Project 2 - MECH 7790

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Part II: Group Portion

Problem 3: LQR State Feedback Regulation

a) Design the controller for the linear model.

Q, R, and K were designed using the following methodology. Q was designed using Bryson's rule to unit scale the max θ requirement of 0.61 rad and an arbitrary maximum x (set at 0.55 m) to limit overshoot. The derivatives of the states in question were set to a value of 1 as their weighting is inherently captured within the θ and x states.

$$Q = \begin{bmatrix} 3.31 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2.69 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

R was set to a value of 1 as there is only 1 input, meaning there is more value in scaling Q relative to R given the performance of your LQR controller is determined by the ratio between the two.

Using `icare()` in MATLAB, the controller K was determined to be the following:

$$K = \begin{bmatrix} -1.82 & -2.70 & 22.32 & 4.30 \end{bmatrix}$$

`icare()` solves the Algebraic Riccati Equation ($A^T P + PA + Q - PBR^{-1}B^T P = 0$) for P which is used in determining the controller K with the following:

$$K = R^{-1}B^TP$$

The following figures depict the outputs and input of the linear system with the above controller:

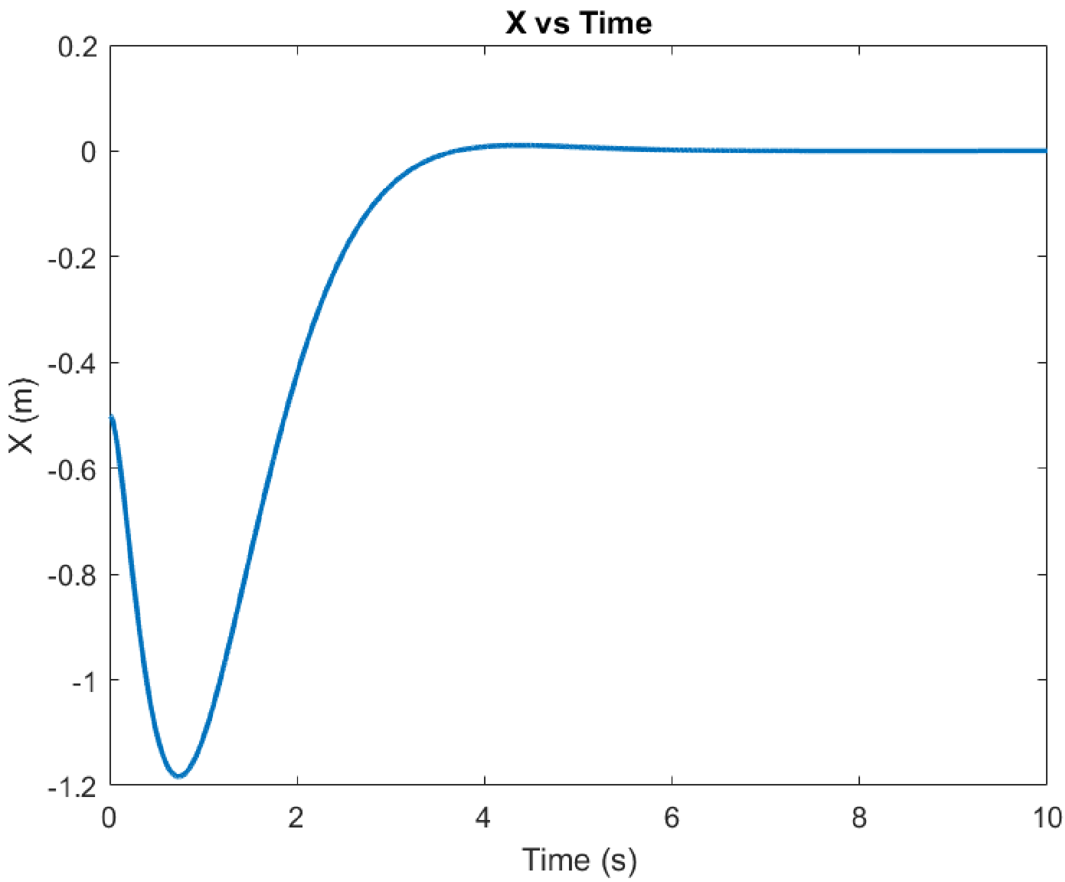

Figure 1: Linear System X Output

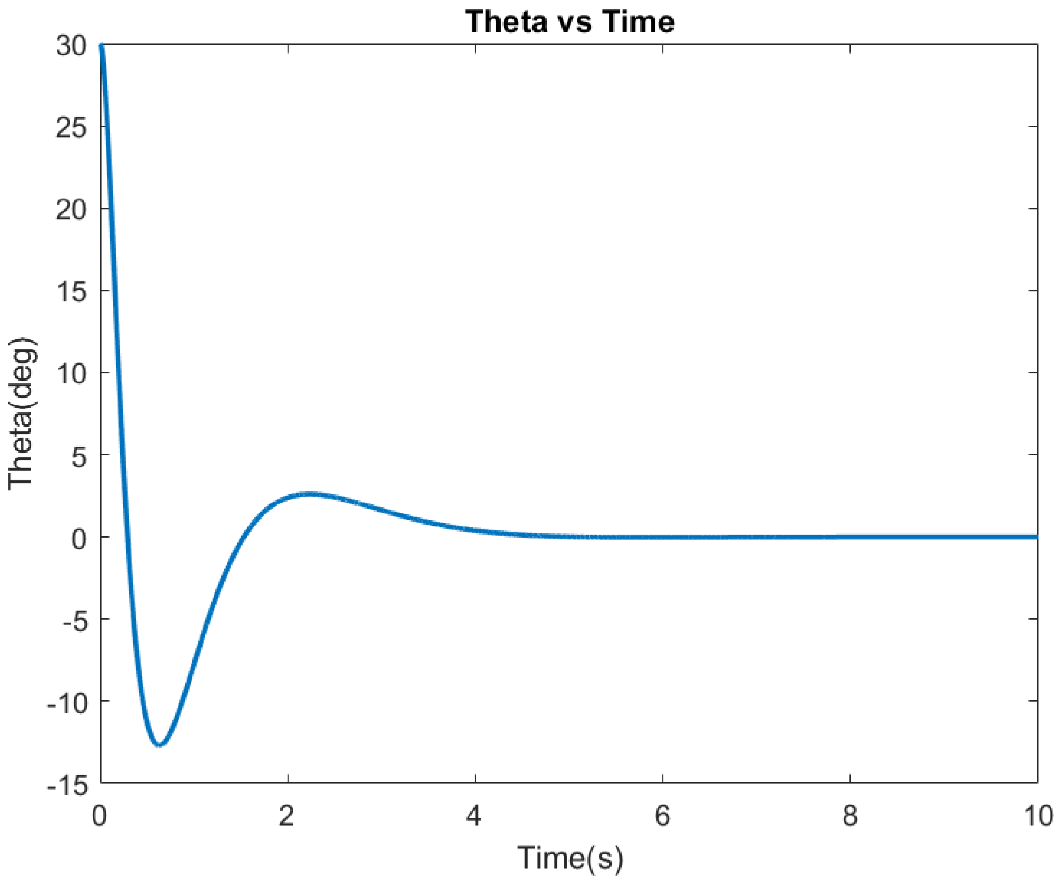

Figure 2: Linear System θ Output

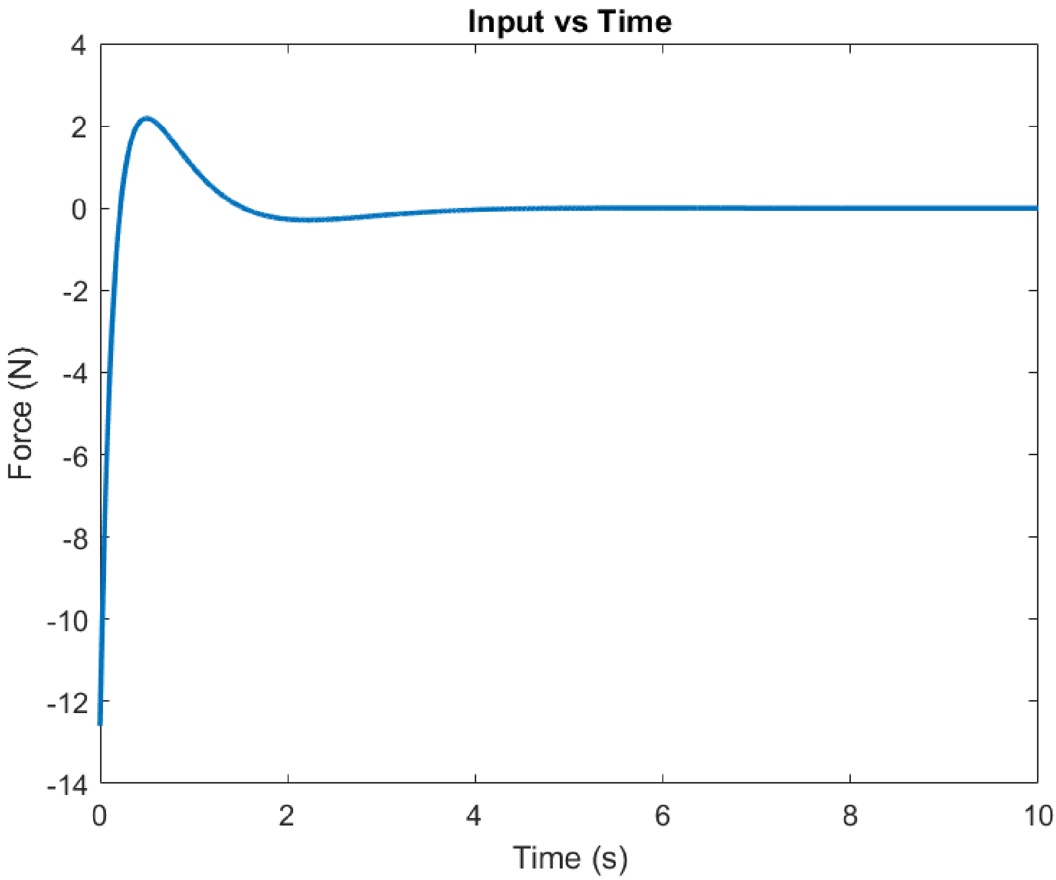

Figure 3: Linear System Input

The closed-loop eigenvalues are also found in the result of `icare()`, but it is simply from $det(sI - (A - BK)) = 0$. They are the following:

