MATH 202 Exam 1 (A)

February 4th, 2020

Student Name: ______________________________

Student ID: ______________________________

Tutorial Section: ______________________________

Instructions:

- Write your name, student ID, and the tutorial section that you are sitting in.
- Organize and show your work. Any unsupported answers will receive no credit.
- The exam has **6 questions** and is out of 100 points.
- If you run out of space for computations, you may continue on the back of the exam. However, you **must** state your answers right in the space below the questions and specify where to find the supporting computations.
- You may use a SHARP EL-510R calculator.

Question 1 (15%). Find the area of the parallelogram determined by $\mathbf{u} = \langle 1, 2, 0\rangle$ and $\mathbf{v} = \langle 1, 0, 1\rangle$.

Question 2. Set $f(x, y) = \sqrt{4 - (x^2 + y^2)}$.

(1) (6%) Find the range of $f(x, y)$.

(2) (7%) Find the domain of $f(x, y)$.

(3) (7%) Sketch the level curve $f(x, y) = \sqrt{3}$ in the x-y plane.

Question 3. Let $\mathbf{u} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j}$. Find

(1) **(7%)** the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$, and

(2) **(8%)** the vector $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Question 4 (15%). Find the equation of the plane such that (i) it has a normal vector perpendicular to $\langle 1, 2, 0\rangle$ and $\langle 0, 1, 4\rangle$, and (ii) it passes through $P = (0, 2, 4)$. Write your answer in the form $ax + by + cz = d$.

Question 5 (20%). Let L be the line which (i) is perpendicular to the plane $P_1 : x + y + z = 0$ and (ii) passes through $(2, -1, -1)$. Find the intersection of L and the plane $P_2 : x + 2y + 3z = 1$.

Question 6 (15%). Evaluate the limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y + x}{x^2 + y^2}.$$

Justify your answer if the limit does not exist.

Proof. $\square$

Solution to Question 1. The area of the parallelogram is given by $|\mathbf{u} \times \mathbf{v}|$. See Section 1.4 Cross Product (of lectures) for the formula. Hence, we compute

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \mathbf{k} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}.$$

It follows that the required area is given by $\sqrt{2^2 + 1 + 2^2} = \boxed{3}$. $\square$

Solution to Question 2. (1) The range of $f(x, y)$ is the set of possible values of $f(x, y)$. The function can take $\boxed{\text{any value between 0 and } \sqrt{4} \text{ (the two values included)}}$.

(2) The domain of $f(x, y)$ is the set of (x, y) such that the formula of $f(x, y)$ is meaningful. Taking into account the square root defining $f(x, y)$, we find that the domain is given by $4 - (x^2 + y^2) \geq 0$, that is $\boxed{x^2 + y^2 \leq 4}$.

(3) Setting $f(x, y) = \sqrt{3}$, we find that $x^2 + y^2 = 1$. This equation is the equation of $\boxed{\text{the circle centered at the origin with radius 1}}$. $\square$

Solution to Question 3. (1) The scalar component is

$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{(-2) \cdot 1 + 1 \cdot (-1) + 1 \cdot 0}{\sqrt{1^2 + (-1)^2}} = \boxed{\frac{-3}{\sqrt{2}}}.$$

Recall that the above formula follows since it is the same as $|\mathbf{u}| \cos\theta$ for θ being the angle between $\mathbf{u}$ and $\mathbf{v}$, whereas $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$. See Section 1.3 Dot Product for this formula and the formula in (2) below.

(2) The projection is given by

$$\text{(the scalar component from part (1))} \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) = \boxed{\frac{-3}{\sqrt{2}} \left(\frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}} \right)}.$$

$\square$

Solution to Question 4. Since the cross product of two vectors is perpendicular to the two vectors, a normal vector of the plane can be chosen to be

$$\langle 1, 2, 0 \rangle \times \langle 0, 1, 4 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \mathbf{k} = 8\mathbf{i} - 4\mathbf{j} + \mathbf{k}.$$

Using this direction and the point assumed to be passed, we know that the equation of the plane is given by $\langle 8, -4, 1 \rangle \cdot \langle x - 0, y - 2, z - 4 \rangle = 0$, or equivalently,

$$\boxed{8x - 4y + z = -4}.$$

See Section 1.5 Lines and Planes for the background. $\square$

Solution to Question 5. Since the line is perpendicular to the plane P_1, any normal vector of P_1 can be used as a direction of the line. A normal vector of P_1 can be chosen to be $\langle 1, 1, 1\rangle$ from coefficients of x, y, z in the equation of P_1. Hence, the vector equation of L is

$$\mathbf{r}(t) = \langle 2, -1, -1\rangle + t\langle 1, 1, 1\rangle = \langle 2 + t, -1 + t, -1 + t\rangle, \quad -\infty < t < \infty. \tag{0.1}$$

Next, the intersection of L and the plane P_2 must satisfy the following four equations:

$$x = 2 + t, \quad y = -1 + t, \quad z = -1 + t, \quad \text{and} \quad x + 2y + 3z = 1,$$

where the first three equations follow from the equation of the line (0.1) and the last equation is the equation of P_2. Applying the first three equations to the last one, we get

$$(2 + t) + 2(-1 + t) + 3(-1 + t) = 1 \Longrightarrow t = 2/3.$$

The point of intersection is given by $\mathbf{r}(2/3)$, and so by the equation (0.1) of the line again, $\boxed{x = 8/3,\ y = -1/3, \text{ and } z = -1/3}$. See Section 1.5 Lines and Planes for the background. □

Solution to Question 6. Write

$$\lim_{(x,y)\to(0,0)} \frac{x^2y + x}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2 + y^2} + \lim_{(x,y)\to(0,0)} \frac{x}{x^2 + y^2}. \tag{0.2}$$

The first limit on the right-hand side of (0.2) is zero by using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2 + y^2} = \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = 0, \tag{0.3}$$

where the last equality follows from the sandwich theorem. The second limit on the right-hand side of (0.2) does not exist by using a line $y = cx$ (for a constant c) since the line passes through $(0, 0)$:

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2 + y^2} = \lim_{x\to 0} \frac{x}{x^2 + c^2x^2} = \lim_{x\to 0} \frac{1}{x(1 + c^2)} = \boxed{\text{DNE}}. \tag{0.4}$$

Applying (0.3) and (0.4) to (0.2), we conclude that the limit considered in this question is $\boxed{\text{DNE}}$.

The alternative method is to use polar coordinates throughout and consider

$$\frac{x^2y + x}{x^2 + y^2} = \frac{r^3\cos^2\theta\sin\theta + r\cos\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \frac{r^3\cos^2\theta\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta} + \frac{r\cos\theta}{r^2\cos^2\theta + r^2\sin^2\theta}.$$

The limit of the last ratio is DNE because that ratio simplifies to

$$\frac{\cos\theta}{r(\cos^2\theta + \sin^2\theta)}.$$

□