Math 3320-3
Take Home Midterm 1

Tannon Warnick

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Question 1

Let X be a metric space with metric d. A function $\mathbb{N} \to X$ is a sequence in X. We denote it as $\{x_n; n \in \mathbb{N}\}$ where x_n is the value of the sequence at n.
A point x_0 in X is a limit of the sequence $\{x_n; n \in \mathbb{N}\}$ if for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $n \geqslant n_0$ implies $d(x_n, x_0) < \epsilon$.
A sequence is called convergent if it has a limit.
Prove that:

(*i*) A convergent sequence $\{x_n; n \in \mathbb{N}\}$ has only one limit x_0. We put $x_0 = \lim x_n$.

(*ii*) Let A be a subset of X. Let $\bar{A}$ denote its closure in the natural topology of X. Prove that $\bar{A}$ is the set of all limits of all convergent sequences in A; i.e., $x \in \bar{A}$ if and only if there exists a convergent sequence $\{x_n; n \in \mathbb{N}\}$ such that $x_n \in A$ for any $n \in \mathbb{N}$ and $x = \lim x_n$.

(*i*) ***Proof:*** From the given problem we know that there is a limit x_0. Assume for the sake for contradiction that there exists x_1 that is also a limit and $x_0 \neq x_1$. From this, let

$$\epsilon = \frac{d(x_1, x_0)}{2} > 0.$$

The definition of the limit only requires a positive ϵ and by assumptions this is. So

$$d(x_0, x_n) < \epsilon, d(x_1, x_n) < \epsilon$$

for some n_0, n_1 large enough, let $N = \max\{n_0, n_1\}$, let $n > N$, By the triangle inequality

$$d(x_0, x_1) \leqslant d(x_0, x_n) + d(x_1, x_n) < 2\epsilon = 2 \cdot \frac{1}{2} d(x_0, x_1) = d(x_0, x_1).$$

this is a contradiction because it is not possible that

$$d(x_0, x_1) < d(x_0, x_1).$$

So, x_1 must be equal to x_0 and since they were chosen arbitrarily, this proves that there only exists one limit to a convergent sequence, and $\lim x_n = x_0$. ☺

(*ii*) ***Proof:*** $\Longrightarrow$ Let $x \in \bar{A} \setminus A$ be a limit of a sequence $x_n$with $x_n \in A$ for all n and $x_n \to x$.
By definition of convergence, for every open set U containing x, there exists some $N \in \mathbb{N}$ such that for all $n \geqslant N$, $x_n \in U$. Since each x_n is in A, it follows that every open set containing x intersects A. Thus, x is in the closure $\bar{A}$, meaning:

$$\{x \in X \mid \exists x_n \in A \text{ such that } x_n \to x\} \subseteq \bar{A}$$

$\Longleftarrow$ Take some point $x \in \bar{A}$. We now define U_n to be some open ball centered at x and of radius $\frac{1}{n}$ we can now construct a sequence x_n where $x_1 \in A \cap U_1, \ldots x_n \in A \cap U_n$ we know that $A \cap U_n$ is nonempty for all n by construction.

By construction $x_n \in A$ and we can look at how this converges,

$$d(x_n, x) < \frac{1}{n}$$

since $\frac{1}{n} \to 0$ $x_n \to x$ this proves that

$$\bar{A} \subseteq \{x \in X \mid \exists x_n \in A \text{ such that } x_n \to x\}$$

which proves that

$$\bar{A} = \{x \in X \mid \exists x_n \in A \text{ such that } x_n \to x\}$$

☺

Question 2

Let f be a continuous map from a topological space X into a topological space Y. Let A be a subset of X. Show that

$$f(\bar{A}) \subset \overline{f(A)}.$$

Also, show by example of a function from $\mathbb{R}$ into $\mathbb{R}$, that $f(\bar{A})$ can be a proper subset of $\overline{f(A)}$.

Proof: Let $f : X \mapsto Y$ be a continuous map and $A \subset X$.
Since f is continuous, for some $U \in Y$ open, $f^{-1}(U)$ is open in X.
Let Z be closed in Y; then $Y \setminus Z$ is open.

$$\begin{aligned} f^{-1}(Y \setminus Z) &= \{x \in X \mid f(x) \in Y \setminus Z\} \\ &= \{x \in X \mid f(x) \notin Z\} \\ &= X \setminus \{y \in X \mid f(y) \in Z\} \\ &= X \setminus f^{-1}(Z). \end{aligned}$$

