Tanner Kogel
TJK 190000

[Problem #1] Consider the initial value problem

$$\dot{x} = x^2, \ x(0) = x_0.$$

Using the method of separation of variables show that the solution of this equation is

$$x(t) = \frac{x_0}{1 - tx_0}.$$

$$\frac{dx}{dt} = x^2$$

$$x^{-2}\,dx = dt$$

$$\int_{x_0}^{x(t)} x^{-2}\,dx = \int_0^t dt$$

$$\left[-x^{-1}\right]_{x_0}^{x(t)} = \left[t\right]_0^t$$

$$\left[\frac{1}{x}\right]_{x(t)}^{x_0} = \left[(t) - (0)\right]$$

$$\left[\left(\frac{1}{x_0}\right) - \left(\frac{1}{x(t)}\right)\right] = t$$

$$\frac{1}{x(t)} = \frac{1}{x_0} - t$$

$$x(t) = \frac{1}{\frac{1}{x_0} - t}$$

$$\boxed{x(t) = \frac{x_0}{1 - tx_0}}$$

[Problem #2] Prove that, if T is a nonsingular matrix, then

$$T^{-1}e^{At}T = e^{T^{-1}ATt}.$$

$$e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!}$$

$$T^{-1}ATt = T^{-1}AtT \text{ (because } t \text{ is a scalar)}$$

$$e^{T^{-1}ATt} = \sum_{i=0}^{\infty} \frac{(T^{-1}ATt)^i}{i!}$$

$$(T^{-1}AtT)^i = \underbrace{(T^{-1}At\cancel{T})(\cancel{T^{-1}}At\cancel{T})\ldots(\cancel{T^{-1}}AtT)}_{i \text{ times}}$$

$$= T^{-1}(At)^i\,T$$

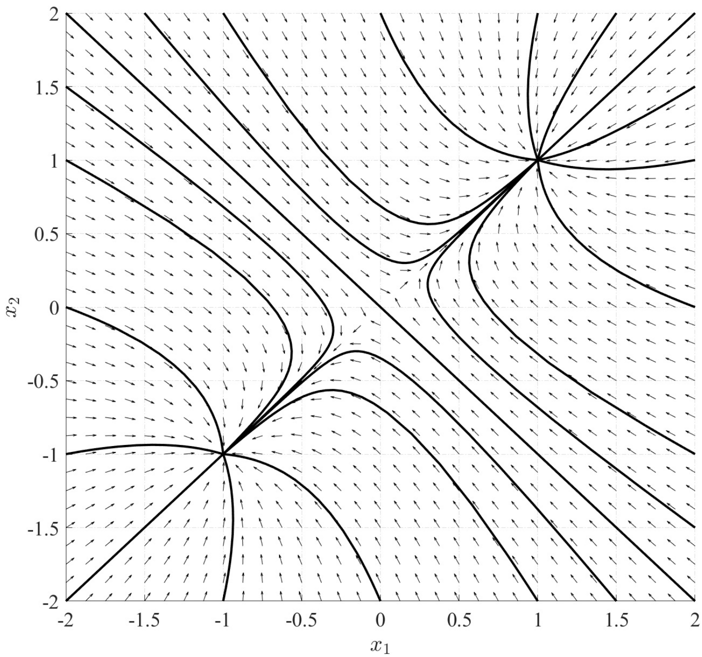
$$= \sum_{i=0}^{\infty} \frac{T^{-1}(At)^i\,T}{i!} = T^{-1}\underbrace{\sum_{i=0}^{\infty} \frac{(At)^i}{i!}}_{e^{At}}\,T = T^{-1}e^{At}\,T$$

$$\boxed{\therefore\ e^{T^{-1}ATt} = T^{-1}e^{At}\,T}$$

[Problem #3] Consider the nonlinear system

$$\dot{x}_1 = -x_1 + \frac{2x_2}{1+x_2^2}$$
$$\dot{x}_2 = -x_2 + \frac{2x_1}{1+x_1^2}.$$

