Problem-90

Problem Statement

(a) Multiply $(1+x)\left(1+x^2\right)$.

(b) Multiply $(1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)$.

(c) Compute $\left(1+x+x^2+x^3\right)^2$.

(d) Compute $\left(1+x+x^2+x^3+\ \ldots\ +x^9+x^{10}\right)^2$.

(e) Find the coefficient of x^{29} and x^{30} in $\left(1+x+x^2+x^3+\ \ldots\ +x^9+x^{10}\right)^3$.

(f) Multiply $(1-x)\left(1+x+x^2+x^3+\ \ldots\ +x^9+x^{10}\right)$.

(g) Multiply $(a+b)\left(a^2-ab+b^2\right)$.

(h) Multiply $\left(1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10}\right)$ by $\left(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}\right)$.

Solution to Part-a

$$\begin{array}{r} x+1 \\ \times\quad x^2+1 \\ \hline x+1 \\ x^3+x^2 \\ \hline x^3+x^2+x+1 \end{array}$$

Solution to Part-b

From part-a we have $(1+x)\left(1+x^2\right)=1+x+x^2+x^3$.

$$\begin{array}{r} x^3+x^2+x+1 \\ \times\qquad\qquad x^4+1 \\ \hline x^3+x^2+x+1 \\ x^7+x^6+x^5+x^4 \\ \hline x^7+x^6+x^5+x^4+x^3+x^2+x+1 \end{array}$$

Similar to the above, $\left(x^7+x^6+x^5+x^4+x^3+x^2+x+1\right)\left(1+x^8\right)=\sum_{n=0}^{15} x^n$.

Solution to Part-c

$$\begin{array}{r}
x^3+x^2+x+1 \\
\times \quad x^3+x^2+x+1 \\
\hline
x^3+x^2+x+1 \\
x^4+x^3+x^2+x \quad\;\; \\
x^5+x^4+x^3+x^2 \qquad\quad\;\; \\
x^6+x^5+x^4+x^3 \qquad\qquad\qquad\; \\
\hline
x^6+2x^5+3x^4+4x^3+3x^2+2x+1
\end{array}$$

In regards to how many times they occur, we note that left and right of x^3 are mirror reflections. For example, there are as many x^5 as there are x. We may write the product as

$$\sum_{n=0}^{2}(n+1)\cdot x^n \;+\; 4\cdot x^3 \;+\; \sum_{n=4}^{6}(6-n+1)\cdot x^n$$

Solution to Part-d

From the observation in part-c, the product here should be

$$\sum_{n=0}^{9}(n+1)\cdot x^n \;+\; 11\cdot x^{10} \;+\; \sum_{n=11}^{20}(20-n+1)\cdot x^n$$

In other words the product is

$$x^{20}+2x^{19}+3x^{18}+\;\dots\;+10x^{11}+11x^{10}+10x^9+9x^8+\;\dots\;+3x^2+2x+1$$

Solution to Part-e

We observe that the coefficient of x^{29} in $\left(1+x+x^2+x^3+\;\dots\;+x^9+x^{10}\right)^3$ is the count of ways we can get 29 from adding three integers between 0 and 10 inclusive. In other words, it is the number of tuples (a,b,c) where $0 \le a,b,c \le 10$ and $a+b+c=29$. There are three such tuples: $(9,10,10)$, $(10,9,10)$, and $(10,10,9)$. So, the coefficient of x^{29} is 3. There is only one tuple whose elements sum to 30, namely the tuple $(10,10,10)$; so the coefficient of x^{30} is 1.

