CMSC740
Advanced Computer Graphics

Fall 2025

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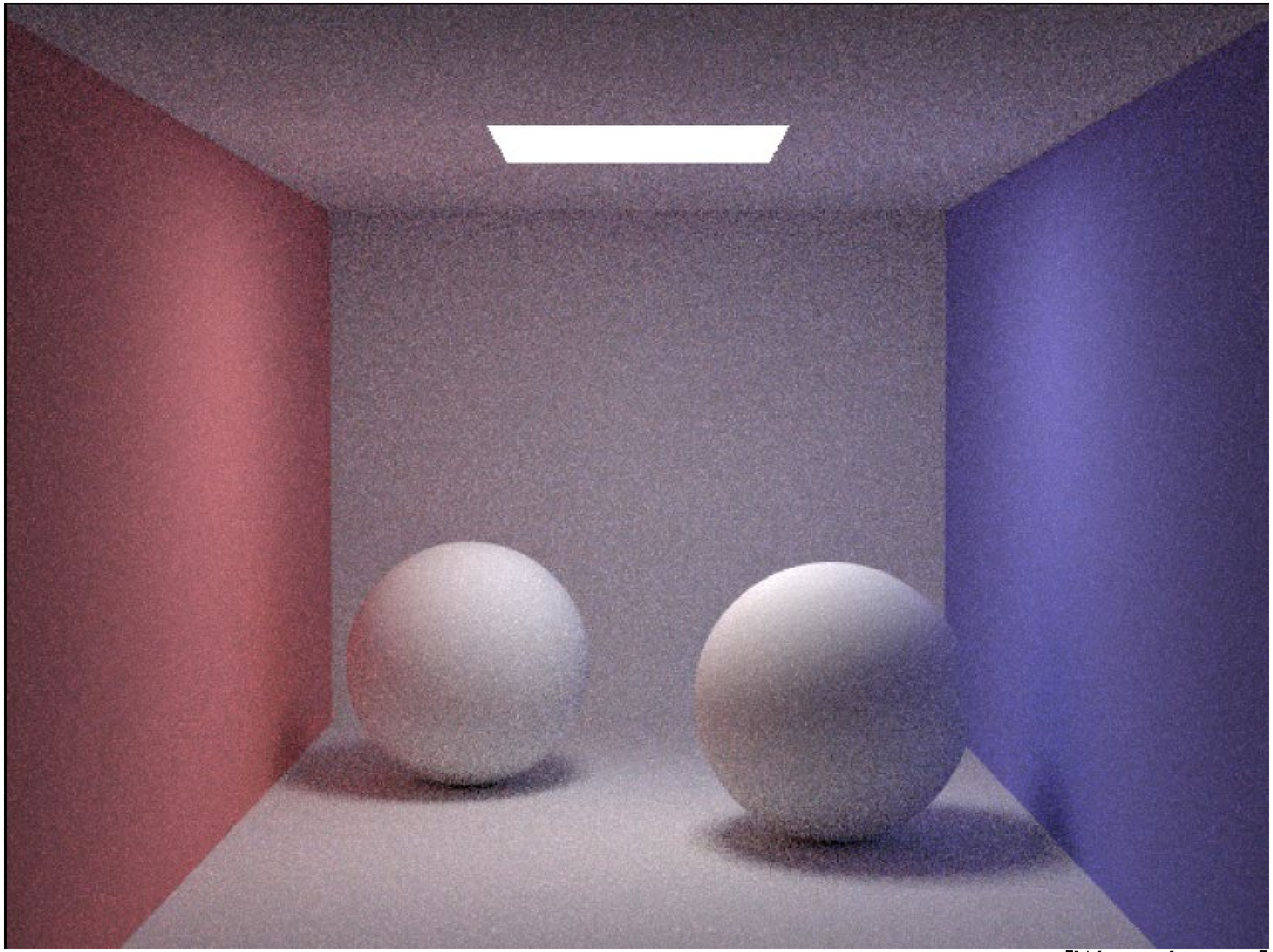
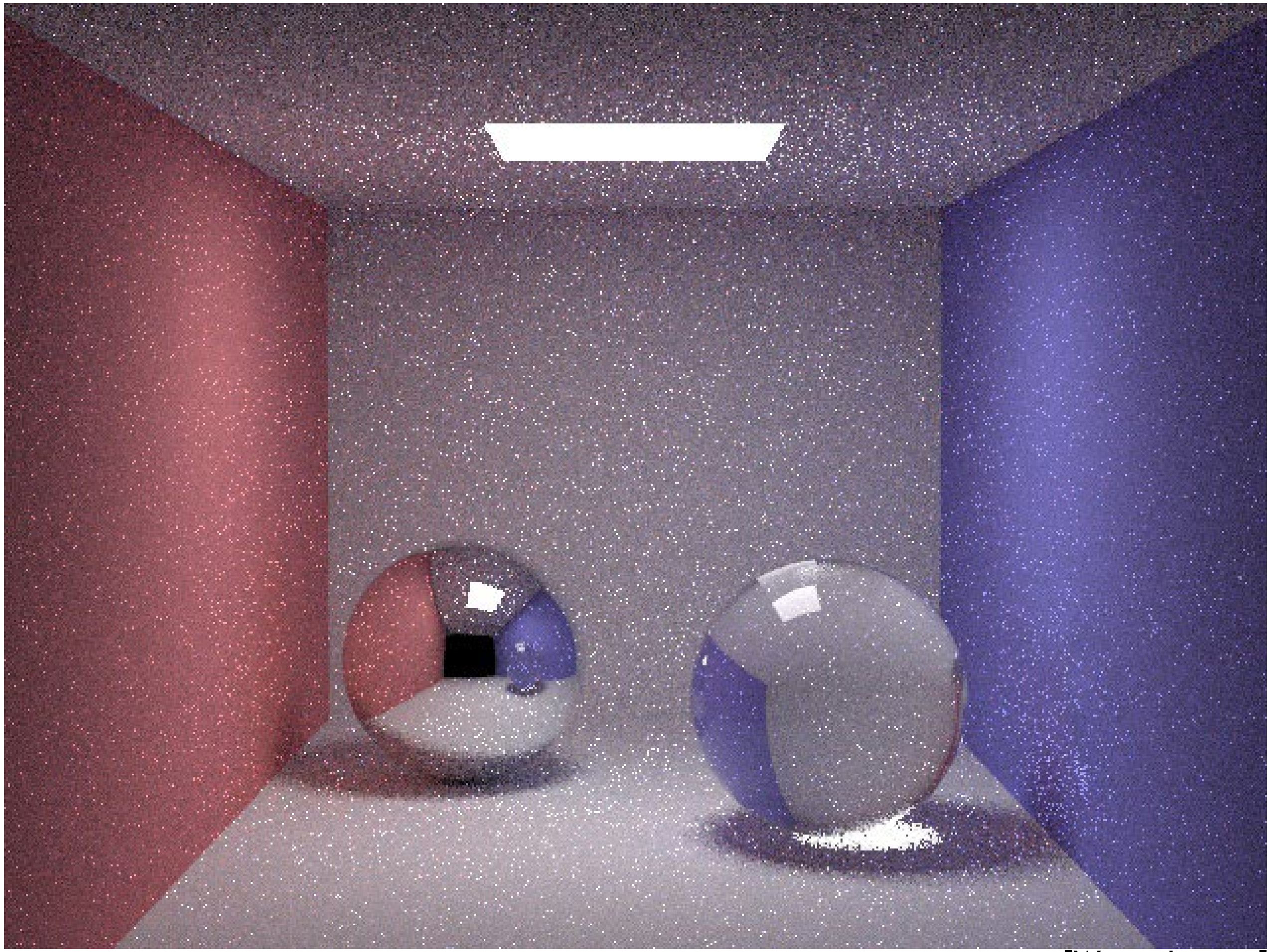
Path tracing



[Wann Jensen]

10 paths/pixel

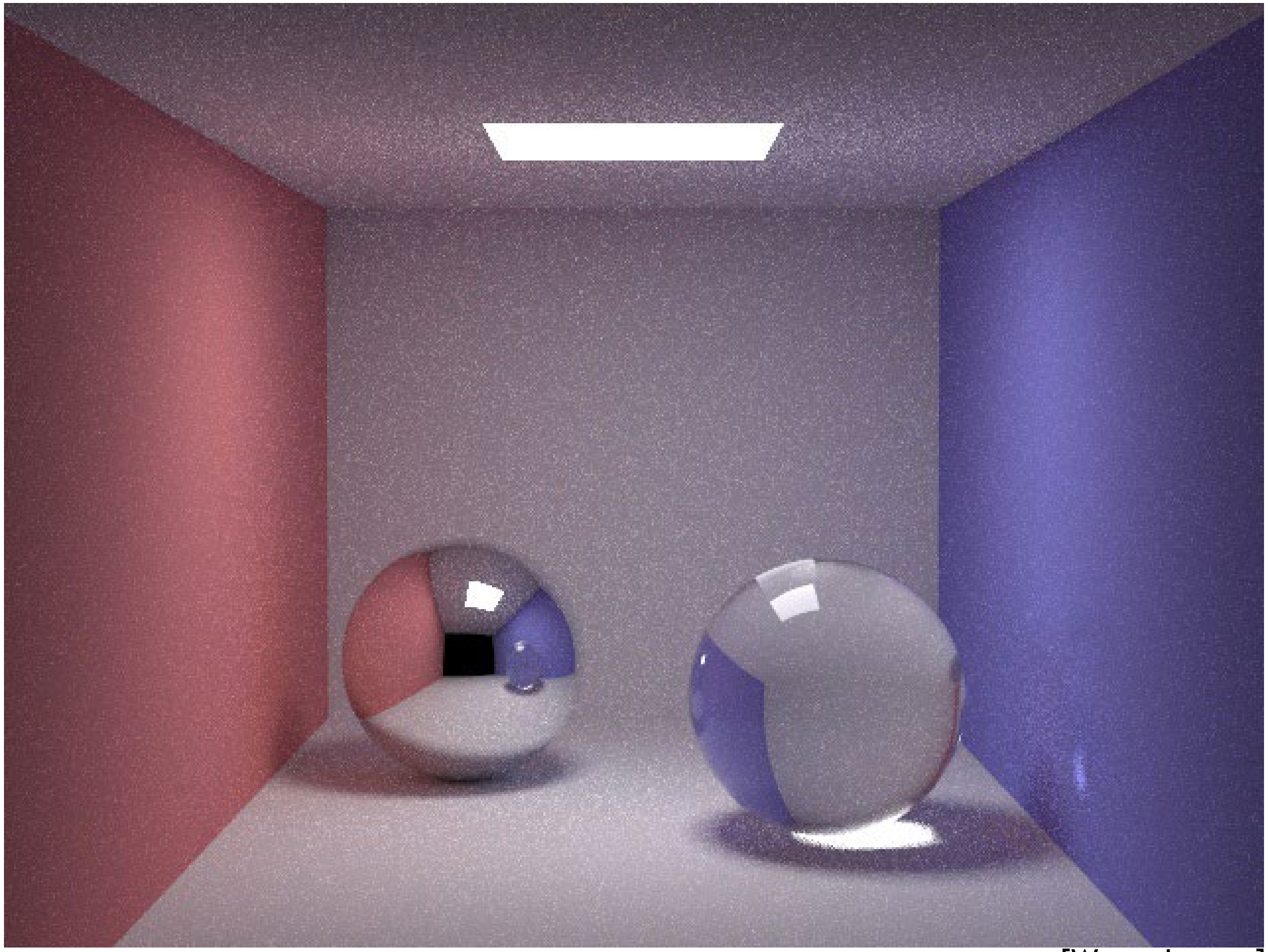
Problems with path tracing



[Wann Jensen]

