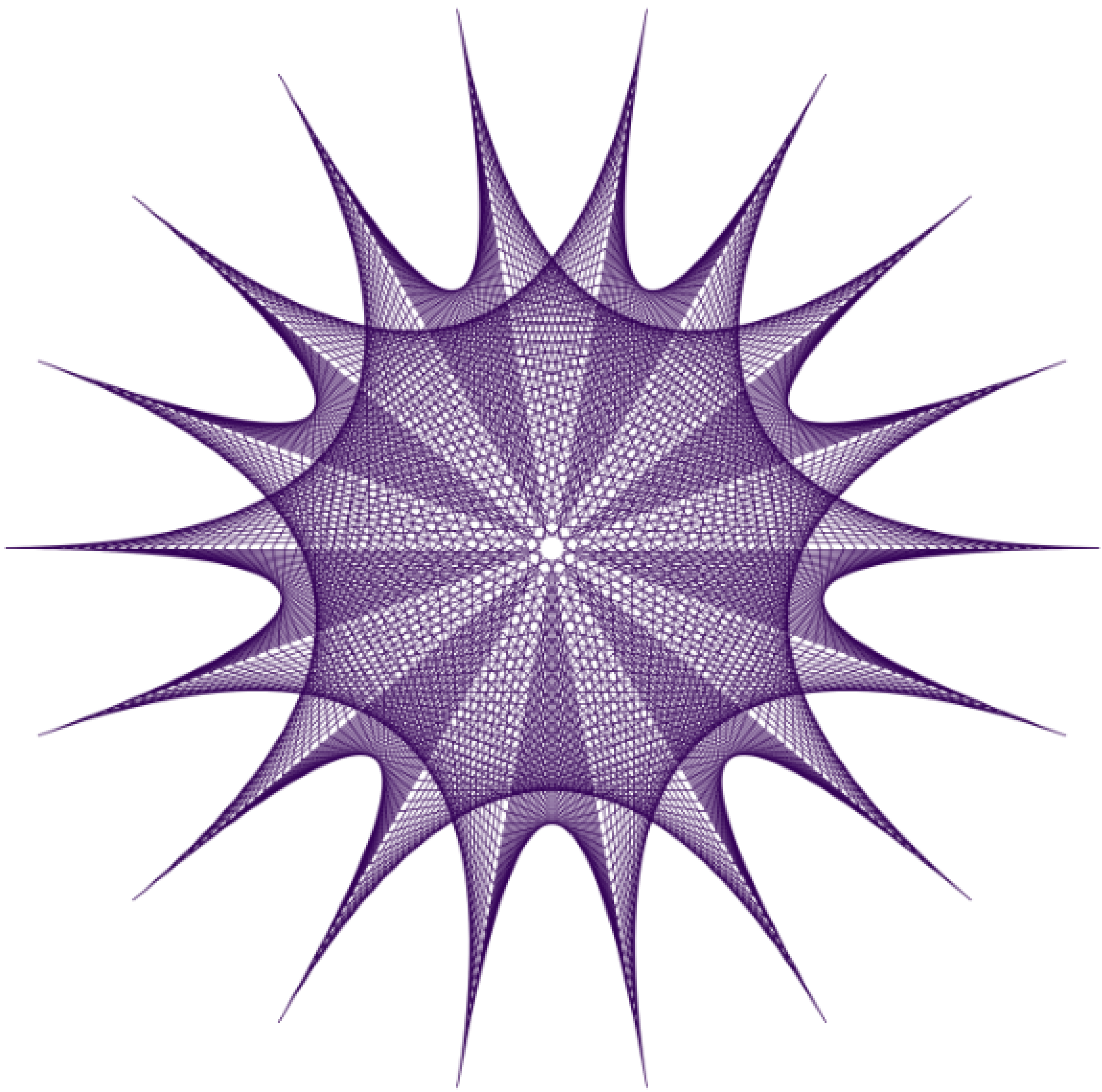
Math 100/109 Lecture Notes Pack

Calculus I



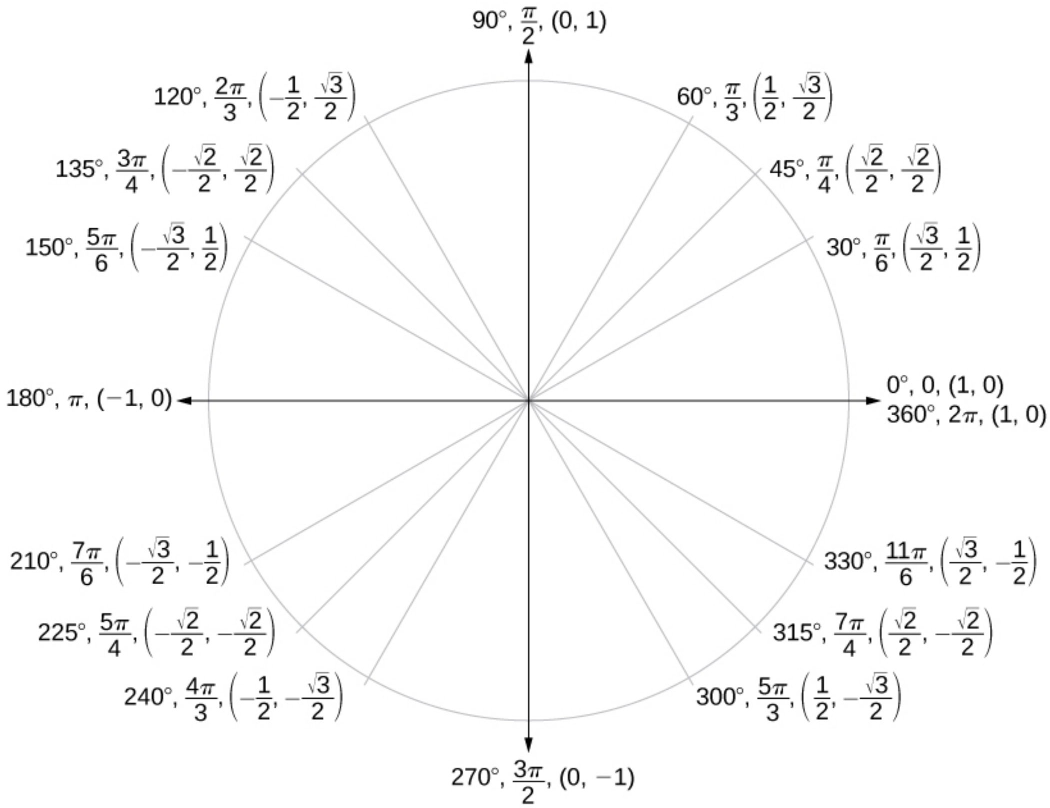
Notes By William Thompson based on
Thomas' Calculus Early Transendentals, Thomas, Weir, Hass and Heil

Contents

Chapter 1

Basic Tables and Formulas

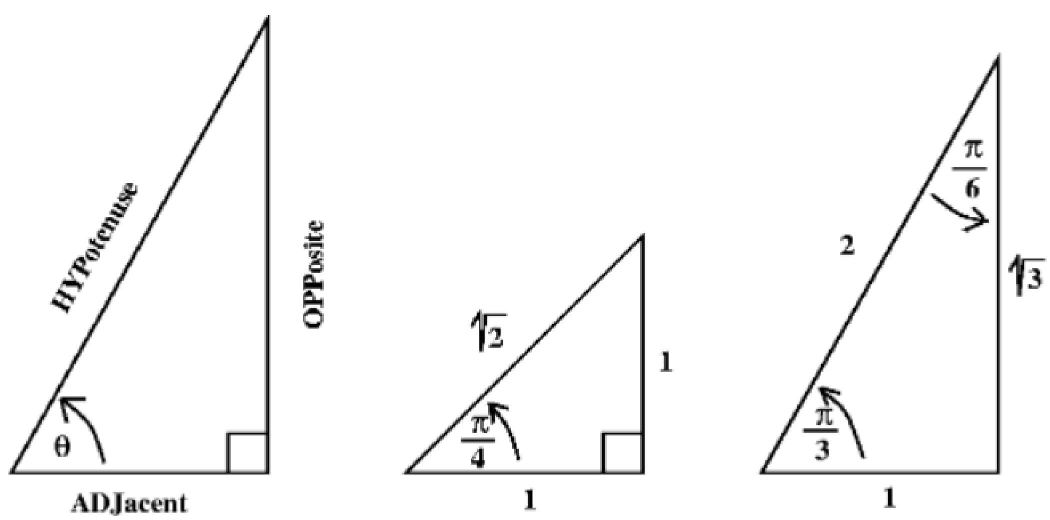
1.1 Unit Circle



1.2 Trigonometric Functions and Reference Triangles

$$\sin(\theta) = \frac{\text{OPP}}{\text{HYP}} \qquad \cos(\theta) = \frac{\text{ADJ}}{\text{HYP}} \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\text{OPP}}{\text{ADJ}}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\text{ADJ}}{\text{OPP}} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{HYP}}{\text{ADJ}} \qquad \csc(\theta) = \frac{\text{HYP}}{\text{OPP}}$$



1.3 Trigonometric Formulae

Trigonometric Formulæ

Basic Identities

- $\cos^2(\theta) + \sin^2(\theta) = 1$
- $\tan^2(\theta) + 1 = \sec^2(\theta)$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

Half Angle Identities

- $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$
- $\cos^2(\theta) = \dfrac{1 + \cos(2\theta)}{2}$

Ptolemy's Identities

- $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
- $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
- $\tan(A + B) = \dfrac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$

Product to Sum

- $\sin(A)\sin(B) = \dfrac{1}{2}\cos(A - B) - \dfrac{1}{2}\cos(A + B)$
- $\cos(A)\cos(B) = \dfrac{1}{2}\cos(A - B) + \dfrac{1}{2}\cos(A + B)$
- $\sin(A)\cos(B) = \dfrac{1}{2}\sin(A - B) + \dfrac{1}{2}\sin(A + B)$

Even and Odd Properties

- $\sin(-\theta) = -\sin(\theta)$
- $\cos(-\theta) = \cos(\theta)$
- $\tan(-\theta) = -\tan(\theta)$

1.4 Table of Derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\cot(x)$	$-\csc^2(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\operatorname{arccot}(x)$	$-\frac{1}{1+x^2}$
$\operatorname{arcsec}(x)$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\operatorname{arccsc}(x)$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
e^x	e^x
a^x	$a^x \ln(a)$ where $a > 0$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x\ln(a)}$

1.5 Derivative Rules

Derivative	**Name**
$(Cf(x))' = Cf'(x)$	Constant Rule
$(f(x)+g(x))' = f'(x)+g'(x)$	Sum Rule
$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$	Product Rule
$\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$	Quotient Rule
$(f(g(x)))' = f'(g(x)) \cdot g'(x)$	Chain Rule
$(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$	Inverse Rule
$\left(\int_a^x f(t)dt\right)' = f(x)$	First Fundamental Theorem of Calculus

1.6 Table of Antiderivatives

$f(x)$	$\int f(x)dx$
x^n	$\frac{1}{n+1}x^{n+1}+C \qquad \text{if } n \neq 1$
$\sin(x)$	$-\cos(x)+C$
$\cos(x)$	$\sin(x)+C$
$\tan(x)$	$\ln\lvert\sec(x)\rvert+C$
$\sec(x)$	$\ln\lvert\sec(x)+\tan(x)\rvert+C$
$\cot(x)$	$\ln\lvert\sin(x)\rvert+C$
$\csc(x)$	$-\ln\lvert\csc(x)+\cot(x)\rvert+C$
$\sec^2(x)$	$\tan(x)+C$
$\csc^2(x)$	$-\cot(x)+C$
$\sec(x)\tan(x)$	$\sec(x)+C$
$\csc(x)\cot(x)$	$-\csc(x)+C$
e^x	e^x+C
$\frac{1}{x}$	$\ln\lvert x\rvert+C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)+C$
$\frac{1}{1+x^2}$	$\arctan(x)+C$
$\frac{1}{x\sqrt{x^2-1}}$	$\operatorname{arcsec}\lvert x\rvert+C$
a^x	$\frac{a^x}{\ln(a)}+C$

Chapter 2

Limits and Continuity

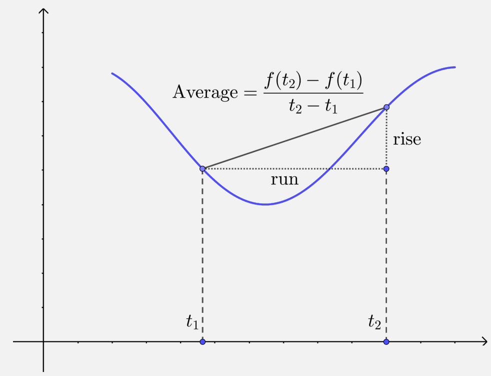
2.1 (Thomas 2.1) Rates of Change (Average)

2.1.1 Average Rate of Change

Definition: Average Rate of Change

The **Average Velocity (Rate of Change)** of a function $f(t)$ over $[t_1, t_2]$ is defined as

$$\text{Average Velocity} = \frac{\text{Displacement}}{\text{Time}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{\Delta f}{\Delta t}$$



Example 1: Suppose that you drop a rock off a cliff. The displacement under free fall is modeled by $f(t) = 4.9t^2$ (meters) where t is measured in seconds.

(a) Determine the average velocity of the rock after 2 seconds.

(b) Determine the average velocity of the rock between 1 second and 2 seconds.

(c) Determine the average velocity of the rock between 1 second and $1 + h$ seconds.

2.1.2 Difference Quotient, Secant Line, and Tangent Line

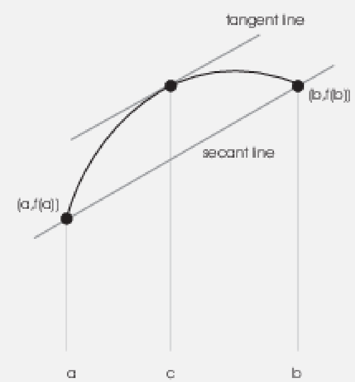
Definition: Difference Quotient

The **Difference Quotient** of a function $f(t)$ at the point t_0 is defined as the average velocity over $[t_0, t_0+h]$. That is, $D(h) = \dfrac{f(t_0+h) - f(t_0)}{h}$.

Definition: Secant Line and Tangent Line

The **Secant Line** of the graph $y = f(x)$ over $[a, b]$ is the line connecting $(a, f(a))$ to $(b, f(b))$.

The **Tangent Line** of the graph $y = f(x)$ AT $x = c$ is the line touching $y = f(x)$ at $x = c$ and does not intersect the graph of $y = f(x)$ elsewhere about $x = c$.



Note: The Slope of the Tangent Line

The slope of the tangent line at $t = t_0$ is given by the difference quotient $D(h)$ as $h \to 0$.

Example 2: Continue with the prior example.

(a) Determine the equation of the secant line to the graph of $y = f(t)$ over $[1, 2]$.

(b) Determine the equation of the tangent line to the graph of $y = f(t)$ at $t_0 = 1$.

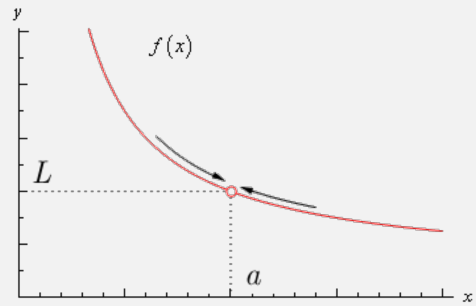
2.2 (Thomas 2.2) Limit of a Function and Limit Laws

2.2.1 (Casual) Definition of a Limit

Definition: The Limit of a Function

We say that the **Limit** of $f(x)$ as x approaches a is L if the values of $f(x)$ approach L as x approaches a.

We denote this by $\lim_{x \to a} f(x) = L$.



Example 1: Consider the functions $f(x) = \begin{cases} x+1, & x \neq 2 \\ 2, & x = 2 \end{cases}$ and $g(x) = x + 1$.

(a) Give a rough sketch of the graphs of both $y = f(x)$ and $y = g(x)$.

(b) Determine $f(2)$ and $g(2)$.

(c) Determine $\lim_{x \to 2} f(x)$ and $\lim_{x \to 2} g(x)$.

Note: Taking a Limit is NOT Evaluation!

By the prior example you'll notice that evaluation and taking a limit are not the same. Limits are what the function is approaching and evaluation is what the function actually is!

2.2.2 Graphical Examples of Limits that Exist

Example 2: Consider the function

$$g(x) = \begin{cases} \dfrac{x^2 - 9}{x + 3} & x \neq -3 \\ 2 & x = -3 \end{cases}$$

Graph $y = g(x)$. Then, use this to determine $\lim_{x \to -3} g(x)$ and $g(-3)$.

Example 3: Consider the function $f(x) = x(x + 2)(x - 2)$. Graph $y = f(x)$. Then, use this to determine $\lim_{x \to 1} f(x)$ and $f(1)$. What's different?

2.2.3 Graphical Examples of Limits that Don't Exist

Example 4: Consider $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Argue that $\lim_{x \to 0} f(x)$ does not exist by drawing a graph. What is $f(0)$?