(a) Sketch the phase portrait in the region $-2 \le x_1 \le 2$, $-2 \le x_2 \le 2$.




```
%% problem 3a

figure(1)
clf
hold on
grid on
[x1,x2]=meshgrid(-3:0.125:3,-2:0.125:3);
x1_dot = -x1 + ( 2 * x2./ (1 + x2.^2) );
x2_dot = -x2 + ( 2 * x1./ (1 + x1.^2) );
L = sqrt(x1_dot.^2 + x2_dot.^2);
q=quiver(x1,x2,x1_dot./L,x2_dot./L,0.4);
q.Color='black';
q.AutoScale='on';
q.MaxHeadSize=0.2;
axis([-2 2 -2 2]);
xlabel('$x_1$','interpreter','Latex'); ylabel('$x_2$','interpreter','Latex')
xic = [2 -2  2 -2 -1  0  1 1.5  2   2 2 2  2 1.5 1 0 -1 -1.5 -2 -2 -2 -2];
yic = [2 -2 -2  2 -2 -2 -2 -2 -1.5 -1 0 1 1.5 2  2 2  2   2  -1  0  1 1.5];
for ic = [xic;yic]
[~,x] = ode45(@system3a,[0,12],ic);
plot(x(:,1),x(:,2),'k');
end
```

```
%ode45 function for system 3a
function dxdt = system3a(~,x)
x1d=x(1);
x2d=x(2);
dxdt = [ (-x1d + ( 2 * x2d / (1 + x2d^2) )) ; (-x2d + ( 2 * x1d / (1 + x1d^2) )) ];
end
```

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + \frac{2x_2}{1+x_2^2} \\
\dot{x}_2 &= -x_2 + \frac{2x_1}{1+x_1^2}.
\end{aligned}
$$

(b) Find the equilibrium points and compute the linear approximation about each equilibrium point.

$$
\begin{cases}
0 = -x_1 + \dfrac{2x_2}{1+x_2^2} \Rightarrow x_1 = \dfrac{2x_2}{1+x_2^2} \\
0 = -x_2 + \dfrac{2x_1}{1+x_1^2} \Rightarrow x_2 = \dfrac{2x_1}{1+x_1^2}
\end{cases}
\Rightarrow x_1 = \frac{2\left(\frac{2x_1}{1+x_1^2}\right)}{1+\left(\frac{2x_1}{1+x_1^2}\right)^2}
$$

$$
x_1 = \frac{\frac{4x_1}{1+x_1^2}}{1+\frac{4x_1^2}{(1+x_1^2)^2}} = \frac{4x_1(1+x_1^2)}{(1+x_1^2)^2+4x_1^2}
$$

$$4x_1^3 + x_1(1+x_1^2)^2 = 4x_1(1+x_1^2)$$

$$\cancel{4x_1^3} + x_1(x_1^4+2x_1^2+1) - \cancel{4x_1^3} - 4x_1 = 0$$

$$x_1^5 + 2x_1^3 + x_1 - 4x_1 = 0$$

$$x_1(x_1^4+2x_1^2-3) = 0$$

$$\underline{x_1 = 0}$$

$$x_1^4 + 2x_1^2 - 3 = 0$$

$$y = x_1^2$$

$$y^2 + 2y - 3 = 0$$

$$(y-1)(y+3) = 0$$

$$y = 1, \quad y = -3$$

$x_1^2 = 1$, $x_1^2 = -3$ ↖ no real solution

$\underline{x_1 = \pm 1}$

$$x_2 = \frac{2x_1}{1+x_1^2}$$

$$x_2 = \frac{2(0)}{1+(0)^2} = \underline{0}$$

$$x_2 = \frac{2(1)}{1+(1)^2} = \frac{2}{1+1} = \underline{1}$$

$$x_2 = \frac{2(-1)}{1+(-1)^2} = \frac{-2}{1+1} = \underline{-1}$$

equilibrium points

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\dot{x}_1 = -x_1 + \frac{2x_2}{1+x_2^2}$$
$$\dot{x}_2 = -x_2 + \frac{2x_1}{1+x_1^2}.$$

(b) Find the equilibrium points and compute the linear approximation about each equilibrium point.

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial f_1}{\partial x_1} = -1$$

$$\frac{\partial f_1}{\partial x_2} = \frac{(1+\bar{x}_2^2)(2)-(2\bar{x}_2)(2\bar{x}_2)}{(1+\bar{x}_2^2)^2} = \frac{2-2x_2^2}{(1+\bar{x}_2^2)^2} = \frac{2}{1+\bar{x}_2^2}$$

$$\frac{\partial f_2}{\partial x_1} = \frac{(1+\bar{x}_1^2)(2)-(2\bar{x}_1)(2\bar{x}_1)}{(1+\bar{x}_1^2)^2} = \frac{2-2x_1^2}{(1+\bar{x}_1^2)^2} = \frac{2}{1+\bar{x}_1^2}$$

