Problem-225

Problem Statement

If in the equation

$$ax^2 + bx + c = 0$$

the coefficient a is equal to zero then the equation takes the form

$$bx + c = 0$$

and has the unique solution

$$x = -\frac{c}{b}$$

Strictly speaking, the last sentence is wrong; when $b = 0$ the quotient $\frac{c}{b}$ is undefined. How are we to correct this error?

Solution

We should require that in the equation $ax^2 + bx + c = 0$, both a and b cannot be zero.

Problem-227

Problem Statement

Prove that $\sqrt{3}$ is irrational.

Solution

The statement '$\sqrt{3}$ is irrational' is equivalent to the statement: $\sqrt{3} \neq \frac{a}{b}$ for any integer a and b. We shall make use of **proof by contradiction**, therefore, we shall assume that $\sqrt{3} = \frac{a}{b}$ and that should lead to a contradiction, completing the proof.

We shall consider four different cases for the pair (a, b) and we shall derive a contradiction for each.

1. a and b both are odd integer. Say $a = 2m+1$ and $b = 2n+1$, where m and n are integers. According to our assumption

$$\begin{aligned}
\sqrt{3} &= \frac{2m+1}{2n+1} \\
3 &= \frac{(2m+1)^2}{(2n+1)^2} \\
3\,(2n+1)^2 &= (2m+1)^2 \\
3\,(4n^2+4n+1) &= 4m^2+4m+1 \\
12n^2+12n+3 &= 4m^2+4m+1 \\
12n^2+12n+2 &= 4m^2+4m \\
6n^2+6n+1 &= 2m^2+2m
\end{aligned}$$

   On the left side we have an odd integer and on the right side we have an even integer—a contradiction. ※

2. a is an odd integer and b is an even integer. Say $a = 2m+1$ and $b = 2n$, where m and n are integers. According to our assumption

$$\begin{aligned}
\sqrt{3} &= \frac{2m+1}{2n} \\
3 &= \frac{(2m+1)^2}{4n^2} \\
12n^2 &= (2m+1)^2
\end{aligned}$$

   On the left side we have an even integer and on the right side we have an odd integer—a contradiction. ※

3. a is an even integer and b is an odd integer. Say $a = 2m$ and $b = 2n+1$, where m and n are integers. According to our assumption

$$\begin{aligned}
\sqrt{3} &= \frac{2m}{2n+1} \\
3 &= \frac{4m^2}{(2n+1)^2} \\
3\,(2n+1)^2 &= 4m^2
\end{aligned}$$

   On the left side we have an odd integer and on the right side we have an even integer—a contradiction. ※

4. a and b both are even integer. In this case, a and b must have 2 as a common factor. We can divide both a and b by 2. We can keep dividing by 2 as long as both remain even integer. At the end, we are in one of the previous three cases.

Problem-229

Problem Statement

Prove that for $a \geq 0,\ b > 0$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Solution

Basically we are being asked to show that $\frac{\sqrt{a}}{\sqrt{b}}$ is the square root of the non-negative integer $\frac{a}{b}$. So, we need to prove the below two:

- $\frac{\sqrt{a}}{\sqrt{b}}$ is non-negative.

  Since a is non-negative, its square root must also be non-negative. Since b is positive, its square root must also be positive. $\frac{\sqrt{a}}{\sqrt{b}}$ is thus the ratio of a non-negative number and a positive number; so, it must be non-negative.

- Squaring $\frac{\sqrt{a}}{\sqrt{b}}$ gives us $\frac{a}{b}$.

$$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b}.$$

Problem-230

Problem Statement

Is the equality $\sqrt{a^2} = a$ true for all a?

Solution

It might seem counter-intuitive that the answer is no. But if we observe closely, the question is actually asking about the equality $+\sqrt{a^2} = a$, where the '+' is implied. When we make the '+' explicit, it is more noticeable that for $a < 0$, the equality cannot hold, because the left side of the equality is always positive. The equality that works for all a is $\sqrt{a^2} = |a|$, where

$$|a| = \begin{cases} a & \text{, if } a \geq 0 \\ -a & \text{, if } a < 0 \end{cases}$$

It is unfortunate that we do not always make the '+' explicit.

Problem-232

Problem Statement

Which is bigger: $\sqrt{1001} - \sqrt{1000}$ or $\frac{1}{10}$?

