Set 6: Conditional Probability

Stat 260 A01: May 18, 2024

Suppose I roll a standard 6-sided die. What is the probability that I roll a "5"?

Now suppose I tell the hint that that I have definitely rolled an odd number. What is the probably that I rolled a "5" now?

Conditional Probability: The probability that an event A occurs given that an event B definitely occurs is:

→ $P(\text{roll a } 5) = \frac{1}{6}$ $S = \{1, 2, 3, 4, 5, 6\}$

→ $P(\text{roll a } 5 \mid \text{roll odd}) = \frac{1}{3}$ $S = \{1, 3, 5\}$

given

Sample Space changes with condition

* knowing extra information can change the probability of the out come

$P(A \mid B)$ → read as "probability of A given B"

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(\underbrace{\text{roll } 5}_{A} \mid \underbrace{\text{roll odd}}_{B}) = \frac{P((\text{roll } 5) \cap (\text{roll odd}))}{P(\text{roll odd})} = \frac{P(\text{roll } 5)}{P(\text{roll odd})} = \frac{1/6}{3/6} = \frac{1}{3}$$

Example 1: Chronic wasting disease (CWD) is a fatal nervous system disease known to infect North American cervids, including moose, mule deer, white-tail deer, and elk. Suppose that a large game ranch is found to have infected populations of both white-tail and mule deer. The number of deer infected and not infected with CWD of each species is recorded below.

	Mule (M)	White-Tail (W)
Infected (D)	48	40
Not infected (D')	72	60

W D

(a) If a random white-tail deer at the ranch is tested, what is the probability that it has CWD?

$$P(D \mid W) = \frac{P(D \cap W)}{P(W)} = \frac{40/220}{100/220} = \frac{40}{100} = \frac{2}{5}$$

M

(b) If a random mule deer at the ranch is tested, what is the probability that it has CWD?

$$P(D \mid M) = \frac{P(D \cap M)}{P(M)} = \frac{48/220}{120/220} = \frac{48}{120} = \frac{2}{5}$$

$$P(D) = \frac{88}{220} = \frac{2}{5} \qquad P(D) = P(D \mid M) = P(D \mid W)$$

(independence)

Diagnostic Testing

In medical testing, a patient may or may not have a certain condition, and the test for that condition may or may not return a positive result.

	True state of Patient	
Test Results	Condition Present	Condition Absent
Test Positive (+)	True Positive	Error: False Positive
Test Negative (–)	Error: False Negative	True Negative

Disease Testing: Suppose that the presence of some disease D in a subject can result in a positive (+) or negative (-) test result.

- The ***false-positive rate*** is the probability that a subject without the disease will (falsely) test positive.

  $\rightarrow P(\text{test } (+) \mid \bar{D})$

- The ***false-negative rate*** is the probability that a subject with the disease will (falsely) test negative.

  $\rightarrow P(\text{test } (-) \mid D)$

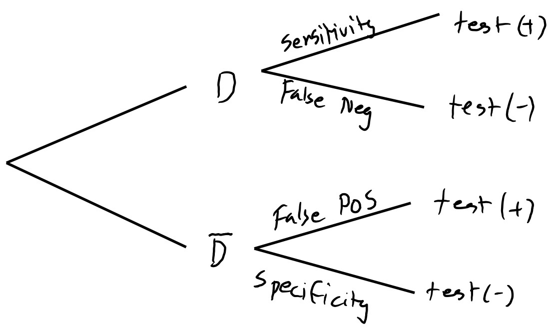
- The ***sensitivity*** is the probability that a subject with the disease will (correctly) test positive.

  $\rightarrow P(\text{test } (+) \mid D)$

→ sometimes extra info changes probability, sometimes it doesn't

- The ***specificity*** is the probability that a subject without the disease will (correctly) test negative. "true negative rate"

$P(\text{test }(-) \mid \bar{D})$



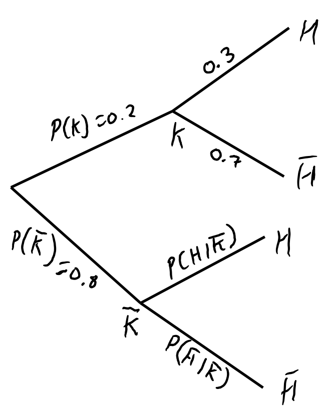
Recall: For events A and B, the probability of A given B is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \qquad P(B) \neq 0$$

Rearrange: $P(A \cap B) = P(A|B) \cdot P(B)$

<u>**Multiplication (Product) Rule for Probabilities**</u>: For events A and B with $Pr(A) \neq 0$ and $Pr(B) \neq 0$,

