MATH 202 Final Exam

December 12th, 2020
2:00 PM–4:00 PM

- This is an open-book exam.
- The exam has 3 different sections. Read the instructions at the beginning of each section carefully.
- The exam has **11 questions** and is out of **100 points**.
- You may use a calculator for elementary arithmetic.
- Submit your solutions in **one file** to Crowdmark. The submission must be complete **within 15 minutes** after this exam.

Question 1 (Academic integrity pledge; mandatory). Students must abide by UVic academic regulations and observe standards of 'scholarly integrity,' (no plagiarism or cheating). Therefore, this do-at-home exam must be taken individually and not with a friend, classmate, or group, nor can you access internet resources while completing the exam. You are also prohibited from sharing any information about the exam with others.

To sign this pledge, write the following sentence with your name and signature on the first page of the file you submit.

I, (name) affirm that I will not give or receive any aid on this exam, that all work will be my own. (Signature).

1 Multiple-Choice Questions (10%)

Instructions

- This section has 2 questions (Questions 2–3).
- Choose **one and only one** answer for each question.

Question 2 (5%). Consider the line in 3D through the point $(1, 2, 3)$ parallel to the x-axis. Choose the answer where *both* of the two descriptions of this line are correct.

(A) Description 1: $y = 2$, $z = 3$.
Description 2: $\langle 1, 2, 3\rangle + t\langle 1, 0, 0\rangle$ for $-\infty < t < \infty$.

(B) Description 1: $y = 2$, $z = 3$.
Description 2: $\langle 1, 0, 0\rangle + t\langle 1, 2, 3\rangle$ for $-\infty < t < \infty$.

(C) Description 1: $x = 1$, $y = 2$.
Description 2: $\langle 1, 2, 3\rangle + t\langle 1, 0, 0\rangle$ for $-\infty < t < \infty$.

(D) Description 1: $x = 1$, $y = 2$.
Description 2: $\langle 1, 0, 0\rangle + t\langle 1, 2, 3\rangle$ for $-\infty < t < \infty$.

Question 3 (5%). Consider the following function of two variables:

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2 - 2}}.$$

Choose the correct set of (x, y) where f is well defined as a real-valued function at *every* point.

(A) $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

(B) $x \geq 1$ and $y \leq -1$.

(C) $x^2 + y^2 = 1$.

(D) $x^2 + y^2 = 4$.

2 Fill-in-The-Blank Questions (30%)

<u>**Instructions**</u>

- This section has 4 questions (Questions 4–7).
- Each question has three blanks.
- **Only the answers** will be reviewed for grading.

Question 4 (7.5%). Let $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j} - \mathbf{k}$.

The area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$ is (4-1) ___________ .

The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$ is (4-2)___________ .

The vector $\text{proj}_{\mathbf{v}}\mathbf{u}$ is (4-3) ___________ .

Fill in (4-1) and (4-2) with numbers, and fill in (4-3) with a vector taking the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Question 5 (7.5%). This question considers the following three limits with *different* ways of passing $(x, y) \to (0, 0)$:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{x^2}{x^2 + y^2} = \text{(5-1) \_\_\_\_\_\_\_\_\_\_\_},$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} \frac{x^2}{x^2 + y^2} = \text{(5-2) \_\_\_\_\_\_\_\_\_\_\_},$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2} = \text{(5-3) \_\_\_\_\_\_\_\_\_\_\_}.$$

Fill in (5-1), (5-2) and (5-3) with numbers or DNE (does not exist).

Question 6 (7.5%). Let

$$f(x, y) = xy^2.$$

This question considers the argument to find the unit vector $\mathbf{u}$ minimizing $D_{\mathbf{u}} f(3, 4)$.

First, given a unit vector $\mathbf{u}$, the directional derivative $D_{\mathbf{u}} f(3, 4)$ is defined as the rate of change of f at $(3, 4)$ in the direction $\mathbf{u}$ so that

$$D_{\mathbf{u}} f(3, 4) = \frac{d}{dt} f\big(\mathbf{r}(t)\big)\Big|_{t=0},$$

where

$$\mathbf{r}(t) = \text{(6-1) \_\_\_\_\_\_\_\_\_\_ } .$$

Then by the chain rule, the directional derivative $D_{\mathbf{u}} f(3, 4)$ is given by

$$D_{\mathbf{u}} f(3, 4) = \mathbf{v} \cdot \mathbf{u}, \quad \text{where } \mathbf{v} = \text{(6-2) \_\_\_\_\_\_\_\_\_\_ } .$$