Example 5: Give a rough graph of the **Topologist's Sine Curve** defined by $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x \leq 0 \end{cases}$. Argue this function does not have a defined limit at the origin. What is $f(0)$?

2.2.4 Limit Laws (Moving From Graphs to Algebra)

Limit Laws

Suppose that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then the following properties hold:

Property	**Name**
$\lim_{x \to a}(f(x) + g(x)) = L + M$	Sum Rule
$\lim_{x \to a}(f(x) - g(x)) = L - M$	Difference Rule
$\lim_{x \to a}(Cf(x)) = CL$	Constant Multiple Rule
$\lim_{x \to a} f(x)g(x) = LM$	Product Rule
$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$	Quotient Rule
$\lim_{x \to a}(f(x))^n = L^n$ if $n > 0$ is an integer	Power Rule
$\lim_{x \to a}(f(x))^{1/n} = L^{1/n}$ if $n > 0$ is an integer	Root Rule
$\lim_{x \to a} P(x) = P(a)$ if $P(x)$ is a polynomial	Continuity of Polynomials

Example 6: Compute $\lim_{x \to 4}(2x^2 + 4x + 5)$.

Example 7: Compute $\lim_{x \to 1} \frac{x^4 + x^2 - 1}{x^2 + 5}$.

Example 8: Compute $\lim_{x\to 1} \dfrac{2x^2 - x - 1}{x^2 - 1}$. *Hint:* $-x = -2x + x$.

2.2.5 Special Trigonometric Limits

Theorem: Special Sine and Cosine Limits

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad\qquad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$$

Example 9: Compute $\lim_{\theta\to 0} \dfrac{\sin(5\theta)}{\theta}$.

Example 10: Compute $\lim_{\psi\to 0} \dfrac{\sin(2\psi)}{\sin(3\psi)}$.

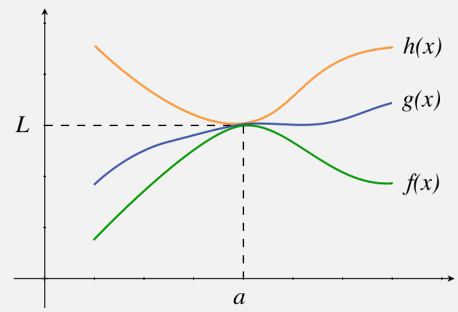
2.2.6 Sandwich (Squeeze) Theorem

Theorem: Sandwich (Squeeze) Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be functions defined on an interval about $x = a$ such that

$$f(x) \leq g(x) \leq h(x) \quad \text{and} \quad \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then $\lim_{x\to a} g(x) = L$.



Example 11: Suppose that $u(z)$ is a function satisfying

$$1 - \frac{z^2}{4} \leq u(z) \leq 1 + \frac{z^2}{4}$$

for all values of z. Determine $\lim_{z\to 0} u(z)$. Make sure to make complete arguments that justify your answer.

Example 12: Prove that $\lim_{x\to 0} x^2 \sin(x) = 0$ using the Squeeze Theorem.

2.3 (Thomas 2.4) One Sided Limits

2.3.1 Definition of the Below (Left) and Above (Right) Limits

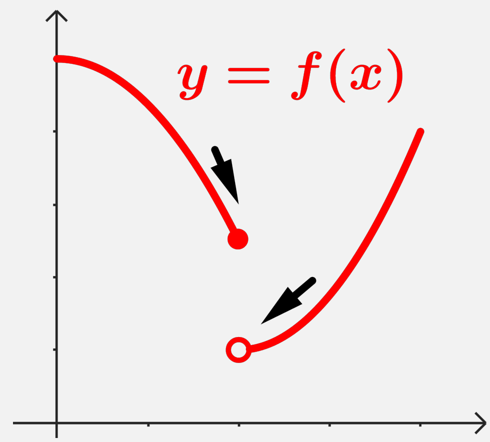
Definition

- **Below:** We say that $f(x)$ approaches M from **below** as x approaches a if $f(x) \to M$ as $x \to a$ for $x < a$ and denote this

$$\lim_{x \to a^-} f(x) = M$$

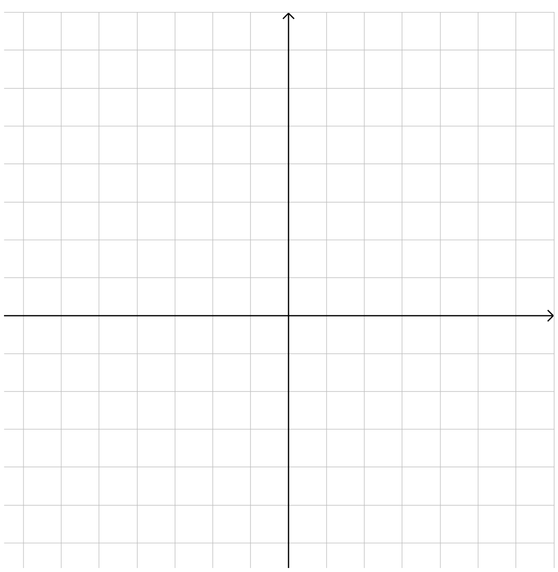
- **Above:** We say that $f(x)$ approaches N from **above** as x approaches a if $f(x) \to N$ as $x \to a$ for $x > a$ and denote this

$$\lim_{x \to a^+} f(x) = N$$



Example 1: Consider the function $f(x) = \dfrac{x}{|x|}$.

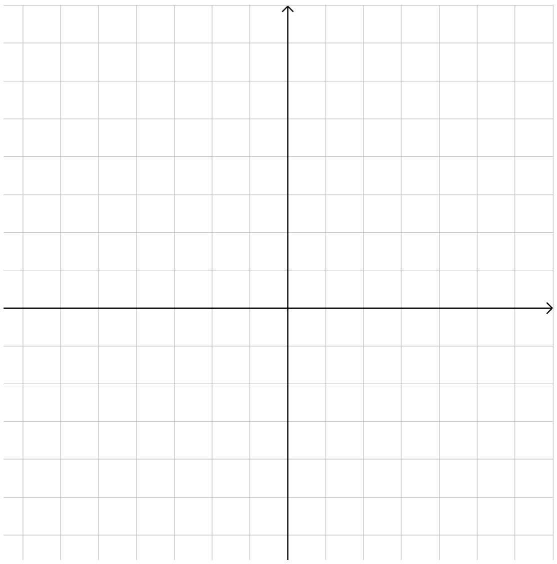
1. Graph $y = f(x)$. *Hint: The absolute function is defined as* $|x| = x$ *if* $x \geq 0$ *and* $|x| = -x$ *if* $x < 0$.



2. Use your graph to evaluate $\lim\limits_{x \to 0^-} f(x)$, $\lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to 0} f(x)$, provided they exist.

Example 2: Consider the **Floor Function** $f(x) = \lfloor x \rfloor$. This function rounds down to the nearest integer and keeps any integer inputs fixed, e.g. $f(2.1) = 2, f(\pi) = 3, f(8.97) = 8, f(5) = 5$.

(a) Graph this function.



(b) Determine $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.

(c) For what values of $x = x_0$ does $f(x_0) = \lim_{x \to x_0} f(x)$?

Theorem: Existence of a Limit

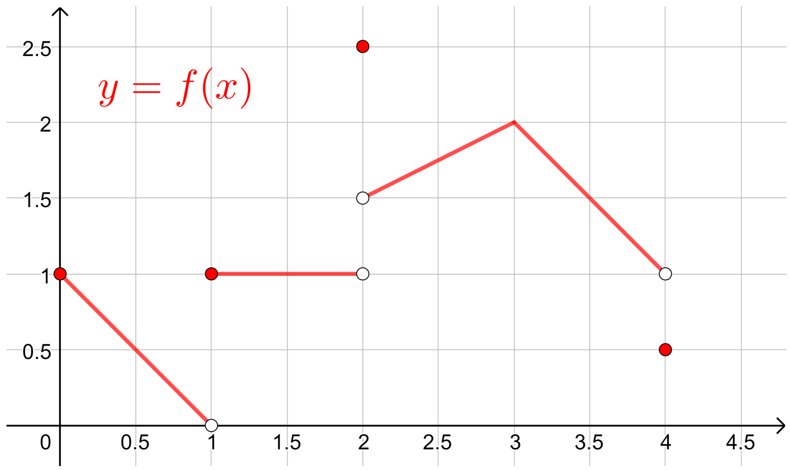
If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$ then the limit of $f(x)$ as $x \to a$ exists and $\lim_{x \to a} f(x) = L$.

Note: Limits at Endpoints of $[a, b]$ of a Domain

Let $f(x)$ be a function with domain $[a, b]$. It makes no sense to talk about a limit from below at $x = a$. Similarly, it makes no sense to talk about a limit from above at $x = b$. Thus by convention we just say that the limit as $x \to a$ or $x \to b$ is just defined to be whatever limit from below or above exists.

2.3.2 A Typical Comprehensive Example

Example 3: Consider the function $y = f(x)$ with a domain of $[0, 4]$ with the following graph:



(a) Fill in the following table on the values of the limits, limits above and below, and the value of the function at given values of x.

	(Below) $\lim\limits_{x\to a^-} f(x)$	(Above) $\lim\limits_{x\to a^+} f(x)$	(Ordinary Limit) $\lim\limits_{x\to a} f(x)$	(Evaluation) $f(a)$
$a = 0$				
$a = 1$				
$a = 2$				
$a = 3$				
$a = 4$				

(b) Determine $\lim\limits_{x\to 3} f(f(x))$ and $f(f(3))$.

From this, can you see why some people prefer the terms '*Above*' and '*Below*' instead of '*Right*' and '*Left*'?

2.4 (Thomas 2.5) Continuity

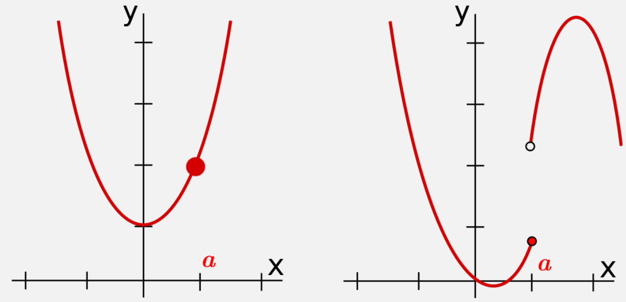
2.4.1 Continuous and Discontinuous Functions

Definition: Points of Continuity and Discontinuity

A function is **continuous at** $x = a$ (in the domain of $f(x)$) if

1. $f(a)$ exists;
2. $\lim_{x \to a} f(x)$ exists; and
3. $\lim_{x \to a} f(x) = f(a)$

If a function is not continuous at $x = a$ we say it is **discontinuous at** $x = a$.

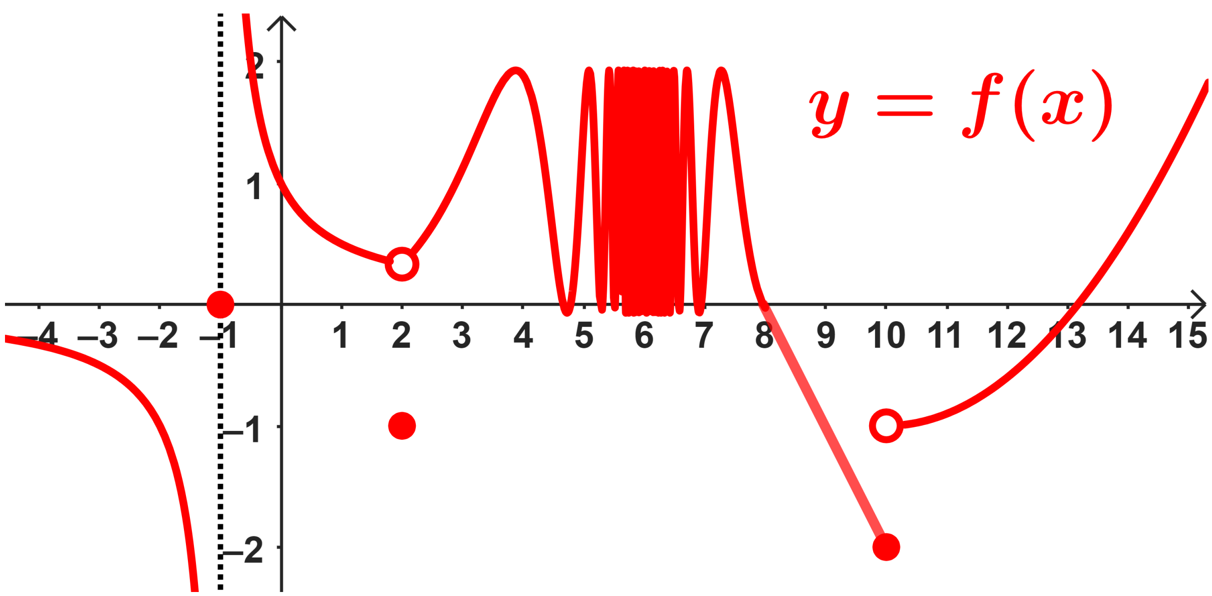


Continuous at x=a *Discontinuous at x=a*

Definition: Continuous Functions

We say that a function $f(x)$ is **Continuous** if it is continuous at every point in its domain.

Example 1: Consider the function $y = f(x)$ below.



Classify the points of discontinuity as either **Removable**, **Infinite**, **Jump**, or **Oscillatory**. Then, state which of the criteria (1), (2), and/or (3) that weren't satisfied in the definition of continuity.

	The Point $x = -1$	The Point $x = 2$	The Point $x = 6$	The Point $x = 10$
Type				
Criteria Failed				

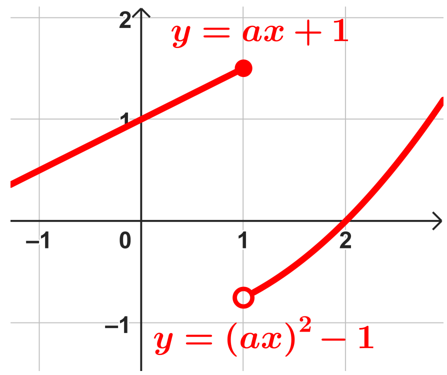
Note: How to '*See*' if a Function is Continuous

Often it is quite simple to tell if a function is continuous. We often can identify a function as being continuous if it is void of discontinuities. Often this is stated as having the property that you can '*draw*' its graph without ever lifting your pencil off the paper (in a finite amount of time).

Example 2: Consider the function

$$f(x) = \begin{cases} ax+1 & x \leq 1 \\ (ax)^2 - 1 & x > 1 \end{cases}.$$

Determine all values of a such that the function is continuous at $x = 1$.



Note: The Domain and Continuity

The domain is important! It makes little sense to talk about discontinuity outside of the domain. That it, what does it mean to talk the features of a location that is non-existent (in our context)?

Example 3: Consider the functions $f(x) = \dfrac{1}{x}$ and $g(x) = \begin{cases} \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Explain why $f(x)$ is continuous, yet $g(x)$ is not.

Note: Another Comment on Domain and Continuity

The above example illustrates the fickle description of discontinuities in the domain versus outside of them. Often people do colloquially just call $f(x)$ discontinuous.

2.4.2 Continuity of Elementary Functions on Their Domain

Note: Continuity of Elementary Functions

All so-called **Elementary Functions** are continuous on their domain.

Elementary Function	**Domain of Continuity**
Constants C	$\mathbb{R}$
x^n where $n > 0$	$\mathbb{R}$
$\frac{1}{x^n}$ where $n > 0$	$(-\infty, 0) \cup (0, \infty)$
Polynomials $P(x)$	$\mathbb{R}$
Rationals $\frac{P(x)}{Q(x)}$	All values where $Q(x) \neq 0$
$\sin(x)$	$\mathbb{R}$
$\cos(x)$	$\mathbb{R}$
$\tan(x)$	All values where $x \neq \frac{n\pi}{2}$ and n is odd
$\cot(x)$	All values where $x \neq n\pi$ and n is an integer
$\sec(x)$	All values where $x \neq \frac{n\pi}{2}$ and n is odd
$\csc(x)$	All values where $x \neq n\pi$ and n is an integer
a^x where $a > 0$	$\mathbb{R}$
$\log_a(x)$ where $a > 0$	$(0, \infty)$

2.4.3 Properties of Continuous Functions

Theorem: Continuous Functions are Preserved Under Algebra

Suppose that $f(x)$ and $g(x)$ are continuous functions at $x = a$. Then all the following expressions are continuous at $x = a$:

1. $f(x) + g(x)$
2. $f(x) - g(x)$
3. $Cf(x)$
4. $f(x) \cdot g(x)$
5. $f(x)/g(x)$ provided $g(a) \neq 0$
6. $f(x)^n$ where n is a positive integer
7. $f(x)^{1/n}$ where n is a positive integer and the expression is defined

Example 4: Explain why the function $h(x) = |x|\sin(x) + x^2 + 2x + 5$ is continuous without graphing it.

Theorem: Continuous Functions are Preserved Under Composition

If $f(u)$ is continuous at $u = g(a)$ and $g(x)$ is continuous at $x = a$ then the composition $f(g(x))$ is continuous at $x = a$.

