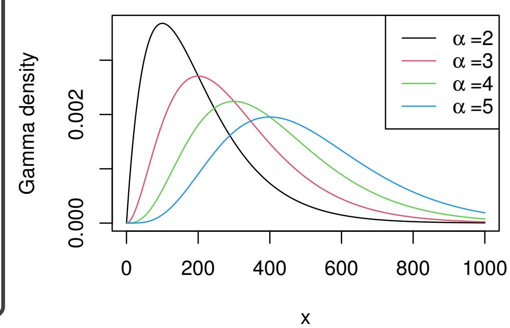
Set 17: The Gamma and Exponential Distributions

Stat 260 A01: June 21, 2024

Gamma Distribution: A continuous random variable X with a **Gamma distribution** and parameters $\alpha > 0$ and $\beta > 0$ has pdf

$$f(x;\alpha,\beta) = \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$$

where $x \geq 0$ and the **gamma function** is defined by $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$.



The gamma function integral is generally intractable (i.e. too hard to solve). Instead we make use of the following facts:

- $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$.
- $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.
- $\Gamma(n) = (n-1)!$ for positive integers n.

Example 1: Suppose that the time (in minutes) it takes a professional to groom a cat is known to be gamma distributed with $\alpha = 4$ and $\beta = 7$. Determine the probability that a random cat takes less than 30 minutes to groom.

If $X \sim Gamma(\alpha, \beta)$, then

$$E[X] = \mu_X = \alpha\beta \qquad V[X] = \sigma_X^2 = \alpha\beta^2 \qquad SD[X] = \sigma_X = \sqrt{\alpha}\beta$$

Example 1 Continued...

The Exponential Distribution

The **exponential distribution** is a one-parameter subfamily of the gamma distribution where $\alpha = 1$ and $\beta = 1/\lambda$.

Exponential Distribution: A continuous random variable X with an **exponential distribution** and parameter $\lambda > 0$ has pdf

where $x \geq 0$. Moreover, it has

$$E[X] = \mu_X = \qquad V[X] = \sigma_X^2 \qquad SD[X] = \sigma_X =$$

The CDF for the exponential(λ) distribution is given by:

The CDF of a exponential r.v. X having parameter $\lambda > 0$ is:

Example 2: Suppose that the lifespan of a certain variety of solid state drive (SSD) is known to follow an exponential distribution with a mean of 6 years. Determine the probability that a random SSD last between 3 and 4.5 years.

Interesting Exponential Fact 1: It is connected to the Poisson distribution.

Suppose that X is a Poisson random variable ($X \sim Poisson(\lambda)$) that counts the number of occurrences of some event in an interval of time/distance/space/etc.

Then, the amount of time/distance/space/etc between those events can also represented as an exponential random variable Y with $Y \sim exponential(\lambda)$.

Example 3: Suppose that the number of students arriving at a computer help desk is known to have a Poisson distribution with average rate of 5 students per hour.

Suppose that student has just arrived at the help desk. What is the probability that it will more than 20 minutes before the next student arrives at the help desk?

Interesting Exponential Fact 2: The exponential distribution has the **memoryless property**:

$$P(X \geq a + b \mid X \geq a) = P(X \geq b).$$

Suppose that we have a very poor password-cracking program that randomly tests passwords on a website login, possibly even repeating failed attempts that it has already tried. The probability that the program takes at least 5 minutes (starting from now) to find the password is $P(X \geq 5)$*. If the program has been running unsuccessfully for a week (10080 minutes), the probability that it takes at least 5 minutes more, given that it has already been running for a full week is* $P(X \geq 10080 + 5 | X \geq 10080)$*. However, since the password guesses are so randomized, this is really just like determining whether the program takes longer than 5 minutes to find the password. That is,* $P(X \geq 10080 + 5 | X \geq 10080) = P(X \geq 5)$*.*

Note: Memorylessness is not the same as independence.

Readings: Swartz 5.3 [EPS 3.11]
Practice problems: EPS 3.89, 3.91, 3.93, 3.95, 3.97, 3.99