Operation Research II

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CONTENT

4. **QUANTITY DISCOUNTS FOR THE EOQ MODEL**
5. **SINGLE-PERIOD INVENTORY MODEL WITH PROBABILISTIC DEMAND**
   - Johnson Shoe Company
   - Nationwide Car Rental
6. **ORDER-QUANTITY,REORDER POINT MODEL WITH PROBABILISTIC DEMAND**
   - The How-Much-to-Order Decision
   - The When-to-Order Decision

PERIODIC REVIEW MODEL WITH PROBABILISTIC DEMAND

- More complex Periodic Review models

INTRODUCTION

Inventory refer to idle goods or materials held by an organization for use sometime in the future. Items carried in inventory include raw materials, purchased parts, components, sub-assemblies, work-in-process, finished goods, and supplies.

Some reasons organization maintain inventory include the difficulties in precisely predicting sales levels, production times, demand, and usage needs.

Thus, inventory serves as a buffer against uncertain and fluctuating usage and keeps a supply of items available in case the items are needed by the organization or its customers. Even though inventory serves an important and essential role, the expense associated with financing and maintaining inventories is a substantial part of the cost of doing business.

In large organizations, the cost associated with inventory can run into millions of dollars. In applications involving inventory, managers must answer two important questions:

1. How much should be ordered when the inventory is replenished?
2. When should the inventory be replenished?
   Virtually every business uses some sort of inventory management model or system to address the preceding questions.

ECONOMIC ORDER QUANTITY (EOQ) MODEL

The **economic order quantity (EOQ)** model is applicable when the demand for an item shows a constant, or nearly constant, rate and when the entire quantity ordered arrives in inventory at one point in time. The **constant demand rate** assumption means that the same number of units is taken from inventory each period of time, such as 5 units every day, 25 units every week, 100 units every four-week period, and so on.

To illustrate the EOQ model, let us consider the situation faced by the R&B Beverage Company. R&B Beverage is a distributor of beer, wine, and soft drink products. From a main warehouse located in Columbus, Ohio, R&B supplies nearly 1000 retail stores with beverage products. The beer inventory, which constitutes about 40% of the company's total inventory, averages approximately 50,000 cases. With an average cost per case of approximately \$8, R & B estimates the value of its beer inventory to be \$400,000.

The warehouse manager decided to conduct a detailed study of the inventory costs associated with Bub Beer, the number one selling R&B beer. The purpose of the study is to establish the how-much-to-order and the when-to-order decisions for Bub Beer that will result in the lowest possible total cost. As the first step in the study, the warehouse manager obtained the following demand data for the past 10 weeks:

Week	Demand (cases)
1	2000
2	2025
3	1950
4	2000
5	2100
6	2050
7	2000
8	1975
9	1900
10	2000
Total cases	20,000
Average cases per week	2000

Strictly speaking, these weekly demand figures do not show a constant demand rate. However, given the relatively low variability exhibited by the weekly demand, inventory planning with a constant demand rate of 2000 cases per week appears acceptable. In practice, you will find that the actual inventory situation seldom, if ever, satisfies the assumptions of the model exactly

The **how-much-to-order decision** involves selecting an order quantity that draws a compromise between

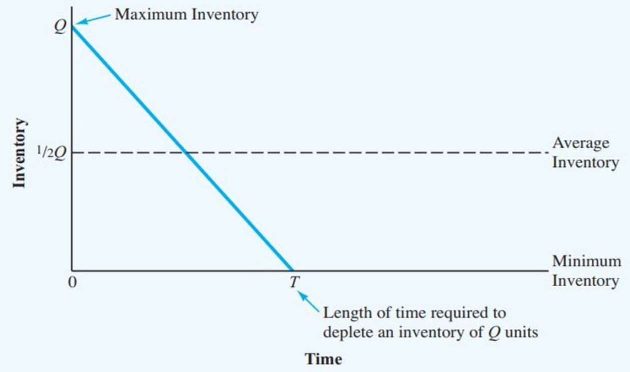
1. keeping small inventories and ordering frequently,
2. and keeping large inventories and ordering infrequently.

The first alternative can result in undesirably high ordering costs, whereas the second alternative can result in undesirably high inventory holding costs

Holding costs are the costs associated with maintaining or carrying a given level of inventory; these costs depend on the size of the inventory. The first holding cost to consider is the cost of financing the inventory investment. When a firm borrows money, it incurs an interest charge. If the firm uses its own money, it experiences an opportunity cost associated with not being able to use the money for other investments. In either case, an interest cost exists for the capital tied up in inventory.

This **cost of capital** is usually expressed as a percentage of the amount invested. The next step in the inventory analysis is to determine the ordering cost. This cost, which is considered fixed regardless of the order quantity, covers the preparation of the voucher, the processing of the order including payment, postage, telephone, transportation, invoice verification, receiving, and so on

FIGURE 10.1 INVENTORY FOR BUB BEER

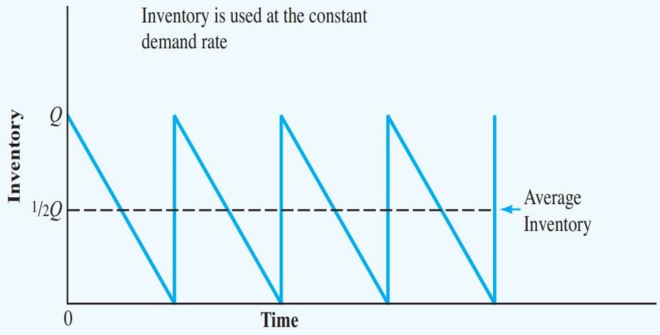


Let

$$I = \text{annual holding cost rate}$$
$$C = \text{unit cost of the inventory item}$$
$$C_h = \text{annual cost of holding one unit in inventory}$$