$$s = -1.32 \pm 0.88, -3.85, -8.34$$

b) Design the controller for the nonlinear model.

The Q, R, and K used for the nonlinear system are the same as for the linear model.

The following figures depict the outputs and input of the nonlinear system with the above controller:

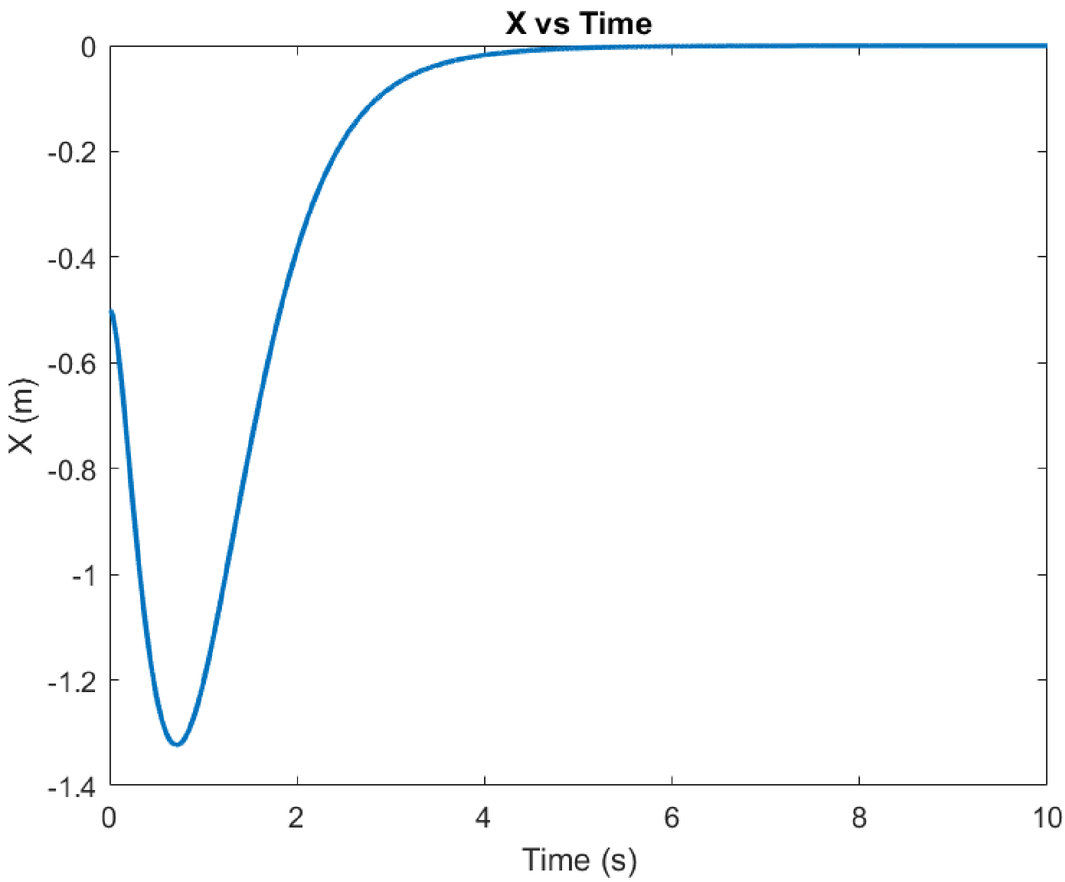

Figure 4: Nonlinear System X Output

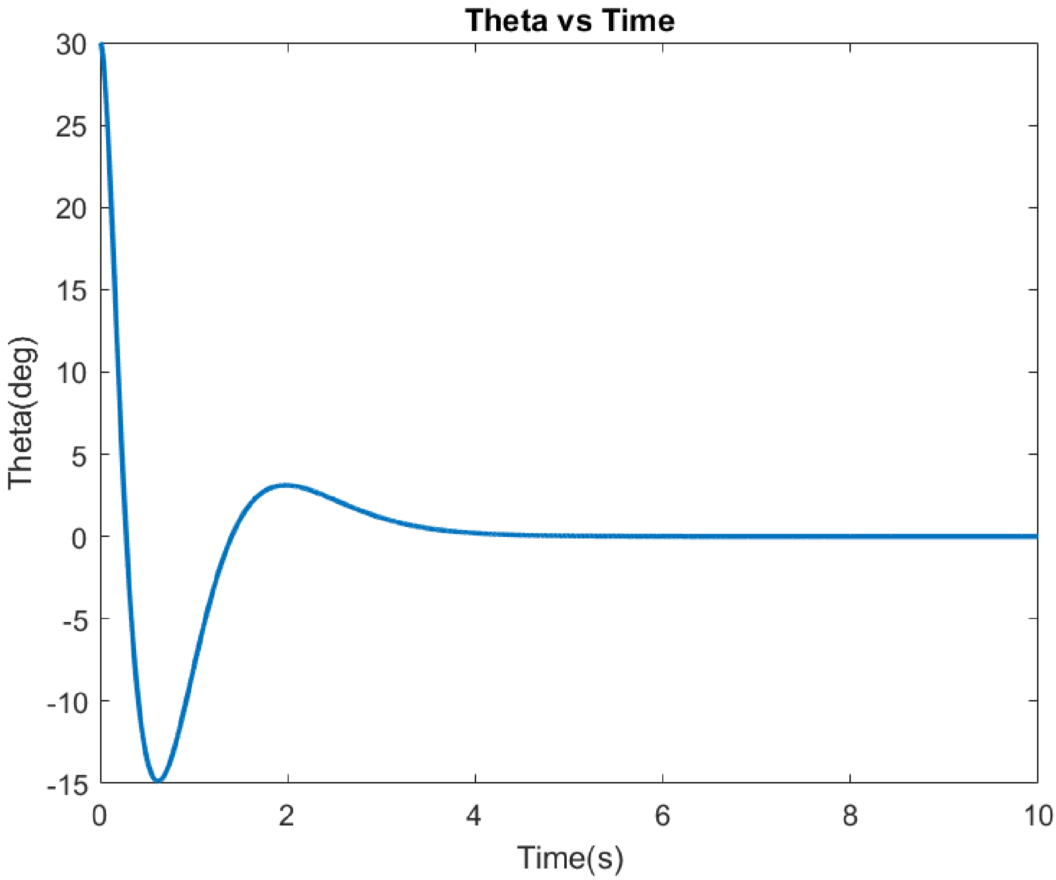

Figure 5: Nonlinear System θ Output

c) Discuss the differences between tuning the model for the linear and non-linear model.

For our particular case, there was no difference in tuning as the Q and R values for the linear model were able to produce a K that yielded stable results that met the design requirements for the nonlinear model. The settling time of the nonlinear response is slightly slower than that for the linear model as expected, but it is still within the performance requirements. If our initial tune was not successful, we would have likely needed to increase our values within Q to account for the nonlinear dynamics.

