

# Optimal Estimation - Homework 3

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## Problem 1

Kalman Filter at its best – simulation (actually the Kalman filter is also quite reliable when we have an excellent model and low noise sensors). Suppose we have a 2nd order system that we are regulating about zero (position and velocity) by wrapping an “optimal” control loop around the system. The new dynamics of the continuous time system are given by the closed-loop A matrix:

$$A_{CL} = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}$$

Suppose our measurement is simply position ( $C=[1 \ 0]$ ). There is a white noise process disturbance (force,  $Bw=[0 \ 1]T$ ) acting on the controlled system.

**Find:**

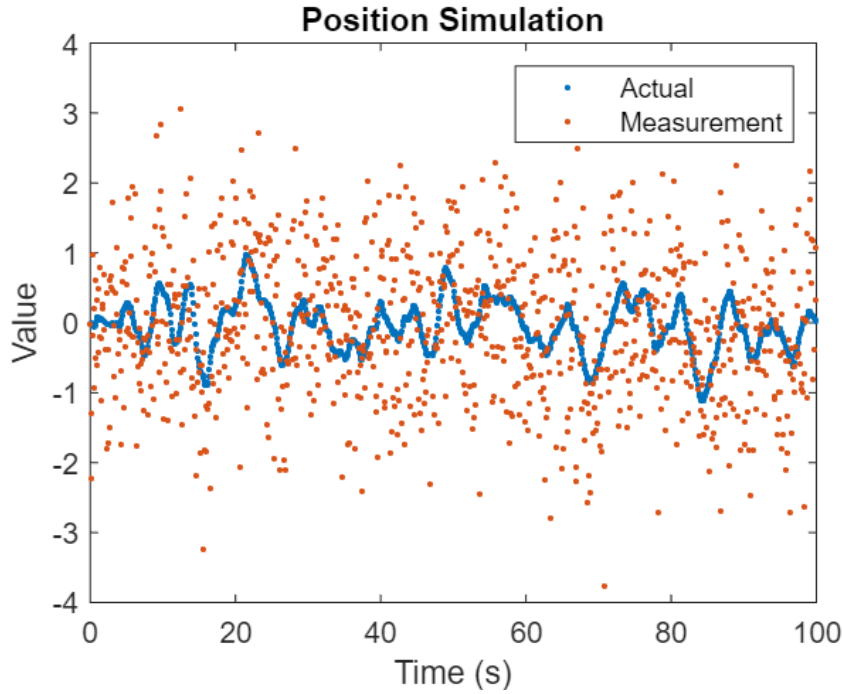
- **Part A:** Simulate the controlled system with the disturbance force ( $1\sigma=2$ ) and a sampled sensor noise ( $1\sigma=1$ ) for 100 seconds at a 10 Hz sample rate.
- **Part B:** What is  $Q$ ,  $Q_D$ , and  $R_D$ ?
- **Part C:** Calculate the steady state Kalman gain for the system. This can be done in one of many ways: iterate the kalman filter until it converges, `dlqe.m`, `dare.m`, `kalman.m`, `dlqr.m` (+ predictor to current estimator trick), etc. What is the steady state covariance of the estimates after the time update,  $P^-$ , as well as after the measurement update,  $P^+$ . Where are the poles of the estimator?
- **Part D:** Now use the steady state kalman filter to generate an estimate ( $\hat{x}$  and  $\dot{\hat{x}}$ ) of the 2 states over time. Calculate the norm of the standard deviation of the errors for each state.

$$N = \sqrt{(std(\dot{x} - \dot{\hat{x}}))^2 + (std(x - \hat{x}))^2}$$

- **Part E:** Change the ratio of the  $Q$  and  $R_D$  weights in the Kalman filter design (and repeat part d with the new Kalman gain but DO NOT regenerate a new  $x$  and  $\dot{x}$ ) and determine the effect on the estimation errors. For what ratio of  $Q$  to  $R_D$  are the errors minimized? Note: Often in practice we do not know the actual  $Q$  and  $R_D$  so these tend to be “tuning” parameters we can use to tune our filter. However, according to Kalman the estimation errors are only minimized if we use the  $Q$  and  $R_D$  of the physical system.

**Solution:**

- **Part A:** The system was simulated with the following code.



