CMSC740
Advanced Computer Graphics

Fall 2025

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Overview

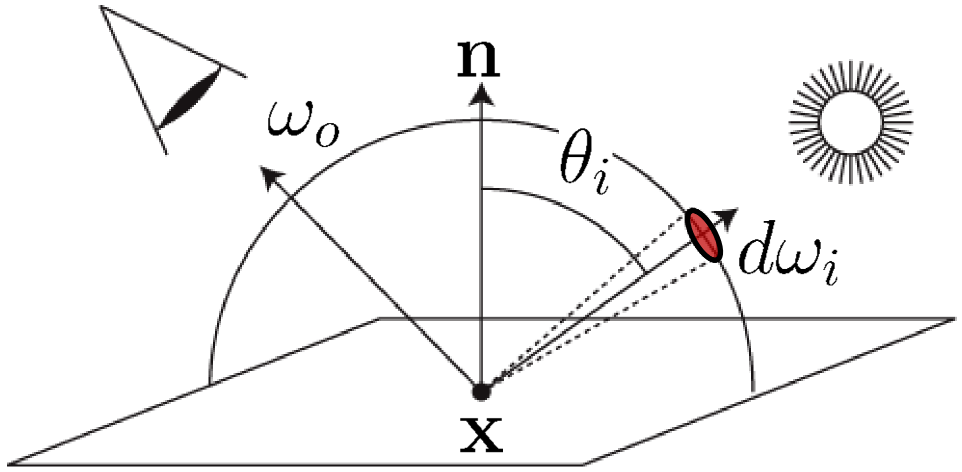
Advanced sampling techniques

- Surface form of hemispherical integrals
- Multiple importance sampling
- Beyond uniform pseudo random numbers

Reflection equation

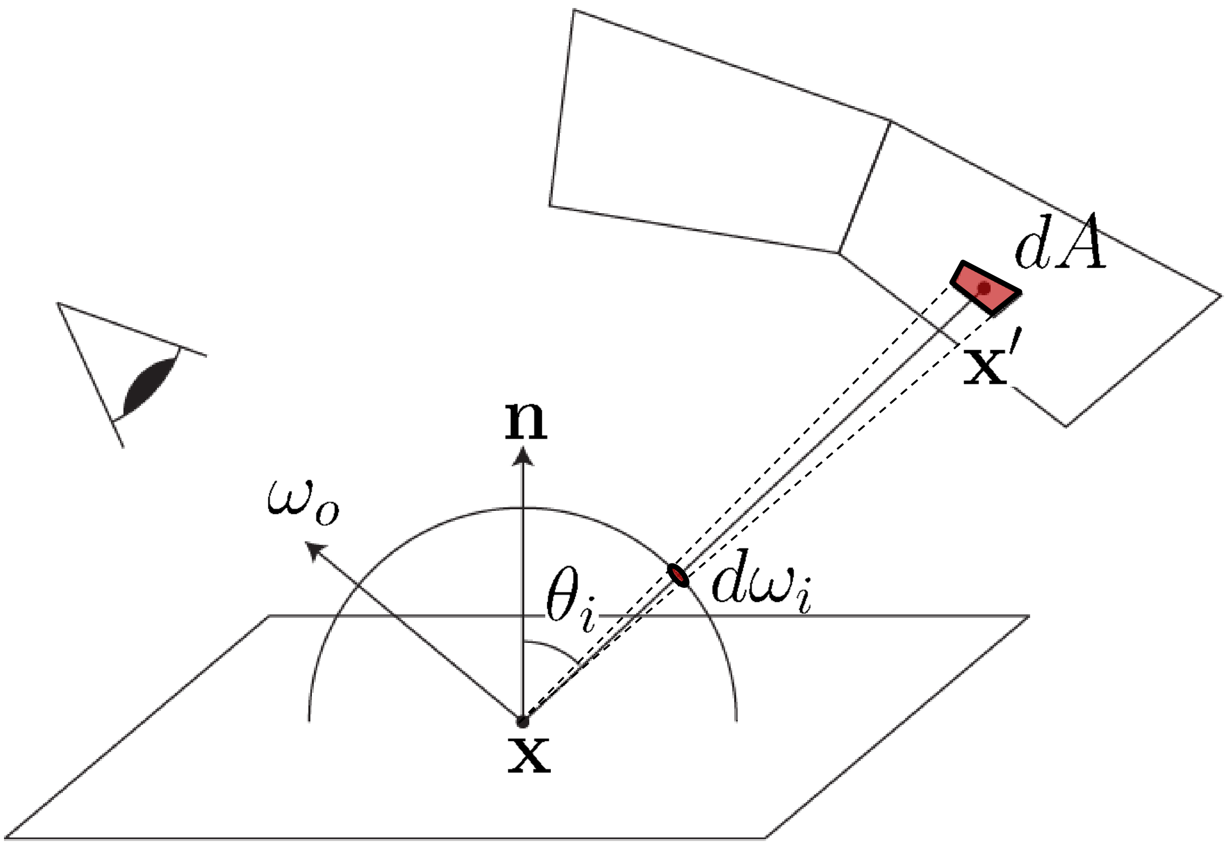
- So far: directional form of reflection equation
- Integration over solid angle

$$L_o(\mathbf{x}, \omega_o) = \int_{\mathcal{H}^2(\mathbf{n})} f(\mathbf{x}, \omega_o, \omega_i) L_i(\mathbf{x}, \omega_i) \cos\theta_i d\omega_i$$



