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MECH 6970: Fundamentals of GPS
Homework 1

Problem 1

Chapter 1, Problem 3

Given: An aircraft moving at constant velocity 360 km/h and constant altitude y_0 broadcasts 100 MHz tone. 3 Doppler shifts are measured on the ground 0.1s apart: -33.1679 Hz, -33.1711 Hz, and -33.1743 Hz.

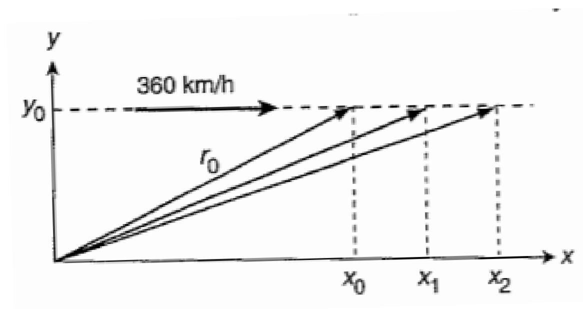

Figure 1: Pseudorange Diagram

Find: Range rates in m/s for each received Doppler shift, 2 linear equations to relate x_1 and x_2 to x_0, and 2 nonlinear equations to relate x_0 and y_0 to the measurements.

Solution: The Doppler shifts were calculated using

$$(f_R - f_T) = -\frac{\dot{r}}{\lambda} \tag{1}$$

where f_R is the received frequency, f_T is the transmitted frequency, $\dot{r}$ is the pseudorange rate, and λ is wavelength which is defined by

$$\lambda = \frac{c}{f_T} \tag{2}$$

where c is the speed of light in m/s. The wavelength of the transmitted signal was determined to be 3 m.

The following is a sample calculation of the pseudorange rates using a rearranged Eq. 1

$$\dot{r_0} = (-33.1679\,\mathrm{Hz})(3\,\mathrm{m}) = 99.5037\,{}^{\mathrm{m}}\!/\mathrm{s}$$

The remaining pseudorange rate values were calculated in the same manner and were determined to be $\dot{r_1} = 99.5133\,{}^{\mathrm{m}}\!/\mathrm{s}$ and $\dot{r_2} = 99.5229\,{}^{\mathrm{m}}\!/\mathrm{s}$.

The linear equations that relate x_1 and x_2 to x_0 were reasoned by establishing a Δx that is a function of the aircraft's constant velocity and the observer's sampling period

$$\Delta x = v\Delta t$$
$$\Delta x = (360\,\mathrm{km/h})(\frac{1000}{1}\,\mathrm{m/km})(\frac{1}{3600}\,\mathrm{h/s})(0.1\,\mathrm{s})$$
$$\Delta x = 10\,\mathrm{m}$$

Considering this, the equations for x_1 and x_2 are just a multiple of Δx added to x_0

$$x_1 = x_0 + \Delta x \tag{3}$$

$$x_2 = x_0 + 2\Delta x \tag{4}$$

The nonlinear equations that relate x_0 and y_0 to the measurements were derived by taking the derivative of the geometric relationships (Pythagorean Theorem) between the pseudoranges, x_0 and y_0. The following is the derivation for $\dot{r_0}$

$$r_0^2 = x_0^2 + y_0^2$$
$$\frac{d}{dt}(r_o^2) = \frac{d}{dt}(x_0^2 + y_0^2)$$
$$2r_0\dot{r_0} = 2x_0\dot{x_0}$$
$$\dot{r_0} = \frac{x_0\dot{x_0}}{r_0}$$

where x_0 is the aircraft's constant velocity and r_0 can be solved for using the Pythagorean Theorem. The corresponding equations for $\dot{r_1}$ and $\dot{r_2}$ are derived in the same manner with Eqs. 3-4 being substituted into the Pythagorean Theorem for r_1 and r_2, respectively.

