Math 3220-3
Take Home Midterm 2

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Question 1

(i) Let V be a vector space with inner product $(\cdot \mid \cdot)$.
Let

$$\|v\| = \sqrt{(v \mid v)}$$

be the corresponding norm. Show that

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

for any $u, v \in V$.

(ii) Consider the normed vector space $C([0,1])$ of continuous real functions on $[0,1]$ with norm $\|f\| = \max_{x\in[0,1]} |f(x)|$. Using (i), show that there is no inner product on $C([0,1])$ which defines this norm.

(i) ***Proofof i:*** In any vector space V, the inner product is linear in the first term and anti-linear in the second. We have:

$$\|u+v\|^2 + \|u-v\|^2 = (u+v \mid u+v) + (u-v \mid u-v)$$

Using the fact above:

$$(u+v \mid u+v) = (u \mid u) + (u \mid v) + (v \mid u) + (v \mid v)$$

$$(u-v \mid u-v) = (u \mid u) - (u \mid v) - (v \mid u) + (v \mid v)$$

$$\implies \|u+v\|^2 + \|u-v\|^2 = (u \mid u) + (u \mid v) + (v \mid u) + (v \mid v) + (u \mid u) - (u \mid v) - (v \mid u) + (v \mid v)$$

$$= 2(u \mid u) + 2(v \mid v) = 2\left(\|u\|^2 + \|v\|^2\right)$$

$$\implies \|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

😄

Proofof ii: The space $C([0,1])$ consists of all continuous real-valued functions on $[0,1]$, equipped with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$:

$$\|f\| = \max_{x\in[0,1]} |f(x)|.$$

Suppose, for contradiction, that there exists an inner product $(\cdot \mid \cdot)$ on $C([0,1])$ such that

$$\|f\|^2 = (f \mid f).$$

From part (i), this inner product must satisfy:

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2.$$

However, for the norm,

$$\|f+g\| = \max_{x\in[0,1]} |f(x) + g(x)|,$$

and

$$\|f - g\| = \max_{x\in[0,1]} |f(x) - g(x)|.$$

It is not generally true that:

$$\max_{x\in[0,1]} |f(x) + g(x)|^2 + \max_{x\in[0,1]} |f(x) - g(x)|^2 = 2 \max_{x\in[0,1]} |f(x)|^2 + 2 \max_{x\in[0,1]} |g(x)|^2.$$

A counterexample is given by choosing:

$$f(x) = x, \quad g(x) = 1 - x.$$

Then,

$$\|f\| = 1, \quad \|g\| = 1,$$

looking at their sum and difference:

$$\|f + g\| = 1, \quad \|f - g\| = 1.$$

Thus,

$$\|f + g\|^2 + \|f - g\|^2 = 1^2 + 1^2 = 2.$$

However,

$$2\|f\|^2 + 2\|g\|^2 = 2(1^2) + 2(1^2) = 4.$$

This is a contradiction because $2 \neq 4$, so there is no inner product that can define this norm. ☺

Question 2

Let V be a complex vector space with inner product. Let $u, v \in V, u \neq 0$ and $v \neq 0$. Show that the following two statements are equivalent:

(i) the vectors u and v are proportional;

(ii)
$$|(u \mid v)| = \|u\| \cdot \|v\|$$

Proof: $\implies$ If u and v are proportional, then there exists some scalar $k \in \mathbb{C}$ such that $u = kv$. Taking the inner product,

$$|(u \mid v)| = |(kv \mid v)| = |k||(v \mid v)| = |k|\|v\|^2 = \|kv\| \cdot \|v\| = \|u\| \cdot \|v\|.$$

$\impliedby$ Suppose $|(u \mid v)| = \|u\| \cdot \|v\|$. Consider the projection of u onto v in their respective Cartesian and polar forms, where θ is the angle between them:

$$\text{Proj}_v(u) = \frac{(u \mid v)}{\|v\|^2} v = \frac{|(u \mid v)|e^{i\theta}}{\|v\|^2} v$$

From our given assumption $|(u \mid v)| = \|u\| \cdot \|v\|$, we can rewrite that as

$$\text{Proj}_v(u) = \frac{|(u \mid v)|e^{i\theta}}{\|v\|^2} v = \frac{\|u\|e^{i\theta}}{\|v\|} v.$$

By definition, the projection of u onto v represents the component of u that lies along v. If $\text{Proj}_v(u) = u$, then u must be entirely in the span of v, meaning that there exists some scalar $k \in \mathbb{C}$ such that

$$u = kv.$$

Now we must prove that indeed $\text{Proj}_v(u) = u$. If $\|\text{Proj}_v(u)\| \neq \|u\|$, then u has an orthogonal component to v, meaning u is not a scalar multiple of v. But if $\|\text{Proj}_v(u)\| = \|u\|$, then u must lie entirely in the span of v, and thus $u = kv$ for some scalar k. Looking at the magnitude of $\text{Proj}_v(u)$:

$$\|\text{Proj}_v(u)\| = \frac{\|u\||e^{i\theta}|}{\|v\|}\|v\| = \|u\||e^{i\theta}| = \|u\|.$$

So it is proven. 😆

Question 3

Let $C\left(S^1\right)$ be the algebra of complex continuous functions on the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ in the complex plane. Consider the subalgebra $\mathcal{A}$ of all functions

$$f\left(e^{i\phi}\right) = \sum_{n=0}^{N} c_n e^{in\phi}$$

for real ϕ. Then $\mathcal{A}$ separates points on S^1 and vanishes at no point of S^1. Show that $\mathcal{A}$ is not dense in $C\left(S^1\right)$. (Hint: Show that $e^{-i\phi}$ is not in the closure of $\mathcal{A}$.)