Java code that follows, gives the below output for $\left(1+x+x^2+x^3+\;\dots\;+x^9+x^{10}\right)^3$ and we see that the coefficients of x^{29} and x^{30} are indeed 3 and 1 respectively.

```
1  + 3 * x^1  + 6 * x^2  + 10 * x^3  + 15 * x^4  + 21 * x^5  + 28 * x^6
+ 36 * x^7  + 45 * x^8  + 55 * x^9  + 66 * x^10  + 75 * x^11  + 82 * x^12
+ 87 * x^13  + 90 * x^14  + 91 * x^15  + 90 * x^16  + 87 * x^17
+ 82 * x^18  + 75 * x^19  + 66 * x^20  + 55 * x^21  + 45 * x^22
+ 36 * x^23  + 28 * x^24  + 21 * x^25  + 15 * x^26  + 10 * x^27
+ 6 * x^28  + 3 * x^29  + 1 * x^30
```

```
  public static void main(String[] args) {
    int[] polynomial1 = new int[11];
    Arrays.fill(polynomial1, 1);
    int[] polynomial2 = new int[11];
    Arrays.fill(polynomial2, 1);
    // we get the square of 1+x+x^2+...+x^10 here
    int[] square = multiply( polynomial1, polynomial2 );
    int[] cube = multiply( square, polynomial1 );
    printPolynomial(cube);
  }
  private static int[] multiply(int[] polynomial1, int[] polynomial2) {
    int m = polynomial1.length;
    int n = polynomial2.length;
    int[] product = new int[m+n];
    for (int i = 0; i < m; ++i) {
      for (int j = 0; j < n; ++j) {
        product[i+j] += polynomial1[i]*polynomial2[j];
      }
    }
    return product;
  }
  private static void printPolynomial(int[] poly) {
    System.out.println();
    System.out.print( String.format( "%d", poly[0] ) );
    for (int i = 1; i < poly.length; ++i) {
      if (poly[i] != 0) {
        System.out.print( String.format( "  + %d * x^%d", poly[i], i ) );
      }
    }
    System.out.println();
  }
```

Solution to Part-f

$$
\begin{aligned}
(1-x)\ \left(1+x+x^2+x^3+\ \dots\ +x^9+x^{10}\right) &= 1+\sum_{n=1}^{10} x^n - \sum_{n=1}^{10} x^n - x^{11}\\
&= 1-x^{11}
\end{aligned}
$$

Solution to Part-g

$$
\begin{aligned}
(a+b)\ \left(a^2-ab+b^2\right) &= (a+b)\ a^2-(a+b)\ ab+(a+b)\ b^2\\
&= (a+b)\ a^2-ba^2+(a+b)\ b^2-ab^2\\
&= a^3+b^3
\end{aligned}
$$

If a, b are positive integers, we can interpret $(a+b)\ a^2$ as a solid with base a^2 from which if we take away a solid of volume ba^2, we are left with a cubic solid with volume a^3. Similarly, from a solid $(a+b)\ b^2$ with base b^2 if we take away a volume of ab^2 we are left with a cubic solid of volume b^3.

Solution to Part-h

We make the following three observations:

1. In the product, the coefficient of an odd power below 10 is coming from the sum of even number of terms, where half of the terms have negative sign and half of the terms have positive sign. So, the product does not contain an odd power below 10. For example, consider the coefficient of x^7. The eight tuples $(0,7)$, $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, $(6,1)$, $(7,0)$ contribute. Here the signs of the tuple alternate between positive and negative, the first one being postive. So, $(1,6)$, $(3,4)$, $(5,2)$, and $(7,0)$ have negative sign.

2. In the product, the coefficient of an even power not exceeded 10 is coming from the sum of odd number of terms, where lesser half of the terms have negative sign and greater half of the terms have positive sign. That leaves one positive term alive. So, in the product, all even powers not exceeding 10 appear exactly once. For example, consider the coefficient of x^8. The nine tuples $(0,8)$, $(1,7)$, $(2,6)$, $(3,5)$, $(4,4)$, $(5,3)$, $(6,2)$, $(7,1), (8,0)$ contribute. Here again the signs of the tuples alternate starting with positive. That leaves the last $(8,0)$ intact, all other vanish.

3. As we noted in part-d, in the product, the coefficients to the left and to the right of x^{10} are mirror reflections, for example x^{13} on the right would have the same coefficient as x^7 on the left, thus x^{13} also would not appear in the prouduct. The coefficient of x^{12} on the right would match with the coefficient of x^8 on the left. Thus there will be exactly one x^{12} in the product.

From the above observations, we have the product:

$$1 + x^2 + x^4 + x^6 + x^8 + x^{10} + x^{12} + x^{14} + x^{16} + x^{18} + x^{20}$$