Problem-42

Problem Statement

Fractions $\frac{a}{b}$ and $\frac{c}{d}$ are called neighbor fractions if their difference $\frac{ad-bc}{bd}$ has numerator ± 1, that is, $ad - bc = \pm 1$. Prove that

a. in this case neither fraction can be simplified (that is, neither has any common factors in numerator and denominator);

b. if $\frac{a}{b}$ and $\frac{c}{d}$ are neighbor fractions, then $\frac{a+c}{b+d}$ is between them and is a neighbor fraction for both $\frac{a}{b}$ and $\frac{c}{d}$; moreover,

c. no fraction $\frac{e}{f}$ with positive integer e and f such that $f < b + d$ is between $\frac{a}{b}$ and $\frac{c}{d}$.

   (In part b. of the statement the book says $\frac{a+b}{c+d}$ instead of $\frac{a+c}{b+d}$ and looks like that is a typo. For, neighbor fractions $\frac{1}{3}$ and $\frac{1}{2}$ the composite fraction $\frac{1+3}{1+2}$ is not in-between.)

Solution to Part a.

We assume that $a < b, c < d$ and if a common factor exists, it is greater than 1. In other words, if the only common factor between say a and b is 1, we consider them not having a common factor at all.

We shall use **proof by contradiction**, therefore we shall assume that two neighbor fractions share a common factor and that assumption will lead to a false conclusion, completing the proof.

Say $a < b$, a and b share a common factor f, and f is greater than 1. Since the common factor is greater than 1, $a > 1$. So, we have $b = f \cdot a$.

From the definition of neighbor fraction given in the problem statement, we have:

$$\begin{aligned} ad - bc &= \pm 1 \\ ad - (f \cdot a)c &= \pm 1 \\ a(d - fc) &= \pm 1 \\ d - fc &= \pm \frac{1}{a} \end{aligned}$$

Since we started with integers on both sides of the equation and ended up with an integer on the left and a proper fraction on the right, we have a contradiction. Similar idea works if we assumed $a > b$ or if we assumed c and d also share a common factor. ■

Solution to Part b.

Say $\frac{a}{b} > \frac{c}{d}$. Let's now interpret the neighbor fractions $\frac{a}{b}$ and $\frac{c}{d}$ as follows:

We have two teams of people. The first team has b people in it and they have a apples in total, therefore equally sharing, each person in the first team gets $\frac{a}{b}$ apples. The second team has d people in it and they have c apples in total, therefore equally sharing, each person in the second team gets $\frac{c}{d}$ apples. If the two teams come together and share their apples between them, then each person in this bigger, combined team gets $\frac{a+c}{b+d}$ apples. It then makes sense that each person in this bigger, combined team would get no less than the smaller of the two ratios $\frac{c}{d}$ and would get no more than the larger of the two ratios $\frac{a}{b}$. Hence, $\frac{a+c}{b+d}$ should be in-between $\frac{c}{d}$ and $\frac{a}{b}$. More interesting is the claim that $\frac{a+c}{b+d}$ is neighbor fraction for each of the two original neighbor fractions.

Proof for $\frac{a+c}{b+d}$ is between $\frac{c}{d}$ and $\frac{a}{b}$

Again, let's assume $\frac{c}{d} < \frac{a}{b}$, therefore from the statement of the problem we have $ad - bc = 1$. The argument that follows can easily be modified so it would hold even if the inequality is reversed. We shall make two comparisons: first between $\frac{a+c}{b+d}$ and $\frac{c}{d}$; and then between $\frac{a+c}{b+d}$ and $\frac{a}{b}$. The result of these two comparisons will give us the order among the three fractions showing that $\frac{a+c}{b+d}$ is between $\frac{c}{d}$ and $\frac{a}{b}$.

Let's compare $\frac{c}{d}$ and $\frac{a+c}{b+d}$:

$$\begin{aligned} \frac{c}{d} &\ ?\ \frac{a+c}{b+d} \\ c(b+d) &\ ?\ d(a+c) \\ bc+cd &\ ?\ ad+cd \\ bc &\ ?\ ad \end{aligned}$$

Since $ad - bc = 1$, we know $bc < ad$, therefore $\frac{c}{d} < \frac{a+c}{b+d}$.

Let's now compare $\frac{a+c}{b+d}$ and $\frac{a}{b}$:

$$\begin{aligned} \frac{a+c}{b+d} &\ ?\ \frac{a}{b} \\ b(a+c) &\ ?\ a(b+d) \\ ab+bc &\ ?\ ab+ad \\ bc &\ ?\ ad \end{aligned}$$

Since $ad - bc = 1$, we know $bc < ad$, therefore $\frac{a+c}{b+d} < \frac{a}{b}$.

Results of the two comparisons combined:

$$\frac{c}{d} < \frac{a+c}{b+d} < \frac{a}{b} \quad \blacksquare$$

Proof for $\frac{a+c}{b+d}$ is a neighbor fraction both for $\frac{a}{b}$ and $\frac{c}{d}$

Let's assume $\frac{c}{d} < \frac{a}{b}$, thus $ad - bc = 1$. From the previous result, we then know $\frac{c}{d} < \frac{a+c}{b+d} < \frac{a}{b}$. If we can show that the numerator is 1 for both of the differences, $\frac{a+c}{b+d} - \frac{c}{d}$ and $\frac{a}{b} - \frac{a+c}{b+d}$, we are done.

For the first difference, the numerator is:

$$\begin{aligned}(a+c)\cdot d - c\cdot(b+d) &= ad + cd - bc - cd \\ &= ad - bc \\ &= 1\end{aligned}$$

For the second difference, the numerator is:

$$\begin{aligned}a\cdot(b+d) - b\cdot(a+c) &= ab + ad - ab - bc \\ &= ad - bc \\ &= 1 \quad \blacksquare\end{aligned}$$

Solution to Part c.

