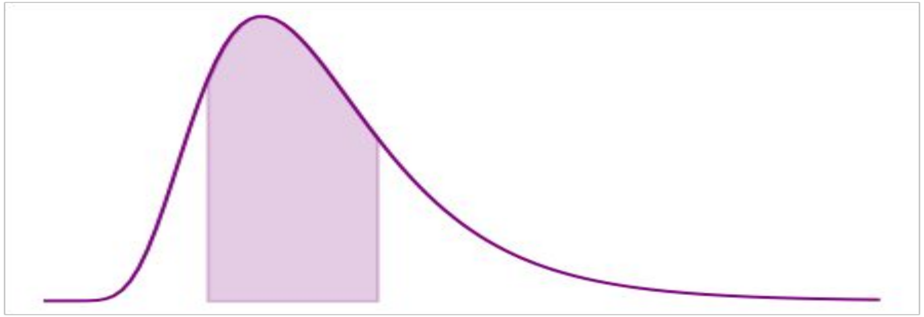
Example: What is the shaded region, in terms of the cdf?

More generally: $$P(a \leq X \leq b) = \int_a^b f(t)\, dt = F(b) - F(a)$$



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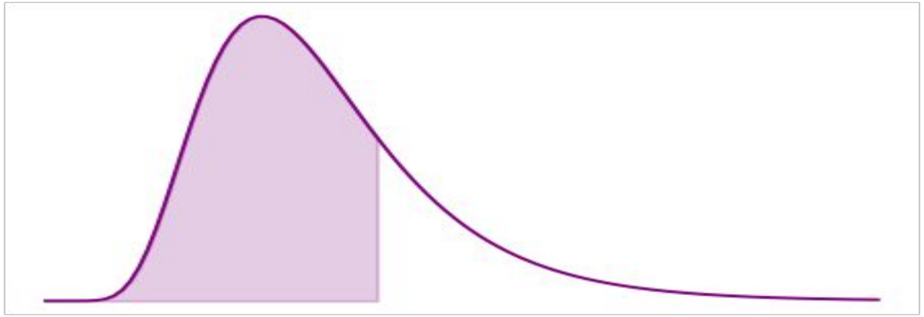


The cumulative distribution function

Can we use the cdf to compute things like $P(a \leq X \leq b)$?

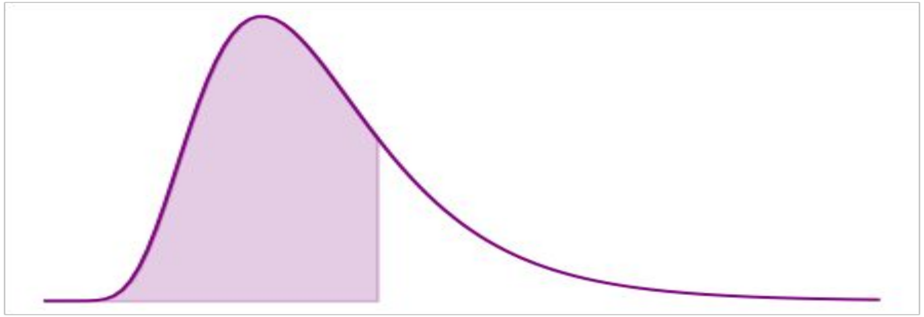
Example: What if we viewed the cdf as a function of x?

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_{-\infty}^{x} f(t)\, dt = f(x) - \text{“}f(-\infty)\text{”} = f(x)$$

$$\frac{d}{dx}F(x) = f(x)$$

THIS is a wildly important and useful relationship between F(x) and f(x)



The Normal distribution

Definition: A continuous random variable X has a **normal (or Gaussian) distribution** with parameters μ and σ^2 if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

We say $X \sim N(\mu, \sigma^2)$

Let's play around with this distribution: https://academo.org/demos/gaussian-distribution/

The Exponential distribution

Sometimes it's easier to first find the **cdf** and then derive the **pdf** by taking a derivative.

Example: Recall that the Poisson distribution describes the number of arrivals (or hits), assuming some constant average rate of arrivals per time period.

A Poisson random variable is discrete because we're counting things
(e.g., number of cars entering a parking garage)

But s'pose each arrival is someone finishing checking out in a grocery store line?

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(e.g., number of cars entering a parking garage)

But s’pose each arrival is someone finishing checking out in a grocery store line?

Now, we are interested in the **amount of time between each arrival**
(so we can choose which line will be the shortest!)

