

## Problem-225

### Problem Statement

If in the equation

$$ax^2 + bx + c = 0$$

the coefficient  $a$  is equal to zero then the equation takes the form

$$bx + c = 0$$

and has the unique solution

$$x = -\frac{c}{b}$$

Strictly speaking, the last sentence is wrong; when  $b = 0$  the quotient  $\frac{c}{b}$  is undefined. How are we to correct this error?

### Solution

We should require that in the equation  $ax^2 + bx + c = 0$ , both  $a$  and  $b$  cannot be zero.

## Problem-227

### Problem Statement

Prove that  $\sqrt{3}$  is irrational.

### Solution

The statement ' $\sqrt{3}$  is irrational' is equivalent to the statement:  $\sqrt{3} \neq \frac{a}{b}$  for any integer  $a$  and  $b$ . We shall make use of **proof by contradiction**, therefore, we shall assume that  $\sqrt{3} = \frac{a}{b}$  and that should lead to a contradiction, completing the proof.

We shall consider four different cases for the pair  $(a, b)$  and we shall derive a contradiction for each.

1.  $a$  and  $b$  both are odd integer. Say  $a = 2m + 1$  and  $b = 2n + 1$ , where  $m$  and  $n$  are integers. According to our assumption

$$\begin{aligned}\sqrt{3} &= \frac{2m+1}{2n+1} \\ 3 &= \frac{(2m+1)^2}{(2n+1)^2} \\ 3(2n+1)^2 &= (2m+1)^2 \\ 3(4n^2+4n+1) &= 4m^2+4m+1 \\ 12n^2+12n+3 &= 4m^2+4m+1 \\ 12n^2+12n+2 &= 4m^2+4m \\ 6n^2+6n+1 &= 2m^2+2m\end{aligned}$$

On the left side we have an odd integer and on the right side we have an even integer—a contradiction. ✖

2.  $a$  is an odd integer and  $b$  is an even integer. Say  $a = 2m + 1$  and  $b = 2n$ , where  $m$  and  $n$  are integers. According to our assumption

$$\begin{aligned}\sqrt{3} &= \frac{2m+1}{2n} \\ 3 &= \frac{(2m+1)^2}{4n^2} \\ 12n^2 &= (2m+1)^2\end{aligned}$$

On the left side we have an even integer and on the right side we have an odd integer—a contradiction. ✖

3.  $a$  is an even integer and  $b$  is an odd integer. Say  $a = 2m$  and  $b = 2n + 1$ , where  $m$  and  $n$  are integers. According to our assumption

$$\begin{aligned}\sqrt{3} &= \frac{2m}{2n+1} \\ 3 &= \frac{4m^2}{(2n+1)^2} \\ 3(2n+1)^2 &= 4m^2\end{aligned}$$

On the left side we have an odd integer and on the right side we have an even integer—a contradiction. ✖

4.  $a$  and  $b$  both are even integer. In this case,  $a$  and  $b$  must have 2 as a common factor. We can divide both  $a$  and  $b$  by 2. We can keep dividing by 2 as long as both remain even integer. At the end, we are in one of the previous three cases.

## Problem-229

### Problem Statement

Prove that for  $a \geq 0$ ,  $b > 0$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

### Solution

Basically we are being asked to show that  $\frac{\sqrt{a}}{\sqrt{b}}$  is the square root of the non-negative integer  $\frac{a}{b}$ . So, we need to prove the below two:

- $\frac{\sqrt{a}}{\sqrt{b}}$  is non-negative.

Since  $a$  is non-negative, its square root must also be non-negative. Since  $b$  is positive, its square root must also be positive.  $\frac{\sqrt{a}}{\sqrt{b}}$  is thus the ratio of a non-negative number and a positive number; so, it must be non-negative.

- Squaring  $\frac{\sqrt{a}}{\sqrt{b}}$  gives us  $\frac{a}{b}$ .

$$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b}.$$

## Problem-230

### Problem Statement

Is the equality  $\sqrt{a^2} = a$  true for all  $a$ ?

