Set 4: Basic Set Theory

Experiment: any activity where we can measure, or observe the results.

Example 1: Flipping a coin three times and noticing the sequence of heads and tails is an experiment.

Outcomes: the observations from our experiment.

Sample space: the set of all possible outcomes. We denote the sample space by S. The sample space may contain a finite or an infinite number of outcomes.

Sample point: A single outcome in the sample space.

Event: Any subset of S (i.e. any collection of outcomes).

Simple event: An event consisting of one outcome.

Compound event: An event consisting of more than one outcomes.

Example 2: Consider the experiment where we flip a coin three times and note the sequence of heads and tails.

For this experiment the sample space is as follows:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Each of these eight elements of S are sample points.

Some examples of events are:

$$A = \{HHH, HHT, HTH, THH\}$$
$$B = \{HHT, HTT, THT, TTT\}$$
$$C = \{HHH, TTT\}$$

Events can sometimes be described in words. For example, B is the event that the third flip is a tail.

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Example 3: What is the event that you observe more heads than tails?

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We say that an event **occurs** if one of its sample points is observed when we carry out the experiment.

Set Theory:

- The **intersection** of A and B, denoted by $A \cap B$.

  A **and** B both occur.

- The **union** of A and B, denoted $A \cup B$.

  A **or** B (or both) occur.

- The **complement** of A, denoted $\overline{A}$ or A' or A^C.

  A does **not** occur.

Note: Throughout, we will use "or" in the inclusive sense (i.e., "this or that *or both*").

Example 4: Suppose we select an integer from 1 to 10 at random. Let A be the event that an even number is selected. Let B be the event that a number 7 or larger is selected.

Find $A \cap B$, $A \cup B$, and $\overline{B}$.

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We call the event S the **guaranteed** or **certain event**, because it will always occur.

The event $\emptyset$, which consists of no outcomes, is called the **impossible event** or **null event**, because it never occurs.

If for events A and B, we have $A \cap B = \emptyset$, then we say that A and B are **disjoint** or **mutually exclusive events**.

We can often use tree diagrams to help us find all possible outcomes.

Example 5: Suppose that a box contains red, blue, and green marbles (several of each colour). Two marbles are selected one at a time from the box, and the sequence of colours is noted. What is the sample space?

Set 5: Introduction to Probability

Probability is used to express the likelihood that some event will or will not occur. We measure probability on a scale from 0 to 1, where 0 indicates that it is *impossible* for the event to occur and 1 indicates that the event is *guaranteed* to occur.

For an event A, we denote the probability that A will occur by $P(A)$.

One way to determine probability is experimentally:

- Repeat an experiment n times.
- Count f, the number of times the event in question occurs.
- Then $P(A) \approx f/n$.

We'll be using the **classical** (theoretical) approach. We will determine the probability of an event occurring by using properties of probability, ideas from set theory, and counting techniques.

Probability Axioms

Probabilities should satisfy the following axioms:

1. $P(S) = 1$.
2. $P(A) \geq 0$ for any event A.
3. $P(A_1 \cup A_2 \cup A_3 \cup \dots) = \sum P(A_i)$ for any countable (could be infinite) collection of mutually exclusive events.

From these axioms, we can derive other properties of probability, including:

- $P(A_1 \cup A_2 \cup \cdots \cup A_k) = \sum P(A_i)$ where the events are all mutually exclusive. This is a special case of axiom 3.
- $P(\emptyset) = 0$
- $P(A) = 1 - P(\overline{A})$ for any event A.
- $P(A) \leq 1$ for any event A.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any events A and B.

- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$ for any events A, B, and C.

The last 2 formulas are called the inclusion-exclusion formulas for probability.

Uniform sample space: a sample space where each outcome is equally likely to occur.

Let $n(S)$ denote the size of the sample space, and let $n(A)$ denote the size of the event A. Here, we are assuming that the sample space is finite, and uniform.

Each of the $n(S)$ simple events must have the same probability, and those probabilities must add to 1.

The probability of each simple event must be $\frac{1}{n(S)}$.

For any event A in the uniform, finite sample space S, we have:

$$P(A) = \frac{n(A)}{n(S)}$$

e.g. Flipping a fair coin once to find the probability of getting a head.

e.g. Rolling a fair die (a cube with six faces, each showing one to six dots) to find the probability of event A, where the number of dots is 5 or greater.

Example 6: There are 80 students in a classroom. We will select one of the 80 students at random to answer a question. Of the 80 students, 7 are sitting in the front row. What is the probability that we select a student who is sitting in the front row?

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Example 7: The 2001 Census found that in Tofino, there were 790 residents who traveled to work. Here are the results of this census question:

Mode of Transportation	**Number (Frequency)**
Car/truck/van	435
Walk/bicycle	250
Other method	105
Total	790

Suppose a Tofino resident who travels to work is selected at random. What is the probability that this resident walks or bikes to work?

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Example 8: Consider the results of the following survey of 250 single-crop farms:

	Wheat	Corn	Soy
Alberta	69	15	16
Saskatchewan	61	65	24

If we select one farm at random, what is the probability that the farm grows wheat, or is in Saskatchewan?

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Example 9: Suppose for some experiment we have two events E and F. $P(E) = 0.4$ and $P(F) = 0.8$. Determine whether each of the following statements are **(A) True** or **(B) False**.

- If we carry out the experiment ten times, we are guaranteed that F will occur twice as often as E does.

- If we carry out the experiment ten times, we are guaranteed that F will occur exactly 8 times.

- If we carry out the experiment ten times, we are guaranteed that E will occur less often than F.

