MATH 202 Midterm 1

October 10th, 2024
3:30 PM – 4:20 PM

Instructions

- In the following box, write your first and last names displayed on Brightspace, and your student ID number. Do not include the V and first two 0's in your student ID number. For example, write 123456 instead of V00123456.

- This exam has **5 questions** and is out of **100 points**.
- Organise and show your work.
- You may use a Sharp EL-510R calculator for elementary arithmetic.
- Any unsupported answers will receive no credit.
- You may ask for scrap paper from the invigilator. The scrap paper will **NOT** be reviewed for grading.
- Some formulas can be found in the last page of this exam.

Do not turn this page over until instructed to do so

Question 1. Let $\vec{u} = \vec{i} + 2\vec{j} + \vec{k}$ and $\vec{v} = 3\vec{i} + 3\vec{j} + 5\vec{k}$ be 3D vectors, where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the standard unit vectors.

(1) (10%) Find the length of $\vec{u} + \vec{v}$.

(2) (10%) Find the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.

Solution. (1) $\vec{u} + \vec{v} = 4\vec{i} + 5\vec{j} + 6\vec{k}$. Hence,

$$|\vec{u} + \vec{v}| = \sqrt{4^2 + 5^2 + 6^2} = \sqrt{77}.$$

(2) The area is given by

$$|\vec{u} \times \vec{v}| = |\langle 7, -2, -3 \rangle| = \sqrt{62}$$

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Answer:

Question 2. **(1) (10%)** Describe the following set with a single equation or with a pair of equations: The line through the point $(-5, -4, 2)$ and parallel to the x-axis.

(2) (10%) Find the limit or show that the limit does not exist:

$$\lim_{(x,y)\to(1,2)} \frac{(x-1)^4}{(x-1)^2 + (y-2)^2}.$$

Solution. (1) $y = -4$ and $z = 2$.

(2) We use the polar coordinates after making the change of variables $a = x - 1$ and $b = y - 2$:

$$\begin{aligned}
\lim_{(x,y)\to(1,2)} \frac{(x-1)^4}{(x-1)^2 + (y-2)^2} &= \lim_{(a,b)\to(0,0)} \frac{a^4}{a^2 + b^2} \\
&= \lim_{r\to 0} \frac{r^4 \cos^4\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \\
&= \lim_{r\to 0} \frac{r^4\cos^4\theta}{r^2} \qquad (\cos^2\theta + \sin^2\theta = 1) \\
&= \lim_{r\to 0} r^2\cos^4\theta \\
&= 0.
\end{aligned}$$

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Answer:

Question 3 (20%). Find an equation of the plane that passes through the point $P = (1, 1, 1)$ and is perpendicular to $\vec{u} = \langle 1, 2, 3\rangle$ and $\vec{v} = \langle 3, 2, 1\rangle$. Write your answer in the form of $ax + by + cz = d$.

Solution. The question was meant to consider the plane as the one that passes through $(1, 1, 1)$ and has a normal vector given by $\vec{u}\times\vec{v}$. In this direction, we compute

$$\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \vec{k} \\
&= -4\vec{i} + 8\vec{j} - 4\vec{k}.
\end{aligned}$$

An equation of the plane is

$$-4x + 8y - 4z = -4 \cdot 1 + 8 \cdot 1 - 4 \cdot 1 = 0.$$

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Answer:

Question 4 (20%). For the functions $w = xy + yz + xz$, $x = 2u$, $y = 3v$, and $z = uv$, find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

Solution. We have

$$\begin{aligned}
\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \\
&= (y+z)\cdot 2 + (x+z)\cdot 0 + (x+y)\cdot v \\
&= (3v+uv)\cdot 2 + 0 + (2u+3v)\cdot v \\
&= 6v + 2uv + 2uv + 3v^2 = 6v + 4uv + 3v^2,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \\
&= (y+z)\cdot 0 + (x+z)\cdot 3 + (x+y)\cdot u \\
&= 0 + (2u+uv)\cdot 3 + (2u+3v)\cdot u \\
&= 6u + 3uv + 2u^2 + 3uv \\
&= 6u + 6uv + 2u^2.
\end{aligned}$$

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Answer:

Question 5 (20%). Let $z = F(x, y) = 2x^2 + 3y^2$, and $P = (1, 1)$.

(1) Find an equation of the line through P and tangent to $F(x, y) = 5$.

(2) Find an equation for the line through P and parallel to $\nabla F(P)$.

Solution. We have $\nabla F = \langle 4x, 6y \rangle$, and so, $\nabla F(P) = \langle 4, 6 \rangle$.

(1) $4x + 6y = 10$.
(2) $\langle 1, 1 \rangle + t\langle 4, 6 \rangle$, $-\infty < t < \infty$. ■

Answer:

Some Formulas

- $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$, where θ is the angle between $\vec{u}$ and $\vec{v}$, with $0 \le \theta \le \pi$.
- $\text{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$
- $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$
- The area of the parallelogram spanned by the 3D vectors $\vec{u}$ and $\vec{v}$ is equal to $|\vec{u}\times\vec{v}|$.
- The volume of the parallelepiped spanned by the 3D vectors $\vec{u}, \vec{v}, \vec{w}$ is $|(\vec{u}\times\vec{v})\cdot\vec{w}|$.
- Some trigonometric identities:

$$\begin{aligned}
&\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos(\theta),\\
&\cos^2\theta + \sin^2\theta = 1,\\
&\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta
\end{aligned}$$