[Problem #1] Consider the first-order system

$$\dot{x} = -g(x)\ ,\ x \in \mathbf{R},$$

where $g(x)$ satisfies

$$g(0) = 0,\ xg(x) > 0.$$

Show that the function

$$V(x) = \int_0^x g(u)du.$$

is a valid Lyapunov function candidate. Using this Lyapunov function candidate, show that the equilibrium $x = 0$ is asymptotically stable.

$$V(0) = \int_0^0 g(u)du = \underline{0} \ \checkmark$$

$$V(x) = \int_0^x g(u)\,du = \underline{xg(x) > 0 \ \forall\, x \neq 0} \ \checkmark$$

$$\dot{V}(x) = \frac{d}{dx}\left(\int_0^x g(u)du\right)\frac{dx}{dt}$$

$$= g(x)\frac{dx}{dt} = g(x)(-g(x)) = \underline{-(g(x))^2 < 0 \ \forall\, x \neq 0} \ \checkmark$$

The equilibrium $\bar{x} = 0$ is asymptotically stable because the Lyapunov function $V(x)$ satisfies the requirements for asymptotic stability:

(i) $V(0) = 0$

(ii) $V(x) > 0 \ \forall\, x \neq 0$

(iii) $\dot{V}(x) < 0 \ \forall\, x \neq 0$

[Problem #2] Consider the second-order system below

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1^3 - x_2.\end{aligned}$$

(a) Is the equilibrium at $(0,0)$ exponentially stable? Explain why or why not.

$\frac{\partial f_1}{\partial x_1} = 0$

$\frac{\partial f_1}{\partial x_2} = 1$

$\frac{\partial f_2}{\partial x_1} = -3x_1^2$

$\frac{\partial f_2}{\partial x_2} = -1$

Linear Approximation about $(0,0)$

$$\dot{v} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} v$$

$$\begin{vmatrix} -\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} = (-\lambda)(-1-\lambda) - \cancel{(0)(1)}$$

$$\lambda^2 + \lambda = 0$$

$$\lambda(\lambda+1) = 0$$

$\lambda = 0$

$\lambda = -1$

Because the linear approximation of the system has eigenvalues of A equal to zero, $\bar{x}=0$ is a non-hyperbolic equilibrium $\therefore$ $\bar{x}=0$ cannot be asymptotically stable.

(b) Use the Lyapunov function $V = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ and LaSalle's Theorem to show that the equilibrium at $(0,0)$ is globally asymptotically stable.

$V(0) = \frac{1}{4}(0)^4 + \frac{1}{2}(0)^2 = 0$ ✓

$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 > 0 \quad \forall x \neq 0$ ✓

$\lim_{\|x\| \to \infty} V(x) = \infty$ ✓

$\dot{V}(x) = x_1^3(\dot{x}_1) + x_2(\dot{x}_2) = x_1^3(x_2) + x_2(-x_1^3 - x_2) = x_1^3 x_2 - x_1^3 x_2 - x_2^2 = -x_2^2 < 0 \quad \forall x \neq 0$ ✓

Because $V(0)=0$, $V(x) > 0 \ \forall x \neq 0$, $\dot{V}(0)=0$, $\dot{V}(x) < 0 \ \forall x \neq 0$, and V is radially unbounded, the system is globally asymptotically stable about the origin.

[Problem #3] Consider the nonlinear system

$$\dot{x} = f(x) + g(x), \qquad x \in R^n.$$

Suppose $x^T f(x) < -||x||^2$ and $||g(x)|| < k||x||$. Show that the origin is asymptotically stable if $k < 1$.

