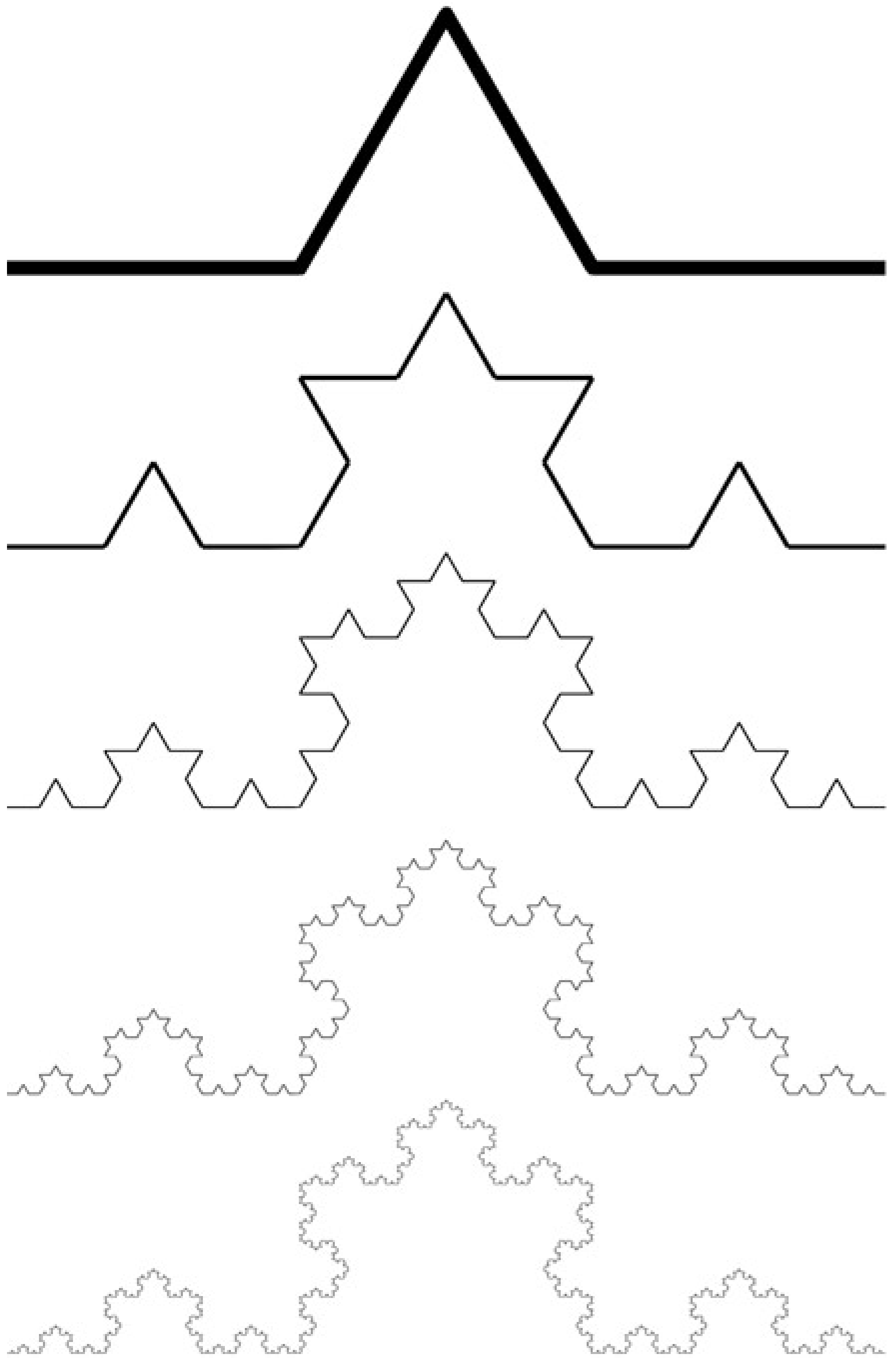
Math 101 Course Pack



Calculus II

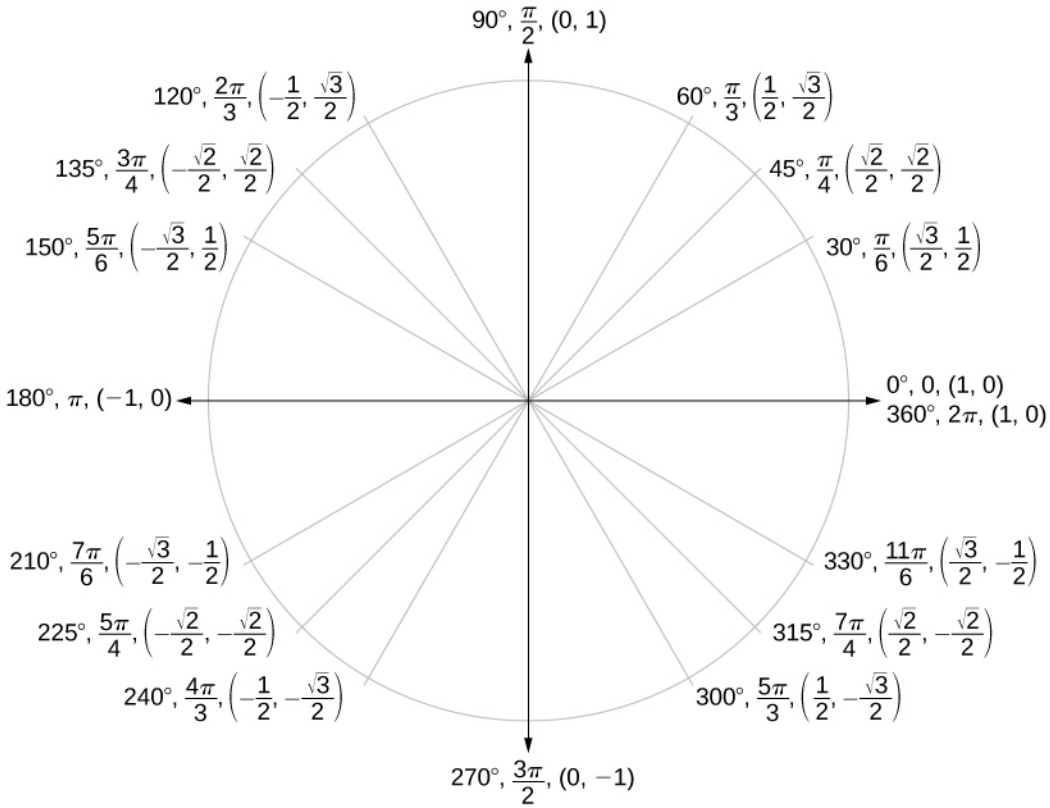
Notes By William Thompson

Contents

Chapter 1

Basic Tables and Formulas

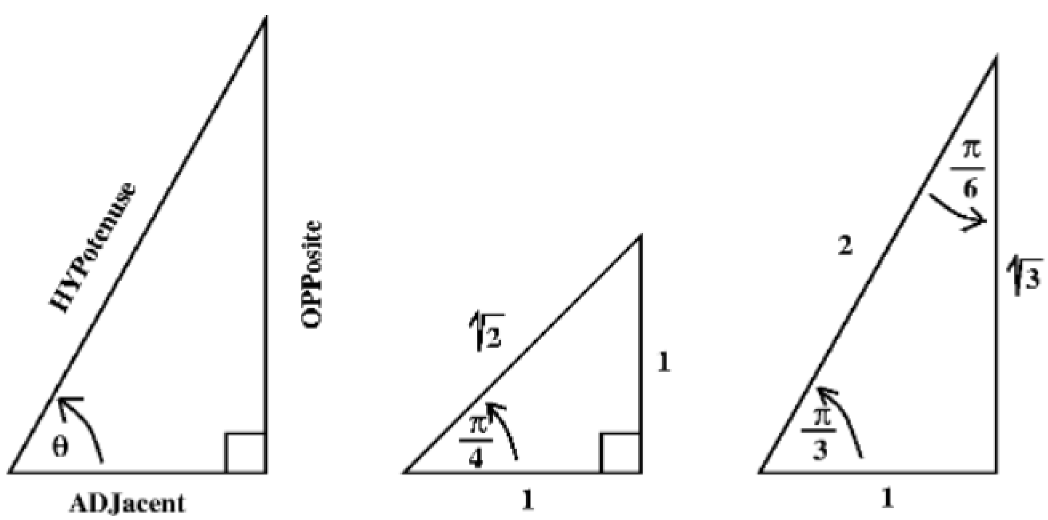
1.1 Unit Circle



1.2 Trigonometric Functions and Reference Triangles

$$\sin(\theta) = \frac{\text{OPP}}{\text{HYP}} \qquad \cos(\theta) = \frac{\text{ADJ}}{\text{HYP}} \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\text{OPP}}{\text{ADJ}}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\text{ADJ}}{\text{OPP}} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{HYP}}{\text{ADJ}} \qquad \csc(\theta) = \frac{\text{HYP}}{\text{OPP}}$$



1.3 Trigonometric Formulae

Trigonometric Formulæ

Basic Identities

- $\cos^2(\theta) + \sin^2(\theta) = 1$
- $\tan^2(\theta) + 1 = \sec^2(\theta)$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

Half Angle Identities

- $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$
- $\cos^2(\theta) = \dfrac{1 + \cos(2\theta)}{2}$

Ptolemy's Identities

- $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
- $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
- $\tan(A + B) = \dfrac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$

Product to Sum

- $\sin(A)\sin(B) = \dfrac{1}{2}\cos(A - B) - \dfrac{1}{2}\cos(A + B)$
- $\cos(A)\cos(B) = \dfrac{1}{2}\cos(A - B) + \dfrac{1}{2}\cos(A + B)$
- $\sin(A)\cos(B) = \dfrac{1}{2}\sin(A - B) + \dfrac{1}{2}\sin(A + B)$

Even and Odd Properties

- $\sin(-\theta) = -\sin(\theta)$
- $\cos(-\theta) = \cos(\theta)$
- $\tan(-\theta) = -\tan(\theta)$

1.4 Table of Derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\cot(x)$	$-\csc^2(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\mathrm{arccot}(x)$	$-\frac{1}{1+x^2}$
$\mathrm{arcsec}(x)$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\mathrm{arccsc}(x)$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
e^x	e^x
a^x	$a^x\ln(a)$ where $a>0$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x\ln(a)}$

1.5 Derivative Rules

Derivative	**Name**
$(Cf(x))' = Cf'(x)$	Constant Rule
$(f(x)+g(x))' = f'(x)+g'(x)$	Sum Rule
$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$	Product Rule
$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$	Quotient Rule
$(f(g(x)))' = f'(g(x)) \cdot g'(x)$	Chain Rule
$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$	Inverse Rule
$\left(\int_a^x f(t)dt\right)' = f(x)$	First Fundamental Theorem of Calculus

1.6 Table of Antiderivatives

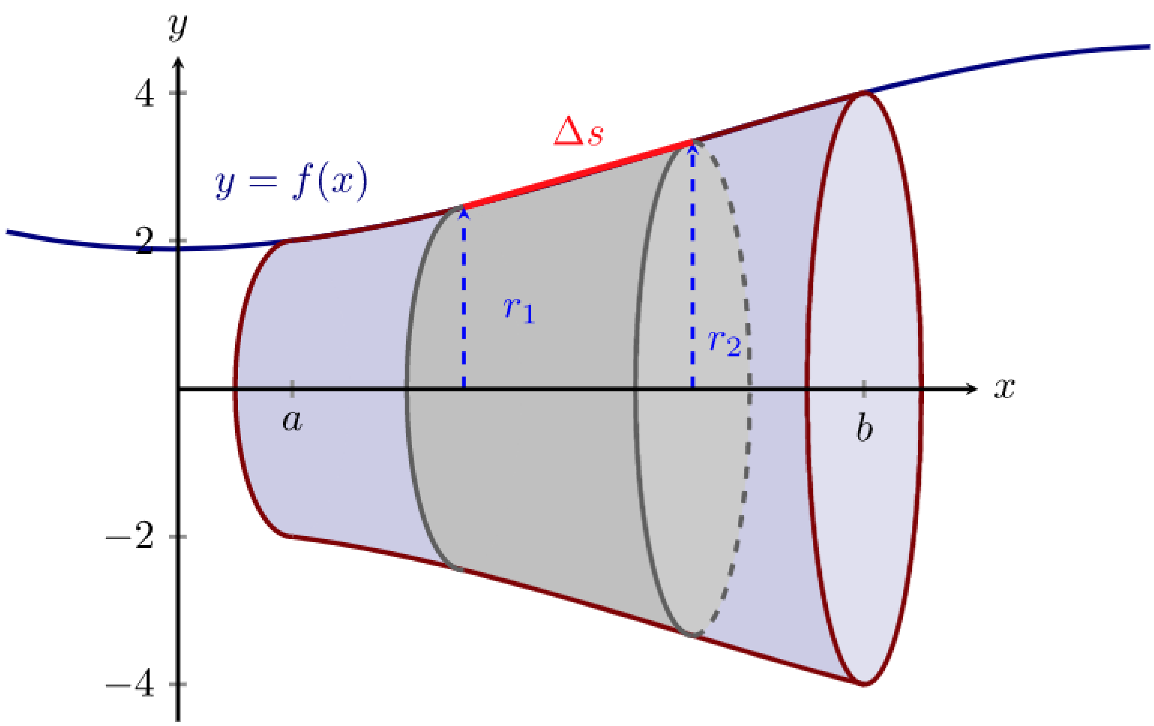
$f(x)$	$\int f(x)dx$
x^n	$\frac{1}{n+1}x^{n+1} + C \qquad \text{if } n \neq 1$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\tan(x)$	$\ln\|\sec(x)\| + C$
$\sec(x)$	$\ln\|\sec(x) + \tan(x)\| + C$
$\cot(x)$	$\ln\|\sin(x)\| + C$
$\csc(x)$	$-\ln\|\csc(x) + \cot(x)\| + C$
$\sec^2(x)$	$\tan(x) + C$
$\csc^2(x)$	$-\cot(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$
$\csc(x)\cot(x)$	$-\csc(x) + C$
e^x	$e^x + C$
$\frac{1}{x}$	$\ln\|x\| + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$
$\frac{1}{1+x^2}$	$\arctan(x) + C$
$\frac{1}{x\sqrt{x^2-1}}$	$\operatorname{arcsec}\|x\| + C$
a^x	$\frac{a^x}{\ln(a)} + C$

