Chapter 1

Introduction to Fuzzy Set

In this chapter, the concept of fuzzy sets and the operations on the fuzzy set are discussed. The concepts are the generalizations of crisp sets. Classical sets are also called 'crisp' sets so as to distinguish them from fuzzy sets. In fact, the Crisp sets can be taken as special cases of fuzzy sets. Let A be a crisp set defined over the Universe X. Then for any element x in X, either x is a member of A or not. In fuzzy set theory, this property is generalized. Therefore, in a fuzzy set, it is not necessary that x is a full Member of the set or not a member. It can be a partial member of the sets.

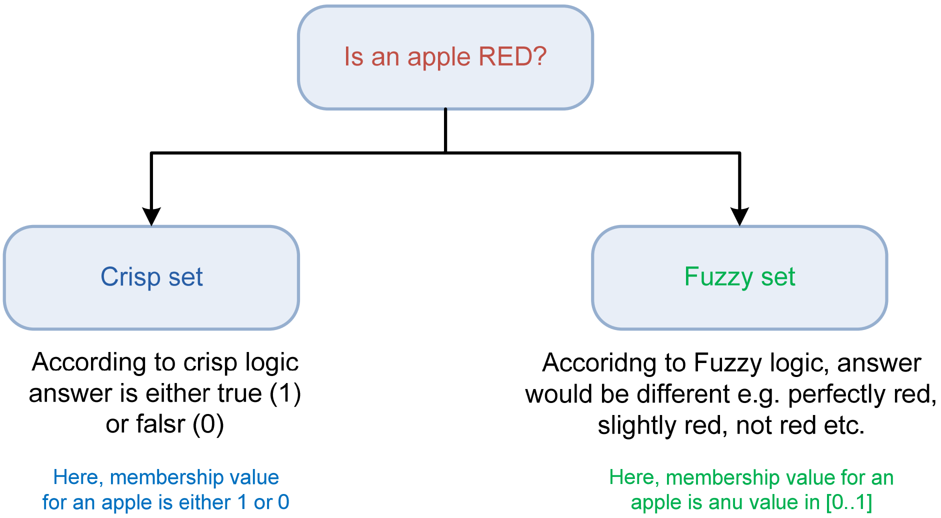


Figure 1: Crisp vs. Fuzzy sets

The generalization is performed as follows: For any crisp set A, it is possible to define a Characteristic function or membership function $\mu A = \{0, 1\}$.i.e. the characteristic function takes either of the values 0 or1 in the classical set. For a fuzzy set, the characteristic function can take any value between zero and one.

Definition

The membership function $\mu_A(x)$ of a fuzzy set A is a function $\mu_A : X \rightarrow [0,1]$
So, every element in x in X has membership degree: $\mu_A(x) \in [0,1]$
A is completely determined by the set of tuples: $A = \{(x, \mu_A(x))\ x \in X\}$

Example: Suppose someone wants to describe the class of cars having the property of being expensive by considering BMW, Rolls Royce, Mercedes, Ferrari, Fiat, Honda and Renault. Some cars like Ferrari and Rolls Royce are definitely expensive and some like Fiat and Renault are not expensive in comparison and do not belong to the set. Using a fuzzy set, the fuzzy set of expensive cars can be described as:

{(Ferrari, 1), (Rolls Royce, 1), (Mercedes, 0.8), (BMW, 0.7), (Honda,0.4)}. Obviously, Ferrari and Rolls Royce have membership value of 1 whereas BMW, which is less expensive, has a Membership value of 0.7 and Honda 0.4.

The Fuzzy set is similar to the super set of the Boolean logic with extra membership functions in between "true" and "false". As its name suggests, it is the logic underlying modes of reasoning which are approximate rather than exact. The importance of fuzzy logic derives from the fact that most modes of human reasoning and especially common sense reasoning are approximate in nature.
The essential characteristics of fuzzy logic are as follows.

- In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning.
- In fuzzy logic everything is a matter of degree.
- Any logical system can be fuzzified
- In fuzzy logic, knowledge is interpreted as a collection of elastic or, equivalently , fuzzy constraint on a collection of variables
- Inference is viewed as a process of propagation of elastic constraints.

