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Homework #1

1) $\ddot{y} - 4\dot{y} + 4y = 0$

a) Show by direct calculation that both $y(t) = e^{2t}$ and $y(t) = te^{2t}$ are solutions

① $y(t) = e^{2t}$

$\dot{y}(t) = 2e^{2t}$

$\ddot{y}(t) = 4e^{2t}$

plug in to system:

$(4e^{2t}) - 4(2e^{2t}) + 4(e^{2t}) = 0$

$\cancel{e^{2t}}(4-8+4) = 0$

$4-8+4 = 0$ ✓

② $y(t) = te^{2t}$

$\dot{y}(t) = 2te^{2t} + e^{2t}$

$\ddot{y}(t) = 4te^{2t} + 2e^{2t} + 2e^{2t} = \underline{4te^{2t} + 4e^{2t}}$

Plug in to system:

$(4te^{2t} + 4e^{2t}) - 4(2te^{2t} + e^{2t}) + 4(te^{2t}) = 0$

$\cancel{e^{2t}}(4t + 4 - 8t - 4 + 4t) = 0$

$t(4-8+4) + (4-4) = 0$ ✓

both $y(t)$ definitions solve the second-order system

b) Write system in state space & find T such that $\bar{A} = T^{-1}AT$ is in Jordan form

$x_1 = y$, $\dot{x}_1 = \dot{y} = x_2$

$x_2 = \dot{x}_1 = \dot{y}$, $\dot{x}_2 = \ddot{y} = 4\dot{y} - 4y = 4x_2 - 4x_1$

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$ $a = 0; b = 1$ $c = -4; d = 4$

$p = a + d = (0) + (4) = 4$

$q = ad - bc = (0)(4) - (1)(-4) = 4$

$\lambda = \frac{1}{2}p \pm \sqrt{p^2 - 4q} = \frac{1}{2}(4) \pm \cancel{\sqrt{(4)^2 - 4(4)}} = \underline{2}$

$(A - \lambda I)x = 0$

$$\left(\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

1b cont) $\begin{cases} -\lambda x_1 + x_2 = 0 \\ -4x_1 + (4-\lambda)x_2 = 0 \end{cases}$ $(\lambda = 2)$ $\begin{cases} -2x_1 + x_2 = 0 \\ -4x_1 + 2x_2 = 0 \end{cases}$

$\underline{2x_1 = x_2}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$(A - \lambda I)y = x$

$$\left(\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\begin{cases} -2y_1 + y_2 = 1 \\ -4y_1 + 2y_2 = 2 \end{cases} \Rightarrow \underline{y_2 = 1 + 2y_1}$ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\boxed{T = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}}$$

$$T^{-1} = \frac{1}{(1)(3) - (1)(2)}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3-2}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

$$\bar{A} = T^{-1}AT = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\boxed{\bar{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}$$

$x = Tz$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$x_1 = z_1 + z_2 = y$

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{aligned}}$$

2) $\dot{x} = Ax$

Prove that if A is nonsingular, $\bar{x} = 0$ is the only equilibrium point

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular, $ad - bc \neq 0$, $\Rightarrow ad \neq bc$

equilibrium is found at $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$ so that $\dot{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$\dot{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

$$\begin{cases} ax_1 + bx_2 = 0 \Rightarrow x_1 = -\frac{b}{a} x_2 \\ cx_1 + dx_2 = 0 \end{cases}$$

$$\downarrow$$

$$-\frac{bc}{a} x_1 = -d x_2$$

$$bcx_1 = adx_2 \quad ①$$

in order for ① to be true, while A is nonsingular ($ad \neq bc$), the only value of x is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

3) $\dot{x} = Ax$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Find the set of all equilibrium points of the system & if they are isolated or non-isolated

$\det(A) = 4 - 4 = 0 \quad \therefore$ A is singular

$\dot{x}_1 = x_1 + 2x_2 = 0 \qquad x_1 = -2x_2$

$\dot{x}_2 = 2x_1 + 4x_2 = 0$

points on the form $x_1 = -2x_2$ are equilibrium points (non-isolated).