10 paths/pixel

Problems with path tracing

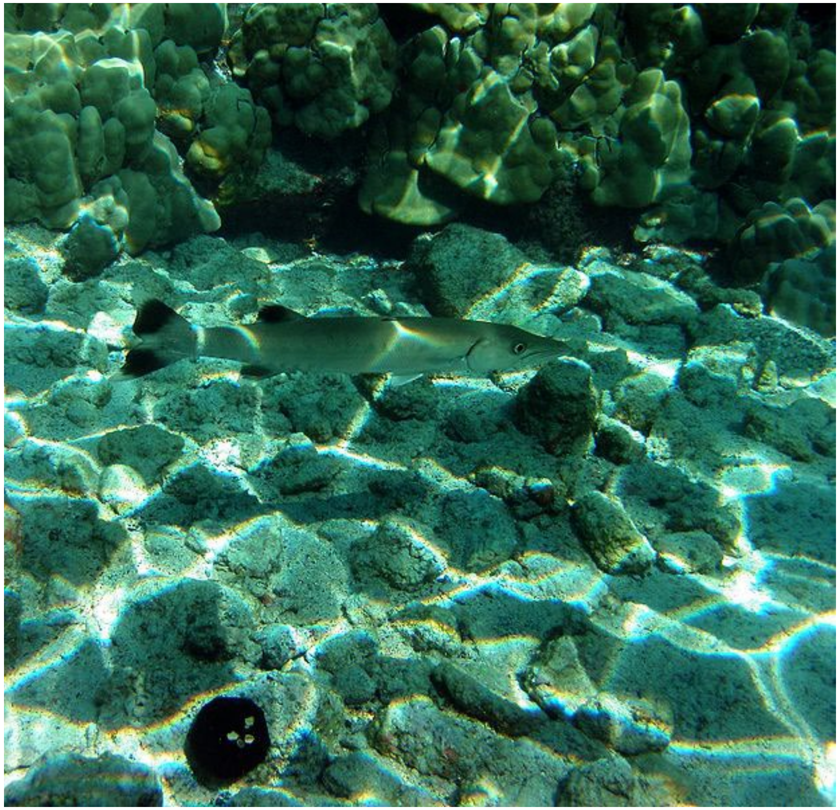


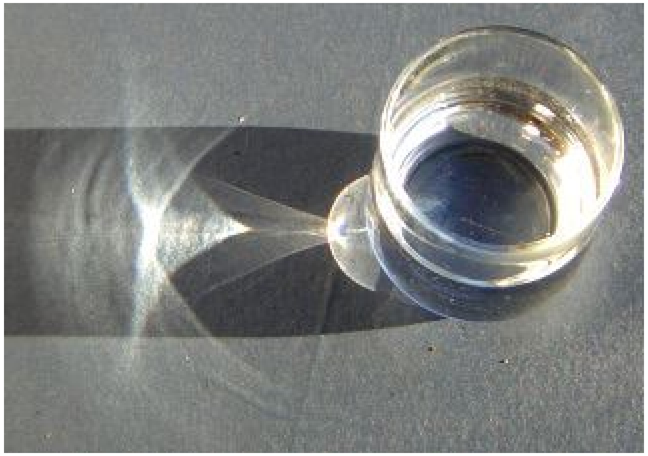
[Wann Jensen]

100 paths/pixel

Caustics

- Focusing of light by reflective and refractive surfaces

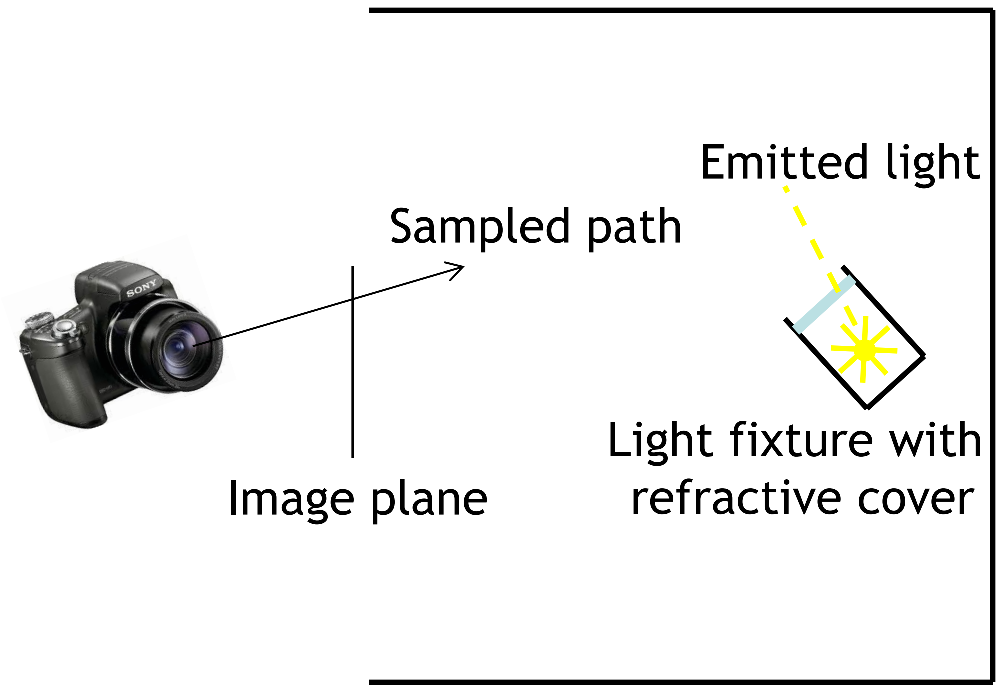




http://en.wikipedia.org/wiki/Caustic_(optics)

Realistic light sources

- Light source encased in fixture, behind glass



Light transport notation

- Path tracing not suitable to render caustics
- Light L, diffuse reflection D, specular reflection S, eye E
- Caustics are paths $\mathrm{L}\{\mathrm{S}\}^{+}\mathrm{DE}$
  - $\{\mathrm{S}\}^{+}$ means one ore more specular bounces
  - Focusing of light through reflections and refractions on diffuse surfaces
  - Realistic light fixtures

Today

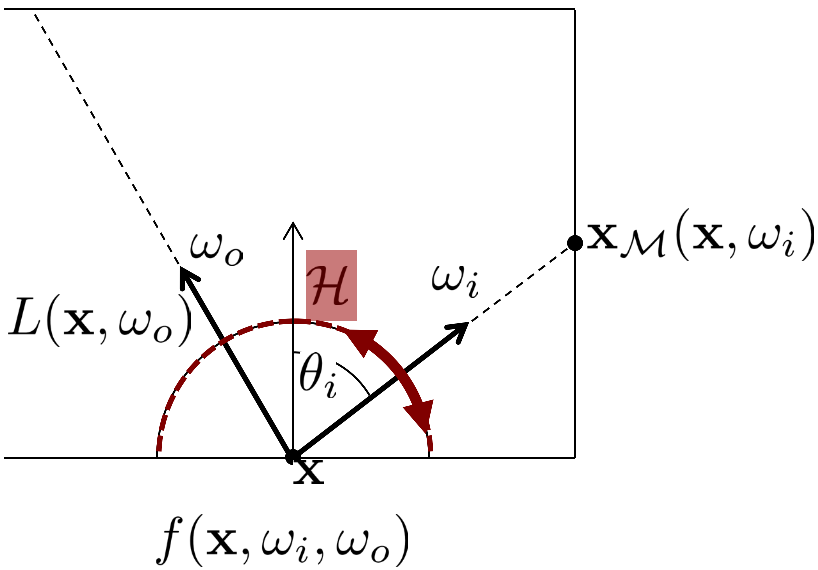
- More sophisticated methods to sample light paths
  - Bidirectional path tracing
  - (Photon mapping)

Overview

- Background
  - Three-point form of rendering equation
  - Measurement equation
- Reformulation of path tracing
- Bidirectional path tracing
  - Multiple importance sampling (MIS) weights

See additional document on ELMS!

