Unit 1 - Vector Geometry

→ Planes, Lines, and circles in 3D

→ Example (planes):

- → $x=0$ is a plane, the y-z plane in 3D
- → 3 equivalent POV's:
  - → Algebraic description ($x=0$)
  - → Geometric Interpretation
  - → Interpretation by words (All the possible values of x, y, z in 3D such that $x=0$, y and z are "$f(x)$")
- → $y=0$ is the x-z plane
- → $z=0$ is the x-y plane

→ Example (Lines):

- → $x=3$ and $y=2$ ⇒ the line defined by the intersection of the planes $x=3$ and $y=2$.
- → eg: $x=3$ and $y^2+z^2=1$
  - → $x=3$ is a plane
  - → $y^2+z^2=1 \iff (y-0)^2+(z-0)^2=1$ is a circle
    $\sqrt{(y-0)^2+(z-0)^2}=1$

→ Sphere:

- → Geometric object in 3D such that all the (x,y,z) has a constant distance to the center.
- → Center: $P(x_0, y_0, z_0)$

→ Radius : $r > 0$

→ Equation of the sphere

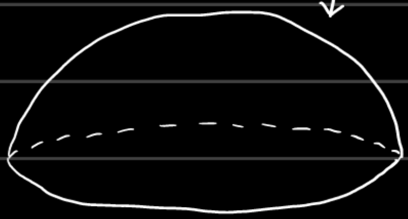
$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = L$$

↑ dist between (x,y,z) and P.

→ eg:

$$\begin{cases} (x-0)^2 + (y-0)^2 + (z-0)^2 = 2^2 \\ z \geqslant 0 \end{cases}$$

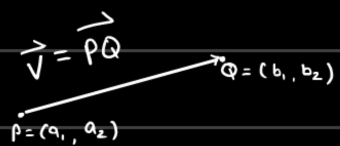
Northern hemisphere



→ <u>Vectors</u>

→ A vector is characterized by its <u>length</u> and its <u>direction</u>

→ <u>2D vectors</u>



→ For $P(a_1, a_2)$ and $Q = (b_1, b_2)$ in 2D,

the vector $\overrightarrow{PQ}$ is defined as:

$$\overrightarrow{PQ} \overset{\text{defn}}{=} \langle b_1 - a_1 \, , \, b_2 - a_2 \rangle \quad \text{(just } Q-P\text{)}$$

→ P is the initial point

→ Q is the terminal point

→ use $\langle \ldots, \ldots \rangle$, angular brackets

→ <u>Length</u> of $\vec{v}$:

$$|\vec{v}| \overset{\text{defn}}{=} \sqrt{(b_1-a_1)^2 + (b_2-a_2)^2} \quad (= \text{dist between P and Q})$$

→ <u>Direction</u> of $\vec{v} = \langle \cos\theta \, , \sin\theta \rangle$ (as in polar coordinates)

(θ is the angle measured counter clock-wise from the positive x-axis.)

→ Examples

(i) $\vec{0} = \langle 0, 0 \rangle$: zero vector

(ii) $P = (0,0)$, $Q = (2,1)$

$\vec{u} = \vec{PQ} = \langle 2-0, 1-0 \rangle = \langle 2, 1 \rangle$

(iii) $P' = (1,1)$ $Q' = (3,2)$

$\vec{u'}\ \vec{P'Q'} = \langle 3-1, 2-1 \rangle = \langle 2, 1 \rangle$

conclusion: $\vec{u} = \vec{u'}$

→ Algebra of vectors:

→ $\vec{u} = \langle u_1, u_2 \rangle$

→ $\vec{v} = \langle v_1, v_2 \rangle$

→ λ = scalar

then:

$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$

$\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2 \rangle$

$\lambda \vec{u} = \langle \lambda u_1, \lambda u_2 \rangle$ (scalar multiplication)

→ Examples:

(i) $\langle 1, 0 \rangle + \langle 0, 2 \rangle = \langle 1+0, 2+0 \rangle$

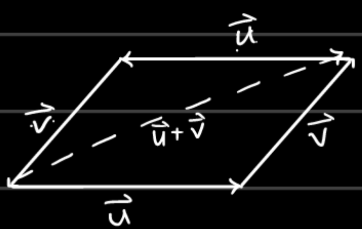
$= \langle 1, 2 \rangle$

(ii) $3\langle 0, 1 \rangle = \langle 3(0), 3(1) \rangle$

$= \langle 0, 3 \rangle$

→ Geometry of vector algebra:

→ parallelogram law for $\vec{u} + \vec{v}$



→ draw $\vec{v}$ from the terminal point of $\vec{u}$, and then complete the triangle

→ draw $\vec{u}$ from the terminal point of $\vec{v}$, and the complete the triangle.

→ Scalar Multiplication:

→ $|\lambda\vec{u}| = |\lambda||\vec{u}|$

→ $|\lambda|$ changes the length of $\vec{u}$ for the length of $\lambda\vec{u}$

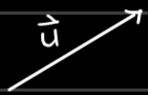
→ Sign of λ (+1 or −1) decides the direction of $\lambda\vec{u}$

→ "+1" for the same direction of $\vec{u}$

→ "−1" for the opposite of $\vec{u}$

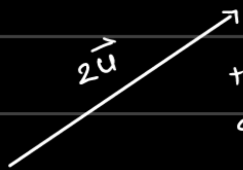
→ Example:

$\vec{u}$



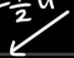
Draw $2\vec{u}$, $-\frac{1}{2}\vec{u}$

$2\vec{u}$



the length is doubled, and the direction is unchanged.

$-\frac{1}{2}\vec{u}$



the length is halved, and the direction is flipped.

→ Application:

→ Given $\vec{v} \neq \vec{0}$, direction of $\vec{v} = \frac{1}{|\vec{v}|}\vec{v}$ (scalar multiplication)

→ Direction of $\vec{v}$ defined before $\langle \cos\theta, \sin\theta \rangle$

$$|\langle \cos\theta, \sin\theta \rangle| = \sqrt{(\cos^2\theta + \sin^2\theta)}$$
$$= \sqrt{1} = 1$$

→ $\therefore \langle \cos\theta, \sin\theta \rangle$ is a unit vector (length = 1)

→ check:

→ $\frac{1}{|\vec{v}|}\vec{v}$ is a unit vector: $\left|\frac{1}{|\vec{v}|}\vec{v}\right| = \frac{1}{|\vec{v}|}|\vec{v}|$

$$= \frac{|\vec{v}|}{|\vec{v}|} = 1$$

$\therefore$ is a unit vector

→ $\frac{1}{|\vec{v}|}\vec{v}$ does not change the direction of $\vec{v}$

→ $\frac{1}{|\vec{v}|}\vec{v}$ and $\langle \cos\theta, \sin\theta \rangle$ have the same direction

→ Conclusion: $\frac{1}{|\vec{v}|}\vec{v} = \langle \cos\theta, \sin\theta \rangle$ for $\vec{v}$

→ Example:

$\vec{u} = \langle 365, 7 \rangle$

direction of $\vec{u} = \langle \cos\theta, \sin\theta \rangle$

$$= \frac{1}{\sqrt{365^2+7^2}} \langle 365, 7 \rangle$$

$$= \left\langle \frac{365}{\sqrt{365^2+7^2}}, \frac{7}{\sqrt{365^2+7^2}} \right\rangle$$

→ Applications

→ Given $\vec{v} \neq 0$, direction of $\vec{v} = \frac{1}{|\vec{v}|}\vec{v}$

→ Standard unit vectors: $\vec{i} = \langle 1, 0 \rangle$ $\vec{j} = \langle 0, 1 \rangle$

