Problem-225

Problem Statement

If in the equation
$$ax^2 + bx + c = 0$$
the coefficient a is equal to zero then the equation takes the form
$$bx + c = 0$$
and has the unique solution
$$x = -\frac{c}{b}$$

Strictly speaking, the last sentence is wrong; when $b = 0$ the quotient $\frac{c}{b}$ is undefined. How are we to correct this error?

Solution

We should require that in the equation $ax^2 + bx + c = 0$, both a and b cannot be zero.

Problem-227

Problem Statement

Prove that $\sqrt{3}$ is irrational.

Solution

The statement '$\sqrt{3}$ is irrational' is equivalent to the statement: $\sqrt{3} \neq \frac{a}{b}$ for any integer a and b. We shall make use of **proof by contradiction**, therefore, we shall assume that $\sqrt{3} = \frac{a}{b}$ and that should lead to a contradiction, completing the proof.

We shall consider four different cases for the pair (a, b) and we shall derive a contradiction for each.

1. a and b both are odd integer. Say $a = 2m+1$ and $b = 2n+1$, where m and n are integers. According to our assumption

$$\begin{aligned}
\sqrt{3} &= \frac{2m+1}{2n+1} \\
3 &= \frac{(2m+1)^2}{(2n+1)^2} \\
3\,(2n+1)^2 &= (2m+1)^2 \\
3\,(4n^2+4n+1) &= 4m^2+4m+1 \\
12n^2+12n+3 &= 4m^2+4m+1 \\
12n^2+12n+2 &= 4m^2+4m \\
6n^2+6n+1 &= 2m^2+2m
\end{aligned}$$

On the left side we have an odd integer and on the right side we have an even integer—a contradiction. ※

2. a is an odd integer and b is an even integer. Say $a = 2m+1$ and $b = 2n$, where m and n are integers. According to our assumption

$$\begin{aligned}
\sqrt{3} &= \frac{2m+1}{2n} \\
3 &= \frac{(2m+1)^2}{4n^2} \\
12n^2 &= (2m+1)^2
\end{aligned}$$

On the left side we have an even integer and on the right side we have an odd integer—a contradiction. ※

3. a is an even integer and b is an odd integer. Say $a = 2m$ and $b = 2n+1$, where m and n are integers. According to our assumption

$$\begin{aligned}
\sqrt{3} &= \frac{2m}{2n+1} \\
3 &= \frac{4m^2}{(2n+1)^2} \\
3\,(2n+1)^2 &= 4m^2
\end{aligned}$$

On the left side we have an odd integer and on the right side we have an even integer—a contradiction. ※

4. a and b both are even integer. In this case, a and b must have 2 as a common factor. We can divide both a and b by 2. We can keep dividing by 2 as long as both remain even integer. At the end, we are in one of the previous three cases.

Problem-229

Problem Statement

Prove that for $a \geq 0,\ b > 0$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Solution

Basically we are being asked to show that $\frac{\sqrt{a}}{\sqrt{b}}$ is the square root of the non-negative integer $\frac{a}{b}$. So, we need to prove the below two:

- $\frac{\sqrt{a}}{\sqrt{b}}$ is non-negative.

  Since a is non-negative, its square root must also be non-negative. Since b is positive, its square root must also be positive. $\frac{\sqrt{a}}{\sqrt{b}}$ is thus the ratio of a non-negative number and a positive number; so, it must be non-negative.

- Squaring $\frac{\sqrt{a}}{\sqrt{b}}$ gives us $\frac{a}{b}$.

$$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b}.$$

Problem-230

Problem Statement

Is the equality $\sqrt{a^2} = a$ true for all a?

Solution

It might seem counter-intuitive that the answer is no. But if we observe closely, the question is actually asking about the equality $+\sqrt{a^2} = a$, where the '+' is implied. When we make the '+' explicit, it is more noticeable that for $a < 0$, the equality cannot hold, because the left side of the equality is always positive. The equality that works for all a is $\sqrt{a^2} = |a|$, where

$$|a| = \begin{cases} a & \text{, if } a \geq 0 \\ -a & \text{, if } a < 0 \end{cases}$$

It is unfortunate that we do not always make the '+' explicit.

Problem-232

Problem Statement

Which is bigger: $\sqrt{1001} - \sqrt{1000}$ or $\frac{1}{10}$?

Solution

It would be easier to compare if we could convert $\sqrt{1001} - \sqrt{1000}$ into a fraction.

$$\begin{aligned} &\sqrt{1001} - \sqrt{1000} \\ &= \frac{\left(\sqrt{1001} - \sqrt{1000}\right)\left(\sqrt{1001} + \sqrt{1000}\right)}{\sqrt{1001} + \sqrt{1000}} \\ &= \frac{\left(\sqrt{1001}\right)^2 - \left(\sqrt{1000}\right)^2}{\sqrt{1001} + \sqrt{1000}} \\ &= \frac{1001 - 1000}{\sqrt{1001} + \sqrt{1000}} \\ &= \frac{1}{\sqrt{1001} + \sqrt{1000}} \\ &< \frac{1}{\sqrt{900} + \sqrt{900}} \end{aligned}$$

Since $\sqrt{900} + \sqrt{900} = 60$, $\sqrt{1001} - \sqrt{1000} < \frac{1}{60} < \frac{1}{10}$. So, $\frac{1}{10}$ is the bigger between the two.

