Problem-183

Problem Statement

The first term of an arithmetic progression is a, the 4th term is b. Find the second and the third terms.

Solution

The n-th term of the arithmetic progression has the form $a+(n-1)d$ where a is the first term ($n=1$) and d is the difference. From the information given in the statement, we have:

$$a+(4-1)d=b$$

Thus $d=\frac{b-a}{3}$. So, the second term is $a+(2-1)d=a+\frac{b-a}{3}=\frac{2a+b}{3}$. The third term is d more than the second term, that is, $\frac{a+2b}{3}$.

Problem-198

Problem Statement

The first term of a geometric progression is a and the third term is b. Find the second term.

Solution

Say the second term is x. Then from the definition of a geometric progression, $\frac{x}{a}=\frac{b}{x}$. Therefore, $x^2=ab$. We shall consider three separate possibilities.

1. When $ab<0$, there is no real-valued x. For example, even if the ratio were negative like -1, the terms of the geometric progression would have alternate signs, so that the first and the third terms would still have the same sign, making $ab<0$ impossible.

2. When $ab=0$, we have $x=0$.

3. When $ab>0$, there are two possibilities for x: $\sqrt{ab}$ and $-\sqrt{ab}$.

Problem-204

Problem Statement

Is it possible that numbers $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$ are (not necessarily adjacent) terms of the same arithmetic progression?

Solution

If the numbers come from the same arithmetic progression, then we can consider them as sorted. Since, in the question, the numbers are already in descending order, we shall go with that order. So, if the three numbers indeed come from the same arithmetic progression, the difference will turn out to be negative. We may consider $\frac{1}{2}$ as the first term of the sub-progression. We thus have:

$$\frac{1}{2} + (m-1)\,d = \frac{1}{3}$$
$$\frac{1}{2} + (n-1)\,d = \frac{1}{5}$$

Where $m \neq 1$, $m < n$ and d $(\neq 0)$ is the difference. We can rearrange and get the below two equations:

$$(m-1)\,d = -\frac{1}{6} \tag{1}$$

$$(n-1)\,d = -\frac{3}{10} \tag{2}$$

Dividing (1) by (2), we have:

$$\frac{m-1}{n-1} = \frac{5}{9} \tag{3}$$

Choosing $(m,n) = (6,10)$ agrees with (3). Using $m = 6$ in (1) gives $d = -\frac{1}{30}$. So, the numbers could be a part of the same arithmetic progression because, $\frac{1}{2} + 5\left(-\frac{1}{30}\right) = \frac{1}{3}$ and $\frac{1}{2} + 9\left(-\frac{1}{30}\right) = \frac{1}{5}$. Hence, starting from $\frac{1}{2}$, the 5-th term is $\frac{1}{3}$ and the 9-th term is $\frac{1}{5}$. Actually, there are other valid choices like $(m,n,d) = \left(11, 19, -\frac{1}{60}\right)$.

Problem-205

Problem Statement

Is it possible that the numbers 2, 3, and 5 are (not necessarily adjacent) terms of a geometric progression?

Solution

If the numbers belong to the same geometric progression, they must be sorted. Since the numbers are in increasing order in the question, we shall go with that order. So, for the sub-progression that starts at 2, we could consider 2 as the first term. We thus have the following:

$$2 \cdot q^m = 3$$
$$q^{mn} = \frac{3^n}{2^n}$$

And,

$$2 \cdot q^n = 5$$
$$q^{mn} = \frac{5^m}{2^m}$$

Here $0 < m < n$ and q is the ratio. We can then write:

$$\frac{3^n}{2^n} = \frac{5^m}{2^m}$$
$$3^n = 2^{n-m} \cdot 5^m$$

On the left side we have an odd number and on the right side we have an even number. We have reached a contradiction. ※

Therefore, the numbers cannot be part of the same geometric progression. Note, we would reach a contradiction even if we considered the numbers in reverse order.

Problem-210

Problem Statement

Is it possible that the second term of a geometric progression is less than its first term and also less than its third term?

Solution

Let's assume that the first three terms of the geometric progression are a, aq, and aq^2. When $a = 0$ or $q = 0$, the answer is negative. So, we shall now consider $a \neq 0$ and $q \neq 0$ case. We shall consider two separate possibilities.

1. When $a < 0$. If $aq < a$, then $q > 1$. If it is also true that $aq < aq^2$, then $q > q^2$. Both $q > 1$ and $q > q^2$ cannot be true at the same time.

2. When $a > 0$. If $aq < a$, we have $q < 1$. If it is also true that $aq < aq^2$, then $q < q^2$. Both $q < 1$ and $q < q^2$ are possible for say $q = -1$.

So, $1,\ -1,\ 1, \ldots$ is a possible geometric progression.

Problem-212

Problem Statement

Is it possible that an infinite arithmetic progression contains exactly two integer terms?

Solution

It is not possible. We shall use **proof by contradiction**. Therefore, we shall assume that exactly two terms of an infinite arithmetic progression are integers and that would lead us to a contradiction, completing the proof.

