

# CHAOTIC MOTION IN DOUBLE PENDULUM SYSTEMS

## A Final Computing Project

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### Abstract

This project investigates the nonlinear and chaotic dynamics of a double pendulum system. Starting from the Lagrangian formulation, we derive the coupled second-order differential equations governing the system's motion. These equations are then transformed into a first-order state-space representation suitable for numerical integration. Using a fourth-order Runge–Kutta (RK4) method implemented in Julia, we simulate the motion and analyze the system's sensitivity to initial conditions. Numerical experiments demonstrate infinitesimal perturbations in the initial angles lead to exponentially divergent trajectories. The results illustrate how a physically simple, deterministic system can exhibit rich, unpredictable behavior.



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## 1. Introduction

A single pendulum is a classical example of simple harmonic motion. When constrained to small angles the pendulum will swing periodically and consistently. They are so predictable that some have been used to keep time. By simply adding a second pendulum at the end of the first, the system transforms into a classical example of chaotic motion. Even though these two systems are governed by the same physical laws of motion and only being acted upon by one force (gravity), a double pendulum is *heavily* dependent on initial conditions. We will use an approximation to solve this equation to view the behavior; the approximation we will use is the fourth order Runge-Kutta method.

## 2. Mathematical Model

To simulate the double pendulum, we must first derive the equations of motion. Because the system involves multiple degrees of freedom, the Lagrangian formulation of mechanics is significantly more efficient than the Newtonian approach [2].

### 2.1. System Setup and Geometry

We consider a system of two point masses,  $m_1$  and  $m_2$ , connected by massless rigid rods of length  $\ell_1$  and  $\ell_2$ . The system is confined to a 2D plane and acted upon by a uniform gravitational field  $g$  pointing downwards. The system can be visualized in Figure 2.1 on this page. Note: unlike in Figure 2.1, the mathematical (and numeric) system will not intersect with itself or other objects.

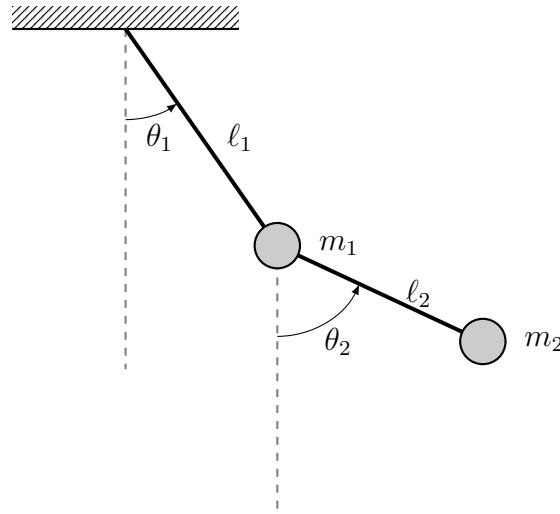


Figure 2.1: Double pendulum system

The system we will construct mathematically and depicted by Figure 2.1 is a simple pendulum, which means that there is a point mass at the end of a massless rod. A compound

pendulum can be similarly derived by just substituting the position of the point masses by the position of the centers of mass [1].

We define the generalized coordinates as the angles  $\theta_1$  and  $\theta_2$ , measured from the vertical axis. The origin  $(0, 0)$  is the pivot point of the first pendulum.

$$x_1 = \ell_1 \sin \theta_1 \quad (2.1)$$

$$y_1 = -\ell_1 \cos \theta_1 \quad (2.2)$$

$$x_2 = \ell_1 \sin \theta_1 + \ell_2 \sin \theta_2 \quad (2.3)$$

$$y_2 = -\ell_1 \cos \theta_1 + \ell_2 \cos \theta_2 \quad (2.4)$$

## 2.2. The Lagrangian Formulation

The Lagrangian  $\mathcal{L}$  is defined as the difference between the kinetic energy ( $T$ ) and the potential energy ( $V$ ) of the system:

$$\mathcal{L} = T - V$$

### 2.2.1. Kinetic Energy ( $T$ )

The total kinetic energy is the sum of the kinetic energies of the individual masses:

$$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

Calculating the velocities squared ( $v^2 = \dot{x}^2 + \dot{y}^2$ ):

$$\begin{aligned} v_1^2 &= (l_1\dot{\theta}_1)^2 \\ v_2^2 &= \dot{x}_2^2 + \dot{y}_2^2 \\ &= l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2(\sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2) \\ &= l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) \end{aligned}$$

