Set 7: Independent Events

In some cases, knowing the outcome of one event does not provide any information about a second event, or affect the probability of the second event occurring.

Example: Suppose we want to roll two dice, and are concerned with the events A, that the first roll is a five, and B, the second roll is a four.

Suppose I know that A has occurred. This doesn't in any way change the likelihood that B will occur. Since the occurrences of A and B do not affect each other, we say that A and B are **independent events**.

More formally, we say that A and B are independent events (and neither event is impossible) if $P(B|A) = P(B)$.

Note that if $P(B|A) = P(B)$, then automatically it will be true that $P(A|B) = P(A)$.

We can also conclude that if A and B are independent events, then

$$P(A \cap B) = P(A)P(B).$$

Proof: If A and B are independent, $P(B|A) = P(B)$. Hence, by the Multiplication Rule,

$$P(A \cap B) = P(B \cap A) = P(A)P(B|A) = P(A)P(B)$$

Note: if A and B are independent events, then A and $\overline{B}$ are independent, $\overline{A}$ and B are independent, and also $\overline{A}$ and $\overline{B}$ are independent.

Be careful: Suppose events A and B are independent, and B and C are independent. It **DOES NOT** imply that A and C are independent. We still need to establish independence by checking the definition or the formula.

Independence vs. Mutually Exclusive:

A and B are mutually exclusive if and only if $P(A \cap B) = 0$.

A and B are independent if and only if $P(A \cap B) = P(A)P(B)$ or $P(B|A) = P(B)$ or $P(A|B) = P(A)$.

Whether or not two events are mutually exclusive is separate from whether or not the two events are independent.

Exercise: If events A and B, with non-zero probabilities, are mutual exclusive, what can we say about their independence?

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Example 1: At a university, it is known that 60% all first-year students are taking a calculus class. It is also known that 70% of all first-year students are taking a history class. It is also known that 80% of all first-year students are taking a calculus class or a history class (or both).

Suppose we select a first-year student at random.

Let A be the event "the first-year student is taking a calculus class".

Let B be the event "the first-year student is taking a history class".

Are events A and B independent? Are events A and B mutually exclusive?

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Example 2: At a particular computer store, every customer who enters purchases exactly one laptop and exactly one printer. The probabilities associated with the various purchases are given below.

	Printer 1	Printer 2	Total
Laptop A	0.12	0.28	0.40
Laptop B	0.12	0.22	0.34
Laptop C	0.06	0.20	0.26
Total	0.30	0.70	

Let D be the event "selected customer buys Printer 1".
Let A be the event "selected customer buys Laptop A".
Let B be the event "selected customer buys Laptop B".

Determine whether each statement is **(A) True** or **(B) False**.

1. D and A are independent events.

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2. D and B are mutually exclusive events.

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3. D and B are independent events.

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4. A and B are mutually exclusive events.

We say that a *collection* of events are independent (more specifically, they are **mutually independent**) if for every subset of this collection, $A_1, A_2, \ldots, A_k$, we have

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2)\ldots P(A_k)$$

A collection of events are called **pairwise independent** if every pair of events are independent (i.e. if $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$.)

Example: For events A, B, and C, they are (mutually) independent if and only if:

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Much like dependent events, we can use tree diagrams to model problems where the events are independent. We find the solution with a tree diagram in the exact same way as we did for dependent events.

Example 3: A machine is made of three components (A,B,C) which function independently. The probability that components A,B,C will need to be repaired today is $0.03, 0.02, 0.08$ respectively. What is the probability that **exactly one** of the three components will need to be repaired today?

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Example 4: Suppose that $P(A) = 0.5, P(B) = 0.4$, and that A and B are independent events. What is the probability that only A (and not B) will occur? That is, what is $P(A \cap \overline{B})$?

Example 5: At a small taco shop, it has been noted that 80% of customers order beef tacos, and the other 20% of customers order veggie tacos. Suppose six customers each orders one taco. Assuming the orders of the customers are independent of each other, what is the probability that **at least one** customer will order a veggie taco?

Bayes' Formula can also be applied in the case of independent events.

Example 6: In a factory, quality-control is done by two inspectors, who function independently of each other. The first inspector correctly identifies defective items with probability 0.95. The second inspector correctly identifies defective items with probability 0.85.