Example 2: A study suggests that 30% of smokers suffer from hypertension (ie high blood pressure). If approximately 20% of Canadians smoke, what is the probability that a randomly selected Canadian smokes and has hypertension? K = Smoking H = BP



→ $P(H \cap K) = P(H|K) \cdot P(K)$

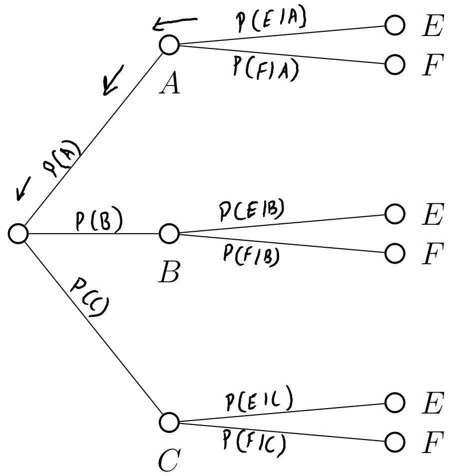
$= (0.2) \cdot (0.6)$

$= 0.12$

* For trees, multiply down the entire branch for the multiplication formula

See diagram below for explanation ↓ (next page)

Probability Tree Diagrams: Given a two-phase experiment, where Part I has outcomes A, B, and C, and Part II has outcomes E and F, we can represent the experiment and its associated probabilities as:



$P(E|A) = P(A) \cdot P(E|A)$

$P(E) = P(E|A) + P(E \cap B) + P(E \cap C)$

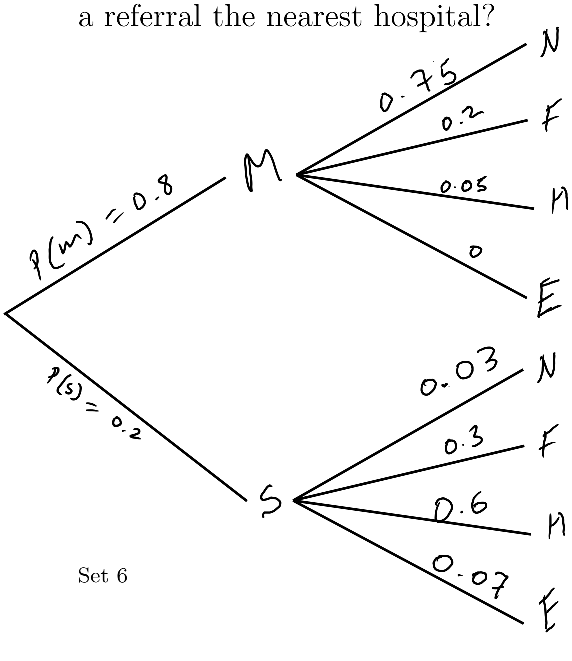
$= P(A) \cdot P(E|A) + P(B) \cdot P(E|B) + P(C) \cdot P(E|C)$

Law of Total Probability:

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) + \cdots = \sum_{i=1}^{\infty} P(A|B_i)P(B_i)$$

Example 3: At a small rural clinic, each patient's health concern is classified as either Minor (M) or Serious (S). Suppose that 80% of health concerns at the clinic are Minor, and the rest are Serious. After analyzing a patient, the local doctor decides upon one of four treatment options: no further action (N), follow-up appointment at clinic (F), referral to nearest hospital (H), or MedEvac to nearest hospital (E). Of patients with a Mild condition, 20% are issued follow-up appointments at the clinic, 5% are referred to the nearest hospital, and no action is required for the remaining 75%. Of those with Serious conditions, 30% are issued follow-up appointments at the clinic, 60% are referred to the hospital, 7% are MedEvaced to the hospital, and no further action is taken for the remaining 3%.

What is the probability that a random patient visiting the clinic is classified as Serious and issued a referral the nearest hospital?