Thus, the total kinetic energy is:

$$T = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) \quad (2.5)$$

### 2.2.2. Potential Energy ( $V$ )

Assuming potential energy is zero at  $y = 0$ , we have:

$$\begin{aligned} V &= m_1gy_1 + m_2gy_2 \\ &= -(m_1 + m_2)gl_1 \cos \theta_1 - m_2gl_2 \cos \theta_2 \end{aligned} \quad (2.6)$$

## 2.3. Equations of Motion

Applying the Euler-Lagrange equation for each coordinate  $\theta_i$ :

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} \right) - \frac{\partial \mathcal{L}}{\partial \theta_i} = 0$$

Solving these derivatives yields a system of two coupled, non-linear second-order differential equations.

For  $\theta_1$ :

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2l_2\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2)g \sin \theta_1 = 0 \quad (2.7)$$

For  $\theta_2$ :

$$m_2l_2\ddot{\theta}_2 + m_2l_1\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2l_1\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2g \sin \theta_2 = 0 \quad (2.8)$$

These two equations fully describe the motion of the double pendulum. In Section 3, we will rearrange these terms to solve for  $\ddot{\theta}_1$  and  $\ddot{\theta}_2$  explicitly to implement the RK4 algorithm.

## 3. Preparing for Numerical Model

Implementing RK4 requires transforming the coupled second-order differential equations derived in Section 2 into a system of first-order differential equations.

### 3.1. Matrix Formulation of Motion

We begin with the coupled Euler-Lagrange equations (2.7) and (2.8). By grouping the angular acceleration terms ( $\ddot{\theta}_1, \ddot{\theta}_2$ ) on the left-hand side and the remaining velocity and potential terms on the right, we obtain the following linear system:

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 = -m_2l_2\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - (m_1 + m_2)g \sin \theta_1, \quad (3.1)$$

$$m_2l_1 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2 = m_2l_1\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - m_2g \sin \theta_2. \quad (3.2)$$

This system can be written in matrix form as  $M(\theta)\ddot{\theta} = b(\theta, \omega)$ , where  $\omega_i = \dot{\theta}_i$ :

$$\underbrace{\begin{pmatrix} (m_1 + m_2)l_1 & m_2l_2 \cos(\theta_1 - \theta_2) \\ m_2l_1 \cos(\theta_1 - \theta_2) & m_2l_2 \end{pmatrix}}_{M(\theta)} \underbrace{\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix}}_{\ddot{\theta}} = \underbrace{\begin{pmatrix} -m_2l_2\omega_2^2 \sin(\theta_1 - \theta_2) - (m_1 + m_2)g \sin \theta_1 \\ m_2l_1\omega_1^2 \sin(\theta_1 - \theta_2) - m_2g \sin \theta_2 \end{pmatrix}}_{b(\theta, \omega)}.$$

### 3.2. Explicit Acceleration Terms

To solve for the angular accelerations  $\ddot{\theta}$ , we invert the mass matrix  $M$ . For a  $2 \times 2$  matrix, the inverse is explicitly given by:

$$\ddot{\theta} = M^{-1}b = \frac{1}{\det M} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Defining the angle difference as  $\Delta\theta = \theta_1 - \theta_2$ , the determinant is  $\det M = m_2 l_1 l_2 (m_1 + m_2 \sin^2 \Delta\theta)$ . Carrying out the matrix multiplication yields the explicit accelerations:

$$\ddot{\theta}_1 = \frac{1}{\det M} [m_2 l_2 b_1 - m_2 l_2 \cos(\Delta\theta) b_2], \quad (3.3)$$

$$\ddot{\theta}_2 = \frac{1}{\det M} [-m_2 l_1 \cos(\Delta\theta) b_1 + (m_1 + m_2) l_1 b_2]. \quad (3.4)$$

Substituting the components of vector  $b$  (from (3.1) and (3.2)) into these expressions provides the functions necessary for the numerical solver.

### 3.3. State-Space Representation

Finally, we convert the two second-order equations into a system of four first-order equations. We introduce the state vector  $X$ :

$$X = (\theta_1 \ \theta_2 \ \omega_1 \ \omega_2)^T.$$

The time evolution of the system is governed by  $\dot{X} = F(X)$ :

$$\dot{X} = \frac{d}{dt} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \ddot{\theta}_1(\theta_1, \theta_2, \omega_1, \omega_2) \\ \ddot{\theta}_2(\theta_1, \theta_2, \omega_1, \omega_2) \end{pmatrix},$$

where  $\ddot{\theta}_1$  and  $\ddot{\theta}_2$  are determined by equations (3.3) and (3.4). This vector  $F(X)$  is passed directly to the RK4 algorithm.

