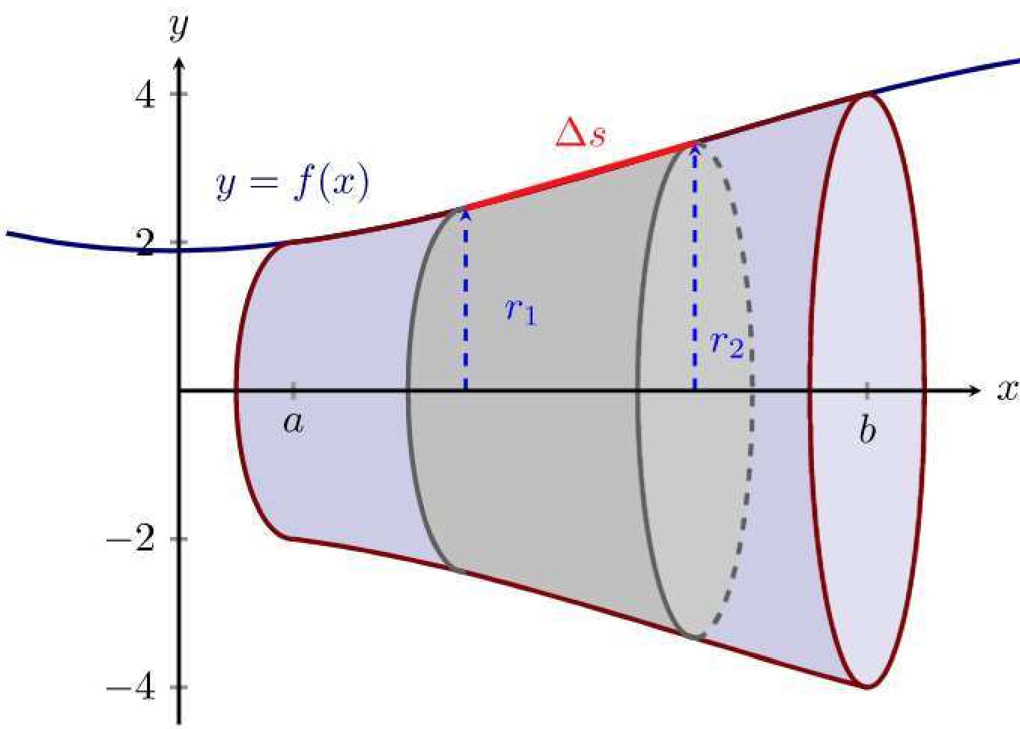
Chapter 2

Integration Techniques and Applications



2.1 (Section 8.1) Some Integration Techniques and Tricks

Note: Algebraic Techniques

A common technique to evaluating an integral is to algebraically manipulate it to an already known expression. It's impossible to cover every scenario in a single lecture, but we give several examples below.

2.1.1 Completing the Square

Example 1: Evaluate $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$.

Complete the square

$$x^2-4x+3 = \left(x^2-4x+\left(-\tfrac{4}{2}\right)^2\right)-\left(-\tfrac{4}{2}\right)^2+3$$

add and subtract half of b squared

x^2+bx+c

$$= \underbrace{(x^2-4x+4)-4+3}$$

$$= (x-2)^2-1$$

$$\Rightarrow I = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}} = \int \frac{du}{u\sqrt{u^2-1}}$$

u-sub $\Rightarrow$ Let $u = x-2$

$du = (1)\,dx$

$$= \operatorname{arcsec}|u| + C$$

$$= \operatorname{arcsec}|x-2| + C$$

2.1.2 Using Trigonometric Identities

Example 2: Evaluate $\displaystyle\int_{-\pi/2}^{0} \sqrt{1-\cos(2\theta)}\; d\theta$ using the identity $2\sin^2(A) = 1-\cos(2A)$.

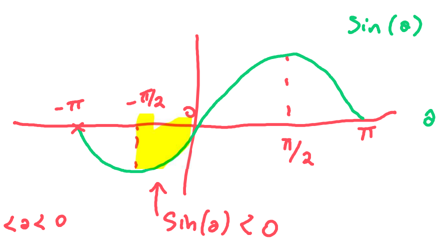
by Hint

$$= \int_{-\pi/2}^{0} \sqrt{2\sin^2(\theta)}\; d\theta$$

$$|A| = \begin{cases} A & A>0 \\ -A & A<0 \end{cases}$$

$$= \int_{-\pi/2}^{0} \sqrt{2}\cdot|\sin(\theta)|\,d\theta$$

$$= \sqrt{2}\int_{-\pi/2}^{0} |\sin(\theta)|\,d\theta$$



$$= \sqrt{2}\int_{-\pi/2}^{0} -\sin(\theta)\,d\theta \quad \because \sin(\theta)<0 \text{ on } -\pi/2<\theta<0$$

$$= \sqrt{2}\cdot\cos(\theta)\Big|_{-\pi/2}^{0}$$

$$= \sqrt{2}\,\left(\cos(0)-\cos(-\pi/2)\right)$$

$$= \sqrt{2}\,(1-0) = \sqrt{2}$$

2.1.3 Long Division

Example 3: Evaluate $\displaystyle\int \frac{x^3+x}{x-1}\,dx$.

$x^3 + 0x^2 + x + 0$

use poly long div.

$$\begin{array}{r} x^2 + x + 2 \\ x - 1 \,\overline{\big)\, x^3 + 0x^2 + x + 0} \\ -(x^3 - x^2) \qquad\qquad \\ x^2 + x \qquad \\ -(x^2 - x) \qquad \\ 2x + 0 \\ -(2x - 2) \\ 2 \end{array}$$

Remainder

$$\Rightarrow I = \int \Big(\underbrace{x^2 + x + 2}_{\text{quotient}} + \frac{2}{x-1} \Big)\,dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + \int \frac{2}{x-1}\,dx$$

Let $u = x - 1$ '$x - a$' shifts

$du = dx$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + \int \frac{2}{u}\,du$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + 2\ln|u| + C$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + 2\ln|x-1| + C \quad //$$

2.1.4 Conjugates

Example 4: Evaluate $\displaystyle\int_0^{\pi/4} \frac{dx}{1-\sin(x)}$.

$(a-b)(a+b) = a^2 - b^2$

$$= \int_0^{\pi/4} \frac{1}{1-\sin(x)} \cdot \frac{1+\sin(x)}{1+\sin(x)}\,dx$$

$$= \int_0^{\pi/4} \frac{1+\sin(x)}{1^2 - \sin^2(x)}\,dx$$

Note: $1^2 - \sin^2(x) = 1 - \sin^2(x)$

$= \cos^2(x)$

$$= \int_0^{\pi/4} \frac{1+\sin(x)}{\cos^2(x)}\,dx$$

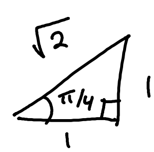
$$= \int_0^{\pi/4} \left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)} \right) dx$$

Note: $\dfrac{\sin(x)}{\cos^2(x)} = \dfrac{\sin(x)}{\cos(x)} \cdot \dfrac{1}{\cos(x)}$

$= \tan(x)\sec(x)$

$$= \int_0^{\pi/4} \left(\sec^2(x) + \tan(x)\sec(x) \right) dx$$

$$= \big(\tan(x) + \sec(x)\big)\Big|_0^{\pi/4}$$



$$= (1 + \sqrt{2}) - (0 + 1)$$

$$= \sqrt{2} \quad //$$

2.1.5 Dumb Luck Substitution

Note: Substitution to Change Form

A better way to think about integral substitution is that it changes the expression you're currently working with. Sometimes you find a substitution that changes it in just the right way that allows you to evaluate.

Example 5: Evaluate $\int \frac{\sqrt{x}}{1+x^3}dx$ using $u = x^{3/2}$.

Let $u^2 = x^3$

$\Rightarrow u = +x^{3/2}$

So $du = \frac{3}{2}x^{1/2}dx$

$= \frac{3}{2}\sqrt{x}\,dx \Rightarrow \sqrt{x}\,dx = \frac{2}{3}du$

$$= \int \frac{2/3}{1+u^2}du$$

$$= \frac{2}{3}\int \frac{1}{1+u^2}du$$

$$= \frac{2}{3}\arctan(u) + C$$

$$= \frac{2}{3}\arctan(x^{3/2}) + C$$

2.1.6 Splitting up an Integral by Linearity

Example 6: Evaluate $\int \frac{3x+2}{\sqrt{1-x^2}}dx$.

