CMSC740
Advanced Computer Graphics

Fall 2025

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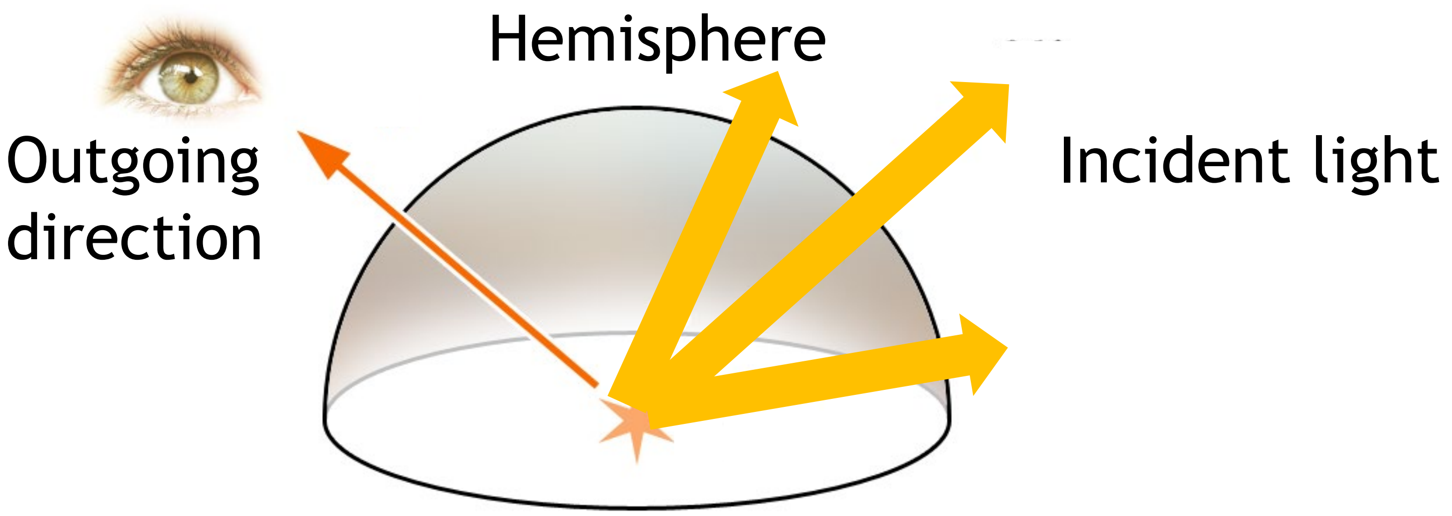
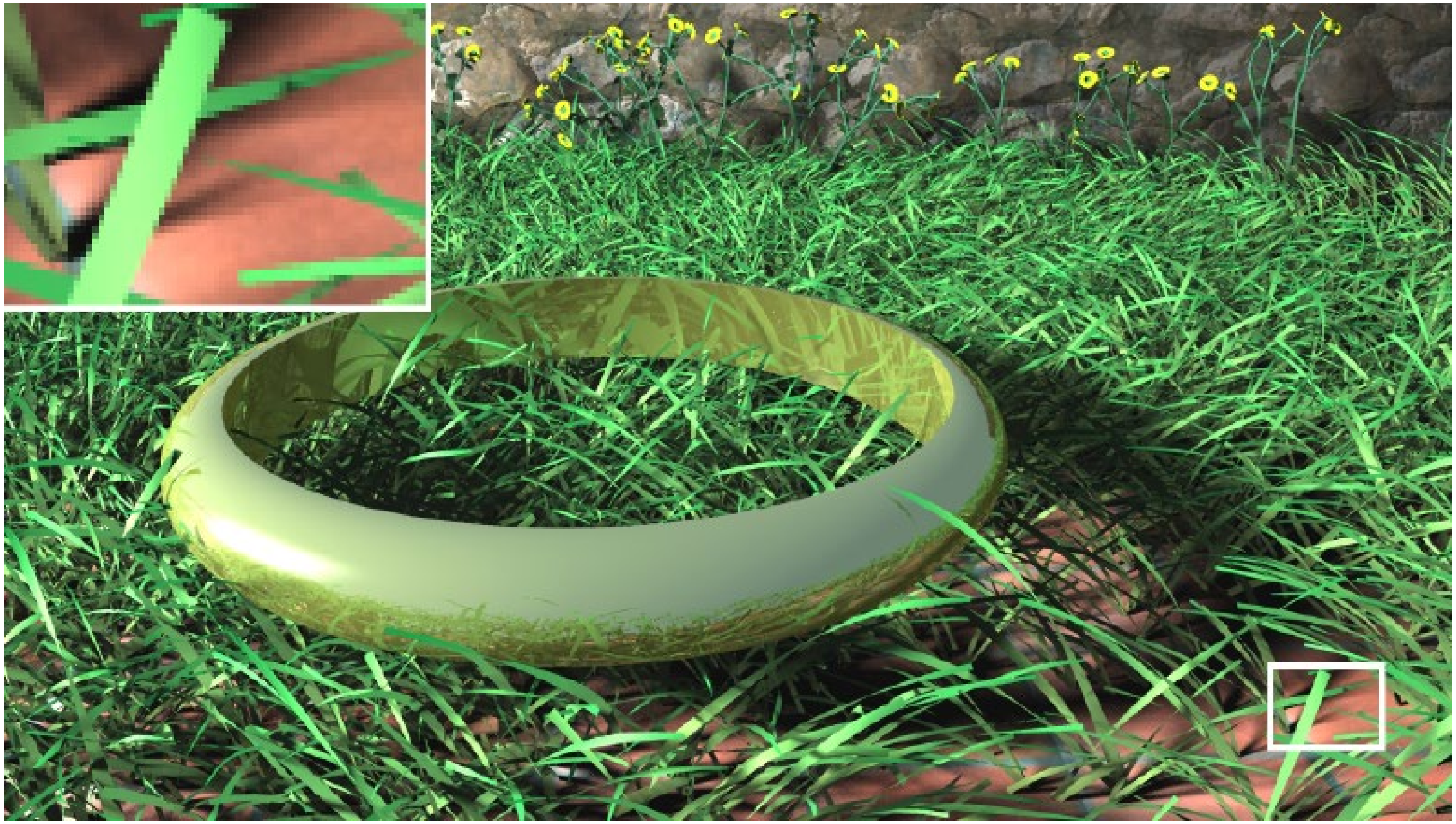
Math challenge

- Integrate functions over the sphere (or hemisphere)
- Application: integrate ("sum up") incident light over the hemisphere at a point on surface to calculate reflected light in some outgoing direction



Challenge

- Complicated integral, terms not known analytically: incident light depends on complex scene geometry



[Laine, Aila, Assarsson, Lehtinen, Akenine-Moeller]

Monte Carlo integration

http://en.wikipedia.org/wiki/Monte_Carlo_integration

- General technique for numerical integration
- Advantages
  - Easy to implement
  - Robust, can integrate almost any function
  - Efficient for high dimensional integrals
- Disadvantages
  - Variance (noise)
  - Slow convergence

