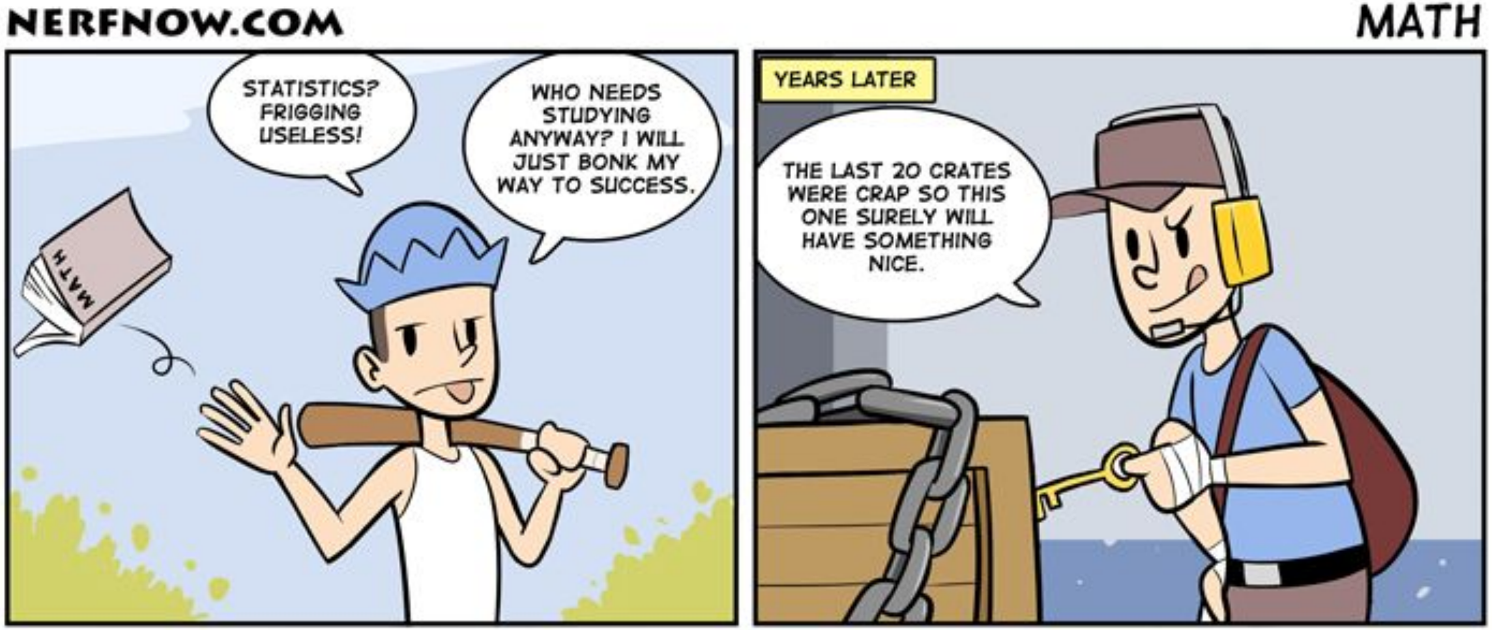
Lecture 8: More Discrete Random Variables and Their Distributions



Announcements and reminders

- HW 2 due Friday at 5 PM

Previously, on CSCI 3022…

Definition: A **discrete random variable** (r.v.) X is a function that maps the elements of the sample space Ω to a finite number of values $a_1, a_2, \ldots, a_n$ or an infinite number of values $a_1, a_2, \ldots$

Definition: A discrete r.v. X ~ *Ber*(*p*), where $0 \le p \le 1$, if its probability mass function is given by

$$f(1) = p_X(1) = P(X=1) = p \quad \textbf{and} \quad f(0) = p_X(0) = P(X=0) = 1-p$$

Definition: A discrete r.v. X ~ *Bin*(*n*, *p*), where *n* = 1, 2, … and $0 \le p \le 1$, if its probability mass function is given by

$$p_X(k) = P(X=k) = \binom{n}{k} p^k \, (1-p)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Binomial-like distributions

There are several discrete distributions that are similar in spirit to the Binomial distribution. We'll look at three of them today:

- Geometric distribution
- Negative Binomial distribution
- Poisson distribution

Binomial-like distributions

Example: You are doing an exit poll outside a voting station on Election Day. As people exit, you ask them questions about their political affiliation, who they voted for, etc.

In particular, you're interested in how registered Independents voted. Being well-prepared, you know that about 20% of registered voters are registered as Independents.

Goal: S'pose you interview 100 people. Let X be a random variable describing the number of actual Independents you encounter.

Distribution: $\text{Bin}(n = 100, \ p = 0.2)$

Binomial-like distributions

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Distribution: Binomial distribution (Bin(n=100, p=0.2))

Binomial-like distributions

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In particular, you're interested in how registered Independents voted. Being well-prepared, you know that about 20% of registered voters are registered as Independents.

Goal: S'pose you talk to a lot of registered Republicans and Democrats, but haven't found an Independent yet. Let X be a random variable describing the number of people you have interviewed up to an including your first registered Independent voter.

Distribution: Geometric Dist.

Binomial-like distributions

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Goal: S'pose you're really interested in talking to a lot of Independents. Let X be the random variable describing the number of people you have to talk to in order to interview exactly 100 registered Independents.

how many until we get 100 successes?

Distribution: Negative Binomial Dist.

Binomial-like distributions

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Goal: You're concerned about being overwhelmed during peak voting times, so you track the number of people arriving in line at the voting station. Let X be a random variable describing the number of voters that arrive at the station over a 15-minute period.

think: # of "hits" or "arrivals"... w/in a certain fixed time interval

Distribution:

Poisson Distr.

Binomial-like distributions

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Distribution: Poisson distribution

The Geometric distribution

Example: S'pose you flip the same biased coin repeatedly. How many times do you flip the coin before you see your first Heads?