Hemispherical form

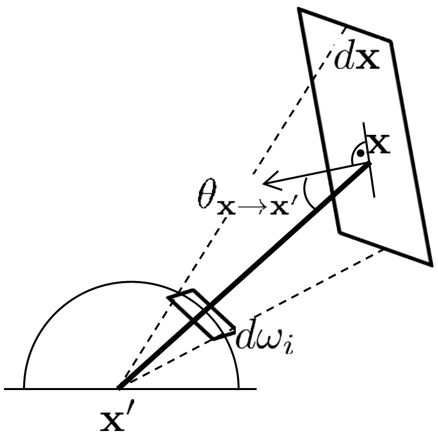


point hit by ray $(\mathbf{x}, \omega_i)$
(formerly denoted $\mathbf{x}'$)

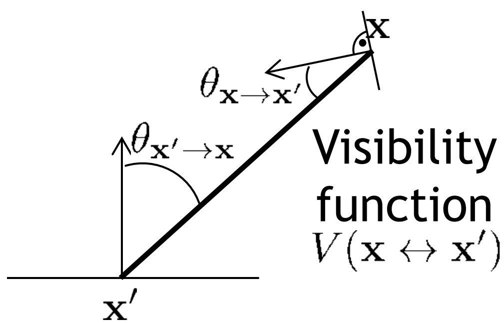
$$L(\mathbf{x}, \omega_o) = L_e(\mathbf{x}, \omega_o) + \int_{\mathcal{H}} L(\mathbf{x}_{\mathcal{M}}(\mathbf{x}, \omega_i), -\omega_i) f(\mathbf{x}, \omega_i, \omega_o) \cos(\theta_i) d\omega_i,$$

Integration over hemisphere H of incident directions ω_i

Integration over surface area



Change of integration variables between directions on hemisphere $d\omega_i$ and points on surfaces $d\mathbf{x}$



Notation for geometry term

Change of integration variables:

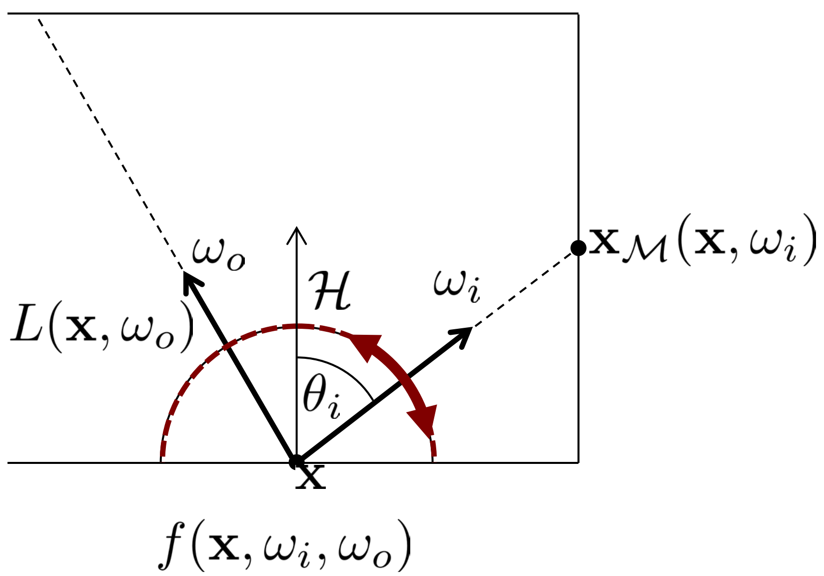
$$d\omega_i = \frac{\cos(\theta_{\mathbf{x}\to\mathbf{x}'})}{\|\mathbf{x}-\mathbf{x}'\|^2} d\mathbf{x}$$

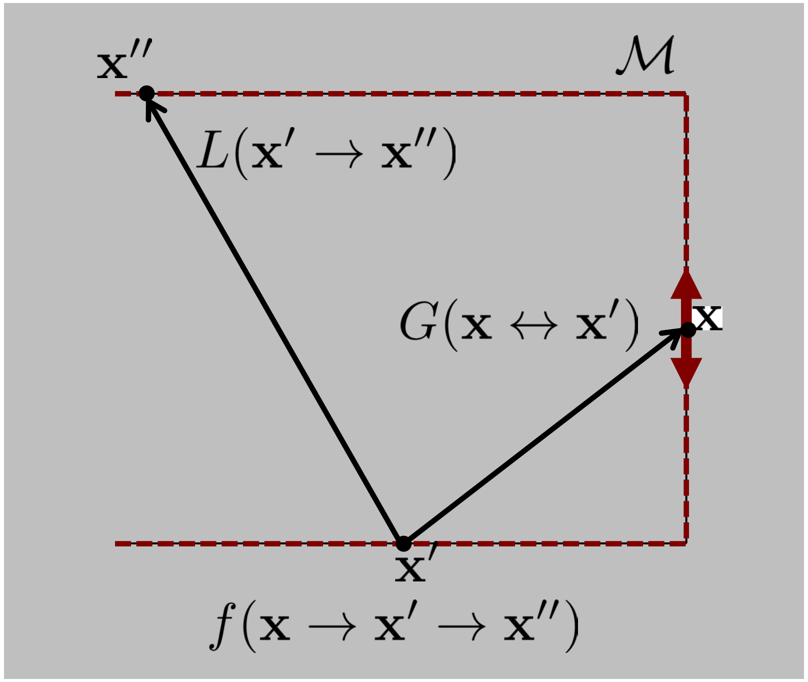
Geometry term:

$$G(\mathbf{x} \leftrightarrow \mathbf{x}') = V(\mathbf{x} \leftrightarrow \mathbf{x}') \cdot \frac{\cos(\theta_{\mathbf{x}'\to\mathbf{x}}) \cos(\theta_{\mathbf{x}\to\mathbf{x}'})}{\|\mathbf{x}-\mathbf{x}'\|^2}$$

(Slide 5, slide set 07 Advanced sampling)

Three point form





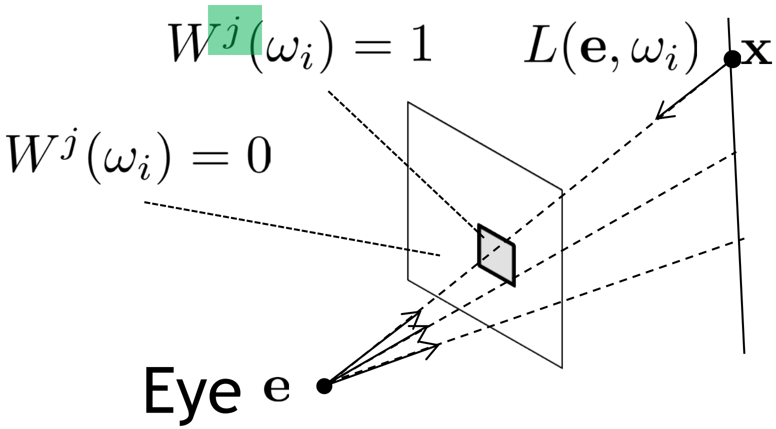
$$L(\mathbf{x}, \omega_o) = L_e(\mathbf{x}, \omega_o) + \int_{\mathcal{H}} L(\mathbf{x}_{\mathcal{M}}(\mathbf{x}, \omega_i), -\omega_i) f(\mathbf{x}, \omega_i, \omega_o) \cos(\theta_i) d\omega_i,$$

$$L(\mathbf{x}' \to \mathbf{x}'') = L_e(\mathbf{x}' \to \mathbf{x}'') + \int_{\mathcal{M}} L(\mathbf{x} \to \mathbf{x}') f(\mathbf{x} \to \mathbf{x}' \to \mathbf{x}'') G(\mathbf{x} \leftrightarrow \mathbf{x}') d\mathbf{x},$$

Three-point form, integration over surface area

Measurement equation

- Expresses pixel value I_j of pixel i



Importance function W^j for pixel j, here a box function (1 inside pixel, 0 elsewhere)

$$I_j = \int_{\mathcal{H}} W^j(\omega_i) L(\mathbf{e}, \omega_i) \cos(\theta_i) d\omega_i$$

Hemispherical integral

$$I_j = \int_{\mathcal{M}} W^j(\mathbf{x} \to \mathbf{e}) L(\mathbf{x} \to \mathbf{e}) G(\mathbf{x} \leftrightarrow \mathbf{e}) d\mathbf{x}$$

Surface area integral

Recursive expansion (3-point form)

Initial guess $L(\mathbf{x}' \to \mathbf{x}'') = L_e(\mathbf{x}' \to \mathbf{x}'')$, plug recursively into

$$L(\mathbf{x}' \to \mathbf{x}'') = L_e(\mathbf{x}' \to \mathbf{x}'') + \int_{\mathcal{M}} L(\mathbf{x} \to \mathbf{x}') f(\mathbf{x} \to \mathbf{x}' \to \mathbf{x}'') G(\mathbf{x} \leftrightarrow \mathbf{x}') d\mathbf{x}$$

Plug result into $I_j = \int_{\mathcal{M}} W^j(\mathbf{x} \to \mathbf{e}) L(\mathbf{x} \to \mathbf{e}) G(\mathbf{x} \leftrightarrow \mathbf{e}) d\mathbf{x}$

$$I_j = \underbrace{\sum_{k=1}^{\infty}}_{\text{Sum over path lengths } k} \underbrace{\int_{\mathcal{M}^k} L_e(\mathbf{x}_0 \to \mathbf{x}_1) G(\mathbf{x}_0 \leftrightarrow \mathbf{x}_1) \left(\prod_{i=1}^{k-1} f(\mathbf{x}_{i-1} \to \mathbf{x}_i \to \mathbf{x}_{i+1}) G(\mathbf{x}_i \leftrightarrow \mathbf{x}_{i+1}) \right) \cdot W^j(\mathbf{x}_{k-1} \to \mathbf{x}_k) d\mathbf{x}_0 \ldots d\mathbf{x}_{k-1}}_{\text{Path contribution function } f_j(\bar{x})}$$