Surface form

- Integration over surface elements dA instead of solid angle $d\omega_i$



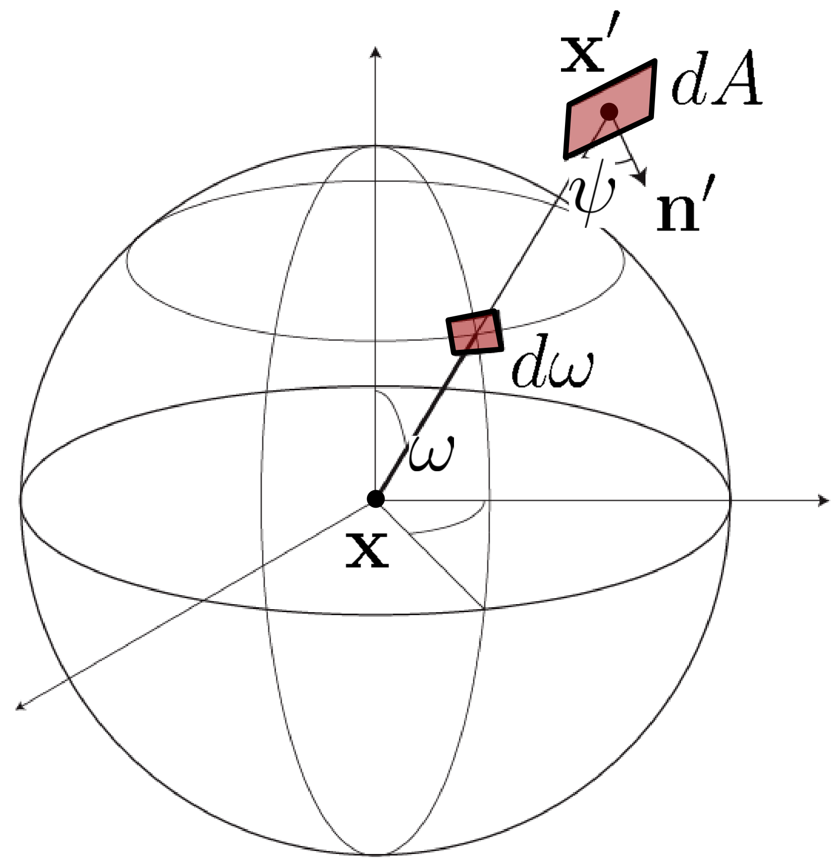
Change of integration variables

- Solid angle spanned by differential surface element dA at location $\mathbf{x}$' with normal $\mathbf{n}$'

$$d\omega = \frac{\cos\psi}{\|\mathbf{x}' - \mathbf{x}\|^2} dA$$

where

$$\cos\psi = \frac{\mathbf{n}' \cdot (\mathbf{x} - \mathbf{x}')}{\|\mathbf{x}' - \mathbf{x}\|}$$

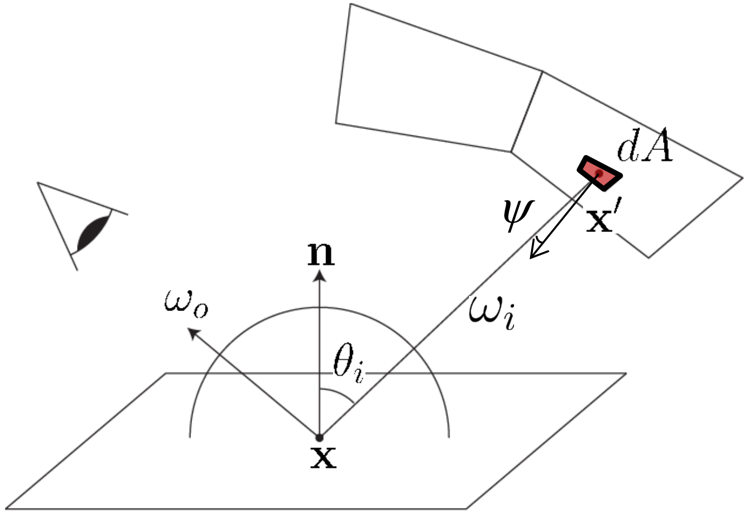


Surface form

- Reflection integral over all visible surface elements

$$L_o(\mathbf{x}, \omega_o) = \int_{\text{all } \mathbf{x}' \text{ visible to } \mathbf{x}} f(\mathbf{x}, \omega_o, \omega_i) L_i(\mathbf{x}, \omega_i) \cos\theta_i \frac{\cos\psi}{\|\mathbf{x}' - \mathbf{x}\|^2} dA$$

where $\omega_i = \dfrac{\mathbf{x}' - \mathbf{x}}{\|\mathbf{x}' - \mathbf{x}\|}$



