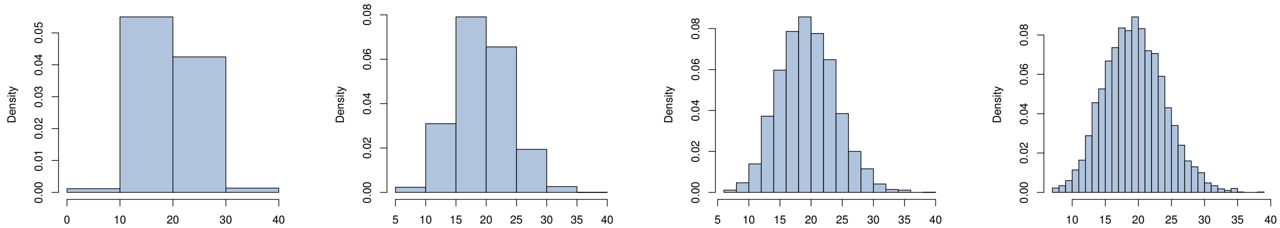
Sets 13 and 14: Continuous Random Variables

Stat 260 A01: June 12, 2024

Recall discrete pmfs:



The smaller the x-intervals (the thinner the blocks), the smoother the curves becomes.

Density Function for a Continuous Random Variable:
Let X be a continuous random variable. The ***density*** for X is a function $f(x)$ defined for all real numbers, such that:

(i)

(ii)

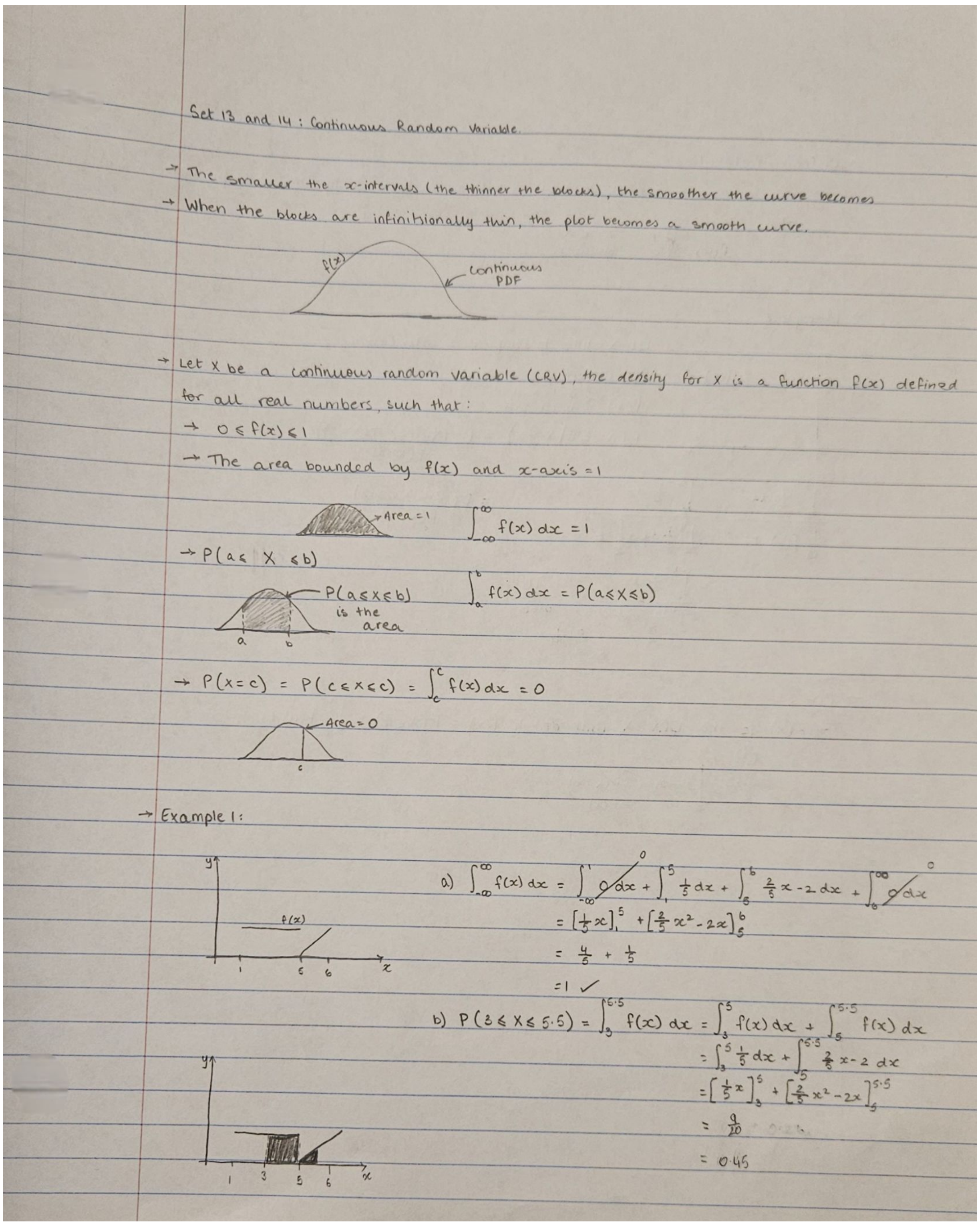
(iii)

Question: If X is a continuous r.v. and c is some constant, what is $P(X = c)$?