$x^T f(x) < -||x||^2$

$x_1 f_1 + x_2 f_2 + \dots + x_n f_n < -x_1^2 - x_2^2 - \dots - x_n^2$

$||g(x)|| < k||x||$

$\sqrt{g_1^2 + g_2^2 + \dots + g_n^2} < k\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

$V(x) = ||x||^2 = x_1^2 + x_2^2 + \dots + x_n^2$

$V(0) = ||0|| = \underline{0}$

$V(x) = x_1^2 + x_2^2 + \dots + x_n^2$ because all values are squared, $V(x) > 0 \ \forall x \neq 0$

$\dot{V}(x) = \frac{d}{dt}(x_1^2 + x_2^2 + \dots + x_n^2) = 2x_1(\dot{x}_1) + 2x_2(\dot{x}_2) + \dots + 2x_n(\dot{x}_n)$

$= 2x_1(f_1 + g_1) + 2x_2(f_2 + g_2) + \dots + 2x_n(f_n + g_n)$

$= 2\sum_{i=1}^{n} x_i f_i + 2\sum_{i=1}^{n} x_i g_i$

$= 2x^T f(x) + 2x^T g(x)$

$= 2[\,x^T f(x) + (x_1 g_1 + x_2 g_2 + \dots + x_n g_n)\,]$

In order to be asymptotically stable, $\dot{V}(x) < 0 \ \forall x \neq 0$, so what is the 'most positive' that this eqution can be?

$\dot{V}(x) = 2[-||x||^2 + k||x||^2]$

Therefore, if $k < 1$, then $\dot{V}(x)$ will be negative for all $x \neq 0$, which satisfies the final criteria for asymptotic stability through Lyapunov theorem.

[Problem #4] Suppose the Jacobian $A = \frac{\partial f}{\partial x}$ for a second-order nonlinear system $\dot{x} = f(x)$ is

$$A = \begin{bmatrix} -x_1 & 1 - x_2 \\ x_1 x_2^3 & -x_2 \end{bmatrix}$$

and suppose that $(0, 0)$ and $(1, 1)$ are both equilibrium points. What can you say about the solution of the nonlinear system near these equilibrium points?

$(0,0)$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = (-\lambda)(-\lambda) - (0)(1) = 0$$

$$\lambda^2 = 0$$

$$\lambda = 0, 0$$

Because $(0,0)$ is a hyperbolic equilibrium point, stability cannot be inferred, but it is known that $\bar{x} = (0,0)$ is not exponentially stable

$(1, 1)$

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\begin{vmatrix} -1-\lambda & 0 \\ 1 & -1-\lambda \end{vmatrix} = (-1-\lambda)(-1-\lambda) - \cancel{(0)(1)} = 0$$

$$(\lambda+1)^2 = 0$$

$$\lambda = -1, -1$$

Because A is a Hurwitz Matrix (i.e. $Re(\lambda) < 0$) $\bar{x} = (1,1)$ is locally exponentially stable

[Problem #5] Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -\sin(x_1)\cos(x_2)\\ \dot{x}_2 &= -\cos(x_1)\sin(x_2)\end{aligned}$$

Show that the origin is locally exponentially stable.

$$\frac{\partial f_1}{\partial x_1} = -\cos(\bar{x}_1)\cos(\bar{x}_2) = -\cos(0)\cos(0) = -1$$

$$\frac{\partial f_1}{\partial x_2} = \sin(\bar{x}_1)\sin(\bar{x}_2) = \sin(0)\sin(0) = 0$$

$$\frac{\partial f_2}{\partial x_1} = \sin(\bar{x}_1)\sin(\bar{x}_2) = \sin(0)\sin(0) = 0$$

$$\frac{\partial f_2}{\partial x_2} = -\cos(\bar{x}_1)\cos(\bar{x}_2) = -\cos(0)\cos(0) = -1$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{vmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = (-1-\lambda)(-1-\lambda) - (0)(0) = 0$$

$$(\lambda+1)^2 = 0$$

$$\boxed{\begin{array}{l} \lambda = -1, -1 \\ \text{the origin is locally exponentially stable because} \\ A \text{ is a Hurwitz Matrix} \\ (Re(\lambda) < 0) \end{array}}$$

