STAT 260 Summer 2024: Written Assignment 6

Due: Upload your solutions to Crowdmark BEFORE 6pm (PT) Friday June 28.

You may upload and change your files at any point up until the due date of Friday June 28 at 6pm (PT).

A 2% per hour late penalty will be automatically applied within Crowdmark. The penalty is applied in such a way so that assignments submitted 6pm to 6:59pm will have 2% deducted, assignments submitted 7pm-7:59pm will have 4% deducted, etc.

Note that if you submit any portion of your assignment before the deadline, Crowdmark will NOT permit you to edit your submission (including make additional uploads) after the 6pm deadline passes. This means that if, for example, you upload only Question 1 before the deadline, you will not be able to upload Question 2 after the deadline. If you intend to submit late (with penalty) you must submit the entire assignment late.

Submission: Solutions are to be uploaded to Crowdmark. Here you will be asked to upload your solutions to each question separately. Your solution to Question 1 must be uploaded in the location for Question 1, your solution to Question 2 must be uploaded in the location for Question 2, etc. If your work is uploaded to the wrong location, the marker will not be able to grade it.

You may hand-write your solution on a piece of paper or tablet. If you wish to use this question sheet and write your solutions on the page, space has been provided below. One of the quickest ways to upload work is by accessing Crowdmark from within a web browser on a smartphone. In the area where you upload work, press the "+" button. This will give you the option of using a file already on your phone, or you can use the phone camera to photograph your work. If you complete your work on a tablet, save the file as a PDF or each question as a jpeg and drag/drop the file into the Crowdmark box. ***Photographs of laptop/tablet screens will not be graded***; take a proper screenshot.

Instructions: For full marks, your work must be neatly written, and contain enough detail that it is clear how you arrived at your solutions. ***You will be graded on correct notation***. Messy, unclear, or poorly formatted work may receive deductions, or may not be graded at all. Only resources presented in lecture or linked to on the Stat 260 Brightspace page are permitted for use in solving these assignments; using outside editors/tutors, and/or software (include AIs) is strictly forbidden. Talking to your classmates about assigned work is a healthy practice that is encouraged. However, in the end, each person is expected to write their own solutions, in their own words, and in a way that reflects their own understanding.

Additional Instructions:

- *For each of the following questions, include the correct notation for the random variable that you are calculating in your solution, not just the numeric answer. For example: "$f(3) = P(X = 3) = 0.157$".*
- *Be extra careful with your integral notation. Marks will be deducted for missing bounds, dx's, etc.*
- *All calculations are to be completed using only your SHARP EL-510 calculator and the* STAT 260 Distribution Tables Booklet *on Brightspace. Do not use R or other software.*

1. **[8 marks]** Suppose that X is a continuous random variable with the following pdf:

$$f(x) = \begin{cases} \frac{2}{34}(5x - x^2) & 2 \leq x < 4 \\ \frac{1}{9} & 4 \leq x < 7 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine $P(3 < X \leq 6)$. *Give your answer rounded to 4 decimal places.*

$$f(x) = \begin{cases} \frac{2}{34}(5x - x^2) & 2 \leq x < 4 \\ \frac{1}{9} & 4 \leq x < 7 \\ 0 & \text{otherwise} \end{cases}$$

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1a) $P(3 < X \leq 6)$:

We split the probability into two integrals like so:

$$\int_3^4 f(x)\,dx + \int_4^6 f(x)\,dx$$

$$\int_3^4 \frac{2}{34}(5x - x^2)\,dx \rightarrow \frac{2}{34} = \frac{1}{17} \text{ so } \int_3^4 (5x - x^2)\,dx$$

$$\rightarrow \frac{1}{17}\left[\int_3^4 5x\,dx - \int_3^4 x^2\,dx\right]$$

$$\rightarrow \frac{1}{17}\left[\left[\frac{5x^2}{2}\right]_3^4 - \left[\frac{x^3}{3}\right]_3^4\right]$$

