Problem-42

Problem Statement

Fractions $\frac{a}{b}$ and $\frac{c}{d}$ are called neighbor fractions if their difference $\frac{ad-bc}{bd}$ has numerator ± 1, that is, $ad - bc = \pm 1$. Prove that

a. in this case neither fraction can be simplified (that is, neither has any common factors in numerator and denominator);

b. if $\frac{a}{b}$ and $\frac{c}{d}$ are neighbor fractions, then $\frac{a+c}{b+d}$ is between them and is a neighbor fraction for both $\frac{a}{b}$ and $\frac{c}{d}$; moreover,

c. no fraction $\frac{e}{f}$ with positive integer e and f such that $f < b + d$ is between $\frac{a}{b}$ and $\frac{c}{d}$.

   (In part b. of the statement the book says $\frac{a+b}{c+d}$ instead of $\frac{a+c}{b+d}$ and looks like that is a typo. For, neighbor fractions $\frac{1}{3}$ and $\frac{1}{2}$ the composite fraction $\frac{1+3}{1+2}$ is not in-between.)

Solution to Part a.

I shall assume that $a < b, c < d$ and if a common factor exists, it is greater than 1. In other words, if the only common factor between say a and b is 1, I consider them not having a common factor at all.

We shall use **proof by contradiction**, therefore we shall assume that two neighbor fractions share a common factor and that assumption will lead to a false conclusion, completing the proof.

Say $a < b$, a and b share a common factor f, and f is greater than 1. Since the common factor is greater than 1, $a > 1$. So, we have $b = f \cdot a$.

From the definition of neighbor fraction given in the problem statement, we have:

$$\begin{aligned}
ad - bc &= \pm 1 \\
ad - (f \cdot a)c &= \pm 1 \\
a(d - fc) &= \pm 1 \\
d - fc &= \pm \frac{1}{a}
\end{aligned}$$

Since we started with integers on both sides of the equation and ended up with an integer on the left and a proper fraction on the right, we have a contradiction. Similar idea works if we assumed $a > b$ or if we assumed c and d also share a common factor. ■

Solution to Part b.

Say $\frac{a}{b} > \frac{c}{d}$. Let's now interpret the neighbor fractions $\frac{a}{b}$ and $\frac{c}{d}$ as follows:

We have two teams of people. The first team has b people in it and they have a apples in total, therefore equally sharing, each person in the first team gets $\frac{a}{b}$ apples. The second team has d people in it and they have c apples in total, therefore equally sharing, each person in the second team gets $\frac{c}{d}$ apples. If the two teams come together and share their apples between them, then each person in this bigger, combined team gets $\frac{a+c}{b+d}$ apples. It then makes sense that each person in this bigger, combined team would get no less than the smaller of the two ratios $\frac{c}{d}$ and would get no more than the larger of the two ratios $\frac{a}{b}$. Hence, $\frac{a+c}{b+d}$ should be in-between $\frac{c}{d}$ and $\frac{a}{b}$. More interesting is the claim that $\frac{a+c}{b+d}$ is neighbor fraction for each of the two original neighbor fractions.

Proof for $\frac{a+c}{b+d}$ is between $\frac{c}{d}$ and $\frac{a}{b}$

Again, let's assume $\frac{c}{d} < \frac{a}{b}$, therefore from the statement of the problem we have $ad - bc = 1$. The argument that follows can easily be modified so it would hold even if the inequality is reversed. We shall make two comparisons: first between $\frac{a+c}{b+d}$ and $\frac{c}{d}$; and then between $\frac{a+c}{b+d}$ and $\frac{a}{b}$. The result of these two comparisons will give us the order among the three fractions showing that $\frac{a+c}{b+d}$ is between $\frac{c}{d}$ and $\frac{a}{b}$.

Let's compare $\frac{c}{d}$ and $\frac{a+c}{b+d}$:

$$\begin{aligned} \frac{c}{d} \; &? \; \frac{a+c}{b+d} \\ c(b+d) \; &? \; d(a+c) \\ bc+cd \; &? \; ad+cd \\ bc \; &? \; ad \end{aligned}$$

Since $ad - bc = 1$, we know $bc < ad$, therefore $\frac{c}{d} < \frac{a+c}{b+d}$.

Let's now compare $\frac{a+c}{b+d}$ and $\frac{a}{b}$:

$$\begin{aligned} \frac{a+c}{b+d} \; &? \; \frac{a}{b} \\ b(a+c) \; &? \; a(b+d) \\ ab+bc \; &? \; ab+ad \\ bc \; &? \; ad \end{aligned}$$

Since $ad - bc = 1$, we know $bc < ad$, therefore $\frac{a+c}{b+d} < \frac{a}{b}$.

Results of the two comparisons combined:

$$\frac{c}{d} < \frac{a+c}{b+d} < \frac{a}{b} \quad \blacksquare$$

Proof for $\frac{a+c}{b+d}$ is a neighbor fraction both for $\frac{a}{b}$ and $\frac{c}{d}$

Let's assume $\frac{c}{d} < \frac{a}{b}$, thus $ad - bc = 1$. From the previous result, we then know $\frac{c}{d} < \frac{a+c}{b+d} < \frac{a}{b}$. If we can show that the numerator is 1 for both of the differences, $\frac{a+c}{b+d} - \frac{c}{d}$ and $\frac{a}{b} - \frac{a+c}{b+d}$, we are done.

For the first difference, the numerator is:

$$\begin{aligned}(a+c)\cdot d - c\cdot(b+d) &= ad + cd - bc - cd\\ &= ad - bc\\ &= 1\end{aligned}$$

For the second difference, the numerator is:

$$\begin{aligned}a\cdot(b+d) - b\cdot(a+c) &= ab + ad - ab - bc\\ &= ad - bc\\ &= 1 \quad \blacksquare\end{aligned}$$

Solution to Part c.

Once again we assume that $\frac{c}{d} < \frac{a}{b}$, thus $ad - bc = 1$. Say $\frac{e}{f}$ is a fraction that lies between $\frac{c}{d}$ and $\frac{a}{b}$, therefore $\frac{c}{d} < \frac{e}{f} < \frac{a}{b}$. We thus have two inequalities: $af > be$ and $de > cf$. Since a, b, c, d, e, f are all positive integers, we could rewrite the two inequalities as: $af - be \geq 1$ and $de - cf \geq 1$.

$$\begin{aligned}\frac{a}{b} - \frac{c}{d} &= \left(\frac{a}{b} - \frac{e}{f}\right) + \left(\frac{e}{f} - \frac{c}{d}\right)\\ \frac{ad-bc}{bd} &= \frac{af-be}{bf} + \frac{de-cf}{df}\\ \frac{1}{bd} &\geq \frac{1}{bf} + \frac{1}{df}\\ \frac{1}{bd} &\geq \frac{b+d}{bdf}\\ f &\geq b+d\end{aligned}$$

So, for any fraction $\frac{e}{f}$ that lies between the pair of neighbor fractions $\frac{c}{d}$ and $\frac{a}{b}$, f cannot be less than $b+d$. $\blacksquare$

In some sense $b+d$ is a hard limit on how low the denominator of such a fraction can get, signifying the closeness between the neighbor fractions themselves. In fact if we use the lowest value of f therefore $f = b+d$ in the above inequality and go backwards we get $af - be = 1$ and $de - cf = 1$. We can then solve for e as follows:

$$\begin{aligned}af - be &= de - cf\\ (b+d)\cdot e &= (a+c)\cdot f\\ e &= a+c\end{aligned}$$

Seems like $\frac{a+c}{b+d}$ is the unique in-between fraction having the lowest possible denominator.