Problem-225

Problem Statement

If in the equation
$$ax^2 + bx + c = 0$$
the coefficient a is equal to zero then the equation takes the form
$$bx + c = 0$$
and has the unique solution
$$x = -\frac{c}{b}$$

Strictly speaking, the last sentence is wrong; when $b = 0$ the quotient $\frac{c}{b}$ is undefined. How are we to correct this error?

Solution

We should require that in the equation $ax^2 + bx + c = 0$, both a and b cannot be zero.

Problem-227

Problem Statement

Prove that $\sqrt{3}$ is irrational.

Solution

The statement '$\sqrt{3}$ is irrational' is equivalent to the statement: $\sqrt{3} \neq \frac{a}{b}$ for any integer a and b. We shall make use of **proof by contradiction**, therefore, we shall assume that $\sqrt{3} = \frac{a}{b}$ and that should lead to a contradiction, completing the proof.

We shall consider four different cases for the pair (a, b) and we shall derive a contradiction for each.

1. a and b both are odd integer. Say $a = 2m + 1$ and $b = 2n + 1$, where m and n are integers. According to our assumption

$$\begin{aligned}
\sqrt{3} &= \frac{2m+1}{2n+1} \\
3 &= \frac{(2m+1)^2}{(2n+1)^2} \\
3\,(2n+1)^2 &= (2m+1)^2 \\
3\,(4n^2+4n+1) &= 4m^2+4m+1 \\
12n^2+12n+3 &= 4m^2+4m+1 \\
12n^2+12n+2 &= 4m^2+4m \\
6n^2+6n+1 &= 2m^2+2m
\end{aligned}$$

   On the left side we have an odd integer and on the right side we have an even integer—a contradiction. ※

2. a is an odd integer and b is an even integer. Say $a = 2m + 1$ and $b = 2n$, where m and n are integers. According to our assumption

$$\begin{aligned}
\sqrt{3} &= \frac{2m+1}{2n} \\
3 &= \frac{(2m+1)^2}{4n^2} \\
12n^2 &= (2m+1)^2
\end{aligned}$$

   On the left side we have an even integer and on the right side we have an odd integer—a contradiction. ※

3. a is an even integer and b is an odd integer. Say $a = 2m$ and $b = 2n + 1$, where m and n are integers. According to our assumption

$$\begin{aligned}
\sqrt{3} &= \frac{2m}{2n+1} \\
3 &= \frac{4m^2}{(2n+1)^2} \\
3\,(2n+1)^2 &= 4m^2
\end{aligned}$$

   On the left side we have an odd integer and on the right side we have an even integer—a contradiction. ※

4. a and b both are even integer. In this case, a and b must have 2 as a common factor. We can divide both a and b by 2. We can keep dividing by 2 as long as both remain even integer. At the end, we are in one of the previous three cases.

Problem-229

Problem Statement

Prove that for $a \geq 0,\ b > 0$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Solution

Basically we are being asked to show that $\frac{\sqrt{a}}{\sqrt{b}}$ is the square root of the non-negative integer $\frac{a}{b}$. So, we need to prove the below two:

- $\frac{\sqrt{a}}{\sqrt{b}}$ is non-negative.

  Since a is non-negative, its square root must also be non-negative. Since b is positive, its square root must also be positive. $\frac{\sqrt{a}}{\sqrt{b}}$ is thus the ratio of a non-negative number and a positive number; so, it must be non-negative.

- Squaring $\frac{\sqrt{a}}{\sqrt{b}}$ gives us $\frac{a}{b}$.

$$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b}.$$

Problem-230

Problem Statement

Is the equality $\sqrt{a^2} = a$ true for all a?

Solution

It might seem counter-intuitive that the answer is no. But if we observe closely, the question is actually asking about the equality $+\sqrt{a^2} = a$, where the '+' is implied. When we make the '+' explicit, it is more noticeable that for $a < 0$, the equality cannot hold, because the left side of the equality is always positive. The equality that works for all a is $\sqrt{a^2} = |a|$, where

$$|a| = \begin{cases} a & \text{, if } a \geq 0 \\ -a & \text{, if } a < 0 \end{cases}$$

It is unfortunate that we do not always make the '+' explicit.

