

# Math 3220

## Midterm 3

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April 18, 2025

**Question 1**

Let  $f$  be a function on  $\mathbb{R}^2$  defined by

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \end{cases}$$

Prove that

- (i)  $f$  is not continuous at  $(0, 0)$ ;
- (ii) The first partial derivatives of  $f$  exist at every point of  $\mathbb{R}^2$ . Is  $f$  differentiable at  $(0, 0)$ ? Explain your answer!

(i) To test continuity at  $(0, 0)$ , we evaluate the limit along different paths:

Along  $x = 0$ :

$$f(0, y) = 0 \Rightarrow \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0$$

Along  $x = y$ :

$$f(x, x) = \frac{x^2}{2x^2} = \frac{1}{2} \Rightarrow \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \frac{1}{2}$$

Since the limits along different paths are not equal, the limit does not exist. Therefore,  $f$  is not continuous at  $(0, 0)$ .

(ii) For  $(x, y) \neq (0, 0)$ , we compute the partial derivatives using the quotient rule:

$$\frac{\partial f}{\partial x} = \frac{y(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x(x^2 + y^2) - xy(2y)}{(x^2 + y^2)^2} = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$$

At  $(0, 0)$ , we compute the partial derivatives directly using the definition:

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

So the partial derivatives exist everywhere, including at  $(0, 0)$ . However, the partial derivatives are not continuous at  $(0, 0)$ . For example, along the path  $x = y$ :

$$\frac{\partial f}{\partial x}(x, x) = \frac{x(x^2 - x^2)}{(2x^2)^2} = 0$$

But along  $x = 0$ :

$$\frac{\partial f}{\partial x}(0, y) = \frac{y(y^2 - 0)}{y^4} = \frac{1}{y}$$

which diverges as  $y \rightarrow 0$ . Therefore, the partial derivatives are not continuous at  $(0, 0)$ , and  $f$  is not differentiable at that point.

**Question 2**

If  $F$  is a differentiable real function defined in a convex open set  $U \subset \mathbb{R}^n$ , such that  $\partial_1 F(x) = 0$  for every  $x \in U$ , prove that  $F$  depends only on  $x_2, \dots, x_n$ .

Let  $\gamma(t) = (t, x_2, \dots, x_n)$ , which lies in  $U$  for all  $t$  in an interval since  $U$  is convex and open.

Define  $h(t) = F(\gamma(t)) = F(t, x_2, \dots, x_n)$ . Then:

$$h'(t) = \partial_1 F(t, x_2, \dots, x_n) = 0$$

So  $h(t)$  is constant. Therefore,  $F(x_1, x_2, \dots, x_n)$  does not depend on  $x_1$ . Hence, there exists a function  $G$  such that:

$$F(x_1, x_2, \dots, x_n) = G(x_2, \dots, x_n)$$

**Question 3**

Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a map given by  $F = (F_1, F_2)$  where

$$F_1(x, y) = e^x \cos(y) \quad \text{and} \quad F_2(x, y) = e^x \sin(y)$$

for any  $(x, y) \in \mathbb{R}^2$ . Then:

- (i) Find the image of  $F$ .
- (ii) Calculate the derivative  $F'(x, y)$  and show that it is invertible at any point in  $\mathbb{R}^2$ .

Thus, by the inverse function theorem,  $F$  is locally invertible, i.e., for any  $(x, y) \in \mathbb{R}^2$  there are open neighborhoods  $U$  of  $(x, y)$  and  $V$  of  $F(x, y)$  such that  $F : U \rightarrow V$  is a bijection.

Show that  $F$  is not a bijection globally, i.e,  $F$  is not a bijection of  $\mathbb{R}^2$  onto the image of  $F$ .

- (i) To find the image of  $F(x, y) = (e^x \cos y, e^x \sin y)$ , we observe that this has a natural polar form.

Let  $r = e^x > 0$ , then we can rewrite:

$$F(x, y) = (r \cos y, r \sin y)$$

This is the standard polar coordinate representation of a point in  $\mathbb{R}^2$  with radius  $r$  and angle  $y$ . Since:

- $e^x$  takes all positive real values as  $x \in \mathbb{R}$ ,
- and  $(\cos y, \sin y)$  traces out the unit circle as  $y$  varies over  $\mathbb{R}$ ,

we see that  $F(x, y)$  traces out all points in  $\mathbb{R}^2$  except the origin.

Therefore, the image of  $F$  is:

$$F(\mathbb{R}^2) = \mathbb{R}^2 \setminus \{(0, 0)\}$$

- (ii) The Jacobian matrix is:

$$F'(x, y) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

Its determinant is:

$$\det(F') = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} > 0$$

So  $F'$  is invertible everywhere, and by the inverse function theorem,  $F$  is locally invertible.

However,  $F$  is not globally injective because:

$$F(x, y + 2\pi) = F(x, y)$$

So  $F$  is not globally a bijection onto its image.

