Set 12: Poisson Distribution

Stat 260 A01: June 12, 2024

A ***Poisson Random Variable*** X counts the number of occurrences of a certain event in a fixed unit of time, space, distance, etc. It has parameter:

μ or $\lambda =$ average number of event occurrences in the interval of interest.

Example 1: Determine whether each of the following is a Poisson r.v.

(i) The number of customers arriving at a store in one hour. - Poisson

(ii) The number of Biology majors in a class of 150 students. - Binomial

(iii) The number of fish on the Endangered Species list. - Discrete

(iv) The number of stop signs in a 1km^2 region. - Poisson

(v) The number of bacteria in 1 cubic cm of sea water. - Poisson

(vi) The number of runners in a marathon currently moving more than 10 km/hr. - Binomial

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Example 2: A 911 operator receives an average of 3 calls per hour. Suppose we want to study the number of calls in a 5 hour period.

Let X = num of calls in a 5 hour period

$X \sim$ Poisson $(\lambda = 15$

$\lambda =$ 3calls per hour (5hours) =15 calls

The **Poisson Random Variable** has a pdf:

Example 2 Continued...

(a) What is the probability of exactly 10 calls in 5 hours?

$\lambda = 15$

$P(x=10) = f(10) = \dfrac{e^{-15} \cdot (15)^{10}}{10!} = 0.048$

λ 1.5 hours

(b) What is the probability of 6 calls in 90 minutes?

$\lambda = \left(\frac{3 \text{ calls}}{\text{hour}}\right)(1.5 \text{ hours}) = 4.5$ calls (in 90 mins)

X = # of calls in 90 mins, $X \sim$ Poisson ($\lambda = 4.5$)

$$P(X=6) = \frac{e^{-4.5}(4.5)^6}{6!} = 0.128$$

(c) What is the probability of at least 4 calls in 2 hours?

$\lambda = \left(\frac{3 \text{ call}}{\text{hour}}\right)(2 \text{ hours}) = 6$ calls (in 2 hours)

X = num of calls in 2 hours $X \sim$ Poisson ($\lambda = 6$)

$$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6) + \ldots \rightarrow$$ goes to infinity too hard to sum all terms so do complement

$$= 1 - (X \leq 3)$$

$$= \underbrace{1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]}_{\text{feasible but tm work}}$$

$$P(X \geq 4) = 1 - P(X \leq 3)$$

$= 1 - 0.1512 \leftarrow$ From poisson CDF Table

$= 0.8488$

The **Poisson Cumulative Distribution Function (CDF)**:

$$F(x) = P(X \leq x) = \sum_{t \leq x} P(X=t) = \sum_{t \leq x} \frac{e^{-\lambda}\lambda^t}{t!}$$

Example 2 Continued...

A **Poisson random variable** X has

$E(x) = \lambda = \mu_x$, $V(x) = \lambda = \sigma_x^2$, $SD(x) = \sqrt{\lambda} = \sigma_x$

Example 2 Continued...

(d) How many calls are expected in a 8 hour workday?

Let x = num of calls in 8 hours

λ = (3 call per hour) (8 hours) = 24 calls $\quad x \sim$ Poisson ($\lambda = 24$)

$E(x) = 24$ calls, $V(x) = 24 \text{ calls}^2$, $SD(x) = \sqrt{24}$ calls

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Extra Example 1: Saddle anemones are randomly distributed around a large coral reef, with an average of 1.75 anemones per 20m^2. A marine biologist examines 5 different 40m^2 plots of reef.

↳ double = λ cause 40m^2

What is the probability that exactly one of these plots has no saddle anemones?

Let X = num of anemones in a 40m^2 plot.

$X \sim$ Poisson ($\lambda = 3.5$)

Let y = num of plots (out of 5) with 0 anemones

$y \sim$ Binomial ($n = 5$, $p = ?$) probability a single 40m^2 plot has 0 anemones

Step 1: Determine probability that a single 40m^2 plot has 0 anemones

$$P(x=0) = \frac{e^{-3.5}(3.5)^0}{0!} = 0.03 = p$$

x is the var that counts anemones

Step 2: Determine prob exactly 1 plot has 0 anemones.

$$p(y=1) = \binom{5}{1}(0.03)^1(1-0.03)^4 = 0.1328$$

The Poisson Approximation to the Binomial Distribution

Example 3: Approximately 4% of all humans have some variety of colour blindness. In a collection of 100 random subjects, what is the probability that at most 3 are colour blind?

Let X = num of colour blind people (in a group of 100)

$X \sim$ Brown $(n=100, p=0.04)$

$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3) \rightarrow$ too long

CDF table does not go to $n=100$

The **Poisson distribution** is a

"large n value small p value"

- good approximation of the **binomial distribution** if: $n \geq 20$, $p \leq 0.05$

  $n \geq 100$, $np \leq 10$

The **Poisson Approximation** uses:

$np = \mu = \lambda$ (Since $E(X) = np$ for a binomial and $E(X) = \lambda$ for a poisson)

Example 3 Continued...

Check $n = 100$ ✓ $np = 100(0.04) = 4 \leq 10$

$\therefore$ Poisson is a very good approximation of binomial.

$\lambda = np = 4$ $X \sim$ Poisson $(\lambda = 4)$

$P(X \leq 3) = 0.4335$ (From Poisson CDF table)

Readings: Swartz 4.4 [EPS 3.5] **Practice problems**: EPS 3.37, 3.41, 3.43, 3.45, 3.47, 3.49, 3.53, 3.55