Fuzzy Sets

Fuzzy Set Theory was formalized by Professor Lofti Zadeh at the University of California in 1965.

He proposed a paradigm shift that first gained acceptance in the Far East and its successful application has ensured its adoption around the world.

A paradigm is a set of rules and regulations which defines boundaries and tells us what to do to be successful in solving problems within these boundaries. For example the use of transistors instead of vacuum tubes is a paradigm shift - likewise the development of Fuzzy Set Theory from conventional bivalent set theory is a paradigm shift.

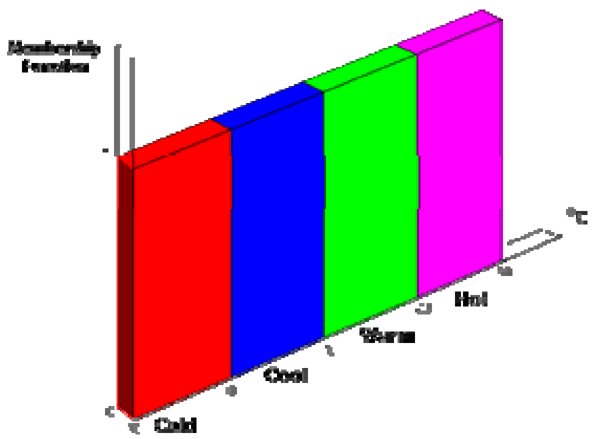


Figure 2: Example of a crisp set

Bivalent Set Theory can be somewhat limiting if we wish to describe a 'humanistic' problem mathematically. For example, Fig.1 below illustrates bivalent sets to characterize the temperature of a room.

The most obvious limiting feature of bivalent sets that can be seen clearly from the diagram is that they are mutually exclusive - it is not possible to have membership of more than one set Clearly, it is not accurate to define a transition from a quantity such as 'warm' to 'hot' by the application of one degree Fahrenheit of heat. In the real world a smooth (unnoticeable) drift from warm to hot would occur.

This natural phenomenon can be described more accurately by Fuzzy Set Theory. Fig.2 below shows how fuzzy sets quantifying the same information can describe this natural drift.

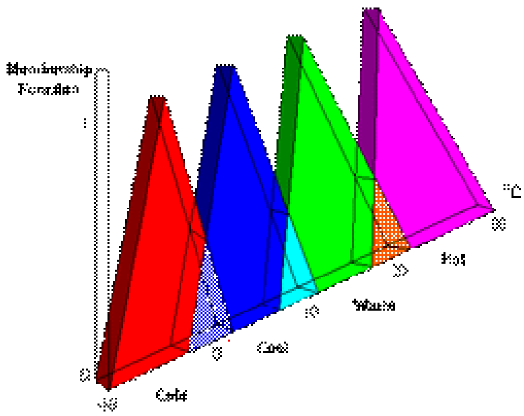


Figure 3: Example of a Fuzzy set

Properties of Fuzzy sets

Fuzzy sets follow the same properties as crisp sets. Since membership values of crisp sets are a subset of the interval [0,1], classical sets can be thought of as generalization of fuzzy sets.

$$
\begin{aligned}
&\text{Commutativity}: \tilde{A} U \tilde{B} = \tilde{B} U \tilde{A}\\
&\qquad \tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}\\
&\text{Associativity}: \tilde{A} \cup (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cup \tilde{C}\\
&\qquad \tilde{A} \cap (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cap \tilde{C}\\
&\text{Distributivity}: \tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})\\
&\text{Idempotency}: \tilde{A} \cup \tilde{A}) = \tilde{A}\\
&\qquad \tilde{A} \cap \tilde{A} = \tilde{A}\\
&\text{Identity}: \tilde{A} \cup \varnothing = \tilde{A} \qquad \tilde{A} \cap X = \tilde{A}\\
&\qquad \tilde{A} \cap \varnothing = \varnothing \qquad \tilde{A} \cup X = X\\
&\text{Transitivity}: \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}) \quad then \quad \tilde{A} \subseteq \tilde{C})\\
&\text{Involution}: \qquad \overline{\overline{\tilde{A}}} = \tilde{A}
\end{aligned}
$$