- **Part B:**  $Q$  is equivalent to  $E[ww^T]$  which is the variance associated with the process noise. Therefore,  $Q = 4$ .  $Q_D$  is equivalent to  $B_w Q B_w^T \delta t$ . Therefore,  $Q_D$  is:

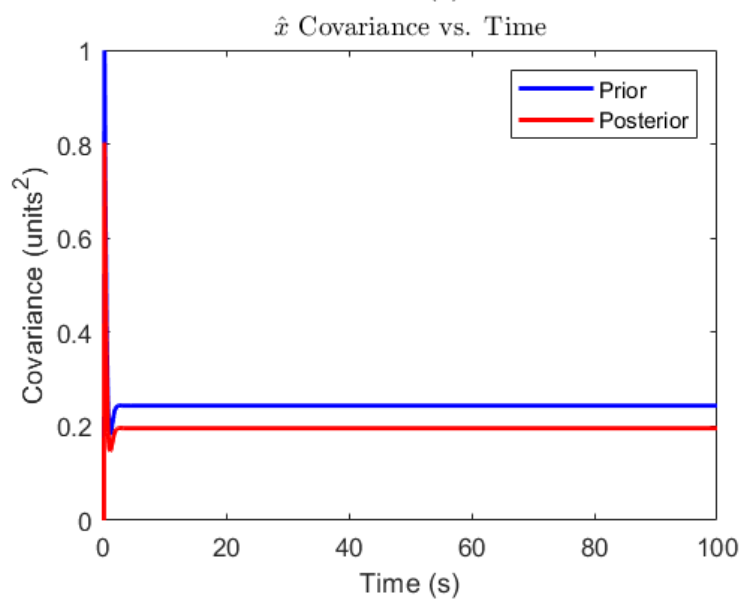
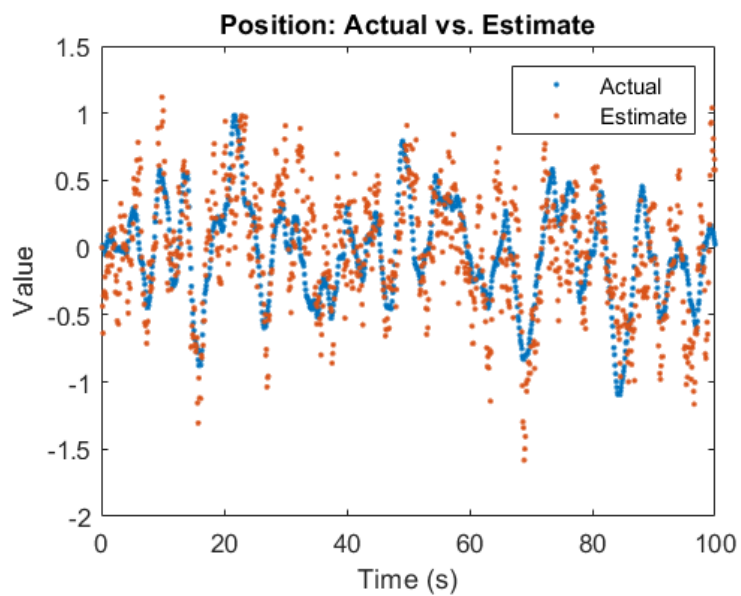
$$Q_D = \begin{bmatrix} 0 & 0 \\ 0 & 0.4 \end{bmatrix}$$

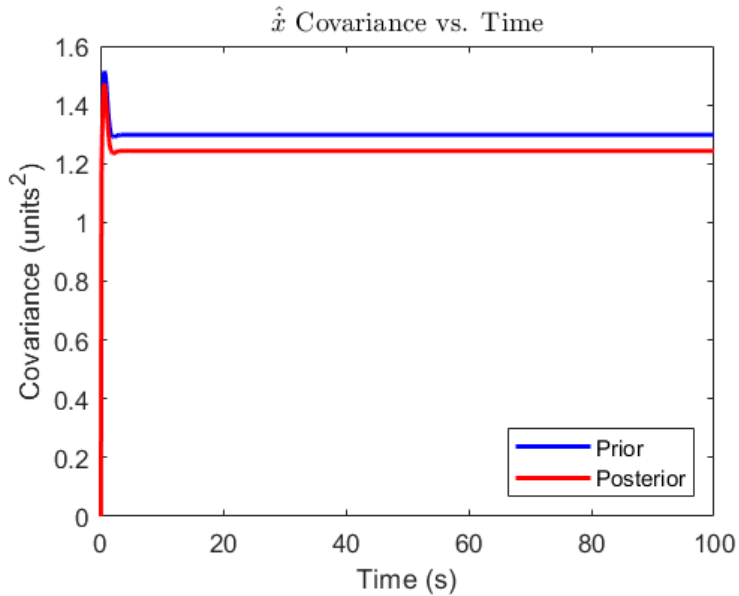
$R_D$  is equivalent to the measurement noise variance. Therefore,  $R_D = 1$ .

- **Part C:** The steady-state Kalman gain was determined by creating the Kalman filter and recording the value of the gain when it converged.

$$K_{ss} = \begin{bmatrix} 0.1959 \\ 0.2094 \end{bmatrix}$$

The following plots display estimate along with the prior and posterior covariance over time for  $\hat{x}$  and  $\hat{\dot{x}}$ . The estimate is very noisy because of the low SNR and the current ratio between process and measurement covariance.