Example 5: Argue that the function $f(x) = \sin\left(\frac{\pi}{2} + x - \sin(x)\right)$ is continuous and use this to determine $\lim_{x\to\pi} \sin\left(\frac{\pi}{2} + x - \sin(x)\right)$. *Hint: If a function is continuous, then by definition of criteria (3),* $\lim_{x\to a} f(x) = f(a)$.

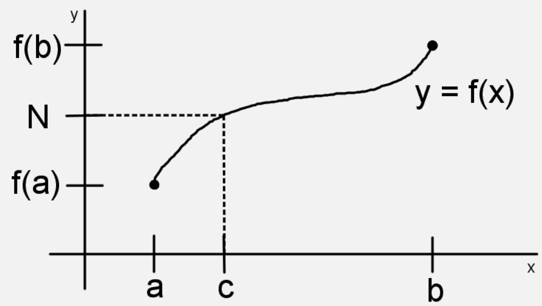
2.4.4 The Intermediate Value Theorem

Theorem: Intermediate Value Theorem (IVT)

If $f(x)$ is continuous on $[a, b]$ and if N satisfies

$$f(a) \leq N \leq f(b)$$

then there exists a point c in $[a, b]$ such that $N = f(c)$.

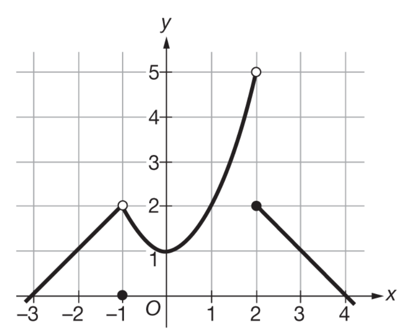


Example 6: Prove that the function $f(x) = x^3 - x - 1$ has a root in the interval $[1, 2]$.

Example 7: Consider the adjacent graph corresponding to the function $y = f(x)$.

Find a value a and b such that $f(a) = 1$ and $f(b) = 4$ (eye out an approximation).

Then, explain why there is no value $x = c$ in the domain such that $f(c) = 3$. What went wrong?



Graph of f

2.5 (Thomas 2.6) Limits Involving Infinity

2.5.1 Defining Infinite Limits

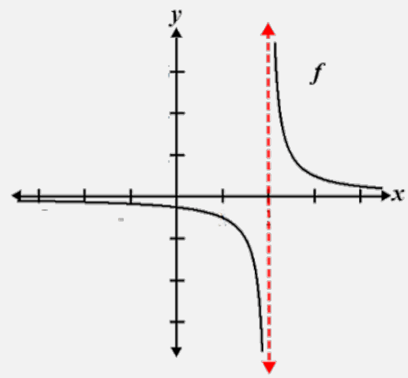
Definition: Limits where $f(x) \to \infty$

If the values of $f(x)$ grow infinitely without bound as $x \to a$ then we write

$$\lim_{x \to a} f(x) = \infty.$$

Similarly, we define $\lim_{x \to a} f(x) = -\infty$ if it grows infinitely negative without bound as $x \to a$.

Graphically, equations of the form $x = a$ represent **Vertical Asymptotes**.



Example 1: Graph the function $f(x) = -\dfrac{1}{x-2}$ and determine $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$.

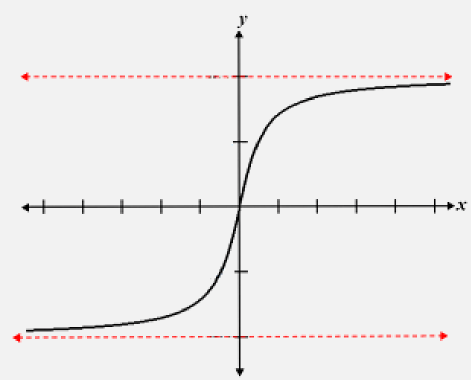
Definition: Limits where $x \to \infty$

If the values of $f(x)$ approach a value L (possibly infinite) as x grows infinitely big then we write

$$\lim_{x \to \infty} f(x) = L.$$

Similarly, we define $\lim_{x \to -\infty} f(x) = L$ if the values of $f(x)$ approach a value L as x becomes infinitely negative.

Graphically, equations of the form $y = L$ represent **Horizontal Asymptotes**.



Example 2: Graph the $f(x) = 1 + e^{-x}$ and determine $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

2.5.2 Working with Infinite Limits Algebraically

Theorem: Limits Laws for Infinite Limits

All limit laws hold when replacing $x \to a$ with $x \to \pm\infty$.

Theorem: A Useful Limit

Let $n > 0$. Then $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$.

Procedure: Evaluating Infinite Limits of Rational Functions

1. Identify the highest degree term in the denominator.
2. Divide both the numerator and denominator by the highest degree term.
3. Invoke $\lim_{x\to\infty} (1/x^n) = 0$ to obtain the end result. Note that $\lim_{x\to\infty} x^m = \infty$ if $m > 0$.

Example 3: Determine $\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2}$.

Example 4: Determine $\lim_{x\to\infty} \frac{3x^2 + 2x}{4x^3 - 5x + 7}$.

Example 5: Determine $\lim_{x\to\infty} \frac{3x^2 + 4x}{x + 2}$.

Note: Infinite Limits of Algebraic Ratios

One must exercise caution when dealing with infinite limits of general algebraic ratios. The following example will illustrate that its possible for $x \to -\infty$ to result in a different limit than $x \to \infty$.

Example 6: Consider the function $f(x) = \dfrac{3x-2}{\sqrt{4x^2+5}}$. *Hint:* $\dfrac{3x-2}{\sqrt{4x^2+5}} = \dfrac{x\left(3-\frac{2}{x}\right)}{\sqrt{x^2\left(4+\frac{5}{x^2}\right)}} = \dfrac{x}{|x|}\dfrac{3-\frac{2}{x}}{\sqrt{4+\frac{5}{x^2}}}$

(a) Determine the limit as $x \to \infty$.

(b) Determine the limit as $x \to -\infty$.

Note: Undefined Expressions of the Form $\infty - \infty$

Just like the form $0/0$, the form $\infty - \infty$ is indeterminate. You must algebraically rewrite the expression to obtain a defined limit.

Example 7: Determine $\lim\limits_{x\to\infty}(x - \sqrt{x^2+16x})$.

2.5.3 Horizontal, Vertical and Slant Asymptotes

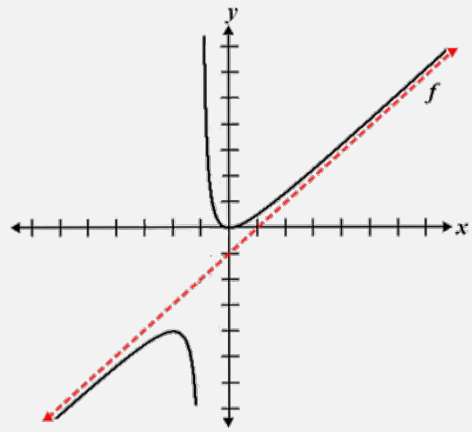
Definition: Oblique (Slant) Asymptotes

A line of the form $y = mx+b$ is an **Oblique (or Slant) Asymptote** of $f(x)$ if

$$\lim_{x\to\infty} (f(x) - (mx+b)) = 0$$

or

$$\lim_{x\to-\infty} (f(x) - (mx+b)) = 0.$$



Example 9: Determine a slant asymptote of $f(x) = 3x + 1 + e^{-x}$.

Example 8: Determine all horizontal, vertical and slant asymptotes of the function $f(x) = \dfrac{x^2-1}{2x+4}$.

Chapter 3

Derivatives

3.1 (Thomas 3.1) The Tangent Line to a Curve

3.1.1 Revisiting Tangent Lines

Note: The Tangent Line (Again)

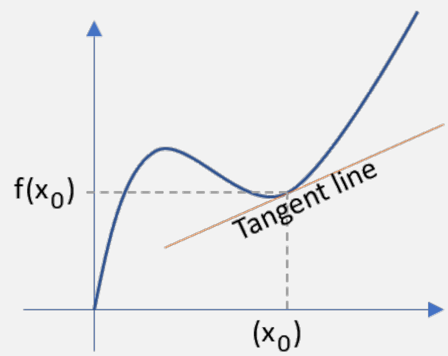
We've already discussed the tangent line in the introductory section of this chapter. The aim of this section and the next one are to formally revisit it with our new found knowledge of limits.

Definition: Slope of a Function and The Tangent Line

The **Slope of the Curve** of $y = f(x)$ at $x = x_0$ is defined as

$$\text{Slope of } f(x) \text{ at } x_0 = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided the limit exists. The **Tangent Line** to $y = f(x)$ at $x = x_0$ is the line with the above slope containing the point $(x_0, f(x_0))$ (called the **Point of Tangency**).



Example 1: Determine the slope of the function $f(x) = \sqrt{x+3}$ at the point $x = 6$ and use this to construct the equation of the corresponding tangent line.

3.2 (Thomas 3.2) The Derivative as a Function

3.2.1 Defining the Derivative

Definition: The Derivative and Differentiability.

The **Derivative** of $f(x)$ with respect to x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. $f'(x)$ is a function whose domain is the interior of the domain of $f(x)$. We say a function is called **Differentiable** at $x = a$ provided $f'(a)$ exists. If the derivative exists at every point of in the interior of the domain of $f(x)$ then we say that $f(x)$ is **Differentiable**.

Note: Notation Used

The **Newton** notation is $f'(x_0)$ and the **Leibniz** notation is $\left.\frac{df}{dx}\right|_{x=x_0}$. These represent the same thing, that being the derivative of f at $x = x_0$.

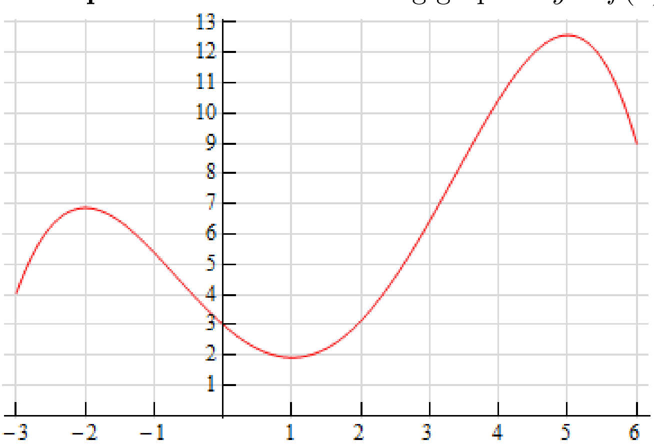
Example 1: Compute the derivative of the function $f(x) = \frac{x}{2x+1}$.

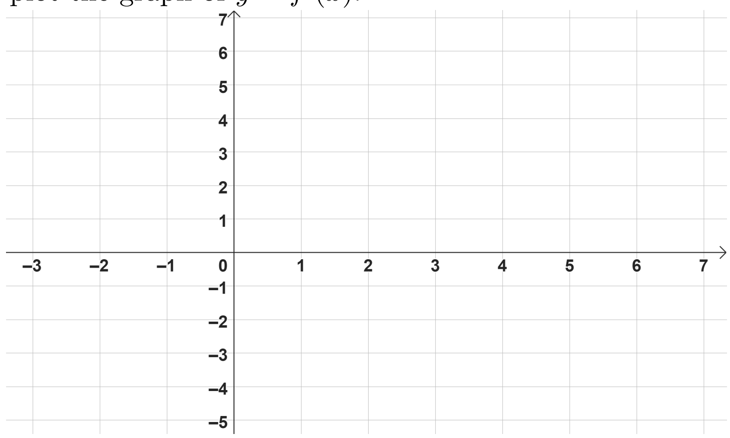
3.2.2 Graphing the Derivative (Interpreting It Geometrically)

Note: Interpreting the Derivative

We've seen two ways to understand the derivative (now that it has a name). One was the (instantaneous) rate of change (**a physical interpretation**), and the other is as the slope of the tangent line (**a geometric interpretation**).

Example 2: Given the following graph of $y = f(x)$, plot the graph of $y = f'(x)$.





3.2.3 Existence of the Derivative

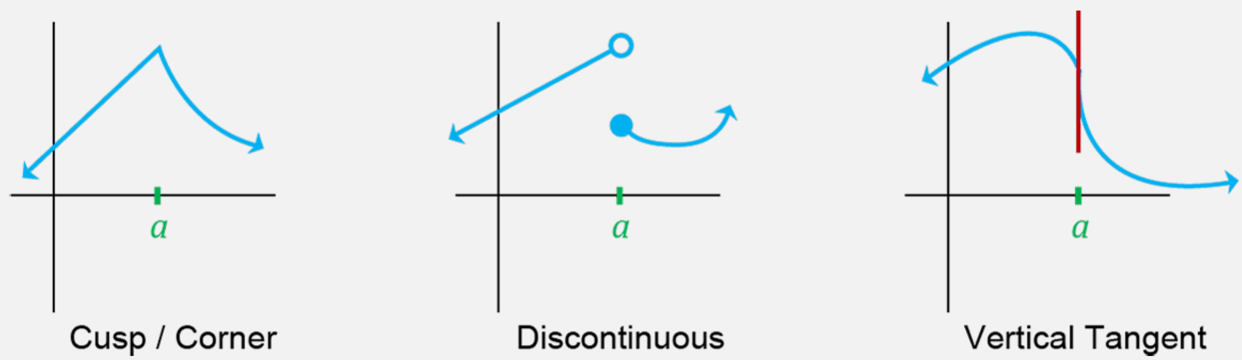
Example 3: Consider the function $f(x) = |x|$.

(a) Using the definition, show that the function is not differentiable at $x = 0$.

(b) What is $f'(x)$ when $x \neq 0$?

Note: When the Derivative Does Not Exist

Graphically, the derivative doesn't exist at one of the three types of points: **A Cusp/Corner**, **A Point of Discontinuity**, and **A Vertical Tangent.**



Theorem: Differentiable Functions are Continuous

All differentiable functions are continuous. The reverse implication (converse) is not necessarily true.

3.3 (Thomas 3.3) Differentiation Rules

3.3.1 The Constant Rule and Power Rule

Theorem: The Constant Rule

If $f(x) = C$ is a constant function then $f'(x) = 0$ for all values of x.

Example 1: Provided $f(x) = 3$ compute $f'(x)$.

Example 2: Provided $f(x) = \ln(e^2 + 1)$ compute $f'(x)$.

Power Rule

If n is any real number and $f(x) = x^n$ then $f'(x) = nx^{n-1}$ for all values of x.

Note: Operator Notation for Differentiating

We denote 'taking' the derivative of $f(x)$ as $\frac{d}{dx}[f(x)]$ or $(f(x))'$. This represents the operation of taking the slope of a function.

Example 3: Compute the derivative of $f(x) = x^3$.

Example 4: Compute $\frac{d}{dz}\left[z^{2/3}\right]$.

Example 5: Compute the derivative of $h(t) = \frac{1}{\sqrt{t}}$.

3.3.2 Constant and Sum Rule (Linearity)

Theorem: Constant and Sum Rule (Linearity)

If C is a constant and $f(x)$ is differentiable then $\dfrac{d}{dx}[Cf(x)] = C\dfrac{d}{dx}[f(x)]$.

If $f(x)$ and $g(x)$ are differentiable functions then $\dfrac{d}{dx}[f(x)+g(x)] = \dfrac{d}{dx}[f(x)] + \dfrac{d}{dx}[g(x)]$.

Example 6: Compute $\dfrac{d}{dx}\left[3x^{1/6} - 7x^{11/9}\right]$.

3.3.3 Exponential Functions

Theorem: Derivative of the Exponential Function

If k is a constant then $\dfrac{d}{dx}\left[e^{kx}\right] = ke^{kx}$. In particular, $\dfrac{d}{dx}[e^{x}] = e^{x}$.

Example 7: Compute $\dfrac{d}{d\theta}\left[\dfrac{6}{\theta} + 4e^{\pi\theta}\right]$.

3.3.4 The Product Rule and Quotient Rule

Theorem: The Product Rule

If $f(x)$ and $g(x)$ are differentiable functions then $\dfrac{d}{dx}[f(x)g(x)] = \dfrac{d}{dx}[f(x)]g(x) + f(x)\dfrac{d}{dx}[g(x)]$.

Example 8: Let $f(t) = t^2e^{3t}$. Compute $f'(t)$.

Quotient Rule

If $f(x)$ and $g(x)$ are differentiable, then for all values of x where $g(x) \neq 0$ we have

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{(g(x))^2}$$

Example 9: Compute the derivative of $h(y) = \dfrac{y}{2y^2+1}$ with respect to y.

3.3.5 Higher Order Derivatives

Higher Order Derivatives

We define higher order derivatives recursively. Specifically,

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left[\frac{df}{dx}\right], \frac{d^3 f}{dx^3} = \frac{d}{dx}\left[\frac{d^2 f}{dx^2}\right], ..., \frac{d^n f}{dx^n} = \frac{d}{dx}\left[\frac{d^{n-1} f}{dx^{n-1}}\right]$$

In Newton's notation we denote higher order derivatives as $f''(x), f'''(x), f^{(4)}(x), ...$ and use the notation $f^{(k)}(x)$ for $k \geq 4$.