Let $u = 1-x^2$

$\Rightarrow du = (-2x)\,dx$

$\Rightarrow -\frac{1}{2}du = x\,dx$

$$= \int \frac{3x}{\sqrt{1-x^2}}dx + \int \frac{2}{\sqrt{1-x^2}}dx$$

$$= -\frac{3}{2}\int \frac{du}{\sqrt{u}} + 2\arcsin(x)$$

$$= -\frac{3}{2}\cdot \frac{u^{-1/2+1}}{-1/2+1} + 2\arcsin(x) + C$$

$$= \left(-\frac{3}{2}\times 2\right)u^{1/2} + 2\arcsin(x) + C$$

$$= -3\sqrt{1-x^2} + 2\arcsin(x) + C \;//$$

2.2 (Section 7.1) Logarithms and Exponentials

Note: Defining the Logarithm as an Integral

The textbook goes through extensive measures to demonstrate how to reconstruct the exponential and all properties of the logarithm if we initially define the natural logarithm by $\ln(x) = \int_1^x \frac{1}{t}dt, \qquad x > 0$ the process is long and worth a read. What we will do is give examples of computing logarithmic and exponential type integrals.

2.2.1 Scenario 1: Integrals Involving the Natural Logarithm

Note: Integrands of the Form $f'(x)/f(x)$

To evaluate the integral $I = \int \frac{f'(x)}{f(x)}dx$ perform the substitution $u = f(x)$ to obtain $I = \ln|f(x)| + C$.

Example 1: Evaluate $\int_{-1}^{0} \frac{3dx}{3x-2}$

$u = 3x - 2 \Rightarrow du = 3dx$

Bounds

- $x = -1 \Rightarrow u = 3(-1) - 2 = -5$
- $x = 0 \Rightarrow u = 3(0) - 2 = -2$

$$= \int_{-5}^{-2} \frac{du}{u}$$

$$= \ln|u| \Big|_{-5}^{-2}$$

$$= \ln|-2| - \ln|-5|$$

$$= \ln(2) - \ln(5) \; //$$

Example 2: Evaluate $\int \frac{1}{\arctan(4x)(1+16x^2)}dx.$

Let $u = \arctan(4x)$

$\Rightarrow du = \frac{1}{1+(4x)^2} \cdot 4\,dx$

$= \frac{4}{1+16x^2}dx$

$\Rightarrow \frac{1}{4}du = \frac{1}{1+16x^2}dx$

$$= \int \frac{1/(1+16x^2)}{\arctan(4x)}dx$$

$$= \int \frac{\frac{1}{4}du}{u}$$

$$= \frac{1}{4}\ln|u| + C$$

$$= \frac{1}{4}\ln|\arctan(4x)| + C \; //$$

2.2.2 Scenario 2: Integrals Involving the Natural Logarithm

Note: Integrands of the Form $f(\ln(x))/x$

If you see a logarithm in your integral, you should try to look for the term $1/x$. If so, then you may use the substitution $u = \ln(x)$ to obtain $\int \frac{f(\ln(x))}{x}dx = \int f(u)du$, which could potentially be evaluated. Naturally, this technique generalizes to integrals of the form $\int \frac{f(\log_a(x))}{x}dx$.

Example 3: Evaluate $\int \frac{\ln(\ln(x))}{x\ln(x)}dx$.

$$= \int \frac{\ln(\ln(x))}{\ln(x)} \cdot \frac{1}{x}dx$$

Let $u = \ln(x)$
$\Rightarrow du = \frac{1}{x}dx$

$$= \int \frac{\ln(u)}{u}du = \int \ln(u) \cdot \frac{1}{u}du$$

Let $v = \ln(u)$
$\Rightarrow dv = \frac{1}{u}du$

$$= \int v\,dv$$

$$= \frac{1}{2}v^2 + C$$

$$= \frac{1}{2}(\ln(u))^2 + C$$

$$= \frac{1}{2}(\ln(\ln(x)))^2 + C$$

Example 4: Evaluate $\int \frac{\csc^2(\log_{10}(x))}{x}dx$. $= \int \csc^2(\log_{10}(x)) \cdot \frac{1}{x}dx$

Reminder: $(\log_a(x))' = \frac{1}{x\ln(a)}$

Let $u = \log_{10}(x)$
$\Rightarrow du = \frac{1}{x \cdot \ln(10)}dx$
$\Rightarrow \ln(10)\,du = \frac{1}{x}dx$

So $I = \int \csc^2(u) \cdot \ln(10)\,du$

$$= \ln(10)\int \csc^2(u)\,du$$

$$= -\ln(10)\cot(u) + C$$

$$= -\ln(10) \cdot \cot(\log_{10}(x)) + C$$

2.2.3 Integrals Involving the Exponential Function

Note: Integrands of the form $f(e^{ax})e^{ax}$

Since the derivative of an exponential is (in a sense) itself, a good strategy when dealing with exponentials is to see if another one is present. Performing the substitution $u = e^{ax}$ transforms the integral as $\displaystyle\int f(e^{ax})e^{ax}dx = \frac{1}{a}\int f(u)du$. Naturally, this technique generalizes to integrals of the form $\displaystyle\int f(b^x)b^x dx$.

Example 5: Evaluate $\displaystyle\int_0^{\ln(\pi)} e^x \cos(e^x)dx$.

$$= \int_0^{\ln(\pi)} \cos(e^x)\cdot e^x\,dx$$

Let $u = e^x$
$\Rightarrow du = e^x dx$

Bounds

$x = 0 \Rightarrow u = e^0 = 1$
$x = \ln(\pi) \Rightarrow u = e^{\ln(\pi)} = \pi$

$$= \int_1^{\pi} \cos(u)\,du$$

$$= \sin(u)\Big|_1^{\pi}$$

$$= \cancel{\sin(\pi)}^{0} - \sin(1)$$

$$= -\sin(1)$$

Example 6: Evaluate $\displaystyle\int \frac{1}{1+e^t}dt$.

$$= \int \frac{1}{1+e^t}\cdot\frac{e^{-t}}{e^{-t}}\,dt$$

$(1+e^t)e^{-t} = e^{-t} + e^{t-t}$
$= 1 + e^{-t}$

$\displaystyle\int \frac{1}{1+e^{-t}}\cdot e^{-t}dt$

$$= \int \frac{e^{-t}}{1+e^{-t}}\,dt$$

Let $u = 1+e^{-t}$
$\Rightarrow du = (-e^{-t})dt$
$\Rightarrow -du = e^{-t}dt$

$$= -\int \frac{du}{u}$$

$$= -\ln|u| + C$$

$$= -\ln|1+e^{-t}| + C$$

2.3 (Section 7.2) Separable Differential Equations and Modeling

2.3.1 Separable Differential Equations

$y' = y$ $y = e^{x}$

Definition: Differential Equations and Separable Equations

A **Differential Equation** is an equation involving an unknown function and its derivatives. A **Separable Differential Equation** is a differential equation of the form $y' = f(x)g(y)$ for some functions f and g.

Procedure: Solving a SDE

1. Bring all the y terms to the LHS and all the x terms to the RHS to obtain the equation $g(y)^{-1}dy = f(x)dx$.

2. Integrate both sides $\int g(y)^{-1}dy = \int f(x)dx$ to get an implicit equation for x and y.

Example 1: Solve the differential equation $(1+x^2)\frac{dy}{dx} = 2x\sqrt{1-y^2}$, $y(0) = 1/2$ explicitly for y.

$$\frac{dy}{dx} = \underbrace{\frac{2x}{1+x^2}}_{f(x)} \cdot \underbrace{\sqrt{1-y^2}}_{g(y)}$$

LHS: All y's
RHS: All x's

$$\Rightarrow \frac{1}{\sqrt{1-y^2}}\,dy = \frac{2x}{1+x^2}\,dx$$

$$\overset{\int}{\Rightarrow} \int \frac{1}{\sqrt{1-y^2}}\,dy = \int \frac{2x}{1+x^2}\,dx$$

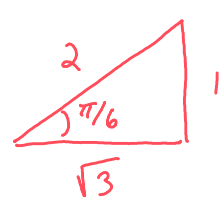
$u = 1+x^2$
$\Rightarrow du = 2x\,dx$

$$\Rightarrow \arcsin(y) = \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln(1+x^2) + C \;//$$

$$\arcsin(y) = \ln(1+x^2) + C$$



Using $y(0) = 1/2 \Rightarrow \arcsin(1/2) = \ln(1) + C$

$$= 0 + C$$

$$\Rightarrow \pi/6 = C$$

So $\arcsin(y) = \ln(1+x^2) + \pi/6$

$$\Rightarrow y(x) = \sin\left(\ln(1+x^2) + \pi/6\right) \;//$$

SDE. $y' = ky = \underbrace{k}_{f(t)} \cdot \underbrace{y}_{g(y)}$

$y(t)$ = population after t years

$y' \propto y$

$\Rightarrow y' = k \cdot y$

2.3.2 Unlimited Growth and Radioactive Decay Models

Model: Exponential Model

Let $y(t)$ measure the amount of substance of an important material at time t with an initial amount $y(0) = y_0$. If the rate of change y is proportional to y itself, then it is given by the separable differential equation

$$y'(t) = ky(t); \quad y(0) = y_0$$

and has the solution $y(t) = y_0 e^{kt}$. The substance grows if $k > 0$ and it decays if $k < 0$.