We know that $f^{-1}(Z)$ is closed in X, since Z is closed in Y. By the same argument, we look at $\overline{f(A)}$. $f^{-1}(\overline{f(A)})$ is closed. Since $A \subset f^{-1}(\overline{f(A)})$, we have $\bar{A} \subset f^{-1}(\overline{f(A)})$ by definition, since $\bar{A}$ is the smallest closed set containing A.
By simply applying f to both sides, we get:

$$f(\bar{A}) \subset \overline{f(A)}.$$

😄

Take, for example, the function $\frac{1}{1+x^2}$ on the domain $A = (0, \infty)$, which can be closed; $\bar{A} = [0, \infty)$.
Looking at this graph in **Figure 1** on this page, you can clearly see that:

$$f(A) = (0, 1),$$
$$f(\bar{A}) = (0, 1],$$

and

$$\overline{f(A)} = [0, 1].$$

This is an example of a function $f : \mathbb{R} \to \mathbb{R}$ where $f(\bar{A}) \subsetneq \overline{f(A)}$.

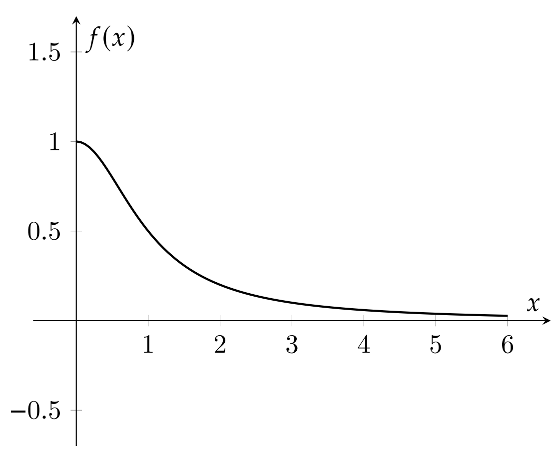

Figure 1: Graph of the function $f(x) = \frac{1}{1+x^2}$

Question 3

Let X be a hausdorff topological space and $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ a decreasing sequence of compact subsets of X. Let U be an open set in X. If $\bigcap_{n=1}^{\infty} K_n \subset U$, show that there exists n_0 such that $K_n \subset U$ for $n \geqslant n_0$.

Proof: For each n, define:

$$V_n = (X \setminus K_n) \cup U.$$

Each V_n is open because $X \setminus K_n$ is open, U is open, and a finite union of open sets is open. Consider the union:

$$\bigcup_{n=1}^{\infty} V_n = \left(X \setminus \bigcap_{n=1}^{\infty} K_n \right) \cup U.$$

This forms an open cover of X, which also implies that it is an open cover of K_1. Since $\bigcap_{n=1}^{\infty} K_n \subset U$, we conclude that

$$K_1 \subset \bigcup_{n=1}^{\infty} V_n.$$

Since we are in a Hausdorff space and K_1 is closed, it is also compact. By compactness, there exists a finite subcover

$$\{V_{n_1}, \ldots, V_{n_m}\}.$$

Since

$$K_n \supset K_{n+1} \implies V_n \subset V_{n+1},$$

there exists a largest index in this finite subcover, say $n_0 = \max(n_1, \ldots, n_m)$, such that

$$K_1 \subset V_{n_0}.$$

Expanding V_{n_0}, we obtain

$$K_1 \subset (X \setminus K_{n_0}) \cup U.$$

Since $K_{n_0} \subset K_1$, it follows that

$$K_{n_0} \subset U.$$

Finally, since K_n is a decreasing sequence, for all $n \geqslant n_0$, we have $K_n \subset U$. 😄

Question 4

Let f be a continuous real function on $[0,1]$ such that

$$\int_0^1 f(x)x^n\,dx = 0$$

for all integers $n \geqslant 0$. Show that $f = 0$.

We will use Weierstrass's theorem for approximating $f(x)$. By the theorem, there exists some polynomial $P(x)$ such that for any $\epsilon > 0$, $\left|f(x) - P(x)\right| < \epsilon$ for all $x \in [0,1]$. Take a sequence of polynomials $P_k(x)$ that converge uniformly to f, and consider the integral of f^2:

$$\begin{aligned}
\int_0^1 f^2(x)\,dx &= \lim_{k\to\infty} \int_0^1 f(x)P_k(x)\,dx \\
&= \lim_{k\to\infty} \int_0^1 f(x) \sum_{n=0}^{k} a_n x^n\,dx \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} a_n \int_0^1 f(x)x^n\,dx
\end{aligned}$$

From the given assumption that $\int_0^1 f(x)x^n\,dx = 0$, we get

$$\int_0^1 f^2(x)\,dx = 0.$$

Since $f^2(x)$ is nonnegative, we conclude that $f \equiv 0$.