Assignment-1

Q.1

In this case, Alice wins 11 matches while Bob wins 88 matches. To calculate total probability, we have to sum all possible cases.

Initial case: Alice (A) wins first round and Bob (B) wins second round. SInce both are attacking, probability of draw is 0.

$$T_1 T_2 = 11 \qquad T_3 T_4 = 88$$

$$T = 99$$

$$P\left(\begin{matrix} A:11W \\ \& B:88W \end{matrix} \,\middle|\, T=99\right) = \frac{P\left(\begin{matrix} & \& B:88W \\ A:11W & \cap T=99 \end{matrix}\right)}{P(T=99)}$$

1) B wins 1 time after A

For example, considering cases where a block of 10 consecutive wins by A is taken at once (AAAAAAAAAA),

$$\overset{+1}{A}\ \overset{+1}{B}\ \underbrace{\overset{+1}{A}\overset{+1}{A}\ldots\overset{+1}{A}}_{10}\ \underbrace{\overset{+1}{B}\overset{+1}{B}\ldots\overset{+1}{B}}_{87}$$

$$\left(\frac{1}{1}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\cdots\left(\frac{1}{11}\right)\left(\frac{11}{12}\right)\left(\frac{11}{13}\right)\cdots\left(\frac{11}{98}\right) \qquad \frac{11^{87}}{98!}$$

Now, we take

1) fixed block of 9 consecutive wins (only) by A.

S = $$\sum_{i=0}^{86} (i+2)^9 . 2^{i+1} . 11^{86-i}$$

2) fixed block of 8 consecutive wins (only) by A

$$S = \sum_{i=0}^{86} 3^{i+1} . (i+2)^8 . 11^{86-i}$$

Therefore, we observe a pattern.

$$A B A \underbrace{B B - - - B}_{87} \underbrace{A A - - - A}_{9}$$

$$\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{2}{4}\right) - - - \left(\frac{2}{89}\right)\left(\frac{88}{90}\right)\left(\frac{88}{90}\right) - - - \left(\frac{88}{98}\right)$$

$$= \frac{2^{87} . 88^{9} . 11^{0}}{98!}$$

$$A B \underline{A} \underline{A} \underbrace{B B - - - B}_{87} \underbrace{A A - - - A}_{8}$$

$$\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{3}{\cancel{4}}\right)\left(\frac{3}{5}\right) - - - \left(\frac{3}{90}\right)\left(\frac{88}{9\cancel{1}}\right) - - - \left(\frac{88}{98}\right)$$

$$= \frac{3^{87} . 88^{8} . 11^{0}}{98!}$$

If we go till just 1 consecutive win by A.

$$S = \sum_{j=2}^{10} \sum_{i=0}^{86} (j)^{i+1} (i+2)^{11-j} . 11^{86-i}$$

By dividing S by 98!, we get probability of this specific case.

$$A B A A \underbrace{B B - - - B}_{86} \underbrace{A A - - - A}_{8} B$$

$$\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{3}{4}\right) - - - \left(\frac{3}{89}\right)\left(\frac{8\cancel{8}}{90}\right) - - - \left(\frac{87}{97}\right)\left(\frac{11}{98}\right)$$

$$= \frac{3^{86} . 87^{8} . 11^{1}}{98!}$$

$$A B A A \underbrace{B B - - - B}_{85} \underbrace{A A - - - A}_{8} B B : \frac{3^{85} . 86^{8} . 11^{2}}{98!}$$

$$A B A A B \underbrace{A A \ldots A}_{8} \underbrace{B B \ldots B}_{86}$$

$$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{2}{5} \cdot \frac{2}{6} \cdots \frac{2}{12} \cdot \frac{11}{13} \cdots \frac{11}{98}$$

$$= \frac{3^1 \cdot 2^8 \cdot 11^{86}}{98!}$$

$$A B \underline{B} A \underline{B} \underbrace{A A \ldots A}_{9} \underbrace{B B \ldots B}_{85}$$

$$\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{2}{4}\right)\left(\frac{3}{5}\right) \cdots \left(\frac{3}{13}\right)\left(\frac{11}{14}\right) \cdots \left(\frac{11}{98}\right)$$

$$= \frac{2 \cdot 2^1 \cdot 3^9 \cdot 11^{85}}{98!} = \frac{2 \cdot 2^1 \cdot 3^9 \cdot 11^{85}}{98!}$$

2) 2 wins by B after A

Similarly, taking fixed blocks of 9, 8, …. 1 consecutive wins by A,

$$\overset{+1}{A} \underbrace{B B}_{2} \underbrace{A A \ldots A}_{10} \underbrace{B B \ldots B}_{86}$$