The Geometric distribution

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$P(H) = p$

1 flip: p

2 flips: $(1-p)\cdot p$

3 flips: $(1-p)^2\cdot p$

P(T on first k-1 flips)

P(H on k^{th} flip)

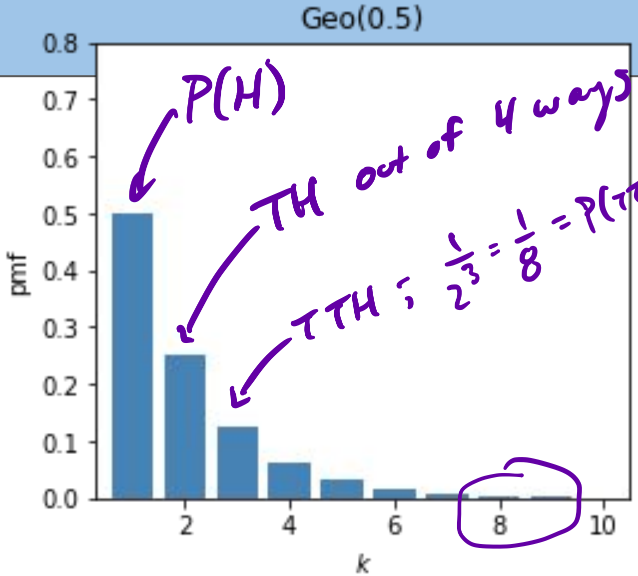
In general: $\underbrace{p_X(k) = (1-p)^k\cdot p}$

The Geometric distribution

Definition: A discrete r.v. X has a **geometric distribution** with parameter p, where $0 \le p \le 1$, if its probability mass function is given by

$$p_X(k) = P(X{=}k) = (1-p)^k \cdot p \quad \text{for } k = 1, 2, 3, \ldots$$

We say that $X \sim Geo(p)$.



$P(H)$

TH out of 4 ways $\to P(TH) = \frac{1}{4}$

$TTH ; \frac{1}{2^3} = \frac{1}{8} = P(TTH)$

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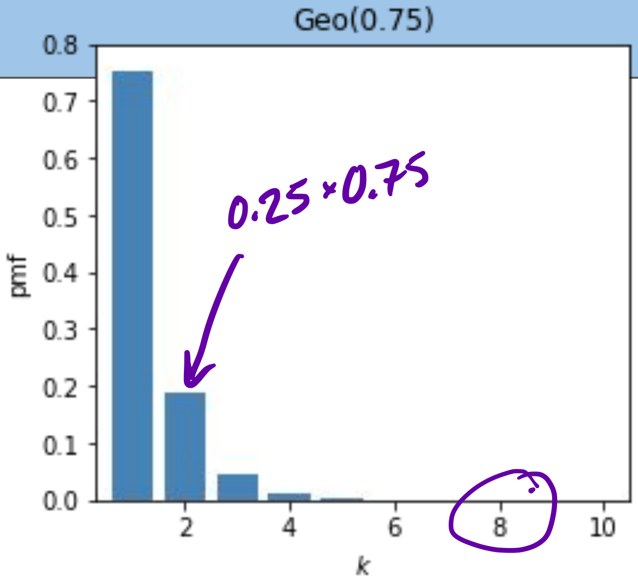

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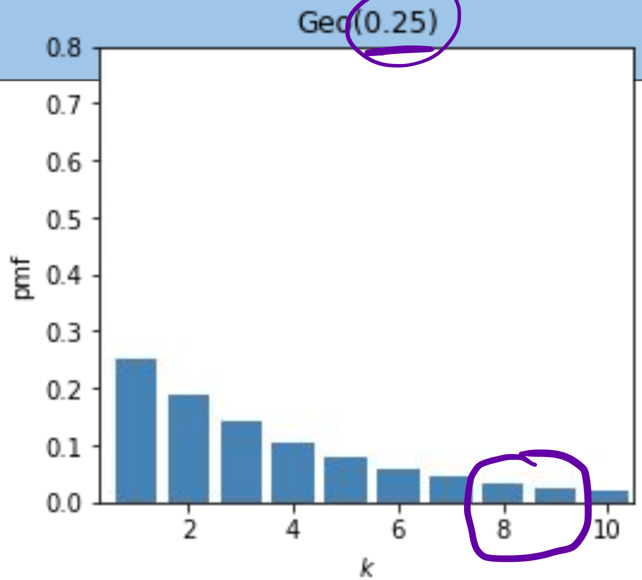
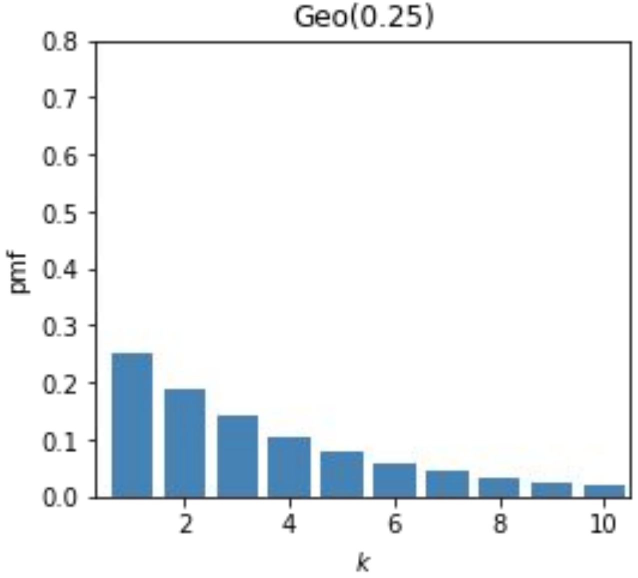


The Geometric distribution

Question: What assumptions did we implicitly make in deriving the Geometric distribution?

