[Problem #1] Write a Matlab script to solve the following problem by the method of steepest descent

$$\text{minimize} \qquad x_1^2 + 4x_1x_2 + 6x_2^2 - 2x_1 + 3x_2$$

$$\begin{array}{l} x_1^2 \\ x_1x_2 \\ x_2^2 \\ x_1 \\ x_2 \end{array} \left\{ \begin{array}{l} 1 = \frac{1}{2} q_{11} \Rightarrow \underline{q_{11} = 2} \\ \underline{4 = q_{12} = q_{21}} \\ 6 = \frac{1}{2} q_{22} \Rightarrow \underline{q_{22} = 12} \\ \underline{-2 = b_1} \\ \underline{3 = b_2} \end{array} \right.$$

$$Q = \begin{bmatrix} 2 & 4 \\ 4 & 12 \end{bmatrix} \quad b = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$f(x) = \frac{1}{2} [x_1 \ \ x_2] \begin{bmatrix} 2 & 4 \\ 4 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [-2 \ \ 3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$f(x) = \frac{1}{2} x^T Q x + b^T x$$

$$= \frac{1}{2} [x_1 \ \ x_2] \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [b_1 \ \ b_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{1}{2} [q_{11}x_1 + q_{21}x_2 \ \ \ q_{12}x_1 + q_{22}x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [b_1x_1 + b_2x_2]$$

$$= \frac{1}{2} [q_{11}x_1^2 + q_{21}x_1x_2 + q_{12}x_1x_2 + q_{22}x_2^2] + [b_1x_1 + b_2x_2]$$

$$f(x) = \frac{1}{2} q_{11} x_1^2 + \frac{1}{2}(q_{21} + q_{12}) x_1x_2 + \frac{1}{2} q_{22} x_2^2 + b_1x_1 + b_2x_2$$

$$x_{k+1} = x_k - \alpha_k g(x_k)$$

$$\alpha_k = \frac{g(x_k)^T g(x_k)}{g(x_k)^T Q g(x_k)}$$

$$g(x_k) = \overbrace{\begin{bmatrix} 2 & 4 \\ 4 & 12 \end{bmatrix}}^{Q} x_k + \overbrace{\begin{bmatrix} -2 \\ 3 \end{bmatrix}}^{b}$$

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local minima found at:
x =

    4.5000
   -1.7500

in 11 iterations
>>
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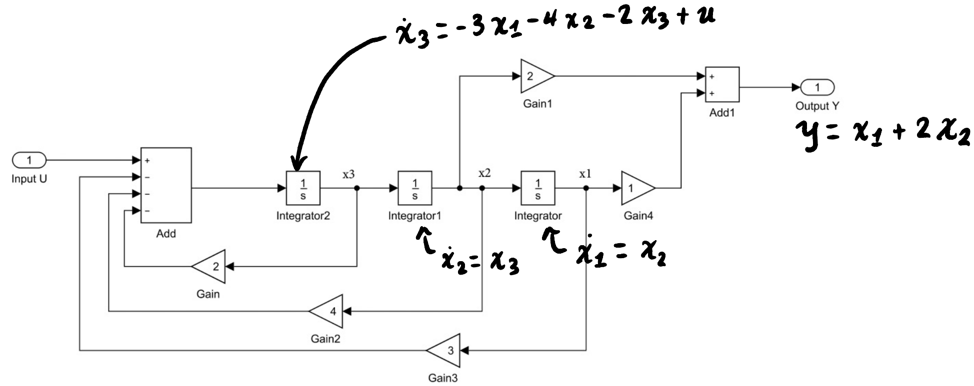
[Problem #2] Consider the simulink model below



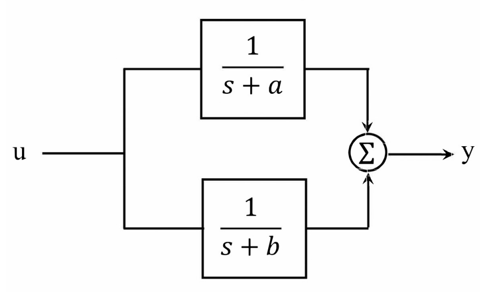
Write the state equations for this model. In other words, define the matrices A, B, and C so that

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -3x_1 - 4x_2 - 2x_3 + u \\ y = x_1 + 2x_2 \end{cases}$$

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}}$$

[Problem #3] Show that the system below is controllable and observable if and only if $a \neq b$.