The annual cost of holding one unit in inventory is $C_h = IC$

FIGURE 10.2 INVENTORY PATTERN FOR THE EOQ INVENTORY MODEL



The general equation for the annual holding cost for the average inventory of $\mathbf{1/2Q}$ units is as follows:
Annual holding cost = (Average inventory)(Annual cost per unit)

$$=\frac{1}{2}QC_h$$

To complete the total cost model, we must now include the annual ordering cost. The goal is to express the annual ordering cost in terms of the order quantity $\mathbf{Q}$.

Let D denote the annual demand for the product.
If Co is the cost of placing one order, the general equation for the annual ordering cost is as follows:
Annual Odering cost = (Number of orders per year)(cost per order)

$$=\left(\frac{D}{Q}\right) C_o$$

Thus, the total annual cost, denoted TC, can be expressed as follows:
Total Annual cost = (Annual holding cost)+(Annual ordering cost)

$$\text{TC} = \frac{1}{2} Q C_h + \frac{D}{Q} \dot{C}_0$$

The How-Much-to-Order Decision

The next step is to find the order quantity Q that will minimize the total annual cost for Bub Beer. Using a trial-and-error approach, we can compute the total annual cost for several possible order quantities. As a starting point, let us consider $Q = 8000$. The total annual cost for Bub Beer is;

$$\mathrm{TC} = \mathrm{Q} + \frac{3,328000}{Q}$$

$$\mathrm{TC} = 8000 + \frac{3,328,000}{8000} = \$8{,}416$$

A trial order quantity of 5000 gives

$$\mathrm{TC} = 5000 + \frac{3,328,000}{5000} = \$5{,}666$$

The results of several other trial order quantities are shown in Table 10.1. It shows the lowest-cost solution to be about 2000 cases.

Graphs of the annual holding and ordering costs and total annual costs are shown in Figure 10.3. The advantage of the trial-and-error approach is that it is rather easy to do and provides the total annual cost for a number of possible order quantity decisions. In this case, the minimum cost order quantity appears to be approximately 2000 cases. The disadvantage of this approach, however, is that it does not provide the exact minimum cost order quantity.

Refer to Figure 10.3. The minimum total cost order quantity is denoted by an order size of Q*. By using differential calculus, it can be shown that the value of Q* that minimizes the total annual cost is given by the formula:

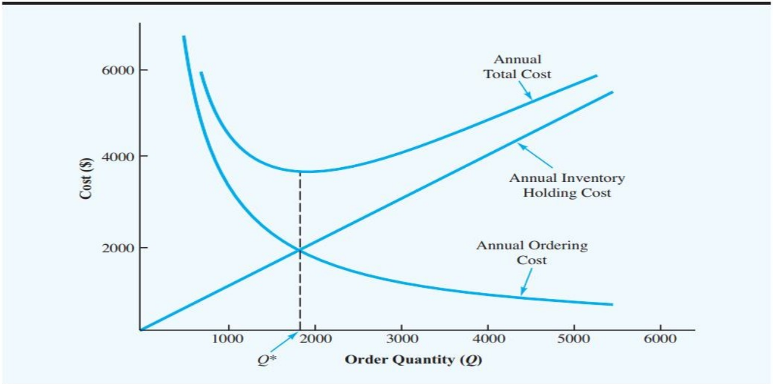
$$Q^* = \sqrt{\frac{2DC_0}{C_h}}$$

This formula is referred to as the economic order quantity (EOQ) formula

TABLE 10.1 ANNUAL HOLDING, ORDERING, AND TOTAL COSTS FOR VARIOUS ORDER QUANTITIES OF BUB BEER

	Annual Cost		
Order Quantity	Holding	Ordering	Total
5000	$5000	$ 666	$5666
4000	$4000	$ 832	$4832
3000	$3000	$1109	$4109
2000	$2000	$1664	$3664
1000	$1000	$3328	$4328

FIGURE 10.3 ANNUAL HOLDING, ORDERING, AND TOTAL COSTS FOR BUB BEER



Using equation (1.5), the minimum total annual cost order quantity for Bub Beer is

$$Q^* = \sqrt{\frac{2(104{,}000)32}{2}} = 1824 \text{ cases}$$

The use of an order quantity of 1824 in equation (1.4) shows that the minimum cost inventory policy for Bub Beer has a total annual cost of \$3649. Note that $Q^* = 1824$ balances the holding and ordering costs.

The When-to-Order Decision

Now that we know how much to order, we want to address the question of when to order. To answer this question, we need to introduce the concept of inventory position. **The inventory position** is defined as the amount of inventory on hand plus the amount of inventory on order. The when-to-order decision is expressed in terms of a **reorder point**-the inventory position at which a new order should be placed.

The manufacturer of Bub Beer guarantees a two-day delivery on any order placed by R&B Beverage. Hence, assuming R&B Beverage operates 250 days per year, the annual demand of 104,000 cases implies a daily demand of $104{,}000/250 = 416$ cases. Thus, we expect (2 days)(416 cases per day) 832 cases of Bub to be sold during the two days it takes a new order to reach the R&B warehouse.