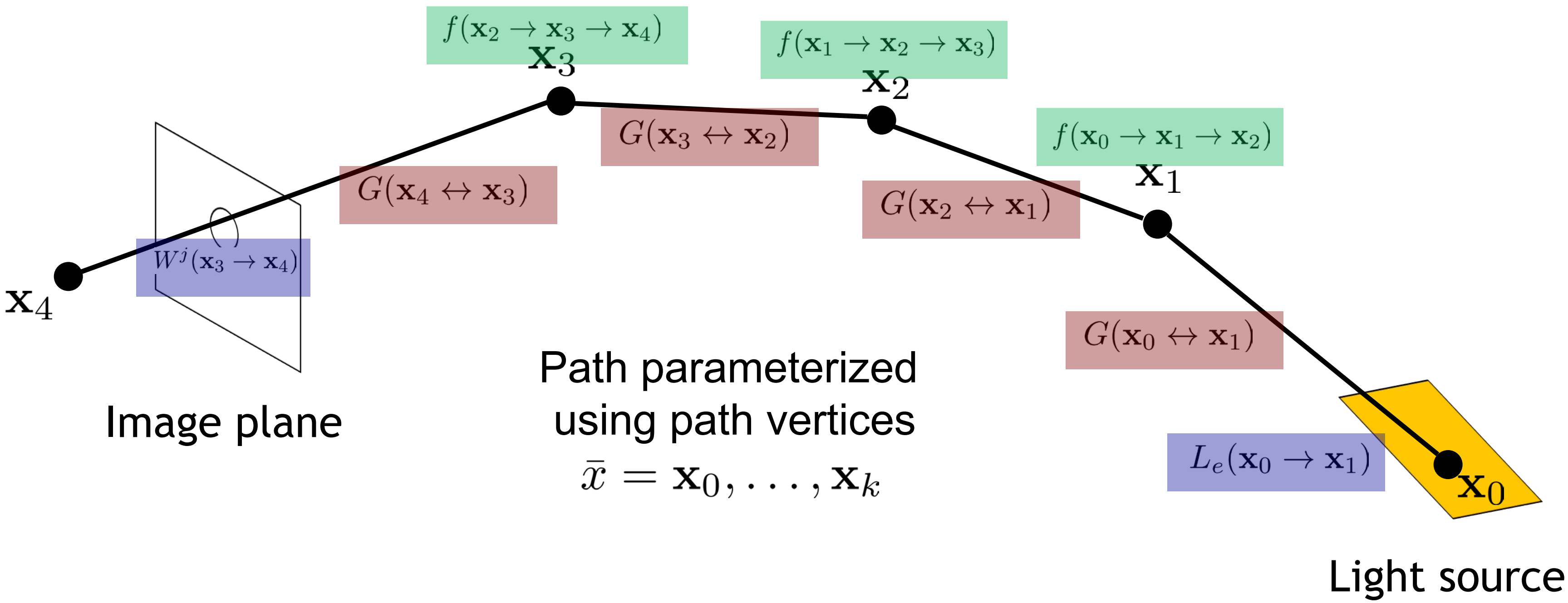
Sum over path lengths k

Path contribution function $f_j(\bar{x})$, almost same as $\sum_{k=1}^{\infty} \mathcal{T}^{k-1}\{L_e\}$

Path $\bar{x} = \mathbf{x}_0, \ldots, \mathbf{x}_k$

Path contribution function

(based on 3-point form)



$$f_j(\bar{x}) = L_e(\mathbf{x}_0 \to \mathbf{x}_1) G(\mathbf{x}_0 \leftrightarrow \mathbf{x}_1) f(\mathbf{x}_0 \to \mathbf{x}_1 \to \mathbf{x}_2) G(\mathbf{x}_1 \leftrightarrow \mathbf{x}_2) f(\mathbf{x}_1 \to \mathbf{x}_2 \to \mathbf{x}_3)$$
$$\cdot\, G(\mathbf{3}_1 \leftrightarrow \mathbf{x}_3) f(\mathbf{x}_2 \to \mathbf{x}_3 \to \mathbf{x}_4) G(\mathbf{x}_3 \leftrightarrow \mathbf{x}_4) W^j(\mathbf{x}_3 \to \mathbf{x}_4)$$

Symmetry wrt. light source and pixel (imp. function)!

Monte Carlo integration

- Estimate

$$I_j \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f_j(\bar{X}_i)}{p(\bar{X}_i)}$$

  - Random paths $\bar{X}_i$
  - Path probabilities (conceptually): product of vertex probabilities $p(x_i)$ (area densities in three point form!) and probability of $p(k)$ for length k

$$p(\bar{X}_i) = \left(\prod_{i=0}^{k} p(\mathbf{x}_i) \right) p(k)$$

- Path contribution using three-point form
  - Note: directly sampling points on surfaces with given density $p(\mathbf{x})$ not a good strategy in practice
  - Instead, construct $p(\mathbf{x})$ via ray tracing

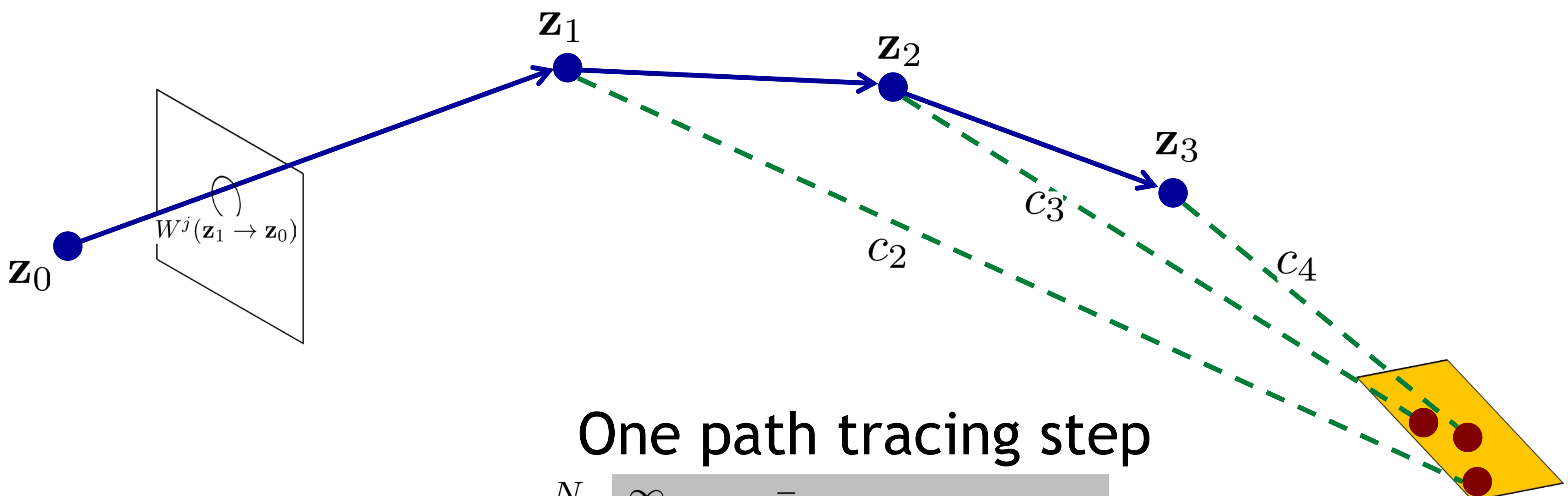
$$\frac{f_j(\bar{X}_i)}{p(\bar{X}_i)} = \frac{L_e(\mathbf{x}_0 \to \mathbf{x}_1) G(\mathbf{x}_0 \leftrightarrow \mathbf{x}_1) \left(\prod_{i=1}^{k-1} f(\mathbf{x}_{i-1} \to \mathbf{x}_i \to \mathbf{x}_{i+1}) G(\mathbf{x}_i \leftrightarrow \mathbf{x}_{i+1}) \right) \cdot W^j(\mathbf{x}_{k-1} \to \mathbf{x}_k)}{\left(\prod_{i=0}^{k} p(\mathbf{x}_i) \right) p(k)}$$

Summary

- Hemispherical form implies sampling of light paths incrementally by starting at camera
- Three-point form enables more general techniques to sample light paths
  - Ray tracing from camera
  - Ray tracing from light sources
  - Combinations of the above
  - (Markov Chain Monte Carlo sampling by iteratively modifying paths)

Path tracing revisited: expressed using 3-point form

- Sample paths incrementally from eye
- At each step connect to light to obtain path $\bar{X}_{k,i}$ of length k
- Terminate path using Russian roulette, implemented with binary random variable R_k
- Conceptually, each path tracing step evaluates sum over all lengths $k=0\ldots1$

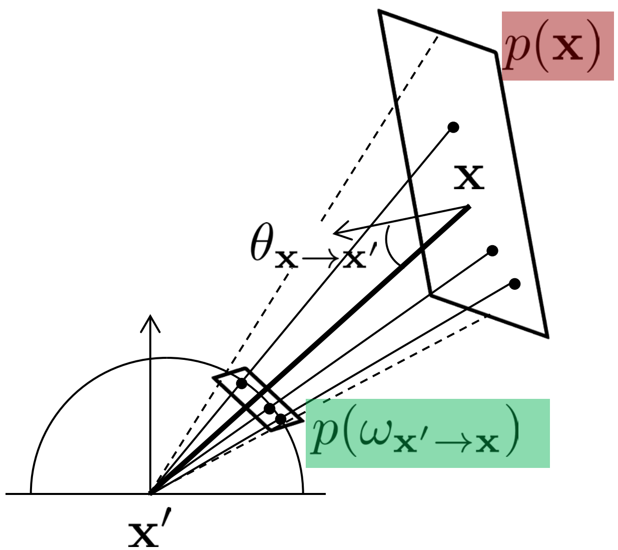


One path tracing step

$$I_j \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{\infty} \frac{f_j(\bar{X}_{k,i})}{p(\bar{X}_{k,i})} \frac{R_k}{p(R_k = 1)}$$

Relation of density on surface vs. hemisphere

- Calculation of area density $p(\mathbf{x})$, if (as in basic path tracing) we sampled x by sampling direction $\omega_{x'->x}$ from x' with density $p(\omega_{\mathbf{x}'->\mathbf{x}})$

