Q1) Let $p(n)$ be the statement: $\sum_{j=1}^{n} j(j+1) = \frac{1}{3} \times n \times (n+1) \times (n+2)$

Base case: $p(1)$

$LHS = 1 \times (1+1) = 1 \times 2 = 2$

$RHS = (1/3) \times 1 \times (1+1) \times (1+2) = \frac{1}{3} \times 1 \times 2 \times 3 = 2$

$\therefore$ LHS = RHS for $p(1)$ or $n=1$.

$\therefore$ $p(1)$ holds

Let $n \geq 1$ Suppose $p(n)$ holds.

$n = k$

$(1\times2) + (2\times3) + \dots + (k \times (k+1)) + ((k+1) \times (k+2))$

Inductive Step:

$(1/3) \times k \times (k+1) + ((k+1) \times (k+2))$

$\rightarrow ((k+1) \times (k+2)) + (k/3 + 1)$

$\rightarrow ((k+1) \times (k+2)) + ((k/3)/3)$

$\rightarrow (1/3) \times (k+1) \times (k+2) \times (k+3)$

Thus $p(k+1)$ or $p(n+1)$ holds.

Hence by PMI, $P(n)$ is true for all $n \geq 1$.

$\therefore$ we have proven that:

$\sum_{j=1}^{n} j(j+1) = \frac{1}{3} \times n \times (n+1) \times (n+2)$

Q2) Let $p(n)$ be the statement $n^5 - n$ is a multiple of 5.

$p(1) = 1^5 - 1 = 0$, 0 is a multiple of 5.

$\therefore$ Base case is true.

Let $n \geq 1$, Suppose $p(n)$ holds.

Assuming that for some natural number k, $k^5 - k$ is a multiple of 5. This means that there exists an integer m such that: $k^5 - k = 5m$.

$p(n+1)$:

$(k+1)^5 - (k+1)$

$= (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - (k+1)$

$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k - k$

$= k^5 - k + 5(k^4 + 2k^3 + 2k^2 + k)$

$= 5m + 5(k^4 + 2k^3 + 2k^2 + k)$

$= 5(m + k^4 + 2k^3 + 2k^2 + k)$

Since $m + k^4 + 2k^3 + 2k^2 + k$ is an integer, we can conclude that $(k+1)^5 - (k+1)$ is a multiple of 5.

$\therefore$ $P(n+1)$ holds.

Hence by PMI, $P(n)$ is true for all natural numbers n.

Q3) a) Equation 1: $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$

$LHS = \left(\frac{1+\sqrt{5}}{2}\right)^2$

$= \frac{(1+\sqrt{5})^2}{4} \rightarrow a^2 + 2ab + b^2$

$= \frac{1^2 + 2 \times 1 \times \sqrt{5} + (\sqrt{5})^2}{4} = \frac{1 - 2\sqrt{5} + 5}{4}$

$= \frac{6 + 2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$

$\therefore$ LHS = RHS

Equation 2: $\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$

$LHS = \left(\frac{1-\sqrt{5}}{2}\right)^2$

$= \frac{(1-\sqrt{5})^2}{4} \rightarrow a^2 - 2ab + b^2$

$= \frac{1^2 - 2 \times 1 \times \sqrt{5} + (\sqrt{5})^2}{4}$

$= \frac{1 - 2\sqrt{5} + 5}{4}$

$= \frac{6 - 2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2}$

$\therefore$ LHS = RHS

$\therefore$ Both equation 1 and 2 hold.

b) Let $p(n)$ be the statement $\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

Base cases $p(1) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right)$

$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right)$

$= \frac{1}{\sqrt{5}}(\sqrt{5})$

$= 1$

Since $f_1 = 1$ $p(1)$ holds

$p(2) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right)$

$= \frac{1}{\sqrt{5}}\left(\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{3-\sqrt{5}}{2}\right)\right)$

$= \frac{1}{\sqrt{5}}(\sqrt{5})$

$= 1$

Since $f_2 = 1$, $p(2)$ holds.

Let $n \geq 2$ Suppose $p(n)$ and $p(n-1)$ hold.

Assume:

$f_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right)$

$f_{k-1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right)$

$P(n+1)$:

$f_{k+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right)$

$f_{k+1} = f_k + f_{k-1}$

Using the assumed formulas:

$f_{k+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right) + \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right)$

$= \frac{1}{\sqrt{5}}\left[\left(\left(\frac{1+\sqrt{5}}{2}\right)^k \times \left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^k \times \left(\frac{1-\sqrt{5}}{2}\right) + \left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$

$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right)$

$\therefore$ $P(k+1)$ holds

Hence by PMI, $P(n)$ is true for all natural numbers n.

Q4) Let $p(n)$ be the statement:

For any $n \geq 18$, n can be expressed as a sum of 4's and 7's.

