GATE 2021 CH-36

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Question: For the ordinary differential equation

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 1,$$

with initial conditions $y(0) = y'(0) = y''(0) = y'''(0) = 0$, the value of

$$\lim_{t\to\infty} y(t) = ?$$

(round off to 3 decimal places). (GATE CH 2021)

Solution:

Parameter	Value	Description
$y(0)$	0	Initial displacement
$y'(0)$	0	First derivative at $t = 0$
$y''(0)$	0	Second derivative at $t = 0$
$y'''(0)$	0	Third derivative at $t = 0$

TABLE 0
Parameters

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 1 \tag{1}$$

Applying the Laplace transform to both sides:

$$s^3Y(s) + 6s^2Y(s) + 11sY(s) + 6Y(s) = \frac{1}{s} \tag{2}$$

$$Y(s)(s^3 + 6s^2 + 11s + 6) = \frac{1}{s} \tag{3}$$

$$\implies Y(s) = \frac{1}{s(s+1)(s+2)(s+3)}; \quad Re(s) > 0 \tag{4}$$

$$Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + \frac{D}{s+3} \tag{5}$$

$$1 = A(s+1)(s+2)(s+3) + Bs(s+2)(s+3) + Cs(s+1)(s+3) + Ds(s+1)(s+2) \tag{6}$$

$$1 = A(s^3 + 6s^2 + 11s + 6) + Bs(s^2 + 5s + 6) + Cs(s^2 + 4s + 3) + Ds(s^2 + 3s + 2) \tag{7}$$

(8)

Comparing the coefficients on both sides

$$A+B+C+D=0 \tag{9}$$
$$6A+5B+4C+3D=0 \tag{10}$$
$$11A+6B+3C+2D=0 \tag{11}$$
$$6A=1 \tag{12}$$
$$A=1/6, B=-11/26, C=5/26, D=5/78 \tag{13}$$

Substitute these values

$$Y(s)=\frac{6}{s}-\frac{11}{26\,(s+1)}+\frac{5}{26\,(s+2)}+\frac{5}{78\,(s+3)} \tag{14}$$

Apply Inverse Laplace Transform

$$y(t)=6\mathcal{L}^{-1}\left(\frac{1}{s}\right)-11\mathcal{L}^{-1}\left(\frac{1}{26(s+1)}\right)+5\mathcal{L}^{-1}\left(\frac{1}{26(s+2)}\right)+5\mathcal{L}^{-1}\left(\frac{1}{78(s+3)}\right) \tag{15}$$
$$y(t)=6-\frac{11}{26}e^{-t}+\frac{5}{26}e^{-2t}+\frac{5}{78}e^{-3t} \tag{16}$$

Consider

$$\lim_{t\to\infty} y(t)=\lim_{t\to\infty}\left(6-\frac{11}{26}e^{-t}+\frac{5}{26}e^{-2t}+\frac{5}{78}e^{-3t}\right) \tag{17}$$
$$=6 \tag{18}$$

Verifying using Final Value Theorem :
The final value theorem states thatif $\lim_{t\to\infty} y(t)$ exists, then $\lim_{s\to 0} sY(s)$ also exists

$$\lim_{s\to 0} sY(s)=\lim_{s\to 0} s\left(\frac{6}{s}-\frac{11}{26(s+1)}+\frac{5}{26(s+2)}+\frac{5}{78(s+3)}\right) \tag{19}$$
$$=\lim_{s\to 0}\left(6-\frac{11s}{26(s+1)}+\frac{5s}{26(s+2)}+\frac{5s}{78(s+3)}\right) \tag{20}$$
$$=6 \tag{21}$$

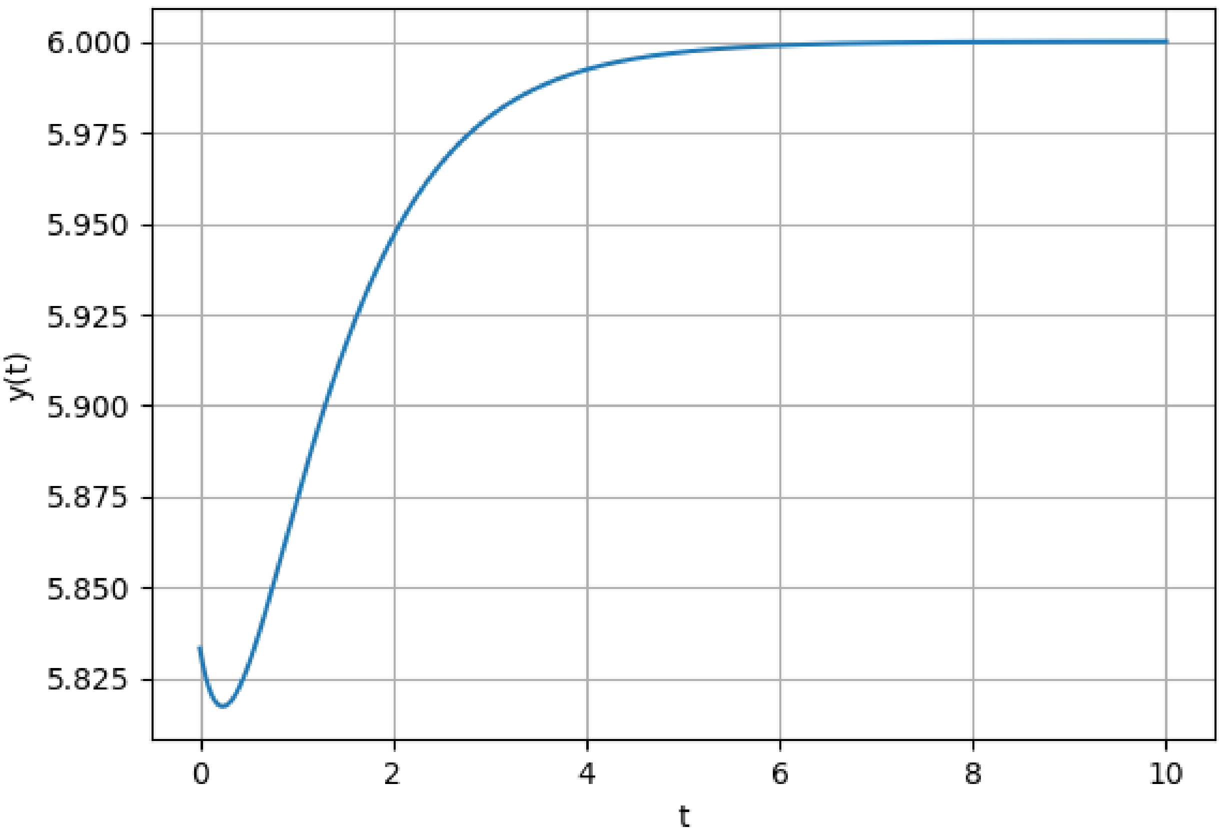

Fig. 0. Plot $y(t)$ vs t