Suppose a defective item has been correctly identified. What is the probability that the second inspector, but not the first inspector, correctly identified this item?

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Example 7: A particular item is made of two independently functioning components. The probability that Component A will need service within a year is 0.05. The probability that Component B will need service within a year is 0.1.

Suppose we know that exactly one component requires service. What is the probability that only Component B will require service?

Set 8: Random Variables

A **random variable** is a function which maps each outcome of an experiment to a number. We often use a random variable when we have an experiment where the outcomes may be counted or measured.

Example 8: Suppose we select ten items at random from a production line. Let the random variable X count the number of flawed items in this collection.

The number of defective items could be $0, 1, \ldots, 10$. Thus, X can take on the values $0, 1, \ldots, 10$.

Then $P(X = 1)$ would be the probability that exactly one item is defective, and $P(X \geq 2)$ would be the probability that at least two items are defective.

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Random variables may be used, even if the outcomes of an experiment are not numerical.

Example 9: We randomly select a student and ask if they have taken Math 122. For this experiment, we define the random variable Y, which takes on two values: 0 and 1.

We can set $Y = 1$ if the answer is "Yes", and $Y = 0$ if the answer is "No".

We can classify random variables based on the set of all possible values they can take, known as **support** or **support set**.

A random variable is **discrete** if the support set is **countable**. Such set may contain a finite number of elements. If the set contains an infinite number of the elements, the elements can easily correspond to the set of integers.

A random variable is **continuous** if the support set contains intervals of real numbers.

Discrete Random Variable

The **probability mass function** (pmf) or **probability distribution** of a discrete random variable X is defined by $f(x) = P(X = x)$ for every number x in the support.

Notation: Some textbooks use $p(x)$ rather than $f(x)$ to refer to the pmf of a random variable X.

In some cases, we display the probability distribution of the discrete random variable X in a **probability distribution table**.

Example 10: Two machines work independently to manufacture an electronic component. The probability that machine 1 will produce a component that works is 0.8, while the probability that machine 2 will produce a component that works is 0.6. We randomly selected a component produced from each machine. Let X be the random variable which counts the number of components that works from the selection. Construct the probability distribution for X.

Note: For any value of x not specified in the table, $f(x) = 0$.

Example 11: What is the probability that no working component will be produced?

Example 12: What is the probability that at least one working component will be produced?

Example 13: Suppose we know that at least one working component is produced. What is the probability that exactly one working component will be produced?

For a discrete random variable X with pmf $f(x)$ and support set $\{x_1, x_2, x_3, \ldots\}$, the **cumulative distribution function** (cdf), $F(x)$, is defined as:

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

Example 14: In Example 10, we found the pmf for X, the number of electronic components that works as shown below:

x	0	1	2
$f(x)$	0.08	0.44	0.48

Find the cdf for this random variable.

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Example 15: Suppose the random variable X has the following probability distribution:

x	1	2	3	4	5
$f(x)$	0.30	0.15	0.05	0.20	0.30

Find the cdf for this random variable.

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Properties of a cdf:

1. $F(x)$ is monotone increasing (non-decreasing).
2. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$.
3. $F(x)$ is right-continuous (continuous at each point $x = k$ where x approaches k from the *right*).

Note: In the previous example, the support for the pmf was $x = 1, 2, 3, 4, 5$. As we've discussed previously, for any x which is not part of the support (i.e. impossible outcomes), the probability of that value of being observed is zero.

The event $X = 3.5$ is impossible. Therefore, $f(3.5) = P(X = 3.5) = 0$.

However, this does not mean the cdf also has a value of zero:

$F(3.5) = P(X \leq 3.5) = P(X \leq 3) = F(3) = 0.5$

Example 16: Write the cdf of the random variable X in Example 14 as a function with real number domain and then graph it.

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Exercise: Write the cdf of the random variable X in Example 15 as a function with real number domain and then graph it.

Example 17: Let the discrete random variable X count the number of classes a randomly selected UVic student is currently taking. The cdf for X is the following:

x	1	2	3	4	5	6	7
$F(x)$	0.15	0.25	0.40	0.60	0.75	0.90	1.00

- What is the probability that the student is taking **no more than** 4 classes?

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- Calculate $F(4.5)$.

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- What is the probability that the student is taking **at least** 3 classes?

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- What is the probability that the student is taking **exactly** 3 classes?