Base cases:

$p(18) = 7+7+4$

$p(19) = 7+4+4+4$

$p(20) = 4+4+4+4+4$

$p(21) = 7+7+7$

Base cases hold true.

Induction Hypothesis:

Assume that for some $k \geq 21$, the number (k) can be expressed by a sum of 4's and 7's. i.e. $(k = 4m + 7n)$

Induction Step:

If $n \geq 1$:

$k+1 = 4m + 7n + 1 = 4(m+2) + 7(n-1)$

$\rightarrow$ nonnegative $\therefore$ the expression holds

If $n \geq 0$:

Since $4m \geq 21 \rightarrow m \geq 5.25$ so $m \geq 6$ as m is an integer.

$k+1 = 4m + 1 = 4(m-5) + 7 + 7 + 7$

Since $m \geq 6$ and $m - 5 \geq 1$ $\therefore$ the expression holds.

Hence by PMI, $p(n)$ holds for all $n \geq 18$.

Q5) Base cases:

For $n=1$: $a_1 = 1 < 3^1 = 3$

$\therefore$ case holds

For $n=2$: $a_2 = 2 < 3^2 = 6$

$\therefore$ case holds

Inductive Hypothesis:

Assume that $a_k < 3^k$ for all $k \leq n$ where $n \geq 2$.

Inductive Step:

prove: $a_{n+1} < 3^{n+1}$

$\rightarrow a_{n+1} = 2a_n + a_{n-1} < 2 \cdot 3^n + 3^{n-1}$ by IH

$= 2 \cdot 3^n + \frac{3n}{3}$

$= 7 \cdot \frac{3^n}{3} < 3 \cdot 3^n$

$= 3^{(n+1)}$

$\therefore$ By PMI, $a_n < 3^n$ for all n.

Q6a) Proof:

Let n be an integer such that $n \geq 3$.

$\frac{(n+1)^2}{n^2} = \frac{n^2 + 2n + 1}{n^2}$

$= \frac{n^2}{n^2} + \frac{2n}{n^2} + \frac{1}{n^2}$

$= 1 + \frac{2}{n} + \frac{1}{n^2}$

Since $n \geq 3$:

$1 + \frac{2}{n} + \frac{1}{n^2} \leq 1 + \frac{2}{3} + \frac{1}{9}$

$= 1 + 0.6666\ldots + 0.1111\ldots$

$\approx 1.7778 < 2$

$\therefore \frac{n+1^2}{n^2} < 2$ for all integers $n \geq 3$.

b) Base case: $n = 5$

$5^2 < 2^5 \rightarrow 25 < 32$

$\therefore$ base case holds true.

Inductive Hypothesis:

Assume that $k^2 < 2^k$ for some integer $k \geq 5$.

Inductive Step:

$(k+1)^2 > 2^{k+1}$

From part a $\rightarrow \frac{(k+1)^2}{k^2} < 2$ whenever $k \geq 3$.

Multiplying both sides by k^2:

$(k+1)^2 < 2k^2$

By IH, $k^2 < 2^k$:

$(k+1)^2 < 2k^2 < 2(2^k) = 2^{k+1}$

$\therefore (k+1)^2 < 2^{k+1}$ holds given $k^2 < 2^k$ holds for some integer $k \geq 5$.

Hence by PMI, $n^2 < 2^n$ holds for all integers $n \geq 5$.

Q7) Let $P(n)$ be the statement "The decimal expansion of 24^n ends in a 6".

Base case:

$p(1) = 24^1 = 24$

The decimal expansion ends in a 4 so,

rewrite 24 as $24 = 10 \times 2 + 4$, $24 \times 6 = (10j + 4)6$

$\rightarrow 60j + 24 \rightarrow 10(6j) + 24 \rightarrow 10(6j+2) + 4$

$10j' + 4$ where $j' = 6j + 2$

$\rightarrow 24$

for some integer ($j = 2$ for base case)

$\therefore$ $P(1)$ or base case holds true.

Inductive Hypothesis:

Assume that $P(k)$ holds true for some positive integer k.

$\therefore$

Q7) Let $p(n)$ " $\exists i$ such that $2^{4n} = 10j + 6$ "

Base case :

$p(1) = 2^4 = 16 = 10(1) + 6$ $p(1)$ is true ✓

Suppose $p(n)$ holds :

$2^{4n+4} = 10i + 6$

$2^{4n} = 10i + 6$

Then $2^{4n+4} \rightarrow 2^{4n} \cdot 2^4$

$(10i + 6)\ 16$

$\rightarrow 160i + \underbrace{96}_{\text{split up}}$

$\rightarrow 160i + 90 + 6$

$\rightarrow 10\,(16i + 9) + 6$

$\therefore p(n+1)$ holds

Hence by PMI , $p(n)$ is true $\forall n \geq 1$.