MATH 202 Midterm 2

November 7th, 2024
3:30 PM – 4:20 PM

Instructions

- In the following box, write your first and last names displayed on Brightspace, and your student ID number. Do not include the V and and the beginning 00 or 01 in your student ID number. For example, write 123456 instead of V00123456.

- This exam has **5 questions** and is out of **100 points**.
- Organise and show your work.
- You may use a Sharp EL-510R calculator for elementary arithmetic.
- Any unsupported answers will receive no credit.
- Some formulas can be found at the end of this exam.

Do not turn this page over until instructed to do so

Question 1. **(1) (10%)** Determine whether the following DE is linear:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y + \frac{1}{1+x^2}.$$

Justify your answer, but do not solve the DE.

(2) (10%) Sketch the phase line of the following DE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -(y-1)(y-2).$$

Solution. (1) The equation is linear because it can be written as

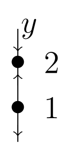
$$x\frac{\mathrm{d}y}{\mathrm{d}x} - x^2 y = \frac{1}{1+x^2}$$

or

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = b(x)$$

for $a_1(x) = x$, $a_0(x) = -x^2$, and $b(x) = 1/(1+x^2)$.

(2) The zeros of $-(y-1)(y-2)$ are $y = 1, 2$. These values define the nodes in the phase line. For $y > 2$, $-(y-1)(y-2) < 0$, so the solution curves are decreasing when $y > 2$. The monotonicity of the solution curves in $1 < y < 2$ and $y < 1$ can be obtained similarly. Hence, the phase line is as follows:



$\blacksquare$

Question 2 (20%). Determine whether the given function is a solution to the given DE:

$$y = x^2, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + 2x^3.$$

Justify your answer.

Solution. With $y = x^2$, we have

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y\frac{\mathrm{d}y}{\mathrm{d}x} = (x^2)'' + (x^2)(x^2)' = 2 + 2x^3.$$

Hence, $y = x^2$ is a solution to the DE under consideration. ■

Question 3. Consider the equation $y'' + 9y' = 0$ for $y = y(x)$.

(1) (10%) Find the associated characteristic polynomial $p(z)$.

(2) (10%) Find the general solutions of the DE.

Solution. (1) $p(z) = z^2 + 9z$.

(2) Since the zeros of $p(z)$ are $0, -9$ and they are distinct, the general solutions are given by $y = C_1 + C_2 \mathrm{e}^{-9x}$. ■

Question 4. Solve

$$(4x^2 - y^2)\mathrm{d}x + (xy - 2x^3y^{-1})\mathrm{d}y = 0 \tag{$\natural$}$$

by taking the following steps:

(1) **(5%)** Determine the type of first-order equation (separable, linear, exact, Bernoulli, or homogeneous) you want to work with.

(2) **(15%)** Use the method of the corresponding type of equation to solve ($\natural$).

Whenever solving a linear DE (if any), derive the integrating factor by solving the DE of the integrating factor.

Solution. (1) By rearrangement, the DE can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x^2 - y^2}{xy - 2x^3y^{-1}}.$$

This is a homogeneous equation since the right-hand side can be written as

$$-\frac{4x^2 - y^2}{xy - 2x^3y^{-1}} = -\frac{4 - (y/x)^2}{y/x - 2(y/x)^{-1}}$$

by dividing the numerator and denominator of the left-hand side by x^2.

(2) With the substitution $u = y/x$, the DE under consideration can be written as

$$\begin{aligned}
x\frac{\mathrm{d}u}{\mathrm{d}x} + u &= -\frac{4 - u^2}{u - 2u^{-1}} = -\frac{4u - u^3}{u^2 - 2} \\
x\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{4u - u^3}{u^2 - 2} - u &= -\frac{4u - u^3 + u^3 - 2u}{u^2 - 2} = -\frac{2u}{u^2 - 2} \\
-\frac{u^2 - 2}{2u}\mathrm{d}u &= \frac{\mathrm{d}x}{x} \\
-\frac{u}{2}\mathrm{d}u + \frac{\mathrm{d}u}{u} &= \frac{\mathrm{d}x}{x} \\
-\frac{1}{4}u^2 + \ln|u| &= \ln|x| + C \\
-\frac{1}{4}|y/x|^2 + \ln|y/x| &= \ln|x| + C.
\end{aligned}$$

In more detail, the last equality of the first line is obtained by multiplying the numerator and denominator of the left-hand side by u, and the last equality uses $u = y/x$. ■

Question 5. Solve the initial value problem (IVP)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2x^8y^2, \quad y(1) = 1, \tag{$\sharp$}$$

by taking the following steps:

(1) **(5%)** Determine the type of first-order equation (separable, linear, exact, Bernoulli, or homogeneous) you want to work with.