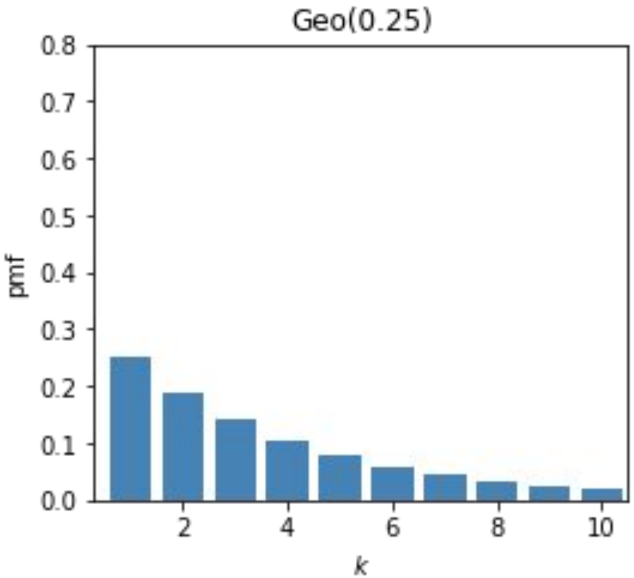


The Geometric distribution

Question: What assumptions did we implicitly make in deriving the Geometric distribution?

same for every trial

- Each trial is **independent**
- Each trial is a Bernoulli r.v. with probability of success *p*



The Negative Binomial distribution

$P(H) = p$

Example: S'pose you flip the same biased coin repeatedly. How many times do you flip the coin before you see 3 Heads total?

X = # trials when we see our 3rd Heads

$X = 3, 4, 5, 6, \ldots$ (no theoretical u.b.)

$$P(X=k) = \underbrace{P(2 \text{ Heads in the first } \underline{k-1} \text{ trials})}_{\text{Binomial distribution}} \times \underbrace{P(H \text{ on } k^{th} \text{ flip})}_{p}$$

$$= \binom{k-1}{2} p^2 (1-p)^{k-1-2} \cdot p$$

$$= \binom{k-1}{2} p^3 (1-p)^{k-3}$$

orig. Binomi

$$\binom{n}{k} p^k (1-p)^{n-k}$$

The Negative Binomial distribution

Example: S'pose you flip the same biased coin repeatedly. How many times do you flip the coin before you see 3 Heads total?

X = random variable representing the number of flips total when we observe our 3rd Heads

$\rightarrow X \in \{3, 4, 5, \ldots\}$

$p_X(k)$ = [probability of 2 Heads in the first k-1 flips] $\times$ [probability of Heads on k^{th} flip]

= [Binomial r.v. with n=k-1, and 2 successes] $\times$ p

$$= \binom{k-1}{2} p^2 (1-p)^{(k-1)-2} \cdot p$$

$$= \binom{k-1}{2} p^3 (1-p)^{k-3}$$ …. Can we generalize this?

The Negative Binomial distribution

Definition: A discrete r.v. X has a **negative binomial distribution** with parameters *r* and *p*, where *r* > 1 and $0 \le p \le 1$, if its probability mass function is given by

$$p_X(k) = P(X = k) = \binom{k-1}{r-1} p^r \, (1-p)^{k-r}$$

We say that $X \sim NB(r, p)$

p = probability of success for each trial

r = number of successes we want to observe (# successes we want to observe is treated as FIXED)

X = number of trials needed before we observe *r* successes (r.v.)

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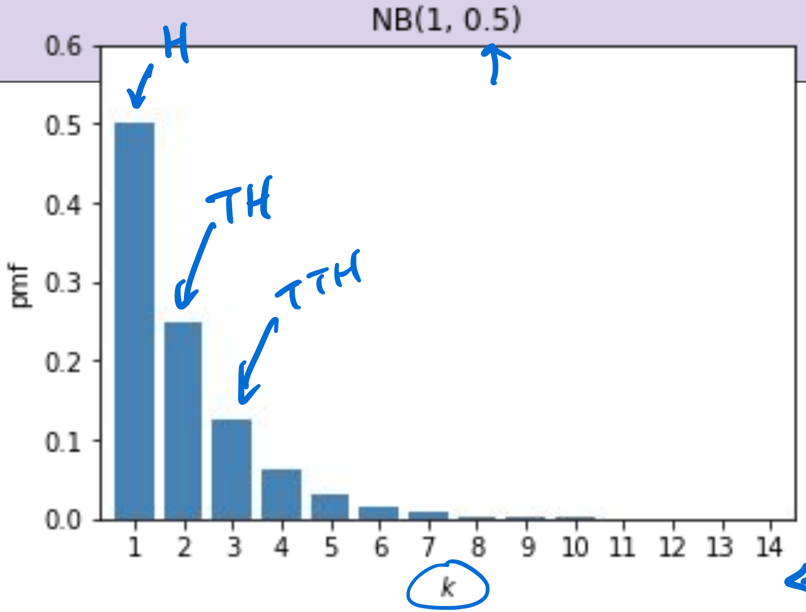
Geo(p=0.5)



trial to see 1st success

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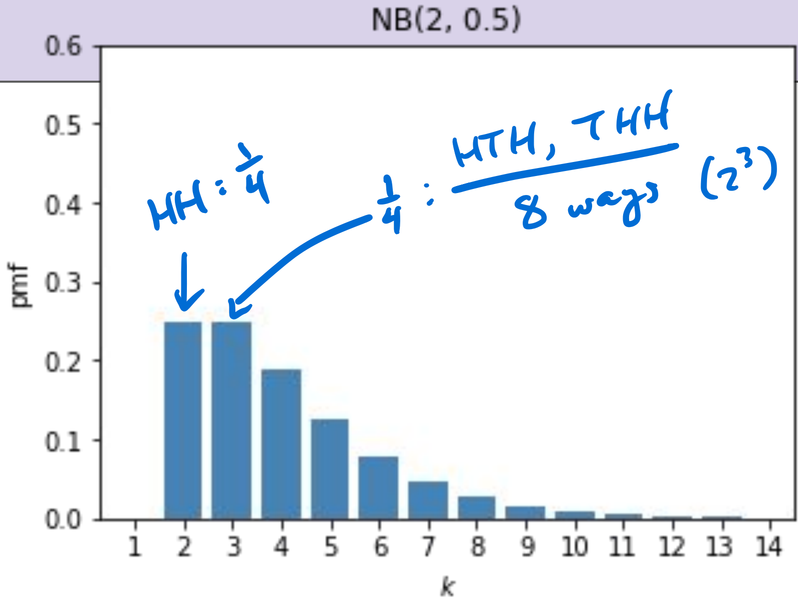