$$\frac{\partial f_2}{\partial x_2} = -1$$

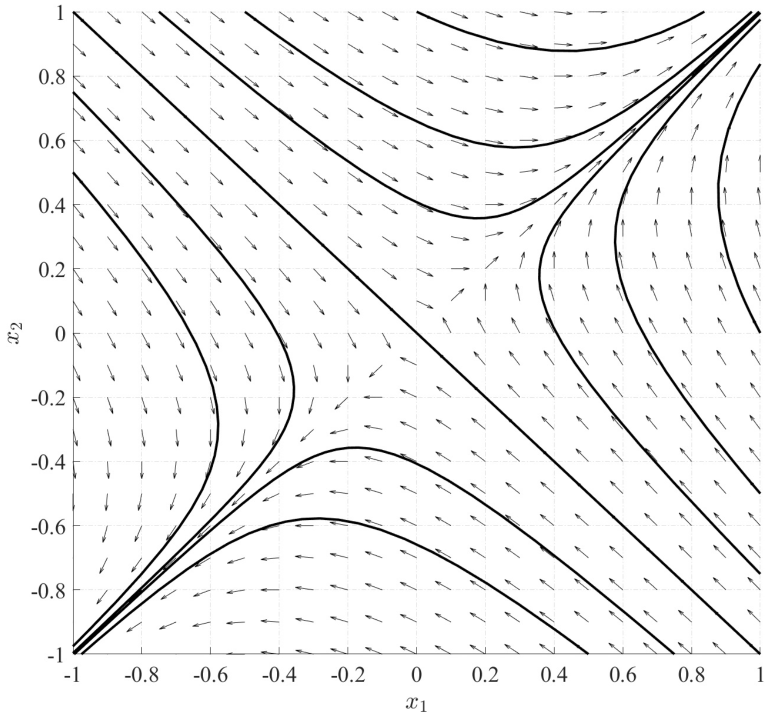
$(0,0)$
$$\dot{v} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} v$$

$(1,1)$
$$\dot{w} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} w$$

$(-1,-1)$
$$\dot{u} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} u$$

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + \frac{2x_2}{1+x_2^2} \\
\dot{x}_2 &= -x_2 + \frac{2x_1}{1+x_1^2}.
\end{aligned}
$$

(c) Sketch the phase portraits about each equilibrium point.

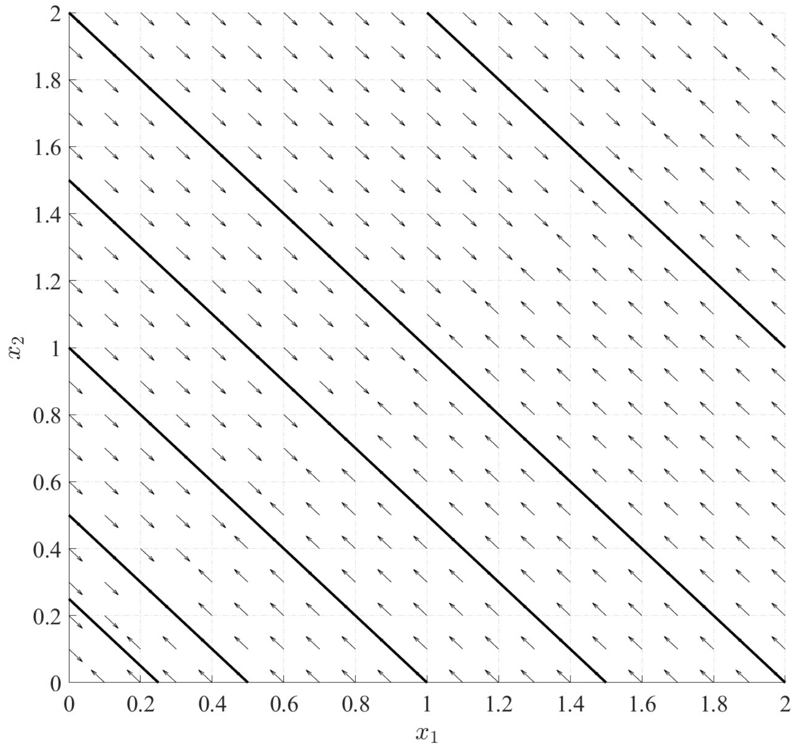



```
%% problem 3c

figure(2)
clf
hold on
grid on
[x1,x2]=meshgrid(-1:0.1:1,-1:0.1:1);
x1_dot = -x1 + 2*x2;
x2_dot = 2*x1 - x2;
L = sqrt(x1_dot.^2 + x2_dot.^2);
q=quiver(x1,x2,x1_dot./L,x2_dot./L,0.4);
q.Color='black';
q.AutoScale='on';
q.MaxHeadSize=0.2;
axis([-1 1 -1 1])
xlabel('$x_1$','interpreter','Latex'); ylabel('$x_2$','interpreter','Latex')
xic = [-1  1 0 1   1     1 -0.5 -0.75 -1  -1 0.75 0.5];
yic = [ 1 -1 1 0 -0.5 -0.75  1     1 0.75 0.5 -1  -1];
for ic = [xic;yic]
[~,x] = ode45(@system3c0,[0,9],ic);
plot(x(:,1),x(:,2),'k');
end
```