Relevant chapters in PBRT book

- See http://www.pbr-book.org/3ed-2018/Monte_Carlo_Integration.html
- Section 13.1, probability background
- Section 13.2, Monte Carlo integration
- Section 13.3, sampling random variables (1D inversion sampling)
- Section 13.5, transforming between distributions (1D)
- Section 13.6, 2D sampling with multidimensional transformations

Probability concepts

- Random variable X
  - Every time you sample ("query") a random variable, it returns a random number
- Cumulative distribution function (CDF) P
  - Fraction of times the value of a sample of the random variable X is below a value x
  - For all CDFs: $0 <= P(x) <= 1$

$$P(x) = Pr\{X \leq x\} \sim \frac{\text{Number of times sample value } X{<=}x}{\text{Number of samples that were taken}}$$

Probability recap

- Probability density function (PDF) p is derivative of cumulative distribution (CDF) P

$$p(x) = \frac{dP}{dx}(x) \quad \int_{-\infty}^{\infty} p(x) = 1$$

- "Given many samples of random variable X, the PDF $p(x)$ is the fraction of these samples that lie in a small area around location x, divided by the size of the area (in the limit)"
- Therefore

$$Pr\{\alpha \leq X \leq \beta\} = \int_{\alpha}^{\beta} p(x)dx = P(\beta) - P(\alpha)$$

Probability recap

- Uniform random variables

Why not 1?

$$p(x) = \frac{dP}{dx}(x) = \boxed{\text{const.}}$$

- Canonical uniform random variable ξ

$$p(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Probability recap

- Random variable $Y = f(X)$, f any function
- Expected value

$$E[Y] = \int_{-\infty}^{\infty} f(x)p(x)dx$$

  - Linearity: A, B random variables, α scalar

$$E[\alpha A + B] = \alpha E[A] + E[B]$$

- Variance

$$V[Y] = E\left[(Y - E[Y])^2\right]$$

$$V[Y] = E[Y^2] - E[Y]^2$$

Monte Carlo integration

http://en.wikipedia.org/wiki/Monte_Carlo_integration

- Want to compute numerically

$$\int_a^b f(x)dx$$

- Given uniform random variables

$$X_i \in [a,b] \qquad p(x) = \begin{cases} 1/(b-a) & x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

- Monte Carlo estimator with N samples

$$F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i) \approx \int_a^b f(x)dx$$

Instead of evaluating the integrand at random positions, it would be better to use equally space positions

A. True

B. False

If we use a non-uniform random variable the MC estimator will not work.

A. True

B. False

Monte Carlo integration

http://en.wikipedia.org/wiki/Monte_Carlo_integration

$$
\begin{aligned}
E[F_N] &= E\left[\frac{b-a}{N}\sum_{i=1}^{N} f(X_i)\right] \\
&= \frac{b-a}{N}\sum_{i=1}^{N} E\left[f(X_i)\right] \\
&= \frac{b-a}{N}\sum_{i=1}^{N} \int_a^b f(x)p(x)dx \\
&= \frac{1}{N}\sum_{i=1}^{N} \int_a^b f(x)dx \\
&= \int_a^b f(x)dx
\end{aligned}
$$

Monte Carlo integration

http://en.wikipedia.org/wiki/Monte_Carlo_integration

Variance of the estimator

$$
\begin{aligned}
\boxed{V[F_N]} &= V\left[\frac{b-a}{N}\sum_{i=1}^{N} f(X_i)\right] \\
&= \frac{(b-a)^2}{N^2} V\left[\sum_{i=1}^{N} f(X_i)\right] \\
&= \frac{(b-a)^2}{N^2} \sum_{i=1}^{N} V[f(X_i)] \\
&= \boxed{\frac{(b-a)^2}{N} V[f(X)]}
\end{aligned}
$$

Monte Carlo integration

http://en.wikipedia.org/wiki/Monte_Carlo_integration

Standard deviation (expected error) σ of the estimator

$$\sigma[F_N] = \sqrt{V[F_N]} = \frac{b-a}{\sqrt{N}}\sigma[f(X)]$$

- The expected error converges with $O(N^{-1/2})$
- To get half the error, we need four times the number of samples
- Slow convergence …

Importance sampling

- Samples with general probability density p

$$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)} \approx \int_a^b f(x)dx$$

  known as "importance sampling"
- Show $E[F_N] = \int_a^b f(x)dx$ as before

Importance sampling

- Standard deviation σ, variance V

$$\sigma[F_N] = \sqrt{V[F_N]} = \frac{1}{\sqrt{N}}\sigma\left[\frac{f(X)}{p(X)}\right]$$

- Convergence still $O(N^{-1/2})$
- Observe: σ proportional to $f(X)/p(X)$
- Idea: choose f and p to be as similar as possible to reduce variance!

Importance sampling

- Imagine we had PDF p exactly proportional to f

$$p(x) \propto f(x), \text{ or } p(x) = cf(x)$$

- Normalization (unit integral) of PDF forces

$$c = 1/\int f(x)dx$$

- Now

$$\frac{f(X)}{p(X)} = \frac{1}{c} = \int f(x)dx$$

- No variance, we could exactly determine the integral with a single sample!
- Unfortunately, we used the unknown integral to compute c…

Practical issues

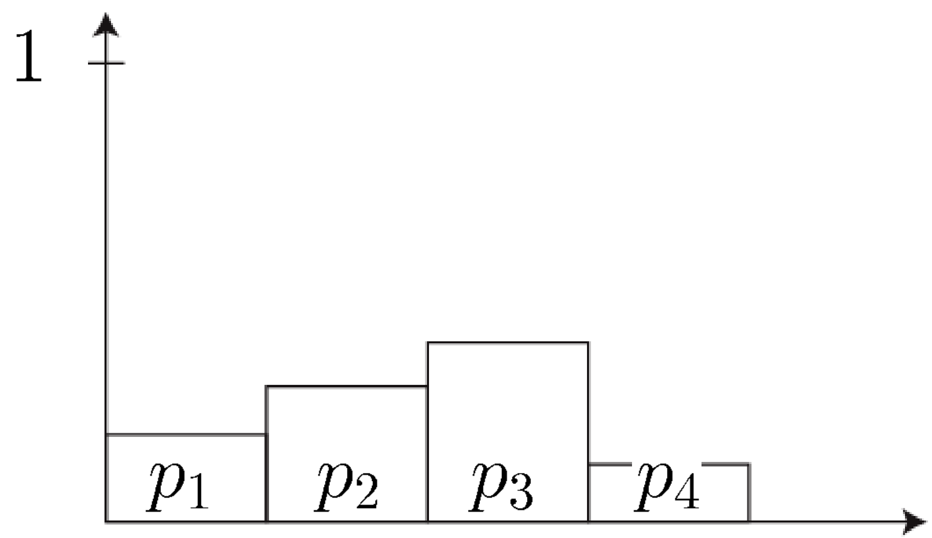
- Conceptual issue: how to choose probability density $p(x)$ (or $p(\omega)$)?
  - Goal: get best result with as few samples as possible, p as similar as possible to integrand
  - More on this later
- Technical problem: how to generate samples X_i given PDF $p(x)$
  - "Sampling the PDF"
  - Today