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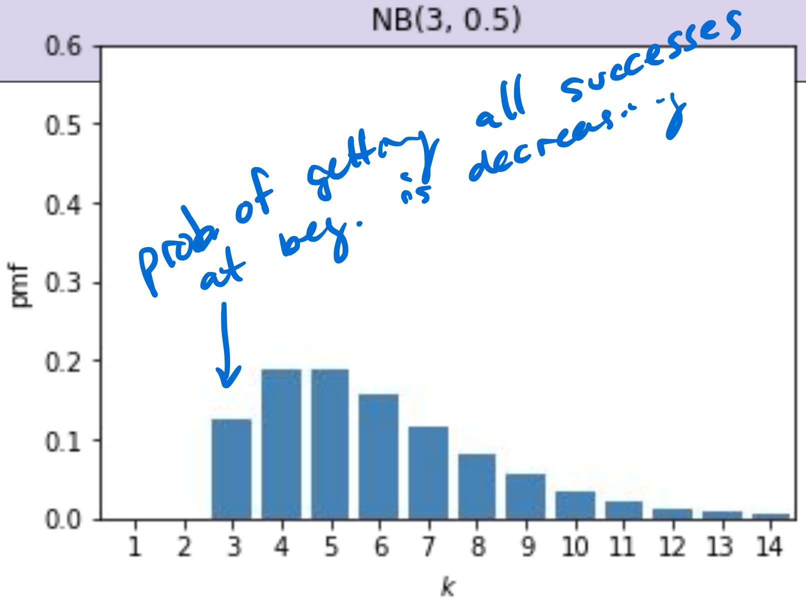
We say that *X* ~ *NB*(*r*, *p*)



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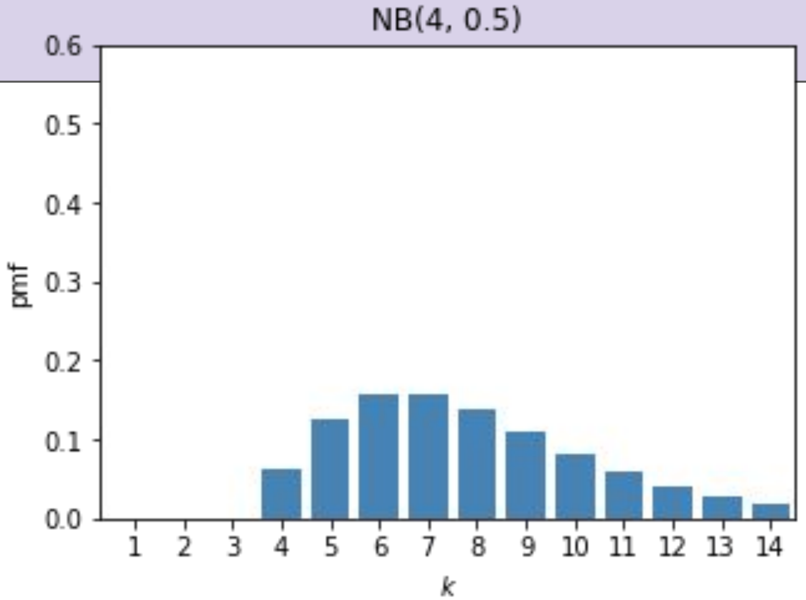

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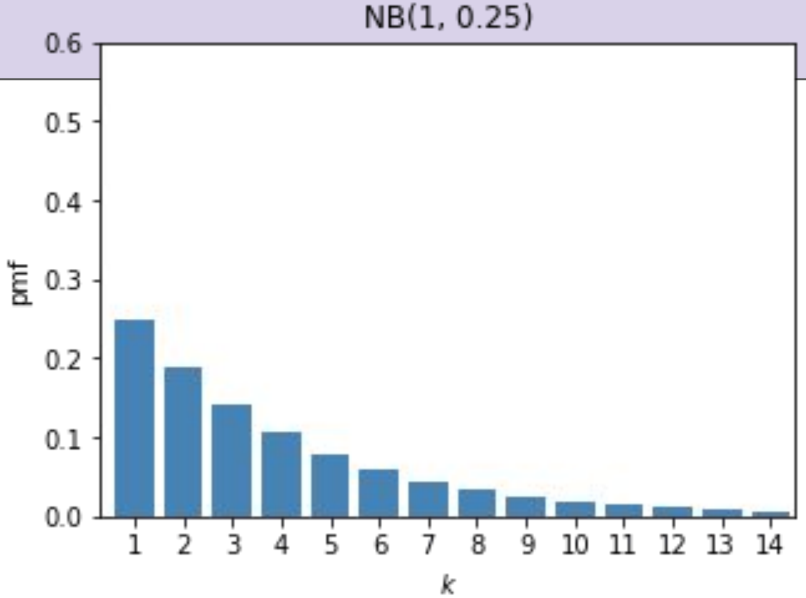

