Math 101 Notes:

Area under the curve: width height

Approx area: $\sum_{i=1}^{n} \Delta x \cdot f(x_i^*)$

In words: The sum of the area of all rectangles Exact

Area: $\lim_{n \to \infty} \sum_{i=1}^{\infty} f(x_i^*) \Delta x = \int_a^b f(x)\,dx$

Reimann Sum

Area Between the curve:

$$\sum_{i=1}^{n} \underbrace{\Delta x}_{\text{width}} \cdot \Big(\underbrace{f(x_i^*)}_{\text{height in } f(x_i)} - \underbrace{g(x_i^*)}_{\text{height for } g(x_i)}\Big)$$

The exact area $= \int_a^b f(x) - g(x)\,dx$

if $f(x) \geq g(x)$ this works, otherwise reverse the subtraction aka switch $f(x) \leftrightarrow g(x)$

Example

Find the area between the curves $y = x^2 + x$ and $y = 3 - x^2$ area $= \int_a^b (3 - x^2) - (x^2 + x)$ → $3 - x^2$ is greater from graph

→ To find limits set equations of curves to eachother

$3 - x^2 = x^2 + x \rightarrow 2x^2 + x - 3 = 0$

$\rightarrow \quad a = 2, \; b = 1, \; c = -3$

$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-1 \pm \sqrt{1^2 - 4(2)(-3)}}{2(2)}$

$= \dfrac{-1 \pm \sqrt{1 - (-24)}}{4} = \dfrac{-1 \pm \sqrt{25}}{4}$

$x = \dfrac{4}{4} = 1 \quad$ or $\quad x = -\dfrac{6}{4} = -\dfrac{3}{2} = -1.5$

$\therefore \int_{-1.5}^{1} 3 - x^2 - x^2 + x \rightarrow \left[3x - \dfrac{x^3}{3} - \dfrac{x^3}{3} + \dfrac{x^2}{2}\right]_{-1.5}^{1}$

$\left[3(1) - \dfrac{1^3}{3} - \dfrac{1^3}{3} + \dfrac{1^2}{2}\right] - \left[3(-1.5) - \dfrac{(-1.5)^3}{3} - \dfrac{(-1.5)^3}{3} + \dfrac{(-1.5)^2}{2}\right]$

$= \dfrac{125}{24}$ or 5.2084

i. e. Area between curves $= \int_a^b$ top curve $-$ bottom curve

$= \int^b f(x) - g(x)\,dx \quad a$

Volume of Solid of Revolution:

Area's

Washer : $\int_a^b \pi r^2_{outer} - \pi r^2_{inner}\,dx$ } Around x-axis Disk:

$\int_a^b \pi r^2 \, dx$ For y-axis replace x with y

Example:

Consider the region bounded by the curve $y = \sqrt[3]{x}$, the x-axis, and the line $x=8$. What is the volume of the solid of revolution formed by rotating this region around the x-axis.

Disks → from graph → $V = \int_0^8 \pi r^2 \, dx$ note: $r = y = \sqrt[3]{x}$

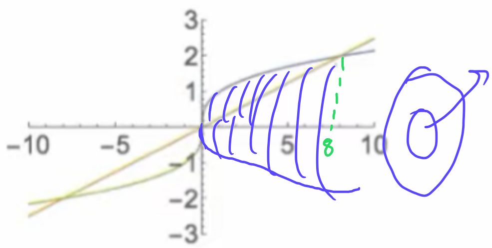
$$\therefore \int_0^8 \pi (\sqrt[3]{x})^2 dx \rightarrow \int_0^8 \pi (x^{1/3})^2 dx$$

$$\rightarrow \int_0^8 \pi \frac{x^{2/3+1}}{2/3+1} dx$$

$$\rightarrow \pi \int_0^8 x^{5/3} \cdot \frac{3}{5} = \left[\pi \cdot \frac{x^{5/3}}{5/3}\right]_0^8$$

$$= 60.3186$$

Example. Consider the region in the first quadrant bounded by the curves $y = \sqrt[3]{x}$ and $y = \frac{1}{4}x$. What is the volume of the solid of revolution formed by rotating this region around the x-axis? The y-axis?



