Math 2280
Problem Set 8

Tannon Warnick

April 11, 2025

Question 1

Let f be a function on $\mathbb{R}^2$ defined by

$$f(x,y)=\begin{cases}0 & \text{if } (x,y)=(0,0)\\ \frac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\end{cases}$$

Prove that

(i) f is not continuous at $(0,0)$;

(ii) The first partial derivatives of f exist at every point of $\mathbb{R}^2$. Is f differentiable at $(0,0)$? Explain your answer!

(i) Looking at the limit of f from different paths we find:

Let $x=0$

$$f(0,y)=\frac{0}{y^2}=0 \implies \lim_{(x,y)\to(0,0)} f(x,y)=0$$

Let $x=y$

$$f(x,x)=\frac{x^2}{2x^2}=\frac{1}{2} \implies \lim_{(x,y)\to(0,0)} f(x,y)=\frac{1}{2}$$

The limits are different on these paths so the limit does not exist so f is not continuous at (0,0) even though it is well defined.

(ii) Taking partial derivatives of f we get:

$$\frac{\partial f}{\partial y}=\frac{x^3-y^2x}{x^4+2x^2y^2+y^4}$$
$$\frac{\partial f}{\partial x}=\frac{y^3-x^2y}{x^4+2x^2y^2+y^4}$$

These are functions that are defined for all $x^2+y^2>0$ and at $x^2+y^2=0$ the derivatives are 0 because of the inputted point at (0,0), so they are defined on $\mathbb{R}^2$. Looking at continuity of the partials we get that

for $x=0$

$$\frac{\partial f}{\partial x}(0,y)=\frac{y^3}{y^4}=\frac{1}{y}$$
$$\implies \lim_{(x,y)\to(0,0)^+} f(x,y)=+\infty \quad \lim_{(x,y)\to(0,0)^-} f(x,y)=-\infty$$

The same is for $y=0$ on the other partial. This tells us that ∂f is not continuous. So f is not differentiable at that discontinuous point (i.e. (0,0)).

Question 2

If F is a differentiable real function defined in a convex open set $U\subset\mathbb{R}^n$, such that $\partial_1 F(x)=0$ for every $x\in U$, prove that F depends only on $x_2,\ldots,x_n$.

We first define

$$\gamma(t)=(t,x_2,\ldots,x_n),\ \text{for } t\in I$$

Where $I\subset\mathbb{R}$ such that $\gamma(t)\in U\ \forall t\in I$, I is a connected interval since U is convex

We now look at

$$h(t)=f(\gamma(t))$$

$h(t)$ is a differentiable function from $I\to\mathbb{R}$. How we have constructed h lets us know that $h'(t)=0$ because $\partial_1 F=0$. Thus $h(t)$ is constant for all $t\in I$. Thus there exists G such that:

$$F(x_1,x_2,\ldots,x_n)=G(x_2,\ldots,x_n)$$

Question 3

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a map given by $F = (F_1, F_2)$ where

$$F_1(x, y) = e^x \cos(y) \quad \text{and} \quad F_2(x, y) = e^x \sin(y)$$

for any $(x, y) \in \mathbb{R}^2$. Then:

(i) Find the image of F.

(ii) Calculate the derivative $F'(x, y)$ and show that it is invertible at any point in $\mathbb{R}^2$.

Thus, by the inverse function theorem, F is locally invertible, i.e., for any $(x, y) \in \mathbb{R}^2$ there are open neighborhoods U of (x, y) and V of $F(x, y)$ such that $F : U \longrightarrow V$ is a bijection.
Show that F is not a bijection globally, i.e, F is not a bijection of $\mathbb{R}^2$ onto the image of F.

Question 4

Let f be a function on $\mathbb{R}$ defined by

$$f(x) = x + 2x^2 \sin\left(\frac{1}{x}\right)$$

for $x \neq 0$ and $f(0) = 0$. Show that

(i) f is continuous on $\mathbb{R}$;

(ii) f is differentiable on $\mathbb{R}$;

(iii) the derivative f' is not continuous at 0 ;

(iv) $f'(0) = 1$;

(v) for any $\epsilon > 0$, the restriction of f to $(-\epsilon, \epsilon)$ is not injective.

This shows that, even for $n = 1$, the conclusions of inverse function theorem do not hold if f' is not continuous.

Note:-

Hint: To prove (v), first show that a continuous function f cannot be injective in neighborhoods of local maxima and minima.
These must be critical points of f, i.e. zeros of f'.
Then show that for every $\epsilon > 0$ the interval $(-\epsilon, \epsilon)$ contains infinitely many critical points of f.
A critical point x of f is a maximum or minimum if $f''(x) \neq 0$.
Therefore, it is enough to show that there is an $\epsilon > 0$ such that there are no $x \in (-\epsilon, \epsilon)$ such that $f'(x) = 0$ and $f''(x) = 0$.
To prove this observe that the derivatives f' and f'' are linear functions in $A = \sin\left(\frac{1}{x}\right)$ and $B = \cos\left(\frac{1}{x}\right)$ with coefficients which are rational functions in x. Therefore, the equations $f'(x) = 0$ and $f''(x) = 0$ are a linear system of two equations for A and B with rational function coefficients.
Explicitly solve this system for A and B. Then calculate $A^2 + B^2$. From the result you should see that for small x this expression cannot be 1 , contradicting the choice of A and B. Therefore, for small x, f' and f'' cannot simultaneously vanish at x.