Problem-120

Problem Statement

Can you factor any other polynomial of the form $a^{2n} + b^{2n}$?

Solution

From the definition of a polynomial, $n \geq 0$. We shall consider two separate cases: (1) when n is even (2) when n is odd.

When n is even, therefore n has the form $2m$ with $m \geq 1$

$$\begin{aligned}
a^{2n} + b^{2n} &= a^{4m} + b^{4m} \\
&= \left(a^{2m}\right)^2 + \left(b^{2m}\right)^2 \\
&= \left(a^{2m}\right)^2 + 2 \cdot a^{2m} \cdot b^{2m} + \left(b^{2m}\right)^2 - 2 \cdot a^{2m} \cdot b^{2m} \\
&= \left(a^{2m} + b^{2m}\right)^2 - \left(\sqrt{2} \cdot a^m \cdot b^m\right)^2 \\
&= \left(a^{2m} + \sqrt{2} \cdot a^m \cdot b^m + b^{2m}\right) \left(a^{2m} - \sqrt{2} \cdot a^m \cdot b^m + b^{2m}\right)
\end{aligned}$$

When n is odd, therefore n has the form $2m+1$ with $m \geq 0$

Let's consider some examples and try to generalize from them.

- When $m = 0$ therefore $n = 1$.

$$\begin{aligned}
a^2 + b^2 &= a^2 - \left(-b^2\right) \\
&= a^2 - \left(\sqrt{-1} \cdot b\right)^2 \\
&= a^2 - (i \cdot b)^2 \\
&= (a + i \cdot b)\,(a - i \cdot b)
\end{aligned}$$

- When $m = 1$ therefore $n = 3$. Our polynomial in this case is $a^6 + b^6$. We notice that if we set $a^2 = -b^2$, the polynomial evaluates to zero. So, we might be able to factor the polynomial into $\left(a^2 + b^2\right)$ (...). We thus try to extract $a^2 + b^2$ from each pair of consecutive terms, introducing temporary terms as required.

$$
\begin{aligned}
a^6+b^6 &= a^6+a^4\cdot b^2-a^4\cdot b^2-a^2\cdot b^4+a^2\cdot b^4+b^6 \\
&= a^4\cdot\left(a^2+b^2\right)-a^2\cdot b^2\cdot\left(a^2+b^2\right)+b^4\cdot\left(a^2+b^2\right) \\
&= \left(a^2+b^2\right)\ \left(a^4-a^2\cdot b^2+b^4\right)
\end{aligned}
$$

- When $m=2$ therefore $n=5$. Our polynomial in this case is $a^{10}+b^{10}$. Again, setting $a^2=-b^2$ makes the polynomial zero. So, a^2+b^2 is a potential factor. We again try to extract a^2+b^2 from each pair of consecutive terms introducing temporary terms as required.

$$
\begin{aligned}
&a^{10}+b^{10} \\
&= \ a^{10}+a^8\cdot b^2-a^8\cdot b^2-a^6\cdot b^4+a^6\cdot b^4+a^4\cdot b^6-a^4\cdot b^6-a^2\cdot b^8+a^2\cdot b^8+b^{10} \\
&= a^8\cdot\left(a^2+b^2\right)-a^6\cdot b^2\cdot\left(a^2+b^2\right)+a^4\cdot b^4\cdot\left(a^2+b^2\right)-a^2\cdot b^6\cdot\left(a^2+b^2\right)+b^8\cdot\left(a^2+b^2\right) \\
&= \left(a^2+b^2\right)\ \left(a^8-a^6\cdot b^2+a^4\cdot b^4-a^2\cdot b^6+b^8\right)
\end{aligned}
$$

The factoring process for $n>1$ looks like below:

$$
\begin{aligned}
a^{2n}+b^{2n} &= \sum_{k=0}^{n-1}(-1)^k\ \left[a^{2(n-k)}\cdot b^{2k}\ +\ a^{2(n-k-1)}\cdot b^{2(k+1)}\right] \\
&= \sum_{k=0}^{n-1}(-1)^k\ a^{2(n-k-1)}\cdot b^{2k}\ \left[a^2+b^2\right] \\
&= \left(a^2+b^2\right)\ \left(\sum_{k=0}^{n-1}(-1)^k\ a^{2(n-k-1)}\cdot b^{2k}\right)
\end{aligned}
$$

From the above three cases, we can generalize the factoring of $a^{2n}+b^{2n}$ when n is odd as follows:

$$
a^{2n}+b^{2n}=\begin{cases}(a+i\cdot b)\ (a-i\cdot b) & \text{, when } n=1\\ \left(a^2+b^2\right)\ \left(\sum_{k=0}^{n-1}(-1)^k\ a^{2(n-k-1)}\cdot b^{2k}\right) & \text{, otherwise}\end{cases}
$$