### 3.4. Numerical Implementation: The RK4 Algorithm

With the system reduced to the first-order form  $\dot{X} = F(X)$ , we can now apply the classical fourth-order Runge-Kutta (RK4) method to numerically integrate the equations of motion. This method is preferred over simpler schemes (like Euler's method) because its local truncation error is of order  $O(h^4)$ , providing the high precision necessary to track chaotic trajectories.

Given the state vector  $X_n$  at time  $t_n$  and a time step  $h$ , the next state  $X_{n+1}$  is computed

as a weighted average of four slope estimates:

$$X_{n+1} = X_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where the intermediate slopes  $k_i$  are defined as:

$$\begin{aligned} k_1 &= F(X_n), \\ k_2 &= F\left(X_n + \frac{h}{2}k_1\right), \\ k_3 &= F\left(X_n + \frac{h}{2}k_2\right), \\ k_4 &= F(X_n + hk_3). \end{aligned}$$

In these equations, the function  $F$  corresponds to the system defined in Section 3.3, which calculates the derivatives  $[\omega_1, \omega_2, \ddot{\theta}_1, \ddot{\theta}_2]^T$  at each step. This algorithm ensures that the non-linearities of the double pendulum are captured accurately even over extended simulation times. The full Julia code implementing this solver can be found in [Appendix A](#) on page [7](#).

## 4. Results and Discussion

Using what we have established in previous sections provides the necessary tools to simulate the double pendulum's complex dynamics. With the equations of motion derived via the Lagrangian and the Runge-Kutta solver implemented in Julia, we now turn to the simulation results.

### 4.1. Numerical Results

To verify the chaotic nature of the double pendulum, we simulated the system's time evolution using the RK4 integrator derived in Section 3. We specifically investigated the system's sensitivity to initial conditions.

We initialized two *almost* identical double pendulum systems (sharing identical lengths  $l_1, l_2$  and masses  $m_1, m_2$ ) with initial angles differing by a microscopic perturbation,  $\epsilon$ :

$$\Theta_A = (\theta_1, \theta_2) \quad \text{and} \quad \Theta_B = (\theta_1 + \epsilon, \theta_2)$$

where  $\epsilon = 10^{-5}$  degrees.

[Figure 4.1](#) on the following page illustrates the trajectories of these two systems over time. During the initial phase of the simulation, the deviation caused by the perturbation appears negligible, and the systems move in unison. However, as the system evolves, the non-linear coupling terms in the equations of motion, specifically those derived in [\(3.1\)](#), cause this infinitesimal difference  $\epsilon$  to propagate and grow exponentially for some time.

By the end of the simulation, the two pendulums exhibit completely uncorrelated states, occupying different regions of the configuration space despite being governed by the same deterministic laws. This confirms that while the double pendulum is deterministic (fully

