$E[x] = \sum x \cdot f(x)$

Set 19: Expectation and Covariance for Joint Distributions

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For discrete random variables X and Y,

$E[xy] = \sum_{\text{all } y} \left(\sum_{\text{all } x} xy \cdot f(x,y) \right)$

Moreover, for a function $g(x, y)$,

$E[g(x,y)] = \sum_{\text{all } y} \left(\sum_{\text{all } x} g(x,y) \cdot f(x,y) \right)$

Example 1: Suppose that X and Y are discrete random variables with the following joint pmf.

		Y		
$p(x,y)$		-3	5	8
X	1	0.21	0.19	0.12
	2	0.17	0.15	0.16

(a) Determine $E[XY]$

$= (1)(-3)(0.21) + (1)(5)(0.19)$
$+ (1)(8)(0.12) + (2)(-3)(0.17)$
$+ (2)(5)(0.15) + (2)(8)(0.16)$
$= 4.32$

(b) Determine $E[\,|2X - Y|\,] = 4.32$

Suppose that X and Y are independent random variables ...

$E(xy) = \sum \sum xy \cdot f(x,y) = \sum \sum xy \cdot \underbrace{f_x(x)\, f_y(y)}_{\text{by independence}}$

$\sum_{\text{all } x} f_x(x) \cdot \sum_{\text{all } y} f_y(y) = E(x) \cdot E(y)$

So $E(xy) = E(x)E(y) \leftarrow$ only if x and y are independent

For discrete random variables X and Y, the **covariance** of X and Y is

$$Cov(X, Y) = \underbrace{E[X - E[X]] \cdot E[Y - E[Y]]}_{\text{in theory}} = \underbrace{E[XY] - E[X]E[Y]}_{\text{use this}}.$$

If X and Y are independent random variables, covariance $(x,y) = 0$ since $E(xy) = E(x)E(y)$

$E(xy) = 4.32$

Example 1 Continued...

(c) Determine $Cov(X,Y)$. $= E(xy) - E(x)\cdot E(y)$

find $f_x(x)$ and $f_y(y)$ ← marginals of x and y basically

x	1	2
$f_x(x)$	0.52	0.48

y	-3	5	8
$f_y(y)$	0.38	0.34	0.28

$E(x) = 1.48$

$E(y) = 2.8$

$Cov(X,y) = E(xy) - E(x)\cdot E(y) = 4.32 - (1.48)(2.8) = 0.176$

$Cov(x,y) > 0$ or positive so some positive relationship between X and y.

For discrete random variables X and Y, the **correlation coefficient** ρ is

$$\rho = Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{V[X]}\,\sqrt{V[Y]}} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}.$$

If X and Y are independent, $Cov(x,y) = 0$

*The correlation coefficient $\rho = Cor(X,Y)$ measures the strength of the **linear** relationship between X and Y. This is the population version of sample correlation coefficient r that we previously saw in Set 3. The same rules apply:*

- $-1 \leq \rho \leq 1$
- ρ close to ± 1 indicates a strong linear relationship.
- ρ close to 0 does not necessarily mean X and Y are not related.

Example 1 Continued... $Cov(x,y) = 0.176$

(d) Determine $Cor(X,Y)$.

$E(x^2) = (1^2)(0.52) + 2^2(0.48) = 2.44$

$V(x) = E(x^2) - E(x)^2 = 2.44 - 1.48^2 = 0.2496$

$E(y^2) = (-3)^2(0.38) + (5^2)(0.34) + (8^2)(0.28)$

$V(y) = E(y^2) - E(y)^2 = 29.84 - 7.84 = 22$

$Cov(x,y) = \dfrac{0.176}{\sqrt{0.2496}\cdot\sqrt{22}} = 0.075$ ← close to 0 so no indication of a linear relationship

For discrete random variables X and Y,

$$V[X+Y] = V[X] + V[Y] + 2Cov(X,Y).$$

If X and Y are independent, $cov(x,y) = 0 \rightarrow V(x+y) = V(x) + V(y)$ ← only if X and y are independent

For discrete random variables X and Y,

$$E[aX + bY + c] = E[aX] + E[bY] + E[c] = aE[X] + bE[Y] + c$$
$$V[aX + bY + c] = V[aX + bY] = a^2V[X] + b^2V[Y] + 2ab \cdot Cov(X,Y)$$

Example 1 Continued...

(e) Determine $E[4X - 3Y + 7]$ and $V[4X - 3Y + 7]$.

$$E[4x - 3y + 7] = E[4x] - E[3y] + 7$$
$$= 4E[x] - 3E[y] + 7$$
$$= 4(1.48) - 3(2.8) + 7$$
$$= 4.52$$

$$V[4x - 3y + \cancel{7}] = 4^2 V(x) + (-3)^2 V(y) + 2(4)(-3) \cdot cov(x,y)$$
$$= 4^2(0.2496) + (-3)^2(22) + 2(4)(-3) \cdot 0.176$$
$$= 197.77$$

$$SD[4x - 3y + 7] = \sqrt{197.77} = 14.063$$

Suppose that $X_1, X_2, \ldots, X_n$ are discrete random variables, then

$$E\left[\sum_{i=1}^{n} a_i X_i + b\right] = \sum_{i=1}^{n} a_i E[X_i] + b$$
$$V\left[\sum_{i=1}^{n} a_i X_i + b\right] = \sum_{i=1}^{n} a_i^2 V[X_i] + 2\sum_{i<j} a_i a_j Cov(X_i, X_j)$$

Readings: Swartz 5.4 (discrete parts only) [EPS the discrete parts of 2.4]
Practice problems: EPS : 2.29, 2.39, 2.43, 2.67. Also, for the joint probability function in 2.67 confirm numerically that Cov(X,Y) = 0