Problem-232

Problem Statement

Which is bigger: $\sqrt{1001} - \sqrt{1000}$ or $\frac{1}{10}$?

Solution

It would be easier to compare if we could convert $\sqrt{1001} - \sqrt{1000}$ into a fraction.

$$\begin{aligned}
&\sqrt{1001} - \sqrt{1000} \\
&= \frac{\left(\sqrt{1001} - \sqrt{1000}\right)\left(\sqrt{1001} + \sqrt{1000}\right)}{\sqrt{1001} + \sqrt{1000}} \\
&= \frac{\left(\sqrt{1001}\right)^2 - \left(\sqrt{1000}\right)^2}{\sqrt{1001} + \sqrt{1000}} \\
&= \frac{1001 - 1000}{\sqrt{1001} + \sqrt{1000}} \\
&= \frac{1}{\sqrt{1001} + \sqrt{1000}} \\
&< \frac{1}{\sqrt{900} + \sqrt{900}}
\end{aligned}$$

Since $\sqrt{900} + \sqrt{900} = 60$, $\sqrt{1001} - \sqrt{1000} < \frac{1}{60} < \frac{1}{10}$. So, $\frac{1}{10}$ is the bigger between the two.

Problem-239

Problem Statement

0.0.1 Solutions to quadratic equation

A quadratic equation $x^2+px+q=0$ can be solved by “completing the square”.

$$x^2+2\cdot x\cdot\frac{p}{2}+\left(\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)=0$$
$$\left(x+\frac{p}{2}\right)^2=\frac{p^2}{4}-q$$
$$x=-\frac{p}{2}+\frac{\pm\sqrt{p^2-4q}}{2}$$

There are three possible cases:

1. $p^2-4q>0$. In this case two distinct solutions $x_1,\ x_2$ exist:
   $x_1=-\frac{p}{2}+\frac{\sqrt{p^2-4q}}{2}$ and $x_2=-\frac{p}{2}-\frac{\sqrt{p^2-4q}}{2}$
2. $p^2-4q=0$. In this case the two solutions coincide; therefore, there is one distinct solution.
   $x_1=-\frac{p}{2}$ and $x_2=-\frac{p}{2}$
3. $p^2-4q<0$. In this case there is no real-valued solution.

Since the sign of p^2-4q determines how many distinct solutions there are, for convenience, we shall call this expression D, therefore, $D=p^2-4q$.

0.0.2 Vieta’s theorem

If a quadratic equation $x^2+px+q=0$, has two distinct solutions α and β then

$$\alpha+\beta=-p$$
$$\alpha\cdot\beta=q$$

0.0.3 Proof-1 of Vieta's theorem

If $x^2 + px + q = 0$ has two distinct solutions, we are in the first case of 0.0.1; thus we can let α and β be as follows:

$$\alpha = -\frac{p}{2} + \frac{\sqrt{D}}{2} \tag{1}$$

$$\beta = -\frac{p}{2} - \frac{\sqrt{D}}{2} \tag{2}$$

$(1)+(2)$ gives:

$$\alpha + \beta = -p$$

$(1)\times(2)$ gives:

$$\begin{aligned}\alpha \cdot \beta &= \frac{p^2}{4} - \frac{D}{4} \\ &= \frac{p^2 - \left(p^2 - 4q\right)}{4} \\ &= q\end{aligned}$$

0.0.4 Proof-2 of Vieta's theorem

We can rewrite $x^2 + px + q = 0$ as $P(x) = 0$ where $P(x) = x^2 + px + q$ is a polynomial with degree two. If α and β are two distinct solutions of $P(x) = 0$ then

$$P(x) = (x-\alpha)\,(x-\beta)\,R(x)$$

Here $R(x)$ must be constant, otherwise right side would have degree more than two. In fact $R(x)$ must be exactly 1, otherwise the coefficient of x^2 would not match with the one in $P(x)$. So, we have

$$\begin{aligned}P(x) &= (x-\alpha)\,(x-\beta) \\ &= x^2 - (\alpha+\beta)\,x + \alpha\cdot\beta\end{aligned}$$

Comparing $P(x) = x^2 + (\ -(\alpha+\beta)\ )\ x + \alpha\cdot\beta$ with $P(x) = x^2 + px + q$ gives us Vieta's theorem.