The Exponential distribution

S’pose the # arrivals follows a Poisson distribution (process) with rate λ arrivals/minute

- Start from $t = 0$ and let T be the random variable describing the first arrival
- What is the probability that the first arrival does not occur in the first t minutes?

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- Start from t = 0 and let T be the random variable describing the first arrival
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$$\begin{aligned} P(T > t) &= P(\text{no hits in } \leq t) \\ &= \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t} \end{aligned}$$

$$\begin{aligned} P(T \leq t) = 1 - P(T > t) = 1 - e^{-\lambda t} \\ F(t) = 1 - e^{-\lambda t} \end{aligned}$$

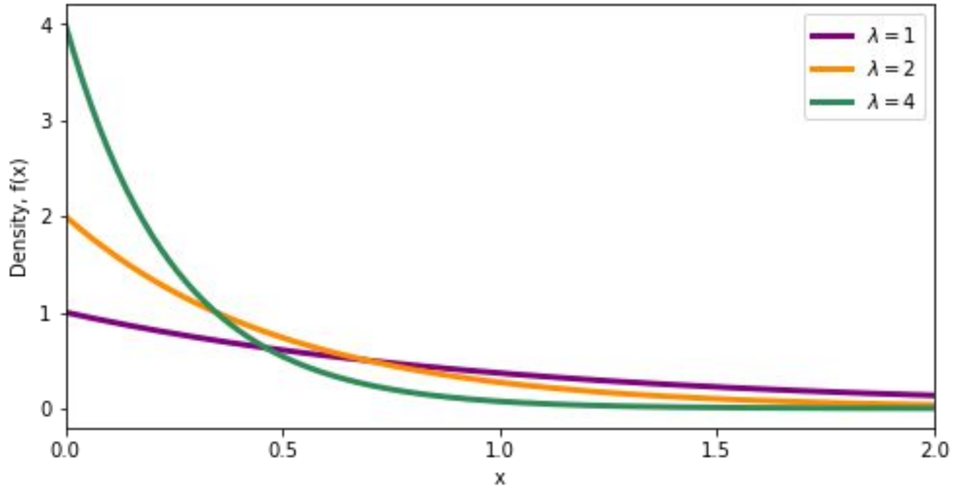
$$\begin{aligned} f(t) &= \frac{d}{dt}F(t) = \frac{d}{dt}\left(1 - e^{-\lambda t}\right) \\ &= \begin{cases} \lambda e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \end{aligned}$$

The Exponential distribution

Definition: A continuous random variable X has an **exponential distribution** with **rate parameter** λ > 0 if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

We say $X \sim Exp(\lambda)$



The Exponential distribution

Now hang on a second!

- That works for the **first** arrival.   But what about the second? Or third?
- We wanted the distribution of time between all arrivals

The Exponential distribution

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Fun fact:
It turns out the Exponential distribution has something called the **memoryless property**

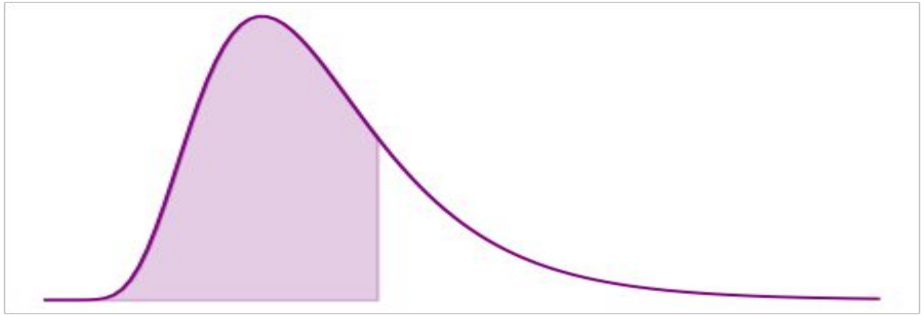
Theorem: (memoryless property)

If $T \sim Exp(\lambda)$, then $P(T > t + t_0 \mid T > t_0) = P(T > t)$

Quartiles and Percentiles

Given everything we have learned since Week 1, consider the following…

Question: How can we compute an x such that, say, $P(X \leq x)$ 75% of the time?



Okay! Let's get to work!

Get in groups, get out laptops, and open **nb09** notebook

Lets…

- Get more practice with probability density and distribution functions
- See how we can simulate normal and exponential random variables with Numpy
- See how we can approximate the density function of a continuous random variable using histograms