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + \frac{2x_2}{1+x_2^2} \\
\dot{x}_2 &= -x_2 + \frac{2x_1}{1+x_1^2}.
\end{aligned}
$$

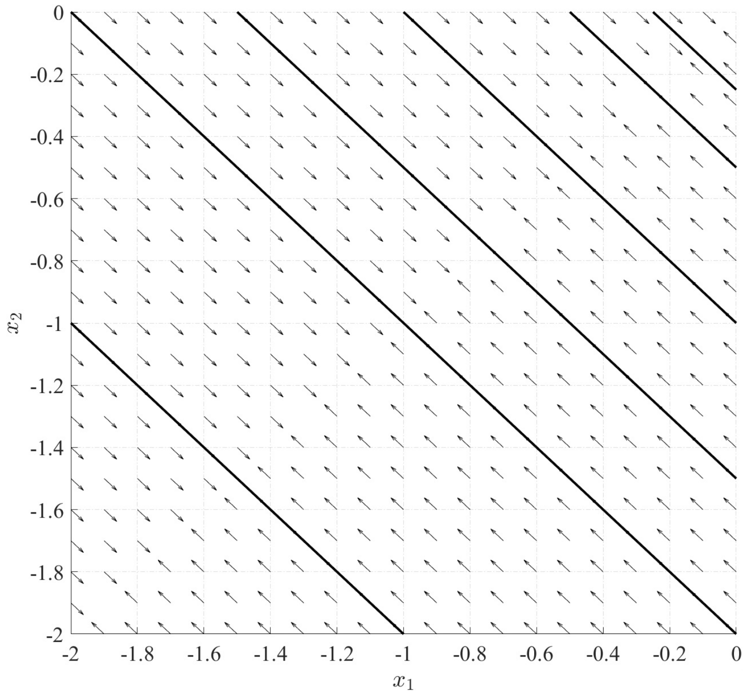
(c) Sketch the phase portraits about each equilibrium point.




```
figure(3)
clf
hold on
grid on
[x1,x2]=meshgrid(0:0.1:2,0:0.1:2);
x1_dot = -x1 + x2;
x2_dot = x1 - x2;
L = sqrt(x1_dot.^2 + x2_dot.^2);
q=quiver(x1,x2,x1_dot./L,x2_dot./L,0.4);
q.Color='black';
q.AutoScale='on';
q.MaxHeadSize=0.2;
axis([0 2 0 2])
xlabel('$x_1$','interpreter','Latex'); ylabel('$x_2$','interpreter','Latex')
xic = [1 0.5 0.25 0   0  0 1.5 2  0  0 2 1 2];
yic = [0  0   0 0.25 0.5 1  0  0 1.5 2 1 2 2];
for ic = [xic;yic]
[~,x] = ode45(@system3c1,[0,9],ic);
plot(x(:,1),x(:,2),'k');
end
```

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + \frac{2x_2}{1+x_2^2} \\
\dot{x}_2 &= -x_2 + \frac{2x_1}{1+x_1^2}.
\end{aligned}
$$

(c) Sketch the phase portraits about each equilibrium point.




```
figure(4)
clf
hold on
grid on
[x1,x2]=meshgrid(-2:0.1:0,-2:0.1:0);
x1_dot = -x1 + x2;
x2_dot = x1 - x2;
L = sqrt(x1_dot.^2 + x2_dot.^2);
q=quiver(x1,x2,x1_dot./L,x2_dot./L,0.4);
q.Color='black';
q.AutoScale='on';
q.MaxHeadSize=0.2;
axis([-2 0 -2 0])
xlabel('$x_1$','interpreter','Latex'); ylabel('$x_2$','interpreter','Latex')
xic = [-1 -0.5 -0.25 0     0   0 -1.5 -2   0   0 -2 -1 -2];
yic = [ 0   0    0 -0.25 -0.5 -1   0   0 -1.5 -2 -1 -2 -2];
for ic = [xic;yic]
[~,x] = ode45(@system3c1,[0,9],ic);
plot(x(:,1),x(:,2),'k');
end
```

```
%ode45 function for system 3c with equilirium point at (0,0)
function dxdt = system3c0(~,x)
x1d=x(1);
x2d=x(2);
dxdt = [ (-x1d + 2*x2d) ; (2*x1d - x2d) ];
end

%ode45 function for system 3c with equilirium point at (-1,-1),(1,1)
function dxdt = system3c1(~,x)
x1d=x(1);
x2d=x(2);
dxdt = [ (-x1d + x2d) ; (x1d - x2d) ];
end
```

$$\dot{x}_1 = -x_1 + \frac{2x_2}{1+x_2^2}$$
$$\dot{x}_2 = -x_2 + \frac{2x_1}{1+x_1^2}.$$