0.0.5 Questions to answer

Can you generalize Vieta's theorem to the case of a quadratic equation having only one root? Are both proofs still valid for this case?

0.1 Solution

0.1.1 Generalization

Generalized version of Vieta's theorem is:

If a quadratic equation $x^2 + px + q = 0$ has two roots α and β (the roots can be equal) then

$$\alpha + \beta = -p$$

$$\alpha \cdot \beta = q$$

0.1.2 Proof-1

Vieta's generalized theorem already works for the first case of 0.0.1. We shall show that it works for the second case as well. In the second case of 0.0.1, $\alpha = \beta = -\frac{p}{2}$.

$$\alpha + \beta = 2\alpha = 2\left(-\frac{p}{2}\right) = -p$$

In the second case of 0.0.1 we also have $p^2 - 4q = 0$; thus $q = \frac{p^2}{4}$.

$$\alpha \cdot \beta = \alpha^2 = \frac{p^2}{4} = q$$

0.1.3 Proof-2

Since $x^2 + px + q = 0$ has two duplicate solutions, we can consider $P(x) = x^2 + px + q$ having two repeated roots, each equal to α. Thus $P(x) = (x - \alpha)(x - \alpha)$. In other words, $P(x) = x^2 - 2\alpha + \alpha^2$. A comparison gives us Vieta's theorem.

Problem-240

Problem Statement

(Vieta's theorem for a cubic equation) Assume that a cubic equation $x^3 + px^2 + qx + r = 0$ has three different roots $\alpha,\ \beta,\ \gamma$. Prove that

$$\begin{aligned} \alpha + \beta + \gamma &= -p \\ \alpha\beta + \alpha\gamma + \beta\gamma &= q \\ \alpha\beta\gamma &= -r \end{aligned}$$

0.2 Solution

The polynomial $P(x) = x^3 + px^2 + qx + r$ can be written as

$$P(x) = (x-\alpha)\,(x-\beta)\,(x-\gamma)\,R(x)$$

To match the coefficient of x^3, $R(x)$ must be 1. Therefore,

$$\begin{aligned} P(x) &= (x-\alpha)\,(x-\beta)\,(x-\gamma) \\ &= x^3 - (\alpha+\beta+\gamma)\,x^2 + (\alpha\cdot\beta+\alpha\cdot\gamma+\beta\cdot\gamma)\,x - \alpha\cdot\beta\cdot\gamma \end{aligned}$$

On comparing against $P(x) = x^3 + px^2 + qx + r$ we have

$$\begin{aligned} \alpha+\beta+\gamma &= -p \\ \alpha\beta+\alpha\gamma+\beta\gamma &= q \\ \alpha\beta\gamma &= -r \end{aligned}$$

Problem-259

Problem Statement

Starting with the graph for $y = x^2$ we can get the graph for $y = a(x+m)^2 + n$ as follows:

(a) Stretching it vertically a times gives us the graph for $y = ax^2$. If $a > 1$, the graph gets steeper. If $0 < a < 1$, the graph gets flatter. If $a < 0$, the graph becomes upside down.

(b) Moving the resulting graph m units to the left gives us the graph for $y = a(x+m)^2$. If $m > 0$, the graph moves to the left. If $m < 0$, the graph moves to the right.

(c) Moving the resulting graph n units up gives us the graph for $y = a(x+m)^2 + n$. If $n > 0$, the graph moves upwards. If $n < 0$, the grpah moves downwards.

There are six possible orderings of operations (a), (b), and (c). Do we get six different graphs or do some of the graphs coincide?

Solution

We do not get six different graphs. The six graphs fall into two distinct groups: one for the function $y = a(x+m)^2 + n$ and the other for the function $y = a(x+m)^2 + an$. Since (b) is the only one that affects $x-$coordinates, (b) does not interfere with (a) and (c). On the other hand, (a) and (c) both affect $y-$coordinates and thus they interfere with each other. Which of these two groups a graph falls into, thus, depends on the order of (a) and (c). If (c) follows (a), we get the graph for the function $y = a(x+m)^2 + n$. If (a) follows (c), we get the graph for the function $y = a(x+m)^2 + an$.

