Set 15: The Normal Distribution

Stat 260 A01: June 18, 2024

A **continuous** random variable X with a **normal probability distribution**, with mean μ and standard deviation σ, has density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \text{for } -\infty < x < \infty$$

- **Parameters:**

- The graph of a normal density function is a:

The exact shape of the curve depends upon μ and σ.

- Changing μ shifts the curve left or right.
- Changing σ makes the curve flatter or taller.

For a normal random variable X, the probability $P(X \leq x)$ is given by the CDF,

$$F(x) = \int_{-\infty}^{x} f(y)dy = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\, dy$$

Standard Normal Distribution: A **standard normal** random variable Z has mean __________ and variance __________.

The pdf of Z is $f(z) = \dfrac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}$.

The CDF of Z is $\Phi(z) = P(Z \leq z) = \displaystyle\int_{-\infty}^{z} f(y)dy = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}\, dy$

Example 1:

(a) Find $P(Z \leq 2.56)$

(b) $P(Z \leq -3.14)$

(c) $P(Z \geq 0.09)$

(d) $P(-1.27 \leq Z \leq 2.67)$

Symmetry of a Normal Distribution

(e) $P(Z \geq 0.09)$

(f) $P(Z \leq -4.00)$

(g) $P(Z = 3.14)$

Notation: z_α is the z value where the area under the standard normal curve to the **right** of z_α is equal to α. That is, so that $P(Z \geq z_\alpha) = \alpha$; z_α is called the **critical value**.

By symmetry, $-z_\alpha$ is the z value where the area under the standard normal curve to the **left** of $-z_\alpha$ is equal to α. That is, so that $P(Z \leq -z_\alpha) = \alpha$.

(h) Find c such that 25% of the standard normal distribution is below c.

(i) Find c such that 30.5% of z-values exceed c.

(j) Find c such that $P(-c \leq Z \leq c) = 0.85$.

Standardizing a Normal Random Variable

If a normal random variable X has mean μ and standard deviation σ then the random variable

$$Z = \frac{X - \mu}{\sigma}$$

has the standard normal distribution.

Example 2: Suppose that X is a normal random variable with $\mu = 10$ and $\sigma = 4$.

(a) Find $P(X \geq 12)$

(b) Find $P(8 \leq X \leq 12)$

(c) Determine the value c such that 5% of the distribution lies above c.

The Empirical Rule: If the population distribution of a variable is (approximately) normal

- ▷ Roughly **68%** of observations are within 1 standard deviation of the mean.
- ▷ Roughly **95%** of observations are within 2 standard deviations of the mean.
- ▷ Roughly **99.7%** of observations are within 3 standard deviations of the mean.

Example 3: Suppose that among a certain population of adult humans, the mean length of the lateral incisor tooth is known to be 8.30mm, with a standard deviation of 0.95mm.

Extra Example 1: In female migraine patients, serum estradiol (E2) levels were found to be normally distributed with mean $\mu = 46$ pg/ml and standard deviation $\sigma = 6$ pg/ml.

(a) What is the probability that the estradiol levels of a random patient are less than 40 pg/ml?

(b) What is the probability that the estradiol levels of a random patient are between 40 pg/ml and 60 pg/ml?

(c) What level of estradiol characterizes the top 5% of the population?

Ans: (a) 0.1587 (b) 0.8314 (c) 55.87

Readings and Practice problems: See Set 16