Say $a + md = x$ and $a + nd = y$ are the only two integral terms where a is the first term of the progression, $0 < m < n$, and d is the difference. We notice that $(n - m)d = y - x$ is an integer. Now consider the term $y + (n - m)d = z$. z must come after y, so z is another term distinct from x and y. By our assumption, z must not be an integer. However, y and $(n - m)d$ are both integers, so should be their sum, i.e., z. We have our contradiction. ※

Problem-214

Problem Statement

In a geometric progression each term is equal to the sum of two preceeding terms. What can be said about the common ratio of this progression?

Solution

Say $a \neq 0$ is the first term and $q \neq 0$ is the common ratio. Three consecutive terms in the progression would be aq^n, aq^{n+1}, and aq^{n+2}. We thus have:

$$\begin{aligned} aq^{n+2} &= aq^{n+1} + aq^n \\ q^2 - q - 1 &= 0 \\ q^2 - 2 \cdot q \cdot \frac{1}{2} + \frac{1}{4} - \frac{5}{4} &= 0 \\ \left(q - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2 &= 0 \\ \left(q - \frac{1+\sqrt{5}}{2}\right)\left(q - \frac{1-\sqrt{5}}{2}\right) &= 0 \end{aligned}$$

There are two possible values for the common ratio q: $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

Problem-215

Problem Statement

The *Fibonacci sequence*

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$$

is defined as follows: The first two terms are equal to 1, and each subsequent term is equal to the sum of the two preceeding terms. Find numbers A and B such that (for all n), the nth term of the Fibonacci sequence is equal to

$$A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solution

For $n = 1, 2$ the terms of the Fibonacci sequence are both 1. We shall use this observation to derive two equations and we should be able to find A and B as a solution-pair of the two equations. Then we shall use induction to show that those values of A and B once plugged into the formula for the nth term, work for all n.

Find values of A and B

From the first two terms ($n = 1,2$) of the Fibonacci sequence, we have the below pair of equations.

$$A\left(\frac{1+\sqrt{5}}{2}\right)+B\left(\frac{1-\sqrt{5}}{2}\right)=1 \tag{1}$$

$$A\left(\frac{1+\sqrt{5}}{2}\right)^2+B\left(\frac{1-\sqrt{5}}{2}\right)^2=1 \tag{2}$$

By massaging (1) we get (3) and by massaging (2) we get (4):

$$A(1+\sqrt{5})^2-4B=2(1+\sqrt{5}) \tag{3}$$

$$A(1+\sqrt{5})^2+B(1-\sqrt{5})^2=4 \tag{4}$$

From $(4)-(3)$ we get:

$$\begin{aligned} B(1-\sqrt{5})^2+4B &= 4-2(1+\sqrt{5}) \\ B &= \frac{2(1-\sqrt{5})}{-2\sqrt{5}(1-\sqrt{5})} \\ B &= -\frac{1}{\sqrt{5}} \end{aligned}$$

Plugging in $B=-\frac{1}{\sqrt{5}}$ in (1) gives us:

$$\begin{aligned} A\left(\frac{1+\sqrt{5}}{2}\right)-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right) &= 1 \\ A\left(\frac{1+\sqrt{5}}{2}\right) &= \frac{1+\sqrt{5}}{2\sqrt{5}} \\ A &= \frac{1}{\sqrt{5}} \end{aligned}$$

So, the potential formula for the nth term of the Fibonacci sequence is as follows:

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Prove that the formula works for all n

We already know that the formula works for $n = 1,2$. We shall use these as our base cases. For $n > 2$ we shall use induction. Therefore, we shall show that adding nth and $(n+1)$th terms yields the $(n+2)$th term. For convenience, let's denote $\frac{1+\sqrt{5}}{2}$ by α and $\frac{1-\sqrt{5}}{2}$ by β. We make the below two observations:

$$\alpha^2 = 1 + \alpha$$

$$\beta^2 = 1 + \beta$$

Now to complete induction, we add up nth and $(n+1)$th terms which should give us the $(n+2)$th term.

$$\begin{aligned}
&\left(\frac{1}{\sqrt{5}}\ \alpha^{n+1} - \frac{1}{\sqrt{5}}\ \beta^{n+1}\right) + \left(\frac{1}{\sqrt{5}}\ \alpha^{n} - \frac{1}{\sqrt{5}}\ \beta^{n}\right) \\
&= \frac{\alpha^n}{\sqrt{5}}\ (1+\alpha) - \frac{\beta^n}{\sqrt{5}}\ (1+\beta) \\
&= \frac{\alpha^n}{\sqrt{5}}\ \alpha^2 - \frac{\beta^n}{\sqrt{5}}\ \beta^2 \\
&= \frac{1}{\sqrt{5}}\ \alpha^{n+2} - \frac{1}{\sqrt{5}}\ \beta^{n+2} \quad \blacksquare
\end{aligned}$$