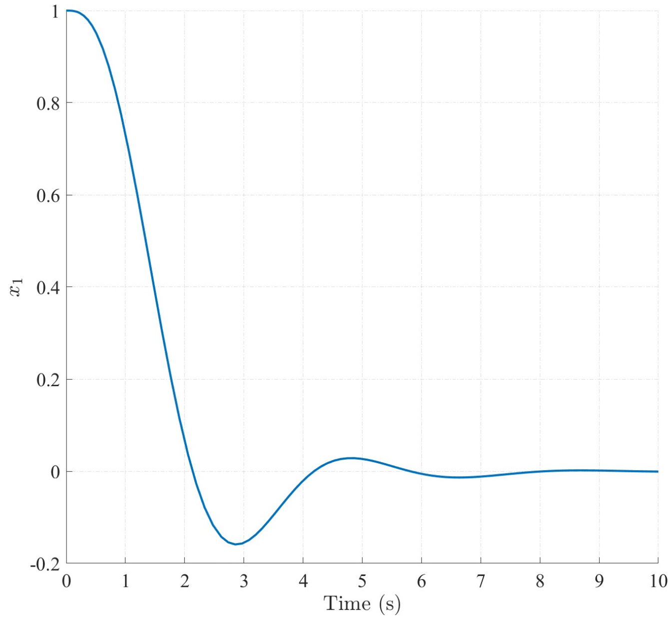
(d) Interpret the results from item (c) in light of the Hartman-Grobman Theorem.

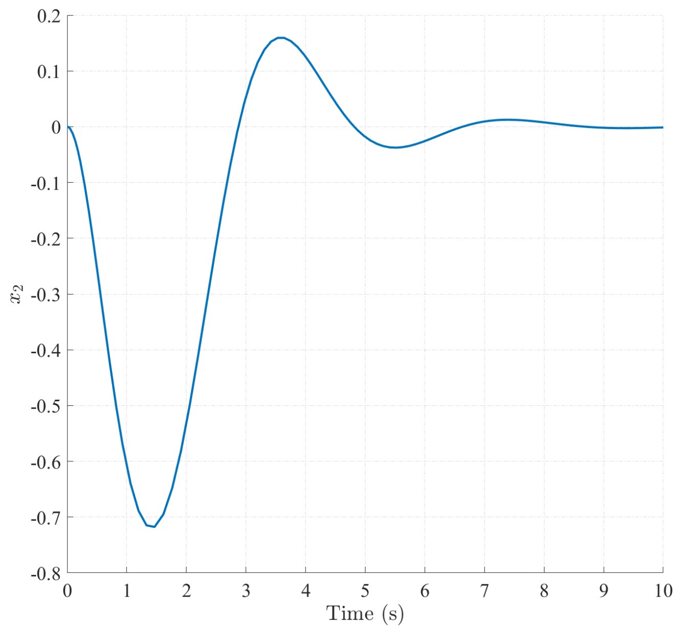
The Hartzman-Grobman Theorem equates the phase portrait near an equilibrium as a distorted version of the phase portrait of the linearized system, but only for hyperbolic equilibrium points. Becuase (1,1) and (-1,-1) are non-hyperbolic equilibrium points, the phase portrait of the linearized system is not a good estimation of the nonlinear system near those equilibrium points. In contrast, the linearized phase portrait at the equilibrium point (0,0) is a good estimation because the equilibrium point is hyperbolic.

[Problem #4] Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= -2x_1 - 4x_2 - 2x_3 - x_1^2 - x_2^2.\end{aligned}$$

(a) Write a Matlab script to simulate this system for 10 seconds starting at the initial conditions $(1, 0, 0)$.

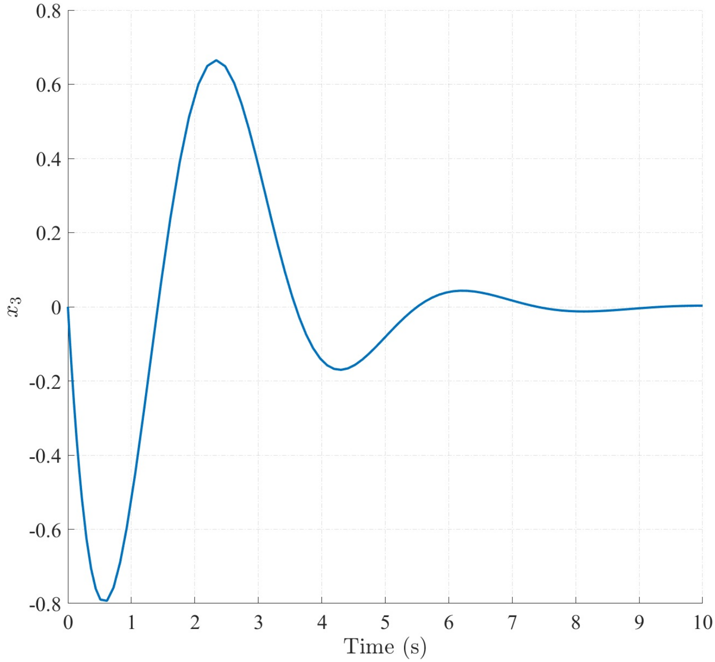




[Problem #4] Consider the nonlinear system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -2x_1 - 4x_2 - 2x_3 - x_1^2 - x_2^2.
\end{aligned}
$$

