Problem-225

Problem Statement

If in the equation

$$ax^2 + bx + c = 0$$

the coefficient a is equal to zero then the equation takes the form

$$bx + c = 0$$

and has the unique solution

$$x = -\frac{c}{b}$$

Strictly speaking, the last sentence is wrong; when $b = 0$ the quotient $\frac{c}{b}$ is undefined. How are we to correct this error?

Solution

We should require that in the equation $ax^2 + bx + c = 0$, both a and b cannot be zero.

Problem-227

Problem Statement

Prove that $\sqrt{3}$ is irrational.

Solution

The statement '$\sqrt{3}$ is irrational' is equivalent to the statement: $\sqrt{3} \neq \frac{a}{b}$ for any integer a and b. We shall make use of **proof by contradiction**, therefore, we shall assume that $\sqrt{3} = \frac{a}{b}$ and that should lead to a contradiction, completing the proof.

We shall consider four different cases for the pair (a, b) and we shall derive a contradiction for each.

1. a and b both are odd integer. Say $a = 2m + 1$ and $b = 2n + 1$, where m and n are integers. According to our assumption

$$\begin{aligned}
\sqrt{3} &= \frac{2m+1}{2n+1} \\
3 &= \frac{(2m+1)^2}{(2n+1)^2} \\
3\,(2n+1)^2 &= (2m+1)^2 \\
3\,(4n^2+4n+1) &= 4m^2+4m+1 \\
12n^2+12n+3 &= 4m^2+4m+1 \\
12n^2+12n+2 &= 4m^2+4m \\
6n^2+6n+1 &= 2m^2+2m
\end{aligned}$$

   On the left side we have an odd integer and on the right side we have an even integer—a contradiction. ※

2. a is an odd integer and b is an even integer. Say $a = 2m + 1$ and $b = 2n$, where m and n are integers. According to our assumption

$$\begin{aligned}
\sqrt{3} &= \frac{2m+1}{2n} \\
3 &= \frac{(2m+1)^2}{4n^2} \\
12n^2 &= (2m+1)^2
\end{aligned}$$

   On the left side we have an even integer and on the right side we have an odd integer—a contradiction. ※

3. a is an even integer and b is an odd integer. Say $a = 2m$ and $b = 2n + 1$, where m and n are integers. According to our assumption

$$\begin{aligned}
\sqrt{3} &= \frac{2m}{2n+1} \\
3 &= \frac{4m^2}{(2n+1)^2} \\
3\,(2n+1)^2 &= 4m^2
\end{aligned}$$

   On the left side we have an odd integer and on the right side we have an even integer—a contradiction. ※

4. a and b both are even integer. In this case, a and b must have 2 as a common factor. We can divide both a and b by 2. We can keep dividing by 2 as long as both remain even integer. At the end, we are in one of the previous three cases.

Problem-229

Problem Statement

Prove that for $a \geq 0, \; b > 0$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Solution

Basically we are being asked to show that $\frac{\sqrt{a}}{\sqrt{b}}$ is the square root of the non-negative integer $\frac{a}{b}$. So, we need to prove the below two:

- $\frac{\sqrt{a}}{\sqrt{b}}$ is non-negative.

  Since a is non-negative, its square root must also be non-negative. Since b is positive, its square root must also be positive. $\frac{\sqrt{a}}{\sqrt{b}}$ is thus the ratio of a non-negative number and a positive number; so, it must be non-negative.

- Squaring $\frac{\sqrt{a}}{\sqrt{b}}$ gives us $\frac{a}{b}$.

$$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b}.$$

Problem-230

Problem Statement

Is the equality $\sqrt{a^2} = a$ true for all a?

Solution

It might seem counter-intuitive that the answer is no. But if we observe closely, the question is actually asking about the equality $+\sqrt{a^2} = a$, where the '+' is implied. When we make the '+' explicit, it is more noticeable that for $a < 0$, the equality cannot hold, because the left side of the equality is always positive. The equality that works for all a is $\sqrt{a^2} = |a|$, where

$$|a| = \begin{cases} a & \text{, if } a \geq 0 \\ -a & \text{, if } a < 0 \end{cases}$$

It is unfortunate that we do not always make the '+' explicit.