Example 2: The biomass of a yeast culture in an experiment is 29g. (1) After 30 min the mass is 37g. (2) Assuming the equation for unlimited population growth models this, (3) how long will it take for the mass to double from it's initial population size?

exponential model

$y(t)$ = population size after t min.

(1) Infer $y(0) = 29g$ $(= y_0)$

(2) $y(30) = 37$

(3) $y' = ky \Rightarrow y(t) = y_0 e^{kt}$

By (1) $\Rightarrow$ $y(t) = 29e^{kt}$

By (2) $\Rightarrow$ $37 = 29e^{30k}$

$\Rightarrow \frac{37}{29} = e^{30k}$

$\ln\left(\frac{37}{29}\right) = 30k$

$\Rightarrow k = \frac{\ln(37/29)}{30}$

$y(t) = 29e^{kt}$ where $k =$

What value of t satisfies $y(t) = 58 = 2 \times y_0$

$\Rightarrow 29e^{kt} = 58$

$\Rightarrow e^{kt} = 58/29 = 2$

$\Rightarrow kt = \ln(2)$

$\Rightarrow t = \frac{\ln(2)}{k} = \frac{30\ln(2)}{\ln(37/29)} \approx 85.36$ min

Example 3: The half-life of carbon-14 is 5730 years (i.e. $y(\cancel{5370})$ 5730 $= y_0/2$). Assuming the exponential model is a suitable model, find the age of a sample in which 10% of the radioactive substance has decayed.

$y(t) = y_0 e^{kt}$

can't determine

use $y(5730) = \frac{y_0}{2}$

$\Rightarrow \cancel{y_0} e^{5730k} = \frac{\cancel{y_0}}{2}$

$\Rightarrow e^{5730k} = \frac{1}{2} \Rightarrow 5730k = \ln(1/2)$

$\Rightarrow k = \frac{\ln(1/2)}{5730}$

as 10% is gone

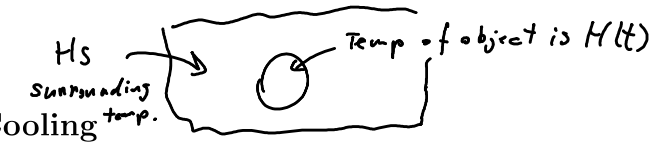
$y(t) = 0.9y_0$

$\Rightarrow \cancel{y_0} e^{kt} = 0.9\cancel{y_0}$

$\Rightarrow e^{kt} = 0.9$

$\Rightarrow kt = \ln(0.9)$

$\Rightarrow t = \frac{\ln(0.9)}{k} = \frac{5730\ln(0.9)}{\ln(1/2)} \approx 870.98$ years.



2.3.3 Heat Transfer: Newton's Law of Cooling

Model: Newton's Law of Cooling

If $H(t)$ is the temperature of an object at time t and H_s is the constant surrounding temperature the Newton's Law of Cooling model is given by

$$H'(t) = -k(H - H_s); \quad H(0) = H_0$$

where $k > 0$. It has the solution $H(t) = H_s + (H_0 - H_s)e^{-kt}$.

Example 4: You are Sherlock Holmes investigating a murder. You examine the cadaver at 1:30 pm and register a temperature of $33°C$. An hour later (at 2:30 pm) you measure a temperature of $30°C$. Given that the temperature of a living body is $37°C$ and the surrounding temperature is a stable $20°C$, determine the time of the murder.

$H_s = 20$ $\qquad$ $t = 0$ corresponds to 1:30 pm

$$H(0) = H_0 = 33$$

$$\text{So } H(t) = 20 + (33 - 20)e^{-kt}$$
$$= 20 + 13e^{-kt}$$

One hour later,

$$H(60) = 30 \Rightarrow 30 = 20 + 13e^{-60k}$$
$$\Rightarrow 10 = 13e^{-60k}$$
$$\Rightarrow \frac{10}{13} = e^{-60k}$$
$$\Rightarrow \ln(10/13) = -60k$$
$$\Rightarrow k = -\frac{\ln(10/13)}{60}$$

Person is last alive when $H(t) = 37$

$$37 = 20 + 13e^{-kt} \qquad \text{where } k = \text{\_\_\_\_\_\_}$$
$$\Rightarrow 17 = 13e^{-kt}$$
$$\Rightarrow \frac{17}{13} = e^{-kt} \Rightarrow -kt = \ln(17/13)$$
$$\Rightarrow t = -\frac{\ln(17/13)}{k}$$
$$= \frac{60\ln(17/13)}{\ln(10/13)}$$
$$\approx -61.35 \text{ min}$$

approximately 61.35 minutes ago was the time of murder.

2.4 (Section 8.2) Integration By Parts (IBP)

2.4.1 Establishing Integration by Parts and LIATE

Theorem: Integration By Parts (IBP)

Let $u(x)$ and $v(x)$ be differentiable functions on (a, b) and continuous on $[a, b]$. Then,

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu. \quad \text{or} \quad \int_a^b uv'dx = uv\Big|_a^b - \int_a^b vu'dx$$

Note: Derivation of IBP

By the product rule, $uv' = (uv)' - vu'$. Integrating both sides yields the IBP theorem.

Example 1: Use IBP to evaluate $\int xe^x dx$. Identify this product as $u = x$ and $dv = e^x dx$. (1)
What happens if $u = e^x$ and $dv = xdx$ instead? (2)

$u = x \qquad du = 1\,dx$

$dv = e^x dx \qquad v = e^x$

can ignore +C here.

$$\int xe^x dx = uv - \int vdu$$
$$= xe^x - \int e^x dx$$
$$= xe^x - e^x + C$$

Check: $\frac{d}{dx}[xe^x - e^x + C]$
$= (1)\cancel{e^x} + x(e^x) - \cancel{e^x} + 0$
$= xe^x$

Try instead?

$u = e^x \qquad du = e^x dx$

$dv = x\,dx \qquad v = \frac{1}{2}x^2$

$$\int xe^x dx = \frac{1}{2}x^2 e^x - \int \frac{1}{2}x^2 e^x dx$$

even worse!

Choice matters!

LIATE

Consider an integral of the form $\int f(x)g(x)dx$ where $f(x)$ and $g(x)$ are either a **L**ogarithmic function, **I**nverse trigonometric function, **A**lgebraic function, **T**rigonometric function, or **E**xponential function. **LIATE** is an acronym that tells you that you set your u term as the first function that is present on the list:

- **L** = Logarithmic function $\ln(x)$, $\log_{10}(x)$
- **I** = Inverse trigonometric function $\arctan(x)$
- **A** = Algebraic function : e.g. $\sqrt{x}$, x^2+1, etc. $x^n + ax^m + \ldots$
- **T** = Trigonometric function $\sin(x)$
- **E** = Exponential function e^x, 2^x, etc.

2.4.2 Examples Using LIATE

First highlight is your 'u'!

LIATE

Example 2: Evaluate $\int_1^e \underbrace{x^3}_{A} \underbrace{\ln(x)}_{L}dx$.

Let $u = \ln(x)$ $\quad du = \frac{1}{x}dx$

$dV = x^3 dx$ $\quad V = \frac{1}{4}x^4$

$$\int_1^e x^3 \ln(x)dx = \frac{1}{4}x^4 \ln(x)\Big|_1^e - \int_1^e \frac{1}{4}x^4 \cdot \frac{1}{x}dx$$

$$= \left(\frac{1}{4}e^4 \overset{1}{\cancel{\ln(e)}} - \frac{1}{4}(1^4)\overset{0}{\cancel{\ln(1)}}\right) - \int_1^e \frac{1}{4}x^3 dx$$

$$= \frac{1}{4}e^4 - \left(\frac{1}{16}x^4\right)\Big|_1^e = \frac{1}{4}e^4 - \left(\frac{1}{16}e^4 - \frac{1}{16}\right)$$

$$= \frac{3}{16}e^4 + \frac{1}{16}$$

Example 3: Evaluate $\int \underbrace{e^x}_{E} \underbrace{\sin(2x)}_{T}dx$.

LIATE, T first

so $u = \sin(2x)$ $\quad du = 2\cos(2x)dx$

$dv = e^x dx$ $\quad v = e^x$

$$\underbrace{\int e^x \sin(2x)dx}_{=I} = e^x \sin(2x) - \int \underbrace{e^x}_{E} \cdot \underbrace{2\cos(2x)}_{T}dx$$

$u = 2\cos(2x)$ $\quad du = -4\sin(2x)dx$

$dv = e^x dx$ $\quad v = e^x$

$$= e^x\sin(2x) - \left(2\cos(2x)e^x - \int e^x(-4\sin(2x))dx\right)$$

$$= e^x\sin(2x) - 2\cos(2x)e^x - 4\int e^x \sin(2x)dx$$

$$\Rightarrow I = e^x\sin(2x) - 2\cos(2x)e^x - 4I$$

$$\Rightarrow I + 4I = e^x\sin(2x) - 2e^x\cos(2x)$$

$$\Rightarrow 5I = e^x\sin(2x) - 2e^x\cos(2x)$$

$$\Rightarrow I = \frac{1}{5}e^x\sin(2x) - \frac{2}{5}e^x\cos(2x) + C$$

Inverse functions: arctrig (x) (e.g. $\arctan(x)$) and logs.