The derivation above for each pseudorange rate provides 3 nonlinear equations, but any two can be chosen to solve for x_0 and y_0. The nonlinear equations for $\dot{r_1}$ and $\dot{r_2}$ are as follows

$$\dot{r_1} = \frac{(x_0 + \Delta x)\dot{x_1}}{\sqrt{(x_0 + \Delta x)^2 + y_o^2}} \tag{5}$$
$$\dot{r_2} = \frac{(x_0 + 2\Delta x)\dot{x_2}}{\sqrt{(x_0 + 2\Delta x)^2 + y_o^2}} \tag{6}$$

Eqs. 5-6 were solved in MATLAB using the nonlinear systems of equations solver `fsolve()`. x_0 and y_0 were determined to be 981.7 m and 98.2 m, respectively.

Chapter 1, Problem 4

Given: An observer is constrained to a line between two pseudolites PL1 and PL2, which are separated by 1000 m. Assume the pseudolites' clocks are perfectly synchronized but the observer's is not.

Find: Determine the observer's position and clock bias given PL1 and PL2 pseudoranges are (a) 550 m and 500 m, respectively, and (b) 400 m and 1400 m, respectively.

Solution: The equation used to determine pseudoranges in a one-dimensional situation is

$$\rho^{(k)} = \sqrt{(x^{(k)} - x)^2} - b \tag{7}$$

where k is a satellite, $x^{(k)}$ is the position of satellite k, x is the observer's position, and b is the observer's clock bias.

Part (a) was solved by substituting the pseudorange and satellite position into Eq. 7 for both PL1 and PL2. Then, the 2 equations were rearranged to solve for x and b

$$\begin{gathered}
\rho^{(PL1)} = 550 = \sqrt{(0 - x)^2} - b \\
\rho^{(PL2)} = 500 = \sqrt{(1000 - x)^2} - b \\
x = 550 + b \\
500 = \sqrt{(1000 - 550 - b)^2} - b \\
500 = 450 - 2b \\
b = -25\,\mathrm{m} \\
x = 525\,\mathrm{m}
\end{gathered}$$

The clock bias was converted to units of time using the following

$$b_{time} = \frac{b_{meters}}{c} \tag{8}$$

where c is the speed of light in $\mathrm{m/s}$.

This means b for part (a) is $-8.333\mathrm{e}-8\,\mathrm{s}$ or the observer's clock is $8.333\mathrm{e}-8\,\mathrm{s}$ fast.

Part (b) was solved by substituting the pseudorange and satellite position into Eq. 7 for both PL1 and PL2. Then, the 2 equations were rearranged to solve for x and b

$$\begin{gathered}
\rho^{(PL1)} = 400 = \sqrt{(0 - x)^2} - b \\
\rho^{(PL2)} = 1400 = \sqrt{(1000 - x)^2} - b \\
x = 400 + b \\
1400 = \sqrt{(1000 - 400 - b)^2} - b \\
1400 = 600 - 2b \\
b = -400\,\mathrm{m} \\
x = 0\,\mathrm{m}
\end{gathered}$$

The clock bias was converted to units of time using Eq. 8 and determined to be $-1.333\mathrm{e}-6\,\mathrm{s}$ or the observer's clock is $1.333\mathrm{e}-6\,\mathrm{s}$ fast.

Problem 2

Given: Generate two random sequences that are 100 long and randomly comprised of +1 and -1. BONUS: Repeat with 1000 long sequences.

Part A

Find: Plot the histogram of each sequence.

Solution: The 2 random signals were generated in MATLAB using `2*ceil(0.1*randn(100,1))-1` and the `histogram()` function was used to plot the histogram of each sequence below

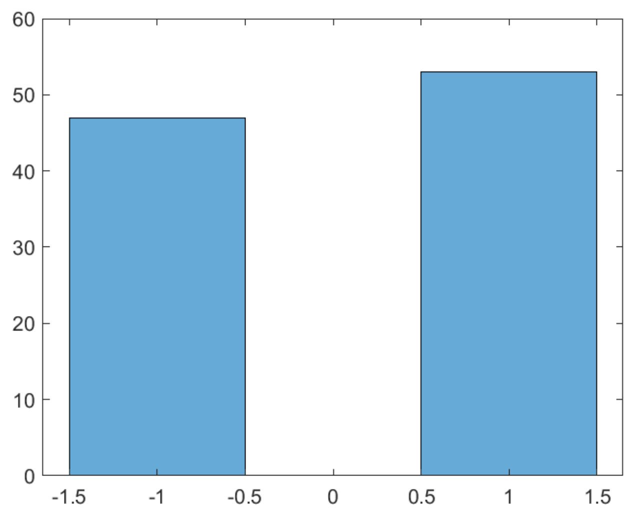

Figure 2: Random Signal 1 Histogram (100 Samples)