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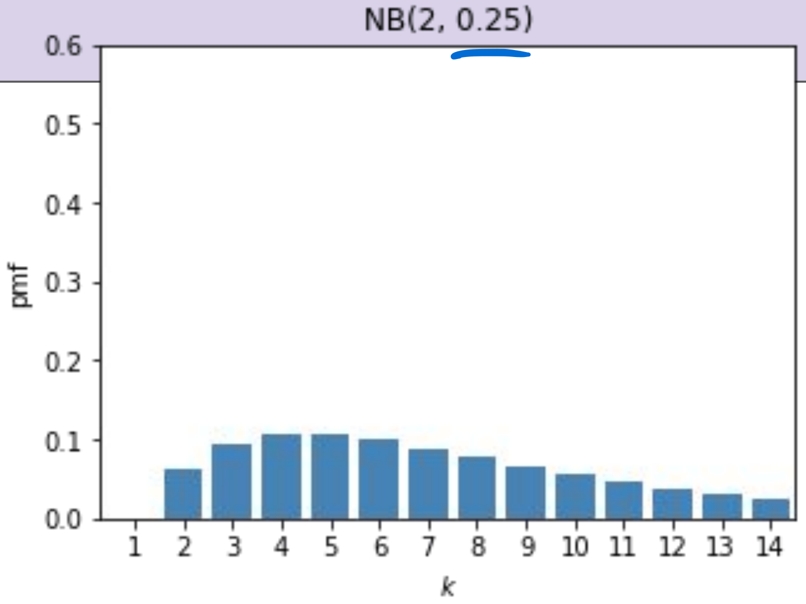

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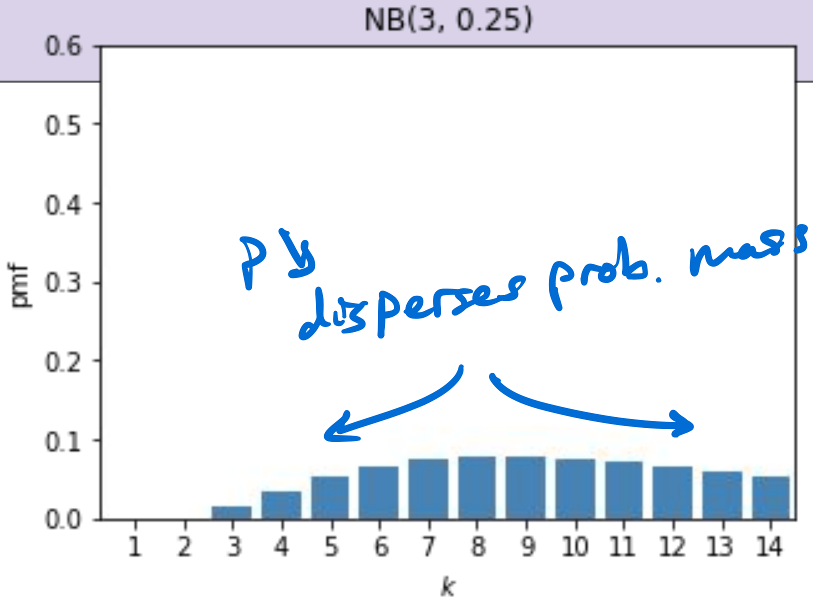
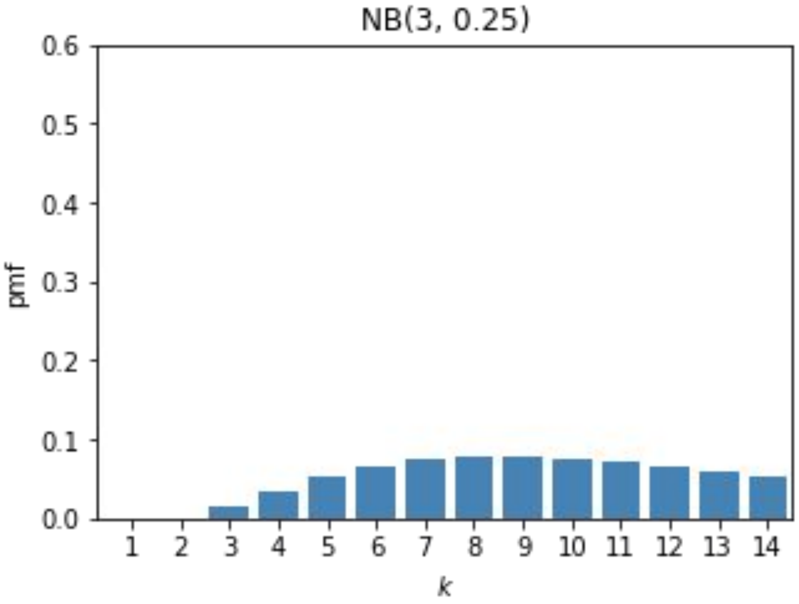


The Negative Binomial distribution

Question: What assumptions did we implicitly make in deriving the Negative Binomial distribution?

- Each trial is a **Bernoulli r.v.** with probability of success *p*
- Each trial is **independent**



The Poisson distribution

μ

Example: A grocery store check-out line moves people through at an average rate of 1 customer per 3 minutes. What is the probability that they check-out:

(i) 1 customer in 5 minutes? (ii) 3 customers in 5 minutes? (iii) 10 customers in 5 minutes?

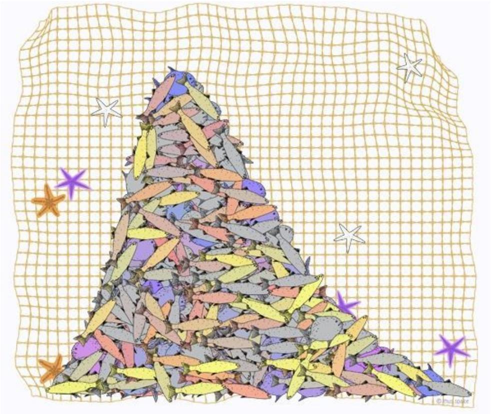
time

0 3 6 9

Dissection:

0 3

n small time slices : $p = \frac{\mu}{n}$ or $\mu = pn$



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Derivation:

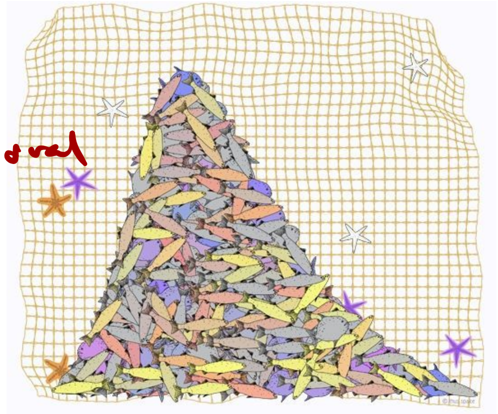
Think of this process as the limit of a Binomial r.v., as we pack more and more trials into a fixed slice of time.

