Set 30: Inference for Two Binomial Samples

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So far we've looked at inferences for p from a single population. However, we often want to compare the proportions, p_1 and p_2, from two different populations

Difference in proportions: In general, hypothesis tests and confidence intervals of $\underline{p_1 - p_2}$ use

population parameters (we want to estimate)

$$\hat{p}_1 - \hat{p}_2 \qquad \text{as a point estimator.}$$

with estimated standard error:

$$ese = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

- **General Requirements:**
  - ▷ Random samples → every inference ever
  - ▷ $n_1 \geq 30$, $\hat{p}_1 n_1 \geq 5$, $n_1(1-\hat{p}_1) \geq 5$, and $n_2 \geq 30$, $\hat{p}_2 n_2 \geq 5$, $n_2(1-\hat{p}_2) \geq 5$.

    same as normal approx to binomial requirements

- **Confidence Interval for $p_1 - p_2$:**

$$\overbrace{(\hat{p}_1 - \hat{p}_2)}^{\text{point estimate}} \pm \underbrace{z_{\alpha/2}}_{\text{critval}} \underbrace{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}_{ese}$$

- **Hypothesis Testing for $p_1 - p_2$**

  A value $(p_1 - p_2)_0$ is proposed for $p_1 - p_2$ (the true population proportions difference). A study/experiment collects data that may support or refute this proposed value.

  - ▷ **Hypotheses:**

Right-Tailed:	**Left-Tailed:**	**Two-Tailed:**
$H_0: p_1 - p_2 = (p_1 - p_2)_0$	$H_0: p_1 - p_2 = (p_1 - p_2)_0$	$H_0: p_1 - p_2 = (p_1 - p_2)_0$
or $H_0: p_1 - p_2 \leq (p_1 - p_2)_0$	or $H_0: p_1 - p_2 \geq (p_1 - p_2)_0$	or $H_0: p_1 - p_2 = (p_1 - p_2)_0$
$H_1: p_1 - p_2 > (p_1 - p_2)_0$	$H_1: p_1 - p_2 < (p_1 - p_2)_0$	$H_1: p_1 - p_2 \neq (p_1 - p_2)_0$

  - ▷ **Test Statistic:**

$$z_{obs} = \frac{\overbrace{(\hat{p}_1 - \hat{p}_2)}^{\text{point estimate}} - \overbrace{(p_1 - p_2)_0}^{\text{parameters from } H_0}}{\underbrace{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}_{ese}}$$

→ For single proportion p use Standard Error (SE) (with p not $\hat{p}$)

  - ▷ **p-values:**

    Right-tailed Test: $p\text{-value} = P(Z > z_{obs})$
    Left-tailed Test: $p\text{-value} = P(Z < z_{obs})$
    Two-tailed Test: $p\text{-value} = 2P(Z < -|z_{obs}|)$

    same as other p-value for z distribution

$\hat{p}_1 = \frac{28}{250}$ $\hat{p}_2 = \frac{26}{320}$

Example 1: Suppose we want to examine the prevalence of tick infestations in European blackbirds (*Turdus merula*) versus wrens (*Troglodytes troglodytes*).

Suppose we examine 250 blackbirds and find that 28 have ticks. We also examine 320 wrens and find that 26 have ticks. Is there evidence to suggest that the proportion of blackbirds with ticks is greater than the proportion of wrens with ticks?

- Let p_1 be the population proportion of blackbirds with ticks.
- Let p_2 be the population proportion of wrens with ticks.

→ define parameters

$H_0 : p_1 - p_2 \leq 0$

same meaning

$H_0 : p_1 - p_2 = 0$

$H_1 : p_1 - p_2 > 0$ → pointing to the right ∴ righttail test

Assumptions:

$n_1 = 250 \geq 30$ ✓ $n_1\hat{p}_1 = 28 \geq 5$ ✓ $n_1(1-\hat{p}_1) = 222 \geq 5$ ✓

$n_2 = 320 \geq 30$ ✓ $n_2\hat{p}_2 = 25 \geq 5$ ✓ $n_2(1-\hat{p}_2) = 294 \geq 5$ ✓

Test Statistic: $\frac{(\text{estimate}) - (\text{param})_0}{\text{ese}} = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} =$

Observed value of test:

$$Z_{obs} = \frac{\left[\frac{28}{250} - \frac{26}{320}\right] - 0}{\frac{\left(\frac{28}{250}\right)\left(1-\frac{28}{250}\right)}{250} - \frac{\left(\frac{26}{320}\right)\left(1-\frac{26}{320}\right)}{320}} = 1.22$$

p-value: $P(Z > Z_{obs}) = P(Z > 1.22) = 1 - P(Z < 1.22)$
$= 1 - 0.8888$
$= 0.1112$

Since p-value > 0.1, little to no evidence against H_0.

No evidence to suggest the proportion of black birds with ticks is greater than the proportion of wrens with ticks.

Example 1 Continued... Determine the 97% confidence interval for $p_1 - p_2$, the difference in proportions of tick-infected blackbirds and wrens.

Post

all sample size equations

margin of error d = (civ) $\cdot$ (ese)

$$d = z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n} - \frac{\hat{p}_2(1-\hat{p}_2)}{n}}$$

solve for n: $\left(\frac{z_{\alpha/2}}{d}\right)^2 \left[\hat{p}_1(1-\hat{p}_1) + \hat{p}_2(1-\hat{p}_2)\right]$

Sample Size for Estimating $p_1 - p_2$

The **common** sample size ($n = n_1 = n_2$) needed to construct a $(1-\alpha)100\%$ confidence interval for $p_1 - p_2$ within margin of error d is give by:

$$d = z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n} + \frac{\hat{p}_2(1-\hat{p}_2)}{n}} \quad \Rightarrow \quad n = \frac{z_{\alpha/2}^2\left[\hat{p}_1(1-\hat{p}_1) + \hat{p}_2(1-\hat{p}_2)\right]}{d^2}$$

* When no estimate $\hat{p}_1 - \hat{p}_2$ is available, use $\hat{p}_1 = \hat{p}_2 = 1/2$.

Example 1 Continued... Suppose we want to conduct a second study on the prevalence of ticks on blackbirds versus wrens. What sample size would be needed to construct a 90% confidence interval for $p_1 - p_2$, $\underbrace{\text{within 2\%}}_{d=0.02}$ of the true difference? Use the data from the previous study for $\hat{p}_1$ and $\hat{p}_2$.

$\hat{p}_1 = \frac{28}{250}$ $\quad \hat{p}_2 = \frac{26}{320}$

90% CI = $\alpha = 0.1 \to \alpha/2 = 0.05 \to Z_{0.05} = 1.645$

$$n = \left(\frac{1.645}{0.02}\right)^2\left(\left(\frac{28}{250}\right)\left(1-\frac{28}{250}\right) + \left(\frac{26}{320}\right)\left(1-\frac{26}{320}\right)\right)$$

$n = 1177.8 \approx 1178$

$n = 1178$

$n_1 = 1178$ black birds
$n_2 = 1178$ wrens

$\to$ Total $= 2356$ birds

Extra Example: A new drug is tested on hypertension patients. Of the 615 patients to receive the medication, 92 eventually had strokes. Of the 700 hypertension patients that received placebos, 56 had strokes. Is there evidence to suggest the medication has more than a 5% increase in stroke incidence than the placebo? Test the hypothesis at the significance level $\alpha = 0.10$. Then state the estimated value of the parameter and the estimated standard error.

Textbook Readings: Swartz 7.3 [EPS 5.11, 6.9]
Practice problems: EPS: 5.51, 5.53, 6.61, 6.63, 6.65