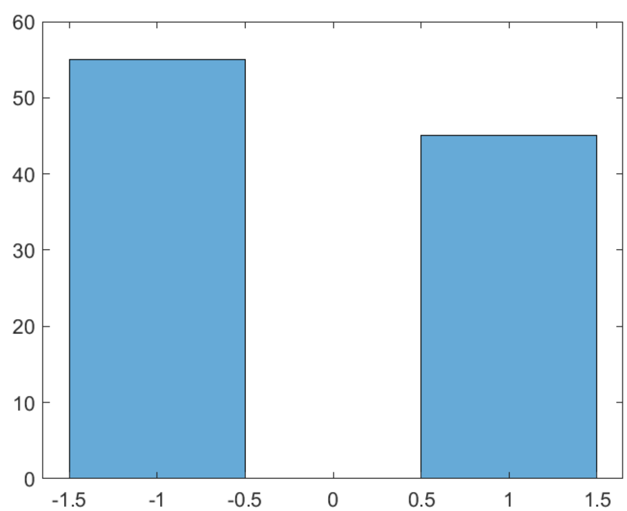

Figure 3: Random Signal 2 Histogram (100 Samples)

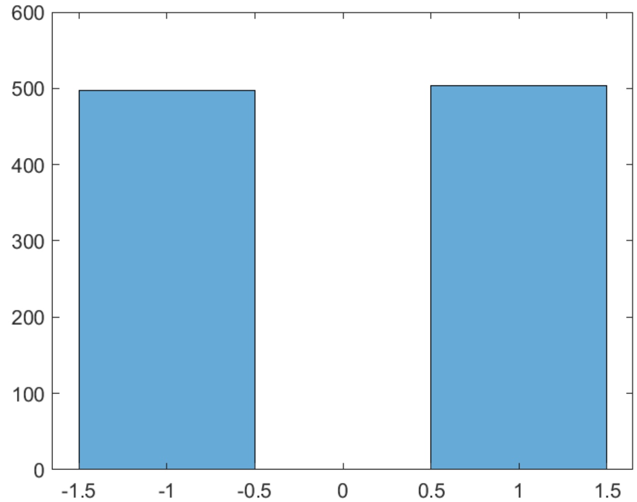

Figure 4: Random Signal 1 Histogram (1000 Samples)

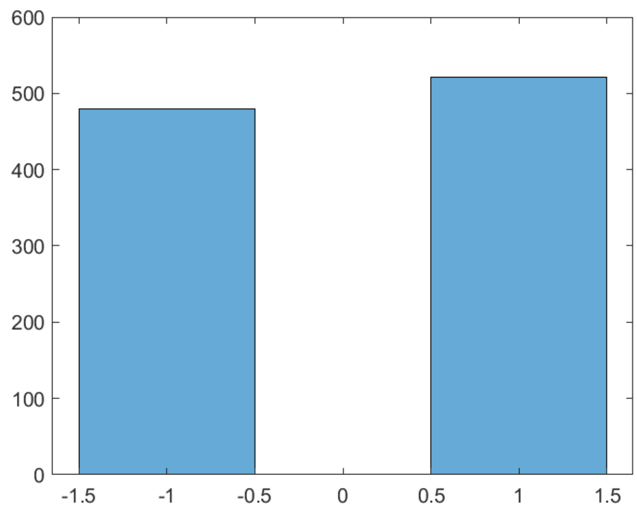

Figure 5: Random Signal 2 Histogram (1000 Samples)

Part B

Find: Plot the spectral analysis of each sequence.

