

Problem-304

Problem Statement

- (a) Find the side of a square having the same perimeter as a rectangle with sides a and b .
- (b) Find the side of a square having the same area as a rectangle with sides a and b .

Solution

- (a) The question could also ask what are the sides of the rectangle having the same perimeter as a rectangle with sides a and b , but has the maximum possible area. From **Problem-263**, we know that the rectangle with the maximum area would be a square. Say the side of the square is x . Then we need $4x = 2(a + b)$, or $x = \frac{a+b}{2}$. Therefore, the side of the square is the arithmetic mean of the sides of the rectangle.
- (b) The question could also ask what are the sides of the rectangle having the same area as a rectangle with sides a and b , but has the minimum possible perimeter. From **Problem-264**, we know that the rectangle with the minimum perimeter would be a square. We need $x^2 = a \cdot b$, or $x = \sqrt{a \cdot b}$. Therefore, the side of the square is the geometric mean of the sides of the rectangle.

Problem-317

Problem Statement

Prove the inequality between arithmetic and geometric means for $n = 4$.

Solution

For non-negative integers a, b, c, d , we need to prove

$$\sqrt[4]{a \cdot b \cdot c \cdot d} \leq \frac{a + b + c + d}{4}$$

We make the below two observations, (1) and (2) which we use during the proof.

$$\begin{array}{rclcl} a \cdot b & ? & \frac{a+b}{2} \cdot \frac{a+b}{2} \\ 4 \cdot a \cdot b & ? & (a+b)^2 \\ 0 & ? & (a-b)^2 \\ 0 & \leq & (a-b)^2 \end{array}$$

Thus we have

$$\frac{a+b}{2} \cdot \frac{a+b}{2} \geq a \cdot b \quad (1)$$

Similarly,

$$\frac{a+b}{2} \cdot \frac{c+d}{2} \quad ? \quad \frac{a+b+c+d}{4} \cdot \frac{a+b+c+d}{4}$$

Let $a+b=x$ and $c+d=y$, and we have

$$\begin{aligned} \frac{x}{2} \cdot \frac{y}{2} & ? \quad \frac{x+y}{4} \cdot \frac{x+y}{4} \\ 4 \cdot x \cdot y & ? \quad (x+y)^2 \\ 0 & ? \quad (x-y)^2 \\ 0 & \leq (x-y)^2 \end{aligned}$$

Thus we have,

$$\frac{a+b+c+d}{4} \cdot \frac{a+b+c+d}{4} \geq \frac{a+b}{2} \cdot \frac{c+d}{2} \quad (2)$$

Now we perform a sequence of transformations on the four numbers (a, b, c, d) . After each transformation, the sum remains $a+b+c+d$ but the product is bigger or equal to $a \cdot b \cdot c \cdot d$.

$$(a, b, c, d) \mapsto \left(\frac{a+b}{2}, \frac{a+b}{2}, c, d \right)$$

In the above transformation, the product increases or remains the same (when $a=b$) because of (1).

$$\left(\frac{a+b}{2}, \frac{a+b}{2}, c, d \right) \mapsto \left(\frac{a+b}{2}, \frac{a+b}{2}, \frac{c+d}{2}, \frac{c+d}{2} \right)$$

In the above transformation, the product increases or remains the same (when $c=d$) because of (1).

$$\left(\frac{a+b}{2}, \frac{a+b}{2}, \frac{c+d}{2}, \frac{c+d}{2} \right) \mapsto \left(\frac{a+b+c+d}{4}, \frac{a+b}{2}, \frac{a+b+c+d}{4}, \frac{c+d}{2} \right)$$

In the above transformation, the product increases or remains the same (when $a+b=c+d$) because of (2).

$$\left(\frac{a+b+c+d}{4}, \frac{a+b}{2}, \frac{a+b+c+d}{4}, \frac{c+d}{2} \right) \mapsto \left(\frac{a+b+c+d}{4}, \frac{a+b+c+d}{4}, \frac{a+b+c+d}{4}, \frac{a+b+c+d}{4} \right)$$

In the above transformation, the product increases or remains the same (when $a+b=c+d$) because of (2). Let $\frac{a+b+c+d}{4} = S$. Then from the last transformation, we have

$$\begin{aligned} a \cdot b \cdot c \cdot d & \leq S \cdot S \cdot S \cdot S \\ \sqrt[4]{a \cdot b \cdot c \cdot d} & \leq \frac{a+b+c+d}{4} \quad \blacksquare \end{aligned}$$

Problem-320

Problem Statement

Prove the inequality between arithmetic and geometric means for $n = 3$.