Operations on Fuzzy sets

The well-known operations which can be performed on fuzzy sets are the operations of union, intersection, complement, algebraic product and algebraic sum. Much research concerning fuzzy sets and their applications to automata theory, logic, control, game, topology, pattern recognition, integral, linguistics, taxonomy, system, decision making, information retrieval and so on, has been earnestly undertaken by using these operations for fuzzy sets.
In addition to these operations, new operations called "bounded-sum" and In addition to these operations, new operations called "bounded-sum" and "bounded-difference" are introduced by Zadeh (1975) to investigate the fuzzy reasoning which provides a way of dealing with the reasoning problems which are too complex for precise solution.

Types of operators

1. Equality
2. Complement
3. Intersection
4. Union
5. Algebraic product
6. Multiplication of fuzzy set with crisp number
7. Power of fuzzy set
8. Algebraic sum
9. Algebraic difference
10. Bounded sum
11. Bounded difference
12. Cartesian product
13. Composition

1. Equal fuzzy sets

Two fuzzy sets A(x) and B(x) are said to be equal, if $\mu_A(x) = \mu_B(x)$ for all $\mathbf{x} \in X$. It is expressed as follows

$$A(x) = B(x), \text{ if } \mu_A(x) = \mu_B(x)$$

Note: Two fuzzy sets A(x) and B(x) are said to be unequal, if $\mu_A(x) \neq \mu_B(x)$ for at least $\mathbf{x} \in X$.

Example:

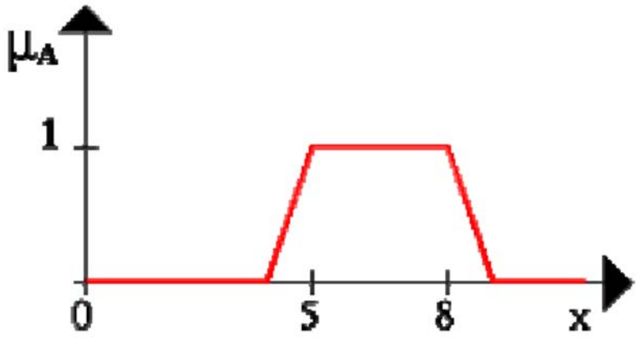
$A(x) = \{(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)\}$
$B(x) = \{(x_1,0.1),(x_2,0.5),(x_3,0.3),(x_4,0.6)\}$

As $\mu_A(x) \neq \mu_B(x)$ for different $x \in X$, $A(x) \neq B(x)$

2. Complement of fuzzy set A(x)

The complement is the opposite of the set. The complement of a fuzzy set is denoted by $\bar{A}(x)$ and is defined with respect to the universal set X as follows:

$$\bar{A}(x) = 1 - A(x) \text{ for all } x \in X$$



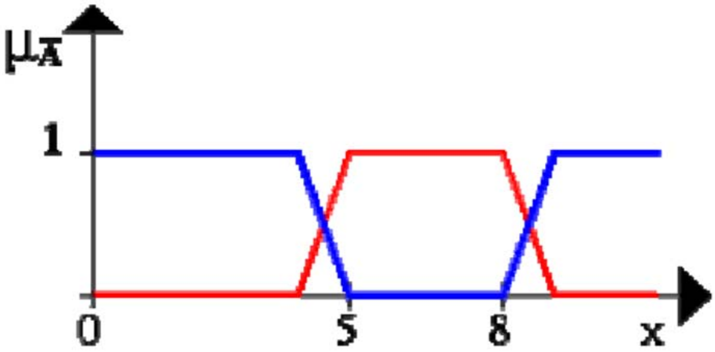


Figure 4: Example of complement operation on a fuzzy set

3. Intersections of fuzzy sets

Inter section of a fuzzy sets define how much of the element belongs to both sets. May have different degrees of membership in each set. The degree of membership is the lower membership in both sets of each element. Let A(x) and B(x) are two fuzzy sets, the intersection of is denoted by **(A∩B)(x)** and the membership function value is given as follows

$$\mu_{(A\cap B)}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Intersection is analogous to logical AND operation