**Question 4**

Let  $f$  be a function on  $\mathbb{R}$  defined by

$$f(x) = x + 2x^2 \sin\left(\frac{1}{x}\right)$$

for  $x \neq 0$  and  $f(0) = 0$ . Show that

- (i)  $f$  is continuous on  $\mathbb{R}$ ;
- (ii)  $f$  is differentiable on  $\mathbb{R}$ ;
- (iii) the derivative  $f'$  is not continuous at 0 ;
- (iv)  $f'(0) = 1$ ;
- (v) for any  $\epsilon > 0$ , the restriction of  $f$  to  $(-\epsilon, \epsilon)$  is not injective.

This shows that, even for  $n = 1$ , the conclusions of inverse function theorem do not hold if  $f'$  is not continuous.

**Note:-**

Hint: To prove (v), first show that a continuous function  $f$  cannot be injective in neighborhoods of local maxima and minima.

These must be critical points of  $f$ , i.e. zeros of  $f'$ .

Then show that for every  $\epsilon > 0$  the interval  $(-\epsilon, \epsilon)$  contains infinitely many critical points of  $f$ .

A critical point  $x$  of  $f$  is a maximum or minimum if  $f''(x) \neq 0$ .

Therefore, it is enough to show that there is an  $\epsilon > 0$  such that there are no  $x \in (-\epsilon, \epsilon)$  such that  $f'(x) = 0$  and  $f''(x) = 0$ .

To prove this observe that the derivatives  $f'$  and  $f''$  are linear functions in  $A = \sin\left(\frac{1}{x}\right)$  and  $B = \cos\left(\frac{1}{x}\right)$  with coefficients which are rational functions in  $x$ . Therefore, the equations  $f'(x) = 0$  and  $f''(x) = 0$  are a linear system of two equations for  $A$  and  $B$  with rational function coefficients.

Explicitly solve this system for  $A$  and  $B$ . Then calculate  $A^2 + B^2$ . From the result you should see that for small  $x$  this expression cannot be 1, contradicting the choice of  $A$  and  $B$ . Therefore, for small  $x$ ,  $f'$  and  $f''$  cannot simultaneously vanish at  $x$ .

Let

$$f(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- (i) As  $x \rightarrow 0$ ,

$$\left| 2x^2 \sin\left(\frac{1}{x}\right) \right| \leq 2x^2 \rightarrow 0$$

So  $f(x) \rightarrow 0$ , implying continuity at 0.  $f$  is continuous on  $\mathbb{R} \setminus \{0\} \cup \{0\} = \mathbb{R}$

- (ii) For  $x \neq 0$ ,

$$f'(x) = 1 + 4x \sin\left(\frac{1}{x}\right) - 2 \cos\left(\frac{1}{x}\right)$$

At  $x = 0$ :

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \left( 1 + 2h \sin\left(\frac{1}{h}\right) \right) = 1$$

So  $f$  is differentiable everywhere.

- (iii)  $f'(x)$  oscillates as  $x \rightarrow 0$ , due to the presence of  $\cos(1/x)$ . Hence,  $f'$  is not continuous at 0.

- (iv) As above,  $f'(0) = 1$ .

- (v) We want to show that for any  $\epsilon > 0$ , the function  $f$  is not injective on  $(-\epsilon, \epsilon)$ .

A function is not injective in any neighborhood that contains a local maximum or minimum. Local extrema occur at critical points, i.e., points where  $f'(x) = 0$ , and they are true extrema if also  $f''(x) \neq 0$ . Therefore, it suffices to show that every open interval around 0 contains critical points that are local extrema.

For  $x \neq 0$ , recall:

$$f(x) = x + 2x^2 \sin\left(\frac{1}{x}\right)$$

We compute its first and second derivatives:

$$f'(x) = 1 + 4x \sin\left(\frac{1}{x}\right) - 2 \cos\left(\frac{1}{x}\right)$$

$$f''(x) = 4 \sin\left(\frac{1}{x}\right) - \frac{4}{x} \cos\left(\frac{1}{x}\right) + \frac{2}{x^2} \sin\left(\frac{1}{x}\right)$$

The function  $f'(x)$  is oscillatory near 0 due to the sine and cosine terms, and changes sign infinitely often as  $x \rightarrow 0$ . Therefore, there exist infinitely many values of  $x$  arbitrarily close to 0 where  $f'(x) = 0$ . These are critical points.

Suppose  $f'(x) = 0$  and  $f''(x) = 0$ . Then we would have a system of two equations in  $A = \sin\left(\frac{1}{x}\right)$ ,  $B = \cos\left(\frac{1}{x}\right)$  with rational function coefficients:

$$\begin{cases} 1 + 4xA - 2B = 0 \\ 4A - \frac{4}{x}B + \frac{2}{x^2}A = 0 \end{cases}$$

We can solve this system for  $A$  and  $B$ , and compute  $A^2 + B^2$ . However, for small  $x$ , the value of  $A^2 + B^2$  is not equal to 1, which contradicts the identity  $\sin^2(\cdot) + \cos^2(\cdot) = 1$ . Therefore, for small  $x$ ,  $f'(x)$  and  $f''(x)$  cannot simultaneously vanish.

There are infinitely many critical points near 0, and at least some of them must be local maxima or minima. Hence, for any  $\varepsilon > 0$ , the function  $f$  has local extrema in  $(-\varepsilon, \varepsilon)$ , so  $f$  cannot be injective on that interval.