μ = np n = time slices; p = prob. of a customer in that time slice

→ What is the probability of seeing *k* successes in that slice of time?

n time intervals, p = prob. of success in each interval
& we want P(k successes out of n intervals)

= Binomial Distribution! $= \binom{n}{k} p^k (1-p)^{n-k}$



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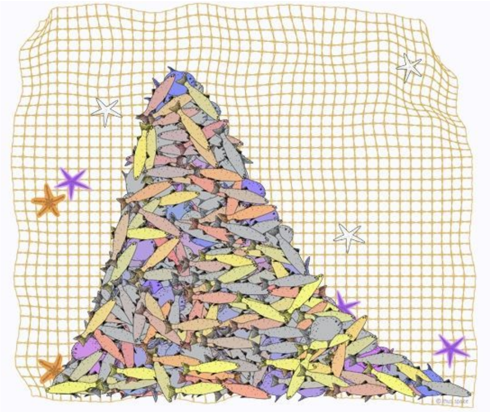


$$= \lim_{n \to \infty} \binom{n}{k} p^k \, (1-p)^{n-k}$$

$$= \lim_{n \to \infty} \frac{n!}{k!(n-k)!} p^k \, (1-p)^{n-k}$$

The Poisson distribution

Recall: $p = \frac{\mu}{n}$ ←

$$\left\{ \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \right.$$

Example: A grocery store check-out line moves people through at an average rate of 1 customer per 3 minutes. What is the probability that they check-out:

(i) 1 customer in 5 minutes? (ii) 3 customers in 5 minutes? (iii) 10 customers in 5 minutes?

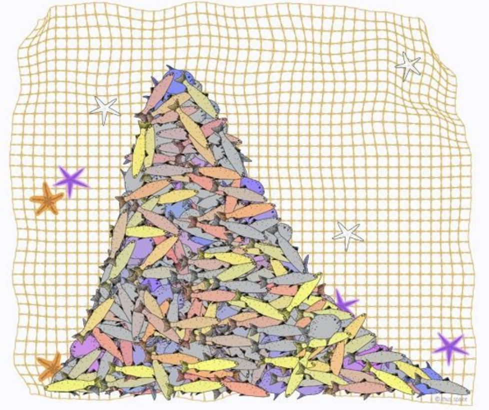
$$\lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = \lim_{n\to\infty} \frac{n!}{(n-k)!\,k!} \left(\frac{\mu}{n}\right)^k \underbrace{\left(1 - \frac{\mu}{n}\right)^{n-k}}$$

$$\frac{\left(1-\frac{\mu}{n}\right)^n}{\left(1-\frac{\mu}{n}\right)^k}$$

$$= \frac{\mu^k}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!\,n^k} \frac{\left(1-\frac{\mu}{n}\right)^n}{\left(1-\frac{\mu}{n}\right)^k}$$

$$\frac{n!}{(n-k)!\,n^k} \xrightarrow{n\to\infty} 1 \qquad \left(1-\frac{\mu}{n}\right)^n \xrightarrow{n\to\infty} e^{-\mu} \qquad \left(1-\frac{\mu}{n}\right)^k \xrightarrow{n\to\infty} 1$$

$$= \frac{\mu^k}{k!} e^{-\mu}$$



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$$= \lim_{n\to\infty} \frac{n!}{k!(n-k)!}\, p^k\, (1-p)^{n-k}$$

$$= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\mu}{n}\right)^k \left(1-\left(\frac{\mu}{n}\right)\right)^{n-k}$$

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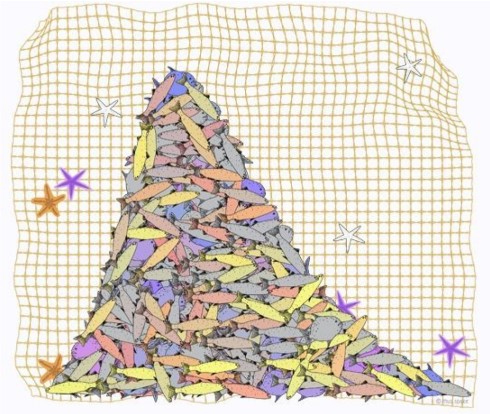
$$= \frac{\mu^k}{k!} \cdot 1 \cdot \frac{e^{-\mu}}{1}$$

$$f(k) = \frac{\mu^k\, e^{-\mu}}{k!}$$



The Poisson distribution

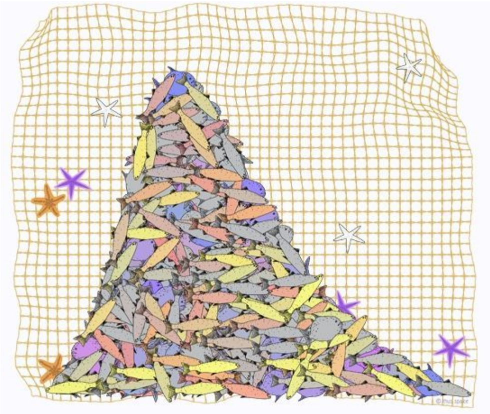
average "hit" rate (counts, arrivals) in a given time interval

Definition: A discrete r.v. X has a **Poisson distribution** with parameter μ, where $\mu > 0$, if its probability mass function is given by

often use λ

$$p_X(k) = P(X = k) = \frac{\mu^k \, e^{-\mu}}{k!} \qquad \text{for } k = 0, 1, 2, \ldots$$