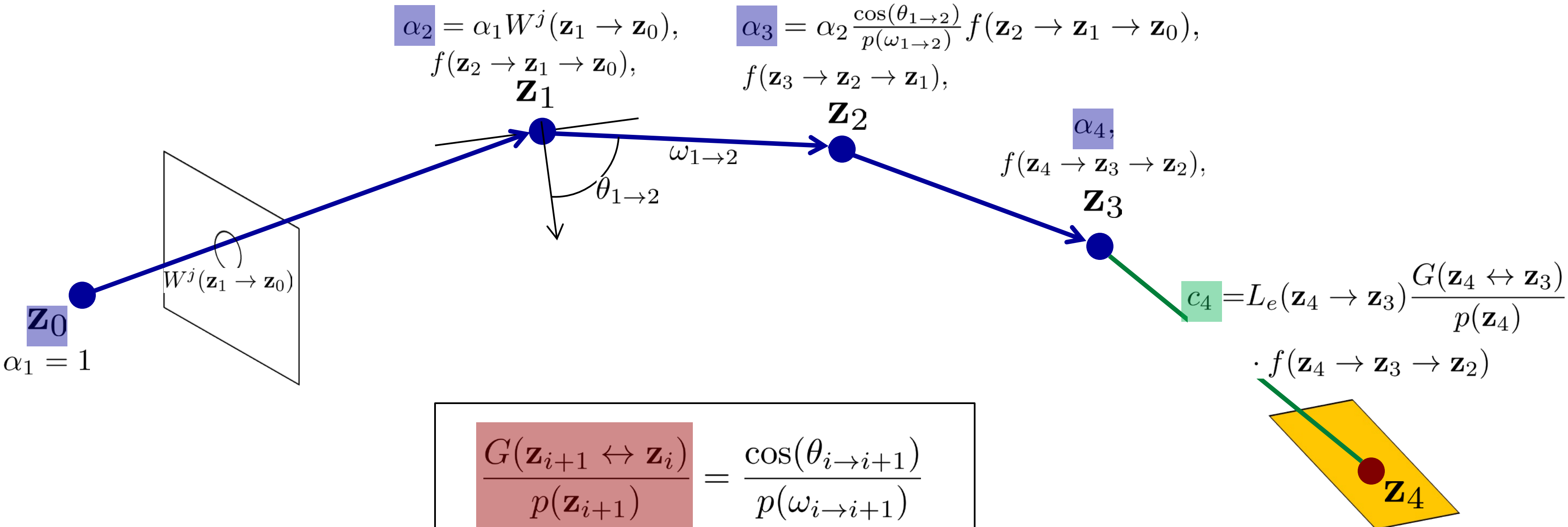


Relation of densities:

$$p(\mathbf{x}) = p(\omega_{\mathbf{x}'\to\mathbf{x}}) \frac{\cos(\theta_{\mathbf{x}\to\mathbf{x}'})}{\|\mathbf{x}-\mathbf{x}'\|^2}$$

$$= p(\omega_{\mathbf{x}'\to\mathbf{x}}) \frac{G(\mathbf{x}' \leftrightarrow \mathbf{x})}{\cos(\theta_{\mathbf{x}'\to\mathbf{x}})}$$

Path tracing: same as before



$$\boxed{\frac{G(\mathbf{z}_{i+1} \leftrightarrow \mathbf{z}_i)}{p(\mathbf{z}_{i+1})} = \frac{\cos(\theta_{i\to i+1})}{p(\omega_{i\to i+1})}}$$

$$\frac{f_j(\bar{X}_{k,i})}{p(\bar{X}_{k,i})} = \underbrace{L_e(\mathbf{z}_k \to \mathbf{z}_{k-1}) \frac{G(\mathbf{z}_k \leftrightarrow \mathbf{z}_{k-1})}{p(\mathbf{z}_k)} f(\mathbf{z}_k \to \mathbf{z}_{k-1} \to \mathbf{z}_{k-2})}_{c_k}$$

$$\underbrace{\prod_{i=1}^{k-2} \frac{G(\mathbf{z}_{i+1} \leftrightarrow \mathbf{z}_i)}{p(\mathbf{z}_{i+1})} f(\mathbf{z}_{i+1} \to \mathbf{z}_i \to \mathbf{z}_{i-1}) \cdot \frac{G(\mathbf{z}_1 \leftrightarrow \mathbf{z}_0)}{p(\mathbf{z}_1)} \frac{W^j(\mathbf{z}_1 \to \mathbf{z}_0)}{p(\mathbf{z}_0)}}_{\alpha_k}$$

$$\alpha_1 = 1$$

$$\alpha_2 = \alpha_1 \frac{G(\mathbf{z}_1 \leftrightarrow \mathbf{z}_0)}{p(\mathbf{z}_1)} W^j(\mathbf{z}_1 \to \mathbf{z}_0) = W^j(\mathbf{z}_1 \to \mathbf{z}_0)$$

$$\alpha_i = \alpha_{i-1} \frac{\cos(\theta_{i-2\to i-1})}{p(\omega_{i-2\to i-1})} f(\mathbf{z}_{i-1} \to \mathbf{z}_{i-2} \to \mathbf{z}_{i-3})$$

As in path tracing pseudo code

Summary

- Formulation of path tracing using three-point form is equivalent to previous approach (based on hemispherical integrals)
  - Many terms in division of geometry term over area density cancel out

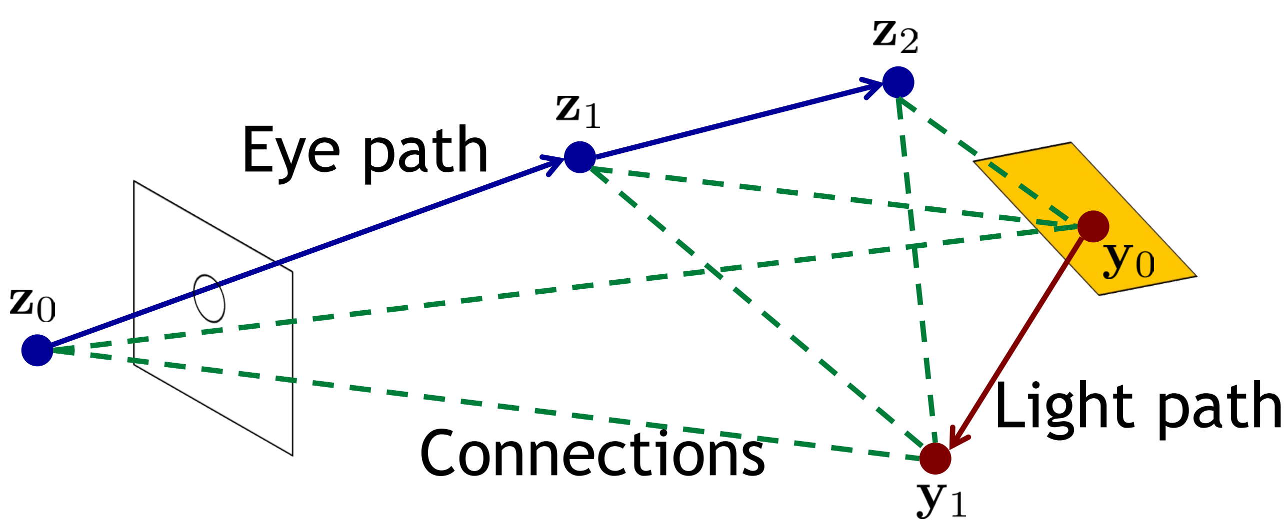
$$\frac{G(\mathbf{z}_{i+1} \leftrightarrow \mathbf{z}_i)}{p(\mathbf{z}_{i+1})} = \frac{\cos(\theta_{i \to i+1})}{p(\omega_{i \to i+1})}$$

Advantage of three-point form

More flexible, allows formulation of other path sampling strategies, beyond incremental path tracing from camera

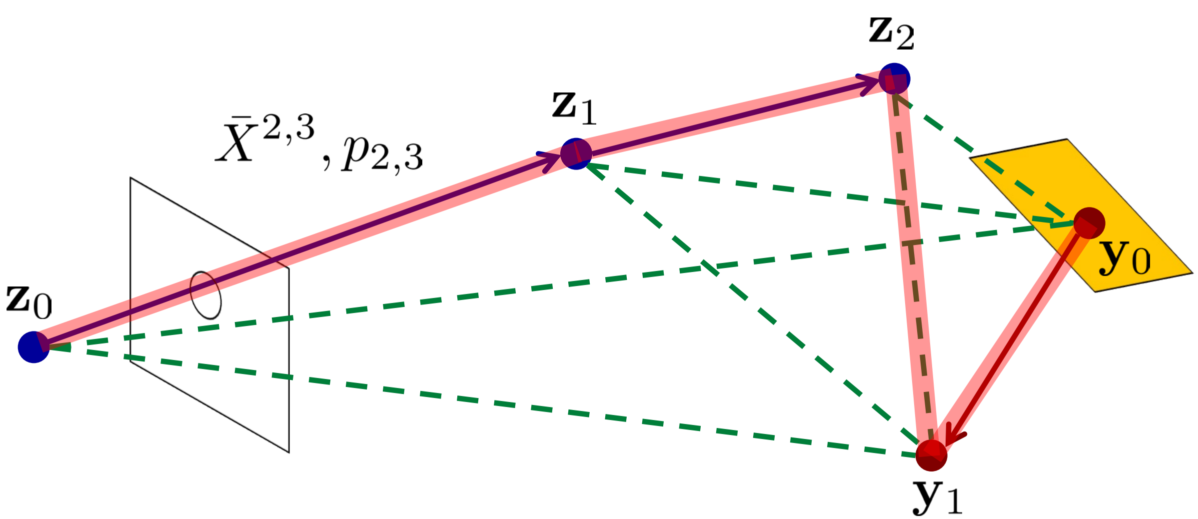
Bidirectional path tracing

- Trace paths both from **eye** and **light** (eye and light subpaths)
  - Terminate each using Russian roulette
- Evaluate all connections and sum up



