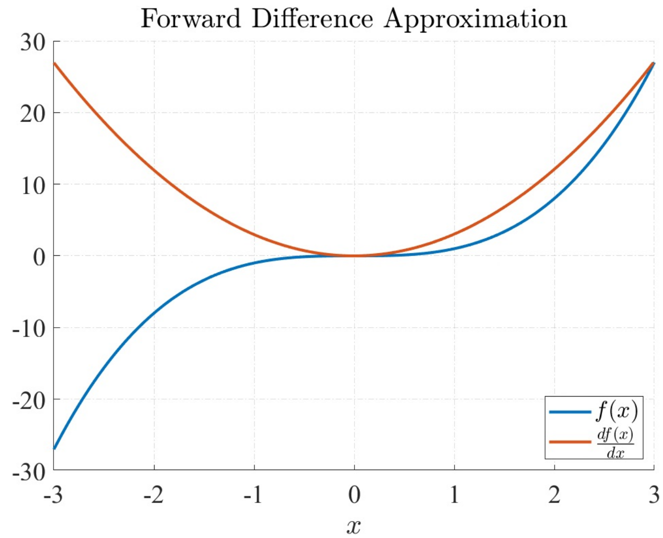
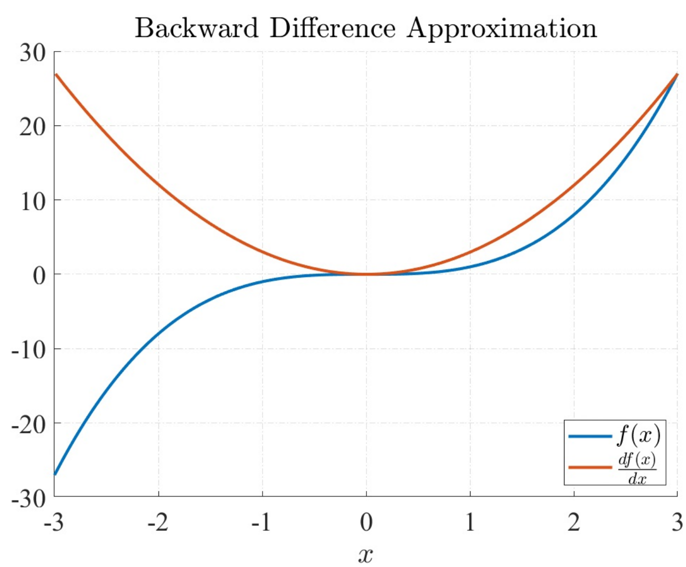
[Problem #1] Write a Matlab script to estimate the derivative of the function $f(x) = x^3$ using

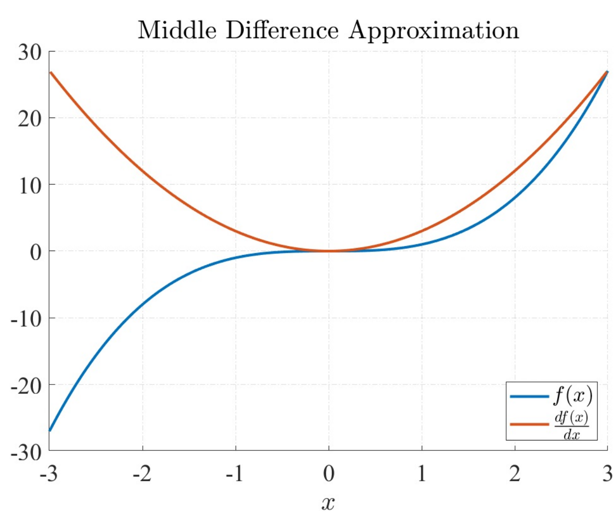
(a) the forward difference approximation



(b) the backward difference approximation



(c) the middle difference approximation.