Solution

We shall reduce the case for $n = 3$ to the case for $n = 4$ and use the result from **Problem-317** to finish it off.

For three non-negative integers a, b, c we are asked to prove

$$\sqrt[3]{a \cdot b \cdot c} \leq \frac{a + b + c}{3}$$

We shall throw in the geometric mean of the three integers and form a group of four non-negative integers: $(a, b, c, \sqrt[3]{a \cdot b \cdot c})$. From **Problem-317** we know

$$\sqrt[4]{abc \sqrt[3]{abc}} \leq \frac{a + b + c + \sqrt[3]{abc}}{4} \quad (1)$$

We note that $\sqrt[4]{abc \sqrt[3]{abc}} = \sqrt[4]{(abc)^1 \cdot (abc)^{\frac{1}{3}}} = \sqrt[4]{(abc)^{\frac{4}{3}}} = \sqrt[3]{abc}$. So, from (1) now we have

$$\begin{aligned} \sqrt[3]{abc} &\leq \frac{a + b + c + \sqrt[3]{abc}}{4} \\ 4\sqrt[3]{abc} &\leq a + b + c + \sqrt[3]{abc} \\ \sqrt[3]{a \cdot b \cdot c} &\leq \frac{a + b + c}{3} \quad \blacksquare \end{aligned}$$

Problem-323

Problem Statement

Prove the inequality between arithmetic and geometric means for all integer $n \geq 2$.

Solution

For $n \geq 2$ non-negative integers a_1, a_2, \dots, a_n we are asked to prove

$$\sqrt[n]{\prod_{k=1}^n a_k} \leq \frac{\sum_{k=1}^n a_k}{n} \quad (1)$$

Proof-1

We can prove (1) for $n = 2^m$ where $m \geq 1$ using the transformation idea from **Problem-317**. Say $n \geq 2$, lies in between 2^p and 2^{p+1} . We already know (1) holds for 2^{p+1} numbers; using that, we can use the idea from **Problem-320** to prove it for $2^{p+1} - 1$ numbers as well. Applying the idea from **Problem-320** in sequence, starting with $2^{p+1} - 1$ numbers and going backwards, we can prove (1) for n .

Proof-2

Let's scale each of the n numbers $\sigma > 0$ times and see what happens to their arithmetic and geometric means. We start off with arithmetic mean:

$$\begin{aligned} & \frac{\sum_{k=1}^n \sigma \cdot a_k}{n} \\ &= \sigma \cdot \frac{\sum_{k=1}^n a_k}{n} \end{aligned}$$

Let's now look at the modified geometric mean:

$$\begin{aligned} & \sqrt[n]{\prod_{k=1}^n \sigma \cdot a_k} \\ &= \sqrt[n]{\sigma^n \prod_{k=1}^n a_k} \\ &= \sigma \sqrt[n]{\prod_{k=1}^n a_k} \end{aligned}$$

We see that both arithmetic and geometric means have been scaled by the same factor σ thus (1) holds for $a'_k = \sigma \cdot a_k$, if it holds for a_k .

Let $\sum_{k=1}^n a_k = \psi$. If all a_k 's are not zero, $\psi > 0$. We can now scale a_k 's to get a'_k 's such that $a'_k = \frac{n}{\psi} \cdot a_k$. Since scaling numbers by the same amount does not change the relation between their arithmetic and geometric means, if we can show (1) for a'_k 's that would be sufficient. Now, observe that $\sum_{k=1}^n a'_k = n$. So, for a'_k 's the inequality (1) takes the below form:

$$\sqrt[n]{\prod_{k=1}^n a'_k} \leq 1$$

We shall now try to prove this derived inequality.