Bidirectional path tracing

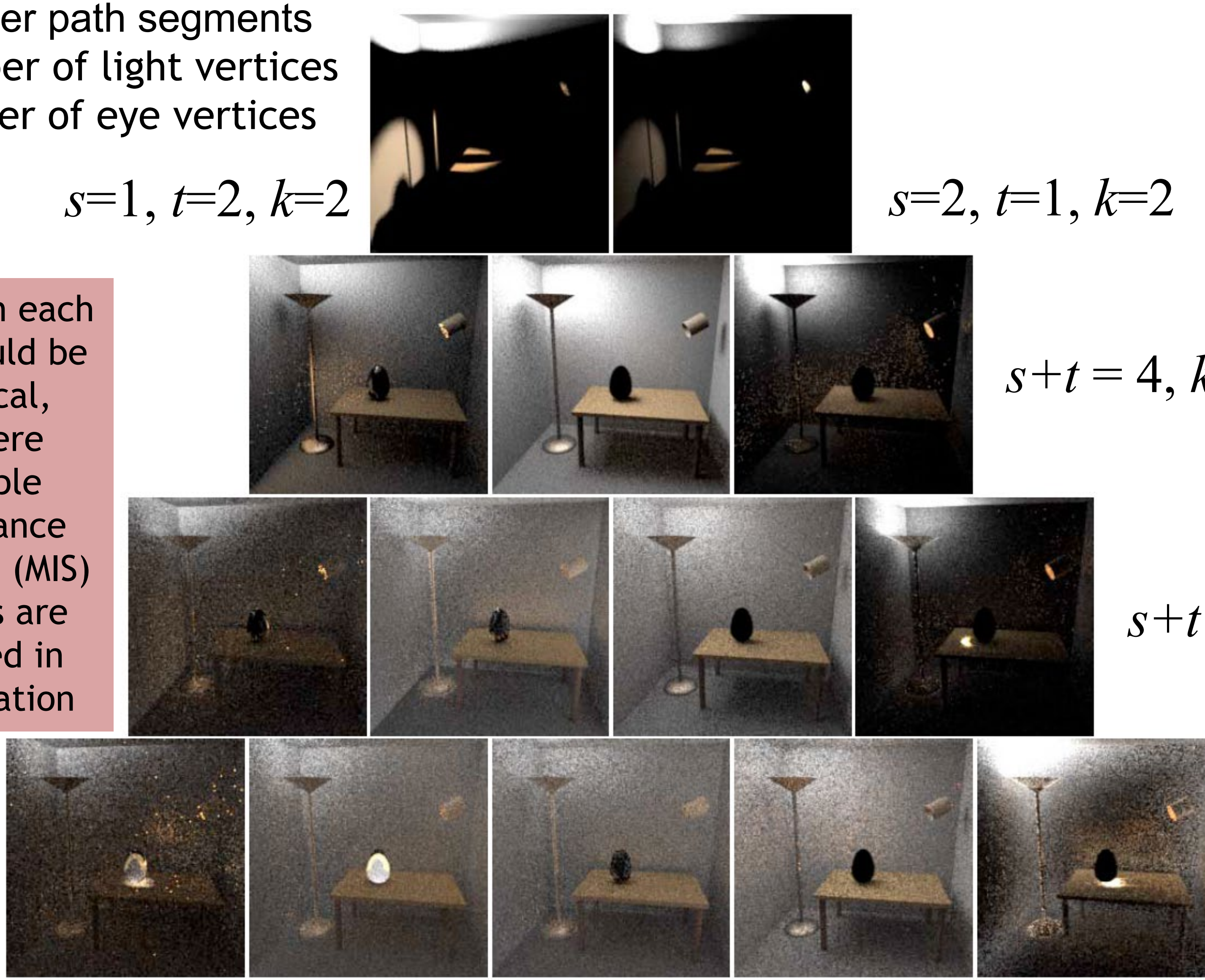
- Path of length k, with $k+1$ vertices
  - Sampled with s vertices from light, t from eye, $s+t = k+1$
  - Path denoted $\bar{X}^{s,t}$
- Each length k can be sampled in $k+2$ ways, or with $k+2$ "sampling techniques"
  - $s = 0,1,\ldots k+1$ and $t = k+1\text{-}s$
  - Probability density for "technique s,t" denoted $p_{s,t}$



Example

k: number path segments
s: number of light vertices
t: number of eye vertices

Images in each row should be identical, but here multiple importance sampling (MIS) weights are included in visualization

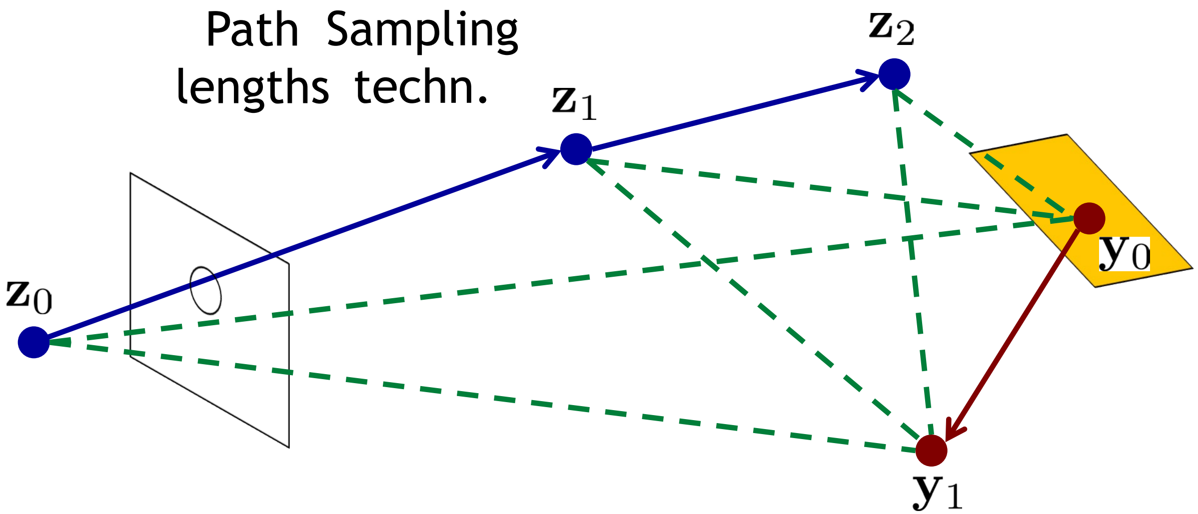


[Veach]

Bidirectional path tracing

- Conceptually, each bidirectional path tracing operation generates one sample for all techniques and for all path lengths
- In practice, many are not evaluated because of Russian roulette, represented with binary variable $R_{s,t}$
- Techniques combined with multiple importance sampling, weights w_{st}

$$I_j \approx \frac{1}{N}\sum_{i=1}^{N} \overbrace{\underbrace{\sum_{k=1}^{\infty}}_{\text{Path lengths}} \underbrace{\sum_{s+t-1=k}}_{\text{Sampling techn.}} \frac{R_{s,t}}{p(R_{s,t}=1)} w_{s,t}(\bar{X}_i^{s,t}) \frac{f_j(\bar{X}_i^{s,t})}{p_{s,t}(\bar{X}_i^{s,t})}}^{\text{One bidirectional path tracing step}}$$



Path contribution

$$\frac{f_j(\bar{X}^{s,t})}{p(\bar{X}^{s,t})} = \underbrace{\frac{L_e(\mathbf{y}_0 \to \mathbf{y}_1)}{p(\mathbf{y}_0)} \frac{G(\mathbf{y}_0 \leftrightarrow \mathbf{y}_1)}{p(\mathbf{y}_1)} \prod_{i=1}^{s-2} f(\mathbf{y}_{i-1} \to \mathbf{y}_i \to \mathbf{y}_{i+1}) \frac{G(\mathbf{y}_i \leftrightarrow \mathbf{y}_{i+1})}{p(\mathbf{y}_{i+1})}}_{\alpha_s^L}$$

Light subpath

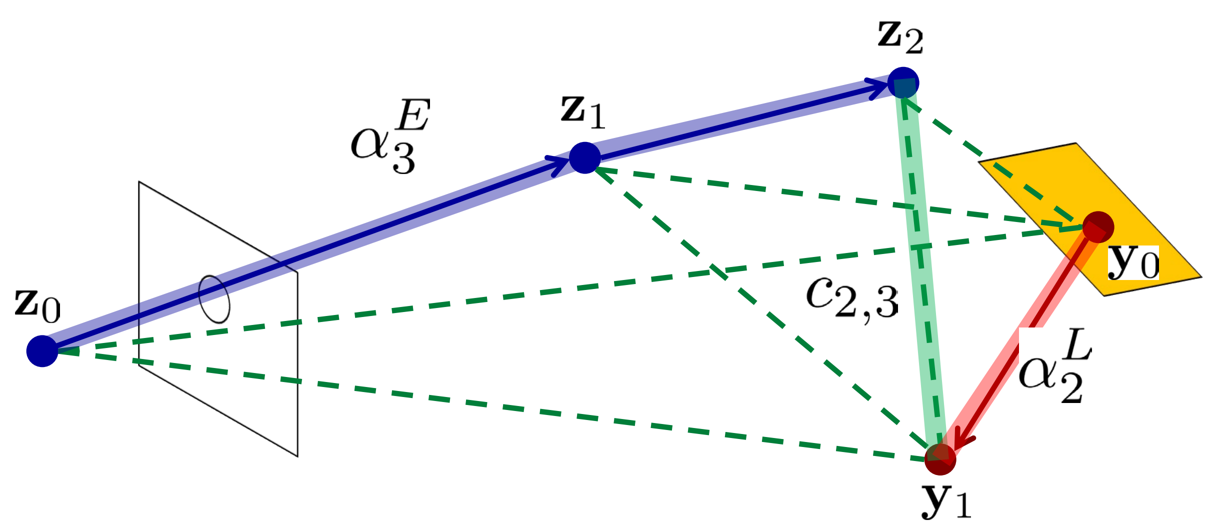
$$\underbrace{f(\mathbf{y}_{s-2} \to \mathbf{y}_{s-1} \to \mathbf{z}_{t-1}) G(\mathbf{y}_{s-1} \leftrightarrow \mathbf{z}_{t-1}) f(\mathbf{y}_{s-1} \to \mathbf{z}_{t-1} \to \mathbf{z}_{t-2})}_{c_{s,t}}$$