1. (c) follows (a):

   i. $y = x^2 \overset{(a)}{\Longrightarrow} y = ax^2 \overset{(b)}{\Longrightarrow} y = a(x+m)^2 \overset{(c)}{\Longrightarrow} y = a(x+m)^2 + n$

   ii. $y = x^2 \overset{(a)}{\Longrightarrow} y = ax^2 \overset{(c)}{\Longrightarrow} y = ax^2 + n \overset{(b)}{\Longrightarrow} y = a(x+m)^2 + n$

   iii. $y = x^2 \overset{(b)}{\Longrightarrow} y = (x+m)^2 \overset{(a)}{\Longrightarrow} y = a(x+m)^2 \overset{(c)}{\Longrightarrow} y = a(x+m)^2 + n$

2. (a) follows (c):

   i. $y = x^2 \overset{(b)}{\Longrightarrow} y = (x+m)^2 \overset{(c)}{\Longrightarrow} y = (x+m)^2 + n \overset{(a)}{\Longrightarrow} y = a(x+m)^2 + an$

   ii. $y = x^2 \overset{(c)}{\Longrightarrow} y = x^2 + n \overset{(a)}{\Longrightarrow} y = ax^2 + an \overset{(b)}{\Longrightarrow} y = a(x+m)^2 + an$

   iii. $y = x^2 \overset{(c)}{\Longrightarrow} y = x^2 + n \overset{(b)}{\Longrightarrow} y = (x+m)^2 + n \overset{(a)}{\Longrightarrow} y = a(x+m)^2 + an$

Problem-260

Problem Statement

How can you determine the signs of $a,\ b,\ c$ by looking at the graph of $y = ax^2 + bx + c$?

0.3 Solution

If $a = 0$ but $b \neq 0$, we have $y = bx + c$, thus our graph is no longer a parabola, but a straight line. The slope and y-intercept of the straight line gives the signs of b and c. On the other hand, if $b = 0$ but $a \neq 0$, we have $y = ax^2 + c$. This is similar to the form $y = a(x+m)^2 + n$, so we would know the signs of a and c easily. If both a and b are zero, we have $y = c$. So, we have a graph that is a straight line and parallel to the $x-$axis. So, if the line lies above the $x-$axis, $c > 0$, else $c < 0$. We shall now consider the more interesting case where $a \neq 0$ and $b \neq 0$. Since we know a lot about the graph of $y = a(x+m)^2 + n$, let's transform $y = ax^2 + bx + c$ into that form.

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x^2 + x \cdot \frac{b}{a}\right) + c \\ &= a\left(x^2 + 2 \cdot x \cdot \frac{b}{2a} + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a^2} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a^2}\right) \end{aligned}$$

Comparing with $y = a(x+m)^2 + n$, we can consider $\frac{b}{2a} = m$.
Below is how we can find out the signs of $a,\ b,\ c$:

- Compared to the graph of $y = x^2$, if our graph is upside down, then $a < 0$; otherwise $a > 0$.

- $\frac{b}{2a}$ determines how much our graph is shifted horizontally compared to the graph of $y = x^2$. So, if we know the sign of a and how the graph is shifted horizontally, then we can deduce the sign of b.

	$a > 0$	$a < 0$
Shifted left	$b > 0$	$b < 0$
Shifted right	$b < 0$	$b > 0$
Unshifted	$b = 0$	$b = 0$

- When $x = 0$, $y = ax^2 + bx + c$ becomes $y = c$. So, looking at where our graph intersects with the line $x = 0$ aka the $y-$axis, we would know the sign of c.

Problem-262

Problem Statement

The sum of two numbers is equal to 1. What is the maximal possible value of their product?

Solution

We see that 347 + (-346) = 1, but their product is negative. Whereas 0 + 1 = 1, and their product is 0 and this is better. Can the product be positive? What about $\frac{1}{10}+\frac{9}{10}$? In this case the product is $\frac{9}{100}$ and is the best among the three examples we have just considered.