$$Y(s) = U(s)\left(\frac{1}{s+a} + \frac{1}{s+b}\right) \Rightarrow \frac{Y(s)}{U(s)} = G(s) = \frac{1}{s+a} + \frac{1}{s+b} = \frac{2s+a+b}{(s+a)(s+b)}$$

$$G(s) = G_2(s)\, G_1(s) = \left(\frac{1}{(s+a)(s+b)}\right)\left(2s+a+b\right)$$

$$u \longrightarrow \boxed{G_2(s)} \xrightarrow{z} \boxed{G_1(s)} \longrightarrow y$$

in: u
out: z

$$G_2(s) = \frac{Z(s)}{U(s)} = \frac{1}{s^2 + (a+b)s + ab} \Rightarrow s^2 Z(s) + (a+b)s Z(s) + ab\, Z(s) = U(s)$$

$$\underline{\ddot{z} + (a+b)\dot{z} + ab\, z = u}$$

$x_1 = z$ $\qquad$ $\dot{x}_1 = \dot{z} = x_2$

$x_2 = \dot{x}_1 = \dot{z}$ $\qquad$ $\dot{x}_2 = -ab\, z - (a+b)\dot{z} + u = -ab\, x_1 - (a+b)x_2 + u$

in: z
out: y

$$G_1(s) = \frac{Y(s)}{Z(s)} = 2s+a+b \Rightarrow Y(s) = 2s Z(s) + (a+b) Z(s)$$

$$y = 2\dot{z} + (a+b)z$$

$$y = (a+b)x_1 + 2x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -ab & -a-b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} a+b & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathcal{E} = [B \;\; AB]$$

$$\mathcal{E} = \begin{bmatrix} 0 & 1 \\ 1 & -a-b \end{bmatrix}$$

> because $\mathcal{E}$ will always be full rank, the system will be controllable for all values of a & b.

$$\Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} a+b & 2 \\ -2ab & -a-b \end{bmatrix}$$

$$\begin{vmatrix} a+b & 2 \\ -2ab & -a-b \end{vmatrix} = (a+b)(-a-b) - (2)(-2ab) = 0$$

$$-a^2 - 2ab - b^2 + 4ab = 0$$

$$-a^2 - b^2 + 2ab = 0$$

$$a^2 + b^2 = 2ab$$

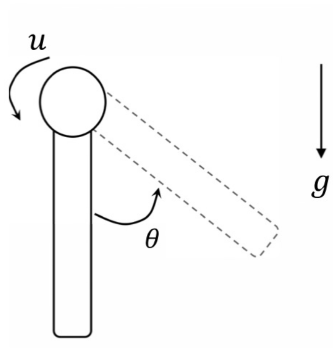
if $a = b$

$$a^2 + a^2 = 2a\,a$$

$$2a^2 = 2a^2$$

> because
> $$\det(\Omega) = \begin{cases} 0 & a = b \\ \neq 0 & \text{otherwise} \end{cases}$$
> the system is observable if $a \neq b$

[Problem #4] Consider a one-link robot arm as shown with input u equal to the torque at the joint, output $y = \theta$ the link angle, and $g = 9.8$, the acceleration due to gravity.