Solution

It would be easier to compare if we could convert $\sqrt{1001} - \sqrt{1000}$ into a fraction.

$$\begin{aligned}
&\sqrt{1001} - \sqrt{1000} \\
&= \frac{\left(\sqrt{1001} - \sqrt{1000}\right)\left(\sqrt{1001} + \sqrt{1000}\right)}{\sqrt{1001} + \sqrt{1000}} \\
&= \frac{\left(\sqrt{1001}\right)^2 - \left(\sqrt{1000}\right)^2}{\sqrt{1001} + \sqrt{1000}} \\
&= \frac{1001 - 1000}{\sqrt{1001} + \sqrt{1000}} \\
&= \frac{1}{\sqrt{1001} + \sqrt{1000}} \\
&< \frac{1}{\sqrt{900} + \sqrt{900}}
\end{aligned}$$

Since $\sqrt{900} + \sqrt{900} = 60$, $\sqrt{1001} - \sqrt{1000} < \frac{1}{60} < \frac{1}{10}$. So, $\frac{1}{10}$ is the bigger between the two.

Problem-239

Problem Statement

0.0.1 Solutions to quadratic equation

A quadratic equation $x^2+px+q=0$ can be solved by “completing the square”.

$$x^2+2\cdot x\cdot\frac{p}{2}+\left(\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)=0$$
$$\left(x+\frac{p}{2}\right)^2=\frac{p^2}{4}-q$$
$$x=-\frac{p}{2}+\frac{\pm\sqrt{p^2-4q}}{2}$$

There are three possible cases:

1. $p^2-4q>0$. In this case two distinct solutions $x_1,\ x_2$ exist:
   $x_1=-\frac{p}{2}+\frac{\sqrt{p^2-4q}}{2}$ and $x_2=-\frac{p}{2}-\frac{\sqrt{p^2-4q}}{2}$
2. $p^2-4q=0$. In this case the two solutions coincide; therefore, there is one distinct solution.
   $x_1=-\frac{p}{2}$ and $x_2=-\frac{p}{2}$
3. $p^2-4q<0$. In this case there is no real-valued solution.

Since the sign of p^2-4q determines how many distinct solutions there are, for convenience, we shall call this expression D, therefore, $D=p^2-4q$.

0.0.2 Vieta’s theorem

If a quadratic equation $x^2+px+q=0$, has two distinct solutions α and β then

$$\alpha+\beta=-p$$
$$\alpha\cdot\beta=q$$

0.0.3 Proof-1 of Vieta's theorem

If $x^2 + px + q = 0$ has two distinct solutions, we are in the first case of 0.0.1; thus we can let α and β be as follows:

$$\alpha = -\frac{p}{2} + \frac{\sqrt{D}}{2} \tag{1}$$

$$\beta = -\frac{p}{2} - \frac{\sqrt{D}}{2} \tag{2}$$

$(1) + (2)$ gives:

$$\alpha + \beta = -p$$

$(1) \times (2)$ gives:

$$\begin{aligned} \alpha \cdot \beta &= \frac{p^2}{4} - \frac{D}{4} \\ &= \frac{p^2 - \left(p^2 - 4q\right)}{4} \\ &= q \end{aligned}$$

0.0.4 Proof-2 of Vieta's theorem

We can rewrite $x^2 + px + q = 0$ as $P(x) = 0$ where $P(x) = x^2 + px + q$ is a polynomial with degree two. If α and β are two distinct solutions of $P(x) = 0$ then

$$P(x) = (x - \alpha)\,(x - \beta)\,R(x)$$

Here $R(x)$ must be constant, otherwise right side would have degree more than two. In fact $R(x)$ must be exactly 1, otherwise the coefficient of x^2 would not match with the one in $P(x)$. So, we have

$$\begin{aligned} P(x) &= (x - \alpha)\,(x - \beta) \\ &= x^2 - (\alpha + \beta)\,x + \alpha \cdot \beta \end{aligned}$$

Comparing $P(x) = x^2 + (\ -(\alpha + \beta)\ )\ x + \alpha \cdot \beta$ with $P(x) = x^2 + px + q$ gives us Vieta's theorem.

0.0.5 Questions to answer

Can you generalize Vieta's theorem to the case of a quadratic equation having only one root? Are both proofs still valid for this case?

0.1 Solution

0.1.1 Generalization

Generalized version of Vieta's theorem is:

If a quadratic equation $x^2 + px + q = 0$ has two roots α and β (the roots can be equal) then

$$\alpha + \beta = -p$$
$$\alpha \cdot \beta = q$$

0.1.2 Proof-1

Vieta's generalized theorem already works for the first case of 0.0.1. We shall show that it works for the second case as well. In the second case of 0.0.1, $\alpha = \beta = -\frac{p}{2}$.

$$\alpha + \beta = 2\alpha = 2\left(-\frac{p}{2}\right) = -p$$

In the second case of 0.0.1 we also have $p^2 - 4q = 0$; thus $q = \frac{p^2}{4}$.

$$\alpha \cdot \beta = \alpha^2 = \frac{p^2}{4} = q$$

0.1.3 Proof-2

Since $x^2 + px + q = 0$ has two duplicate solutions, we can consider $P(x) = x^2 + px + q$ having two repeated roots, each equal to α. Thus $P(x) = (x - \alpha)(x - \alpha)$. In other words, $P(x) = x^2 - 2\alpha + \alpha^2$. A comparison gives us Vieta's theorem.