[Problem #6] Consider the linear non-homogeneous equation

$$8y(k+2) - 6y(k+1) + 2y(k) = 1, \text{ with } y(1) = 2,\ y(2) = 0.$$

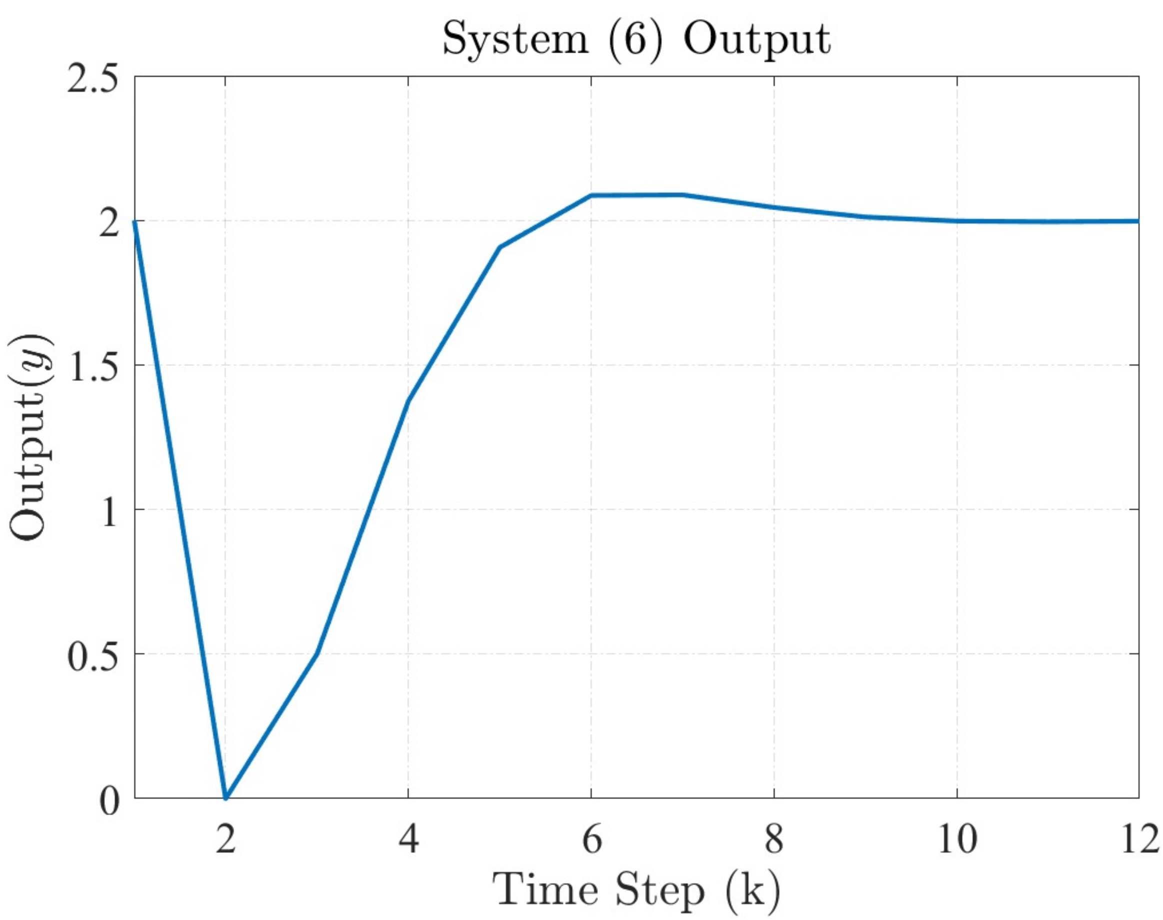
Write a Matlab script to simulate this system and plot the results.