2.4.3 Integrals of Lonely Logarithms and Inverse Functions (Ninja's)

Note: Integrals of Inverse Functions

One may identify the integral $\displaystyle\int f^{-1}(x)dx = \int f^{-1}(x) \cdot 1dx$ where $u = f^{-1}(x)$ and $dv = 1dx$.

Example 4: Evaluate $\displaystyle\int \arctan(x)dx.$ $= \displaystyle\int \underset{I}{\arctan(x)} \cdot \underset{A}{1}\, dx$

first

LIATE

Let $u = \arctan(x)$ $\quad du = \dfrac{1}{1+x^2}dx$

$dv = 1 \cdot dx$ $\quad v = x$

$$I = x\arctan(x) - \int \frac{x}{1+x^2}dx$$

different 'u'.

Let $u = 1+x^2$

$\Rightarrow du = 2x dx \Rightarrow \frac{1}{2}du = x dx$

$$= x\arctan(x) - \frac{1}{2}\int \frac{du}{u}$$

$$= x\arctan(x) - \frac{1}{2}\ln|u| + C$$

$$= x\arctan(x) - \frac{1}{2}\ln|1+x^2| + C.$$

2.4.4 The Tabular Method

Note: The Tabular Method

The tabular "*method*" is just a way to organize information when performing IBP multiple times (like in a prior example). It's better seen by demonstration.

Example 5: Evaluate $\displaystyle\int \underset{A}{x^2}\,\underset{T}{\cos(x)}dx$ using the tabular method.

First

LIATE

$u = x^2$

$dv = \cos(x)dx$

+/−	u	V'
+	x^2	$\cos(x)$
−	$2x$	$\sin(x)$
+	2	$-\cos(x)$
−	0	$-\sin(x)$

$$= x^2\sin(x) - \int \underset{A}{2x}\,\underset{T}{\sin(x)}dx$$

+	u	dv
−	du	V

$u = 2x \quad dv = \sin(x)dx$

$$= x^2\sin(x) - (2x)(-\cos(x)) + \int 2(-\cos(x))dx$$

$$= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + 0 + C$$

2.5 (Section 8.3) Trigonometric Integrals

2.5.1 Products (Without Powers) and Roots

Procedure: Integrand of the Form $\sin(mx)\sin(nx)$, $\sin(mx)\cos(nx)$, or $\cos(mx)\cos(nx)$

If your integral is one of the three forms $\int \sin(mx)\sin(nx)dx$; $\int \sin(mx)\cos(nx)dx$; or $\int \cos(mx)\cos(nx)dx$ use the *Product to Sum* trigonometric formulas found in the course pack.

Example 1: Evaluate $\int \cos(\underbrace{2x}_{B=2x})\sin(\underbrace{4x}_{A=4x})dx$.

$\sin(A)\cos(B) = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)$

$$= \int \left(\frac{1}{2}\sin(\underbrace{2x}_{A-B}) + \frac{1}{2}\sin(\underbrace{6x}_{A+B})\right)dx$$

$$= -\frac{1}{4}\cos(2x) - \frac{1}{12}\cos(6x) + C$$

Procedure: Integrand of the Form $\sqrt{1 \pm \cos(ax)}$

If your integrand contains $\sqrt{1 \pm \cos(ax)}$ try using the *Half Angle Identities*.

Example 2: Evaluate $\int_{\pi}^{2\pi} \sqrt{1-\cos(2x)}dx$.

$\sin^2(A) = \frac{1-\cos(2A)}{2}$

$\cos^2(B) = \frac{1+\cos(2B)}{2}$

use $1-\cos(2A) = 2\sin^2(A)$ where $A=x$,

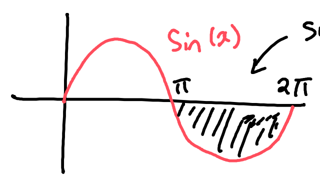
$$= \int_{\pi}^{2\pi} \sqrt{2\sin^2(x)}\,dx$$

$$= \sqrt{2}\int_{\pi}^{2\pi} |\sin(x)|dx$$

$$= \sqrt{2}\int_{\pi}^{2\pi} (-\sin(x))dx = \sqrt{2}\cos(x)\Big|_{\pi}^{2\pi}$$

$$= \sqrt{2}(1-(-1)) = 2\sqrt{2}$$

Remember: $(\sqrt{c})^2 = c$, $\sqrt{c^2} = |c|$



$\sin(x) < 0 \Rightarrow |\sin(x)| = -\sin(x)$

Procedure: Integrand of the Form $\sqrt{1 \pm \sin(ax)}$

If your integrand contains $\sqrt{1 \pm \sin(ax)}$ multiply by the conjugate and use $1-\sin^2(ax) = \cos^2(ax)$.

Example 3: Evaluate $\int_0^{\pi/2} \frac{1}{2}\sqrt{1+\sin(x)}dx$.

$(a+b)(a-b) = a^2 - b^2$

$\sin^2(x) + \cos^2(x) = 1$

use $\Rightarrow 1-\sin^2(x) = \cos^2(x)$

$$= \int_0^{\pi/2} \frac{1}{2}\sqrt{(1+\sin(x))\cdot\frac{(1-\sin(x))}{1-\sin(x)}}\,dx$$

$$= \int_0^{\pi/2} \frac{1}{2}\sqrt{\frac{1-\sin^2(x)}{1-\sin(x)}}\,dx$$

$$= \int_0^{\pi/2} \frac{1}{2}\frac{\sqrt{\cos^2(x)}}{\sqrt{1-\sin(x)}}\,dx$$

$$= \int_0^{\pi/2} \frac{1}{2}\cdot\frac{|\cos(x)|}{\sqrt{1-\sin(x)}}\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\cos(x)}{\sqrt{1-\sin(x)}}\,dx$$

Let $u = 1-\sin(x)$

$du = -\cos(x)dx$

$\Rightarrow -du = \cos(x)dx$

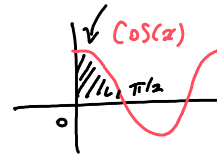
Bounds

$x=0 \Rightarrow u = 1-\sin(0) = 1$

$x=\pi/2 \Rightarrow u = 1-\sin(\pi/2) = 0$

(sorry!)

$\cos(x) > 0 \Rightarrow |\cos(x)| = \cos(x)$



$\int_a^b = -\int_b^a$ $\quad I = -\frac{1}{2}\int_1^0 \frac{du}{\sqrt{u}} = \int_0^1 \frac{du}{2\sqrt{u}} = \sqrt{u}\Big|_0^1 = \sqrt{1} - \sqrt{0}$

$= 1$

2.5.2 Products of Sines and Cosines, Same Input, Raised to a Power

Procedure: Integral of the Form $\int \sin^m(x)\cos^n(x)dx$

The procedure depends on whether either m and n are both even, or one of them is odd.

- m or n (or both) are odd: Take off a single term of the odd power and join it with dx to form your du. Convert the remaining terms using $\sin^2(x) + \cos^2(x) = 1$ and finish the u-substitution.
- m and n are even: Convert all terms using the *Half Angle Identities*. If any new even power terms appear, repeat the prior step procedure, else use the odd power procedure.

Example 4: Evaluate $\int \sin^3(x)\cos^4(x)dx$.

odd

take one odd off and group with dx

This will be your du'ish

$$= \int \sin^2(x)\cos^4(x)\cdot \sin(x)dx \qquad \text{use } \sin^2(x) = 1-\cos^2(x)$$

$$= \int (1-\cos^2(x))\cos^4(x)\cdot \sin(x)dx \qquad \text{Let } u = \cos(x)$$

$du = -\sin(x)dx$

$\Rightarrow -du = \sin(x)dx$

$$= \int (1-u^2)u^4(-du)$$

$$= \int (u^6 - u^4)du$$

$$= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$$

$$= \frac{1}{7}\cos^7(x) - \frac{1}{5}\cos^5(x) + C$$

Example 5: Evaluate $\int \sin^2(x)\cos^2(x)dx$. use half angle

$$\sin^2(A) = \frac{1-\cos(2A)}{2}$$

$$\cos^2(B) = \frac{1+\cos(2B)}{2}$$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)\cdot\left(\frac{1+\cos(2x)}{2}\right)dx \qquad (1-c)(1+c) = 1^2 - c^2$$

$$= \frac{1}{4}\int (1-\cos^2(2x))dx \qquad \text{use half angle again! } B = 2x$$

$$= \frac{1}{4}\int \left(1 - \left(\frac{1+\cos(4x)}{2}\right)\right)dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos(4x)\right)dx$$

