Set 9: Expectation

Stat 260 A01: May 31, 2024

Probability tells us how likely an outcome is. The natural extension is "expectation" which tells us a long-run average of our random variable.

The ***expected value*** of a random variable X, denoted $E[X]$, is the long run *theoretical* average value of X.

For a ***discrete r.v.*** with pmf $f(x)$, the expected value, or mean value of X, is

$E(x) = \mu_x = \sum x_i \cdot f(x_i) = \sum x_i \cdot P(x = x_i)$

Example 1: Suppose that you play a game where you win the face-value of your die roll (i.e. if you roll a **2**, you win \$2). How much should you expect to win in a single round?

let X = winnings from one round of the dice game

x	1\$	2\$	3\$	4\$	5\$	6\$
$f(x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

→ equally likely outcomes

$E(x) = \mu_x = (1)(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6)$

$= \$3.50$

Notice: \$3.50 isnt a possible outcome and that's okay since $E(x)$ is the theoretical long run average of many rounds of the game

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Example 2: Suppose that a certain breed of sheep may give birth to 0, 1, 2, or 3 lambs each spring, with the following probabilities:

Number of offspring (x_i)	0	1	2	3
$f(x_i) = P(X = x_i) = p_i$	0.10	0.25	0.60	0.05

let X = no of lambs from a ewe

$E(x) = 0(0.1) + 1(0.25) + 2(0.6) + 3(0.05)$

$= 1.6$ lambs

Sometimes we want to know the expected value a ***function*** *of a r.v. rather than of the r.v. itself.*

The ***expectation of a function*** $g(x)$ corresponding to a discrete random variable X with pmf $f(x)$ is

$E[g(x)] = \sum g(x) \cdot f(x) = \sum g(x) \cdot P(X = x)$

Example 2 Continu-ewed... How much should he excpect from 1 ewe

After this season, the farmer wants to sell his ewes and all their lambs. He will get \$50 for the ewe and \$30 for each lamb.

Let X = no of lambs from a ewe

y = amount of money the farmer earns

$y = 30X + 50$

$$E[y] = \underbrace{(30(0)+50)}_{g(0)} \cdot \underbrace{(0.1)}_{f(0)} + \underbrace{(30(1)+50)}_{g(1)}\underbrace{(0.25)}_{f(1)} + \underbrace{(30(2)+50)}_{g(2)}\underbrace{(0.6)}_{f(2)} + \underbrace{(30(3)+50)}_{g(3)}\underbrace{(0.05)}_{f(3)} = \$98$$

alternative shorter method that works sometimes:

$E[y] = E[30x+50] = 30 \cdot E[x] + 50$

$= 30(1.6) + 50$

$= \$98$

Rules for Expectation: For a constant c,

(i) $E[X+c] = E[x] + E[c] = E[x] + c$

(ii) $E[c] = c$

(iii) $E[cX] = c\,E[x]$

- $E[aX+b] = a\,E[x] + b$

warning: $E[x^2] \neq E[x]^2$

Example 2 Continu-ewed...

Number of offspring (x_i)	0	1	2	3
$f(x_i) = P(X = x_i) = p_i$	0.10	0.25	0.60	0.05

Determine the following for the Sheep pmf:

(a) $E[X^2]$ $= (0)^2(0.01) + (1)^2(1.25) + (2)^2(0.6) + (3)^2(0.05)$

$= 3.1$ not equal $E(x)^2 \neq E(x^2)$

$(E[x])^2 = (1.6)^2 = 2.56$

(b) $E[X^3 - 2X + 5]$

$E[x^3 - 2x + 5] = [0^3 - 2(0) + 5](0.01) + [1^3 - 2(1) + 5](0.25) + [2^3 - 2(2) + 5](0.6) + [3^3 - 2(3) + 5](0.05)$

$= 8.2$

Example 2 Continu-ewed...

Suppose the farmer has 8 ewes and that this year, they have the following number of lambs:

$$3, 2, 0, 1, 2, 2, 1, 1$$

What is the mean of this sample? What does it have to do with $E[X]$?

mean $= 1.5$ lambs → a single sample mean just 1 farmer not all in the long run
$\bar{x}$

$E(X) = 1.6$ close

↳ Theoretical (not an actual sample) long run average (for many sheep, not just 8 sheep)

Textbook Readings and Practice Problems: *See Set 10*