Problem 1

Given,
A program A that generates a number in [0, 1] uniformly at random. Also we have a discrete random variable **X** whose probability distribution is :

P(X = 1) = 0.4
P(X = 0) = 0.4
P(X = -1) = 0.2

To draw random samples of X we can use A. This can be done by distributing the numbers in A as follows

$$\left\{ \begin{array}{ll} X = 1, & \text{for A in } 0 \leq a < \frac{2}{5} \\ X = 0, & \text{for A in } \frac{2}{5} \leq a < \frac{4}{5} \\ X = -1, & \text{for A in } \frac{4}{5} \leq a \leq 1 \end{array} \right\}$$

Problem 2

Given,
Number of red balls = 3
Number of blue balls =4
Number of green balls = 5
Total number of balls in the urn = 12

- Probability of Alice drawing a green ball: $Pr(A_G) = \frac{5}{12}$
- Probability of Alice not drawing a green ball: $Pr(A'_G) = \frac{7}{12}$
- Probability of Bob drawing a green ball given Alice has drawn a green ball : $Pr(B_G|A_G) = \frac{4}{11}$
- Probability of Bob drawing a green ball given Alice has not drawn a green ball : $Pr(B_G|A'_G) = \frac{5}{11}$

Then,
Probability that Bob drew a green ball :

$Pr(B_G) = \sum_{A_G} Pr(B_G|A_G)Pr(A_G)$

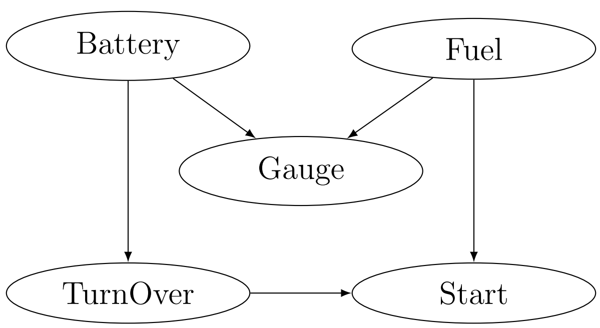
$= Pr(B_G|A_G)Pr(A_G) + Pr(B_G|A'_G)Pr(A'_G)$

$= \frac{4}{11} * \frac{5}{12} + \frac{5}{11} * \frac{7}{12} = \frac{55}{132} = 0.4166$

Problem 3

Battery(B)	
B=bad	B=good
0.02	0.98

Fuel	
F=empty	F=not empty
0.05	0.95



		Gauge(G)	
Battery(B)	Fuel(F)	empty	not empty
good	not empty	0.04	0.96
good	empty	0.97	0.03
bad	not empty	0.10	0.90
bad	empty	0.99	0.01

	Turn Over(T)	
Battery	T=false	T=true
good	0.03	0.97
bad	0.98	0.02

		Start(S)	
TurnOver(T)	Fuel(F)	F	T
true	not empty	0.01	0.99
true	empty	0.92	0.08
false	not empty	1.00	0.00
false	empty	0.99	0.01

Let $P(F = empty) = P(f)$;$P(S = false) = P(s)$;$P(T = false) = P(t)$; $P(G = empty) = P(g)$

Then,
$P(F = empty|S = false) = P(f|s) = P(f, s)/P(s)$

From the bayesian network, we can write the expression as:
$P(f, s) = \sum_B \sum_G \sum_T P(f, s, B, G, T)$ and $P(s) = \sum_F \sum_B \sum_G \sum_T P(s, F, B, G, T) = \alpha$
$P(f|s) = \alpha P(f, s)$

$= \alpha \sum_B \sum_G \sum_T P(f, s, B, G, T)$

$= \alpha \sum_B \sum_G \sum_T P(B)P(f)P(G|B, f)P(T|B)P(s|T, f)$

$= \alpha P(f) \sum_B P(B) \sum_G P(G|B, f) \sum_T P(T|B)P(s|T, f)$

$f_1(B) = \sum_T P(T|B)P(s|T, f)$
$f_2(B) = \sum_G P(G|B, f)$
$f_3(B) = \sum_B P(B)$

$f_1(B) = \sum_T P(T|B)P(s|T, f)$
$= [P(t|B)P(s|t, f) + P(t'|B)P(s|t', f)]$

$= \begin{pmatrix} 0.03 \\ 0.98 \end{pmatrix} * 0.99 + \begin{pmatrix} 0.97 \\ 0.02 \end{pmatrix} * 0.92$
$= \begin{pmatrix} 0.9221 \\ 0.9886 \end{pmatrix}$