(a) Write a Matlab script to simulate this system for 10 seconds starting at the initial conditions $(1, 0, 0)$.




```
%% problem 4a

x0 = [1; 0; 0]; % define initial condition

[t,x] = ode45(@system4,[0,10],x0);  % simulate response of the system

% plot figures
figure(5)
clf
hold on
grid on
plot(t,x(:,1));
xlabel('Time (s)'); ylabel('$x_1$','interpreter','Latex')

figure(6)
clf
hold on
grid on
plot(t,x(:,2));
xlabel('Time (s)'); ylabel('$x_2$','interpreter','Latex')

figure(7)
clf
hold on
grid on
plot(t,x(:,3));
xlabel('Time (s)'); ylabel('$x_3$','interpreter','Latex')
```

```
%ode45 function for system 4
function dxdt = system4(~,x)
x1=x(1);
x2=x(2);
x3=x(3);
dxdt = [ x2 ; x3 ; -2*x1 - 4*x2 - 2*x3 - x1^2 - x2^2];
end
```

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -2x_1 - 4x_2 - 2x_3 - x_1^2 - x_2^2.$$

(b) Find the equilibrium points and computer the linear approximation about each equilibrium point.

$$\begin{cases} 0 = x_2 \\ 0 = x_3 \\ 0 = -2x_1 - 4x_2 - 2x_3 - x_1^2 - x_2^2 \end{cases} \Rightarrow \quad x_1^2 + 2x_1 = 0$$

$$x_1(x_1+2) = 0$$
$$x_1 = 0$$
$$x_1 = -2$$

equilibrium points

$x_{1,0} = 0$	$x_{1,0} = -2$
$x_{2,0} = 0$	$x_{2,0} = 0$
$x_{3,0} = 0$	$x_{3,0} = 0$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}$$

$\frac{\partial f_1}{\partial x_1} = 0$

$\frac{\partial f_1}{\partial x_2} = 1$

$\frac{\partial f_1}{\partial x_3} = 0$

$\frac{\partial f_2}{\partial x_1} = 0$

$\frac{\partial f_2}{\partial x_2} = 0$

$\frac{\partial f_2}{\partial x_3} = 1$

$\frac{\partial f_3}{\partial x_1} = -2x_{1,0} - 2$

$\frac{\partial f_3}{\partial x_2} = -2x_{2,0} - 4$

$\frac{\partial f_3}{\partial x_3} = -2$

about $(0,0,0)$

$$\dot{v} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -2 \end{bmatrix} v$$

about $(-2,0,0)$

$$\dot{w} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -4 & -2 \end{bmatrix} w$$

[Problem #5] Consider the singularly perturbed system

$$\begin{aligned}\dot{x} &= -x + z \\ \epsilon \dot{z} &= -2x^3 - z\end{aligned}$$