Importance sampling in 1D

- Problem statement: draw random samples, such that their density matches a given distribution (PDF)
- “Draw samples”,”sampling” the PDF: generate samples with appropriate probability densities
- Challenge: pseudo random number generators only produce uniform density
- First: 1D case

Sampling arbitrary 1D PDFs

- Discrete example: sample 4 events with given probabilities $p_1 \; p_2 \; p_3 \; p_4$
- “Generate events with appropriate probabilities“

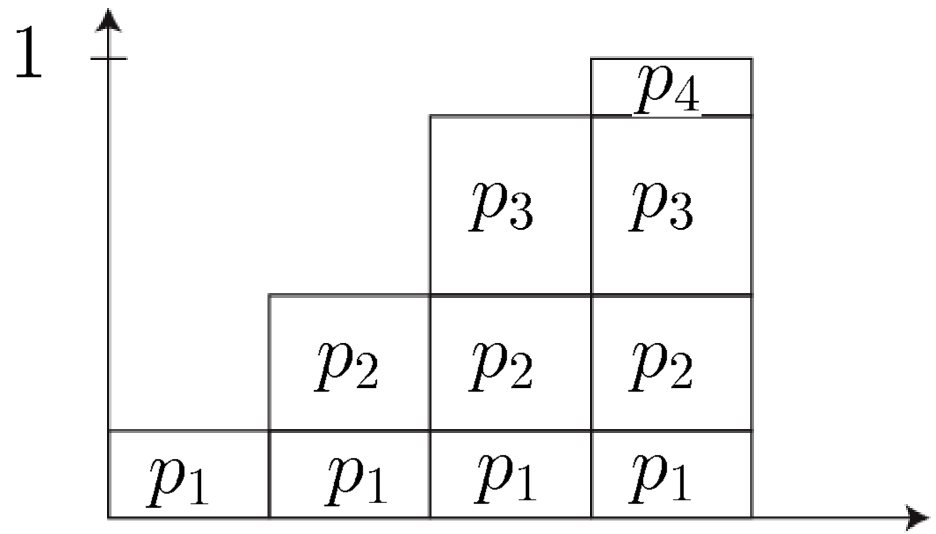


Discrete PDF

The inversion method

http://en.wikipedia.org/wiki/Inverse_transform_sampling

1. Compute the discrete CDF (cumulative density function)

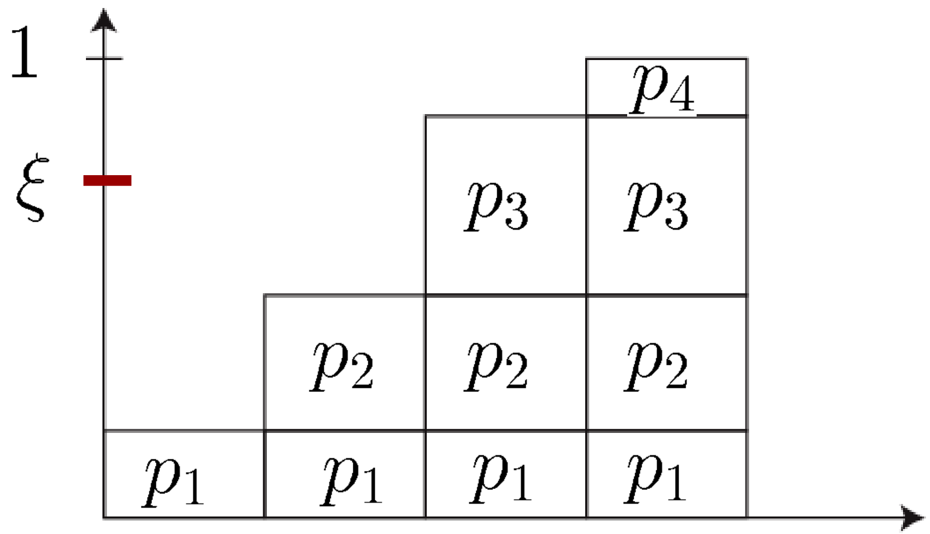


Discrete CDF

The inversion method

http://en.wikipedia.org/wiki/Inverse_transform_sampling

1. Compute the discrete CDF (cumulative density function)
2. Sample canonic random variable ξ

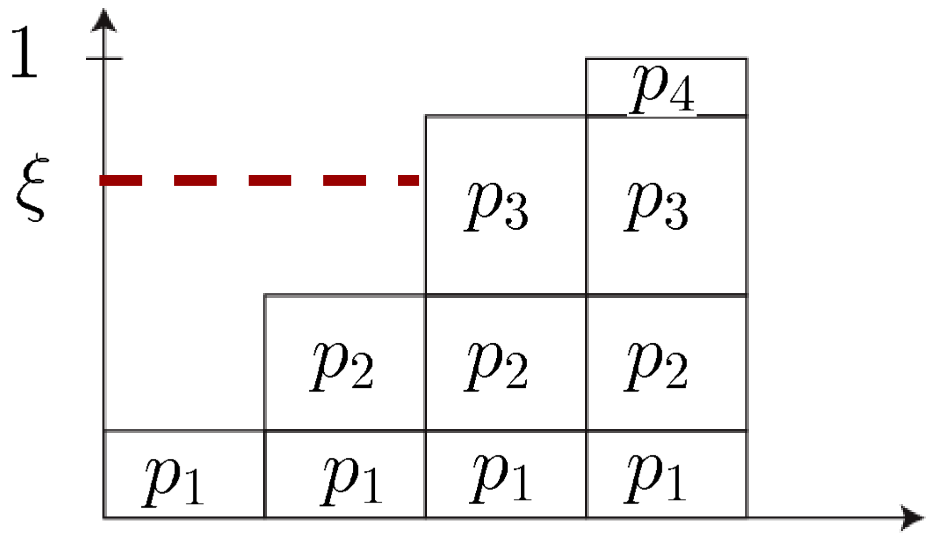


Discrete CDF

The inversion method

http://en.wikipedia.org/wiki/Inverse_transform_sampling

1. Compute the discrete CDF (cumulative density function)
2. Sample canonic random variable ξ
3. Generate event intersected by ξ



Discrete CDF

Continuous case: inverse transform sampling

http://en.wikipedia.org/wiki/Inverse_transform_sampling

Goal: sample from arbitrary PDF $p(x)$

1. Compute the CDF $P(x) = \int_0^x p(x')dx'$
2. Compute its inverse $P^{-1}(x)$
3. Obtain a uniformly distributed random number ξ
4. Compute sample $X = P^{-1}(\xi)$

