Problem-304

Problem Statement

(a) Find the side of a square having the same perimeter as a rectangle with sides a and b.

(b) Find the side of a square having the same area as a rectangle with sides a and b.

Solution

(a) The question could also ask what are the sides of the rectangle having the same perimeter as a rectangle with sides a and b, but has the maximum possible area. From **Problem-263**, we know that the rectangle with the maximum area would be a square. Say the side of the square is x. Then we need $4x = 2(a+b)$, or $x = \frac{a+b}{2}$. Therefore, the side of the square is the arithmetic mean of the sides of the rectangle.

(b) The question could also ask what are the sides of the rectangle having the same area as a rectangle with sides a and b, but has the minimum possible preimeter. From **Problem-264**, we know that the rectangle with the minimum perimeter would be a square. We need $x^2 = a \cdot b$, or $x = \sqrt{a \cdot b}$. Therefore, the side of the square is the geometric mean of the sides of the rectangle.

Problem-317

Problem Statement

Prove the inequality between arithmetic and geometric means for $n = 4$.

Solution

For non-negative integers a, b, c, d, we need to prove

$$\sqrt[4]{a \cdot b \cdot c \cdot d} \leq \frac{a+b+c+d}{4}$$

We make the below two observations, (1) and (2) which we use during the proof.

$$\begin{aligned} a \cdot b &\quad ? \quad \frac{a+b}{2} \cdot \frac{a+b}{2} \\ 4 \cdot a \cdot b &\quad ? \quad (a+b)^2 \\ 0 &\quad ? \quad (a-b)^2 \\ 0 &\quad \leq \quad (a-b)^2 \end{aligned}$$

Thus we have

$$\frac{a+b}{2} \cdot \frac{a+b}{2} \geq a \cdot b \tag{1}$$

Similarly,

$$\frac{a+b}{2} \cdot \frac{c+d}{2} \quad ? \quad \frac{a+b+c+d}{4} \cdot \frac{a+b+c+d}{4}$$

Let $a+b=x$ and $c+d=y$, and we have

$$\begin{aligned} \frac{x}{2} \cdot \frac{y}{2} \quad &? \quad \frac{x+y}{4} \cdot \frac{x+y}{4} \\ 4 \cdot x \cdot y \quad &? \quad (x+y)^2 \\ 0 \quad &? \quad (x-y)^2 \\ 0 \quad &\leq \quad (x-y)^2 \end{aligned}$$

Thus we have,

$$\frac{a+b+c+d}{4} \cdot \frac{a+b+c+d}{4} \geq \frac{a+b}{2} \cdot \frac{c+d}{2} \tag{2}$$

Now we peform a sequence of transformations on the four numbers (a,b,c,d). After each transformation, the sum remains $a+b+c+d$ but the product is bigger or equal to $a \cdot b \cdot c \cdot d$.

$$(a,b,c,d) \mapsto \left(\frac{a+b}{2}, \frac{a+b}{2}, c, d\right)$$

In the above transformation, the product increases or remains the same (when $a=b$) because of (1).

$$\left(\frac{a+b}{2}, \frac{a+b}{2}, c, d\right) \mapsto \left(\frac{a+b}{2}, \frac{a+b}{2}, \frac{c+d}{2}, \frac{c+d}{2}\right)$$

In the above transformation, the product increases or remains the same (when $c=d$) because of (1).

$$\left(\frac{a+b}{2}, \frac{a+b}{2}, \frac{c+d}{2}, \frac{c+d}{2}\right) \mapsto \left(\frac{a+b+c+d}{4}, \frac{a+b}{2}, \frac{a+b+c+d}{4}, \frac{c+d}{2}\right)$$

In the above transformation, the product increases or remains the same (when $a+b=c+d$) because of (2).

$$\left(\frac{a+b+c+d}{4}, \frac{a+b}{2}, \frac{a+b+c+d}{4}, \frac{c+d}{2}\right) \mapsto \left(\frac{a+b+c+d}{4}, \frac{a+b+c+d}{4}, \frac{a+b+c+d}{4}, \frac{a+b+c+d}{4}\right)$$

In the above transformation, the product increases or remains the same (when $a+b=c+d$) because of (2). Let $\frac{a+b+c+d}{4} = S$. Then from the last transformation, we have

$$\begin{aligned} a \cdot b \cdot c \cdot d &\leq S \cdot S \cdot S \cdot S \\ \sqrt[4]{a \cdot b \cdot c \cdot d} &\leq \frac{a+b+c+d}{4} \qquad \blacksquare \end{aligned}$$

Problem-320

Problem Statement

Prove the inequality between arithmetic and geometric means for $n = 3$.