4) a) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1^4 + x_1^3 + 2x_1^2 + 2x_1 - x_2 \end{cases}$

$\begin{cases} \cancel{0 = x_2} \\ 0 = x_1^4 + x_1^3 + 2x_1^2 + 2x_1 - \cancel{x_2} \end{cases}$ $(x_2 = 0)$ <u>all</u>

$\underline{x_1^3 + x_1^2 + 2x_1 + 2 = 0}$ $(x_1 = 0)$

$f(-1) = (-1)^3 + (-1)^2 + 2(-1) + 2 = -1 \cancel{+1} \cancel{-2+2} = 0$

$f(-2) = (-2)^3 + (-2)^2 + 2(-2) + 2 = -8 + \cancel{4} \cancel{-4} + 2 = -6$

$(x_1 + 1) \Rightarrow (x_1 = -1)$

$$\begin{array}{r|l} & x_1^2 + 2 \\ \hline x_1 + 1 & x_1^3 + x_1^2 + 2x_1 + 2 \\ & -(x_1^3 + x_1^2) \\ \hline & 2x_1 + 2 \\ & -(2x_1 + 2) \\ \hline & 0 \end{array}$$

$(x_1 + 1)(x_1^2 + 2) = 0$

$(x_1^2 + 2) = 0$

$x_1 = \pm\sqrt{-2}$ $(x_1 = \pm i\sqrt{2})$

$\boxed{\text{equilibriums: all isolated} \quad \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} i\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -i\sqrt{2} \\ 0 \end{bmatrix}}$

4a cont) linearization

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial f_1}{\partial x_1} = 0$$

$$\frac{\partial f_1}{\partial x_2} = 1$$

$$\frac{\partial f_2}{\partial x_1} = 4\bar{x}_1^3 + 3\bar{x}_1^2 + 4\bar{x}_1 + 2$$

$$\frac{\partial f}{\partial x_2} = -1$$

about $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x$$

about $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x$$

about $\begin{bmatrix} i\sqrt{2} \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 \\ -4-4i\sqrt{2} & -1 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -4-4i\sqrt{2} & -1 \end{bmatrix} x$$

about $\begin{bmatrix} -i\sqrt{2} \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 \\ -4+4i\sqrt{2} & -1 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -4+4i\sqrt{2} & -1 \end{bmatrix} x$$

4b) $\begin{cases} \dot{x}_1 = 2x_1 - x_1 x_2 \\ \dot{x}_2 = -\frac{1}{2}x_2 + x_1 x_2 \end{cases}$

$\begin{cases} 0 = 2x_1 - x_1 x_2 \\ 0 = x_2 - 2x_1 x_2 \end{cases} \Rightarrow \begin{cases} x_1 x_2 = 2x_1 \\ x_1 x_2 = \frac{1}{2}x_2 \end{cases}$

$\underline{(0,0)}$

$(x_2 = a)$

$\begin{cases} x_1 a = 2x_1 \Rightarrow a = 2 \\ x_1 a = \frac{1}{2}a \Rightarrow x_1 = \frac{1}{2} \end{cases}$

$(x_1 = a)$

$\begin{cases} a x_2 = 2a \Rightarrow x_2 = 2 \\ a x_2 = \frac{1}{2}x_2 \Rightarrow a = \frac{1}{2} \end{cases}$ $\underline{(\frac{1}{2}, 2)}$

$$\boxed{\text{equilibriums: all isolated} \quad \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}}$$

linearization:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$\frac{\partial f_1}{\partial x_1} = 2 - \bar{x}_2$

$\frac{\partial f_1}{\partial x_2} = -\bar{x}_1$

$\frac{\partial f_2}{\partial x_1} = \bar{x}_2$

$\frac{\partial f_2}{\partial x_2} = -\frac{1}{2} + \bar{x}_1$

about $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$

$\dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} x$

about $\begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}$

$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix} x$

4c) $\begin{cases} \dot{x}_1 = 2x_1(1-x_1) - x_1 x_2 \\ \dot{x}_2 = -\frac{1}{2}x_1 + x_1 x_2 \end{cases}$

