Q1)

a) Let $p(n)$ be the statement ∋

$\exists n_0$ such that $\forall n \geq n_0$, $n = 6a + 10b + 15c$

Base cases:

$30 = 6(0) + 10(0) + 15(2)$
$31 = 6(1) + 10(1) + 15(1)$
$32 = 6(2) + 10(2) + 15(0)$
$33 = 6(3) + 10(0) + 15(1)$
$34 = 6(4) + 10(1) + 15(0)$
$35 = 6(0) + 10(2) + 15(1)$

Assume $p(30), \ldots, p(n)$ holds.

$p(n+1) = 6a + 10b + 15c$

$p(n-5)$ is also true

$n - 5 + 6 = 6a + 6 + 10b + 15c$

$n + 1 = 6(a+1) + 10b + 15c$

$\therefore p(n+1)$ holds

Hence by PMI, $p(n)$ holds for all $n \geq n_0$.

b) The smallest possible value of $n_0 = 30$.

c) Case 1: 6's and 10's:

$10 \div 6 = 1 \text{ r } 4$
$6 \div 4 = 1 \text{ r } \boxed{2} \leftarrow \gcd$
$4 \div 2 = 2 \text{ r } 0$

$\gcd(10, 6) = 2$ thus they aren't coprime, this means that not all integers can be expressed in this form.

$\therefore$ only even numbers ≥ 30 can be expressed in the form $6a + 10b$

Case 2: 6's and 15's:

$15 \div 6 = 2 \text{ r } \boxed{3} \leftarrow \gcd$
$6 \div 3 = 2 \text{ r } 0$

$\gcd(15, 6) = 3$ thus they aren't coprime, this means that only multiples of 3 can be expressed in this form. n_0 smallest for which every multiple of $3 \geq 6 \times 3 = 18$ can be expressed in the form $6a + 15c$.

Case 3: 10's and 15's:

$15 \div 10 = 1 \text{ r } 5 \leftarrow \gcd$
$10 \div 5 = 2 \text{ r } 0$

$\gcd(10, 15) = 5$ thus they aren't coprime, this means that only multiples of 5 can be expressed in this form. n_0 smallest for which every multiple of $5 \geq 5 \cdot 5 = 25$ can be expressed in the form $10b + 15c$.

Q2)

a) Let $p(n)$ be the statement "for all $k \geq 1$, $k \times k! + (k-1) \times (k-1)! + \ldots 2 \times 2! + 1 \times 1! = (k+1)! - 1$"

Base case: $p(1)$ $k = 1$

$1 \times 1!$ and $(1+1)! - 1 = 2! - 1 = 2 - 1 = 1$ $\therefore p(1)$ holds

Inductive Hypothesis: Assume $p(n)$ holds

Inductive Step: $p(n+1)$

$\Rightarrow (k+1) \times (k+1)! + k \times k! + (k-1) \times (k-1)! + \ldots + 2 \times 2! + 1 \times 1!$

$\Rightarrow (k+1) \times (k+1)! + ((k+1)! - 1) \rightarrow$ via IH

$\Rightarrow (k+1) \times (k+1)! + (k+1)! - 1 = (n+2) \times (n+1)! - 1$
$= (n+2)! - 1$

$\therefore p(n+1)$ holds

By PMI $p(n)$ holds for all $k > 1$.

b) Let $p(n)$ be the statement "every n has a factorial expansion"

Base case: $p(0) \Rightarrow 0 \times 1! = 0$ holds

Assume $p(n)$ holds $\rightarrow$ IH

Inductive Step: $p(n+1)$, Let k be the largest integer such that $k! \leq n+1$

$n+1 = d_k \times k! + r$ where $0 \leq r \leq k!$ and $d_k \in \{1, \ldots, k\}$

$\Rightarrow r$ has a factorial expansion by IH

so, $n+1 = d_k \times k! + d_{k-1} \times (k-1)! + \ldots + d_1 \times 1!$

$\therefore p(n+1)$ holds

By PMI, $p(n)$ holds for all n.

Q3)

Let $p(n)$ be the statement "$f_{2n} + f_{2n-2} + \ldots f_4 + f_2 = f_{2n+1} - 1$ for each $n \geq 1$"

Base case: $p(1)$

$\Rightarrow$ LHS: $f_{2 \cdot 1} = f_2 = 1$

RHS: $f_{2(1)+1} = f_3 - 1 = 2 - 1 = 1$

$\therefore$ since LHS = RHS, base case holds

Assume $p(n)$ holds $\rightarrow$ IH

Inductive Step: $p(n+1)$

$= f_{2(n+1)} + f_{2(n+1)-2} + \cdots f_4 + f_2$
$= f_{2n+2} + (f_{2n} + f_{2n-2} + \cdots f_4 + f_2)$
$= f_{2n+2} + (f_{2n+1} - 1) \rightarrow$ by IH
$= (f_{2n+1} + f_{2n}) + f_{2n+1} - 1$
$= f_{2n+1} + f_{2n+1} + f_{2n} - 1$
$= f_{2n+2} - 1$

$\therefore p(n+1)$ holds

Hence by PMI, $p(n)$ holds for every $n \geq 1$.