In inventory terminology, the two-day delivery period is referred to as the **lead time** for a new order, and the 832-case demand anticipated during this period is referred to as the **lead-time demand**. Thus, R&B should order a new shipment of Bub Beer from the manufacturer when the inventory reaches 832 cases. For inventory systems using the constant demand rate assumption and a fixed lead time, the reorder point is the same as the lead-time demand. For these systems, the general expression for the reorder point is as follows:

$r = dm$

r = recorder point

d = demand per day

m = lead time for a new order in days

The question of how frequently the order will be placed can now be answered. The period between orders is referred to as the cycle time. Previously in equation (10.2), we defined D/Q as the number of orders that will be placed in a year. Thus, $D/Q^* = 104{,}000/1824 = 57$ is the number of orders R&B Beverage will place for Bub Beer each year.

If R&B places 57 orders over 250 working days, it will order approximately every $250/57 = 4.39$ working days. Thus, the cycle time is 4.39 working days. The general expression for a cycle time of T days is given by $\mathsf{T} = \frac{250}{D/Q*} = \frac{250Q*}{D}$

Sensitivity Analysis for the EOQ Model

Even though substantial time may have been spent in arriving at the cost per order ($32) and the holding cost rate (25%), we should realize that these figures are at best good estimates. Thus, we may want to consider how much the recommended order quantity would change with different estimated ordering and holding costs. To determine the effects of various cost scenarios, we can calculate the recommended order quantity under several different cost conditions.

TABLE 10.2 OPTIMAL ORDER QUANTITIES FOR SEVERAL COST POSSIBILITIES

Possible Inventory Holding Cost (%)	Possible Cost per Order	Optimal Order Quantity (Q^*)	Projected Total Annual Cost Using Q^*	Projected Total Annual Cost Using $Q = 1824$
24	\$30	1803	\$3461	\$3462
24	34	1919	3685	3690
26	30	1732	3603	3607
26	34	1844	3835	3836

Table 10.2 shows the minimum total cost order quantity for several cost possibilities. As you can see from the table, the value of Q* appears relatively stable, even with some variations in the cost estimates. Based on these results, the best order quantity for Bub Beer is in the range of 1700-2000 cases. If operated properly, the total cost for the Bub Beer inventory system should be close to \$3400 to \$3800 per year.

We also note that little risk is associated with implementing the calculated order quantity of 1824. For example, if holding cost rate = 24%, $C_o = \$34$, and the true optimal order quantity Q* = 1919, R&B experiences only a $5 increase in the total annual cost; that is, "$3690-$3685 ="$5, with Q = 1824. From the preceding analysis, we would say that this EOQ model is insensitive to small variations or errors in the cost estimates. This insensitivity is a property of EOQ models in general, which indicates that if we have at least reasonable estimates of ordering cost and holding cost, we can expect to obtain a good approximation of the true minimum cost order quantity.

EXCEL SOLUTION OF THE EOQ MODEL

Inventory models such as the EOQ model are easily implemented with the aid of worksheets. The Excel EOQ worksheet for Bub Beer is shown in Figure 10.4. The formula worksheet is in the background; the value worksheet is in the foreground. Data on annual

FIGURE 10.4 WORKSHEET FOR THE BUB BEER EOQ INVENTORY MODEL

	A	B	C
1	**Economic Order Quantity**		
2			
3	Annual Demand	104,000	
4	Ordering Cost	$32.00	
5	Annual Inventory Holding Rate %	25	
6	Cost per Unit	$8.00	
7	Working Days per Year	250	
8	Lead Time (Days)	2	
9			
10			
11	**Optimal Inventory Policy**		
12			
13	Economic Order Quantity	=SQRT(2*B3*B4/(B5/100*B6))	
14	Annual Inventory Holding Cost	=(1/2)*B13*(B5/100*B6)	
15	Annual Ordering Cost	=(B3/B13)*B4	
16	Total Annual Cost	=B14+B15	
17	Maximum Inventory Level	=B13	
18	Average Inventory Level	=B17/2	
19	Reorder Point	=(B3/B7)*B8	
20	Number of Orders per Year	=B3/B13	
21	Cycle Time (Days)	=B7/B20	

	A	B
1	**Economic Order Quantity**	
2		
3	Annual Demand	104,000
4	Ordering Cost	$32.00
5	Annual Inventory Holding Rate %	25
6	Cost per Unit	$8.00
7	Working Days per Year	250
8	Lead Time (Days)	2
9		
10		
11	**Optimal Inventory Policy**	
12		
13	Economic Order Quantity	1824.28
14	Annual Inventory Holding Cost	$1,824.28
15	Annual Ordering Cost	$1,824.28
16	Total Annual Cost	$3,648.56
17	Maximum Inventory Level	1824.28
18	Average Inventory Level	912.14
19	Reorder Point	832.00
20	Number of Orders per Year	57.01
21	Cycle Time (Days)	4.39

THE EOQ MODEL ASSUMPTIONS

1. Demand D is deterministic and occurs at a constant rate.
2. The order quantity Q is the same for each order. The inventory level increases by Q units each time an order is received.
3. The cost per order, C_0, is constant and does not depend on the quantity ordered.
4. The purchase cost per unit, C, is constant and does not depend on the quantity ordered.
5. The inventory holding cost per unit per time period, Ch, is constant. The total inventory holding cost depends on both Ch and the size of the inventory.
6. Shortages such as stockouts or backorders are not permitted.
7. The lead time for an order is constant.
8. The inventory position is reviewed continuously. As a result, an order is placed as soon as the inventory position reaches

ECONOMIC PRODUCTION LOT SIZE MODEL

The inventory model presented in this section is similar to the EOQ model in that we are attempting to determine how much we should order and when the order should be placed. We again assume a constant demand rate. However, instead of assuming that the order arrives in a shipment of size Q*, as in the EOQ model, we assume that units are supplied to inventory at a constant rate over several days or several weeks.