$f_2(B) = \sum_G P(G|B, f)$
$= [P(g|B, f) + P(g'|B, f)]$
$= \begin{pmatrix} 0.97 \\ 0.99 \end{pmatrix} + \begin{pmatrix} 0.03 \\ 0.01 \end{pmatrix} = \begin{pmatrix} 1.00 \\ 1.00 \end{pmatrix}$

$f_2(B)X\ \ f_1(B) = \begin{pmatrix} 0.9221 \\ 0.9886 \end{pmatrix} \text{X} \begin{pmatrix} 1.00 \\ 1.00 \end{pmatrix} = \begin{pmatrix} 0.9221 \\ 0.9886 \end{pmatrix}$

$f_3(B) = \sum_B P(B)$
$= \begin{pmatrix} 0.02 \\ 0.98 \end{pmatrix}$

$f_3(B)X(f_2(B) \text{ X } \mathrm{f}_1(B)) = \sum_B P(B)(f_2(B) \text{ X } \mathrm{f}_1(B)) = [p(b)X(f_2(b) \text{ X } \mathrm{f}_1(b)) + p(b')X(f_2(b')$ $\text{X } \mathrm{f}_1(b'))\ ]$

After substituting values from the conditional probability tables we get:
$P(f|s) = \alpha P(f)f_3(B)f_2(B)f_1(B)$
$= \alpha * 0.05 * \begin{pmatrix} 0.02 \\ 0.98 \end{pmatrix} * \begin{pmatrix} 1.00 \\ 1.00 \end{pmatrix} * \begin{pmatrix} 0.9221 \\ 0.9886 \end{pmatrix} = \alpha * 0.05 * \begin{pmatrix} 0.018442 \\ 0.968828 \end{pmatrix} = \alpha \begin{pmatrix} 0.0009221 \\ 0.968828 \end{pmatrix}$

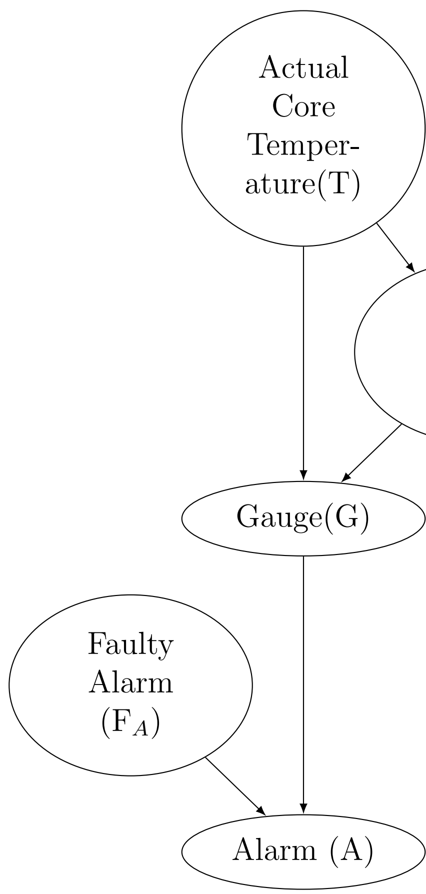
After normalizing we get:

$P(f|s) = < 0.0009221/(0.0009221 + 0.968828)\ 0.968828/(0.0009221 + 0.968828) > =$
$P(f|s) = < 0.00095\ 0.999049 >$

Hence, the probability $P(F = empty|S = false) = 0.00095$

Problem 4

(a) Bayesian network given the gauge is more likely to fail when the core temperature gets too high is:



(b) Given,
Two possible actual and measured temperatures: normal and high. Also it is mentioned that the probability that the gauge gives the correct temperature when it is working is 'x'. And the probability that the gauge gives the correct temperature when it is faulty is 'y'.

Conditional probability table for Gauge(G) is :

Conditional Probability table for Gauge(G)			
P(G)	Faulty Gauge(F_G)	Actual Core Temperature(T)	Gauge(G)
y	true	normal	normal
1-y	true	normal	high
1-y	true	high	normal
y	true	high	high
x	false	normal	normal
1-x	false	normal	high
1-x	false	high	normal
x	false	high	high

(c) Given,
Alarm works correctly unless it is faulty. If the alarm is faulty, the alarm never sounds.