Sampling 2D PDFs (sample warping)

Goal

- Generate samples on a desired 2D domain (unit circle, triangle, hemisphere, sphere, etc.) with a desired density (uniform, desired nonuniform density)
- Approach
  1. Start with samples from canonic random variables (pseudo-random number generator)
  2. Warp samples to desired domain, density

Sampling PDFs (sample warping)

Two examples

- Sampling directions with cosine distribution on the hemisphere
- (Sampling the specular highlight with a Blinn microfacet distribution)

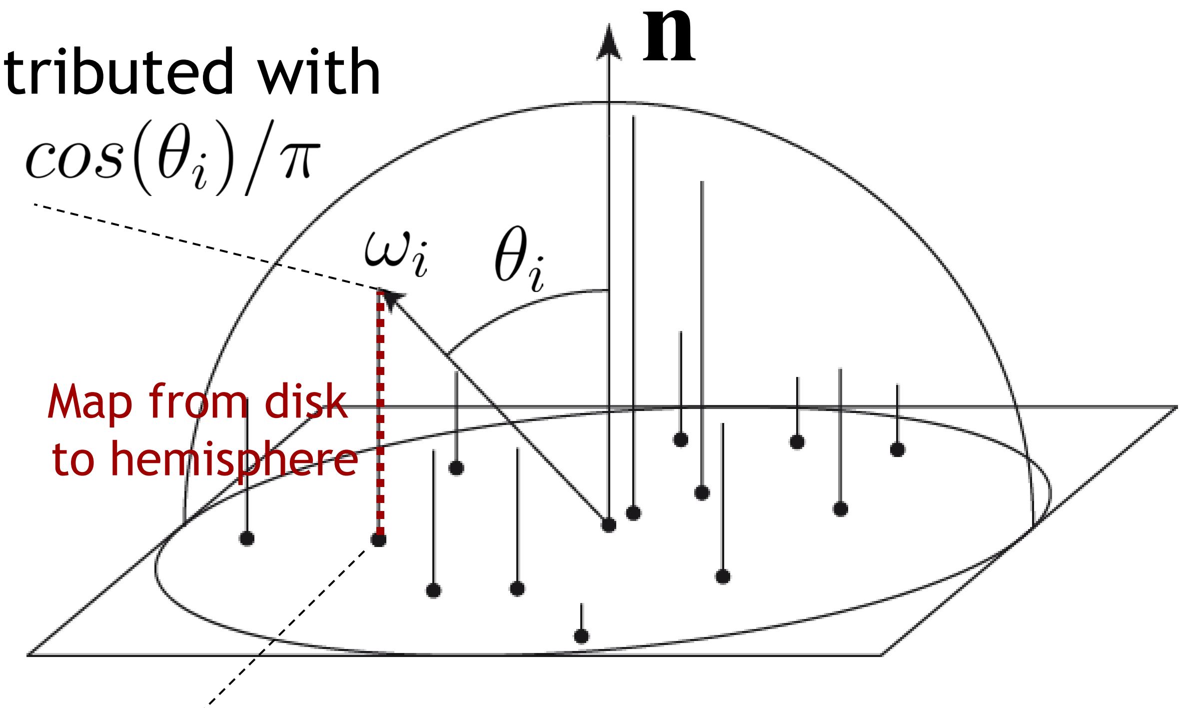
Sampling directions with cosine distribution

- Will be used to construct random rays for ray tracing
- "Sampling the random variable" means obtaining a random ray direction ω, with a desired density p
- Here: desired density $p(\omega) = \cos(\theta)/\pi$, where θ is angle between ω and normal $\mathbf{n}$
- Ray direction = point on unit (hemi-)sphere

Sampling directions with cosine distribution

- Trick: uniform samples on disk can be mapped to directions with $p(\omega_i) = \cos(\theta_i)/\pi$

Direction distributed with
$p(\omega_i) = cos(\theta_i)/\pi$



Uniform samples (indexed with i) on disk

Uniformly sampling a disk

- Uniform probability density on a unit disk

$$p(x,y) = \begin{cases} 1/\pi & x^2+y^2<1 \\ 0 & \text{otherwise.} \end{cases}$$

- Goal: draw samples X_i, Y_i inside the disk that are distributed with desired pdf p

$$(X_i, Y_i) \sim p(x,y)$$

Uniformly sampling a disk

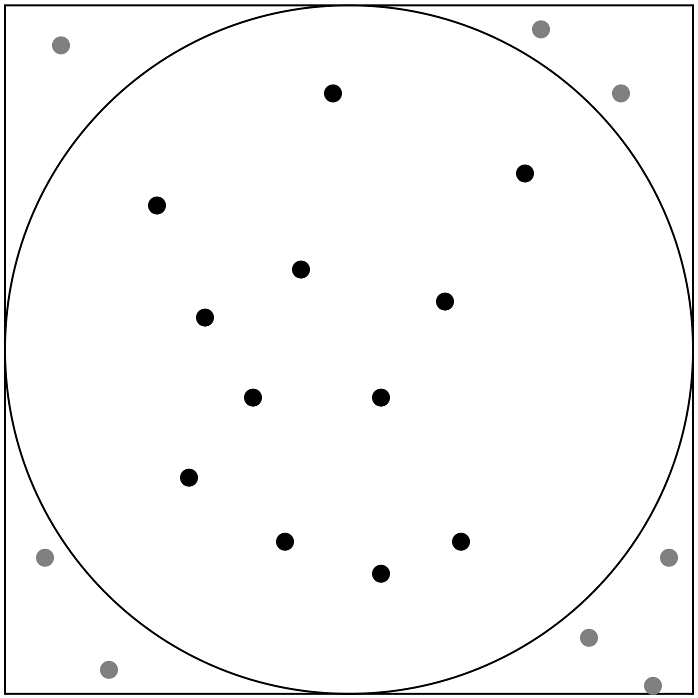
- Problem: pseudo random number generators allow us only to draw samples ξ_i from the canonical distribution

$$\xi_i \sim p(\xi) = \begin{cases} 1 & \xi \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Rejection sampling

http://en.wikipedia.org/wiki/Rejection_sampling

- Simple solution for our problem
- Draw samples ξ_1, ξ_2
- Reject if $\xi_1^2 + \xi_2^2 > 1$
- Drawback: inefficient



- rejected
- accepted

Sample warping

- Idea: transform (warp) samples to polar coordinates (radius r, angle θ)

$$r = \xi_1 \quad \theta = 2\pi\xi_2$$

  where ε_1, ε_2 canonical uniform random variables
- Advantage: sample is always in unit circle
- Problem?