$$\rightarrow \frac{1}{17}\left[\left[\frac{5\cdot(4)^2}{2} - \frac{5\cdot(3)^2}{2}\right] - \left[\frac{4^3}{3} - \frac{3^3}{3}\right]\right]$$

$$\rightarrow \frac{1}{17}\left[\left[\frac{80}{2} - \frac{45}{2}\right] - \left[\frac{64}{3} - \frac{27}{3}\right]\right]$$

$$\rightarrow \frac{1}{17}\left[\frac{35}{2} - \frac{37}{3}\right] = \frac{1}{17}\left[\frac{31}{6}\right] = \frac{31}{102} \rightarrow \text{1st integral}$$

$$\int_4^6 \frac{1}{9}\,dx \rightarrow \frac{1}{9}\int_4^6 dx \rightarrow \frac{1}{9}\left[[x]_4^6\right] \rightarrow \frac{1}{9}(6-4) = \frac{2}{9}$$

$$\text{So, } P(3 < X \leq 6) = \int_3^4 \frac{2}{34}(5x - x^2)\,dx + \int_4^6 \frac{1}{9}\,dx$$

$$= \frac{31}{102} + \frac{2}{9} = \frac{161}{306} \text{ or } \approx 0.5261$$

Question 1 continues on next page...

(b) Determine μ_X and σ_X^2.
Give μ_X as a reduced fraction. You may give σ_X^2 as a fraction or as rounded to 4 decimal positions.

$$f(x) \begin{cases} \frac{2}{34}(5x - x^2) & 2 \le x < 4 \\ \frac{1}{9} & 4 \le x < 7 \\ 0 & \text{otherwise} \end{cases}$$

1b) $\mu_x = \int_{-\infty}^{\infty} x f(x)\,dx$

f(x) is non zero only on [2,7) so we split the integral:

$$\mu_x = \overbrace{\int_2^4 x \cdot \frac{2}{34}(5x - x^2)\,dx}^{1} + \overbrace{\int_4^7 x \cdot \frac{1}{9}\,dx}^{2}$$

integral 1) $\frac{1}{17}\int_2^4 x(5x - x^2)\,dx$

$$= \frac{1}{17}\left[\frac{5x^3}{3} - \frac{x^4}{4}\right]_2^4 = \frac{1}{17}\left[\left(\frac{5(4)^3}{3} - \frac{4^4}{4}\right) - \left(\frac{5(2)^3}{3} - \frac{2^4}{4}\right)\right]$$

$$= \frac{1}{17}\left(\frac{128}{3} - \frac{28}{3}\right) = \frac{1}{17}\left(\frac{100}{3}\right) = \frac{100}{51}$$

integral 2) $\frac{1}{9}\int_4^7 x\,dx = \frac{1}{9}\left[\frac{x^2}{2}\right]_4^7 = \frac{1}{9}\left[\left(\frac{7^2}{2}\right) - \left(\frac{4^2}{2}\right)\right]$

$$= \frac{1}{9}\left(\frac{33}{2}\right) = \frac{11}{6}$$

$\therefore \mu x = \frac{100}{51} + \frac{11}{6} \Rightarrow \mu x = \frac{129}{34}$

$$\sigma_x^2 = \int_2^7 x^2 f(x)\,dx - (\mu x)^2 = \frac{1}{17}\int_2^4 x^2 \cdot \frac{2}{34}(5x - x^2)\,dx + \int_4^7 x^2 \cdot \frac{1}{9}\,dx - (\mu x$$

$$= \frac{1}{17}\left[\frac{5x^4}{4} - \frac{x^5}{5}\right]_2^4 + \frac{1}{9}\left[\frac{x^3}{3}\right]_4^7 - (\mu x)^2$$

$$= \frac{1}{17}(101.6) + \frac{1}{9}(93) - \left(\frac{129}{34}\right)^2 = \frac{508}{85} + \frac{31}{3} - \left(\frac{129}{34}\right)^2$$