Surface form

- Reflection integral over all surfaces using visibility function

$$L_o(\mathbf{x}, \omega_o) =$$

Geometry term

$$\int_{\text{all } \mathbf{x}'} f(\mathbf{x}, \omega_o, \omega_i) L_i(\mathbf{x}, \omega_i) \boxed{\frac{v(\mathbf{x}, \mathbf{x}') \cos\theta_i \cos\psi}{\|\mathbf{x}' - \mathbf{x}\|^2}} dA$$

where the visibility function is

$$v(\mathbf{x}, \mathbf{x}') = \begin{cases} 1 & \text{if } \mathbf{x} \text{ and } \mathbf{x}' \text{ are mutually visible} \\ 0 & \text{otherwise.} \end{cases}$$

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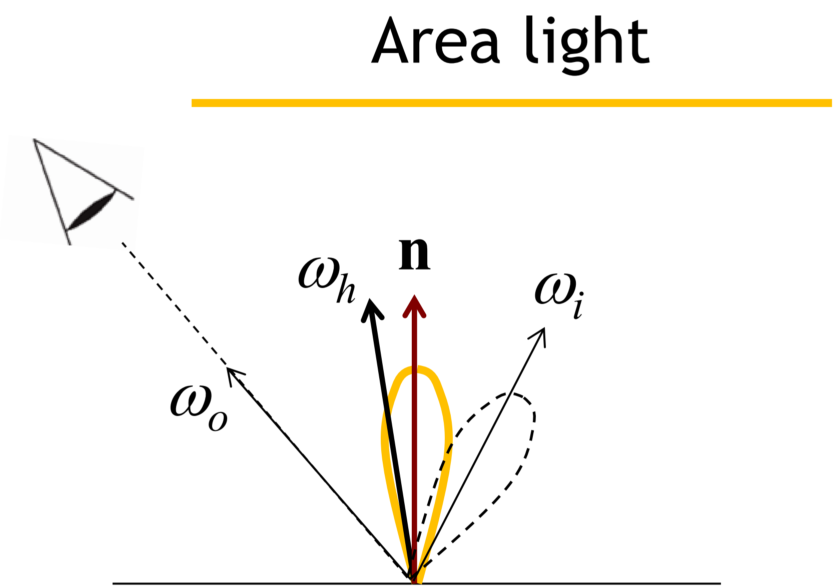
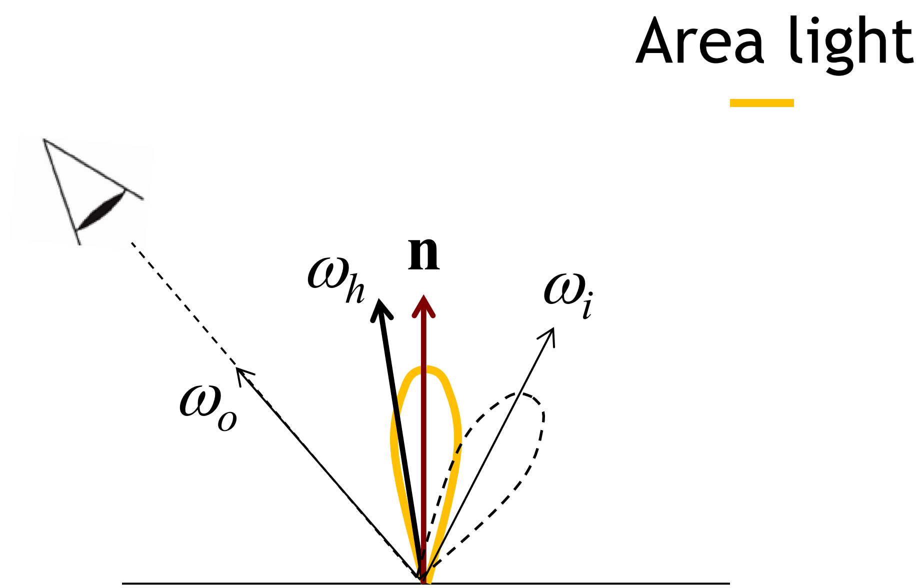
Sampling direct illumination

- How to best distribute samples?



