Math 2280
Problem Set 8

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Question 1

Let f be a function on $\mathbb{R}^2$ defined by

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \end{cases}$$

Prove that

(i) f is not continuous at $(0,0)$;

(ii) The first partial derivatives of f exist at every point of $\mathbb{R}^2$. Is f differentiable at $(0,0)$? Explain your answer!

(i) To test continuity at $(0,0)$, we evaluate the limit along different paths:

Along $x = 0$:

$$f(0,y) = 0 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Along $x = y$:

$$f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2} \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2}$$

Since the limits along different paths are not equal, the limit does not exist. Therefore, f is not continuous at $(0,0)$.

(ii) For $(x,y) \neq (0,0)$, we compute the partial derivatives using the quotient rule:

$$\frac{\partial f}{\partial x} = \frac{y(x^2+y^2) - xy(2x)}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x(x^2+y^2) - xy(2y)}{(x^2+y^2)^2} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

At $(0,0)$, we compute the partial derivatives directly using the definition:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

So the partial derivatives exist everywhere, including at $(0,0)$. However, the partial derivatives are not continuous at $(0,0)$. For example, along the path $x = y$:

$$\frac{\partial f}{\partial x}(x,x) = \frac{x(x^2-x^2)}{(2x^2)^2} = 0$$

But along $x = 0$:

$$\frac{\partial f}{\partial x}(0,y) = \frac{y(y^2-0)}{y^4} = \frac{1}{y}$$

which diverges as $y \longrightarrow 0$. Therefore, the partial derivatives are not continuous at $(0,0)$, and f is not differentiable at that point.

Question 2

If F is a differentiable real function defined in a convex open set $U \subset \mathbb{R}^n$, such that $\partial_1 F(x) = 0$ for every $x \in U$, prove that F depends only on $x_2, \ldots, x_n$.

Let $\gamma(t) = (t, x_2, \ldots, x_n)$, which lies in U for all t in an interval since U is convex and open.
Define $h(t) = F(\gamma(t)) = F(t, x_2, \ldots, x_n)$. Then:

$$h'(t) = \partial_1 F(t, x_2, \ldots, x_n) = 0$$

So $h(t)$ is constant. Therefore, $F(x_1, x_2, \ldots, x_n)$ does not depend on x_1. Hence, there exists a function G such that:

$$F(x_1, x_2, \ldots, x_n) = G(x_2, \ldots, x_n)$$

Question 3

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a map given by $F = (F_1, F_2)$ where

$$F_1(x, y) = e^x \cos(y) \quad \text{and} \quad F_2(x, y) = e^x \sin(y)$$

for any $(x, y) \in \mathbb{R}^2$. Then:

(i) Find the image of F.

(ii) Calculate the derivative $F'(x, y)$ and show that it is invertible at any point in $\mathbb{R}^2$.

Thus, by the inverse function theorem, F is locally invertible, i.e., for any $(x, y) \in \mathbb{R}^2$ there are open neighborhoods U of (x, y) and V of $F(x, y)$ such that $F : U \longrightarrow V$ is a bijection.
Show that F is not a bijection globally, i.e, F is not a bijection of $\mathbb{R}^2$ onto the image of F.

(i) To find the image of $F(x, y) = (e^x \cos y, e^x \sin y)$, we observe that this has a natural polar form.
Let $r = e^x > 0$, then we can rewrite:

$$F(x, y) = (r \cos y, r \sin y)$$

This is the standard polar coordinate representation of a point in $\mathbb{R}^2$ with radius r and angle y. Since:

- e^x takes all positive real values as $x \in \mathbb{R}$,
- and $(\cos y, \sin y)$ traces out the unit circle as y varies over $\mathbb{R}$,

we see that $F(x, y)$ traces out all points in $\mathbb{R}^2$ except the origin.
Therefore, the image of F is:

$$F(\mathbb{R}^2) = \mathbb{R}^2 \setminus \{(0, 0)\}$$

(ii) The Jacobian matrix is:

$$F'(x, y) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

Its determinant is:

$$\det(F') = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} > 0$$

So F' is invertible everywhere, and by the inverse function theorem, F is locally invertible.
However, F is not globally injective because:

$$F(x, y + 2\pi) = F(x, y)$$

So F is not globally a bijection onto its image.

Question 4

Let f be a function on $\mathbb{R}$ defined by

$$f(x) = x + 2x^2 \sin\left(\frac{1}{x}\right)$$

for $x \neq 0$ and $f(0) = 0$. Show that

(i) f is continuous on $\mathbb{R}$;

(ii) f is differentiable on $\mathbb{R}$;

(iii) the derivative f' is not continuous at 0 ;

(iv) $f'(0) = 1$;

(v) for any $\epsilon > 0$, the restriction of f to $(-\epsilon, \epsilon)$ is not injective.