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- What is the probability that the student is taking **at least** 2 but **no more than** 5 classes?

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Set 9: Expected Value

Let X be a discrete random variable. The **expected value**, or **mean**, of X, denoted by $E(X)$, or μ_X, or just μ if there is only one random variable being considered, is

$$E(X) = \mu = \sum_{\text{all } x} x \cdot f(x)$$

where $f(x)$ is the pmf of X.

Interpretation: $E(X)$ is the population mean or *average* of all observed values of X, over all repetitions of the experiment.

If we were to repeat the experiment many times, the *long-run average* of the observed values of X would approximately equal $E(X)$.

Example 18: Suppose that X has the following distribution.

x	5	15	100
$f(x)$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{5}{12}$

Find $E(X)$.

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Example 19: Approximately 40% of all laptops of a particular brand will need a battery replacement within 3 years of purchase. Two laptops of this brand are selected at random.

What is the expected number of laptops (in each group of two laptops) which will need a battery replacement within 3 years of purchase?

If X is a random variable, and $Y = g(X)$, then Y is also a random variable and :

$$E(Y) = E(g(X)) = \sum g(x)P(X = x) = \sum g(x)f(x)$$

<u>Example 20:</u> Using the pmf from the previous example, find $E(X + 2)$, and $E(X^2)$.

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Note: In general, $E\big(g(X)\big) \neq g\big((E(X)\big).$

In the above example, $E(X^2) =$

$[E(X)]^2 =$

Laws of Expected Value: (a, b are constants)

1. $E(b) = b$
2. $E(X + b) = E(X) + b$
3. $E(aX) = aE(X)$

Notation: We may also express $E(aX + b)$ as μ_{aX+b}.

Example 21: If the random variable X is known to have expected value 3.8, find $E(7X + 3)$.

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Example 22: For the laptop experiment, the cost for a replacement battery is $\$30$ per laptop. What is the expected cost for each group of 2 laptops? (Assume that each laptop will need at most one replacement battery.)

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Suppose a random variable X has the following cdf:

x	1	2	3
$F(x)$	0.3	0.8	1.0

- Find $E(X)$.

  Select the closest to your unrounded answer:

  (A) 2 (B) 3 (C) 4 (D) 5

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- Find $E(X^2)$.

  Select the closest to your unrounded answer:

  (A) 3.5 (B) 4 (C) 4.5 (D) 5

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Set 10: Variance

Let X be a discrete random variable. The **variance** of X, denoted by $V(X)$, or σ_X^2, or σ^2, is defined as:

$$\sigma^2 = V(X) = E[(X - \mu)^2]$$

The **standard deviation** of X, written σ, is $\sigma = \sqrt{\sigma^2} = \sqrt{V(X)}$.

We can interpret $V(X)$ in a similar way to $E(X)$: If we were to carry out the experiment many times, and each time keeping track of the observed value of X, then the variance of these observed values would approach $V(X)$, as the number of repetitions of the experiment approaches infinity.

Computational (shortcut) Formula for Variance:

$$\sigma^2 = V(X) = E(X^2) - \mu^2$$

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Example 23: Find the variance and standard deviation of X for the laptop experiment, where X is the number of batteries (out of two) that would need replacement.

Recall that the pmf was:

x	0	1	2
$f(x)$	0.36	0.48	0.16

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Laws of Variance: (a, b are constants)

1. $V(b) = 0$
2. $V(X + b) = V(X)$
3. $V(aX) = a^2 V(X)$

Example 24: If the random variable X has $V(X) = 2$, then

$V(3X + 1) = (3)^2 V(X) = 9(2) = 18.$

Notation: We may write the variance of $aX + b$ as either $V(aX + b)$ or as σ^2_{aX+b}.

We would write the standard deviation of $aX + b$ as σ_{aX+b}.

Important: These laws apply to variance, and not to standard deviation.

Example 25: If the random variable X has $\sigma = 5$, find σ_{-2X+1}.

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<u>Example 26:</u> Suppose the random variable X has $E(X) = 1.9$ and $V(X) = 0.5$.

- Find $E(3X + 2)$.

  Select the closest to your unrounded answer:

  (A) 2 (B) 4 (C) 6 (D) 8

- Find $V(-4X + 8)$.

  Select the closest to your unrounded answer:

  (A) -8 (B) 0 (C) 8 (D) 16