- We would expect that in a long-term average, E will occur about 40% of the time.

- We would expect that in a long-term average, F will occur approximately twice as often as E does.

Set 6: Conditional Probability

In some cases, we may perform an experiment, and already know something about the outcome.

Example 10: Suppose we have a bag containing 5 red marbles and 5 green marbles. We draw two marbles from the bag, one at a time, without replacement.

- If the first marble is red, then the probability that the second marble will be red will be $\frac{4}{9}$.
- If the first marble is green, then the probability that the second marble will be red will be $\frac{5}{9}$.

These probabilities are known as **conditional probabilities**. They are the probabilities of an event occurring, taking into account some information about the experiment (the **condition** or **conditioning event**).

Notation: $P(B|A)$ denotes the probability that B will occur *given that* A occurs.

Definition: $$P(B|A) = \frac{P(B \cap A)}{P(A)}$$ provided $P(A) > 0$.

For uniform finite sample space, $$P(B|A) = \frac{n(B \cap A)}{n(A)}$$

Example 11: Consider the results of the following survey of 250 single-crop farms:

	Wheat	Corn	Soy	**Total**
Alberta	69	15	16	
Saskatchewan	61	65	24	
Total				

Suppose that a single-crop farm is selected at random. If the farm is in Alberta, what is the probability the farm grows soy?

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Example 12: If a farm which grows soy is selected, what is the probability that the farm is in Alberta.

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Example 13: Suppose 80% of all Canadians exercise one or more days a week (event A). It is also known that 20% of all Canadians exercise at five or more days a week (event B). If we randomly select a Canadian who exercises at least one day a week, what is the probability that this Canadian exercises five or more days a week?
Hint: $B \subseteq A$, i.e., B is a subset of A.

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Interpreting word problems:

An ice cream store sells chocolate and vanilla ice cream, which is served in either a regular cone or a waffle cone.

Example 14: Suppose we would like to know the probability that someone orders chocolate ice cream waffle cone. Which of the following are we trying to find:

(A) $P(\text{Chocolate} \,|\, \text{Waffle})$

(B) $P(\text{Chocolate} \cap \text{Waffle})$

(C) $P(\text{Chocolate} \cup \text{Waffle})$

(D) $P(\text{Waffle} \,|\, \text{Chocolate})$

Example 15: Suppose we would like to know the probability that someone who wants a waffle cone will order chocolate ice cream. Which of the following are we trying to find:

(A) $P(\text{Chocolate} \,|\, \text{Waffle})$

(B) $P(\text{Chocolate} \cap \text{Waffle})$

(C) $P(\text{Chocolate} \cup \text{Waffle})$

(D) $P(\text{Waffle} \,|\, \text{Chocolate})$

The Multiplication Rule: Rearranging the equation for condition probability, we get:

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \qquad \Longleftrightarrow \qquad P(B \cap A) = P(A)P(B|A)$$

$\square$

Definition: A collection of events $A_1, A_2, \ldots, A_k$ are **exhaustive** if they completely cover the sample space. That is, if these events are exhaustive, then $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

Law of Total Probability: If $A_1, A_2, \ldots, A_k$ are a collection of mutually exclusive and exhaustive events, then for any event B we have:

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_k)$$

$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_k)P(A_k)$$

This forms of the conditional probability formulas are useful when using a tree diagram to model an experiment.

Example 16: Suppose that 30% of all students drive to school, 50% take the bus, and 20% walk. Of those who drive, 20% are usually late for their first class of the day. Of those who take the bus, 10% are usually late for their first class of the day. Of those who walk, 15% are usually late for their first class of the day. What is the probability that a randomly selected student is usually late for their first class?

Example 17: The probability of an item on a certain production line being defective is 0.1. If an item is defective, the probability that the inspector will remove it from the line is 0.9. If an item is not defective, the probability that the inspector will remove it from the line is 0.2.

What is the probability that a randomly selected item will be removed from the production line?

Bayes' Formula/Rule: If $A_1, A_2, \ldots, A_k$ are a collection of mutually exclusive and exhaustive events, then for any event B (where $P(B) \neq 0$) we have the following, for $1 \leq i \leq k$:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)}$$

$$= \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_k)P(A_k)}$$

Note: You can use a tree diagram to compute this conditional probability without having to memorize the formula if you can visualize it.

Example 18: Suppose that 30% of all students drive to school, 50% take the bus, and 20% walk. Of those who drive, 20% are usually late for their first class of the day. Of those who take the bus, 10% are usually late for their first class of the day. Of those who walk, 15% are usually late for their first class of the day. Suppose that a student is late for class. What is the probability that this student walks to school?

Example 19: The probability of an item on a certain production line being defective is 0.1. If an item is defective, the probability that the inspector will remove it from the line is 0.9. If an item is not defective, the probability that the inspector will remove it from the line is 0.2.

If an item is removed from the production line, what is the probability that it isn't actually defective?

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If an item is not removed by the inspector, what is the probability that it is actually defective?

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Example 20: At a factory, 80% of all items produced are judged to need minor repairs, and 20% need major repairs. Of the items needing minor repairs, 90% are eventually sold, and the remainder are destroyed. Of the items needing major repairs, 50% eventually sold, and the remainder are destroyed.
What is the probability that a randomly selected item will be destroyed?

Select the closest to your unrounded answer:

(A) 0.2

(B) 0.4

(C) 0.6

(D) 0.8

If a randomly selected item was destroyed, what is the probability it needed major repairs?

Select the closest to your unrounded answer:

(A) 0.2

(B) 0.4

(C) 0.6

(D) 0.8