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Assignment-2: Ruin to Returns

Q.1 (a)
Transition matrix: (for N=4)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let P_k be the probability of winning (reaching N) when current wealth is k.
Therefore, we get the following recursive relation
$P_k = pP_{k+1}+qP_{k-1}$,
where boundary conditions are $P_0 = 0$ and $P_N = 1$.
Assuming solution to be of the form $P_k = ar^k + b$, we get a = $\frac{1}{r^N-1}$,

r = $\frac{q}{p}$ $(solving\ quadratic\ equation:\ roots\ =\ \frac{q}{p}\ and\ 1\ (rejected))$ and b = $-\frac{1}{r^N-1}$

Therefore, $P_k = \frac{1-(\frac{q}{p})^k}{1-(\frac{q}{p})^N}$. If p = q = 0.5, then by L'Hopital's Rule $P_k = \frac{k}{N}$.

(b) The limiting probability (N tends to infinity) comes out to be $P_k = 1 - (\frac{q}{p})^k$, only when q<p as $(q/p)^N$ tends to zero.

For code, I used matrix method, i.e. (I-B)P = A, where P = $(I-B)^{-1}A$ gives the probabilities for non-absorbing states
The transition matrix (after removing absorbing states) becomes

$$B = \begin{array}{ccc} 1 & 2 \ldots & N-1 \\ 0 & p & 0 \\ q & 0 & p \\ 0 & q & 0 \end{array} \qquad A = \begin{array}{cc} 0 & N \\ \hline q & 0 \\ 0 & 0 \\ 0 & p \end{array}$$

This gives values of P_k (both winning ($P_k[1]$) and losing ($P_k[0]$))

(c) For game duration we can calculate the expectation value of the number of rounds until the gambler wins (reaches N) or loses (reaches 0). The recurrence relation for the same comes out to be
$E_k = 1 + pE_{k+1} + qE_{k-1}$, with boundary conditions $E_0 = 0$ and $E_N = 0$.
Solving homogeneous part i.e. $E_k = pE_{k+1}+qE_{k-1}$, and then for particular solution,

For p not equal to q, $E_k = \frac{N}{p-q}\frac{1-(\frac{q}{p})^k}{1-(\frac{q}{p})^N} + \frac{k}{q-p}$,

For p = q: $E_k = Nk - k^2$

For code,
$(I - F)E = (1)_N$ (column vector of 1's), where F is the transition matrix excluding the absorbing states.
Hence, $E = (I - F)^{-1}(1)_{N-1}$
$(k-1)^{th}$ element of E gives game duration when initial wealth is k.

P_T =

$$
\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} & N \\
\hline
0 & 1 & 0 & 0 & & & & & 0 \\
1 & q & 0 & p & 0 & 0 \;\ldots & 0 \;\ldots & & 0 \\
2 & 0 & q & 0 & p & 0 \;\ldots & 0 \;\ldots & & 0 \\
. & 0 & 0 & q & 0 & p \;\ldots & 0 \;\ldots & & 0 \\
. & & & & & & & & \\
. & & & & & & & & \\
. & & & & & & & & \\
N & 0 & 0 & 0 & 0 & \multicolumn{3}{c}{\ldots\ldots 0 \ldots\ldots} & 1
\end{array}
$$

B/F =

$$
\begin{array}{c|cccccc}
 & 1 & 2 & \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} & N-1 \\
\hline
1 & 0 & p & 0 & 0 \;\ldots & 0 \;\ldots & 0 \\
2 & q & 0 & p & 0 \;\ldots & 0 \;\ldots & 0 \\
. & 0 & q & 0 & p \;\ldots & 0 \;\ldots & 0 \\
. & & & & & & \\
. & & & & & & \\
. & & & & & & \\
N-1 & 0 & 0 & 0 & \ldots\ldots q & 0 & p
\end{array}
$$

A =

	0	N
0	1	0
1	q	0
2	0	0
.		
.		
N-1	0	p
N	0	1

2. (a) The transition matrix will be defined as P[i][0]=q, P[i][2*i]=p, for i less than ceiling value of N/2, for i more than ceiling value of N/2, p1[i][2*i-N]=q, p1[i][N]=p.
k-(N-k) k+(N-k)

The recurrence relation for the same is $P_k = qP_0+pP_{2k} = pP_{2k}$ for k<ceil(N/2).
$P_k = qP_{2k-N}+pP_N = p+qP_{2k-N}$ for k >=ceil(N/2).

Here, $P = (I-B)^{-1}A$, where B is the transition matrix excluding absorbing states, and A is the matrix mapping non-absorbing states to absorbing states.