Say one number is x and the other is $1-x$. So, the product is the polynomial $x(1-x) = -x^2+x = -\left(x-\frac{1}{2}\right)^2+\frac{1}{4}$. So, the graph of the product polynomial is a parabola that is upside down, $\frac{1}{2}$ units shifted right; therefore the vertex or crest is at $x=\frac{1}{2}$ and that is where maximal value occurs. So, the maximal value is $\frac{1}{2}\left(1-\frac{1}{2}\right)=\frac{1}{4}$.

Problem-263

Problem Statement

Prove that a square has the maximum area of all rectangles having the same perimeter.

Solution

Say the perimeter is p. If one side of the rectangle is x, then the other side must be $\frac{p}{2}-x$. Now the area is a polynomial: $x\left(\frac{p}{2}-x\right) = -x^2+x\cdot\frac{p}{2} = -\left(x-\frac{p}{4}\right)^2+\frac{p^2}{16}$. Thus the area has the maximum value when $x=\frac{p}{4}$. But in that case the other side is also $\frac{p}{4}$, making it a square.

Problem-264

Problem Statement

Prove that a square has the minimum perimeter of all rectangles having the same area.

Solution

So we need to prove that among all rectangles with an area say a, the square has the minimum perimeter. Say x is one side of a rectangle with area a. The other side then is $\frac{a}{x}$. The perimeter is a polynomial $P(x) = 2\left(x + \frac{a}{x}\right)$. Consider the below equation, where $c > 0$.

$$2\left(x + \frac{a}{x}\right) = c \tag{1}$$

We shall first find out for which $c > 0$, the equation (1) has solutions. Then from those values of $c > 0$, we shall find out the minimum, because that gives us the minimum periemeter for a given area a. We shall then see, for that minimum $c > 0$, what value x takes. Once we know how x and the other side $\frac{a}{x}$ look, we would know what type of rectangle has this minimum perimeter. Note, since this is a rectangle we are talking about, we are interested in $x > 0$ solutions.

$$\begin{aligned}
2\left(x + \frac{a}{x}\right) &= c \\
x^2 - x \cdot \frac{c}{2} + a &= 0 \\
\left(x - \frac{c}{4}\right)^2 &= \frac{c^2}{16} - a \\
x &= \frac{c}{4} + \frac{\pm\sqrt{c^2 - 16a}}{16}
\end{aligned}$$

So, (1) has solutions when $c^2 - 16a \geq 0$. In other words, when $c \geq 4\sqrt{a}$ or $c \leq -4\sqrt{a}$, the equation (1) has solutions. Hence, the minimum positive value of c for which (1) has a solution is $4\sqrt{a}$ and at this value of c, we have $x = \frac{c}{4} = \frac{4\sqrt{a}}{4} = \sqrt{a}$. The other side is $\frac{a}{x} = \frac{a}{\sqrt{a}} = \sqrt{a}$. Therefore, the rectangle with area a that has the minimum perimeter is a square.

In another way, say a square with area a has perimeter p. Now, from **Problem-263**, we know that a is the maximum possible area with the perimeter p. For all other rectangles, if they want to achieve the area a, their perimeter must be bigger than p. So, with area a, p is the minimum possible perimeter for a rectangle and that happens when the rectangle is a square.

Problem-265

Problem Statement

Find the minimal value of the expression $x + \frac{2}{x}$ for positive x.

Solution

We may consider x and $\frac{2}{x}$ as the two sides of a rectangle with area 2. With that interpretation, $2\left(x + \frac{2}{x}\right)$ is the perimeter of the rectangle which, according to **Problem-264**, is minimal when the rectangle is a square. Therefore, $2\left(x + \frac{2}{x}\right)$ is minimal when $x = \frac{2}{x}$. This is also the condition for our expression $x + \frac{2}{x}$, which is half the perimeter, to have its minimum value for $x > 0$. So, when $x = \sqrt{2}$, the perimeter and also our expression $x + \frac{2}{x}$ assume their minimal values. So minimal value of $x + \frac{2}{x}$ is $\sqrt{2} + \frac{2}{\sqrt{2}} = 2\sqrt{2}$.