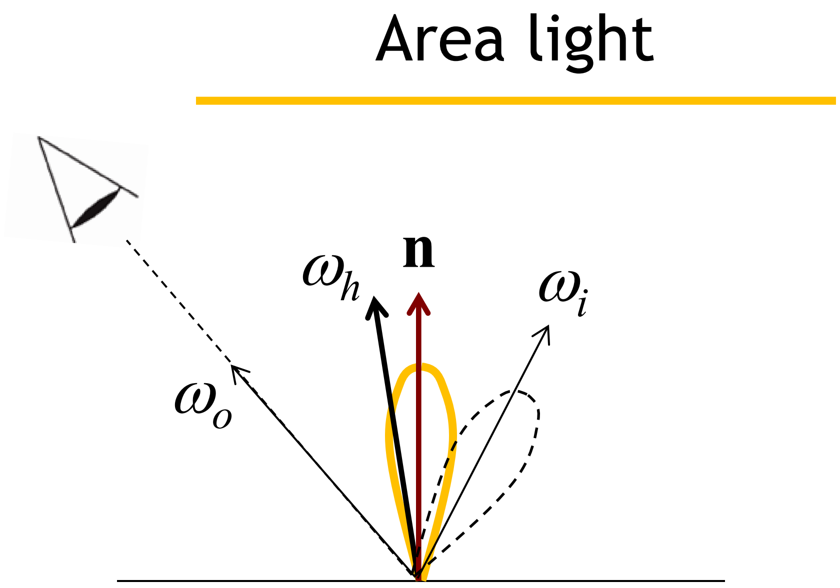


BSDF vs. are light sampling

- Importance sample the BSDF: place more samples where BSDF is large
- Importance sample the light source: place more samples where emission is large (non-zero)

BSDF vs. are light sampling

A. It is better to importance sample using the BSDF
B. It is better to importance sample the light source



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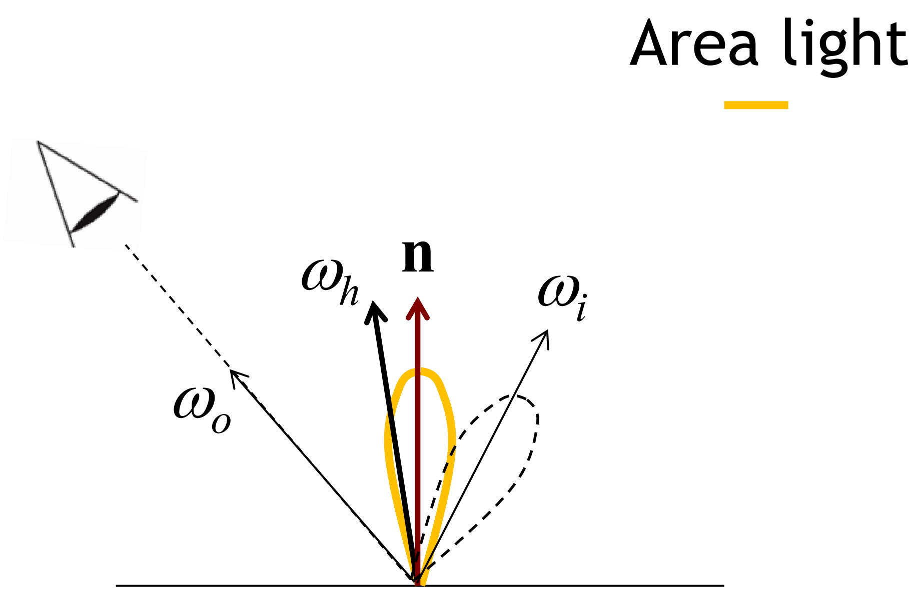
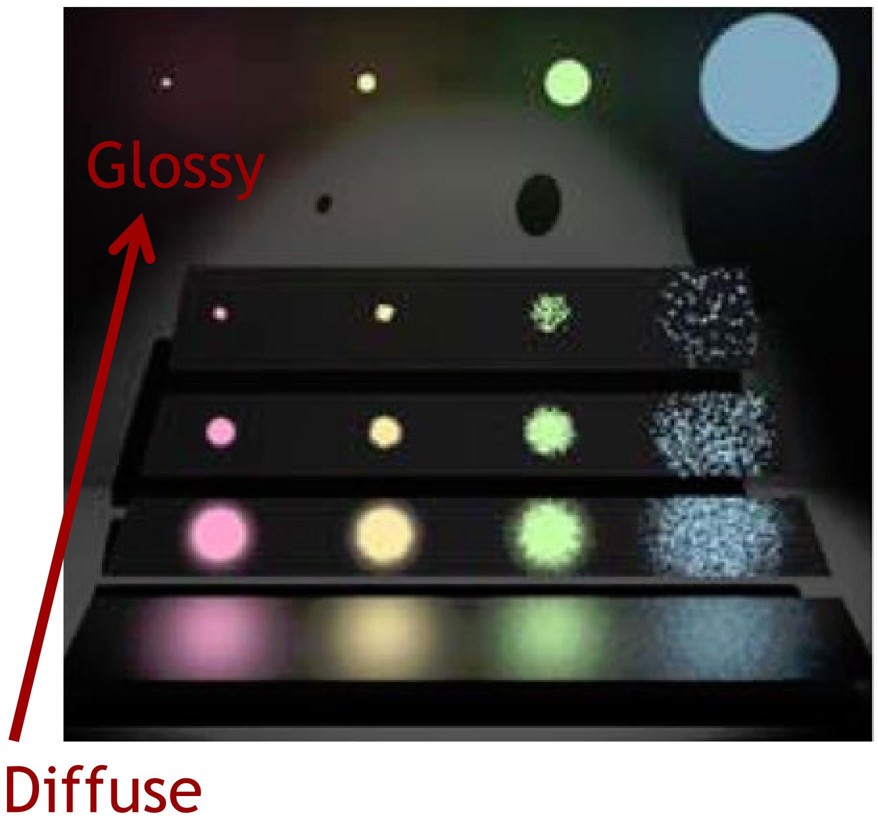
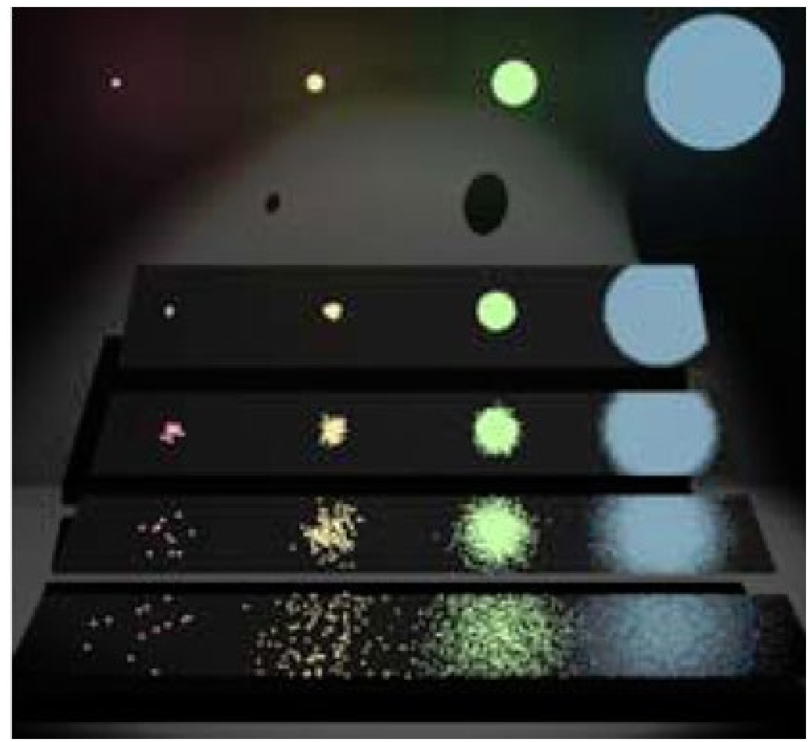