Connection

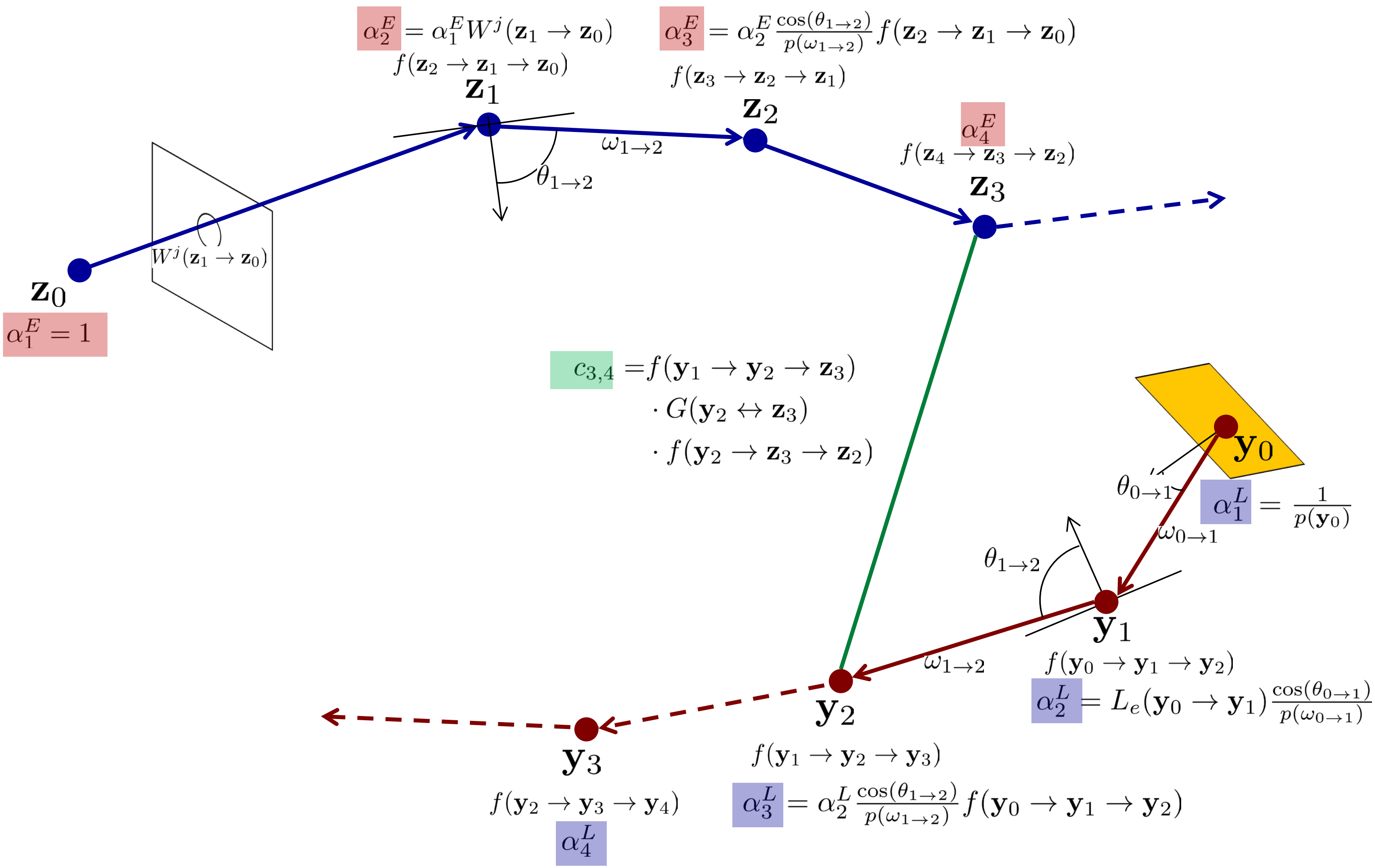
$$\underbrace{\prod_{i=1}^{t-2} \frac{G(\mathbf{z}_{i+1} \leftrightarrow \mathbf{z}_i)}{p(\mathbf{z}_{i+1})} f(\mathbf{z}_{i+1} \to \mathbf{z}_i \to \mathbf{z}_{i-1}) \frac{G(\mathbf{z}_1 \leftrightarrow \mathbf{z}_0)}{p(\mathbf{z}_1)} \cdot W^j(\mathbf{z}_1 \to \mathbf{z}_0)}_{\alpha_t^E}$$

Eye subpath

$$= \alpha_s^L c_{s,t} \alpha_t^E$$

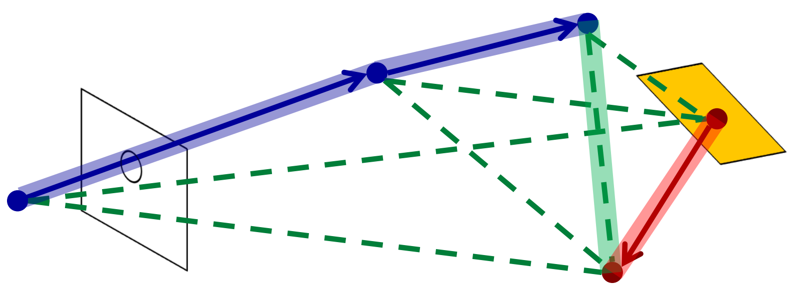


Incremental computation

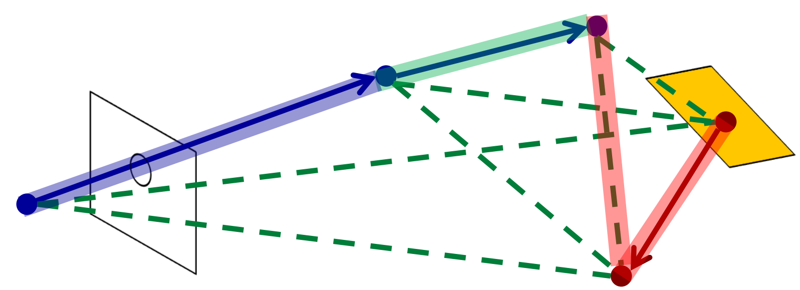


Multiple importance sampling

- For each path, need to evaluate, for each technique, probability that path would have been sampled with that technique
- Example:



$p_{2,3}$



$p_{3,2}$

Probability to sample same path with technique $p_{3,2}$ (instead of $p_{2,3}$)

MIS weights

- Balance heuristics

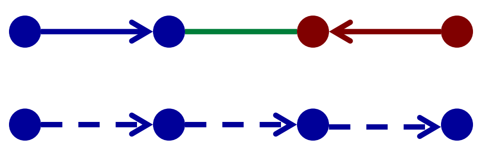
$$w_{s,t} = \frac{p_{s,t}}{\sum_{0<i\leq s} p_{s-i,t+i} + p_{s,t} + \sum_{0<i\leq t} p_{s+i,t-i}}$$
$$= \frac{1}{\sum_{0<i\leq s} \frac{p_{s-i,t+i}}{p_{s,t}} + 1 + \sum_{0<i\leq t} \frac{p_{s+i,t-i}}{p_{s,t}}}$$

Probabilities to sample path with all other techniques

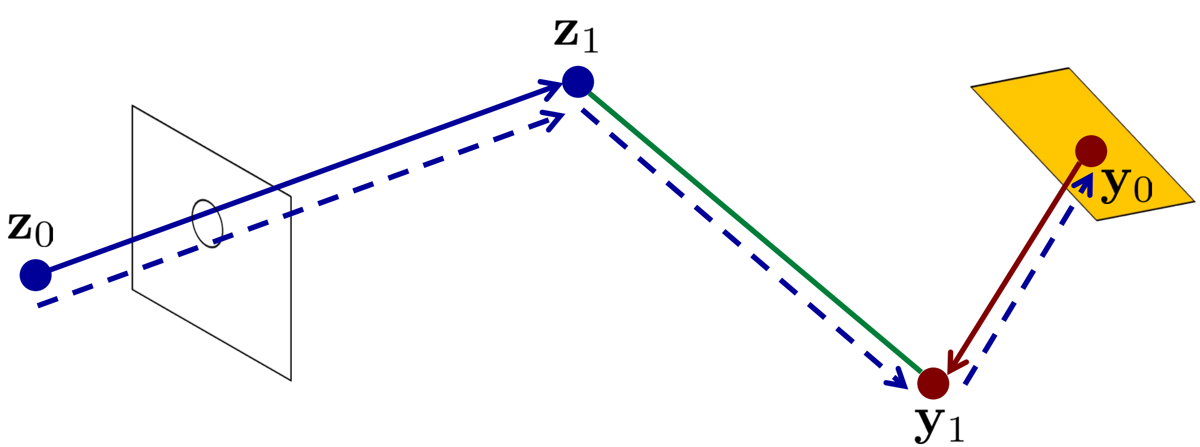
- Example with s=2, t=2

$p_{2,2}$

$p_{2-2,2+2} = p_{0,4}$



$s = 2, t = 2$

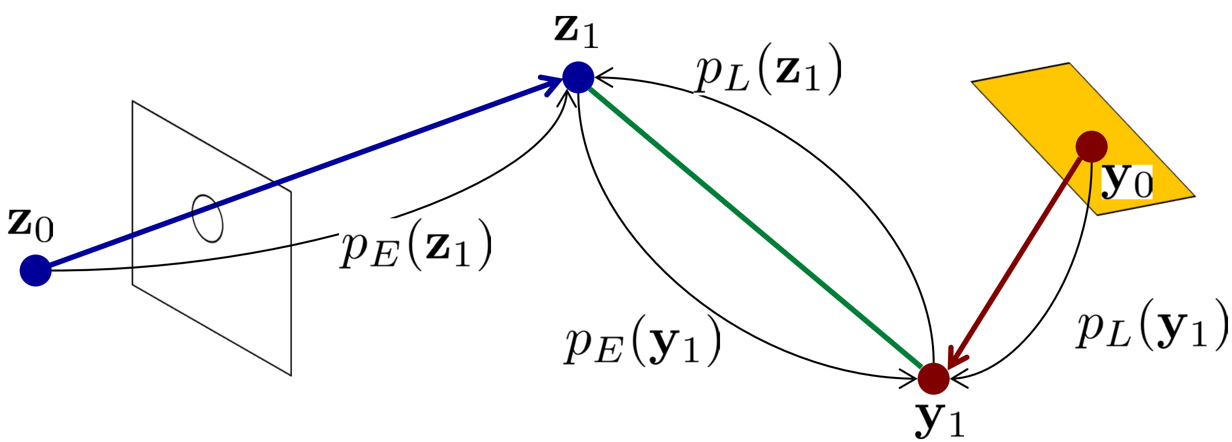


MIS weights, implementation

- Idea: given a path sampled with technique s,t, incrementally compute probabilities for $s\text{-}i,t+i$ step by step for i=0,1,2,…
  - Compute $p_{s\text{-}1,t+1}$ using $p_{s,t}$; then $p_{s\text{-}2,t+2}$ using $p_{s\text{-}1,t+1}$, etc.
- A bit hairy to implement in practice