$i \neq (1, 0)$ $\quad$ $j \neq (0, 1)$

then for $\vec{v} = \langle v_1, v_2 \rangle$

$\vec{v} = v_1\vec{i} + v_2\vec{j}$ b/c $\vec{v} = \langle v_1, v_2 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle = v_1 \langle 0, 1 \rangle + v_2 \langle 1, 0 \rangle$

→ 3D vectors:

→ Given $P = (a_1, a_2, a_3)$ and $Q = (b_1, b_2, b_3)$

$\vec{PQ} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$

→ Length of $\vec{PQ}$: $|\vec{PQ}| = \sqrt{(b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2}$

→ Direction of $\vec{PQ} \neq \vec{0} = \frac{1}{|\vec{PQ}|}\vec{PQ}$

→ <u>Example</u>: given $\vec{v} = \langle 1, 2, 3 \rangle$

$$|\vec{v}| = \sqrt{1^2+2^2+3} = \sqrt{1+4+9} = \sqrt{14}$$

<u>direction</u> $= \dfrac{1}{|\vec{v}|}\vec{v} = \dfrac{1}{\sqrt{14}}\langle 1, 2, 3 \rangle$

$$= \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$$

→ Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, λ = scalar

→ $\vec{u} + \vec{v} = \langle u_1+v_1, u_2+v_2, u_3+v_3 \rangle$

→ $\lambda\vec{u} = \langle \lambda u_1, \lambda u_2, \lambda u_3 \rangle$

→ Standard unit vectors for 3D:

→ $\vec{i} = \langle 1, 0, 0 \rangle$

→ $\vec{j} = \langle 0, 1, 0 \rangle$

→ $\vec{k} = \langle 0, 0, 1 \rangle$

→ <u>Dot product in 3D</u>:

→ $\vec{u} = \langle u_1, u_2, u_3 \rangle$

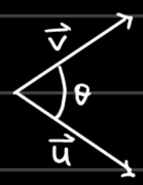
$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ → scalar

→ Geometric Interpretation: (IMPORTANT)

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}$$

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}\right), \quad 0 \le \theta \le \pi$$



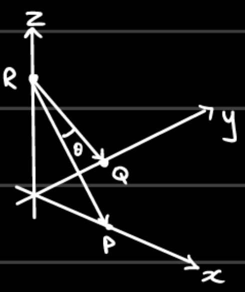
→ Example: given $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, 4)$

Question: find the angle θ between $\overrightarrow{RQ}$ and $\overrightarrow{RP}$.

solution:

$\overrightarrow{RP} = \langle 1-0, 0-0, 0-4 \rangle = \langle 1, 0, -4 \rangle$

$\overrightarrow{RQ} = \langle 0-0, 1-0, 0-4 \rangle = \langle 0, 1, -4 \rangle$



$\overrightarrow{RP} \cdot \overrightarrow{RQ} = (1 \cdot 0) + (0 \cdot 1) + (-4 \cdot -4) = 16$

$|\overrightarrow{RP}| = \sqrt{1^2+0^2+4^2} = \sqrt{17}$

$|\overrightarrow{RQ}| = \sqrt{0^2+1^2+4^2} = \sqrt{17}$

$$\cos\theta = \frac{16}{\sqrt{17}\cdot\sqrt{17}} = \frac{16}{(\sqrt{17})^2} = \frac{16}{17}$$

$$\theta = \cos^{-1}\left(\frac{16}{17}\right)$$

$$\approx 0.3447$$

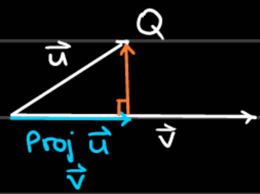
→ Further geometric interpretations:

→ $\vec{u}\cdot\vec{v} > 0 \iff \cos\theta > 0 \iff \theta < \pi/2$ (acute)

→ $\vec{u}\cdot\vec{v} = 0 \iff \cos\theta = 0 \iff \theta = \pi/2$ ($\vec{u} \perp \vec{v}$) ← Perpendicular