1.7 Taylor Series

Integral Function	**Name**	**Domain**
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$	Exponential Series	$\mathbb{R}$
$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$	Sine Series	$\mathbb{R}$
$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	Cosine Series	$\mathbb{R}$
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$	Reciprocal Series	$\lvert x\rvert < 1$
$\ln(x+1) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$	Logarithm Series	$-1 < x \leq 1$
$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$	Arctangent series	$\lvert x\rvert \leq 1$
$\arcsin(x) = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} x^{2k+1}$	Arcsine series	$\lvert x\rvert \leq 1$
$(1+x)^m = 1 + \sum_{k=0}^{\infty} \binom{m}{k} x^k$	Binomial Series	$\lvert x\rvert < 1$
$f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$	Series Generated by $f(x)$	Depends

Chapter 2

Integration Techniques and Applications



2.1 (Section 8.1) Some Integration Techniques and Tricks

Note: Algebraic Techniques

A common technique to evaluating an integral is to algebraically manipulate it to an already known expression. It's impossible to cover every scenario in a single lecture, but we give several examples below.

2.1.1 Completing the Square

Example 1: Evaluate $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$.

2.1.2 Using Trigonometric Identities

Example 2: Evaluate $\displaystyle\int_{-\pi/2}^{0} \sqrt{1-\cos(2\theta)}\ \, d\theta$ using the identity $2\sin^2(A) = 1 - \cos(2A)$.

2.1.3 Long Division

Example 3: Evaluate $\int \frac{x^3+x}{x-1}dx$.

2.1.4 Conjugates

Example 4: Evaluate $\int_0^{\pi/4} \frac{dx}{1-\sin(x)}$.

2.1.5 Dumb Luck Substitution

Note: Substitution to Change Form

A better way to think about integral substitution is that it changes the expression you're currently working with. Sometimes you find a substitution that changes it in just the right way that allows you to evaluate.

Example 5: Evaluate $\int \frac{\sqrt{x}}{1+x^3}dx$ using $u = x^{3/2}$.

2.1.6 Splitting up an Integral by Linearity

Example 6: Evaluate $\int \frac{3x+2}{\sqrt{1-x^2}}dx$.

2.2 (Section 7.1) Logarithms and Exponentials

Note: Defining the Logarithm as an Integral

The textbook goes through extensive measures to demonstrate how to reconstruct the exponential and all properties of the logarithm if we initially define the natural logarithm by $\ln(x) = \int_1^x \frac{1}{t}dt, \qquad x > 0$ the process is long and worth a read. What we will do is give examples of computing logarithmic and exponential type integrals.

2.2.1 Scenario 1: Integrals Involving the Natural Logarithm

Note: Integrands of the Form $f'(x)/f(x)$

To evaluate the integral $I = \int \frac{f'(x)}{f(x)}dx$ perform the substitution $u = f(x)$ to obtain $I = \ln|f(x)| + C$.

Example 1: Evaluate $\int_{-1}^{0} \frac{3dx}{3x-2}$

Example 2: Evaluate $\int \frac{1}{\arctan(4x)(1+16x^2)}dx$.

2.2.2 Scenario 2: Integrals Involving the Natural Logarithm

Note: Integrands of the Form $f(\ln(x))/x$

If you see a logarithm in your integral, you should try to look for the term $1/x$. If so, then you may use the substitution $u = \ln(x)$ to obtain $\displaystyle\int \frac{f(\ln(x))}{x}dx = \int f(u)du$, which could potentially be evaluated. Naturally, this technique generalizes to integrals of the form $\displaystyle\int \frac{f(\log_a(x))}{x}dx$.

Example 3: Evaluate $\displaystyle\int \frac{\ln(\ln(x))}{x\ln(x)}dx$.

Example 4: Evaluate $\displaystyle\int \frac{\csc^2(\log_{10}(x))}{x}dx$.

2.2.3 Integrals Involving the Exponential Function

Note: Integrands of the form $f(e^{ax})e^{ax}$

Since the derivative of an exponential is (in a sense) itself, a good strategy when dealing with exponentials is to see if another one is present. Performing the substitution $u = e^{ax}$ transforms the integral as $\displaystyle\int f(e^{ax})e^{ax}dx = \frac{1}{a}\int f(u)du$. Naturally, this technique generalizes to integrals of the form $\displaystyle\int f(b^x)b^x dx$.

Example 5: Evaluate $\displaystyle\int_0^{\ln(\pi)} e^x \cos(e^x)dx$.

Example 6: Evaluate $\displaystyle\int \frac{1}{1+e^t}dt$.

2.3 (Section 7.2) Separable Differential Equations and Modeling

2.3.1 Separable Differential Equations

Definition: Differential Equations and Separable Equations

A **Differential Equation** is an equation involving an unknown function and its derivatives. A **Separable Differential Equation** is a differential equation of the form $y' = f(x)g(y)$ for some functions f and g.

Procedure: Solving a SDE

1. Bring all the y terms to the LHS and all the x terms to the RHS to obtain the equation $g(y)^{-1}dy = f(x)dx$.
2. Integrate both sides $\displaystyle\int g(y)^{-1}dy = \int f(x)dx$ to get an implicit equation for x and y.

Example 1: Solve the differential equation $(1+x^2)\dfrac{dy}{dx} = 2x\sqrt{1-y^2}$, $y(0) = 1/2$ explicitly for y.

2.3.2 Unlimited Growth and Radioactive Decay Models

Model: Exponential Model

Let $y(t)$ measure the amount of substance of an important material at time t with an initial amount $y(0) = y_0$. If the rate of change y is proportional to y itself, then it is given by the separable differential equation

$$y'(t) = ky(t); \quad y(0) = y_0$$

and has the solution $y(t) = y_0 e^{kt}$. The substance grows if $k > 0$ and it decays if $k < 0$.

Example 2: The biomass of a yeast culture in an experiment is 29g. After 30 min the mass is 37g. Assuming the equation for unlimited population growth models this, how long will it take for the mass to double from it's initial population size?

Example 3: The half-life of carbon-14 is 5730 years (i.e. $y(5370) = y_0/2$). Assuming the exponential model is a suitable model, find the age of a sample in which 10% of the radioactive substance has decayed.

2.3.3 Heat Transfer: Newton's Law of Cooling

Model: Newton's Law of Cooling

If $H(t)$ is the temperature of an object at time t and H_s is the constant surrounding temperature the Newton's Law of Cooling model is given by

$$H'(t) = -k(H - H_s); \quad H(0) = H_0$$

where $k > 0$. It has the solution $H(t) = H_s + (H_0 - H_s)e^{-kt}$.

Example 4: You are Sherlock Holmes investigating a murder. You examine the cadaver at 1:30 pm and register a temperature of $33°C$. An hour later (at 2:30 pm) you measure a temperature of $30°C$. Given that the temperature of a living body is $37°C$ and the surrounding temperature is a stable $20°C$, determine the time of the murder.

2.4 (Section 8.2) Integration By Parts (IBP)

2.4.1 Establishing Integration by Parts and LIATE

Theorem: Integration By Parts (IBP)

Let $u(x)$ and $v(x)$ be differentiable functions on (a, b) and continuous on $[a, b]$. Then,

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du.$$

Note: Derivation of IBP

By the product rule, $uv' = (uv)' - vu'$. Integrating both sides yields the IBP theorem.

Example 1: Use IBP to evaluate $\displaystyle\int xe^x dx$. Identify this product as $u = x$ and $dv = e^x dx$. What happens if $u = e^x$ and $dv = xdx$ instead?

LIATE

Consider an integral of the form $\displaystyle\int f(x)g(x)dx$ where $f(x)$ and $g(x)$ are either a **L**ogarithmic function, **I**nverse trigonometric function, **A**lgebraic function, **T**rigonometric function, or **E**xponential function. **LIATE** is an acronym that tells you that you set your u term as the first function that is present on the list:

- **L** = Logarithmic function
- **I** = Inverse trigonometric function
- **A** = Algebraic function
- **T** = Trigonometric function
- **E** = Exponential function

2.4.2 Examples Using LIATE

Example 2: Evaluate $\int_1^e x^3 \ln(x)dx$.

Example 3: Evaluate $\int e^x \sin(2x)dx$.

2.4.3 Integrals of Lonely Logarithms and Inverse Functions (Ninja's)

Note: Integrals of Inverse Functions

One may identify the integral $\int f^{-1}(x)dx = \int f^{-1}(x) \cdot 1dx$ where $u = f^{-1}(x)$ and $dv = 1dx$.

Example 4: Evaluate $\int \arctan(x)dx$.

2.4.4 The Tabular Method

Note: The Tabular Method

The tabular "*method*" is just a way to organize information when performing IBP multiple times (like in a prior example). It's better seen by demonstration.

Example 5: Evaluate $\int x^2 \cos(x)dx$ using the tabular method.

2.5 (Section 8.3) Trigonometric Integrals

2.5.1 Products (Without Powers) and Roots

Procedure: Integrand of the Form $\sin(mx)\sin(nx)$, $\sin(mx)\cos(nx)$, or $\cos(mx)\cos(nx)$

If your integral is one of the three forms $\int \sin(mx)\sin(nx)dx$; $\int \sin(mx)\cos(nx)dx$; or $\int \cos(mx)\cos(nx)dx$ use the *Product to Sum* trigonometric formulas found in the course pack.

Example 1: Evaluate $\int \cos(2x)\sin(4x)dx$.

Procedure: Integrand of the Form $\sqrt{1 \pm \cos(ax)}$

If your integrand contains $\sqrt{1 \pm \cos(ax)}$ try using the *Half Angle Identities*.

Example 2: Evaluate $\int_{\pi}^{2\pi} \sqrt{1-\cos(2x)}dx$.

Procedure: Integrand of the Form $\sqrt{1 \pm \sin(ax)}$

If your integrand contains $\sqrt{1 \pm \sin(ax)}$ multiply by the conjugate and use $1-\sin^2(ax) = \cos^2(ax)$.

Example 3: Evaluate $\int_{0}^{\pi/2} \frac{1}{2}\sqrt{1+\sin(x)}dx$.

2.5.2 Products of Sines and Cosines, Same Input, Raised to a Power

Procedure: Integral of the Form $\int \sin^m(x)\cos^n(x)dx$

The procedure depends on whether either m and n are both even, or one of them is odd.

- m or n (or both) are odd: Take off a single term of the odd power and join it with dx to form your du. Convert the remaining terms using $\sin^2(x) + \cos^2(x) = 1$ and finish the u-substitution.
- m and n are even: Convert all terms using the *Half Angle Identities*. If any new even power terms appear, repeat the prior step procedure, else use the odd power procedure.

Example 4: Evaluate $\int \sin^3(x)\cos^4(x)dx$.

Example 5: Evaluate $\int \sin^2(x)\cos^2(x)dx$.

2.5.3 Lonely Powers of Tangent or Secant

Procedure: Integral of the Form $\int \tan^m(x)dx$ or $\int \sec^n(x)dx$

- $n \geq 3$:

  1. Pull off a $\sec^2(x)$ term and perform IBP with $dv = \sec^2(x)dx$.
  2. Convert tangent terms using $\tan^2(x) = \sec^2(x) - 1$, provided they are existent.
  3. Complete the integration if you are at a basic integral of $\sec^2(x)$ or $\sec(x)$. Otherwise, collect the common (original) integral on both sides and solve for it. You may have integrals of secants in your new integral still.
  4. If the remaining secant integrals are $\sec(x)$ or $\sec^2(x)$, finish evaluation. Else, repeat the above steps to continue the reduction.

- $m \geq 3$ is odd:

  1. Pull off $\tan(x)$ and multiply by $\frac{\sec(x)}{\sec(x)}$ to form $\int \tan^m(x)dx = \int \tan^{m-1}(x)\frac{\sec(x)\tan(x)}{\sec(x)}dx$.
  2. Construct $du = \sec(x)\tan(x)dx$ and convert the remaining tangent terms to secants using $\tan^2(x) = \sec^2(x) - 1$.
  3. Complete your u-substitution using $u = \sec(x)$ and integrate.