Sampling direct illumination

Sampling according to the light (importance sampling light source areas)



Sampling according to the BSDF (importance sampling the BSDF)



[Veach, Guibas]

Which sampling technique?

- Importance sample light source (area form)

$$F_a = \frac{1}{N}\sum_j \frac{f(\mathbf{x}, \omega_o, \omega_{i,j}) L_i(\mathbf{x}, \omega_{i,j}) v(\mathbf{x}, \mathbf{x}'_j) \cos\theta_{i,j} \cos\psi_j}{\|\mathbf{x}'_j - \mathbf{x}\|^2 p(\mathbf{x}'_j)}$$

PDF proportional to emission

Geometry term

- Importance sample BSDF (directional form)

$$F_d = \frac{1}{N}\sum_j \frac{f(\mathbf{x}, \omega_o, \omega_{i,j}) L_i(\mathbf{x}, \omega_{i,j}) \cos\theta_i}{p(\omega_{i,j})}$$

PDF proportional to BSDF

Multiple importance sampling (MIS)

- Naive approach: use both sampling techniques and average, that is, $(F_a+F_d)/2$
  - Variance is bounded by larger variance of two techniques
- MC estimator F with multiple importance sampling
  - Weighted average, with weights that reduce variance in optimal way
  - n sampling techniques p_i, weights w_i
  - N samples from each technique, j-th sample of i-th technique $X_{i,j}$

$$F = \underbrace{\frac{1}{N} \sum_{j=1}^{N}}_{N \text{ samples}} \underbrace{\sum_{i=1}^{n} w_i(X_{i,j})}_{\text{Weighted average of } n \text{ techniques}} \frac{f(X_{i,j})}{p_i(X_{i,j})}$$

N samples Weighted average of n techniques

Requirement: partition of unity $\sum_{i=1}^{n} w_i(x) = 1 \quad \text{for all } x$

Multiple importance sampling (MIS)

- Unbiased estimator of integral, as before!
- Weights for provable variance reduction
  - Balance heuristics

$$w_i(x) = \frac{p_i(x)}{\sum_{k=1}^{n} p_k(x)}$$

  - Power heuristics

$$w_i(x) = \frac{p_i^2(x)}{\sum_{k=1}^{n} p_k^2(x)}$$

- Details, proofs http://graphics.stanford.edu/papers/veach_thesis/thesis.pdf

Multiple importance sampling

Naïve sampling

Optimal sampling

[Veach, Guibas]

Same number of samples

Implementation

- Multiple importance sampling for making connection to light source (shadow rays, also called “next event estimation”) is simple extension of standard unidirectional path tracing
- Take two samples for each next event estimation
  - One by sampling the BRDF/BSDF (next randomly sampled ray direction)
  - One by sampling the light
  - Combine using MIS weights