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{2}{4}\right) \cdots \left(\frac{2}{12}\right)\left(\frac{11}{13}\right) \cdots \left(\frac{11}{98}\right)$$

$$= \frac{2^{10} \cdot 11^{86}}{98!}$$

$$A\,\underline{B}\,\underline{B}\,A\,\underline{B}\,\underbrace{AA\cdots A}_{9}\,\underbrace{BB\cdots B}_{85}$$

$$\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{2}{4}\right)\left(\frac{3}{5}\right)\cdots\left(\frac{3}{13}\right)\left(\frac{11}{14}\right)\cdots\left(\frac{11}{98}\right)$$

$$= \frac{2.2^1.3^9.11^{85}}{98!} = \frac{2.2^1.3^9.11^{85}}{98!}$$

$$A\,\underline{B}\,\underline{B}\,\underline{A}\,BB\,\underbrace{AA\cdots A}_{9}\,\underbrace{BB\cdots B}_{84}$$

$$\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{2}{4}\right)\left(\frac{2}{5}\right)\left(\frac{4}{6}\right)\cdots\left(\frac{4}{14}\right)\left(\frac{11}{15}\right)\cdots\left(\frac{11}{98}\right)$$

$$= \frac{2^2.2^2.4^9.11^{84}}{98!}$$

$$A\,\underline{B}\,\underline{B}\,\underline{A}\,BBB\,\underbrace{AA\cdots A}_{9}\,\underbrace{BB\cdots B}_{83}$$

$$\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{2}{4}\right)\left(\frac{2}{5}\right)\left(\frac{2}{6}\right)\left(\frac{5}{7}\right)\cdots\left(\frac{5}{15}\right)\left(\frac{11}{16}\right)\cdots\left(\frac{11}{98}\right)$$

$$= \frac{2^1.2^3.5^9.11^{83}}{98!}$$

3) 3 wins by B after A

$$\overset{+1}{\underbrace{A}_{1}}\quad \overset{+1\,+1\,+1}{\underbrace{BBB}_{3}}\quad \underbrace{AA\ldots A}_{10}\quad \underbrace{BB\ldots B}_{85}$$

$$\left(\frac{1}{1}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)\left(\frac{3}{5}\right)\cdots\left(\frac{3}{13}\right)\left(\frac{11}{14}\right)\cdots\left(\frac{11}{98}\right)$$

$$= \frac{3^{10}\cdot 11^{85}}{98!}$$

…. 88) 88 wins by B after A.

$$\overset{+1}{\underbrace{A}_{1}}\quad \overset{+1\,+1}{\underbrace{BB\ldots B}_{88}}\quad \underbrace{AA\ldots A}_{10}$$

$$\left(\frac{1}{1}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\cdots\left(\frac{1}{88}\right)\left(\frac{88}{89}\right)\left(\frac{88}{90}\right)\cdots\left(\frac{88}{98}\right)$$

$$= \frac{(88)^{10}\ 88^{0}}{98!}$$

$$S = 11^{87} + 2^{10}\cdot 11^{86} + 3^{10}\cdot 11^{85} + \cdots + 87^{10}\cdot 11^{1} + 88^{10}\, 11^{0}$$

So, final probability would be S divided by 98!

The probability of each scenario can be generalised as $i^{10}\ 11^{88-i}$, where i signifies the number of matches won by B after A (second round onwards).

However, not all cases can be accounted by manual analysis.

Hence, we might simulate all $^{99}C_{11}$ cases in the code. By computing all possible permutations and updating values of nA and nB for each case, the total probability may be calculated. However, this would take prohibitively long. Hence, we can simulate each round and compute approximate probability through Monte Carlo by iterating over 10^4 possibilities.

1 b) We can sum conditional probability for each round, since

$x_i = 1$ A W

$x_i = 0$ D

$x_i = -1$ AL

$$E\left[\sum_{i=1}^{T} x_i\right] = \sum_{i=1}^{T} E(x_i) = \sum_{i=1}^{T} x_i P(X = x_i)$$

$$= \sum_{j=1}^{T} x_j P(X = x_j)$$

We assume that all events are independent. Hence,

$$P\left(A\right) = \sum_{i} P\left(A\,||B_i\right)P\left(B_i\right)$$

For example, at 6th round,

$$P(A|ABAAA\_) = \frac{P(A \cap ABAAA\_)}{P(ABAAA\_)} = \frac{P(ABAAA\_|A)P(A)}{P(ABAAA\_)}$$

$$P(B_J|A) = \frac{P(A|B_J)P(B_J)}{\sum P(A|B_i)P(B_i)} \Rightarrow \frac{P(B_J|A)}{P(B_J)} = \frac{P(A|B_J)}{\sum P(A|B_i)P(B_i)}$$

A, B _ _ _ A (+1) B (−1)

$${}^3C_3 + {}^3C_2 + {}^3C_1 + {}^3C_0 = 2^3$$

$$P(A|ABAAA) = \frac{1}{5} \quad ; \quad P(B|ABAAA\_) = \frac{4}{5}$$

$$P(A|B_J) = \frac{P(A)\, P(A|B_J)}{\sum P(A|B_i)\, P(B_i)} \Rightarrow P(A) = \sum P(A|B_i) P(B_i)$$

$$P(A|ABAAB\_) = \frac{2}{5} \qquad \frac{1}{8}\left[\frac{1}{5}\, {}^{3}C_{3} + \frac{2}{5}\, {}^{3}C_{2} + \frac{3}{5}\, {}^{3}C_{1} + \frac{4}{5}\, {}^{3}C_{0}\right]$$