washer

notice line also revolves ($\frac{1}{4}x$)

$$V = \int_0^8 \pi \, r_{outer}^2 - \pi \, r_{inner}^2$$

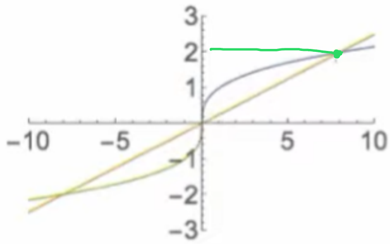
we get by $\sqrt[3]{x} = \frac{1}{4}x$ or from seeing where graph intersects $\therefore$

$$V = \int_0^8 \pi \left(\sqrt[3]{x}\right)^2 - \pi \left(\frac{1}{4}x\right)^2$$

$$\Rightarrow \quad V = \pi \int_0^8 x^{2/3} - \frac{1}{16}x$$

$$\Rightarrow \quad V = \pi \left[\frac{x^{5/3}}{5/3} - \frac{x^3}{48} \right]_0^8 = 26.81$$

Example. Consider the region in the first quadrant bounded by the curves $y = \sqrt[3]{x}$ and $y = \frac{1}{4}x$. What is the volume of the solid of revolution formed by rotating this region around the x-axis? The y-axis?



$$\int_0^2 \pi (r\,outer)^2 - \pi (r\,inner)^2$$

outer from graph is x-coordinate on the line $\frac{1}{4}x$

So $y = \frac{1}{4}x$, $x = 4y$ $\quad r_{outer} = 4y$

So $y = \sqrt[3]{x}$, $x = y^3$ $\quad r_{inner} = y^3$

$$\therefore \quad \pi \int_0^8 (4y)^2 - (y^3)^2 \, dy \Rightarrow \pi \int_0^8 16y^2 - y^6 \Rightarrow \pi \left[\frac{16y^3}{3} - \frac{y^7}{7} \right]$$

$V = \frac{512}{21}\pi$ or 76.5950

Volumes Using Cross Sections

Find the volume of the solid whose base is an ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and whose cross sections perpendicular to the x-axis are squares

Do sample later

Arc Length

If the curve goes from point a to point b then: arc length $= \int_a^b \sqrt{1+(f'(x))^2}\,dx$

Integration by Parts

$\int u\,dv = uv - \int v\,du$

Example: $\int x e^x\,dx$ $\quad u = x \quad dv = e^x dx$
$\quad du = dx \quad v = e^x$

$= x \cdot e^x - \int e^x dx$

$= xe^x - e^x + C$

Example Question (Trig): $\int \sin^4(x)\cos(x)\,dx$

$u = \sin x \quad du = \cos(x)\,dx \rightarrow \int u^4\,du$

$\rightarrow \frac{u^5}{5} + C$

$$\rightarrow \quad \frac{\sin^5(x)}{5} + C$$

Example Question (Trig): $\int \sin^4(x)\cos^3(x)\,dx$

$\rightarrow \int \sin^4(x)\cos^2(x)\cos(x)\,dx \qquad u = \sin(x) \quad du = \cos(x)dx$

$\rightarrow \int u^4 \cdot (1-u^2)\,du \qquad \cos^2(x) = 1 - \sin^2(x)$

$\rightarrow \int u^4 - u^6\,du = \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C$

Example: $\int \sin^5(x)\cos^2(x)\,dx = \int \sin^4(x)\cos^2(x)\sin(x)\,dx$ u

$= \cos(x) \quad du = -\sin x\,dx \quad -du = \sin x\,dx$

we know $\sin^2(x) = 1 - \cos^2(x)$ and $\sin^4(x) = (\sin^2(x))^2$

$\rightarrow \int (1-\cos^2(x))^2 \cdot \cos^2(x)\sin(x)\,dx$

$\rightarrow \int (1-u^2)^2 \cdot u^2(-du) \rightarrow (a-b)^2 = a^2 - 2ab + b^2$

so $\rightarrow \int (1 - 2u^2 + u^4)\cdot u^2(-du) \rightarrow -\int u^2 - 2u^4 + u^6\,du$