What is the probability that it rains 4.2324568674531235 45mm today?

Example 1: Let

$$f(x) = \begin{cases} \frac{1}{5} & 1 < x < 5 \\ \frac{2}{5}x - 2 & 5 \le x \le 6 \\ 0 & \text{otherwise.} \end{cases}$$



Set 13 and 14 : Continuous Random Variable

→ The smaller the x-intervals (the thinner the blocks), the smoother the curve becomes

→ When the blocks are infinitisionally thin, the plot becomes a smooth curve.

f(x)

continuous PDF

→ Let X be a continuous random variable (CRV), the density for X is a function f(x) defined for all real numbers, such that:

→ $0 \leq f(x) \leq 1$

→ The area bounded by f(x) and x-axis = 1

Area = 1 $\quad \int_{-\infty}^{\infty} f(x)\,dx = 1$

→ $P(a \leq X \leq b)$

$P(a \leq X \leq b)$ is the area $\quad \int_a^b f(x)\,dx = P(a \leq X \leq b)$

→ $P(X=c) = P(c \leq X \leq c) = \int_c^c f(x)\,dx = 0$

Area = 0

→ Example 1:

a) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{1} \cancel{0}\,dx + \int_1^5 \frac{1}{5}\,dx + \int_5^6 \frac{2}{5}x - 2\,dx + \int_6^{\infty} \cancel{0}\,dx$

$= \left[\frac{1}{5}x\right]_1^5 + \left[\frac{2}{5}x^2 - 2x\right]_5^6$

$= \frac{4}{5} + \frac{1}{5}$

$= 1$ ✓

b) $P(3 \leq X \leq 5.5) = \int_3^{5.5} f(x)\,dx = \int_3^5 f(x)\,dx + \int_5^{5.5} f(x)\,dx$