$$= \frac{1}{8}\int (1-\cos(4x))dx$$

$$= \frac{1}{8}\left(x - \frac{1}{4}\sin(4x)\right) + C$$

Sometimes more than one way:

$$\int \sec^4(x)dx = \int \sec^2(x)\sec^2(x)dx \qquad \sec^2(x) = 1+\tan^2(x)$$

$$= \int (1+\tan^2(x))\sec^2(x)dx \qquad u = \tan(x) \quad du = \sec^2(x)dx$$

$$= \int (1+u^2)du$$

$$= u + \tfrac{1}{3}u + C$$

$$= \tan(x) + \tfrac{1}{3}\tan^3(x) + C$$

2.5.3 Lonely Powers of Tangent or Secant

Procedure: Integral of the Form $\int \tan^m(x)dx$ or $\int \sec^n(x)dx$

- $n \geq 3$:
  1. Pull off a $\sec^2(x)$ term and perform IBP with $dv = \sec^2(x)dx$.
  2. Convert tangent terms using $\tan^2(x) = \sec^2(x) - 1$, provided they are existent.
  3. Complete the integration if you are at a basic integral of $\sec^2(x)$ or $\sec(x)$. Otherwise, collect the common (original) integral on both sides and solve for it. You may have integrals of secants in your new integral still.
  4. If the remaining secant integrals are $\sec(x)$ or $\sec^2(x)$, finish evaluation. Else, repeat the above steps to continue the reduction.

- $m \geq 3$ is odd:
  1. Pull off $\tan(x)$ and multiply by $\frac{\sec(x)}{\sec(x)}$ to form $\int \tan^m(x)dx = \int \tan^{m-1}(x)\frac{\sec(x)\tan(x)}{\sec(x)}dx$.
  2. Construct $du = \sec(x)\tan(x)dx$ and convert the remaining tangent terms to secants using $\tan^2(x) = \sec^2(x) - 1$.
  3. Complete your u-substitution using $u = \sec(x)$ and integrate.

- $m \geq 2$ is even:
  1. Convert all your tangent terms into secants using the identity $\tan^2(x) = \sec^2(x) - 1$.
  2. If the result is a basic integral, integrate it. Otherwise, expand the result to construct a sum of powers of secants.
  3. Integrate the basic integral terms of $\sec^2(x)$, $\sec(x)$ and constants, then use the procedure of integrating higher powers of secants mentioned above.

Example 6: Evaluate $\int \sec^4(x)dx$. pull off $\sec^2(x)$.

use IBP with $dv = \sec^2(x)dx$

So $u = \sec^2(x) \quad du = 2\sec(x)\sec(x)\tan(x)dx$

$dv = \sec^2(x)dx \quad v = \tan(x)$

$$= \int \sec^2(x)\sec^2(x)dx$$

$$= \sec^2(x)\tan(x) - \int 2\sec^2(x)\tan^2(x)dx$$

$$= \sec^2(x)\tan(x) - \int 2\sec^2(x)(\sec^2(x)-1)dx \qquad \text{use } \tan^2(x)+1 = \sec^2(x)$$

$$= \sec^2(x)\tan(x) - 2\int \sec^4(x)dx + 2\int \sec^2(x)dx$$

Have $I = \sec^2(x)\tan(x) - 2I + 2\tan(x) \Rightarrow I + 2I = \text{stuff}$

$\Rightarrow I = \frac{1}{3}\text{stuff} + C$

Solve $I = \frac{1}{3}\sec^2(x)\tan(x) + \frac{2}{3}\tan(x) + C$ //

Example 7: Evaluate $\int \tan^3(x)dx$.

$$= \int \tan^2(x) \times \frac{\tan(x)\sec(x)}{\sec(x)} dx$$

$\tan^2(x) = \sec^2(x) - 1$

$$= \int (\sec^2(x) - 1) \times \frac{\tan(x)\sec(x)}{\sec(x)} dx$$

Let $u = \sec(x)$

$du = \sec(x)\tan(x)dx$

$$= \int \frac{\sec^2(x) - 1}{\sec(x)} \times \tan(x)\sec(x)dx$$

$$= \int \frac{u^2 - 1}{u} du$$

$$= \int \left(u - \frac{1}{u}\right) du$$

$$= \frac{1}{2}u^2 - \ln|u| + C$$

$$= \frac{1}{2}\sec^2(x) - \ln|\sec(x)| + C$$

2.5.4 Powers of Tangent and Secant with Same Input

Procedure: Integral of the Form $\int \tan^m(x)\sec^n(x)dx$

- m is odd:
  1. Pull off a single $\tan(x)$ and $\sec(x)$ term and construct $du = \sec(x)\tan(x)dx$.
  2. Convert the remaining tangent terms using the identity $\tan^2(x) = \sec^2(x) - 1$.
  3. Complete the u-substitution with $u = \sec(x)$ and integrate.
- n is even:
  1. Pull of a $\sec^2(x)$ term and construct $du = \sec^2(x)dx$.
  2. Convert the remaining secant terms using the identity $\sec^2(x) = \tan^2(x) + 1$.
  3. Complete the u-substitution with $u = \tan(x)$ and integrate.
- m is even and n is odd: (evil case)
  1. Convert all the tangent terms to secants using the identity $\tan^2(x) = \sec^2(x) - 1$.
  2. Expand your integrand to get a sum of powers of secants.
  3. Use the appropriate integration techniques for integrating lonely powers of secants.

$n=1$
$m=3$ Case m is odd

Example 7: Evaluate $\int \sec(x)\tan^3(x)dx$.

pull off a $\tan(x)\sec(x)$

$$= \int \tan^2(x)\cdot \underbrace{\tan(x)\sec(x)\,dx}_{du}$$

use $\tan^2(x) = \sec^2(x) - 1$

$$= \int (\sec^2(x)-1)\cdot \tan(x)\sec(x)dx$$

Let $u = \sec(x)$
$\Rightarrow du = \sec(x)\tan(x)dx$

$$= \int (u^2-1)du$$

$$= \frac{1}{3}u^3 - u + C$$

$$= \frac{1}{3}\sec^3(x) - \sec(x) + C$$

$m=2,\ n=4$ Case n is even

Example 8: Evaluate $\int \tan^2(x)\sec^4(x)dx$.

pull off $\sec^2(x)$

$$= \int \tan^2(x)\sec^2(x)\cdot \underbrace{\sec^2(x)dx}_{= du\text{'ish}}$$

Convert $\sec^2(x) = \tan^2(x)+1$

$$= \int \tan^2(x)(\tan^2(x)+1)\cdot \sec^2(x)\,dx$$

Let $u = \tan(x)$
$du = \sec^2(x)dx$

$$= \int u^2(u^2+1)du$$

$$= \int (u^4+u^2)du$$

$$= \frac{1}{5}u^5 + \frac{1}{3}u^3 + C$$

$$= \frac{1}{5}\tan^5(x) + \frac{1}{3}\tan^3(x) + C$$

2.6 (Section 8.4) Trigonometric Substitution

2.6.1 Forms of Trigonometric Subsitution

Procedure: Integrands Containing $u^2 \pm a^2$ or $a^2 - u^2$ where $u = f(x)$.

The procedure for all the following is essentially the same. Use the substitution $u = a \times$ (Appropriate Trig Function), use techniques of trigonometric integrals to complete the integration, then convert back.

- Containing $a^2 - u^2$:

  1. Let $u = a\sin(\theta)$ and compute $du = a\cos(\theta)d\theta$.
  2. Complete the substitution and use an appropriate integration technique to compute the resulting trigonometric integral.
  3. Convert all terms back. If you encounter a lonely θ use the identity $\theta = \arcsin(u/a)$. If you encounter a double angle, reduce to a single angle using a double angle identity. If you encounter a non-sine function, represent the identity $\sin(\theta) = \dfrac{u}{a}$ as a triangle and solve for the values of other trigonometric functions.

- Containing $a^2 + u^2$:

  1. Let $u = a\tan(\theta)$ and compute $du = a\sec^2(\theta)d\theta$.
  2. Complete the substitution and use an appropriate integration technique to compute the resulting trigonometric integral.
  3. Convert all terms back. If you encounter a lonely θ use the identity $\theta = \arctan(u/a)$. If you encounter a double angle, reduce to a single angle using a double angle identity. If you encounter a non-sine function, represent the identity $\tan(\theta) = \dfrac{u}{a}$ as a triangle and solve for the values of other trigonometric functions.

- Containing $u^2 - a^2$:

  1. Let $u = a\sec(\theta)$ and compute $du = a\sec(\theta)\tan(\theta)d\theta$.
  2. Complete the substitution and use an appropriate integration technique to compute the resulting trigonometric integral.
  3. Convert all terms back. If you encounter a lonely θ use the identity $\theta = \text{arcsec}(u/a)$. If you encounter a double angle, reduce to a single angle using a double angle identity. If you encounter a non-sine function, represent the identity $\sec(\theta) = \dfrac{u}{a}$ as a triangle and solve for the values of other trigonometric functions.