Want: $P(S \cap H) = P(S) \cdot P(H|S)$ or multiply down the tree

$= 0.2 \times 0.6$

$= 0.12$

b) Sample question $= P(M) \cdot P(H|M) + P(S) \cdot P(H|S)$ or multiply down the tree

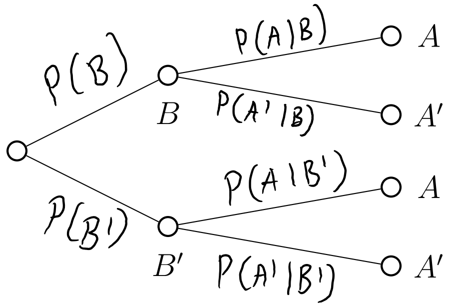
$= (0.8 \times 0.05) + (0.2 \times 0.6)$ "Law of total probability"

$= 0.04 + 0.12$

$= 0.16$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

Bayes' Theorem



Recall: $P(B|A) = \dfrac{P(A\cap B)}{P(A)}$ ← Conditional Probability

$P(A\cap B) = P(B)\cdot P(A|B)$ ← Product Rule

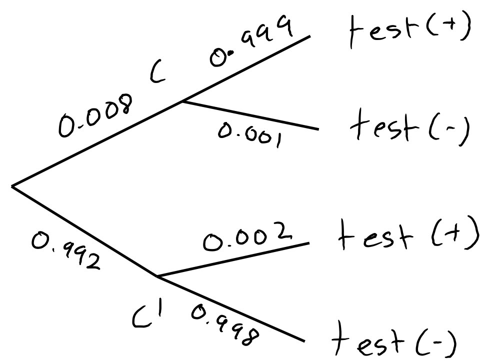
$P(A) = P(B)\,P(A|B) + P(B')\,P(A|B')$ ← Law of total Probability

Bayes' Theorem ($n = 2$): Let A and B be events in S, then

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(B)\cdot P(A|B)}{P(B)\cdot P(A|B) + P(B')\,P(A|B')}$$

Example 4: A study puts cocaine usage among a certain population at 0.8%. Suppose a pharmaceutical company claims to have developed a new test for cocaine usage. The company claims that the test can detect the presence of the drug in 99.9% of cocaine users. Conversely, the drug is advertised as having only a 0.2% false-positive rate.

(a) What is they probability that a randomly tested subjected is a cocaine user and tests positive?



$P(C\cap \text{test}(+)) = (0.008)(0.999)$
$= 0.007992$

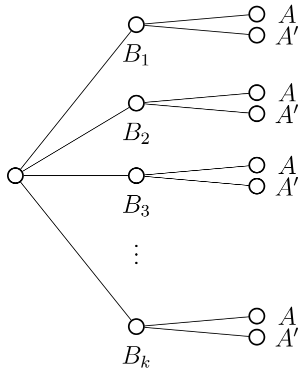
(b) What is the probability that a randomly selected subject tests positive?

$P(\text{test}(+)) = P(C\cap \text{test}(+)) + P(C'\cap \text{test}(+))$
$= (0.008)(0.999) + (0.992)(0.002)$
$= 0.009976$

(c) A random subject is selected and tests positive. What is the probability that they actually use cocaine?

$$\underbrace{P(C|\text{test}(+)) = \frac{P(C\cap \text{test}(+))}{P(\text{test}(+))}}_{\text{Bayes theorm}} = \frac{0.007992}{0.009976} = 0.8011$$

Bayes' Theorem $(n = k)$: Let $B_1, B_2, \ldots, B_k$ be ***mutually exclusive*** events such that $B_1 \cup B_2 \cup \cdots \cup B_k = S$, then for any event A in S,



$$P(B_r|A) = \frac{P(B_r \cap A)}{Pr(A)} = \frac{P(B_r)P(A|B_r)}{P(A)}$$

Product Rule

$$= \frac{P(B_r)P(A|B_r)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + \cdots + P(B_k)P(A|B_k)}$$

Law of total probability

$$= \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{\infty} P(A|B_i)P(B_i)}$$

Extra Example 1: To achieve a grade of "Canada No. 1" (N) by the Canadian Food Inspection Agency, peaches are inspected for a variety of quality restrictions, including the size and presence of bruising, limb rub marks, russetting, mildew, etc. The "Canada Domestic" (D) grade has slightly relaxed requirements, and those falling below both grades are rejected (R). Suppose that a distributor receives shipments of peaches from three orchards with the following distribution: Orchard A (40%), Orchard B (25%), and Orchard C (35%). The grading of each shipment is given below.

answer :

$P(A|N) = 0.31$

- Orchard A: 50% are Canada No.1 and 40% are Canada Domestic,
- Orchard B: 80% are Canada No.1 and 15% are Canada Domestic,
- Orchard C: 70% are Canada No.1 and 20% are Canada Domestic.

If a random peach is graded as Canada No. 1, what is the probability that it came from Orchard A?

Textbook Readings: Swartz 3.4 [EPS 1.8]
Practice problems: Swartz 1.59, 1.61, 1.63, 1.65(a,b), 1.73, 1.75, 1.77, 1.79