$\rightarrow -\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right] \rightarrow -\frac{\cos^3(x)}{3} + \frac{2\cos^5(x)}{5} - \frac{\cos^7(x)}{7} + C$

Integrals of Rational Functions (Partial Fractions)

$\int \frac{3x+2}{x^2+2x-3} \rightarrow x^2 + 2x - 3 = (x-1)(x+3)$

Making partial fraction: $\frac{A}{x-1} + \frac{B}{x+3} = \frac{3x+2}{(x-1)(x+3)}$

Multiply LCD by partial fraction:

$$(x-1)(x+3)\left[\frac{A}{x-1}+\frac{B}{x+3}\right]=\cancel{(x-1)(x+3)}\left[\frac{3x+2}{\cancel{(x-1)(x+3)}}\right]$$

$(x+3)A+(x-1)B=3x+2$

$Ax+3A+Bx-B=3x+2$

$(A+B)x+3A-B=3x+2$ we need $A+B=3$ and $3A-B=2$

$A+\cancel{B}=3 + 3A-\cancel{B}=2 \rightarrow 4A=5$ or $A=\frac{5}{4}$

So $B \Rightarrow \frac{5}{4}+B=3 \rightarrow B=\frac{7}{4}$

$$\therefore \int \frac{5/4}{x-1}+\frac{7/4}{x+3}\,dx=\int\frac{3x+2}{(x-1)(x+3)}$$

$$\therefore \frac{5}{4}\int\frac{1}{x-1}+\frac{7}{4}\int\frac{1}{x+3} \rightarrow \frac{5}{4}(\ln|x-1|)+\frac{7}{4}(\ln|x+3|)+C$$

Improper Integrals: Type 1

Find $\int_1^{\infty}\frac{1}{x^2}\,dx \therefore \lim_{n\to\infty}\int_1^n \frac{1}{x^2}\,dx$

$\lim_{n\to\infty}\left[-x^{-1}\right]_1^n=\lim_{n\to\infty}\left[-\frac{1}{x}\right]_1^n=\lim_{n\to\infty}\left[\underset{0}{-\frac{1}{n}}-\left(\frac{-1}{1}\right)\right]=1$ $\therefore$ Integral converges

Example2: $\int_{-\infty}^{-1}\frac{1}{x}\,dx$

$\lim_{n\to\infty}\int_{-n}^{-1}\frac{1}{x}\,dx \rightarrow \lim_{n\to\infty}\left[\ln|x|\right]_{-n}^{-1}$

$\rightarrow \lim_{n\to\infty}\left[\underset{0}{\cancel{\ln|-1|}}-\underset{\infty}{\cancel{\ln|-n|}}\right]=-\infty$

The integral diverges

Improper Integrals Type 2

Find the area under the curve $y = \frac{x}{\sqrt{x^2-1}}$ between the lines $x=1$ and $x=2$.

$$\text{Area} = \int_1^2 \frac{x}{\sqrt{x^2-1}} \quad u = x^2-1 \quad du = 2x\,dx \rightarrow \frac{du}{2} = x\,dx$$

$$\rightarrow \lim_{n\to 1^+} \int_n^2 \frac{x}{\sqrt{x^2-1}}\,dx \rightarrow \int_n^2 \frac{1/2\,du}{\sqrt{u}}$$ new bounds $x = n \rightarrow n^2-1$, $x=2 \rightarrow u=3$

$$= \lim_{n\to 1^+} \frac{1}{2}\int_{n^2-1}^{3} \frac{1}{u^{1/2}}\,du \rightarrow \lim_{n\to 1^+} \frac{1}{2}\int u^{-1/2} \rightarrow 3^{1/2} - \cancel{(n^2+1)^{1/2}}_{0}\ n^2-1 =$$
$\sqrt{3}$

$$\rightarrow \left[(3^2-1)^{1/2} - ((n^2-1)^2-)^{1/2}\right] \rightarrow \sqrt{8} - (n^4-2n^2+1)^{1/2}$$

