COUNTING

Basic Combinatorial Results:

The number of **permutations** (arrangements) of n distinct items is $n!$, which we read as "n factorial".

For positive integers, $n! = n(n-1)(n-2)\dots(2)(1)$. Define $0!$ to be 1.

Example 1: The number of different ways to arrange 3 people for a photograph is $3! = 6$.

The number of arrangements of r items taken from a collection of n distinct items is:

$$P(n,r) = {}_nP_r = n^{(r)} = \frac{n!}{(n-r)!}$$

Example 2: Suppose a class has 20 students. The number of ways we can **select and arrange** 4 of these students for a photograph is:

$${}_{20}P_4 = \frac{20!}{16!} = 116280$$

The number of **combinations** (selections) of r items taken from a collection of n distinct items is:

$$C(n, r) = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{{}_nP_r}{r!}$$

Example 3: Suppose a class has 20 students. The number of ways we can **select, but not arrange,** 4 of these students is:

$$\binom{20}{4} = \frac{20!}{4!16!} = 4845$$

Example 4: A box contains slips of paper, numbered $1, 2, \ldots, 30$. Three slips are selected at random without replacement. What is the probability that all three slips show a number which is 9 or less (event A)?

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Set 11: Binomial Distribution

Bernoulli Process: An experiment consisting of one or more trials, each having the following properties.

1. Each trial has exactly two outcomes, which we call **success** and **failure**.
2. The trials are independent of each other.
3. For all trials, the probability of success, p, is constant.

A **binomial experiment** is a Bernoulli process where n, the number of trials, is fixed in advance.

Let X count the number of successes in a binomial experiment. Then X is a **binomial random variable**, and we write $X \sim Bin(n, p)$, where n is the number of trials, and p is the probability of successes. For a binomial random variable, n and p are its parameters.

Example 5: In a manufacturing process, each item has a probability of 0.05 of being defective, independent of all other items. Suppose 12 items are selected at random, and we let W denote the number of defective items.

Binomial Probability Distribution:

$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad\qquad x = 0, 1, 2, \ldots, n$$

Example 6: On a multiple choice test, there are 10 questions, each with 8 possible responses. The test taker completes the test by randomly selecting answers. What is the probability that they will get (exactly) one question correct?

Example 7: In the manufacture of lithium batteries, it is found that 7% of all batteries are defective. Suppose that we test 6 randomly selected batteries. What is the probability that at least two batteries are defective?

Expected Value and Variance: If $X \sim Bin(n, p)$, then:

$$E(X) = \mu = np \quad \text{and} \quad V(X) = \sigma^2 = np(1-p)$$

Example 8: What is the expected number of defective lithium batteries per batch of 6? What is the variance?

Cumulative Distribution Tables: These tables give $P(X \leq x)$ for "nice" values of n and p.

Example 9: It is known that 20% of all tablet computers will need the touch-screen repaired within the first two years of use. Suppose we select 15 tablet computers at random.

What is the probability that no more than 6 tablets will need repairs to the touch-screen within the first two years of use?

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Example 10: What is the probability that exactly 5 tablets will need touch-screen repairs?

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Example 11: What is the probability that at least 2 tablets will need touch-screen repairs?

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<u>Example 12:</u> It is known that 30% of all laptops of a certain brand experience hard-drive failure within 3 years of purchase. Suppose that 20 laptops are selected at random. Let the random variable X denote the number of laptops which have experienced hard-drive failure within 3 years of purchase.

If it is known that at least 3 laptops experience hard-drive failure, what is the probability that no more than 6 laptops will experience hard-drive failure?

Set 12: Poisson Distribution

Poisson Process: Consider counting the number of successes or occurrences over an interval of time or space, or other appropriate intervals. In a Poisson process or experiment, we make the following assumptions:

1. The number of successes that occur in any interval is independent of the number of successes occurring in any other non-overlapping interval.
2. The probability of one success in a small interval is proportional to the size of the interval. The probability of having more than one success in this small interval is negligible.
3. If two non-overlapping intervals have the same size, then the probabilities of successes are the same for both intervals.

Poisson Random Variable: In a Poisson experiment, let X counts the number of successes that occur in *one* interval of time or space. Under this scenario, X is a Poisson random variable with parameter λ.

