Some Formulas

- $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$, where θ is the angle between $\vec{u}$ and $\vec{v}$, with $0 \le \theta \le \pi$.
- $\mathrm{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$
- $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$
- The area of the parallelogram spanned by the 3D vectors $\vec{u}$ and $\vec{v}$ is equal to $|\vec{u}\times\vec{v}|$.
- The volume of the parallelepiped spanned by the 3D vectors $\vec{u}, \vec{v}, \vec{w}$ is $|(\vec{u}\times\vec{v})\cdot\vec{w}|$.
- Some trigonometric identities:

$$\begin{aligned}
&\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos(\theta),\\
&\cos^2\theta + \sin^2\theta = 1,\\
&\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta
\end{aligned}$$

- **Linear equation.** The equation takes the form of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x).$$

  This equation can be solved by using the following formulas:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = p(x)z \quad \text{(integrating factor)}; \quad y(x) = \frac{1}{z(x)}\int z(x)f(x)\mathrm{d}x.$$

- **Exact equation.** The equation takes the form of

$$N(x,y) + M(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

  such that

$$\frac{\partial}{\partial y}N(x,y) = \frac{\partial}{\partial x}M(x,y).$$

  This equation can be solved by $F(x, y(x)) = C$ for an implicit solution $y = y(x)$ such that $F(x,y)$ satisfies

$$N(x,y) = \frac{\partial F}{\partial x}(x,y), \quad M(x,y) = \frac{\partial F}{\partial y}(x,y).$$

- **Bernoulli equation.** The equation takes the form of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x)y^n \quad (n \neq 1).$$

This equation can be solved as the linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)p(x)u = (1-n)f(x)$$

by the substitution $u = 1/y^{n-1}$.

- **Homogeneous equation.** The equation takes the form of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(y/x).$$

This equation can be solved as the separable equation

$$x\frac{\mathrm{d}u}{\mathrm{d}x} + u = F(u)$$

by the substitution $u = y/x$. Also, a function $f(x,y)$ can be expressed as $f(x,y) = F(y/x)$ if $f(tx,ty) = f(x,y)$ for all $t \neq 0$ and all x, y.

- **Second-order homogeneous linear equations with constant coefficients.** For $ay'' + by' + cy = 0$ with real coefficients $a \neq 0, b, c$ and characteristic polynomial $p(z)$, the solutions are given as follows:

  - If $p(z)$ has distinct real roots $r_1 \neq r_2$, then $y = C_1\mathrm{e}^{r_1 x} + C_2\mathrm{e}^{r_2 x}$.
  - If $p(z)$ has complex roots $\alpha \pm \beta$ß for $\beta \neq 0$, then $y = C_1\mathrm{e}^{\alpha x}\cos(\beta x) + C_2\mathrm{e}^{\alpha x}\sin(\beta x)$.
  - If $p(z)$ has a double root $r = r_1 = r_2$, then $y = C_1\mathrm{e}^{rx} + C_2 x\mathrm{e}^{rx}$.

Also, recall that for the quadratic polynomial $Ax^2 + Bx + C$, the roots are given by $\frac{-B\pm\sqrt{B^2-4AC}}{2A}$.

- **Method of undetermined coefficients.** (1) The equation

$$p(D)y = \mathrm{e}^{\gamma x}, \quad p(z) = az^2 + bz + c,$$

can be solved by

$$y(x) = \frac{\mathrm{e}^{\gamma x}}{p(\gamma)} \quad \text{if } p(\gamma) \neq 0.$$

(2) The equations

$$p(D)y = \mathrm{e}^{\alpha x}\cos(\beta x), \quad p(z) = az^2 + bz + c,$$
$$p(D)y = \mathrm{e}^{\alpha x}\sin(\beta x), \quad p(z) = az^2 + bz + c,$$

for real α, β, have solutions take the form of

$$y(x) = A\mathrm{e}^{\alpha x}\cos(\beta x) + B\mathrm{e}^{\alpha x}\sin(\beta x).$$

(3) The equation

$$ay'' + by' + cy = Ax^2 + Bx + C$$

has a solution taking the form of $y(x) = \alpha x^2 + \beta x + \gamma$, where $a, b, c, A, B, C, \alpha, \beta, \gamma$ are constants.

- **Variation of parameters.** To find a particular solution y_p for the equation $ay'' + by' + cy = f(x)$ by using the method of variation of parameters, choose the two functions y_1, y_2 defining the complementary solutions $y_c = C_1 y_1 + C_2 y_2$, and then set $y_p = v_1 y_1 + v_2 y_2$, where $v_1(x)$ and $v_2(x)$ satisfy

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0, \\ v_1' y_1' + v_2' y_2' = f/a. \end{cases}$$

- **Cauchy–Euler equation.** The equation takes the form of

$$a_2 x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 x \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = g(x).$$

The equation can be solved by using the following substitution:

$$x = \mathrm{e}^t, \text{ which implies } x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} \quad \text{and} \quad x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}.$$

- **Laplace transforms.**

$\mathscr{L}\{t^n\}(s) = \dfrac{n!}{s^{n+1}},\ s > 0;\ \ n = 0, 1, 2 \cdots$
$\mathscr{L}\{\sin(\theta t)\}(s) = \dfrac{\theta}{s^2 + \theta^2},\ s > 0$
$\mathscr{L}\{\cos(\theta t)\}(s) = \dfrac{s}{s^2 + \theta^2},\ s > 0$
$\mathscr{L}\{\mathrm{e}^{at} f(t)\}(s) = \mathscr{L}\{f(t)\}(s - a),\ s > a$
$\mathscr{L}\{t^n f(t)\}(s) = (-1)^n \dfrac{\mathrm{d}^n}{\mathrm{d}s^n} \mathscr{L}\{f(t)\}(s)$
$\mathscr{L}\{f'(t)\}(s) = -f(0) + s\mathscr{L}\{f(t)\}(s)$
$\mathscr{L}\{f''(t)\}(s) = -f'(0) - sf(0) + s^2\mathscr{L}\{f(t)\}(s)$