Conditional probability table for Alarm(A) is :

Conditional Probability table for Alarm(A)			
P(A)	Faulty Alarm(F_A)	Gauge(G)	Alarm(A)
0	true	normal	sound
1	true	normal	no sound
0	true	high	sound
1	true	high	no sound
0	false	normal	sound
1	false	normal	no sound
1	false	high	sound
0	false	high	no sound

(d) Given,
Alarm and gauge are working and the alarm sounds. The temperature of the core is too high.
Since alarm works correctly, then : Faulty Alarm (F_A) = False and let $P(FaultyAlarm(F_A) = False) = P(f'_a)$
Also,Since gauge works correctly, then : Faulty Gauge (F_G) = False and let $P(FaultyGauge(F_G) = False) = P(f'_g)$
And also temperature of the core is too high, then : Actual Core Temperature (T) = high and let $P(T = high) = P(t)$
And alarm sounds, then : Alarm (A) = sound and let $P(A = sound) = P(a)$

So an expression for the probability that the temperature of the core is too high, in terms of the various conditional probabilities in the network can be written as:

P(t | f_a',f_g',a)

Then we can write this expression as:
$P(t|f_a$',f_g',a) = $\sum_G P(t, f_a$',f_g',a ,G) / $\sum_T \sum_G P(T, f_a$',f_g',a ,G)

We already know that the alarm and the gauge are working. So if this is the case then if the temperature of the core is high, then we can say that the gauge also shows the temperature as high.

Since the temperatue in the gauge is high,then : Gauge (G) = high and let $P(G = high) = P(g)$. Then the expression becomes :
$P(t|f_a$',f_g',a) = $P(t|f_a$',f_g',a ,g) = $P(t, f_a$',f_g',a ,g) / $P(f_a$',f_g',a ,g) = P(t ,f_a',f_g',a ,g) / $\sum_T P(T, f_a$',f_g',a ,g)

From the Bayesian network we can write the expression as:
$= P(t)P(f'_g|t)P(g|f'_g, t)P(a|g)P(f_a)/ \sum_T P(T)P(f'_g|T)P(g|f'_g, T)P(a|g)P(f_a)$

Cancelling out common terms from the numerator and denominator we get:
$= P(t)P(f'_g|t)P(g|f'_g, t)/ \sum_T P(T)P(f'_g|T)P(g|f'_g, T)$

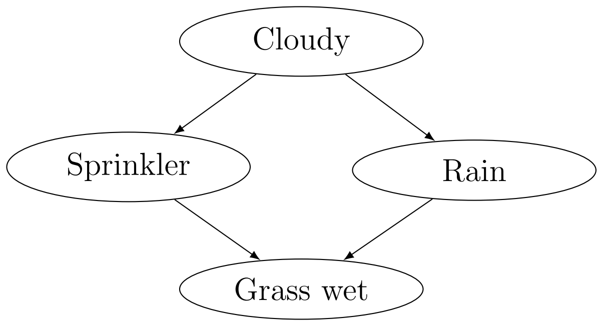
Hence the expression for the probability that the temperature of the core is too high, in terms of the various conditional probabilities in the network can be written as:

$$P(t|f_a\text{'},f_g\text{'},a) = P(t)P(f'_g|t)P(g|f'_g, t)/ \sum_T P(T)P(f'_g|T)P(g|f'_g, T)$$

Problem 5

Cloudy	
T	F
0.5	0.5

	Sprinkler	
Cloudy	T	F
T	0.10	0.90
F	0.50	0.50



	Rain	
Cloudy	T	F
T	0.80	0.20
F	0.20	0.80

Sprinkler	rain	Grass wet T	Grass wet F
T	T	0.99	0.01
T	F	0.90	0.10
F	T	0.90	0.10
F	F	0.01	0.99

(a) Conditional probability of a variable given its Markov Blanket is proportional to the probability of the variable given its parents times the probability of its children given their respective parents :

$P(x|MB(x)) = \alpha P(x|Parents(x)) \prod_{y \epsilon Children(x)} P(y|Parents(y))$

Given the evidence variables "Sprinkler" and "Wet Grass" are true:
P(Sprinkler = true) = P(s) and P(Wet Grass = true) = P(w)
Let, P(Cloudy = true) = P(c) and P(Rain = true) = P(r)