Theory: sampling 2D PDFs

- Need to solve two problems
  1. Sampling arbitrary, non-uniform PDFs
  2. Transforming PDFs between 2D (or higher dimensional) coordinate systems
- Combine both steps for efficient sampling of PDFs
  - Will show example of uniformly sampling a disc

Sampling 2D PDFs: special case

- Draw samples (X, Y) from 2D PDF $p(x, y)$

- Special case
  If $p(x, y)$ is separable, i.e., $p(x, y) = p_x(x)p_y(y)$
  we can independently sample $p_x(x)$, $p_y(y)$
  using inversion method, as described
  earlier

Sampling 2D PDFs: general

- Can express any 2D PDF as product of marginal and conditional densities
  http://en.wikipedia.org/wiki/Conditional_probability_distribution
  http://en.wikipedia.org/wiki/Marginal_density

$$p(x, y) = p(y|x)p(x)$$

- Marginal density

$$p(x) = \int_{-\infty}^{\infty} p(x, y)dy$$

- Conditional density

$$p(y|x) = \frac{p(x, y)}{p(x)}$$

- Procedure: first sample $X_i \sim p(x)$, then $Y_i \sim p(y|x)$ using inversion method

Recap

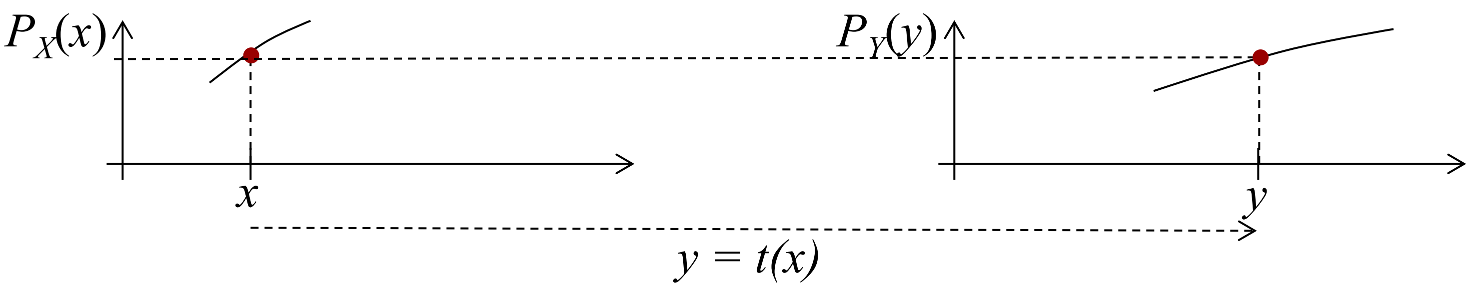
- So far, solved Problem 1: how to sample arbitrary, non-uniform 2D PDFs via marginal, conditional 1D PDFs; inversion sampling
- Problem 2 remains open: want to generate samples in convenient coordinates systems
  - Our example: density given in Euclidean coordinates, but need to sample in polar coordinates to place samples in desired domain (unit disc)
- Need to transform (warp) random variables between coordinate systems
  - „Express same random variable in different coordinate systems“
  - Transformation between coordinate systems represented by a 1-to-1 mapping
- But: when random variable is warped, how are densities before/after warping related?

Transforming 1D random variables

- Given $X \sim p_x(x)$
- PDF of warped random variable $Y = t(X)$?
  - Transformation $y = t(x)$ must be one-to-one
- This implies

$$Pr\{X \leq x\} = Pr\{Y \leq y\}\} = Pr\{Y \leq t(x)\}$$

$$P_X(x) = P_Y(y) = P_Y(t(x))$$



But what is probability density of $t(X)$?

Transforming 1D random variables

- Differentiating $P_Y(y) = P_Y(t(x)) = P_X(x)$ w.r.t x using the chain rule

$$p_Y(y)\frac{dy}{dx} = p_Y(t(x))\frac{dt(x)}{dx} = p_X(x)$$

- Therefore, PDF of Y related to PDF of X as

$$p_Y(y) = \left(\frac{dt(x)}{dx}\right)^{-1} p_X(x)$$

Transforming n-D random variables

- With n-dimensional random variable X
- Bijective mapping $Y = T(X)$ from $\mathbf{R}^n$ to $\mathbf{R}^n$
- Densities

$$p_Y(y) = p_Y(T(x)) = \frac{1}{|J_T(x)|} p_X(x)$$

where $|J_T| = |det(J_T)|$

"How much area is stretched/squeezed by mapping $T(X)$"

- The Jacobi matrix is

https://en.wikipedia.org/wiki/Jacobian_matrix_and_determinant

$$J_T = \begin{bmatrix} \partial T_1/\partial x_1 & \cdots & \partial T_1/\partial x_n \\ \vdots & & \vdots \\ \partial T_n/\partial x_1 & \cdots & \partial T_n/\partial x_n \end{bmatrix}$$

Transforming random variables

- Same mechanism as for change of variables in multi-dimensional integration
  http://en.wikipedia.org/wiki/Integration_by_substitution#Substitution_for_multiple_variables
  http://en.wikipedia.org/wiki/Integration_by_substitution#Application_in_probability

Recipe to sample PDFs

1. Express the desired PDF in a convenient coordinate system (e.g. Euclidean)
2. Transform the PDF to different coordinate system suitable for sampling (depending on domain, e.g., unit disc)
   - Find transformation T
   - Compute determinant of Jacobian T
3. Sample (multi-dimensional) PDF
   - Compute marginal and conditional 1D PDFs
   - Sample 1D PDFs using the inversion method

Uniformly sampling a disk

1. Desired distribution in Euclidean coords.

$$p(x,y) = \begin{cases} 1/\pi & x^2+y^2<1 \\ 0 & \text{otherwise.} \end{cases}$$

2. Transform PDF to polar coordinates r,θ suitable for sampling
   - Transformation $(x,y)=T(r,\theta)$ is
     $x = r\cos\theta, y = r\sin\theta$
   - Jacobi matrix

$$J_T = \begin{bmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Uniformly sampling a disk

- Determinant is $r(\cos^2\theta + \sin^2\theta) = r$
- Density in polar coordinates

$$p(x,y) = p(r,\theta)/r$$

$$p(r,\theta) = rp(x,y) = r/\pi$$

3. Sample 2D PDF $p(r,\theta)$
   - Marginal $p(r) = \int_0^{2\pi} p(r,\theta)d\theta = 2r$
   - Conditional $p(\theta|r) = \frac{p(r,\theta)}{p(r)} = \frac{1}{2\pi}$