$4-3 \quad 4-2 \quad 4-1 \quad 4-0$

Page..........................

$$2^4 = 2^{7-3}$$

$$\sum P(ABAAAA\_) P(A|\underbrace{ABAAAA}_{7}\_) = \left(\frac{1}{16} \cdot {}^{4}C_{4} \cdot \frac{1}{6}\right.$$

$5-4 \to (7-2)-4$; $5-1$

$$\sum P(ABAAAB\_) P(A|ABAAAB\_) = \frac{1}{16} \cdot {}^{4}C_{3}\, \frac{2}{6}$$

$8-3$; $5-3 \to (7-2)-3$; $5-2$; $(7-2)-2$

III

$$P(A|AB\_) \Rightarrow \frac{\cancel{2}1}{\cancel{2}} \cdot {}^{2}C_{0}\, \frac{1}{2} + \frac{\cancel{2}1}{\cancel{2}} \cdot \frac{1}{2}(-1) = 0$$

$$P(B|AB\_)$$

IV

$4-3 \to {}^{3-2}C_{1}$; $2-1 \to (3-1)-1$

$$P(A|ABA\_) \Rightarrow {}^{1}C_{1}\, \frac{1}{2} \cdot \frac{1}{3}\cancel{(+1)} = \frac{1}{6} \qquad \frac{1}{2}$$

${}^{3-2}C_{0}$; $2-0 \to (3-1)-0$

$$P(A|ABB\_) \Rightarrow {}^{1}C_{0}\, \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

$(4-2)-0$

$$P(B|ABA\_) = {}^1C_1 \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

$$P(A|ABAA\_) \Rightarrow \frac{1}{2^2} \cdot {}^2C_0 \cdot \frac{1}{4} = \frac{1}{16}$$

$$P(A|ABBA\_) \Rightarrow \frac{{}^2C_1}{2^2} \cdot \frac{\cancel{2}}{4} = \frac{1}{4}$$

$$P(A|ABBB\_) \Rightarrow \frac{{}^2C_2}{2^2} \cdot \frac{3}{4} = \frac{3}{16}$$

$$\boxed{\frac{1}{2}}$$

Since all events are independent,
$Cov(X_i, X_j) = 0$

Hence, $$Var\left(\sum_i X_i\right) = \sum_i Var(X_i) = \sum_i \left(E\left(X_i^2\right) - E^2(X_i)\right)$$

2. a)

		Bob		
		Attack	*Balanced*	*Defence*
	Attack	$(\frac{n_B}{n_A+n_B}, 0, \frac{n_A}{n_A+n_B})$	$(\frac{7}{10}, 0, \frac{3}{10})$	$(\frac{5}{11}, 0, \frac{6}{11})$
Alice	*Balanced*	$(\frac{3}{10}, 0, \frac{7}{10})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$(\frac{3}{10}, \frac{1}{2}, \frac{1}{5})$
	Defence	$(\frac{6}{11}, 0, \frac{5}{11})$	$(\frac{1}{5}, \frac{1}{2}, \frac{3}{10})$	$(\frac{1}{10}, \frac{4}{5}, \frac{1}{10})$

We defined payoff_matrix:

```
prob={'Attack':([nA/(nA+nB), 0, nB/(nA+nB)], [0.7, 0, 0.3], [5/11, 0, 6/11]),
        'Balanced': ([0.3, 0, 0.7], [1/3, 1/3, 1/3], [0.3, 0.5, 0.2]),
        'Defensive': ([6/11, 0, 5/11], [0.2, 0.5, 0.3], [0.1, 0.8, 0.1])}
```

We could also define it as expectation value for Alice.

If Bob won the previous round, then he chooses to play defensively.
If the previous round resulted in a draw, then he plays balanced.
If he lost the previous round, then he plays aggressively.

Therefore, at every step, Alice may choose the step that maximises either expectation value, or the probability of winning.

b) For non-greedy strategy, many mixed strategies can be chosen, for example Attack: 0.4, Balanced: 0.25, and Defence: 0.35, since Attack usually has highest probability of winning than Defence, followed by Balanced. However, we noticed that greedy strategy usually works more effectively.

c) To estimate E[tau], we defined function simulate_round and iterated over it 10^5 times, where the expectation is incremented by 1 while the number of wins by Alice is less than 88.

3. a) To maximise her points, Alice may choose the strategy that maximises the expectation value i.e. 1*v[Bob().play_move()][0]+0.5*v[Bob().play_move()][1].

b) For deciding optimal strategy, Alice can analyse the potential future expectation value and the current expectation value to select the one strategy that will yield the maximum points.

Based on the random choice that Bob makes, points can be incremented and the expectation value then analysed.