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CHAOTIC MOTION IN DOUBLE PENDULUM SYSTEMS

A Final Computing Project

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Abstract

This project explores the chaotic dynamics of a double pendulum system, a classic example of how simple physical laws can yield complex, unpredictable behavior. Starting from the Lagrangian formulation, we derive the coupled non-linear equations of motion governing the system. We implement the fourth-order Runge-Kutta (RK4) numerical method to simulate the system's trajectory over time. The study demonstrates the system's extreme sensitivity to initial conditions the "butterfly effect" where infinitesimal differences in starting states lead to exponentially diverging outcomes.

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1. Introduction

A single pendulum is a classical example of simple harmonic motion. When constrained to small angles the pendulum will swing periodically and consistently. By simply adding a second pendulum at the end of the first, the system transforms into a classical example of chaotic motion. Even though these two systems are governed by the same physical laws of motion and only being acted upon by one force (gravity), a double pendulum is *heavily* dependent on initial conditions. We will use an approximation to solve this equation to view the behavior; the approximation we will use is the fourth order Runge-Kutta method.

2. Mathematical Model

To simulate the double pendulum, we must first derive the equations of motion. Because the system involves multiple degrees of freedom, the Lagrangian formulation of mechanics is significantly more efficient than the Newtonian approach.

2.1. System Setup and Geometry

We consider a system of two point masses, m_1 and m_2, connected by massless rigid rods of length l_1 and l_2. The system is confined to a 2D plane and acted upon by a uniform gravitational field g pointing downwards. The system can be visualized in Figure 2.1 on this page. Note that the system allows for θ_1 to be greater than 90° and not intersect with the first pivot.

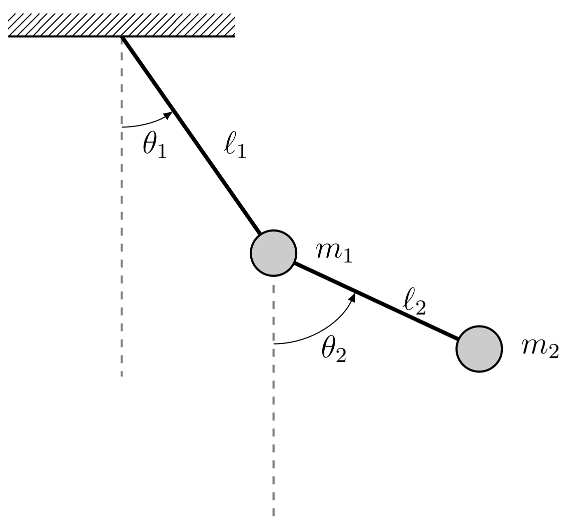

Figure 2.1: Double pendulum system

We define the generalized coordinates as the angles θ_1 and θ_2, measured from the vertical axis. The origin $(0, 0)$ is the pivot point of the first pendulum.

The Cartesian coordinates (x_1, y_1) of the center of mass m_1 and (x_2, y_2) of the center of mass m_2 can be expressed in terms of the angles:

$$x_1 = \frac{\ell_1}{2} \sin\theta_1 \tag{2.1}$$

$$y_1 = -\frac{\ell_1}{2} \cos\theta_1 \tag{2.2}$$

$$x_2 = \ell \left(\sin\theta_1 + \frac{1}{2} \sin\theta_2 \right) \tag{2.3}$$

$$y_2 = -\ell \left(\cos\theta_1 + \frac{1}{2} \cos\theta_2 \right) \tag{2.4}$$

2.2. The Lagrangian Formulation

The Lagrangian $\mathcal{L}$ is defined as the difference between the kinetic energy (T) and the potential energy (V) of the system:

$$\mathcal{L} = T - V$$

2.2.1. Kinetic Energy (T)

The total kinetic energy is the sum of the kinetic energies of the individual masses:

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Calculating the velocities squared ($v^2 = \dot{x}^2 + \dot{y}^2$):

$$\begin{aligned} v_1^2 &= (l_1 \dot\theta_1)^2 \\ v_2^2 &= \dot{x}_2^2 + \dot{y}_2^2 \\ &= l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 (\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2) \\ &= l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2) \end{aligned}$$

Thus, the total kinetic energy is:

$$T = \frac{1}{2}(m_1 + m_2) l_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 l_2^2 \dot\theta_2^2 + m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2) \tag{2.5}$$

2.2.2. Potential Energy (V)

Assuming potential energy is zero at $y = 0$, we have:

$$\begin{aligned} V &= m_1 g y_1 + m_2 g y_2 \\ &= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \end{aligned} \tag{2.6}$$

2.3. Equations of Motion

Applying the Euler-Lagrange equation for each coordinate θ_i:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i} = 0$$

Solving these derivatives yields a system of two coupled, non-linear second-order differential equations.

For θ_1:

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin\theta_1 = 0 \tag{2.7}$$

For θ_2:

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0 \tag{2.8}$$

These two equations fully describe the motion of the double pendulum. In Section 3, we will rearrange these terms to solve for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ explicitly to implement the RK4 algorithm.

3. Numerical Model

The full Julia code used to execute this method used in this derivation can be found in Appendix A on the current page.

4. Results

5. Conclusion

A. Julia Code