Problem-240

Problem Statement

(Vieta's theorem for a cubic equation) Assume that a cubic equation $x^3 + px^2 + qx + r = 0$ has three different roots α, β, γ. Prove that

$$\begin{aligned} \alpha + \beta + \gamma &= -p \\ \alpha\beta + \alpha\gamma + \beta\gamma &= q \\ \alpha\beta\gamma &= -r \end{aligned}$$

0.2 Solution

The polynomial $P(x) = x^3 + px^2 + qx + r$ can be written as

$$P(x) = (x-\alpha)\,(x-\beta)\,(x-\gamma)\,R(x)$$

To match the coefficient of x^3, $R(x)$ must be 1. Therefore,

$$\begin{aligned} P(x) &= (x-\alpha)\,(x-\beta)\,(x-\gamma) \\ &= x^3 - (\alpha+\beta+\gamma)\,x^2 + (\alpha\cdot\beta + \alpha\cdot\gamma + \beta\cdot\gamma)\,x - \alpha\cdot\beta\cdot\gamma \end{aligned}$$

On comparing against $P(x) = x^3 + px^2 + qx + r$ we have

$$\begin{aligned} \alpha+\beta+\gamma &= -p \\ \alpha\beta + \alpha\gamma + \beta\gamma &= q \\ \alpha\beta\gamma &= -r \end{aligned}$$

Problem-259

Problem Statement

Starting with the graph for $y = x^2$ we can get the graph for $y = a(x+m)^2 + n$ as follows:

(a) Stretching it vertically a times gives us the graph for $y = ax^2$. If $a > 1$, the graph gets flatter. If $0 < a < 1$, the graph gets steeper. If $a < 0$, the graph becomes upside down.

(b) Moving the resulting graph m units to the left gives us the graph for $y = a(x+m)^2$. If $m > 0$, the graph moves to the left. If $m < 0$, the graph moves to the right.

(c) Moving the resulting graph n units up gives us the graph for $y = a(x+m)^2 + n$. If $n > 0$, the graph moves upwards. If $n < 0$, the grpah moves downwards.

There are six possible orderings of operations (a), (b), and (c). Do we get six different graphs or do some of the graphs coincide?

Solution

We do not get six different graphs. The six graphs fall into two distinct groups: one for the function $y = a(x+m)^2 + n$ and the other for the function $y = a(x+m)^2 + an$. The order of (b) and (c) does not matter. Which of these two groups a graph falls into, depends on the order of (a) and (c). If (c) follows (a), we get the graph for the function $y = a(x+m)^2 + n$. On the other hand, if (a) follows (c), we get the graph for the function $y = a(x+m)^2 + an$.

1. (c) follows (a):

   i. $y = x^2 \overset{\text{(a)}}{\Longrightarrow} y = ax^2 \overset{\text{(b)}}{\Longrightarrow} y = a(x+m)^2 \overset{\text{(c)}}{\Longrightarrow} y = a(x+m)^2 + n$

   ii. $y = x^2 \overset{\text{(a)}}{\Longrightarrow} y = ax^2 \overset{\text{(c)}}{\Longrightarrow} y = ax^2 + n \overset{\text{(b)}}{\Longrightarrow} y = a(x+m)^2 + n$

   iii. $y = x^2 \overset{\text{(b)}}{\Longrightarrow} y = (x+m)^2 \overset{\text{(a)}}{\Longrightarrow} y = a(x+m)^2 \overset{\text{(c)}}{\Longrightarrow} y = a(x+m)^2 + n$

2. (a) follows (c):

   i. $y = x^2 \overset{\text{(b)}}{\Longrightarrow} y = (x+m)^2 \overset{\text{(c)}}{\Longrightarrow} y = (x+m)^2 + n \overset{\text{(a)}}{\Longrightarrow} y = a(x+m)^2 + an$

   ii. $y = x^2 \overset{\text{(c)}}{\Longrightarrow} y = x^2 + n \overset{\text{(a)}}{\Longrightarrow} y = ax^2 + an \overset{\text{(b)}}{\Longrightarrow} y = a(x+m)^2 + an$

   iii. $y = x^2 \overset{\text{(c)}}{\Longrightarrow} y = x^2 + n \overset{\text{(b)}}{\Longrightarrow} y = (x+m)^2 + n \overset{\text{(a)}}{\Longrightarrow} y = a(x+m)^2 + an$