Solution: The `periodogram()` function in MATLAB was used to plot the spectral density of each sequence below

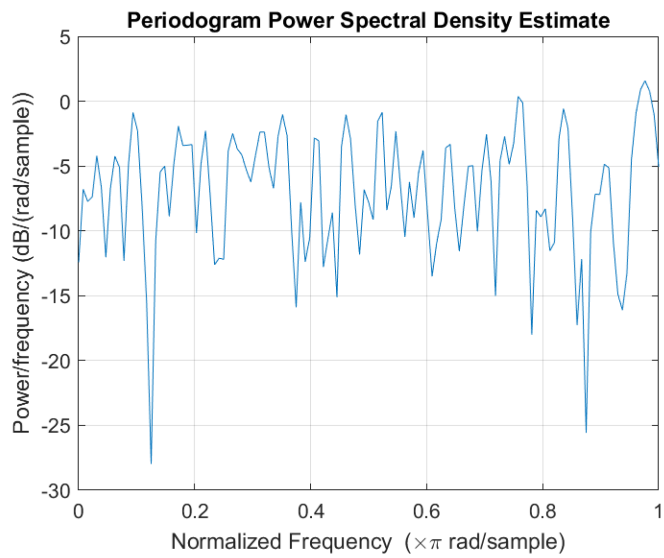

Figure 6: Random Signal 1 Spectral Density (100 Samples)

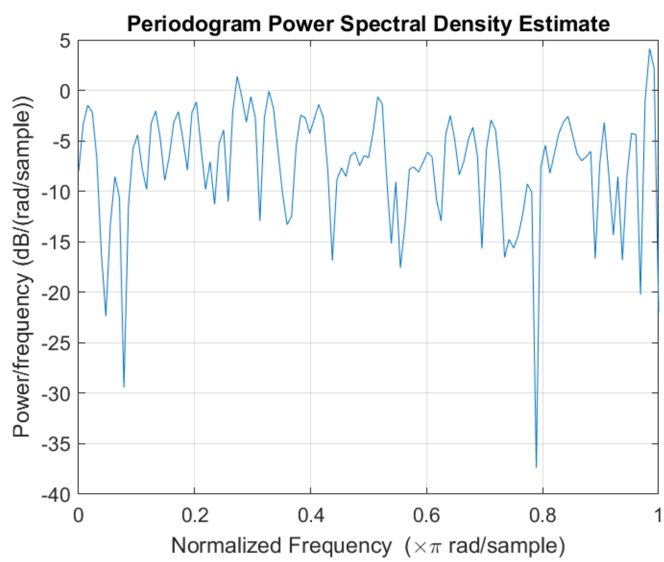

Figure 7: Random Signal 2 Spectral Density (100 Samples)

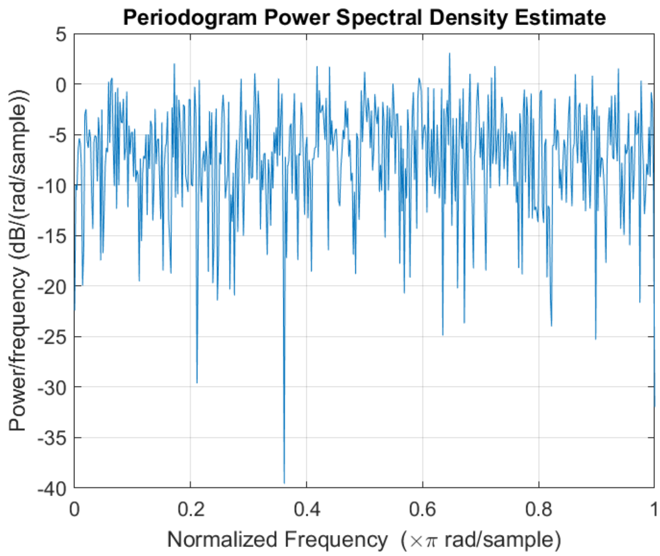

Figure 8: Random Signal 1 Spectral Density (1000 Samples)

