Problem-72

Problem Statement

There is a rule that allows us to square any number with the last digit 5, namely, "Drop this last digit out and get some n; multiply n by $n+1$ and add the digits 2 and 5 to the end." For example, for 35^2, we delete 5 and get 3, multiplying 3 and 4 we get 12, adding "2" and "5" we get the answer: 1225. Explain why this rule works.

Solution

Let's say we want to find the square of an n digits decimal number $d_{n-1}d_{n-2}\ldots d_1 d_0$ with $d_0 = 5$. Following expansion justifies the rule:

$$
\begin{aligned}
d_{n-1}d_{n-2}\ldots d_1 5^2 &= (d_{n-1}d_{n-2}\ldots d_1 0 + 5)^2 \\
&= d_{n-1}d_{n-2}\ldots d_1 0^2 + 2 \times d_{n-1}d_{n-2}\ldots d_1 0 \times 5 + 25 \\
&= d_{n-1}d_{n-2}\ldots d_1 0 \times (d_{n-1}d_{n-2}\ldots d_1 0 + 10) + 25 \\
&= d_{n-1}d_{n-2}\ldots d_1 0 \times d_{n-1}d_{n-2}\ldots d_1' 0 + 25 \quad [d_1' = d_1 + 1 \text{ with carry}] \\
&= d_{n-1}d_{n-2}\ldots d_1 \times d_{n-1}d_{n-2}\ldots d_1' \times 100 + 25
\end{aligned}
$$

Note, $d_{n-1}d_{n-2}\ldots d_1$ and $d_{n-1}d_{n-2}\ldots d_1'$ are consecutive integers. ∎