We write $X \sim Poisson(\lambda)$, where λ is the average number of successes **per interval or region**.

Note: Some textbooks will use μ rather than λ for the parameter of the Poisson random variable.

Example 13: At a bank, customers use the bank machine at an average rate of 40 customers per hour. Let X count the number of customers that use the machine in a 30-minute interval.

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Example 14: At a busy intersection, it is noted that on average 5 cars pass through the intersection per minute. Let X count the number of cars which pass through the intersection in an hour.

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Example 15: Suppose that a typist makes on average 10 errors while typing 300 pages of text. Let X count the number of errors on one page of text.

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Example 16: We examine ten pages of text. Let Y count the number of pages with at least one error. The random variable Y is **not** Poisson. Why?

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Probability mass function (pmf) for $X \sim Poisson(\lambda)$:

$$f(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad\qquad x = 0, 1, 2, \ldots$$

Example 17: Suppose a machine makes defective items at an average rate of 5 defective items per hour. What is the probability that the machine will make exactly 4 defective items in an hour?

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Expected Value and Variance: If $X \sim Poisson(\lambda)$, then:

$$E(X) = \mu = \lambda \quad \text{and} \quad V(X) = \sigma^2 = \lambda$$

Example 18: What is the expected number of defective items made by the machine in an hour? What is the variance?

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Cumulative Distribution Tables: These tables give $P(X \leq x)$ for "nice" values of $\lambda = \mu$.

Example 19: Recall that the machine makes on average 5 defective items per hour. Suppose the machine is observed for three hours. What is the probability that it will make no more than 12 defective items?

Example 20: What is the probability that at least 6 defective items will be made?

Example 21: What is the probability that exactly 13 defective items will be made?

Example 22: Suppose that a typist makes on average of 2 errors per page. Suppose the typist is creating a ten-page document. What is the probability that exactly three of the pages do not contain any errors?

Poisson approximation to Binomial: If X is a binomial random variable where n is very large and p is very small then X can be approximated with a Poisson distribution with $\lambda = np$.

Rule of thumb for this course: If $\mathbf{n \geq 20}$ **and** $\mathbf{p \leq 0.05}$, then Poisson approximation to binomial is appropriate.

In practice, if $n \geq 100$ and $np \leq 10$, then the approximation will be quite good.

Example 23: Brugada syndrome is a rare disease which afflicts 0.02% of the population. Suppose 10,000 people are selected at random and tested for Brugada syndrome. What is the probability that no more than 3 of the tested people will have Brugada syndrome?

Sets 13 and 14: Continuous Random Variable

Continuous Random Variable: A random variable which can assume an uncountable number of values (here we will deal with intervals of real numbers only).

We can describe a continuous random variable with a **probability density function** (pdf) $f(x)$ satisfying:

1. $f(x) \geq 0$ for all real number x; and
2. $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1$.

We define:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

for any two numbers a and b (with $a \leq b$).

Note: Since a valid pdf must be non-negative, graphically $P(a \leq X \leq b)$ is just the area below $f(x)$ and above the x-axis on the interval $[a, b]$.

Uniform Probability Distribution: For a uniform probability distribution, the pdf is:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ \\ 0 & \text{otherwise} \end{cases}$$

The graph of $f(x)$ is a horizontal line segment from a to b with height $1/(b-a)$.

For $a \le x_1 \le x_2 \le b$,

$$P(x_1 \le X \le x_2) = (height) \times (width) = \left(\frac{1}{b-a}\right)(x_2 - x_1)$$

Some further consequences for a valid pdf:

1. $P(X = a) = 0$ for any a.

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2. $P(X \ge a) = P(X > a)$ and $P(X \le a) = P(X < a)$

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3. $P(X \geq a) = 1 - P(X \leq a)$

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4. $P(a \leq X \leq b) = P(X \leq b) - P(X \leq a)$ for $a \leq b$

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Example 24: Suppose that the continuous r. v. X has the following pdf:

$$f(x) = \begin{cases} \dfrac{4}{609}x^3 & 2 \leq x \leq 5 \\ \\ 0 & \text{otherwise} \end{cases}$$

Find $P(3 \leq X \leq 4)$.