## Solution

It might seem counter-intuitive that the answer is no. But if we observe closely, the question is actually asking about the equality  $+\sqrt{a^2} = a$ , where the '+' is implied. When we make the '+' explicit, it is more noticeable that for  $a < 0$ , the equality cannot hold, because the left side of the equality is always positive. The equality that works for all  $a$  is  $\sqrt{a^2} = |a|$ , where

$$|a| = \begin{cases} a & , \text{ if } a \geq 0 \\ -a & , \text{ if } a < 0 \end{cases}$$

It is unfortunate that we do not always make the '+' explicit.

## Problem-232

### Problem Statement

Which is bigger:  $\sqrt{1001} - \sqrt{1000}$  or  $\frac{1}{10}$ ?

### Solution

It would be easier to compare if we could convert  $\sqrt{1001} - \sqrt{1000}$  into a fraction.

$$\begin{aligned} & \sqrt{1001} - \sqrt{1000} \\ &= \frac{(\sqrt{1001} - \sqrt{1000})(\sqrt{1001} + \sqrt{1000})}{\sqrt{1001} + \sqrt{1000}} \\ &= \frac{(\sqrt{1001})^2 - (\sqrt{1000})^2}{\sqrt{1001} + \sqrt{1000}} \\ &= \frac{1001 - 1000}{\sqrt{1001} + \sqrt{1000}} \\ &= \frac{1}{\sqrt{1001} + \sqrt{1000}} \\ &< \frac{1}{\sqrt{900} + \sqrt{900}} \end{aligned}$$

Since  $\sqrt{900} + \sqrt{900} = 60$ ,  $\sqrt{1001} - \sqrt{1000} < \frac{1}{60} < \frac{1}{10}$ . So,  $\frac{1}{10}$  is the bigger between the two.

## Problem-239

### Problem Statement

#### 0.0.1 Solutions to quadratic equation

A quadratic equation  $x^2 + px + q = 0$  can be solved by “completing the square”.

$$\begin{aligned}x^2 + 2 \cdot x \cdot \frac{p}{2} + \left(\frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right) &= 0 \\ \left(x + \frac{p}{2}\right)^2 &= \frac{p^2}{4} - q \\ x &= -\frac{p}{2} + \frac{\pm\sqrt{p^2 - 4q}}{2}\end{aligned}$$

There are three possible cases:

1.  $p^2 - 4q > 0$ . In this case two distinct solutions  $x_1, x_2$  exist:

$$x_1 = -\frac{p}{2} + \frac{\sqrt{p^2 - 4q}}{2} \text{ and } x_2 = -\frac{p}{2} - \frac{\sqrt{p^2 - 4q}}{2}$$

2.  $p^2 - 4q = 0$ . In this case the two solutions coincide; therefore, there is one distinct solution.

$$x_1 = -\frac{p}{2} \text{ and } x_2 = -\frac{p}{2}$$

3.  $p^2 - 4q < 0$ . In this case there is no real-valued solution.

Since the sign of  $p^2 - 4q$  determines how many distinct solutions there are, for convenience, we shall call this expression  $D$ , therefore,  $D = p^2 - 4q$ .

#### 0.0.2 Vieta's theorem

If a quadratic equation  $x^2 + px + q = 0$ , has two distinct solutions  $\alpha$  and  $\beta$  then

$$\alpha + \beta = -p$$

$$\alpha \cdot \beta = q$$

### 0.0.3 Proof-1 of Vieta's theorem

If  $x^2 + px + q = 0$  has two distinct solutions, we are in the first case of 0.0.1; thus we can let  $\alpha$  and  $\beta$  be as follows:

$$\alpha = -\frac{p}{2} + \frac{\sqrt{D}}{2} \quad (1)$$

$$\beta = -\frac{p}{2} - \frac{\sqrt{D}}{2} \quad (2)$$

(1) + (2) gives:

$$\alpha + \beta = -p$$

(1)  $\times$  (2) gives:

$$\begin{aligned} \alpha \cdot \beta &= \frac{p^2}{4} - \frac{D}{4} \\ &= \frac{p^2 - (p^2 - 4q)}{4} \\ &= q \end{aligned}$$