The dot product $\mathbf{v} \cdot \mathbf{u} = |\mathbf{v}||\mathbf{u}| \cos(\theta)$, for $0 \le \theta \le \pi$, is minimized when the angle θ between $\mathbf{v}$ and $\mathbf{u}$ is π. Hence, $D_{\mathbf{u}} f(3, 4)$ is minimized if we choose

$$\mathbf{u} = \text{(6-3) \_\_\_\_\_\_\_\_\_\_ } .$$

Fill in (6-1) with a function explicit in t. Fill in (6-2) and (6-3) with vectors taking the form $\langle u_1, u_2 \rangle$.

Question 7 (7.5%). This question looks for general solutions of the linear equation:

$$y'' - \frac{7}{x} y' + \frac{9}{x^2} y = 0, \quad y = y(x). \tag{*}$$

This equation can be solved by using the general solutions of homogeneous linear equations with constant coefficients if we use an appropriate substitution. Specifically, the homogeneous linear equation with constant coefficients corresponding to (*) is

$$\text{(7-1) \_\_\_\_\_\_\_\_\_\_ },$$

and the substitution is

$$t = \text{(7-2) \_\_\_\_\_\_\_\_\_\_}.$$

Accordingly, the general solutions of the original equation (*) are given by

$$\text{(7-3) \_\_\_\_\_\_\_\_\_\_}.$$

Fill in (7-1) with an equation of the form $A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0$ for y after the substitution and for numbers A, B, C. Fill in (7-2) and (7-3) with expressions explicit in x.

3 Short-Answer Questions (60%)

Instructions

- This section has 4 questions (Questions 8–11).
- Answer each question as you did in the quizzes and the midterm exams.
- Organise and show your work. **Any unsupported answers will receive no credit.**

Question 8. Let P be a plane in 3D such that (i) P is perpendicular to the plane $P' : x + 2y + 3z = 0$, (ii) P is parallel to the vector $\langle 3, 2, 1\rangle$, and (iii) P passes through $(1, 6, 8)$.

(1) (4%) Find two vectors from the above statement such that they are perpendicular to *any* normal vector of P.

(2) (8%) Find a nonzero normal vector of P.

(3) (3%) Find the equation of P taking the form $ax + by + cz = d$.

Question 9. Consider the first-order equation:

$$\frac{dy}{dx} = \frac{y}{x} - \frac{4x}{y}, \quad y = y(x).$$

(1) (6%) Determine whether the equation is separable and whether the equation is exact. Justify your answer for *each* of these types of equations by showing how the differential equation can be written in the required form or why it cannot be written so.

(2) (9%) Solve the above equation as a homogeneous equation. (Solutions by other methods will not be given any credit.)

Question 10. Consider the differential equation

$$y'' - 10y' + 25y = 3e^{5x}, \quad y = y(x).$$

(1) (6%) Find the complementary solutions.

(2) (7%) Find a particular solution.

(3) (2%) Find the general solutions.

(Solutions by using Laplace transforms will not be given any credit.)

Question 11. Consider the equation

$$y'' + 4y = e^t, \quad y = y(t)$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

(1) (7%) Find the Laplace transform $Y(s) = \mathscr{L}\{y(t)\}(s)$.

(2) (8%) Solve $y(t)$ by inverting the Laplace transform from (1).

Justify your solutions for (1) and (2) by using the following formulas:

$\mathscr{L}\{e^{at}t^n\}(s) = \dfrac{n!}{(s-a)^{n+1}}, \quad s > a;\ n = 0, 1, 2\cdots$
$\mathscr{L}\{e^{at}\sin(\theta t)\}(s) = \dfrac{\theta}{(s-a)^2+\theta^2}, \quad s > a$
$\mathscr{L}\{e^{at}\cos(\theta t)\}(s) = \dfrac{s-a}{(s-a)^2+\theta^2}, \quad s > a$
$\mathscr{L}\{e^{at}f(t)\}(s) = \mathscr{L}\{f\}(s-a)$
$\mathscr{L}\{f'(t)\}(s) = -f(0) + s\mathscr{L}\{f(t)\}(s)$
$\mathscr{L}\{f''(t)\}(s) = -f'(0) - sf(0) + s^2\mathscr{L}\{f(t)\}(s)$