Parametric Equations

Example $x = 1-2t$, $y = t^2+4$

Cartesian equation $\rightarrow x = 1-2t$ |

$\rightarrow x-1 = -2t$

$\rightarrow 2t = 1-x$

$\rightarrow t = \frac{1-x}{2}$

so $y = \left(\frac{1-x}{2}\right)^2 + 4$

Example write the following in parametric equations: 1) $y = \sqrt{x^2 - x}$ $x = t$ for $t \leq 0$ and $t \geq 1 \rightarrow$ part of quation so $y = \sqrt{t^2 - t}$ is the parametric equation

2) $25x^2 + 36y^2 = 900 \rightarrow \frac{25x^2}{900} + \frac{36y^2}{900} = 1$

simplifies to $\rightarrow \frac{x^2}{36} + \frac{y^2}{25} = 1$

rewrite $\rightarrow \left(\frac{x}{6}\right)^2 + \left(\frac{y}{5}\right)^2$

now if we take $\frac{x}{6} = \cos t$ and $\frac{y}{5} = \sin t$

$\therefore$ $x = 6\cos t$, $y = 5\sin t$

$\rightarrow$ basically an ellipse

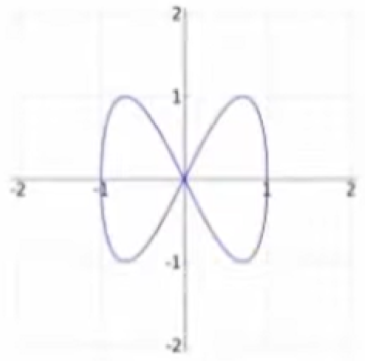
Tangent Lines for Parametric Equations

Remember: $\frac{dy}{dx} = \frac{f'(t)}{g'(t)}$

Example. For the Lissajous figure:

$$x = \cos(t), y = \sin(2t) \qquad 0 \le t \le 2\pi$$

1. Find the slopes of the tangent lines at the center point $(0, 0)$.
2. Find where the tangent line is horizontal.



$x = \cos(t)$, $x' = -\sin(t)$, $y = \sin(2t)$, $y' = 2\cos(2t)$

$\dfrac{dy}{dx} = \dfrac{2\cos(2t)}{-\sin(t)}$ we want to find for $(0,0)$ so $x = 0$ when $\cos(t) = 0$

which is $t = \pi/2$ and $t = 3\pi/2$

so we find $\dfrac{dy}{dx}$ for $t = \pi/2$ and $t = 3\pi/2$

$\therefore \quad \dfrac{2\cos(2(\pi/2))}{-\sin(\pi/2)}$ and $\dfrac{2\cos(3\pi/2)}{-\sin(3\pi/2)}$

$= 2 \qquad\qquad = -2$

So the slopes are 2 and -2

Find where the tangent line is horizontal

$$\frac{dy}{dx} = 0 \quad \therefore \quad \frac{2\cos(2t)}{-\sin(t)} = 0 \Rightarrow \cos(2t) = 0$$

$$\therefore 2t = \frac{\pi}{2} + \pi n$$

$$t = \frac{\pi}{4} + \pi n$$

So $t = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

Area under a Parametric Curve

Example. For the Lissajous figure:

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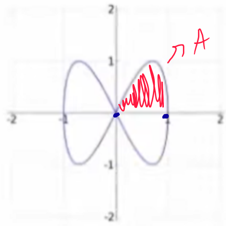


$\therefore$ Area $= 4A$

$y = \sin(2t)$

$x = \cos(t)$

$A = \int y \, dx$

$dx = x' = -\sin(t) dt$

Right point happens at $(1,0)$ for $(1,0)$ t must be 0 to get $(1,0)$, similarly $(0,0)$ will happen at $t = \pi/2$

So from $t=0$ and $t=\pi/2$

$$A = \int_0^{\pi/2} \sin(2t)(-\sin(t))\,dt \rightarrow -\int_0^{\pi/2} 2\sin(t)\cos(t)\sin(t)\,dt$$

$\rightarrow$ use double angle identity $\Big| \rightarrow -2\int_0^{\pi/2} \sin^2(t)\cos(t)\,dt$

$\rightarrow$ $u = \sin(t)$ $\quad du = \cos(t)\,dt$ $\quad$ new bounds $t=0 \rightarrow u=0$, $t=\frac{\pi}{2} \rightarrow$