Implementation

- For MIS weights w_i, need to compute probability densities of taking each sample with both techniques
- Need to express both densities in same parameterization (surface area or solid angle)

$$p_{solid\ angle} = p_{area} / cos\ *\ r\text{^}2$$

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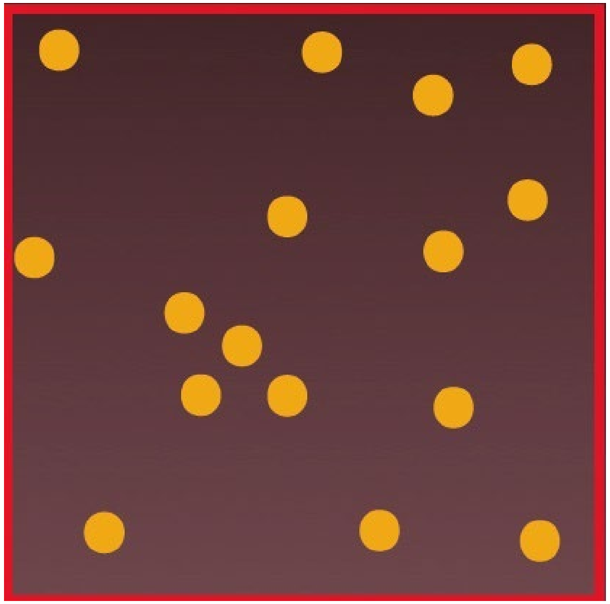
Observation

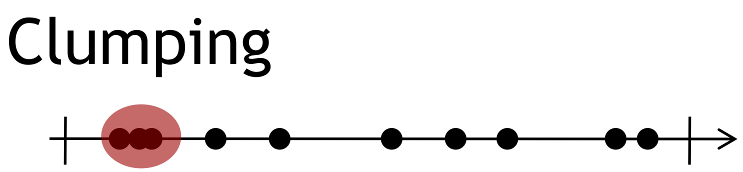
- Sample distributions from uniform random numbers can have large gaps and dense clusters
- „Sample space is covered quite non-uniformly“
  - Leads to variance in Monte Carlo estimates



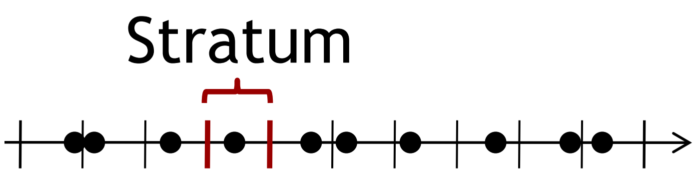
2D samples from uniform random numbers

Stratified sampling

- Intuitition: clumping of samples is bad
- Instead of canonic uniform random variables, generate variables in strata
  - One sample per stratum
  - **Everything else stays the same**



Uniform

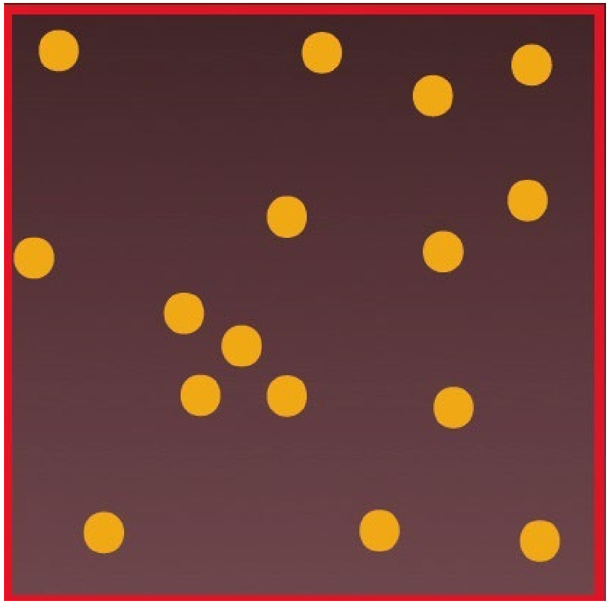


Stratified

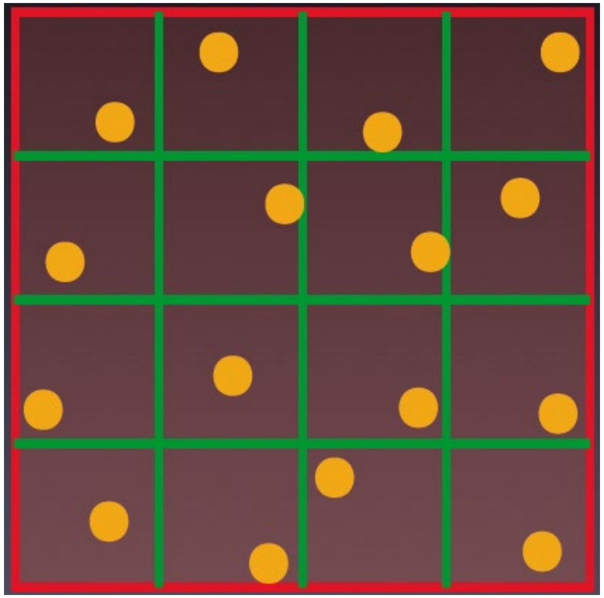
Jittered sampling

- Stratification using uniform grid
- „Jittered“ sample randomly in each cell
- Not suitable for higher dimensions (curse of dimensionality)