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Example 25: Find an expression for $P(X \leq b)$, where b is some number in $[2, 5]$.

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Note: We can conclude that:

- $P(X \leq x) = x^4/609 - 16/609$ for all x in the interval $[2, 5]$.
- If x is less than 2, $P(X \leq x) = 0$.
- If x is greater than 5, $P(X \leq x) = 1$.

Using this, we can write the **cumulative distribution function**, $F(x)$, for the given pdf.

$$F(x) = \begin{cases} 0, & x < 2 \\ \dfrac{x^4}{609} - \dfrac{16}{609}, & 2 \leq x \leq 5 \\ 1, & x > 5 \end{cases}$$

Note: The fundamental theorem of calculus tells us that for every x at which $F'(x)$ exists, that $F'(x) = f(x)$.

Example 26: Suppose the random variable X has the following cdf:

$$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x}{x+1}, & x \geq 0 \end{cases}$$

Find the pdf for the random variable X.

Let p be a number between 0 and 1. The $100p^{th}$ **percentile** or **quantile** of a continuous random variable is the value ϵ such that $F(\epsilon) = p$.

Example 27: For the random variable from the previous example, find the 90^{th} percentile.

Example 28: Suppose the random variable X has pdf

$$f(x) = \begin{cases} 2e^{-2x} & 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the median of the distribution.

Note: The **median**, $\tilde{\mu}$ of a continuous random variable is the 50^{th} percentile.

The **expected value** or **mean** of a continuous random variable X with pdf $f(x)$ is:

$$E(X) = \mu = \int_{-\infty}^{\infty} x f(x) dx$$

(provided this integral converges)

The **variance** of a continuous random variable X with pdf $f(x)$ is:

$$V(X) = \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

(provided this integral converges)

and the standard deviation, $\sigma = \sqrt{\sigma^2}$.

As with discrete random variables, we have the following:

- $V(X) = E(X^2) - \mu^2$
- $E(aX + b) = aE(X) + b$
- $V(aX + b) = a^2 V(X)$

Example 29: Suppose a random variable X is uniformly distributed over the interval [a, b]. Recall the pdf of X is:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and the variance.

Differences between discrete and continuous RV's

Discrete	Continuous

Sets 15 and 16: The Normal Distribution

Normal Density Function: If X is normally distributed with mean μ and standard deviation σ, then we write $X \sim N(\mu, \sigma)$. The pdf of X is:

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Properties of Normal Curves:

- All normal curves are defined on $(-\infty, \infty)$ and are bell-shaped.
- There is a single peak at $x = \mu$ and the curve is symmetric about this peak.
- The mean, median, and mode are all μ; the variance is σ^2.
- There are points of inflection at $\mu - \sigma$ and $\mu + \sigma$
- As μ increases, the peak moves further to the right. As μ decreases, the peak moves further to the left. (μ is a **location parameter**)
- As σ increases, the peak becomes lower, and the curve becomes flatter. As σ decreases, the curve becomes more abruptly peaked, and the peak becomes taller. (σ is a **scale** parameter).

Note: All normal curves are bell-shaped. Not all bell-shaped curves are normal.

Standard Normal Distribution: The standard normal random variable has mean $\mu = 0$ and standard deviation $\sigma = 1$. We use the letter Z to denote the standard normal distribution.

$$f(z; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

The standard normal curve is:

- has its peak at 0, and is symmetric about the y-axis
- has points of inflection at 1 and -1.

If our random variable is Z, then we denote the cdf $P(Z \leq z)$ by $\Phi(z)$, pronounced as capital phi of z.

Cumulative Distribution Tables: These tables give $P(Z \leq z) = \Phi(z)$ for a range of $-3.5 < z < 3.5$.

Example 30: Find $P(Z \leq 2.56)$

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Symmetry Property: Since the random variable Z is symmetric about $Z = 0$, then for any α:

$$P(Z \leq \alpha) = P(Z \geq -\alpha)$$

Note: This property works for any distribution that is symmetric about 0.

Example 31: Calculate $P(Z \geq 0.16)$.

