Sample mean $\bar{x}$ is a point estimator of μ (population mean)

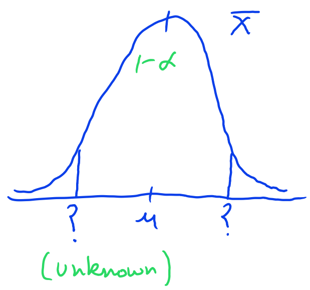
Set 22: Confidence Intervals

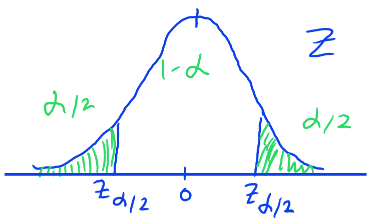
Stat 260 A01: July 10, 2024

We have already seen ***point estimators****, which are* ***single*** *valued statistics, used to estimate population parameters. Point estimators can be useful, but they don't give any indication of how accurate the estimate is. Instead, we will now try to estimate μ on an* ***interval****.*

From Standardization Theorem or CLT

Recall from Set 21: If a random variable X is normally distributed, or n is large ($n \geq 30$), then $\overline{X}$ is approximately normally distributed, and $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$ is approximately standard normal (i.e. $Z \sim Normal(0, 1)$). Then, using the definition of our $z_{\alpha/2}$ critical values, we have that:





Goal: create a bound around the unknown μ.

$$P\left(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}\right) = 1-\alpha$$

$$= P\left(-z_{\alpha/2} \leq \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) = 1-\alpha$$

$$= P\left(-z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \leq \overline{X} - \mu \leq z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$= P\left(-\overline{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq -\mu \leq -\overline{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$= P\left(\overline{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \geq \mu \geq \overline{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$= P\left(\underset{rv}{\overline{X}} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \underset{const}{\mu} \leq \underset{rv}{\overline{X}} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

Thus, with repeated sampling from this population, the proportion of values of $\overline{X}$ for which the interval $\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$, $\overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ includes the population mean μ is $1 - \alpha$.

Confidence Interval Estimator of μ When σ is Known (Case 1)*

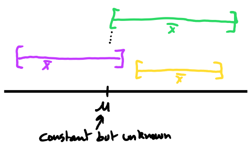
$$[L_1, L_2] = \left[\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] \qquad \text{or} \qquad \overline{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- The probability $1 - \alpha$ is the ***confidence level***.
- $L_1 = \overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ is the ***Lower Confidence Limit (LCL)***.
- $L_2 = \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ is the ***Upper Confidence Limit (UCL)***.

**Note: this case is nice in theory, but somewhat unrealistic as it is unlikely σ is known if μ is not.*

A ***confidence interval (CI)*** is an ***interval estimate*** of a parameter.
We are often concerned with the $(1-\alpha)100\%$ CI for the population mean μ.

95% CI $= (1-0.05)\,100\%$ CI $\quad \alpha = 0.05$



- everytime experiment is repeated, sample mean $\bar{x}$ changes slightly and Confidence Interval (which may or may not contain μ)
- The 95% CI means that if we repeated experiment 100 times and made a 100 CI's, 95 should (secretly) contain μ.

In general, a confidence interval has the form:

$$\begin{pmatrix}\text{point}\\ \text{estimate}\end{pmatrix} \pm \begin{pmatrix}\text{critical}\\ \text{value}\end{pmatrix}\begin{pmatrix}\text{standard error}\\ \text{or}\\ \text{estimated standard error}\end{pmatrix}$$

Example 1: Suppose we want to study the mean weight of the rufous hummingbird (*Selasphorus rufus*), which are known to be normally distributed and to have a population standard deviation of 1.1 grams. Suppose that we collect a sample of 100 rufous hummingbirds, and find a sample mean weight of 3.9 grams. Determine the 95% CI and the 99% CI for this sample.

$$\bar{x} \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

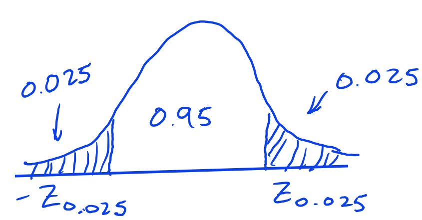
$\sigma = 1.1$ grams, $n = 100$, $\bar{x} = 3.9$ grams

95% CI $= (1-\alpha)\,100$ CI $\Rightarrow 1-\alpha = 0.95$

$\rightarrow \alpha = 0.05$, $\rightarrow \alpha/2 = 0.025$

$Z_{\alpha/2} = Z_{0.025} = -1.96$ → but this is wrong as the val must be pos so we ignore the -ve