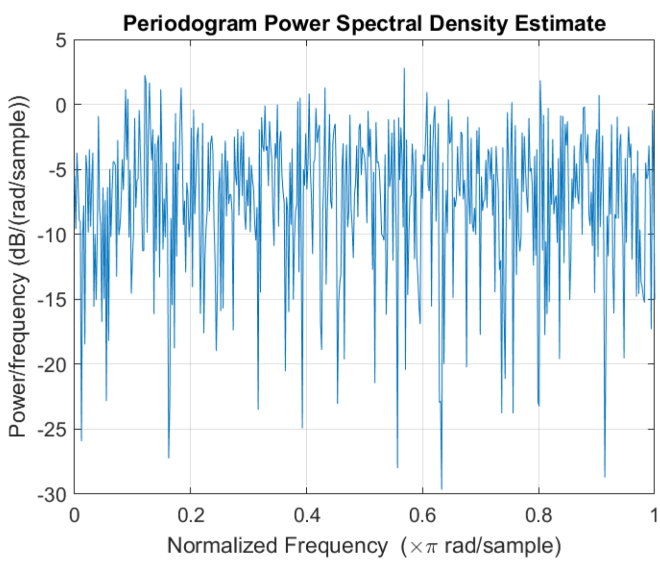

Figure 9: Random Signal 2 Spectral Density (1000 Samples)

Part C

Find: Plot the autocorrelation of each sequence.

Solution: The `autocorr()` function in MATLAB was used to plot the autocorrelation of each sequence below

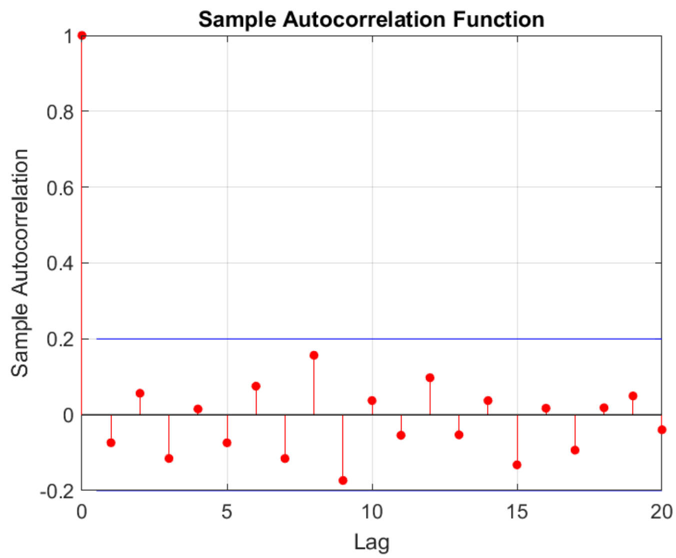

Figure 10: Random Signal 1 Autocorrelation (100 Samples)

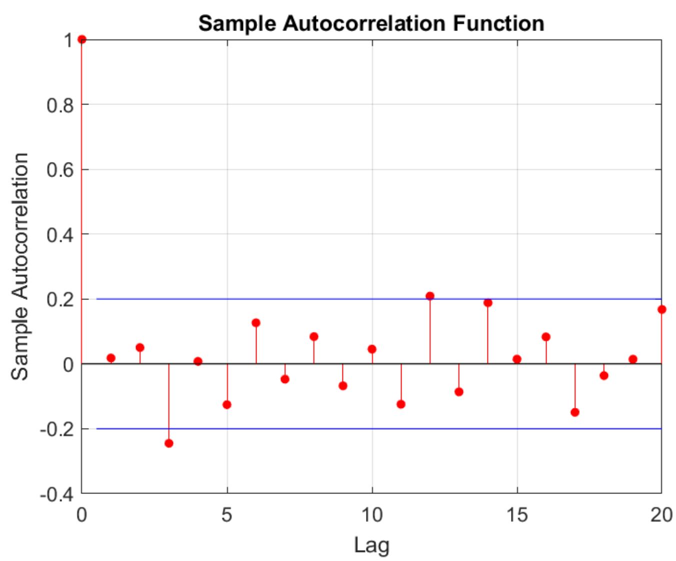

Figure 11: Random Signal 2 Autocorrelation (100 Samples)