Example

$A(x)=\{(x_1,0.7),(x_2,0.3),(x_3,0.9),(x_4,0.1)\}$
$B(x)=\{(x_1,0.2),(x_2,0.5),(x_3,0.7),(x_4,0.4)\}$

$\mu_{(A\cap B)}(x_1)= \min\{\mu_A(x_1),\mu_B(x_1)\} = \min\{0.7,0.2\} = 0.2$
$\mu_{(A\cap B)}(x_2)= \min\{\mu_A(x_2),\mu_B(x_2)\} = \min\{0.3,0.5\} = 0.3$
$\mu_{(A\cap B)}(x_3)= \min\{\mu_A(x_3),\mu_B(x_3)\} = \min\{0.9,0.7\} = 0.7$
$\mu_{(A\cap B)}(x_4)= \min\{\mu_A(x_4),\mu_B(x_4)\} = \min\{0.1,0.4\} = 0.1$

The graphical representation of the intersection operator is given below

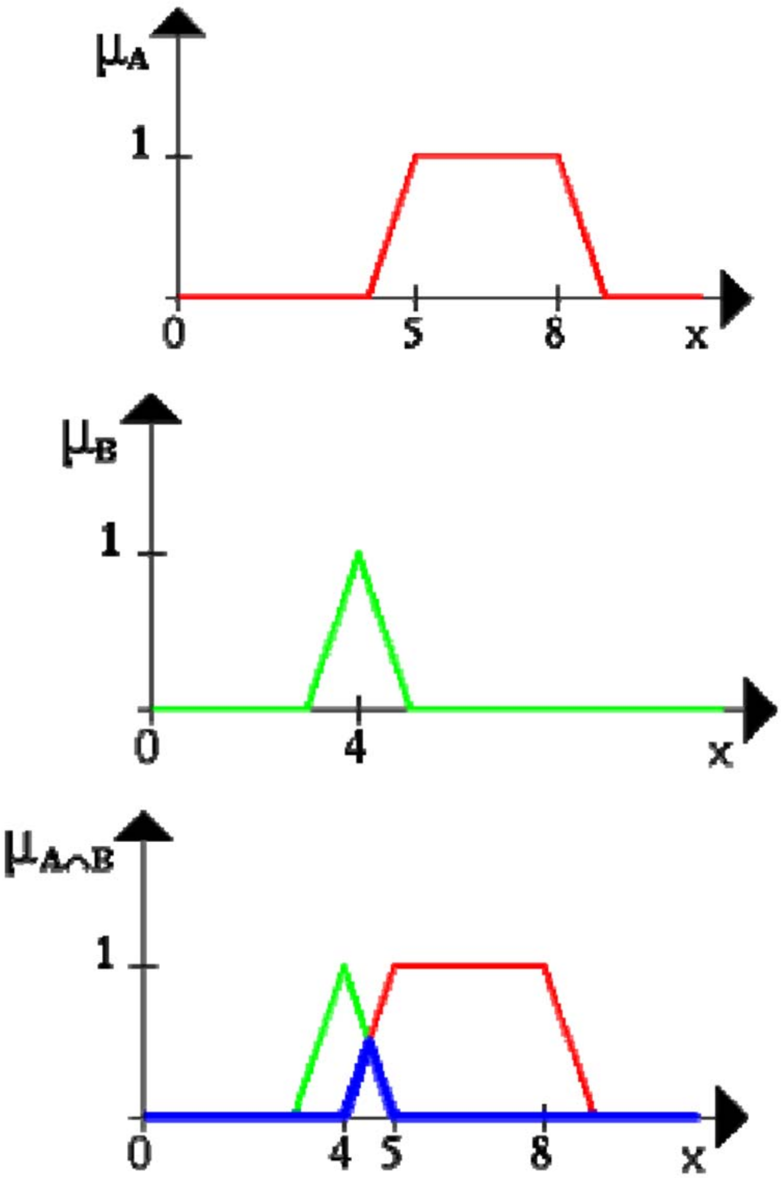


Figure 5: Example of intersection operation on a fuzzy set

4. Union of fuzzy sets

Union of fuzzy sets consists of every element that falls into either set. The value of the membership value is will be the largest membership value of the element in either set