- $m \geq 2$ is even:

  1. Convert all your tangent terms into secants using the identity $\tan^2(x) = \sec^2(x) - 1$.
  2. If the result is a basic integral, integrate it. Otherwise, expand the result to construct a sum of powers of secants.
  3. Integrate the basic integral terms of $\sec^2(x)$, $\sec(x)$ and constants, then use the procedure of integrating higher powers of secants mentioned above.

Example 6: Evaluate $\int \sec^4(x)dx$.

Example 7: Evaluate $\int \tan^3(x)dx$.

2.5.4 Powers of Tangent and Secant with Same Input

Procedure: Integral of the Form $\int \tan^m(x)\sec^n(x)dx$

- m is odd:
  1. Pull off a single $\tan(x)$ and $\sec(x)$ term and construct $du = \sec(x)\tan(x)dx$.
  2. Convert the remaining tangent terms using the identity $\tan^2(x) = \sec^2(x) - 1$.
  3. Complete the u-substitution with $u = \sec(x)$ and integrate.
- n is even:
  1. Pull of a $\sec^2(x)$ term and construct $du = \sec^2(x)dx$.
  2. Convert the remaining secant terms using the identity $\sec^2(x) = \tan^2(x) + 1$.
  3. Complete the u-substitution with $u = \tan(x)$ and integrate.
- m is even and n is odd:
  1. Convert all the tangent terms to secants using the identity $\tan^2(x) = \sec^2(x) - 1$.
  2. Expand your integrand to get a sum of powers of secants.
  3. Use the appropriate integration techniques for integrating lonely powers of secants.

Example 7: Evaluate $\int \sec(x)\tan^3(x)dx$.

Example 8: Evaluate $\int \tan^2(x)\sec^4(x)dx$.

2.6 (Section 8.4) Trigonometric Substitution

2.6.1 Forms of Trigonometric Subsitution

Procedure: Integrands Containing $u^2 \pm a^2$ or $a^2 - u^2$ where $u = f(x)$.

The procedure for all the following is essentially the same. Use the substitution $u = a \times$ (Appropriate Trig Function), use techniques of trigonometric integrals to complete the integration, then convert back.

- Containing $a^2 - u^2$:

  1. Let $u = a\sin(\theta)$ and compute $du = a\cos(\theta)d\theta$.
  2. Complete the substitution and use an appropriate integration technique to compute the resulting trigonometric integral.
  3. Convert all terms back. If you encounter a lonely θ use the identity $\theta = \arcsin(u/a)$. If you encounter a double angle, reduce to a single angle using a double angle identity. If you encounter a non-sine function, represent the identity $\sin(\theta) = \dfrac{u}{a}$ as a triangle and solve for the values of other trigonometric functions.

- Containing $a^2 + u^2$:

  1. Let $u = a\tan(\theta)$ and compute $du = a\sec^2(\theta)d\theta$.
  2. Complete the substitution and use an appropriate integration technique to compute the resulting trigonometric integral.
  3. Convert all terms back. If you encounter a lonely θ use the identity $\theta = \arctan(u/a)$. If you encounter a double angle, reduce to a single angle using a double angle identity. If you encounter a non-sine function, represent the identity $\tan(\theta) = \dfrac{u}{a}$ as a triangle and solve for the values of other trigonometric functions.

- Containing $u^2 - a^2$:

  1. Let $u = a\sec(\theta)$ and compute $du = a\sec(\theta)\tan(\theta)d\theta$.
  2. Complete the substitution and use an appropriate integration technique to compute the resulting trigonometric integral.
  3. Convert all terms back. If you encounter a lonely θ use the identity $\theta = \operatorname{arcsec}(u/a)$. If you encounter a double angle, reduce to a single angle using a double angle identity. If you encounter a non-sine function, represent the identity $\sec(\theta) = \dfrac{u}{a}$ as a triangle and solve for the values of other trigonometric functions.

2.6.2 Sine Substitutions ($a^2 - u^2$)

Note: Domain Restriction for Sine Substitutions

In these integrals we always have a domain restriction of $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Example 1: Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{16 - x^2}}$.

2.6.3 Tangent Substitutions $(a^2 + u^2)$

Note: Domain Restriction for Tangent Substitutions

In these integrals we always have a domain restriction of $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.

Example 2: Evaluate $\displaystyle\int \frac{dx}{\sqrt{9x^2 + 4}}$.

2.6.4 Secant Substitutions $(u^2 - a^2)$

Note: Domain Restriction for Secant Substitutions

In these integrals we always have a domain restriction of $0 < \theta < \dfrac{\pi}{2}$ if $u > a$ or $\dfrac{\pi}{2} < \theta < \pi$ if $u < -a$.

Example 3: Evaluate $\displaystyle\int \frac{dx}{\sqrt{16x^2 - 1}}$ if $x < -\dfrac{1}{4}$.

2.7 (Section 8.5) Partial Fractions

Example 1: Use the fact that $\dfrac{3x+11}{x^2-x-6} = \dfrac{4}{x-3} - \dfrac{1}{x+2}$ to evaluate $\displaystyle\int \frac{3x+11}{x^2-x-6}dx$.

2.7.1 Procedure to "Un-simplifying" Rational Functions

Procedure: Partial Fractions

Suppose that $f(x) = \dfrac{P(x)}{Q(x)}$.

1. Make sure the degree of the numerator is less than the denominator. If not, perform long division.
2. Factor $Q(x)$ into linear terms and irreducible quadratic terms. A reminder that a quadratic is irreducible (i.e. does not factor further) if for ax^2+bx+c we gave $b^2-4ac<0$.
3. If a linear term $ax+b$ has maximal power m then suggest the following terms
$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$
in the un-simplification where $A_1, A_2, ..., A_m$ are unknown constants.
4. If a an irreducible quadratic term ax^2+bx+c has maximal power n then suggest the following terms
$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$
in the un-simplification where $B_1, .., B_n$ and $C_1, ..., C_n$ are unknown constants.
5. Once you have fully suggested an equation, rid the entire expression of denominators by cross multiplying everything properly. Then, expand everything.
6. Match coefficients of polynomials to form a system of equations for all unknown constants and solve this system.

Example 2: Suggest a form for the partial fraction decomposition of $f(x) = \dfrac{3x+4}{(x-1)^3(x-2)(x^2+x+1)^2}$.

2.7.2 Simplest Case: All Linear Factors, No Repeats

Example 3: Evaluate $\displaystyle\int \frac{1}{x^2-x-6}\,dx$.

2.7.3 Using Partial Fractions to Evaluate an Integral

Example 4: Evaluate $\displaystyle\int \frac{x^5+x^3+x+1}{x^4+x^2}dx$. Notice that the degree of the numerator is not less than the denominator.

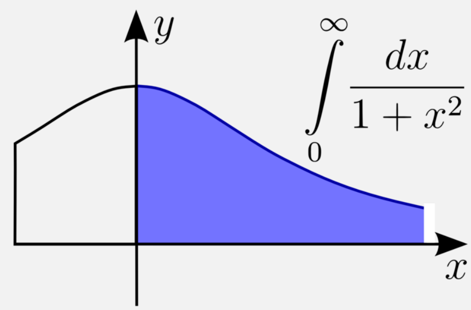
2.8 (Section 8.8) Improper Integrals

2.8.1 Type I Improper Integrals and Convergence

Definition: Type I Improper Integrals

An integral with an infinite bound is defined in the limit sense and called a **Type I Integral**. Specifically,

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$



Example 1: Evaluate $\displaystyle\int_0^\infty \frac{1}{1+x^2}dx$.

Example 2: Evaluate $\displaystyle\int_e^\infty \frac{1}{x\ln(x)}dx$.

Definition: Convergence and Divergence

If an integral results in a finite value we say the integral is **Convergent**. Otherwise, if an integral is not convergent we say it is **Divergent**.

2.8.2 p-Integrals

Note: Studying Nature vs. Obtaining Values

Most integrals can't be evaluated in closed form. Furthermore, if we want to evaluate an integral numerically, we may always code a script on a computer to do so. What we care about from a human perspective is whether a finite value is obtainable or not (i.e. whether or not it converges) and not about what that specific value is. This is done by constructing a **Proof Argument**, usually formatted as to compare the nature of an integral to a simpler well known one.

Definition and Theorem: Convergence of p-Integrals

A Type I p-**Integral** is an integral of the form $I = \displaystyle\int_a^{\infty} \frac{dx}{x^p}$ where $a > 0$. Furthermore,

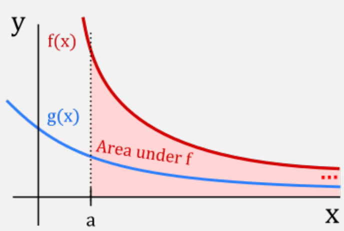
- If $p > 1$ then I converges; and
- If $p \leq 1$ then I diverges.

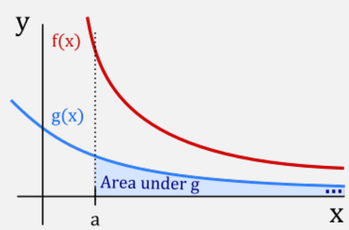
2.8.3 Direct Comparison Test

Theorem: Direct Comparison Test (DCT)

Let f and g be continuous functions on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Let $F = \displaystyle\int_a^{\infty} f(x)dx$ and $G = \displaystyle\int_a^{\infty} g(x)dx$.

- If G converges then F converges; and
- If F diverges then G diverges.





Example 3: Determine whether or not the integral $\displaystyle\int_1^{\infty} \frac{dx}{1+x^4}$ converges or diverges. Write a complete proof of your claim using an appropriate theorem in your argument.

2.8.4 Limit Comparison Test

Limit Comparison Test

Let f and g be continuous and positive functions on $[a, \infty)$ with $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ where L is positive and finite, i.e. $0 < L < \infty$. Then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge together or diverge together.

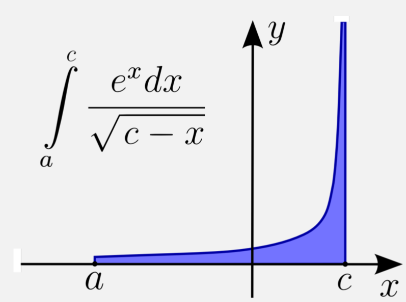
Example 4: Determine whether or not the integral $\int_1^\infty \frac{1 - e^{-x}}{x} dx$ converges or diverges. Write a complete proof of your claim using an appropriate theorem in your argument.

2.8.5 Type II Improper Integral

Definition: Type II Improper Integrals

An integral with a summation about a discontinuity is defined in the limit sense and is called a **Type II Integral**. To illustrate, if $x = a$ is a vertical asymptote of $f(x)$ then

$$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$$



Example 5: Evaluate $\int_0^4 \frac{1}{2\sqrt{x}} dx$.

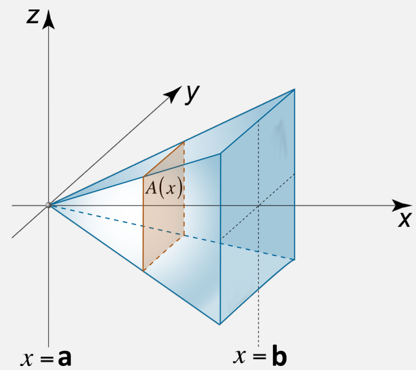
2.9 (Section 6.1) Volumes Using Cross Sections

2.9.1 Defining Volumes by Cross Sections

Definition: Defining the Volume of a Solid by Cross Sections

The **Volume** of a solid with cross sectional area $A(x)$ at each $a \leq x \leq b$ is defined as $V = \int_a^b A(x)dx$.

Note: This makes sense as integration is just summation. You're adding up the product of area with length, giving volume.

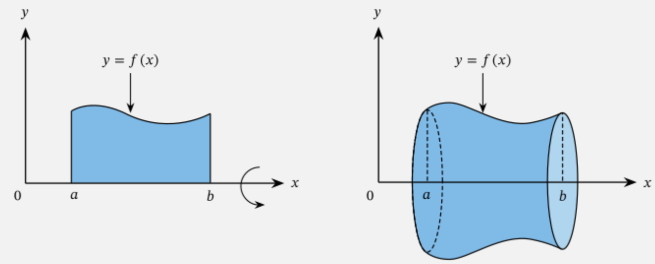


Example 1: Setup (but do not evaluate) an integral representing the volume of a cheese wedge cut from a circular cylinder of radius r if the angle between the top and the bottom is $\pi/6$ radians.

2.9.2 Solids of Revolution and their Volume by Washers

Definition: Solid of Revolution

A **Solid of Revolution** is a solid obtained by rotating a region trapped by a curve about a line.



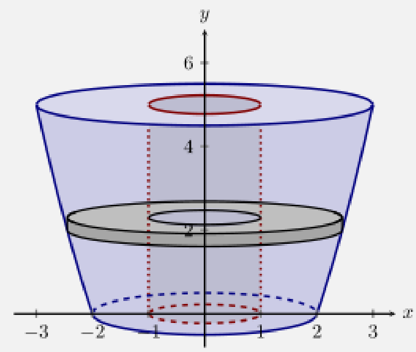
Note: Cross Sections of Solids of Revolutions

The cross section of a solid of revolution is a **Washer**. This is a punctured disc.

Let r be the radius of the inner hole of the cross section and let R be the radius of the outer circle of the cross section. The cross sectional area of the washer is then given by the formula

$$A = \pi R^2 - \pi r^2$$

where r and R need to be derived from a diagram.