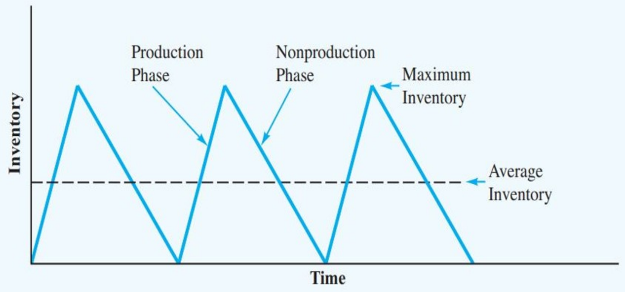
The **constant supply rate** assumption implies that the same number of units is supplied to inventory each period of time (e.g., 10 units every day or 50 units every week). This model is designed for production situations in which, once an order is placed, production begins and a constant number of units is added to inventory each day until the production run has been completed.

If we have a production system that produces 50 units per day and we decide to schedule 10 days of production, we have a $50(10) = 500$-unit production lot size. The lot size is the number of units in an order. In general, if we let Q indicate the production **lot size**, the approach to the inventory decisions is similar to the EOQ model; that is, we build a holding and ordering cost model that expresses the total cost as a function of the production lot size. Then we attempt to find the production lot size that minimizes the total cost.

One other condition that should be mentioned at this time is that the model only applies to situations where the production rate is greater than the demand rate; the production system must be able to satisfy demand. For instance, if the constant demand rate is 400 units per day, the production rate must be at least 400 units per day to satisfy demand.

In fact, in a production situation the ordering cost is more correctly referred to as the production setup cost. This cost, which includes labor, material, and lost production costs incurred while preparing the production system for operation, is a fixed cost that occurs for every production run regardless of the production lot size

FIGURE 10.5 INVENTORY PATTERN FOR THE PRODUCTION LOT SIZE INVENTORY MODEL



Total Cost Model

Let us begin building the production lot size model by writing the holding cost in terms of the production lot size Q. Again, the approach is to develop an expression for average inventory and then establish the holding costs associated with the average inventory. We use a one-year time period and an annual cost for the model

In the EOQ model the average inventory is one-half the maximum inventory or $\frac{1}{2}Q$. Figure 10.5 shows that for a production lot size model a constant inventory buildup rate occurs during the production run and a constant inventory depletion rate occurs during the nonproduction period; thus, the average inventory will be one-half the maximum inventory.

However, in this inventory system the production lot size Q does not go into inventory at one point in time, and thus the inventory never reaches a level of Q units. To show how we can compute the maximum inventory, let

$$\begin{aligned} d &= \text{daily demand rate} \\ p &= \text{daily production rate} \\ t &= \text{number of days for a production run} \end{aligned}$$

Because we are assuming that **p** will be larger than **d**, the daily inventory buildup rate during the production phase is **p = d**. If we run production for t days and place **p - d** units in inventory each day, the inventory at the end of the production run will be **(p - d)t**. From Figure 10.5 we can see that the inventory at the end of the production run is also the maximum inventory.

Thus,

Maximum memory = (p - d)t

If we know we are producing a production lot size of Q units at a daily production rate of p units, then Q = pt, and the length of the production run t must be

$$t = Q/P \text{ days}$$

Maximum inventory $= (\text{p - d})\text{t}{=}(\text{p - d})\frac{Q}{P}$
$=\left(1-\frac{d}{p}\right)Q$

The average inventory, which is one-half the maximum inventory, is given by

Average inventory $= \frac{1}{2}\left(1-\frac{d}{p}\right)Q$

With an annual per unit holding cost of C_h, the general equation for annual holding cost is as follows:
Annual holding cost $= \frac{1}{2}\left(1 - \frac{d}{p}\right) QC_h$

If D is the annual demand for the product and Co is the setup cost for a production run, then the annual setup cost, which takes the place of the annual ordering cost in the EOQ model, is as follows: Annual setup cost $= \frac{D}{Q} C_0$

Thus, the total annual cost (TC) model is

$$\mathrm{TC} = \frac{1}{2}\left(1 - \frac{d}{p}\right) QC_h + \frac{D}{Q} C_0$$

Suppose that a production facility operates 250 days per year. Then we can write daily demand d in terms of annual demand D as follows:

$$\mathrm{d} = \frac{D}{250}$$

Now let P denote the annual production for the product if the product were produced every day. Then

$$\mathrm{P} = 250\mathrm{p} \text{ and } \mathrm{p} = \frac{P}{250}$$

Thus

$$\frac{d}{p} = \frac{D/250}{P/250} = \frac{D}{P}$$

Therefore, we can write the total annual cost model as follows:

$$\mathrm{TC} = \frac{1}{2}\left(1 - \frac{D}{P}\right) QC_h + \frac{D}{Q} C_0$$

Equations (10.14) and (10.15) are equivalent. However, equation (10.15) may be used more frequently because an annual cost model tends to make the analyst think in terms of collecting annual demand data (D) and annual production data (P) rather than daily data.

