Problem-90

Problem Statement

(a) Multiply $(1+x)\ \left(1+x^2\right)$.

(b) Multiply $(1+x)\ \left(1+x^2\right)\ \left(1+x^4\right)\ \left(1+x^8\right)$.

(c) Compute $\left(1+x+x^2+x^3\right)^2$.

(d) Compute $\left(1+x+x^2+x^3+\ \dots\ +x^9+x^{10}\right)^2$.

(e) Find the coefficient of x^{29} and x^{30} in $\left(1+x+x^2+x^3+\ \dots\ +x^9+x^{10}\right)^3$.

(f) Multiply $(1-x)\ \left(1+x+x^2+x^3+\ \dots\ +x^9+x^{10}\right)$.

(g) Multiply $(a+b)\ \left(a^2-ab+b^2\right)$.

(h) Multiply $\left(1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10}\right)$ by $\left(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}\right)$.

Solution to Part-a

$$
\begin{array}{lr}
 & x+1 \\
\times & x^2+1 \\
\hline
 & x+1 \\
x^3+x^2 & \\
\hline
x^3+x^2+x+1 &
\end{array}
$$

Solution to Part-b

From part-a we have $(1+x)\ \left(1+x^2\right)=1+x+x^2+x^3$.

$$
\begin{array}{lr}
 & x^3+x^2+x+1 \\
\times & x^4+1 \\
\hline
 & x^3+x^2+x+1 \\
x^7+x^6+x^5+x^4 & \\
\hline
x^7+x^6+x^5+x^4+x^3+x^2+x+1 &
\end{array}
$$

Similar to the above, $\left(x^7+x^6+x^5+x^4+x^3+x^2+x+1\right)\ \left(1+x^8\right)=\sum_{n=0}^{15} x^n$.

Solution to Part-c

$$
\begin{array}{r}
x^3+x^2+x+1 \\
\times \quad x^3+x^2+x+1 \\
\hline
x^3+x^2+x+1 \\
x^4+x^3+x^2+x \\
x^5+x^4+x^3+x^2 \\
x^6+x^5+x^4+x^3 \\
\hline
x^6+2x^5+3x^4+4x^3+3x^2+2x+1
\end{array}
$$

In regards to how many times they occur, we note that left and right of x^3 are mirror reflections. For example, there are as many x^5 as there are x. We may write the product as

$$\sum_{n=0}^{2}(n+1)\cdot x^n \ + \ 4\cdot x^3 \ + \ \sum_{n=4}^{6}(6-n+1)\cdot x^n$$

Solution to Part-d

From the observation in part-c, the product here should be

$$\sum_{n=0}^{9}(n+1)\cdot x^n \ + \ 11\cdot x^{10} \ + \ \sum_{n=11}^{20}(20-n+1)\cdot x^n$$

In other words the product is

$$x^{20}+2x^{19}+3x^{18}+\ \dots\ +10x^{11}+11x^{10}+10x^9+9x^8+\ \dots\ +3x^2+2x+1$$