$\begin{cases} 0 = -2x_1^2 + 2x_1 - x_1 x_2 \\ 0 = x_1 - 2x_1 x_2 \end{cases} \Rightarrow \begin{cases} 2x_1(1-x_1) = x_1 x_2 \\ x_1 = 2x_1 x_2 \end{cases}$

$\underline{(0, \bar{x}_2)}$

$(x_1 = a)$

$\begin{cases} -2a^2 + 2a = a x_2 \Rightarrow x_2 = 2 - 2a \Rightarrow \frac{1}{2} = 2 - 2a \\ \cancel{a} = 2\cancel{a} x_2 \Rightarrow x_2 = \frac{1}{2} \end{cases}$

$1 = 4 - 4a$

$4a = 4 - 1 = 3$

$a = \frac{3}{4}$ $\qquad \underline{(\frac{3}{4}, \frac{1}{2})}$

equilibriums:

$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{x}_2 \end{bmatrix}, \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$

non-isolated $\qquad$ isolated

linearization:

$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$

$\frac{\partial f_1}{\partial x_1} = -4\bar{x}_1 + 2 - \bar{x}_2$

$\frac{\partial f_1}{\partial x_2} = -\bar{x}_1$

$\frac{\partial f_2}{\partial x_1} = -\frac{1}{2} + \bar{x}_2$

$\frac{\partial f_2}{\partial x_2} = \bar{x}_1$

about $\begin{bmatrix} 0 \\ \bar{x}_2 \end{bmatrix}$

$A = \begin{bmatrix} 2 - \bar{x}_2 & 0 \\ \bar{x}_2 - \frac{1}{2} & 0 \end{bmatrix}$

$\dot{x} = \begin{bmatrix} 2 - \bar{x}_2 & 0 \\ \bar{x}_2 - \frac{1}{2} & 0 \end{bmatrix} x$

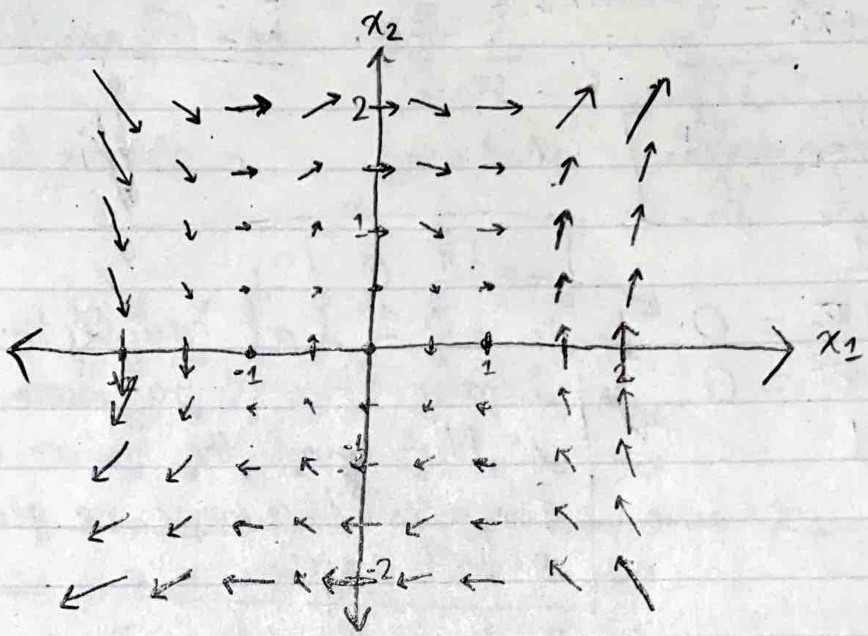
about $\begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$

$A = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{4} \\ 0 & \frac{3}{4} \end{bmatrix}$

$\dot{x} = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{4} \\ 0 & \frac{3}{4} \end{bmatrix} x$

5) $\ddot{y} + y - y^3 = 0$

$x_1 = y$
$x_2 = \dot{x}_1 = \dot{y}$ $\Rightarrow$ $\boxed{\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1^3 - x_1 \end{cases}}$ eq. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}$



