Sets 28-29: Inferences for the Means of Two Samples

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So far, we've looked at inferences (confidence intervals and hypothesis tests) for μ from a single population. However, we often want to compare the means from two different populations.

Suppose that we want to examine the difference in mean weights of koalas from the Bongil Bongil National Park verus those from the Murray Valley National Park.

- ▷ Let μ_1 be the population mean weight of koalas in Bongil Bongil National Park.
- ▷ Let μ_2 be the population mean weight of koalas in Murray Valley National Park.

We want to study want is the difference between the yield of the two populations: $\mu_1 - \mu_2$.

After we gather our data, we end up with two samples:

- ▷ **Sample 1** of the Bongil Bongil koalas: with sample size n_1, sample mean $\overline{x_1}$, and sample standard deviation s_1.
- ▷ **Sample 2** of the Murray Valley koalas: with sample size n_2, sample mean $\overline{x_2}$, and sample standard deviation s_2.

How we build a confidence interval or complete a hypothesis test for $\mu_1 - \mu_2$ will depend upon sample size, population distribution, and whether the populations have equal/unequal variances. Four possible cases arise:

Case 1: σ_1 and σ_2 are known, and both populations are normally distributed.

Z-test (using σ_1 and σ_2)

Case 2: Both samples are large ($n_1 \geq 30$ AND $n_2 \geq 30$).

Z - test (using s_1 and s_2)

Case 3: At least one sample is small (< 30), and the populations have equal variances ($\sigma_1^2 = \sigma_2^2$).

That is, $\left[\dfrac{\text{larger variance}}{\text{smaller variance}} \leq 2 \text{ or } \dfrac{\text{larger standard deviation}}{\text{smaller standard deviation}} \leq 1.4\right]$ Rule of thumb $1.4 \approx \sqrt{2}$

assume $\sigma_1^2 = \sigma_2^2$

use: Pooled t-test

Case 4: At least one sample is small (< 30), and the populations have unequal variances ($\sigma_1^2 \neq \sigma_2^2$).

That is, $\left[\dfrac{\text{larger variance}}{\text{smaller variance}} > 2 \text{ or } \dfrac{\text{larger standard deviation}}{\text{smaller standard deviation}} > 1.4\right]$

assume $\sigma_1^2 \neq \sigma_2^2$

use unpooled T-test

Example 1: Adult koalas (*Phascolarctos cinereus*) are captured for tagging for two different Australian national parks: Bongil Bongil and Murray. During the tagging, their vital statistics are recorded, including their weights, as recorded below:

	Sample Size	**Sample Mean Weight**	**Sample Std Dev**
Bongil Bongil (B):	$n_1 = 14$,	$\bar{x}_1 = 15$kg,	$s_1 = 2.5$kg
Murray Valley (M):	$n_2 = 15$,	$\bar{x}_2 = 12$kg,	$s_2 = 1.0$kg

Let μ_1 be the mean weight of koalas in Bongil Bongil National Park.
Let μ_2 be the mean weight of koalas in Murray Valley National Park.
parameter definition

(a) Is there evidence to suggest that the mean weight of Bongil Bongil koalas is more than 2.0kg greater than Murray Valley koalas?

σ_1 and σ_2 are unknown, small samples ($n_1, n_2 < 30$) → use T-test

Rule of thumb: $\frac{s_{big}}{s_{small}} = \frac{2.5}{1.0} = 2.5 > 1.4$ → use an unpooled t-test (assume $\sigma_1^2 \neq \sigma_2^2$)

$H_0 = \mu_1 - \mu_2 \leq 2.0$, $H_1: \mu_1 - \mu_2 > 2.0$

Test Statistic:

$$T = \frac{(\text{estimate}) - (\text{param from } H_0)}{ese} = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

from H_0

$$\text{dof } v = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}} = \frac{\left(\frac{2.5^2}{14} + \frac{1.0^2}{15}\right)^2}{\frac{\left(\frac{2.5^2}{14}\right)^2}{14-1} + \frac{\left(\frac{1.0^2}{15}\right)^2}{15-1}} = 16.\underline{82}$$

ignore decimal values

$v = 16$

Use T_{16} unpooled test statistic:

$$t_{obs} = \frac{(15 - 12) - 2}{\sqrt{\frac{2.5^2}{14} + \frac{1.0^2}{15}}} = 1.396$$

$p\text{-value} = P(T_{16} > 1.396)$

$0.05 < p\text{-value} < 0.1$

Moderate evidence against H_0

Moderate evidence the mean difference $(\mu_1 - \mu_2)$ is not at most 2kg.