We say that $X \sim Pois(\mu)$



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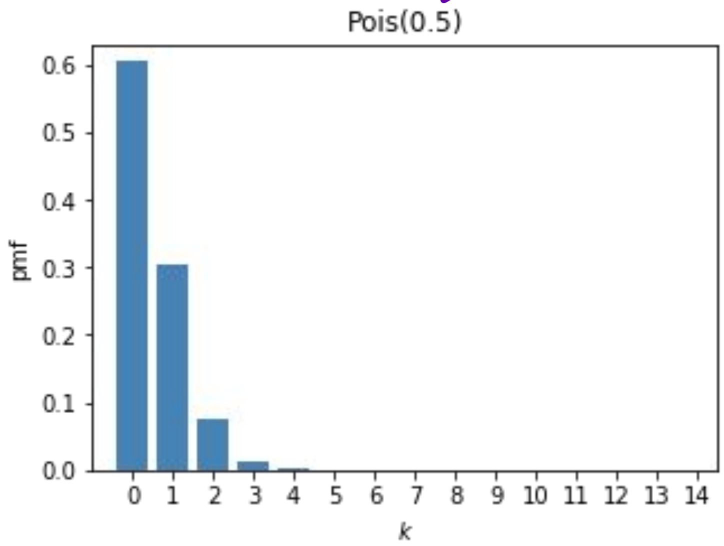
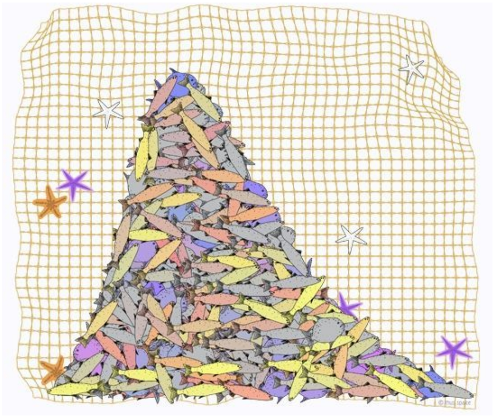
μ = 0.5 counts per [time interval]





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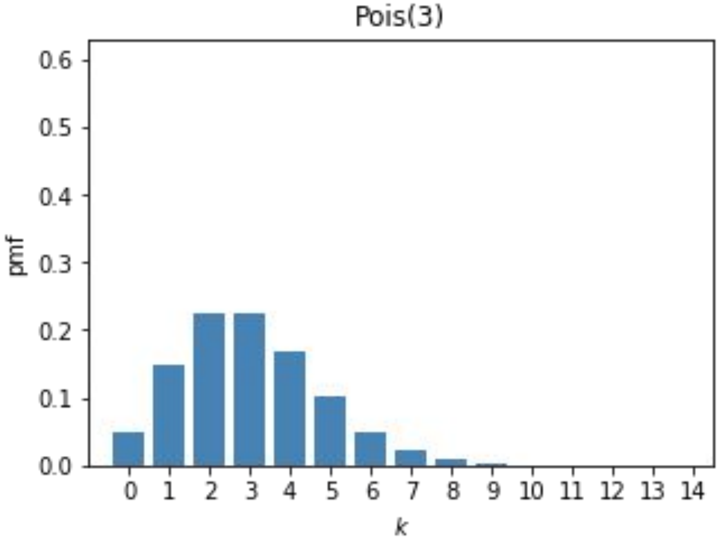
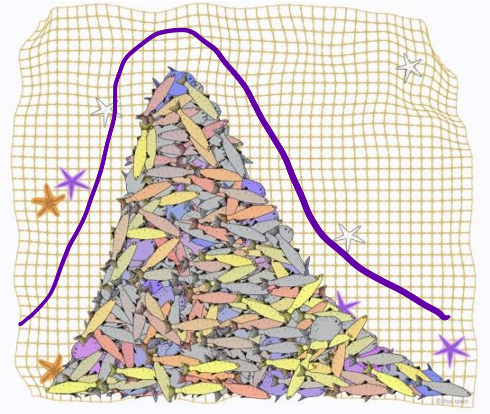


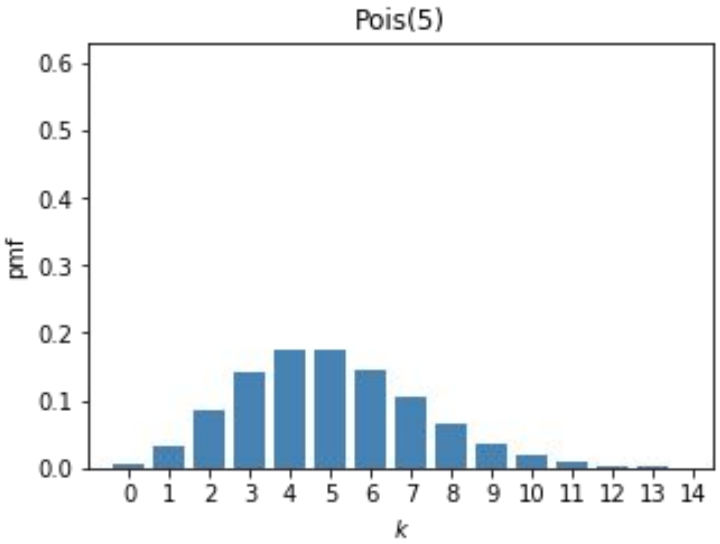


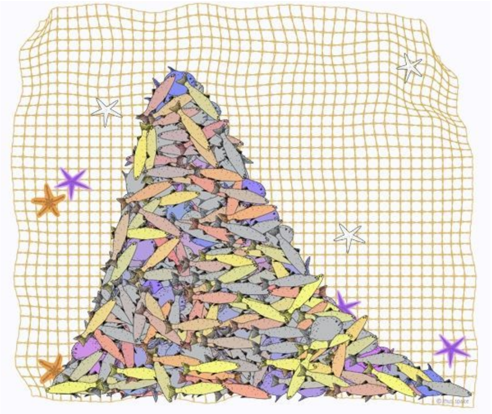
The Poisson distribution

Definition: A discrete r.v. X has a **Poisson distribution** with parameter *μ*, where *μ* > 0, if its probability mass function is given by

$$p_X(k) = P(X = k) = \frac{\mu^k \ e^{-\mu}}{k!} \qquad \text{for } k = 0, 1, 2, \ldots$$