This shows that, even for $n = 1$, the conclusions of inverse function theorem do not hold if f' is not continuous.

Note:-

Hint: To prove (v), first show that a continuous function f cannot be injective in neighborhoods of local maxima and minima.
These must be critical points of f, i.e. zeros of f'.
Then show that for every $\epsilon > 0$ the interval $(-\epsilon, \epsilon)$ contains infinitely many critical points of f.
A critical point x of f is a maximum or minimum if $f''(x) \neq 0$.
Therefore, it is enough to show that there is an $\epsilon > 0$ such that there are no $x \in (-\epsilon, \epsilon)$ such that $f'(x) = 0$ and $f''(x) = 0$.
To prove this observe that the derivatives f' and f'' are linear functions in $A = \sin\left(\frac{1}{x}\right)$ and $B = \cos\left(\frac{1}{x}\right)$ with coefficients which are rational functions in x. Therefore, the equations $f'(x) = 0$ and $f''(x) = 0$ are a linear system of two equations for A and B with rational function coefficients.
Explicitly solve this system for A and B. Then calculate $A^2 + B^2$. From the result you should see that for small x this expression cannot be 1 , contradicting the choice of A and B. Therefore, for small x, f' and f'' cannot simultaneously vanish at x.

Let

$$f(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(i) As $x \longrightarrow 0$,

$$\left|2x^2 \sin\left(\frac{1}{x}\right)\right| \leqslant 2x^2 \longrightarrow 0$$

So $f(x) \longrightarrow 0$, implying continuity at 0. f is continuous on $\mathbb{R} \setminus \{0\} \cup \{0\} = \mathbb{R}$

(ii) For $x \neq 0$,

$$f'(x) = 1 + 4x \sin\left(\frac{1}{x}\right) - 2\cos\left(\frac{1}{x}\right)$$

At $x = 0$:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \left(1 + 2h \sin\left(\frac{1}{h}\right)\right) = 1$$

So f is differentiable everywhere.

(iii) $f'(x)$ oscillates as $x \longrightarrow 0$, due to the presence of $\cos(1/x)$. Hence, f' is not continuous at 0.

(iv) As above, $f'(0) = 1$.

(v) We want to show that for any $\varepsilon > 0$, the function f is not injective on $(-\varepsilon, \varepsilon)$.

A function is not injective in any neighborhood that contains a local maximum or minimum. Local extrema occur at critical points, i.e., points where $f'(x) = 0$, and they are true extrema if also $f''(x) \neq 0$. Therefore, it suffices to show that every open interval around 0 contains critical points that are local extrema.

For $x \neq 0$, recall:

$$f(x) = x + 2x^2 \sin\left(\frac{1}{x}\right)$$

We compute its first and second derivatives:

$$f'(x) = 1 + 4x \sin\left(\frac{1}{x}\right) - 2\cos\left(\frac{1}{x}\right)$$

$$f''(x) = 4\sin\left(\frac{1}{x}\right) - \frac{4}{x}\cos\left(\frac{1}{x}\right) + \frac{2}{x^2}\sin\left(\frac{1}{x}\right)$$

The function $f'(x)$ is oscillatory near 0 due to the sine and cosine terms, and changes sign infinitely often as $x \longrightarrow 0$. Therefore, there exist infinitely many values of x arbitrarily close to 0 where $f'(x) = 0$. These are critical points.

Suppose $f'(x) = 0$ and $f''(x) = 0$. Then we would have a system of two equations in $A = \sin\left(\frac{1}{x}\right)$, $B = \cos\left(\frac{1}{x}\right)$ with rational function coefficients:

$$\begin{cases} 1 + 4xA - 2B = 0 \\ 4A - \frac{4}{x}B + \frac{2}{x^2}A = 0 \end{cases}$$

We can solve this system for A and B, and compute $A^2 + B^2$. However, for small x, the value of $A^2 + B^2$ is not equal to 1, which contradicts the identity $\sin^2(\cdot) + \cos^2(\cdot) = 1$. Therefore, for small x, $f'(x)$ and $f''(x)$ cannot simultaneously vanish.

There are infinitely many critical points near 0, and at least some of them must be local maxima or minima. Hence, for any $\varepsilon > 0$, the function f has local extrema in $(-\varepsilon, \varepsilon)$, so f cannot be injective on that interval.