(2) **(10%)** Solve $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2x^8y^2$.

(3) **(5%)** Use the general solutions you find in (2) to solve the IVP in ($\sharp$).

Whenever solving a linear DE (if any), derive the integrating factor by solving the DE of the integrating factor.

Solution. (1) The DE is a Bernoulli equation because by dividing both sides by x, it can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = 2x^7y^2. \tag{0.1}$$

(2) By clearing y^2 on the right-hand side, (0.1) can be written as

$$\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}\frac{1}{y} = 2x^7. \tag{0.2}$$

Hence, with the substitution $u = 1/y$, the foregoing equation becomes

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{x}u = -2x^7.$$

The integrating factor solves $\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{1}{x}z$, so $\frac{\mathrm{d}z}{z} = -\frac{\mathrm{d}x}{x}$, $\ln|z| = -\ln|x| + C$, so we can take $z = 1/x$. It follows that

$$\frac{1}{y} = u = \frac{1}{1/x}\int \frac{1}{x}\cdot(-2x^7)\mathrm{d}x = x\int -2x^6\mathrm{d}x = x(-\frac{2}{7}x^7 + C),$$

that is $y = \frac{1}{x(-\frac{2}{7}x^7+C)}$.

(3) With $y(1) = 1$ and the general solution derived in (2), we get

$$1 = y(1) = \frac{1}{-2/7 + C} \Rightarrow C = \frac{9}{7},$$

so that the required solution is $y = \frac{1}{x(-\frac{2}{7}x^7+\frac{9}{7})}$. ■

Some Formulas

- **Linear equation.** The equation takes the form of
$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x).$$
This equation can be solved by using the following formulas:
$$\frac{\mathrm{d}z}{\mathrm{d}x} = p(x)z \quad \text{(integrating factor);} \quad y(x) = \frac{1}{z(x)} \int z(x)f(x)\mathrm{d}x.$$

- **Exact equation.** The equation takes the form of
$$N(x,y) + M(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
such that
$$\frac{\partial}{\partial y}N(x,y) = \frac{\partial}{\partial x}M(x,y).$$
This equation can be solved by $F(x, y(x)) = C$ for an implicit solution $y = y(x)$ such that $F(x,y)$ satisfies
$$N(x,y) = \frac{\partial F}{\partial x}(x,y), \quad M(x,y) = \frac{\partial F}{\partial y}(x,y).$$

- **Bernoulli equation.** The equation takes the form of
$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x)y^n \quad (n \neq 1).$$
This equation can be solved as the linear equation
$$\frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)p(x)u = (1-n)f(x)$$
by the substitution $u = 1/y^{n-1}$.

- **Homogeneous equation.** The equation takes the form of
$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(y/x).$$
This equation can be solved as the separable equation
$$x\frac{\mathrm{d}u}{\mathrm{d}x} + u = F(u)$$
by the substitution $u = y/x$. Also, a function $f(x,y)$ can be expressed as $f(x,y) = F(y/x)$ if $f(tx,ty) = f(x,y)$ for all $t \neq 0$ and all x, y.

- **Second-order homogeneous linear equations with constant coefficients.** For $ay'' + by' + cy = 0$ with real coefficients $a \neq 0, b, c$ and characteristic polynomial $p(z)$, the solutions are given as follows:
  - If $p(z)$ has distinct real roots $r_1 \neq r_2$, then $y = C_1 \mathrm{e}^{r_1 x} + C_2 \mathrm{e}^{r_2 x}$.
  - If $p(z)$ has complex roots $\alpha \pm \beta \mathrm{i}$ for $\beta \neq 0$, then $y = C_1 \mathrm{e}^{\alpha x} \cos(\beta x) + C_2 \mathrm{e}^{\alpha x} \sin(\beta x)$.
  - If $p(z)$ has a double root $r = r_1 = r_2$, then $y = C_1 \mathrm{e}^{rx} + C_2 x \mathrm{e}^{rx}$.

  Also, recall that for the quadratic polynomial $Az^2 + Bz + C$, the roots are given by $\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$.