Q1) $\exists x, \forall y, x \leq y$

Translated: There exists an x such that for all y, x is less than or equal to y.

a) For universe $\in \mathbb{N}$ the statement is true.
universe $\mathbb{N} = \{1, 2, 3, \ldots\}$

Let x be 1, the smallest natural number.
If $x = 1$ then, $1 \leq y$ is always true since any value of y within the universe would be greater than or equal to x.
Moreover, since 1 is the smallest natural number $x=1$ automatically makes the statement true $\square$

b) Theorem: $\exists x, \forall y, x \leq y$ universe $\in \mathbb{Q}$

Proof: Let universe $= \mathbb{Q}$ (rational numbers)

$\therefore$ There can never be a "smallest" rational number.

Consider 2 cases:

Case 1:
$x = 0$
find $y = -1 < 0$

Case 2:
$x = -100$
$y = -101 < -100$

Case 3:
$x = \frac{22}{7}$
$y = -\frac{22}{7} < \frac{22}{7}$

Given that there is no such case where $y < x$ we can conclude that therefore,
There is no rational number x, for all y, such that $x \leq y$. Thus, the statement $\exists x, \forall y, x \leq y$ for univ $= \mathbb{Q}$ is false $\square$

c) The statement is true for universe $(0, 1]$
Reasoning: The universe does not include 0 but it does include 1 as the largest number in the set.
If $x = 1$ then, for any y in the universe $(0,1]$, we have $1 \leq y$, since 1 is the maximum value.
Therefore there does exist an x ($x=1$) such that for all y in universe $(0,1]$, $x \leq y$.
Thus the statement is true given univ $= (0,1]$ $\square$

d) For the statement:
There exists an x such that for all y, x is less than or equal to y.
The universe must have a minimum / smallest / lowest element for the statement to be true.

2)

a) If $q(X)$ is the statement that student X got the highest mark (possibly the equal highest mark) on each of the tests, we can express it using quantifiers as,
$q(x) = \forall t, \forall y, p(t, x, y)$
"For all tests t and for all students y, student X scored higher than or equal to student y on test t."
In other words, for student X to have gotten the highest mark on each test, they must have scored at least as high as every other student on every test.

b) $r = \neg \exists x\ \forall t\ \forall y\ p(t, x, y)$

"There does not exist a student X such that for all tests t and for all students y, student X scored higher than or equal to student y on test t"

3a) $\exists t, \forall N, \exists n, \exists m, (m \geq N \land n \geq N) \land |x_n - x_m| \geq t$
$\neg\forall = \exists$, $\neg\exists = \forall$, $\neg(p \to q) = p \land \neg q$
in words:

"There exists a positive real number t such that for all natural numbers N, there exist natural numbers n and m greater than or equal to N, such that the absolute difference between Xn and Xm is greater than or equal to t."

3b)

Property Y states that for any positive real t we can find a natural number N such that for all natural numbers n and m greater than or equal to N, the absolute difference $|X_n - X_m|$ is less than t.

In this sequence for any even index n and odd index m (or vice versa), $|X_n - X_m| = 1$ regardless of how large n and m are. So, if we choose $t = 0.5$, there is no N such that for all n, m $\geq$ N, $|X_n - X_m| < 0.5$

No matter how far we go in the sequence, there will always be pairs of terms (one even index, one odd index) whose absolute difference is 1, which is not less than 0.5

Therefore, the sequence (1, 0, 1, 0, 1, 0, ...) does not have property Y.

4a) For S the subsets are: $\emptyset, \{1\}, \{2\}, \{1,2\}$
$\therefore P(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

4b) $S = \{1,2\}$ Find $P(P(S))$
Num of elements is $= 2^n$ but since $P(P(S))$
num elements $= 2^{n^n} \to 2^{2^2} = 2^4 = 16$ elements

subsets of $P(S)$: $\emptyset, \{\{1\}\}, \{\{2\}\}, \{\{1,2\}\}, \{\emptyset, \{1\}\}, \{\emptyset, \{2\}\}, \{\emptyset, \{1,2\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{1,2\}\}, \{\{2\}, \{1,2\}\}, \{\emptyset, \{1\}, \{2\}\}, \{\emptyset, \{1\}, \{1,2\}\}, \{\emptyset, \{2\}, \{1,2\}\}, \{\{1\}, \{2\}, \{1,2\}, \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