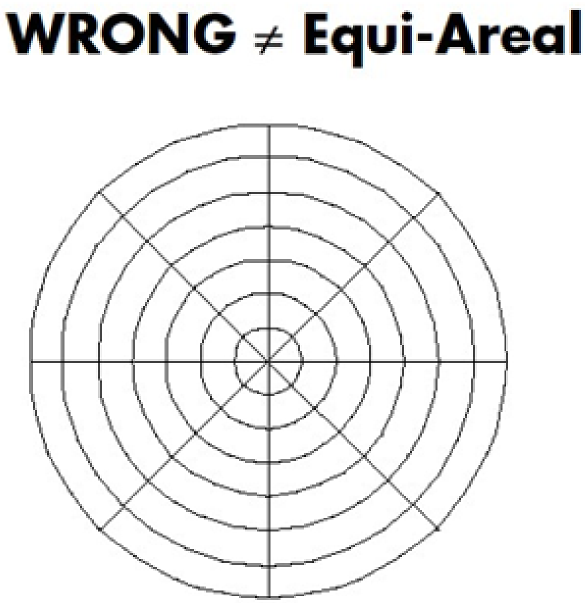
Uniformly sampling a disk

- Sample from $p(r)$ and $p(\theta|r)$ using inversion method
  - From before: $p(r) = 2r, p(\theta|r) = 1/(2\pi)$
  - Get CDFs $P(r)$, $P(\theta)$ by integrating PDFs and invert

$$\xi_1 = P(r) = r^2 \Rightarrow \boxed{r = P^{-1}(\xi_1) = \sqrt{\xi_1}}$$

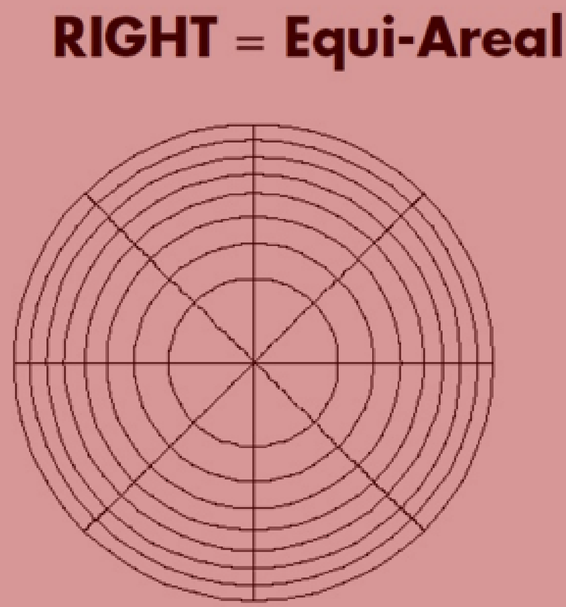
$$\xi_2 = P(\theta) = \theta/(2\pi) \Rightarrow \boxed{\theta = P^{-1}(\xi_2) = 2\pi\xi_2}$$

Uniformly sampling a disk



Polar coordinates

$$r = \xi_1$$
$$\theta = 2\pi\xi_2$$



Polar coords.

$$r = \sqrt{\xi_1}$$
$$\theta = 2\pi\xi_2$$

Euclidean coords.

$$x = \cos(2\pi\xi_2)\sqrt{\xi_1}$$
$$y = sin(2\pi\xi_2)\sqrt{\xi_1}$$

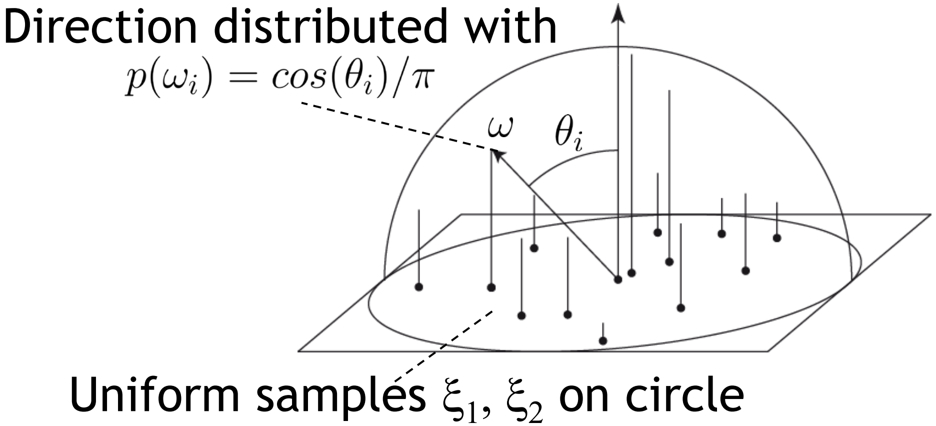
Recap: sampling directions

- Want: directions ω_i with $p(\omega_i) = \cos(\theta_i)/\pi$
- Given: uniform random variables ξ_1, ξ_2
- Get x,y,z coordinates of direction ω_i using

$$x = \cos(2\pi\xi_2)\sqrt{\xi_1}$$

$$y = sin(2\pi\xi_2)\sqrt{\xi_1}$$

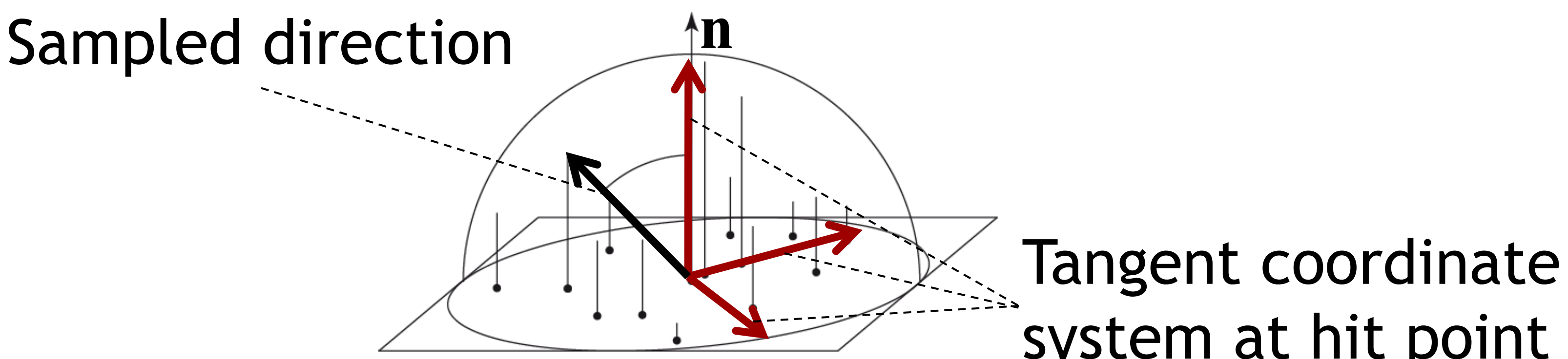
$$z = \sqrt{1 - \xi_1}$$



- Note: $p(\omega_i) = \cos(\theta_i)/\pi$
  - Factor $1/\pi$ to normalize PDF to unit integral

Implementation note

- Sampled direction is relative to local coordinate system at hit point (ray-surface intersection point)
  - Tangent frame, constructed using surface normal $\mathbf{n}$
- Transform to world coordinates for further ray tracing
  - See notes in Assignment 2



Other useful examples

Sampling distributions on

- Rectangles
- Triangles
- Spheres
- Hemispheres
- Details see for example PBRT book
  http://www.pbr-book.org/3ed-2018/Monte_Carlo_Integration/2D_Sampling_with_Multidimensional_Transformations.html