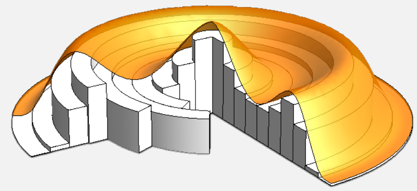
Example 2: Setup (but do not evaluate) the volume of the solid of revolution obtained by revolving the region bounded by $y = x^2$, $x = -1$, $x = 1$, and $y = 0$ about the x-axis.

Example 3: Setup (but do not evaluate) the volume of the solid of revolution obtained by revolving the region bounded by $y = x^2$ and $y = \sqrt{x}$ about the line $x = -1$.

2.10 (Section 6.2) The Shell Method

Note: Shell Method

Instead of visualizing a solid of revolution as a layering of washers, another way to imagine them is as a nesting of cylinders. Think of it like visualizing a solid of revolution like an shells that make an onion instead of the layers that make a cake.

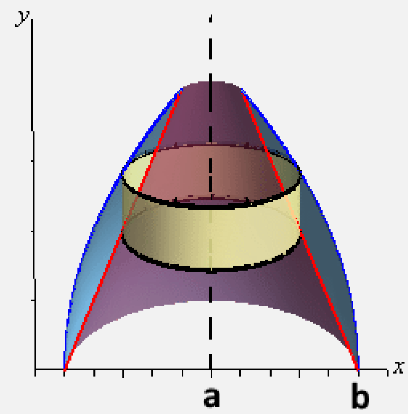


Note: Surface Area of Shells

The process of setting up shells to determine volume is similar in theory (but not in practice). Since the surface area of a shell of radius r and height h is $A = 2\pi rh$, then adding up all the shells that we layer from $a \leq x \leq b$ yields the volume

$$V = \int_a^b 2\pi rh dx$$

where we assume we are adding along the x-axis in this case.



Example 1: Consider the solid of revolution obtained by revolving the region bounded by $y = x^2$ and $y = x$ about the y-axis.

(a) Setup (but do not evaluate) the volume of the solid using Washers.

(b) Setup (but do not evaluate) the volume of the solid using Shells.

Example 2: Setup (but do not evaluate) the volume of the solid obtained by revolving the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = 6$.

Example 3: Determine the volume of the solid obtained by rotating the region bounded by $x = (y-2)^2$ and $y = x$ about the line $y = -1$. Why is using Washers in this circumstance a bad idea?

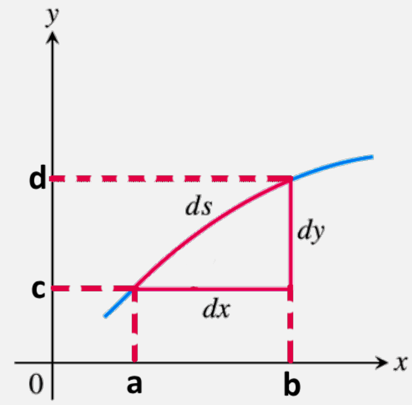
2.11 (Section 6.3) Arc-Length

Definition: Arc Length

Let $y = f(x)$ (or $x = g(y)$) be a differentiable function on the interval $a \leq x \leq b$ (resp. $c \leq y \leq d$). The **Arc-Length** of a curve is defined as

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx \ \left(\text{resp. } \int_c^d \sqrt{1 + (g'(y))^2}dy\right)$$

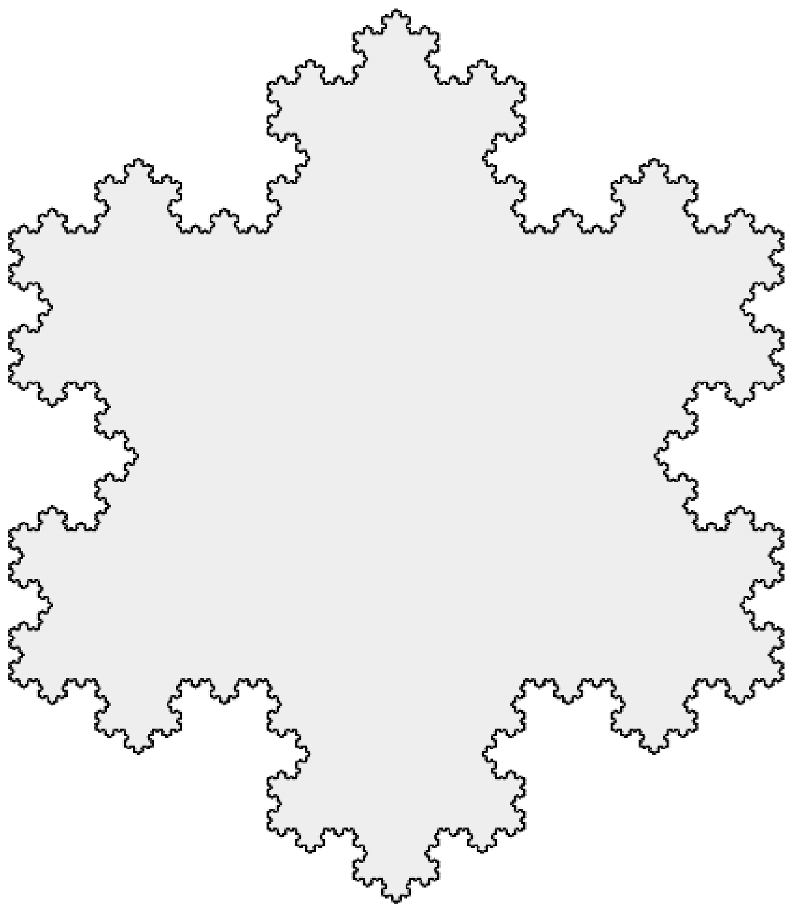
Note: Both definitions come from the diagram and using the fact that a tiny length of a segment of a curve is given by $\sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}dy.$



Example 1: Find the arc-length of $y = \dfrac{x^2}{2} - \dfrac{\ln(x)}{4}$ over $1 \leq x \leq 3$.

Chapter 3

Series



3.1 (Section 10.1) Introduction to Sequences

3.1.1 Definition and Convergence of Sequences

Definition: Sequence

A **Sequence** is an ordered list $a_1, a_2, a_3, ...$ and may be abbreviated as $\{a_i\}$ or $\{a_i\}_{i=1}^{\infty}$. If $\lim_{i\to\infty} a_i$ exists and is finite, we say that the sequence $\{a_i\}$ converges. Else, we say the sequence diverges.

Example 1: Write out the first five terms of the sequence $a_n = \dfrac{n+1}{n}$ where $n \geq 1$. Does this sequence converge?

Example 2: Determine the nature of convergence of the sequence $a_n = \cos(n\pi)$ where $n \geq 0$.

Example 3: Consider the sequence given **recursively** by $a_{n+1} = \sqrt{2 + a_n}$, for $n \geq 1$ with $a_1 = \sqrt{2}$.

(a) Write out the first four terms of the sequence.

(b) Given that the sequence a_n converges, what must it converge to?

3.1.2 Useful Results

Theorem: Useful Limits to Know

These are useful limits to memorize. As $n \to \infty$,

- $\ln(n)/n \to 0$
- $(\ln(n))^{1/n} \to 1$
- $a^n \to 0$ if $|a| < 1$
- $a^n/n! \to 0$
- $n^{1/n} \to 1$
- $a^{1/n} \to 1$ if $a > 0$
- $\left(1 + \dfrac{a}{n}\right)^n \to e^a$

Example 4: Consider $a_n = \left(\dfrac{n}{n+1}\right)^n$ for $n \geq 1$. Determine the limit as $n \to \infty$.

3.1.3 Monotonic Sequences

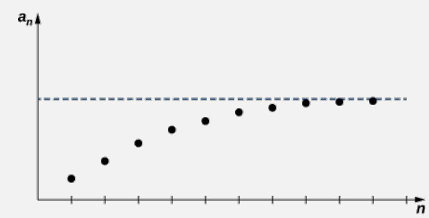
Definition: Monotonic Sequences

A sequence that is either strictly increasing or strictly decreasing is called **Monotonic**.

Definition: Bounded Sequences

A sequence is $\{a_n\}$ is said to be **Bounded Above** if $a_n \leq M$ for some M and all allowable n. A similar definition follows for **Bounded Below**.

If a sequence is both bounded above and below, we say that it is **Bounded**.



Example 5: Argue that the sequence $a_n = \dfrac{1}{n}$ for $n \geq 1$ is a sequence that is both monotonically decreasing and bounded below.

Theorem: Monotonic Convergence Theorem (MCT)

Every bounded monotonic sequence converges. More specifically, every monotonically increasing (resp. decreasing) sequence bounded above (resp. below) converges.

Example 7: Consider the recursive sequence $x_{n+1} = 2 - \dfrac{1}{x_n}$ where $n \geq 1$.

(a) Argue that the sequence is monotonically increasing. You may merely illustrate this fact by writing out the first few terms without a formal proof.

(b) Argue that the sequence is bounded above. Use this to conclude that the sequence converges.

(c) Find the limit $\lim\limits_{n\to\infty} x_n$.

3.2 (Section 10.2) Infinite Series

3.2.1 Defining an Infinite Sum

Definition: Series

A **Series** is an infinite sum. It is defined as the limit of a sequence. Specifically, given a sequence $\{a_k\}$ as above we define the sequence of **Partial Sums** $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$.

We say that the series $\sum_{k=1}^{\infty} a_k$ converges to L if $\lim_{n\to\infty} S_n = L$. If the Sequence $\{S_n\}$ diverges we say the series diverges.

Example 1: Consider the series given by $\sum_{k=1}^{\infty} \frac{1}{k}$.

(a) Argue that the sequence $a_k = \frac{1}{k}$ (not series!) converges.

(b) Write out the first three terms of the partial sums.

(c) You are given the fact that the partial sums $S_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfy the inequality $S_n \geq \ln(n) + \frac{1}{n}$. Conclude the nature of convergence of the series.

(d) What does part (a) and (c) tell you? Is your "*intuition*" a safe thing to rely on?

Note: Harmonic Series

The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is called the **Harmonic Series**.

3.2.2 Telescoping Series

Definition: Telescoping Series

A **Telescoping Series** is a series of the form $\sum_{k=1}^{\infty}(a_{k-m} - a_{k-l})$.

Example 2: Determine whether the series $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ converges or diverges. If it converges, find the sum.

Note: To save on lecture time, $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ *by partial fractions.*

Example 3: Determine if the series $\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)$ converges or diverges. If it converges find it's sum.

3.2.3 Geometric Series

Definition: Geometric Series

A **Geometric Series** is a series of the form $\displaystyle\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$.

Theorem: Convergence of Geometric Series

The geometric series of the above form converges as $\displaystyle\lim_{n\to\infty} S_n = \frac{a}{1-r}$ provided $|r| < 1$. If $|r| \geq 1$ then the series diverges.

Example 4: Determine the nature of convergence of $\displaystyle\sum_{k=0}^{\infty} \left(\frac{e}{\pi}\right)^k$. Prove your claim. If it converges find its sum.

Example 5: Determine a closed fraction form of 1.11111111.....

3.2.4 The n-th Term Divergence Test

Note: Nature of Series

Determining the closed sum of a general series is nearly impossible. Just like the nature of improper integrals, we care mostly about whether it converges or not and instead leave it up to numerical methods to find approximations if we need them.

Theorem: n-th Term Divergence Test

If $\lim_{n\to\infty} a_n$ is non-zero or $\{a_n\}$ diverges (as a sequence) then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 6: Determine the nature of convergence of $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{2}\right)$.

Example 7: Determine the nature of convergence of $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{\pi}{n}\right)^{-n}$.

3.2.5 Common Student Misconceptions

Example 8: A friend is was tested on determining the nature of the series $\sum_{k=1}^{\infty} \frac{4k^2 - k^4}{10 + 2k^4}$. This friend determined that $\lim_{k \to \infty} \frac{4k^2 - k^4}{1 + 2k^4} = -\frac{1}{2} \neq 0$. Following this, they have written down $\sum_{k=1}^{\infty} \frac{4k^2 - k^4}{10 + 2k^4} = -\frac{1}{2}$ but earned a failing mark on their answer when they had their test returned. Explain why.

Example 9: The same friend was tested on determining the nature of the series $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$. They determined that $\lim_{n \to \infty} \frac{1}{n \ln(n)} = 0$ and concluded that the series converges by the n'th term test. However, they (rightfully) earned a failing mark on their answer when they had their test returned. Explain why.

3.2.6 Properties of Series

Theorem: Properties of Series

Let $\sum a_n$ and $\sum b_n$ be two convergent series and let k be a constant. Then...

- $\sum(a_n+b_n) = \sum a_n + \sum b_n$
- $\sum(a_n-b_n) = \sum a_n - \sum b_n$
- $\sum ka_n = k\sum a_n$

and consequently all the above expressions are convergent series.

Example 10: The friend from the previous page states that $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) = \sum_{k=1}^{\infty}\frac{1}{k} - \sum_{k=1}^{\infty}\frac{1}{k+1}$. Explain why they earned a zero.

Theorem

1. Every non-zero constant multiple of a divergent series is also divergent.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then both $\sum(a_n+b_n)$ and $\sum(a_n-b_n)$ diverge.