Economic Production Lot Size

Given estimates of the holding cost (C_h), setup cost (C_o), annual demand rate (D), and annual production rate (P), we could use a trial-and-error approach to compute the total annual cost for various production lot sizes (Q). However, trial and error is not necessary; we can use the minimum cost formula for Q* that has been developed using differential calculus

$$\text{Q*} = \sqrt{\frac{2DC_0}{\left(1 - \frac{D}{P}\right) C_h}}$$

Example

Beauty Bar Soap is produced on a production line that has an annual capacity of 60,000 cases. The annual demand is estimated at 26,000 cases, with the demand rate essentially constant throughout the year. The cleaning, preparation, and setup of the production line cost approximately \$135. The manufacturing cost per case is \$4.50, and the annual holding cost is figured at a 24% rate. Thus, $C_h = IC = 0.24(\$4.50) = \$1.08$. What is the recommended production lot size? Using equation (10.16), we have

$$Q^* = \sqrt{\frac{2(26{,}000)(135)}{\left(1-\frac{26{,}000}{60{,}000}\right)(1.08)}}$$

The total annual cost using equation (1.15) and $Q^* = 3387$ is \$2073. Other relevant data include a five-day lead time to schedule and set up a production run and 250 working days per year. Thus, the lead-time demand of $(26{,}000/250)(5) = 520$ cases is the reorder point. The cycle time is the time between production runs. Using equation (1.7), the cycle time is $T = 250Q^*/D$ $[(250)(3387)]/26{,}000$, or 33 working days. Thus, we should plan a production run of 3387 units every 33 working days.

INVENTORY MODEL WITH PLANNED SHORTAGES

A *shortage*, or **Stockout**, is a demand that cannot be supplied. In many situations, shortages are undesirable and should be avoided if at all possible. However, in other cases it may be desirable-from an economic point of view-to plan for and allow shortages. In practice, these types of situations are most commonly found where the value of the inventory per unit is high and hence the holding cost is high.

An example of this type of situation is a new car dealer's inventory. Often a specific car that a customer wants is not in stock. However, if the customer is willing to wait a few weeks, the dealer is usually able to order the car.
The model developed in this section takes into account a type of shortage known as a **backorder**.

If we let S indicate the number of backorders that are accumulated when a new shipment of size Q is received, then the inventory system for the backorder case has the following characteristics:

- If S backorders exist when a new shipment of size Q arrives, then S backorders are shipped to the appropriate customers, and the remaining Q - S units are placed in inventory. Therefore, Q - S is the maximum inventory.

- The inventory cycle of T days is divided into two distinct phases: t_1 days when inventory is on hand and orders are filled as they occur, and t_2 days when stockouts occur and all new orders are placed on backorder.

The inventory pattern for the inventory model with backorders, where negative inventory represents the number of backorders, is shown in Figure 10.6. With the inventory pattern now defined, we can proceed with the basic step of all inventory models-namely, the development of a total cost model. For the inventory model with backorders, we encounter the usual holding costs and ordering costs. We also incur a backorder cost in terms of the labor and special delivery costs directly associated with the handling of the backorders

Another portion of the backorder cost accounts for the loss of goodwill because some customers will have to wait for their orders. Because the **goodwill cost** depends on how long a customer has to wait, it is customary to adopt the convention of expressing backorder cost in terms of the cost of having a unit on backorder for a stated period of time.

This method of costing backorders on a time basis is similar to the method used to compute the inventory holding cost, and we can use it to compute a total annual cost of backorders once the average backorder level and the backorder cost per unit per period are known.

Let us begin the development of a total cost model by calculating the average inventory for a hypothetical problem. If we have an average inventory of two units for three days and no inventory on the fourth day, what is the average inventory over the four-day period? It is

$$\frac{2units(3days) + 0units(1day)}{4days} = \frac{6}{4} = 1.5units$$

Refer to Figure 10.6. You can see that this situation is what happens in the backorder model. With a maximum inventory of **Q -S** units, the t_1 days we have inventory on hand will have an average inventory of **(Q - S)/2**. No inventory is carried for the t_2 days in which we experience backorders. Thus, over the total cycle time of T $= t_1 + t_2$ days, we can compute the average inventory as follows:

$$\text{Average Inventory} = \frac{0.5(Q-S)t_1 + 0t_5}{t_1 + t_2} = \frac{0.5(Q-S)t_1}{T}$$

Can we find other ways of expressing t1 and T? Because we know that the maximum inventory is **Q - S** and that d represents the constant daily demand, we have

$$t_1 = \frac{(Q-S)}{d} days$$

That is, the maximum inventory of **Q - S** units will be used up in **(Q - S)/d** days. Because Q units are ordered each cycle, we know the length of a cycle must be

$$T = \frac{Q}{d} days$$

Combining equations (1.18) and (1.19) with equation (1.17), we can compute the average inventory as follows:

$$\text{Average b} = \frac{\frac{1}{2}(Q-S)[(Q-S)/d]}{Q/d} = \frac{(Q-S)^2}{2Q}$$

Thus, the average inventory is expressed in terms of two inventory decisions: how much will order (Q) and the maximum number of backorders(S)

The formula for the annual number of orders placed using this model is identical to that for the EOQ model. With D representing the annual demand, we have

Average number of orders $= \frac{Q}{D}$

The next step is to develop an expression for the average backorder level. Because we know the maximum for backorders is S, we can use the same logic we used to establish average inventory in finding the average number of backorders.