Once again we assume that $\frac{c}{d} < \frac{a}{b}$, thus $ad - bc = 1$. Say $\frac{e}{f}$ is a fraction that lies between $\frac{c}{d}$ and $\frac{a}{b}$, therefore $\frac{c}{d} < \frac{e}{f} < \frac{a}{b}$. We thus have two inequalities: $af > be$ and $de > cf$. Since a, b, c, d, e, f are all positive integers, we could rewrite the two inequalities as: $af - be \geq 1$ and $de - cf \geq 1$.

$$\begin{aligned}\frac{a}{b} - \frac{c}{d} &= \left(\frac{a}{b} - \frac{e}{f}\right) + \left(\frac{e}{f} - \frac{c}{d}\right) \\ \frac{ad - bc}{bd} &= \frac{af - be}{bf} + \frac{de - cf}{df} \\ \frac{1}{bd} &\geq \frac{1}{bf} + \frac{1}{df} \\ \frac{1}{bd} &\geq \frac{b+d}{bdf} \\ f &\geq b + d\end{aligned}$$

So, for any fraction $\frac{e}{f}$ that lies between the pair of neighbor fractions $\frac{c}{d}$ and $\frac{a}{b}$, f cannot be less than $b + d$. $\blacksquare$

In some sense $b + d$ is a hard limit on how low the denominator of such a fraction can get, signifying the closeness between the neighbor fractions themselves. In fact if we use the lowest value of f therefore $f = b + d$ in the above inequality and go backwards we get $af - be = 1$ and $de - cf = 1$. We can then solve for e as follows:

$$\begin{aligned}af - be &= de - cf \\ (b+d)\cdot e &= (a+c)\cdot f \\ e &= a + c\end{aligned}$$

Seems like $\frac{a+c}{b+d}$ is the unique in-between fraction having the lowest possible denominator.

Problem-43

Problem Statement

A stick is divided by red marks into 7 equal segments and by green marks into 13 equal segments. Then it is cut into 20 equal pieces. Prove that any piece (except the two end pieces) contain exactly one mark (which may be red or green).

0.0.1 Proof for two end pieces do not contain any mark

Say we are putting the marks from left to right. There would be 6 red marks dividing the stick into 7 equal segments. The 6 red marks come at distances $\frac{1}{7}, \frac{2}{7}, \ldots, \frac{5}{7}, \frac{6}{7}$ from left to right. Similarly there would be 12 green marks dividing the stick into 13 equal segments. The 12 green marks come at distances $\frac{1}{13}, \frac{2}{13}, \ldots, \frac{11}{13}, \frac{12}{13}$ from left to right. Say we cut the stick from left to right at 19 points and get 20 equal pieces. The cuts happen at distances $\frac{1}{20}, \frac{2}{20}, \ldots, \frac{18}{20}, \frac{19}{20}$ from left to right. At the left end we find $\frac{1}{20} < \frac{1}{13} < \frac{1}{7}$. So the left-end piece does not contain any mark. At the right end we find $\frac{6}{7} < \frac{12}{13} < \frac{19}{20}$. So, the right end-piece does not contain any mark either. ■

0.0.2 Proof for none of the remaining 18 pieces contain more than one mark

Looking at the fractions $\frac{1}{13}$, $\frac{1}{20}$, and $\frac{1}{7}$ we see that $7+13=20$ looks similar to $b+d$ for the three fractions $\frac{c}{d}$, $\frac{a+c}{b+d}$, and $\frac{a}{b}$ from problem-42. So, there is an opportunity to use the property that $\frac{a+c}{b+d}$ comes in between $\frac{c}{d}$ and $\frac{a}{b}$. We also observe that for any two fractions $\frac{c}{d} < \frac{a}{b}$ with $a < b,\ c < d$, we have $\frac{c}{d} < \frac{a+c}{b+d} < \frac{a}{b}$; because, $c \cdot (b+d) < d \cdot (a+c)$ and $(a+c) \cdot b < (b+d) \cdot a$.

From the above obervations, we conclude that between any pair of red and green marks we always have a cut in-between; because, $\frac{c}{13} < \frac{a+c}{20} < \frac{a}{7}$. Similarly, between any pair of green and red marks we always have a cut in-between; because, $\frac{c'}{7} < \frac{a'+c'}{20} < \frac{a'}{13}$. Also, between two red marks or two green marks there always is an in-between cut; because the distance between these pairs are larger than $\frac{1}{20}$. ■

Note, since the maximum value of a is 6 and the maximum value of c is 12, we have $a+c < 20$.

0.0.3 Proof for each of the remaining 18 pieces has at least one mark

We need to put 6 red and 12 green, total 18 marks. Since $\frac{1}{20} < \frac{1}{13} < 2 \cdot \frac{1}{20} < \frac{1}{7}$, the first red mark comes on the second piece from the left. If we now diregard the first two pieces and the last piece from the right-end, we are left with $20-3=17$ pieces to house the remaining $18-1=17$ marks. If any of these 17 pieces is devoid of mark, then we would have 16 or less pieces to house the remaining 17 marks. By **Pigeonhole Principle**, at least one of these pieces would then contain more than one mark, which in 0.0.2 we showed cannot happen. So, there isn't a piece that is devoid of a mark. We have the following two:

1. Every piece except the two end-pieces has $\leq$ 1 mark. (From 0.0.2)
2. Every piece except the two end-pieces has $\geq$ 1 mark. (From 0.0.3)

So, every piece except the two end-pieces has exactly 1 mark. ■