$x_1(k) = y(k)$ $\quad$ $x_1(k+1) = y(k+1) = x_2(k)$

$x_2(k) = y(k+1)$ $\quad$ $x_2(k+1) = y(k+2) = -\frac{1}{4} y(k) + \frac{3}{4} y(k+1) + 1 = -\frac{1}{4} x_1(k) + \frac{3}{4} x_2(k) + 1$

$$x(1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad x(k+1) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$



[Problem #7] Consider the discrete-time system

$$y_{k+3} + 0.2y_{k+2} + 0.5y_{k+1} - 0.3y_k = 0.$$

Define a state vector x and write this system as

$$x_{k+1} = Ax_k.$$

Write a Matlab script to simulate and plot the solution of the system starting at a given initial condition x_0. Plot the solution for the initial condition $[10\ 1\ -1]^T$.

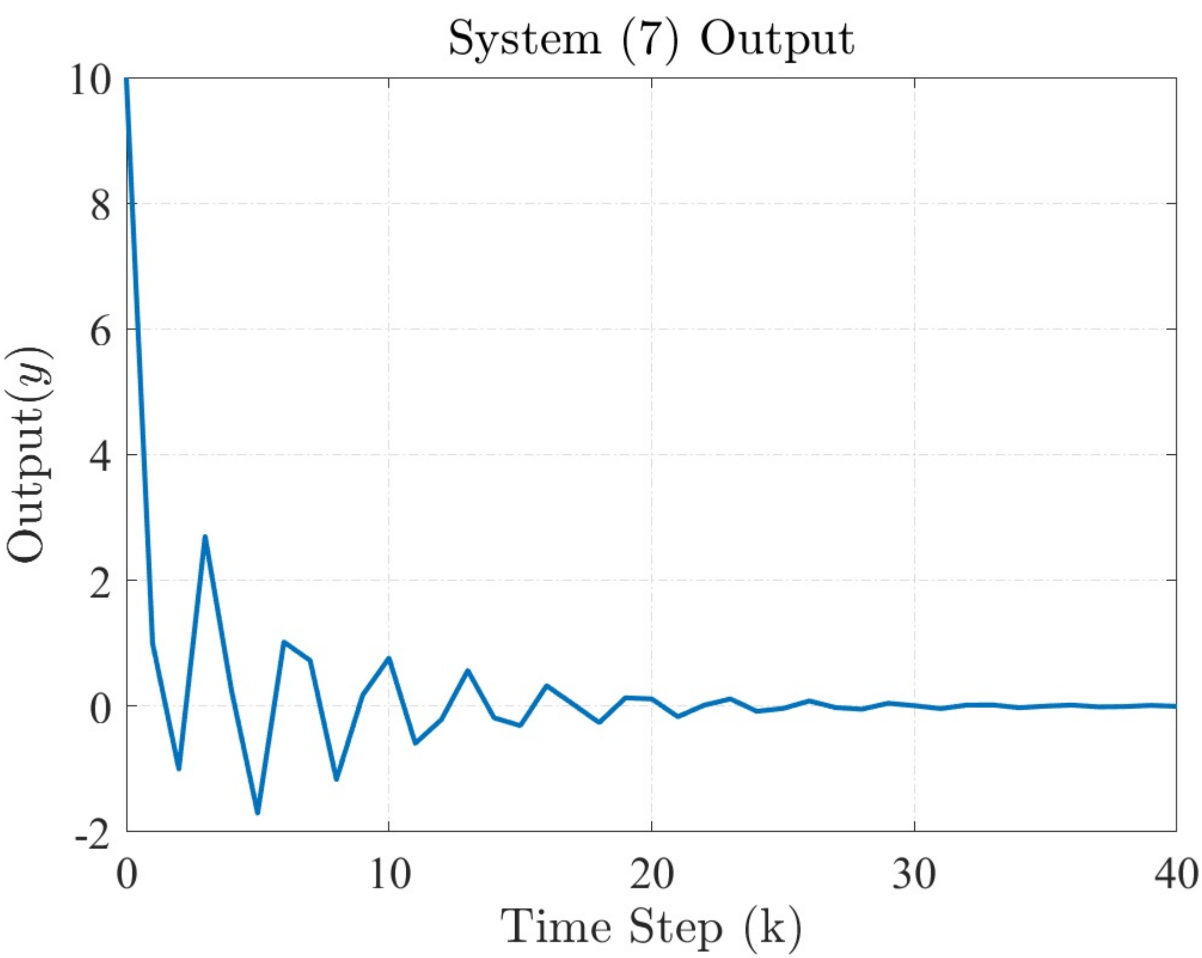
$$y_{k+3} + 0.2y_{k+2} + 0.5y_{k+1} - 0.3y_k = 0$$