- **Part D:**
- **Part E:**

## Problem 2

Download the data hw3\_2 from the website. The data is in the form  $[t \ y]$ . Suppose we want to design an estimator to estimate the bias in the measurement  $y$ . We believe that the bias ( $x$ ) is constant so we use the model given by:

$$\dot{x} = 0$$

$$y_k = x_k + \nu_k$$

$$\nu \sim N(0, 1)$$

### Find:

- **Part A:** Run the Kalman filter estimator with  $Q_D = 0$ . What happens at  $t > 100$  seconds? Why? Calculate the steady state Kalman gain  $L_{SS}$ . Plot  $L(k)$ . This is known as the filter “going to sleep” (becomes a least squares estimator).
- **Part B:** To offset this problem we will “tune”  $Q_D$  to track the bias. What is the effect of changing  $Q_D$  on the ability to track the step change in the bias? (try values of  $Q_D$  from 0.0001 to 0.01 and plot  $L(k)$  as well as the estimate of the bias ( $\hat{x}$ ). What is the tradeoff?
- **Part C:** Now filter the measurement using the first order low-pass filter

$$H(z) = \frac{\sqrt{Q_D}}{z - (1 - \sqrt{Q_D})}$$

- **Part D:** How does this compare to the Kalman filter solution?

## Solution:

- Part A:
- Part B:
- Part C:
- Part D:

## Problem 3

Design a “Navigation” type Kalman filter to estimate the states [East, North, Radar\_Bias, Psi, Gyro\_Bias]. (Note: this is a non-linear problem that requires an Extended Kalman Filter (EKF) to do correctly. However we can solve the problem in one of two ways: (i) linearize the equations about the nominal operating point and produce a constant  $A$  matrix for that operating point, or (ii) simply update the  $A$  matrix at every time step with our measurements or estimates. Download the data hw3\_3 from the website and run the filter sampled at 5 Hz.

## Find:

- **Part A:** How did you choose the covariance values for  $Q_D$  (especially for the radar and gyro biases).
- **Part B:** How does the bias estimation compare to a Least Squares Solution. How does the bias estimate compare to the Recursive Least Squares solution if you make the covariance ( $Q_d$ ) of the bias estimates equal to zero.
- **Part C:** Integrate the last 40 seconds of data to see how well you have estimated the biases. This can simply be done by “turning off” the measurements in the observation matrix! Why do the bias estimates remain constant during the 40 second “outage”?

## Solution:

- Part A:
- Part B:
- Part C:

## Problem 4

Estimator for Vehicle Dynamics. The yaw dynamics of a car (for a stability control system) can be described by the following model (at 25 m/s):

$$\dot{x} = \begin{bmatrix} -2.62 & 12 \\ -0.96 & -2 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \beta \end{bmatrix} + \begin{bmatrix} 14 \\ 1 \end{bmatrix} \delta$$

$$y_k = [1 \quad 0]x_k + \nu_k$$

where:

$$\dot{\psi} = \text{Vehicle Yaw Rate}$$

$\beta$  = Vehicle Side Slip Angle

$\delta$  = Steer Angle

$\nu_k$  = Sample Sensor Noise

### Find:

- **Part A:** Assuming we can only measure the yaw rate ( $\nu_k \sim N(0, 0, 1^2)$ ), design a Kalman filter to do full state estimation (select a reasonable  $Q_D$ ). Provide a unit step steer input and estimate both states. On one page plot the actual states and estimated states (use subplot(2,2,n) for each of the two states). Where are the steady state poles of the estimator?

Now, somebody has loaded the trunk of the vehicle with bricks, changing the CG of the vehicle so now the actual model (at 25 m/s) is:

$$\dot{x} = \begin{bmatrix} -2.42 & 4 \\ -0.99 & -2 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \beta \end{bmatrix} + \begin{bmatrix} 18 \\ 1 \end{bmatrix} \delta$$

$$y_k = [1 \quad 0]x_k + \nu_k$$

- **Part B:** Redo Part A. Can you estimate the slip angle correctly? Try different  $Q_D$ .
- **Part C:** Now lets say we have a noisy measurement of the slip angle ( $\eta_k \sim N(0, 0.5^2)$ ).

$$y_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \nu_k \\ \eta_k \end{bmatrix}$$

Assuming the sensor noises are uncorrelated, what is  $R$ ?

- **Part D:** Redo Part A. What is the effect of changing the element of  $Q_D$  associated with the slip angle estimate? What must  $Q_D$  equal to ensure an unbiased estimate of the states (How much filtering does that provide)?

### Solution:

- **Part A:**
- **Part B:**
- **Part C:**

$$R = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.25 \end{bmatrix}$$

- **Part D:**