$= 1.96$



a) 95% CI: $3.9 \pm (1.96)\cdot\frac{1.1}{\sqrt{100}}$

$L_1: \bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 3.9 - 1.96\cdot\frac{1.1}{\sqrt{100}} = 3.6844\text{g}$

$L_2: \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 3.9 + 1.96\cdot\frac{1.1}{\sqrt{100}} = 4.1156\text{g}$

95% CI: $[3.6844\text{g}, 4.1156\text{g}]$

b) 99% CI $\rightarrow 1-\alpha = 0.99 \rightarrow \alpha = 0.01 \rightarrow \alpha/2 = 0.005 \rightarrow Z_{0.005} \rightarrow$ 2.57 and 2.58 → split difference → 2.575

Nice in theory but unrealistic because we rarely know σ

CI for μ (because we don't know μ)

1. **When can we use the CI formula**: $\bar{x} \pm z_{\alpha/2} \dfrac{\sigma}{\sqrt{n}}$?

   Answer: Whenever σ is known and $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$ is normally distributed. That is,

   (*i*) $X_1, X_2, \ldots, X_n$ are normally distributed and σ is known, or ← standardization theorem

   (*ii*) $X_1, X_2, \ldots, X_n$ has any distribution, n is large ($n \geq 30$), and σ is known. ← CLT (Central Limit Theorem)

2. **What if σ is unknown?**

   Answer: We can approximate the ***standard error (se)*** $\dfrac{\sigma}{\sqrt{n}}$ with the ***estimated standard error (ese)*** $\dfrac{s}{\sqrt{n}}$. Then the $(1-\alpha)100\%$ CI for μ is:

   → $\bar{x} \pm z_{\alpha/2} \dfrac{s}{\sqrt{n}}$

   ▷ We can use this whenever $\dfrac{\overline{X} - \mu}{s/\sqrt{n}}$ is normally distributed. That is,

   - $X_1, X_2, \ldots, X_n$ has any distribution, n is large ($n \geq 30$), and σ is unknown.

   ▷ **Warning**: even if $X_1, X_2, \ldots, X_n$ has a normal distribution, if n is small ($n < 30$) and σ is unknown, $\dfrac{\overline{X} - \mu}{s/\sqrt{n}}$ is better approximated by the $\underbrace{T\text{-distribution}}_{\text{set 23}}$ than the normal distribution.

Basically ⇒ if you know σ then use it, otherwise use s.

Example 2: A sample of 200 random adult American bison (*Bison bison*) yielded an average tale length of 35cm, with a sample standard deviation of 7cm.

(a) Find the 88% CI for American bison tail lengths.

($\bar{x} = 35$ cm, $s = 7$ cm, $n = 200$) $n = 200 \geq 30$, so use $z_{\alpha/2}$ critical value

$88\%\text{CI} = (1-\alpha)100\,\text{CI} \Rightarrow 1-\alpha = 0.88$

$\to \alpha = 0.12$, $\to \alpha/2 = 0.06$

$z_{0.06} = 1.555$ ← split values from tables

$35 \pm (1.555)\dfrac{7}{\sqrt{200}} = 35 \pm 0.77$cm

or $[34.23\text{cm}, 35.77\text{cm}]$

Example 2 Continued...

(b) Suppose we had sampled 500 bison instead. Would the 88% CI for 500 bison be wider than the 88% CI for 200 bison?

$\bar{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}$ as n increases the estimated standard error decreases

$\therefore$ 500 bison CI is narrower

Warning: Suppose we are working an example for the mean length of a certain variety of rabbit ears in cms, and find a 95% CI for μ to be $[4.5, 8.5]$. Does this mean that there is a 95% chance that μ is in the interval $[4.5, 8.5]$?

i.e. $P(4.5 \leq \underset{\substack{\uparrow \\ \text{constant}}}{\mu} \leq 8.5) = 0.95$?

No, μ is a constant and its either in $[4.5, 8.5]$ $(P(4.5 < \mu < 8.5) = 1)$ or its not $(P(4.5 < \mu < 8.5) = 0)$.

Readings: Swartz 6.1, 6.1.1, [EPS bottom half of p. 198 – top half of p. 203]
Practice problems: EPS 5.1, 5.3, 5.5, 5.7
Devore 7ed: Readings 7.1 (Practice Problems 1, 3, 5, 7)