We say that *X* ~ *Pois*(*μ*)





The Poisson distribution

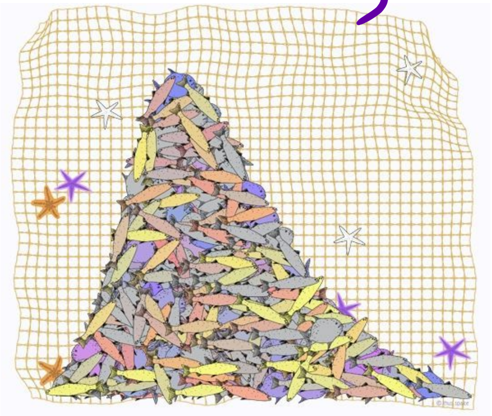
$\lambda = \frac{1}{3} \frac{\text{cust.}}{\text{min}}$

Example: A grocery store check-out line moves people through at an average rate of 1 customer per 3 minutes. What is the probability that they check-out:
(i) 1 customer in 5 minutes? (ii) 3 customers in 5 minutes? (iii) 10 customers in 5 minutes?

μ = ⅓ customer/min $\longrightarrow$ $\lambda = 5 \times \frac{1}{3} \frac{\text{cust}}{\text{5 min. block}}$

(extrapolating up to 5 min blocks from 3 prob. okay, but maybe not 3 min to 5 hours)

→ 5/3 cust./5 min block

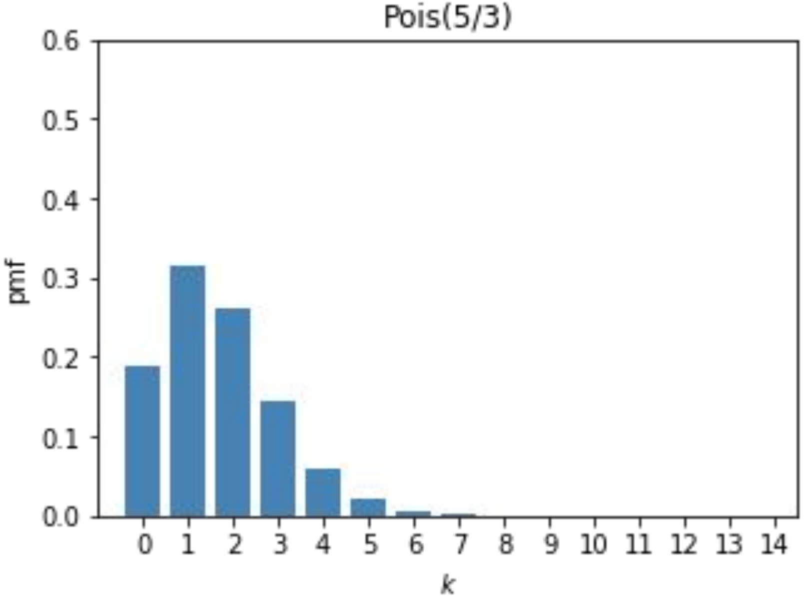


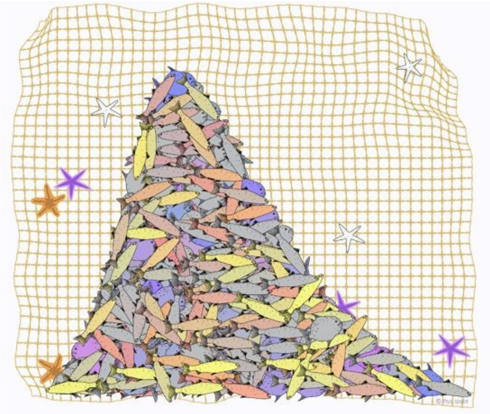
The Poisson distribution

Example: A grocery store check-out line moves people through at an average rate of 1 customer per 3 minutes. What is the probability that they check-out:
(i) 1 customer in 5 minutes? (ii) 3 customers in 5 minutes? (iii) 10 customers in 5 minutes?

μ = ⅓ customer/min

→ 5/3 cust./5 min block



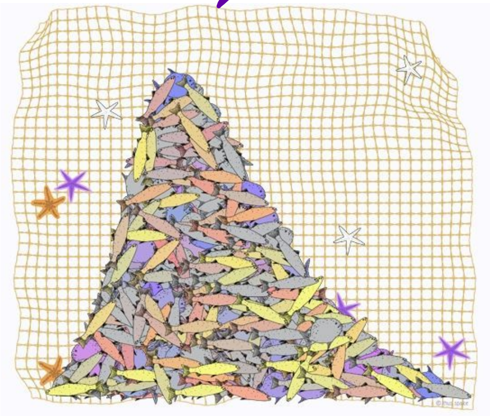


The Poisson distribution

Question: What assumptions did we implicitly make in deriving the Poisson distribution?

- Probability of observing a single event over a small interval is **proportional to** the size of the interval
- Each event/arrival is **independent**

→ so using smaller blocks makes perfectly good sense (but need to be careful when using larger time scales)



Riddle me this…

Example: You and a friend want to go to a concert, but only 1 ticket is available, and it is being sold by The Riddler.

The Riddler will toss a coin until Heads appears. In each toss, Heads appears with probability p ($0 < p < 1$), independent of each of the previous tosses. If the number of tosses needed is odd, your friend is allowed to buy the ticket; otherwise, you can buy it.

Should you agree to this arrangement?

Okay! Let's get to work!

Get in groups, get out laptops, and open **nb08** notebook

Let's…

- Practice identifying applications for the distributions we've learned
- Confirm our theoretical distributions with some simulations
- Look at the *Challenger* disaster
- Determine whether or not we should accept The Riddler's offer!