We have an average number of backorders during the period t2 of $\frac{1}{2}$ the maximum number of backorders, or $\frac{1}{2}$S. We do not have any backorders during the t_1 days we have inventory; therefore, we can calculate the average backorders in a manner similar to equation (1.17). Using this approach, we have

Average backorders $= \frac{0t_1+(\frac{S}{2})t_2}{T} = \frac{(\frac{S}{2})t_2}{T}$

When we let the maximum number of backorders reach an amount S at a daily rate of d, the length of the backorder portion of the inventory cycle is

$$t_2 = \frac{s}{d}$$

Using equations (10.23) and (10.19) in equation (10.22), we have

$$= \frac{S^2}{2Q}$$

Let
C_h = cost to maintain one unit in inventory for one year
C_0 = cost per order C_b = cost to maintain one unit on backorder for one year.
The total annual cost (TC) for the inventory model with backorders becomes

$$TC = \frac{(Q-S)^2}{2Q}C_h + \frac{D}{Q}C_0 + \frac{S^2}{2Q}C_b$$

Given C_h, C_o, and C_b and the annual demand D, differential calculus can be used to show that the minimum cost values for the order quantity Q* and the planned backorders S* are as follows:

$$\text{Q*} = \sqrt{\tfrac{2DC_0}{C_h}\left(\tfrac{C_h+C_b}{C_b}\right)}$$

$$\text{S*} = \text{Q*}\left(\frac{C_h}{C_h + C_b}\right)$$

EXAMPLE

Suppose that the Higley Radio Components Company has a product for which the assumptions of the inventory model with backorders are valid. Information obtained by the company is as follows:

D = 2000 units per year, I = 20%

, C = \$50 per unit Ch = IC = (0.20)(\$50) = \$10 per unit per year

C_0 = \$25 per order

The company is considering the possibility of allowing some backorders to occur for the product. The annual backorder cost is estimated to be \$30 per unit per year. Using equations (1.26) and (1.27), we have

$$Q* = \sqrt{\frac{2(2000)(25)}{10}\left(\frac{10+30}{20}\right)} = 115.47$$

$$S^* = 115.47\left(\frac{10}{10+30}\right) = 28.87$$

If this solution is implemented, the system will operate with the following properties:

Maximum inventory = Q - S =115.47 - 28.87= 86.6

Cycle time = T=$\frac{Q}{D}$ (250)= $\frac{115.47}{2000}$ (250) = 14.43

The total annual cost is

$$\text{Holding cost} = \frac{(86.6)^2}{2(115.47)}(10) = \$325$$

$$\text{Ordering cost} = \frac{(2000)}{2(115.47)}(25) = \$433$$

$$\text{Backorder cost} = \frac{(28.87)^2}{2(115.47)}(30) = \$108$$

Total cost = \$866

If the company chooses to prohibit backorders and adopts the regular EOQ model, the recommended inventory decision would be

$$Q* = \sqrt{\frac{2(2000)(25)}{10}} = Q* = \sqrt{10,000} = 100$$

This order quantity would result in a holding cost and an ordering cost of **$500** each, or a total annual cost of **$1000**. Thus, in this problem, allowing backorders is projecting a **$1000 - $866 = $134** or **13.4%**
savings in cost from the no-stockout EOQ model. The preceding comparison and conclusion are based on the assumption that the backorder model with an annual cost per backordered unit of **$30** is a valid model for the actual inventory situation.

If the company is concerned that stockouts might lead to lost sales, then the savings might not be enough to warrant switching to an inventory policy that allowed for planned shortages.

QUANTITY DISCOUNTS FOR THE EOQ MODEL

Quantity discounts occur in numerous situations in which suppliers provide an incentive for large order quantities by offering a lower purchase cost when items are ordered in larger quantities. In this section we show how the EOQ model can be used when quantity discounts are available. Assume that we have a product in which the basic EOQ model (see Table 10.3) is applicable. Instead of a fixed unit cost, the supplier quotes the following discount schedule:

Discount Category	Order Size	Discount (%)	Unit Cost
1	0 to 999	0	\$5.00
2	1000 to 2499	3	4.85
3	2500 and over	5	4.75

The 5% discount for the 2500-unit minimum order quantity looks tempting. However, realizing that higher order quantities result in higher inventory holding costs, we should prepare a thorough cost analysis before making a final ordering and inventory policy recommendation.

Example

Suppose that the data and cost analyses show an annual holding cost rate of **20%**, an ordering cost of **\$49** per order, and an annual demand of 5000 units; what order quantity should we select? The following three-step procedure shows the calculations necessary to make this decision. In the preliminary calculations, we use **Q1** to indicate the order quantity for discount category 1, **Q2** for discount category 2, and Q3 for discount category 3.

Step 1

For each discount category, compute a Q* using the EOQ formula based on the unit cost associated with the discount category. Recall that the EOQ model provides the Q* $= \sqrt{2DC_0/C_h}$ Where $=$ IC $=$ (0.20)C. With three discount categories providing three Different units costs **C**, we obtain:

$$Q_1* = \sqrt{\frac{2(5000)49}{(0.20)(5.00)}} = 700$$

$$Q_2* = \sqrt{\frac{2(5000)49}{(0.20)(4.85)}} = 711$$

$$Q_3* = \sqrt{\frac{2(5000)49}{(0.20)(4.75)}} = 718$$

Because the only differences in the EOQ formulas come from slight differences in the holding cost, the economic order quantities resulting from this step will be approximately the same.