Let A(x) and B(x) are two fuzzy sets for all **x** ∈ X, Union of fuzzy sets is denoted by **(AUB)(x)** and the membership function value is determined as follows

$$\mu_{(AUB)}(x)= \max\{\mu_A(x),\mu_B(x)\}$$

Example:

$A(x)=\{(x_1,0.7),(x_2,0.3),(x_3,0.9),(x_4,0.1)\}$

$B(x)=\{(x_1,0.2),(x_2,0.5),(x_3,0.7),(x_4,0.4)\}$

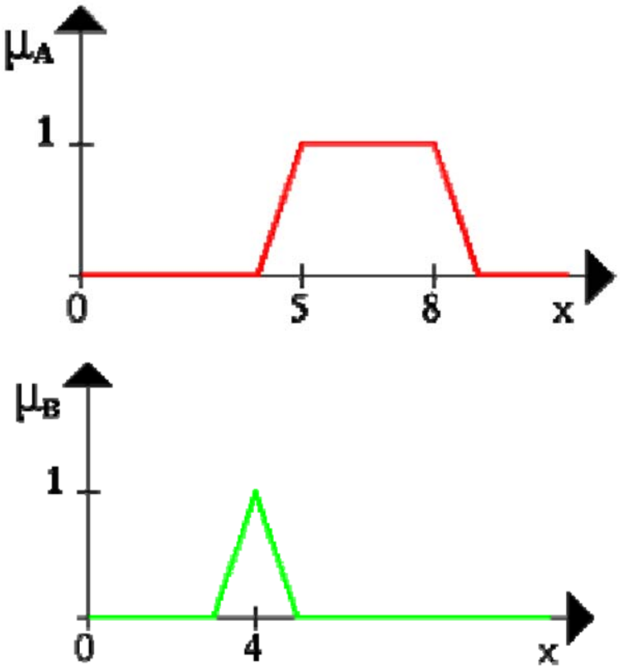
$\mu_{(AUB)}(x_1)= \max\{\mu_A(x_1),\mu_B(x_1)\} = \max\{0.7,0.2\} = 0.7$

$\mu_{(AUB)}(x_2)= \max\{\mu_A(x_2),\mu_B(x_2)\} = \max\{0.3,0.5\} = 0.5$

$\mu_{(AUB)}(x_3)= \max\{\mu_A(x_3),\mu_B(x_3)\} = \max\{0.9,0.7\} = 0.9$

$\mu_{(AUB)}(x_4)= \max\{\mu_A(x_4),\mu_B(x_4)\} = \max\{0.1,0.4\} = 0.4$

Note: Union is analogous to logical OR operation.



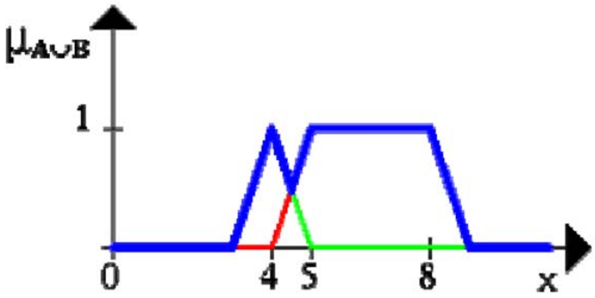


Figure 6: Example of union operation on a fuzzy set

5. Algebraic product of fuzzy sets

The Algebraic product of two fuzzy sets A(x) and B(x) for all **x** $\in$ X, is denoted by **A(x).B(x)** and defined as follows

$$A(x).B(x) = \{(x, \mu_A(x).\mu_B(x)), x \in X\}$$

Example

A(x) = {(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)}
B(x) = {(x_1,0.5),(x_2,0.7),(x_3,0.8),(x_4,0.9)}

A(x).B(x) = {(x_1,0.05),(x_2,0.14),(x_3,0.24),(x_4,0.36)}

6. Multiplication of fuzzy sets by a crisp number

The product of fuzzy set A(x) and a crisp number 'd' is expressed as follows

$$A(x).B(x)= \{(x, d \,.\, \mu_A(x)), x \in X\}$$

Example

Let us consider a fuzzy set A(x) such that

A(x)={(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)}
d = 0.2
then d.A(x) = {(x_1,0.02),(x_2,0.04),(x_3,0.06),(x_4,0.08)}