Sampling the Blinn distribution

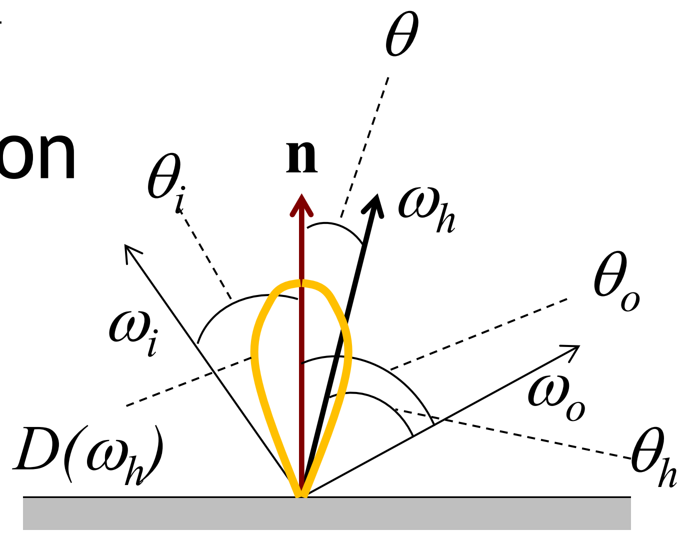
- Recall Torrance-Sparrow BRDF

$$f(\omega_o, \omega_i) = \frac{D(\omega_h)G(\omega_o,\omega_i)F_r(\omega_o)}{4\cos\theta_o\cos\theta_i}$$

- Blinn Microfacet distribution

$$\boxed{D(\omega_h)} = \frac{e+2}{2\pi}(\omega_h \cdot \mathbf{n})^e$$

  - Shininess coefficient e



- Microfacet distribution D determines BRDF, try to distribute samples proportionally to D
- Details see also http://www.pbr-book.org/3ed-2018/Light_Transport_I_Surface_Reflection/Sampling_Reflection_Functions.html#MicrofacetBxDFs

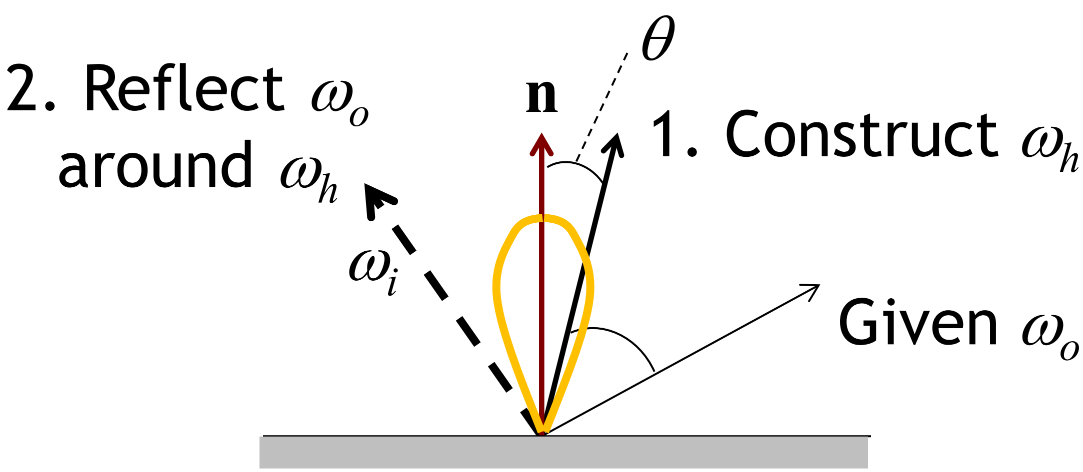
Sampling the Blinn distribution

1. Sample half-angle direction ω_h according to

$$p(\omega_h) = \frac{e+1}{2\pi}(\omega_h \cdot \mathbf{n})^e$$

   - Slightly different normalization than $D(\omega_h)$, since $D(\omega_h)$ is normalized w.r.t. projected solid angle, not solid angle

2. Use ω_h to find ω_i by reflecting ω_o around ω_h



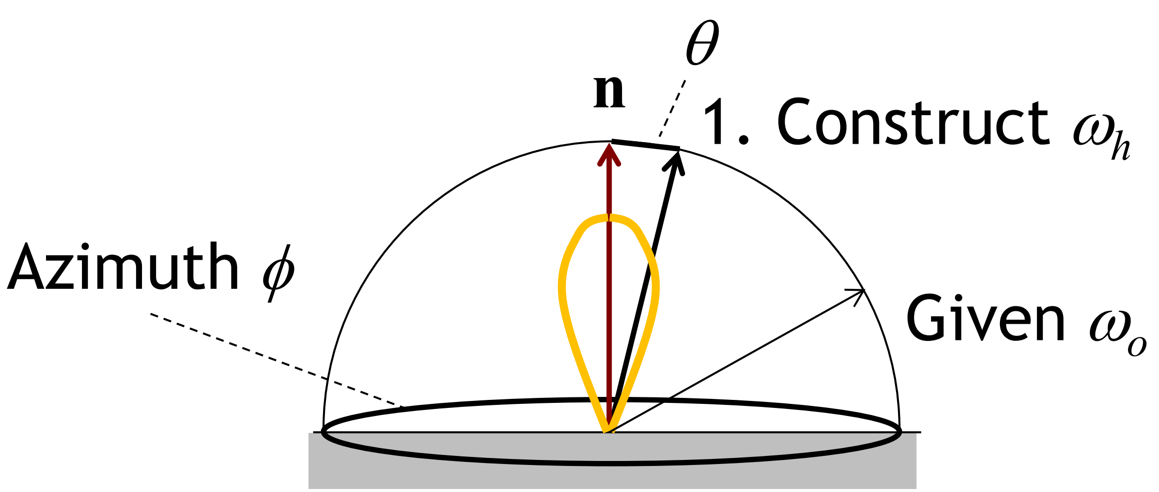
Sampling the Blinn distribution

- Sample ω_h in spherical coordinates θ, ϕ centered around normal
- Change of coordinates to spherical coordinates