★ All Subs : $u = a \times \text{Trig}$ for an appropriate trig.

$16 - 16\sin^2(\theta) = 16(1-\sin^2(\theta))$
$= 16\cos^2(\theta)$

2.6.2 Sine Substitutions $(a^2 - u^2)$

Note: Domain Restriction for Sine Substitutions

In these integrals we always have a domain restriction of $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Example 1: Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{16-x^2}}$. $a = 4$, $u = x$

Let $u = a \times \text{Trig}$ ← have $a^2 - u^2$ which is a sine sub.
$\Rightarrow x = 4 \cdot \sin(\theta)$

Compute $dx = 4 \cdot \cos(\theta)\, d\theta$

$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx = \int \frac{1}{(4\sin(\theta))^2\sqrt{16-(4\sin(\theta))^2}}\,4\cos(\theta)\,d\theta$$

$$= \int \frac{4\cos(\theta)}{16\sin^2(\theta)\sqrt{16(1-\sin^2(\theta))}}\,d\theta$$

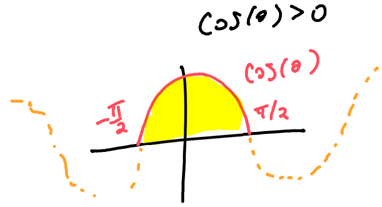
$$= \frac{4}{16 \cdot 4}\int \frac{\cos(\theta)}{\sin^2(\theta)\sqrt{\cos^2(\theta)}}\,d\theta$$

$$= \frac{1}{16}\int \frac{\cos(\theta)}{\sin^2(\theta)|\cos(\theta)|}\,d\theta$$

$\cos(\theta) > 0$

$$= \frac{1}{16}\int \frac{\cancel{\cos(\theta)}}{\sin^2(\theta)\cancel{\cos(\theta)}}\,d\theta$$

$$= \frac{1}{16}\int \frac{1}{\sin^2(\theta)}\,d\theta = \frac{1}{16}\int \csc^2(\theta)\,d\theta = -\frac{1}{16}\cot(\theta) + C$$



Implied domain rest. for sine sub within $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

★ Unacceptable answer

$x = 4\sin(\theta)$
$\sin(\theta) = x/4 \Rightarrow \theta = \arcsin(x/4)$

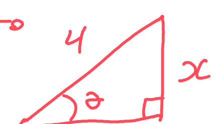
$\therefore -\frac{1}{16}\cot(\theta) + C$
$= -\frac{1}{16}\cot(\arcsin(\frac{x}{4})) + C$

Don't do this!!!

Acceptable answer

$x = 4\sin(\theta) \Rightarrow \sin(\theta) = \frac{x}{4}$

map to



pythagoras → $\sqrt{16-x^2}$

So $I = -\frac{1}{16}\cot(\theta) + C$
$= -\frac{1}{16} \times \frac{\text{Adj}}{\text{opp}} + C$
$= -\frac{1}{16} \times \frac{\sqrt{16-x^2}}{x} + C$

2.6.3 Tangent Substitutions $(a^2 + u^2)$

Note: Domain Restriction for Tangent Substitutions

In these integrals we always have a domain restriction of $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Example 2: Evaluate $\int \frac{dx}{\sqrt{9x^2+4}}$. $= \int \frac{dx}{\sqrt{(3x)^2 + 2^2}}$

$a = 2$
$u = 3x$

$u = a \times \text{Trig}$ u^2+a^2 is tan

$\Rightarrow 3x = 2\tan(\theta)$

$\Rightarrow x = \frac{2}{3}\tan(\theta)$

Then $dx = \frac{2}{3}\sec^2(\theta)\,d\theta$

$$\int \frac{1}{\sqrt{(3x)^2+4}}\,dx = \int \frac{1}{\sqrt{(2\tan(\theta))^2+4}} \cdot \frac{2}{3}\sec^2(\theta)\,d\theta$$

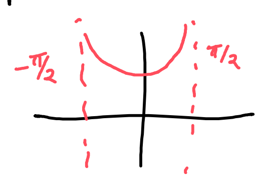
$$= \frac{2}{3}\int \frac{1}{\sqrt{4(\tan^2(\theta)+1)}}\sec^2(\theta)\,d\theta$$

$\tan^2(\theta) + 1 = \sec^2(\theta)$

$$= \frac{2}{3}\cdot\frac{1}{2}\int \frac{1}{\sqrt{\sec^2(\theta)}}\sec^2(\theta)\,d\theta$$

$$= \frac{1}{3}\int \frac{\sec^2(\theta)}{|\sec(\theta)|}\,d\theta$$

Implied domain $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$



$\sec(\theta) > 0$

$$= \frac{1}{3}\int \frac{\sec^2(\theta)}{\sec(\theta)}\,d\theta$$

$$= \frac{1}{3}\int \sec(\theta)\,d\theta$$

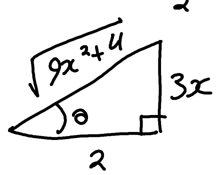
$$= \frac{1}{3}\ln|\sec(\theta) + \tan(\theta)| + C$$

$$= \frac{1}{3}\ln\left|\frac{\sqrt{9x^2+4}}{2} + \frac{3x}{2}\right| + C$$

Look at $3x = 2\tan(\theta)$

$\Rightarrow \tan(\theta) = \frac{3x}{2}$

solve



2.6.4 Secant Substitutions $(u^2 - a^2)$

Note: Domain Restriction for Secant Substitutions

In these integrals we always have a domain restriction of $0 < \theta < \frac{\pi}{2}$ if $u > a$ or $\frac{\pi}{2} < \theta < \pi$ if $u < -a$.

Example 3: Evaluate $\displaystyle\int \frac{dx}{\sqrt{16x^2 - 1}}$ if $x < -\frac{1}{4}$.

$$= \int \frac{dx}{\sqrt{(4x)^2 - 1}}$$

$x < -\frac{1}{4} \overset{\text{implied}}{\Longrightarrow} \frac{\pi}{2} < \theta < \pi$

$u = 4x \qquad a = 1$

So $u = a \times \text{Trig}$

$4x = \sec(\theta)$

$\Rightarrow x = \frac{1}{4}\sec(\theta)$

$\Rightarrow dx = \frac{1}{4}\sec(\theta)\tan(\theta)\,d\theta$

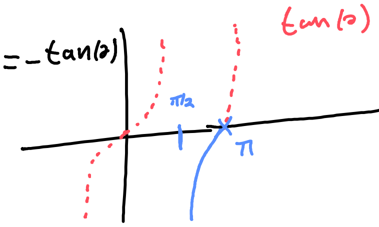
$$= \int \frac{\frac{1}{4}\sec(\theta)\tan(\theta)}{\sqrt{\sec^2(\theta) - 1}}\,d\theta$$

$\sec^2(\theta) - 1 = \tan^2(\theta)$

$$= \frac{1}{4}\int \frac{\sec(\theta)\tan(\theta)}{|\tan(\theta)|}\,d\theta$$

$|\tan(\theta)| = -\tan(\theta)$

$\tan(\theta) < 0$



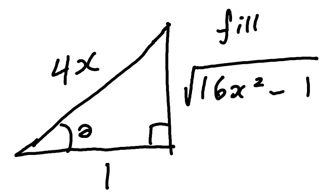
$$= \frac{1}{4}\int \frac{\sec(\theta)\cancel{\tan(\theta)}}{-\cancel{\tan(\theta)}}\,d\theta$$

$$= -\frac{1}{4}\int \sec(\theta)\,d\theta$$

$$= -\frac{1}{4}\ln|\sec(\theta) + \tan(\theta)| + C$$

$$= -\frac{1}{4}\ln\left|4x + \sqrt{16x^2 - 1}\right| + C$$

use $4x = \sec(\theta)$



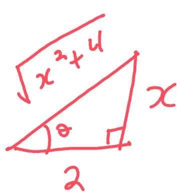
Example: Someone does work using $x = 2\tan(\theta)$ to get

$$\int f(x)\,dx = \theta + \sin(2\theta) + C$$

double angle

Sub back.

Triangle $\frac{x}{2} = \tan(\theta)$



use $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$$I = \theta + 2\sin(\theta)\cos(\theta) + C$$

$$= \theta + 2 \cdot \frac{x}{\sqrt{x^2+4}} \cdot \frac{2}{\sqrt{x^2+4}} + C$$

$$= \arctan\left(\frac{x}{2}\right) + \frac{4x}{x^2+4} + C$$

2.7 (Section 8.5) Partial Fractions

Example 1: Use the fact that $\dfrac{3x+11}{x^2-x-6} = \dfrac{4}{x-3} - \dfrac{1}{x+2}$ to evaluate $\displaystyle\int \frac{3x+11}{x^2-x-6}dx$.

$$\frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$\text{check} = \frac{4(x+2)}{(x-3)(x+2)} - \frac{(x-3)}{(x-3)(x+2)} = \frac{4x+8-x+3}{x^2-x-6} = \frac{3x+11}{x^2-x-6}$$

$$\int \frac{3x+11}{x^2-x-6}dx \overset{\text{unsimplify}}{=} \int \left(\frac{4}{x-3} - \frac{1}{x+2}\right)dx$$

$$= 4\ln|x-3| - \ln|x+2| + C$$

$$\int \frac{1}{x-a}dx \qquad u = x-a \Rightarrow du = dx$$

$$= \int \frac{1}{u}du = \ln|u| + C = \ln|x-a| + C$$

2.7.1 Procedure to "Un-simplifying" Rational Functions

Procedure: Partial Fractions

Suppose that $f(x) = \dfrac{P(x)}{Q(x)}$.