Solution

We shall reduce the case for $n = 3$ to the case for $n = 4$ and use the result from **Problem-317** to finish it off.

For three non-negative integers $a,\ b,\ c$ we are asked to prove

$$\sqrt[3]{a\cdot b\cdot c} \le \frac{a+b+c}{3}$$

We shall throw in the geometric mean of the three integers and form a group of four non-negative integers: $(a, b, c, \sqrt[3]{a\cdot b\cdot c})$. From **Problem-317** we know

$$\sqrt[4]{abc\ \sqrt[3]{abc}} \le \frac{a+b+c+\sqrt[3]{abc}}{4} \tag{1}$$

We note that $\sqrt[4]{abc\ \sqrt[3]{abc}} = \sqrt[4]{(abc)^1\cdot(abc)^{\frac{1}{3}}} = \sqrt[4]{(abc)^{\frac{4}{3}}} = \sqrt[3]{abc}$. So, from (1) now we have

$$\begin{aligned}
\sqrt[3]{abc} &\le \frac{a+b+c+\sqrt[3]{abc}}{4}\\
4\sqrt[3]{abc} &\le a+b+c+\sqrt[3]{abc}\\
\sqrt[3]{a\cdot b\cdot c} &\le \frac{a+b+c}{3} \qquad \blacksquare
\end{aligned}$$

Problem-323

Problem Statement

Prove the inequality between arithmetic and geometric means for all integer $n \ge 2$.

Solution

For $n \ge 2$ non-negative integers $a_1,\ a_2,\ \ldots,\ a_n$ we are asked to prove

$$\sqrt[n]{\prod_{k=1}^{n} a_k} \le \frac{\sum_{k=1}^{n} a_k}{n} \tag{1}$$

Proof-1

We can prove (1) for $n = 2^m$ where $m \geq 1$ using the transformation idea from **Problem-317**. Say $n \geq 2$, lies in between 2^p and 2^{p+1}. We already know (1) holds for 2^{p+1} numbers; using that, we can use the idea from **Problem-320** to prove it for $2^{p+1} - 1$ numbers as well. Applying the idea from **Problem-320** in sequence, starting wtih $2^{p+1} - 1$ numbers and going backwards, we can prove (1) for n.

Proof-2

Let's scale each of the n numbers $\sigma > 0$ times and see what happens to their arithmetic and geometric means. We start off with arithmetic mean:

$$
\begin{aligned}
&\frac{\sum_{k=1}^{n} \sigma \cdot a_k}{n} \\
&= \sigma \cdot \frac{\sum_{k=1}^{n} a_k}{n}
\end{aligned}
$$

Let's now look at the modified geometric mean:

$$
\begin{aligned}
&\sqrt[n]{\prod_{k=1}^{n} \sigma \cdot a_k} \\
&= \sqrt[n]{\sigma^n \prod_{k=1}^{n} a_k} \\
&= \sigma \sqrt[n]{\prod_{k=1} n\ a_k}
\end{aligned}
$$

We see that both arithmetic and geometric means have been scaled by the same factor σ thus (1) holds for $a'_k = \sigma \cdot a_k$, if it holds for a_k.

Let $\sum_{k=1}^{n} a_k = \psi$. If all a_k's are not zero, $\psi > 0$. We can now scale a_k's to get a'_k's such that $a'_k = \frac{n}{\psi} \cdot a_k$. Since scaling numbers by the same amount does not change the relation between their arithmetic and geometric means, if we can show (1) for a'_k's that would be sufficient. Now, observe that $\sum_{k=1}^{n} a'_k = n$. So, for a'_k's the inequality (1) takes the below form:

$$
\sqrt[n]{\prod_{k=1}^{n} a'_k} \leq 1
$$

We shall now try to prove this derived inequality.

1. For $n = 2$. We thus have $a'_1 + a'_2 = 2$. If both numbers are not equal to 1, we can let $a'_1 = 1 - \delta$ and $a'_2 = 1 + \delta$ with $\delta > 0$. Now the product $a'_1 \cdot a'_2 = (1 - \delta)(1 + \delta) = 1 - \delta^2 \leq 1$. That is what we needed.