Select the closest to your unrounded answer from the following:

(A) 0.2 (B) 0.4 (C) 0.6 (D) 0.8

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Example 32: Calculate $P(-1.22 \leq Z \leq 1.73)$.

Select the closest to your unrounded answer from the following:

(A) 0.2 (B) 0.4 (C) 0.6 (D) 0.8

Notation: z_α is the number such that $P(Z > z_\alpha) = \alpha$. Alternately, z_α is the $100(1-\alpha)$ percentile of the standard normal distribution.

Example 33: Find the 97.5^{th} percentile of the standard normal distribution.

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Example 34: Find $z_{0.05}$.

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Example 35: Find $z_{0.10}$.

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Example 36: Find $z_{0.005}$.

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Normal Random Variable $X \sim N(\mu, \sigma)$ with $\mu \neq 0$ and/or $\sigma \neq 1$

Example 37: Suppose that the heights of Andean flamingos are normally distributed with a mean of $105\ cm$ and a standard deviation of $2\ cm$. Let the random variable X denote the height of a randomly selected Andean flamingo.

What is the **median** Andean flamingo height?

Select the closest to your unrounded answer:

(A) $105\ cm$

(B) Not enough information to answer this question.

Example 38: Is $P(X \geq 100) = P(X \leq -100)$?

(A) Yes

(B) No

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General Symmetry Property: If a random variable X is symmetric about $x = a$, then for any real number k:

$$P(X \leq a - k) = P(X \geq a + k)$$

Example 39: What is $P(X = 105)$?

Select the closest answer:

(A) 0

(B) 0.5

(C) 1

(D) Not enough information available.

Standardizing a Normal Random Variable

If X is normally distributed with mean μ and standard deviation σ, i.e., $X \sim N(\mu, \sigma)$, then:

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

Example 40: The masses of a certain type of bolt is approximately normally distributed with $\mu = 15$ g, and $\sigma = 2$ g. What is the probability that a randomly selected bolt has a mass between 14.3 g and 17.1 g?

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Example 41: What is the probability that at a randomly selected bolt will have a mass of at least 20 g?

Example 42: What is the minimum mass of the heaviest 5% of bolts?

The **empirical rule** states that if the distribution of a variable is **approximately normal**, then:

1. About 68% of values lie within σ of the mean.
2. About 95% of values lie within 2σ of the mean.
3. About 99.7% of values lie within 3σ of the mean.

From this, we can conclude that almost all bolts will have a mass within 6 g of the mean 15 (i.e. about 99.7% will have a a mass between 9 g and 21 g).

Approximating a Binomial Distribution with Normal Distribution:

Rule of Thumb: Suppose $X \sim Bin(n, p)$ where np and $n(1-p)$ are both at least 5. Then $X \approx N(\mu = np, \sigma^2 = np(1-p))$.
This means that:

$$P(X \leq x) \approx P\left(Z \leq \frac{x - np}{\sqrt{np(1-p)}}\right)$$

Since we are using a continuous distribution to approximate a discrete one, this approximation will be slightly off. If we wish to get a better approximation, use the following, with a **continuity correction**:

$$P(X \leq x) \approx P(X \leq x + 0.5)$$

$$\approx P\left(Z \leq \frac{x - np + 0.5}{\sqrt{np(1-p)}}\right)$$

Note: To compute other probabilities, always convert to the form involving $P(X \leq x)$ in the binomial distribution first, then add the continuing correction.

Example 43: Suppose it is known that 20% of batteries have a lifespan shorter than the advertised lifespan. Suppose that 100 batteries are selected at random. What is the approximate probability (using the continuity correction) that at least 10 batteries will have a short lifespan?

Example 44: Suppose it is known that the reaction time of type of voice-activated robot is normally distributed with $\mu = 6.3$ microseconds, and $\sigma = 2$ microseconds.

If one voice-activated robot is randomly selected, what is the probability that its reaction time is between 5 and 7 microseconds? Report your answer to three decimal places.

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Example 45: Suppose that five robots are tested. Assume the reaction time of each robot is independent of the other robot reaction-times. What is the probability that exactly three of the robots will have a reaction time between 5 and 7 microseconds? Report your answer to three decimal places.

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