1. Make sure the degree of the numerator is less than the denominator. If not, perform long division.
2. Factor $Q(x)$ into linear terms and irreducible quadratic terms. A reminder that a quadratic is irreducible (i.e. does not factor further) if for ax^2+bx+c we gave $b^2-4ac<0$.
3. If a linear term $ax+b$ has maximal power m then suggest the following terms

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

   in the un-simplification where $A_1, A_2, ..., A_m$ are unknown constants.

4. If a an irreducible quadratic term ax^2+bx+c has maximal power n then suggest the following terms

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

   in the un-simplification where $B_1, .., B_n$ and $C_1, ..., C_n$ are unknown constants.

5. Once you have fully suggested an equation, rid the entire expression of denominators by cross multiplying everything properly. Then, expand everything.
6. Match coefficients of polynomials to form a system of equations for all unknown constants and solve this system.

and Linear irreducible

x^2+x+1
$a=1,\ b=1,\ c=1$
$b^2-4ac = 1-4<0$
irreducible

Example 2: Suggest a form for the partial fraction decomposition of $f(x) = \dfrac{3x+4}{(x-1)^3(x-2)(x^2+x+1)^2}$.

Linear $\times 3$

$$f(x) = \underbrace{\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}}_{\text{Corresponds to } (x-1)^3} + \underbrace{\frac{D}{x-2}}_{\text{to } (x-2)}$$

$$+ \underbrace{\frac{Ex+F}{x^2+x+1} + \frac{Gx+H}{(x^2+x+1)^2}}_{\text{Corr. to } x^2+x+1 \text{ irreducible}}$$

2.7.2 Simplest Case: All Linear Factors, No Repeats

Example 3: Evaluate $\displaystyle\int \frac{1}{x^2-x-6}\,dx$.

$x^2-x-6 = (x-3)(x+2)$

b^2-4ac
$= (-1)^2-4(1)(-6)$
$= 1+24>0$
Not irreducible

So we suggest

$$\frac{1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$\times (x-3)(x+2)$

$$\Rightarrow\quad 1 = \frac{A}{\cancel{(x-3)}}\cancel{(x-3)}(x+2) + \frac{B}{\cancel{x+2}}(x-3)\cancel{(x+2)}$$

$$\Rightarrow\quad 1 = A(x+2) + B(x-3)$$

Technique #1 (always works)

expand $\Rightarrow 1 = Ax + 2A + Bx - 3B$

Collect terms $\Rightarrow 1 = (A+B)x + (2A-3B)$

$\Rightarrow 0x + 1 = (A+B)x + (2A-3B)$

$$\begin{cases} A+B=0 \\ 2A-3B=1 \end{cases}$$

Solve a system of linear equations

etc...

Technique #2 (better, when it works)

$1 = A(x+2) + B(x-3)$ holds for all x $\Rightarrow$ holds for any x we pick.

pick $x=3$,

$1 = A(3+2) + B(0) \Rightarrow A = 1/5$

pick $x=-2$,

$1 = A(0) + B(-2-3) \Rightarrow B = -1/5$

Thus $\dfrac{1}{(x-3)(x+2)} = \dfrac{1}{5}\cdot\dfrac{1}{x-3} - \dfrac{1}{5}\cdot\dfrac{1}{x+2}$

$$\Rightarrow I = \frac{1}{5}\int\left(\frac{1}{x-3} - \frac{1}{x+2}\right)dx = \frac{1}{5}\left(\ln|x-3| - \ln|x+2|\right)+C$$

2.7.3 Using Partial Fractions to Evaluate an Integral

Example 4: Evaluate $\int \frac{x^5+x^3+x+1}{x^4+x^2}dx$. Notice that the degree of the numerator is not less than the denominator.

$\frac{P(x)}{Q(x)}$ Note $\deg(P) = 5 > 4 = \deg(Q)$. use long division first

x ← quotient

$x^4+x^2 \,\big)\, x^5+0x^4+x^3+0x^2+x+1$

$-(\cancel{x^5}+0x^4+\cancel{x^3})$

$0+0x^2+x+1$

Can't reduce further

ie $x+1$ = Remainder

$$\text{Thus } I = \int\left(x + \frac{x+1}{x^4+x^2}\right)dx = \frac{1}{2}x^2 + \int\frac{x+1}{x^4+x^2}dx$$

$\deg(P) = 1$

$\deg(Q) = 4$

$$\Rightarrow \frac{x+1}{x^4+x^2} \overset{\text{factor}}{=} \frac{x+1}{x^2(x^2+1)} = \frac{A}{x-0} + \frac{B}{(x-0)^2} + \frac{Cx+D}{x^2+1}$$

$(x-0)^2$ Linear repeated

irreducible

$\times x^2(x^2+1)$

$$\Rightarrow \frac{x+1}{\cancel{x^2(x^2+1)}} \times \cancel{x^2(x^2+1)} = \frac{A}{\cancel{x}} \times x^{\cancel{2}}(x^2+1) + \frac{B}{\cancel{x^2}} \times \cancel{x^2}(x^2+1) + \frac{Cx+D}{\cancel{x^2+1}} \times x^2\cancel{(x^2+1)}$$

$$\Rightarrow x+1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$ holds for all x.

Try to pick x to solve for some constants

$\Rightarrow$ pick $x=0 \Rightarrow 0+1 = 0 + B(0^2+1) + 0 \Rightarrow B = 1$

update $\Rightarrow x+\cancel{1} = Ax(x^2+1) + (x^2+\cancel{1}) + (Cx+D)x^2$

$\Rightarrow -x^2+x = Ax(x^2+1) + (Cx+D)x^2$

$\div x \Rightarrow -x+1 = A(x^2+1) + (Cx+D)x$

pick $x=0 \Rightarrow 0+1 = A(0+1)+0 \Rightarrow A = 1$

update $\Rightarrow -x+\cancel{1} = x^2+\cancel{1} + (Cx+D)x$

$\div x \Rightarrow -x^2-x = (Cx+D)x$

$\Rightarrow -x-1 = Cx+D$

pick $x=0 \Rightarrow 0-1 = 0+D \Rightarrow D = -1$

Thus update $-x-\cancel{1} = Cx-\cancel{1}$

$\Rightarrow -x = Cx$

$\Rightarrow C = -1$

u-sub $u = x^2+1$

$$\text{Thus } \int\frac{x+1}{x^4+x}dx = \int\left(\frac{1}{x} + \frac{1}{x^2} + \frac{-x}{x^2+1} + \frac{-1}{x^2+1}\right)dx$$

Continue...

$$= \ln|x| - \frac{1}{x} - \frac{1}{2}\ln|x^2+1| - \arctan(x) + C$$

So overall

$$I = \frac{1}{2}x^2 + \ln|x| - \frac{1}{x} - \frac{1}{2}\ln|x^2+1| - \arctan(x) + C$$

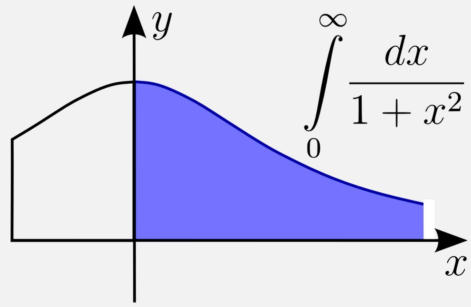
2.8 (Section 8.8) Improper Integrals

2.8.1 Type I Improper Integrals and Convergence

Definition: Type I Improper Integrals

An integral with an infinite bound is defined in the limit sense and called a **Type I Integral**. Specifically,

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$



Example 1: Evaluate $\int_0^\infty \frac{1}{1+x^2}dx$.