Example 11: Explain why the series $\displaystyle\sum_{k=1}^{\infty}\left(\left(\frac{2}{3}\right)^k - \frac{k}{k+1}\right)$ diverges.

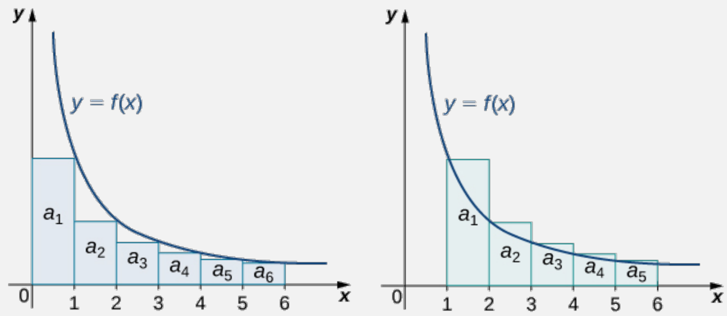
3.3 (Section 10.3) Integral Test

3.3.1 Constructing the Integral Test

Note: Relating a Series to the Area Under a Similar Curve

By considering the curve $f(k) = a_k$ going through points $(1, a_1)$, $(2, a_2)$, ... one may find a relationship between $\sum a_k$ and $\int f(x)dx$. Specifically,

$$\int_1^\infty f(x)dx \leq \sum_{k=1}^{\infty} a_k \leq a_1 + \int_1^\infty f(x)dx.$$



Example 2: Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$.

(a) Argue that the partial sums form a monotonically increasing sequence.

(b) Use the above inequality to derive an upper bound for the sequence of partial sums. Use this to argue the nature of the series.

Theorem: The Integral Test

Let $f(x)$ be continuous, positive, and decreasing on $[a, \infty)$. Then, $\displaystyle\int_a^\infty f(x)dx$ and $\displaystyle\sum_{n=a}^{\infty} f(n)$ either both converge or both diverge.

Note: p Series

Since $f(x) = 1/x^p$ satisfies the conditions of the integral test on $x \geq 1$, then the convergence results for the Type I p-integrals is the same as for the corresponding p-series. That is, $\displaystyle\sum \frac{1}{n^p}$ converges only if $p > 1$.

3.3.2 Integral Test Examples

Example 2: Determine the nature of convergence for $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$.

Example 3: Determine the nature of convergence for $\displaystyle\sum_{n=2}^{\infty} ne^{-n}$.

3.3.3 Another Common Student Misconception

Example 4: The same friend of the prior section states that $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} = \int_0^{\infty} 2^{-x} dx = \frac{1}{\ln(2)}$ and concluded that the series converges, and specifically to this value. They're shocked to get their test back and earned a failing grade on this question despite getting the nature correct (it converges) and the integral correct. Explain why they (rightfully) earned a failing grade.

3.3.4 Error Estimation of Approximating Series that Converge by Integral Test

Theorem: Error Inequality for Estimating Series Convergent by an Integral Test

Let $f(x)$ be a function satisfying the conditions of the integral test with convergent series. Then the error (difference between true value and approximation) using N terms satisfies

$$\int_{N+1}^{\infty} f(x)dx \leq R_N \leq \int_{N}^{\infty} f(x)dx.$$

Example 5: Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. How many terms do we use if we want an error less than 10^{-4}?

3.4 (Section 10.4) Comparison Tests

Note: Comparison Tests for Series

Since integration is merely a type of summation (and similar in form to a series), it's not surprising that the comparison tests of improper integrals translate to series.

3.4.1 The Direct Comparison Test for Series

Theorem: Direct Comparison Test (DCT)

Let $\sum a_n$ and $\sum b_n$ be series with $0 \leq a_n \leq b_n$ for all n. Then

- If $\sum b_n$ converges then $\sum a_n$ converges
- If $\sum a_n$ diverges then $\sum b_n$ diverges

Example 1: Determine and prove the nature of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3 + \pi n}$.

3.4.2 The Limit Comparison Test for Series

Theorem: Limit Comparison Test (LCT)

Suppose $a_n > 0$ and $b_n > 0$ and let $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. Then provided

- $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or diverge
- $L = 0$, then if $\sum b_n$ converges then $\sum a_n$ converges
- $L = \infty$, then if $\sum b_n$ diverges then $\sum a_n$ diverges

Example 2: Determine and prove the nature of convergence of the series $\sum_{n=2}^{\infty} \frac{1 + n\ln(n)}{n^2 + 5}$.

3.4.3 And, Another Common Student Error

Example 3: Your friend from a prior page wrote the following on their exam paper when tested on the nature of $\sum_{n=1}^{\infty} \frac{n}{1+n^3}$.

$$\text{“}\textit{Since } \sum_{n=1}^{\infty} \frac{n}{1+n^3} \approx \sum_{n=1}^{\infty} \frac{1}{n^2} \textit{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \textit{ converges then by LCT the series converges.”}$$

However, they earned a failing mark when the test was returned to them. They're confused as they "*referenced*" a test and got the nature "*correct*". Explain why they earned a failing mark.

3.5 (Section 10.5) Absolute Convergence, Ratio and Root Tests

3.5.1 The Absolute Convergence Test

Definition: Alternating Series

An **Alternating Series** is a series of the form $\sum (-1)^n a_n$ where $a_n \geq 0$.

Example 1: Approximate $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{2^n}$ using the first four terms.

Theorem: Absolute Convergence Test (ACT)

If $\sum |a_n|$ converges then $\sum a_n$ converges.

Example 2: Determine the nature of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$.

Definition: Absolute Convergence

If $\sum |a_n|$ is convergent we say $\sum a_n$ is **Absolutely Convergent**.

3.5.2 One More Student Error

Example 3: Back to your friend. They were asked about determining the nature of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. They concluded that since $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges then the original series must diverge. Explain why they failed the question.

3.5.3 The Ratio Test

Theorem: The Ratio Test

Let $\sum a_n$ be any series and suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$
Then if...

- $L < 1$ the series converges absolutely.
- $L > 1$ the series diverges.
- $L = 1$ then the test is inconclusive. You must apply a different test as this one does not work.

Note: Series that Work Well with Ratio Test

This test is useful for **FACTORIALS** (especially!!!), polynomials, and simple exponents.

Example 4: Determine the nature of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{3^n(n+1)}{n!}$.

Example 5: Determine the nature of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$.

3.5.4 The Root Test

Theorem: The Root Test

Let $\sum a_n$ be any series and suppose $\lim_{n\to\infty} |a_n|^{1/n} = L$.
then if...

- $L < 1$ then the series converges absolutely.
- $L > 1$ then the series diverges.
- $L = 1$ then the test is inconclusive. You must use a different test as this one does not work.

Example 6: Determine the nature of convergence of $\displaystyle\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$.

Example 7: Determine the nature of convergence of $\displaystyle\sum_{n=3}^{\infty} \frac{2^{n^2}}{n^2}$.

3.6 (Section 10.6) Alternating Series Test

3.6.1 Conditional Convergence and the Alternating Series Test

Theorem: The Alternating Series Test

The series $\sum(-1)^{n+1}a_n$ converges if all the following are satisfied:

- All $a_n \geq 0$;
- All terms in $\{a_n\}$ are eventually all non-increasing;
- $\lim_{n\to\infty} a_n = 0$.

Example 1: Determine the nature of convergence of the alternating Harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Justify your claim with a proper proof.

Definition: Conditional Convergence

If a series $\sum b_n$ converges but $\sum |b_n|$ diverges then series $\sum b_n$ is called conditionally convergent.

Example 2: Argue that the alternating series of the previous example is conditionally convergent.

Example 3: Determine the nature of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ with proof. Specify whether it is absolutely convergent, conditionally convergent, or divergent.

Example 4: Determine the nature of convergence of $\sum_{n=5}^{\infty} (-1)^{n-3} \frac{n}{n^2-4}$. Is it conditionally or absolutely convergent?

3.6.2 Error Estimation of Series that Converge by the AST

Theorem: Error Estimation on Convergent Series by AST

Let $\sum(-1)^{n+1}a_n$ be an alternating series that converges by the AST. Let $S = \sum(-1)^{n+1}a_n$, then S always lies between two successive sums, i.e.

$$S_N \le S \le S_{N+1} \quad \text{or} \quad S_{N+1} \le S \le S_N.$$

Furthermore, let $R_N = |S - S_N|$ be the error using N terms. Then,

$$R_N = |S - S_N| \le a_{N+1}.$$

Example 5: Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

(a) Determine the number of terms required to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ to within four decimal accuracy, i.e. $|S - S_N| \le 10^{-4}$.

(b) Determine the number of terms required to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ to within four decimal accuracy.

(c) What do you notice about the difference between conditionally convergent and absolutely convergent series?

3.7 (Section 10.7) Power Series

3.7.1 Power Series Functions

Definition: Power Series

A **Power Series centered about** $x = a$ is a function of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where all c_n are constant.

Note: Power Series are a New Function

Power series are a function to add to your list of known functions. Just like polynomials, trigonometric functions, logarithms, etc, you plug in a number and (provided it converges) you get a number out. They can be graphed, differentiated, etc.

Example 1: Consider the function $f(x) = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k x^k$.

(a) What are the coefficients c_k? Where is the function (power series) centered about?

(b) Write out the first three terms of $f(x)$.

(b) Evaluate $f(0)$ and $f(1)$. What is the issue with evaluating $f(4)$?

3.7.2 The Domain of a Power Series Function (Interval of Convergence)

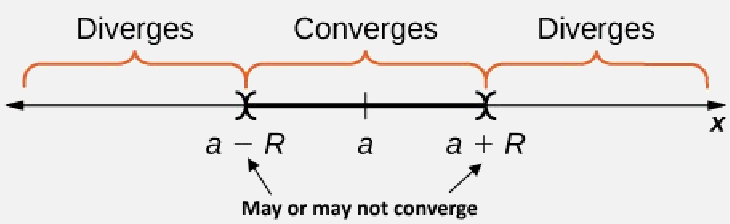
Theorem: Interval of Convergence

The domain of a power series function is always an interval. Furthermore, if $f(x)$ is a power series centered about $x = a$ then a is the midpoint of the interval, i.e. the interior of the interval is given by the inequality $|x-a| < R$. We call the domain the **Interval of Convergence** and we call R the **Radius of Convergence**.

Procedure: Determining the Domain

You determine the domain by forming the L in the ratio test or root test.

- When $L < 1$ the series converges absolutely and will yield an interval of the form $|x - a| < R$ (unless $L = 0$ in which case the interval is all real numbers);
- When $L > 1$ the series will diverge and will yield a region of the form $|x - a| > R$; and
- When $L = 1$ the test is inconclusive and will yield the endpoints of the interval $|x - a| = R$. You will need to apply a different test to determine their convergence.



Example 2: Determine the interval of validity (domain) of the power series function $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Example 3: Determine the interval of validity (domain) of the power series function $f(x) = \sum_{k=1}^{\infty} \frac{(x-3)^k}{k \cdot 2^k}$.

3.7.3 Algebraic Operations of Power Series

Theorem: Addition and Scalar Multiplication

Let $\sum b_n(x-a)^n$ and $\sum c_n(x-a)^n$ be power series functions that converge for $|x-a| < R$ and start their sum at the same value of n. Let A be a constant. Then,

- $\sum b_n(x-a)^n + \sum c_n(x-a)^n = \sum(b_n + c_n)(x-a)^n$; and
- $\sum Ab_n(x-a)^n = A\sum b_n(x-a)^n$

where each series formed converges over $|x-a| < R$ as well.

Theorem: Substitution

If $f(x) = \sum c_n(x-a)^n$ converges for $|x-a| < R$ and $g(x)$ is continuous then $f(g(x)) = \sum c_n(g(x)-a)^n$ converges for all values x such that $|g(x)-a| < R$.

Example 4: You are given the fact that $f(x) = \sum_{k=0}^{\infty} x^k$ converges for $|x| < 1$. Use this to determine where the power series $\sum_{k=0}^{\infty} \frac{2^k(x-3)^k}{5^k}$ converges.

3.7.4 Calculus of Power Series

Theorem: Differentiation of Power Series

If $f(x)$ is a power series, it is differentiable on the interior of its domain. Specifically, if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ and converges on $|x-a| < R$ then $f'(x) = \sum_{n=1}^{\infty} c_n \cdot n(x-a)^{n-1}$ and converges on $|x-a| < R$. That is, you use the power rule. It may or may not converge at the endpoints.

Example 5: Let $f(x) = \sum_{k=0}^{\infty} \frac{k+1}{(2k)!}(x-2)^k$.

(a) Write out the first 3 terms of $f(x)$.

(b) Compute $f'(x)$ and write out the first 3 terms of $f'(x)$.

Theorem: Integration of Power Series

If $f(x)$ is a power series, it is integrable on its domain. Specifically, if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ and converges on $|x-a| < R$ then $\int f(x)dx = \left(\sum_{n=0}^{\infty} c_n \cdot \frac{1}{n+1}(x-a)^{n+1} \right) + C$ and converges on $|x-a| < R$. That is, you use the power rule. It may or may not converge at the endpoints.

Example 6: Let $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} x^{2k}$. Evaluate $\int f(x)dx$ and write out the first 3 terms of $\int f(x)dx$.

