NCERT Physics 11.15 Q14

EE23BTECH11212 - ABBURI TANUSHA*

Question: A wire stretched between two rigid supports vibrates in its fundamental mode with a frequency of 45 Hz. The mass of the wire is 3.5×10^{-2} kg, and its linear mass density is 4.0×10^{-2} kg/m. The length of the wire is 0.875 m. Determine the speed of a transverse wave on the string and the tension in the string.

Solution: The following information is provided in the question:

Parameters	Values
Mass per unit length	4.0×10^{-2} kg/m
Frequency	45 Hz
Mass	3.5×10^{-2} kg

TABLE I
PARAMETERS

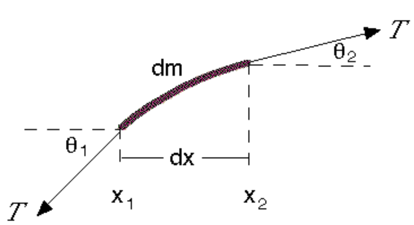

Fig. 1. Tensions acting on the small element of the string

The angle θ between the string and the x-direction is much smaller than 1, so $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Let's apply Newton's second law in the vertical y direction:

$$F_y = ma_y \tag{1}$$

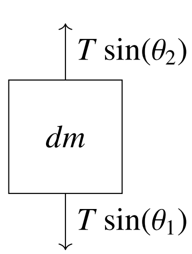

Fig. 2. Forces acting on the body in Y-direction

The sum of forces in the Y direction is

From Figure Fig. 2

$$F_y = T\sin\theta_2 - T\sin\theta_1 \tag{2}$$

Using the small angle approximation, $\sin\theta \approx \tan\theta = \frac{\partial y}{\partial x}$. So we may write:

$$F = T\left(\frac{dy}{dx}\bigg|_2 - \frac{dy}{dx}\bigg|_1\right) \tag{3}$$

The acceleration in the y direction is the rate of change in the y velocity, so $a_y = \frac{\partial v_y}{\partial t} = \frac{\partial^2 y}{\partial t^2}$.Now,Newton's second law in the y direction

$$F = T\left(\frac{dy}{dx}\bigg|_2 - \frac{dy}{dx}\bigg|_1\right) = \mu dx\frac{d^2y}{dt^2} \tag{4}$$

$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\left(\frac{dy}{dx}\bigg|_2 - \frac{dy}{dx}\bigg|_1\right) \tag{5}$$

Now we have been using the subscript 1 to identify the position x, and 2 to identify the position $(x+dx)$.

$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{d^2y}{dx^2} \tag{6}$$

So the acceleration is proportional to the tension T and inversely proportional to the mass per unit length μ. It is also proportional to $\frac{\partial^2 y}{\partial x^2}$. $y = A\sin(kx-\omega t)$, Wave speed in a stretched string.

However, to be a solution to

$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{d^2y}{dx^2}, \tag{7}$$

The second partial derivatives of $y = A\sin(kx-\omega t)$ with respect to x and t are given by:

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx-\omega t), \tag{8}$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx-\omega t). \tag{9}$$

In other words, $y = A\sin(kx-\omega t)$ is a solution provided that

$$\frac{T}{\mu} = \frac{\omega^2}{k^2}. \tag{10}$$

In Travelling waves II, we saw that $\frac{\omega}{k}$ was the wave speed v, so we now have an expression for the speed of a wave in a stretched string:

$$v = \sqrt{\frac{T}{\mu}} \tag{11}$$

This completes the derivation of the wave speed using first principles.

1. Find the wavelength (λ) for the fundamental mode:

$$\lambda = 2L \quad \Rightarrow \quad \lambda = 2 \times 0.875\,\text{m} = 1.75\,\text{m} \tag{12}$$

2. Use the frequency and wavelength to find the wave speed (v):

$$v = f \cdot \lambda \quad \Rightarrow \quad v = 45\,\text{Hz} \times 1.75\,\text{m} = 78.75\,\text{m/s} \tag{13}$$

3. Use the wave speed (v) to find the tension (T) using the wave equation:

$$T = \mu{\cdot}v^2 \quad \Rightarrow \quad T = (4{\times}10^{-2}\,\text{kg/m}){\times}(78.75\,\text{m/s})^2 = 123.1875\,\text{N} \tag{14}$$

Therefore, the speed of the transverse wave on the string is 78.75 m/s, and the tension in the string is 123.1875 N.