$$p(\omega_h) = \frac{e+1}{2\pi}(\omega_h \cdot \mathbf{n})^e$$

$$p(\theta, \phi) = \frac{e+1}{2\pi}(\cos\theta)^e \sin\theta$$

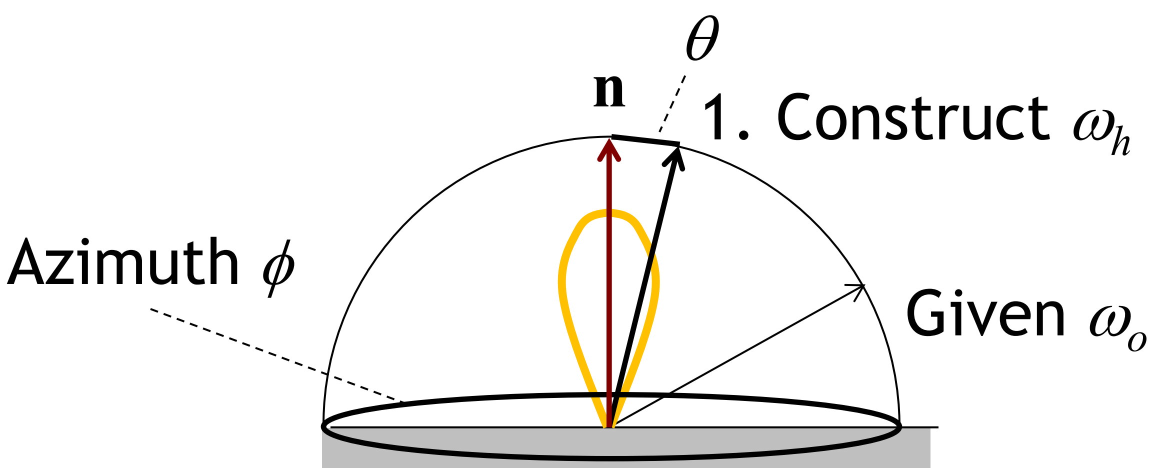
$$p(\omega_h) = \frac{p(\theta,\phi)}{\sin\theta}$$



Sampling the Blinn distribution

- Given canonical random variables ξ_1, ξ_2
- Sample $\phi = 2\pi\,\xi_2$, and $\cos\theta = \xi_1^{1/(e+1)}$
  - Derivation see handwritten notes, PBRT book
- PDF of sampled direction

$$p(\omega_h) = \frac{e+1}{2\pi}(\cos\theta)^e$$



http://www.pbr-book.org/3ed-2018/Light_Transport_I_Surface_Reflection/Sampling_Reflection_Functions.html#MicrofacetBxDFs

Derivation: Sampling the Blinn PDF

Desired density:

$$p(\omega_h) = \frac{e+1}{2\pi}(\omega_h \cdot n)^e$$

Change of coordinates to spherical angles:

$$p(\theta,\phi) = \frac{e+1}{2\pi} \cos\theta^e \sin\theta$$

$$p(\omega_h) = \frac{p(\theta,\phi)}{\sin\theta}$$

Marginal & conditional distributions:

$$\underline{p(\theta)} = \int_0^{2\pi} \frac{e+1}{2\pi} \cos\theta^e \sin\theta \, d\phi$$

$$= \cancel{2\pi} \cdot \frac{e+1}{\cancel{2\pi}} \underline{\cos\theta^e \sin\theta}$$

marginal density $p(\theta)$

$$\underline{p(\phi|\theta)} = \frac{p(\theta,\phi)}{p(\theta)} = \underline{\frac{1}{2\pi}}$$

conditional density $p(\phi|\theta)$

Inversion sampling of θ

$$P(\theta) = \int_0^\theta p(\theta')\,d\theta' = e+1\int_0^\theta \cos(\theta)^e \sin\theta \, d\theta$$

Substitution: $u = \cos\theta$, $\frac{du}{d\theta} = -\sin(\theta)$

$d\theta = -du/\sin(\theta)$

$$P(\cos\theta) = P(u) = e+1\int_0^u u'^e \, du' = u^{e+1} = \cos\theta^{e+1}$$

Inversion method, given uniform random variable ε_1

$$\varepsilon_1 = P(\cos\theta) = \cos\theta^{e+1} \Rightarrow \underline{\cos\theta} = P^{-1}(\varepsilon_1) = \underline{\sqrt[e+1]{\varepsilon_1}}$$

Inversion method for ϕ with random var. ε_2

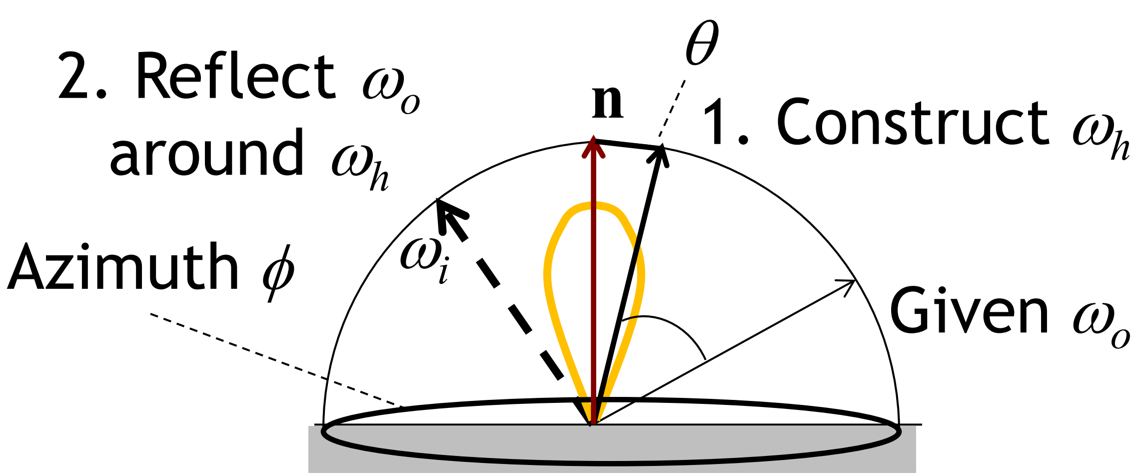
$$p(\phi|\theta) = \frac{1}{2\pi} \Rightarrow P(\phi) = \phi/2\pi = \varepsilon_2$$

$$\Rightarrow \underline{\phi} = P^{-1}(\varepsilon_2) = \underline{\varepsilon_2 \cdot 2\pi}$$

Sampling the Blinn distribution

- Note: in the end, we sample direction ω_i, and we need $p(\omega_i)$, not $p(\omega_h)$
- Can show $p(\omega_i) = p(\omega_h)$ / $(4\ \omega_o$ ¢$\omega_h)$
  - See PBRT book, http://www.pbr-book.org/3ed-2018/Light_Transport_I_Surface_Reflection/Sampling_Reflection_Functions.html#MicrofacetBxDFs
- Hence, PDF of sampled direction ω_i

$$p(\omega_i) = \frac{1}{4\omega_o \cdot \omega_h} \frac{e+1}{2\pi} (\cos\theta)^e$$



Finally

- Use sampled directions and their PDF in Monte Carlo estimate

$$L_o(\mathbf{x}, \omega_o) \approx \frac{1}{N} \sum_j \frac{f(\mathbf{x}, \omega_o, \omega_{i,j}) L_i(\mathbf{x}, \omega_{i,j}) \cos\theta_{i,j}}{p(\omega_{i,j})}$$

Implementation notes

- To convert sampled spherical angles θ, ϕ to direction ω_h, need local tangent coordinate system
- If sampled ω_i direction is below horizon (angle with normal larger than 90 degrees), return 0 for BRDF value