$\sigma_x^2 = 1.914$

2. **[6 marks]** Suppose that X is a continuous random variable with the following pdf:

$$f(x) = \begin{cases} e^{-5x} + \dfrac{1}{5e^5} & 0 \le x < 1 \\ \frac{1}{8}(x-1)^4 & 1 \le x < 3 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the CDF of X.

$$f(x) = \begin{cases} e^{-5x} + \frac{1}{5e^5} & 0 \le x < 1 \\ \frac{1}{8}(x-1)^4 & 1 \le x < 3 \\ 0 & \text{otherwise} \end{cases}$$

2a) $F(x) = \int_{-\infty}^{x} f(y)\,dy$

For $x < 0$: $F(x) = 0$ as corresponding pdf is 0 for $x < 0$

For $0 \le x < 1$:

$$F(x) = \int_0^x \left(e^{-5y} + \frac{1}{5e^5}\right) dy = \int_0^x \left(e^{-5y}\right) dy + \int_0^x \left(\frac{1}{5e^5}\right) dy$$

$$= \left[-\frac{1}{5}e^{-5y}\right]_0^x + \left[\frac{1}{5e^5}\right]_0^x$$

$$F(x) = -\frac{1}{5}e^{-5x} + \frac{1}{5} + \frac{1}{5e^5}x$$

For $1 \le x < 3$:

$$F(x) = F(1) + \int_1^x \frac{1}{8}(y-1)^4\,dy \qquad u = t-1 \quad du = dt$$

$$\Rightarrow \frac{1}{8}\int_1^x u^4\,du = \frac{1}{8}\left[\frac{u^5}{5}\right]_0^{x-1} = \frac{(x-1)^5}{40}$$

$$F(x) = \frac{1}{5} + \frac{(x-1)^5}{40}$$

For $x \ge 3$:

$$F(x) = F(3) \quad \therefore \quad F(3) = \frac{1}{5} + \frac{(3-1)^5}{40} = \frac{1}{5} + \frac{32}{40} = 1$$

$$F(x) = 1$$

$\therefore$ Final CDF =

$$f(x) \begin{cases} 0 & x < 0 \\ -\frac{1}{5}e^{-5x} + \frac{1}{5} + \frac{1}{5e^5}x & 0 \le x < 1 \\ \frac{1}{5} + \frac{(x-1)^5}{40} & 1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

Question 2 continues on next page...

(b) Find the 65-th percentile of X. That is, find η such that $P(X \leq \eta) = 0.65$

$$F(x) = \begin{cases} 0 & x < 0 \\ -\frac{1}{5}e^{-5x} + \frac{1}{5} + \frac{1}{5e^5}x & 0 \leq x < 1 \\ \frac{1}{5} + \frac{(x-1)^5}{40} & 1 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

2b) Using our previously found CDF:

We need $F(\eta) = 0.65$ thus we start by checking intervals:

Case 1: $0 \leq \eta < 1$

$$F(\eta) = -\frac{1}{5}e^{-5\eta} + \underbrace{\frac{1}{5} + \frac{1}{5e^5}\eta} = 0.65$$

Since the interval for ↑ ranges from 0.2 to 0.2, evidently 0.65 must be in $[1,3)$

Case 2: $1 \leq \eta < 3$

$$\overset{0.2}{\frac{1}{5}} + \frac{(\eta-1)^5}{40} = 0.65 \quad \therefore \quad \frac{(\eta-1)^5}{40} = 0.65 - 0.2$$

$$\therefore (\eta-1)^5 = 0.45 \times 18 \rightarrow \eta - 1 = \sqrt[5]{18}$$

$$\rightarrow \eta = 1 + \sqrt[5]{18}$$

$$\rightarrow \eta = 1 + 1.782602458$$

$$\rightarrow \eta \approx 2.783$$