### 0.0.4 Proof-2 of Vieta's theorem

We can rewrite  $x^2 + px + q = 0$  as  $P(x) = 0$  where  $P(x) = x^2 + px + q$  is a polynomial with degree two. If  $\alpha$  and  $\beta$  are two distinct solutions of  $P(x) = 0$  then

$$P(x) = (x - \alpha)(x - \beta)R(x)$$

Here  $R(x)$  must be constant, otherwise right side would have degree more than two. In fact  $R(x)$  must be exactly 1, otherwise the coefficient of  $x^2$  would not match with the one in  $P(x)$ . So, we have

$$\begin{aligned} P(x) &= (x - \alpha)(x - \beta) \\ &= x^2 - (\alpha + \beta)x + \alpha \cdot \beta \end{aligned}$$

Comparing  $P(x) = x^2 + (-(\alpha + \beta))x + \alpha \cdot \beta$  with  $P(x) = x^2 + px + q$  gives us Vieta's theorem.

### 0.0.5 Questions to answer

Can you generalize Vieta's theorem to the case of a quadratic equation having only one root? Are both proofs still valid for this case?

## 0.1 Solution

### 0.1.1 Generalization

Generalized version of Vieta's theorem is:

If a quadratic equation  $x^2 + px + q = 0$  has two roots  $\alpha$  and  $\beta$  (the roots can be equal) then

$$\alpha + \beta = -p$$

$$\alpha \cdot \beta = q$$

### 0.1.2 Proof-1

Vieta's generalized theorem already works for the first case of 0.0.1. We shall show that it works for the second case as well. In the second case of 0.0.1,  $\alpha = \beta = -\frac{p}{2}$ .

$$\alpha + \beta = 2\alpha = 2\left(-\frac{p}{2}\right) = -p$$

In the second case of 0.0.1 we also have  $p^2 - 4q = 0$ ; thus  $q = \frac{p^2}{4}$ .

$$\alpha \cdot \beta = \alpha^2 = \frac{p^2}{4} = q$$

### 0.1.3 Proof-2

Since  $x^2 + px + q = 0$  has two duplicate solutions, we can consider  $P(x) = x^2 + px + q$  having two repeated roots, each equal to  $\alpha$ . Thus  $P(x) = (x - \alpha)(x - \alpha)$ . In other words,  $P(x) = x^2 - 2\alpha x + \alpha^2$ . A comparison gives us Vieta's theorem.

## Problem-240

### Problem Statement

(Vieta's theorem for a cubic equation) Assume that a cubic equation  $x^3 + px^2 + qx + r = 0$  has three different roots  $\alpha$ ,  $\beta$ ,  $\gamma$ . Prove that

$$\alpha + \beta + \gamma = -p$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = q$$

$$\alpha\beta\gamma = -r$$

## 0.2 Solution

The polynomial  $P(x) = x^3 + px^2 + qx + r$  can be written as

$$P(x) = (x - \alpha)(x - \beta)(x - \gamma)R(x)$$

To match the coefficient of  $x^3$ ,  $R(x)$  must be 1. Therefore,

$$\begin{aligned} P(x) &= (x - \alpha)(x - \beta)(x - \gamma) \\ &= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma \end{aligned}$$

On comparing against  $P(x) = x^3 + px^2 + qx + r$  we have

$$\begin{aligned} \alpha + \beta + \gamma &= -p \\ \alpha\beta + \alpha\gamma + \beta\gamma &= q \\ \alpha\beta\gamma &= -r \end{aligned}$$

## Problem-259

### Problem Statement

Starting with the graph for  $y = x^2$  we can get the graph for  $y = a(x + m)^2 + n$  as follows:

- (a) Stretching it vertically  $a$  times gives us the graph for  $y = ax^2$ . If  $a > 1$ , the graph gets steeper. If  $0 < a < 1$ , the graph gets flatter. If  $a < 0$ , the graph becomes upside down.
- (b) Moving the resulting graph  $m$  units to the left gives us the graph for  $y = a(x + m)^2$ . If  $m > 0$ , the graph moves to the left. If  $m < 0$ , the graph moves to the right.
- (c) Moving the resulting graph  $n$  units up gives us the graph for  $y = a(x + m)^2 + n$ . If  $n > 0$ , the graph moves upwards. If  $n < 0$ , the graph moves downwards.

There are six possible orderings of operations (a), (b), and (c). Do we get six different graphs or do some of the graphs coincide?