- P(c | r,w,s) = $\alpha P(c|Parents(c)) \prod_{y \epsilon Children(c)} P(y|Parents(y))$

Since 'Wet Grass' is not a child of 'Cloudy', it is not part of its Markov Blanket. Hence the probability expression is written as:

P(c | r,w,s) = P(c | r,s) = $\alpha P(c)P(r|c)P(s|c)$

Substituting with the values in the conditional probability tables we get :
P(c | r,w,s) = α * 0.50 * 0.80 * 0.10
= α 0.04

Similarly,
P(c' | r,w,s) = P(c' | r,s) = $\alpha P(c')P(r|c')P(s|c')$

Substituting with the values in the conditional probability tables we get :
P(c' | r,w,s) = α * 0.50 * 0.20 * 0.50
= α 0.05

After normalizing P(c | r,w,s) and P(c' | r,w,s) we get:

P(C | r,w,s) = $< 0.04/(0.04 + 0.05)\ 0.05/(0.04 + 0.05) > = < 0.444\ 0.555 >$

Hence,
P(c | r,w,s) = 0.444
P(c' | r,w,s) = 0.555

- P(c | r',w,s) = $\alpha P(c|Parents(c)) \prod_{y \epsilon Children(c)} P(y|Parents(y))$

  Since 'Wet Grass' is not a child of 'Cloudy', it is not part of its Markov Blanket. Hence the probability expression is written as:

  P(c | r',w,s) = P(c | r',s) = $\alpha P(c)P(r'|c)P(s|c)$

  Substituting with the values in the conditional probability tables we get :
  P(c | r',w,s) = α * 0.50 * 0.20 * 0.10
  = α 0.01

  Similarly,
  P(c' | r',w,s) = P(c' | r',s) = $\alpha P(c')P(r'|c')P(s|c')$

  Substituting with the values in the conditional probability tables we get :
  P(c' | r,w,s) = α * 0.50 * 0.80 * 0.50
  = α 0.20

After normalizing P(c | r',w,s) and P(c' | r',w,s) we get:
After normalizing we get:

P(C | r',w,s) = $< 0.01/(0.01 + 0.20)\ 0.01/(0.01 + 0.020) > = < 0.047\ 0.952 >$

Hence,
P(c | r',w,s) = 0.047
P(c' | r',w,s) = 0.952

- P(r | c,w,s) = $\alpha P(r|Parents(r)) \prod_{y \epsilon Children(r)} P(y|Parents(y))$

  P(r | c,w,s) = $\alpha P(r|c)P(w|s,r)$

  Substituting with the values in the conditional probability tables we get :
  P(r | c,w,s) = α * 0.80 * 0.99
  = α 0.792

  Similarly,
  P(r' | c,w,s) = P(r' | c,w,s) = $\alpha P(r'|c)P(w|r', s)$

  Substituting with the values in the conditional probability tables we get :
  P(r' | c,w,s) = α * 0.2 * 0.90
  = α 0.180

  After normalizing P(r | c,w,s) and P(r' | c,w,s) we get:

  P(R | c,w,s) = $< 0.792/(0.792 + 0.180)\ 0.180/(0.792 + 0.180) > = < 0.814\ 0.185 >$

  Hence,
  P(r | c,w,s) = 0.814
  P(r' | c,w,s) = 0.185

- P(r | c',w,s) = $\alpha P(r|Parents(r)) \prod_{y \epsilon Children(r)} P(y|Parents(y))$

  P(r | c',w,s) = $\alpha P(r|c')P(w|s,r)$

Substituting with the values in the conditional probability tables we get :
P(r | c',w,s) = α * 0.20 * 0.99
= α 0.198

Similarly,
P(r' | c',w,s) = P(r' | c',w,s) = $\alpha P(r'|c')P(w|r', s)$

Substituting with the values in the conditional probability tables we get :
P(r' | c',w,s) = α * 0.80 * 0.90
= α 0.720

After normalizing P(r | c',w,s) and P(r' | c',w,s) we get:

P(R | c',w,s) = $< 0.198/(0.198 + 0.720)\ 0.720/(0.198 + 0.720) > = < 0.215\ 0.784 >$

Hence,
P(r | c',w,s) = 0.215
P(r' | c',w,s) = 0.784

(b) Programming Assignment.