Notation

- Probability $p_L(\mathbf{y}_i)$ to sample vertex $\mathbf{y}_i$ on light path from light
- Probability $p_E(\mathbf{y}_i)$ to sample from eye
- Similar: $p_E(\mathbf{z}_i)$ and $p_L(\mathbf{z}_i)$

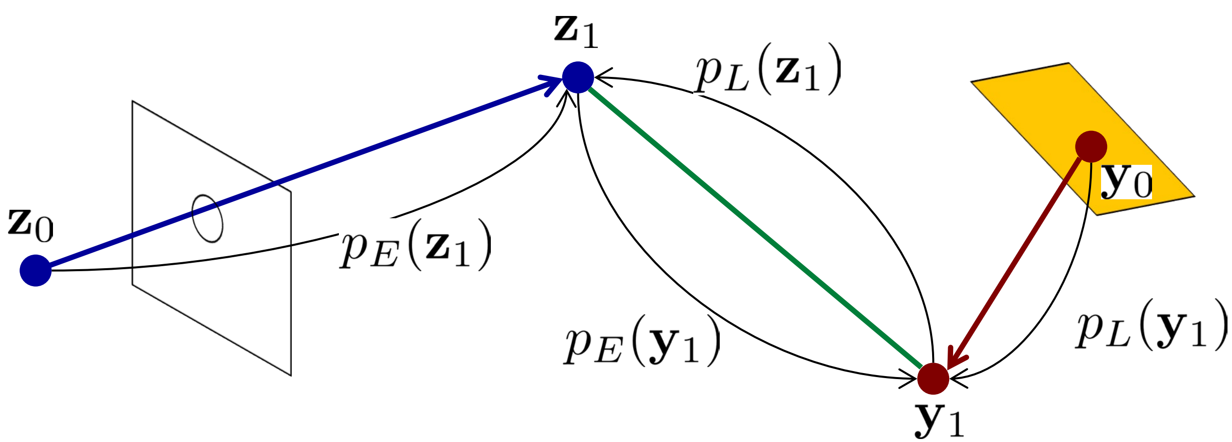


Notation

- Computation of probabilities (using relation of densities on surface vs. hemisphere)

$$p_L(\mathbf{y}_i) = p(\mathbf{y}_{i-1} \to \mathbf{y}_i) / \cos(\theta_{i \to i-1}) G(\mathbf{y}_{i-1} \leftrightarrow \mathbf{y}_i)$$

$$p_E(\mathbf{y}_{i-1}) = p(\mathbf{y}_i \to \mathbf{y}_{i-1}) / \cos(\theta_{i-1 \to i}) G(\mathbf{y}_i \leftrightarrow \mathbf{y}_{i-1})$$



MIS weights

- Note

$$\frac{p_{s-i,t+i}}{p_{s,t}} = \prod_{j=1}^{i} \frac{p_E(\mathbf{y}_{s-j})}{p_L(\mathbf{y}_{s-j})}, \quad 0 < i \le s$$

„Change i vertices from light to eye“

$$\frac{p_{s+i,t-i}}{p_{s,t}} = \prod_{j=1}^{i} \frac{p_L(\mathbf{z}_{t-j})}{p_E(\mathbf{z}_{t-j})}, \quad 0 < i \le t$$

„Change i vertices from eye to light“

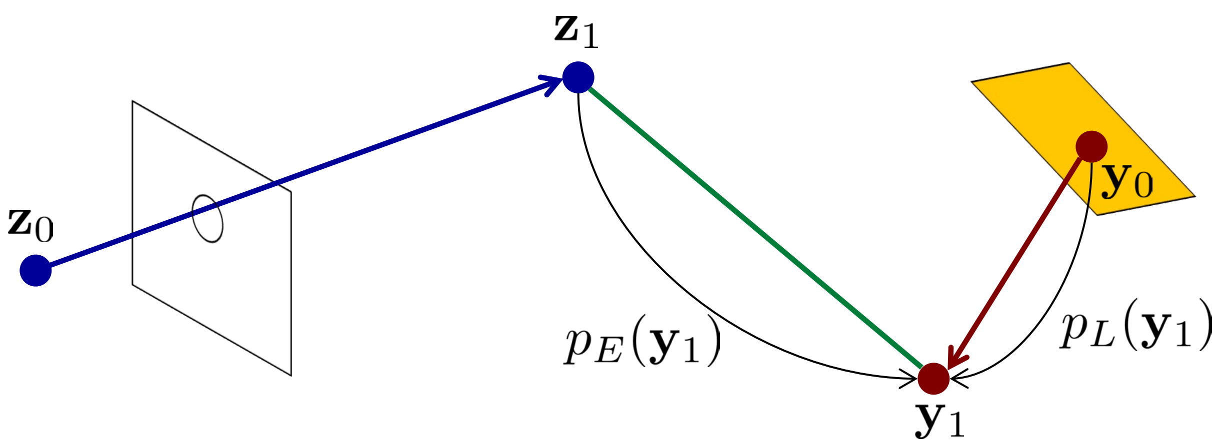
Change 1 vertex from light to eye

$$\frac{p_{s-1,t+1}}{p_{s,t}} = \frac{p_E(\mathbf{y}_{s-1})}{p_L(\mathbf{y}_{s-1})}$$

Change 2 vertices from light to eye

$$\frac{p_{s-2,t+2}}{p_{s,t}} = \frac{p_E(\mathbf{y}_{s-1})}{p_L(\mathbf{y}_{s-1})} \frac{p_E(\mathbf{y}_{s-2})}{p_L(\mathbf{y}_{s-2})}$$

$s = 2, t = 2$



MIS weights

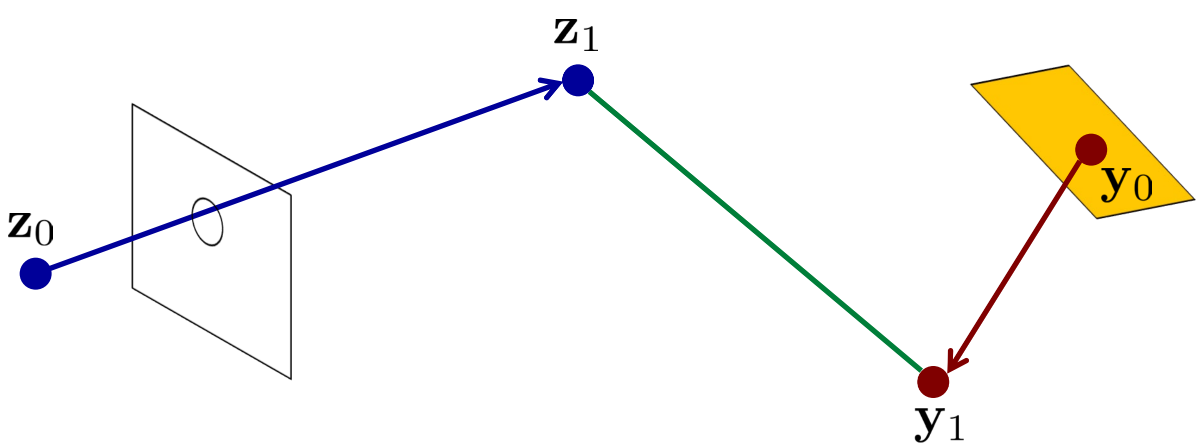
- Need to compute

$$w_{s,t} = \frac{1}{\sum_{0<i\le s} \prod_{j=1}^{i} \frac{p_E(\mathbf{y}_{s-j})}{p_L(\mathbf{y}_{s-j})} + 1 + \sum_{0<i\le t} \prod_{j=1}^{i} \frac{p_L(\mathbf{z}_{t-j})}{p_E(\mathbf{z}_{t-j})}}$$

„Change 1 to s vertices from light to eye“

„Change 1 to t vertices from eye to light“

$s=2, t=2$



Example

- Same render time

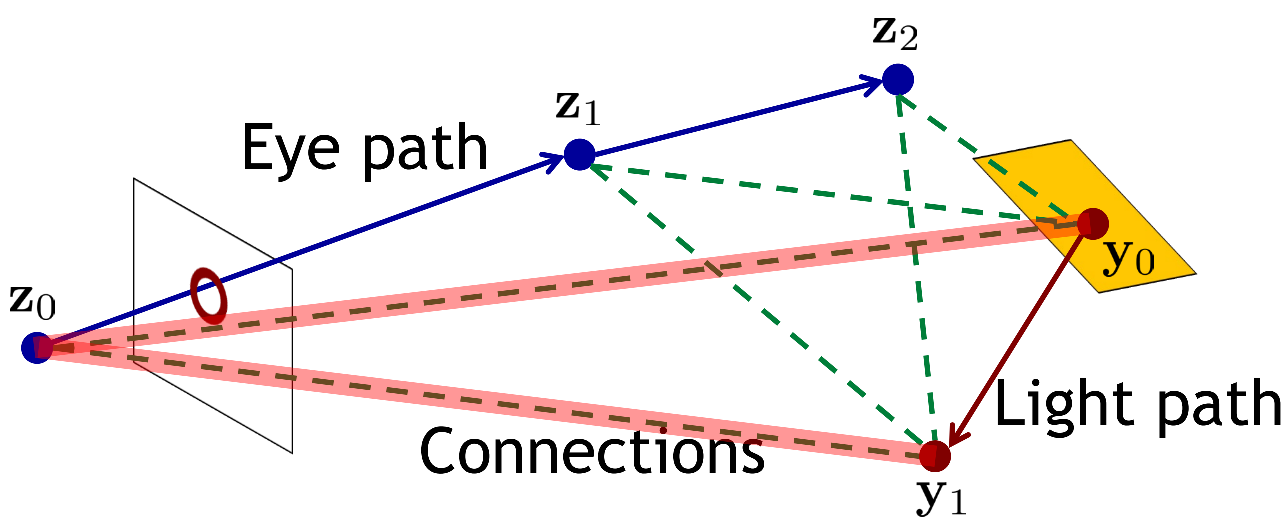
[Veach]

Bidirectional with multiple importance sampling

Standard path tracing

Implementation details

- Short paths ($s,t<=2$) need special attention
  - See definition of α^E, α^L values and connection terms $c_{s,t}$ in additional document on ELMS
- For $t=1$, paths do not necessarily go through sampled pixel!
  - Accumulate in separate image buffer, see document



Next time

- Participating media