Why $-3 \times -5 = 15$?

0.1 Assuming negative numbers follow certain basic rules

To show $-3 \times -5 = 15$, we need the following assumptions:

1. Distributive property holds even for negative integers.
2. For integers a and b, if $a + b = 0$, then for a given a there is exactly one choice for b. For example, if $4 + x = 0$ then x can only be -4.
3. $x \times 0 = 0$ holds even when x is a negative integer.

We first need to prove $-3 \times 5 = -15$. Consider the below:

$$\begin{aligned} -3 \times 5 + 15 &= -3 \times 5 + 3 \times 5 \\ &= (-3 + 3) \times 5 \quad \text{/* Using first assumption */} \\ &= 0 \times 5 \\ &= 0 \end{aligned}$$

So, -3×5 is a number when added to 15 gives 0. From the second assumption we can thus say $-3 \times 5 = -15$.

Now consider the below:

$$\begin{aligned} 0 &= -3 \times 0 \quad \text{/* Using third assumption */} \\ &= -3 \times (-5 + 5) \\ &= (-3 \times -5) + (-3 \times 5) \quad \text{/* Using first assumption */} \\ &= (-3 \times -5) + (-15) \end{aligned}$$

So, -3×-5 is a number when added to -15 gives 0. From the second assumption, we can thus say $-3 \times -5 = 15$. ■

0.2 From pattern of series

Consider the below series:

$$\ldots \ldots 1, 2, 3, 4, 5, \ldots \ldots \tag{1}$$

Lets multiply the series by 3. Between each consecutive pair, the left one is 3 less than the right one.

$$\ldots \ldots 3, 6, 9, 12, 15, \ldots \ldots \tag{2}$$

Lets reveal the second series more to the left.

$$\ldots \ldots -15, -12, -9, -6, -3, 0, 3, 6, 9, 12, 15, \ldots \ldots \tag{3}$$

We can now show a correspondence between the first and the third series which essentially is a correspondence between x and $3x$.

$$\begin{array}{cccccccccccccc} \ldots & \ldots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \ldots & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \ldots & \ldots & -15 & -12 & -9 & -6 & -3 & 0 & 3 & 6 & 9 & 12 & 15 & \ldots & \ldots \end{array}$$

We see that 3×-5 corresponds to -15, thus we can say $3 \times -5 = -15$.

Now we do the same routine but this time we multiply the first series by -3 and get the fourth series below. Between every pair of consecutive integers, the left one is 3 greater than the right one.

$$\ldots \ldots -3,\ -6,\ -9,\ -12,\ -15,\ \ldots \ldots \tag{4}$$

If we reveal the above series more towards left, we find below:

$$\ldots \ldots 15,\ 12,\ 9,\ 6,\ 3,\ 0,\ -3,\ -6,\ -9,\ -12,\ -15\ \ldots \ldots \tag{5}$$

Now we establish a correspondence between the first and the fifth series below, which actually is the correspondence between x and $-3x$.

$$\begin{array}{cccccccccccccc} \ldots & \ldots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \ldots & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \ldots & \ldots & 15 & 12 & 9 & 6 & 3 & 0 & -3 & -6 & -9 & -12 & -15 & \ldots & \ldots \end{array}$$

Above, -5 corresponds to 15, so we can say $-3 \times -5 = 15$. ■