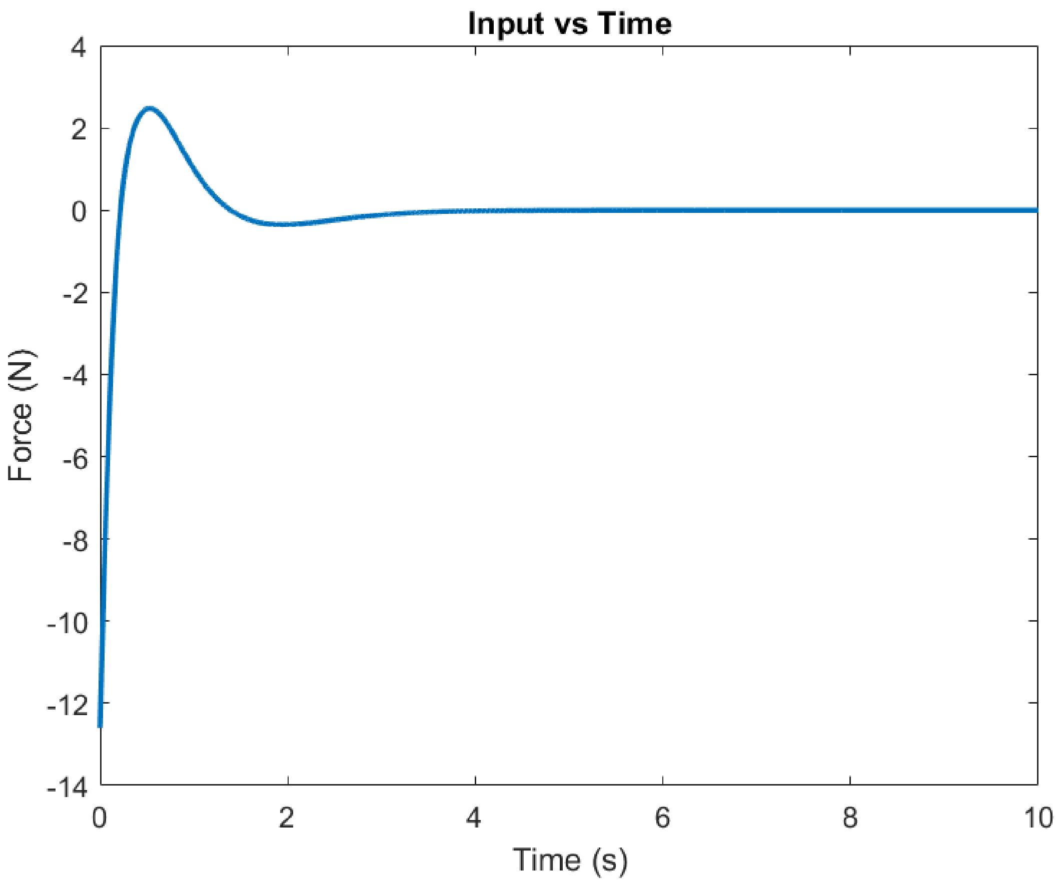

Figure 6: Nonlinear System Input

Problem 4: LQR State Feedback Regulation

The following table outlines the 5 different combinations of Q and R with the corresponding K we tested on the linear model. The values for Q in the table indicate the diagonal terms with the off-diagonal terms being 0. The Q for Combination 1 was derived using $Q = C^T Q_1 C$.

Table 1: Q & R Combinations with Resulting K & s

#	Q	R	K	s
1	$diag\left(\begin{bmatrix}3.31\\0\\2.69\\0\end{bmatrix}\right)$	1	$\begin{bmatrix}-1.82 & -2.38 & 20.85 & 3.89\end{bmatrix}$	-1.14 ± 1.11 -5.62 ± 0.67
2	$diag\left(\begin{bmatrix}33.06\\10\\26.87\\10\end{bmatrix}\right)$	1	$\begin{bmatrix}-5.75 & -6.92 & 38.02 & 7.96\end{bmatrix}$	-1.96 -2.37 ± 1.44 -17.07
3	$diag\left(\begin{bmatrix}3.31\\1\\2.69\\1\end{bmatrix}\right)$	10	$\begin{bmatrix}-0.58 & -1.27 & 17.57 & 3.25\end{bmatrix}$	-0.68 ± 0.59 -4.95 -6.33
4	$diag\left(\begin{bmatrix}3.31\\1\\2.69\\1\end{bmatrix}\right)$	1	$\begin{bmatrix}-1.82 & -2.70 & 22.32 & 4.30\end{bmatrix}$	-1.32 ± 0.88 -3.85 -8.34
5	$diag\left(\begin{bmatrix}0.17\\0.05\\0.13\\0.05\end{bmatrix}\right)$	1	$\begin{bmatrix}-0.41 & -1.04 & 16.84 & 3.10\end{bmatrix}$	-0.57 ± 0.51 -5.13 -6.10

The following figures depict the outputs and input of the linear system with the above controllers from the 5 combinations:

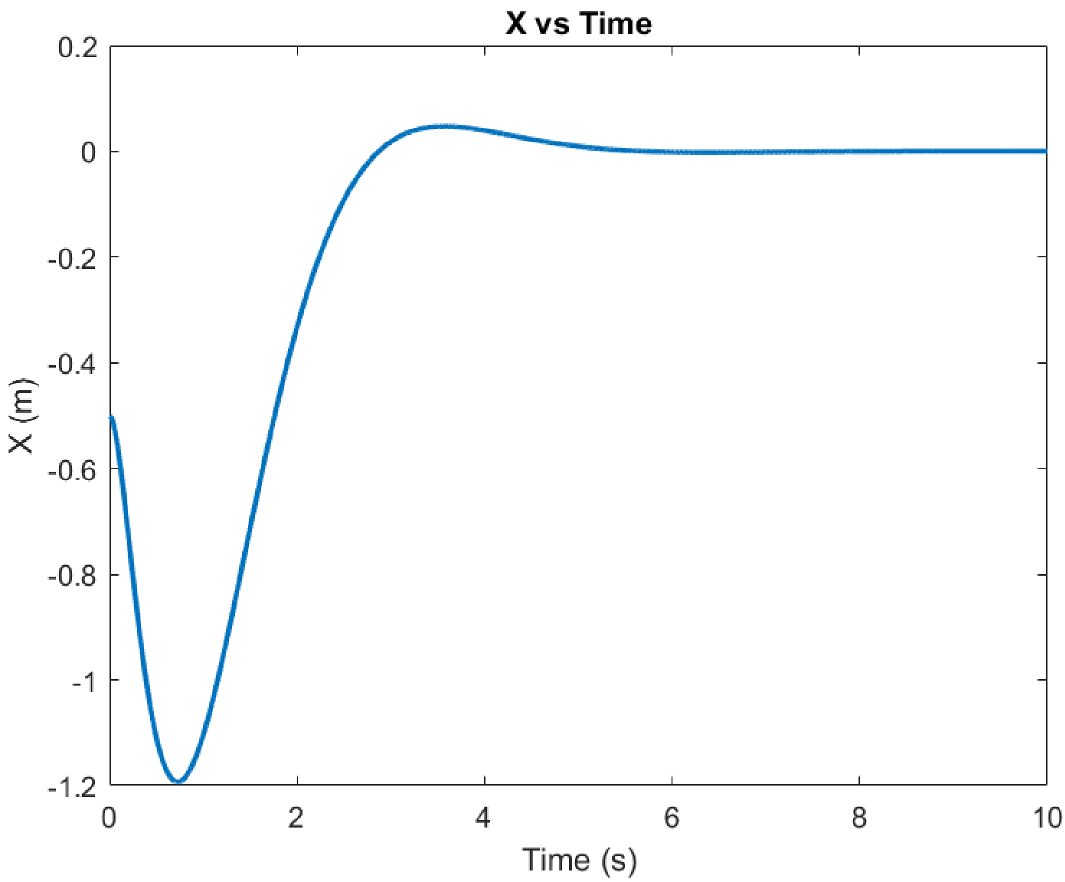

Figure 7: Combination #1 - Linear System X Output

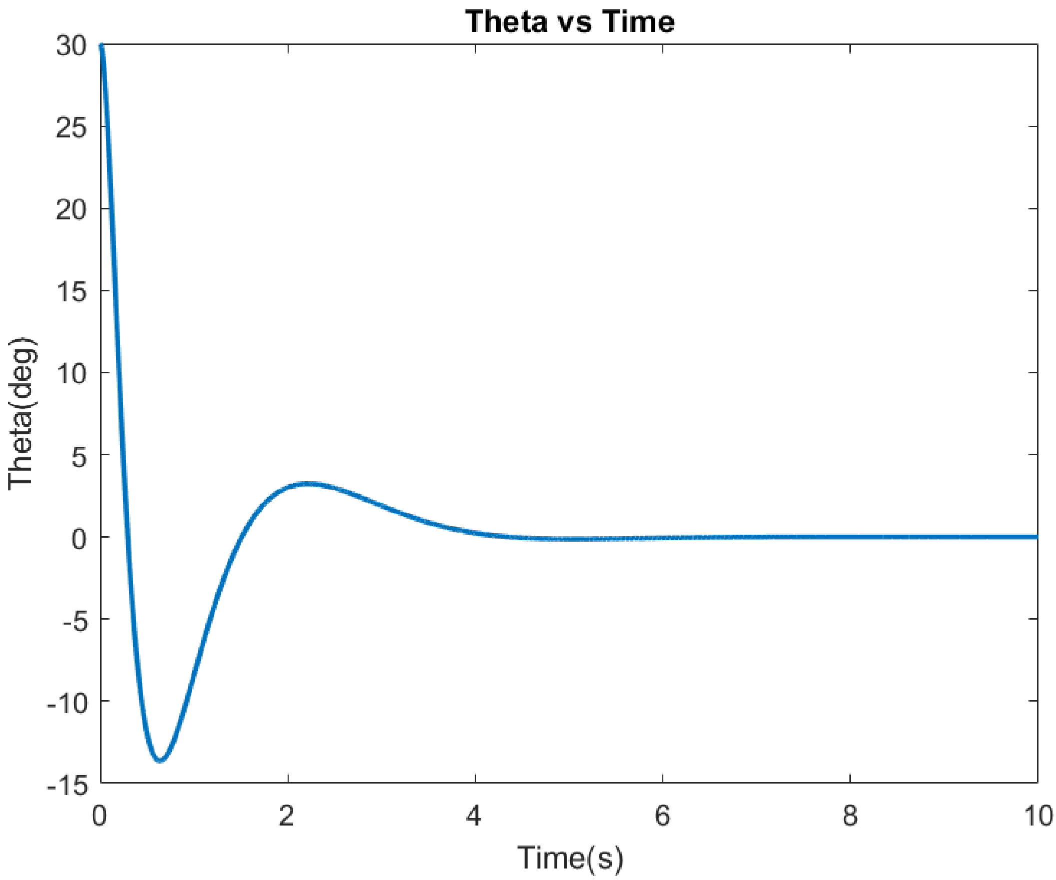

Figure 8: Combination #1 - Linear System θ Output

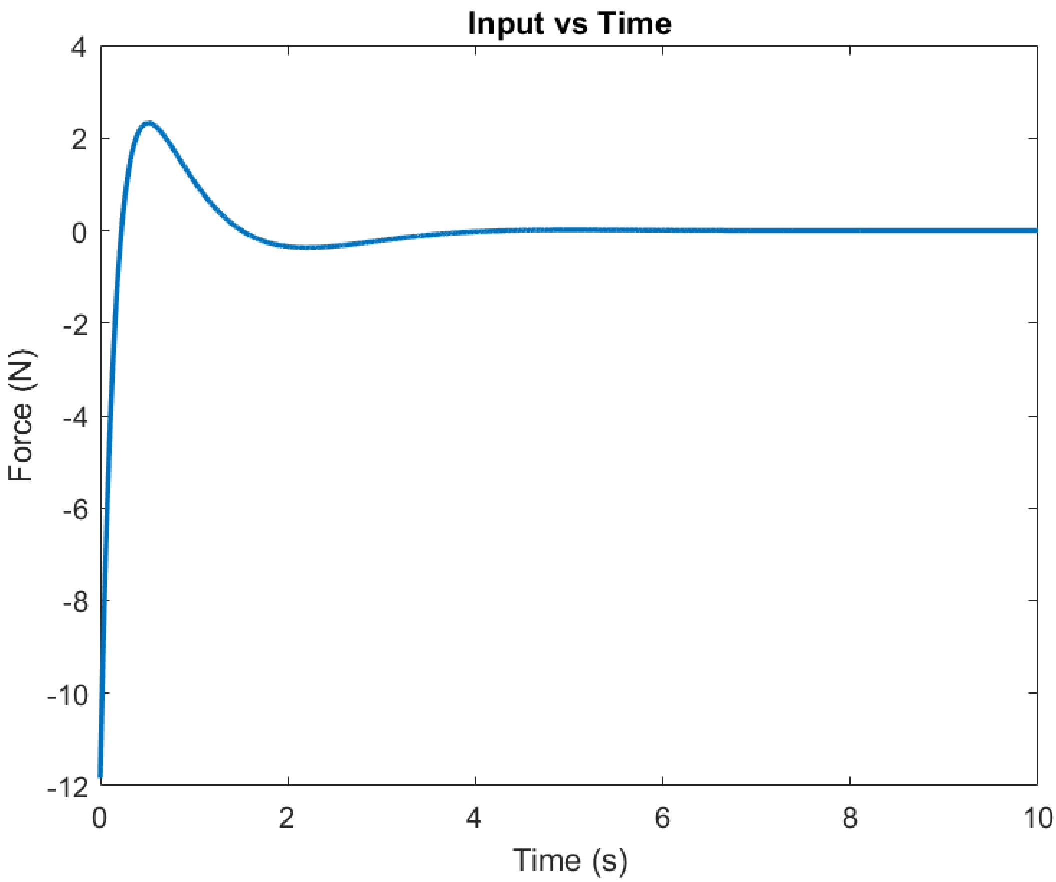

Figure 9: Combination #1 - Linear System Input

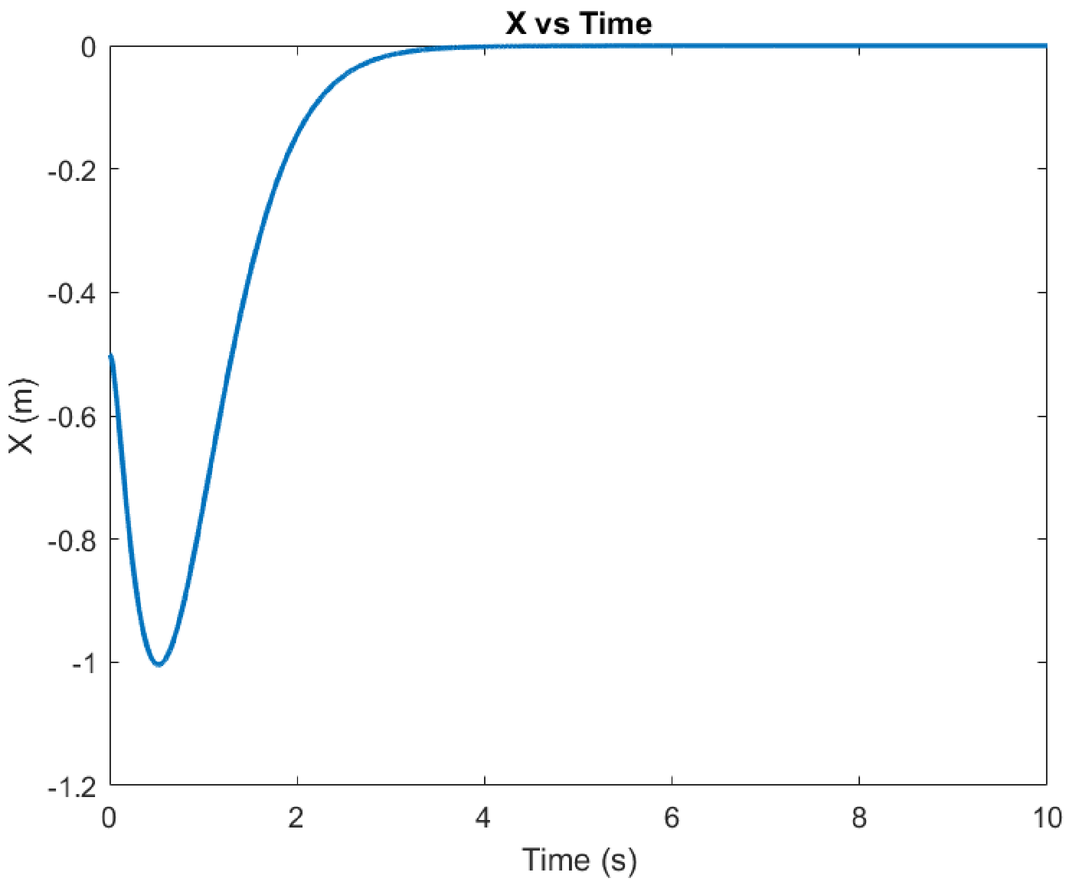

Figure 10: Combination #2 - Linear System X Output

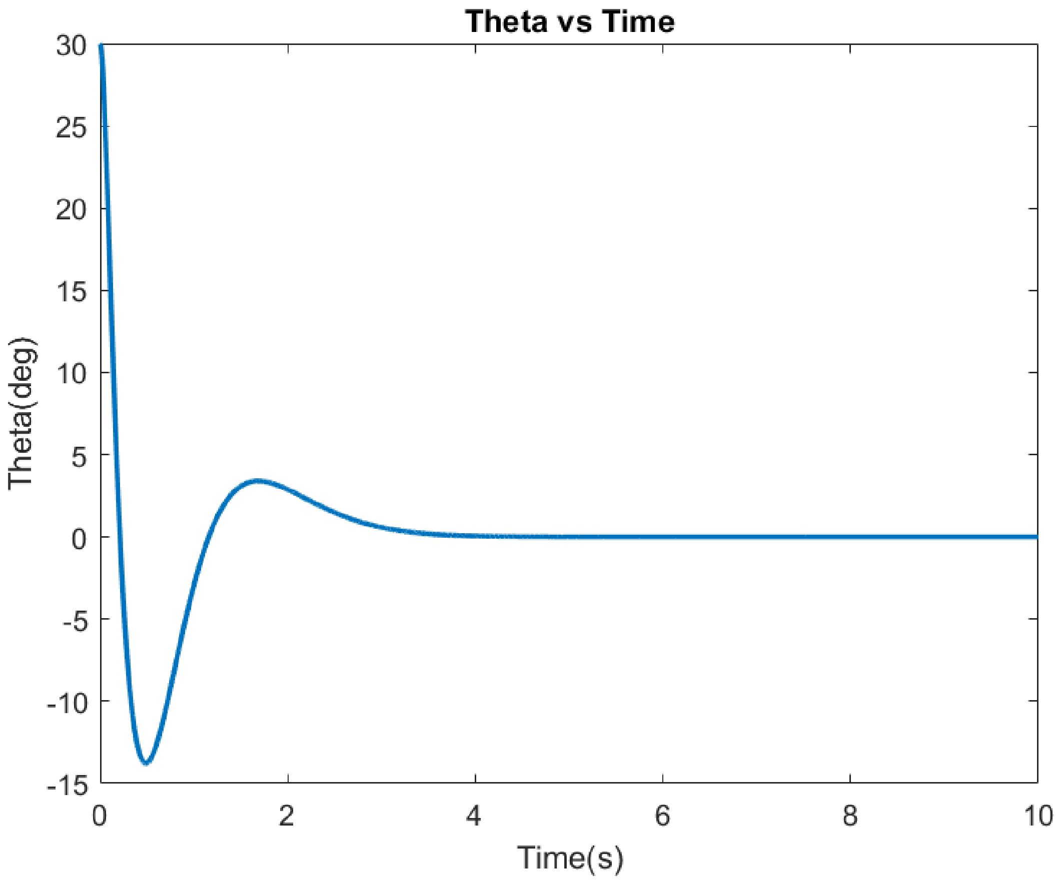

Figure 11: Combination #2 - Linear System θ Output

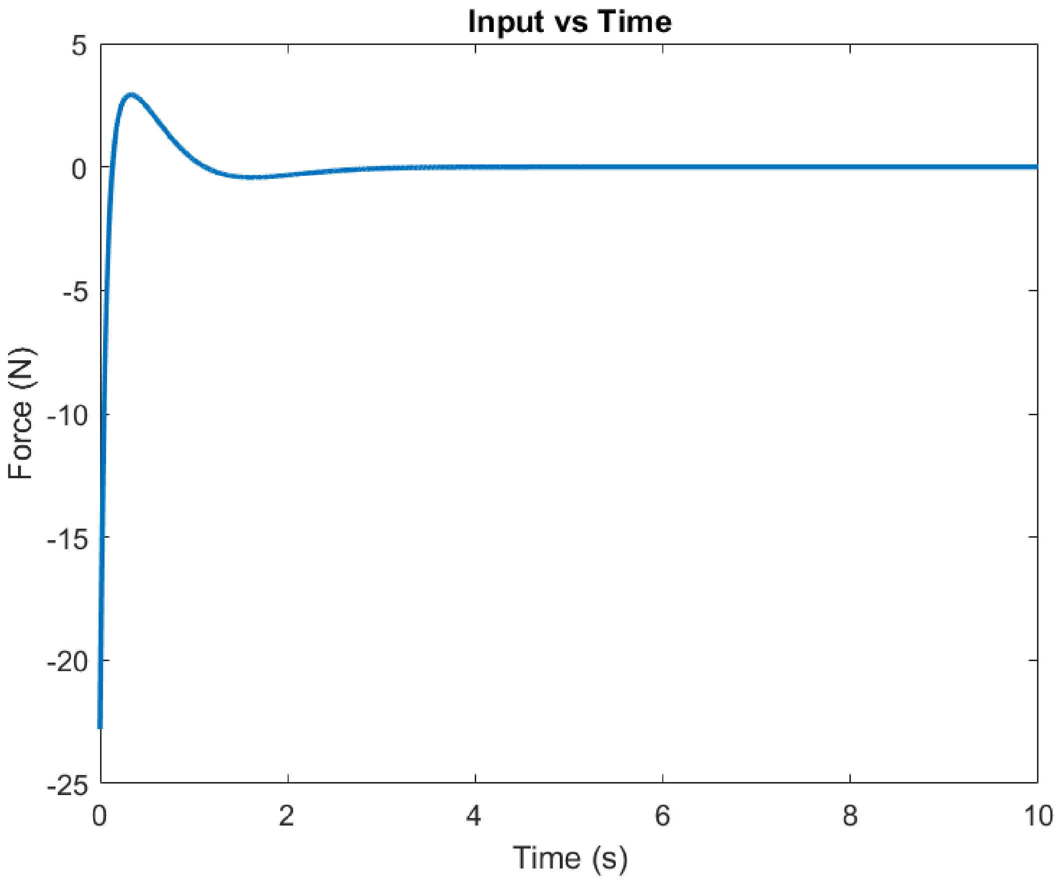

Figure 12: Combination #2 - Linear System Input

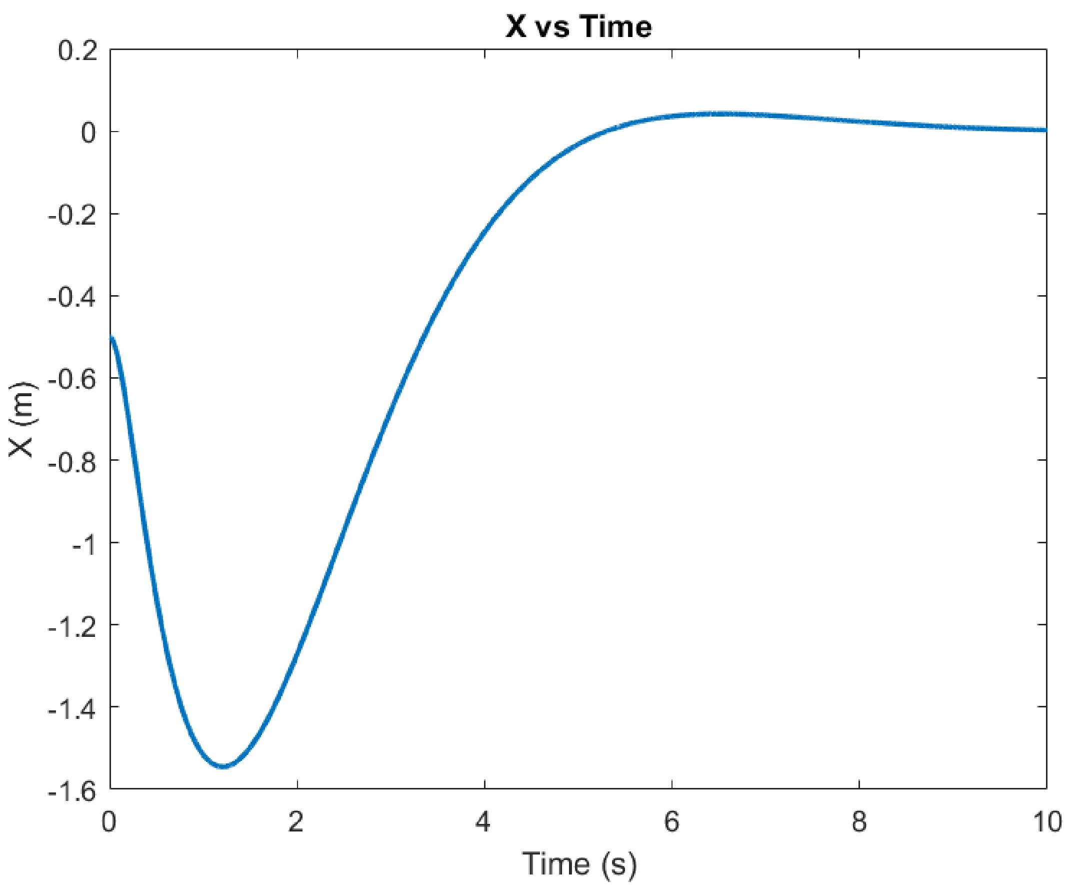

Figure 13: Combination #3 - Linear System X Output

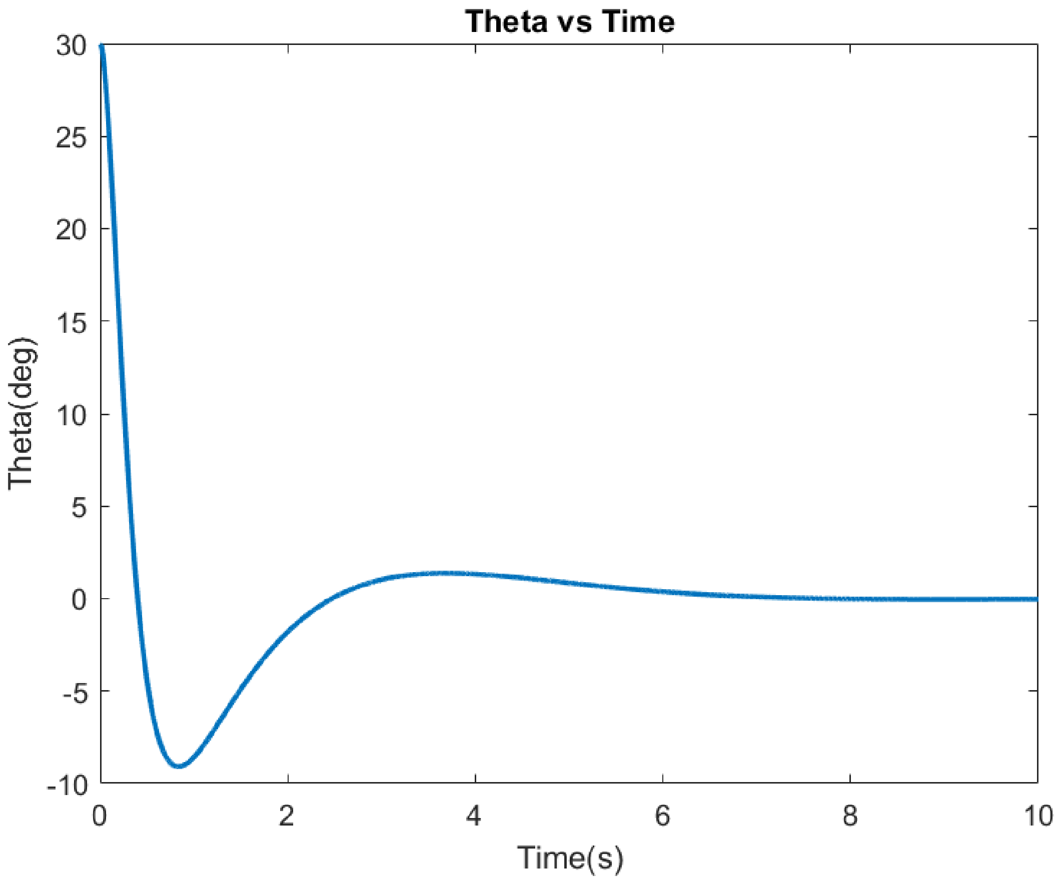

Figure 14: Combination #3 - Linear System θ Output

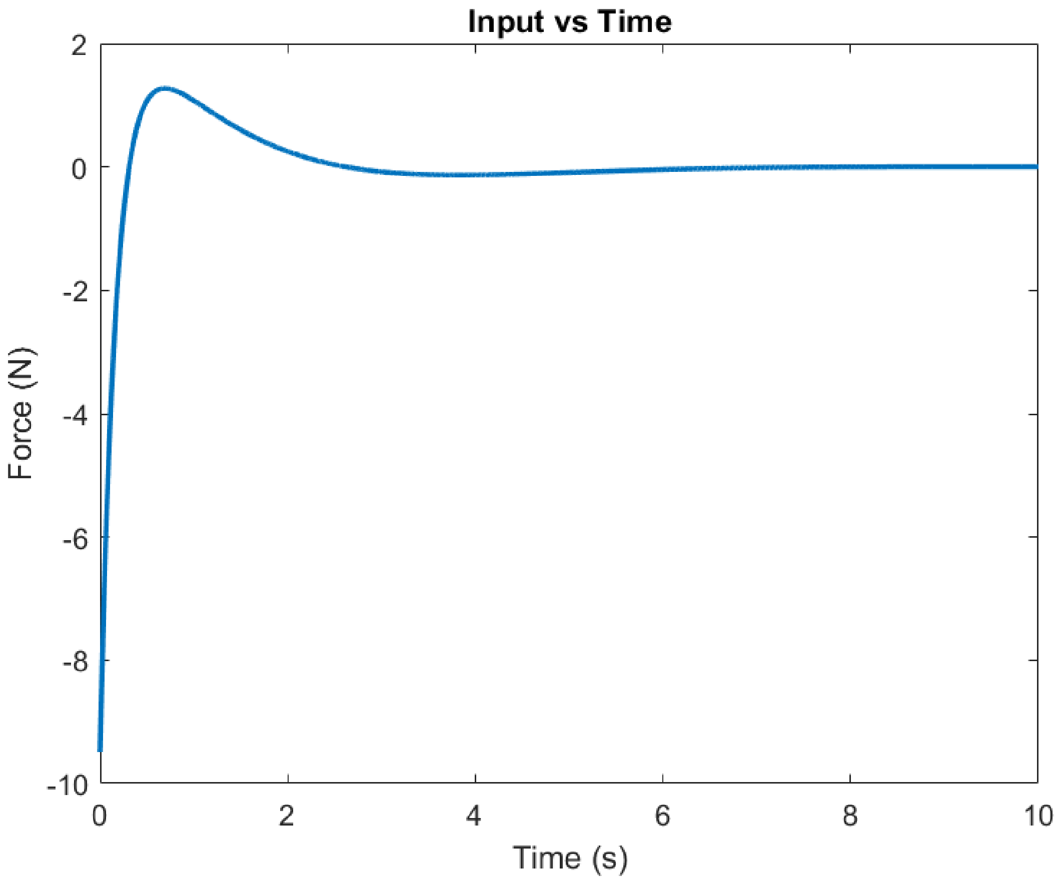

Figure 15: Combination #3 - Linear System Input

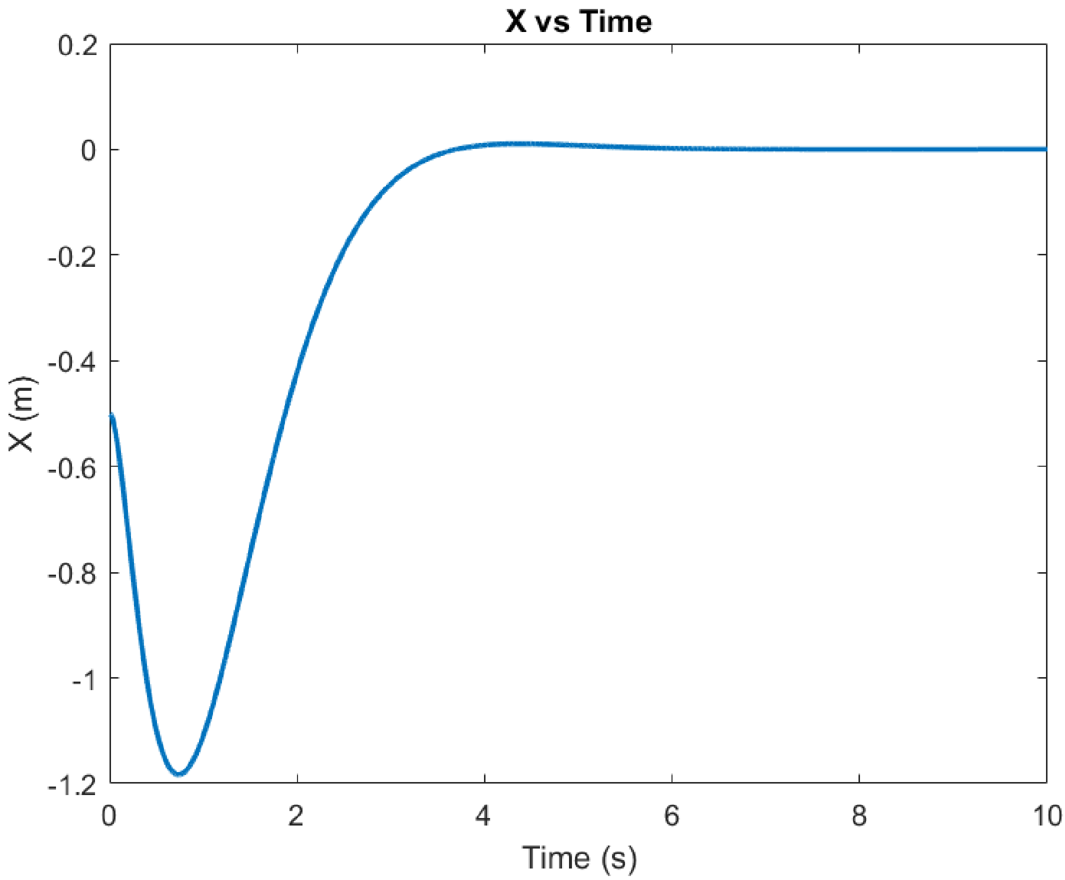

Figure 16: Combination #4 - Linear System X Output

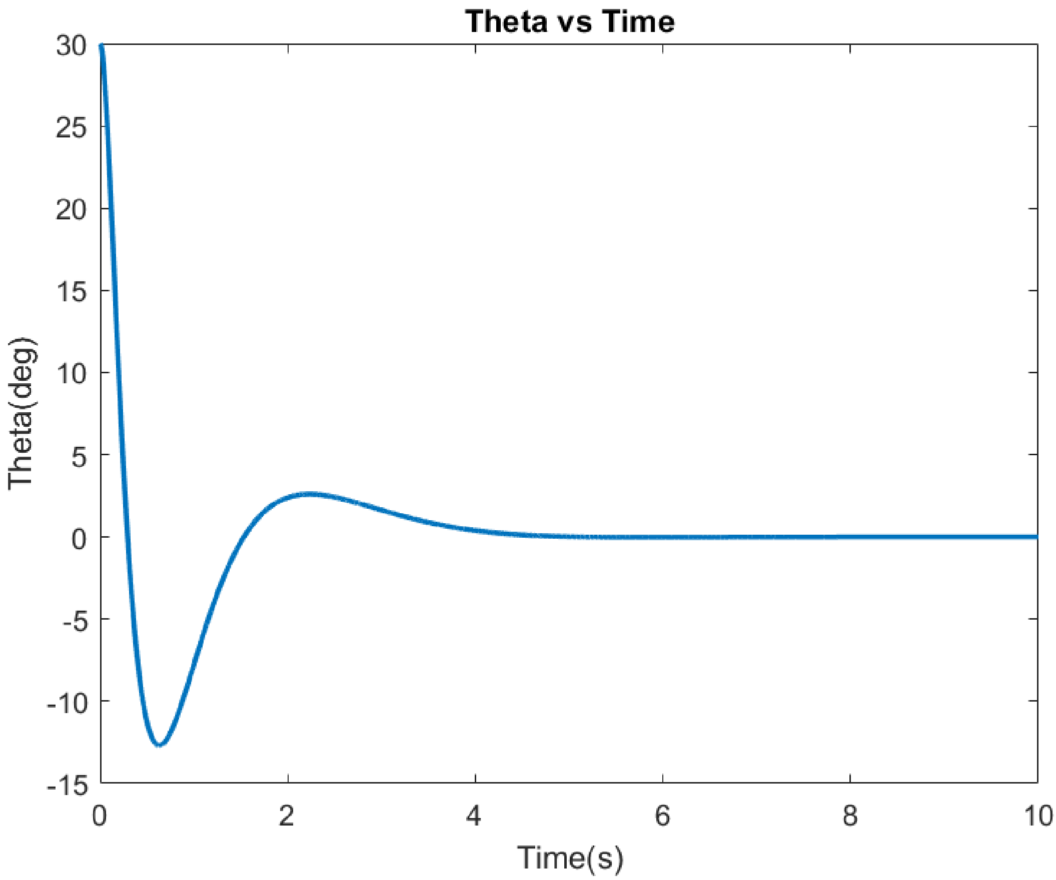

Figure 17: Combination #4 - Linear System θ Output

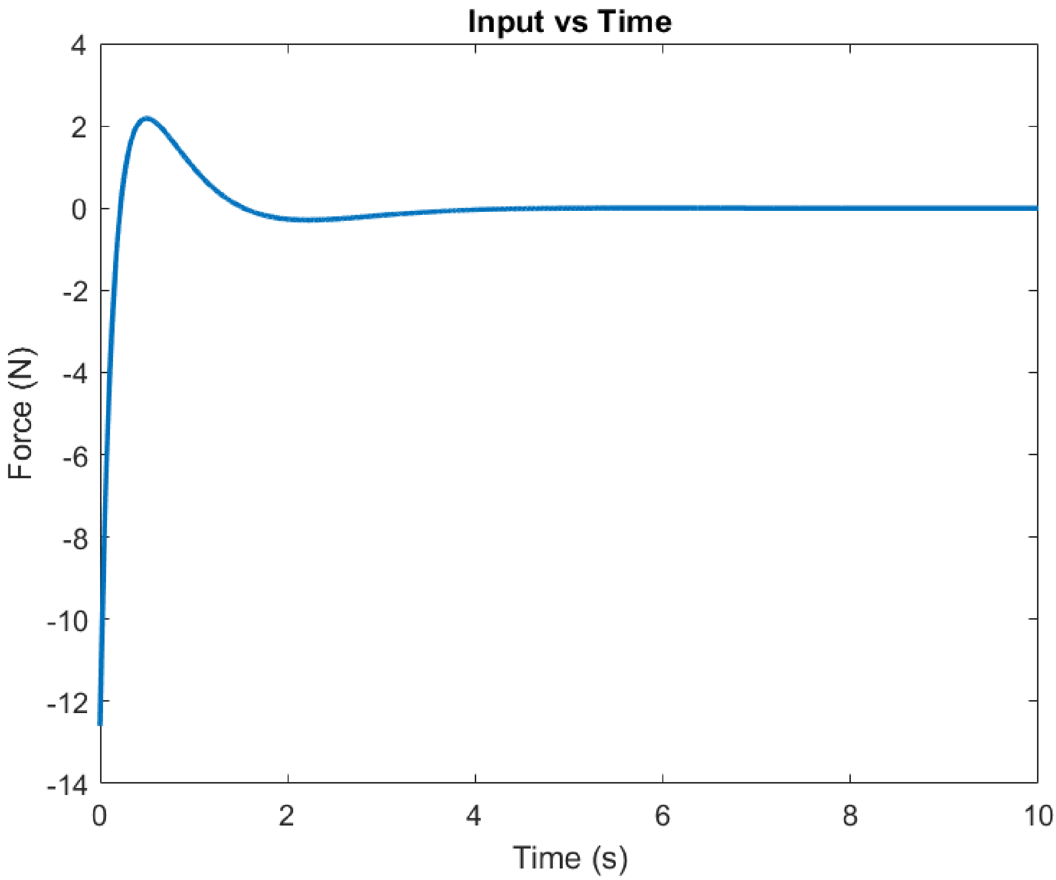

Figure 18: Combination #4 - Linear System Input

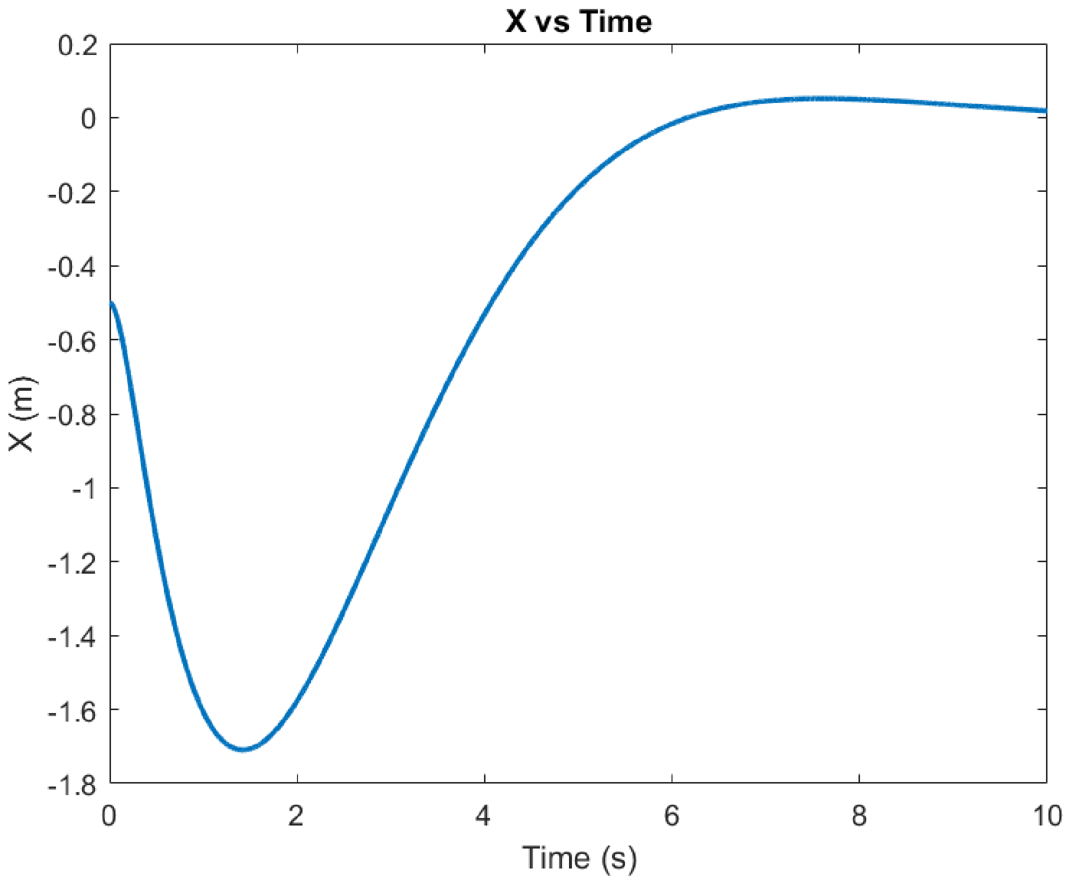

Figure 19: Combination #5 - Linear System X Output

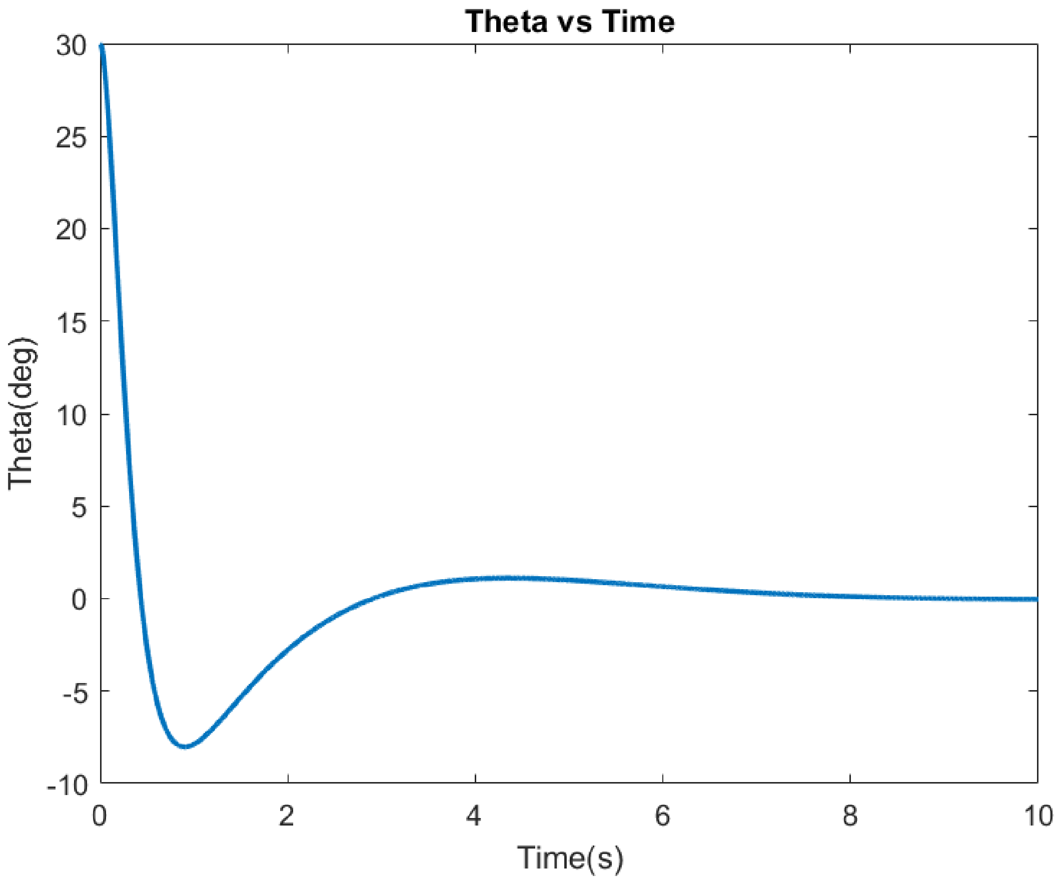

Figure 20: Combination #5 - Linear System θ Output