[Problem #2] Write a Matlab script file to compute $\int_{x_0}^{x_f} f(x)dx$ using Simpson's Rule. Test your file on the following function

$$f(x) = x^3 + 2x \text{ with } x_0 = 0 \text{ and } x_f = 1.$$

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>> %% Problem 2

% define standard values
x_0 = 0;                     % begin point
x_f = 1;                     % end point
n = 10;                      % number of steps
h = (x_f - x_0) / n;         % time step value
x = linspace(x_0,x_f,n+1);   % array of x values
f = x.^3 + 2 * x;            % function value at each time step
summation = 0;               % initialize summation value at 0

for i = 0:n
    if i == 0 || i == n      % first and last time steps
        a = 1;               % coeficcient of 1
    elseif mod(i,2) == 0     % even time steps
        a = 2;               % coefficient of 2
    else                     % odd time steps
        a = 4;               % coefficient of 4
    end
    summation = summation + a*f(i+1);    % add apropriate value to summation
end
area = h * summation / 3
```

```
area =

    1.2500
```

[Problem #3] Consider the nonautonomous initial-value problem

$$\dot{x} = -tx^2,\ x(0) = 2.$$

Solve this problem on the interval $t = [0, 1]$ using Euler's method and the Euler predictor-corrector method and compare the results.

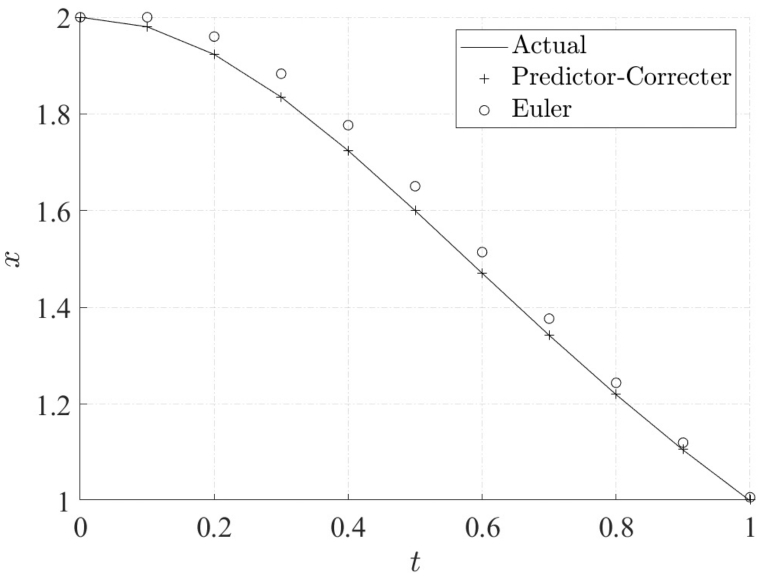
$$f(x) = \dot{x} = -t x^2$$

Euler's Method

$$x_{k+1} = x_k + h f(x_k)$$
$$x_{k+1} = x_k + h(-t x_k^2)$$
$$\underline{x_{k+1} = -h t x_k^2 + x_k}$$

$$x(t=1) = -(1)(0)(2)^2 + (2)$$
$$\underline{x(1) = 2}$$

use MATLAB to compare results over time



Euler Predictor-Corrector

$$\begin{cases} \tilde{x}_{k+1} = x_k + h f(x_k) \\ x_{k+1} = x_k + \frac{h}{2}\left[f(x_k) + f(\tilde{x}_{k+1}) \right] \end{cases}$$

$$\tilde{x}_{k+1} = x_k + h(-t x_k^2) = \underline{-h t x_k^2 + x_k}$$

$$x_{k+1} = x_k + \frac{h}{2}\left[(-t x_k^2) + (-h t x_k^2 + x_k) \right]$$