Example 10: Let $f(x) = x^3 - 3x^2 + 2$. Compute $f'''(x)$.

Example 11: Let $f(x)$ be a twice differentiable function such that $f(1) = 1$, $f'(1) = 2$ and $f''(1) = 3$ and let $g(x) = xf(x)$. Compute $g''(1)$.

3.4 (Thomas 3.4) Derivative as the Rate of Change

3.4.1 The Physical Interpretation of the Derivative

Note: Understanding the Derivative Physically

We've already gone over this before, but we may understand the derivative as a rate of change. Specifically, suppose we measure f in an arbitrary unit called *utils* and x in, say, dollars. Then the units of $f'(x)$ are

$$\left[\frac{df}{dx}\right] = \left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right] = \frac{\text{Utils}}{\text{Dollar}}.$$

That is, Utils-per-dollar. We interpret it as how much the output changes simultaneously with the input.

Example 1: We blow a bubble and model the volume of it at any given time t. We find that the volume of this bubble is given by $V(t) = \dfrac{4}{3}\pi t^{8/3}$. How fast is the volume changing at $t = 2$ seconds?

3.4.2 Simple Physics and the Derivative

Definition: Velocity, Speed, Acceleration and Jerk

Let $s = f(t)$ represent the displacement of an object from some position. Assume that s is measured in meters and t is measured in seconds.

- **Velocity:** This is defined as $v(t) = s'(t)$ and is measured in meters-per-second. This represents the change in displacement.
- **Speed:** This is defined as $v^+(t) = |v(t)|$, or in other words $v^+(t) = |s'(t)|$. This is measured in (absolute) meters-per-second. This represents the directionless change in displacement.
- **Acceleration:** This is defined as $a(t) = v'(t)$, or in other words $a(t) = s''(t)$. This is measured in meters-per-squared-second. This represents the change in velocity.
- **Jerk:** This is defined as $j(t) = a'(t)$, or in other words $j(t) = s'''(t)$. This is measured in meters-per-cubed-second. This represents the change in acceleration.

Note: Interpreting the Sign of the Velocity

The above definition also tells us a sense of 'direction' of a moving object in that $v(t) > 0$ means the object is moving away from us and $v(t) < 0$ means the object is moving back to us.

Example 2: At time $t \geq 0$, the velocity of a body moving along the horizontal s-axis is $v(t) = t^3 - 6t^2 + 9t$.

(a) Determine the body's acceleration each time the velocity is zero.

(b) When is the body moving forward? Backward?

(c) When is the body's velocity increasing? Decreasing?

Example 3: You decide to move the Leaning Tower of Pisa to the planet Neptune and drop bowling balls from it. Using standard equations from mechanics you find that the displacement of the ball s meters from the ground is given by $s(t) = 55 - 5.6t^2$.

What is the ball's velocity at the moment of impact with the ground?

3.4.3 Derivatives in Economics

Note: Marginals

In Economic theory derivatives are called **Marginals**. For example, if $c(x)$ is the cost of producing x units then $\frac{dc}{dx}$ is called the **Marginal cost of production**.

Example 4: Suppose that it costs $C(x) = x^3 - 6x^2 + 15x$ to produce x radiators where $8 \leq x \leq 30$. Suppose further that the revenue is given by $R(x) = x^3 - 3x^2 + 12x$.

(a) If you currently produce and sell 10 a day, what is the increased profit by selling one more a day? *Hint: Profit = Revenue - Cost and you want to know how this is increasing at $x = 10$.*

(b) Compute the actual change in profit given by $P(11) - P(10)$. Is part (a) fairly close?

3.5 (Thomas 3.5) Derivatives of Trigonometric Functions

3.5.1 Derivatives of Sine and Cosine

Theorem: Derivatives of Sine and Cosine

The sine and cosine functions are differentiable with

$$\frac{d}{dx}[\sin(x)] = \cos(x) \qquad\qquad \frac{d}{dx}[\cos(x)] = -\sin(x)$$

Example 1: Compute $\frac{d}{dx}[x^{2/3}\cos(x) + 2x]$

Example 2: Compute $\frac{d}{dx}\left[\frac{2+\sin(x)}{\cos(x)}\right]$.

Example 3: Determine all points where the function $f(x) = \sin(x) + \frac{1}{2}x$ has a horizontal tangent line.

Example 4: A damped mass-spring system measures the displacement of an object attached to the spring from equilibrium position by $s(t) = e^{-2t}\cos(t)$. Determine the acceleration $a(t)$.

Example 5: Find a general pattern for $\dfrac{d^n}{dx^n}[\sin(x)]$ and use this to compute $\dfrac{d^{231}}{dx^{231}}[\sin(x)]$.

3.5.2 The Derivatives of Secant, Tangent, Cosecant, and Cotangent

Note: Deriving the Other Trigonometric Derivatives

All remaining trigonometric functions are related to sine and cosine through algebraic operations. Hence, their derivatives are related to sine and cosine through derivative rules. We list the end results below.

Theorem: Derivatives of Secant, Tangent, Cosecant, and Cotangent

All the trigonometric functions are differentiable on their domains with

$$\frac{d}{dx}[\tan(x)] = \sec^2(x) \qquad \frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\cot(x)] = -\csc^2(x) \qquad \frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$

Example 5: Find the tangent line to $f(x) = \sec(x)\tan(x)$ at $x = \frac{\pi}{6}$.

Example 6: Construct the second derivative of the function $f(x) = \cot(x)$ at $x = \frac{\pi}{4}$.

3.6 (Thomas 3.6) Chain Rule

3.6.1 Introducing the Chain Rule

Theorem: The Chain Rule

If $f(u)$ is differentiable at $u = g(x)$ and $g(x)$ is differentiable at x then

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

Note: Visualizing the Chain Rule

Notice the pattern above. This result is commonly stated as the '*derivative of the "outside" times the derivative of the "inside"*'.

Example 1: Compute the derivative of $f(x) = \cos(x^2 + 1)$ with respect to x.

Example 2: Compute the derivative of $h(z) = \sec(z - \sin(z))$ with respect to z.

Example 3: Compute $\dfrac{d}{dt}\left[\dfrac{\sin(t^2)}{t}\right]$.

3.6.2 Differentiating $e^{f(x)}$ and $(f(x))^n$

Note: Differentiating $e^{f(x)}$ and $(f(x))^n$

It's easy enough to see the 'inside' and 'outside' function for expressions like $\sin(f(x))$. However, it is less so for expressions like $e^{f(x)}$ and $(f(x))^n$. We state them below,

- $\dfrac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)} f'(x)$
- $\dfrac{d}{dx}\left[(f(x))^n\right] = n(f(x))^{n-1} f'(x)$

Example 4: Compute $\dfrac{d}{dx}\left[\dfrac{1}{8}x^{2/3} + 3e^{\sin(x)}\right]$.

Example 5: Compute $\dfrac{d}{d\theta}\left[\sin^2(\theta)\cos(\theta)\right]$.

Example 6: Compute $\dfrac{d}{dx}\left[\left(1 + xe^{-x}\right)^2\right]$.

3.6.3 Layered Chain Rule

Note: Layered Chain Rule

It is not uncommon to encounter functions within functions within... etc. For example, the expression $\cos\left(e^{x^2}\right)$ is the composition $f(g(h(x)))$ where $f(x) = \cos(x)$, $g(x) = e^x$ and $h(x) = x^2$.

When differentiating these, just handle the outside-times-inside rule slowly one step at a time. We provide examples below.

Example 7: Compute $\dfrac{d}{dx}\left[\sqrt{1+\cos(x^2)}\right]$.

Example 8: Compute $\dfrac{d}{dt}\left[\tan^2(\sin^3(t))\right]$.

3.7 (Thomas 3.7) Implicit Differentiation

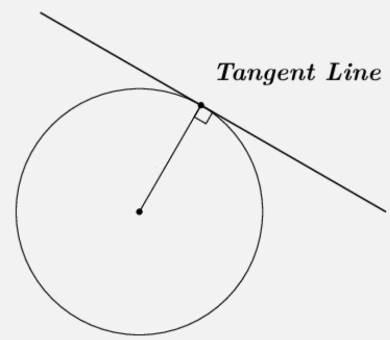
3.7.1 Implicit Expressions

Note: Slopes of Curves

Not all curves are of the form $y = f(x)$. For example, a circle is given by $x^2 + y^2 = 1$.

It is still makes sense (physically) to talk about a tangent line (with slope $\frac{dy}{dx}$) to the curve.

To do so, we use **Implicit Differentiation** (which requires the chain rule).



Definition: Implicit and Explicit Expressions

We say that an expression is written **Explicitly** if it is of the form $y = f(x)$. Otherwise, we say an expression is written **Implicitly** if it is of the form $F(x, y) = 0$ for some function of both x and y.

Example 1: Consider the expressions below. Underline the expressions that are implicit. Assume the independent variable is x.

$$y = \sin(x), \quad ye^{xy} = xy, \quad x^2 + y^2 = 1, \quad \sin(y) = x, \quad x^2 + e^x = y, \quad y^2 e^x = \sin(x), \quad y = \sec(x)\ln(x)$$

Note: Implicit Differentiation

In an implicit expression, $y = y(x)$ is treated like a function that is unsolved for (or quite literally can't be solved for). Derivatives of expressions depending on y follow the chain rule,

$$\frac{d}{dx}[g(y)] = \frac{d}{dx}[g(y(x))] = g'(y(x))y'(x).$$

Example 2: Compute $\frac{d}{dx}[\sin(y)]$ assuming y is a function of x.

Example 3: Compute $\frac{d}{dx}[xe^y]$ assuming y is a function of x.

3.7.2 Slopes and Implicit Differentiation

Procedure: Implicit Differentiation

We start with an implicit expression $F(x, y) = 0$ (possibly rearranged).

1. **Differentiation Both Sides:** Apply a derivative to both sides treating y as a function of x.
2. **Use the Chain Rule:** When you differentiate a function depending on y use the chain rule appropriately, e.g. $\frac{d}{dx}[g(y)] = g'(y)y'$.
3. **Evaluate at a Point:** If you are evaluating at a point $P(x_0, y_0)$, plug this in prior to solving for $\frac{dy}{dx}$.
4. **Solve Explicitly for the Slope:** Algebraically solve the equation for $\frac{dy}{dx}$ to obtain the slope.

Example 4: Consider the unit circle $x^2 + y^2 = 1$ and the point $P(1/\sqrt{2}, 1/\sqrt{2})$ on it. Determine the equation of the tangent line to the circle at P.

Example 5: Determine the slope of the tangent line to $x^2 + xy - y^2 = 1$ at the point $P(2, 3)$.

Example 6: Determine the slope of the curve $x^4 + \sin(y) = x^3y^2$ at any general point $P(x_0, y_0)$.

Example 7: Determine the slope of the curve $y \sin\left(\frac{1}{y}\right) = 1 - xy$ at any general point $P(x_0, y_0)$.

3.8 (Thomas 3.8) Derivatives of Inverse Functions

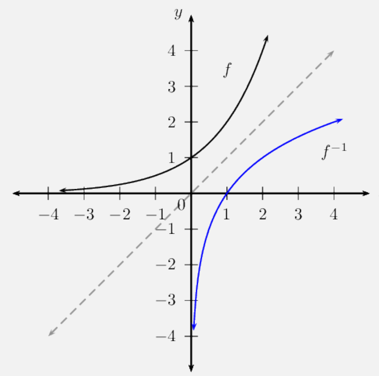
3.8.1 Inverse Derivative Formula

Recap: Inverse Functions

Let $y = f(x)$ be a function. We say that $y = g(x)$ is the **Inverse Function** of $f(x)$ provided whenever $f(a) = b$ we have $g(b) = a$. We denote this $g(x) = f^{-1}(x)$.

If $f(x)$ and $g(x)$ are inverses of each other then $g(f(x)) = f(g(x)) = x$ for all allowable x.

The graphs of $y = f(x)$ and $y = f^{-1}(x)$ look like mirror images of each other reflected across the line $y = x$.



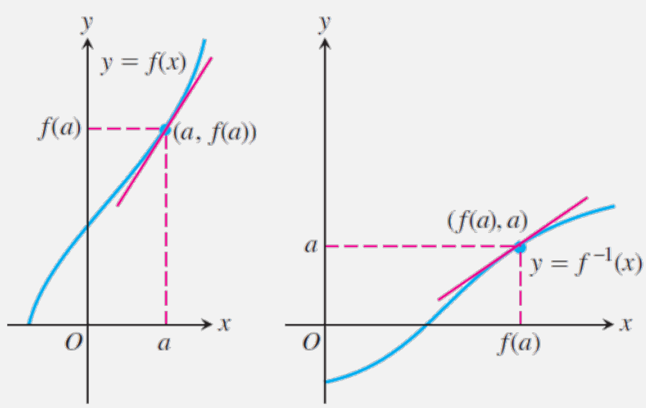
Theorem: Inverse Derivative Rule

If f is differentiable with inverse f^{-1} then

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$$

whenever $f'(f^{-1}(x)) \neq 0$.

That is, the slope of $f^{-1}(x)$ at $b = f(a)$ is the reciprocal of the slope of $f(x)$ at $x = a$ (as one can see by the adjacent picture).



The slopes are reciprocal: $\left.\dfrac{df^{-1}}{dx}\right|_{f(a)} = \dfrac{1}{\left.\dfrac{df}{dx}\right|_{a}}$

Example 1: Suppose that $f(x) = e^x + x^2 + 1$. Determine the derivative of $f^{-1}(x)$ when $x = 2$.

3.8.2 The Natural Logarithm and Exponential

Theorem: The Derivative of the Natural Logarithm

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x} \qquad \text{whenever } x > 0$$

Note: Obtaining the Derivative of $\ln(x)$

The inverse of the function $f(x) = e^x$ is $f^{-1}(x) = \ln(x)$ by definition. Therefore, you obtain the above result using the inverse derivative formula.

Example 2: Let $f(x) = \ln(\sec(x^2 + 1))$. Compute $f'(x)$.

Example 3: Consider the function $f(x) = \ln\left(\frac{\sqrt{1+x}}{x^3}\right)$. Determine $f'(x)$. *Hint: Use logarithmic properties.*

Definition: The Exponential Function

The function e^{kx} is defined as $e^{kx} = \lim_{k\to\infty}\left(1 + \frac{k}{x}\right)^x$.

3.8.3 The Exponential and Logarithm in other Bases

Theorem: The Derivative of General Exponentials and Logarithms

Let $a > 0$ be a constant. Then,

$$\frac{d}{dx}[a^x] = a^x \ln(a) \qquad\qquad \frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln(a)}$$

Note: Obtaining the General Derivatives

Through the identities $a^x = e^{\ln(a^x)} = e^{x\ln(a)}$ and $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$ we may construct the derivatives above using ordinary derivative rules.

Example 4: Let $r(z) = z \cdot 3^{\sin(z)}$. Compute $r'(z)$.

Example 5: Let $h(t) = \pi^{t^2} + \log_2(t^2+1)$. Compute $h'(t)$.

3.8.4 Logarithmic Differntiation

Recap: Logarithmic Properties

We remind the reader that logarithmic functions have three important properties:

1. $\ln(ab) = \ln(a) + \ln(b)$ provided $a, b > 0$
2. $\ln(a/b) = \ln(a) - \ln(b)$ provided $a, b > 0$
3. $\ln(a^b) = b\ln(a)$ provided $a > 0$

Note: Logarithmic Differentiation

Logarithms greatly simplify any expression plentiful with products division and exponents. Thus if you have an expression $y = f(x)$ you need to differentiate with this problem, you may take a logarithm of both sides $\ln(y) = \ln(f(x))$ and perform implicit differentiation.

Example 6: Let $f(x) = \dfrac{x^{2/3}}{x-1}$. Use logarithmic differentiation to compute $f'(x)$.

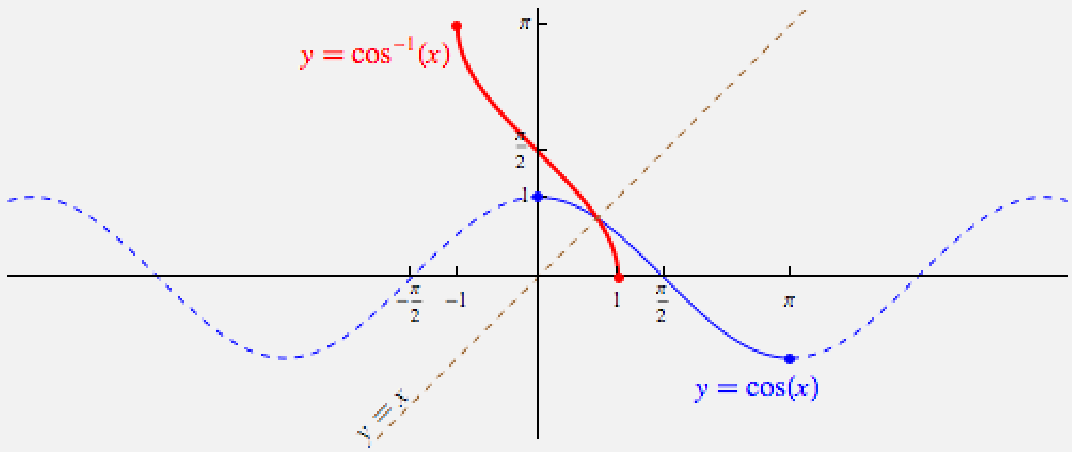
Example 7: Let $f(x) = x^{x+1}$. Compute $f'(x)$.