Problem-239

Problem Statement

0.0.1 Solutions to quadratic equation

A quadratic equation $x^2+px+q=0$ can be solved by "completing the square".

$$x^2+2\cdot x\cdot\frac{p}{2}+\left(\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)=0$$
$$\left(x+\frac{p}{2}\right)^2=\frac{p^2}{4}-q$$
$$x=-\frac{p}{2}+\frac{\pm\sqrt{p^2-4q}}{2}$$

There are three possible cases:

1. $p^2-4q>0$. In this case two distinct solutions $x_1,\ x_2$ exist:
   $x_1=-\frac{p}{2}+\frac{\sqrt{p^2-4q}}{2}$ and $x_2=-\frac{p}{2}-\frac{\sqrt{p^2-4q}}{2}$
2. $p^2-4q=0$. In this case the two solutions coincide; therefore, there is one distinct solution.
   $x_1=-\frac{p}{2}$ and $x_2=-\frac{p}{2}$
3. $p^2-4q<0$. In this case there is no real-valued solution.

Since the sign of p^2-4q determines how many distinct solutions there are, for convenience, we shall call this expression D, therefore, $D=p^2-4q$.

0.0.2 Vieta's theorem

If a quadratic equation $x^2+px+q=0$, has two distinct solutions α and β then

$$\alpha+\beta=-p$$
$$\alpha\cdot\beta=q$$

0.0.3 Proof-1 of Vieta's theorem

If $x^2 + px + q = 0$ has two distinct solutions, we are in the first case of 0.0.1; thus we can let α and β be as follows:

$$\alpha = -\frac{p}{2} + \frac{\sqrt{D}}{2} \tag{1}$$

$$\beta = -\frac{p}{2} - \frac{\sqrt{D}}{2} \tag{2}$$

$(1)+(2)$ gives:

$$\alpha + \beta = -p$$

$(1)\times(2)$ gives:

$$\begin{aligned}\alpha \cdot \beta &= \frac{p^2}{4} - \frac{D}{4} \\ &= \frac{p^2 - \left(p^2 - 4q\right)}{4} \\ &= q\end{aligned}$$

0.0.4 Proof-2 of Vieta's theorem

We can rewrite $x^2 + px + q = 0$ as $P(x) = 0$ where $P(x) = x^2 + px + q$ is a polynomial with degree two. If α and β are two distinct solutions of $P(x) = 0$ then

$$P(x) = (x-\alpha)\,(x-\beta)\,R(x)$$

Here $R(x)$ must be constant, otherwise right side would have degree more than two. In fact $R(x)$ must be exactly 1, otherwise the coefficient of x^2 would not match with the one in $P(x)$. So, we have

$$\begin{aligned}P(x) &= (x-\alpha)\,(x-\beta) \\ &= x^2 - (\alpha+\beta)\,x + \alpha\cdot\beta\end{aligned}$$

Comparing $P(x) = x^2 + (\ -(\alpha+\beta)\ )\ x + \alpha\cdot\beta$ with $P(x) = x^2 + px + q$ gives us Vieta's theorem.

0.0.5 Questions to answer

Can you generalize Vieta's theorem to the case of a quadratic equation having only one root? Are both proofs still valid for this case?

0.1 Solution

0.1.1 Generalization

Generalized version of Vieta's theorem is:

If a quadratic equation $x^2 + px + q = 0$ has two roots α and β (the roots can be equal) then

$$\alpha + \beta = -p$$
$$\alpha \cdot \beta = q$$

0.1.2 Proof-1

Vieta's generalized theorem already works for the first case of 0.0.1. We shall show that it works for the second case as well. In the second case of 0.0.1, $\alpha = \beta = -\frac{p}{2}$.

$$\alpha + \beta = 2\alpha = 2\left(-\frac{p}{2}\right) = -p$$

In the second case of 0.0.1 we also have $p^2 - 4q = 0$; thus $q = \frac{p^2}{4}$.

$$\alpha \cdot \beta = \alpha^2 = \frac{p^2}{4} = q$$

0.1.3 Proof-2

Since $x^2 + px + q = 0$ has two duplicate solutions, we can consider $P(x) = x^2 + px + q$ having two repeated roots, each equal to α. Thus $P(x) = (x - \alpha)(x - \alpha)$. In other words, $P(x) = x^2 - 2\alpha + \alpha^2$. A comparison gives us Vieta's theorem.

Problem-240

Problem Statement

(Vieta's theorem for a cubic equation) Assume that a cubic equation $x^3 + px^2 + qx + r = 0$ has three different roots $\alpha,\ \beta,\ \gamma$. Prove that

$$\begin{aligned} \alpha + \beta + \gamma &= -p \\ \alpha\beta + \alpha\gamma + \beta\gamma &= q \\ \alpha\beta\gamma &= -r \end{aligned}$$

0.2 Solution

The polynomial $P(x) = x^3 + px^2 + qx + r$ can be written as

$$P(x) = (x - \alpha)\,(x - \beta)\,(x - \gamma)\,R(x)$$

To match the coefficient of x^3, $R(x)$ must be 1. Therefore,

$$\begin{aligned} P(x) &= (x - \alpha)\,(x - \beta)\,(x - \gamma) \\ &= x^3 - (\alpha + \beta + \gamma)\,x^2 + (\alpha \cdot \beta + \alpha \cdot \gamma + \beta \cdot \gamma)\,x - \alpha \cdot \beta \cdot \gamma \end{aligned}$$

On comparing against $P(x) = x^3 + px^2 + qx + r$ we have

$$\begin{aligned} \alpha + \beta + \gamma &= -p \\ \alpha\beta + \alpha\gamma + \beta\gamma &= q \\ \alpha\beta\gamma &= -r \end{aligned}$$