## Solution

We do not get six different graphs. The six graphs fall into two distinct groups: one for the function  $y = a(x + m)^2 + n$  and the other for the function  $y = a(x + m)^2 + an$ . Since (b) is the only one that affects  $x$ -coordinates, (b) does not interfere with (a) and (c). On the other hand, (a) and (c) both affect  $y$ -coordinates and thus they interfere with each other. Which of these two groups a graph falls into, thus, depends on the order of (a) and (c). If (c) follows (a), we get the graph for the function  $y = a(x + m)^2 + n$ . If (a) follows (c), we get the graph for the function  $y = a(x + m)^2 + an$ .

1. (c) follows (a):

- i.  $y = x^2 \xrightarrow{(a)} y = ax^2 \xrightarrow{(b)} y = a(x + m)^2 \xrightarrow{(c)} y = a(x + m)^2 + n$
- ii.  $y = x^2 \xrightarrow{(a)} y = ax^2 \xrightarrow{(c)} y = ax^2 + n \xrightarrow{(b)} y = a(x + m)^2 + n$
- iii.  $y = x^2 \xrightarrow{(b)} y = (x + m)^2 \xrightarrow{(a)} y = a(x + m)^2 \xrightarrow{(c)} y = a(x + m)^2 + n$

2. (a) follows (c):

- i.  $y = x^2 \xrightarrow{(b)} y = (x + m)^2 \xrightarrow{(c)} y = (x + m)^2 + n \xrightarrow{(a)} y = a(x + m)^2 + an$
- ii.  $y = x^2 \xrightarrow{(c)} y = x^2 + n \xrightarrow{(a)} y = ax^2 + an \xrightarrow{(b)} y = a(x + m)^2 + an$
- iii.  $y = x^2 \xrightarrow{(c)} y = x^2 + n \xrightarrow{(b)} y = (x + m)^2 + n \xrightarrow{(a)} y = a(x + m)^2 + an$

## Problem-260

### Problem Statement

How can you determine the signs of  $a$ ,  $b$ ,  $c$  by looking at the graph of  $y = ax^2 + bx + c$ ?

### 0.3 Solution

If  $a = 0$  but  $b \neq 0$ , we have  $y = bx + c$ , thus our graph is no longer a parabola, but a straight line. The slope and  $y$ -intercept of the straight line gives the signs of  $b$  and  $c$ . On the other hand, if  $b = 0$  but  $a \neq 0$ , we have  $y = ax^2 + c$ . This is similar to the form  $y = a(x + m)^2 + n$ , so we would know the signs

of  $a$  and  $c$  easily. If both  $a$  and  $b$  are zero, we have  $y = c$ . So, we have a graph that is a straight line and parallel to the  $x$ -axis. So, if the line lies above the  $x$ -axis,  $c > 0$ , else  $c < 0$ . We shall now consider the more interesting case where  $a \neq 0$  and  $b \neq 0$ . Since we know a lot about the graph of  $y = a(x + m)^2 + n$ , let's transform  $y = ax^2 + bx + c$  into that form.

$$\begin{aligned}
 y &= ax^2 + bx + c \\
 &= a \left( x^2 + x \cdot \frac{b}{a} \right) + c \\
 &= a \left( x^2 + 2 \cdot x \cdot \frac{b}{2a} + \frac{b^2}{4a^2} \right) + c - \frac{b^2}{4a^2} \\
 &= a \left( x + \frac{b}{2a} \right)^2 + \left( c - \frac{b^2}{4a^2} \right)
 \end{aligned}$$

Comparing with  $y = a(x + m)^2 + n$ , we can consider  $\frac{b}{2a} = m$ .

Below is how we can find out the signs of  $a$ ,  $b$ ,  $c$  :

- Compared to the graph of  $y = x^2$ , if our graph is upside down, then  $a < 0$ ; otherwise  $a > 0$ .
- $\frac{b}{2a}$  determines how much our graph is shifted horizontally compared to the graph of  $y = x^2$ . So, if we know the sign of  $a$  and how the graph is shifted horizontally, then we can deduce the sign of  $b$ .