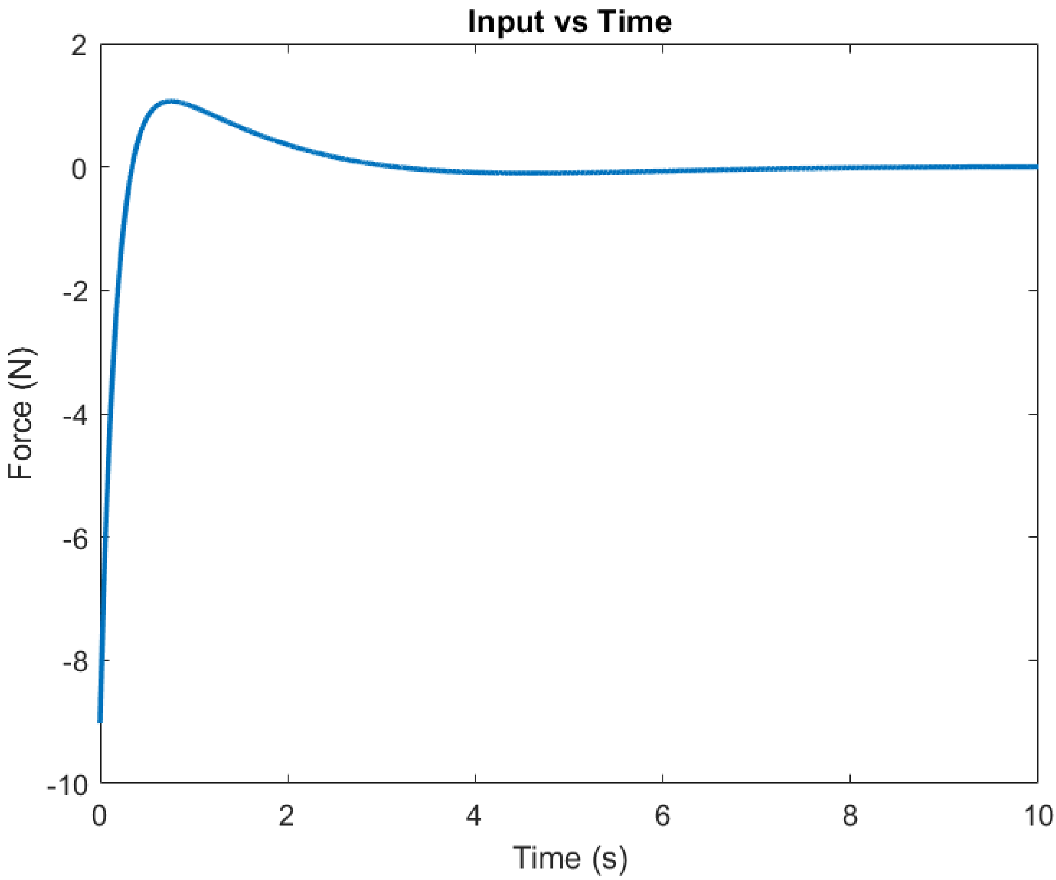

Figure 21: Combination #5 - Linear System Input

Overall, increasing the values of Q made the state response quicker without drastically changing the input (however it slightly increases) and decreasing Q makes the response slower, but doesn't drastically change the input. On the other hand, increasing R decreased the input and made the response slower. Decreasing R did the opposite. Using the modified Q in Combination 1 eliminates the consideration for the derivative states which makes the response slightly slower while still being somewhat similar.

Problem 5: LQR State Feedback Setpoint Tracking

a) Design a state feedback controller to drive $\phi \to 0$ and $z \to 0.5$ m and to satisfy the aforementioned design requirements.

$\bar{A}$, $\bar{B}$, and $\bar{C}$ were determined in Project 1 for the State Feedback Setpoint Tracking, therefore, their derivation will not be reiterated here.

$\bar{Q}$, $\bar{R}$, and $\bar{K}$ were designed using the following methodology. The addition of a new error state for setpoint tracking demands the Q matrix be expanded to a 5×5 matrix. This means there is now a diagonal term that unit scales the max error using Bryson's rule. Through trial and error, the terms in this new $\bar{Q}$ were set by multiplying the existing values from Problem 3 and setting the new diagonal value to 100. This is shown in the following:

$$\bar{Q} = \begin{bmatrix} 33.06 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 26.87 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 100 \end{bmatrix}$$

The value for $\bar{R}$ was left a 1 as in Problem 3. The controller performance was initially tested with the values of Q from Problem 3 while setting the last diagonal equal to 1, but this configuration did not comfortably meet the design requirements. This is why the above changes were made. The resulting K using the same methods in Problem 3 is the following:

$$\bar{K} = \begin{bmatrix} -16.0787 & -11.3733 & 48.8080 & 10.0091 \end{bmatrix}$$

The following figures depict the outputs and input of the linear system with the above controller:

Figure 22: Linear System X Output

Figure 23: Linear System θ Output

Figure 24: Linear System Input

The closed-loop eigenvalues are found from $det(sI - (A - BK)) = 0$. They are the following:

$$s = -0.57 \pm 0.51, -5.13, -6.10$$

b) Design the controller for the nonlinear model.

The $\bar{Q}$, $\bar{R}$, and $\bar{K}$ used for the nonlinear system are the same as for the linear model.

The following figures depict the outputs and input of the nonlinear system with the above controller:

Figure 25: Nonlinear System X Output

Figure 26: Nonlinear System θ Output

Figure 27: Nonlinear System Input

Problem 6: LQR Output Feedback Regulation (Full Order)

a) Design the observer for the linear model.

Given LQR is a form of control, not much changed when designing the Output Feedback Observer. In fact, the same methodology from Project 1 was used to design the observer. The same LQR controller from Problem 3 was used and the closed-looped eigenvalues of that controller were multiplied by 7.5 to be used in designing the observer. These new scaled eigenvalues were used in the `place()` function in MATLAB to get the following observer gains:

$$L = \begin{bmatrix} 73.95 & -6.51 \\ 719.20 & -226.84 \\ 2.66 & 37.18 \\ 268.62 & 289.14 \end{bmatrix}$$

The closed-loop eigenvalues are found from $det(sI - (A - LC)) = 0$. They are the following:

$$s = -9.91 \pm 6.60, -28.90, -62.59$$

This design satisfies the requirements still as we can perfectly observe the output states with fast enough eigenvalues given there is no noise in our measurements.

The following figures depict the outputs and input of the linear system with the above controller:

Figure 28: Linear System X Output

Figure 29: Linear System X Error

Figure 30: Linear System θ Output

Figure 31: Linear System θ Error Input

Figure 32: Linear System Input

b) Design the observer for the nonlinear model.

The controller and observer used for the nonlinear system are the same as for the linear model. There is enough design buffer to compensate for the nonlinear dynamics much like in Problem 3.

The following figures depict the outputs and input of the nonlinear system with the above controller:

Figure 33: Nonlinear System X Output

Figure 34: Nonlinear System X Error

Figure 35: Nonlinear System θ Output

Figure 36: Nonlinear System θ Error Input

Figure 37: Nonlinear System Input

c) Compare the responses of the system under state feedback to the responses of the system under output feedback.

The outputs for each of these cases look relatively similar as there is no noise on the "measurements" we're using in our observer. The transient responses for each look slightly different on a small scale until the observer tracks with zero error; however, it is hard to notice the difference in the transients graphically. Once tracking, the responses look the same as the controller eigenvalues are the same for state and output feedback. This is the case for the linear and nonlinear models where the only difference is the slightly slower response in the nonlinear case.