However, these order quantities will usually not all be of the size necessary to qualify for the discount price assumed. In the preceding case, both and are insufficient order quantities to obtain their assumed discounted costs of **$4.85** and **$4.75**, respectively. For those order quantities for which the assumed price cannot be obtained, the following procedure must be used:

Step 2

For the Q* that is too small to qualify for the assumed discount price, adjust the order quantity upward to the nearest order quantity that will allow the product to be purchased at the assumed price
In our example, this adjustment causes us to set

$$Q_2 * = 1000$$

and

$$Q_3 * = 2500$$

If a calculated **Q*** for a given discount price is large enough to qualify for a bigger discount, that value of **Q*** cannot lead to an optimal solution. Although the reason may not be obvious, it does turn out to be a property of the EOQ quantity discount model. In the previous inventory models considered, the annual purchase cost of the item was not included because it was constant and never affected by the inventory order policy decision.

However, in the quantity discount model, the annual purchase cost depends on the order quantity and the associated unit cost. Thus, annual purchase cost (annual demand D x unit cost C) is included in the equation for total cost, as shown here:

$$TC = \frac{Q}{2}C_h + \frac{D}{Q}C_0 + DC$$

TABLE 10.4 :TOTAL ANNUAL COST CALCULATIONS FOR THE EOQ MODEL WITH QUANTITY DISCOUNTS

TABLE 10.4 TOTAL ANNUAL COST CALCULATIONS FOR THE EOQ MODEL WITH QUANTITY DISCOUNTS

Discount Category	Unit Cost	Order Quantity	Annual Cost Holding	Annual Cost Ordering	Annual Cost Purchase	Annual Cost Total
1	$5.00	700	$ 350	$350	$25,000	$25,700
2	4.85	1000	$ 485	$245	$24,250	$24,980
3	4.75	2500	$1188	$ 98	$23,750	$25,036

Using this total cost equation, we can determine the optimal order quantity for the EOQ discount model in step 3. . For each order quantity resulting from steps 1 and 2, compute the total annual cost using the unit price from the appropriate discount category and equation (1.28). The order quantity yielding the minimum total annual cost is the optimal order quantity.

The step 3 calculations for the example problem are summarized in **Table 10.4.** As you can see, a decision to order 1000 units at the 3% discount rate yields the minimum cost solution. Even though the 2500-unit order quantity would result in a 5% discount, its excessive holding cost makes it the second-best solution. **Figure 10.7** shows the total cost curve for each of the three discount categories. Note that Q* 1000 provides the minimum cost order quantity.

SINGLE-PERIOD INVENTORY MODEL WITH PROBABILISTIC DEMAND

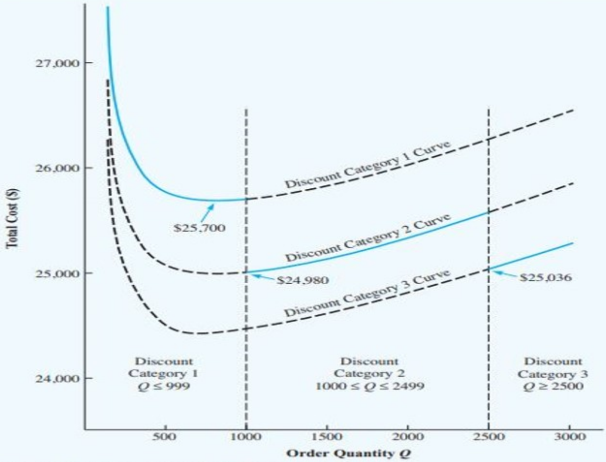
The inventory models discussed thus far were based on the assumption that the demand rate is constant and **deterministic** throughout the year. We developed minimum cost order quantity and reorder point policies based on this assumption. In situations in which the demand rate is not deterministic, other models treat demand as **probabilistic** and best described by a probability distribution. In this section we consider a **single-period inventory model** with probabilistic demand.

The single-period inventory model refers to inventory situations in which one order is placed for the product; at the end of the period, the product has either sold out or a surplus of unsold items will be sold for a salvage value. The single-period inventory model is applicable in situations involving seasonal or perishable items that cannot be carried in inventory and sold in future periods. Seasonal clothing (such as bathing suits and winter coats) is typically handled in a single-period manner.

In these situations, a buyer places one preseason order for each item and then experiences a stockout or holds a clearance sale on the surplus stock at the end of the season. No items are carried in inventory and sold the following year. Newspapers are another example of a product that is ordered one time and is either sold or not sold during the single period.

Although newspapers are ordered daily, they cannot be carried in inventory and sold in later periods. Thus, newspaper orders may be treated as a sequence of single-period models; that is, each day or period is separate, and a single-period inventory decision must be made each period (day). Because we order only once for the period, the only inventory decision we must make is how much of the product to order at the start of the period.

FIGURE 10.7 TOTAL COST CURVES FOR THE THREE DISCOUNT CATEGORIES



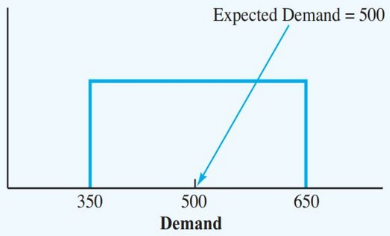
The overall minimum cost of $24,980 occurs at $Q^* = 1000$.

Obviously, if the demand were known for a single-period inventory situation, the solution would be easy; we would simply order the amount we knew would be demanded. However, in most single-period models, the exact demand is not known. In fact, forecasts may show that demand can have a wide variety of values. If we are going to analyze this type of inventory problem in a quantitative manner, we need information about the probabilities associated with the various demand values. Thus, the single-period model presented in this section is based on probabilistic demand.

