Sequence

$\{f(n)\}_{n=k}^{\infty} = f(k), f(k+1), \ldots, f(n), \ldots$

Some basic lim rules:

$$\lim_{n\to\infty} \sqrt[n]{n} = 1 \qquad \lim_{n\to\infty} \frac{\ln n}{n} = 0 \qquad \lim_{n\to 0} \frac{\sin n}{n} = 1$$

$$\lim_{n\to\infty} x^n = 0, |x| < 1 \qquad \lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x$$

$$\lim_{n\to\infty} x^{\frac{1}{n}} = 1, x > 0 \qquad \lim_{n\to\infty} \frac{x^n}{n!} = 0$$

Series

Geometric series: $S_\infty = \frac{a}{1-r}$, $|r| < 1$; a: first term, r: multiplicative factor. Btw, $S_n = \sum_{n=0}^{\infty} a(r)^n$

Telescoping series: Plug in numbers and evaluate.

Test	Conditions	Conclusion	Use this…
n^{th} term test *Always try this first*	$\lim_{n\to\infty} a_n$	If $\lim_{n\to\infty} a_n \neq 0$, the series $\sum a_n$ **diverges**.	to show a series diverges. We can't use this test to show a series converges.
Integral test	$f(x)$ is continuous, positive, and decreasing.	$\sum_{n=k}^{\infty} f(n)\ and \int_k^{\infty} f(x)dx$ either **both converge** or **both diverge**.	
** p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Series converges **if and only if** $p > 1$.	
Comparison test	$0 \leq a_n \leq b_n$	If $\sum_{n=k}^{\infty} a_n$ **di** $\sum_{n=k}^{\infty} b_n$ **di** If $\sum_{n=k}^{\infty} b_n$ **con** $\sum_{n=k}^{\infty} a_n$ **con**	Find the essential behavior and use this to bound the sequence (above and below).

Limit comparison test	$a_n, b_n > 0$ $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$	- If $L > 0$, then S_a and S_b **both** converge or diverge. - If $L = 0$, *and* S_b converges, then S_a **converges**. - If $L = \infty$, *and* S_b diverges, then S_a diverges.	Note: a_n is the given function and b_n is the function you are comparing it to.
Ratio test	$a_n > 0$ $\rho = \lim\limits_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert$	If $\rho < 1$, then S_n **converges**. If $\rho > 1$, then S_n **diverges**. If $\rho = 1$, **inconclusive**.	Great for series involving factorials.
Root test	$a_n > 0$ $\rho = \lim\limits_{n\to\infty} (a_n)^{\frac{1}{n}}$	If $\rho < 1$, then S_n **converges**. If $\rho > 1$, then S_n **diverges**. If $\rho = 1$, **inconclusive**.	Great when a_n is something to the power of n.
Alternating series test	i. a_n is **positive**. ii. $a_{n+1} \leq a_n$. iii. a_n **decreases to 0** as $n \to \infty$.	$\sum\limits_{n=N}^{\infty} (-1)^n a_n$ **converges**. For a series $\sum\limits_{n=N}^{\infty} a_n$, Absolutely converge if, $\lvert S_n\rvert$ con. Conditionally converge if, S_n con and $\lvert S_n\rvert$ di. Absolutely diverge if, S_n di.	Use when there's sometime alternating, i.e $(-1)^n$ and $(-1)^{n+1}$. ** Be careful when seeing something like $\sin/\cos(n\boldsymbol{\pi})$, they might alternate too!

*** = important; con = converge; di = diverge*

Error analysis: $$\int_{N+1}^{\infty} f(x)\,dx \leq \left\lvert \sum_{n=k}^{\infty} f(n) - \sum_{n=k}^{N} f(n) \right\rvert \leq \int_{N}^{\infty} f(x)\,dx$$

<u>Calculus of power series</u>

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = \sum_{n=0}^{\infty} c_n n (x-a)^{n-1}$$

$$\int \left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C$$. To find C we can evaluate at $x = a$.

Taylor series

Taylor series for $f(x)$ at $x = a$, $f(a) \approx \sum_{k=0}^{\infty} f^{(k)}(a)\frac{(x-a)^k}{k!}$

, for $|x| < 1$ When $a = 0$, it is called Maclaurin series.

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + f^{(k)}(a)\frac{(x-a)^k}{k!}$$

Order: 0 1 2 3 ... k

Important Taylor series (or Maclaurin series because $a = 0$):

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, |x| < 1 \qquad sinx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad cosx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \ \binom{\alpha}{k} = \frac{\alpha^{\underline{k}}}{k!} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-(k-1))}{k!}$$

$$\frac{1}{\sqrt{1+x}} = \sum_{k=0}^{\infty} \frac{(-1)^k(2k)!}{4^k(k!)^2}x^k \qquad tan^{-1}x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, |x| \le 1$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{k}, -1 < x \le 1$$

Error approximation

$$|ACTUAL - ESTIMATE| = |ERROR|$$

$$|f(x) - P_n(x)| = |R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!}$$, $M = \max |f^{(n+1)}(t)|$ between x and a.