3.7.5 Representing Functions as Power Series

Note: Representation as Power Series

Some functions can be expressed in the form of a Power Series. This isn't unusual. We have expressed functions in terms of other functions in the past. For example, in an earlier section it is mentioned that one description of $\ln(x)$ is as $\ln(x) = \displaystyle\int_1^x \frac{1}{t}dt$.

Example 7: Remember that from Geometric series that $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ provided that $|r| < 1$.

(a) Use the above result to find a Power Series that is equivalent to the function $f(x) = \dfrac{1}{1-x}$. Over what domain are they equal?

(b) Use part (a) to construct a power series function that is equivalent to the function $\dfrac{1}{4+x^2}$. Over what domain are they equal?

(c) Use part (a) to construct a power series function that is equivalent to the function $\dfrac{x}{(1-x)^2}$. *Hint: Differentiation.* Over what domain are they equal?

3.8 (Section 10.8) Taylor and MacLaurin Series

3.8.1 Taylor Series

Definition: Taylor Series Generated by a Function

Let $f(x)$ be a function with derivatives of all orders about an interval containing $x = a$. Then the **Taylor Series** about $x = a$ of $f(x)$ is the power series function

$$P(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

where $c_n = \dfrac{f^{(n)}(a)}{n!}$. Here $f(x)$ is called the **Generating Function** of $P(x)$. If $a = 0$ it is called a **Maclaurin Series**.

Example 1: Determine the Taylor Series generated by $f(x) = e^x$ about the point $x = 0$. Leave your answer in sigma form.

Theorem: Equivalence of Taylor Series with Generating Function

If a function can be represented as a power series, then it is equal to its Taylor Series over an appropriate interval.

Note: Ability to Represent Functions as Taylor Series

The prior theorem presupposes that a function <u>can</u> be represented as a power series. Is this a reasonable assumption? For many function, the answer is yes. But for many functions, it is not. For example, the function $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$ cannot be represented as a Power series. It is not equal to its generating Taylor series about the origin.

Example 2: Determine the Taylor Series generated by $f(x) = \sin(x)$ about the point $x = \pi/2$. Leave your answer in sigma form.

3.8.2 Taylor Polynomials of Order N

Definition

The **Taylor Polynomial of order** N to $f(x)$ at $x = a$ is

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

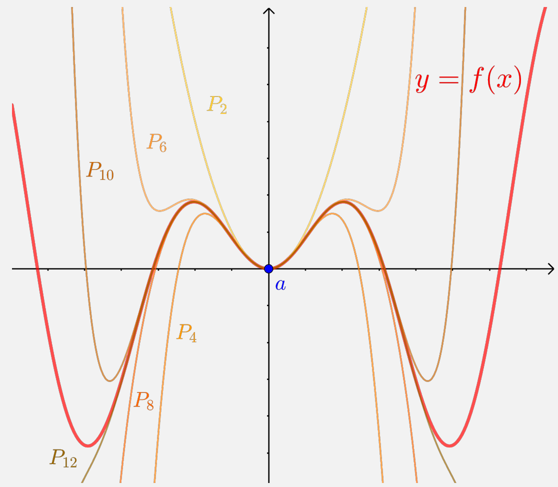
Example 3: Determine the second order Taylor series polynomial of $\tan(x)$ centered about $a = \pi/4$. Graph the two functions in Geogebra. How good of an approximation is this function? What happens if you use more terms?

3.9 (Section 10.9) Convergence of Taylor Series

3.9.1 Taylor's Theorem and Approximating Functions

Note: Representing Functions as Power Series

In this section we work towards answering the question *" When is a function equal to its Taylor Series"*. That is, *"When can we represent a function as a power series?"* We saw that by an example of the prior section, the more terms we use the better an approximation we obtain. This illustrates that for some functions, there's a relationship between the function itself and the partial sums of the Taylor Series $P(x)$.



Theorem: Taylor's Theorem

Suppose that $f(x)$ has $N+1$ continuous derivatives on an open interval containing $x=a$. Then for every x in the interval

$$f(x) = P_N(x) + R_{N+1}(x)$$

where $P_N(x)$ is the N'th order Taylor polynomial generated by $f(x)$ and $R_{N+1}(x)$ is the error term at each x using $P_N(x)$ to approximate $f(x)$. We infer that $R_{N+1}(x)$ satisfies

$$R_{N+1}(x) = \frac{f^{N+1}(c)}{(N+1)!}(x-a)^{N+1}$$

for some constant c between a and x.

Example 1: Determine the form of the remainder function for the function $f(x) = e^x$ when forming an N'th order Taylor approximation.

Example 2: Using the fact that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for every value of x, prove that e^x is equal to its Taylor Series.

Theorem: Equality of $f(x)$ and its Taylor Series

Let $f(x)$ be an continuously infinitely differentiable function and let $R_{N+1}(x)$ be as described by Taylor's Theorem. If $R_{N+1}(x) \to 0$ as $N \to \infty$ for every x then $f(x) = P(x)$ over a suitable interval. If a function is equal to its Taylor Series we say that it is **Analytic**.

Example 3: Consider $\sin(x)$. Use the fact that the derivatives of $\sin(x)$ are all bounded to prove that $\sin(x)$ is equal to its Taylor Series.

3.9.2 Important Analytic Functions and Generating Others Using a List

Theorem: Important Taylor Series Expression

The following functions are analytic with their corresponding Maclaurin series and interval where they are equal.

Function	**Taylor Series**	**Interval**
e^x	$\sum_{k=0}^{\infty} \frac{x^n}{n!}$	$\mathbb{R}$
$\cos(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$	$\mathbb{R}$
$\sin(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$	$\mathbb{R}$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$	$(-1, 1)$

Note: Constructing Other Taylor Series

The above list is a good list to use as a foundation for generating most everything else you need. Thus, we memorize this list and use techniques to generate other function we require.

Example 4: Construct the Maclaurin Series for the function $f(x) = x^3 e^{-x^2}$.

Example 5: Construct the Maclaurin Series for the function $f(x) = \arctan(x)$.
Hint: $\arctan(x) + C = \displaystyle\int \frac{1}{1+x^2} dx$.

Example 6: Find $P_3(x)$ of $f(x) = e^x \cos(x)$.

3.10 (Section 10.10) Binomial Series and Applications of Taylor Series

3.10.1 Binomial Series

Theorem: Binomial Series

The function $f(x) = (1+x)^m$ is analytic on the interval $|x| < 1$ with corresponding Maclaurin series

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$$

where $\binom{m}{1} = m$, $\binom{m}{2} = \dfrac{m(m-1)}{2}$, and $\binom{m}{k} = \dfrac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$.

Example 1: Consider the function $f(x) = \sqrt{1+x}$.

(a) Determine the first five non-zero terms of the $f(x)$.

(b) Use your result from part (a) to determine an approximation for $\sqrt{5}$.

3.10.2 Applications of Taylor Series: Evaluating Integrals

Note: Existence of an Antiderivative

Not every function has an (elementary) antiderivative. For example, you will never be able to express $\int e^{-x^2}dx$ in terms of elementary functions. However, with the use of Taylor series, an approximation becomes quite simple.

Example 2: Approximate $\displaystyle\int_0^{1/2} e^{-x^2}dx$ using the first four non-zero terms of a Taylor series approximation.

Example 3: Approximate $\displaystyle\int_0^1 x\cos(x^3)dx$ to within 3 decimal accuracy. *Hint: Alternating Series Estimation. Pick N until $a_{N+1} \leq 10^{-3}$.*

3.10.3 Applications of Taylor Series: Constructing Formulas

Note: Imaginary Numbers

The **imaginary unit** is defined as $i = \sqrt{-1}$. It can't be measured on a ruler in the traditional sense, but we can still use it structurally in equations.

Example 4: Prove Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

3.10.4 Applications of Taylor Series: Approximating Values

Note: Expressing Values as Series

We had previously approximated $\sqrt{5}$ using Taylor series in a prior example. Provided that $f(x) = P(x)$ we may use partial sums of $P(x)$ to approximate the values of $f(x)$.

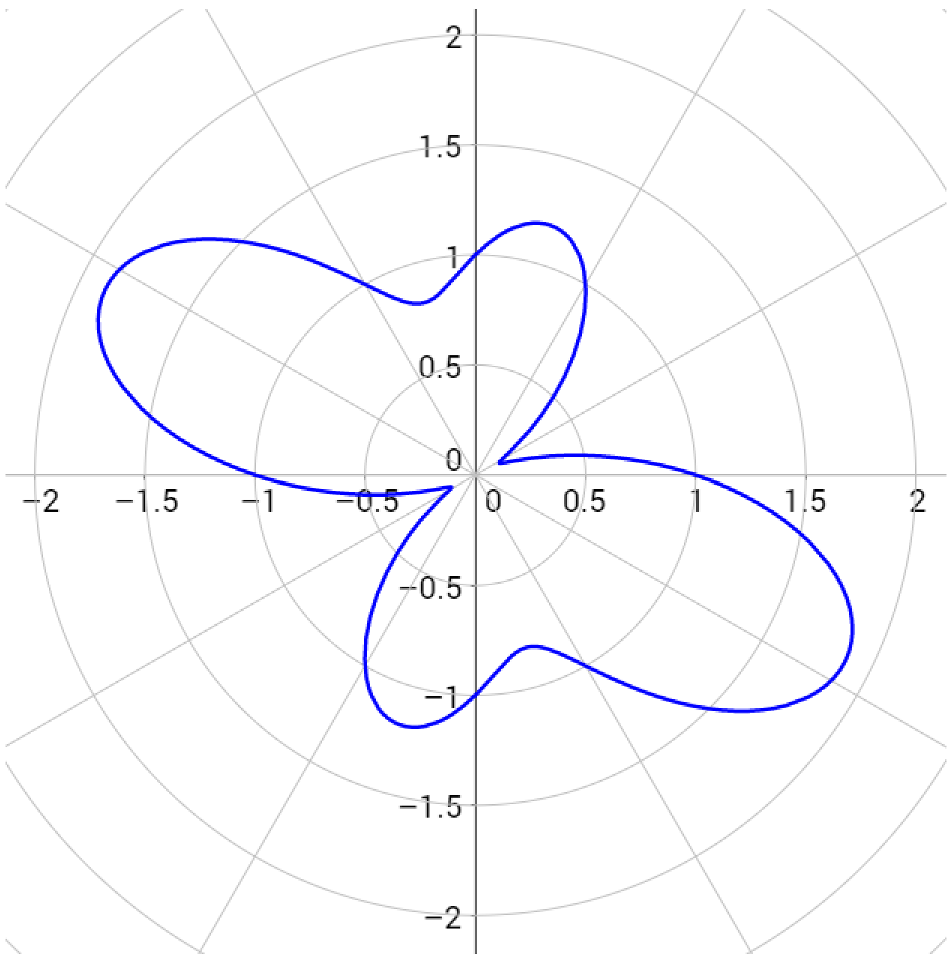
Example 5: Consider the Maclaurin series of $\arctan(x)$ obtained in a prior section, $\arctan(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$. Prove that the series converges at $x = 1$ and use this to form an approximation of π.

3.10.5 Applications of Taylor Series: Evaluating Limits

Example 6: Evaluate $\lim_{x \to 0} \dfrac{e^x + \cos(x) - 2 - x - x^2}{\sin(x) - x}$.

Chapter 4

Coordinate Systems



4.1 (Appendix A7) Complex Numbers

4.1.1 Introducing Complex Numbers

Note: Extending the Reals

For many mathematical purposes, the real numbers are too restrictive. Indeed, in the past we have rejected solutions of polynomial equations merely because we desired the solution to look a certain way. While this might make sense for practical purposes, there are mathematical advantages to developing a language beyond the reals.

Definition: Imaginary Unit

The **Imaginary Unit** is the object i and defined by $i = \sqrt{-1}$. It satisfies the property that $i^2 = -1$.

Example 1: Solve the equation $z^2 + 4z + 5 = 0$ for all values of z.

Definition: Complex Numbers

The **Complex Numbers** (abbreviated $\mathbb{C}$) consist of all expressions of the form $a + bi$ where a and b are real numbers. They are an **algebraic extension** of $\mathbb{R}$ in that every algebraic real equation has a solution given by something in $\mathbb{C}$.

4.1.2 Conjugate and Modulus

Definition: Conjugate

The **Conjugate** of a complex number $z = a + bi$ is given by $\bar{z} = a - bi$.

Definition: Modulus

The **Modulus** of a complex number $z = a + bi$ is given by $|z| = \sqrt{a^2 + b^2}$.

Example 2: Compute $|3 - 4i|$ and $\overline{2 + 5i}$.

4.1.3 Algebraic Operations of Complex Numbers

Note: Algebraic Operations

The goal of simplifying any expression involving complex numbers is to write them in the form $a + bi$ in the end. Usually you need to use properties such as: multiplying by a conjugate, grouping like terms, using $i^2 = -1$, etc. We give an example of each.

Example 2 (Addition): Simplify $(3 + 4i) - (-1 + 2i)$.