Johnson Shoe Company

Let us consider a single-period inventory model that could be used to make a how-muchto-order decision for the Johnson Shoe Company. The buyer for the Johnson Shoe Company decided to order a men's shoe shown at a buyers' meeting in New York City. The shoe will be part of the company's spring–summer promotion and will be sold through nine retail stores in the Chicago area. Because the shoe is designed for spring and summer months, it cannot be expected to sell in the fall.

FIGURE 10.8 UNIFORM PROBABILITY DISTRIBUTION OF DEMAND FOR THE JOHNSON SHOE COMPANY PROBLEM



Johnson plans to hold a special August clearance sale in an attempt to sell all shoes not sold by July 31. The shoes cost \$40 a pair and retail for \$60 a pair. At the sale price of \$30 a pair, all surplus shoes can be expected to sell during the August sale. If you were the buyer for the Johnson Shoe Company, how many pairs of the shoes would you order? An obvious question at this time is, What are the possible values of demand for the shoe?

We need this information to answer the question of how much to order. Let us suppose that the uniform probability distribution shown in Figure 10.8 can be used to describe the demand for the size 10D shoes. In particular, note that the range of demand is from 350 to 650 pairs of shoes, with an average, or expected, demand of 500 pairs of shoes.

Incremental analysis is a method that can be used to determine the optimal order quantity for a single-period inventory model. Incremental analysis addresses the howmuch-to-order question by comparing the cost or loss of ordering one additional unit with the cost or loss of not ordering one additional unit. The costs involved are defined as follows:

Co = cost per unit of overestimating demand. This cost represents the loss of ordering one additional unit and finding that it cannot be sold.

cu = cost per unit of underestimating demand. This cost represents the opportunity loss of not ordering one additional unit and finding that it could have been sold. In incremental analysis, we consider the possible losses associated with an order quantity of 501(ordering one additional unit) and an order quantity of 500 (not ordering one additional unit). The order quantity alternatives and the possible losses are summarized here:

Order Quantity Alternatives	Loss Occurs If	Possible Loss	Probability Loss Occurs
$Q = 501$	Demand overestimated; the additional unit *cannot* be sold	$c_o = \$10$	$P(\text{demand} \leq 500)$
$Q = 500$	Demand underestimated; an additional unit *could have* been sold	$c_u = \$20$	$P(\text{demand} > 500)$

By looking at the demand probability distribution in Figure 10.8, we see that P(demand $>$ 500) $=$ 0.50 and that P(demand $>$ 500) $=$ 0.50. By multiplying the possible losses, Co $=$ \$10 and Cu $=$ \$20, by the probability of obtaining the loss, we can compute the expected value of the loss, or simply the expected loss (EL), associated with the order quantity alternatives. Thus,

$$\text{EL(Q} = \text{501)} = C_o P(demand \leq 500) = \$10(0.50) = \$5$$
$$\text{EL(Q} = \text{500)} = \text{CuP(demand} > \text{500)} = \$20(0.50) = \$10$$

Although we could continue this unit-by-unit analysis, it would be time-consuming and cumbersome. We would have to evaluate $Q = 502$, $Q = 503$, $Q = 504$, and so on, until we found the value of Q where the expected loss of ordering one incremental unit is equal to the expected loss of not ordering one incremental unit; that is, the optimal order quantity Q^* occurs when the incremental analysis shows that;

$$EL\ (Q^* + 1) = EL(Q^*)$$

When this relationship holds, increasing the order quantity by one additional unit has no economic advantage. Using the logic with which we computed the expected losses for the order quantities of 501 and 500, the general expressions for $EL(Q^* + 1)$ and $EL(Q^*)$ can be written

EL(Q* + 1) $= C_0 P(demand \leq Q*)$ ...(1.30)
EL(Q*) $= CuP(demand > Q*)$ ..(1.31)

Because we know from basic probability that

P(demand $\leq$) + P(demand > Q*) = 1 .. (1.32)

We can write that

P(demand > Q*) = 1 - P(demand $\leq$ Q*) ..(1.33)

Using this expression, equation (1.31) can be rewritten as

EL(Q*) $=$ Cu[1 - P(demand $\leq$ Q*)]

Equations (1.30) and (1.34) can be used to show that

EL(Q* $+$ 1) $=$ EL(Q*) whenever
CoP(demand $\leq$ Q*) $=$ Cu[$1 - P(demand \leq Q*)$] .. (1.35)

Solving for P(demand Q*), we have

$$\text{P(demand} \leq \text{Q*)} = \frac{Cu}{Cu + Co}$$

This expression provides the general condition for the optimal order quantity Q* in the single-period inventory model. In the Johnson Shoe Company problem $C_o = 10$ and $C_u = 20$. Thus, equation (10.36) shows that the optimal order size for Johnson shoes must satisfy the following condition:

$$\text{P(demand} \leq \text{Q*)} = \frac{20}{20 + 10} = \frac{2}{3}$$

We can find the optimal order quantity Q* by referring to the probability distribution shown in Figure 10.8 and finding the value of Q that will provide $P(\text{demand} \leq Q) = \frac{2}{3}$. To find this solution, we note that in the uniform distribution the probability is evenly distributed over the entire range of 350 to 650 pairs of shoes. Thus, we can satisfy the expression for **Q*** by moving two-thirds of the way from 350 to 650. Because this range is 650 - 350 = 300, we move 200 units from 350 toward 650. Doing so provides the optimal order quantity of 550 pairs of shoes.