7. Power of a fuzzy set

The p-th power of a fuzzy set A(x) yields another fuzzy set $A^P(x)$, whose membership value can be determined as follows

$\mu_{A^P}(x) = \{\mu_A(x)\}^P$, $\mathbf{x} \in X$ }

p >= 1 → $A^P(x)$ is called concentration
p < 1 → $A^P(x)$ is called dilation

Example:

Let us consider a fuzzy set A(x)

$A(x)=\{(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)\}$

P=2

Then $A^2(x) = \{(x_1,0.01),(x_2,0.04),(x_3,0.09),(x_4,0.16)\}$

8. Algebraic sum of two fuzzy sets

The Algebraic sum of two fuzzy sets A(x) and B(x) for all $\mathbf{x} \in X$, is denoted by **A(x)+B(x)** and defined as follows

$$A(x)+B(x)= \{(x,\mu_{A+B}(x), x \in X\}$$

Where $\mu_{A+B}(x) = \mu_A(x)+\mu_B(x) - \mu_A(x).\mu_B(x)$

Example:

$A(x)=\{(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)\}$
$B(x)=\{(x_1,0.5),(x_2,0.7),(x_3,0.8),(x_4,0.9)\}$

Now (x)+B(x) = $\{(x_1,0.55),(x_2,0.76),(x_3,0.86),(x_4,0.94)\}$

9. Bounded sum of two fuzzy sets

The bounded sum of two fuzzy sets A(x) and B(x) for all $\mathbf{x} \in X$, is denoted by $A(x) \oplus \boldsymbol{B(x)}$ and defined as follows

$$A(x) \oplus \boldsymbol{B(x)} = \{(x,\mu_{A \oplus B}(x), \mathbf{x} \in X\}$$

Where $\mu_{A \oplus B}(x)= \min\{1,\mu_A(x)+\mu_B(x)\}$

Example:

$A(x)=\{(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)\}$
$B(x)=\{(x_1,0.5),(x_2,0.7),(x_3,0.8),(x_4,0.9)\}$

$A(x) \oplus \boldsymbol{B}(\boldsymbol{x})$ = {(x_1,0.6),(x_2,0.9),(x_3,1.0),(x_4,1.0)}

10. Algebraic deference of two fuzzy sets

The Algebraic deference of two fuzzy sets A(x) and B(x) for all **x** ∈ X, is denoted by **A(x)+B(x)** and defined as follows

$$A(x) - B(x) = \{(x, \mu_{A-B}(x), x \in X\}$$

Where $\mu_{A-B}(x) = \mu_{A \cap \bar{B}}(x)$

Example:

A(x)={(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)}
B(x)={(x_1,0.5),(x_2,0.7),(x_3,0.8),(x_4,0.9)}

$\bar{B}$(x) = {(x_1,0.5),(x_2,0.3),(x_3,0.2),(x_4,0.1)}

A(x) - B(x) = {(x_1,0.1),(x_2,0..2),(x_3,0.2),(x_4,0.1)}

11. Bounded sum of two fuzzy sets

The bounded difference of two fuzzy sets A(x) and B(x) for all **x** ∈ X, is denoted by **A(x)** ⊖ **B(x)** and defined as follows

$$A(x) \ominus \boldsymbol{B}(\boldsymbol{x}) = \{(x, \mu_{A \ominus B}(x), \mathbf{x} \in X\}$$

Where $\mu_{A \ominus B}(x) = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$

Example:
A(x)={(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)}
B(x)={(x_1,0.5),(x_2,0.7),(x_3,0.8),(x_4,0.9)}

$A(x) \ominus \boldsymbol{B}(\boldsymbol{x})$ = {(x_1,0),(x_2,0),(x_3,0.1),(x_4,0.3)}