1. For $n = 2$. We thus have $a'_1 + a'_2 = 2$. If both numbers are not equal to 1, we can let $a'_1 = 1 - \delta$ and $a'_2 = 1 + \delta$ with $\delta > 0$. Now the product $a'_1 \cdot a'_2 = (1 - \delta)(1 + \delta) = 1 - \delta^2 \leq 1$. That is what we needed.
2. For $n = 3$. Now we have $a'_1 + a'_2 + a'_3 = 3$. If all numbers are not equal to 1 (if they are, we are done), one should be less than 1 and another should be

greater than 1. Say $a'_1 < 1$ and $a'_2 > 1$. So, $a'_1 - 1 < 0$ and $a'_2 - 1 > 0$.

$$\begin{aligned}
& (a'_1 - 1)(a'_2 - 1) < 0 \\
& a'_1 a'_2 - a'_1 - a'_2 + 1 < 0 \\
& a'_1 a'_2 + 1 < a'_1 + a'_2 \\
& a'_1 a'_2 + 1 + a'_3 < a'_1 + a'_2 + a'_3 \\
& a'_1 a'_2 + 1 + a'_3 < 3 \\
& a'_1 a'_2 + a'_3 < 2
\end{aligned}$$

We have now back to the first case where we have two non-negative numbers $a'_1 a'_2$ and a'_3 and they sum to 2. So, we know how to go about the proof henceforth.

3. For $n = 4$. We have $a'_1 + a'_2 + a'_3 + a'_4 = 4$. With a similar argument to the second case, we arrive at the below inequality:

$$a'_1 a'_2 + a'_3 + a'_4 < 3$$

This takes us back to the second case and we know the rest.

For any $n > 3$, we can use the reducing idea in the second and third cases to go back to our base case, namely the case with $n = 2$ which we have proved already.

Proof-3

Let's prove an identity. We start off with $(a + b + c)^3$:

$$\begin{aligned}
& (a + b + c)^3 \\
&= a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3b^2c + 3bc^2 + 3c^2a + 3ca^2 + 6abc \\
&= a^3 + b^3 + c^3 - 3abc + 3a^2b + 3ab^2 + 3abc + 3abc + 3b^2c + 3bc^2 + 3ca^2 + 3abc + 3c^2a \\
&= a^3 + b^3 + c^3 - 3abc + 3(a + b + c)(ab + bc + ca)
\end{aligned}$$

We can now rearrange the two sides as follows:

$$\begin{aligned}
& a^3 + b^3 + c^3 - 3abc \\
&= (a + b + c)^3 - 3(a + b + c)(ab + bc + ca) \\
&= (a + b + c)[(a + b + c)^2 - 3(ab + bc + ca)] \\
&= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\
&= \frac{1}{2}(a + b + c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca) \\
&= \frac{1}{2}(a + b + c)[(a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)] \\
&= \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]
\end{aligned}$$

We see that for non-negative numbers a, b, c , the right side cannot be negative. So,

$$a^3 + b^3 + c^3 - 3abc \geq 0$$

$$abc \leq \frac{a^3 + b^3 + c^3}{3}$$

Now if we make the replacements: $a = \sqrt[3]{p}$, $b = \sqrt[3]{q}$, $c = \sqrt[3]{r}$, then we have our required inequality between geometric and arithmetic means of three non-negative numbers:

$$\sqrt[3]{p \cdot q \cdot r} \leq \frac{p + q + r}{3}$$

If we have more than 3 numbers for which we need to establish (1), we can always group (and replace the group with the group-sum) some of the numbers so that we end up with three numbers and appeal to our just-established fact.

Problem-326

Problem Statement

Assume that a_1, \dots, a_n are positive numbers. Prove that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n$$

Solution

Consider the below product:

$$\frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \dots \cdot \frac{a_{n-1}}{a_n} \cdot \frac{a_n}{a_1}$$

Each a_i with $1 \leq i \leq n$ appears exactly once above and below, so the product evaluates to 1. So, we have:

$$\sqrt[n]{\frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \dots \cdot \frac{a_{n-1}}{a_n} \cdot \frac{a_n}{a_1}} = 1$$

Thus the geometric mean of the positive numbers $\frac{a_1}{a_2}, \frac{a_2}{a_3}, \dots, \frac{a_{n-1}}{a_n}, \frac{a_n}{a_1}$ is 1. Their arithmetic mean cannot be less than 1. We thus have:

$$\frac{\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}}{n} \geq 1$$

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n \quad \blacksquare$$