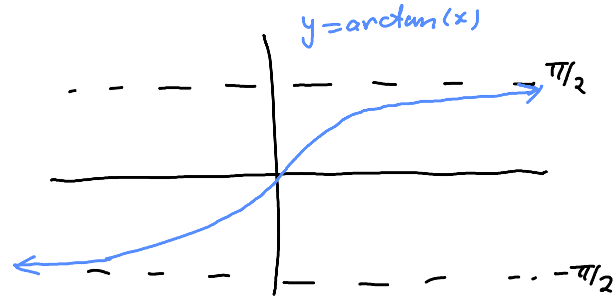
area from $x=0$ to $x=b$.

$$= \lim_{b\to\infty}\left(\int_0^b \frac{1}{1+x^2}dx\right)$$

$$= \lim_{b\to\infty}\left(\arctan(x)\Big|_0^b\right)$$

$$= \lim_{b\to\infty}\arctan(b) - \arctan(0)$$

$$= \frac{\pi}{2} - 0 = \pi/2$$



Example 2: Evaluate $\int_e^\infty \frac{1}{x\ln(x)}dx$.

$$= \lim_{b\to\infty}\int_e^b \frac{1}{x\ln(x)}dx$$

Let $u = \ln(x)$

$du = \frac{1}{x}dx$

Bounds

$x = e \Rightarrow u = \ln(e) = 1$

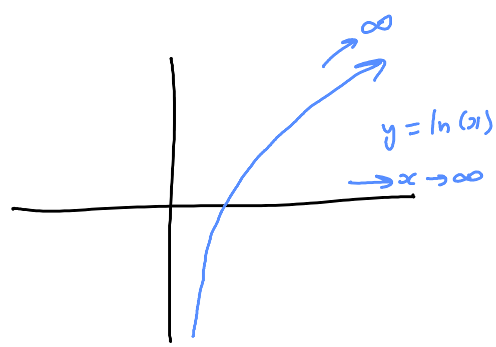
$x = b \Rightarrow u = \ln(b)$

$$= \lim_{b\to\infty}\int_1^{\ln(b)} \frac{1}{u}du$$

$$= \lim_{b\to\infty}\left(\ln|u|\Big|_1^{\ln(b)}\right)$$

$$= \lim_{b\to\infty}\left(\ln(\ln(b)) - \overset{0}{\ln(1)}\right)$$

$$= \lim_{b\to\infty}\ln(\underbrace{\ln(b)}_{\to\infty}) = \lim_{a\to\infty}\ln(a) = \infty$$



Definition: Convergence and Divergence

If an integral results in a finite value we say the integral is **Convergent**. Otherwise, if an integral is not convergent we say it is **Divergent**.

2.8.2 p-Integrals

Note: Studying Nature vs. Obtaining Values

Most integrals can't be evaluated in closed form. Furthermore, if we want to evaluate an integral numerically, we may always code a script on a computer to do so. What we care about from a human perspective is whether a finite value is obtainable or not (i.e. whether or not it converges) and not about what that specific value is. This is done by constructing a **Proof Argument**, usually formatted as to compare the nature of an integral to a simpler well known one.

Definition and Theorem: Convergence of p-Integrals

e.g. $\int_5^\infty \frac{dx}{x^2}$ is a 2-integral $= -\frac{1}{x}\Big|_5^\infty = -\frac{1}{\infty} + \frac{1}{5} = 0 + \frac{1}{5} = \text{finite.!}$

A Type I **p-Integral** is an integral of the form $I = \int_a^\infty \frac{dx}{x^p}$ where $a > 0$. Furthermore,

- If $p > 1$ then I converges; and
- If $p \leq 1$ then I diverges.

e.g. $\int_8^\infty \frac{dx}{\sqrt{x}} = \int_8^\infty \frac{dx}{x^{1/2}}$ $\quad p = 1/2 \leq 1$ satisfies $\therefore$ Diverges

2.8.3 Direct Comparison Test

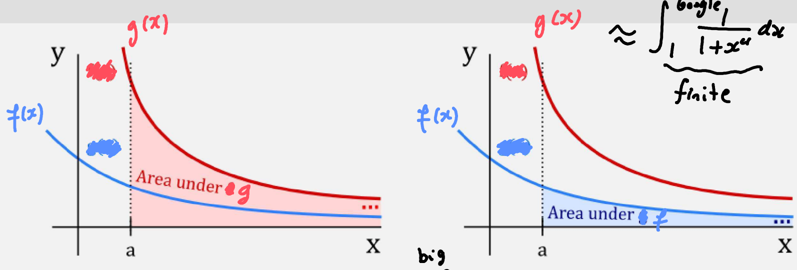
$\int_1^\infty \frac{1}{1+x^4}dx = \int_1^{\text{Google}} \frac{1}{1+x^4}dx + \int_{\text{Google}}^\infty \frac{1}{1+x^4}$

$\approx \underbrace{\int_1^{\text{Google}} \frac{1}{1+x^4}dx}_{\text{finite}} + \int_{\text{Google}}^\infty \frac{1}{x^4}dx$

Theorem: Direct Comparison Test (DCT)

Let f and g be continuous functions on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Let $F = \int_a^\infty f(x)dx$ and $G = \int_a^\infty g(x)dx$.

- If G converges then F converges; and
- If F diverges then G diverges.



Intuition: $\frac{1}{1+x^4} \overset{\text{big } x \approx \infty}{\sim} \frac{1}{x^4}$ $\quad p = 4 > 1$ converges

Example 3: Determine whether or not the integral $\int_1^\infty \frac{dx}{1+x^4}$ converges or diverges. Write a complete proof of your claim using an appropriate theorem in your argument.

Note that since $\underset{\text{Bigger}}{x^4+1} \geq \underset{\text{Smaller}}{x^4}$ then $\frac{1}{1+x^4} \leq \frac{1}{x^4}$. Note as well that $0 \leq \frac{1}{1+x^4}$.

Form $G = \int_1^\infty \frac{1}{x^4}dx$ and $F = \int_1^\infty \frac{1}{1+x^4}dx$. Since $0 \leq \frac{1}{1+x^4} \leq \frac{1}{x^4}$ on $x \geq 1$

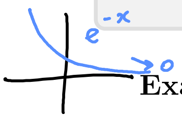
and $G = \int_1^\infty \frac{1}{x^4}dx$ is a p-integral with $p = 4 > 1$, and thus converges, then

by the DCT $F = \int_1^\infty \frac{1}{1+x^4}dx$ converges as well $\blacksquare$

2.8.4 Limit Comparison Test

Limit Comparison Test

Let f and g be continuous and positive functions on $[a, \infty)$ with $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ where L is positive and finite, i.e. $0 < L < \infty$. Then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge together or diverge together.



Example 4: Determine whether or not the integral $\int_1^\infty \frac{1-e^{-x}}{x}dx$ converges or diverges. Write a complete proof of your claim using an appropriate theorem in your argument.

Intuition: $f(x) = \frac{1-e^{-x}}{x} \overset{x\to\infty}{\sim} \frac{1-0}{x} = \frac{1}{x}$, (Not worth marks)

$f(x) = \frac{1-e^{-x}}{x}$ and $g(x) = \frac{1}{x}$ are continuous and positive on $x \geq 1$

Form,

$$L = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1-e^{-x}}{x} \div \frac{1}{x}$$
$$= \lim_{x\to\infty} \frac{1-e^{-x}}{\cancel{x}} \times \frac{\cancel{x}}{1} = \lim_{x\to\infty} (1-e^{-x}) = 1-0 = 1$$

as $L=1$ is positive and finite then by the LCT both $F = \int_1^\infty \frac{1-e^{-x}}{x}dx$ and $G = \int_1^\infty \frac{1}{x}dx$ either both converge or diverge. Since G is a p-integral with $p = 1 \underset{\leq 1}{\text{satisfies}}$ so it diverges.

$\therefore F = \int_1^\infty \frac{1-e^{-x}}{x}dx$ diverges as well ∎

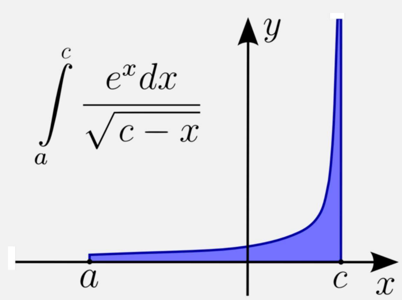
2.8.5 Type II Improper Integral

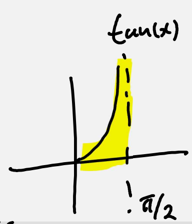
Definition: Type II Improper Integrals

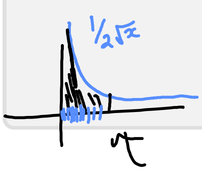
An integral with a summation about a discontinuity is defined in the limit sense and is called a **Type II Integral**. To illustrate, if $x = a$ is a vertical asymptote of $f(x)$ then

$$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$$

$$\int_a^c \frac{e^x dx}{\sqrt{c-x}}$$







what is $\int_0^{\pi/2} \tan(x)dx$?

Example 5: Evaluate $\int_0^4 \frac{1}{2\sqrt{x}}dx$.

$$= \lim_{a\to 0^+} \int_a^4 \frac{1}{2\sqrt{x}}dx = \lim_{a\to 0^+} \sqrt{x}\Big|_a^4$$
$$= \sqrt{4} - \lim_{a\to 0^+} \sqrt{a}$$
$$= 2 - \sqrt{0} = 2$$