	$a > 0$	$a < 0$
Shifted left	$b > 0$	$b < 0$
Shifted right	$b < 0$	$b > 0$
Unshifted	$b = 0$	$b = 0$

- When  $x = 0$ ,  $y = ax^2 + bx + c$  becomes  $y = c$ . So, looking at where our graph intersects with the line  $x = 0$  aka the  $y$ -axis, we would know the sign of  $c$ .

## Problem-262

### Problem Statement

The sum of two numbers is equal to 1. What is the maximal possible value of their product?

## Solution

We see that  $347 + (-346) = 1$ , but their product is negative. Whereas  $0 + 1 = 1$ , and their product is 0 and this is better. Can the product be positive? What about  $\frac{1}{10} + \frac{9}{10}$ ? In this case the product is  $\frac{9}{100}$  and is the best among the three examples we have just considered.

Say one number is  $x$  and the other is  $1 - x$ . So, the product is the polynomial  $x(1 - x) = -x^2 + x = -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}$ . So, the graph of the product polynomial is a parabola that is upside down,  $\frac{1}{2}$  units shifted right; therefore the vertex or crest is at  $x = \frac{1}{2}$  and that is where maximal value occurs. So, the maximal value is  $\frac{1}{2} \left(1 - \frac{1}{2}\right) = \frac{1}{4}$ .

## Problem-263

### Problem Statement

Prove that a square has the maximum area of all rectangles having the same perimeter.

## Solution

Say the perimeter is  $a$ . If one side of the rectangle is  $x$ , then the other side must be  $\frac{a}{2} - x$ . Now the area is a polynomial:  $x \left(\frac{a}{2} - x\right) = -x^2 + x \cdot \frac{a}{2} = -\left(x - \frac{a}{4}\right)^2 + \frac{a^2}{16}$ . Thus the area has the maximum value when  $x = \frac{a}{4}$ . But in that case the other side is also  $\frac{a}{4}$ , making it a square.

## Problem-264

### Problem Statement

Prove that a square has the minimum perimeter of all rectangles having the same area.

## Solution

So we need to prove that among all rectangles with an area say  $a$ , the square has the minimum perimeter. Say  $x$  is one side of a rectangle with area  $a$ . The

other side then is  $\frac{a}{x}$ . The perimeter is a polynomial  $P(x) = 2 \left(x + \frac{a}{x}\right)$ . Consider the below equation, where  $c > 0$ .

$$2 \left(x + \frac{a}{x}\right) = c \quad (1)$$

We shall first find out for which  $c > 0$ , the equation (1) has solutions. Then from those values of  $c > 0$ , we shall find out the minimum, because that gives us the minimum perimeter for a given area  $a$ . We shall then see, for that minimum  $c > 0$ , what value  $x$  takes. Once we know how  $x$  and the other side  $\frac{a}{x}$  look, we would know what type of rectangle has this minimum perimeter. Note, since this is a rectangle we are talking about, we are interested in  $x > 0$  solutions.

$$\begin{aligned} 2 \left(x + \frac{a}{x}\right) &= c \\ x^2 - x \cdot \frac{c}{2} + a &= 0 \\ \left(x - \frac{c}{4}\right)^2 &= \frac{c^2}{16} - a \\ x &= \frac{c}{4} + \frac{\pm \sqrt{c^2 - 16a}}{16} \end{aligned}$$

So, (1) has solutions when  $c^2 - 16a \geq 0$ . In other words, when  $c \geq 4\sqrt{a}$  or  $c \leq -4\sqrt{a}$ , the equation (1) has solutions. Hence, the minimum positive value of  $c$  for which (1) has a solution is  $4\sqrt{a}$  and at this value of  $c$ , we have  $x = \frac{c}{4} = \frac{4\sqrt{a}}{4} = \sqrt{a}$ . The other side is  $\frac{a}{x} = \frac{a}{\sqrt{a}} = \sqrt{a}$ . Therefore, the rectangle with area  $a$  that has the minimum perimeter is a square.

In another way, say a square with area  $a$  has perimeter  $p$ . Now, from **Problem-263**, we know that  $a$  is the maximum possible area with the perimeter  $p$ . For all other rectangles, if they want to achieve the area  $a$ , their perimeter must be bigger than  $p$ . So, with area  $a$ ,  $p$  is the minimum possible perimeter for a rectangle and that happens when the rectangle is a square.