The equation governing this system is

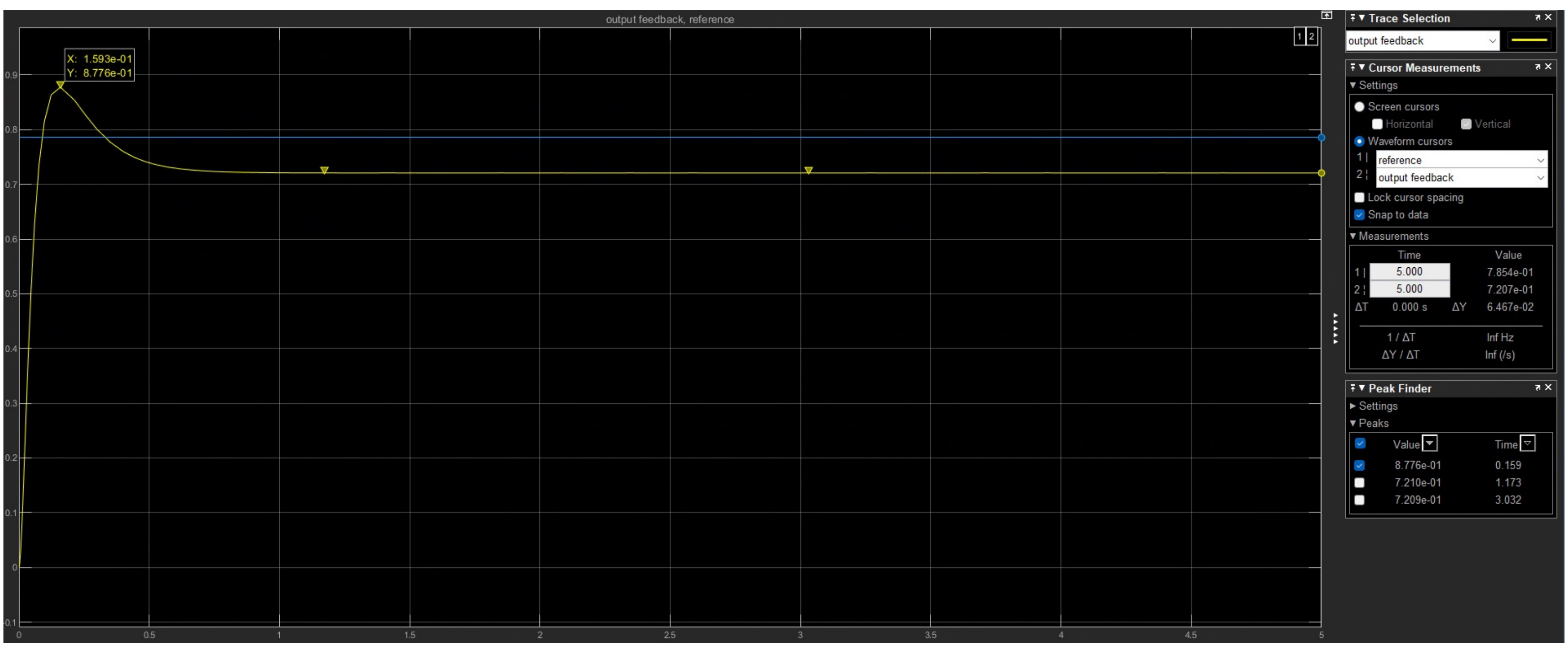
$$\ddot{\theta} + g\sin(\theta) = u$$

Using Simulink, model this system and design a PD control to move the arm to 45 degrees in one second. What is the steady-state error using your controller?

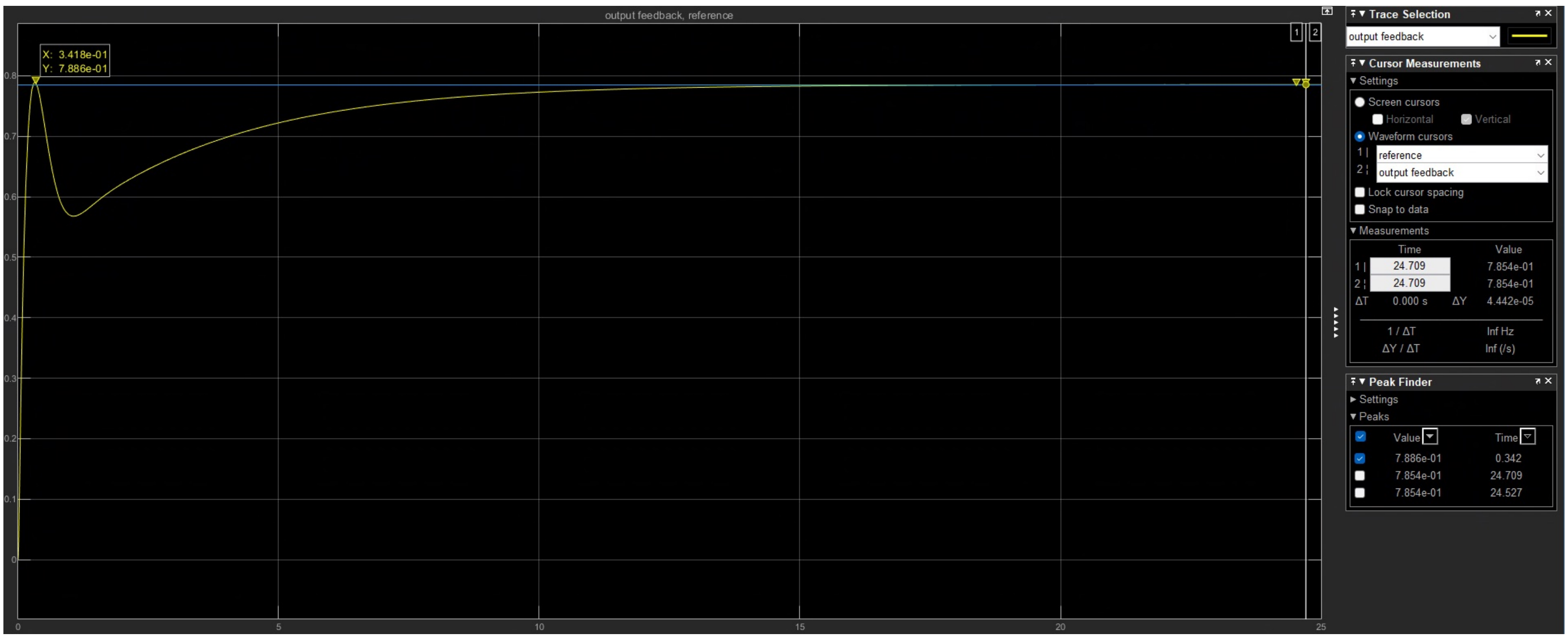
Now add an integral control to make the steady-state error zero.

$x_1 = \theta \qquad \dot{x}_1 = \dot{\theta} = x_2$

$x_2 = \dot{\theta} \qquad \dot{x}_2 = \ddot{\theta} = u - g\sin(\theta) = u - g\sin(x_1)$



Steady-state error: -0.0647 rad


output feedback, reference
X: 3.418e-01
Y: 7.886e-01
Trace Selection
output feedback
Cursor Measurements
Settings
Screen cursors
Horizontal
Vertical
Waveform cursors
1 reference
2 output feedback
Lock cursor spacing
Snap to data
Measurements
Time
Value
1 24.709 7.854e-01
2 24.709 7.854e-01
ΔT 0.000 s ΔY 4.442e-05
1 / ΔT Inf Hz
ΔY / ΔT Inf (/s)
Peak Finder
Settings
Peaks
Value
Time
7.886e-01 0.342
7.854e-01 24.709
7.854e-01 24.527