3.9 (Thomas 3.9) Inverse Trigonometric Functions

3.9.1 Review of Inverse Cosine, Sine, and Tangent

Note: Existence of Inverse Trigonometric Functions

The functions $\sin(x)$, $\cos(x)$ and $\tan(x)$ can't have inverses because they don't pass the horizontal line test. For example $\cos(0) = \cos(2\pi) = \cos(4\pi) = \cdots = 1$, thus what is $\cos^{-1}(1)$? To talk about the inverse you need to restrict the domain so that you only have one possible answer for the output.



Definition: Inverse Cosine, Sine and Tangent

We defined the inverse trigonometric functions as follows on the given domains:

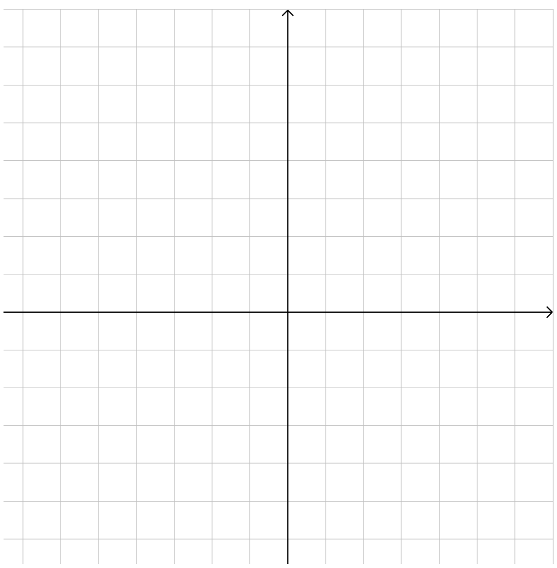
1. $y = \tan^{-1}(x) = \arctan(x)$ is the inverse of $\tan(x)$ restricted to $(-\pi/2, \pi/2)$.
2. $y = \sin^{-1}(x) = \arcsin(x)$ is the inverse of $\sin(x)$ restricted to $[-\pi/2, \pi/2]$.
3. $y = \cos^{-1}(x) = \arccos(x)$ is the inverse of $\cos(x)$ restricted to $[0, \pi]$.

Example 1: Compute the value of $\arccos\left(\frac{\sqrt{3}}{2}\right)$.

Example 2: Compute the value of $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

Example 3: Compute the value of $\arctan(-1)$.

Example 4: Graph the function $y = \arctan(x)$ and use this to compute $\lim_{x\to\infty} \arctan(x)$.



Example 5: Simplify the composition $\sin(\arccos(x))$.

3.9.2 Review of Inverse Secant, Cosecant, and Cotangent

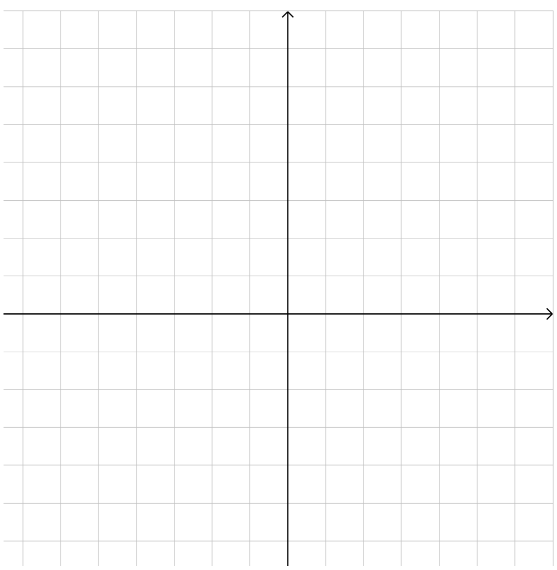
Definition: Inverse Secant, Cosecant and Cotangent

We defined the inverse trigonometric functions as follows on the given domains:

1. $y = \cot^{-1}(x) = \text{arccot}(x)$ is the inverse of $\cot(x)$ restricted to $(0, \pi)$.
2. $y = \sec^{-1}(x) = \text{arcsec}(x)$ is the inverse of $\sec(x)$ restricted to (0π).
3. $y = \csc^{-1}(x) = \text{arccsc}(x)$ is the inverse of $\csc(x)$ restricted to $[-\pi/2, \pi/2]$.

Example 6: Compute the value of $\text{arccsc}\,(-2)$.

Example 7: Graph the function $y = \text{arcsec}(x)$ and use this to compute $\lim\limits_{x\to-\infty} \text{arcsec}(x)$.



Example 8: Simplify the composition $\sin(\text{arccot}(x))$

3.9.3 Derivatives of Inverse Trigonometric Functions

Theorem: Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\arccos(x)] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2} \qquad \frac{d}{dx}[\text{arccot}(x)] = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}[\text{arcsec}(x)] = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}[\text{arccsc}(x)] = -\frac{1}{|x|\sqrt{x^2-1}}$$

Note: Derivation of the Inverses Trigonometric Derivatives

As all the inverse trigonometric functions are inverse functions, their formulas are obtained through using the inverse derivative rule.

Example 9: Determine the equation of the tangent line to $y = \arccos(x)$ at $x = -\frac{1}{2}$.

Example 10: Compute $\frac{d}{dt}[\text{arccsc}(t) - 4\text{arccot}(t)]$.

Example 11: Compute $\frac{d}{dx}[5x^6 - \text{arcsec}(x)]$.

Example 12: Let $f(w) = \sin(w) + w^2 \arctan(w)$. Compute $f'(w)$.

Example 13: Compute $\dfrac{d}{dx}\left[\dfrac{\sin^{-1}(x)}{1+x^2}\right]$.

Example 14: Let $f(x) = 4\arccos(x^2) - 10\arctan(x)$. Compute $f'(x)$.

Example 15: Let $h(t) = \operatorname{arccsc}(\log_2(t))$. Compute $h'(t)$.

Example 16: Let $r(z) = \log_\pi(z^2 + 4) - z\arctan(z/2)$. Compute $r'(z)$.

3.10 (Thomas 3.10) Related Rates

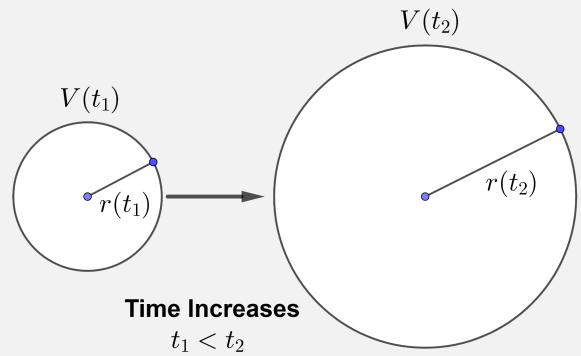
3.10.1 Introducing Related Equations

Note: Related Equations and Rates

Sometimes you have relationships between functions. For example, if you blow a bubble the radius $r(t)$ and $V(t)$ both increase with time. They are related by the volume of a sphere formula

$$V(t) = \frac{4}{3}\pi(r(t))^3.$$

Since there is a relationship between $V(t)$ and $r(t)$, then there is a relationship between $V'(t)$ and $r'(t)$. This is called **Related Rates**.



Example 1: You blow a (spherical) bubble. You blow air consistently into the bubble at a rate of 50 cubic-centimeters-per-second. At the moment when the radius is 5 centimeters, how fast is the radius increasing?

Procedure: Strategy to Solving Related Rates Problems

1. **Draw a Picture:** Make an illustration. Name your variables and use t for time. Assume your functions depend on t.
2. **Write Down What You Are Given:** Translate all given information from English to Math, in terms of the symbols you're using.
3. **Form a Relationship Between Variables:** Write down an equation that relates everything.
4. **Construct the Related Rates Equation:** Differentiate the equation that related everything to obtain an equation that relates the derivatives to one another.
5. **Compile Everything:** Use the combined information of all the above to solve for the desired quantity.

3.10.2 Examples of Related Rates

Example 2: A 13 foot ladder is leaning against a wall when suddenly the base of the ladder slides away from the wall. By the time its base is 12 feet from the wall its base is moving at a rate of 5 feet per second.

(a) How fast is the top of the ladder sliding down?

(b) How fast is the angle changing at this moment in time? Round your answer to two decimals.

Example 3: A light is at the top of a 16 ft pole. A 6 ft tall man walks away from the pole at a rate of 5 feet-per-second. How fast is the tip of his shadow moving when he is 20 feet from the pole?

3.11 (Thomas 3.11) Linearization and Differentials

3.11.1 Linearization and the Tangent Line

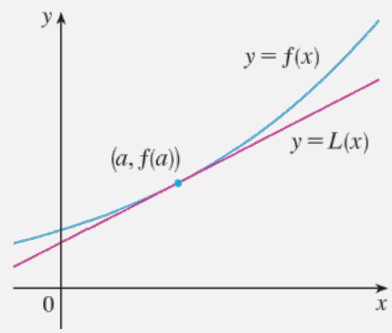
Note: The Equation of the Tangent Line

Since the tangent line to $y = f(x)$ at $x = a$ is the line through $(a, f(a))$ with slope $f'(a)$ then the general equation of the line is $y = f(a) + f'(a)(x - a)$. Since the tangent line is *tangent* to the curve at $(a, f(a))$, the y-values of this line are close to the y-values of $y = f(x)$ about this point.

Definition: Linearization

Let $f(x)$ be a differentiable function at $x = a$. The **Linearization** of f at a is defined to be the tangent line of $f(x)$ at $x = a$. That is,

$$L(x) = f(a) + f'(a)(x - a).$$



Example 1: Approximate $\sqrt{4.1}$ using an appropriate linearization.

Example 2: Use the linearization of $f(x) = (1 + x)^k$ to construct an approximating function that holds for all small values of x.

3.11.2 Differentials

Note: Approximations of $f(x)$ Using $f'(x)$

We have for small values of Δx that

$$f'(a) \approx \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

and thus $f(a+\Delta x) - f(a) \approx f'(a)\Delta x$. That is, the rise is approximately the product of the slope and the run. Out of tradition, we denote $df = f(a+h) - f(a)$ (or dy) and $dx = \Delta x$.

Definition: Differentials

Let $f(x)$ be a differentiable function. We define the differential of the function f to be $df = f'(x)dx$.

Example 3: The demand for grass seed (in thousands of pounds) at a price of p dollars is $D(p) = -3p^3 - 2p^2 + 1500$. Use differentials to approximate the change in demand if the price changes from $p = \$2$ to $p = \$2.10$. Interpret your answer.

Example 4: Approximate $\sqrt{50}$ using differentials.

3.11.3 Sensitivity to Change

Note: Sensitivity to Change

The equation $df = f'(x)dx$ illustrates how *sensitive* the output of f is to change in input at different values of x. That is, the larger the value of f' at x, the greater the effect of a given change dx.

Example 5: You drop a rock off a cliff and measure the displacement using the free-fall equation $s(t) = 4.9t^2$ measured in meters where t is measured in seconds. Given an error in measurement of $dt = 0.1$ seconds, determine the error in measurement of $s(t)$ when $t = 2$ seconds and $t = 5$ seconds.

Definition: Measuring Sensitivity of Change

There are three common ways of measuring the sensitivity of change of $f(x)$.

	True Measurement	Approximation
Absolute Change	$\Delta f = f(a + \Delta x) - f(a)$	$df = f'(a)dx$
Relative Change	$\dfrac{\Delta f}{f(a)}$	$\dfrac{df}{f(a)}$
Percentage Change	$\dfrac{\Delta f}{f(a)} \times 100\%$	$\dfrac{df}{f(a)} \times 100\%$

Example 6: An astronaut using a camera measures the radius of Earth as 4000 miles with an error of $dr = \pm 80$ miles. Use differentials to approximate the relative and percentage error of this radius measurement to measure the volume of the Earth, assuming the planet is a perfect sphere.

Chapter 4

Applications of Derivatives

4.1 (Thomas 4.1) Extreme Values of Functions

4.1.1 Absolute (Global) Extrema and the Extreme Value Theorem (EVT)

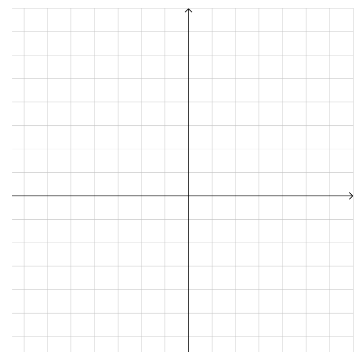
Definition: Absolute (Global) Extrema

Let $f(x)$ be a function with domain D.

- **Absolute (Global) Maximum:** f has an absolute maximum at $c \in D$ if $f(x) \leq f(c)$ for all $x \in D$.
- **Absolute (Global) Minimum:** f has an absolute minimum at $c \in D$ if $f(x) \geq f(c)$ for all $x \in D$.

Example 1: This example illustrates the importance of the shape of the domain. Consider the function $f(x) = 1 - x^2$. Determine all absolute extrema of $f(x)$ on the following domains.

Domain	Absolute Maximum	Absolute Minimum
$(-\infty, \infty)$		
$[-1, 2]$		
$[-1, 2)$		
$(-1, 2)$		



Theorem: Extreme Value Theorem (EVT)

If $f(x)$ is continuous on a closed interval $[a, b]$ then $f(x)$ obtains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$ and $f(x_2) = M$ and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

Example 2: Consider the function $f(x)$ defined on the domain $[0, 4]$ given by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 2 \\ (x-3)^2 - 2, & 2 < x \leq 4 \end{cases}.$$

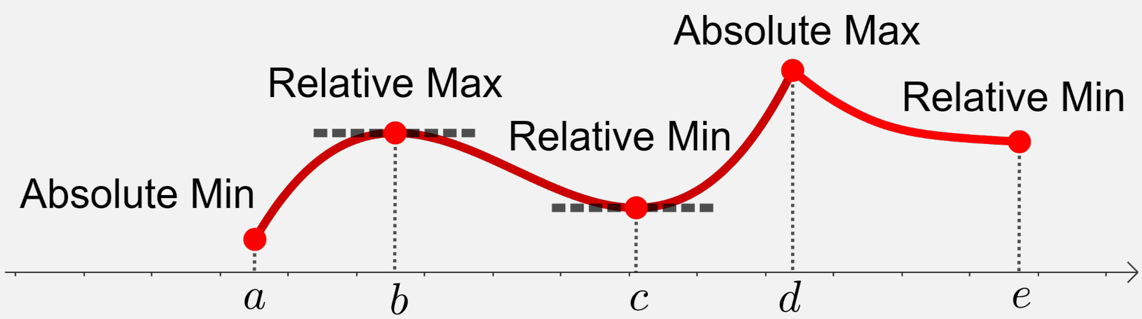
Graph this function and demonstrate that it obtains both an absolute maximum and minimum despite not being continuous. Does this contradict the extreme value theorem?

4.1.2 Relative (Local) Extrema

Definition: Relative (Local) Extrema

Let $f(x)$ be a function with domain D.

- **Relative (Local) Maximum:** f has a relative maximum at $c \in D$ if $f(x) \leq f(c)$ for all x close to c.
- **Relative (Local) Minimum:** f has a relative minimum at $c \in D$ if $f(x) \geq f(c)$ for all x close to c.



Note: All Global Extrema are Local Extrema

A global extremum is a local extremum. The reverse implication is not necessarily true.

4.1.3 Finding Global Extrema

Theorem: First Order Condition

If f has a local extremum at $x = c$ and $f'(c)$ exists then $f'(c) = 0$.

Definition: Critical Points

An interior point of the domain of f where f' is zero or undefined is called a **Critical Point** of f.

Example 3: Find all critical points of the function $f(x) = 2x - 24x^{1/3}$.

Procedure: Determining Global Extrema of a Continuous Function on a Closed Interval

Let $f(x)$ be a **Continuous Function** on a **Closed Interval** $[a, b]$. Then, a global extrema is either a critical point or an endpoint of the interval. Find these points and compare their values.

Note: Existence of the Global Extrema

The global extrema in the prior procedure is guaranteed to exist by the extreme value theorem. Thus, all we have to do is invoke that procedure to find them.

Example 4: Consider the function $f(x) = 3x(x-4)^{2/3}$. Determine the minimum and maximum values of this function on $[1, 9/2]$.

Hint: To save on time, you are given the fact that $f'(x) = \dfrac{5x-12}{(x-4)^{1/3}}$.

Example 5: Suppose that the population (in thousands) of a certain kind of insect after t months is given by

$$P(t) = 2t + \sin(4t) + 100$$

Determine the minimum and maximum population within the first 2 months.