Therefore $P(P(S)) = \{\emptyset, \{\{1\}\}, \{\{2\}\}, \{\{1,2\}\}, \{\emptyset, \{1\}\}, \{\emptyset, \{2\}\}, \{\emptyset, \{1,2\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{1,2\}\}, \{\{2\}, \{1,2\}\}, \{\emptyset, \{1\}, \{2\}\}, \{\emptyset, \{1\}, \{1,2\}\}, \{\emptyset, \{2\}, \{1,2\}\}, \{\{1\}, \{2\}, \{1,2\}, \{\emptyset, \{1\}, \{2\}, \{1,2\}\}\}$

5) Given $A \subsetneq B$, $B \subsetneq C$ Prove $A \subsetneq C$
Proof:
Step 1) By proper subset definition, $A \subsetneq B$ means that $A \subseteq B$ and $A \neq B$.
Step 2) $B \subseteq C$ means that every element in B is also an element of C.
Step 3) Let x be an arbritary element of A.
Using that we know if $x \in A$ then $x \in B$ and if $x \in B$ the $x \in C$.
Therefore $A \subseteq C$
Step 4) To prove $A \subsetneq C$, $A \neq C$
- Suppose $A = C$ then since $A \subseteq B$, $C \subseteq B$.
- Given $B \subseteq C$ and if both $B \subseteq C$ and $C \subseteq B$ then $B = C$.
- However this is a contradiction to $A \subsetneq C$ which implies $A \neq B$. Since $A = C$ by our assumption, and $B = C$ we would get $A = B$ which is the contradiction.

Step 5) Therefore by contradiction we can conclude that $A \neq C$.
Step 6) Thus $A \subsetneq C$ $\square$

6)

a) Let n be an odd integer.
Then $n = 2k+1$ where k is an integer.

$n^3 = (2k+1)^3$
$= 8k^3 + 12k^2 + 6k + 1$
$= 2(4k^3 + 6k^2 + 3k) + 1$

Since $2(4k^3 + 6k^2 + 3k)$ is even and one is odd n^3 is odd.
Therefore, by contrapositive, if n^3 is even then n must be even.

b) Assuming $\sqrt[3]{2}$ is rational it can be written in the form p/q.
We can also assume p and q have no common factors since the fraction is in its lowest form of p/q.
So $(\sqrt[3]{2})^3 = \left(\frac{p}{q}\right)^3$
$\to 2 = \frac{p^3}{q^3}$
$\to 2q^3 = p^3$
$\to$ Let $p = 2k$ since $2q^3$ is even p^3 must also be even.
$\to 2q^3 = (2k)^3$
$\to 2q^3 = 8k^3$
$\to q^3 = 4k^3$
$\therefore q^3$ is even thus contradicting our assumptions that p and q have no common factors.

$\therefore$ By contradiction we can conclude $\sqrt[3]{2}$ is in fact irrational.

7a) $\exists k \in \mathbb{Z}\ (n = 2k+1)$

In other words, there exists an integer k such that n equals 2k+1. This is the definition of an odd integer — it leaves a remainder of 1 when divided by 2.

7b) Let m, n be arbitary odd integers.
$\exists k_1, k_2 \in \mathbb{Z}$ such that $m = 2k_1 + 1$ $n = 2k_2 + 1$
$m + n = (2k_1 + 1) + (2k_2 + 1)$
$= 2k_1 + 2k_2 + 2$
$= 2(k_1 + k_2 + 1)$
Let $k_3 = k_1 + k_2 + 1$
Then $m + n = 2k_3$
This makes $m+n$ even.
Since m, n were arbitary:
$\forall m, \forall n \in \mathbb{Z}, [(m \text{ odd}) \land (n \text{ odd})] \to (m+n \text{ even})$ $\square$

7c) Let m, n be arbitary odd integers.
$\exists k_1, k_2 \in \mathbb{Z}$ such that $m = 2k_1 + 1$ $n = 2k_2 + 1$
$mn = (2k_1 + 1)(2k_2 + 1)$
$= 4k_1k_2 + 2k_1 + 2k_2 + 1$
$= 2(2k_1k_2 + k_1 + k_2) + 1$
Let $k_3 = 2k_1k_2 + k_1 + k_2$ and $mn = 2k_3 + 1$
Since m and n were arbitary,
$\forall m, \forall n \in \mathbb{Z}, [(m \text{ odd}) \land (n \text{ odd})] \to (mn \text{ odd})$ $\square$