

Problem-120

Problem Statement

Can you factor any other polynomial of the form $a^{2n} + b^{2n}$?

Solution

We shall consider two separate cases: (1) when n is even (2) when n is odd.

When n is even, therefore n has the form $2m$ with $m \geq 1$

$$\begin{aligned} a^{2n} + b^{2n} &= a^{4m} + b^{4m} \\ &= (a^{2m})^2 + (b^{2m})^2 \\ &= (a^{2m})^2 + 2 \cdot a^{2m} \cdot b^{2m} + (b^{2m})^2 - 2 \cdot a^{2m} \cdot b^{2m} \\ &= (a^{2m} + b^{2m})^2 - (\sqrt{2} \cdot a^m \cdot b^m)^2 \\ &= (a^{2m} + \sqrt{2} \cdot a^m \cdot b^m + b^{2m}) (a^{2m} - \sqrt{2} \cdot a^m \cdot b^m + b^{2m}) \end{aligned}$$

When n is odd, therefore n has the form $2m + 1$ with $m \geq 0$

Let's consider some examples and try to generalize from them.

- When $m = 0$ therefore $n = 1$.

$$\begin{aligned} a^2 + b^2 &= a^2 - (-b^2) \\ &= a^2 - (\sqrt{-1} \cdot b)^2 \\ &= a^2 - (i \cdot b)^2 \\ &= (a + i \cdot b) (a - i \cdot b) \end{aligned}$$

- When $m = 1$ therefore $n = 3$. Our polynomial in this case is $a^6 + b^6$. We notice that if we set $a^2 = -b^2$, the polynomial evaluates to zero. So, we might be able to factor the polynomial into $(a^2 + b^2)$ (...). We thus try to extract $a^2 + b^2$ from each pair of consecutive terms, introducing

temporary terms as required.

$$\begin{aligned}
a^6 + b^6 &= a^6 + a^4 \cdot b^2 - a^4 \cdot b^2 - a^2 \cdot b^4 + a^2 \cdot b^4 + b^6 \\
&= a^4 \cdot (a^2 + b^2) - a^2 \cdot b^2 \cdot (a^2 + b^2) + b^4 \cdot (a^2 + b^2) \\
&= (a^2 + b^2) (a^4 - a^2 \cdot b^2 + b^4)
\end{aligned}$$

- When $m = 2$ therefore $n = 5$. Our polynomial in this case is $a^{10} + b^{10}$. Again, setting $a^2 = -b^2$ makes the polynomial zero. So, $a^2 + b^2$ is a potential factor. We again try to extract $a^2 + b^2$ from each pair of consecutive terms introducing temporary terms as required.

$$\begin{aligned}
a^{10} + b^{10} &= a^{10} + a^8 \cdot b^2 - a^8 \cdot b^2 - a^6 \cdot b^4 + a^6 \cdot b^4 + a^4 \cdot b^6 - a^4 \cdot b^6 - a^2 \cdot b^8 + a^2 \cdot b^8 + b^{10} \\
&= a^8 \cdot (a^2 + b^2) - a^6 \cdot b^2 \cdot (a^2 + b^2) + a^4 \cdot b^4 \cdot (a^2 + b^2) - a^2 \cdot b^6 \cdot (a^2 + b^2) + b^8 \cdot (a^2 + b^2) \\
&= (a^2 + b^2) (a^8 - a^6 \cdot b^2 + a^4 \cdot b^4 - a^2 \cdot b^6 + b^8)
\end{aligned}$$

The factoring process for $n > 1$ looks like below:

$$\begin{aligned}
a^{2n} + b^{2n} &= \sum_{k=0}^{n-1} (-1)^k \left[a^{2(n-k)} \cdot b^{2k} + a^{2(n-k-1)} \cdot b^{2(k+1)} \right] \\
&= \sum_{k=0}^{n-1} (-1)^k a^{2(n-k-1)} \cdot b^{2k} [a^2 + b^2] \\
&= (a^2 + b^2) \left(\sum_{k=0}^{n-1} (-1)^k a^{2(n-k-1)} \cdot b^{2k} \right)
\end{aligned}$$

From the above three cases, we can generalize the factoring of $a^{2n} + b^{2n}$ when n is odd as follows:

$$a^{2n} + b^{2n} = \begin{cases} (a + i \cdot b) (a - i \cdot b) & , \text{ when } n = 1 \\ (a^2 + b^2) \left(\sum_{k=0}^{n-1} (-1)^k a^{2(n-k-1)} \cdot b^{2k} \right) & , \text{ otherwise} \end{cases}$$