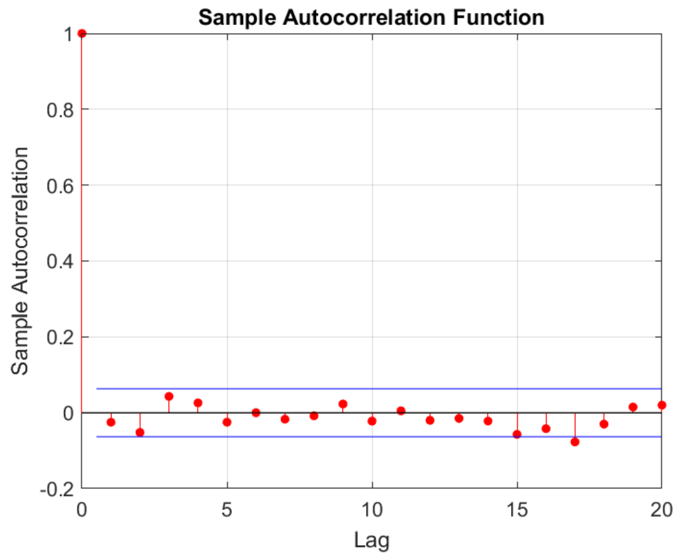

Figure 12: Random Signal 1 Autocorrelation (1000 Samples)

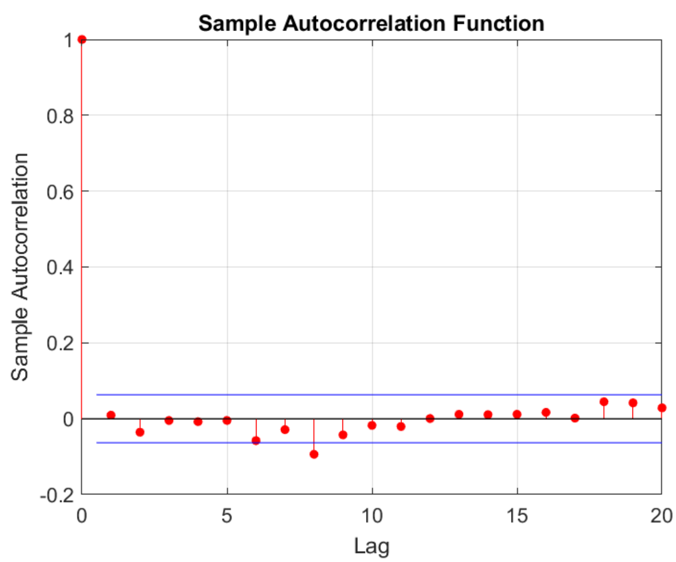

Figure 13: Random Signal 2 Autocorrelation (1000 Samples)

Part D

Find: Plot the crosscorrelation of the sequences.

Solution: The `crosscorr()` function in MATLAB was used to plot the crosscorrelation of each sequence below

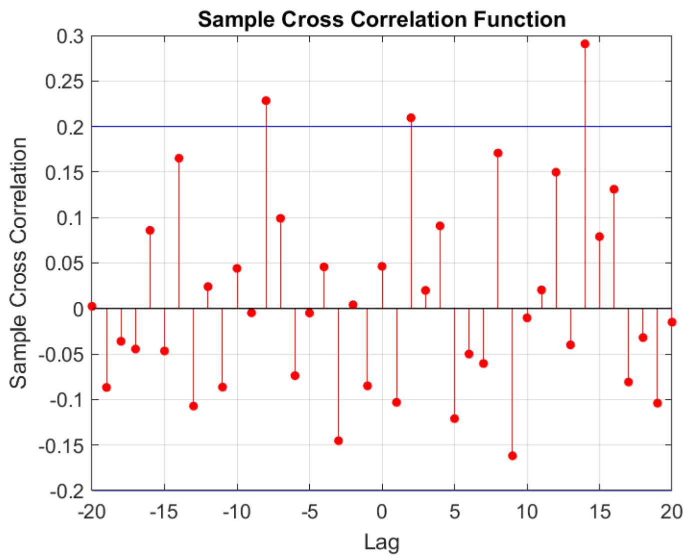

Figure 14: Signal Crosscorrelation (100 Samples)