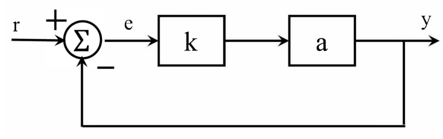
$$x_{k+1} = x_k - \frac{ht}{2} x_k^2 - \frac{h^2 t}{2} x_k^2 + \frac{h}{2} x_k$$

$$\underline{x_{k+1} = x_k \left(1 + \frac{h}{2} - \frac{ht}{2}\left(1 + h\right) x_k \right)}$$

$$x(t=1) = 2\left(1 + \frac{(1)}{2} - \frac{(1)(0)}{2}\left(1 + (1)\right)(2) \right)$$
$$= 2\left(\frac{3}{2}\right) = 3$$

$$\underline{x(1) = 3}$$

[Problem #4] For the system below, show that $y \to r$ as $k \to \infty$, in other words, the error $r - y$ can be made arbitrarily small independent of the plant parameter a.



$$y = e\,ka$$

$$y = (r - y)\,ka$$

$$y = r\,ka - y\,ka$$

$$y(1 + ka) = r\,ka$$

$$y = \frac{ka}{1 + ka}\,r$$

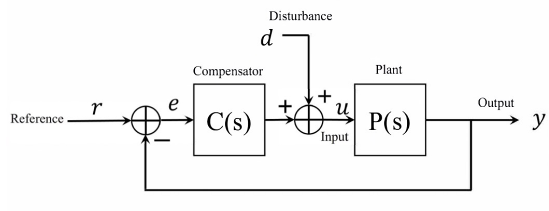
$$\lim_{k \to \infty} \left(\frac{ka}{1 + ka} \right) = 1 \quad \therefore \quad \boxed{\lim_{k \to \infty} y = r}$$

[Problem #5] Consider the system below and assume this represents a servo control system where

$$P(s) = \frac{0.5}{s(s+2)}$$

is the dynamics of a servomotor, and the controller is the gain of a power amplifier

$$C(s) = K_a.$$



(a) Determine the transfer function from Reference to Output and show that it is a second-order system.

$$Y(s) = P(s)U(s) = P(s)\left(C(s)E(s) + \cancel{D(s)}\right) = P(s)C(s)E(s) = P(s)C(s)\left(R(s) - Y(s)\right)$$

$$Y(s)\left(1 + P(s)C(s)\right) = P(s)C(s)R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{\frac{0.5}{s(s+2)}K_a}{1 + \frac{0.5}{s(s+2)}K_a} = \frac{0.5K_a}{s(s+2) + 0.5K_a}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{0.5K_a}{s^2 + 2s + 0.5K_a}}$$

(b) Find the power amplifier gain that results in the fastest step response with no overshoot.

$$\frac{Y(s)}{R(s)} = \frac{0.5K_a}{s^2 + 2s + 0.5K_a} \qquad \therefore\ \omega^2 = 0.5K_a\ ;\ 2\zeta\omega = 2$$

The system w/ the fastest response time & no overshoot is critically damped ($\zeta = 1$).

$$2\zeta\omega = 2$$
$$(1)\omega = 1$$
$$\omega = 1$$