4.2 (Thomas 4.2) The Mean Value Theorem

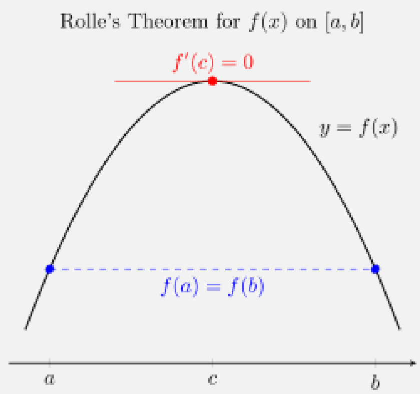
4.2.1 Rolle's Theorem

Theorem: Rolle's Theorem

Suppose that $f(x)$ is...

- Continuous over the closed interval [a,b];
- Differentiable on the interior (a,b); and
- $f(a) = f(b)$

Then, there exists a point $c \in (a, b)$ such that $f'(c) = 0$.



Example 1: Consider the function $f(x) = 4x^5 + x^3 + 7x - 2$. Use both Rolle's Theorem and the Intermediate Value Theorem to prove that $f(x)$ has exactly one (i.e. no more than one) real root.

4.2.2 The Mean Value Theorem (MVT)

Note: Transitioning from Rolle's Theorem to the Mean Value Theorem

We obtain the following result by relaxing the assumption that $f(a) = f(b)$. If you omit this restriction, then you instead obtain a point in the interval where the corresponding tangent line is parallel to the secant line of the interval.

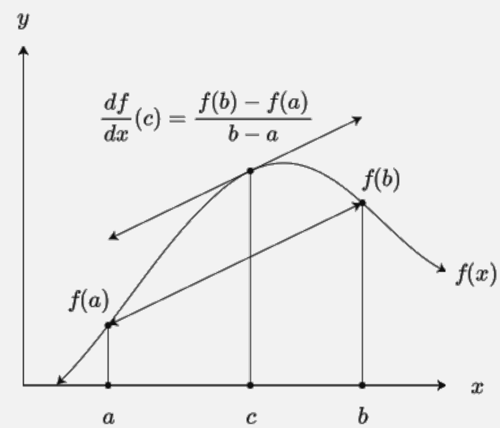
Theorem: The Mean Value Theorem (MVT)

Suppose that $f(x)$ is...

- Continuous over the closed interval [a,b]; and
- Differentiable on the interior (a,b);

Then, there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Example 2: Consider the function $f(x) = x^3 + 2x^2 - x$ on $[-1, 2]$. Determine all numbers in the interval where the corresponding tangent line is parallel to the secant line over $[-1, 2]$.

Example 3: Suppose that $f(x)$ is a function that is continuous on $[3, 10]$ and differentiable $(3, 10)$. Furthermore, suppose that you are given that $f(3) = -2$ and $f'(x) \leq 20$ for all values of $x \in (3, 10)$. Determine the largest possible value of $f(10)$.

4.2.3 Consequences of the Mean Value Theorem

Corollary

If $f'(x) = 0$ for all x in an interval I then f is constant on I.

Example 4: Use the Mean Value Theorem to prove the above corollary.

Corollary

If $f'(x) = g'(x)$ for all x in an interval I then there exists a constant C such that $f(x) = g(x) + C$ on I.

Example 5: Define the function $h(x) = f(x) - g(x)$ and use this to prove the above corollary using the Mean Value Theorem.

Example 6: The acceleration of an object under free fall is given by $a(t) = -9.81$ meters-per-squared-second. Determine the displacement of a function thrown upwards at a velocity of 2 meters-per-second from a height 20 meters off the ground.

4.3 (Thomas 4.3) Monotonicity and the First Derivative Test

4.3.1 Monotonicity and the Derivative

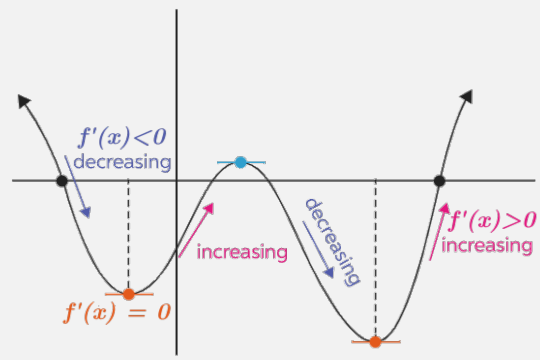
Note: Monotonicity

We assume the reader is familiar with what it means for a function to be increasing or decreasing on an interval. If such a function is either increasing or decreasing on an interval, we say it is **Monotonic** on that interval.

Theorem: The Derivative and Monotonicity

Suppose that f is continuous and differentiable on (a, b).

- **Increasing:** If $f'(x) > 0$ at every $x \in (a, b)$ then f is increasing on (a, b).
- **Decreasing:** If $f'(x) < 0$ at every $x \in (a, b)$ then f is decreasing on (a, b).



Note: Partitioning the Domain into Intervals of Increasing and Decreasing

Notice in the above picture that the intervals for which $f'(x) > 0$ and $f'(x) < 0$ are separated by the points where $f'(x) = 0$. It further includes points where $f'(x)$ does not exist, i.e. overall the critical points.

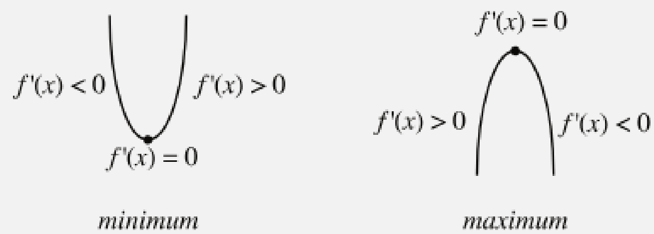
Example 1: Let $f(x) = \dfrac{x}{1+x^2}$. Determine the intervals of increasing and decreasing for $f(x)$.

4.3.2 The First Derivative Test

Theorem: First Derivative Test

Let $f(x)$ be a continuous function and let $x = c$ be a critical point of f.

- **Relative Minimum:** Let x be close to c. If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$ then $x = c$ corresponds to a relative minimum.
- **Relative Maximum:** Let x be close to c. If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ then $x = c$ corresponds to a relative maximum.



Example 2: Let $f(x)$ be a continuous function such that the derivative is given by

$$f'(x) = \frac{(x+1)^2(x-1)}{x^{2/3}}.$$

Classify all critical points.

Example 3: You need to be cautious about the domain when using the first derivative test. Consider the function $f(x) = \dfrac{x}{(x+1)^2}$. Given that the derivative of this function is $f'(x) = \dfrac{1-x}{(x+1)^3}$.

Example 4: Consider the function $g(t) = (t^2 + 3t + 3)e^t$. Find and classify all critical points.

4.4 (Thomas 4.4) Curve Sketching and The Second Derivative Test

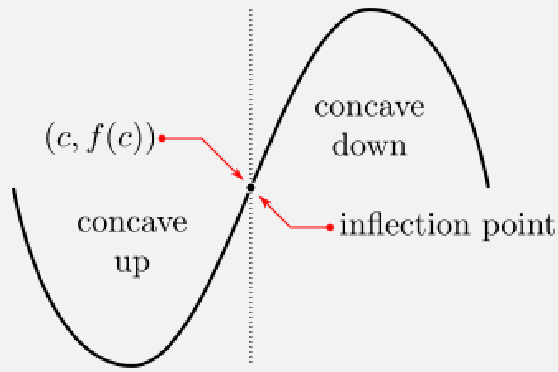
4.4.1 Concavity

(Casual) Definition: Concavity and Inflection Points

Consider the graph of a function $y = f(x)$.

- **Concave Up:** We say an arc of the graph is concave up on an interval if its shape is '*Bowl Shape Up*' on that interval.
- **Concave Down:** We say the graph of a function is concave down if its shape is '*Bowl Shape Down*'.

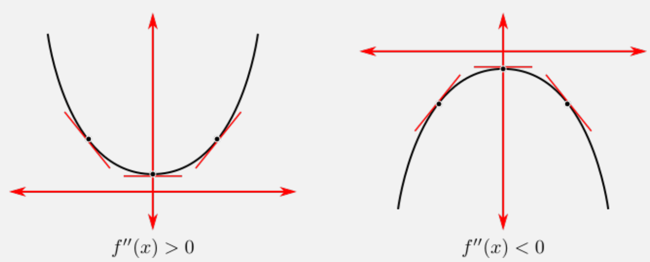
Furthermore, a point where a function changes concavity is called an **Inflection Point**.



Theorem: The Second Derivative and Concavity

Suppose that f is a twice differentiable function on (a, b).

- **Concave Up:** If $f''(x) > 0$ at every $x \in (a, b)$ (i.e. f' is increasing) then f is concave up on (a, b).
- **Concave Down:** If $f''(x) < 0$ at every $x \in (a, b)$ (i.e. f' is decreasing) then f is concave down on (a, b).



Example 1: Determine all intervals of concavity and inflection points for the polynomial $f(x) = x^3 - 3x^2$.

4.4.2 The Second Derivative Test

Theorem: The Second Derivative Test

Let $f(x)$ be a function such that $f'(c) = 0$ and $f''(c)$ exists.

- **Relative Minimum:** If $f''(c) > 0$ then $x = c$ corresponds to a relative minimum.
- **Relative Maximum:** If $f''(c) < 0$ then $x = c$ corresponds to a relative maximum.
- **Inconclusive:** If $f''(c) = 0$ then the second derivative test is inconclusive.

Note: Use the First Derivative Test if Inconclusive

If the second derivative test is inconclusive then you must instead resort to the first derivative test. Inconclusive is not a classification, it merely means that the theorem won't help and you then need to resort to other measures.

Example 2: Find and classify all critical points of the function $f(x) = x^4 - 2x^2 + 3$.

Example 3: Find and classify all critical points of $f(x) = \tan^2(x)$. *Note: There will be an infinite number of critical points.*

4.4.3 Curve Sketching

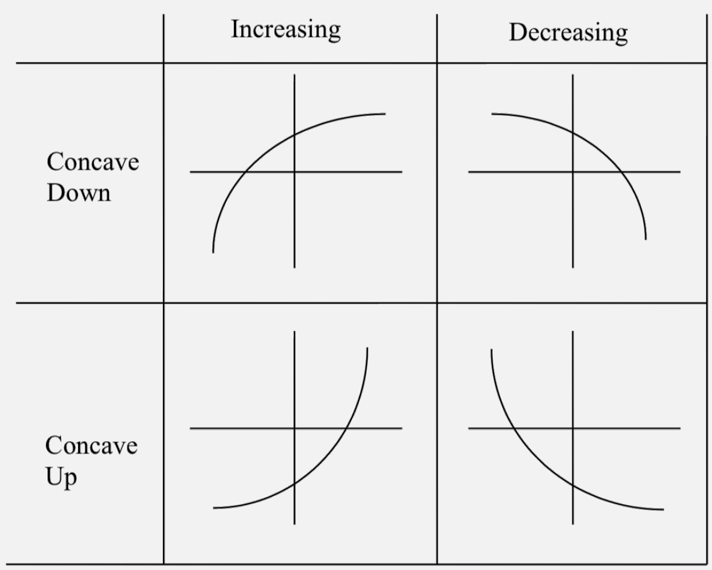
Procedure: Curve Sketching Using Calculus

To sketch the graph of a function f:

1. **Identify the Domain:** This is your canvas. Note any restrictions and regions that are off limits.
2. **Identify Intercepts:** Determine all x and y intercepts. These help position your curve in space.
3. **Determine all Asymptotes:** Determine all horizontal, vertical, oblique, etc asymptotes of your function.
4. **Determine Intervals of Increasing:** Compute $f'(x)$ and locate any critical points. Use these to determine the intervals of increasing or decreasing.
5. **Determine Intervals of Concavity:** Compute $f''(x)$ and locate all points where $f'''(x) = 0$ or does not exist. Use these to determine the intervals of concavity.
6. **Plot Key Points and Asymptotes:** Plot the intercepts and the asymptotes obtained above. If you want greater accuracy, plot even more key points such as relative extrema.
7. **Connect Key Points and Asymptotes:** Connect the points with a piecewise smooth curve (or smooth curve depending) through the key points using the correct concavity and increasing properties obtained prior.

Note: Shape of Arcs

Any 'nice' function is comprised of one of the following four arcs.



Which arc is produced is given by the combination of the increasing or decreasing nature with concavity.

Example 4: Let f be a continuous function satisfying the following:

- The domain is all real numbers;
- f has a single x-intercept at $x = 3$ and a single y-intercept at $y = 2$;
- f has not vertical asymptote, and $\lim_{x \to \infty} f(x) = 1$, $\lim_{x \to -\infty} f(x) = -1$;
- f is increasing on $(-\infty, 1) \cup (3, 5)$ and decreasing on $(-1, 3)$;
- f is concave upward on $(-\infty, -1) \cup (-1, 4) \cup (6, \infty)$ and concave downward on $(4, 6)$; and
- $f(-1) = 3$, $f(3) = 1$, $f(4) = 2$, $f(5) = 4$, and $f(6) = 2$;

Graph this function and label the local extrema along with the points of inflection.

Example 5: Use curve sketching techniques to graph the function $f(x) = \dfrac{1}{x^2+1} + \dfrac{9}{5}$.

Hint: To save on time, you are given that $f'(x) = -\dfrac{2x}{(x^2-1)^2}$ *and* $f''(x) = \dfrac{2(1+3x^2)}{(x^2-1)^3}$.

(Continued...)

4.5 (Thomas 4.5) l'Hôpitals Rule

4.5.1 Indeterminate Forms and L'Hôpitals Rule

Note: Indeterminate Forms

In prior sections we've encountered limits of the form $0/0$, ∞/∞, and $\infty - \infty$ along with possibly others. We've found that we've had to use algebraic tricks to determine their value. That is, the result was dependent on their form and thus they were **Indeterminate Forms**.

There are precisely **Seven Indeterminate Forms**:

$$\frac{0}{0}; \qquad \frac{\infty}{\infty}; \qquad \infty - \infty; \qquad 0 \cdot \infty; \qquad 0^0; \qquad \infty^0; \qquad 1^\infty$$

We will individually cover the techniques required for each form in the following subsections. These techniques around some clever manipulation of **L'Hôpitals Rule**.

Theorem: L'Hôpitals Rule

Suppose that $f(x)$ and $g(x)$ are differentiable on an open interval containing $x = a$. Then provided either

- $f(a) = g(a) = 0$ and $g'(x) \neq 0$ when $x \neq a$; or
- $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to \infty} g(x) = \infty$; then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

The above result holds when $x \to a$ is replaced with $x \to \pm\infty$ provided the appropriate conditions are satisfied.

Example 1: Consider the limit $\lim\limits_{x \to 1} \dfrac{2x^2 - x - 1}{x^2 - 1}$.

(a) Compute the limit using algebraic techniques from an earlier chapter. *Hint:* $-x = -2x + 1$.

(b) Compute the limit using L'Hôpital's Rule. Make sure to check the appropriate conditions are satisfied.

4.5.2 Indeterminate Expressions of the Form $0/0$ and ∞/∞

Procedure: The forms $0/0$ and ∞/∞

Just apply l'Hôpitals Rule.

Example 2: Determine $\lim\limits_{t\to 0} \dfrac{t-\sin(t)}{t^3}$.

Example 3: Determine $\lim\limits_{y\to 0} \dfrac{y^2}{\ln(\sec(y))}$.

Example 4: Determine $\lim\limits_{\theta\to\left(\frac{\pi}{2}\right)^-} \dfrac{\sec(\theta)}{1+\tan(\theta)}$.

4.5.3 Indeterminate Expressions of the Form $\infty - \infty$

Procedure: The form $\infty - \infty$

Join both terms into a single term and check to see if l'Hôpitals rule applies.

Example 5: Determine $\lim_{x\to 0}\left(\frac{1}{\sin(x)} - \frac{1}{x}\right)$.

4.5.4 Indeterminate Expressions of the Form $0 \cdot \infty$

Procedure: The form $0 \cdot \infty$

Suppose that $f(x) \cdot g(x) \to 0 \cdot \infty$. Rewrite the expression either as

$$\frac{g(x)}{1/f(x)} \to \frac{\infty}{\infty} \qquad \text{or} \qquad \frac{f(x)}{1/g(x)} \to \frac{0}{0}$$

then apply l'Hôpitals Rule.

Example 6: Determine $\lim_{\theta\to\infty} \theta \sin(1/\theta)$.

4.5.5 Indeterminate Expressions of the Form 0^0, ∞^0 and 1^∞

Procedure: The forms 0^0, ∞^0 and 1^∞

The trick is to handle the exponent by using a logarithm. Start with $\lim\limits_{x\to a} f(g)^{g(x)}$...

1. Let $y = f(x)^{g(x)}$ and take a natural logarithm of **BOTH** sides to obtain $\ln(y) = g(x)\ln(f(x))$.
2. The expression is now of the form $0 \cdot \infty$. Invoke the prior procedure to write this as
$$\frac{\ln(f(x))}{1/g(x)}$$
and apply l'Hôpitals Rule upon taking the limit of both sides.
3. Solve for the original limit by taking the exponential of both sides, that is
$$\lim_{x\to a} y = \lim_{x\to a} f(x)^{g(x)} = \exp\left(\lim_{x\to a} \frac{\ln(f(x))}{1/g(x)}\right)$$

Example 7: Determine $\lim\limits_{x\to 0^+} (1+x)^{1/x}$.