P_T =

	0	1	2	………………			N
0	1	0	0	0	0 …..	0 …..	0
1	q	0	p	0	0 …..	0 …..	0
2	q	0	0	0	p …..	0 …..	0
N/2	q	0	0	0	0 …..	0 …..	p
N/2+1	0	0	q	0	0		p
.							p
.							.
N	0	0	0	0	……….0……		1

B =

	1	2	………………			N-1
1	0	p	0	0 …..	0 …..	0
2	q	0	0	p …..	0 …..	0
.	q	0	0	0 ..p..	0 …..	0
.						
.						
.						
N-1	0	0	0	……	q 0	p

A =

	0	N
0	1	0
1	q	0
2	q	0
.	..	…
.	..	p
N-1	0	p
N	0	1

(b) The recurrence relation for the expected duration of the game is

$E_k = 1+qE_0 + pE_{2k} = 1+pE_{2k}$ for k<ceil(N/2).

$E_k = 1+qE_{2k-N} + pE_N = 1+p+qE_{2k-N}$ for k>=ceil(N/2).

We can solve for E by $E = (I - F)^{-1}(1)_{N-1}$

Where F is the transition matrix excluding the absorbing states. The value corresponding to k (or k-1 index) will give the expected duration of game when initial wealth is k.

3. In this case, $E_i = 1+qE_{i-1}+pE_{i+1}$ for $i<k+W$.

$E_t = 0$, $E_{k+W} = 1+E_{k+W-1}$

where E is expected number of rounds before the game ends (or gambler is ruined).

To define the transition matrix, number of rows and columns will be k+W (maximum wealth), while probability at t^{th} row and t^{th} column will be 1, and values of all other elements in the t^{th} row become 0. Hence, 0 and t are absorbing states, and we exclude these values in F.

P_T =

	0	1	2	……… t……			k+W-1	k+W
0	1	0	0				0	0
1	q	0	p	0	0 …..	0 …..	0	0
2	0	q	0	p	0 …..	0 …..	0	0
.	0	0	q	0	p …..	0 …..	0	0
.								
t	0	0	0	0	1	0	0	0
.								
k+W	0	0	0	0	……….0……		1	0

F = (excludes 0^{th} and t^{th} row and column)

		1	2	…………			k+W-1	k+W
0 ->	1	0	p	0	0 …..	0 …..	0	0
	2	q	0	p	0 …..	0 …..	0	0
	.	0	q	0	p …..	0 …..	0	0
k-2 ->	k							
	.							
	k+W	0	0	0	……….0……		1	0

k^{th} row becomes $(k-2)^{th}$ row since 0 and t<k row have been removed.

4. (a) The stationary distribution can be calculated by $\pi = \pi P$, or $\pi(P\text{-}I) = 0$ or $(P^T\text{-}I)\pi = 0$.

By finding eigenvector corresponding to eigenvalue 1 subject to $\sum \pi = 1$, we can find stationary distribution. I implemented this in python through singular value decomposition in numpy, where matrix A = $\mathbf{U\Sigma V^T}$, and eigenvector is given by column vector of matrix V^T corresponding to eigenvalue 1, and then normalised the values of V by dividing each term in the solution by the sum of all the elements, yielding π.

The transition matrix is:

	0	1	2	………………			N
0	r[0]	p[0]	0				0
1	q[1]	r[1]	p[1]	0	0	….. 0 …..	0
2	0	q[2]	r[2]	p[2]	0	….. 0 …..	0
.	0	0	q[3]	r[3]	p[3]	…..0 …..	0
.							
.							
.							
N	0	0	0	0	……….0..	q[N]	r[N]

(b) The recurrence relation for expected amount of time will now become:
$E_k = 1 + p[k]\, E_{k+1} + q[k]\, E_{k-1} + r[k]\, E_k$

Here, $E = (I\text{-}F)^{-1}B$, since there are no absorbing states, and we will consider matrix only till wealth b-1. The index corresponding to a in E column vector corresponds to the expected number of steps from a to b.
Here, column vector B of length b consists of 1.

$E_k - (p[k]\, E_{k+1} + q[k]\, E_{k-1} + r[k]\, E_k) = 1$
In matrix form:
$E - FE = (1)_N$ where F is transition matrix (probability matrix).
$(I - F)E = (1)_N$

F =

	0	1	2	………………					b-1	b
0	r[0]	p[0]	0						0	0
1	q[1]	r[1]	p[1]	0	0	…..	0	…..	0	0
2	0	q[2]	r[2]	p[2]	0	…..	0	…..	0	0
.	0	0	q[3]	r[3]	p[3]	…..	0	…..	0	0
.										
.										
.										
b-1	0	0	0	0	……….		0..	q[b-1]	r[b-1]	p[b-1]