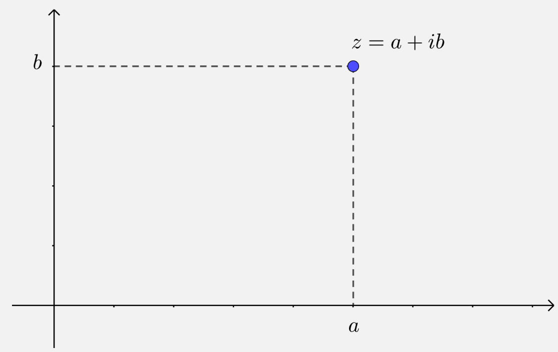
Example 3 (Multiplication): Simplify $(2 + i)(1 + 3i)$.

Example 4 (Division): Simplify $\dfrac{2 + 3i}{1 - 7i}$.
Hint: Multiply by the conjugate.

4.1.4 Argand Diagrams and Polar Coordinates

Note: Graphing Complex Numbers

Graphing something like $x = 5$ on the number line is introduced in elementary school. You just circle the spot 5 on the number line. An expression like $5 + 2i$ is more complicated in that there are two components to measure, the 5 and the 2. Thus we require two rulers.



Definition: Aargand Diagram

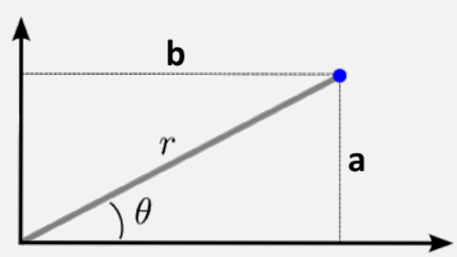
Let $z = a + bi$. Plotting it as the point (a, b) is called an **Aargand Diagram**.

Example 5: Plot the complex number $z = 1 + i$. Determine the distance of z from the origin (r) and the angle it makes with the horizontal axis (θ).

Note: Relationship Between a, b, r and θ

One may see that a relationship between r, θ, a and b given by $\sin(\theta) = \frac{a}{r}$ and $\cos(\theta) = \frac{b}{r}$. This gives us an expression for a and b entirely in terms of r and θ,

$$\begin{cases} a = r\cos(\theta) \\ b = r\sin(\theta) \end{cases}$$



Definition: Polar Form

Since $a = r\cos(\theta)$ and $b = r\sin(\theta)$ we call $z = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta))$ the **Polar Representation** of z.

4.1.5 Exponential Representations of Complex Numbers

Theorem: Euler's Formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Definition: Exponential Form

Since the polar form of $z = a + bi$ is $z = r(\cos(\theta) + i\sin(\theta))$ we have by Euler's formula $z = re^{i\theta}$. This is called **Exponential Form**.

Note: Exponential Properties

The exponential form of a complex number is highly desirable as it allows us to use exponential properties.

Example 6: Consider the numbers $z_1 = 2e^{5\pi i/4}$ and $z_2 = 1 - \sqrt{3}i$.

(a) Express the number z_2 in exponential form.

(b) Compute the product $z_1 z_2$. Leave your answer in polar form.

(c) Compute the division z_1/z_2. Leave your answer in polar form.

Theorem: De Moivre's Formula

By exponential properties $(e^{i\theta})^n = e^{in\theta}$, thus

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$

4.1.6 Radicals of Complex Numbers

Note: Using Exponential Form to Compute Radicals

While one may state something like $(1+i)^{1/2}$, it does little to help understand it. For example, what is the Aargand plot of it? What angle does it make with the horizontal axis? However, if one uses the exponential form $1+i = \sqrt{2}e^{\pi i/4}$ then one sees that

$$(1+i)^{1/2} = (2^{1/2}e^{\pi i}4)^{1/2} = 2^{1/4}e^{\pi i/8}$$

which we can plot. We make one more adjustment in that radicals of complex numbers are **multi-valued expressions** meaning that there is more than one answer to simplifying $(1+i)^{1/2}$. We illustrate this by example.

Example 7: Find all fourth roots of -16. That is, solve $z^4 = -16$ for all complex values of z.

4.2 (Section 11.1) Parametric Equations

4.2.1 Defining Parametric Curves

Definition

If C is a curve and (x, y) is any point on C then provided there exists functions such that

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

we call it a **Parametric Representation** of C. The direction along the curve that corresponds to increasing t is called the **Orientation** of the curve.

Example 1: Consider the curve given by the parametric equations $x(t) = t^2$, $y(t) = \sqrt{t}$. Determine the point on the curve corresponding to $t = 4$.

Example 2: Determine the curve described by the parametric equations $x = t^2 + t$, $y = 2t - 1$. Sketch the curve and determine the orientation of the curve.

Note: Graphing Using Software

Using GeoGebra Classic, you can graph parametric curves given by $x = f(t)$, $y = g(t)$ for $a \leq t \leq b$ using the command `Curve(f(t),g(t),t,a,b)`.

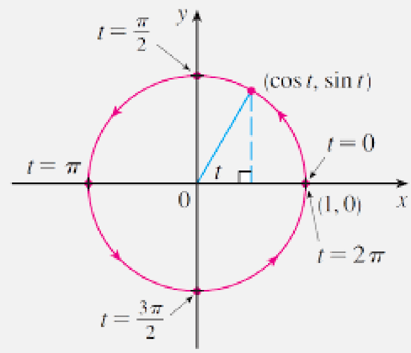
4.2.2 The Standard Representation of a Circle

Definition: Parametric Equations of a Circle

The **Standard Representation** of a circle of radius R is given by

$$\begin{cases} x(t) = R\cos(t) \\ y(t) = R\sin(t) \end{cases}$$

where $0 \leq t \leq 2\pi$. It traces out the circle once in a counter-clockwise orientation. Here, t measures the angle from the positive x-axis.



Example 3: Demonstrate that the standard representation of a circle satisfies the Cartesian equation of a circle $x^2 + y^2 = R^2$.

Example 4: Construct the parametric equations of a unit circle oriented counter-clockwise that traces out the circle twice within $0 \leq t \leq 2\pi$.

Example 5: Construct the parametric equations of a circle of radius 5 that is oriented counter-clockwise and centered about the point $(2, 3)$.

4.3 (Section 11.2) Calculus of a Parameterized Curve

4.3.1 The Slope and Concavity of a Parametric Curve

Note: Notation for Differentiation

We will use the following notation that Newton used: $\dot{u} = du/dt$ and $u' = du/dx$

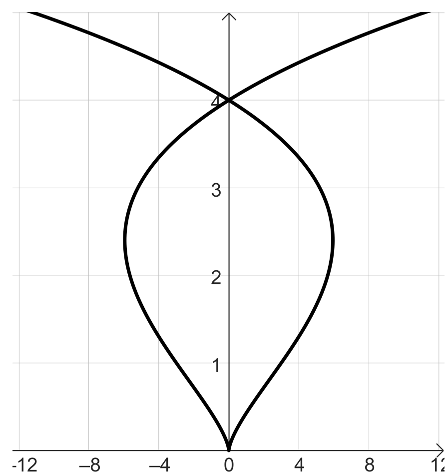
Theorem: Slope of a Parametric Curve

Let $x(t)$ and $y(t)$ be differentiable functions, then the slope of the curve corresponding to the parametrization at $(x(t), y(t))$ is given by $y' = \dot{y}/\dot{x}$.

Example 1: Find the slope(s) of

$$x = t^5 - 4t^3; y = t^2$$

at the point $(0, 4)$.



Theorem: Concavity of a Parametric Curve

Let $x(t)$ and $y(t)$ be twice differentiable functions, then the concavity of the curve corresponding to the parametrization at $(x(t), y(t))$ is given by

$$y'' = \frac{\frac{d}{dt}[y']}{\dot{x}}$$

Example 2: Consider $x(t) = 1 - t^2$, $y(t) = t^7 + t^5$. Compute y''.

4.3.2 Vertical and Horizontal Tangency of a Parametric Curve

Theorem: Vertical and Horizontal Slopes

Let $x(t)$ and $y(t)$ be differentiable functions.

- All points of horizontal tangency correspond to $\dot{y} = 0$ (provided $\dot{x} \neq 0$); and
- All points of vertical tangency correspond to $\dot{x} = 0$ (provided $\dot{y} \neq 0$).

Example 3: Determine the (x, y) coordinates of the points where the following parametric equations

$$x(t) = t^3 - 3t; y = 3t^2 - 9$$

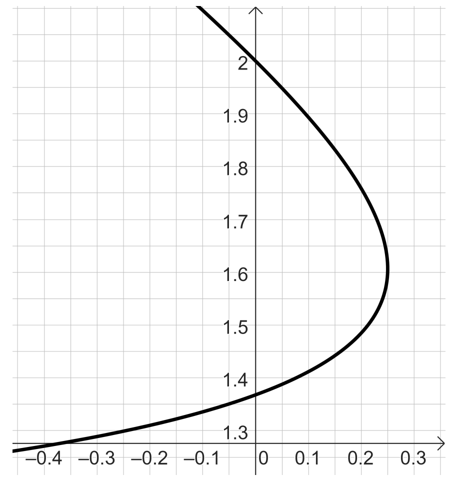
will have horizontal or vertical tangents. Compare your answer to a graph in Geogebra.

4.3.3 Integration of a Parametric Curve

Definition: Area "Under" a Parametric Curve

- The signed area between $y(x)$ and the x-axis over $a \leq t \leq b$ is $\displaystyle\int_{t=a}^{t=b} y dx = \int_{t=a}^{t=b} y(t)x'(t)dt$.
- The signed area between $x(y)$ and the y-axis over $c \leq t \leq d$ is $\displaystyle\int_{t=c}^{t=d} x dy = \int_{t=c}^{t=d} x(t)y'(t)dt$.

Example 4: Find the area enclosed by the y-axis and the curve $x = t - t^2$, $y = 1 + e^{-t}$.



4.3.4 Arc-Length of a Parametric Curve

Definition: Arc-Length of a Parametric Curve

Let $x(t)$ and $y(t)$ be continuously differentiable. The arc-length of a parametrized curve $x(t)$, $y(t)$ over $a \leq t \leq b$ is

$$L = \int_{t=a}^{t=b} \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

Example 5: Compute the arc-length of the curve $x = \ln(\sec(t) + \tan(t)) - \sin(t)$, $y = \cos(t)$ over $0 \leq t \leq \pi/3$.

4.4 (Section 11.3) Polar Coordinates

4.4.1 Revisiting Polar Coordinates and Non-Uniqueness

Note: Revisiting Polar Coordinates

We've already discussed polar coordinates when introducing *Polar Form* under the section on Complex Numbers. Specifically the point $a + ib$, or in other words $(x, y) = (a, b)$ is related to the angle from the positive x axis θ and the distance from the origin (radius) r by

$$\begin{cases} a = r\cos(\theta) \\ b = r\sin(\theta) \end{cases}$$

Example 1: Express the Polar point $(r; \theta) = (2, \pi/6)$ in Cartesian coordinates.

Example 2: Express the Cartesian point $(x, y) = (1, -\sqrt{3})$ in Polar coordinates. What is different with trying to represent objects in polar?

4.4.2 Polar Curves

Note: Polar Curves of the Form $r = f(\theta)$

Think about how Cartesian functions of the form $y = f(x)$ work. You plug in a value x to return a value y giving you the point (x, y) on the curve. Similarly, if you have a **Polar Function** of the form $r = f(\theta)$ you plug in an angle θ to return a value r giving you a Polar point (r, θ). The collection of all points is the curve.

Example 3: What is the point on the curve $r = 1 + \cos(\theta)$ corresponding to $\theta = \pi$?

4.4.3 Converting Cartesian Equations to Polar Equations

Procedure: Converting to Polar

Converting a Cartesian equation to a Polar one is quite simple. Simply use the fact that $x = r\cos(\theta)$, $y = r\sin(\theta)$ and simplify.

Example 4: Find the polar curve equation of $(x^2 + y^2 + x)^2 = x^2 + y^2$.

4.4.4 Converting Cartesian Equations to Polar Equations

Procedure: Converting to Cartesian

Converting back to Cartesian is trickier. The trick (usually) is to construct terms of the form $r\cos(\theta)$ and $r\sin(\theta)$, sometimes even introducing terms to make it so, then converting back to x and y. Similarly, constructing terms of the form r^2 to convert using $r^2 = x^2 + y^2$.

Example 5: Convert the equation $r = -8\cos(\theta)$ to Cartesian.

Example 6: Convert the equation $r = \dfrac{4}{2\cos(\theta) - \sin(\theta)}$ to Cartesian.

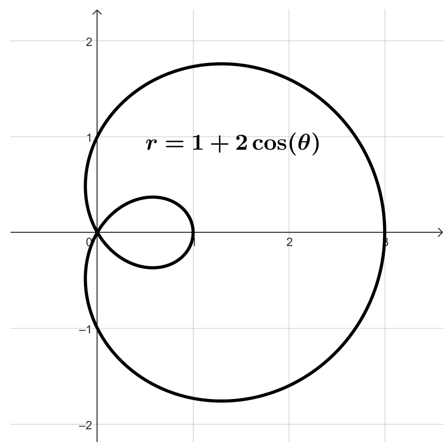
4.5 (Section 11.4) Graphing Polar Curves in the Cartesian Plane

4.5.1 Symmetry of Polar Curves

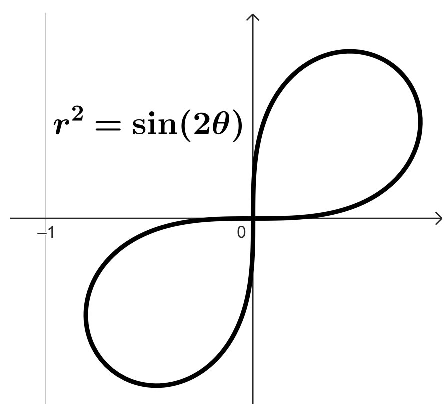
Theorem: Symmetry Tests of Polar Curves

- **Symmetry about the x-axis**: If the point (r, θ) lies on the graph then either $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph.
- **Symmetry about the y-axis:** If the point (r, θ) lies on the graph then either $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph.
- **Symmetry about the origin:** If the point (r, θ) lies on the graph then either $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph.