(a) Compute the quasi-steady-state system

$A_{11} = -1 \qquad A_{12} = 1$

$A_{21} = -2x^2 \qquad A_{22} = -1$

$\dot{x} = (A_{11} - A_{12} A_{22}^{-1} A_{21})x$

$= (-1 - (1)(-1)^{-1}(-2x^2))x = (-1 - (1)(-1)(-2x^2))x$

$= (-1 - 2x^2)x = -2x^3 - x$

$\boxed{\dot{x} = -2x^3 - x}$

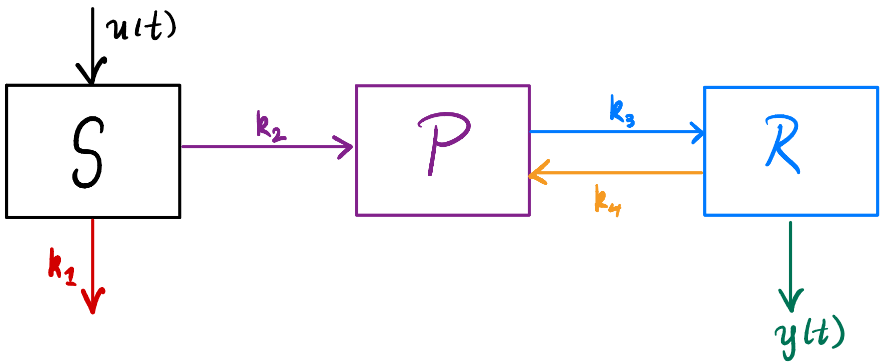
(b) Compute the boundary-layer system

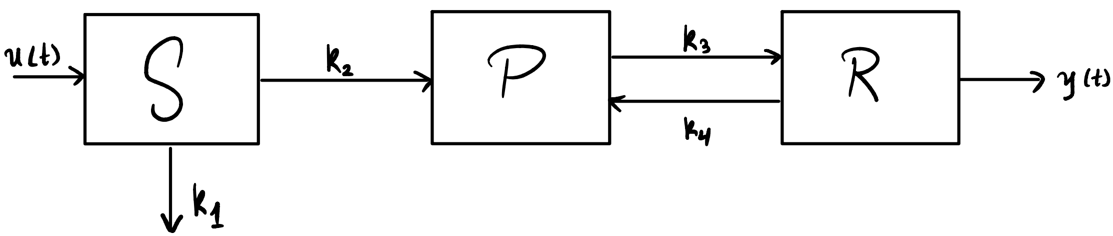
$y = z + A_{22}^{-1} A_{21} x = z + (-1)^{-1}(-2x^2)x = z + 2x^3 \Rightarrow \underline{y = z + 2x^3}$

$\frac{dy}{d\tau} = A_{22} y = (-1) y \Rightarrow \boxed{\frac{dy}{d\tau} = -y}$

[Problem #6] Consider a supply chain with suppliers (S), producers (P), and retailers (R). The supplier takes in raw material at input rate $u(t)$. Raw material that does not meet standards is discarded at rate k_1 and raw material meeting standards is shipped to the producer at rate k_2. The producer manufactures various products, which are sent to retailers at rate k_3. The retailers sell products at output rate $y(t)$ and return products to the producer at rate k_4.

(a) Draw a three-compartment model to represent this supply chain.





(b) Write the differential equations for this model in terms of states x_s, x_p, and x_r.

$$\boxed{\begin{aligned} \dot{x}_s &= -k_1 x_s - k_2 x_s + u(t) \\ \dot{x}_p &= k_2 x_s - k_3 x_p + k_4 x_R \\ \dot{x}_R &= k_3 x_p - k_4 x_R - y(t) \end{aligned}}$$

$$\boxed{\begin{aligned} \dot{x}_s &= -k_1 x_s - k_2 x_s + u(t) \\ \dot{x}_p &= k_2 x_s - k_3 x_p + k_4 x_R \\ \dot{x}_R &= k_3 x_p - k_4 x_R + y(t) \end{aligned}}$$

[Problem #7] Using the method of separation of variables derive the solution of the logistic equation

$$\frac{dN}{dt} = rN(1-\frac{N}{K})$$

with $N(0) = N_0$ as

$$N(t) = \frac{K}{1+Ae^{-rt}}, \text{ where } A = \frac{K-N_0}{N_0}.$$

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$$

$$\frac{dN}{dt} = rN - \frac{r}{K}N^2$$

$$\left(rN - \frac{r}{K}N^2\right)^{-1} dN = dt$$

$$\int_{N_0}^{N(t)} \frac{K}{rKN - rN^2}\, dN = \int_0^t dt$$

$$\frac{K}{r}\int_{N_0}^{N(t)} \frac{1}{KN-N^2}\, dN = \Big[t\Big]_0^t$$

$$\frac{1}{KN-N^2} = \frac{1}{N(K-N)} = \frac{B}{N} + \frac{C}{K-N}$$

$$\Rightarrow \quad 1 = B(K-N) + C(N)$$

$$(N=0) \qquad\qquad (N=K)$$

$$1 = BK \qquad\qquad 1 = CK$$

$$\underline{B = K^{-1}} \qquad\qquad \underline{C = K^{-1}}$$

$$\frac{K}{r}\int_{N_0}^{N(t)} \frac{1}{KN} + \frac{1}{K(K-N)}\, dN = \Big[(t)-(0)\Big]$$