Example 8: Determine $\lim_{x\to\infty} x^{1/x}$.

Example 9: Determine $\lim_{x\to\infty} \ln(x)^{1/x}$.

4.6 (Thomas 4.6) Applied Optimization

4.6.1 The Critical Point Theorem

Theorem: The Critical Point Theorem

Suppose a function f is continuous on an interval I and that f has **exactly one critical number** in the **interval** I. If this critical number corresponds to a relative extrema then it is a global extrema.

Example 1: The total profit $P(x)$ (in thousands of dollars) from the sale of x hundred-thousand automobile tires is approximated by

$$P(x) = -x^3 + 9x^2 + 120x - 400$$

where $x \geq 5$. Determine the number of hundred-thousand tires that must be sold to maximize profit. Find the maximum profit.

Note: Determining Global Extrema

We have two techniques to determine global extrema:

- Using the Critical Point Theorem; or
- Use the procedure for a continuous function on a closed interval.

Each of these are equipped with conditions that need to be satisfied in order to conclude the desired result, so be cautious.

4.6.2 Optimization Word Problems

Procedure: Solving Problems Using Optimization

1. Identify what the problem is asking.
2. Draw a picture.
3. Introduce and name variables
4. Write an equation for the unknown
5. Test critical and endpoints of the unknown variable in its domain
6. Conclude your answer in written English

Example 1: Show among all 8 meter perimeter rectangles that the one of largest area is a square.

Example 2: A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides. Given 100 meters of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

Example 3: A supplier of bolts wants to create open boxes for the bolts by cutting a square from each corner of a 12 inch by 12 inch piece of metal and then folding up the sides. What size square should be cut from each corner to produce a box of maximum volume?

Example 4: You are a trained lifeguard enjoying your day off at Cadboro-Gyro park, relaxing on the shore. To your dismay, you suddenly see a swimmer in the distance being attacked by a polar bear.

The swimmer is roughly 120 feet out directly from the shore line and you are roughly 300 feet down the beach from the nearest point on the shore to the swimmer. You can run at 13 feet-per-second along the beach, and you can swim at 5 feet-per-second. Given that you want to reach the swimmer as quickly as possible, how far down the beach do you run and how far do you swim?

Given that it takes a polar bear about a minute to finish a meal, are you able to save them?

Example 5: A window is being built and the bottom is a rectangle and the top is a semicircle. If there is 12 meters of framing material, what dimensions must the window be to let in the most light?

4.7 (Thomas 4.7) Newton's Method

Procedure: Newton's Method (Approximating Roots)

The following procedure approximates a root of $f(x)$ given by x^*.

1. Guess a first approximation x_0 to a solution of $f(x) = 0$.
2. Use the recursive formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ provided $f'(x_n) \neq 0$ to obtain a sequence of values $x_0, x_1, x_2, x_3, \ldots$ that approach x^*.

Example 1: Construct the recursive formula for a root of $f(x) = x^2 - 2$ (i.e. $\sqrt{2}$) and use it to fill in the table below. Use an initial guess of $x_0 = 1$.

Current Value of x_n	**Error**$= \lvert x_n - \sqrt{2}\rvert$	**Number of Correct Digits**
1		
1.5		

Example 2: Attempt to use Newton's Method to find a root of $f(x) = x^3 - 2x + 2$ with an initial guess of $x_0 = 1$. Explain why it fails and illustrate with a graph.

4.8 (Thomas 4.8) Antiderivatives

4.8.1 Antiderivatives

Definition: Antiderivative

A function F is an **Antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example 1: Construct an antiderivative $F(x)$ of $f(x) = \cos(x)$ such that $F(\pi) = 2$.

Example 2: Construct an antiderivative $G(s)$ of $g(s) = s^2 + 2s + 3$ such that $G(0) = 0$.

Note: Uniqueness of an Antiderivative

We remind the reader that as a consequence of the mean value theorem that if $F_1(x)$ and $F_2(x)$ are two antiderivatives of $f(x)$, i.e. $F_1'(x) = F_2'(x) = f(x)$, then $F_1(x) = F_2(x) + C$. Thus the only difference between any two antiderivatives of a function is that they differ by a constant, meaning antiderivatives are unique up to a family $F(x) + C$.

4.8.2 The Indefinite Integral

Definition

If $F(x)$ is an antiderivative of $f(x)$ on I then $F(x) + C$ where C is a constant (called a **constant of integration**) is called the **most general antiderivative** of $f(x)$. This is denoted

$$\int f(x)dx = F(x) + C$$

The expression $\displaystyle\int f(x)dx$ is called the **integral of** $f(x)$ and $f(x)$ is called the **integrand**. The term dx is called the **infinitesimal** or **differential** of x.

Theorem: Linearity of the Integral

The integral satisfies the **Linearity Condition**. That is,

$$\int (af(x) + bf(x))dx = a\int f(x)dx + b\int g(x)dx$$

for all constants a and b along with functions $f(x)$ and $g(x)$ that have an antiderivative.

Example: Determine $\int \left(5e^{3x} + 4\csc^2(x)\right) dx$.

Example: Determine $\int \left(\frac{1}{\sqrt{1-t^2}} - \pi^2 \sec(2t)\tan(2t)\right) dt$.

Note: Existence of a Closed Form Antiderivative

Not all functions have an antiderivative (at least not in the conventional sense of representing it by any of the functions you know). For example, there is no $F(x)$ you will be able to find such that

$$\int e^{-x^2} dx = F(x) + C$$

which was proven in around 1833 by Liouville.

Note: Table of Antiderivatives

There is a table of antiderivatives at the beginning of this course pack. It is highly recommended you have these all either memorized or are able to swiftly determine them from memory of reverse differentiation.

Note: Using Algebraic Tricks

You may have to use algebraic methods or trigonometric identities to turn unfamiliar integrands into a scaled sum (linear combination) of familiar ones.

Example: Determine $\int \frac{2z^2+5}{z^2+1} dz$.

Example: Determine $\int (2\sin(\theta/2)\cos(\theta/2) + \tan^2(\theta) + 2) d\theta$.

Chapter 5

Integrals

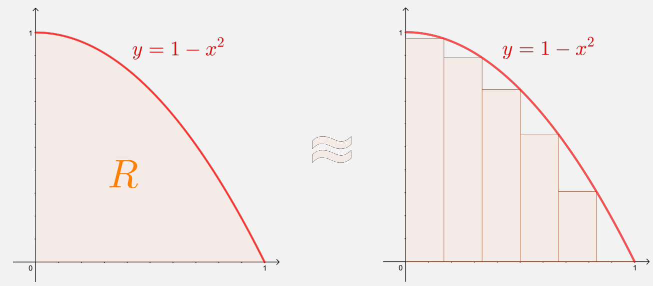
5.1 (Thomas 5.1) Approximating Area Under a Curve

5.1.1 Riemann Sums and Using Rectangles

Note: Using Rectangles to Approximate Area

As an illustration, consider the region R bounded between the x-axis and $y = 1 - x^2$ for $0 \leq x \leq 1$.

To construct an approximation of the area of R, we may cleverly place rectangles about the curve and take the total sum of their area to obtain this approximation.



Definition: Riemann Sums

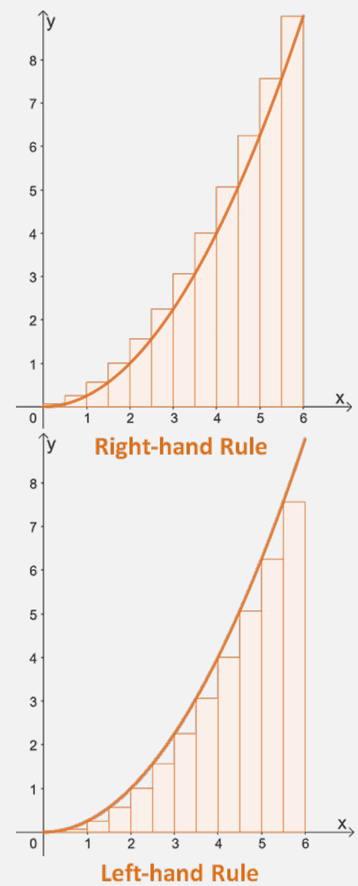
Suppose that $f(x)$ is a continuous on the interval $[a, b]$ and suppose that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is a sequence of points within $[a, b]$. We say that

$$S_n = \sum_{k=1}^{n} f(\xi_k)\Delta x_k = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \cdots + f(\xi_n)\Delta x_n$$

is called a **Riemann Sum** using n rectangles where $\Delta x_k = x_k - x_{k-1}$ represents the width of the k'th rectangle.

- If $\xi_k = x_{k-1}$ we say it is a sum using **Left Endpoints**.
- If $\xi_k = x_k$ we say it is a sum using **Right Endpoints**.
- If $\xi_k = \frac{1}{2}(x_k + x_{k-1})$ we say it is a sum using **Midpoints**.

The over-approximation is referred to as an **Upper Sum** and the under-approximation is referred to as a **Lower Sum**.



Note: Rectangles of Equal Width

The most common approach (out of simplicity) is to construct every rectangle so that they're the same width. If each rectangle is of equal width then $\Delta x_k = \dfrac{b-a}{n}$ for each k. When this occurs, we simply denote it as Δx instead of Δx_k as it no longer depends on which rectangle we're examining.

Example 1: Construct a left endpoint, right endpoint and midpoint Riemann Sum of the function $f(x) = 16 - x^2$ over the interval $[0, 4]$ using $n = 4$ rectangles. Compare this to the actual area of $A = \dfrac{128}{3}$.

5.1.2 Distance Traveled

Note: Distance Traveled Using Velocity and Time

We have established velocity as the rate of change of displacement. Specifically, over a small enough time interval $[t^*, t^* + \Delta t]$ and $\xi \in [t^*, t^* + \Delta t]$ we have

$$v(\xi) \approx \frac{\Delta s}{\Delta t} \Longrightarrow \Delta s \approx v(\xi)\Delta t.$$

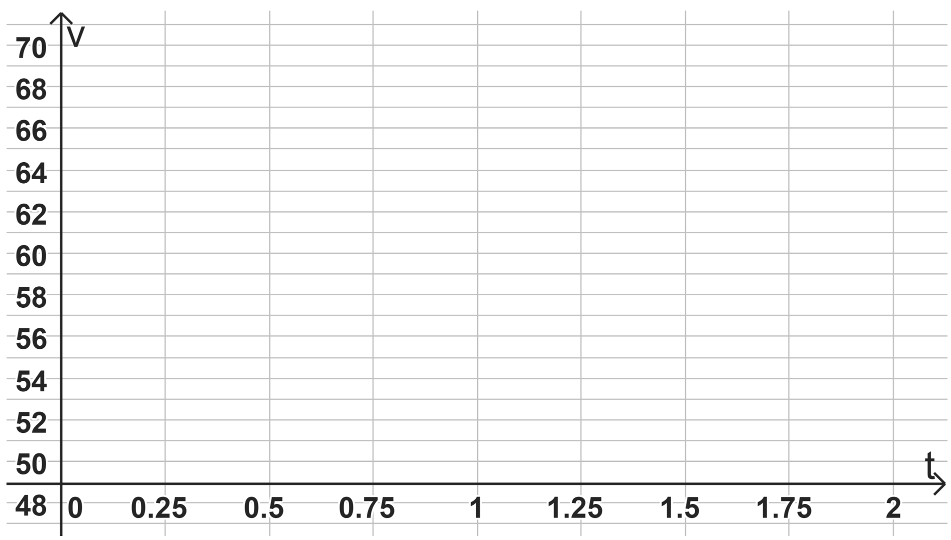
If we subdivide a velocity-time graph into intervals of size Δt then we get that the area underneath the curve represents the total distance traveled:

$$\text{Total Distance Traveled } \approx v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_n)\Delta t.$$

Example 2: The velocities of an automobile moving along a straight highway over a two hour period are given in the following table.

Time (Hours)	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
Velocity (Miles-per-hour)	50	50	58	60	60	66	68	68	70

(a) Plot the key points on a velocity-time graph.



(b) For an upper sum approximation of the distance traveled by the automobile.

5.2 (Thomas 5.2) Sigma Notation and Limits of Finite Sums

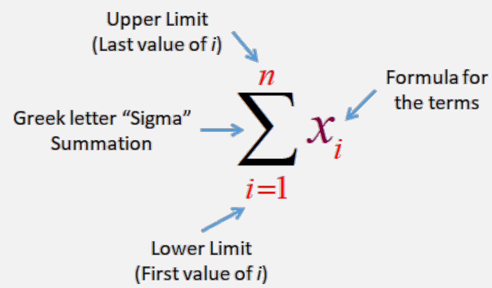
5.2.1 Sigma Notation

Recap: Sigma Notation

Sigma notation allows us to rewrite sum in a more compact form.

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n)$$

the term k is called the **Index**.



Example 1: Compute $\sum_{r=3}^{7} r^2$. Does $\sum_{p=3}^{7} p^2 = \sum_{r=3}^{7} r^2$? What about $\sum_{q=0}^{4} (q+3)^2$?

(a) Compute $\sum_{r=2}^{5} r^2$.

(b) Compute $\sum_{p=2}^{5} p^2$. Does this equal $\sum_{r=2}^{5} r^2$?

(c) Compute $\sum_{q=0}^{3} (q+2)^2$. What do you notice?

Example 2: Use the fact that all odd numbers are of the form $2n+1$ (e.g. $3 = 2(1)+1$, $5 = 2(2)+1$,...) to write the sum $1+3+5+\cdots+51$ in Sigma notation.

5.2.2 Closed Forms and Summation Properties

Theorem: Properties of Finite Sums

Property	**Name**
$\sum_k (f(k)+g(k)) = \sum_k f(k) + \sum_k g(k)$	Sum Rule
$\sum_k Cf(k) = C\sum_k f(k)$	Constant Rule

Theorem: Important Sums

1. $\displaystyle\sum_{k=1}^{n} = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$
2. $\displaystyle\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\displaystyle\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$
4. $\displaystyle\sum_{k=1}^{n} 1 = 1+1+1+\cdots+1 = n$
5. $\displaystyle\sum_{k=1}^{n} r^{k-1} = 1+r+r^2+\cdots+r^{n-1} = \frac{1-r^n}{1-r}$

Example 3: Compute $\sum_{m=1}^{3}(m+5)$ using properties and known important sums

Example 4: Compute $\sum_{s=1}^{n}\frac{s}{n}$ using properties and known important sums

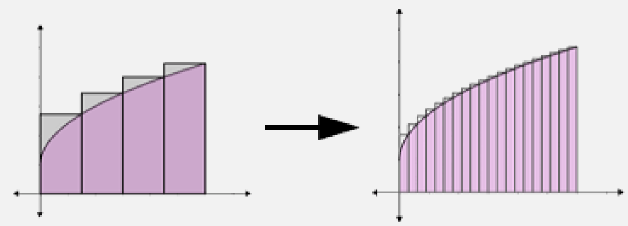
Example 5: Compute $\sum_{p=1}^{n}(3p^2+2)\frac{1}{n}$ using properties and known important sums

5.2.3 Riemann Sums Revisited

Note: Limit of Riemann Sums

As the number of rectangles increase, the approximation for the area under a curve improves. Naturally, the closest we get to a definition for area is the limit,

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x.$$



Example 6: Construct a general formula for the left endpoint sum of $f(x) = 16 - x^2$ over the interval $[0, 4]$. Find a closed form of the sum and take the limit $n \to \infty$ to obtain the area.

5.3 (Thomas 5.3) The Definite Integral

Definition: The Definite (Riemann) Integral

Provided it exists, the **Definite (Riemann) Integral** is denoted and defined as

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\Delta x\right)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$. If this exists we say that $f(x)$ is **(Riemann) Integrable** on $[a, b]$.

Theorem: Integrability Conditions for a Function

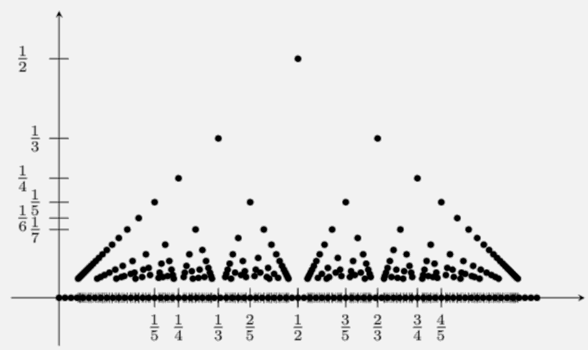
If $f(x)$ is continuous on $[a, b]$ with possible exception of having a finite number of jump discontinuities then f is integrable on $[a, b]$.