Example 1: Demonstrate that the function $r = 1 + 2\cos(\theta)$ is symmetric about the x-axis.



Example 2: Demonstrate that the function $r^2 = \sin(2\theta)$ is not symmetric about the y-axis by considering the polar point $(r; \theta) = (1, \pi/4)$.



4.5.2 Graphing Polar Curves Using GeoGebra

GeoGebra Polar Graphing

To graph the curve $r = f(\theta)$ over the region $a \leq \theta \leq b$ use the command

```
Curve(f(t);t,t,a,b)
```

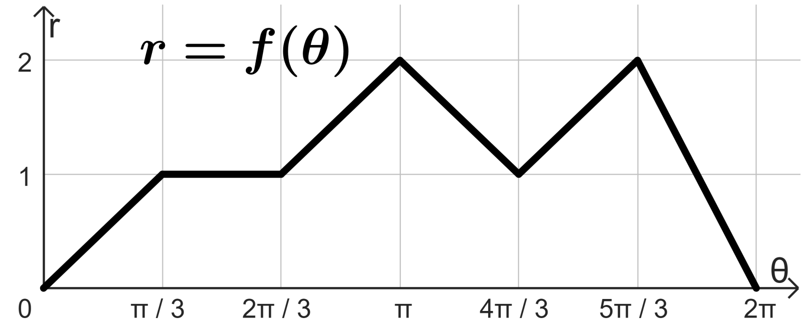
Note that the semicolon is important!

4.5.3 Plotting Curves in the xy-Plane Based on Their $r\theta$-Graph

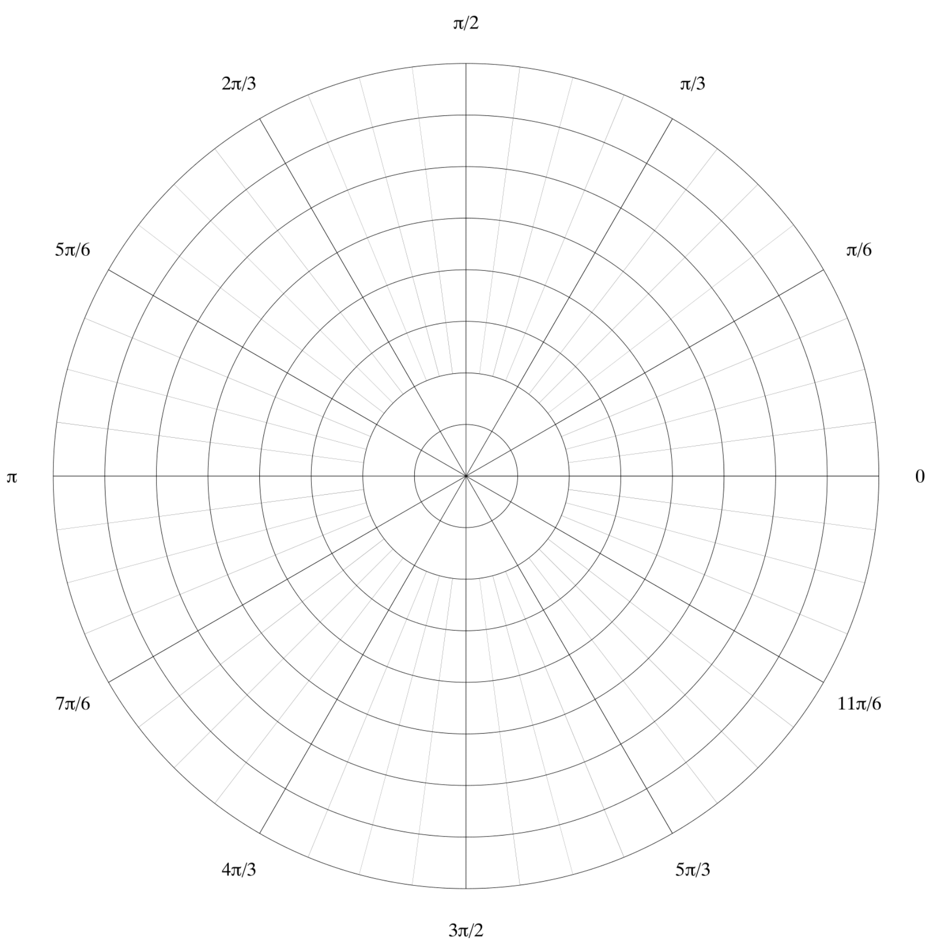
Procedure: Graphing Polar Curves

When graphing functions of the form $r = f(\theta)$ in the Cartesian plane, the best way is to follow along by graphing it initially in the $r\theta$-plane and then the xy-plane. This is best seen by example.

Example 3: Let $r = f(\theta)$ be a curve with graph in the $r\theta$-plane given by the following sketch.



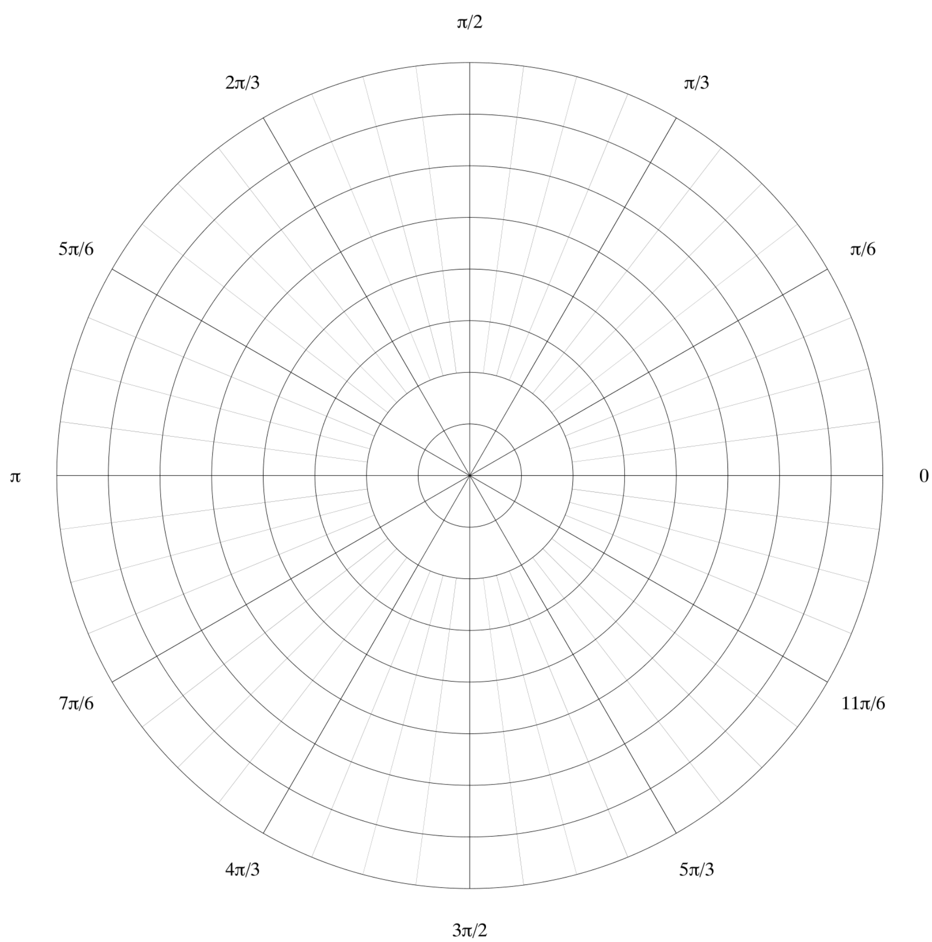
Use this to sketch the curve in the xy-plane.



Example 4: Consider the polar curve $r = 1 - \cos(\theta)$.

(a) Graph the curve in the $r\theta$-plane.

(b) Use the sketch in part (a) to graph the curve in the xy-plane.



Example 5: Graph $r = 1 + 2\sin(\theta)$ in the Cartesian plane. Start by sketching the $r\theta$-graph.

4.5.4 Calculus in Polar Coordinates

Note: Slope of a Polar Curve and Transforming Polar to Parametric

To find the slope of a polar curve of the form $r = f(\theta)$ simply convert from polar to parametric using

$$\begin{cases} x(\theta) = r\cos(\theta) = f(\theta)\cos(\theta) \\ y(\theta) = r\sin(\theta) = f(\theta)\sin(\theta) \end{cases}$$

and use parametric differentiation.

Example 6: Compute the slope of the polar curve $r = 1 + 2\cos(\theta)$ at the point $\theta = \pi/4$.

Theorem: Arc-Length of Polar Curve

Let $r = f(\theta)$ be a differentiable curve. Then the arc-length of the curve over $\alpha \leq \theta \leq \beta$ is given by $L = \displaystyle\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

Example 7: Compute the arc-length of $r = 1 - \cos(\theta)$ for $0 \leq \theta \leq 2\pi$.

4.6 (Section 11.5) Area Trapped by Polar Curves

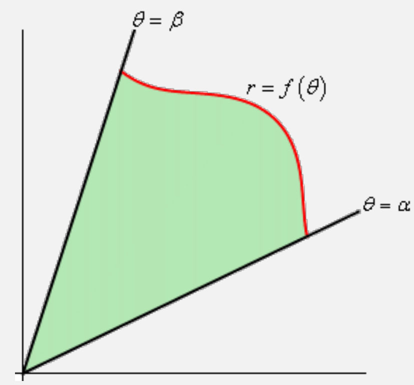
4.6.1 Area "Swept" by Polar Curves

Theorem: Area in Polar Coordinates

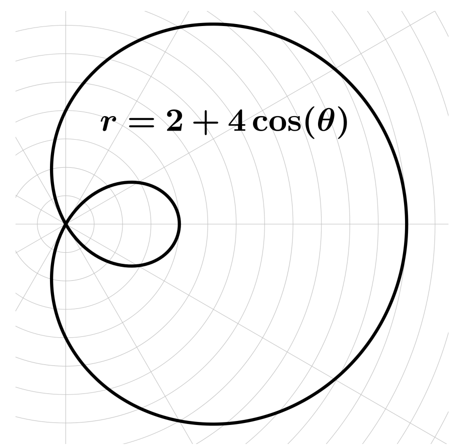
The area between the polar curve $r = f(\theta)$ and the origin as a wedge over $\alpha \leq \theta \leq \beta$ is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta.$$

Note: This originates from the fact that we are adding up wedges instead of rectangles to get the area. The area of a wedge of angle θ and side lengths r is given by $\frac{1}{2}r^2\theta$. You repeat a similar argument to Riemann sums using this area instead.

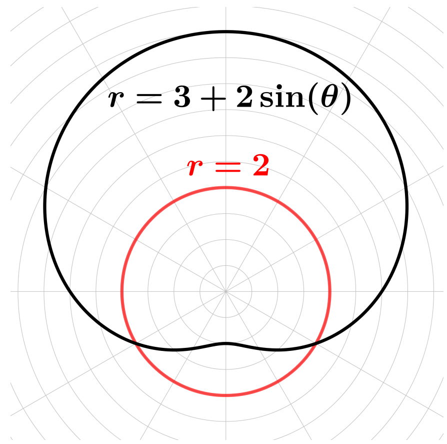


Example 1: Determine the area trapped by the inner loop of $r = 2 + 4\cos(\theta)$.



4.6.2 Area Trapped Between Two or More Polar Curves

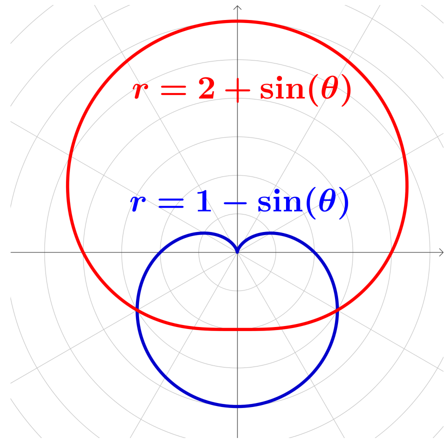
Example 2: Determine the area that lies inside $r = 3 + 2\sin(\theta)$ and outside $r = 2$.



Example 3: Refer to the curves and the sketch of the prior example. That is $r = 3 + 2\sin(\theta)$ and $r = 2$. Determine the area outside of $r = 3 + 2\sin(\theta)$ and inside $r = 2$.

Example 4: Refer once again back to the curves $r = 3 + 2\sin(\theta)$ and $r = 2$. Determine the area that is the overlap of the area within both $r = 3 + 2\sin(\theta)$ and $r = 2$.

Example 5: Determine the area that is common to both the interior of $r = 1 - \sin(\theta)$ and $r = 2 + \sin(\theta)$.



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