(b) Compute a 95% confidence interval for the difference in Bongil Bongil and Murray Valley koala weights.

(Use an unpooled T with $r=16$ dof as in part (c))

95% CI $\Rightarrow \alpha = 0.05 \Rightarrow \alpha/2 = 0.025$

Critical value $\Rightarrow t_{16,0.025} = 2.120$

(estimate) ± (critical value)(ese) $= (\bar{x}_1 - \bar{x}_2) \pm t_{16,0.025}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$

$= (15-12) \pm (2.120)\sqrt{\frac{2.5^2}{14} + \frac{1.0^2}{15}} = 3 \pm 1.519\text{kg}$

$[1.481\text{kg}, 4.519\text{kg}]$

Example 2: Suppose that we want to examine the difference in crop yields of pea plants (*Pisum sativum*) treated with inoculant (a culture of the nitrogen fixing bacteria, *Rhizobium leguminosarum*), versus those treated with a standard commercial nitrogen-phosphate-potash fertilizer.

We grow 22 pea plants, and treat 10 with the inoculant, and 12 with the fertilizer. The mean yield for the sample of inoculant-treated plants is 22g with a standard deviation of 5g. The mean yield for the sample of fertilized plants is 20g with a standard deviation of 4g.

▷ Let μ_1 be the population mean yield of pea plants treated with inoculant.

▷ Let μ_2 be the population mean yield of pea plants treated with the commercial fertilizer.

(a) Compute a 98% Confidence Interval for difference in means.

Inoculant: $n_1 = 10$ $\bar{x}_1 = 22\text{g}$ $s_1 = 5\text{g}$

Fertilzer: $n_2 = 12$ $\bar{x}_2 = 20\text{g}$ $s_2 = 4\text{g}$

σ_1 and σ_2 are unknown, small samples $(n_1, n_2 \leq 30) \Rightarrow T$

Test for equal variances

$\frac{s_{big}}{s_{small}} = \frac{5}{4} = 1.25 < 1.4$ use a pooled - T

dof: $r = n_1 + n_2 - 2 = 10 + 12 - 2 = 20$, $t_{20,0.01} = 2.528$

98% CI $\Rightarrow (22-20) \pm (2.528)\sqrt{\frac{(10-1)5^2 + (12-1)4^2}{10+12-2}\left(\frac{1}{10} + \frac{1}{12}\right)} = 4.847\text{g}$

$\Rightarrow 2 \pm 4.847\text{g}$

Use → pooled-T with $r = 20$ dof

(b) Test the claim that the mean crop yield of pea plants treated with the inoculant is different than the mean crop yield from plants treated with the fertilizer.

$H_0: \mu_1 - \mu_2 = 0$ $(\mu_1 = \mu_2)$

$H_1: \mu_1 - \mu_2 \neq 0$ $(\mu_1 \neq \mu_2)$

Extra Example 1: A study on the effects of nicotinic acid (niacin) in rat total cholesterol (TC) levels found that in a sample of 50 rats given regular treatments of nicotinic acid, their mean TC level after 42 days was 149 mg/dl, with a standard deviation of 27 mg/dl. Among the placebo group of 65 rats, the mean TC level after 42 days was 213 mg/dl, with a standard deviation of 24 mg/dl.

Let μ_1 be the mean TC level after 42 days for rats treated with nicotinic acid.
Let μ_2 be the mean TC level after 42 days for rats given the placebo.

(a) Is there evidence to suggest that the mean difference between rats treated with nicotinic acid and placebo is more than 50 mg/dl?

(b) Compute a 90% confidence interval for the difference in mean TC level between rats treated with nicotinic acid group and those given a placebo.

always use z

CI: (est) ± (crit val) (ese) → margin of error

Sample Size for Estimating $\mu_1 - \mu_2$

The common ($n = n_1 = n_2$) sample size needed to construct a $(1-\alpha)100\%$ confidence interval within margin of error d (with length $2d$) for $\mu_1 - \mu_2$ is given by

$$d = z_{\alpha/2}\sqrt{\frac{s_1^2}{n} + \frac{s_2^2}{n}} \qquad \Rightarrow \qquad \underbrace{n}_{\text{round up}} = \left(\frac{z_{\alpha/2}}{d}\right)^2 (s_1^2 + s_2^2)$$

Example 2 Continued ... Suppose we want to repeat our study on pea plant yields. How many pea plants would we need to sample in order to construct a 98% confidence interval for the difference in crops yields with inoculant and plants with fertilizer, within 2 grams?

Readings and Practice problems: See PDF on Brightspace for detailed instructions.