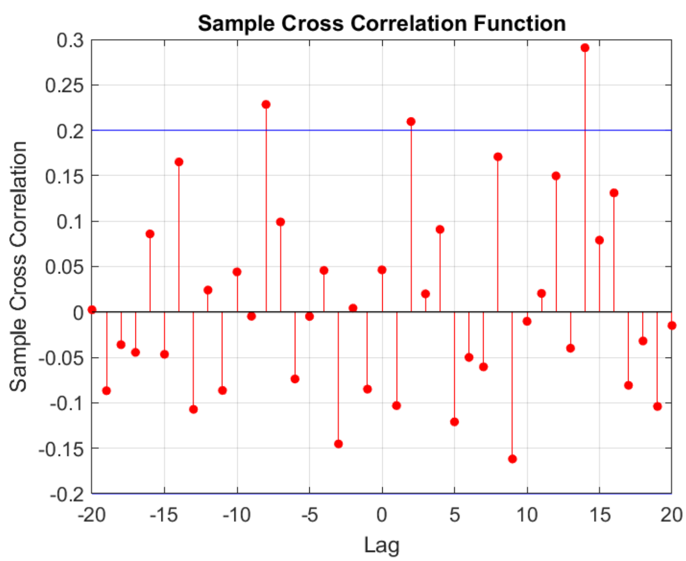

Figure 15: Signal Crosscorrelation (1000 Samples)

Problem 3

Given: Generate 3 1000 long sequences in MATLAB using:

```
A = 3+3*randn(1000,1)
B = 5+5*randn(1000,1)
C = A+B
DATA = [A   B   C]
```

Part A

Find: The mean and variance of A, B, and C.

Solution: The mean of each random signal was found using the `mean()` function in MATLAB. The mean of A was 3.1148, the mean of B was 5.1366, and the mean of C was 8.2515.

The variance of each random signal was found using the `std()` function in MATLAB. The variance of A was 3.0164, the variance of B was 4.9768, and the variance of C was 5.8349.

Part B

Find: The mean of DATA.

Solution: The mean of DATA was found using the `mean()` function in MATLAB with the `'all'` flag. The mean of DATA was 5.5010.

Part C

Find: The covariance matrix of DATA.

Solution: The covariance matrix of DATA was found using the `cov()` function in MATLAB. The covariance matrix was

$$COV_{DATA} = \begin{bmatrix} 9.0988 & 0.0892 & 9.1881 \\ 0.0892 & 24.7683 & 24.8575 \\ 9.1881 & 24.8575 & 34.0456 \end{bmatrix}$$

Problem 4

Find: The Taylor Series linearized approximation of

$$r(x, y) = \sqrt{(x-a)^2 + (y-b)^2} \tag{9}$$

Solution: The 1^{st} degree Taylor Series linearized approximation of multivariate functions is

$$f(x, y) \approx f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$$

where (a, b) is the point the approximation is centered around.

Assuming (a, b) is (x_0, y_0), the following comprise the 1^{st} degree Taylor Series approximation of Eq. 9

$$\begin{aligned}
f(x_0, y_0) &= \sqrt{(x_0 - a)^2 + (y_0 - b)^2} \\
f_x(x_0, y_0) &= \frac{x_0 - a}{\sqrt{(x_0 - a)^2 + (y_0 - b)^2}} \\
f_y(x_0, y_0) &= \frac{y_0 - b}{\sqrt{(x_0 - a)^2 + (y_0 - b)^2}}
\end{aligned}$$

Combining these provides the following approximation

$$f(x, y) \approx \sqrt{(x_0 - a)^2 + (y_0 - b)^2} + \frac{x_0 - a}{\sqrt{(x_0 - a)^2 + (y_0 - b)^2}}(x - x_0) + \frac{y_0 - b}{\sqrt{(x_0 - a)^2 + (y_0 - b)^2}}(y - y_0) \tag{10}$$