$$(\omega^2 = 0.5K_a)$$
$$\sqrt{0.5K_a} = 1$$
$$0.5K_a = 1$$
$$\boxed{K_a = 2}$$

(c) If the amplifier gain is selected to be half or twice this value, will the step response be under damped or over damped? Explain why.

case: $K_a = 4$

$\omega^2 = 0.5 K_a = 2$

$2\zeta\omega = 2$

$\zeta\omega = 1$

$\zeta\sqrt{2} = 1$

$\zeta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\boxed{\text{if the amplifier gain is doubled, the system will be underdamped as } \zeta = \frac{\sqrt{2}}{2} < 1.}$

case: $K_a = 1$

$\omega^2 = 0.5 K_a = \frac{1}{2}$

$2\zeta\omega = 2$

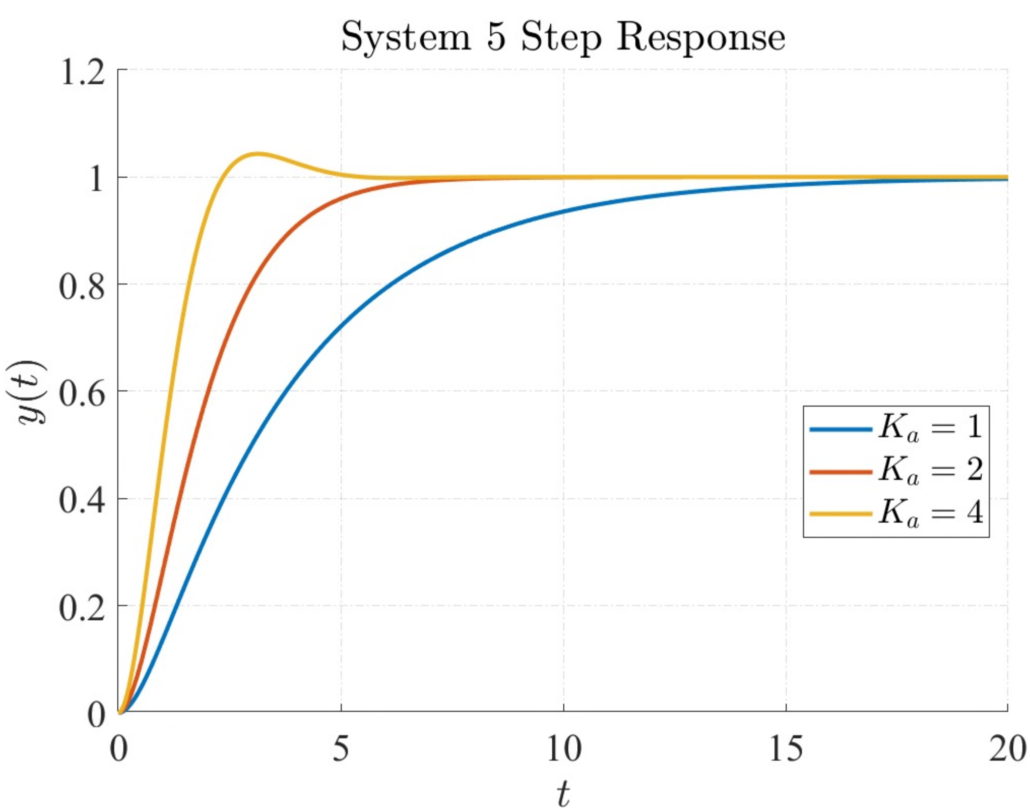
$\zeta\omega = 1$

$\zeta\sqrt{\frac{1}{2}} = 1$

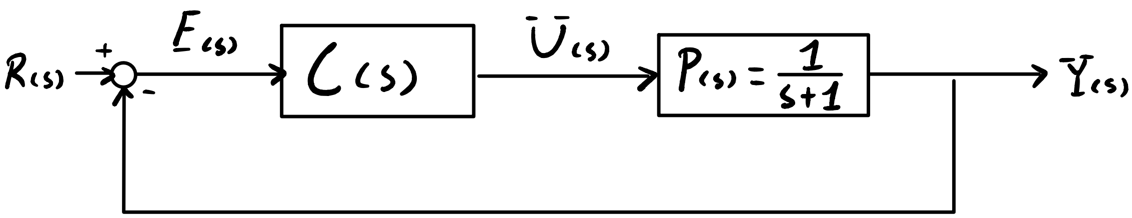
$\zeta = \sqrt{2}$

$\boxed{\text{if the amplifier gain is halved, the system will be over damped as } \zeta = \sqrt{2} > 1.}$

(d) Write a Matlab code to plot step response of the system for the three values of K_a obtained above.



[Problem #6] Consider a first-order system with transfer function $\frac{1}{s+1}$. Write a Matlab script using the functions `tf` and `step` to simulate the step response for the system using a PI controller $u = k_p s + \frac{k_i}{s}$. Choose the PI gains so that the closed loop system reaches steady state in less than two seconds with minimal overshoot. Plot the step response from 0 to 2 seconds.



$$Y(s) = P(s)\,U(s) = P(s)\,C(s)\,E(s) = P(s)\,C(s)\,(R(s) - Y(s))$$

$$\frac{Y(s)}{R(s)} = \frac{P(s)\,C(s)}{1 + P(s)\,C(s)} = \frac{\frac{1}{s+1}\left(k_p + \frac{k_i}{s}\right)}{1 + \frac{1}{s+1}\left(k_p + \frac{k_i}{s}\right)} = \frac{\frac{k_p}{s+1} + \frac{k_i}{s(s+1)}}{1 + \frac{k_p}{s+1} + \frac{k_i}{s(s+1)}} = \frac{k_p s + k_i}{s(s+1) + k_p s + k_i}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{k_p s + k_i}{s^2 + s(k_p+1) + k_i}}$$

$$2\zeta\omega = k_p + 1 \qquad \omega^2 = k_i$$

$$2(1)\sqrt{k_i} = k_p + 1 \qquad \omega = \sqrt{k_i}$$

$$k_p = 2\sqrt{k_i} - 1$$