$x_{1,k} = y_k \qquad x_{1,k+1} = y_{k+1} = x_{2,k}$

$x_{2,k} = y_{k+1} \qquad x_{2,k+1} = y_{k+2} = x_{3,k}$

$x_{3,k} = y_{k+2} \qquad x_{3,k+1} = y_{k+3} = -0.2y_{k+2} - 0.5y_{k+1} + 0.3y_k = 0.3x_{1,k} - 0.5x_{2,k} - 0.2x_{3,k}$

$$\boxed{x_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.3 & -0.5 & -0.2 \end{bmatrix} x_k}$$



[Problem #8] (Repeated Roots to the Characteristic Polynomial) Consider the second-order difference equation

$$y(k+2) - 2ay(k+1) + a^2y(k) = 0.$$

a) Show that both $y(k) = a^k$ and $y(k) = ka^k$ are solutions.

(i) $y(k) = a^k$

$y(k+1) = a^{k+1} = a(a^k)$

$y(k+2) = a^{k+2} = a^2(a^k)$

$y(k+2) - 2ay(k+1) + a^2y(k) = 0$

$a^{k+2} - 2a(a(a^k)) + a^2(a^k) = 0$

$a^{k+2} - 2a^{k+2} + a^{k+2} = 0$

$1 - 2 + 1 = 0$

(ii) $y(k) = ka^k$

$y(k+1) = (k+1)a^{k+1} = ka^{k+1} + a^{k+1}$

$y(k+2) = (k+2)a^{k+2} = ka^{k+2} + 2a^{k+2}$

$y(k+2) - 2ay(k+1) + a^2y(k) = 0$

$ka^{k+2} + 2a^{k+2} - 2a(ka^{k+1} + a^{k+1}) + a^2(ka^k) = 0$

$ka^{k+2} + 2a^{k+2} - 2ka^{k+2} - 2a^{k+2} + ka^{k+2} = 0$

$k + \cancel{2} - 2k \cancel{-2} + k = 0$

$k - 2k + k = 0$

$1 - 2 + 1 = 0$

b) Find the solution to this equation with initial conditions $y(0) = 1$ and $y(1) = 0$.

(i) $y(k) = a^k$

$y(0) = a^0$

$y(0) = 1$

$y(1) = a^1$

$y(1) = a = 0$

$\boxed{y(k) = 0^k}$

(ii) $y(k) = ka^k$

$y(0) = (0)a^{(0)} = 0$ X

[Problem #9] Consider the second-order difference equation

$$y(k+2) + y(k) = 0.$$

a) Write the system in state as $x(k+1) = Ax(k)$.

$x_1(k) = y(k) \qquad x_1(k+1) = y(k+1) = x_2(k)$

$x_2(k) = y(k+1) \qquad x_2(k+1) = y(k+2) = -y(k) = -x_1(k)$

$$\boxed{x(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(k)}$$

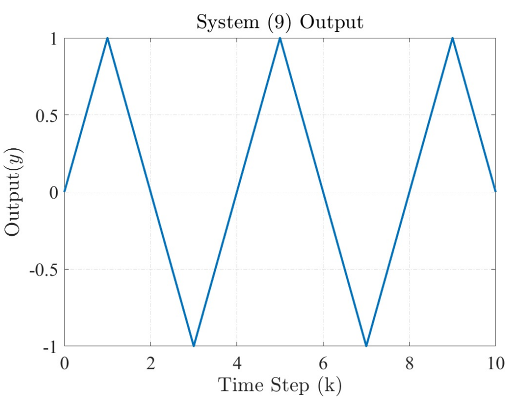
b) What are the eigenvalues of the coefficient matrix A?