12. Cartesian product of two fuzzy sets

Let us consider two fuzzy sets A(x) and B(y) defined on the Universal sets X and Y, respectively.The Cartesian product of fuzzy sets A(x) and B(y), is denoted by **A(x)** X **B(x), such that x** ∈ X, y∈ Y. It is determined, so that the following conditions satisfy

$$\mu_{(AXB)}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$$

Example:

$A(x)=\{(x_1,0.2),(x_2,0.3),(x_3,0.5),(x_4,0.6)\}$
$B(y)=\{(y_1,0.8),(y_2,0.6),(y_3,0.3)\}$

$\min\{\mu_A(x_1),\mu_B(y_1)\} = \min\{0.2,0.8\} = 0.2$ $\min\{\mu_A(x_1),\mu_B(y_2)\} = \min\{0.2,0.6\} = 0.2$
$\min\{\mu_A(x_1),\mu_B(y_3)\} = \min\{0.2,0.3\} = 0.2$

$\min\{\mu_A(x_2),\mu_B(y_1)\} = \min\{0.3,0.8\} = 0.3$ $\min\{\mu_A(x_2),\mu_B(y_2)\} = \min\{0.3,0.6\} = 0.3$
$\min\{\mu_A(x_2),\mu_B(y_3)\} = \min\{0.3,0.3\} = 0.3$

$\min\{\mu_A(x_3),\mu_B(y_1)\} = \min\{0.5,0.8\} = 0.5$ $\min\{\mu_A(x_3),\mu_B(y_2)\} = \min\{0.5,0.6\} = 0.5$
$\min\{\mu_A(x_3),\mu_B(y_3)\} = \min\{0.5,0.3\} = 0.3$

$\min\{\mu_A(x_4),\mu_B(y_1)\} = \min\{0.6,0.8\} = 0.6$ $\min\{\mu_A(x_4),\mu_B(y_2)\} = \min\{0.6,0.6\} = 0.6$
$\min\{\mu_A(x_4),\mu_B(y_3)\} = \min\{0.6,0.3\} = 0.3$

$$\mathbf{A \times B} = \begin{bmatrix} \mathbf{0.2} & \mathbf{0.2} & \mathbf{0.2} \\ \mathbf{0.3} & \mathbf{0.2} & \mathbf{0.3} \\ \mathbf{0.5} & \mathbf{0.5} & \mathbf{0.3} \\ \mathbf{0.6} & \mathbf{0.6} & \mathbf{0.3} \end{bmatrix}$$

Physical Significance of Fuzzy Operations

The physical significance of the operators on fuzzy sets can be explained with the help of an example as given below:

Example: A simple hollow shaft is 1-m radius and has a wall thickness of ($1/2\pi$) m. The shaft is built up stacking a ductile section and a brittle section. A downward force P and a torque T are simultaneously applied to the shaft. The failure properties of the two sections can be described by the following fuzzy sets A and B for the ductile and brittle sections as follows:

$$\tilde{A} = \{\frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5}\} \quad \text{and} \quad \tilde{B} = \{\frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5}\}$$

We can see the following:

1. The set of loadings for which either material B **or** material D will be "safe" can be obtained by getting $A \cup B$.
2. The set of loadings for which one expects that both material B and material D are "safe" can be obtained by forming $A \cap B$.
3. The complements A and B represent the set of loadings for material D and B are unsafe.
4. $A \mid B$ gives the set of loadings for which the ductile material is safe but the brittle is not.
5. $B \mid A$ gives the set of loadings for which the brittle material is safe but the ductile not.
6. De Morgans laws can be used to find which asserts that the loadings that are not safe with respect to both materials are the union of that are unsafe with respect to the brittle material with those that are unsafe for with respect to the ductile material.
7. De Morgans asserts that the loads that are safe for neither material D nor material B are the intersection of those that are unsafe for material D with those that are unsafe for material B.

Consequently, we can find the following:

$$\tilde{A} = \{\frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5}\} \qquad \tilde{B} = \{\frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5}\}$$

Compliments

$$\overline{\tilde{A}} = \{\frac{1}{1} + \frac{0}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5}\} \qquad \overline{\tilde{B}} = \{\frac{0.5}{2} + \frac{0.3}{3} + \frac{0.8}{4} + \frac{0.6}{5}\}$$

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