Proof: In order for an algebra to be dense in some space, it must fulfill the requirements of the Stone-Weierstrass theorem. In this complex space, they are:

(1) $\mathcal{A}$ separates all points.

(2) $\mathcal{A}$ contains constant functions.

(3) $\mathcal{A}$ is closed under complex conjugation.

Consider the inner product $(\cdot \mid \cdot)$ on $C(S^1)$:

$$(f \mid g) = \int_0^{2\pi} f(e^{i\phi})\overline{g(e^{i\phi})}\, d\phi.$$

The Fourier basis functions satisfy the orthogonality relation:

$$\int_0^{2\pi} e^{in\phi}e^{-im\phi}\, d\phi = \begin{cases} 2\pi, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

Suppose, for contradiction of (3), that $e^{-i\phi}$ were in the closure of $\mathcal{A}$. Then there would exist a sequence $f_N \in \mathcal{A}$ such that

$$(f_N \mid e^{-i\phi}) \to (e^{-i\phi} \mid e^{-i\phi}) = \int_0^{2\pi} e^{-i\phi}e^{i\phi}\, d\phi = 2\pi.$$

However, since every function in $\mathcal{A}$ consists only of nonnegative Fourier terms, the inner product $(f_N \mid e^{-i\phi})$ always vanishes. Expanding the inner product:

$$(f_N \mid e^{-i\phi}) = \int_0^{2\pi} \left(\sum_{n=0}^{N} c_n e^{in\phi}\right) e^{i\phi}\, d\phi.$$

By linearity of integration, we can swap the sum and the integral:

$$\sum_{n=0}^{N} c_n \int_0^{2\pi} e^{i(n+1)\phi}\, d\phi.$$

Using the fundamental integral property of exponentials:

$$\int_0^{2\pi} e^{i(n+1)\phi}\, d\phi = \left[\frac{e^{i(n+1)\phi}}{i(n+1)}\right]_0^{2\pi} = 0, \quad \text{for all } n \geqslant 0.$$

Thus, every term in the sum evaluates to zero, implying that

$$(f_N \mid e^{-i\phi}) = 0 \quad \text{for all } f_N \in \mathcal{A}.$$

Since for all $f_N \in \mathcal{A}$, the inner product $(f_N \mid e^{-i\phi})$ vanishes, it does not fulfill (3). Therefore, by the Stone-Weierstrass theorem, $\mathcal{A}$ is not dense. 😄

Question 4

Let f be a continuous function on $\mathbb{R}$ periodic with period 2π, given by $f(x) = |x|$ for $-\pi \leqslant x \leqslant \pi$. Using Bessel's equality for its Fourier coefficients, prove that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

Proof: The function $f(x) = |x|$ is an *even* function, so its Fourier series will look like:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

The coefficients will be:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(nx)\, dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\, dx$$

We must use integration by parts to solve this integral:
Let

$$\begin{aligned} u &= x & du &= dx \\ dv &= \cos(nx) & v &= \tfrac{\sin(nx)}{n} \end{aligned}$$

Using this substitution, we get:

$$\frac{2}{\pi}\int x\cos(nx)\, dx = \frac{2}{\pi}\left[\frac{x\sin(nx)}{n} - \int \frac{\sin(nx)}{n}\, dx\right]$$

This simplifies to:

$$\frac{2}{\pi}\left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]$$

Evaluating at the bounds:

$$\frac{2}{\pi}\int_0^{\pi} x\cos(nx)\, dx = \frac{2}{\pi}\left[\frac{\pi\sin(n\pi)}{n} + \frac{\cos(n\pi)-1}{n^2}\right] = \frac{2}{\pi}\cdot\frac{\cos(n\pi)-1}{n^2}$$

Thus, the Fourier coefficients are:

$$a_n = \frac{2}{\pi}\cdot\frac{(-1)^n - 1}{n^2}$$

For *odd* n:

$$a_{2m+1} = -\frac{4}{\pi n^2}$$

For *even* n:

$$a_{2m} = 0$$

So only the odd coefficients contribute in the sum.
Applying Bessel's equality:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |x|^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

Computing the left side:

$$\frac{2}{\pi}\int_{0}^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

Computing the right side:

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 = \frac{a_0^2}{2} + \sum_{m=1}^{\infty}\left(\frac{-4}{\pi(2m+1)^2}\right)^2 = \frac{a_0^2}{2} + \frac{16}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4}$$

Using our first integral to find a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\cos(0\cdot x)\,dx = \frac{\pi^2}{2}\cdot\frac{2}{\pi} = \pi$$

Equating the sides and solving for the sum:

$$\frac{2\pi^2}{3} = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4}$$

Subtracting $\frac{\pi^2}{2}$:

$$\frac{\pi^2}{6} = \frac{16}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4}$$

Multiplying by $\frac{\pi^2}{16}$:

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4} = \frac{\pi^4}{96}$$

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