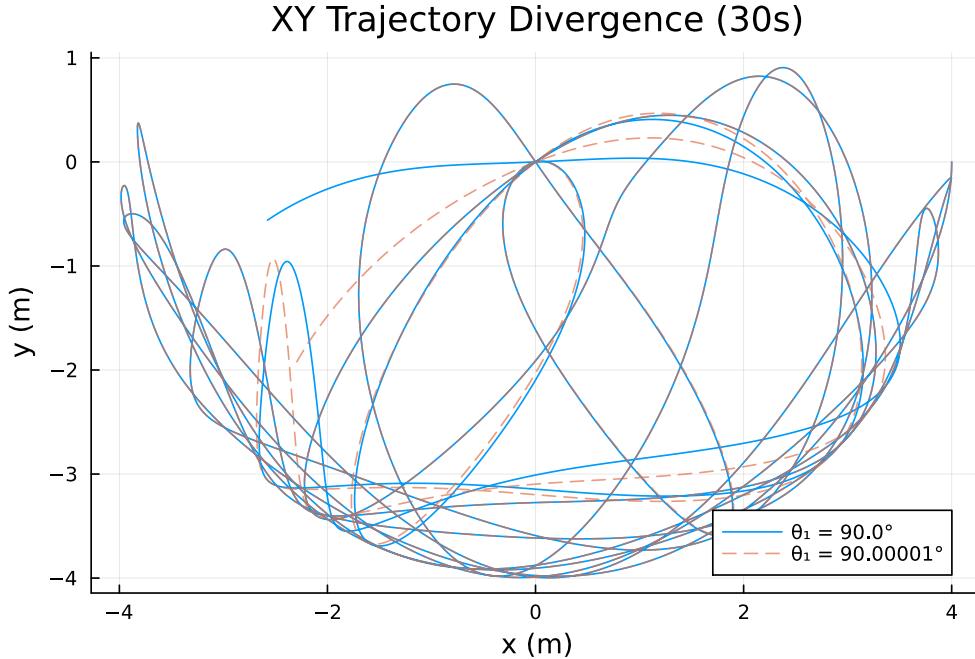


Figure 4.1: Divergence of trajectories with initial conditions differing by  $10^{-5}$  degrees.

described by Equations 3.3 and 3.4), it is practically unpredictable over long time horizons due to this extreme sensitivity.

Plotting the angular difference  $\Delta\theta$  between the two systems further illustrates this chaotic dissociation. As shown in Figure 4.2 on the next page, the error does not grow linearly but exhibits rapid, unpredictable fluctuations, the peaks of which grow exponentially, characteristic of the butterfly effect.

## 4.2. Conclusion

In this project, we modeled the dynamics of a double pendulum, moving from the theoretical physical setup to a full numerical simulation. By utilizing the Lagrangian formulation, we avoided the vector complexity of Newtonian mechanics and directly derived the coupled non-linear equations of motion.

We transformed these second-order differential equations into a state-space representation suitable for computational solving. The implementation of the fourth-order Runge-Kutta (RK4) method allowed us to accurately approximate the solution, maintaining numerical stability even as the system entered chaotic regimes.

The results validate that simple physical systems can exhibit complex, non-linear behavior. The visualization of the trajectory divergence serves as a concrete demonstration of deterministic chaos, satisfying the primary objective of this study. Future work could involve solving triple or  $n$ -pivot pendulums. This would require much higher dimension vector fields to incorporate into the Runge-Kutta methods.

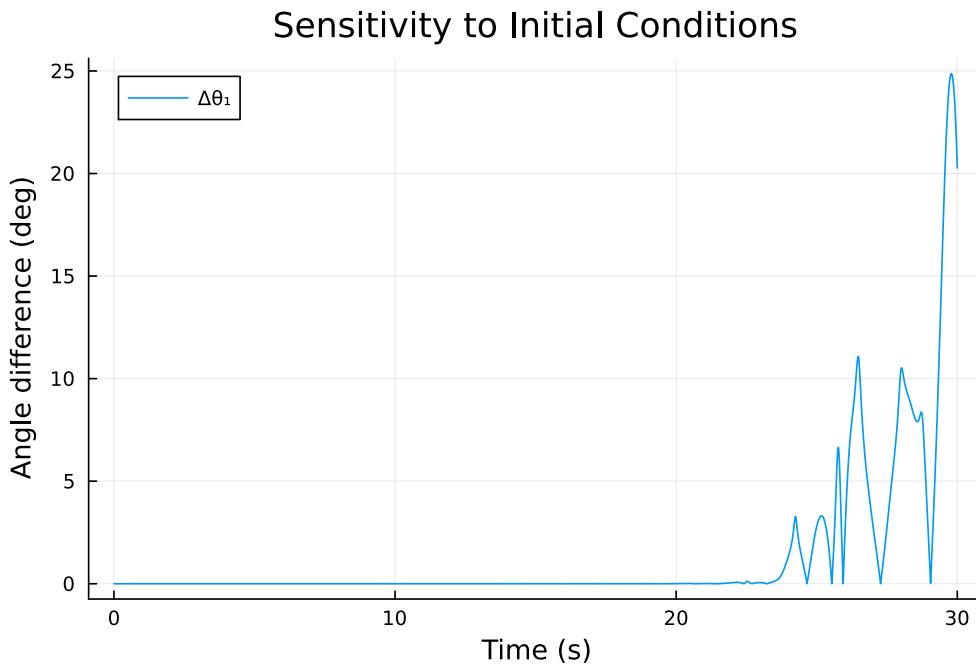


Figure 4.2: Evolution of the angular difference  $\Delta\theta$  between the two systems over time.