6) $\begin{cases} \dot{x}_1 = x_2^2 + x_1 \\ \dot{x}_2 = x_2 + x_1^3 \end{cases}$

equilibriums: $(0,0)$

$\begin{cases} 0 = x_2^2 + x_1 \\ 0 = x_2 + x_1^3 \end{cases} \Rightarrow (x_2 = a) \begin{cases} x_1 = -a^2 \\ x_1^3 = a \end{cases}$ no other equillibrium

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$\frac{\partial f_1}{\partial x_1} = 1$

$\frac{\partial f_1}{\partial x_2} = 2\bar{x}_2 = 0$

$\frac{\partial f_2}{\partial x_1} = 3\bar{x}_1^2 = 0$

$\frac{\partial f_2}{\partial x_2} = 1$

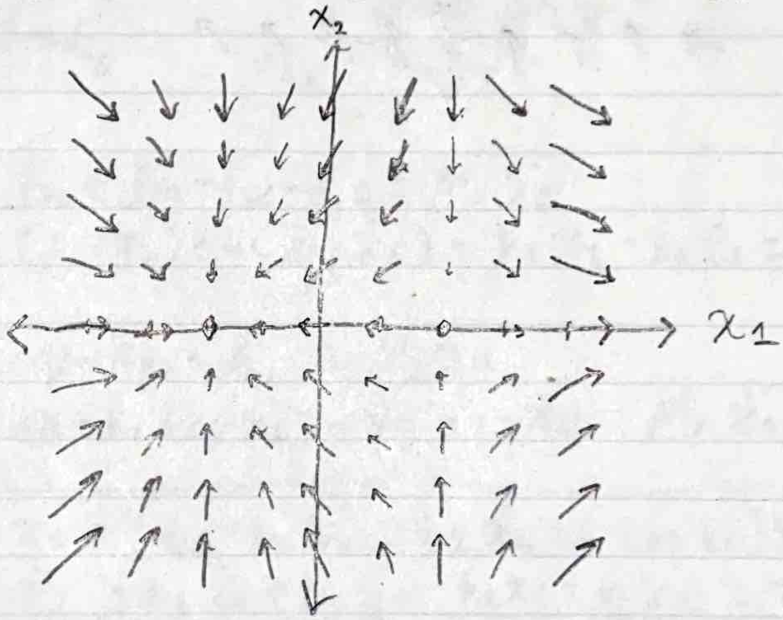
$J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ at equillibrium
because there is no parameter μ that can cause the system to change from a sink to a source, no periodic orbit exists

7) $\begin{cases} \dot{x}_1 = \beta + x_1^2 \\ \dot{x}_2 = -x_2 \end{cases}$

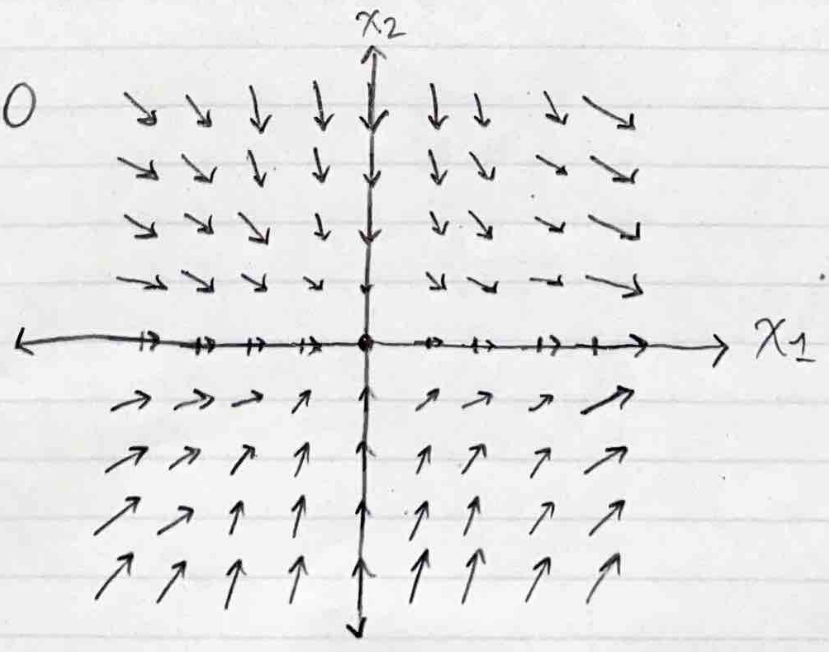
equilibrium points:

$\beta = -1$	$\beta = 0$	$\beta = 1$
$(\pm 1, 0)$	$(0,0)$	$(\pm i, 0)$
nonhyperbolic	hyperbolic	nonhyperbolic

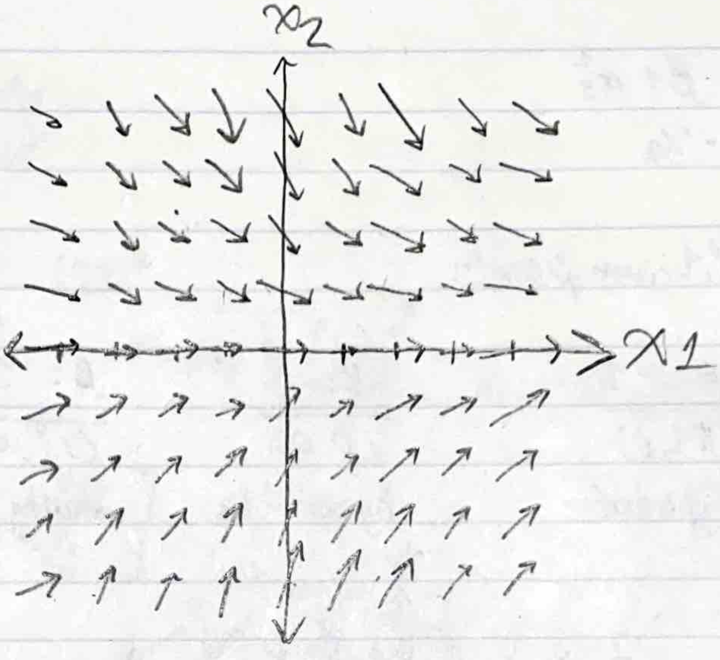
$\beta = -1$



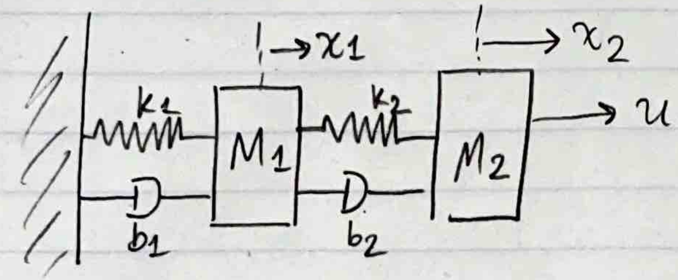
$\beta = 0$



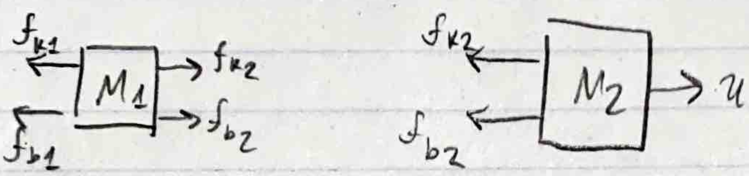
7 cont)



8)



a) Derive EOMs



$$\Sigma F_{M_1} = f_{k2} + f_{b2} - f_{k1} - f_{b1} = M_1 \ddot{x}_1$$

$$k_2(x_2 - x_1) + b_2(\dot{x}_2 - \dot{x}_1) - k_1 x_1 - b_1 \dot{x}_1 = M_1 \ddot{x}_1$$

$$\Sigma F_{M_2} = u - f_{k2} - f_{b2} = M_2 \ddot{x}_2$$

$$u - k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1) = M_2 \ddot{x}_2$$

$$\boxed{\begin{cases} M_1 \ddot{x}_1 + (b_1 + b_2)\dot{x}_1 - b_2 \dot{x}_2 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \\ M_2 \ddot{x}_2 - b_2 \dot{x}_1 + b_2 \dot{x}_2 - k_2 x_1 + k_2 x_2 = u \end{cases}}$$