→ $\vec{u}\cdot\vec{v} < 0 \iff \cos\theta < 0 \iff \theta > \pi/2$ (obtuse)

→ Vector projections:



→ data fitting

find best approx of Q

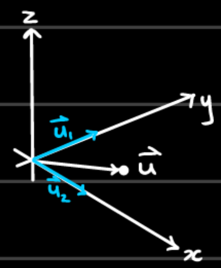
in the direction of $\vec{v}$

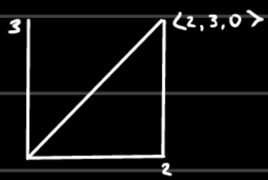
→ physics: effective force

→ Implied decomposition of a vector ($\vec{u}$):

→ $\vec{u} = \underbrace{\text{proj}_{\vec{v}}\vec{u}}_{\text{Parallel to } \vec{v}} + \underbrace{(\vec{u} - \text{proj}_{\vec{v}}\vec{u})}_{\text{Perpendicular to } \vec{v}}$

→ Example: $\vec{u} = \langle 2, 3, 0 \rangle$

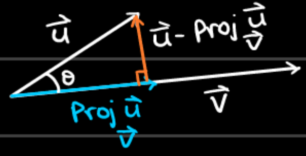




$\text{proj}_{\vec{i}}\vec{u} = \langle 2, 0, 0 \rangle$

$\text{proj}_{\vec{j}}\vec{u} = \langle 0, 3, 0 \rangle$

→ Deviation of $\text{proj}_{\vec{v}}\vec{u}$:



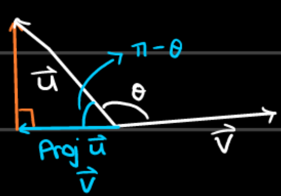
$\text{proj}_{\vec{v}}\vec{u} = (|\vec{u}|\cos\theta) \cdot \dfrac{\vec{v}}{|\vec{v}|}$

Length direction

$= |\cancel{\vec{u}}| \dfrac{\vec{u}\cdot\vec{v}}{|\cancel{\vec{u}}||\vec{v}|} \cdot \dfrac{\vec{v}}{|\vec{v}|}$

$\boxed{\text{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\,\vec{v}}{|\vec{v}|^2} \cdot \vec{v}}$

for acute Angle



$\text{proj}_{\vec{v}}\vec{u} = |\vec{u}|\cos(\pi - \theta) \cdot \left(-\dfrac{\vec{v}}{|\vec{v}|}\right)$

length direction

$= |\vec{u}|(-\cos(\theta))\left(-\dfrac{\vec{v}}{|\vec{v}|}\right)$

$= |\vec{u}|\cos(\theta)\dfrac{\vec{v}}{|\vec{v}|}$ (same as acute!)

for obtuse Angle

→ No matter if θ is acute or obtuse, we always have:

$\boxed{\text{proj}_{\vec{v}}\vec{u} = |\vec{u}|\cos(\theta)\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\vec{u}\,\vec{v}}{|\vec{v}|^2} \cdot \vec{v}}$

→ Example: $\vec{u} = \langle 1, 2, 3 \rangle$

$\vec{u} \cdot \vec{i} = \langle 1, 2, 3 \rangle \cdot \langle 1, 0, 0 \rangle$ $\qquad |\vec{i}| = \sqrt{1^2+0^2+0^2}$

$= 1 \times 1 + 2 \times 0 + 3 \times 0$ $\qquad = \sqrt{1} = 1$

$= 1$

$$\text{proj}_{\vec{i}} \vec{u} = \frac{\vec{u} \cdot \vec{i}}{|\vec{i}|^2} \vec{i} = \frac{1}{1^2} \langle 1, 0, 0 \rangle = \langle 1, 0, 0 \rangle$$

→ Dot product in 2D:

→ Given $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$

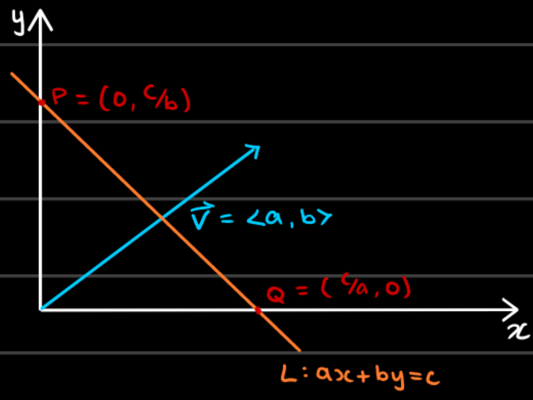
→ $\vec{u} \cdot \vec{v} = u_1 \times v_1 + u_2 \times v_2$

→ $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}$; $0 \leq \theta \leq \pi$

→ Application:

→ $\vec{V} = \langle a, b \rangle \perp L: ax + by = c$

(the vector v and L are perpendicular to each other)



$$\vec{PQ} = \left\langle \frac{c}{a} - 0, 0 - \frac{c}{b} \right\rangle = \left\langle \frac{c}{a}, -\frac{c}{b} \right\rangle$$

$$\vec{V} \cdot \vec{PQ} = \langle a, b \rangle \cdot \left\langle \frac{c}{a}, -\frac{c}{b} \right\rangle$$

$$= \left(a \cdot \frac{c}{a}\right) + \left(b \cdot -\frac{c}{b}\right)$$

$$= c - c = 0$$

$\therefore$ angle between $\vec{V}$ and $\vec{PQ}$ is $\pi/2$, i.e. $\vec{V} \perp \vec{PQ}$

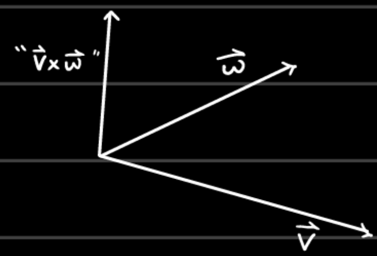
→ Cross Product:

→ $\vec{u} \times \vec{v}$

→ Applications:

→ Define a new coordinate system in 3D.

→ Compute areas and volumes in 3D.



→ Tools for the definition of cross product:

→ $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = +a_1 b_2 - a_2 b_2$

→ $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

→ Definition of cross product:

→ $\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ if $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$

Note: $\vec{v} \times \vec{w}$ is not the same as $\vec{w} \times \vec{v}$

$$= + \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \vec{k}$$

$$= \underbrace{(v_2 w_3 - v_3 w_2)}_{\text{coefficient of } \vec{i}} \vec{i} - \underbrace{(v_1 w_3 - v_3 w_1)}_{\text{coefficient of } \vec{j}} \vec{j} + \underbrace{(v_1 w_2 - v_2 w_1)}_{\text{coefficient of } \vec{k}} \vec{k}$$

→ Example:

→ given $P = (1,1,1)$, $Q = (2,3,4)$, $R = (5,6,7)$

find $\vec{PQ} \times \vec{PR}$

→ solution:

Step 1: $\vec{PQ} = \langle 2-1, 3-1, 4-1 \rangle = \langle 1, 2, 3 \rangle$

$\vec{PR} = \langle 5-1, 6-1, 7-1 \rangle = \langle 4, 5, 6 \rangle$

<u>Step 2:</u> $\overrightarrow{PQ} \times \overrightarrow{PR} = \vec{v} \times \vec{w}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \vec{k}$$

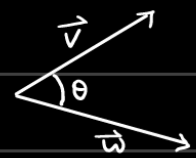
$$= (12-15)\vec{i} - (6-12)\vec{j} + (5-8)\vec{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = -3\vec{i} + 6\vec{j} - 3\vec{k}$$

$$= \langle -3, 6, -3 \rangle$$

→ Characterization of $\vec{v} \times \vec{w}$:

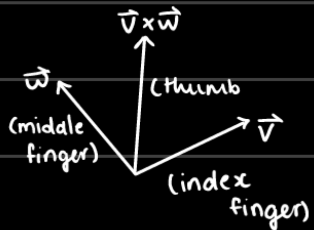
→ Length:



$|\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot \sin(\theta)$, $0 \leq \theta \leq \pi$

here θ is the angle between $\vec{v}$ and $\vec{w}$

→ Direction:



use right-hand thumb rule

→ $\vec{v} \times \vec{w} \perp \vec{v}$ and $\vec{v} \times \vec{w} \perp \vec{w}$

→ $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$ (change of direction)

→ Verification:

→ $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ have the same length

$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$, $|\vec{w} \times \vec{v}| = |\vec{w}||\vec{v}| \sin\theta$

→ direction of $\vec{w} \times \vec{v}$ is the opposite of $\vec{v} \times \vec{w}$ by the right-hand thumb rule.

→ Example:

find $\vec{i} \times \vec{j}$

$|\vec{i}| = 1$, $|\vec{j}| = 1$, $\sin(\frac{\pi}{2}) = 1$

→ length: $1 \cdot 1 \cdot \sin(\pi/2)$

→ direction: the same as $\vec{k}$ (pointing to the +ve z-axis)

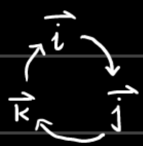
$\therefore \vec{i} \times \vec{j} = \vec{k}$

→ Overall relations:

→ $\vec{i} \times \vec{j} = \vec{k}$

→ $\vec{j} \times \vec{k} = \vec{i}$

→ $\vec{i} \times \vec{k} = \vec{j}$



for example: when running from $\vec{i}$ to $\vec{j}$, it gives you $\vec{k}$

→ Example:

$(2\vec{i} + 3\vec{j}) \times k = \langle 2, 3, 0 \rangle \times \langle 0, 0, 1 \rangle$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ ← method 1 (long)

$= (2\vec{i} + 3\vec{j}) \times \vec{k}$

$= 2\vec{i} \times \vec{k} + 3\vec{j} \times \vec{k}$ → use same order when expanding!

$= 2(\vec{i} \times \vec{k}) + 3(\vec{j} \times \vec{k})$

$= -2(\vec{k} \times \vec{i}) + 3(\vec{j} \times \vec{k})$

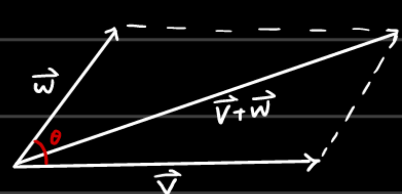
$= -2(\vec{j}) + 3(\vec{i})$

method 2 (quicker)

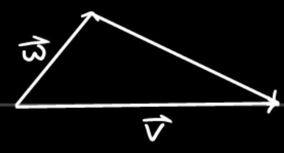
→ Applications to Areas and volumes:

→ Area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$

$= |\vec{v} \times \vec{w}|$ $(= |\vec{v}||\vec{w}| \sin(\theta))$



base $= |\vec{v}|$ height $= |\vec{w}| \sin\theta$



→ Area of the triangle spanned by

$\vec{v}, \vec{w} = \frac{1}{2}$ (area of parallelogram)

$= \frac{1}{2} (|\vec{v} \times \vec{w}|)$

→ Example (2D in 3D):

Find the area of the parallelogram spanned by $\vec{v} = \langle 1, 2 \rangle$ and $\vec{w} = \langle 3, 4 \rangle$.