$= \int_3^5 \frac{1}{5}\,dx + \int_5^{5.5} \frac{2}{5}x - 2\,dx$

$= \left[\frac{1}{5}x\right]_3^5 + \left[\frac{2}{5}x^2 - 2x\right]_5^{5.5}$

$= \frac{9}{20}$

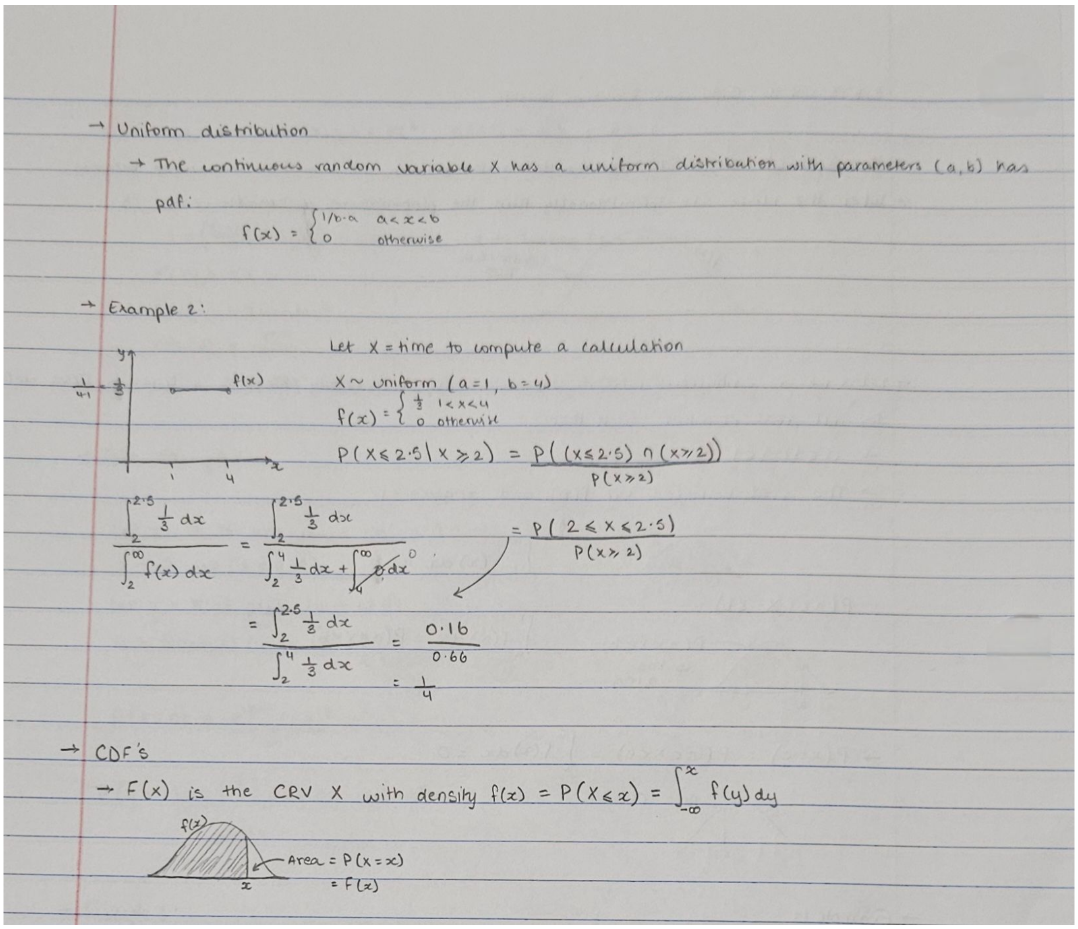
$= 0.45$

The Uniform Distribution

The continuous random variable X has a **Uniform Distribution** with parameters (a, b) if it has pdf

$$f(x) = \begin{cases} 1/(b-a) & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Example 2: The time it takes for a program to complete a complex calculation of a certain class is known to be uniformly distributed between 1 and 4 seconds. If a certain calculation takes at least 2 seconds to complete, what is the probability that it takes at most 2.5 seconds to complete?



→ Uniform distribution

→ The continuous random variable X has a uniform distribution with parameters (a, b) has pdf:

$$f(x) = \begin{cases} 1/b-a & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

→ Example 2:

Let X = time to compute a calculation

$X \sim$ uniform $(a=1, b=4)$

$$f(x) = \begin{cases} \frac{1}{3} & 1 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$P(X \leq 2.5 \mid X \geq 2) = \frac{P((X \leq 2.5) \cap (X \geq 2))}{P(X \geq 2)} = \frac{P(2 \leq X \leq 2.5)}{P(X \geq 2)}$$

$$\frac{\int_2^{2.5} \frac{1}{3}\,dx}{\int_2^{\infty} f(x)\,dx} = \frac{\int_2^{2.5} \frac{1}{3}\,dx}{\int_2^4 \frac{1}{3}\,dx + \int_4^{\infty} \cancel{0}\,dx}$$

$$= \frac{\int_2^{2.5} \frac{1}{3}\,dx}{\int_2^4 \frac{1}{3}\,dx} = \frac{0.16}{0.66} = \frac{1}{4}$$

→ CDF's

→ F(x) is the CRV X with density $f(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\,dy$

CDFs, Percentiles, and the Median of Continuous Random Variables

A continuous r.v. X with density $f(x)$ has cumulative distribution function (cdf):

and is the area under $f(x)$ to the left of (and including x):

The **100p-th percentile** of a continuous distribution with CDF $F(x)$ is the value of $\eta(p)$ such that

$$p = F(\eta(p)) = P(X \leq \eta(p))$$

The **median** $\widetilde{u}$ is the 50-th percentile.

Example 3: Find the CDF and the median of the following pdf:

$$f(x) = \begin{cases} \frac{1}{5} & 1 < x < 5 \\ \frac{2}{5}x - 2 & 5 \leq x \leq 6 \\ 0 & \text{otherwise.} \end{cases}$$

Expectation and Variance of Continuous Random Variables

Recall: For a discrete r.v.: $E[X] = \sum x f(x)$

For a continuous random variable X,

Geometrically $E[X]$ is the value that would "balance" the graph of $f(x)$...

For a continuous random variable X,

- $E[g(X)]$
- $V[X]$

Example 4: Determine the expected value and variance of the following:

$$f(x) = \begin{cases} \frac{3}{2}(1-x^2) & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Extra Example 1:

(a) Let X be a continuous random variable with a uniform distribution on the interval $[a, b]$. Determine the CDF.

(b) A train is equally likely to arrive at a certain station any time between 10:00am and 10:30am. What is the probability that the train arrives between 10:15am and 10:25am?

Readings: Swartz 5.1 [EPS 2.3, continuous parts of 2.5 and 2.6]
Practice problems: EPS : 2.7, 2.15, 2.19, 2.21, 2.23, 2.25, 2.57, 2.59, 2.69, 2.71, 2.73, 2.79, 2.81, 2.83