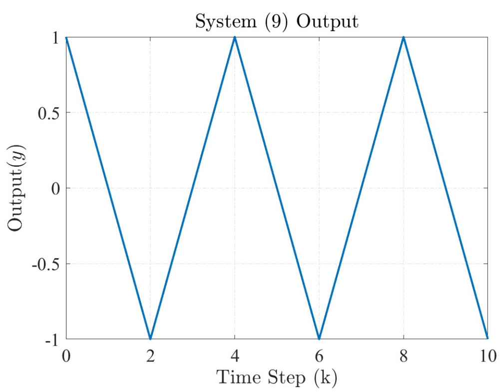
$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - (-1)(1) = 0$$

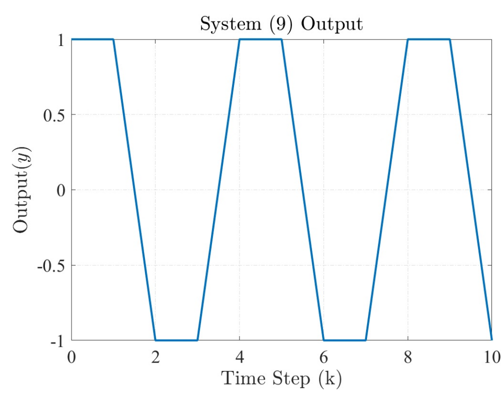
$$\lambda^2 + 1 = 0$$

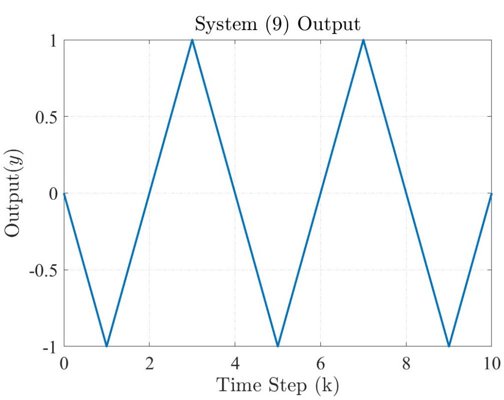
$$\lambda^2 = -1 \qquad \boxed{\lambda = \pm i}$$

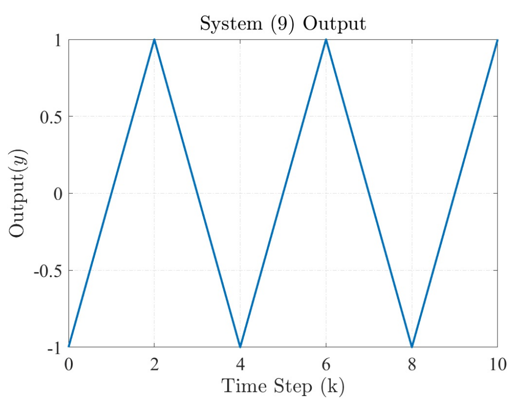
c) Write a Matlab script to simulate the system for various initial conditions and plot the solution. Note the analogy to the harmonic oscillator differential equation.