Soln: View the vectors as vectors in the x-y plane in 3D

$\vec{v} = \langle 1, 2 \rangle \longrightarrow \vec{v}_1 = \langle 1, 2, 0 \rangle$

$\vec{w} = \langle 3, 4 \rangle \longrightarrow \vec{w}_1 = \langle 3, 4, 0 \rangle$

$\Rightarrow$ Area $= |\vec{v}_1 \times \vec{w}_1|$ //

→ Volume of the parallelopiped spanned by $\vec{u}, \vec{v}, \vec{w}$

$\Rightarrow$ the volume $= |\vec{u} \times \vec{v} \cdot \vec{w}|$

b/c → Area of base $= |\vec{u} \times \vec{v}|$

→ height $= |\vec{w}| \cos(\theta)$

$\Rightarrow$ volume = (Area of base) (height)

$= |\vec{u} \times \vec{v}| \; |\vec{w}| \cos(\theta)$

$= |(\vec{u} \times \vec{v}) \cdot \vec{w}|$

→ Different formula for Volume:

→ volume $= |(\vec{u} \times \vec{v}) \cdot \vec{w}|$

$$= \left| \left(\begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} \vec{i} - \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} \vec{j} + \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \vec{k} \right) \cdot \langle w_1, w_2, w_3 \rangle \right|$$

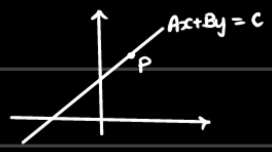
$$= \left| w_1 \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} - w_2 \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} + w_3 \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right|$$

$$= \left| \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \right| = \left| \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right|$$ //

→ Lines and Planes:

→ (Slope, point) - characterization of a line

$L: ax + by = c \quad , \quad -\infty < x < \infty$



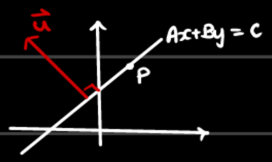
→ Slope

→ Point ($P = (x_0, y_0)$)

→ (Normal vector, point) - to characterize a line

→ Normal vector: $\vec{u} = \langle a, b \rangle$

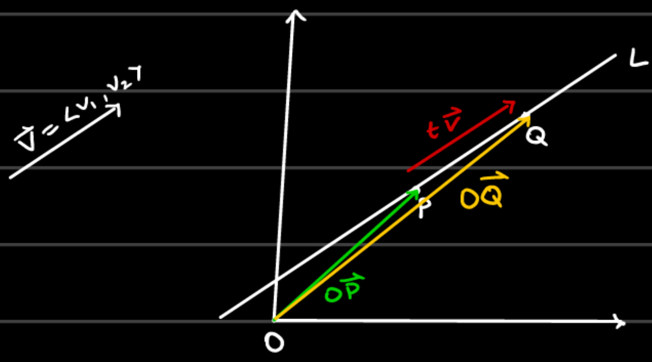
($\vec{u} \perp L$ shown before)



→ Point ($P = (x_0, y_0)$)

($Ax + By = c$)

→ (Parallel vector, Point) - to characterize a line



$\vec{OQ} = \vec{OP} + \vec{PQ}$

by parallelogram law

$= \langle x_0, y_0 \rangle + t\vec{v}$

$= \langle x_0, y_0 \rangle + t\langle v_1, v_2 \rangle$

$= \langle x_0 + tv_1, y_0 + tv_2 \rangle$

→ Parallel vector

($\vec{v} = \langle v_1, v_2 \rangle$)

→ Point

($P = (x_0, y_0)$)

$\therefore \vec{OQ} = \langle x_0 + tv_1, y_0 + tv_2 \rangle$

$Q = (x_0 + tv_1, y_0 + tv_2)$

the same applies if $t < 0$

→ Vector equation of a line

$$x = x_0 + v_1$$

$$y = y_0 + v_1 \quad , \quad -\infty < t < \infty$$

→ Parametric equation of a line:

$$x = x_0 + tv_1$$

$$y = y_0 + tv_2 \quad , \quad -\infty < t < \infty$$

→ Vector equation of a line:

$$L = \vec{r}(t) = \langle x_0 , y_0 \rangle + t\vec{v}$$

$$= \langle x_0 + t\vec{v}_1 , y_0 + tv_2 \rangle \quad , \quad -\infty < t < \infty$$

→ Example:

Find the parametric equation for the line passing through

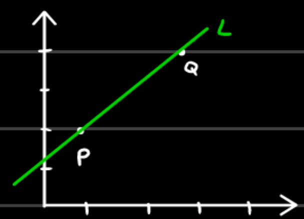
$P = (1, 2)$, $Q = (3, 4)$

base point $= (1, 2)$

Parallel vector $= \vec{PQ}$

$$= \langle 3-1 , 4-2 \rangle$$

$$= \langle 2, 2 \rangle$$



$$x = 1 + t_1(2)$$

$$y = 2 + t_1(2) \quad , \quad -\infty < t_1 < \infty$$

→ Lines in 3D:

$$\left.\begin{aligned} \to L = \vec{r}(t) &= \langle x_0 , y_0 , z_0 \rangle + t\vec{v} \\ &= \langle x_0 , y_0 , z_0 \rangle + t \langle v_1 , v_2 , v_3 \rangle \\ &= \langle x_0 + tv_1 , y_0 + tv_2 , z_0 + tv_3 \rangle , \quad -\infty < t < \infty \end{aligned}\right\} \text{Vector Equations}$$

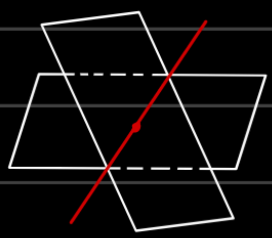
→ Parallel vector: $\vec{v} = \langle v_1 , v_2 , v_3 \rangle$

→ Base Point: $p = (x_0 , y_0 , z_0)$

$$\rightarrow \left.\begin{aligned} x &= \boxed{x_0} + \boxed{tv_1} \\ y &= \boxed{y_0} + \boxed{tv_2} \\ z &= \boxed{z_0} + \boxed{tv_3} \end{aligned}\right\} \text{Parametric Equations}$$

Point Vector

→ <u>Planes</u>:

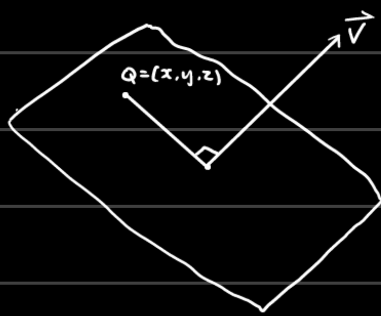


<u>PARALLEL VECTORS WON'T WORK</u>

→ characterization of a plane in 3D

→ Base Point : $P = (x_0, y_0, z_0)$

→ Normal vector : $\vec{v} = \langle a, b, c \rangle$



For any $Q = (x, y, z)$ on the plane :

→ $\vec{v} \perp \overrightarrow{PQ}$ or equivalently,

→ $0 = \vec{v} \cdot \overrightarrow{PQ} = \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$

$0 = a(x - x_0) + b(y - y_0) + c(z - z_0)$

$0 = ax - ax_0 + by - by_0 + cz - cz_0$

$$\boxed{ax_0 + by_0 + cz_0 = ax + by + cz}$$

Base Point

→ (x_0, y_0, z_0) satisfies this equation

→ Coefficients of $x, y, z = \vec{v} = \langle a, b, c \rangle$

→ Equivalent form :

→ $ax + by + cz = d$

→ $\langle a, b, c \rangle$ defines the normal vector $(\vec{v})$

→ Any $P(x_0, y_0, z_0)$ satisfies $d = ax_0 + by_0 + cz_0$

→ Example:

Q :- Find an equation of the plane such that it is perpendicular to $L = t\langle 1, 2, 3\rangle$, $-\infty < t < \infty$, and passing through the point $P(-1, -2, -3)$.

parallel vector to L

Soln: Base Point : $P(-1, -2, -3)$

Normal vector : $\vec{v} = \langle 1, 2, 3\rangle$

make point P.

equation of plane : $1x + 2y + 3z = 1\cdot(-1) + 2(-2) + 3(-3)$

make normal vector.