$$\frac{1}{r}\int_{N_0}^{N(t)} N^{-1} + (K-N)^{-1}\, dN = t$$

$$\int_{N_0}^{N(t)} N^{-1} dN + \int_{N_0}^{N(t)} \frac{1}{K-N}\, dN = rt$$

$$\ln\left(\frac{N(t)}{N_0}\right) + \ln\left(\frac{K-N_0}{K-N(t)}\right) = rt$$

$$\ln\left(\frac{N(t)(K-N_0)}{KN_0 - N_0N(t)}\right) = rt$$

$$\frac{(K-N_0)N(t)}{KN_0 - N_0N(t)} = e^{rt}$$

$$N(t) = e^{rt}\left(\frac{KN_0 - N_0N(t)}{K-N_0}\right)$$

$$N(t) = N_0e^{rt}\left(\frac{K-N(t)}{K-N_0}\right)$$

$$N(t) = N_0e^{rt}\left(\frac{K}{K-N_0}\right) - \frac{N_0e^{rt}}{K-N_0}N(t)$$

$$N(t)\left(1 + \frac{N_0e^{rt}}{K-N_0}\right) = \frac{KN_0e^{rt}}{K-N_0}$$

$$N(t)\left(1 + \frac{1}{Ae^{-rt}}\right) = \frac{K}{Ae^{-rt}}$$

$$N(t)\left(\frac{Ae^{-rt}+1}{\cancel{Ae^{-rt}}}\right) = \frac{K}{\cancel{Ae^{-rt}}}$$

$$N(t) = \frac{K}{Ae^{-rt}+1}$$

$$\boxed{N(t) = \frac{K}{1+Ae^{-rt}}}$$

[Problem #7] Using the method of separation of variables derive the solution of the logistic equation

$$\frac{dN}{dt} = rN(1 - \frac{N}{K})$$

with $N(0) = N_0$ as

$$N(t) = \frac{K}{1 + Ae^{-rt}}, \text{ where } A = \frac{K - N_0}{N_0}.$$

$$\int_{N_0}^{N(t)} N^{-1}\, dN = \Big[\ln(N)\Big]_{N_0}^{N(t)} = \Big[\ln(N(t)) - \ln(N_0)\Big] = \boxed{\ln\left(\frac{N(t)}{N_0}\right)}$$

$$\int_{N_0}^{N(t)} \frac{1}{K-N}\, dN \qquad u = K - N$$

$$\frac{du}{dN} = -1$$

$$\int_{K-N_0}^{K-N(t)} u^{-1}(-du) \qquad du = -dN$$

$$-\int_{K-N_0}^{K-N(t)} u^{-1}\, du = \int_{K-N(t)}^{K-N_0} u^{-1}\, du = \Big[\ln(u)\Big]_{K-N(t)}^{K-N_0} = \Big[\ln(K-N_0) - \ln(K-N(t))\Big]$$

$$\boxed{= \ln\left(\frac{K-N_0}{K-N(t)}\right)}$$

[Problem #8] Consider the predator-prey model

$$\begin{aligned}\dot{x} &= ax - bxy\\ \dot{y} &= -cy + dxy.\end{aligned}$$

Show that the equilibrium points are $(0,0)$ and $(c/d, a/b)$ and verify that $(0,0)$ is a saddle point and $(c/d, a/b)$ is a center for the linear approximations about these equilibrium points.

$$\begin{cases} 0 = ax - bxy \Rightarrow x = \frac{b}{a}xy \Rightarrow \frac{b}{a}y = 1 \\ 0 = -cy + dxy \Rightarrow y = \frac{d}{c}xy \end{cases} \qquad \underline{y = \frac{a}{b}}$$

$$\Downarrow$$

$$\frac{d}{c}x = 1$$

$$\underline{x = \frac{c}{d}}$$

$\therefore \left(\frac{c}{d}, \frac{a}{b}\right)$ is an equilibrium point

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$\frac{\partial f_1}{\partial x} = a - b\bar{y} \qquad \frac{\partial f_2}{\partial x} = d\bar{y}$$

$$\frac{\partial f_1}{\partial y} = -b\bar{x} \qquad \frac{\partial f_2}{\partial y} = -c + d\bar{x}$$

$$\dot{v} = \begin{bmatrix} a - b\bar{y} & d\bar{y} \\ -b\bar{x} & d\bar{x} - c \end{bmatrix} v$$

$(0,0)$

$$\dot{v} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} v$$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - a & 0 \\ 0 & \lambda + c \end{bmatrix} = (\lambda - a)(\lambda + c) = 0$$

$\underline{\lambda = a, -c}$

$(0,0)$ is a saddle point because it has both positive & negative eigenvalues

$\left(\frac{c}{d}, \frac{a}{b}\right)$

$$\dot{w} = \begin{bmatrix} 0 & \frac{ad}{b} \\ -\frac{bc}{d} & 0 \end{bmatrix} w$$

$$\det \begin{vmatrix} \lambda & -\frac{ad}{b} \\ \frac{bc}{d} & \lambda \end{vmatrix} = \lambda^2 + ac = 0$$

$\underline{\lambda = \pm\sqrt{ac}\, i}$

$\left(\frac{c}{d}, \frac{a}{b}\right)$ is a center because the eigenvalues are a purely imaginary conjugate pair