Let $p(n)$ be the statement "$f_{2n+1} + f_{2n-1} + \ldots f_5 + f_3 = f_{2n+2} - 1$ for each $n \geq 1$"

Base case: $p(1)$

LHS $= f_{2(1)+1} = f_3 = 2$

RHS $= f_{2(1)+2} - 1 = f_4 - 1 = 3 - 1 = 2$

$\therefore$ since LHS = RHS, base case holds

Assume $p(n)$ holds $\rightarrow$ IH

Inductive Step: $p(n+1)$

$= f_{2(n+1)+1} + f_{2(n+1)-1} + \ldots f_5 + f_3$
$= f_{2n+3} + (f_{2n+1} + f_{2n-1} + \cdots f_5 + f_3)$
$= f_{2n+3} + (f_{2n+2} - 1) \rightarrow$ By IH
$= 2f_{2n+2} + f_{2n+1} - 1$
$= f_{2n+4} - 1 = f_{2(n+1)+2} - 1$

So, $f_{2n+1} + f_{2n-1} + \ldots f_5 + f_3 = f_{2(n+1)+2} - 1$

$\therefore p(n+1)$ holds

Hence by PMI, $p(n)$ holds for every $n \geq 1$.

Q4) $d = \lfloor \log_b n \rfloor + 1$

we need base 2: $d = \lfloor \log_2 (75!) \rfloor + 1$

$\log_2 n = \dfrac{\log_{10}(n)}{\log_{10}(2)} \Rightarrow \log_2(75!) = \dfrac{\log_{10}(75!)}{\log_{10}(2)}$

$\log(75!)$ rewritten as $\rightarrow \log_{10}(69!) + \log_{10}(75 \cdot 74 \cdot 73 \cdot 72 \cdot 71 \cdot 70)$

$\log_{10}(75 \cdot 74 \cdot 73 \cdot 72 \cdot 71 \cdot 70) = \log_{10}(75) + \log_{10}(74) + \log_{10}(73) + \log_{10}(72) + \log_{10}(71) + \log_{10}(70)$
$= 1.875 + 1.869 + 1.863 + 1.857 + 1.851 + 1.845$
$= 11.161$

$\Rightarrow \log_{10}(75!) = \log_{10}(69!) + 11.161$
$= 98.233 + 11.161$
$= 109.395$

$\log_2(75!) = \dfrac{109.395}{\log_{10}(2)}$
$= 363.4 \rightarrow \lfloor 363.4 \rfloor + 1$
$\rightarrow 363 + 1$
$\rightarrow 364$ digits

Q5)

$\overset{\binom{1}{0}}{724} \div \overset{\binom{0}{1}}{417} = 1 \text{ r } \overset{\binom{1}{-1}}{307}$

$\overset{\binom{0}{1}}{417} \div \overset{\binom{1}{-1}}{307} = 1 \text{ r } \overset{\binom{-1}{2}}{110}$

$\overset{\binom{1}{-1}}{307} \div \overset{\binom{-1}{2}}{110} = 2 \text{ r } \overset{\binom{3}{-5}}{87}$

$\overset{\binom{-1}{2}}{110} \div \overset{\binom{3}{-5}}{87} = 1 \text{ r } \overset{\binom{-4}{7}}{23}$

$\overset{\binom{3}{-5}}{87} \div \overset{\binom{-4}{7}}{23} = 3 \text{ r } \overset{\binom{15}{-26}}{18}$

$\overset{\binom{-4}{7}}{23} \div \overset{\binom{15}{-26}}{18} = 1 \text{ r } \overset{\binom{-19}{33}}{5}$

$\overset{\binom{15}{-26}}{18} \div \overset{\binom{-19}{33}}{5} = 3 \text{ r } \overset{\binom{72}{-125}}{3}$

$\overset{\binom{-19}{33}}{5} \div \overset{\binom{72}{-125}}{3} = 1 \text{ r } \overset{\binom{-91}{158}}{2}$

$\overset{\binom{72}{-125}}{3} \div \overset{\binom{-91}{158}}{2} = 1 \text{ r } \overset{\binom{163}{-283}}{\boxed{1}} \leftarrow \gcd$

$\overset{\binom{-91}{158}}{2} \div \overset{\binom{163}{-283}}{1} = 2 \text{ r } 0$

$\gcd(724, 417) = 1$

$724(163) + 417(-283) = 1$

$\therefore a = 163$ and $b = -283$

Q6) $n = qb + r$ where $b > 1 \leftarrow$ division algorithm
$r \in \{0, 1, \ldots, b-1\}$

$m = ab + 1$
$n = cb + 1$

$m \cdot n = (ab+1)(cb+1)$
$= acb^2 + ab + cb + 1$
$= b(acb + a + c) + 1 \rightarrow$ follows form $qb + r$ where $r = 1$

mn has remainder 1 when divided by b.

Bonus)

Let $p(k)$ be the statement "all numbers n in the range f_k up to $f_{k+1} - 1$ have a fibonacci expansion with loading term f_k"

Base case: $p(2)$

the range of n is $[f_2, f_3 - 1] = [1, 1]$

$1 = f_2$

Thus $p(2)$ holds

Assume $p(k)$ holds for $k \geq 2 \rightarrow$ IH

Inductive Step: $p(k+1)$

Consider the range $[f_{k+1}, f_{k+2} - 1]$

$f_{k+2} = f_{k+1} + f_k$

so for n in range $[f_{k+1}, f_{k+1} + f_k - 1]$ can be written as $f_{k+1} + m$ where $1 \leq m \leq f_k - 1$

By the IH m has a fibonacci expansion starting with a term $\leq f_k$. So, $n = f_{k+1} + m$ has a fibonacci expansion starting with f_{k+1}

for n in range $[f_{k+1} + f_k, f_{k+2} - 1]$ can be written as $f_{k+2} - 1 = f_{k+1} + f_k - 1$

By IH, $n - f_{k+1}$ is in the range f_k up to $f_{k+1} - 1$, and thus has an expansion starting from f_k.

$\therefore n$ has a fibonacci expansion starting at f_{k+1} so $p(k+1)$ holds

By PMI, $p(k)$ holds for all $k \geq 2$.