2. For $n = 3$. Now we have $a'_1 + a'_2 + a'_3 = 3$. If all numbers are not equal to 1 (if they are, we are done), one should be less than 1 and another should be

greater than 1. Say $a_1' < 1$ and $a_2' > 1$. So, $a_1' - 1 < 0$ and $a_2' - 1 > 0$.

$$\begin{aligned}
\left(a_1' - 1\right)\left(a_2' - 1\right) &< 0 \\
a_1'a_2' - a_1' - a_2' + 1 &< 0 \\
a_1'a_2' + 1 &< a_1' + a_2' \\
a_1'a_2' + 1 + a_3' &< a_1' + a_2' + a_3' \\
a_1'a_2' + 1 + a_3' &< 3 \\
a_1'a_2' + a_3' &< 2
\end{aligned}$$

We have now back to the first case where we have two non-negative numbers $a_1'a_2'$ and a_3' and they sum to 2. So, we know how to go about the proof henceforth.

3. For $n = 4$. We have $a_1' + a_2' + a_3' + a_4' = 4$. With a similar argument to the second case, we arrive at the below inequality:

$$a_1'a_2' + a_3' + a_4' < 3$$

This takes us back to the second case and we know the rest.

For any $n > 3$, we can use the reducing idea in the second and third cases to go back to our base case, namely the case with $n = 2$ which we have proved already.

Proof-3

Let's prove an identity. We start off with $(a+b+c)^3$:

$$\begin{aligned}
&(a+b+c)^3 \\
&= a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3b^2c + 3bc^2 + 3c^2a + 3ca^2 + 6abc \\
&= a^3 + b^3 + c^3 - 3abc + 3a^2b + 3ab^2 + 3abc + 3abc + 3b^2c + 3bc^2 + 3ca^2 + 3abc + 3c^2a \\
&= a^3 + b^3 + c^3 - 3abc + 3\,(a+b+c)\,(ab+bc+ca)
\end{aligned}$$

We can now rearrange the two sides as follows:

$$\begin{aligned}
&a^3 + b^3 + c^3 - 3abc \\
&= (a+b+c)^3 - 3\,(a+b+c)\,(ab+bc+ca) \\
&= (a+b+c)\left[(a+b+c)^2 - 3(ab+bc+ca)\right] \\
&= (a+b+c)\,\left(a^2 + b^2 + c^2 - ab - bc - ca\right) \\
&= \frac{1}{2}\,(a+b+c)\left(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca\right) \\
&= \frac{1}{2}\,(a+b+c)\left[\left(a^2 - 2ab + b^2\right) + \left(b^2 - 2bc + c^2\right) + \left(c^2 - 2ca + a^2\right)\right] \\
&= \frac{1}{2}\,(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]
\end{aligned}$$

We see that for non-negative numbers $a,\ b,\ c$, the right side cannot be negative. So,

$$a^3 + b^3 + c^3 - 3abc \geq 0$$

$$abc \leq \frac{a^3 + b^3 + c^3}{3}$$

Now if we make the replacements: $a = \sqrt[3]{p},\ b = \sqrt[3]{q},\ c = \sqrt[3]{r}$, then we have our required inequality between geometric and arithmetic means of three non-negative numbers:

$$\sqrt[3]{p \cdot q \cdot r} \leq \frac{p + q + r}{3}$$

If we have more than 3 numbers for which we need to establish (1), we can always group (and replace the group with the group-sum) some of the numbers so that we end up with three numbers and appeal to our just-established fact.

Problem-326

Problem Statement

Assume that $a_1,\ \ldots,\ a_n$ are positive numbers. Prove that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n$$

Solution

Conside the below product:

$$\frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \cdots \cdot \frac{a_{n-1}}{a_n} \cdot \frac{a_n}{a_1}$$

Each a_i with $1 \leq i \leq n$ appears exactly once above and below, so the product evaluates to 1. We thus have:

$$\sqrt[n]{\frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \cdots \cdot \frac{a_{n-1}}{a_n} \cdot \frac{a_n}{a_1}} = 1$$

In other words, the geometric mean of the positive numbers $\frac{a_1}{a_2},\ \frac{a_2}{a_3},\ \ldots,\ \frac{a_{n-1}}{a_n},\ \frac{a_n}{a_1}$ is 1. Their arithmetic mean cannot be less than 1. So, we have:

$$\frac{\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}}{n} \geq 1$$

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n \qquad \blacksquare$$