Uniform



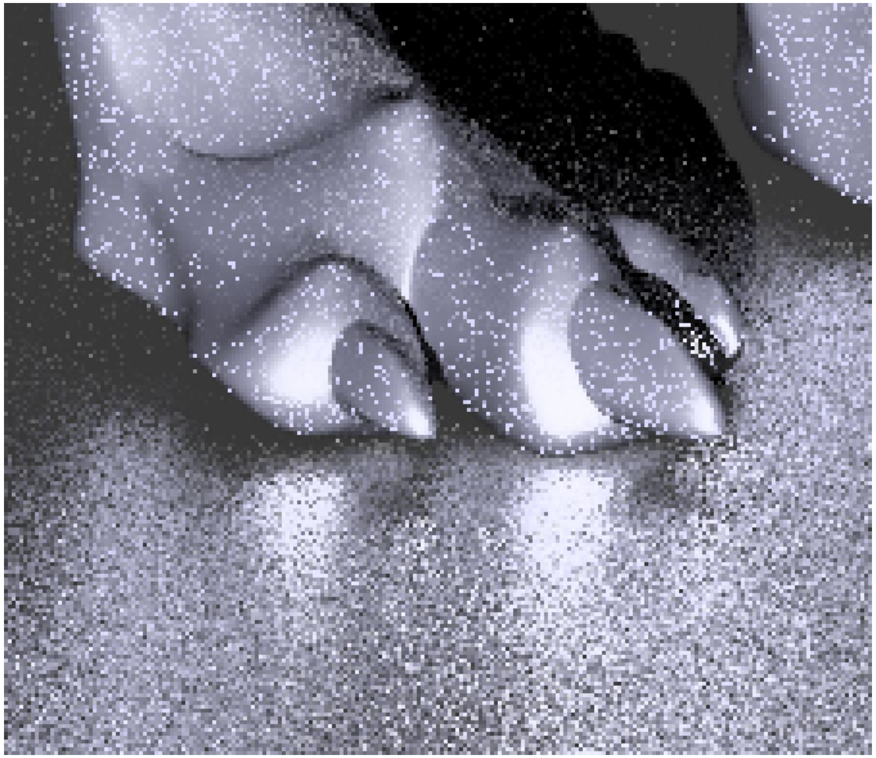
Jittered



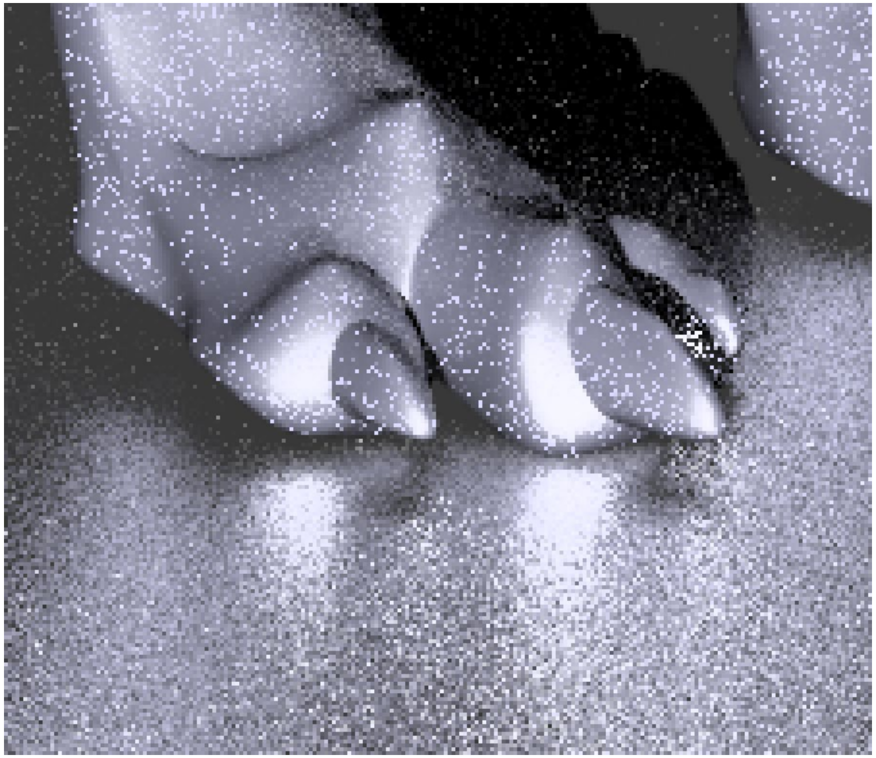
Comparison

- Sampling glossy reflection

Uniform



Stratified



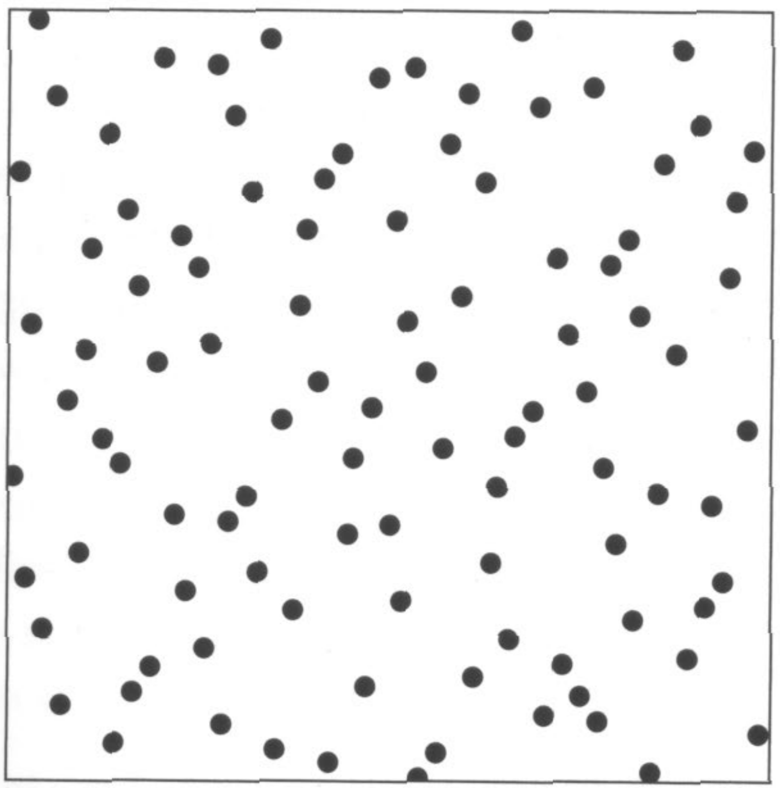
[Pharr, Humphreys]

Other stratified sampling patterns

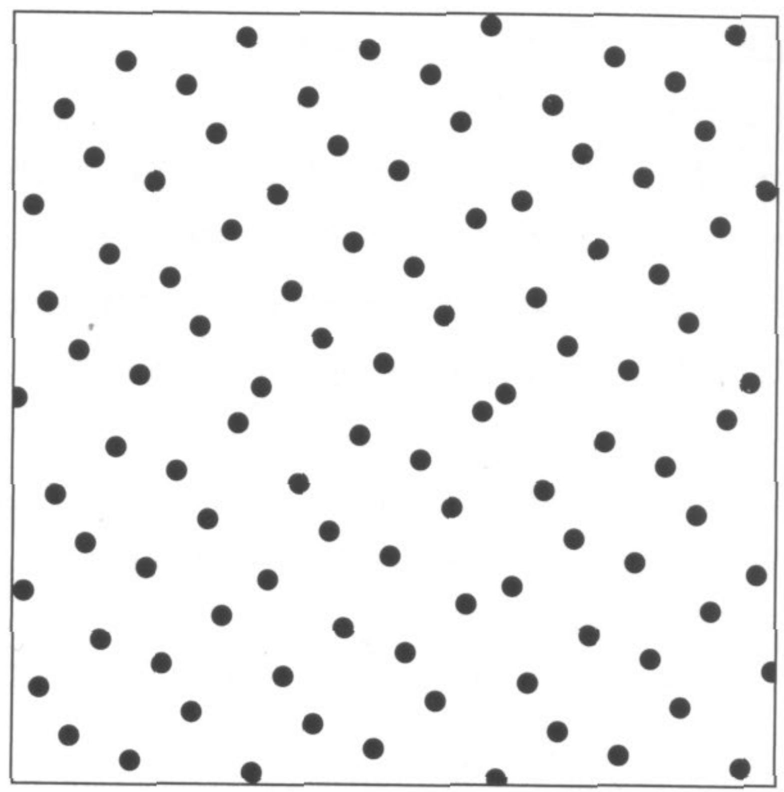
- N-rooks (Latin hypercube) sampling
  http://en.wikipedia.org/wiki/Latin_hypercube_sampling
  - Advantage over jittered grid: can generate any number n of samples, not restricted to n x m grid
- Quasi-Monte Carlo
  http://en.wikipedia.org/wiki/Quasi-Monte_Carlo_method
  - Based on low-discrepancy instead of pseudo-random samples
    http://en.wikipedia.org/wiki/Low-discrepancy_sequence
  - Low-discrepancy sequences have more uniform distribution of points, less clumping, also in higher dimensions
  - Can show theoretically that convergence improves

Low-discrepancy sequences

- First 100 points of 2D sequences
  http://en.wikipedia.org/wiki/Constructions_of_low-discrepancy_sequences



Halton sequence



Hammerseley sequence

Note on implementation

- Any of the improved sampling sequences (jittered, Latin hypercube, low-discrepancy) yield sequences of points $(\xi_1, \xi_2, \ldots \xi_n)$ in unit hypercube $[0,1]^n$
- Can directly substitute these for pseudo-random points $(\xi_1, \xi_2, \ldots \xi_n)$
  - No need to change rest of implementation

Next time

- Generalized path sampling (three-point form or rendering equation, BDP)