Note: What a Non-Integrable Function Looks Like

There are many ways to encounter non-integrable functions. You could either break the continuity condition or the number of finite jump discontinuities. For example, the function $f(x) = 1/x$ is non-integrable on $[-1, 1]$. Another function which is non-integrable is the infamous **Thomae Function**

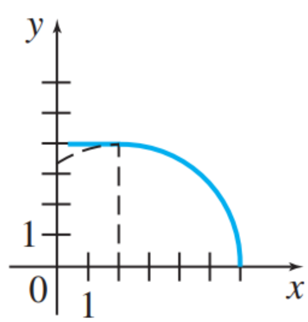
$$\delta(x) = \begin{cases} 1/b & x = a/b \text{ reduced rational} \\ 0 & x \text{ is irrational} \end{cases}$$

which has an infinite number of jump discontinuities.

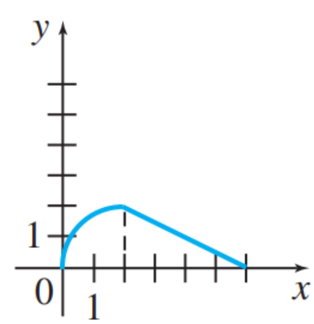


Example 1: For each of the following graphs corresponding to a function $y = f(x)$, evaluate $\int_0^6 f(x)dx$. Assume the curves are comprised of line segments and circular arcs.

(a)



(b)



5.3.1 Properties of Definite Integrals

Properties of the Definite Integral

Property 1	$\int_a^b f(x)dx = -\int_b^a f(x)dx$
Property 2	$\int_a^a f(x)dx = 0$
Property 3	$\int_a^b Cf(x)dx = C\int_a^b f(x)dx$
Property 4	$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
Property 5	$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
Property 6	$\min_{[a,b]}(f(x))(b-a) \leq \int_a^b f(x)dx \leq \max_{[a,b]}(f(x))(b-a)$
Property 7	If $f(x) \leq g(x)$ on $[a,b]$ then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

Example: Suppose that $\int_{-1}^{1} f(x)dx = 5$ and $\int_{1}^{4} f(x)dx = -2$. Compute the following.

(a) $\int_{1}^{-1} f(x)dx$.

(b) $\int_{1}^{4} 5f(x)dx$.

(c) $\int_{-1}^{4} f(x)dx$.

Theorem

For all positive integers n,

$$\int_a^b 1dx = b - a \qquad \text{and} \qquad \int_a^b x^n dx = \frac{1}{n+1}(b^{n+1} - a^{n+1})$$

Example: Determine $\int_2^3 (x^2 + 1)dx$.

Definition

The **average value** of $f(x)$ over $[a, b]$ is $\text{AVG}(f) = \frac{1}{b-a}\int_a^b f(x)dx$.

Example: Find the average value of $f(x) = 3x^2 + 2x + 1$ over the interval $[1, 2]$.

5.4 (Thomas 8.7) Numerical Integration

5.4.1 Replacing Rectangles with other Shapes

Note: Replacing Rectangles with other Shapes

We arbitrarily chose to approximate the area under a curve using rectangles. With good reason of course, as it was a simple choice and it all works out to the same as we take the limit $n \to \infty$. However, if we are not taking $n \to \infty$ and aim to numerically approximate, there are shapes to choose which lead to better approximations of area using less of them. We cover two main types: **The Trapezoidal Rule** and **Simpsons Rule**.

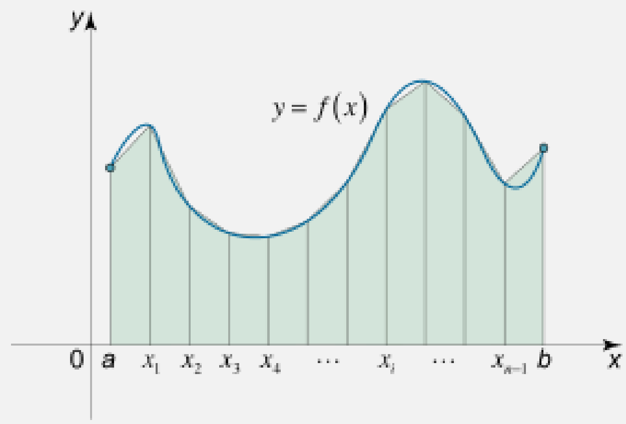
5.4.2 Trapezoidal Rule

Definition: Trapezoidal Approximations

Assuming all trapezoidal widths are equal and given by $\Delta x = \dfrac{b-a}{n}$ for n trapezoids, the area under the curve $y = f(x)$ over $[a, b]$ is approximately

$$T = \frac{\Delta x}{2}(f(a) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(b))$$

where $x_k = a + k\Delta x$. This method is called the **Trapezoidal Rule**.



Example 1: Estimate $\displaystyle\int_1^5 x^2 dx$ using $n = 4$ steps and the Trapezoidal Rule.

Note: A Comment on Approximation

Practically speaking, an approximation is utterly useless if you don't have some way of describing how accurate it is and to what degree it is inaccurate. Hence, it is vital that we are able to describe the error.

Theorem: Error on Trapezoidal Rule

If f'' is continuous and M is a constant such that $|f''(x)| \leq M$ on $[a, b]$, then the error E_T from the trapezoidal approximation of $\displaystyle\int_a^b f(x)dx$ using n steps satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}$$

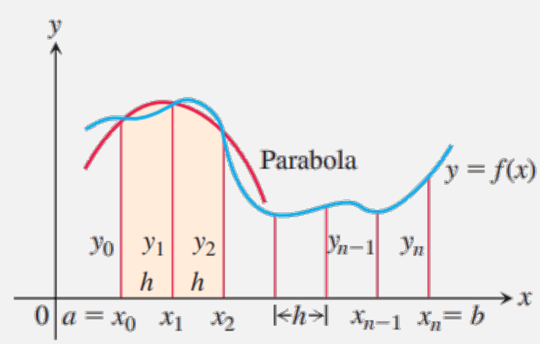
Example 2: Determine an bound on the error E_T of the previous example.

5.4.3 Simpson's Rule (Using Parabolas)

Note: Obtaining Simpson's Rule

Simpson's Rule takes the Trapezoidal Rule a step further and adds some curvature to the roof of the trapezoids.

Specifically, we use parabolas. The derivation is arduous, and we leave it to the reader to satiate their curiosity by reading the textbook.



Definition: Simpson's Rule

Assuming all parabolic shape widths are equal and given by $\Delta x = \dfrac{b-a}{n}$ for n trapezoids, the area under the curve $y = f(x)$ over $[a, b]$ is approximately

$$S = \frac{\Delta x}{3}(f(a)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(b))$$

where n **must be even** and $x_k = a + k\Delta x$.
This method is called **Simpson's Rule**.

Example 3: Estimate $\int_0^2 5x^4 dx$ using $n = 4$ steps and Simpson's Rule.

Theorem: Error on Trapezoidal Rule

If $f^{(4)}$ is continuous and M is a constant such that $|f^{(4)}(x)| \leq M$ on $[a, b]$, then the error E_S from using Simpson's Rule to approximate $\int_a^b f(x)dx$ using n steps satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^5}{180n^4}$$

Example 4: Determine a bound on the error E_S of the previous example.

5.5 (Thomas 5.4) The Fundamental Theorem(s) of Calculus

5.5.1 Integral Functions

Definition: Integral Functions

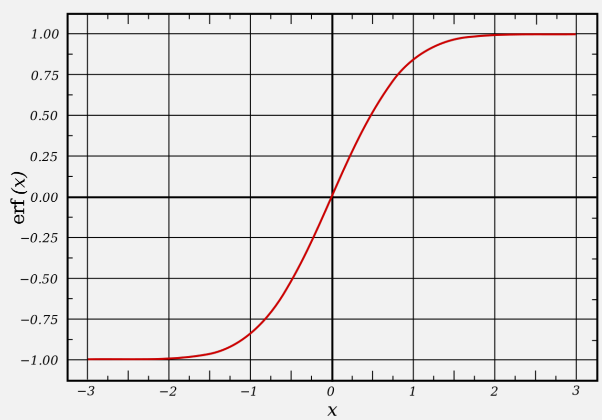
An **Integral Function** is a function of the form $G(x) = \int_a^x g(t)dt$.

Note: Well Known Integral Functions

There are several well known integral functions whose utility is cardinal to scientific theory and modeling. For example, in statistics the area underneath a bell curve is given by the **Error Function**:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Your calculator isn't sophisticated enough to compute values of this function, but programs like GeoGebra or Wolfram (or even typing it into Google) will give you values of this function.



5.5.2 The First Fundamental Theorem

Theorem: The First Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a, b]$ then the integral function $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) = f(x)$.

Example 1: Consider the function $f(x) = \int_0^x \cos(t^2)dt$. Determine the equation of the tangent line to $y = f(x)$ at $x = \sqrt{\pi}$.

Example 2: Let $f(x) = \int_1^{\sin(x)} e^{-t^2}dt$. Determine $f'(x)$.

Example 3: Let $f(x) = x\int_1^{x^2} \sqrt{1-2\sin^2(t)}dt$. Determine $f'(x)$.

Example 4: Let $f(x) = \int_{-x}^{x^3} \frac{e^{-t}}{t}dt$. Determine $f'(x)$. *Note: The lower bound is not constant.*

5.5.3 The Second Fundamental Theorem

Theorem: The Second Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$ then $\int_a^b f(x)dx = F(b) - F(a)$.

Note: Notation for the Net Difference

The net difference $F(b) - F(a)$ is often denoted as $F(x)\Big|_{x=a}^{x=b}$.

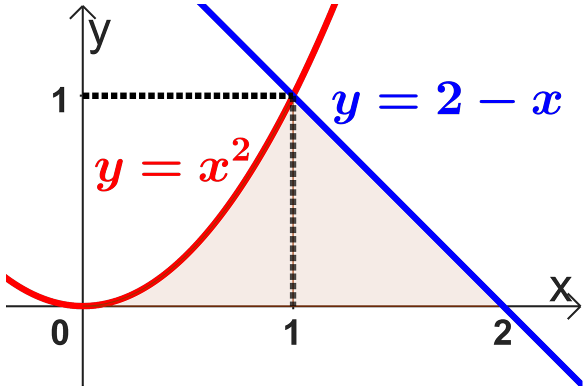
Example 5: Compute $\int_0^{\pi/4} \sec(x)\tan(x)dx$.

Example 6: Compute $\int_0^1 \frac{dz}{z^2 + 1}$.

Example 7: Compute $\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2} ds$.

5.5.4 Determining 'Area' Using the Definite Integral

Example 8: Find the area trapped between $y = x^2$, $y = 2 - x$ and the x-axis.

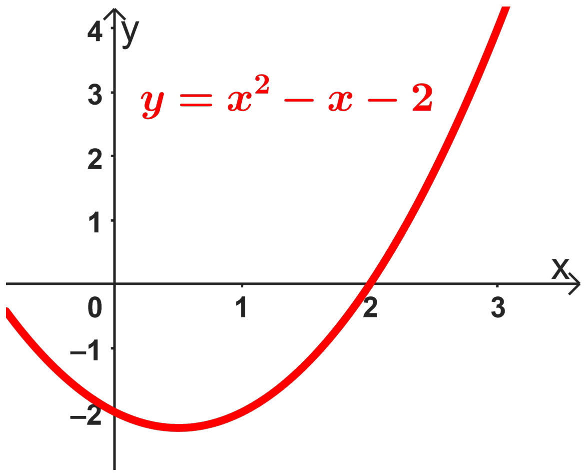


Note: Signed Area

The integral gives the **signed area** between the function and the x-axis. That is, if $f(x) < 0$ on $[a, b]$ then so will $\int_a^b f(x)dx < 0$.

Example 9: Compute $\int_0^{2\pi} \sin(x)dx$ and explain your result.

Example 10: Compute the **Area** between $y = x^2 - x - 2$ and the x-axis on $[0, 3]$, i.e. compute $\int_0^3 |x^2 - x - 2|dx$.



5.6 (Thomas 5.5) Substitution

5.6.1 Indefinite Integral Substitution

Theorem: The Indefinite Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Procedure: Using Integral Substitution to find an Antiderivative

1. Substitute $u = g(x)$. Compute and replace $du = g'(x)dx$ to obtain $\int f(u)du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

Example 1: Compute $\int (x^3 + x)^5(3x^2 + 1)dx$.

Example 2: Compute $\int 2xe^{x^2}dx$.

Example 3: Compute $\int \frac{\ln(t)}{t}dt$.

Example 4: Compute $\int \sin^{50}(\theta)\cos(\theta)d\theta$.

Example 5: Compute $\int \psi \tan^2(\psi^2)d\psi$.

Example 6: Compute $\int \sqrt{1+\sin^2(\phi)}\,\sin(\phi)\cos(\phi)d\phi$.

5.6.2 Integrals of $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$

Theorem: Integrals of Trigonometric Functions

$$\int \tan(x)dx = \ln|\sec(x)| + C \qquad \int \cot(x)dx = \ln|\sin(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C \qquad \int \csc(x)dx = -\ln|\csc(x) + \cot(x)| + C$$

Note: Repeated Substitution

Sometimes repeated uses of substitution is necessary to determine an antiderivative.

Example 7: Compute $\displaystyle\int 3e^{3x+5}\sec\left(e^{3x+5}\right)dx$.

Example 8: Compute $\displaystyle\int \frac{18\tan^2(\theta)\sec^2(\theta)}{(2+\tan^3(\theta))^2}d\theta$.

5.6.3 Trickier Substitutions

Note: Trickier Substitutions

Integration is often more art than science. Sometimes you have to be quite creative with your substitution in order to determine an antiderivative. Below are some illustrations.

Example 9: Evaluate $\int \frac{2x}{x^4+1}dx$.

Example 10: Evaluate $\int \frac{1}{1+e^{-t}}dt$

Example 11: Evaluate $\int z\sqrt{2z-3}dz$.

5.7 (Thomas 5.6) Definite Integral Substitutions and the Area Between Curves

5.7.1 Definite Integral Substitution

Note: Changing the Bounds

Definite integral substitution is mostly the same as indefinite integrals. The one difference being that you transform the bounds.

Theorem: The Definite Substitution Rule

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$ then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 1: Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$.

Example 2: Evaluate $\displaystyle\int_{\pi/4}^{\pi/2} \cot(\theta)\csc^2(\theta)d\theta$.

Example 3: Let $f(x)$ be a continuous function satisfying the following table.

	$\int_2^6 f(x)dx$	$\int_2^{25} f(x)dx$	$\int_4^5 f(x)dx$	$\int_4^{25} f(x)dx$	$\int_4^6 f(x)dx$
$\int_a^b f(x)dx =$	1	2	3	4	5

Use this table to evaluate $\int_2^5 4xf(x^2)dx$

Example 4: Evaluate $\int_{\frac{1}{2\sqrt{2}}}^{3} \frac{4t}{2-8t^2}dt$.

Example 5: Evaluate $\int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^4}}dx$.

5.7.2 Integrals of Even and Odd Functions

Definition: Odd and Even Functions

- A function $f(x)$ is called **Odd** if $f(-x) = -f(x)$ for all x in the domain.
- A function $f(x)$ is called **Even** if $f(-x) = f(x)$ for all x in the domain.

Theorem: Integrals of Odd and Even Functions

Let f be continuous on the **Symmetric Interval** $[-a, a]$.

- If $f(x)$ is an even function then $\displaystyle\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.
- If $f(x)$ is an odd function then $\displaystyle\int_{-a}^{a} f(x)dx = 0$.

Example 6: Evaluate $\displaystyle\int_{-5}^{5} |x|dx$ and $\displaystyle\int_{-5}^{5} x|x|dx$.

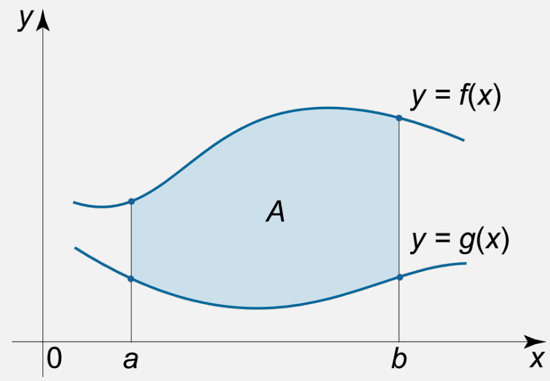
Example 7: Evaluate $\displaystyle\int_{-\pi}^{\pi} \cos^2(x)dx$. *Hint:* $\cos^2(x) = \dfrac{1}{2}\left(1 + \cos(2x)\right)$.

5.7.3 The Area Between Curves

Definition: The Area Between Curves

If $f(x)$ and $g(x)$ are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then **Area of the Region Between the Curves** $y = f(x)$ **and** $y = g(x)$ **from** a **to** b is the integral

$$\text{Area} = \int_a^b (f(x) - g(x))dx.$$



Example 8: Find the area of the region bounded by $f(x) = 1 - x^2$, $g(x) = 2x + 4$, $x = -1$ and $x = 2$.

Example 9: Determine the area of the region bounded by the curves $y = \frac{1}{2}x^2 - 2$ and $y = x + 2$.

Example 10: Determine the area between the curves $y = x^2 - 2x$ and $y = x$ on $[0, 4]$. *Hint: Notice there is a crossover in the region of integration.*

Example 11: Determine the area of the region bounded by $x = 10 - y^2$ and $x = (y-2)^2$.