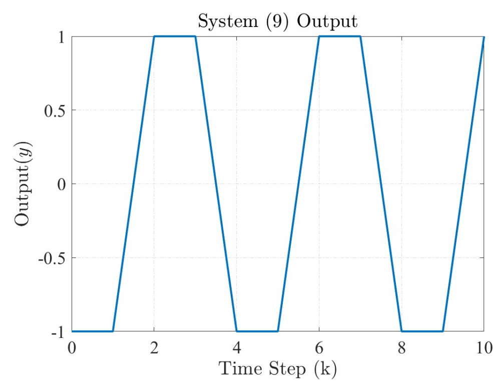


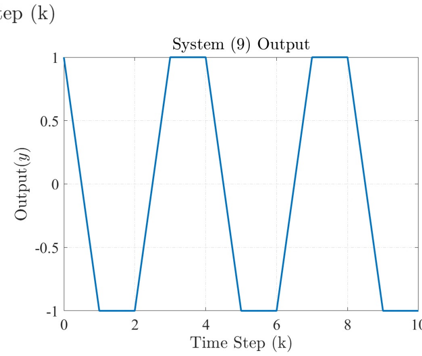


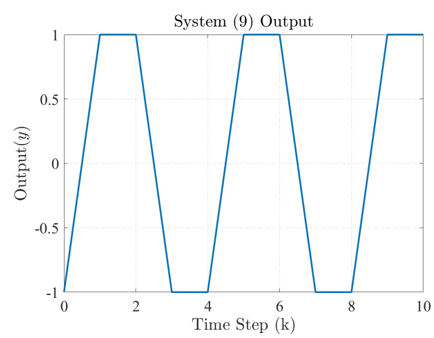












[Problem #10] Consider the discrete-time system

$$x(k+1) = ax(k)^n,$$

where n is an integer. Find the equilibrium points and investigate stability of the system in terms of a and n. Investigate both analytically and experimentally using Matlab.

$\bar{x} = a\bar{x}^n$

$\bar{x}\,(a\bar{x}^{n-1} - 1) = 0$

$\bar{x} = 0 \qquad a\bar{x}^{n-1} = 1$

$\bar{x}^{n-1} = a^{-1}$

$\bar{x} = (a^{-1})^{\frac{1}{n-1}}$

$\bar{x} = a^{\frac{1}{1-n}}$

case: $n = 1$

$\bar{x} = a\bar{x}$

$\bar{x}\,(a-1) = 0$

$\bar{x} = 0$ or $a = 1$

case: $n = 0$

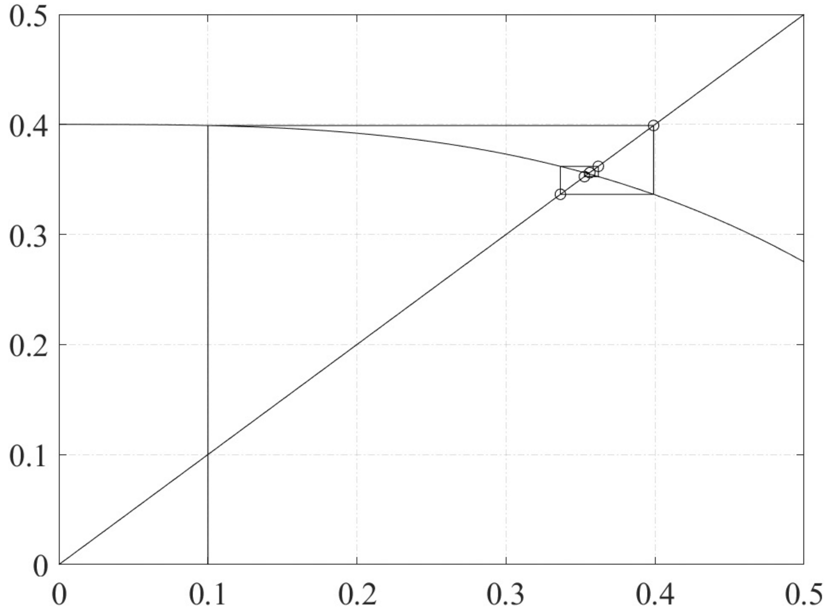
$\bar{x} = a$

$$\boxed{\text{eq. points} \begin{cases} \bar{x} = 0 & \text{if } n \neq 0 \\ \bar{x} = a^{\left(\frac{1}{1-n}\right)} & \text{if } n \neq 1 \\ \bar{x} = \mathbb{R} & \text{if } a = n = 1 \end{cases}}$$

[Problem #11] Consider the discrete-time system

$$x(k+1) = -2x(k)^3 + 0.4.$$

Find the equilibrium points and plot 5 points of a cobweb diagram for the system.



The equilibrium point is $\bar{x} = 0.329$