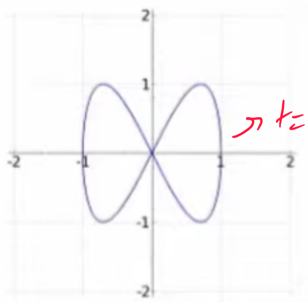
$u=1$

$$\therefore -2\int_0^1 u^2\,du \rightarrow -2\int_0^1 \frac{u^3}{3} \rightarrow \left[-\frac{2u^3}{3}\right]_0^1 = -\frac{2}{3}$$

Arclength of Parametric Curve:

Formula: $\displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Example. Set up an integral to express the arclength of the Lissajous figure $x = \cos(t), y = \sin(2t)$.



$\rightarrow t=0$ or $t=2\pi$

$$\frac{dx}{dt} = -\sin(t) \qquad \frac{dy}{dt} = 2\cos(2t)$$

so arclength $= \displaystyle\int \sqrt{(-\sin(t))^2 + (2\cos(2t))^2}\,dt$

$\rightarrow \displaystyle\int_0^{2\pi} \sqrt{(-\sin(t))^2 + (2\cos(2t))^2}\,dt$ $\rightarrow$ cant actually integrate by hand too difficult.

Polar Coordinates

Example

$\left(8, -\frac{2\pi}{3}\right)$ Negative angle θ go clockwise

$r \quad \theta$ Negative radius r go to other side of origin

or take pos r and add π to θ

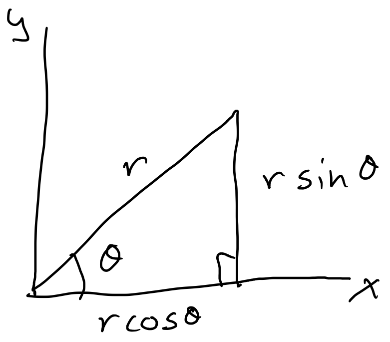
$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$r = \sqrt{x^2 + y^2}$

$\tan\theta = \frac{y}{x}$



Convert $\left(5, -\frac{\pi}{6}\right)$ from polar to Cartesian

$x = r\cos\theta \qquad y = r\sin\theta$

$x = 5\cos(-\pi/6) \qquad y = 5\sin\left(-\frac{\pi}{6}\right)$

$x = \frac{5\sqrt{3}}{2} \qquad y = -\frac{5}{2}$

Convert $(-1, -1)$ from cartesian to polar

$r^2 = (-1)^2 + (-1)^2 = 2$

$\tan\theta = \frac{-1}{-1} = 1$

we know we need values in 3rd quadrant as $(-1,-1)$ lies there

so $r = \sqrt{2}$ or $-\sqrt{2}$

$\theta = \pi/4,\ 5\pi/4 + 2\pi k$

but only combination that works is

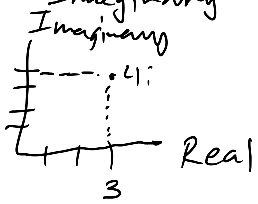
$(r, \theta) = (-\sqrt{2}, \pi/4)$ or $(\sqrt{2}, 5\pi/4)$
$+ 2\pi k \qquad\qquad + 2\pi k$

Complex Numbers

$a + bi$ where a = Real and bi = Imaginary

can be graphed like so:

$3 + 4i$

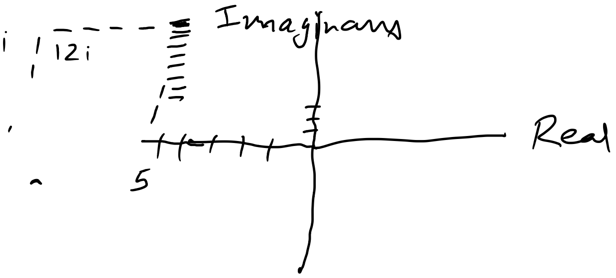


also called modulus

Finding absolute value: $|a+bi| = \sqrt{a^2+b^2}$

So $|3+4i| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$

Example 2: $-5 + 12i$

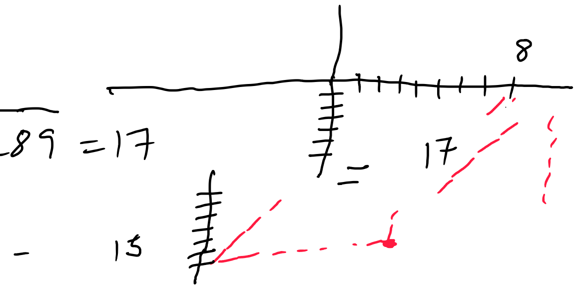


$|-5+12i| = \sqrt{(-5)^2 + (12)^2} = \sqrt{25+144} = \sqrt{169} = 13$

Example 3: $8 - 15i$

$|8-15i| = \sqrt{8^2+15^2}$

$\to \sqrt{64+225} = \sqrt{289} = 17$



$\sqrt{4} = 2$ but $\sqrt{-4} = 2i$ as $i = \sqrt{-1}$

$\sqrt{9} = 3$ but $\sqrt{-9} = \sqrt{9} \cdot \sqrt{-1} = 3i$

similarly $\sqrt{25} = 5$, $\sqrt{-25} = 5i$, $-\sqrt{25} = -5$, $-\sqrt{-25} = -5i$

for $\sqrt{-18} = \sqrt{9} \cdot \sqrt{2} \cdot \sqrt{-1}$

$= 3\sqrt{2}\, i$

- $\sqrt{-50} = \sqrt{25} \cdot \sqrt{2} \cdot \sqrt{-1}$

$= 5\sqrt{2}\, i$

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = +1$

Addition: $(5+2i)+(3+7i)$

$\to 8+9i$

$(3+4i)-(-1+2i) \to 3 - -1 + 6i$

$\to 4+2i$

Multiplication: $(2+i)(1+3i) = 2 + 6i + i + 3i^2$

$= 2 + 7i - 3 \quad (i^2 = -1) = -1 + 7i$

Division: $\frac{2+3i}{1-7i} \rightarrow \frac{2+3i}{1-7i} \times \frac{1+7i}{1+7i}$

$\rightarrow \frac{2 + 14i + 3i - 21i^2}{(1-7i)(1+7i)}$

$\rightarrow \frac{2 + 17i - 21}{1 + \cancel{7i} - \cancel{7i} - 49i^2} \rightarrow \frac{2 + 17i - 21}{1 + 49}$

$\frac{-19 + 17i}{50} \rightarrow \frac{-19}{50} + \frac{17i}{50}$

Conjugate is just opposit site $z = a + bi$

conjugate $\rightarrow z = a - bi$

Solve the equation: $z^2 + 4z + 5 = 0$ forallvalofz

$a = 1 \quad b = 4 \quad c = 5$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)}$

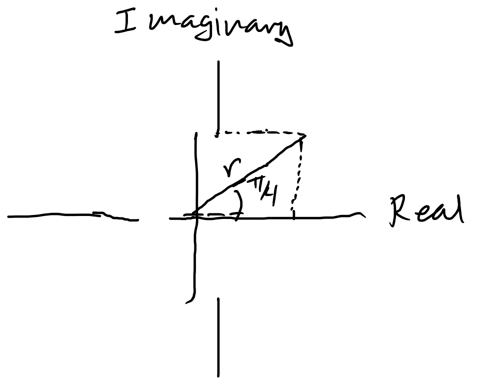
So we get $\frac{-4 \pm \sqrt{-4}}{2} \rightarrow \frac{-4 \pm 2i}{2} = -2 \pm i$

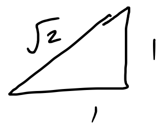
conjugate pairs $\begin{cases} -2 + i = z_1 \\ \end{cases}$

$- \quad 2 - i \quad = z_2$

Argand Diagrams and Polar Coordinates:

Plot the complex number $z = 1+i$. Determine the distance of z from the origin (r) and the angle it makes with the horizontal axis (θ).





$\sqrt{1^2+1^2} = \sqrt{2}$

By triangle $r = \sqrt{2}$

Note: $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

Exponential Representation of Complex Numbers Course Pack Section 4.1.5