## A. Julia Code

Here is the compilation of all Julia code needed to solve the system in this paper.

```

1  using Plots
2  gr() # Set plotting backend
3
4  # parameters
5  G = 9.8
6  lengths = [2.0, 2.0]
7  masses = [2.0, 2.0]
8  time_span = (0.0, 30.0) # 30 seconds
9  n_steps = 15000 # RK4 steps
10
11 # ====== RK4 Solver ===== #
12 function rk4(IVP, n)
13     a, b = IVP.tspan
14     h = (b - a)/n
15     t = [a + i*h for i in 0:n]
16
17     u0 = IVP.u0
18     m = length(u0)
19     u = zeros(m, n+1)
20     u[:,1] .= u0
21
22     for i in 1:n
23         k1 = h * IVP.f(u[:,i], IVP.p, t[i])

```

```

24         k2 = h * IVP.f(u[:,i] + k1/2, IVP.p, t[i] + h/2)
25         k3 = h * IVP.f(u[:,i] + k2/2, IVP.p, t[i] + h/2)
26         k4 = h * IVP.f(u[:,i] + k3, IVP.p, t[i] + h)
27         u[:,i+1] = u[:,i] + (k1 + 2*k2 + 2*k3 + k4)/6
28     end
29
30     return t, u
31 end
32
33 # Double Pendulum ODE (returns vector)
34 function double_pendulum(u, p, t)
35     m1, m2, l1, l2, g = p
36     theta1, omega1, theta2, omega2 = u
37
38     Delta = theta2 - theta1
39     denom1 = (m1 + m2)*l1 - m2*l1*cos(Delta)^2
40     denom2 = (l2/l1)*denom1
41
42     du = zeros(4)
43     du[1] = omega1
44     du[2] = ( m2*l1*omega1^2*sin(Delta)*cos(Delta) +
45                 m2*g*sin(theta2)*cos(Delta) +
46                 m2*l2*omega2^2*sin(Delta) -
47                 (m1+m2)*g*sin(theta1) ) / denom1
48
49     du[3] = omega2
50     du[4] = ( -m2*l2*omega2^2*sin(Delta)*cos(Delta) +
51                 (m1+m2)*g*sin(theta1)*cos(Delta) -
52                 (m1+m2)*l1*omega1^2*sin(Delta) -
53                 (m1+m2)*g*sin(theta2) ) / denom2
54
55     return du
56 end
57
58 # IVP Struct
59 struct ODEProblem
60     f::Function
61     u0::Vector{Float64}
62     tspan::Tuple{Float64, Float64}
63     p::Tuple
64 end
65
66 # Simulation Function
67 function simulate_pendulum(theta1_deg, theta2_deg; omega1=0.0, omega2=0.0)
68     u0 = deg2rad.([theta1_deg, omega1, theta2_deg, omega2])
69     p = (masses[1], masses[2], lengths[1], lengths[2], G)
70     IVP = ODEProblem(double_pendulum, u0, time_span, p)

```

```
71     t, u = rk4(IVP, n_steps)
72     return t, u
73 end
74
75 # Plotting Multiple Initial Conditions
76 initial_conditions = [
77     (90.0, 90.0),
78     (91.0, 90.0),
79     (90.0, 91.0)
80 ]
81
82 plot(title="Double Pendulum Theta vs Time", xlabel="Time (s)", ylabel="Angle (deg")
83 )
84 for (theta1, theta2) in initial_conditions
85     t, u = simulate_pendulum(theta1, theta2)
86     plot!(t, rad2deg.(u[1,:]), label="theta1 start=(theta1,theta2)")
87     plot!(t, rad2deg.(u[3,:]), label="theta2 start=(theta1,theta2)")
88 end
89
90 t, u1 = simulate_pendulum(90.0, 90.0)
91 t, u2 = simulate_pendulum(90.00001, 90.0)
92 x_vals = @. lengths[1] * sin(u[1,:]) + lengths[2] * sin(u[3,:])
93 y_vals = @. -lengths[1] * cos(u[1,:]) - lengths[2] * cos(u[3,:])
94
95 # Plotting Delta Theta graph
96 plot(t, rad2deg.(abs.(u1[1,:] .- u2[1,:])), label="Delta theta1", xlabel="Time (s",
97     ), ylabel="Angle difference (deg)", title="Sensitivity to Initial Conditions
98 ")
99 # Plotting (x,y) graph
100 plot(x_vals, y_vals, xlabel="x (m)", ylabel="y (m)", title="Pendulum Trajectory",
101     aspect_ratio=:equal, label="")
```

## References

- [1] Double pendulum *Wikipedia*. [https://en.wikipedia.org/wiki/Double\\_pendulum](https://en.wikipedia.org/wiki/Double_pendulum). Accessed: 2025-12-11.
- [2] John R. Taylor. *Classical Mechanics*. University Science Books, 2005.